Honors Analysis II

Math 396

University of Michigan

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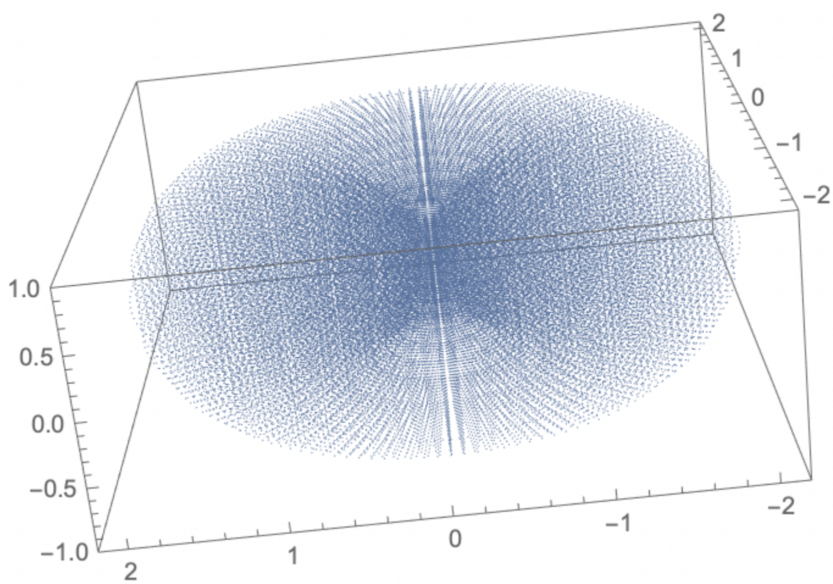


Contents

INTRODUCTION & MOTIVATION

Textbooks:

(i) Munkres, Analysis on Manifolds

(ii) Spivak, Calculus on Manifolds.

(iii) (Possibly) Fourier Analysis, an Introduction.

Content:

Manifolds are k-dimensional objects embedded in ambient n-dimensional space. We will be interested in integration over manifolds. Next, we will study differential forms which are generalizations of functions and vector fields. We will then integrate differential forms on manifolds which will lead us to the celebrated **Stokes Theorem**. Stokes Theorem describes the relationship between the integral over a manifold and its boundary. We will study many classical examples.

EUCLIDEAN DIFFERENTIABLE MANIFOLDS

Motivation

Informally, a **topological manifold** is a topological space that is **homeomorphic** to Euclidean space. This means a manifold, locally, looks like $\mathbb{R}^k$.

For example, $\mathbb{S}^1$ is a manifold because when we "zoom into" the circle it looks like a line. Also $\mathbb{S}^1 \times \mathbb{S}^1$ is a manifold because donuts look locally like a plane (see front cover).

We want to do calculus on manifolds, so we need to add more structure. A **differentiable manifold** is a special type of topological manifold that is "smooth."

Proposition. (Volume of a Parallelepiped)
If $v_1, \ldots, v_n \in \mathbb{R}^n$ are linearly independent. The volume of the parallelepiped generated by $v_1, \ldots, v_n$ is the absolute value of

$$\det \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix}$$

We want to determine the k-dimensional volume of a parallelepiped determined by k vectors in $\mathbb{R}^n$. Since there are only k-vectors, appending them into a matrix will form a non-square matrix—we cannot use the determinant.

Definition. (Volume of Parallelepiped)
Let $k \leq n$, Let $M(n,k)$ be the space of $n \times k$ matrices. Define $V : M(n,k) \to [0, \infty)$ by

$$V(X) = \sqrt{\det(X^T X)}$$

Suppose $x_1, \ldots, x_k \in \mathbb{R}^n$ are linearly independent. We define the **k-dimensional volume** of the parallelepiped generated by $x_1, \ldots, x_k$ to be $V(X)$ where

$$X = \begin{bmatrix} | & & | \\ x_1 & \cdots & x_k \\ | & & | \end{bmatrix}$$

This is well defined because $X^T X$ is a positive definite matrix (it has positive determinant).

Examples:

(i) $k = n$ (should agree with previous proposition)

$$V(X) = \sqrt{\det(X^T X)} = \sqrt{\det(X) \cdot \det(X)} = |\det(X)|.$$

(ii) $k = 1$ (should agree with length of vector)

$$\sqrt{v^T v} = \|v\|.$$

(iii) $k = 2$ and $n = 3$ (should agree with cross product of the rectangular sides)

$$X = \begin{bmatrix} | & | \\ a & b \\ | & | \end{bmatrix} \implies X^T X = \begin{bmatrix} a^T a & a^T b \\ b^T a & b^T b \end{bmatrix} \implies \det(X^T X) = \det \begin{bmatrix} \|a\|^2 & a \cdot b \\ a \cdot b & \|b\|^2 \end{bmatrix}$$

$$= \|a\|^2 \|b\|^2 - (a \cdot b)^2 = \|a\|^2 \|b\|^2 \sin \theta$$

So $\det(X^T X) = \|a \times b\|^2$.

Generalizing this, we obtain an interesting fact about the determinant of the product of matrices.

Definition. (Ascending k-tuple)
Let $k \leq n$.

(a) An **ascending k-tuple** from the set $[n]$ is $I = (i_1, \ldots, i_k)$ satisfying $1 \leq i_1 \leq \cdots \leq i_k$.

(b) Denote by $\mathrm{ASC}_{k,n}$ the set of all ascending k-tuples from $[n]$.
So $|\mathrm{ASC}_{k,n}| = \binom{n}{k}$ and $\mathrm{ASC}_{k,n} \cong \binom{[n]}{k}$.

Theorem. (Cauchy-Binet Identity)
Let $k \leq n$. If $A \in M(k, n)$ and $B \in M(n, k)$, then

$$\det(AB) = \sum_{\mathrm{ASC}_{k,n}} \det(A^I) \det(B_I)$$

where for $I = (i_1, \ldots, i_k)$, A^I is the $k \times k$ submatrix of A containing the columns $i_1, \ldots, i_k$ and B_I, is the $k \times k$ submatrix of A containing the rows $i_1, \ldots, i_k$.

Corollary. For $k \leq n, X \in M(n, k)$

$$V(X)^2 = \det(X^T X) = \sum_{\mathrm{ASC}_{k,n}} (\det X_I)^2$$

The corollary generalizes the Pythagorean Theorem.

Check directly for a 2×3 matrix.

Proof.
We will prove for $k = 2$ and n arbitrary.

$$\det(AB) = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ \vdots & \vdots \\ b_{n1} & b_{n2} \end{bmatrix}$$

So,

$$\begin{aligned}
\det(AB) &= \det \begin{bmatrix} \sum_{i\in[n]} a_{1i}b_{i1} & \sum_{i\in[n]} a_{1i}b_{i2} \\ \sum_{j\in[n]} a_{2j}b_{j1} & \sum_{j\in[n]} a_{2j}b_{j2} \end{bmatrix} && \text{matrix product} \\
&= \sum_{i\in[n]}\sum_{j\in[n]} \det \begin{bmatrix} a_{1i}b_{i1} & a_{1i}b_{i2} \\ a_{2j}b_{j1} & a_{2j}b_{j2} \end{bmatrix} && \text{det is multilinear} \\
&= \sum_{i\in[n]}\sum_{j\in[n]} a_{1i}a_{2j} \det \begin{bmatrix} b_{i1} & b_{i2} \\ b_{j1} & b_{j2} \end{bmatrix} && \text{det is multilinear} \\
&= \sum_{i\in[n]}\sum_{j\in[n]} \delta_{ij} \cdot a_{1i}a_{2j} \det \begin{bmatrix} b_{i1} & b_{i2} \\ b_{j1} & b_{j2} \end{bmatrix} && \text{det is alternating} \\
&= \sum_{i\in[n]}\sum_{i<j} a_{1i}a_{2j} \det \begin{bmatrix} b_{i1} & b_{i2} \\ b_{j1} & b_{j2} \end{bmatrix} + \sum_{i\in[n]}\sum_{i>j} a_{1i}a_{2j} \det \begin{bmatrix} b_{i1} & b_{i2} \\ b_{j1} & b_{j2} \end{bmatrix} && \text{expansion of sum} \\
&= \sum_{i\in[n]}\sum_{i<j} a_{1i}a_{2j} \det \begin{bmatrix} b_{i1} & b_{i2} \\ b_{j1} & b_{j2} \end{bmatrix} + \sum_{j\in[n]}\sum_{j>i} a_{1j}a_{2i} \det \begin{bmatrix} b_{j1} & b_{j2} \\ b_{i1} & b_{i2} \end{bmatrix} && \text{permute } i \text{ and } j \\
&= \sum_{i\in[n]}\sum_{i<j} a_{1i}a_{2j} \det \begin{bmatrix} b_{i1} & b_{i2} \\ b_{j1} & b_{j2} \end{bmatrix} - \sum_{j\in[n]}\sum_{j>i} a_{1j}a_{2i} \det \begin{bmatrix} b_{i1} & b_{i2} \\ b_{j1} & b_{j2} \end{bmatrix} && \text{det is alternating} \\
&= \sum_{i\in[n]}\sum_{i<j} (a_{1i}a_{2j} - a_{1j}a_{2i}) \det \begin{bmatrix} b_{i1} & b_{i2} \\ b_{j1} & b_{j2} \end{bmatrix} && \text{factor} \\
&= \sum_{(i,j)\in \mathrm{ASC}_{2,n}} \det \begin{bmatrix} a_{i1} & a_{i2} \\ a_{j1} & a_{j2} \end{bmatrix} \det \begin{bmatrix} b_{i1} & b_{i2} \\ b_{j1} & b_{j2} \end{bmatrix} && \text{definition of det.}
\end{aligned}$$

$\square$

Parametrized Manifolds

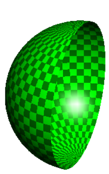
We will almost always use n to denote the dimension of the ambient space and k the subspace. We now turn to study manifolds given by a single patch, called parametrized manifolds.

Definition. (Parametrized Manifold)
Let $k \leq n$ and $A \subseteq \mathbb{R}^k$ be open. Let $\alpha : A \subseteq \mathbb{R}^k \to \mathbb{R}^n$ be a C^1 map. Put $Y = \alpha(A)$.
The pair $Y_\alpha = (Y, \alpha)$ is called a **parametrized manifold** of dimension k.

Examples:

(a) $\alpha : (0, 3\pi) \subseteq \mathbb{R} \to \mathbb{R}^2$ given by $\alpha(t) = (2\cos t, 2\sin t)$. Think of this manifold not as a circle but the trajectory of a particle that moves around the circle 1.5 times.

(b) $\alpha : (0, \pi) \times (0, \pi) \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ given by $\alpha(\theta, \phi) = (2\cos\theta\sin\phi, 2\sin\theta\sin\phi, 2\cos\phi)$. This is the portion of $\mathbb{S}^2$ in the positive x quadrant.



(c) Let $\Omega \subseteq \mathbb{R}^n$ be open. Let $h : \Omega \to \mathbb{R}$ be a C^1 function. Put $\alpha : \Omega \to \mathbb{R}^{n+1}$ with $\alpha(x) = (x, h(x))$. Then (G_h, α) is a parametrized manifold.

We want to compute the k-dimensional volume of parametrized manifolds, and in general compute integrals over them. We now define reasonable notions of length, area, and volume.

Take a rectangle in A with vertex at p and lengths $\Delta x_1, \Delta x_2$. Then it should be that the volume of this rectangle in the image is $\alpha(p + (\Delta x_i)e_i) - \alpha(p) \approx \frac{\partial \alpha}{\partial x_i}\Delta x_i$. So the volume in the image should be approximately the volume of the parallelepiped determined by $\frac{\partial \alpha}{\partial x_1}(p)\Delta x_1, \dots, \frac{\partial \alpha}{\partial x_k}(p)\Delta x_k$ which is equal to $V(D\alpha(p))\Delta x_1 \Delta x_2 \cdots \Delta x_k$. Where

$$D\alpha = \begin{bmatrix} | & & | \\ \frac{\partial \alpha}{\partial x_1} & \cdots & \frac{\partial \alpha}{\partial x_k} \\ | & & | \end{bmatrix}.$$

This motivates the following definition

Definition. (Volume of Parametrized Manifold)
Let $k \le n$, $A \subseteq \mathbb{R}^k$ be open, $\alpha : A \to \mathbb{R}^n$ be C^1. Set $Y = \alpha(A)$ and $Y_\alpha = (Y, \alpha)$.
Define the **volume** of Y_α as

$$v(Y_\alpha) = \int_A V(D\alpha)$$

For a continuous function $f : Y \to \mathbb{R}$, define the **integral** of f over Y_α as

$$\int_{Y_\alpha} f dV = \int_A (f \circ \alpha) V(D\alpha)$$

if the RHS exists[a].

[a] Here we are using the concept of a Pullback.

Examples:

(1) $\alpha : (0, 3\pi) \subseteq \mathbb{R} \to \mathbb{R}^2$ given by $\alpha(t) = (2\cos t, 2\sin t)$.

$$D\alpha = \begin{bmatrix} -2\sin t \\ 2\cos t \end{bmatrix} \implies V(D\alpha) = \sqrt{4} = 2 \implies v(Y_\alpha) = \int_0^{3\pi} 2 = 6\pi$$

(2) For $k = 2, n = 3$ and $\alpha : A \subseteq \mathbb{R}^2 \to \mathbb{R}^3$.

$$D_\alpha = \begin{bmatrix} | & | \\ \frac{\partial \alpha}{\partial x} & \frac{\partial \alpha}{\partial y} \\ | & | \end{bmatrix} \implies V(D\alpha) = \left\| \frac{\partial \alpha}{\partial x} \times \frac{\partial \alpha}{\partial y} \right\| \implies v(Y_\alpha) = \int_A \left\| \frac{\partial \alpha}{\partial x} \times \frac{\partial \alpha}{\partial y} \right\|$$

More generally,

$$\int_{Y_\alpha} f dV = \int_A (f \circ \alpha) \left\| \frac{\partial \alpha}{\partial x} \times \frac{\partial \alpha}{\partial y} \right\|$$

(3) $\alpha : (0, \pi) \times (0, \pi) \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ given by $\alpha(\theta, \phi) = (2\cos\theta \sin\phi, 2\sin\theta\sin\phi, 2\cos\phi)$.
Check that $V(D\alpha) = 4\sin\phi$.

(4) Let $\alpha : \Omega \to \mathbb{R}^{n+1}$ be given by $\alpha(x) = (x, g(x))$ for C^1 g.
Check that
$$v(D\alpha) = \sqrt{1 + \sum_{i \in [n]} \left(\frac{\partial g}{\partial x_i}\right)^2}$$

Now we show that integrals over parametrized manifolds are invariant under reparametrization.

For a parametrized manifold to exist there is one α the following theorem says any β diffeomorphic to α will agree on integrals. It does not say anything about two "randomly" chosen maps which define the same parametrized manifold.

Theorem. (Reparametrization Invariance)
Let $A, B \subseteq \mathbb{R}^k$ be open. Let $g : A \to B$ be a diffeomorphism. Let $\beta : B \to \mathbb{R}^n$ be a C^1 map. Let $\alpha = \beta \circ g : A \to \mathbb{R}^n$. Put $Y = \beta(B) = \alpha(A)$. Diagrammatically,

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[dr, bend right, "\alpha"'] & B \arrow[d, "\beta"] \\ & Y \end{tikzcd}$$

For a continuous function $f : Y \to \mathbb{R}$, f is integrable on $Y_\alpha \iff$ f is integrable on Y_β. If so,
$$\int_{Y_\alpha} f dV = \int_{Y_\beta} f dV.$$

Proof.
We need to show
$$\int_A (f \circ \alpha) V(D\alpha) = \int_B (f \circ \beta) V(D\beta) \qquad (\star)$$
This amounts to change of variables in $\mathbb{R}^k$.
$$\int_A (f \circ \alpha) V(D\alpha) = \int_B f(\beta(y)) V(D\beta(y)) = \int_A f(\beta(g(x))) V(D\beta(g(x))) \cdot |\det Dg(x)|.$$
By the Chain rule
$$D\alpha(x) = D\beta(g(x)) Dg(x)$$
$$\implies V(D\alpha(x))^2 = \det(D\alpha(x)^T D\alpha(x)) = \det\Big([D\beta(g(x))Dg(x)]^T D\beta(g(x))Dg(x)\Big)$$
$$= \det\Big(Dg(x)^T D\beta(g(x))^T D\beta(g(x)) Dg(x)\Big) = \det(Dg(x))^2 V(D\beta(g(x)))^2$$
The last step follows from the multiplicativity of the determinant and commutativity[1].
Taking square roots gives $(\star)$. □

[1]Get used to this proof. It's techniques will show up often.

Manifolds Without Boundary

Definition. (Homeomorphism)
Let X and Y be topological spaces (such as subsets of Euclidean spaces). A map $f : X \to Y$ is called a **homeomorphism** provided that f is bijective, continuous, and f^{-1} is continuous (equivalently f is an open map). If there is a homeomorphism between X and Y we say that they are **homeomorphic**.

Examples:[2]

(a) $(0, 1)$ and the unit square minus the point $(0, 1)$ are homeomorphic.

(b) $f(x) = (\cos x, \sin x)$ with $f : [0, 2\pi) \to \mathbb{S}^1$ is a continuous bijective map. However $[0, 2\pi)$ and $\mathbb{S}^1$ are *not* homeomorphic because f^{-1} is not continuous (this makes sense because their fundamental groups are different).

Recall the definition of the **subspace topology**.

Definition. (Differentiable Manifold)
Let $k \leq n$. Let $M \subseteq \mathbb{R}^n$. We call M a **differentiable k-manifold without boundary** in $\mathbb{R}^n$ provided that $\forall p \in M$, there is
(i) a set $\mathcal{U} \subseteq \mathbb{R}^k$, that is open in $\mathbb{R}^k$, (local homeomorphism)
(ii) a set $\mathcal{V} \subseteq M$, containing p, that is open in M, and (open containment)
(iii) a diffeomorphism $\alpha : \mathcal{U} \to \mathcal{V}$ with $\operatorname{rank} D\alpha(x) = k$, $\forall x \in \mathcal{U}$. (rank condition)

If α is C^r we say M is of class C^r. If α is C^∞ then we say M is **smooth**.
The **manifold** is the set M together with its coordinate patches (atlas). A manifold without the rank condition is called a **topological manifold**.

Terminology: We call the map $\alpha : \mathcal{U} \subseteq \mathbb{R}^k \to \mathcal{V} \subseteq M$ a **coordinate patch** (**coordinate system**) on M about p. The map $\varphi = \alpha^{-1} : \mathcal{V} \subseteq M \to \mathcal{U} \subseteq \mathbb{R}$ is called a **coordinate chart**. The collection of coordinate charts $(\varphi_\lambda, \mathcal{V}_\lambda)$ such that $\bigcup_\lambda \mathcal{V}_\lambda = M$ is called an **atlas**.

Intuition: the rank condition assures the linear independence of the columns of

$$D\alpha = \begin{bmatrix} | & & | \\ \frac{\partial \alpha}{\partial x_1} & \cdots & \frac{\partial \alpha}{\partial x_k} \\ | & & | \end{bmatrix} \qquad \text{where} \qquad \frac{\partial \alpha}{\partial x_i} = \lim_{\varepsilon \to 0} \frac{\alpha(x + \varepsilon e_i) - \alpha(x)}{\varepsilon}$$

when it exists is the **tangent vector** to M at $\alpha(x)$. So, rank condition means that there is a k-dimensional tangent "plane" to M at every point.

Examples:

(a) Let M be $\mathbb{S}^1 \subseteq \mathbb{R}^2$ (the unit circle). For every $p \in M \setminus \{(-1, 0)\}$, put $V = M \setminus \{(-1, 0)\}$, $\mathcal{U} = (-\pi, \pi) \subset \mathbb{R}$, and $\alpha(t) = (\cos t, \sin t)$. α is clearly C^∞, onto, 1-1, continuous inverse, and the rank of $D\alpha(t)$ is 1 $\forall t$.

[2] Algebraic Topology is the study of classifying topological spaces invariant under homeomorphism.

For the point $p = (-1, 0)$, put $V = M \setminus \{(1, 0)\}$, $\mathcal{U} = (0, 2\pi) \subset \mathbb{R}$, and $\alpha(t) = (\cos t, \sin t)$. α is clearly C^∞, onto, 1-1, continuous inverse, and the rank of $D\alpha(p)$ is 1.

So $\mathbb{S}^1$ is a differentiable manifold. We showed this by considering a covering of $\mathbb{S}^1$ whose constituents are homeomorphic to $\mathbb{R}$.

