Math 566

Algebraic Combinatorics

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Contents

Chapter 1
Algebraic Graph Theory

Notes from the Syllabus:

- No Exams.
- There will be five or six problem sets.
- Office Hours: Tuesdays and Friday at 1pm.
- There will be almost no calculus or analysis.
- You should expect this class to make frequent use of linear algebra.
- You need to solve all but two problems correctly to get 100%.
- There will be very little partial credit given.
- You can either bring in a printed copy of the homework to class.

1.1 Linear Algebraic Preliminaries

Definition 1.1.1 (Characteristic Polynomial)

Let M be a $p \times p$ matrix in $\mathbb{C}$, the **monic characteristic polynomial** is

$$\det(tI - M) = \prod_{k=1}^{p} (t - \lambda_k)$$

where $\lambda_1, ..., \lambda_k \in \mathbb{C}$ are the p eigenvalues with multiplicity.

Lemma 1.1.2 (Eigenvalues of Matrix Polynomial)

If $f(t) \in \mathbb{C}[t]$ then $f(M)$ has eigenvalues $f(\lambda_1), ..., f(\lambda_k)$.

Proof.

M is diagonalizable, so conjugation commutes with taking powers and therefore computation of a polynomial. The statement is trivial up to general nonsense consideration.

Diagonalizable matrices are dense in the set of matrices. A matrix is only diagonalizable if there are multiple equal eigenvalues. Thus this is a subvariety within the set of matrices (obtained by imposing an algebraic condition of equal eigenvalues) which has dimension less than the set of matrices.

Now we can take a limit within the set of diagonalizable matrices and the limit converges to the general matrix. □

This statement can be extended to more general functions (those with converging power series).

Lemma 1.1.3 (Trace of Matrix)

The trace of a matrix M is the sum of its eigenvalues.

Proof.
The coefficient of t^{p-1} in $\det(tI - M)$ is

$$-\operatorname{tr}(M) = \sum_{i=1}^{p} -\lambda_k.$$

□

Combining this fact with Lemma 1.1.2 gives us that

$$\operatorname{tr}\left(M^\ell\right) = \sum_{k=1}^{p} \lambda_k^\ell.$$

Thus it is easy to compute the sum of powers of eigenvalues of M. This leads to an algorithm of finding roots of polynomials from power sums.
We can recover the multi-set $\left\{\lambda_1, ..., \lambda_p\right\}$ from the traces $\operatorname{tr}(M), \operatorname{tr}\left(M^2\right), ...$.

Theorem 1.1.4 (Multiset Recovery)

Let $\alpha_1, ..., \alpha_r, \beta_1, ..., \beta_s \in \mathbb{C}$ such that $\forall \ell \in \mathbb{Z}^+$

$$\sum_{i=1}^{r} \alpha_i^\ell = \sum_{i=1}^{s} \beta_i^\ell \quad (\star)$$

then $r = s$ and the β's are permutations of the α's.

If the values where over the reals this would be easy since we could just observe the asymptotic behavior and remove each leading term. This cannot be applied to complex numbers since some may have the same modulus

Proof.
We will use the method of generating functions in a noncombinatorial sense. First multiply $(\star)$ by t^ℓ and sum giving,

$$\sum_{i=1}^{r} \frac{\alpha_i t}{1 - \alpha_i t} = \sum_{i=1}^{s} \frac{\beta_i t}{1 - \beta_i t}.$$

We use that $\sum_{k=1}^{\infty} = \frac{x}{1-x}$. Pick $\gamma \in \mathbb{C}$, and multiply by $1 - \gamma t$ and set $t = \frac{1}{\gamma}$. Take the limit $t \to \frac{1}{\gamma}$. Each term will become one or zero depending on whether the corresponding α_k or β_k equals γ or not.

