Combinatorial Reciprocity & Möbius Functions

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WHAT IS COMBINATORIAL RECIPROCITY?

(a) Introduced by Richard Stanley in his paper *Combinatorial Reciprocity Theorems* in 1974. We will cover the first about the first 5 of 50 pages of this paper.

(b) Counting functions are often restrictions of polynomials to the natural numbers. It makes sense algebraically—though not *a priori* combinatorially—to evaluate these functions outside of their domain.

(c) They reveal surprising facts that may seem mystical or coincidental, but there is a hidden duality.

(d) There is generally geometric reasons underpinning these statements. Hyperplane arrangements, order-cones, and polytopes are the natural tools to reveal these geometric underpinnings. We will not discuss them today.

(e) We will give purely combinatorial proofs

RECIPROCITY FOR BINOMIAL COEFFICIENTS

Some Combinatorial Reciprocity statements are simple.

Proposition. (Combinatorial Reciprocityfor Binomial Coefficients)
Recall that the function $n \mapsto \binom{n}{k}$ is a polynomial in n of degree k. By evaluating this polynomial at negative integers, we obtain the formal result:

$$(-1)^k \binom{-n}{k} = \binom{n+k-1}{k}$$

<u>Claim</u>: the right hand side counts the number Of k-multisubsets on [n].

The claim follows by considering stars and bars. Take $n-1$ $|$s and k $\star$s. Then write $1|2|\cdots|n$. Taking a permutation of the stars and bars and lining up the bars gives you the bijection.

Figure 1: Stars & Bars Method.

Proof.

$$\begin{aligned}
(-1)^k \binom{x}{k} &= (-1)^k \frac{(x)!}{k!(x-k)!} \\
&= \frac{(-1)^k}{k!}(x)(x-1)\cdots(x-k+1) \\
&= \frac{(-1)^k}{k!}(-n)(-n-1)\cdots(-n-k+1) \\
&= \frac{1}{k!}(n)(n+1)\cdots(n+k-1) \\
&= \frac{(n+k-1)!}{k!(n-1)!} \\
&= \binom{n+k-1}{k}
\end{aligned}$$

POSETS

We now define partially ordered sets and introduce terminology for the talk.

> **Definition**. (Poset & Chain)
> A **poset** is a set Π together with a binary relation $\leq$ which obeys:
> (i) $x \leq x$ for all $x \in \Pi$ *(reflexivity)*
> (ii) $x \leq y \leq z \implies x \leq z$ *(transitivity)*
> (iii) $(x \leq y \ \wedge \ y \leq x) \implies x = y$ *(anti-symmetry)*
> We sometimes write $\leq_\Pi$ to be clear that the $\leq$-relation is relative to the structure Π.
> A **chain** is a poset in which every two elements are comparable. An **antichain** is a poset in which no two elements are comparable. A **multichain** is an ordered multi-subset of a chain.

Notation: We write $[n] = \{1, 2, \ldots, n\}$.

> **Definition**. (Intervals)
> An **interval** is a subset of a partially ordered set Π.
> The closed interval $[x, y] = \{t \in \Pi : x \leq t \leq y\}$.
> We similarly define open intervals, half-open intervals, etc.
> Note that if x and y are incomparable then $[x, y] = \varnothing$ also note $[x, y]$ may not be a chain.
>
> We say that a poset Π is **locally finite** if $[x, y]$ is finite for all $x, y \in \Pi$.

