Supplemental material: Qubit decoherence induced by master clock instabilities

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I. DEFINEING THE PSD & CONNECTING PHASE AND FREQUENCY NOISE

A. Bilateral PSD Definition - $[0, T]$ Window Convention

Let $y(t)$ be a continuous-time real-valued stochastic process, and assume

(a) the ensemble mean is zero everywhere: $\mathbb{E}\left[y(t)\right] = 0, \forall t$

(b) $y(t)$ is ergodic

(c) $y(t)$ is wide-sense stationary: the two-sided ensemble autocorrelation function $C_y(\tau)$ is invariant under time-translations of τ

By ergodic we mean that, in the limit of infinite ensemble sizes, the ensemble mean of $y(t)$ – or the ensemble mean of a function of $y(t)$ – approaches the corresponding *time-average* over a given instance of $y(t)$:

$$\mathbb{E}\left[f(y(t))\right] \rightarrow \langle f(y(t))\rangle_t \tag{1}$$

In particular, the autocorrelation function may therefore be specified either in terms of the ensemble average, or the time average:

$$C_y(\tau) = \mathbb{E}\left[y(t)y(t+\tau)\right] = \langle y(t)y(t+\tau)\rangle_t \tag{2}$$

where the function is described completely by the time difference τ due to the assumption of wide-sense stationarity. Define the (one-sided) truncated Fourier transform of $y(t)$ by

$$Y_T(\omega) = \int_0^T y(t)e^{-i\omega t}dt \tag{3}$$

using the *angular frequency, non-unitary* convention for Fourier transforms. The power spectral density (PSD) of $y(t)$ is then defined

$$S_y(\omega) = \lim_{T\rightarrow\infty}\frac{1}{T}\mathbb{E}\left[|Y_T(\omega|^2\right] \tag{4}$$

This definition is used to avoid the problem that the integral of $y(t)$ for a single run need not will converge over $t \in [0, \infty)$. From the above definitions we get

$$\begin{aligned}
S_y(\omega) &= \lim_{T\rightarrow\infty}\frac{1}{T}\mathbb{E}\left[\int_0^T y^*(t)e^{i\omega t}dt\int_0^T y(t')e^{-i\omega t}dt'\right] && (5)\\
&= \lim_{T\rightarrow\infty}\frac{1}{T}\int_0^T\int_0^T \mathbb{E}\left[y(t)y(t')\right]e^{i\omega(t-t')}dtdt' && (6)\\
&= \lim_{T\rightarrow\infty}\frac{1}{T}\int_0^T\int_0^T C_y(\tau)e^{-i\omega\tau}dtdt' && (7)
\end{aligned}$$

Here we have used the fact that $y(t)$ is real, and defined $\tau = t' - t$. Transforming the integral domain, we get

$$S_y(\omega) = \lim_{T\to\infty} \frac{1}{T} \int_{-T}^{T} (T-\tau) C_y(\tau) e^{-i\omega\tau} d\tau \tag{8}$$

$$= \lim_{T\to\infty} \int_{-T}^{T} C_y(\tau) e^{-i\omega\tau} d\tau + \lim_{T\to\infty} \frac{1}{T} \int_{-T}^{T} \tau C_y(\tau) e^{-i\omega\tau} d\tau \tag{9}$$

However the second term vanishes in the limit $T \to \infty$, yielding

$$S_y(\omega) = \int_{-\infty}^{\infty} C_y(\tau) e^{-i\omega\tau} d\tau \qquad \Longleftrightarrow \qquad C_y(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_y(\omega) e^{i\omega\tau} d\omega \tag{10}$$

That is, $S_y(\omega)$ and $C_y(\tau)$ form a Fourier transform pair, which is a statement of the Wiener-Khintchine theorem.

B. Bilateral PSD Definition - $[-T, T]$ Window Convention

The definition of the bilateral PSD may be formulated also with a symmetric window for the truncated Fourier transform:

$$Y_T(\omega) = \int_{-T}^{T} y(t) e^{-i\omega t} dt \tag{11}$$

The power spectral density (PSD) of $y(t)$ is then defined

$$S_y(\omega) = \lim_{T\to\infty} \frac{1}{2T} \mathbb{E}\left[|Y_T(\omega|^2 \right] \tag{12}$$

Hence

$$S_y(\omega) = \lim_{T\to\infty} \frac{1}{2T} \mathbb{E}\left[\int_{-T}^{T} y^*(t) e^{i\omega t} dt \int_{-T}^{T} y(t') e^{-i\omega t} dt' \right] \tag{13}$$

$$= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \int_{-T}^{T} \mathbb{E}\left[y(t) y(t') \right] e^{i\omega(t-t')} dt dt' \tag{14}$$

$$= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \int_{-T}^{T} C_y(\tau) e^{-i\omega\tau} dt dt' \tag{15}$$

again using the fact that $y(t)$ is real, and defined $\tau = t' - t$. Transforming the integral domain, we get

$$S_y(\omega) = \lim_{T\to\infty} \frac{1}{2T} \int_{-2T}^{2T} (2T - |\tau|) C_y(\tau) e^{-i\omega\tau} d\tau \tag{16}$$

$$= \lim_{T\to\infty} \int_{-2T}^{2T} C_y(\tau) e^{-i\omega\tau} d\tau + \lim_{T\to\infty} \frac{1}{2T} \int_{-2T}^{2T} |\tau| \, C_y(\tau) e^{-i\omega\tau} d\tau \tag{17}$$

The second term vanishes in the limit $T \to \infty$, yielding

$$S_y(\omega) = \lim_{T\to\infty} \int_{-2T}^{2T} C_y(\tau) e^{-i\omega\tau} d\tau = \int_{-\infty}^{\infty} C_y(\tau) e^{-i\omega\tau} d\tau \tag{18}$$

and again we recover the Fourier transform pair:

$$S(\omega) = \int_{-\infty}^{\infty} C_y(\tau) e^{-i\omega\tau} d\tau, \qquad C_y(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_y(\omega) e^{i\omega\tau} d\omega \tag{19}$$

C. Two-Sided and One-Sided PSDs

The power spectral density formulated in Eq. 12 is by definition a non-negative even function whose domain is all real $\omega \in (-\infty, \infty)$. This can also be seen from the Wiener-Khintchine theorem Eq. 18 and observing the autocorrelation function $C_y(\tau)$ is invariant in the sign of τ (*i.e.* an even function). This is known as the *two-sided* PSD, with the total mean-square fluctuation given by the total integrated power over both positive and negative frequencies

$$P_{\text{tot}} = \mathbb{E}\left[y(t)^2\right] = C_y(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_y(\omega)d\omega = \int_{-\infty}^{\infty} S_y(\nu)d\nu, \qquad \omega = 2\pi\nu \tag{20}$$