After that, LHS will be the number of α's equal to γ and the RHS will be the number of β's equal to γ. This shows the multisets are equal. $\square$

And what if I put a little more text

1.2 Enumerating Walks with Eigenvalues

We assume all basic vocabulary of basic graph theory. All graphs will be assumed finite. We allow loops & multiple edges.

Lemma 1.2.1 (Walks on a Graph)

Let G be a graph on the vertex set $[p]$. Then put $M = A(G)$, the **adjacency matrix** of G. So M_{ij} is the number of edges from i to j. Then M is symmetric, with entries in $\mathbb{Z}_{\geq 0}$. The number of walks of length ℓ from i to j is $\left(M^{\ell}\right)_{ij}$.

Proof.

Follows from matrix multiplication (also works for directed graphs). $\square$

This suggests that there must be a way of counting walks using eigenvalues of the graph. But we will have to restrict to closed paths. A walk can move anywhere in the graph, a path cannot repeat vertices. A **marked closed walk** is one that starts and begins at the same vertex.

Proposition 1.2.2 (Marked Closed Walks)

The number of marked closed walks of length ℓ on G is $\sum_{k=1}^{p} \lambda_k^{\ell}$.

Proof.

The LHS equals $\mathrm{tr}\left(M^{\ell}\right)$ be observation and Lemma 1.1.3 gives the equality with the RHS. $\square$

Example: Let $G = K_p$ (the complete graph).

Let J be the $p \times p$ matrix with all entries equal to 1. Then $J - I = A(G)$. Then $\mathrm{rank}(J) = 1$. So the eigenvalues of J are $0, \ldots, 0$ ($p - 1$ times) and p. Now $J - I$ is applying the polynomial $t - 1$ to J. Therefore by Lemma 1.1.2, its eigenvalues are $-1, \ldots, -1$ ($p - 1$ times) and $p - 1$. Thus the number of marked closed walks in G is

$$(p-1)^{\ell} + (p-1)(-1)^{\ell}.$$

Restatement: The number of marked closed walks of length ℓ in G is the number of $(\ell + 1)$-letter words in a p-symbol alphabet such that consecutive letters are distinct and first letter equals the last.

Homework 1.1

Show that the number of walks of length ℓ between two distinct vertices in K_p differs by one from the number of closed walks of length ℓ starting and ending at a given vertex. Use a simple combinatorial argument.

We can also compute eigenvalues via counting walks. Using combinatorics for linear algebra!

Example: Suppose $G = K_{m,n}$, the complete bipartite graph. Let's count the marked closed walks in G. This number is

$$\left(\sqrt{mn}\right)^{\ell} + \left(-\sqrt{mn}\right)^{\ell} + 0^{\ell} + \cdots + 0^{\ell} = \begin{cases} 0 \text{ if } \ell \bmod 2 = 0 \\ 2(mn)^{\frac{\ell}{2}} \text{ if } \ell \bmod 2 = 1. \end{cases}$$

Now we conclude from Theorem 1.1.4 that $K_{m,n}$ has eigenvalues $-\sqrt{mn}, \sqrt{mn}, 0, \ldots, 0$. No linear algebra involved!

Homework 1.2

Prove that the **diameter** (the largest minimal pairwise distance of a connected graph is strictly less than the number of distinct eigenvalues.

Inequalities for Maximal Eigenvalues

Proposition 1.2.6 (Max)

Let G be a graph on $[p]$. Denote $\lambda_{\max} = \max_i |\lambda_i|$. The Perron-Frobenius Theorem ensures that the largest eigenvalue is positive (hence equal to $\lambda_{\max}$).

$$\lambda_{\max} \leq \max_{v \in G} \deg(v)$$

Perron-Frobenius holds because a matrix with nonnegative entries can be though of in probabilistic terms (so each row adds to one). Now we have a Markov Chain and taking powers means considering all possible walks in the state-space. As the matrix is raised to higher powers the largest eigenvalue will dominate. The steady state must be positive (since we are taking powers of nonnegative entries) and have eigenvalue one since it is "steady."