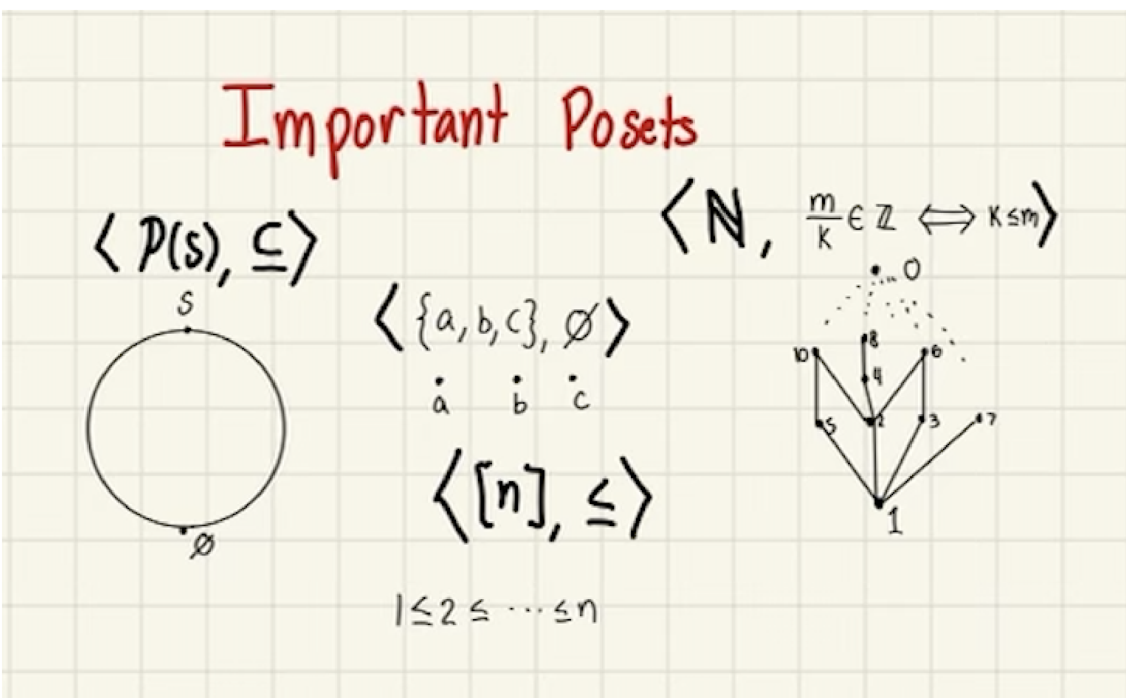

Figure 2: Important Posets

ORDER POLYNOMIALS

Order preserving maps are a natural functions to study when studying posets.[1]

Definition. (Order Preserving Maps)
We say that $\phi : \Pi \to \Pi'$ is order preserving if

$$x \leq_{\Pi} y \implies \phi(x) \leq_{\Pi'} \phi(y)$$

$$\forall x, y \in \Pi.$$

We define strictly order preserving maps similarly.

Definition. (Order Polynomial)
We now define a function that enumerates order preserving maps. Let

$$\Omega_{\Pi}(n) = |\{\phi : \Pi \to [n] \text{ order preserving.}\}|$$

Let $\Omega^{\circ}_{\Pi} : \mathbb{N} \to \mathbb{N}$ enumerate strictly order preserving maps.

Proposition. (The Order Polynomial is a Polynomial)
If Π is a finite, then $\Omega_{\Pi}, \Omega^{\circ}_{\Pi}$ are restrictions of polynomials with rational coefficients to $\mathbb{N}$.

This proof is a bit of a detour. For now I give the main ideas:

Proof.

Factor ϕ then argue that σ and ι are polynomials using binomial coefficients, so their composition is a polynomial.

$$\begin{tikzcd}
\Pi \arrow[rr, "\phi"] \arrow[dr, two heads, "\sigma"'] & & {[n]} \\
& \phi(\Pi) \arrow[ur, hook, "\iota"'] &
\end{tikzcd}$$

Figure 3: Factor ϕ into a injection and surjection.

$\square$

We will give a much simpler proof shortly.

[1]Order preserving maps are actually the natural morphisms between posets in the categorical sense.