(b) Let M be $\mathbb{S}^1 \subseteq \mathbb{R}^2$ (the unit circle). For every p in the upper half of M, put $\alpha_1 : (-1, 1) \to V_1$ given by $\alpha_1(t) = (t, \sqrt{1 - t^2})$. Do the same with the lower half of M. Then do the same with the right and left hand sides of M but with $\alpha_3 : (-1, 1) \to V_3$ given by $\alpha_3(t) = (-\sqrt{1 - t^2}, t)$.

(c) Let $M = \mathbb{R}^n$. Then M is a smooth n-manifold without boundary ($\alpha = \mathrm{Id}$).

(d) Finite dimensional vector space W. Let $v_1, \ldots, v_k$ be a basis of W.
Then,

$$W = \left\{ \sum_{i \in [k]} c_i v_i : c_1, \ldots, c_k \in \mathbb{R} \right\}.$$

Let $\alpha : \mathbb{R}^k \to W$ such that

$$\alpha(x) = \sum_{i \in [k]} x_i v_i.$$

Then

$$D\alpha(x) = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_k \\ | & & | \end{bmatrix}$$

has rank k.

(e) Translates and dilates of a manifold (any diffeomorphism). If $M \subseteq \mathbb{R}^n$ and $p \in \mathbb{R}^n$ such that M is a manifold then $N = M + p_0$ is a manifold. The translation map is continuous and has rank 0. $N = rM$ is also a manifold.

(f) Spheres. $\mathbb{S}^{n-1}\{x \in \mathbb{R}^n : \|x\| = 1\}$ is a smooth manifold without boundary of dimension $n - 1$. Consider all $2n$ half spheres of $\mathbb{S}^{n-1}$ and consider the patch

$$\alpha_1(x_1, \ldots, x_{n-1}) = \left(x_1, \ldots, x_{n-1}, \sqrt{1 - \sum_{i \in [n]} x_i^2} \right).$$

(g) Open subsets of a manifold (**submanifold**). The restriction of C^r maps are C^r. Therefore, open sets in $\mathbb{R}^n$ are differentiable manifolds without boundary. Any open sets in $\mathbb{S}^{n-1}$ are differentiable manifolds without boundary. $\mathrm{GL}(n, \mathbb{R})$ the set of $n \times n$ invertible manifolds is an n^2-manifold without boundary, this is an open subset of $\mathbb{R}^{n^2}$.

(h) **Product manifold**. For $i \in [\ell]$, M_i an k_i-manifold without boundary in $\mathbb{R}^{n_i}$. Then

$$M = \prod_{i \in [\ell]} M_i$$

is a manifold of dimension $\sum_{i \in [\ell]} k_i$.

The coordinate patches are the products of coordinate patches. $T^n = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$ is an n-torus which is a smooth n-manifold without boundary in $\mathbb{R}^{2n}$. So $\mathbb{S}^1 \times \mathbb{S}^1$ is a 4-manifold but we can clearly embed it in $\mathbb{R}^3$ because we all have seen 3-dimensional donuts coated in sprinkles (this is called the edibility[3] question). This is because we can realize the torus as a quotient manifold.

(i) Singletons or discrete sets are by definition 0-dimensional manifolds.

(j) Quotient manifold (not covered).

Non-Examples:

(a) $\alpha : (0, \pi) \to \mathbb{R}^2$ given by $\alpha(t) = \sin(2t) \begin{bmatrix} |\cos t| \\ \sin t \end{bmatrix}$. Then α is 1-1 and onto but the inverse is not continuous.

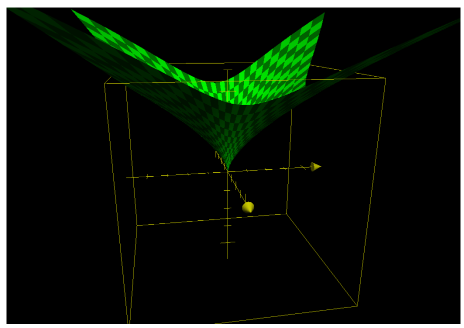


Figure 1: Not a manifold.

Why is the cross not a manifold.

(b) $\alpha : \mathbb{R}^2 \to \mathbb{R}^3$ given by $\alpha(x, y) = (x(x^2 + y^2), y(x^2 + y^2), x^2 + y^2)$. Put $M = \alpha(\mathbb{R}^2)$. α is C^∞, a homeomorphism (check!), but

$$D\alpha(0, 0) = \vec{0}_{3\times 2}$$

so rank $D\alpha(0, 0) \neq 2$.

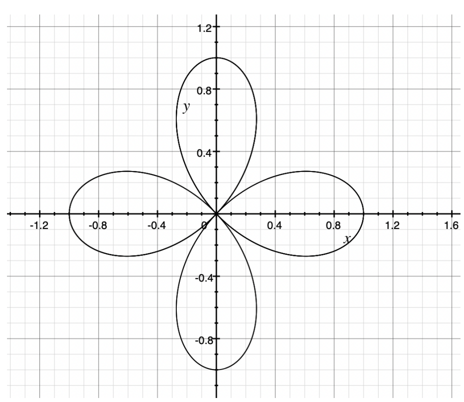


Figure 2: Not a manifold.

At all other point $D\alpha$ has rank two. So M is not a manifold. The surface looks like a parabolic funnel. The set does not have a two dimensional tangent plane at the origin.

[3]It is really called the embedibility question, but I guess AutoCorrect was hungry.

(c) Put $\alpha(t) = (t, |t|)$ and $M = \alpha(\mathbb{R})$. This α does not give rise to a (differentiable) manifold. Put $\beta(t) = (t^3, t^2|t|)$. Note that

$$f(x) = t^2|t| = \begin{cases} t^3 & t \geq 0 \\ -t^3 & t < 0 \end{cases}$$

is C^1.

Since

$$f'(x) = \begin{cases} 3t^2 & t > 0 \\ 0 & t = 0 \\ -3t^2 & t < 0 \end{cases}$$

But the rank condition still fails because $\operatorname{rank} D\beta(0) = \operatorname{rank} \vec{0} \neq 1$.

Moral of the story: if you try to be clever, the rank condition will kick in and you will fail.

Is the topologist's sine curve a manifold?
What topology is generated by using the euclidean topology on $\mathbb{R}$ and then considering a space filling curve.

Now we generalize the notion of C^r to maps with differing dimension and codimension.

Definition. (Continuous Differentiability)
Let $S \subseteq \mathbb{R}^\ell$. A function $f : S \to \mathbb{R}^m$ is said to be C^r **on** S provided that f extends to a C^r function on an open set in $\mathbb{R}^\ell$ containing S. There is an open $\Omega \subseteq \mathbb{R}^\ell$ with $\Omega \supseteq S$ and $\tilde{f} : \Omega \to \mathbb{R}^m$, such that $\tilde{f}$ is C^r and $\tilde{f} \upharpoonright S = f$.

Example:

(a) Let $f : S \to \mathbb{R}$ where $S = \operatorname{Span}(\{e_1 + e_2\})$ and $f(x, y) = xy$ then f is C^∞ on S.

Lemma. (Local $C^r \implies C^r$)
Let $S \subseteq \mathbb{R}^\ell$ and $f : S \to \mathbb{R}^m$. Suppose that $\forall x \in S$ f is locally C^r near x (i.e. $\exists S_x$ open in S such that $x \in S_x$ and f is C^r on S_x), then f is C^r on S.

Proof.
We did this in the 395 homework using Partitions of Unity. □

Lemma. (Coordinate Charts are C^r)
Let M be a differentiable k-manifold without boundary in $\mathbb{R}^n$ of class C^r. Let $\alpha : \mathcal{U} \subseteq \mathbb{R}^k \to \mathcal{V} \subseteq M$ be a coordinate patch. Then $\alpha^{-1} : \mathcal{V} \to \mathcal{U}$ is C^r on $\mathcal{V}$ and is a coordinate chart.

Proof.
It suffices to prove locally. Choose $p_0 \in V$ with $x_0 = \alpha^{-1}(p_0)$.

Since $\operatorname{rank} D\alpha(x_0) = k$ (and row rank equals column rank) there are k linearly independent rows. Without loss of generality we assume that the first k rows of $D\alpha(x_0)$ are linearly independent. Let $\pi : \mathbb{R}^n \to \mathbb{R}^k$ be the projection map onto $\mathbb{R}^k$ (the indices of the k independent rows).

Note π is C^∞ and $D\pi = \left[\begin{array}{c|c} I_k & \vec{0}_{k\times(n-k)} \end{array}\right]$.

Define $g = \pi \circ \alpha$. Then g is C^r and the chain rule gives us

$$Dg(x_0) = D\pi(p_0)D\alpha(x_0) = \left[\begin{array}{c|c} I_k & \vec{0}_{k\times(n-k)} \end{array}\right] \begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \\ \heartsuit & \heartsuit & \heartsuit \\ \heartsuit & \heartsuit & \heartsuit \end{bmatrix} = \begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$$

which is invertible (by rank condition).

By the Inverse Function Theorem, g is a diffeomorphism locally near x_0 and g^{-1} is C^r near $\pi(p_0)$.
Note that $\alpha^{-1} = \pi \circ g^{-1}$, so α^{-1} is C^r.[4] □

Theorem. (Coordinate Patches Overlap Differentiably)
Let M be a differentiable k-manifold without boundary in $\mathbb{R}^n$ of class C^r. Let α_1, α_2 be coordinate patches from $\mathcal{U}_1, \mathcal{U}_2$ to $\mathcal{V}_1, \mathcal{V}_2$ respectively with $\mathcal{V}_1 \cap \mathcal{V}_2 \neq \varnothing$.
The map $\alpha_2^{-1} \circ \alpha_1 : \mathcal{W}_1 \to \mathcal{W}_2$ is C^r where $W_i = \alpha_i^{-1}(\mathcal{V}_1 \cap \mathcal{V}_2)$ are open in $\mathbb{R}^k$.

Proof.
Easy. The lemma above tells us that α_2 is C^r and composition of C^r maps is C^r by the Chain Rule.
The map $\alpha_2^{-1} \circ \alpha_1$ is called a **transition map**[5]. □

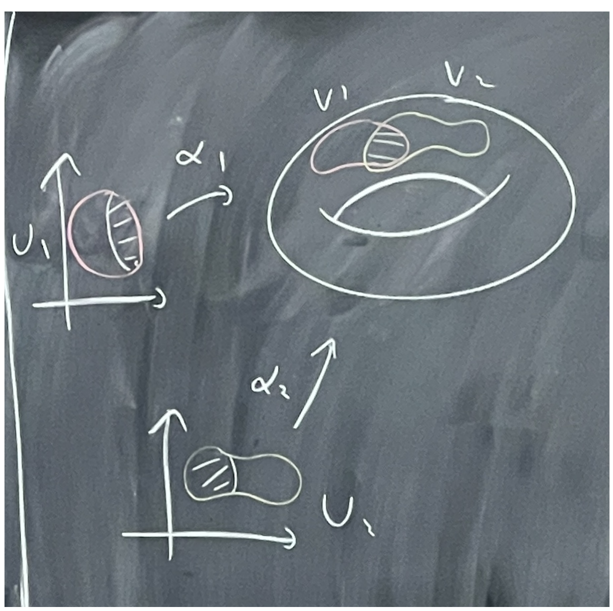


Figure 3: Overlapping coordinate patches.

[4] This step needs more thinking!

[5] In more abstract manifold theory we take the existence of transition maps as the definition of a differentiable manifold.

Manifolds With Boundary

Someone should make a hat with a donut on the top!

Notation: $\mathbb{H}^k = \{x \in \mathbb{R}^k : x_k \geq 0\}$ and $\mathbb{H}^k_+ = \{x \in \mathbb{R}^k : x_k > 0\}$.

Lemma. (Differentiability on Boundary)
Let $\mathcal{U} \subseteq \mathbb{H}^k$ be open in $\mathbb{H}^k$ but not in $\mathbb{R}^k$. Suppose $\alpha : \mathcal{U} \to \mathbb{R}^n$ is C^r. Let $\tilde{\alpha} : \tilde{\mathcal{U}} \to \mathbb{R}^n$ be a C^r extension of α where $\tilde{\mathcal{U}} \supset \mathcal{U}$ is open in $\mathbb{R}^k$, then $\forall x \in \mathcal{U}$, $D\tilde{\alpha}(x)$ depends only on α. As a consequence, $D\alpha(x)$ is well defined.

Proof.
Note that

$$\frac{\partial \tilde{\alpha}}{\partial x_i} = \lim_{\varepsilon \to 0} \frac{\tilde{\alpha}(x + \varepsilon e_i) - \tilde{\alpha}(x)}{\varepsilon}$$

exists by the assumption that α is C^r.

Since the limit exists, it is unique and equal for every path (so we can always approach from within $\mathbb{H}^k$. By taking $\varepsilon > 0$ we see that

$$\frac{\partial \alpha}{\partial x_i} = \lim_{\varepsilon \downarrow 0} \frac{\alpha(x + \varepsilon e_i) - \alpha(x)}{\varepsilon}$$

□

Definition. (Differentiable Manifold with Boundary)
A differentiable k-manifold (**with boundary**) in $\mathbb{R}^n$ of class C^r is a set $M \subseteq \mathbb{R}^n$ such that $\forall p \in M$, $\exists \alpha : \mathcal{U} \to \mathcal{V}$ where

(1) $\mathcal{U}$ is open in either $\mathbb{R}^k$ or $\mathbb{H}^k$,
(2) $\mathcal{V}$ is open in M,
(3) α is a C^r homeomorphism, and $\operatorname{rank} D\alpha(x) = k$ for all $x \in \mathcal{U}$.

Note that any manifold without boundary is necessarily a manifold with boundary.

Examples:

(a) $\mathbb{S}^1 \cap \mathbb{H}^k_+$ has manifold structure (without boundary). Consider $\alpha(t) = (\cos t, \sin t)$.

(b) $\mathbb{S}^1 \cap \mathbb{H}^k$ has manifold structure (with boundary).
For $p \in M \setminus \{(-1, 0)\}$, $\alpha : [0, \pi) \subseteq \mathbb{H}^1 \to M \setminus \{(-1, 0)\}$ given by $\alpha(t) = (\cos t, \sin t)$ is a coordinate patch.
For $p \in M \setminus \{(1, 0)\}$, $\alpha : [0, \pi) \subseteq \mathbb{H}^1 \to M \setminus \{(-1, 0)\}$ given by $\alpha(t) = (\cos(\pi - t), \sin(\pi - t))$ is a coordinate patch.
So this is a manifold with boundary.

(c) The convex hull of $\mathbb{S}^1$ (considered as a subset of $\mathbb{R}^2$) is a manifold with boundary (this is the closed unit disk about the origin).

(d) The portion of the closed unit disk that lies in the closed first quadrant does not have a differentiable manifold structure.

Lemma. (Coordinate Charts are C^r)
Let M be a differentiable k-manifold in $\mathbb{R}^n$ of class C^r. Let $\alpha : \mathcal{U} \subseteq \mathbb{R}^k \to \mathcal{V} \subseteq M \subseteq \mathbb{R}^m$ be a coordinate path on M. Then $\alpha^{-1} : \mathcal{V} \to \mathcal{U}$ is C^r on $\mathcal{V}$.

Theorem. (Transition Maps are Differentiable)
Let M be a differentiable k-manifold in $\mathbb{R}^n$ of class C^r. Let α_1, α_2 be coordinate patches from $\mathcal{U}_1, \mathcal{U}_2$ to $\mathcal{V}_1, \mathcal{V}_2$ respectively with $\mathcal{V}_1 \cap \mathcal{V}_2 \neq \varnothing$.
The map $\alpha_2^{-1} \circ \alpha_1 : \mathcal{W}_1 \to \mathcal{W}_2$ is C^r where $W_i = \alpha_i^{-1}(\mathcal{V}_1 \cap \mathcal{V}_2)$ are open in $\mathbb{R}^k$ or $\mathbb{H}^k$.

Definition. (Interior and Boundary of Manifold)
Let M be a k-manifold in $\mathbb{R}^n$. Take $p \in M$.

(a) p is called an **interior point** of M if there is a coordinate patch $\alpha : \mathcal{U} \to \mathcal{V}$ on M about p such that $\mathcal{U}$ is open in $\mathbb{R}^n$.
(b) p is called an **boundary point** of M if p is not an interior point. The set of boundary points of M is denoted by ∂M.

We want a condition to characterize boundary points.

Lemma. (Restrictions of Coordinate Patches)
Let M be a manifold and $\alpha : \mathcal{U} \to \mathcal{V}$ a coordinate patch. If $\mathcal{U}_0 \subseteq \mathcal{U}$ is open in $\mathcal{U}$, then $\alpha \restriction \mathcal{U}_0 : \mathcal{U}_0 \to \alpha(\mathcal{U}_0)$ is also a coordinate patch.

Proof.
Easy. Restrictions of diffeomorphisms are diffeomorphisms onto their image. □

Definition. (Conditions for Boundary and Interior)
Let M be a k-manifold in $\mathbb{R}^k$ and $\alpha : \mathcal{U} \to \mathcal{V}$ a coordinate patch on M about p.

(1) $\mathcal{U}$ is open in $\mathbb{R}^k \implies p$ is an interior point of M.
(2) $\mathcal{U}$ is open in $\mathbb{H}^k$ and $p = \alpha(x_0)$ for some $x_0 \in \mathbb{H}^k_+ \implies p$ is an interior point of M.
(3) $\mathcal{U}$ is open in $\mathbb{H}^k$ and $p = \alpha(x_0)$ for some $x_0 \in \mathbb{R}^{k-1} \times \{0\} \implies p$ is a boundary point.

Proof.
(1) is clear by definition. (2) Put $\mathcal{U}_0 \coloneqq \mathcal{U} \cap \mathbb{H}^k_+$, which is open in $\mathbb{R}^k$. Now restrict α to $\mathcal{U}_0$ which witnesses that p is an interior point.
(3)[6] Suppose, for the sake of contradiction, p is an interior point. Then, $\exists \beta : \mathcal{U}' \to \mathcal{V}'$ with $\mathcal{U}'$ open in $\mathbb{R}^k$. Consider $\mathcal{U} \cap \mathcal{U}'$ the transition map $\gamma = \alpha^{-1} \circ \beta : \mathcal{W}_1 \to \mathcal{W}_2$ is C^r, a homeomorphism, and $D\gamma(x)$ has rank k for all $x \in \mathcal{W}_1$.

So $\gamma : \mathcal{W}_1 \subseteq \mathbb{R}^k \to \mathbb{R}^k$ so γ should be an *open map*. Therefore $\mathcal{W}_2 = \gamma(\mathcal{W}_1)$ is open in $\mathbb{R}^k$. Contradiction! Since $x_0 \in \mathcal{W}_2$ and $x_0 \in \mathbb{R}^{k-1} \times \{0\}$. □

<u>Example</u>:

(i) $\partial(\mathbb{S}^1 \cap \mathbb{H}^K_+) = \{(1,0), (-1,0)\}$.

[6] You should be able to do this.

(ii) $\partial\mathbb{H}^k = \mathbb{R}^{k-1} \times \{0\}$.

Here's a cute theorem!

Theorem. (Boundary Manifold)
Let M be a k-manifold of class C^r in $\mathbb{R}^n$. If $\partial M \neq \varnothing$, then ∂M is $(k-1)$-manifold without boundary of class C^r in $\mathbb{R}^n$.

Proof.
Read the book. Use the boundary coordinate patches and project them onto $\mathbb{R}^{k-1}$. □

Here is a workhorse theorem:

Theorem. (Condition for Level Set Manifold)
Let $\mathcal{O} \subseteq \mathbb{R}^n$ be open and $f : \mathcal{O} \to \mathbb{R}$ be C^r. Define $N \coloneqq \{x \in \mathcal{O} : f(x) \geq 0\}$ and $M \coloneqq \{x \in \mathcal{O} : f(x) = 0\}$. We say that M is a **level set** of f. Suppose $M \neq \varnothing$ and $\operatorname{rank} Df(x) = 1$ for all $x \in M$. Then, N is a C^r n-manifold in $\mathbb{R}^n$ and $M = \partial N$.

Proof.
Suppose $p \in N$ and $f(p) > 0$. Let $M_0 = \{x \in \mathcal{O} : f(x) > 0\}$, which is open in $\mathbb{R}^n$. Put $\alpha : \mathcal{U} \subseteq \mathbb{R}^n \to \mathcal{U} \subseteq N$, $\alpha = \text{Id}$. Then α is a coordinate patch about p.