The two-sided PSD is useful in pure mathematical analysis involving Fourier transformations, and satisfies the Wiener-Khinchine theorem. However the *one-sided* PSD is more typically used in engineering and metrological applications. For clarity, in this document we use the following notation to keep the usage of one- and two-sided PSDs distinct:

$$S_y^{(2)}(\omega) \qquad \text{two-sided PSD} \tag{21}$$

$$S_y^{(1)}(\omega) \qquad \text{one-sided PSD} \tag{22}$$

where the one-sided PSD is defined by

$$S_y^{(1)}(\omega) = \begin{cases} 2S_y^{(2)}(\omega) & \text{for } \omega \geq 0 \\ 0 & \text{for } \omega < 0 \end{cases} \tag{23}$$

With these definitions the total power in the signal is given by

$$P_{\text{tot}} = \int_{-\infty}^{\infty} S_y^{(2)}(\nu)d\nu = 2\int_{0}^{\infty} S_y^{(2)}(\nu)d\nu = \int_{0}^{\infty} S_y^{(1)}(\nu)d\nu \tag{24}$$

D. PSDs for Phase and Frequency Noise

Consider a local oscillator

$$\Omega(t)\cos[\omega_{\rm LO}t+\phi_N(t)] \tag{25}$$

with instantaneous frequency

$$\omega(t)\equiv\frac{d}{dt}\big[\omega_{\rm LO}t+\phi_N(t)\big]=\omega_{\rm LO}+\dot{\phi}_N(t) \tag{26}$$

or

$$\omega(t)=\omega_{\rm LO}+\delta\nu(t) \tag{27}$$

where the (angular) frequency noise takes the form

$$\delta\nu(t)=\dot{\phi}_N(t) \tag{28}$$

1. *PSD using the* $[0,T]$ *Window Convention*

Following Eq. 3 define the truncated Fourier transforms for phase and frequency noise fields

$$Y_{T,\phi_N}(\omega)=\int_0^T\phi_N(t)e^{-i\omega t}dt,\qquad Y_{T,\delta\nu}(\omega)=\int_0^T\delta\nu(t)e^{-i\omega t}dt \tag{29}$$

Then the phase and frequency noise PSD are defined by

$$S_{\phi_N}(\omega)=\lim_{T\to\infty}\frac{1}{T}\mathbb{E}\left[|Y_{T,\phi_N}(\omega|^2\right],\qquad S_{\delta\nu}(\omega)=\lim_{T\to\infty}\frac{1}{T}\mathbb{E}\left[|Y_{T,\delta\nu}(\omega|^2\right] \tag{30}$$

But using the definition of $Y_{T,\phi_N}(\omega)$; the fact that $\delta\nu(t)=\dot{\phi}_N(t)$; and integrating by parts we find

$$Y_{T,\delta\nu}(\omega)=i\omega Y_{T,\phi_N}(\omega)+R,\qquad R\equiv\phi_N(T)e^{-i\omega T}-\phi_N(0) \tag{31}$$

where the residual term R comes from the integration limits after integrating by parts. Consequently

$$\left|Y_{T,\delta\nu}(\omega)^2\right|=\omega^2\left|Y_{T,\phi_N}(\omega)\right|^2+R' \tag{32}$$

where

$$R'=|R|^2+[i\omega Y_{T,\phi_N}(\omega)R^*]+\left[-i\omega Y^*_{T,\phi_N}(\omega)R\right] \tag{33}$$

contains cross terms due to taking the modulus square of $i\omega Y_{T,\phi_N}(\omega)+R$. Consequently

$$S_{\delta\nu}(\omega)=\lim_{T\to\infty}\frac{1}{T}\mathbb{E}\left[\omega^2\left|Y_{T,\phi_N}(\omega)\right|^2+R'\right] \tag{34}$$

$$=\omega^2\left\{\lim_{T\to\infty}\frac{1}{T}\mathbb{E}\left[\left|Y_{T,\phi_N}(\omega)\right|^2\right]\right\}+\lim_{T\to\infty}\frac{1}{T}\mathbb{E}\left[R'\right] \tag{35}$$

$$\tag{36}$$

However, unlike the term as $|Y_{T,\phi_N}(\omega)|^2$, the residual term involving R' does not grow fast enough with T to be non-vanishing in the limit and disappear after $T\to\infty$. We therefore set

$$\lim_{T\to\infty}\frac{1}{T}\mathbb{E}\left[R'\right]=0 \tag{37}$$

from which it follows

$$S_{\delta\nu}(\omega) = \omega^2 S_{\phi_N}(\omega) \tag{38}$$

Now define a new noise field

$$\beta(t) \equiv \alpha\delta\nu(t) = \alpha\dot{\phi_N}(t) \tag{39}$$

by scaling the frequency noise by some factor α. Substituting this into the above derivation we obtain the slightly more general PSD relation

$$S_\beta(\omega) = \alpha^2 S_{\delta\nu} = \alpha^2\omega^2 S_{\phi_N}(\omega) \tag{40}$$

These relations apply equally to two-sided and one-sided PSDs. Hence, we have omitted this specification in the result.

2. *PSD using the $[-T, T]$ Window Convention*

Using the symmetric window convention:

$$Y_{T,\phi_N}(\omega) = \int_{-T}^{T} \phi_N(t)e^{-i\omega t}dt, \qquad Y_{T,\delta\nu}(\omega) = \int_{-T}^{T} \delta\nu(t)e^{-i\omega t}dt \tag{41}$$

Then the phase and frequency noise PSD are defined by

$$S_{\phi_N}(\omega) = \lim_{T\to\infty} \frac{1}{2T}\mathbb{E}\left[\left|Y_{T,\phi_N}(\omega)\right|^2\right], \qquad S_{\delta\nu}(\omega) = \lim_{T\to\infty} \frac{1}{2T}\mathbb{E}\left[\left|Y_{T,\delta\nu}(\omega)\right|^2\right] \tag{42}$$