Proof.

Informal: As you count your walks your number of choices at each step is at most $\operatorname{maxdeg}(G) = \left(\max_v \deg(v)\right)^{\ell}$. So the inequality must be in this direction since otherwise Lemma 1.2.1 cannot hold.

Formal: For any vector set $x = (x_i)$,

$$\max_j |(A(G)x)_j| = \max_j \left| \sum_{\text{edge } ij} x_i \right| \leq \text{maxdeg}(G) \cdot \max_k |x_k|.$$

Now assume that x is an eigenvector of $A(G)$, with eigenvalue λ. Then this inequality becomes $|\lambda| \cdot \max_k |x_k| \leq \text{maxdeg(G)} \cdot \max_k |x_k| \Rightarrow |\lambda| \leq \text{maxdeg(G)}$. This is just mathematical legalese (the argument is philosophical). $\square$

Homework 1.3

Prove that

$$\lambda_{\max} \geq \frac{1}{p} \sum_{v \in G} \deg(v).$$

This is like a kind of convexity or Jensson Inequality.

Hint: Use that for symmetric real M,

$$\lambda_{\max} = \max_{|x| = 1} x^T M x$$

Corollary: The number of closed walks grows exponentially in ℓ. With rate at least equal to the average degree.

1.3 Eigenvalues of Special Graphs

We now study the eigenvalues of graphs which take certain specified forms.

Block Anti-Diagonal Matrices

Now let's discuss in more detail the bipartite graphs. Whose matrices take the block form $\begin{bmatrix} 0 & * \\ * & 0 \end{bmatrix}$. We will use implicitly the concept of singular values.

Lemma 1.3.2 (Eigenvalues of Block Anti-Diagonal Matrix)

For a real matrix B, $M = \begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix}$ has nonzero eigenvalues $\pm\sqrt{\mu_i}$ where $\mu_1, \mu_2, \ldots$ are the eigenvalues of $B^T B$ counted with multiplicity. Note that $B^T B$ is real, symmetric, and most importantly positive semidefinite. Since $\langle B^T Bx, x \rangle = \langle Bx, Bx \rangle \geq 0$ and the signature of a quadratic form is given by the signs of eigenvalues. Note that the $\mu_1, \mu_2, \ldots$ are the singular values of B.

The proof is going to be the worst kind of proof in mathematics.

For a $p \times p$ matrix X we will denote $F_X(t) := \det\big(tI_p - X\big)$.

Proof.

Suppose B is $n \times m$. Then

$$\begin{bmatrix} tI_n & -B \\ -B^T & tI_m \end{bmatrix} \begin{bmatrix} I_n & B \\ 0 & tI_m \end{bmatrix} = \begin{bmatrix} tI_n & 0 \\ -B^T & t^2 I_m - B^T B \end{bmatrix}.$$

Taking determinants gives

$$F_M(t) \cdot t^m = t^n \cdot F_{B^T B}(t^2).$$

Now we are after the eigenvalues which is precisely when $F_M(t)$ vanishes. Now we know that t^2 is an eigenvalue of $B^T B$ which means that t is a singular value of B. There exists a proof from the book of this fact but it is much longer. □

Example: Let $G = K_{m,n}$. Let B be the $m \times n$ matrix of all ones. Then $B^T B$ is an $m \times m$ matrix of all n's. This is a rank one matrix so it has one nonzero eigenvalue and it is $\text{tr}(A(G)) = nm$. Thus the eigenvalues of $A(G)$ are $\pm\sqrt{mn}, 0, ..., 0$.

Homework 1.4

Let G be the graph obtained by removing n disjoint edges from $K_{n,n}$. Find the eigenvalues of G.

Example: Let G be a $2n$-cycle, thus G is bipartite. Denote $A(G) = M_{2n}$ Then this matrix has the form $\begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix}$. Then $B^T B = 2I_n + M_n$ with the appropriate labeling. This is intuitive because we can return to our original vertex in two ways: counter clockwise or clockwise. [Why?]