Suppose $p \in N$ and $f(p) = 0$ (i.e. $p \in M$). Since $\operatorname{rank} Df(p) = 1$, at least one of $\frac{\partial f}{\partial x_i}(p) \neq 0$ for $i \in [n]$. Without loss of generality, we may assume $\frac{\partial f}{\partial x_n}(p) \neq 0$. Define $F : \mathcal{O} \to \mathbb{R}^n$, $F(x) = (x_1, \ldots, x_{n-1}, f(x))$. F is C^r and

$$DF = \left[\begin{array}{ccc|c} & & & 0 \\ & I_{n-1} & & \vdots \\ & & & 0 \\ \hline \dfrac{\partial f}{\partial x_1} & \cdots & \dfrac{\partial f}{\partial x_{n-1}} & \dfrac{\partial f}{\partial x_n} \end{array}\right] \implies \det DF(p) = \frac{\partial f}{\partial x_n}(p) \neq 0$$

The Inverse Function Theorem guarantees that F is a diffeomorphism locally near p. Meaning, there exists open $A, B \subseteq \mathbb{R}^n$ with $p \in (A)$ such that $F : A \to B$ is a C^r diffeomorphism and $F(A)$ is identically zero. Let $\mathcal{U} = B \cap \mathbb{H}^n$, $\mathcal{V} = A \cap N$, $\alpha = F^{-1} : \mathcal{U} \to \mathcal{V}$. α is a coordinate patch. Hence, N is a C^r n-manifold. This computation also shows us that $M = \partial N$. □

<u>Example:</u>
Let $f : \mathbb{R}^n \to \mathbb{R}$ be given by $f(x) = a^2 - \sum_{i \in [n]} x_i^2$. Then $N = B_a^n(0)$ or $\mathbb{B}^n(a)$ and $M = \mathbb{S}^{n-1}(a)$. $Df(x) = -2\vec{x}^T$ is not the zero vector for $x \in \mathbb{S}^{n-1}(a)$. Thus, $\mathbb{B}^n(a)$ is a smooth n-manifold in $\mathbb{R}^n$ of class C^∞ and $\partial\mathbb{B}^n(a) = \partial\mathbb{S}^{n-1}(a)$.

Integration of Scalar Functions on Manifolds

Later we will integrate vector fields and differential forms over manifolds. For now, we will just be integrating scalar valued functions over a manifold. For simplicity of presentation, we will only consider integration over **compact manifolds**, meaning a closed and bounded subset of $\mathbb{R}^n$ which has manifold structure.

Suppose $f : M \to \mathbb{R}$ where M is a manifold with boundary. Suppose $\operatorname{supp} f$ is contained in a single coordinate patch.

Definition. (One Patch Integral over Manifold)
Let M be a compact k-manifold in $\mathbb{R}^n$. Let $f : M \to \mathbb{R}$ be continuous. Suppose there is a coordinate patch $\alpha : \mathcal{U} \to \mathcal{V}$ such that $\operatorname{supp} f \subseteq \mathcal{V}$. Note that since $\alpha^{-1}(\operatorname{supp} f)$ is compact in $\mathbb{R}^k$, we may choose $\mathcal{U}$ to be bounded.
Define

$$\int_M f\, dV = \int_{\operatorname{Int}\mathcal{U}} (f \circ \alpha) \cdot V(D\alpha)$$

Note that $\operatorname{Int}\mathcal{U} = \mathcal{U}$ if $\mathcal{U}$ is open in $\mathbb{R}^k$ and $\operatorname{Int}\mathcal{U} = \mathcal{U} \cap \mathbb{H}^k_+$ if $\mathcal{U}$ is open in $\mathbb{H}^k$.
Lemma. The RHS is ordinary integrable.
Lemma. $\int_M f\, dV$ does not depend on the choice of α.

Check that the integral is patch-independent and the integral is well defined (recall theorem 13.5 of Munkres).

Example: Suppose $M = \{(x, y) : (x, y) \in \mathbb{S}^1(3),\ x \le 0 \vee y \ge 0\}$. Put

$$f(x, y) = \begin{cases} y & y \ge 0 \\ 0 & y < 0 \end{cases}$$

Then $\operatorname{supp} f = \mathbb{S}^1(3) \cap \mathbb{H}^2$. We can find one coordinate patch "to rule them all."
Put $\alpha : [0, \frac{3\pi}{2}) \subseteq \mathbb{H}^1 \to M \setminus \{(0, -3)\}$, $\alpha(t) = (3\cos t, 3\sin t)$. We have,

$$\int_M f\, dV = \int_0^{\frac{3\pi}{2}} \alpha \circ f(3\cos t, 3\sin t) \cdot 3 = \int_0^{\pi} 9\sin t = 18.$$

Recall the definition of a Partition of Unity subordinate to $\mathscr{A}$.

Lemma. (Partition of Unity on a Manifold)
Let M be a compact k-manifold in $\mathbb{R}^n$. Given a covering of M by coordinate patches, there is a finite collection of C^∞ $\phi_i : \mathbb{R}^n \to \mathbb{R}$ $i \in [\ell]$ such that

(i) $\phi_i(x) \ge 0$, $\forall x \in \mathbb{R}^n$, $\forall i \in [\ell]$.
(ii) $\sum_{i \in [\ell]} \phi_i(p) = 1$, $\forall p \in M$
(iii) $\forall i \in [\ell]$, there is a coordinate patch $\alpha_i : \mathcal{U}_i \to \mathcal{V}_i$ such that $\operatorname{supp} \phi_i \cap M \subseteq \mathcal{V}_i$.

Proof.
Read the book. □

Definition. (Integral over Manifold)
Let M be a compact k-manifold in $\mathbb{R}^n$ and $f : M \to \mathbb{R}$ continuous.
Define

$$\int_M f\, dV = \sum_{i \in [\ell]} \int_M (\phi_i \cdot f) = \sum_{i \in [\ell]} \int_{\mathcal{U}_i} ((\phi_i \cdot f) \circ \alpha) \cdot V(D\alpha)$$

for a partition of unity $\{\phi_i\}_{i \in [\ell]}$ of M.

We need to check:

(a) If supp f lies in one coordinate patch, then the two definitions agree.

(b) $\int_M f\, dV$ is independent of the choice of partition of unity on M.

(c) $\int_M (\alpha f + \beta g)\, dV = \alpha \int_M f\, dV + \beta \int_M g\, dV$ and monotonicity in the domain.

Now we need a practical way to compute the integral over a manifold. We will extend the notion of measure zero on a manifold.

Definition. (Measure Zero Sets in a Manifold)
Let $M \subseteq \mathbb{R}^n$ be a compact k-manifold. $D \subseteq M$ is said to have **measure zero in** M provided that D can be covered by at most countably many coordinate patches $\alpha_i : \mathcal{U}_i \to \mathcal{V}_i$ such that

$$\bigcup_{i \in \mathbb{N}} \alpha_i^{-1}(D \cap \mathcal{V}_i)$$

has measure zero in $\mathbb{R}^k$.

<u>Example</u>:

$M = \mathbb{S}^2(a) \subseteq \mathbb{R}^3$ and $D = \mathbb{S}^1(a) \times \{0\}$. Let α be the stereographic projection from the north pole. Then, $\alpha^{-1}(D)$ is a circle in $\mathbb{R}^2$.

Theorem. (Measure Zero Sets Do Not Affect Integrals)
Let $M \subseteq \mathbb{R}^n$ be a compact k-manifold and $f : M \to \mathbb{R}$ continuous. Suppose $\alpha_i : A_i \to M_i$ for $i \in [\ell]$ are coordinate patches such that $M_1, \ldots, M_N$ are disjoint and

$$M = \left(\bigcup_{i \in [\ell]} M_i \right) \cup K$$

where K is of measure zero in M.

Then,

$$\int_M f\, dV = \sum_{i \in [\ell]} \int_{M_i} f\, dV$$

Proof.
Since both sides of the equation are linear in f, it is enough to show

$$\int_M f\, dV = \sum_{i \in [N]} \int_{A_i} (f \circ \alpha_i) \cdot V(D\alpha)$$

coordinate patch. Hence WLOG we may assume supp f lies in one coordinate patch.

Then the equation to prove becomes

$$\int_{\operatorname{Int} \mathcal{U}} (f \circ \alpha) \cdot V(D\alpha) = \sum_{i \in [N]} \int_{M_i} f\, dV$$

Put $L = \alpha^{-1}(K \cap \mathcal{V})$. Put $\mathcal{W}_i = \alpha^{-1}(M_i \cap \mathcal{V})$, which is open in $\mathbb{R}^k$ or $\mathbb{H}^k$. Try to prove that L is measure zero in $\mathbb{R}^k$ (HW), you should use that C^1 maps take measure zero sets to measure zero sets.

$$\int_{\operatorname{Int}\mathcal{U}} (f\circ\alpha)\cdot V(D\alpha) = \int_{\operatorname{Int}\mathcal{U}\setminus L}(f\circ\alpha)\cdot V(D\alpha) = \sum_{i\in[N]}\int_{\mathcal{W}_i}(f\circ\alpha)\cdot v(D\alpha)$$
$$= \sum_{i\in[N]}\int_{\alpha_i^{-1}(M_i\cap\mathcal{V})}(f\circ\alpha_i)\cdot v(D\alpha_i)$$

The last equality follows from change of variables and the fact that $\operatorname{supp}\alpha_i$ lies almost entirely in A_i.

$\square$

$M = \mathbb{S}^2(a)\subseteq\mathbb{R}^3$. Let's compute $v(M)$. $K = \{(x,y,z)\in M : y=0, x\geq 0\}$ (half the meridian). Let $\alpha : (0,2\pi)\times(0,\pi)\to M\setminus K$ be given by $\alpha(\theta,\phi) = (a\sin\phi\cos\theta, a\sin\phi\sin\theta, a\cos\phi)$.

$$v(M) = \int_{M\setminus K} 1\,dV = \int_{(0,2\pi)\times(0,\pi)} 1\cdot V(D\alpha) = \int_0^{2\pi}\int_0^{\pi} a^2\sin\phi\,d\phi d\theta = a^2(2)(2\pi) = 4\pi a^2$$

Let's compute again with another method: **Cavalieri's Principle**. $\alpha : (-a,a)\times(0,2\pi)\to M\setminus K$. $\alpha(z,\theta) = (\sqrt{a^2-z^2}\cos\theta, \sqrt{a^2-z^2}\sin\theta, z)$. Check that α is a coordinate patch and $V(D\alpha)=a$.

$$v(M) = \int_{M\setminus K} 1\,dV = \int_{-a}^{a}\int_0^{2\pi} a\,d\theta dz = 4\pi a^2$$

A similar computation will give you $v\left(\mathbb{S}^1(a)\right) = 2\pi a$. What about the surface area of $\mathbb{S}^k(\alpha)$?

DIFFERENTIAL FORMS

Tensors & Alternating Tensors

Definition. (Tensor Product)
Let E, F, T, H be vector spaces and $\phi : E \times F \to H$ a bilinear map. There is a bilinear map $\otimes$ called the **tensor product** which is unique, up to isomorphism, and obeys the **universal property** that for every bilinear $\phi : E \times F \to H$, $\exists! f : T \to H$ which is linear and makes

$$\begin{tikzcd} E \times F \arrow[r, "\phi"] \arrow[d, "\otimes"'] & H \\ T \arrow[ur, "f"'] & \end{tikzcd}$$

commutes.

For $f \in \mathcal{L}^k(V)$ and $g \in \mathcal{L}^\ell(V)$ we define

$$(f \otimes g)(v_1, \ldots, v_{k+\ell}) = f(v_1, \ldots, v_k) \cdot g(v_{k+1}, \ldots, v_{k+\ell})$$

Tensors generalize vectors and matrices.

Definition. (Tensor)
Suppose V is a vector space of dimension n with basis $\{b_i : i \in [n]\}$. A k**-tensor** is a function $f : V^K \to \mathbb{R}$ that is **multilinear**.
We write $\mathcal{L}^k(V)$ as the set of k-tensors on V. $\mathcal{L}^k(V)$ forms a vector space with basis $\{\phi_{\sigma(1)} \otimes \cdots \otimes \phi_{\sigma(n)} : \sigma \in [n]^k\}$ where $\{\phi_i : i \in [n]\}$ is the standard basis for V^*.
We call k the **order** of the tensor. $g \otimes h$ has order $\ell + m$ if $\mathcal{L}^\ell(V)$ and $h \in \mathcal{L}^m(V)$

Example:

(a) $\mathcal{L}^1(V) = V^*$ is the dual space of V.

(b) $\mathcal{L}^2(V) = \left\{ f : V^2 \to \mathbb{R} : f \mapsto (f(a_i, a_j))_{i,j \in [n]} \right\}$ is isomorphic to $\mathrm{Hom}(V, V)$.

(c) Let $V = \mathbb{R}^n$ let $f \in \mathcal{L}^2(\mathbb{R}^n)$. Then for $x, y \in \mathbb{R}^n$,

$$f(x, y)) = \left(\sum_{i,j \in [n]} c_{ij} (\phi_i \otimes \phi_j) \right)(x, y) = \sum_{i,j \in [n]} c_{ij} \cdot \phi_i(x) \cdot \phi_j(y) = \sum_{i,j \in [n]} c_{ij} x_i y_j = x^T C y$$

since $\phi_i(e_j) = \delta_{ij}$.

Definition. (Alternating Tensor)
$f \in \mathcal{L}^k(V)$ is said to be **alternating** provided that for every $i \in [n-1]$,

$$f(\cdots, v_i, v_{i+1}, \cdots) = -f(\cdots, v_{i+1}, v_i, \cdots)$$

We write $\Lambda^k(V)$ (or $\Lambda_k(V)$ $\mathcal{A}^k(V)$) as the set of alternating k-tensors on V.
$\Lambda^k(V)$ forms a vector subspace of dimension $\binom{n}{k}$ with basis

$$\left\{ \bigwedge_{j \in [k]} \phi_{\sigma(j)} \mid \sigma \in \mathrm{ASC}_{k,n} \right\}$$

Where $\{\phi_i : i \in [n]\}$ is the standard basis for V^* and the $\bigwedge$ denotes the wedge product.
Note that the space $\Lambda^n(V)$ is one dimensional. When $k > n$, $\Lambda^{\dim(V)}(V) = \{0\}$.

Example:

(a)
$$\begin{aligned} f \in \Lambda^k(V)) &\iff f(x,y) = x^T C y \ \wedge \ f(x,y) = -f(y,x) \\ &\iff x^T C y = -y^T C x = (y^T C x)^T = -x^T C^t y \\ &\iff x^T(C + C^T) y = 0, \ \forall x, y \in \mathbb{R}^n \\ &\iff f(x,y) = x^T C y \ \wedge \ C = -C^T \end{aligned}$$

So $\Lambda^k(V)$ is isomorphic so the set of **skew-symmetric** matrices (note the diagonal must be zero).

(b) $\Lambda^2(\mathbb{R}^3)$ has basis $\{\omega_{12}, \omega_{23}, \omega_{13}\}$ where $\omega_{ij}(x,y) = x_i y_j - x_j y_i = (\phi_i \otimes \phi_j - \phi_j \otimes \phi_i)(x,y)$.

Check that for $\omega \in \Lambda^k(V)$ and $\sigma \in S_k$,

$$\omega(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) = \mathrm{sgn}(\sigma) \cdot \omega(v_1, \ldots, v_k)$$

Note that for $\omega \in \Lambda^k(V)$ then

$$\omega(\cdots, v, \ldots, v, \cdots) = 0$$

Definition. (Alternization)
Define $\mathrm{Alt} : \mathcal{L}^k(V) \to \Lambda^k(V)$ by

$$\mathrm{Alt}(f)(v_1, \ldots, v_k) = \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma) \cdot f(v_{\sigma(1)}, \ldots, v_{\sigma(k)})$$

Lemma.
(1) Alt is a linear map and $f \in \mathcal{L}^k(V) \implies \mathrm{Alt}(f) \in \Lambda^k(V)$.
(2) If $\omega \in \Lambda^k(V)$, then $\mathrm{Alt}(\omega) = k! \cdot \omega$.
(3) $f \in \mathcal{L}^k(V) \implies \mathrm{Alt}(\mathrm{Alt}(f)) \in k! \,\mathrm{Alt}(f)$.
(4) $f \in \mathcal{L}^k(V), g \in \mathcal{L}^\ell(V) \implies \mathrm{Alt}(f \otimes g) = (-1)^{k+\ell} \,\mathrm{Alt}(g \otimes f)$.

Proof.
We have

(1) It suffices to show $\tau \in S_k$,

$$\mathrm{Alt}(f)(v_{\tau(1)}, \ldots, v_{\tau(k)}) = \mathrm{sgn}(\tau) \cdot \omega(v_1, \ldots, v_k)$$

for all $\tau \in S_k$.

Fix $\tau \in S_k$ then

$$\begin{aligned}
&\mathrm{Alt}(f)(v_{\tau(1)}, \ldots, v_{\tau(k)}) \\
&= \sum_{\sigma \in S_k} (\operatorname{sgn} \sigma) f(v_{\sigma\circ\tau(1)}, \ldots, v_{\sigma\circ\tau(k)}) && \text{definition of Alt} \\
&= \sum_{\pi \in S_k} (\operatorname{sgn} \sigma) f(v_{\pi(1)}, \ldots, v_{\pi(k)}) && \text{with } \pi = \sigma \circ \tau \\
&= \sum_{\pi \in S_k} (\operatorname{sgn} \pi)(\operatorname{sgn} \tau) f(v_{\pi(1)}, \ldots, v_{\pi(k)}) && \text{transposition properties} \\
&= (\operatorname{sgn} \tau) \sum_{\pi \in S_k} (\operatorname{sgn} \pi) f(v_{\pi(1)}, \ldots, v_{\pi(k)}) && \text{distributivity} \\
&= (\operatorname{sgn} \tau) \mathrm{Alt}(f)) && \text{definition of Alt}
\end{aligned}$$

(2) Easy

(3) $(1) \wedge (2) \implies (3)$.

(4) Let $\pi \in S_{k+1}$ for $\pi(i) = \ell + i, \pi(j+1) = j, i \in [k], j \in [\ell]$

$$\begin{aligned}
&\mathrm{Alt}(f \otimes g)(v_1, \ldots, v_{k+\ell}) \\
&= \sum_{\sigma \in S_{k+\ell}} (\operatorname{sgn} \sigma) f(v_{\sigma(1)}, \ldots, v_{\sigma(k+\ell)}) && \text{definition of Alt} \\
&= \sum_{\sigma \in S_{k+\ell}} (\operatorname{sgn} \sigma) f(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) g(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+\ell)}) && \text{definition of Alt} \\
&= \sum_{\sigma \in S_{k+\ell}} (\operatorname{sgn} \tau \circ \pi) f(v_{\tau\circ\pi(1)}, \ldots, v_{\sigma(k)}) g(v_{\tau\circ\pi(k+1)}, \ldots, v_{\tau\circ\pi(k+\ell)}) && \sigma \coloneqq \tau \circ \pi \\
&= \sum_{\sigma \in S_{k+\ell}} (\operatorname{sgn} \tau \circ \pi) f(v_{\tau(\ell+1)}, \ldots, v_{\tau(k+\ell)}) g(v_{\tau(1)}, \ldots, v_{\tau(\ell)}) && \text{definition of } \pi \\
&= (\operatorname{sgn} \pi) \mathrm{Alt}(g \otimes f) && \text{definition of Alt}
\end{aligned}$$

The Wedge Product

Definition. (Wedge Product)
For $\omega \in \Lambda^k(V), \eta \in \Lambda^\ell(V)$ define the **wedge product**

$$\omega \wedge \eta = \frac{1}{k!\ell!} \mathrm{Alt}(\omega \otimes \eta)$$

Lemma.

(1) If ω and η are of the same order, then $(\omega \wedge \eta) \wedge \theta = \omega \wedge \theta + \eta \wedge \theta$.
(2) $\wedge : \Lambda^k(V) \times \Lambda^\ell(V) \to \Lambda^{k+\ell}(V)$ is a bilinear map.
(3) $\omega \wedge \eta = (-1)^{k \cdot \ell}(\eta \wedge \omega)$.

Proof.
Done in IBL $\square$

Lemma. (Associativity of Wedge Product)
(1) If $f \in \mathcal{L}^k(V), g \in \mathcal{L}^\ell(V)$, and $\operatorname{Alt}(f) = 0$, then $\operatorname{Alt}(f \otimes g) = 0$.
(2) If $f \in \mathcal{L}^k(V)$ and $\theta \in \Lambda^m(V)$ then $\operatorname{Alt}(f) \wedge \theta = \frac{1}{m!} \operatorname{Alt}(f \otimes \theta)$.
(3) If $\omega \in \Lambda^k(V), \eta \in \Lambda^\ell(V), \theta \in \Lambda^m(V)$, then

$$(\omega \wedge \eta) \wedge \theta = \omega \wedge (\eta \wedge \theta) = \frac{1}{k!\ell!m!} \operatorname{Alt}(\omega \otimes \eta \otimes \theta)$$

Proof.
(1) This is the difficult part. Note that

$$\operatorname{Alt}(f) = 0 \iff \sum_{\pi \in S_k} f(w_{\pi(1)}, \ldots, w_{\pi(k)} = 0$$

For each $I \in [k+\ell]^\ell$. Let G_I be the set of permutations $\sigma \in S_{k+\ell}$ satisfying $\sigma(k+j) = I(j)$ for each $j \in [\ell]$.