But using the definition of $Y_{T,\phi_N}(\omega)$; the fact that $\delta\nu(t) = \dot{\phi}_N(t)$; and integrating by parts we find

$$Y_{T,\delta\nu}(\omega) = i\omega Y_{T,\phi_N}(\omega) + R, \qquad R \equiv \phi_N(T)e^{-i\omega T} - \phi_N(-T)e^{i\omega T} \tag{43}$$

where the residual term R comes from the integration limits after integrating by parts. Consequently

$$\left|Y_{T,\delta\nu}(\omega)^2\right| = \omega^2\left|Y_{T,\phi_N}(\omega)\right|^2 + R' \tag{44}$$

where

$$R' = |R|^2 + \left[i\omega Y_{T,\phi_N}(\omega)R^*\right] + \left[-i\omega Y^*_{T,\phi_N}(\omega)R\right] \tag{45}$$

contains cross terms due to taking the modulus square of $i\omega Y_{T,\phi_N}(\omega) + R$. Consequently

$$S_{\delta\nu}(\omega) = \lim_{T\to\infty} \frac{1}{2T}\mathbb{E}\left[\omega^2\left|Y_{T,\phi_N}(\omega)\right|^2 + R'\right] \tag{46}$$

$$= \omega^2\left\{\lim_{T\to\infty} \frac{1}{2T}\mathbb{E}\left[\left|Y_{T,\phi_N}(\omega)\right|^2\right]\right\} + \lim_{T\to\infty}\frac{1}{2T}\mathbb{E}\left[R'\right] \tag{47}$$

$$\tag{48}$$

However, unlike the term as $|Y_{T,\phi_N}(\omega)|^2$, the residual term involving R' does not grow fast enough with T to be non-vanishing in the limit $T \to \infty$. We therefore set

$$\lim_{T\to\infty}\frac{1}{2T}\mathbb{E}\left[R'\right] = 0 \tag{49}$$

from which it follows

$$S_{\delta\nu}(\omega) = \omega^2 S_{\phi_N}(\omega) \tag{50}$$

Now define a new noise field

$$\beta(t) \equiv \alpha\delta\nu(t) = \alpha\dot{\phi_N}(t) \tag{51}$$

by scaling the frequency noise by some factor α. Substituting this into the above derivation we obtain the slightly more general PSD relation

$$S_\beta(\omega) = \alpha^2 S_{\delta\nu} = \alpha^2\omega^2 S_{\phi_N}(\omega) \tag{52}$$

These relations apply equally to two-sided and one-sided PSDs. Hence, we have omitted this specification in the result.

E. Interpreting $S_\phi(\omega)$: Sideband & Modulation Theory

F. Connecting Phase Noise PSD with $\mathcal{L}(\omega)$

The prevailing measure of phase noise among manufacturers and users of frequency standards is (cf. Eq. 26, p. 210 of [**?**])

$$\mathcal{L}(\omega) = \frac{1}{2} S^{(1)}_{\phi_N}(\omega) \tag{53}$$

where $S^{(1)}_{\phi_N}(\omega)$ is the *one-sided* PSD for phase noise. However the number, $\tilde{\mathcal{L}}(\omega)$, typically quoted by manufacturers is $\mathcal{L}(\omega)$ expressed *decibels* (cf. Eq. 27, p. 210 of [**?**]):

$$\tilde{\mathcal{L}}(\omega) = 10 \log_{10} \left[\frac{1}{2} S^{(1)}_{\phi_N}(\omega) \right] = 10 \log_{10} \left[\mathcal{L}(\omega) \right] \tag{54}$$

Thus

$$S^{(1)}_{\phi_N}(\omega) = 2 \cdot 10^{\frac{\tilde{\mathcal{L}}(\omega)}{10}} \tag{55}$$

II. MAPPING PHASE NOISE ON A CARRIER TO DEPHASING HAMILTONIAN FOR A DRIVEN QUBIT

Here we derive the effective Hamiltonian describing a qubit driven by a local oscillator with phase noise. The form of this effective Hamiltonian allows us to calculate operational fidelities using the filter transfer function formalism. We start with the following assumptions of our qubit system:

(a) 2 level system

(b) Dipole approximation

(c) $\hbar = 1$

A. Preliminaries

Some Notation & Identities

We set up some notation and identities. Start by defining the Hamiltonian system

$$H_\alpha = \alpha(t)\hat{\sigma}_z, \qquad \text{s.t.} \qquad i\dot{U}_\alpha(t) = U_\alpha(t)H_\alpha \tag{56}$$

Hence

$$\frac{\dot{U}_\alpha}{U_\alpha} = -i\alpha(t)\hat{\sigma}_z \iff \ln[U_\alpha] = -i\int \alpha(t)\hat{\sigma}_z \tag{57}$$

Hence, up to a constant,

$$U_\alpha(t) = e^{-i\tilde{\alpha}(t)\hat{\sigma}_z}, \qquad \tilde{\alpha}(t) := \int_0^t dt'\alpha(t') \tag{58}$$

For example, if $\alpha(t) \equiv \alpha$ (constant), $\tilde{\alpha}(t) = \alpha t$, and $U_\alpha(t) = \exp[-i\alpha t\hat{\sigma}_z]$. We introduce the shorthand

$$c := \cos\big(\tilde{\alpha}(t)\big),\ s := \sin\big(\tilde{\alpha}(t)\big) \tag{59}$$

$$\bar{c} := \cos\big(2\tilde{\alpha}(t)\big), \bar{s} := \sin\big(2\tilde{\alpha}(t)\big). \tag{60}$$

Hence

$$U_\alpha = e^{-i\tilde{\alpha}(t)\hat{\sigma}_z} = \mathbf{I}c - i\hat{\sigma}_z s \tag{61}$$

$$U_\alpha^\dagger = e^{+i\tilde{\alpha}(t)\hat{\sigma}_z} = \mathbf{I}c + i\hat{\sigma}_z s \tag{62}$$

We therefore have the following identities:

$$U_\alpha^\dagger\hat{\sigma}_x U_\alpha = \bar{c}\hat{\sigma}_x - \bar{s}\hat{\sigma}_y \tag{63}$$

$$U_\alpha^\dagger\hat{\sigma}_y U_\alpha = \bar{s}\hat{\sigma}_x + \bar{c}\hat{\sigma}_y \tag{64}$$

Proof:

$$
\begin{aligned}
U_\alpha^\dagger \hat\sigma_x U_\alpha &= (Ic + i\hat\sigma_z s)\hat\sigma_x(Ic - i\hat\sigma_z s) && (65)\\
&= c^2\hat\sigma_x - isc\hat\sigma_x\hat\sigma_z + isc\hat\sigma_z\hat\sigma_x - i^2s^2\hat\sigma_z\hat\sigma_x\hat\sigma_z && (66)\\
&= \hat\sigma_x c^2 - isc(-i\hat\sigma_y) + isc(i\hat\sigma_y) + s^2\hat\sigma_z(-i\hat\sigma_y) && (67)\\
&= c^2\hat\sigma_x - 2sc\hat\sigma_y - is^2(-i\hat\sigma_x) && (68)\\
&= c^2\hat\sigma_x - s^2\hat\sigma_x - 2sc\hat\sigma_y && (69)\\
&= \bar c\hat\sigma_x - \bar s\hat\sigma_y && (70)\\
& && (71)
\end{aligned}
$$