This implies that the eigenvalues of G are $\pm\sqrt{2 + \lambda_i}$ where λ_i are the eigenvalues of M_n. Now compute the eigenvalues of M_n for small n (say 4). So the eigenvalues of 2^r-cycle are $\sqrt{2 \pm \sqrt{2 \pm \cdots}}$.

Circulant Graphs

We are talking about matrices which are act upon the set of basis vectors as a cyclic group.

Definition 1.3.5 (Circulant Matrix)

A square matrix is called **circulant** provided that

$$C = \begin{bmatrix} s_0 & s_1 & \cdots & s_{p-1} \\ s_{p-1} & s_0 & \cdots & s_{p-2} \\ \vdots & \vdots & \ddots & \vdots \\ s_1 & s_2 & \cdots & s_0 \end{bmatrix}$$

.

Lemma 1.3.6 (Eigenvalues of Circulant Matrices)

$$\lambda_k = \sum_{j=0}^{p-1} s_j e^{\frac{2\pi i}{p} jk}$$

for $k = 0, ..., p-1$.

Remark: Let $s(x) = \sum_{j=0}^{p-1} s_j x^j$, then the eigenvalues of C are the values of $s(x)$ at the p-th roots of unity.

Proof.
Let T be the circulant matrix such that $e_1^T T = [0, 1, 0, ..., 0]$. T acts on the basis $e_1, ..., e_p$ by cyclically shifting. So T^k will again be a circulant matrix. Then $C = s(T)$. Thus finding the eigenvalues of T will give us the eigenvalues of C. The eigenvalues of C are the p-th roots of unity. By observation, $\det(tI_p - T) = t^p - 1$. Also $T^p = I_p$ so then T itself is a p-th root of unity and so it must be that the eigenvalues of T are p-th roots of unity.
The claim follows. □

Definition 1.3.7 (Circulant Graph)

A graph G is **circulant** if $A(G)$ is circulant for some labeling of G.
Corollary: The eigenvalues of the p-cycle are $2\cos\left(\frac{2\pi k}{p}\right)$ for $k = 0, ..., p-1$.

Proof.
Let G be a p-cycle. So $A(G) = T + T^{-1} = T + T^{p-1}$. Nowa, Lemma 1.1.2 implies that the eigenvalues of G are

$$e^{\frac{2\pi i}{p} k} + e^{\frac{2\pi i}{p} k(p-1)} = e^{\frac{2\pi i}{p} k} + e^{-\frac{2\pi i}{p} k} = 2 \cdot \Re\left(e^{\frac{2\pi i}{p} k}\right) = 2\cos\left(\frac{2\pi k}{p}\right).$$

□

Recall: The eigenvalues of a $2n$-cycle are $\pm\sqrt{2 + \mu_i}$ where μ_i is an eigenvalue of the n-cycle. Note that this is consistent with the more general corollary above via the cosine double angle identity.

Homework 1.5

Find the eigenvalues of the graph obtained by removing n disjoint edges from K_{2n}. Note that this graph is the 1-skeleton of the n-dimensional **cross polytope** (i.e. the **hyperoctahedron**).

The hyperoctahedron is the n-dimensional polytope which is dual to the n-cube. Thus it is the convex hull of the set $\{\pm e_i : i \in [n]\}$. The only non-adjacent vertices are the vertices at e_i and $-e_i$. Counting walks in the hyperoctahedron counts walks on the faces of the hypercube.

Cartesian Products of Graphs

Cartesian products is a general categorical construction.

Definition 1.3.10 (Product of Graphs)

Let G, H be graphs. Then $G \times H$ is the **Cartesian product** of G and H with vertex set $V_{G\times H} = V_G \times V_H$ and edges of two kinds $(g,h)(g',h)$ for edge $gg' \in E_G$. and $(g,h)(g,h')$ for edge $hh' \in E_H$.