For example $G_I = \{(14352), (41352)\}$ for $I = (3, 5, 2)$.

We have

$$\begin{aligned} k!\ell! \operatorname{Alt}(f \otimes g)(v_1, \ldots, v_{k+\ell}) &= \sum_{\sigma \in S_{k+\ell}} f(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) \cdot g(v_{I(1)}, \ldots, v_{I(\ell)}) \\ &= \sum_{I \in [k+\ell]^\ell} \sum_{\sigma \in G_I} f(v_1, \ldots, v_{k+\ell}) \cdot g(v_{k+1}, \ldots, v_{k+\ell}) \\ &= \sum_{I \in [k+\ell]^\ell} \left[\sum_{\sigma \in G_I} f(v_1, \ldots, v_{k+\ell}) \right] \cdot g(v_{k+1}, \ldots, v_{k+\ell}) \end{aligned}$$

Now fix I. Note that if $\sigma, \tau \in G_I$, then $\{\sigma(j) : j \in [k]\} = \{\tau(j) : j \in [k]\}$.

Denote $\{v_{\sigma(j)} : j \in [k]\} = \{w_j : j \in [k]\}$. Then,

$$\begin{aligned} &= \sum_{\sigma \in G_I} f(v_1, \ldots, v_{k+\ell}) \\ &= \pm \sum_{\pi \in S_k} (\operatorname{sgn} \pi) f(w_{\pi(1)}, \ldots, w_{\pi(k)}) \\ &= \pm \operatorname{Alt} f(w_1, \ldots, w_k) \\ &= 0 \end{aligned}$$

(2) Put $F = \operatorname{Alt}(f) - k!f$. Then, $\operatorname{Alt}(F) = 0$. Use (1) with $f \coloneqq F$. Then

$$\begin{aligned} &\implies \\ &\implies \operatorname{Alt}(F \otimes \theta) = 0 \\ &\implies \operatorname{Alt}(\operatorname{Alt}(f) \otimes \theta) = k! \operatorname{Alt}(f \otimes \theta) \\ &\implies k!m! \operatorname{Alt}(f) \wedge \theta && \text{definition of } \wedge \end{aligned}$$

(3) In two parts,

$$\begin{aligned}
(\omega \wedge \eta) \wedge \theta &= \frac{1}{k!\ell!} \operatorname{Alt}(\omega \otimes \eta) \wedge \theta \\
&= \frac{1}{k!\ell!m!} \operatorname{Alt}((\omega \otimes \eta) \otimes \theta) \\
&= \frac{1}{k!\ell!m!} \operatorname{Alt}(\omega \otimes \eta \otimes \theta) && \otimes \text{ is associative}
\end{aligned}$$

For part 2,

$$\begin{aligned}
\theta \wedge (\omega \wedge \eta) &= (-1)(\omega \wedge \eta) \wedge \theta \\
&= \frac{(-1)^{k(\ell+m)}}{k!\ell!m!} \operatorname{Alt}(\eta \otimes \theta \otimes \omega) \\
&= \frac{1}{k!\ell!m!} \operatorname{Alt}(\omega \otimes \eta \otimes \theta)
\end{aligned}$$

Fields & Forms on Euclidean Space

Definition. (Fields)
Let $A \subseteq \mathbb{R}^n$ be open. We define a
(i) a **scalar field** on A is a function $f : A \to \mathbb{R}$.
(ii) a **vector field** on A is a function $F : A \to \mathbb{R}^n$ (note the dimension).
(iii) a k-**tensor field** is a function $F : A \to \mathcal{L}^k(\mathbb{R}^n)$.
(iv) a (**differential**) k-**form** is a function $F : A \to \Lambda^k(\mathbb{R}^n)$.

Example:

(i) $F : \mathbb{R}^2 \to \mathbb{R}^2$ given by $F(x, y) = xe_1 + ye_1$ is a vector field that describes radial growth.

(ii) $F : \mathbb{R}^2\{\vec{0}\} \to \mathbb{R}^2$ given by $F(x, y) = \frac{xe_2 - ye_1}{\sqrt{x^2+y^2}}$ is a vector field that describes counterclockwise rotation at unit speed.

(iii) $F : \mathbb{R}^3 \to \Lambda^2(\mathbb{R}^2)$ given by

$$\omega(x, y, z) = xy(\phi_1 \wedge \phi_2) + xz(\phi_1 \wedge \phi_3) + yz(\phi_2 \wedge \phi_3) \quad \longleftrightarrow \quad \begin{bmatrix} 0 & xy & xz \\ -xy & 0 & yz \\ -xz & -yz & 0 \end{bmatrix}$$

Generally, a k-form on $\mathbb{R}^n$ can be interpreted as an $\underbrace{n \times n \times \cdots \times n}_{k \text{ times}}$ array-valued function.

Definition. (Zero Form)
A 0-**form** is a scalar field.

Let $F : A \subseteq \mathbb{R}^n \to \mathcal{L}^k\left(\mathbb{R}^k\right)$, a k-tensor field on A, then for all $x \in A$ and $v_1, \ldots, v_k \in \mathbb{R}^n$

$$F(x)(v_1, \ldots, v_k) \in \mathbb{R}$$

Definition. (Smooth Tensor Fields)
A k-tensor field F on A (open in $\mathbb{R}^n$) is said to be C^r the function $A \times \mathbb{R}^n \times \cdots \times \mathbb{R}^n \to \mathbb{R}$ given by $(x, v_1, \ldots, v_k) \mapsto F(x)(v_1, \ldots, v_k)$ is C^r.

(a) $F : \mathbb{R}^3 \to \Lambda^2(\mathbb{R}^2)$ given by

$$\omega(x, y, z) = xy(\phi_1 \wedge \phi_2) + xz(\phi_1 \wedge \phi_3) + yz(\phi_2 \wedge \phi_3)$$

$$\omega(x, y, z)(v, w) = xy(v_1w_2 - v_2w_1) + xz(v_1w_3 - v_3w_1) + yz(v_2w_3 - v_3w_2)$$

is C^∞ since it is a polynomial.

Remark: It is enough to check the coefficients (of the tensor space basis) are C^r functions. Meaning, $\omega : A \to \Lambda^k(\mathbb{R}^n)$ we can write

$$\omega(x) = \sum_{I \in \mathrm{ASC}_{k,n}} \omega_I(x) \bigwedge_{j \in [k]} \phi_{I(j)}$$

and ω_I is smooth for each $I \in \mathrm{ASC}_{k,n}$.

Notation: For open $A \subseteq \mathbb{R}^n$, $\Omega^k(A)$ will denote the set of smooth k-forms on A. $\Omega^0(A) = C^\infty(A)$. If $k > n$, then $\Omega^k(A) = \{0\}$. Note that $\Omega^k(A)$ is a vector space.

We will take the convention that all forms are smooth on their domain and worry about continuity in an *ad hoc* manner. This is more fun than always worrying about continuity.

Definition. (Differential of a 0-Form)
Let $A \subseteq \mathbb{R}^n$ be open and $f \in \Omega^0(A)$. Define $df \in \Omega^1(A)$ by

$$df(x) = \sum_{i \in [n]} D_i f(x) \phi_i$$

Example:

(a) Consider $f(x, y, z) = xyz \implies df(x, y, z) = yz\phi_1 + xz\phi_2 + xy\phi_3$. Then

$$df(x)(v) = \sum_{i \in [n]} D_i f(x) v_i = Df(x) \cdot v$$

which is the directional derivative.

(b) The projection function $\pi_i : \mathbb{R}^n \to \mathbb{R}$ given by $\pi_i(x) = x_i \implies d\pi_i = \phi_i$.

Notation: Whenever we see ϕ_i it is often more convenient to write $\phi_i = d\pi_i$ as dx_i. This is a formal notation and has no meaning. So dx_i is the 1-form satisfying $dx_i(e_j) = \delta_{ij}$. The standard basis of $\Lambda^k(\mathbb{R}^n)$ is the set

$$\{dx_{i_1} \wedge \cdots \wedge dx_{i_k} : 1 \leq i_1 \leq \cdots \leq i_k \leq n\}$$

(a) We can rewrite $f(x, y, z) = xyz \implies df(x, y, z) = yz\, dx + xz\, dy + xy\, dz$.

Propsotition. (Properties of Differentials)
Differentials obey the following properties:

(i) $d(fg) = g\, df + f\, dg$

(ii) $d : \Omega^k(A) \to \Omega^{k+1}(A)$ is linear.

(iii)

Definition. (Differential of a Form)
Let $A \subseteq \mathbb{R}^n$ be open and $\omega \in \Lambda^k(A)$. We write

$$\omega = \sum_{I \in \mathrm{ASC}_{k,n}} \omega_I \left(\bigwedge_{j \in [k]} dx_{I(j)} \right) \in \Omega^k(A)$$

Define $d\omega \in \Omega^{k+1}(A)$ the **differential** (or **exterior derivative**) of ω by

$$d\omega = \sum_{I \in \mathrm{ASC}_{k,n}} (d\omega_I) \wedge \left(\bigwedge_{j \in [k]} dx_{I(j)} \right) \in \Omega^k(A)$$

Example:

(a) Consider

$$\begin{aligned}\omega &= xy\,dx + 3\,dy - yz\,dz \in \Omega^1(\mathbb{R}^3)\\ &\Downarrow\\ d\omega &= d(xy)\wedge dx + d(3)\wedge dy + d(-yz)\wedge dz\\ &= (y\,dx + x\,dy + 0\,dz)\wedge dx + 0\wedge\ dy - (z\,dy + y\,dz)\wedge dz\\ &= -(x\,dx\wedge dy + z\,dx\wedge dz)\end{aligned}$$

(b) Consider

$$\begin{aligned}\omega &= (x+z)\,dx\wedge dy - y\,dx\wedge dz + (x^2+y^2)\,dy\wedge dz \in \Omega^2(\mathbb{R}^3)\\ &\Downarrow\\ d\omega &= (dx+dz)\wedge dx\wedge dy - dy\wedge dx\wedge dz + (dx + 2y\,dy)\wedge dy\wedge dz\\ &= dx\wedge dy\wedge dz\end{aligned}$$

Understanding the differential more carefully we have

$$\begin{aligned}\omega &= \sum_{i\in[n]}\omega_i\,dx_i \in \Omega^1(\mathbb{R}^n)\\ &\Downarrow\\ d\omega &= \sum_{i,j\in[n]}((D_j\omega_i)\,dx_j)\wedge dx_i\\ &= \sum_{1\le i<j\le n}((D_j\omega_i - D_i\omega_j)\,dx_i\wedge dx_j)\end{aligned}$$

So this is equivalent to a matrix whose diagonal is zero and ij-entry is $(D_j\omega_i - D_i\omega_j)$. There are n^2 possible partial derivatives of order 2, but we are only choosing $\frac{n(n-1)}{2}$. We will discuss the importance of this choice to follow. Differential forms vastly generalize the notation of gradient, divergence, and curl one encounters in Calc III.

Definition. (Gradient, Divergence, & Curl) We define the following

(1) For $f:\mathbb{R}^n\to\mathbb{R}$, the **gradient** of f is

$$\nabla f = \sum_{i\in[n]}(D_if)e_i.$$

(2) For $f:\mathbb{R}^n\to\mathbb{R}^n$ the **divergence** of F is

$$\operatorname{div} F = \nabla\cdot F = \sum_{i\in[n]} D_iF_i$$

(3) For $R:\mathbb{R}^3\to\mathbb{R}^3$ the **curl** of F can be written formally

$$\operatorname{curl} F = \det\begin{bmatrix} e_1 & e_2 & e_3\\ D_1 & D_2 & D_3\\ F_1 & F_2 & F_3\end{bmatrix}$$

So grad maps smooth scalar fields to vector fields. Similarly, $d:\Omega^0(\mathbb{R}^n)\to\Omega^1(\mathbb{R}^n)$. There is a bijection α_1 between vector fields on $\mathbb{R}^n$ and $\Omega^1(\mathbb{R}^n)$ given by $\alpha_1(f) = \alpha_1\left(\sum_{i\in[n]} F_i e_i\right) = \sum_{i\in[n]} F_i\, dx_i$. Also $\alpha_0(f) = f$ maps smooth scalar fields to $\Omega^0(\mathbb{R}^n)$. α_0, α_1 are isomorphisms (of vector spaces). Furthermore, the diagram commutes, (i.e. $d \circ \alpha_0 = \alpha_1 \circ \operatorname{grad}$ or $\operatorname{grad} = \alpha_1^{-1} \circ d \circ \alpha_0$). We say that the gradient operator is equivalent to d modulo conjugation of α_1^{-1} and α_0.

For $\omega \in \Omega^{n-1}(\mathbb{R}^n)$,

$$\begin{aligned}
\omega &= \sum_{i\in[n]} \omega_i \left(\bigwedge_{j\in[n]\setminus\{i\}} dx_j\right) \\
&\Downarrow \\
d\omega &= \sum_{i\in[n]} (D_i\omega_i)\, dx_i \wedge \left(\bigwedge_{j\in[n]\setminus\{i\}} dx_j\right) \\
&= \sum_{i\in[n]} (-1)^{i-1} (D_i\omega_i) \left(\bigwedge_{j\in[n]} dx_j\right)
\end{aligned}$$

$$\begin{bmatrix} \operatorname{Vec}(\mathbb{R}^n) & \\ \operatorname{div}\downarrow & \\ C^\infty(\mathbb{R}^n, \mathbb{R}) & \end{bmatrix}$$

Put $\beta_n(f) = f \bigwedge_{i\in[n]} dx_i$ and

$$\beta_{n-1}(F) == \sum_{i\in[n]} (-1)^{i-1} (F_i) \left(\bigwedge_{j\in[n]\setminus\{i\}} dx_j\right)$$

For $\omega \in \Omega^1(\mathbb{R}^3)$,

$$\begin{aligned}
\omega &= \omega_1\, dx + \omega_2\, dy + \omega_3\, dz \\
&\Downarrow \\
d\omega &= (D_1\omega_2 - D_2\omega_1)\, dx \wedge dy + (D_1\omega_3 - D_3\omega_1)\, dx \wedge dz + (D_2\omega_3 - D_3\omega_2)\, dy \wedge dz
\end{aligned}$$

$$\begin{tikzcd}
V \arrow[r, "\mathrm{Id}"] \arrow[rrr, bend left, "T"] & V \arrow[r, "T"] & W \arrow[r, "\mathrm{Id}\,W"] \arrow[d, "i^{-1}\beta"] & W \arrow[d, "i^{-1}\beta'"] \\
F^n \arrow[u, "i\alpha'"] \arrow[r, "f_\alpha\,\mathrm{Id}\,\alpha'"'] \arrow[rrr, bend right, "f_{\alpha'}T\beta'"'] & F^n \arrow[u, "i\alpha"] \arrow[r, "f_\beta T\alpha"'] & F^m \arrow[r, "f\beta'\,\mathrm{Id}\,\beta"'] & F^m
\end{tikzcd}$$

Proposition. (Wedge Product of Dual Basis)

$$\bigwedge_{j=1}^{k} dx_{i_j} = \det B_I$$

where B_I is the $k \times k$ matrix obtained from the rows of $i_1, \ldots, i_k$ of

$$B = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} \in \mathbb{R}^{n \times k}$$

Proposition. (Properties of Differential)

(a) $d\,f = \sum_{i=1}^{n} (D_i f) d\,x_i$ for all $f \in \Omega^0(\mathbb{R}^n)$.
(b) $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$
(c) $d(d\omega) = 0$ for all $\omega \in \Lambda^k(V)$.

For $k \in [n]$, d is the **only** linear transformation from $\Omega^k(\mathbb{R}^n)$ to $\Omega^{k+\ell}(\mathbb{R}^n)$ satisfying these properties.

Proof.

(a)

(b)

(c) By the linearity d, it is enough to consider $\omega = f\,dx_I$.

$$\begin{aligned} d\omega &= df \wedge dx_I \\ &= \left(\sum_{i \in [n]} (D_i f) dx_i \right) \wedge dx_I \\ &= \sum_{i \in [n]} ((D_i f) dx_i) \wedge dx_i \wedge dx_I \\ &= \Downarrow \\ d(d\omega) &= \qquad\qquad = \sum_{i \in [n]} (d(D_i f) dx_i) \wedge dx_i \wedge dx_I \\ &= \sum_{i \in [n]} \left(d \sum_{j \in [n]} (D_j D_i f\, dx_j \right) dx_i \wedge dx_i \wedge dx_I \\ &= \left[\sum_{i,j \in [n]} (D_j D_i f)\, dx_i \wedge d_j \right] \wedge dx_I \\ &= 0 \end{aligned}$$

<u>Note</u>: In principle these properties are all we need to compute $d\omega$. So $f \in \Omega^0(\mathbb{R}^n)$ we interpret

$$f\,d\omega = f \wedge d\omega$$

Example:

$$\begin{aligned} \omega = f\,dz \wedge dy &= f \wedge dz \wedge dy \\ &\Downarrow (2) \\ d\omega &= df \wedge dz \wedge dy + (-1)^0 f \wedge d(dz \wedge dy) \\ &\Downarrow \\ d\omega &= df \wedge dz \wedge dy \\ &= (D_1 f\,dx + D_2 f\,dy + D_3 f\,dz) \wedge dz \wedge dy \end{aligned}$$

Note: $d^2 = 0$ implies, when $n = 3$, $\text{curl}(\text{grad}(f)) = 0,\ \text{div}(\text{curl}(F)) = 0 \qquad \forall F, f.$

Fields & Forms on Manifolds

We want to integrate differential forms over manifolds (and potatoes). To do so we must define tangent spaces.

Suppose $\gamma : (a, b) \subseteq \mathbb{R} :: \mathbb{R}^3$ is a local parametrization of a 1-manifold in $\mathbb{R}^3$. so

$$\gamma(t) = x(t)e_1 + y(t)e_2 + z(t)e_3 \implies DD\gamma(t) = \begin{bmatrix} x'(t) \\ y'(t) \\ z'(t) \end{bmatrix}$$

So $D\gamma(t_0)$ is a tangent vector to the curve at $p = \gamma(t_0)$. So the tangent line to the curve at p is $\{(D\gamma(t_0))\, v_1 \mid v_1 \in \mathbb{R}\}$ (we consider this to be a vector space with p as the origin).

Suppose $\alpha : D \subseteq \mathbb{R}^2 :: \mathbb{R}^3$ is a local parametrization of a 2-manifold in $\mathbb{R}^3$. For $v \in \mathbb{R}^2$, consider $\gamma(t) = \alpha(x_0 + tv) : \mathbb{R} \to \mathbb{R}^3$ and the image of γ is a curve on the surface. Then $\gamma'(0) = (D\alpha(x_0))\, v$ is tangent to the embedded curve, hence tangent to the surface at $p = \alpha(x_0)$. The tangent place is $\{D\alpha(x_0)v \mid v \in \mathbb{R}^2\}$ (again up to translation).

Definition. (Tangent Space of a Manifold)
Let M be a (smooth) k-manifold in $\mathbb{R}^n$. Let $p \in M$ and let $\alpha : \mathcal{U} \to \mathcal{V}$ be a coordinate patch about p. Define the **tangent space** to M at p as

$$T_p(M) = \left\{(p, D\alpha(x_0)v) : v \in \mathbb{R}^k\right\}$$

where $x_0 = \alpha^{-1}(p)$.

Remark: The definition does not depend on the choice of coordinate patch. Meaning,

$$\left\{D\alpha(x_0)v : v \in \mathbb{R}^k\right\} = \left\{D\beta(x_0)v : v \in \mathbb{R}^k\right\}.$$

This holds since the transition map $\gamma := \beta^{-1} \circ \alpha$ is smooth and so $\beta \circ \gamma$ is a diffeomorphism.

Definition. (Tangent Bundle)
Taking all tangent spaces over M gives,

$$T(M) = \bigsqcup_{p \in M} T_p(M) \cong \bigcup_{p \in M} (p, T_p(M))$$

which is called the **tangent bundle** of M.

Definition. (Differential Forms on Manifolds)
We may define an ℓ-form on M as a function ω such that for all $p \in M$, $\omega(p) \in \Lambda^\ell(T_p(M))$. In other words, $\omega(p)(v_1, \ldots, v_\ell)$ makes sense for $v_1, \ldots, v_\ell \in T_p(M)$ but not necessarily for all $v_1, \ldots, v_\ell \in \mathbb{R}^n$.

Although it is not trivial, every form on a manifold can be locally extended to $\mathbb{R}^n$.

Note: If we have an ℓ-form ω on $\mathbb{R}^n$, $\omega \restriction M$ is an ℓ-form on M since $\omega(p)(v_1, \ldots, v_\ell)$ makes sense for $v_1, \ldots, v_\ell \in T_p(M)$.