$$
\begin{aligned}
U_\alpha^\dagger \hat\sigma_y U_\alpha &= (Ic + i\hat\sigma_z s)\hat\sigma_y(Ic - i\hat\sigma_z s) && (72)\\
&= c^2\hat\sigma_y - isc\hat\sigma_y\hat\sigma_z + isc\hat\sigma_z\hat\sigma_y - i^2s^2\hat\sigma_z\hat\sigma_y\hat\sigma_z && (73)\\
&= c^2\hat\sigma_y - isc(i\hat\sigma_x) + isc(-i\hat\sigma_x) - i^2s^2(-i\hat\sigma_x)\hat\sigma_z && (74)\\
&= c^2\hat\sigma_y + sc\hat\sigma_x + sc\hat\sigma_x - i^2(-i)s^2(-i\hat\sigma_y) && (75)\\
&= c^2\hat\sigma_y - s^2\hat\sigma_y + 2sc\hat\sigma_x && (76)\\
&= \bar s\hat\sigma_x + \bar c\hat\sigma_y && (77)
\end{aligned}
$$

Define the usual operators

$$\hat\sigma_+ = \hat\sigma_x + i\hat\sigma_y, \qquad \hat\sigma_- = \hat\sigma_x - i\hat\sigma_y. \tag{78}$$

We therefore have the following identities

$$
\begin{aligned}
U_\alpha^\dagger \hat\sigma_+ U_\alpha &= \hat\sigma_+ e^{+i2\tilde\alpha(t)} && (79)\\
U_\alpha^\dagger \hat\sigma_- U_\alpha &= \hat\sigma_- e^{-i2\tilde\alpha(t)} && (80)
\end{aligned}
$$

Proof: Using Eqs. 63 and 64, and due to the linearity of $\hat\sigma^\pm$ in $\hat\sigma_x$ and $\hat\sigma_y$, we have

$$
\begin{aligned}
U_\alpha^\dagger \hat\sigma_+ U_\alpha &= (\bar c\hat\sigma_x - \bar s\hat\sigma_y) + i(\bar s\hat\sigma_x + \bar c\hat\sigma_y) && (81)\\
&= \bar c(\hat\sigma_x + i\hat\sigma_y) + \bar s(-\hat\sigma_y + i\hat\sigma_x) && (82)\\
&= \bar c(\hat\sigma_x + i\hat\sigma_y) + i\bar s(\hat\sigma_x + i\hat\sigma_y) && (83)\\
&= \hat\sigma_+(\bar c + i\bar s) && (84)\\
&= \hat\sigma_+ e^{i2\tilde\alpha(t)} && (85)\\
& && (86)
\end{aligned}
$$

$$
\begin{aligned}
U_\alpha^\dagger \hat\sigma_- U_\alpha &= (\bar c\hat\sigma_x - \bar s\hat\sigma_y) - i(\bar s\hat\sigma_x + \bar c\hat\sigma_y) && (87)\\
&= \bar c(\hat\sigma_x - i\hat\sigma_y) + \bar s(-\hat\sigma_y - i\hat\sigma_x) && (88)\\
&= \bar c(\hat\sigma_x - i\hat\sigma_y) - i\bar s(\hat\sigma_x - i\hat\sigma_y) && (89)\\
&= \hat\sigma_-(\bar c - i\bar s) && (90)\\
&= \hat\sigma_- e^{-i2\tilde\alpha(t)} && (91)\\
& && (92)
\end{aligned}
$$

B. Lab Frame (Schrodinger Picture)

$$
\begin{aligned}
H_S &= \frac{\omega_a}{2}\hat\sigma_z + \Omega(t)\cos(\omega_\mu t + \phi(t))\hat\sigma_x && (93)\\
\Omega(t) &= \Omega_C(t) + \Omega_N(t) && (94)\\
\phi(t) &= \phi_C(t) + \phi_N(t). && (95)
\end{aligned}
$$

Pull out component due to detuning of carrier freqency from qubit frequency:

$$H_S = \left(\frac{\omega_a - \omega_\mu}{2}\right)\hat{\sigma}_z + \frac{\omega_\mu}{2}\hat{\sigma}_z + \Omega(t)\cos(\omega_\mu t + \phi(t))\hat{\sigma}_x \tag{96}$$

C. Rotating Frame (Interaction Picture w.r.t. Carrier)

Define the (drive) carrier Hamiltonian H_{ω_μ} satisfying

$$H_{\omega_\mu} = \frac{\omega_\mu}{2}\hat{\sigma}_z, \qquad (\alpha \equiv \frac{\omega_\mu}{2};\quad 2\tilde{\alpha}(t) \equiv \omega_\mu t) \tag{97}$$

$$i\dot{U}_{\omega_\mu}(t) = U_{\omega_\mu}(t)H_{\omega_\mu} \tag{98}$$

$$U_{\omega_\mu}(t) = \exp[-i\frac{\omega_\mu}{2}t\hat{\sigma}_z]. \tag{99}$$

Next define the interaction picture with respect to H_{ω_μ} by

$$H_I^{(\omega_\mu)} = U^\dagger_{\omega_\mu} H_S U_{\omega_\mu} - H_{\omega_\mu} \tag{100}$$

Term 1 & 2 of H_S

Since $\hat{\sigma}_z$ commutes with H_{ω_μ}, and hence with $U_{\omega_\mu}(t)$, the rotating operator $U^\dagger_{\omega_\mu}\hat{\sigma}_z U_{\omega_\mu} = \hat{\sigma}_z$, and hence terms 1 & 2 of Eq. 96 are unaffected by moving to the interaction picture. Hence term 2 is subtracted away the $-H_{\omega_\mu}$ term in Eq. 100.

