Enumerative Combinatorics Exam Notes

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COEFFICIENTS

Binomial Coefficients

Binomial Definitions
For nonnegative integer n and k,

$$\binom{n}{k} = \frac{n!}{k!(n-k!)} \quad \text{and} \quad \binom{n}{k} = 0 \quad \text{for } k > n$$

$$\binom{n}{k} = \binom{n}{n-k}$$

Multinomial Coefficients

Multinomial Definitions
Note that nonnegative, integer $n = \sum_{i=1}^{k} m_i$,

$$\binom{n}{m_1, \ldots, m_k} = \frac{n!}{m_1! \cdots m_k!}$$

$$\binom{n}{k} = \binom{n}{k, n-k}$$

Two Important Theorems

Binomial Theorem
For nonnegative integer n,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Multinomial Theorem
For nonnegative integer n,

$$(x_1 + \cdots + x_k)^n = \sum_{m_1+\cdots+m_k \; = \; n}^{n} \binom{n}{m_1, \ldots, m_k} x_1^{m_1} \cdots x_k^{m_k}$$

Binomial Identities

Pascal's Recurrence

To construct Pascal's Triangle use,

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

Binomial Relations

For integer $n,\ m$, and k,

$$\sum_{i=0}^{n} \binom{n}{i} = 2^n$$

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} = 0$$

$$\sum_{i=0}^{l} \binom{n}{i} \binom{m}{l-i} = \binom{n+m}{l}$$

$$\sum_{i=0}^{n} \binom{n}{i}^2 = \binom{2n}{n}$$

$$\sum_{j=0}^{n} j \binom{n}{j} = n2^{2n-1}$$

For nonnegative integer k and n such that $k \leq n$,

$$\sum_{j=0}^{n} \binom{j}{k} = \sum_{j=k}^{n} \binom{j}{k} = \binom{n+1}{k+1}$$

GENERATING FUNCTIONS

Binomial Coefficients

Binomial Definitions
For integer k,

$$\binom{-n}{k} = (-1)^k \binom{n+k-1}{k}$$

Generalized Theorems

Newton's Binomial Theorem
For nonnegative integer n,

$$(x+y)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k y^{\alpha-k}$$

holds whenever $y^{\alpha-k}$ and x^b is uniquely defined.

Standard Generating Functions

Fundamental Formal Power Series

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k \qquad h_n = 1$$

$$\frac{1}{1+x} = \sum_{k=0}^{\infty} (-1)^k x^k \qquad h_n = (-1)^n$$

$$x \cdot \frac{d}{dx}\left(\frac{1}{1-x}\right) = \sum_{k=0}^{\infty} k x^k \qquad h_n = n$$

$$(1+x)^n = \sum_{k=0}^{\infty} \binom{n}{k} x^k \qquad h_n = \binom{n}{k}$$

$$(1-x)^{-n} = \sum_{k=0}^{\infty} \binom{n+k-1}{n-1} x^k \qquad h_n = \binom{n+k-1}{n-1}$$

Finite Geometric Series
For nonnegative integer n,

$$\sum_{k=0}^{n} x^k = \frac{1-x^{n+1}}{1-x}$$

Weight Functions

Definition
Let $\Omega : A \to \mathbb{Z}_{\geq 0}$ be a "weight function" on a set A.
For each value k, let

$$h_k = \text{ number of elements } a \in A \text{ with } \Omega(a) = k$$

Then the generating function

$$H(x) = \sum_k h_k x^k = \sum_{a \in A} x^{\Omega(a)}.$$

Note that $h(1) = |A|$.

Multiplication Principle
Suppose there is a bijection of the form

$$A \longleftrightarrow B \times C \times D \times \cdots$$

$$a \longleftrightarrow (b, c, d, \cdots).$$

and that

$$\Omega : A \to \mathbb{Z}_{\geq 0}$$
$$\beta : B \to \mathbb{Z}_{\geq 0}$$
$$\gamma : C \to \mathbb{Z}_{\geq 0}$$
$$\delta : D \to \mathbb{Z}_{\geq 0}$$
$$\vdots$$

are weight functions satisfying the additivity condition

$$\Omega(a) = \beta(b) + \gamma(c) + \delta(d) + \cdots$$

Then,

$$\sum_{a \in A} x^{\Omega(a)} = \left(\sum_{b \in B} x^{\beta(b)} \right) \left(\sum_{c \in C} x^{\gamma(c)} \right) \left(\sum_{d \in D} x^{\delta(d)} \right) \cdots$$

LINEAR RECURRENCES

Homogenous Recurrences

Characteristic Expression
For a homogenous linear recurrence of the form,

$$h_n + a_1 h_{n-1} + a_2 h_{n-2} + \cdots + a_k h_{n-k} = 0$$

the characteristic expression is

$$q^k + a_1 q^{k-1} + a_2 a^{k-2} + \cdots + a_k = 0.$$

The roots and initial conditions gives the closed form of the recurrence.