Theorem. (Reciprocity for Order Polynomials)
Let Π be a finite poset, then
$$(-1)^{|\Pi|}\Omega_\Pi(-n) = \Omega_\Pi^\circ(n)$$

This is actually a generalization of the reciprocity statement for binomial coefficients! Let $\Pi = [k]$, then $\Omega_\Pi^\circ(n) = \binom{n}{k}$ and $\Omega_\Pi(n)$ counts the number of k-multisubsets on $[n]$.
We will give a combinatorial proof after introducing some technology.

We have another way to think about order preserving maps: via order ideals.
An order ideal is analogous to an ideal in ring theory.

Definition. (Order Ideals)
An order ideal I is a subset of a locally finite poset Π satisfying
$$y \in I \ \wedge \ x \leq y \implies x \in I.$$
The set of order ideals of a locally-finite poset Π forms a lattice under $\subseteq$.
We call this the **Birkhoff Lattice** and write $\mathcal{J}(\Pi)$.

Proposition. (Order Bijection)
Order preserving maps $\phi : \Pi \to [n]$ are in bijection with multichains of order ideals of size n.
A multichain of order ideals is a chain of order ideals possibly with repetition:
$$\varnothing = I_0 \subseteq I_1 \subseteq \cdots \subseteq I_n = \Pi$$
The map ϕ is strictly order preserving whenever $I_j \setminus I_{j-1}$ is an antichain $\forall j \in [n]$.
We call this the **antichain condition**.
So $\Omega_\Pi(n)$ enumerates multichains of order ideals in $\mathcal{J}(\Pi)$, and $\Omega_\Pi^\circ(n)$ enumerates multichains of order ideals with the antichain condition.

Proof.
Every order ideal of $[n]$ is principal. Let $k \leq n$, then $\phi^{-1}([k])$ is an order ideal of Π.

In other words $\phi(x) = k \iff x \in I_k - I_{k-1}$.

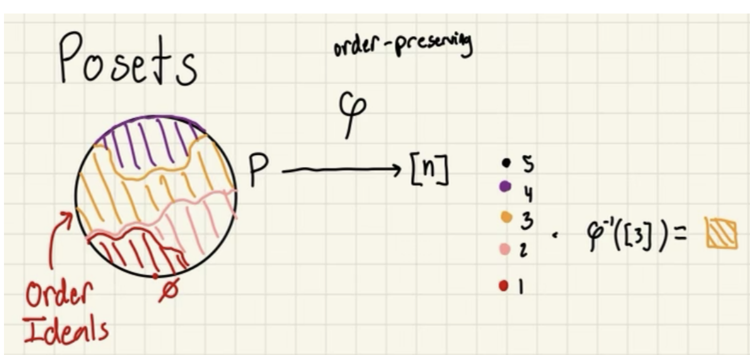

Figure 4: Order Ideals

INCIDENCE ALGEBRA

One is often led to the study of Incidence Algebras by the generalization of Inclusion-Exclusion methods. To do so, we will establish a "difference calculus" relative to an arbitrary partially ordered set.

We will obtain analogs of the Fundamental Theorem of Calculus.

Definition. (Incidence Algebra)
Let Π be a locally finite poset. The **incidence algebra** on Π is the algebra of functions $f : \Pi^2 \to \mathbb{C}$. We write $I(\Pi)$ to denote the incidence algebra and δ to denote the identity. The product of $f, g \in I(\Pi)$ is defined to be the **convolution product**:

$$h(x, y) = (f \star g)(x, y) = \sum_{t \in [x,y]} f(x, t)g(t, y).$$

This is well-defined since $[x, y]$ is finite.

Meaning if $\{x, y\}$ is an antichain then $f(x, y) = (f \cdot \delta)(x, y) = 0$ since the empty sum is 0.

Examples: $\zeta, \delta, \eta \in I(\Pi)$ where ζ, δ, η are the indicator functions for $\leq, =, <$ respectively.

Lemma. (Counting multichains)
Let Π be locally finite and $x, y \in \Pi$.
Then $\zeta^n(x, y)$ counts the number of multichains of length n in $[x, y]$.