Fact. (ℓ-forms Can Be Extended)
An ℓ-form on M can be extended to an ℓ-form on an open set in $\mathbb{R}^n$ containing M.

Proof.
The proof is not trivial (and deep).[7] □

Convention:
Given this, we will only consider forms that are defined in an open neighborhood of M in $\mathbb{R}^n$

Suppose $\omega \in \Omega^2(\mathbb{R}^2)$ and α is a coordinate patch. We need to define the dual transform $\alpha^*\omega \in \Omega^2(\mathbb{R}^2)$ (which is a pullback).

Definition. (Dual Transforms)
The **dual transform** of a linear map $T : V \to W$ is the map $T^* : \mathcal{L}^k(W) \to \mathcal{L}^k(V)$ given by the form for $f \in \mathcal{L}^k(W)$,

$$(T^* f)(v_1, \ldots, v_k) = f(Tv_1, \ldots, Tv_k)$$

Lemma. (Properties of Dual Transforms)
(i) $T^* \in \operatorname{Hom}\left(\mathcal{L}^k(W), \mathcal{L}^k(V)\right)$.
(ii) $T^*(f \otimes g) = T^* f \otimes T^* g$.[a]
(iii) $(S \circ T)^* = T^* \circ S^*$.
(iv) $f \in \Lambda^k(W) \implies T^* f \in \Lambda^k(V)$.
(v) $T^*(\omega \wedge \eta) = (T^*\omega) \wedge (T^*\eta)$.

[a] The dual operation and the tensor operation commute.

[7] see pg. 244-249 of Munkres

Proof.

(i) easy.

(ii) easy.

(iii) easy.

(iv) easy.

(v) See homework to prove $T^* \circ \operatorname{Alt} = \operatorname{Alt} \circ T^*$. Thus

$$T^*(\omega \wedge \eta) = T^*\left(\frac{1}{k!\ell!}\operatorname{Alt}(\omega \otimes \eta\right) = \frac{1}{k!\ell!}\operatorname{Alt}(T^*\omega \otimes T^*\eta) = (T^*\omega) \wedge (T^* \wedge \eta)$$

Recall: For a coordinate patch $\alpha : \mathcal{U} \subseteq \mathbb{R}^k \to \mathcal{V} \subseteq M \subseteq \mathbb{R}^n$. Then $D\alpha(x_0)$ (is an $n \times k$ matrix and) is a linear map from $\mathbb{R}^k$ centered at x_0 to the affine tangent space centered at p. Therefore this map is linear when x_0 is held constant.

Definition. (Dual of a Continuous Map)
Let $\alpha : A \subseteq \mathbb{R}^k \to B \subseteq \mathbb{R}^n$ be a smooth map for A, B open sets. Define the **dual transform** of forms $\alpha^* : \Omega^\ell(B) \to \Omega^\ell(A)$ for $\ell = 0, 1, 2, \ldots$ by

$$(\alpha^* f)(x) = \begin{cases} f \circ \alpha(x) & \ell = 0 \\ f \circ \alpha(x)\Big(D\alpha(x)(v_1), \ldots, D\alpha(x)v_k\Big) & \ell > 0 \end{cases}$$

where $f \in \Omega^\ell(B),\ x \in A,\ v_1, \ldots, v_\ell \in \mathbb{R}^k$.

Lemma.

(a) Let $A \subseteq \mathbb{R}^k \to^\alpha B \subseteq \mathbb{R}^n \to^\beta C \subseteq \mathbb{R}^m$ $\beta^*(a\omega + b\eta) = a\beta^*\omega + b\beta^*\eta$.

(b) $(\beta \circ \alpha)^* = \alpha^* \circ \beta^*$.

(c) $\beta^*(\omega \wedge \eta) = (\beta^*\omega) \wedge (\beta^*\eta)$.

Proof.
Easy. □

Example: Let $\alpha : \mathbb{R}^2 \to \mathbb{R}^3$ given by $\alpha(s,t) = (st, s+t+1, t^3)$. Then,

$$D\alpha(s,t) = \begin{bmatrix} t & s \\ 1 & 1 \\ 0 & 3t^2 \end{bmatrix}$$

Let $\omega \in \Lambda^1(\mathbb{R}^3)$ given by $\omega(x,y,z) = yz\,dy + x\,dz$. We now compute $(\alpha^*\omega)(0,1)$. Note $\alpha(0,1) = (0,2,1)$ and

$$D\alpha(0,1) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 3 \end{bmatrix}$$

For $v = (v_1, v_2) \in \mathbb{R}^2$,

$$\begin{aligned}
(\alpha^*\omega)\,(0,1)v &= \omega(0,2,1)\Big(v_1e_1, v_1+v_2, 3v_2\Big) \\
&= (2\,dy + 0\,dz)\,\Big(v_1e_1 + (v_1+v_2)e_2 + 3v_2e_3\Big) \\
&= 2(v_1+v_2) \\
&\Downarrow \\
(\alpha^*\omega)\,(0,1) &= 2\,ds + 2\,dt.
\end{aligned}$$

Now let's consider a general point (s,t).

$$\begin{aligned}
(\alpha^*\omega)\,(s,t)v &= \omega(st, s+t+1, t^3)\Big((tv_1+sv_2), (v_1+v_2), 3t^2v_2\Big) \\
&= \Big((s+t+1)t^3\,dy + st\,dz\Big)\,\Big((tv_1+sv_2), (v_1+v_2), 3t^2v_2\Big) \\
&= (s+t+1)t^3(v_1+v_2) + (st)(3t^2v_2) \\
&\Downarrow \\
(\alpha^*\omega)\,(s,t) &= (s+t+1)t^3(ds+dt) + (st)(3t^2)dt.
\end{aligned}$$

This strategy works in general, the coefficients in the end are the differentials of the coefficients ???

Theorem. (Computation of Dual Map)
Take $x = (x_1, \ldots, x_k) \in \mathbb{R}^k$ and $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$. Let $A \subseteq \mathbb{R}^k$ and $B \subseteq \mathbb{R}^n$ be open and $\alpha : A \to B$ a smooth map.
Take a 1-form, then

$$\alpha^*(dy_i) = d\alpha_i = \sum_{i=1}^{k} (D_j x_i)\; dx_j.$$

For a k-form we have. For $I \in \mathrm{ASC}_{k,n}$.

$$\alpha^*\left(\bigwedge_{j\in[k]} dy_{i_j}\right) = \left(\det \frac{\partial \alpha_I}{\partial x}\right)\left(\bigwedge_{j\in[k]} dy_{i_j}\right)$$

where

$$\frac{\partial \alpha_I}{\partial x} = \frac{\partial\left(\alpha_{i_1}, \ldots, \alpha_{i_k}\right)}{\partial(x_1, \ldots, x_k)}$$

Proof.
1-form case. Take $v \in \mathbb{R}^k$ then

$$\begin{aligned}
\alpha^*(dy_i)(x)(v) &= dy_i \circ \alpha(x)(D\alpha(x)v) \\
&= D\alpha(x)(v)e_1 \\
&= \sum_{j=1}^{k} D_j\alpha_i(x)v_j \\
&= \sum_{j=1}^{k} (D_j\alpha_i)\,dx_j.
\end{aligned}$$

k-form case.

$$\begin{aligned}
\alpha^*\left(\bigwedge_{j\in[k]} dy_{i_j}\right) &= \bigwedge_{j\in[k]} \alpha^*\left(dy_{i_j}\right) && \text{dual distributes over } \wedge \\
&= \bigwedge_{j\in[k]} d\alpha_{i_j} \\
&= \bigwedge_{\ell=1}^{k}\left(\sum_{j=1}^{k}(D_j\alpha_{i_\ell})\,dx_j\right) && \text{1-form case} \\
&= \bigwedge_{\ell=1}^{k}\left(\sum_{b=1}^{k} C_{\ell b}\,dx_b\right) && \text{putting } C_{ab} = D_b\alpha_{i_a} \\
&= \left(\det(C_{ab})_{a,b\in[k]}\right) && \text{by homework}
\end{aligned}$$

Theorem. (The Differential and Dual Commute)
Let $A \subseteq \mathbb{R}^k$ and $B \subseteq \mathbb{R}^n$ be open and $\alpha : A \to B$ smooth. Then,

$$\alpha^*(d\omega) = d\left(\alpha^*\omega\right)$$

$\forall \omega \in \Omega^\ell(B)$.

Proof.
$\ell = 0$ case: read the book.
$\ell > 0$ By linearity it is enough to consider $\omega = f\,dy_I$, for $I \in \mathrm{ASC}_{\ell,n}$.

$$\alpha^*(d\omega) = \alpha^*(df \wedge dy_I) = \alpha^*(df) \wedge \alpha^*(dy_I)$$

$$d(\alpha^*\omega) = d(\alpha^*(f \wedge dy_I)) = d(\alpha^* f \wedge \alpha^* dy_I) = d\alpha^* f \wedge \alpha^*(dy_I) = (-1)^0(\alpha^* f) \wedge d(\alpha^*(dy_I))$$

But $d(\alpha^*(dy_I)) = d(d\alpha_{i_1} \wedge \cdots \wedge d\alpha_{i_k})$ is zero since d^2 is zero.

By the 1-form case we have $\alpha^*(df) = d\left(\alpha^* f\right)$.

INTEGRATION OF FORMS ON MANIFOLDS

Integration on Parametrized Manifolds

Definition. (Integration over Parametrized Manifold)
Suppose $\eta \in \Omega^k(A)$ for $A \subseteq \mathbb{R}^k$ open. For $\eta = f\, dx_1 \wedge \cdots \wedge dx_k$, define

$$\int_A \eta = \int_A f$$

Definition. (Integral of a Form on a Parametrized Manifold)
Let $\alpha : A \subseteq \mathbb{R}^k \to \mathbb{R}^n$ be smooth. Let $Y = \alpha(A)$ and $B \subseteq \mathbb{R}^n$ be open and contain Y. Then Y_α is the manifold parametrized by α. For $\omega \in \Omega^k(B)$,

$$\int_{Y_\alpha} \omega = \int_A \alpha^* \omega$$

which equals $\int_A f$ provided that $\omega = f\, dx_1 \wedge \cdots \wedge dx_k$,

Example: Let $\alpha : (0, \pi) \times (0, 1) \to \mathbb{R}^3$ be given by $\alpha(\theta, t) = (2\cos\theta, 2\sin\theta, t)$. Then Y_α looks like half the label of a pop can. Let $\omega = xz\, dy \wedge dz - yz\, dx \wedge dz$. This form is identified with the vector field $F(x, y, z) = (xz, yz, 0)$ by the musical isomorphism $\sharp$. This form acts outwards on the manifold and gains magnitude as z increases.

$$\begin{aligned}
\int_{Y_\alpha} \omega &= \int_{Y_\alpha} \alpha^* \omega \\
&= \int_A \alpha^*(xz)\alpha^*(dy) \wedge \alpha^*(dz) - \alpha^*(yz)\alpha^*(dx) \wedge \alpha^*(dz) \\
&= \int_A (2\cos\theta)(t)(2\cos\theta\, d\theta) \wedge dt - (2\sin\theta)(t)(-2\sin\theta\, d\theta) \wedge dt \\
&= \int_A 2t\, d\theta \wedge dt \\
&= \int_0^\pi \int_0^1 2t\, d\theta dt \\
&= 2\pi
\end{aligned}$$

where $d\theta dt$ is a formal expression.

Now we do the same computation with $\beta(t, \theta) = (2\cos\theta, 2\sin\theta, t)$. Then

$$\begin{aligned}
\int_{Y_\alpha} \omega &= \int_A 2t\, dt \wedge d\theta \\
&= \int_A 2t\, d\theta \wedge dt \\
&= -2\pi
\end{aligned}$$

Even though $Y_\beta = Y_\alpha$ as a set, the integrals are equal only up to sign.

Now we do the same computation with $\gamma : (-2, 2) \times (0, 1) \to \mathbb{R}^3$ given by $\gamma(u, v) = (u, \sqrt{4 - u^2}, v)$. Then again $Y_\gamma \cong Y_\alpha$

$$\begin{aligned}\int_{Y_\gamma} \omega &= \int_{(-2,2)\times(0,1)} \frac{4u}{\sqrt{4-u^2}}\, du \wedge dv \\ &= \int_{-2}^{2}\int_{0}^{1} \frac{4u}{\sqrt{4-u^2}}\, dudv \\ &= -2\pi\end{aligned}$$

using substitution with $u = \sin(t)$.

Theorem. (Reparametrization Up To Sign)
Let $A, B \subseteq \mathbb{R}^k$ be open. If $g : A \to B$ is a diffeomorphism and $\det Dg$ does not change sign on A. Let $\beta : B \to \mathbb{R}^n$ be smooth and put $\alpha = \beta \circ g$. Let $O \subseteq \mathbb{R}^n$ be open and contain $Y = \alpha(A) = \beta(B)$. Then for every $\omega \in \Omega^k(O)$

$$\int_{Y_\alpha} \omega = \pm \int_{Y_\beta}$$

the minus sign is obtained only when $\det Dg < 0$ on A.

Remark: if A is connected, the condition "$\det Dg$ does not change sign" is automatically satisfied.

Proof.

$$\begin{aligned}
\int_{Y_\beta} &= \int_B \beta^*\omega && \beta^*\omega = f\, d_{y_1} \wedge \cdots \wedge d_{y_k} \\
&= \int_B f \\
&= \\
&= \int_{Y_\alpha} \omega \\
&= \int_A \alpha^*\omega \\
&= \int_A g^*\left(\beta^*\omega\right) && \alpha \coloneqq \beta \circ g \\
&= \int_A g^*\left(f\, d_{y_1} \wedge \cdots \wedge d_{y_k}\right) && \text{properties of pullback} \\
&= \int_A (f \circ g) \cdot (\det Dg)\ \ d_{y_1} \wedge \cdots \wedge d_{y_k} \\
&= \pm \int_B f && \text{change of variables}
\end{aligned}$$

Theorem. (Computation of Integrals of Forms)
Suppose $A \subseteq \mathbb{R}^k$ is open and $\alpha : A \to \mathbb{R}^n$ is smooth. For $I \in \mathrm{ASC}_{k,n}$,

$$\int_{Y_\alpha} f\, dz_I = \int_A (f \circ \alpha) \cdot \left(\det \frac{\partial \alpha_I}{\partial x}\right)$$

Proof.
Do it!

Remark: If $A = (a,b) \subset \mathbb{R}$.

$$\int_{(a,b)} f\,dx = \int_{(a,b)} f = \int_a^b f(x)\,dx$$

where the dx on the LHS has rigorous meaning and the dx on the RHS has formal meaning. So the formal notation dx in a one dimensional integral immediately makes formal sense.

This does not hold for any higher dimensions.

About the integral of a form question on the IBL.[8]

Integration on Orientable Manifolds

In a parametrized manifold, the manifold is a one-patch manifold and the patch is given. For general differential manifolds we need several patches (this issue is easy to handle with a partition of unity). We also have a freedom of choice of parametrization about any given points.

A reasonable definition would be

$$\int_M \omega = \int_{\operatorname{Int}\mathcal{U}} \alpha^*\omega$$

for some patch $\alpha : \mathcal{U} \to M$ where $\mathcal{U}$ is open in $\mathbb{R}^k$ or $\mathbb{H}^k$.

Unfortunately this definition causes sign issues, which we seek to solve here.

Definition. (Sign of Coordinate Overlap)
Let $M \subseteq \mathbb{R}^n$ be a k-manifold. Let $\alpha_1 : \mathcal{U}_1 \to \mathcal{V}_1$ and $\alpha_2 : \mathcal{U}_2 \to \mathcal{V}_2$ be two coordinate matches with non-empty overlap (i.e. $\mathcal{V}_1 \cap \mathcal{V}_2 \neq \varnothing$). Recall the transition map $\alpha_2^{-1} \circ \alpha_1$ is a diffeomorphism (sufficiently restricted). We say that α_1 and α_2 **overlap positively** if

$$\det\left(\alpha_2^{-1} \circ \alpha_1\right) > 0$$

everywhere it is defined.
We define **overlap negatively** similarly (these are the only cases).

Definition. (Orientable Manifold)
Let $M \subseteq \mathbb{R}^n$ be a k-manifold in $\mathbb{R}^n$. We say M is **orientable** if it can be covered by coordinate patches which pairwise overlap positively. Otherwise, we say M is **non-orientable**.

Example: Consider $\mathbb{S}^1 \subseteq \mathbb{R}^2$. For $i \in [4]$, let $\alpha_i : (-1,1) \to \mathbb{R}^2$ be given by

$$\begin{aligned}
\alpha_1(t) &= \left(t, \sqrt{1-t^2}\right)\\
\alpha_2(t) &= \left(t, -\sqrt{1-t^2}\right)\\
\alpha_3(t) &= \left(\sqrt{1-t^2}, t\right)\\
\alpha_4(t) &= \left(-\sqrt{1-t^2}, t\right)
\end{aligned}$$

[8] "I was hoping this was a trick question, but it took you all 40 minutes to figure it out."

The overlap of α_1 and α_3 is in the first quadrant of the coordinate plane. Thus α_1, α_3 overlap negatively. This occurs because the "trajectory" of these maps is opposite.

$$\left(\alpha_3^{-1} \circ \alpha_1\right)(t) = \sqrt{1-t^2}$$

for $0 < t < 1$.

Thus,

$$D\left(\alpha_3^{-1} \circ \alpha_1\right)(t) = \frac{-t}{\sqrt{1-t^2}} < 0$$

$\forall t \in (0,1)$.

Instead, for $i \in [4]$, put $\beta_i : (-1,1) \to \mathbb{R}^2$

$$\begin{aligned}
\beta_1(t) &= \alpha_1(t) \\
\beta_2(t) &= \alpha_1(-t) \\
\beta_3(t) &= \alpha_1(-t) \\
\beta_4(t) &= \alpha_1(t)
\end{aligned}$$

Then all transition maps overlap positively. Thus, $\mathbb{S}^1$ is orientable.

Example: Let $\alpha : (0,\pi) \times (0,1) \to \mathbb{R}^3$ be given by $\alpha(\theta,t) = (2\cos\theta, 2\sin\theta, t)$. Y_α looks like half the label of a pop can. Since Y_α can be covered by a single patch, Y_α is orientable.

Let $\gamma : (-2,2) \times (0,1) \to \mathbb{R}^3$ be given by $\gamma(u,v) = (u, \sqrt{4-u^2}, v)$. Then again $Y_\gamma \cong Y_\alpha$

Check that α, γ overlap negatively.

Example: The Möbius band is non-orientable and so is the Klein Bottle. The Klein bottle is a 2-manifold in $\mathbb{R}^4$. The Klein Bottle contains the Möbius strip as a subspace.

Do we have a way of showing the Möbius band as a set is a manifold.

Definition. (Orientation of a Manifold)
Let $M \subseteq \mathbb{R}^n$ be an orientable k-manifold. An **orientation** of M is a "maximal collection" of positively overlapping coordinate patches.
An **oriented manifold** is an orientable manifold together with a specific orientation.

There are simpler descriptions of orientation of manifolds in $\mathbb{R}^n$ for three cases $k = 1, (n-1), n$ that correspond nicely with intuition.

Definition. (One Dimensional Orientation of a Manifold)
Let $M \subseteq \mathbb{R}^n$ be a 1-manifold. For $p \in M$, let $\alpha : \mathcal{U} \to \mathcal{V}$ be a patch about p belonging to the orientation. Define $T(p) := \left(p, \frac{D\alpha(t_0)}{\|D\alpha(t_0)\|}\right)$ where $t_0 = \alpha^{-1}(p)$ for some α in the orientation of M. We call $T : M \to T(M)$ the **unit tangent field** corresponding to the orientation of M.
Remark: $T(p)$ is well defined. Thus for $k = 1$ we just need to give our curve a direction.
Let $\mathbb{L} := \mathbb{R}^1 \setminus \mathbb{H}^1_+$ then we can remedy the case for k-manifolds with boundary by allowing patches from $\mathbb{R}, \mathbb{H}$, or $\mathbb{L}$.
Fact: Every 1-manifold is orientable.

Definition. (Orientation of a Hyper-Manifold)
Let $M \subseteq \mathbb{R}^n$ be an oriented $(n-1)$-manifold in $\mathbb{R}^n$. Take $p \in M$ and a α a coordinate patch about p belonging to the orientation. Put $x_0 = \alpha^{-1}(p)$.
Let $N(p)$ be the unit vector such that

(i) $N(p)$ is perpendicular (or normal) to $T_p(M)$, meaning

$$N(p) \cdot \frac{\partial \alpha}{\partial x_i}(x_0) = 0$$

$\forall i \in [n-1]$.

(ii)

$$\det \left[\begin{array}{c|c} N(p) & D\alpha(x_0) \end{array}\right] > 0$$

Remark: $N : M \to \mathbb{R}^n$ is well defined and smooth.
N is called the **unit normal field** corresponding to the orientation of M.