Example:

- Let $G = \bullet - \bullet - \bullet$ and $H = \bullet - \bullet - \bullet - \bullet$. Then $G \times H$ is the 3×4 grid graph.
- The n-cube graph (skeleton of the n-cube) is isomorphic to $\underbrace{K_2 \times \cdots \times K_2}_{n \text{ times}}$.

Proposition 1.3.11 (Eigenvalues of Direct Product)

Suppose G have eigenvalues $\lambda_1, \lambda_2, \ldots$ and H has eigenvalues $\mu_1, \mu_2, \ldots$ then $G \times H$ has eigenvalues $\lambda_i + \mu_j$ for all i, j.

We will give two proofs. The first is highbrow. The second is elementary.

Proof.

Take two vectors u, v in the space of formal linear combinations of the vertices of G and H (i.e. functions on the vertex space of the graphs). Put $u = \sum \alpha_g g$ and $v = \sum \beta_h h$. Define the tensor product of two vectors

$$u \otimes v \overset{\text{def}}{=} \sum_g \sum_h \alpha_g \beta_h (g,h).$$

Then we claim that the matrix

$$A(G \times H)(u \otimes v) = A(G)u \otimes v + u \otimes A(H)v.$$

This is true for the basis vectors of the tensor space $V_G^* \otimes V_H^*$ thus it holds for any u and v by linearity. Now if u and v are eigenvectors with eigenvalues λ and μ for G and H respectively, then, we have

$$A(G \times H)(u \otimes v) = \lambda u \otimes + u \otimes \mu v = (\lambda + \mu)(u \otimes v).$$

So $u \otimes v$ is an eigenvector for $G \times H$ with eigenvalue $\lambda + \mu$. $\square$

Proof.

A marked closed walk of length ℓ in $G \times H$ is a **shuffle** (i.e. interleaving) of marked closed walks of length k and $\ell - k$ in G and H, respectively. Hence, the number of such walks is

$$\sum_{k=0}^{\ell} \binom{\ell}{k} \left(\sum_i \lambda_i^k \right) \left(\sum_j \mu_j^{\ell-k} \right) = \sum_i \sum_j \sum_k \binom{\ell}{k} \lambda_i^k \mu_j^{\ell-k} = \sum_i \sum_j \left(\lambda_i + \mu_j \right)^{\ell}.$$

Where the first equality follows from nontrivial summation rearrangement and the second follows from the binomial theorem. Now by Theorem 1.1.4 we have found the eigenvalues. $\square$

The first proof is better because there is no computation. Also the second proof does not give us the eigenvectors.

Homework 1.6

Find the number of marked closed walks of length ℓ in the 3×3 grid graph. Note that this graph is bipartite so we should get zero for odd ℓ.

Homework 1.7

Find the eigenvalues of the 1-skeleton of an octagonal prism. Note $G = C_2 \times C_8$.

Eigenvalues of the Cube

Let $Q_n = (K_2)^n$ be the 1-skeleton of the n-cube.

Proposition 1.3.15 (Eigenvalues of Cube Graph)

The eigenvalues of Q_n are

$$\left\{ \left(n - 2k; \binom{n}{k} \right) : k = 0, \ldots, n \right\}.$$

Corollary: The number of marked closed walks in Q_n of length ℓ is

$$\sum_{k=}^{\ell} \binom{n}{k} (n - 2k)^{\ell}.$$

Also this number must be divisible by 2^n and zero for odd ℓ since Q_n is bipartite.

Proof.

The eigenvalues of K_2 are ± 1. Now recalling Proposition 1.3.11, we see that the eigenvalues of Q_n are all possible sums of the form $\underbrace{\pm 1 \pm 1 \pm 1 \cdots \pm 1}_{n \text{ times}}$. □

Homework 1.8

Find a direct proof of the formula using generating functions.