Term 3 of H_S

Defining $2\tilde{\alpha} \equiv \omega_\mu t$ so that

$$\bar{c} = \cos(2\tilde{\alpha}(t)) = \cos(\omega_\mu t), \qquad \bar{s} = \sin(2\tilde{\alpha}(t)) = \sin(\omega_\mu t), \tag{101}$$

and using Eq. 63, Term 3 of Eq. 96 transforms according to

$$\text{Term 3 of 96} \longrightarrow \Omega(t)\cos(\omega_\mu t + \phi(t))U^\dagger_{\omega_\mu}\hat{\sigma}_x U_{\omega_\mu}. \tag{102}$$

Now,

$$U^\dagger_{\omega_\mu}\hat{\sigma}_x U_{\omega_\mu} = \bar{c}\hat{\sigma}_x - \bar{s}\hat{\sigma}_y \tag{103}$$

$$= \frac{1}{2}\Big[e^{i\omega_\mu t} + e^{-i\omega_\mu t}\Big]\hat{\sigma}_x - \frac{1}{2i}\Big[e^{i\omega_\mu t} - e^{-i\omega_\mu t}\Big]\hat{\sigma}_y \tag{104}$$

$$= \frac{1}{2}\Big\{e^{i\omega_\mu t}(\hat{\sigma}_x + i\hat{\sigma}_y) + e^{-i\omega_\mu t}(\hat{\sigma}_x - i\hat{\sigma}_y)\Big\} \tag{105}$$

$$= \frac{1}{2}\Big\{e^{i\omega_\mu t}\hat{\sigma}_+ + e^{-i\omega_\mu t}\hat{\sigma}_-\Big\} \tag{106}$$

Hence

$$\text{Term 3} = \frac{1}{2}\Omega(t)\cos(\omega_\mu t + \phi(t))\Big\{e^{i\omega_\mu t}\hat{\sigma}_+ + e^{-i\omega_\mu t}\hat{\sigma}_-\Big\} \tag{107}$$

$$= \frac{1}{4}\Omega(t)\Big\{e^{i\omega_\mu t}e^{i\phi(t)} + e^{-i\omega_\mu t}e^{-i\phi(t)}\Big\}\Big\{e^{i\omega_\mu t}\hat{\sigma}_+ + e^{-i\omega_\mu t}\hat{\sigma}_-\Big\} \tag{108}$$

$$\overset{\text{RWA}}{\approx} \frac{1}{4}\Omega(t)\Big\{e^{-i\phi(t)}\hat{\sigma}_+ + e^{i\phi(t)}\hat{\sigma}_-\Big\} \tag{109}$$

In the last equality we have used the rotating wave approximation, ignoring terms that oscillate at $2\omega_\mu$ (i.e. assuming $\Omega(t) \ll \omega_\mu$).

Hamiltonian in Interaction Picture w.r.t. Carrier

Thus the Hamiltonian in this interation picture becomes

$$H_I^{(\omega_\mu)} = \Big(\frac{\omega_a - \omega_\mu}{2}\Big)\hat{\sigma}_z + \frac{1}{4}\Omega(t)\Big\{e^{-i\phi(t)}\hat{\sigma}_+ + e^{i\phi(t)}\hat{\sigma}_-\Big\} \tag{110}$$

D. Rotating Frame w.r.t. Phase Noise $\dot{\phi}_N(t)$

We rewrite Eq. 110

$$H_I^{(\omega_\mu)} = \Big(\frac{\omega_a - \omega_\mu}{2}\Big)\hat{\sigma}_z + \frac{1}{4}\Omega(t)\Big\{e^{-i\phi_C(t)}e^{-i\phi_N(t)}\hat{\sigma}_+ + e^{i\phi_C(t)}e^{i\phi_N(t)}\hat{\sigma}_-\Big\} \tag{111}$$

formally seperating the control and noise components of the modulated phase. Now define a stochastic detuning Hamiltonian $H_{\dot{\phi}_N}$ satisfying

$$H_{\dot{\phi}_N} = \frac{1}{2}\dot{\phi}_N(t)\hat{\sigma}_z, \qquad \Big(\alpha(t) \equiv \frac{1}{2}\dot{\phi}_N(t); \quad 2\tilde{\alpha}(t) \equiv \phi_N(t)\Big) \tag{112}$$

$$i\dot{U}_{\dot{\phi}_N}(t) = U_{\dot{\phi}_N}(t)H_{\dot{\phi}_N} \tag{113}$$

$$U_{\dot{\phi}_N}(t) = \exp[-i\frac{\phi_N(t)}{2}\hat{\sigma}_z]. \tag{114}$$

Next define the interaction picture with respect to $H_{\dot{\phi}_N}$ by

$$H_I^{(\omega_\mu,\dot{\phi}_N)} = U^\dagger_{\dot{\phi}_N} H_I^{(\omega_\mu)} U_{\dot{\phi}_N} - H_{\dot{\phi}_N} \tag{115}$$

Term 1 of Eq. 111

Since $\hat{\sigma}_z$ commutes with $H_{\dot{\phi}_N}$, and hence with $U_{\dot{\phi}_N}(t)$, the rotating operator $U^\dagger_{\dot{\phi}_N}\hat{\sigma}_z U_{\dot{\phi}_N} = \hat{\sigma}_z$, and hence term 1 of 111 is unaffected by moving to the interaction picture w.r.t. $H_{\dot{\phi}_N}$. We pick up a term in the new Hamiltonian equal to $-H_{\dot{\phi}_N}$ due subtracting $H_{\dot{\phi}_N}$ from $U^\dagger_{\dot{\phi}_N} H_I^{(\omega_\mu)} U_{\dot{\phi}_N}$ in Eq. 115.

Term 2 $\{\cdot\}$ of Eq. 111

Defining $2\tilde{\alpha} \equiv \phi_N(t)$ so that

$$\bar{c} = \cos(2\tilde{\alpha}(t)) = \cos(\phi_N(t)), \qquad \bar{s} = \sin(2\tilde{\alpha}(t)) = \sin(\phi_N(t)), \tag{116}$$

and using Eqs. 79 and 80, we may make the substitutions

$$\hat{\sigma}_+ \longrightarrow \hat{\sigma}_+ e^{+i\phi_N(t)} \tag{117}$$

$$\hat{\sigma}_- \longrightarrow \hat{\sigma}_- e^{-i\phi_N(t)} \tag{118}$$

in the terms in the $\{\cdot\}$ brackets in Eq. 111 in moving to the interaction picutre defined by Eq. 115. Hence

$$\text{Term 2 of Eq. 111} \longrightarrow \frac{1}{4}\Omega(t)\Big\{e^{-i\phi_C(t)}e^{-i\phi_N(t)}e^{+i\phi_N(t)}\hat{\sigma}_+ + e^{i\phi_C(t)}e^{i\phi_N(t)}e^{-i\phi_N(t)}\hat{\sigma}_-\Big\} \tag{119}$$

$$= \frac{1}{4}\Omega(t)\Big\{e^{-i\phi_C(t)}\hat{\sigma}_+ + e^{i\phi_C(t)}\hat{\sigma}_-\Big\} \tag{120}$$

$$\tag{121}$$

Interaction Picture Hamiltonian $H_I^{(\omega_\mu,\dot{\phi}_N)}$

Thus the Hamiltonian in this interation picture becomes

$$H_I^{(\omega_\mu,\dot{\phi}_N)} = \Big(\frac{\omega_a - \omega_\mu}{2}\Big)\hat{\sigma}_z - \frac{1}{2}\dot{\phi}_N(t)\hat{\sigma}_z + \frac{1}{4}\Omega(t)\Big\{e^{-i\phi_C(t)}\hat{\sigma}_+ + e^{i\phi_C(t)}\hat{\sigma}_-\Big\} \tag{122}$$