Example: Let $\alpha : (-\pi, \pi) \times (0,1) \to \mathbb{R}^3$ be given by $\alpha(\theta, t) = (2\cos\theta, 2\sin\theta, t)$.

$$D\alpha(\theta, t) = \begin{bmatrix} -2\sin\theta & 0 \\ 2\cos\theta & 0 \\ 0 & 1 \end{bmatrix} \qquad N(\alpha(\theta, t)) = \pm \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix}$$

We choose the positive vector.

Check that

$$N(\gamma(u,v)) = -\frac{1}{2}\begin{bmatrix} u \\ \sqrt{4-u^2} \\ 0 \end{bmatrix}$$

Definition. (Orientation of an n-Manifold)
Let $M \subseteq \mathbb{R}^n$ be an n-manifold. Define the **natural orientation** of M to be the collection of coordinate patches $\alpha : \mathcal{U} \to \mathcal{V}$ satisfying $\det D\alpha > 0$.

Definition. (Reverse Orientation)
Define $r : \mathbb{R}^k \to \mathbb{R}^k$ by $r(x_1, \ldots, x_k) = (-x_1, x_2, \ldots, x_k)$.
Note $r : \mathbb{H}^1 \to \mathbb{L}^1$ and for $k > 1$ we have $r : \mathbb{H}^k \to \mathbb{H}^k$.

Lemma.
Let M be an oriented manifold in $\mathbb{R}^n$, for a patch $\alpha : \mathcal{U} \to \mathcal{V}$ belonging to the orientation $\mathscr{A}$, let $\beta = \alpha \circ r : r(\mathcal{U}) \to \mathcal{V}$. Then β is a coordinate patch and overlaps with α negatively. So β does not belong to the orientation. If $\alpha_1, \alpha_2 \in \mathscr{A}$, then $r(\alpha_1), r(\alpha_2)$ overlap positively. The collection $r(\mathscr{A})$ is called the **reverse** (or **opposite**) **orientation**.

This shows that every manifold has an even number of orientations. For a connected manifold, we will show that M admits at most 2 orientations. A manifold with n connected components admits at most 2^n orientations. Furthermore, if a manifold admits an orientation, then it admits the maximal number.

Induced Orientation on Boundary Manifolds

An orientation on a manifold induces (by restriction) an orientation on the boundary manifold. We now turn to study this relationship. Later we will see how to induce this orientation without using restriction.

Theorem. (Induced Orientation on ∂M)
Let $k > 1$, if $M \subseteq \mathbb{R}^n$ is an oriented k-manifold with non-empty boundary, then ∂M is orientable. The proof will construct an orientation on ∂M.

Proof.
Let $p \in \partial M$, there is a patch (about p) given by $\alpha : \mathcal{U} \to \mathcal{V}$ that belongs to the orientation of M. Note that $\mathcal{U} \subseteq \mathbb{H}^k$. Let $b : \mathbb{R}^{k-1} \to \mathbb{R}^k$ be given by $b(x_1, \ldots, x_{k-1}) = (x_1, \ldots, x_{k-1}, 0)$. Put $\alpha_0 \coloneqq \alpha \circ b : \mathcal{U}_0 \to \mathcal{V} \cap \partial M$. Recall that α_0 is a coordinate patch on ∂M about p.

Let α, β be two coordinate patches of M about p, in the orientation of M. So, $\det D\left(\beta^{-1} \circ \alpha\right) > 0$.

We will show that $\det D\left(\beta_0^{k-1} \circ \alpha_0\right) > 0$. Put $g \coloneqq \beta^{-1} \circ \alpha$ and $h \coloneqq \beta_0^{-1} \circ \alpha_0$.

$h(x_1, \ldots, x_{k-1}) = \Big(g_1(x_1, \ldots, x_{k-1}, 0), \ldots, g_{k-1}(x_1, \ldots, x_{k-1}, 0)\Big)$. Let $\mathcal{W}_1 = \alpha^{-1} \circ \beta(\mathcal{U}_2)$ and $\mathcal{W}_2 = \beta^{-1} \circ \alpha(\mathcal{U}_1)$. For all $x \in \mathcal{W}_1,\ g)k(x) \geq 0$ and $\forall x_0 \in \mathcal{W}_1 \cap \partial\mathbb{H}^k,\ g_k(x) = 0$. Let $x_0 \in \mathcal{W}_1 \cap \partial\mathbb{H}^k$,

$$\frac{\partial g_k}{\partial x_i} = \lim_{t \to 0^+} \frac{g_k(x_0 + te_i) - g_k(x_0)}{t} = \lim_{t \to 0^+} \frac{g_k(x_0 + te_i)}{t} \geq 0$$

$\forall i \in [k]$ and equality holds $\forall i \in [k-1]$.

$$Dg(x_0) = \frac{\partial(g_1, \ldots, g_k)}{\partial(x_1, \ldots, x_k)}(x_0) = \left[\begin{array}{c|c} \dfrac{\partial(g_1, \ldots, g_{k-1})}{\partial(x_1, \ldots, x_{k-1})} & \begin{matrix} * \\ \vdots \\ * \end{matrix} \\ \hline 0 \ \cdots \ 0 & \dfrac{\partial g_k}{\partial x_k}(x_0) \end{array}\right]$$

$$\implies \det Dg(b(q)) = (Dh(q))\left(\frac{\partial g_k}{\partial x_k}(b(q))\right)$$

Since $\frac{\partial g_k}{\partial x_k}(b(q)) > 0$, we find $\det Dg(b(q)) > 0$. Thus α_0, β_0 overlap positively.

$\square$

<u>Notation</u>: Given an orientation of M, we will call the orientation on ∂M obtained in the way above the **restriction orientation**.[9]

Definition. (Induced Orientation)
Let $M \subseteq \mathbb{R}^n$, be an oriented k-manifold with $\partial M \neq \varnothing$. The **induced orientation** of ∂M is
- (i) the restriction orientation if k is even, and
- (ii) the reverse orientation of the restriction orientation if k is odd.

[9] No one calls it this.

Example: Let $M = \mathbb{S}^2 \cap \mathbb{H}^3$. Then $\partial M = \mathbb{S}^1 \subseteq \mathbb{R}^3$. Let the orientation of M contain the patch $\alpha : (-1,1) \times [0,1) \to \mathbb{R}^3$ given by $\alpha(u,v) = \left(u, \sqrt{1-u^2-v^2}, v\right)$. Te restriction of α on ∂M is $\alpha_0 : (-1,1) \to \mathbb{R}^3$ given by $\alpha_0 = \alpha(u,0) = \left(u, \sqrt{1-u^2}, 0\right)$.

Then $T(\alpha_0(u)) = \frac{D\alpha_0(u)}{\|D\alpha_0(u)\|} = (\sqrt{1-u^2}, -u, 0)$. In particular, $T(0,1,0) = T(\alpha_0(0)) = (1,0,0)$.

$$N(\alpha(u,v)) = \frac{\frac{\partial \alpha}{\partial u} \times \frac{\partial \alpha}{\partial v}}{\left\| \frac{\partial \alpha}{\partial u} \times \frac{\partial \alpha}{\partial v} \right\|} = (-u, -\sqrt{1-u^2-v^2}, -v) = -\alpha(u,v)$$

In particular, $N(0,1,0) = N(\alpha(0,0)) = (0,-1,0)$.

Let $W(p)$ be the unit vector tangent to M at p that points "into" M. This $\left(N(p), T(p), W(p)\right)$ system form the **right-handed frame** in $\mathbb{R}^3$. Even if you do the reverse orientation, you still get the right-handed frame.

Example: Put $M = \mathbb{B}^3$. Then $\partial \mathbb{B}^3 = \mathbb{S}^2$. Since M is a 3-dimensional manifold in $\mathbb{R}^3$, it has a natural orientation. For a patch $\alpha : \mathcal{U} \subseteq \mathbb{H}^3 \to \mathcal{V}$ in the orientation, the natural orientation guarantees that

$$0 < \det D\alpha = \det \begin{bmatrix} \frac{\partial \alpha}{\partial x_1} & \frac{\partial \alpha}{\partial x_2} & \frac{\partial \alpha}{\partial x_3} \end{bmatrix}.$$

Now consider the induced orientation on $\partial M = \mathbb{S}^2$. This amounts to asking where does the Normal vector point?

Recall that we have

$$\det \begin{bmatrix} N & \frac{\partial \beta}{\partial x_1} & \frac{\partial \beta}{\partial x_2} \end{bmatrix} > 0$$

for some patch β.

Using the restriction of α (and since we use the opposite of the restricted orientation), we find that N should satisfy

$$\det \begin{bmatrix} -N & \frac{\partial \alpha}{\partial x_1} & \frac{\partial \alpha}{\partial x_2} \end{bmatrix} > 0.$$

Since

$$0 < \det D\alpha = \det \begin{bmatrix} \frac{\partial \alpha}{\partial x_3} & \frac{\partial \alpha}{\partial x_1} & \frac{\partial \alpha}{\partial x_2} \end{bmatrix}$$

We obtain that $-N$ and $\frac{\partial \alpha}{\partial x_3}$ point the same side of the tangent plane. Thus N points "outward."

Integration of forms on Oriented Manifolds

We proceed just as before. We begin with the single patch case.

Definition. (Integration of form over Single Patch Oriented Manifold)
Let $M \subseteq \mathbb{R}^n$ be a compact, oriented k-manifold. Let ω be a k-form on and open $B \subseteq \mathbb{R}^n$ containing M. Let $C = M \cap \operatorname{supp}(\omega)$ be compact. Suppose there is a coordinate patch in the orientation $\alpha : \mathcal{U} \to \mathcal{V}$ such that $V \supseteq C$ and $\mathcal{U}$ is bounded. Define

$$\int_M \omega = \int_{\operatorname{Int} \mathcal{U}} \alpha^* \omega = \int_{\operatorname{Int} \mathcal{U}} f$$

where $\alpha^* \omega = f\, dx_1 \wedge \cdots \wedge dx_k$.

We can integrate over $\text{Int}\,\mathcal{U}$ because we are at most removing $\partial\mathbb{H}^k$ which is measure zero.

We need to check that the RHS is ordinary integrable, satisfies properties like linearity, and does not depend on the choice of coordinate patch. These concerns are satisfied by the Change of Variables Theorem since

$$\int_{\text{Int}\,\mathcal{U}_1} f \circ \alpha_1 = \pm \int_{\text{Int}\,\mathcal{U}_2} f \circ \alpha_2$$

But since α_1 and α_2 overlap positively these equations are equal.

Note that we often write $-M$ to indicate the oriented manifold with the reverse orientation. Then,

$$\int_{-M} \omega = -\int_M \omega$$

Definition. (Integration of form over Oriented Manifold)
Let $M \subseteq \mathbb{R}^n$ be a compact, oriented k-manifold. Let ω be a k-form on and open $B \subseteq \mathbb{R}^n$ containing M. Let $\{\phi_i\}_{i\in[\ell]}$ be a partition of unity on M dominated by the orientation. Define,

$$\int_M \omega = \sum_{i\in[\ell]} \int_M \phi_i \cdot \omega$$

One can check this is well-defined, satisfies linearity, and sign flip for orientation reversal.

Theorem. (Computation of Integral of a Form)
Let $M \subseteq \mathbb{R}^n$ be a compact, oriented k-manifold. Let ω be a k-form on and open $B \subseteq \mathbb{R}^n$ containing M. Suppose for each $i \in [\ell]$ there is a coordinate patch $\alpha_i : \mathcal{A}_i \to \mathcal{M}_i$ of M in the orientation such that

$$M = \left(\bigcup_{i\in[\ell]} \mathcal{M}_i\right) \cup K$$

for measure zero $K \subseteq M$.

Then,

$$\int_M \omega = \sum_{i\in[\ell]} \int_{\mathcal{A}_i} \alpha_i^* \omega.$$

Proof.
Skip! $\square$

Example:
Let $M = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1,\ 0 \le z \le 1\}$ with orientation so that $N\left(1, 0, \frac{1}{2}\right) = (-1, 0, 0)$. We compute $\int_M z\, dx \wedge dy$. Note that $N(x, y, z) = (-x, -y, 0)$ for all $(x, y, z) \in M$. Now we look for a nice patch. Put $\alpha : (-\pi, \pi) \times (0, 1) \to \mathbb{R}^3$ then $\alpha(\theta, t) = (\cos\theta, \sin\theta, t)$.

To find the orientation of α, we compute the cross product

$$\frac{\partial\alpha}{\partial\theta} \times \frac{\partial\alpha}{\partial t} = \begin{bmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = (\cos\theta, \sin\theta, 0).$$

Whoops! Our sign was wrong, so we choose the opposite orientation. Putting $\gamma(\theta, t) = \alpha(-\theta, t)$ or $\gamma(\theta, t) = \alpha(t, \theta)$. We will use the later method. So $\beta : (0,1) \times (-\pi, \pi) \to \mathbb{R}^3$ with $\beta(\theta, t) = \alpha(t, \theta)$ is a coordinate patch in the orientation.

$$\begin{aligned}\int_M z\,dx \wedge dy &= \int_{(0,1)\times(-\pi,\pi)} \beta^*(z\,dx \wedge dy) \\ &= \int_{(0,1)\times(-\pi,\pi)} t\,d\beta_1 \wedge d\beta_2 \\ &= \int_0^1 \int_{-\pi}^{\pi} t \cdot 0 \\ &= 0\end{aligned}$$

The exam is up to here.

<u>Intuition</u>: We now consider the meaning of integrating a form over an $(n-1)$ manifold.

Let $\Omega \subseteq \mathbb{R}^2$ be open and $g : \Omega \to \mathbb{R}$ smooth. Put $M \coloneqq G_g$, the graph of g. Note that M is an oriented manifold (it has a single patch). Fix $\alpha(x, y) = (x, y, g(x, y))$, so α is a patch that covers M. Consider the orientation of M that contains α. Since $3 - 1 = 2$, we can describe the orientation using the unit normal field.

$$\frac{\partial \alpha}{\partial x} \times \frac{\partial \alpha}{\partial y} = \begin{bmatrix} 1 \\ 0 \\ \frac{\partial g}{\partial x} \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ \frac{\partial g}{\partial y} \end{bmatrix} = \left(-\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, 1 \right)$$

So the unit normal vector, will be this vector but normalized. Thus N points "upwards."

Let $x = u, y = v$, and $z = g(x, y)$ and we compute,

$$\begin{aligned}&\int_M F_1\,dy \wedge dz - F_2\,dx \wedge dz + F_3\,dx \wedge dy \\ &= \int_\Omega (F_1 \circ \alpha)\,dv \wedge \left(\frac{\partial g}{\partial u} du + \frac{\partial g}{\partial v} dv \right) - (F_2 \circ \alpha)\,u \wedge \left(\frac{\partial g}{\partial u} du + \frac{\partial g}{\partial v} dv \right) + (F_3) \circ \alpha)\,du \wedge dv \\ &= \int_\Omega \left(-(F_1 \circ \alpha) \frac{\partial g}{\partial u} - (F_2 \circ \alpha)\,\frac{\partial g}{\partial v} + (F_3) \circ \alpha) \right)\,du \wedge dv \\ &= \int_\Omega (F_1 \circ \alpha, F_3 \circ \alpha, F_2 \circ \alpha) \cdot \left(-\frac{\partial g}{\partial u}, \frac{\partial g}{\partial v}, 1 \right) (\star)\end{aligned}$$

Recall that for any $h : M \to \mathbb{R}$

$$\int_M h\,dV = \int_\Omega (h \circ \alpha)\,V(D\alpha)$$

For our case,

$$V(D\alpha) = \sqrt{\det\left(D\alpha^T D\alpha\right)} = \sqrt{1 + \left(\frac{\partial g}{\partial u}\right)^2 + \left(\frac{\partial g}{\partial v}\right)^2}$$

Multiplying $(\star)$ by $1 = \frac{\sqrt{1+\left(\frac{\partial g}{\partial u}\right)^2+\left(\frac{\partial g}{\partial v}\right)^2}}{\sqrt{1+\left(\frac{\partial g}{\partial u}\right)^2+\left(\frac{\partial g}{\partial v}\right)^2}}$, we obtain

$$\int_M F_1\, dy \wedge dz - F_2\, dx \wedge dz + F_3\, dx \wedge dy$$
$$= \int_\Omega (F \cdot N)\ dV$$

This result still holds if M is an oriented 2-manifold in $\mathbb{R}^3$. More generally, this holds for an oriented $(n-1)$-manifold in $\mathbb{R}^n$. We can interpret this integral as a flux integral, we are finding the flux of the vector field F across M.

Example: Let $M = \mathbb{S}^2(a)$. Orient M such that $N(x,y,z) = \frac{(x,y,z)}{a}$ points outward. Compute

$$\int_{\mathbb{S}^2(a)} x\, dy \wedge dz - y\, dx \wedge dz + z\, dx \wedge dy = \int_{\mathbb{S}^2(a)} (x,y,z) \cdot \frac{(x,y,z)}{a}\, dV$$
$$= \int_{\mathbb{S}^2(a)} \frac{x^2+y^2+z^2}{a} 4\pi a^2 = 4\pi a^3.$$

Intuition: Let M be an oriented 1-manifold in $\mathbb{R}^n$ and suppose $\exists \alpha : (a,b) \to \mathbb{R}^n$, a coordinate patch in the orientation such that $M \setminus \alpha((a,b))$ has measure zero in M.

$$\begin{aligned}
\int_M \sum_{i\in[n]} F_i\, dx_i &= \int_{(a,b)} \sum_{i\in[n]} (F_i \circ \alpha)(t) \alpha_i'(t)\, dt\, dx_i \\
&= \int_{(a,b)} F(\alpha(t)) \cdot D\alpha(t) \\
&= \int_{(a,b)} F(\alpha(t)) \cdot \frac{D\alpha(t)}{\|D\alpha(t)} \|D\alpha(t)\| \\
&= \int_{(a,b)} (F \cdot T)(\alpha(t)) \|D\alpha(t)\|D\alpha(t)\| \\
&= \int_{(a,b)} (F \cdot T)(\alpha(t))\, dV
\end{aligned}$$

because in the one manifold case $\|D\alpha(t)\| = V(D\alpha)$.

This can be interpreted as a work integral, meaning the total work done by F along M. If the manifold is homeomorphic so $\mathbb{S}^1$, then this is called a "circulation" of the "velocity field" F along M.

REVIEW

Up until now we have learned 11 things.

(1) We computed the volume of a parallelepiped.
(2) Integration of a scalar function over a parametrized manifold.
(3) Differentiable manifolds without boundary.
(4) Differentiable manifolds with boundary.
(5) Integration of a scalar function over a manifold. We did so by defining measure zero sets on a manifold.
(6) Tensors.
(7) Differential forms (alternating tensor valued functions).
(8) Tangent spaces to a manifold.
(9) Dual transform.
(10) Integration of differential forms on parametrized manifolds.
(11) Orientations of a manifold.
(12) Integration of differential forms on orientable manifolds.
(13) The relationship of integrals of differential forms on an oriented manifold and its boundary.
(14) Special cases of Generalized Stoke's Theorem.

Note that 12-14 will not be covered on the Midterm.

Afterwards we will get to study Fourier Analysis . . . which "we are very excited to study".

INTEGRATION OF BOUNDARY MANIFOLDS

Generalized Stokes' Theorem

Recall the following theorem from Calculus

Theorem. (Fundamental Theorem of Calculus)

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

Interpreting this as an integration of a form over a manifold we obtain

$$\int_{\partial[a,b]} df = f(b) - f(a).$$

Let $I^k = [0,1]^k$ be the closed unit cube in $\mathbb{R}^k$. So, $\operatorname{Int} I^k = (0,1)^k$ and $\operatorname{Bd} I^k = I^k \setminus \operatorname{Int} I^k$. We will frequently be interested in the set $\operatorname{Int} I^{k-1} \times \{0\}$.

Lemma. ()
Let $k > 1$ and $\eta \in \Omega^{k-1}(A)$ for some open $A \subseteq \mathbb{R}^k$ which contains I^k. Suppose η vanishes at all points of $\operatorname{Bd} I^k$, except possibly at points of $\operatorname{Int} I^{k-1} \times \{0\}$.
Then,

$$\int_{\operatorname{Int} I^k} d\eta = (-1)^k \int_{\operatorname{Int} I^{k-1}} b^*\eta$$

where $b : \mathbb{R}^{k-1} \hookrightarrow \mathbb{R}^k$.