Now we have enough background to consider **random walks** in the cube. We will consider Brownian motion within a graph. When G is a **regular** graph of degree d on p vertices, a **simple** random walk on G proceeds by choosing (uniformly at random) an adjacent vertex and moving into it.

Instead of regularity, we will assume the stronger condition that any two vertices are related by an automorphism of G, meaning the group of automorphisms of G acts transitively on G.

Assuming that the random walk originates at v,

$$\begin{aligned}\mathbb{P}\{\text{after } \ell \text{ steps we return to } v\} &= d^{-\ell} \cdot \#\{\text{closed walks of length } \ell \text{ starting at } v\} \\ &= \frac{1}{d^\ell p} \cdot \sum_{i=1}^{p} \lambda_i^\ell.\end{aligned}$$

Example: For Q_n we put $d = n$ and $p = 2^n$, the probability of return after ℓ steps is

$$\mathbb{P}\{\text{return after } \ell \text{ steps}\} = \frac{1}{n^\ell 2^n} \sum_{k=0}^{n} \binom{n}{k} (n-2k)^\ell.$$

Homework 1.9

Find the analogous probability of return for the discrete torus with nm vertices (the product of an n-cycle and an m-cycle). Alternatively, this is the quotient of the grid graph by its “boundary” graph.

Note that the discrete torus satisfies the automorphism transitivity assumption.

Eigenvalues of a Chain Graph

Let G be the n-cycle with one edge removed. This graph is not regular.

$$A \stackrel{\text{def}}{=} A(G) = \begin{bmatrix} 0 & 1 & & \\ 1 & 0 & 1 & \\ & 1 & 0 & 1 \\ & & & \ddots \end{bmatrix}$$

Proposition 1.3.19 (Eigenvalues of Chain Graph)

The eigenvalues of A_n are $2 \cdot \cos\left(\frac{\pi k}{n+1}\right)$.

Proof.

The characteristic polynomial of A_n is $T_{n(t)} = \det(tI_n - A_n)$. It turns out this determinant is closely related to Chebyshev Polynomials.

Consider a more general determinant (of Toeplitz Matrices)

$$h_n(a,b) = \begin{bmatrix} a+b & -ab & & \\ -1 & a+b & -ab & \\ & -1 & a+b & -ab \\ & & & \ddots \\ & & & \end{bmatrix}.$$

Exploring small values we see that

$$\begin{aligned} h_1(a,b) &= a+b \\ h_2(a,b) &= a^2+ab+b^2 \\ h_3(a,b) &= (a+b)^3 - 2ab(a+b) \\ &= a^3+a^2b+ab^2+b^3. \end{aligned}$$

This suggests the pattern that

$$h_{n(a,b)} = \sum_{k=0}^{n} a^{n-k}b^k = \frac{a^{n+1}-b^{n+1}}{a-b}.$$

We will use a recurrence to prove this formula. We expand the determinant in the last row

$$h_n(a,b) = (a+b)h_{n-1}(a,b) - ab \cdot h_{n-2}(a,b).$$

The proof follows from induction on n. All that must be done is to plug the formula into the recurrence and check that it holds. This trick of expanding the determinant into a recurrence is common for (tri)-diagonal matrices.

Now set $b := \frac{1}{a}$. Then

$$T_n(a+a^{-1}) = h_n(a, a^{-1}) = \frac{a^{n+1} - a^{-(n+1)}}{a - a^{-1}}.$$

This vanishes when $a^{2(n+1)} = 1$ but $a^2 \neq 1$. That is, $a = e^{\frac{2\pi i k}{2(n+1)}} = e^{\frac{\pi i k}{n+1}}$ for $k \in [n]$ since 1 and $n+1$ are forbidden by our vanishing rules.

Summing $t = a + a^{-1} = 2 \cdot \cos\left(\frac{\pi k}{n+1}\right)$. $\square$

1.4 Enumeration of Matching and Tilings