Math 566

Combinatorial Theory II

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Contents

Chapter 1

Algebraic Graph Theory

Syllabus

No Exams.

There will be five or six problem sets.

Office Hours: Tuesdays and Friday at 1pm.

There will be almost no calculus or analysis.

You should expect this class to make frequent use of linear algebra.

1.2 Linear Algebraic Preliminaries

Definition 1.2.1 (Characteristic Polynomial)

Let M be a $p \times p$ matrix in $\mathbb{C}$, the **monic characteristic polynomial** is

$$\det(tI - M) = \prod_{k=1}^{p} (t - \lambda_k)$$

where $\lambda_1, ..., \lambda_k \in \mathbb{C}$ are the p eigenvalues with multiplicity.

Lemma 1.2.2 (Eigenvalues of Matrix Polynomial)

If $f(t) \in \mathbb{C}[t]$ then $f(M)$ has eigenvalues $f(\lambda_1), ..., f(\lambda_k)$.

Proof.

M is diagonalizable, so conjugation commutes with taking powers and therefore computation of a polynomial. The statement is trivial up to general nonsense consideration.

Diagonalizable matrices are dense in the set of matrices. A matrix is only diagonalizable if there are multiple equal eigenvalues. Thus this is a subvariety within the set of matrices (obtained by imposing an algebraic condition of equal eigenvalues) which has dimension less than the set of matrices.

Now we can take a limit within the set of diagonalizable matrices and the limit converges to the general matrix. □

This statement can be extended to more general functions (those with converging power series).

Lemma 1.2.3 (Trace of Matrix)

The trace of a matrix M is the sum of its eigenvalues.

Proof.
The coefficient of t^{p-1} in $\det(tI - M)$ is

$$-\operatorname{tr}(M) = \sum_{i=1}^{p} -\lambda_k.$$

□

Combining this fact with Lemma 1.2.2 gives us that

$$\operatorname{tr}\left(M^\ell\right) = \sum_{k=1}^{p} \lambda_k^\ell.$$

Thus it is easy to compute the sum of powers of eigenvalues of M. This leads to an algorithm of finding roots of polynomials from power sums.
We can recover the multi-set $\{\lambda_1, ..., \lambda_p\}$ from the traces $\operatorname{tr}(M), \operatorname{tr}(M^2), ...$.

Theorem 1.2.4 (Multiset Recovery)

Let $\alpha_1, ..., \alpha_r, \beta_1, ..., \beta_s \in \mathbb{C}$ such that $\forall \ell \in \mathbb{Z}^+$

$$\sum_{i=1}^{r} \alpha_i^\ell = \sum_{i=1}^{s} \beta_i^\ell \quad (\star)$$

then $r = s$ and the β's are permutations of the α's.

If the values where over the reals this would be easy since we could just observe the asymptotic behavior and remove each leading term. This cannot be applied to complex numbers since some may have the same modulus

Proof.
We will use the method of generating functions in a noncombinatorial sense. First multiply $(\star)$ by t^ℓ and sum giving,

$$\sum_{i=1}^{r} \frac{\alpha_i t}{1 - \alpha_i t} = \sum_{i=1}^{s} \frac{\beta_i t}{1 - \beta_i t}.$$

We use that $\sum_{k=1}^{\infty} = \frac{x}{1-x}$. Pick $\gamma \in \mathbb{C}$, and multiply by $1 - \gamma t$ and set $t = \frac{1}{\gamma}$. Take the limit $t \to \frac{1}{\gamma}$. Each term will become one or zero depending on whether the corresponding α_k or β_k equals γ or not.
After that, LHS will be the number of α's equal to γ and the RHS will be the number of β's equal to γ. This shows the multisets are equal. □

1.3 Graph Eigenvalues

We assume all basic vocabulary of basic graph theory. All graphs will be assumed finite. We allow loops & multiple edges.

Lemma 1.3.1 (Walks on a Graph)

Let G be a graph on the vertex set $[p]$. Then put $M = A(G)$, the **adjacency matrix** of G. So M_{ij} is the number of edges from i to j. Then M is symmetric, with entries in $\mathbb{Z}_{\geq 0}$. The number of walks of length ℓ from i to j is $\left(M^{\ell}\right)_{ij}$.

Proof.
Follows from matrix multiplication (also works for directed graphs). □

This suggests that there must be a way of counting walks using eigenvalues of the graph. But we will have to restrict to closed paths. A walk can move anywhere in the graph, a path cannot repeat vertices. A **marked closed walk** is one that starts and begins at the same vertex.

Lemma 1.3.2 (Marked Closed Walks)

The number of marked closed walks of length ℓ on G is $\sum_{k=1}^{p} \lambda_k^{\ell}$.

Proof.
The LHS equals $\operatorname{tr}\left(M^{\ell}\right)$ be observation and Lemma 1.2.3 gives the equality with the RHS. □

Example: Let $G = K_p$ (the complete graph).
Let J be the $p \times p$ matrix with all entries equal to 1. Then $J - I = A(G)$. Then $\operatorname{rank}(J) = 1$. So the eigenvalues of J are $0, ..., 0$ ($p - 1$ times) and p. Now $J - I$ is applying the polynomial $t - 1$ to J. Therefore by Lemma 1.2.2, its eigenvalues are $-1, ..., -1$ ($p - 1$ times) and $p - 1$. Thus the number of marked closed walks in G is

$$(p-1)^{\ell} + (p-1)(-1)^{\ell}.$$

Restatement: The number of marked closed walks of length ℓ in G is the number of $(\ell + 1)$-letter words in a p-symbol alphabet such that consecutive letters are distinct and first letter equals the last.

Homework 1.1

Show that the number of walks of length ℓ between two distinct vertices in K_p differs by one from the number of closed walks of length ℓ starting and ending at a given vertex. Use a simple combinatorial argument.