General Solution for Distinct Roots
If the characteristic polynomial of a homogenous linear recurrence has distinct roots $\{q_1, \ldots, q_k\}$, then its general solution is

$$h_n = \sum_{i=1}^{k} c_i q_i^n$$

General Solution for Multiplicitous Roots
If a homogenous linear recurrence has characteristic polynomial with root q_0 of multiplicity m, then

$$h_n = n^w q_0^n \quad \text{for} \quad w \in \{0, 1, \ldots, m-1\}$$

satisfies the recurrence.

In general, let $\{q_0, \ldots, q_r\}$ be roots of the characteristic equation with respective multiplicities $\{m_0, \ldots, m_r\}$, then

$$h_n = C_1(n) q_0^n + \cdots + C_r(n) q_r^n$$

where each $C_i(n)$ is a polynomial in n of degree less than m_i

Non-homogenous Recurrences

General Solution
Given a sequence $(b_n) = (b_0, b_1, b_2, \dots)$, the linear non-homogenous recurrence is

$$\sum_{i=0}^{k} a_i h_{n-i} = b_n$$

Strategy

1. Find roots $\{q_1, \dots, q_r\}$ of the characteristic expression for the homogenous recurrence.
2. Guess solutions of the form $h_n = c \cdot b_n,\ c_1 \cdot n \cdot b_n$, etc.
3. Fit $c_1, \dots, c_r$ to initial conditions for $c_1(q_1)^n + c_2(q_2)^n + \cdots + c_r(q_r)^n + \{\text{GUESS}\}$

Difference Sequences

Difference Sequences of h_n
The k-th difference sequence of h_n is given by,

$$\sum_{i=0}^{k} (-1)^i \binom{k}{i} h_{n-i}$$

SPECIAL COUNTING SEQUENCES

Stirling Numbers

Stirling Numbers of the Second Kind
Are defined by,

$$S(n,k) = \frac{1}{k!}\sum_{i=0}^{k}(-1)^i \binom{k}{i}(k-i)^n \quad \text{equivalently,} \quad S(n,k) = \frac{(-1)^k}{k!}\sum_{i=0}^{k}(-1)^i \binom{k}{i} i^n$$

$$S(n,k) = S(n-1,k-1) + k \cdot S(n,k-1)$$

Signless Stirling Numbers of the First Kind
Follow the recurrence relation,

$$c(n,k) = c(n-1,k-1) + (n-1)\cdot c(n-1,k)$$

Stirling Numbers of the First Kind
Are defined by,

$$s(n,k) = (-1)^{n-k}\cdot c(n,k)$$

Formal Power Series Involving Stirling Numbers
Stirling numbers provide closed forms for powers of x and falling factorials,

$$x^n = \sum_{k=1}^{n} S(n,k)\cdot k!\binom{x}{k} = \sum_{k=1}^{n} S(n,k)\cdot (x)_k$$

$$(x)_n = x(x-1)(x-2)\cdots(x-n+1) = n!\binom{x}{n} = \sum_{k=1}^{n} s(n,k)x^k$$

Interestingly,

$$\begin{bmatrix} S(1,1) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ S(n,1) & \cdots & S(n,k) \end{bmatrix}\begin{bmatrix} s(1,1) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ s(n,1) & \cdots & s(n,k) \end{bmatrix} = I_n$$

This Sequence Counts . . .
$k!\cdot S(n,k)$ enumerates:

- Partitions of $\{1,\ldots,n\}$ into k distinguished, non-empty boxes.
  Divide by $k!$ for identical boxes.
- Surjections $f:\{1,\ldots,n\}\to\{1,\ldots,k\}$.

$c(n,k)$ enumerates:

- Permutations of $\{1,\ldots,n\}$ with k disjoint cycles.

