Graph Theory Exam Notes

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GRAPH BASICS

Definitions

Simple Graph
A simple graph is one with no multiple edges or loops.

Walk
A walk is an ordered set of vertices in which any two contiguous vertices are adjacent.

Path
A path is a walk with no repeated vertices.
$\{\text{paths of } G\} \subseteq \{\text{walks of } G\}$.

Clique
A clique is a subset of vertices in which any two vertices are non-adjacent. Equivalently, a clique is a K_n subgraph of G. The clique number $\omega(G)$ is the largest clique size in G.

Independent Set
An independent set is a subset of vertices such that no two vertices are adjacent.
The largest cardinality of an independent set in G is denoted $\alpha(G)$

Vertex Cover
A vertex cover is a subset of vertices that contains at least one endpoint of every edge.
The minimum size of a vertex cover is

$$|V| - \alpha(G)$$

For any connected graph let $\rho(G)$ be the size of a minimum edge cover. Then

$$\rho(G) + \alpha(G) = |V|$$

The First Theorem of Graph Theory

Binomial Definitions
Let $G = (V, E)$ be a graph. Then,

$$\sum_{v \in V} \deg(v) = 2|E|$$

PLANARITY

Graph Characteristics

Zeroth Betti Number
denoted $b_0 = b_0(G)$ is the number of connected components of a graph.

First Betti Number
denoted $b_1 = b_1(G)$ is equivalent to any of the following:

- the maximal number of edges whose removal does not increase b_0
- the number of edge deletions required to make G a tree
- the number of independent cycles in a graph

Other Betti Numbers
All other Betti Numbers of a graph are zero. $b_i(G) = 0 \quad \forall i \in \mathbb{N} \setminus \{0, 1\}$

Euler Characteristic
The Euler Characteristic of $G = (V, E)$ is given by

$$|V| - |E| = b_0 - b_1$$

It follows that any three of the following imply the remaining statement:

(i) G is connected $\quad b_0 = 1$
(ii) G is acyclic $\quad b_1 = 0$
(iii) G has n vertices $\quad |V| = n$
(iv) G has $n-1$ edges $\quad |E| = n - 1$

Famous Theorems

Euler's Formula
The Euler Characteristic of $G = (V, E)$ is given by

$$|V| - |E| + |F| = b_0 + 1 \quad \text{(Planar Graph)}$$
$$|V| - |E| + |F| = 2 \quad \text{(Planar, Connected Graph)}$$

Corollaries
The following inequality is true for all connected planar graphs:

$$|E| \leq 3|V| - 6$$

The following inequality is true for all connected, triangle free planar graphs with $|V| \geq 3$:

$$|E| \leq 2|V| - 4$$

The folllowing inequality is true for the 1-skeleton of any polyhedra:

$$2|E| \geq 3|F|$$

Kuratowski's Theorem
Let $G = (V, E)$ be a simple graph, then

G is planar is and only if G contains no subgraph
—or can be subdivided to be— isomorphic to K_5 or $K_{3,3}$.

SPECIAL WALKS

Hamiltonicity

Definition
Let $G = (V, E)$ be a graph. Then,

$$K_n \text{ has } \frac{(n-1)!}{2} \text{ Hamiltonian cycles}$$

$$\text{A Hamiltonian graph has at least } \binom{n-1}{2} + 1 \text{ edges.}$$

Grinberg's Theorem
Let $G = (V, E)$ be a planar graph. Let C be a Hamiltonian cycle in G. Let F_1 denote the set of faces inside C, and F_2 outside. For a face $f \in F = F_1 \cup F_2$, let $b(f)$ denote the number of edges bounding f. Then,

$$\sum_{f \in F_1} (b(f) - 2) = \sum_{f \in F_2} (b(f) - 2) = |V| - 2$$

Gray Codes
A k-digit Gray Code is a Hamiltonian cycle in Q_k the hypercube graph. This corresponds to all k-digit binary strings listed cyclically such that contiguous strings differ by only one bit.

Eulerian Walks

Necessary Conditions for Graphs
A connected graph G has an Eulerian walk if and only if:

every vertex in G has even degree
OR
G has at most two vertices of odd degree

A connected graph G is Eulerian (has a closed trail) if and only if:
every vertex in G has even degree

Necessary Conditions for Digraphs
A digraph $G = (V, E)$ without isolated vertices the following are equivalent:

- G has a closed Eulerian walk (Each directed edge can be traversed exactly once)
- G is connected (disregarding orientation) and $\text{indeg}(v) = \text{outdeg}(v)$ for all $v \in V$
- For any vertices $u, v \in V$ there is a walk that starts at u and ends at v and $\text{indeg}(v) = \text{outdeg}(v)$ for all $v \in V$

COLORINGS

Definitions

Proper Coloring
A proper coloring (coloring) of G is a labelling of its vertices such that all adjacent vertices have distinct labels. It follows that for any graph $G(V, E)$,

$$|E| \geq \binom{\mathcal{X}(G)}{2}$$

Chromatic Number
The chromatic number of a graph $\mathcal{X}(G)$ is the least number of colors needed to color G.