Proof.
We can write

$$\eta = \sum_{j=1}^{k} f_j\, dx_1 \wedge \cdots \wedge \hat{dx_j} \wedge \cdots \wedge dx_k.$$

By the linearity of integrals, it is enough to check when η is of the form $f\, dx_1 \wedge \cdots \wedge \hat{dx_j} \wedge \cdots \wedge dx_k$.

$$\begin{aligned} d\eta &= \left(\sum_{i\in[k]} (D_i f)\, dx_i \right) \wedge dx_1 \wedge \cdots \wedge \hat{dx_j} \wedge \cdots \wedge dx_k \\ &= (-1)^{j-1} (D_j f)\, dx_1 \wedge \cdots \wedge dx_j \wedge \cdots \wedge dx_k \end{aligned}$$

$$
\begin{aligned}
\int_{\operatorname{Int} I^k} d\eta &= \int_{\operatorname{Int} I^k} (-1)^{j-1} D_j f \\
&= (-1)^{j-1} \int_{I^k} D_j f \\
&= (-1)^{j-1} \int_{v \in I^{k-1}} \int_{x_j \in [0,1]} D_j f(x_1, \dots, x_k) && \text{where } v \coloneqq (x_1, \dots, \hat{x}_j, \dots, x_k) \\
&= (-1)^{j-1} \int_{v \in I^{k-1}} f(x_1, \dots, 1, \dots, x_k) - f(x_1, \dots, 0, \dots, x_k) && \text{1,0 in the } j\text{-th spot} \\
&= (-1)^{j-1} \int_{v \in I^{k-1}} f(x_1, \dots, 0, \dots, x_k) && \text{by assumption} \\
&= (-1)^{j-1} \int_{v \in I^{k-1}} f(x_1, \dots, 0, \dots, x_{k-1}, 0) \\
&= (-1)^{j-1} \int_{v \in I^{k-1}} (f \circ b)(x_1, \dots, 0, \dots, x_{k-1}) \\
&= (-1)^{j-1} \int_{\operatorname{Int} I^{k-1}} b^* f
\end{aligned}
$$

Now we consider the RHS. We compute $b^*\eta = (f \circ b)\, db_1 \wedge \cdots \wedge \hat{db_j} \wedge \cdots \wedge db_k$. Is zero unless $j = k$. In which case $b^*\eta = (f \circ b)\, dx_1 \wedge \cdots \wedge dx_{k-1}$. Therefore,

$$
(-1)^k \int_{\operatorname{Int} I^{k-1}} b^*\eta = \begin{cases} 0 & j < k \\ \int_{\operatorname{Int} I^{k-1}} f \circ b & j = k \end{cases}
$$

□

This is half the proof for Generalized Stokes Theorem.

Definition. (Generalized Stokes' Theorem)
Let $k > 1$ and $M \subseteq \mathbb{R}^n$ be a compact, oriented k-manifold. Let ω be a $(k-1)$-form in an open neighborhood of M. Then,

$$
\int_M d\omega = \int_{\partial M} \omega
$$

where ∂M is given the induced orientation.

In the case that $\partial M = \varnothing$, then the RHS is zero.

Proof.

Cover M in patches in the following manner:

For $p \in M \setminus \partial M$. Let $\alpha : \mathcal{U} \to \mathcal{V}$ about p. Restrict, translate, and scale the domain of α to be $\operatorname{Int} I^k$. Note that α remains in the orientation of M.

For $p \in \partial M$, we do a similar maneuver but set the domain of α to be $\left(\operatorname{Int} I^k\right) \cup \left(\operatorname{Int} I^{k-1} \times \{0\}\right)$ which is open in $\mathbb{H}^k$, then $p \in \alpha\left(\operatorname{Int} I^{k-1} \times \{0\}\right)$.

"How do you do?"

Let $\{\phi_i\}_{i \in [\ell]}$ be a partition of unity dominated by the collection of patches constructed above.

$$
\int_M d\omega = \sum_{i \in [\ell]} \int_M \phi_i d\omega
$$

If we show the following identity,

$$\int_M d(\phi_i\omega) = \int_{\partial M} \phi_i\omega$$

$\forall i \in [\ell]$ Then,

$$\sum_{i\in[\ell]} \int_M \phi_i d(\phi_i\omega) = \int_{\partial M} \sum_{i\in[\ell]} \phi_i\omega = \int_{\partial M} \phi_i\omega.$$

Computing the LHS gives

$$\begin{aligned}
\sum_{i\in[\ell]} \int_M \phi_i d(\phi_i\omega) &= \sum_{i\in[\ell]} \int_M \Big(d\phi_i \wedge \omega + (-1)^0\phi_i d\omega\Big) \\
&= \int_M \left(d\left(\sum_{i\in[\ell]} \phi_i\right) \wedge \omega + \sum_{i\in[\ell]} \phi_i d\omega\right) \\
&= \int_M \phi_i d\omega
\end{aligned}$$

By 2, it is enough to prove the theorem for ω such that $\operatorname{supp}\omega \cap M$ is contained in one coordinate patch constructed earlier.

Suppose $W = \operatorname{Int} I^k$ and $\alpha : W \to M$ is a coordinate patch.

$$\begin{aligned}
\int_M d\omega &= \int_{\operatorname{Int} I^k} \alpha^*(d\omega) \\
&= \int_{\operatorname{Int} I^k} d \circ \alpha^*(\omega) \\
&= \int_{\operatorname{Int} I^k} d \circ \alpha^*(\omega) \\
&= (-1)^k \int_{\operatorname{Int} I^k} b^* \circ \alpha^*(\omega) \\
&= 0
\end{aligned}$$

where the third equality follows from the Lemma. The last equality follows since $\operatorname{supp}\alpha^*\omega \subseteq \operatorname{Int} I^k$.

Similarly,

$$\int_{\partial M} \omega = 0$$

since $\operatorname{supp}\omega \cap \partial M = \varnothing$.

Now the last case. Suppose $W = \Big(\operatorname{Int} I^k\Big) \cup \Big(\operatorname{Int} I^k \times \{0\}\Big)$.

Then,

$$\begin{aligned}
\int_M d\omega &= \int_W \alpha^*(d\omega) \\
&= \int_{\operatorname{Int} I^k} \alpha^*(d\omega) \\
&= (-1)^k \int_{\operatorname{Int} I^{k-1}} b^*\alpha^*(\omega)
\end{aligned}$$

Put $\beta = \alpha \circ b : \operatorname{Int} I^{k-1} \to \partial M$. Then β is a coordinate patch given by a restriction of α. Therefore β belongs to the orientation of ∂M if k is even. If k, is odd then we choose the reverse orientation of β.[10]

Thus,

$$\int_{\partial M} \omega = (-1)^k \int_{\operatorname{Int} I^{k-1}} \beta^* \omega = (-1)^k \int_{\operatorname{Int} I^{k-1}} b^* \alpha^*(\omega)$$

$\square$

Example: Set $M := \{(x, y, 9 - x^2 - y^2) : x^2 + y^2 \leq 9\}$. Then M looks like a cut off rotated paraboloid (a beanie). The orientation can be specified by the normal vector. Choose the normal vector to have a positive z-coordinate. Put $\omega = (2z - y)\, dx + (x + z)\, dy + (3x - 2y)\, dz$.

Let $\alpha : \operatorname{Int} \mathbb{B}^2(3) \to \mathbb{R}^3$ be given by $\alpha(u, v) = (u, v, 9 - u^2 - v^2)$.

$$\begin{aligned}
\int_M d\omega &= \int_M 2\, dx \wedge dy + dx \wedge dz - 3dy \wedge dz \\
&= \int_{\operatorname{Int} \mathbb{B}^2(3)} 2\, du \wedge dv + du \wedge (-2u - 2v\, dv) - 3\, dv \wedge (-2u\, du - 2v\, dv) \\
&= \int_{\operatorname{Int} \mathbb{B}^2(3)} (2 - 2v - 6u)\, du \wedge dv \\
&= \int_{\operatorname{Int} \mathbb{B}^2(3)} (2 - 2v - 6u) \\
&= \int_{\operatorname{Int} \mathbb{B}^2(3)} 2 \\
&= 2\pi 3^2 \\
&= 18\pi
\end{aligned}$$

The integral simplifies because integrals of odd functions over symmetric domains are zero. Now let's compute in another way. Recall since 2 is even, the restriction orientation is the induced orientation.

The rule is that when travelling on a 2-d object in $\mathbb{R}^3$ travel so that the manifold stays on your left.

Let $\beta : (0, 2\pi) :\to \mathbb{R}^2$ be given by $\beta(t) = (3\cos(t), 3\sin(t), 0)$.

$$\begin{aligned}
\int_{\partial M} \omega &= \int_{(0,2\pi)} (0 - 3\sin t)(-2\sin t\, dt) + (3\cos t + 0)(3\cos t) + 0 \\
&= \int_{(0,2\pi)} 9\, dt \\
&= \int_0^{2\pi} 9\, dt \\
&= 18\pi
\end{aligned}$$

So they are equal, as we have proved!

Now we prove the Generalized Stokes' Theorem for $k = 1$.

> **Definition**. (Arc)
> A (**smooth**) **arc** in $\mathbb{R}^n$ is an oriented 1-manifold $M \subseteq \mathbb{R}^n$ such that there is a patch $\alpha : [a, b] \to M$. If M is oriented and $\alpha \restriction (a, b)$ belongs to the orientation, we call $p = \alpha(a)$ the **initial point** and $q = \alpha(b)$ the **final point** of M.

[10]This is the sole reason for our definition of the induced orientation.

Theorem. (One Dimensional Stokes' Theorem)
Let $M \subseteq \mathbb{R}^n$ be an arc with initial point p and final point q. If $f : A \to \mathbb{R}$ is a zero form (i.e. a scalar function) where $A \supseteq M$ is open,

$$\int_M df = f(q) - f(p)$$

Proof.
Do it! Use the Fundamental Theorem of Calculus when you need it. $\square$

Example: Set $M := \{(t, t^2) : -1 \leq t \leq 1\}$ with initial point $(-1, 1)$ and final point $(1, 1)$. Let $\omega = y^2\, dx + 2xydy$ compute $\int_M \omega$. First we show that ω is exact. By inspection, $d\left(xy^2\right) = \omega$.[11]

$$\int_M \omega = 1(1)^2 - (-1)(1)^2 = 2.$$

Example: Let

$$F(x, y, z) = \frac{-(x, y, z)}{\left(x^2 + y^2 + z^2\right)^{3/2}}$$

$(x, y, z) \in \mathbb{R}^3 \setminus \{\vec{0}\}$. So,

$$F(\vec{x}) = \frac{1}{\|\vec{x}\|^2} \cdot \frac{\vec{x}}{\|\vec{x}\|^2}$$

Thus F is the gravitational force field due to a point source at the origin.

Now we want to compute total work done by F required to move a particle along a given trajectory that does not pass through the origin. Suppose the initial point is $(1, 0, 1)$ and the final point is $(1, 0, 0)$. Then $\int_M F \cdot T\, dV$ gives the total work.

$$\begin{aligned} \int_M F \cdot T\, dV &= \int_M F_1\, dx + F_2\, dy + F_3\, dz \\ &= f(1, 0, 1) - f(1, 0, 0) \qquad \text{where } f(\vec{x}) = \frac{1}{\|\vec{x}\|} \end{aligned}$$

The notion that the form being exact gives rise to conservative vector fields. Pretty interesting!

Special Cases of Generalized Stokes' Theorem

Corollary. (Green's Theorem)
For a type 3 region $D \subseteq \mathbb{R}^2$ we have

$$\int_{\partial D} P\, dx + Q\, dy = \int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$$

Proof.
Check that $d\omega = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \wedge dy$.

[11] The question of exactness gives rise to homology and cohomology theory.

Then we apply Stokes' Theorem with $n = 2, k = 2$

Corollary. (Green's Theorem)
Suppose $M \subseteq \mathbb{R}^2$ is a 2-manifold. Let $\omega = P\,dx + Q\,dy$, then $d\omega = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \wedge dy$. Generalized Stokes' Theorem tells us

$$\int_M \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \wedge dy = \int_{\partial M} P\,dx + Q\,dy$$

Corollary. (Divergence Theorem)
Suppose $M \subseteq \mathbb{R}^n$ is a n-manifold. For the case of $n = 3$, Let $\omega = P\,dy \wedge dz - Q\,dx \wedge dz + R\,dx \wedge dy$, then $d\omega = \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dx \wedge dy \wedge dz = \operatorname{div}(F)$ where $F = (P, Q, R)$. Generalized Stokes' Theorem tells us,

$$\int_M \operatorname{div}(F)\,dx \wedge dy \wedge dz = \int_{\partial M} P\,dy \wedge dz - Q\,dx \wedge dz + R\,dx \wedge dy = \int_{\partial M} (F \cdot N)\ dV$$

This theorem generalizes for all $n \in \mathbb{N}$.

Corollary. (Classical Stokes' Theorem)
Let $M \subseteq \mathbb{R}^3$ be a 2-manifold. Let $\omega = P\,dx + Q\,dy + R\,dz$, then

$$d\omega = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) dy \wedge dz - \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) dx \wedge dz + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \wedge dy.$$

Note that the coefficients are given by $\operatorname{curl}(F)$. Where

$$\operatorname{curl}(F) = \det \begin{bmatrix} e_1 & e_2 & e_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{bmatrix}.$$

Then Generalized Stokes' Theorem tells us,

$$\int_M d\omega = \int_{\partial M} P\,dx + Q\,dy + R\,dz,$$

Thus,

$$\int_M (\operatorname{curl}(F) \cdot N)\,dV = \int_{\partial M} (F \cdot T)\,dV.$$

We are done with analysis on manifolds!

FOURIER ANALYSIS

Our textbook will be Stein & Shakarchi “Fourier Analysis.” Another book is Dym & McKean “Fourier Series & Integrals.” The latter book is advanced and contains quite good examples. Finally Katnelson “Introduction to Harmonic Analysis” is the most difficult.

Fourier Analysis can be thought of as a subset of Harmonic Analysis. Fourier Analysis deals with representing functions as series of trigonometric functions and has applications to differential equations, number theory, probability, and almost all applied math areas.

Fourier series represent functions on a finite domain say $[-\pi, \pi]$. There are also Fourier Transforms or Fourier Integrals as well as Fourier analysis on groups (think Lie groups).

Complex Numbers

Definition. (Complex Numbers)
A complex number $z \in \mathbb{C}$ is a pair in $\mathbb{R}^2$ and written $z = a + ib$.
The complex numbers are formally defined as the field $\mathbb{C} = \mathbb{R}[i]$ where $i^2 = -1$. They are represented in the Euclidean plane by $z = (x, y) = x + iy$. There are two square roots of negative one in $\mathbb{C}$. The **modulus** of a complex number is $|z| = \sqrt{x^2 + y^2}$.

Since $\mathbb{C}$ is a field, we can divide complex numbers

$$\frac{a+ib}{c+id} = (a+ib)\frac{1}{c^2+d^2}\left(c - id\right).$$

Note that $|z| = 0 \iff z = 0$.

There is a Galois automorphism for the field extension $\mathbb{C} = \mathbb{R}[i]$ which is an involution called complex conjugation $z \mapsto \overline{z}$ where $\overline{z} = x - iy$. Note that $z\overline{z} = |z|^2$. Check that conjugation is a homomorphism, meaning $\frac{\overline{z_1}}{\overline{z_2}} = \overline{\left(\frac{z_1}{z_2}\right)}$ and $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$.

Also check that $|z_1 z_2| = |z_1||z_2$ and $\frac{1}{|z|} = \left|\frac{1}{z}\right|$

Definition. (Complex Exponential)
Define

$$e^{a+ib} = \exp(a+ib) = e^a\left(\cos b + i \sin b\right)$$

So $e^{2+i\frac{\pi}{4}} = e^2(\cos\pi/4 + \sin\pi/4) = \frac{e^2}{\sqrt{2}}(1+i)$.

<u>Motivation</u>: The typical properties of the exponential function are extended to $\mathbb{C}$ by this definition. We want

$$e^{ib} = \sum_{k=0}^{\infty} \frac{(ib)^k}{k!} = \sum_{k=0}^{\infty}\left(\frac{b^{2k}}{(2k)!} + i\left(\frac{b^{2k+1}}{(2k+1)!}\right)\right) = \cos b + i \sin b$$

It turns out this works.

Check that $e^{z+w} = e^z e^w$ for $\forall z, w \in \mathbb{C}$.

Every $z \in \mathbb{C}$ can be written as $re^{i\theta}$ for some $r, \theta \in \mathbb{R}$. This representation is not necessarily unique.

This is all algebra, we want to do analysis.

Definition. (Integral of Complex Functions)
Suppose $f : [a, b] \to \mathbb{C}$ where $f(x) = u(x) + iv(x)$. f is C^r if and only if u and v are C^r. Supposing that u, v are continuous almost everywhere then we define

$$\int_a^b f(x)\,dx := \int_a^b u(x)\,dx + i\int_a^b v(x)\,dx$$

Remark: This is different from the complex integral of a function $g : \mathbb{C} \to \mathbb{C}$. But this is quite different from the "Integral of a complex function."

Check that

$$\int_{-\pi}^{\pi} e^{ikx} = \begin{cases} 2\pi & k = 0 \\ 0 & k \in \mathbb{Z} \setminus \{0\}. \end{cases}$$

Motivation: Let S be the set of continuous $\mathbb{C}$-valued functions on $[-\pi, \pi]$. Then S is a vector space over $\mathbb{C}$. S can be made an inner product space via

$$\langle f, g\rangle = \int_{-\pi}^{\pi} f(x)\overline{g(x)}\,dx$$

Recall that given a spanning set of any vector space (of any dimension) the Graham Schmidtt process gives an orthonormal basis.

Let $e_n(x) = e^{inx}$ for $x \in [-\pi, pi],\ n \in \mathbb{Z}$. We compute

$$\langle e_n, e_m\rangle = \int_{-\pi}^{\pi} e^{inx}\overline{e^{imx}}\,dx = \int_{-\pi}^{\pi} e^{i(n-m)x}\,dx = \begin{cases} 2\pi & k = 0 \\ 0 & k \in \mathbb{Z} \setminus \{0\}. \end{cases}$$

So $\{e_n\}_{n\in\mathbb{Z}}$ is an orthogonal family. Is this a spanning set? Eventually, we will conclude yes. This means for every $f \in S$ we can write

$$f = \sum_{n\in\mathbb{Z}} a_n e_n$$

Questions

(a) What is a_n? Well obviously $\langle f, e_m\rangle = a_m 2\pi$. So,

$$a_m = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-imx}\,dx.$$

(b) Does the series converge?

(c) The series is equal to $f(x)$ for every x?

(d) How will this be useful at all?

"Now let's get into actual things."

Recall. (Notions of Continuity)
On $[0, L]$, piecewise continuous functions are functions that are continuous on $[0, L] \setminus D$ for some finite set of discontinuities $D \subseteq [0, L]$. The (Riemann) integrable functions are those which are bounded and continuous almost everywhere.

Definition. (Fourier Series)
Let f be an integrable function on $[-\pi, \pi]$.

(a) The n**-th Fourier Coefficient** of f is

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) e^{-in\theta} \, d\theta$$

$n \in \mathbb{Z}$.

(b) The **Fourier Series** of f is (formally)

$$f(\theta) \sim \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{in\theta}.$$

we know nothing about convergence yet.

Example: Let $f(\theta) = \theta$ for $\theta \in [-\pi, \pi]$.

Then integrating by parts,

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \theta e^{-in\theta} \, d\theta = \begin{cases} 0 & n = 0 \\ \frac{(-1)^{n+1}}{in} & n \neq 0. \end{cases}$$

So then

$$f(\theta) \sim \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{(-1)^{n+1}}{in} e^{in\theta} = \sum_{n \in \mathbb{Z}^+} \left[\frac{(-1)^{n+1}}{in} e^{in\theta} + \frac{(-1)^{-n+1}}{i(-n)} e^{i(-n)\theta} \right]$$

$$= \sum_{n \in \mathbb{Z}^+} \left[\frac{(-1)^{n+1}}{in} \left(e^{in\theta} + e^{-in\theta} \right) \right] = 2 \sum_{n \in \mathbb{Z}^+} \left[\frac{(-1)^{n+1}}{in} \left(\sin(n\theta) \right) \right]$$

This sum will never converge on $\{-\pi, \pi\}$.

Motivating Applications

Definition. ()
Let $\mathscr{R}$ be the set of (Riemann) integrable functions on $[-\pi, \pi]$. Then, $\mathscr{R}$ is a vector space over $\mathbb{C}$ and an inner product space with

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(x)\overline{g(x)}\, dx$$

which gives a norm,

$$\|f\| = \sqrt{\langle f, f \rangle} = \sqrt{\int_{-\pi}^{\pi} |f(x)|^2\, dx}$$

Facts. Analogues of linear algebra facts in finite dimensions.

(i) $\left\{e_n(\theta) = e^{in\theta}\right\}_{n\in\mathbb{Z}}$ is an orthonormal basis of $\mathscr{R}$.[a]

(ii) $\|f\|^2 = 2\pi \sum_{n\in\mathbb{Z}} \left|\hat{f}(n)\right|^2$. This is called **Parseval's Identity**.