We may reexpress

$$\Big\{e^{-i\phi_C(t)}\hat{\sigma}_+ + e^{i\phi_C(t)}\hat{\sigma}_-\Big\} = 2\Big\{\cos[\phi_C(t)]\hat{\sigma}_x + \sin[\phi_C(t)]\hat{\sigma}_y\Big\} \tag{123}$$

then setting $\omega_\mu = \omega_a$ and defining the engineered detuning noise $\eta_z(t) \equiv -\frac{1}{2}\dot{\phi}_N(t)$ we obtain

$$H_I^{(\omega_\mu,\dot{\phi}_N)} = \eta_z(t)\hat{\sigma}_z + \frac{1}{2}\Omega(t)\Big\{\cos[\phi_C(t)]\hat{\sigma}_x + \sin[\phi_C(t)]\hat{\sigma}_y\Big\} \tag{124}$$

We simplify this further by considering the special case where the control phase is held constant, $\phi_C(t) \equiv 0$. In this case the qubit sees the effective Hamiltonian

$$H_{\text{eff}}(t) = H_c(t) + H_0(t) \tag{125}$$

$$H_c(t) = \frac{1}{2}\Omega(t)\hat{\sigma}_x \tag{126}$$

$$H_0(t) = \eta_z(t)\hat{\sigma}_z, \qquad \eta_z(t) \equiv -\frac{1}{2}\dot{\phi}_N(t) \tag{127}$$

The evolution operator for the control Hamiltonian is given by

$$U_c(\tau) = e^{-i\frac{\int_0^\tau \Omega(t')dt'}{2}\hat{\sigma}_x} \tag{128}$$

Qubit rotations about $\hat{\sigma}_x$ are then determined by

$$\Theta(\tau) = \int_0^\tau \Omega(t')dt' \tag{129}$$

For instance, if $\Theta(\tau) = \pi$, the evolution operator reduces to $U_c(\tau) = \hat{\sigma}_x$, with the action of flipping the quibt state from $|1\rangle$ to $|0\rangle$ (a π rotation).

III. FIDELITY APPROXIMATION VIA FILTER FUNCTION FORMALISM

Consider the following qubit-frame Hamiltonian:

$$H(t) = H_c(t) + H_0(t) \tag{130}$$
$$H_c(t) = \frac{1}{2}\boldsymbol{\Omega(t)\sigma} \tag{131}$$
$$H_0(t) = \beta(t)\hat{\sigma}_z, \tag{132}$$

where $\boldsymbol{\Omega(t)}$ is a row vector of control fields, and $\boldsymbol{\sigma}$ is a column vector of Pauli matrices. The operational fidelity in the presence of the dephasing noise field $\beta(t)$ is derived in [**?**] taking the form

$$\mathcal{F}_{av}(\tau) \approx \frac{1}{2}\Big[1 + e^{-2\langle a_1^2\rangle}\Big] \tag{133}$$
$$\tag{134}$$

Here the first-order Magnus contribution to infidelity is expressed

$$\langle a_1^2\rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{d\omega}{\omega^2}S_z^{(2)}(\omega)F_z(\omega) \tag{135}$$

as an overlap integral between the *two-sided* dephasing-noise power spectral density (PSD) $S_z^{(2)}(\omega)$ and the filter function $F_z(\omega)$ describing the qubit's susceptibility to noise interaction in this quadrature. Both are given in angular frequency units. Assuming wide-sense stationarity of the dephasing field $\beta(t)$, the PSD and autocorrelation function satisfy (Eq.18)

$$S_z^{(2)}(\omega) \equiv \int_{-\infty}^{\infty} d\tau C_\beta(\tau)e^{-i\omega\tau} \qquad \Longleftrightarrow \qquad C_\beta(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega C_\beta(\tau)e^{i\omega\tau} \tag{136}$$

where the autocorrelation function is defined

$$C_\beta(\tau) \equiv \langle\beta(t)\beta(t+\tau)\rangle_t \tag{137}$$

Since both $S_z^{(2)}(\omega)$ and $F_z(\omega)$ are even functions:

$$S_z^{(2)}(\omega) = S_z^{(2)}(-\omega) \tag{138}$$
$$F_z(\omega) = F_z(-\omega) \tag{139}$$

We may write

$$\langle a_1^2\rangle = 2\left(\frac{1}{2\pi}\right)\int_0^{\infty}\frac{d\omega}{\omega^2}S_z^{(2)}(\omega)F_z(\omega) \tag{140}$$

Thus, defining

$$\chi(\tau) \equiv 2\langle a_1^2\rangle \tag{141}$$

we write

$$\mathcal{F}_{av}(\tau) \approx \frac{1}{2}\Big[1 + e^{-\chi(\tau)}\Big] \tag{142}$$

where $\chi(\tau)$ takes the computational form

$$\chi(\tau) = 2\times 2\left(\frac{1}{2\pi}\right)\int_0^{\infty}\frac{d\omega}{\omega^2}S_z^{(2)}(\omega)F_z(\omega) \tag{143}$$

Or, in terms of the *one-sided* PSD $S_z^{(1)}(\omega) = 2S_z^{(2)}(\omega)$:

$$\chi(\tau) = 2\left(\frac{1}{2\pi}\right)\int_0^\infty \frac{d\omega}{\omega^2} S_z^{(1)}(\omega) F_z(\omega) \tag{144}$$

IV. CONNECTING PHASE NOISE WITH QUBIT INFIDELITY

Here we bring the above derivations together to relate the quantities of interest. Our qubit is driven (resonantly) by the local oscillator

$$\Omega(t)\cos(\omega_\mu t+\phi(t)), \qquad \omega_\mu=\omega_{\mathrm{LO}}=\omega_a \tag{145}$$

as described by the Hamiltonian in Eq. 93. The phase fluctuations $\phi(t)$ generally consist of both noise and control components

$$\phi(t)=\phi_N(t)+\phi_C(t), \tag{146}$$

but we set $\phi_C(t)\equiv 0$, so only noisy phase fluctuations are considered. The effective Hamiltonian in the qubit interaction picture is then given by Eq. 127

$$H_{\text{eff}}(t)=\frac{1}{2}\Omega(t)\hat{\sigma}_x+\eta_z(t)\hat{\sigma}_z \qquad \eta_z(t)\equiv-\frac{1}{2}\dot{\phi}_N(t) \tag{147}$$

