Math 217 Exam Two Notes

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COORDINATES

Coordinates Matrices

Book Theorem 4.3.2

The $\mathfrak{B}$-matrix of T

$$[T]_{\mathfrak{B}} = \begin{bmatrix} | & | & & | \\ [T(\vec{v}_1)]_{\mathfrak{B}} & [T(\vec{v}_2)]_{\mathfrak{B}} & \cdots & [T(\vec{v}_n)]_{\mathfrak{B}} \\ | & | & & | \end{bmatrix}$$

$$[T]_{\mathfrak{B}} = S_{\mathfrak{U}\to\mathfrak{B}}[T]_{\mathfrak{U}}S_{\mathfrak{B}\to\mathfrak{U}}$$

Book Theorem 4.3.2

Change of Basis Matrix $S_{\mathfrak{U}\to\mathfrak{B}}$

$$S_{\mathfrak{U}\to\mathfrak{B}} = (S_{\mathfrak{B}\to\mathfrak{U}})^{-1}$$

$$S_{\mathfrak{U}\to\mathfrak{B}} = \begin{bmatrix} | & | & & | \\ [\vec{u}_1]_{\mathfrak{B}} & [\vec{u}_2]_{\mathfrak{B}} & \cdots & [\vec{u}_n]_{\mathfrak{B}} \\ | & | & & | \end{bmatrix}$$

Book Theorem 4.3.4

Change of Basis in a subspace of $\mathbb{R}^n$

$$\begin{bmatrix} | & | & & | \\ \vec{b}_1 & \vec{b}_2 & \cdots & \vec{b}_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_n \\ | & | & & | \end{bmatrix} S_{\mathfrak{B}\to\mathfrak{U}}$$

Coordinate Theorems

Worksheet 16 Problem 3

Suppose that $\mathfrak{U} = (\vec{u}_1, \vec{u}_2, ..., \vec{u}_n)$ is an orthonormal ordered basis for $\mathbb{R}^n$

$$[\,\vec{x}\,]_{\mathfrak{U}} = \begin{bmatrix} \vec{x}\cdot\vec{u}_1 \\ \vec{x}\cdot\vec{u}_2 \\ \vdots \\ \vec{x}\cdot\vec{u}_n \end{bmatrix}$$

ORTHOGONALITY

Orthogonal Complement Theorems

Book Theorem 5.4.1

$$(\operatorname{im} A)^{\perp} = \ker(A^T)$$
$$\operatorname{im}(A^T) = (\ker A)^{\perp}$$

Book Theorem 5.1.8
If V is a subspace, then

$$V^{\perp} \text{ is a subspace}$$
$$(V^{\perp})^{\perp} = V$$
$$n = \dim(V) + \dim(V^{\perp})$$
$$V \cap V^{\perp} = \vec{0}$$

Homework 7 Problem 4
If V is a subspace, then

$$V^{\perp} + W^{\perp} = (V \cap W)^{\perp}$$
$$(V + W)^{\perp} = V^{\perp} \cap W^{\perp}$$

Orthogonal Matrix Characterizations

Book Theorem 5.3.8

$$A^T A = I_n$$
$$A^{-1} = A^T$$
$$||A\vec{x}|| = ||\vec{x}||$$
$$(A\vec{x}) \cdot (A\vec{y}) = \vec{x} \cdot \vec{y}$$

Book Theorem 5.3.4
if A and B are orthogonal matrices

$$AB \text{ is orthogonal}$$
$$A^{-1} \text{ is orthogoanl}$$

Graham Schmidt

Book Theorem 5.2.1
Construct an orthonormal basis $(\vec{u}_1, ..., \vec{u}_n)$
from any basis $(\vec{v}_1, ..., \vec{v}_n)$

$$\vec{u}_j = \frac{\vec{v}_j^{\perp}}{||\vec{v}_j^{\perp}||}$$

$$\vec{v}_j^{\perp} = \vec{v}_j - (\vec{u}_1 \cdot \vec{v}_j)\vec{u}_1 - \cdots - (\vec{u}_{j-1} \cdot \vec{v}_j j)\vec{u}_{j-1}$$

QR Factorization

Book Theorem 5.2.2
QR factorization of a matrix M

$$\begin{array}{ccc} M = & Q & R \\ \begin{bmatrix} | & | & & | \\ \vec{b}_1 & \vec{b}_2 & \cdots & \vec{b}_n \\ | & | & & | \end{bmatrix} = & \begin{bmatrix} | & | & & | \\ \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_n \\ | & | & & | \end{bmatrix} & \begin{bmatrix} ||\vec{v}_1|| & \vec{u}_1 \cdot \vec{v}_2 & \vec{u}_1 \cdot \vec{v}_3 & \cdots & \vec{u}_1 \cdot \vec{v}_n \\ 0 & ||\vec{v}_2^{\perp}|| & \vec{u}_2 \cdot \vec{v}_3 & \cdots & \vec{u}_2 \cdot \vec{v}_n \\ 0 & 0 & ||\vec{v}_3^{\perp}|| & \cdots & \vec{u}_3 \cdot \vec{v}_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & ||\vec{v}_n^{\perp}|| \end{bmatrix} \end{array}$$