Bounds on the Chromatic Number

Lower Bounds
Let $G = (V, E)$ be a simple graph, let $\omega(G)$ be the clique number of G, and let $\alpha(G)$ be the independence number of G. Then,

$$\mathcal{X}(G) \geq \omega(G)$$
$$\mathcal{X}(G) \geq \frac{|V|}{\alpha(G)}$$

Upper Bounds
Let $G = (V, E)$ be a graph which is neither complete nor an odd cycle. Then,

$$\mathcal{X}(G) \leq \max_{v \in V} \deg(v)$$

Chromatic Polynomials

Definition
Let G be a graph. Then $p_G : \mathbb{Z}_{\geq} \to \mathbb{Z}$ is defined by:

$$p_G(k) = \text{ the number of colorings of } G \text{ with } k \text{ colors}$$

Consequences

(i) The coefficient of k^{n-1} in $p_G(k)$ is $-|E|$.
(ii) $p_G(1) = 0$ if and only if G has no edges.
(iii) $\mathcal{X}(G) = \min\{k \mid p_G(k) \neq 0\}$

Examples

If G is a forest with c connected components, then	$p_G(k) = \quad k^c(k-1)^{n-c}$
If $G = C_4$, then	$p_{C_4}(k) = \quad k(k-1)Jk^2 - 3k + 3)$
If G has no edges, then	$p_{C_4}(k) = \quad k^n$

Recursive Formulas
Let $G = (V, E)$ be a simple graph and $e \in E$ an edge. Then,
$G - e$ is the graph obtained from G by deleting e.
G/e is the graph obtained from G by contracting e.

$$p_G(k) = p_{G-e}(k) - p_{G/e}(k)$$

Let $G \sqcup H$ be the disjoint union of two graphs H and G. Then,

$$p_{G \sqcup H}(k) = p_G(k) \cdot p_H(k)$$

Let $G + H$ be the join of two graphs H and G. Then,

$$p_{G+H}(k) = \sum_a \sum_b \hat{p}_G(a)\hat{p}_H(b)\binom{a+b}{a}\binom{k}{a+b}$$

Using Exactly k Colors
Let $G = (V, E)$ be a simple graph and let $\hat{p}_G(k)$ denote the number of proper colorings of G using exactly k colors.

$$p_G(k) = \sum_{a=0}^{|V|} \hat{p}_G(k) \binom{k}{a}$$

note that $\hat{p}_G(k) \neq 0 \implies \mathcal{X}(G) \leq a \leq |V|$

Note that $\hat{p}_G(k)$ is the leftmost entry in the a-th row
of the difference table for the sequence $b = p_G(0), p_G(1), \ldots$

DIGRAPHS & FLOWS

Definitions

Transportation Network
Let $G = (V, E)$ be a graph. Then,

Flow
A flow in a transportation network is a function $f : E \to \mathbb{R}$ that satisfies two conditions,

1. feasibility:

$$0 \leq f(e) \leq c(e) \quad \forall e \in E$$

2. conservatiton law:

$$\sum_{\bullet \xrightarrow{e} v} f(e) = \sum_{v \xrightarrow{e} \bullet} f(e) \quad \forall v \in V \setminus \{s, t\}$$

Max-Flow Min-Cut Theorem
The maximum value of a flow in a transportation network
is equal to the minimum capacity of a cut:

$$\max_f |F| = \min_{(X,Y)} c(X, Y)$$

Matchings

SPICY EXTRAS

Pigeon Hole Principle

Statement
If $|A| > |B|$, there are no injections $f : A \to B$.

Corollaries

(i) Congruence classes

(ii)

Posets

Definition

A partially ordered set (or poset) P is a set endowed with a binary relation denoted $\leq$, that satisfies three conditions:

(i) reflexivity:

$$\forall x \in P, \quad x \leq x$$

(ii) antisymmetricity:

$$x \leq y \land y \leq x \implies x = y$$

(iii) transitivity:

$$x \leq y \land y \leq z \implies x \leq z$$

Combinatorial Reciprocity Theorems

Acyclic Orientations

The number of directed graphs obtained by orienting the edges of $G = (V, E)$ without creating a directed cycle is

$$ao(G) = (-1)^{|V|} \cdot p_G(-1)$$

And defined by the recursion

$$ao(G) = ao(G - e) + ao(G/e)$$

Note that if G is a tree, then $ao(G) = 2^{|E|}$
If $G = C_n$ is a cycle, then $ao(G) = 2^n - 2$
If $G = K_n$, then $ao(G) = n!$

Automorphisms