Corollary.

$$\Omega_\Pi(n) = \zeta^n(\varnothing, \Pi)$$

Proof.
Induction on the length of multichain.

$$\zeta^n(x, z) = (\zeta^{n-1} \star \zeta)(x, z) = \sum_{x \leq y \leq z} \zeta^{n-1}(x, y)\zeta(y, z)$$

The summands on the right contributes when there is a multi-chain $x \to y$ of size $n - 1$ that can be extended to z. □

This gives us another proof that $\Omega_\Pi(n)$ is a polynomial. The Binomial Theorem holds for elements that commute in a ring, so

$$\zeta^n(x, y) = (\delta + \eta)^n(x, y) = \sum_{k=0}^{n} \binom{n}{k} \eta^k(x, y)$$

also η is nilpotent.

We would like to compute $\Omega_\Pi(-n) = \zeta^{-n}_{\mathcal{J}(\Pi)}(\varnothing, \Pi)$. Note that the LHS is a counting function that agrees with a polynomial, so by saying $\Omega_\Pi(-n)$ we mean the evaluation of this polynomial at $-n$. We hope that this equals $\zeta^{-n}(\varnothing, \Pi)$. So we need to invert ζ and take powers to check.

Proposition. (Möbius Functions)
The ζ function of a locally finite poset Π is invertible. We call this the **Möbius Function** of Π and write $\mu = \zeta^{-1}$. By computing μ recursively from $\mu \star \zeta = \delta$ we obtain

$$\mu(x,z) = \begin{cases} 1 & x = z \\ -1 \sum_{x \leq y < z} \mu(x,y) & x < z \end{cases}$$

By considering a linear extension of a finite poset Π we can identify $I(\Pi)$ with the space of complex upper triangular matrices. This gives us an easy invertibility condition on $\beta \in I(\Pi)$, namely only when $\beta(x,x) \neq 0$ for all $x \in \Pi$.

Remark: If you've taken a course in Combinatorics then you've likely encountered the Möbius Inversion Formula (the Inclusion-Exclusion principle is the most simple yet typical form of this statement). The Möbius function that occurs in this statement is the same as the one here! The Inversion Formula is the typical method to introduce Möbius functions and not the Incidence algebra.

Theorem. (Möbius Inversion Formula)
Suppose $f_=, f_\leq : \Pi \to \mathbb{C}$. Then,

$$f_\leq(b) = \sum_{a \leq b} f_=(a) \iff f_=(c) = \sum_{b \leq c} f_\leq(b)\mu(b,c).$$

Example: What does $(2\delta - \zeta)^{-1}(x,y)$ count? Well $2\delta - \zeta = \delta - \eta$ and

$$(\delta - \eta)^{-1} = \delta + \eta + \eta^2 + \cdots$$

total strict chains x to y.

To prove the Order Reciprocity Theorem we want to evaluate $\Omega_\Pi(-n) = \zeta^{-n}(\varnothing, \Pi)$.

Lemma. (Nothing Explodes Lemma)

$$\Omega_\Pi(-n) = \zeta^{-n}(\varnothing, \Pi) = \mu^n(\varnothing, \Pi).$$

The RHS equality is by definition.
The LHS equality we must prove (though it has been heavily implied by our notation).

Proof.
Let $d = |\Pi|$. Then,

$$\begin{aligned} \zeta^{-1} = (\delta + \eta)^{-1} &= \delta - \eta + \eta^2 + \cdots + (-1)^d \eta^d + \cdots \\ &= \delta - \eta + \eta^2 + \cdots + (-1)^d \eta^d \end{aligned}$$

Taking powers of ζ^{-1} gives us

$$\zeta^{-n} = \sum_{k=0}^{d} (-1)^k \binom{n+k-1}{k} \eta^k$$

Recalling that we can recognize $\zeta^n(\varnothing, \Pi)$ as the polynomial $\sum_{k=0}^d \binom{n}{k}\eta^k$ together with the Reciprocity for Binomial Coefficients gives us the equality. $\square$

COMPUTING MÖBIUS FUNCTIONS

Before we compute $\mu_{\mathcal{J}(\Pi)}$, let's compute a simpler Möbius Function. Consider B_d, the powerset of $[d]$.