This effective Hamiltonian is formally equivalent to Eq. 132, with the respective noise fields related by

$$\beta(t)\equiv\eta_z(t)\equiv-\frac{1}{2}\dot{\phi}_N \tag{148}$$

To proceed with quibt fidelity calculations using Eq. 144 we need the form of the dephasing PSD:

$$S_z(\omega)\equiv S_\beta(\omega)\equiv S_{\eta_z}(\omega) \tag{149}$$

But from Eqs. 51 we see

$$\beta(t)=\alpha\delta\nu(t)=\alpha\dot{\phi}(t), \qquad \alpha=-\frac{1}{2} \tag{150}$$

and so from Eq. 52 we find

$$S_\beta(\omega)=\alpha^2 S_{\delta\nu}=\alpha^2\omega^2 S_{\phi_N}(\omega) \tag{151}$$

This relation holds for both one-sided and two-sided PSDs, hence

$$S_z^{(1)}(\omega)=\frac{1}{4}\omega^2 S_{\phi_N}^{(1)}(\omega) \tag{152}$$

Substituting this into Eq. 144 we therefore have

$$\begin{aligned}
\chi(\tau) &= 2\left(\frac{1}{2\pi}\right)\int_0^\infty \frac{d\omega}{\omega^2} S_z^{(1)}(\omega)F_z(\omega) && (153)\\
&= 2\left(\frac{1}{2\pi}\right)\int_0^\infty \frac{d\omega}{\omega^2}\left(\frac{1}{4}\omega^2 S_{\phi_N}^{(1)}(\omega)\right)F_z(\omega) && (154)\\
&= 2\times\frac{1}{4}\times\left(\frac{1}{2\pi}\right)\int_0^\infty d\omega S_{\phi_N}^{(1)}(\omega)F_z(\omega) && (155)\\
&= 2\times\frac{1}{4}\times\left(\frac{1}{2\pi}\right)\int_0^\infty d\omega\left(2\cdot 10^{\frac{\tilde{\mathcal{L}}(\omega)}{10}}\right)F_z(\omega) && (156)\\
&= 2\times 2\times\frac{1}{4}\times\left(\frac{1}{2\pi}\right)\int_0^\infty d\omega\left(10^{\frac{\tilde{\mathcal{L}}(\omega)}{10}}\right)F_z(\omega) && (157)
\end{aligned}$$

where in the last two equalities we have used Eq. 55. Thus

$$\chi(\tau)=\frac{1}{2\pi}\int_0^\infty d\omega 10^{\frac{\tilde{\mathcal{L}}(\omega)}{10}}F_z(\omega)=\frac{1}{4\pi}\int_0^\infty d\omega S_{\phi_N}^{(1)}(\omega)F_z(\omega) \tag{158}$$

V. ANALYTIC FORMS OF FILTER FUNCTIONS CONSIDERED

We consider 4 specific control scenarios:

(a) Free evolution

(b) Spin echo (bang-bang limit)

(c) Primitive, finite π pulse

(d) WAMF π gate

Analytic filter functions for these cases are given below:

A. Free evolution

$$F(\omega) = 4\sin\left(\frac{\omega\tau}{2}\right)^2 \tag{159}$$

B. Spin echo

$$F(\omega) = |1 + e^{i\omega\tau} - 2e^{i\frac{\omega\tau}{2}}|^2 = 16\sin\left(\frac{\omega\tau}{4}\right)^4 \tag{160}$$

C. Primitive π

$$F(\omega) = \left|R_{zy}(\omega)\right|^2 + \left|R_{zz}(\omega)\right|^2 \tag{161}$$

where

$$R_{zy}(\omega) = \frac{i\omega\Omega}{\omega^2 - \Omega^2}\left[e^{i\omega\tau} + 1\right] \tag{162}$$

$$R_{zz}(\omega) = \frac{\omega^2}{\omega^2 - \Omega^2}\left[e^{i\omega\tau} + 1\right] \tag{163}$$

$$\Omega = \frac{\pi}{\tau} \tag{164}$$

D. WAMF π

$$F(\omega) = \left|R_{zy}(\omega)\right|^2 + \left|R_{zz}(\omega)\right|^2 \tag{165}$$

where

$$R_{zy}(\omega) = \frac{i4\pi\tau\omega e^{i\frac{\tau\omega}{2}}\left[\left(\tau^2\omega^2 + 8\pi^2\right)\cos\left(\frac{\tau\omega}{4}\right) + 2\left(\tau^2\omega^2 - 4\pi^2\right)\cos\left(\frac{\tau\omega}{2}\right)\right]}{\tau^4\omega^4 - 20\pi^2\tau^2\omega^2 + 64\pi^4} \tag{166}$$

$$R_{zz}(\omega) = \frac{2\tau^2\omega^2 e^{i\frac{\tau\omega}{2}}\left[12\pi^2\cos\left(\frac{\tau\omega}{4}\right) + \left(\tau^2\omega^2 - 4\pi^2\right)\cos\left(\frac{\tau\omega}{2}\right)\right]}{\tau^4\omega^4 - 20\pi^2\tau^2\omega^2 + 64\pi^4} \tag{167}$$

VI. NOISE FLOOR

Here we consider the contribution to infidelity from the fundamental limit associated with hitting the thermal noise floor. The theoretical minimum noise floor for an oscillator into 50 Ω is kTB for bandwidth B, and normalized to a 1Hz BW this is expressed in dBm/Hz. For instance

$$290\ \text{K} \implies 4\times 10^{-21}\ \text{W/Hz} \implies -174\ \text{dBm/Hz} \tag{168}$$
$$4\ \text{K} \implies -192\ \text{dBm/Hz} \tag{169}$$

We assume an oscillator at 0 dBm where the phase noise hits the white floor (for powers < 0 dBm, you can do no better than this floor in phase noise). Of course there is always *some* finite bandwidth, generally given by e.g. cable bandwidths. Therefore we impose an upper bandwidth cutoff ω_c to the white thermal noise. For instance, this might be 10GHz associated with the high-f cutoff for TEM modes in typical coax cable. Hence, the thermal noise floor may be characterized by

$$\tilde{\mathcal{L}}(\omega) = \begin{cases} L & 0 \leq \omega \leq \omega_c \\ 0 & \omega > \omega_c \end{cases} \tag{170}$$

From Eqs. 142 and 158 the infidelity $\mathcal{I}$ associated with this noise floor is given by

$$\mathcal{I} = 1 - \frac{1}{2}\left[1 + e^{-\chi(\tau)}\right] \tag{171}$$

where

$$\chi(\tau) = \frac{1}{2\pi}\int_0^\infty d\omega 10^{\frac{\tilde{\mathcal{L}}(\omega)}{10}} F(\omega) = \frac{1}{2\pi} 10^{\frac{L}{10}} \int_0^{\omega_c} d\omega F(\omega) \tag{172}$$