Fibonacci Numbers

Definition
For nonnegative integer n,

$$f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

$$f_n = f_{n-1} + f_{n-2}$$

This Sequence Counts ...
f_{n+1} enumerates:

- Binary strings of length $n-1$ that do not contain consecutive 1s.
- Representations of n as an ordered sum of 1s and 2s.
- Tilings of a $2 \times n$ board with dominoes.

Derangement Numbers

Definition
For nonnegative integer n,

$$D_n = n\sum_{k=0}^{n} \frac{(-1)^k}{k!}$$

$$D_n = n \cdot D_{n-1} + (-1)^n$$

$$D_n = \left\{\text{closest integer to } \frac{n!}{e}\right\}$$

This Sequence Counts ...
D_n enumerates:

- Permutations of $\{1, \ldots, n\}$ with no one-cycles.
- Equivalently, permutations of $\{1, \ldots, n\}$ with no number in its natural position.

Catalan Numbers

Definition
For nonnegative integer n,

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{1}{2n+1}\binom{2n+1}{n}$$

$$C_n = \sum_{k+l=n-1}^{n-1} C_k \cdot C_l \quad \text{equivalently} \quad C_{n+1} = \sum_{k=0}^{n} C_k \cdot C_{n-k}$$

This Sequence Counts . . .
C_n enumerates:

- Ballot sequences of length $2n$.
- Dyck Paths from $(0,0)$ to $(2n,0)$ above $y=0$.
- Lattice Paths from $(0,0)$ to (n,n) above $y=x$.
- Triangulations of a convex $(n+2)$-gon by non-crossing diagonals.
- Ordered trees on $n+1$ unlabeled vertices.
- Complete (binary) ordered trees with $n+1$ leaves (or equivalently n internal vertices).
- Syntactically correct bracketings of a sequence of $n+1$ letters.

Partition Numbers

Definition
For nonnegative integer k,

$$\sum_{k=1}^{\infty} p(k)x^k = \prod_{k=1}^{\infty} \frac{1}{1-x^k}$$

$$\sum_{k=1}^{\infty} p_d(k)x^k = \prod_{k=1}^{\infty} (1+x^k)$$

This Sequence Counts . . .
$p(n)$ enumerates:

- All weakly decreasing strong compositions of n.
- All (unordered) multisets of positive integers whose sum is n.
- Nonnegative integer solutions of $m_1 + 2m_2 + 3m_3 + \cdots = n$.

$p_d(n)$ enumerates:

- Partitions with distinct parts.

COMMON THINGS TO COUNT

Enumerating Sets with Intersections

Inclusion-Exclusion Principle
Let S be a finite set and $A_1, \ldots, A_r \subseteq S$.
Then,

$$\left| S - \bigcup_{i=1}^{r} A_i \right| = \sum_{i=1}^{r} (-1)^i n_i$$

where $n_0 = |S|$ and, for $i \in \{1, \ldots, r\}$

$$n_i = \sum_{1 \leq j_1 < \cdots < j_i \leq} |A_{j_1} \cap \cdots \cap A_{j_i}|$$

the sum ranges over all subsets of $A_1, \ldots, A_r$.

Euler's Totient Function
Let $\{p_1, \ldots, p_r\}$ be the set of distinct prime divisors of n.
Let $\phi : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ be the mapping from n to the number of integers coprime to n.
Then,

$$\phi(n) = n \prod_{j=1}^{r} \left(1 - \frac{1}{p_j}\right)$$

Compositions

Weak Compositions
Enumerates the nonnegative integer solutions of $x_1 + \cdots + x_k = n$ is

$$\binom{n+k-1}{k-1}$$

Strong Compositions
Enumerates the positive integer solutions of $x_1 + \cdots + x_k = n$ is

$$\binom{n-1}{k-1}$$

$$\sum_{k=1}^{n} \binom{n-1}{k-1} = 2^{n-1}$$

Lattice Paths

Set Partitions
Enumerates the paths from $(0, 0)$ to (n, m) with moves $(1, 0)$ and $(0, 1)$

$$\binom{n+m}{n} \quad \text{equivalently} \quad \binom{n+m}{m}$$

Subsets

Set Partitions
Enumerates the number of subsets of a set A such that $|A| = n$

$$2^n$$

Finite Strings

Alphabets
The number of k-digit strings over an n-element alphabet is

$$n^k$$

The number of k-digit strings over an n-element alphabet,
in which no letter is used more than once, is

$$(n)_k = \frac{n!}{(n-k)!}$$