Proposition. (Möbius Function for Products of Posets)
Let $\Pi = P \times Q$ for posets P and Q and define $\leq_\Pi$ to be $\leq_P \times \leq_Q$. Then,

$$\mu_\Pi((a,c),(b,d)) = \mu_P(a,b)\mu_Q(c,d)$$

Proof.
Since the Möbius Function of a poset is unique it suffices to show

$$\sum_{(a,b)\leq(x,y)\leq(c,d)} \mu_P(a,b)\mu_Q(c,d) = \begin{cases} 1 & (a,b)=(c,d) \\ 0 & \text{otherwise.} \end{cases}$$

$$\begin{aligned}
\sum_{(a,b)\leq(x,y)\leq(c,d)} \mu_P(a,x)\mu_Q(c,y) &= \sum_{a\leq x\leq c}\left(\sum_{b\leq y\leq d} \mu_P(a,x)\mu_Q(c,y)\right) \\
&= \left(\sum_{a\leq x\leq c} \mu_P(a,x)\right)\left(\sum_{b\leq y\leq d} \mu_Q(c,y)\right) \\
&= \begin{cases} 1 & a=c \ \wedge \ b=d \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

Recognizing that $B_d = [2]^d$ we see that $\mu_{B_d}(T,S) = (-1)^{|S\setminus T|}$ if $T \subseteq S$ and 0 otherwise.

Proposition. (The Möbius Function of $\mathcal{J}(\Pi)$)
Let $K, M \in \mathcal{J}(\Pi)$ and $K \subseteq M$, then

$$\mu(K,M) = \begin{cases} (-1)^{|M\setminus K|} & M\setminus K \text{ is an antichain} \\ 0 & \text{otherwise.} \end{cases}$$

Proof.
Suppose $K \setminus M$ is an antichain. Then $A \cup K$ is an order ideal for any $A \subseteq M$. So $[K,M] \subseteq \mathcal{J}(\Pi)$ is isomorphic to B_d where $d = |M \setminus K|$.

Now we need to show $\mu(K,M) = 0$ when $M \setminus K$ contains comparable elements. We induct on the length of the interval $[K,M]$. For the base case this is 2. We have

$$\mu(K,M) = -\left(\mu(K,K) + \mu(K, K\cup\{x\})\right) = -(1-1) = 0.$$

Now we inductively assume $\mu(K,L)$ is zero for all $[K,L]$ of length $< n$ unless $|L\setminus K|$ is an antichain. Consider all L with $K \subseteq L \subset M$ with $L \setminus K$ an antichain. This number is seen to be even by partitioning the set into those that contain $\{m\}$ and those that do not. For some minimal element $m \in M \setminus K$. So the sum cancels. $\square$

THE PROOF

To prove the Order Polynomial Reciprocity:

(a) Expand $\mu^n(\varnothing, \Pi)$ as $\sum \mu(I_0, I_1) \cdots \mu(I_{n-1}, I_n)$ where the sum is over all multichains of size n in $\mathcal{J}(\Pi)$.

(b) This sum is $(-1)^{|\Pi|}$ times the number of multichains in $\mathcal{J}(\Pi)$ such that $I_k \setminus I_{k-1}$ is a multichain since $\sum_{k=1}^{n} |I_k \setminus I_{k-1}| = |\Pi|$ when $(I_0, \ldots, I_n)$ is a chain of order ideals from $\varnothing$ to Π such that $I_k \setminus I_{k-1}$ is an antichain.

Taking this together with the Order Bijection and the Nothing Explodes Lemma concludes the proof. $\square$