DOT PRODUCT & PROJECTIONS

Dot Product Theorems

Book Theorem 5.1.1

$$||\vec{v}|| = \sqrt{\vec{v}\cdot\vec{v}}$$
$$\vec{v}\cdot\vec{w} = 0 \iff \vec{v}\perp\vec{w}$$
$$\vec{u}\cdot\vec{u} = 1 \iff \vec{u} \text{ is a unit vector}$$

Book Theorem 5.3.6

$$A\vec{v}\cdot\vec{w} = \vec{v}\cdot A^T\vec{w}$$

Book Theorem 5.1.9 - 5.1.12

$||\vec{x}+\vec{y}||^2 = ||\vec{x}||^2 + ||\vec{y}||^2$ **Pythagorean Theorem** *for orthogonal vectors

$||\text{proj}_V\vec{x}|| \leq ||\vec{x}||$ **Magnitude of proj$_V\vec{x}$**

$|\vec{x}\cdot\vec{y}| \leq ||\vec{x}||\;||\vec{y}||$ **Cauchy-Schwarz Inequality**

$\cos\theta = \dfrac{\vec{x}\cdot\vec{y}}{||\vec{x}||\;||\vec{y}||}$ **Angle between two vecors**

Projections

Book Theorem 5.1.5

If V is a subspace of $\mathbb{R}^n$ with orthonormal basis $\mathfrak{U} = (\vec{u}_1, \vec{u}_2, ..., \vec{u}_n)$ then

$$\text{proj}_V\vec{x} = (\vec{u}_1\cdot\vec{x})\vec{u}_1 + (\vec{u}_2\cdot\vec{x})\vec{u}_2 + \cdots + (\vec{u}_n\cdot\vec{x})\vec{u}_n$$

for all $\vec{x}\in\mathbb{R}^n$

Orthogonal Projection

Book Theorem 5.1.5

If V is a subspce of $\mathbb{R}^n$ with orthogonal basis $\mathfrak{B} = (\vec{v}_1, \vec{v}_2, ..., \vec{v}_n)$ then

$$\text{proj}_V\vec{x} = \frac{\vec{v}_1\cdot\vec{x}}{\vec{v}_1\cdot\vec{v}_1}\vec{v}_1 + \frac{\vec{v}_2\cdot\vec{x}}{\vec{v}_2\cdot\vec{v}_2}\vec{v}_2 + \cdots + \frac{\vec{v}_n\cdot\vec{x}}{\vec{v}_n\cdot\vec{v}_n}\vec{v}_n$$

for all $\vec{x}\in\mathbb{R}^n$

Matrix of an Orthogonal Projection

Book Theorem 4.3.4
Projection onto a subspace $V \subseteq \mathbb{R}^n$ with orthonormal basis $\mathfrak{U} = (\vec{u}_1, \vec{u}_2, ..., \vec{u}_d)$ and basis $\mathfrak{B} = (\vec{b}_1, \vec{b}_2, ..., \vec{b}_d)$

$$P = QQ^T \quad \text{where} \quad Q = \begin{bmatrix} | & | & & | \\ \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_d \\ | & | & & | \end{bmatrix}$$

$$P = A(A^TA)^{-1}A^T \quad \text{where} \quad A = \begin{bmatrix} | & | & & | \\ \vec{b}_1 & \vec{b}_2 & \cdots & \vec{b}_d \\ | & | & & | \end{bmatrix}$$

Book Theorem 5.4.6
$A\vec{x} = \vec{b}$ has the unique solution

$$\vec{x}^* = (A^TA)^{-1}A^T\vec{b}$$

TRANSPOSE & LEAST SQUARES

Transpose Theorems

Book Theorem 5.3.9

$$(A+B)^T = A^T + B^T$$
$$(kA)^T = kA^T$$
$$(AB)^T = B^T A^T$$
$$\text{rank}(A) = \text{rank}(A^T)$$
$$(A^T)^{-1} = (A^{-1})^T$$

Homework 8 Problem 3

$$\ker(A) = \ker(A^T A)$$
$$\text{im}(A) = \text{im}(AA^T)$$

Least Squares Definition

Book Theorem 5.4.5

$A\vec{x} = \vec{b}$ has the least-squares solutions of the consistent system

$$A^T A\vec{x} = A^T\vec{b}$$

Book Theorem 5.4.6

If $\ker(A) = \{\vec{0}\}$, then linear system

$A\vec{x} = \vec{b}$ has the unique solution

$$\vec{x}^* = (A^T A)^{-1} A^T \vec{b}$$

Least Squares Theorems

Book Theorem 5.4.6

If $\ker(A) = \{\vec{0}\}$, then linear system

$A\vec{x} = \vec{b}$ has the unique solution

$$\vec{x}^* = (A^T A)^{-1} A^T \vec{b}$$

GEOMETRIC TRANSFORMATIONS

Projections

Rotations

Reflections