That is, the infidelity is given by the integral of the filter function over the band $[0, \omega_c]$ scaled by a constant factor associated with the strength of the thermal noise floor. We can obtain analytic expressions for this integral, for the filter functions in Section V using the assumption that

$$\omega_c \gg \frac{1}{\tau} \iff \tau\omega_c \gg 1 \tag{173}$$

That is, the bulk of the thermal noise floor is in the bandwidth beyond the stopband of the filter function. For instance, from Eq. 159, in the case of free evolution we have

$$\int_0^{\omega_c} 4\sin\left(\frac{\omega\tau}{2}\right)^2 d\omega = 2\left(\omega_c - \frac{\sin(\tau\omega_c)}{\tau}\right) \tag{174}$$

Defining $z = \tau\omega_c$, we therefore have

$$\int_0^{\omega_c} d\omega F(\omega) = 2\omega_c\left(1 - \frac{\sin(z)}{z}\right) \tag{175}$$

But since $z = \tau\omega_c \gg 1$, the sinc function on the right limits to zero, yielding

$$\int_0^{\omega_c} d\omega F(\omega) = 2\omega_c \tag{176}$$

Consequently

$$\chi(\tau) = \frac{1}{2\pi} 10^{\frac{L}{10}} (2\omega_c) \qquad\qquad \textit{free evolution} \tag{177}$$

is proportional to the cutoff, with proportionality factor associated with strength of the thermal noise floor. Similar results hold for the other filter functions. For spin echo it is just as straightforward to derive

$$\int_0^{\omega_c} d\omega F(\omega) = 6\omega_c \tag{178}$$

Hence

$$\chi(\tau) = \frac{1}{2\pi} 10^{\frac{L}{10}} (6\omega_c) \qquad \textit{spin echo} \tag{179}$$

The form of the filter function gets rapidly more complicated for finite-width pulses. However we can still calculate the integrals in the limiting regime as follows. Consider the primitive π pulse. From Eqs. 161, 162 and 163 one can calculate

$$F(\omega) = \frac{2\pi^2 z^2}{(\pi^2 - z^2)^2} + \frac{2z^4}{(\pi^2 - z^2)^2} + \frac{2\pi^2 z^2 \cos(z)}{(\pi^2 - z^2)^2} + \frac{2z^4 \cos(z)}{(\pi^2 - z^2)^2}, \qquad z = \omega\tau \tag{180}$$

When we integrate out to $\omega_c \gg 1/\tau$, unless τ is *very* small (e.g. < 10 ns for ω_c =100 MHz) the main contribution to the integral over the band $[0, \omega_c]$ comes from those terms in Eq. 180 which survive the limit $z \to \infty$. We find

$$\lim_{z\to\infty} \frac{2\pi^2 z^2}{(\pi^2 - z^2)^2} = 0 \tag{181}$$

$$\lim_{z\to\infty} \frac{2z^4}{(\pi^2 - z^2)^2} = 2 \tag{182}$$

$$\lim_{z\to\infty} \frac{2\pi^2 z^2 \cos(z)}{(\pi^2 - z^2)^2} \to \left[\lim_{z\to\infty} \frac{2\pi^2 z^2}{(\pi^2 - z^2)^2}\right] \cos(z) = 0 \tag{183}$$

$$\lim_{z\to\infty} \frac{2z^4 \cos(z)}{(\pi^2 - z^2)^2} \to \left[\lim_{z\to\infty} \frac{2z^4}{(\pi^2 - z^2)^2}\right] \cos(z) = 2\cos(z) \tag{184}$$

$$\tag{185}$$

where in the last two terms we have allowed the oscillating (and bounded) cosine term to be removed from the limit in order to extract the *functional* form of the surviving term. Consequently, we write

$$\int_0^{\omega_c} F(\omega) d\omega = \lim_{z\to\infty} \int_0^{\omega_c} 2\left(1 + \cos(\tau\omega)\right) d\omega, \qquad \tau = z/\omega_c \tag{186}$$

$$= \lim_{z\to\infty} \omega_c \left(2 + \frac{\sin(z)}{z}\right) \tag{187}$$

$$= 2\omega_c \tag{188}$$

Hence

$$\chi(\tau) = \frac{1}{2\pi} 10^{\frac{L}{10}} (2\omega_c) \qquad \textit{PRIM-}\pi \tag{189}$$

Similarly, expanding the filter function for WAMF-π and taking the limit, the only surviving term takes the form

$$\lim_{z\to\infty} \frac{4z^8 \cos(\frac{z}{2})^2}{(64\pi^4 - 20\pi^2 z^2 + z^4)^2} \to 4\cos^2\left(\frac{z}{2}\right) \tag{190}$$

The filter function therefore integrates to

$$\int_0^{\omega_c} F(\omega) d\omega = \lim_{z\to\infty} \int_0^{\omega_c} 4\cos^2\left(\frac{\tau\omega}{2}\right) d\omega, \qquad \tau = z/\omega_c \tag{191}$$

$$= \lim_{z\to\infty} 2\omega_c \left(1 + \frac{\sin(z)}{z}\right) \tag{192}$$

$$= 2\omega_c \tag{193}$$

Hence

$$\chi(\tau) = \frac{1}{2\pi} 10^{\frac{L}{10}} (2\omega_c) \qquad \textit{WAMF-}\pi \tag{194}$$

A. Summary

In summary: Let the thermal noise floor be white with a high-frequency cutoff

$$\tilde{\mathcal{L}}(\omega) = \begin{cases} L & 0 \leq \omega \leq \omega_c \\ 0 & \omega > \omega_c \end{cases} \tag{195}$$

Then, for the control sequences considered (Ramsey, spin-echo, primitive-π, WAMF-π), and assuming the total sequence duration satisfies

$$\omega_c \gg \frac{1}{\tau} \iff \omega_c \tau \gg 1 \tag{196}$$

the contribution to the infidelity is *independent* of τ, and takes the form

$$\mathcal{I} = 1 - \mathcal{F}_{av} = 1 - \frac{1}{2}\Big[1 + e^{-\chi(\tau)}\Big] \tag{197}$$

where

$$\chi(\tau) = \frac{\gamma \omega_c}{2\pi} 10^{\frac{L}{10}} \tag{198}$$

where

$$\gamma = 2 \qquad (\textit{Ramsey}) \tag{199}$$
$$\gamma = 6 \qquad (\textit{spin echo}) \tag{200}$$
$$\gamma = 2 \qquad (\textit{PRIM-}\pi) \tag{201}$$
$$\gamma = 2 \qquad (\textit{WAMF-}\pi) \tag{202}$$
$$\tag{203}$$