(iii) $\langle f, g \rangle = \sum_{n\in\mathbb{Z}} \hat{f}(n)\overline{\hat{g}(n)}$.

[a] The set of Lebesgue integrable function is the completion of this basis.

Proof.
We will do these proofs later. □

Application One: Infinite series
Let $f(\theta) = \theta$ for $\theta \in [-\pi, \pi]$. Earlier we computed that

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \theta e^{-in\theta}\, d\theta = \begin{cases} 0 & n = 0 \\ \frac{(-1)^{n+1}}{in} & n \neq 0. \end{cases}$$

By Parseval's identity,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \theta^2\, d\theta = \sum_{n\neq 0} \frac{1}{n^2}$$

The LHS is $\frac{2}{3}\pi^3$ and the RHS is $2\sum_{n=1}^{\infty} \frac{1}{n^2}$. This gives us that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi}{6}.$$

Using a different f we find $\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^2}{90}$. For all even powers there is a closed form. For odd powers it is much harder, for example n^3 is not known.

Application Two: The isoperimetric problem
Of all simple closed C^1 planar curves (i.e. loops) of equal length, which curve encloses the most area A? The answer is intuitively the circle, we will prove this with Fourier Analysis.

Let C be a curve given by $C = \alpha([-\pi, \pi])$ where $\alpha : [-\pi, \pi] \to \mathbb{R}^2$ with $\alpha(\theta) = (x(\theta), y(\theta))$ is a continuous function satisfying:

(a) closed meaning $\alpha(-\pi) = \alpha(\pi)$,

(b) simple meaning $\alpha(\theta_1) \neq \alpha(\theta_2)$ for distinct $\theta_1, \theta_2 \in (-\pi, \pi)$,

(c) smooth x, y are C^1,

(d) the length is 2π so
$$\int_{-\pi}^{\pi} \sqrt{(x'(\theta))^2 + (y'(\theta))^2}\, d\theta = 2\pi,$$

(e) arc length parametrization is used meaning
$$\|\alpha'(\theta)\| = \sqrt{(x'(\theta))^2 + (y'(\theta))^2} = 1$$
$$\forall \theta \in [-\pi, \pi].$$

Such a parametrization always exists for a curve C because $t = t(\theta)$ is the inverse function of
$$\theta = \theta(t) = \int_{-\pi}^{t} \|\beta'(s)\|\, ds$$
so then $\alpha(\theta) = \beta(t(\theta))$ is a unit speed parametrization. By the Chain rule,
$$\frac{\partial \alpha}{\partial \theta} = \frac{\partial \beta}{\partial t}\frac{\partial t}{\partial \theta} = \frac{\partial \beta}{\partial t}\left(\frac{\partial \theta}{\partial t}\right)^{-1} = \left(\frac{\partial \beta}{\partial t}\right)\frac{1}{\|\beta'(t)\|}$$
So every parametrization of β can be scaled to be a unit speed parametrization.

Claim. (Solution to Isoperimetric Problem)
The enclosed area $A \leq \pi = v\left(\mathbb{S}^2(1)\right)$. Furthermore, if $A = \pi$ then C is a circle.

The following proof is due to Hurwitz (1931).

Proof.
By Green's Theorem,
$$A = \frac{1}{2}\left|\int_C x\, dy - y\, dx\right| \Leftarrow \pm \int_{\partial D} -y\, dx + x\, dy = \int\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = \int\int_D 2 = 2 \cdot A(D).$$
$$\begin{aligned} A &= \frac{1}{2}\left|\int_C x\, dy - y\, dx\right| \\ &= \frac{1}{2}\left|\int_{-\pi}^{\pi} x(\theta)y'(\theta) - y(\theta)x'(\theta)\, d\theta\right| \\ &= \frac{1}{2}\left|\int_{-\pi}^{\pi} x(\theta)\overline{y'(\theta)} - y(\theta)\overline{x'(\theta)}\, d\theta\right| \\ &== \frac{1}{2}\left|\langle x, y'\rangle - \langle y, x'\rangle\right| \\ &== \frac{1}{2}\left|\sum_{n\in\mathbb{Z}} \left(\hat{x}(n), \overline{\hat{y'}(n)} - \hat{y}(n), \overline{\hat{x'}(n)}\right)\right| \end{aligned}$$
Now denote $\hat{x}(n) = a_n$ and $\hat{y}(n) = b_n$. Note that via integration by parts
$$\hat{x'}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} x'(\theta)e^{-in\theta}\, d\theta = \left[\frac{1}{2\pi}x(\theta)e^{-in\theta}\right]_{-\pi}^{\pi} - \frac{1}{2\pi}(-in)\int_{-\pi}^{\pi} x(\theta)e^{-in\theta}\, d\theta$$

Since $x(\pi) = x(-\pi)$, we have

$$\hat{x'}(n) = (in)\hat{x}(n)$$

this is very useful.

Restoring our computation above

$$\begin{aligned}
A &= \pi \left| \sum_{n\in\mathbb{Z}} \left(\hat{x}(n), \overline{\hat{y'}(n)} - \hat{y}(n), \overline{\hat{x'}(n)} \right) \right| \\
&= \pi \left| \sum_{n\in\mathbb{Z}} (-in)(a_n\overline{b_n} - b_n\overline{a_n}) \right| \\
&= \pi \left| \sum_{n\in\mathbb{Z}} n(a_n\overline{b_n} - b_n\overline{a_n}) \right| && -i \text{ has modulus one.} \\
&\leq \pi \sum_{n\in\mathbb{Z}} |n| \left| a_n\overline{b_n} - b_n\overline{a_n} \right| && \text{by triangle inequality.} \\
&= \pi \sum_{n\in\mathbb{Z}} |n| \left| a_n\overline{b_n} \right| + |b_n\overline{a_n}| && \text{by triangle inequality.} \\
&= \pi \sum_{n\in\mathbb{Z}} 2|n||a_n||b_n| && \text{modulus of conjugate is modulus.} \\
&\leq \pi \sum_{n\in\mathbb{Z}} |n| \left(|a_n|^2 + |b_n|^2 \right) && \text{by AM-GM inequality.}
\end{aligned}$$

The only property we have not used is the arc length parametrization. Since we used smoothness, closedness, and simpleness for Green's Theorem.

Note that the unit speed parametrization condition implies that

$$\|x'\|^2 + \|y'^2\|^2 = 1 \implies \int_{-\pi}^{\pi} (x'(\theta))^2 + (y'(\theta))^2 \, d\theta = 2\pi$$

By the unit speed condition we can remove the square root.

Now by Parseval's Identity we have,

$$\sum_{n\in\mathbb{Z}} n^2 \left(|a_n|^2 + |b_n|^2 \right) = 1$$

Maximize

$$\pi \sum_{n\in\mathbb{Z}} |n| \left(|a_n|^2 + |b_n|^2 \right)$$

among all sequences $\{a_n\}_{n\in\mathbb{Z}}, \{b_n\}_{n\in\mathbb{Z}}$ subject to the condition

$$\sum_{n\in\mathbb{Z}} n^2 \left(|a_n|^2 + |b_n|^2 \right) = 1.$$

We can solve this easily. This seems like a Lagrange Multiplier case but for infinite dimensions.

Note that $|n| \leq n^2 \ \forall n \in \mathbb{Z}$. Thus,

$$A \leq \pi \sum_{n\in\mathbb{Z}} n^2 \left(|a_n|^2 + |b_n|^2 \right) \leq \pi.$$

□

To show that $A = \pi \implies C$ is a circle. We will skip this proof, but here are some steps involved. If $A = \pi$ that means that $|n| = n^2$ but this only happens for $n = -1, 0, 1$. This gives us that

$$x(\theta) \sim \sum_{n \in \mathbb{Z}} a_n e^{in\theta} = a_{-1} e^{-i\theta} + a_0 + a_1 e^{i\theta}.$$

You will obtain $x(\theta) = a_0 + \frac{1}{2}\cos(\theta + \alpha)$. A similar thing happens for $y(\theta)$. Read [S&S] section 4.1 for the full proof.

Application Three: Temperature of the Earth.
We interpret the variable t to denote time and x the position starting at the surface of the earth and boring down. Find a function $u(t, x)$ that is C^1 on $(t, x) \in \mathbb{R} \times [0, \infty)$ such that

(a) $\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2}$ is the heat equation. This is a partial differential equation.

(b) $u(t, 0) = f(t) = \sin(t)$ due to seasonal effects on the surface.

(c) $u(t + 2\pi, x) = u(t, x)$ meaning seasonal effects on the surface will permeate into the ground.

(d) $|u(t, x)| \leq \max_t |f(t)| = 1$ (we ignore effects from Earth's core).

It makes sense to think of $u(t, x)$ as a Fourier Series

$$u(t, x) = \sum_{n \in \mathbb{Z}} c_n(x) e^{int}$$

Solution. (Heat Equation on a Rod)

$$u(t, x) = e^{-x} \sin(t - x)$$

If we actually use the correct numbers then you find that you should build your wine cellar 4 meters deep.

Check out Sommerfeld PDEs for Physics.

Uniqueness of Fourier Series

We will consider integrable 2π-periodic functions $f(\theta + 2\pi) = f(\theta)$. Therefore, it is enough to consider functions on $\mathbb{S}^1 \subseteq \mathbb{C}$ in these sense that $g\left(e^{i\theta}\right) = f(\theta)$. We denote $\mathbb{S}^1 \subseteq \mathbb{C}$ considered as a group by $\mathbb{T}$. We say that f is a **function on the circle** to mean that g is a 2π-periodic function. Wee say that f is **integrable on the circle** to mean that f is locally integrable and 2π-periodic.

Definition. (Partial Sums of Fourier Series)
Given integrable f on the circle, we define a new integrable function on the circle,

$$S_n(f)(\theta) = \sum_{n=-N}^{N} \hat{f}(n) e^{in\theta}$$

is the N-th partial sum of the Fourier series of f.

Now we want to consider the map $f \mapsto \left(\hat{f}(n)\right)_{n\in\mathbb{Z}}$ from integrable functions on the circle to $\mathbb{Z}$-sequences of integrable functions. For example we want to know whether this mapping is 1-1, i.e. $\hat{f}(n) = 0,\ \forall n \in \mathbb{Z} \implies f \equiv 0$? Obviously, Thomae's function has zero Fourier coefficients for all n but is not the zero function.

So one to oneness does not hold for the domain of integrable functions on the circle. What about if we rethink the domain to be equivalence classes of integrable functions? This requires Lebesgue's Theory of integration, so instead we will just restrict our domain to continuous functions on $\mathbb{T}$.

Theorem. (Uniqueness of Fourier Series)
Suppose f is integrable on $\mathbb{T}$ and $\hat{f}(n) = 0,\ \forall n \in \mathbb{Z}$. Then, $f(\theta_0) = 0$ provided that f is continuous at θ_0.[a]

Corollary. If f is continuous on $\mathbb{T}$ and $\hat{f}(n) = 0,\ \forall n \in \mathbb{Z}$, then $f \equiv 0$.

[a] This is not equivalent to the statement f is zero almost everywhere.

Proof.
We divide the proof into two steps.

(1) If the theorem holds for all $\mathbb{R}$-valued functions, the it holds for for all $\mathbb{C}$-valued functions.

(2) Assume f is $\mathbb{R}$-valued and $\hat{f}(n) = 0,\ \forall n \in \mathbb{Z}$. Let θ_0 be a point of continuity of f.

Step One
Assuming (1) we can prove the theorem as follows. Let $f(\theta) = u(\theta) + iv(\theta)$ satisfying $\hat{f}(n) = 0,\ \forall n \in \mathbb{Z}$. Let θ_0 be a point of continuity of f. Note that $\overline{f(\theta)} = u(\theta) - iv(\theta)$ so $u(\theta) = \frac{1}{2}\left(f(\theta) + \overline{f(\theta)}\right)$ implying $\hat{u}(n) = \frac{1}{2}\left(\hat{f}(n) + \hat{\overline{f}}(n)\right) = \frac{1}{2}\left(\hat{f}(n) + \overline{\hat{f}(-n)}\right) = 0$ by the assumption of the Theorem.

Similarly, $v(\theta) = \frac{1}{2i}\left(f(\theta) - \overline{f(\theta)}\right)$. So a similar computation gives us the same conclusion.

Step Two
Now assume (2). Suppose $f(\theta_0) \neq 0$, we will derive a contradiction. WLOG, assume $\theta_0 = 0$ and $c \coloneqq f(0) > 0$.

Since f is continuous at zero, $\exists \delta \in \left(0, \frac{\pi}{2}\right)$ s.t. $f(\theta) \geq \frac{c}{2},\ \forall \theta \in B_\delta(0)$. Put $\varepsilon = \frac{2}{3}(1 - \cos\delta) > 0$. Define $p(\theta) \coloneqq \varepsilon + \cos\theta$. For $\delta \leq |\theta| \leq \pi$, $p(\theta) \leq p(\delta) = \varepsilon + \cos\delta = \varepsilon + \left(1 - \frac{3}{2}\varepsilon\right) = 1 - \frac{\varepsilon}{2}$.

On the other hand, $p(0) = 1 + \varepsilon > 1$ and p is continuous at zero, $\implies \exists \eta \in (0, \delta)$ such that $p(\theta) \geq 1 + \frac{\varepsilon}{2}$ provided that $|\theta| \leq \eta$.

For each $k \in \mathbb{N}$, define $p_k(\theta) = p(\theta)^k = \left(\varepsilon + \frac{e^{i\theta} + e^{-i\theta}}{2}\right)$ which is a linear combination of the functions $e^{im\theta}$ for $m \in \mathbb{Z} \cap [-k, k]$.

Consider
$$\int_{-\pi}^{\pi} f(\theta) p_k(\theta)\, d\theta = 0$$
being a linear combination of $\hat{f}(m),\ m \in \mathbb{Z} \cap [-k, k]$.

On the other hand we can see that this integral is positive by partitioning the integral. Near zero, the integral will "blow up" and away from zero it be small for large k.

$$\int_{-\pi}^{\pi} f(\theta)p_k(\theta)\,d\theta = \int_{|\theta|\le\eta} + \int_{\eta\le|\theta|\le\delta} + \int_{\delta\le|\theta|\le\pi} \qquad \text{Partition of integral}$$

$$\int_{\eta\le|\theta|\le\delta} f(\theta)p_k(\theta)\,d\theta \ge 0$$

$$\int_{|\theta|\le\eta} f(\theta)p_k(\theta)\,d\theta \ge \frac{c}{2}\left(1+\frac{\varepsilon}{2}\right)^k \cdot 2\eta$$

$$\left|\int_{\delta\le|\theta|\le\pi} f(\theta)p_k(\theta)\,d\theta\right| \le \int_{\delta\le|\theta|\le\pi} |f(\theta)||p_k(\theta)|\,d\theta \le M\left(1-\frac{\varepsilon}{2}\right)^k 2\pi \quad f \text{ is bounded by } M$$

$$\Downarrow$$

$$\int_{-\pi}^{\pi} f(\theta)p_k(\theta)\,d\theta \ge \frac{c}{2}\left(1+\frac{\varepsilon}{2}\right)^k \cdot 2\eta - M\left(1-\frac{\varepsilon}{2}\right)^k 2\pi$$

Contradicting the fact that the integral is zero.

This technique is very common. To introduce a term which amplifies the contribution of a certain area of the integral and minimizes other portions of the integral.[12]

□

Recall:

(1) Suppose $\{g_n\}_{n\in\mathbb{N}}$ is a sequence of continuous functions such that $g_n \to g$ uniformly in x as $n\to\infty$. Then, g is continuous.

(2) Suppose $\{g_n\}_{n\in\mathbb{N}}$ is a sequence of continuous functions on $[a,b]$ and $S(x) = \sum_{n\in\mathbb{N}} g_n(x)$ converges uniformly in $x\in[a,b]$. Then,

$$\int_a^b S(x)\,dx = \lim_{n\to\infty}\int_a^b g_n(x)\,dx.$$

(3) **Corollary**. Suppose f is continuous on the circle and $\sum_{n\in\mathbb{Z}}\left|\hat{f}(n)\right| < \infty$, then

$$\lim_{N\to\infty} S_N(f)(\theta) = f(\theta)$$

uniformly in θ for all $\theta\in\mathbb{T}$.

Proof.
For every θ, define

$$g(\theta) = \sum_{n\in\mathbb{Z}} \hat{f}(n)e^{in\theta}.$$

By (2), this series converges absolutely and uniformly in θ by the Weierstrauss M-test. So g is a continuous function on $\mathbb{T}$ by (1). Furthermore,

$$\hat{g}(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\theta)e^{-ik\theta}\,d\theta = \frac{1}{2\pi}\sum_{n\in\mathbb{Z}}\int_{-\pi}^{\pi}\hat{f}(\theta)e^{in\theta}e^{-ik\theta}\,d\theta = \begin{cases} 2\pi\hat{f}(k) & n=k \\ 0 & n\neq k.\end{cases}$$

Since f and g are continuous on $\mathbb{T}$ such that $\hat{f}(k) = \hat{g}(k),\ \forall k\in\mathbb{Z}$ we conclude by (2) that $f=g$.

[12]I want you to know how to do this."

Now we ask for which functions f, can we say $\sum_{n\in\mathbb{Z}} \left|\hat{f}(n)\right| < \infty$?

Lemma. (Three short Lemmas)

(i) Suppose f is C^1 on $\mathbb{T}$ then $\hat{f}'(n) = (in)\hat{f}(n),\ \forall n \in \mathbb{Z}$.

(ii) If $\exists B > 0$ such that $|f(\theta)| \leq B,\ \forall\theta$. Then, $|\hat{f}(\theta)| \leq B,\ \forall n \in \mathbb{Z}$.

(iii) Let $k \geq 1$. If f is C^k on $\mathbb{T}$, then $\exists B > 0$ such that $|\hat{f}(\theta)| \leq \frac{B}{|n|^k},\ \forall n \neq 0$.

Corollary. If f is C^2 on $\mathbb{T}$, then

$$\lim_{N\to\infty} S_N(f)(\theta) = f(\theta)$$

uniformly $\forall\theta$.

(i) Via integration by parts,

$$\hat{f}'(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f'(\theta)e^{-in\theta}\,d\theta = \frac{1}{2\pi}\left(f(\pi)e^{-in\pi} - f(-\pi)e^{in\pi}\right) + (in)\hat{f}(n)$$

(ii)

$$|\hat{f}(\theta)| = \frac{1}{2\pi}\left|\int_{-\pi}^{\pi} f(\theta)e^{-in\theta}\,d\theta\right| \leq \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|f(\theta)e^{-in\theta}\right|\,d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi}|f(\theta)|\,d\theta \leq B$$

(iii) By combining the previous two lemmas, if f is C^2, $\hat{f}''(n) = (in)^2\hat{f}(n) = -n^2\hat{f}(n)$.

So if f is C^k, $\hat{f^{(k)}}(n) = (in)^k\hat{f}(n)$. Thus,

$$\hat{f}(n) = \frac{\hat{f^{(k)}}(n)}{(in)^k}$$

$\forall k \in \mathbb{N}$.

Since f is C^k on $f^{(k)}$ is continuous and bounded on $\mathbb{T}$, so Lemma (ii) gives us a bound B such that $\left|\hat{f^{(k)}}(n)\right| \leq B$ for all $n \in \mathbb{Z}$. Combining this with the equation

The corollary follows by Lemma (iii) and the convergence of $\sum_{n\in\mathbb{N}} \frac{1}{n^2}$. $\square$

<u>Note</u>: A smooth function has Fourier coefficients that decay faster than any polynomial but not necessarily exponentially. A real analytic function has exponential decay of its Fourier coefficients.

<u>Note</u>: Even if f is not C^2, the convergence still holds if we could show

$$\sum_{n\in\mathbb{Z}} \left|\hat{f}(n)\right| < \infty$$

For example, consider $f(\theta) = |x|$ for $\theta \in [-\pi, \pi]$ (and periodically extended outside this domain).

However, computing

$$\begin{aligned}\hat{f}(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} |\theta| e^{-in\theta}\,d\theta \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} |\theta|\cos(-n\theta)\,d\theta \\ &= \frac{1}{\pi}\int_{0}^{\pi} |\theta|\cos(n\theta)\,d\theta \\ &= \begin{cases} \frac{\pi}{2} & n = 0 \\ \frac{(-1)^n - 1}{\pi n^2} & n \neq 0 \end{cases}\end{aligned}$$

So f still decays quadratically even though it is not C^2.

From the Uniqueness theorem obtain

$$f(\theta) = \sum_{n\in\mathbb{Z}} \hat{f}(n)e^{in\theta} = \frac{\pi}{2} - \frac{2}{\pi}\sum_{n\in\mathbb{Z}\setminus(2\mathbb{Z})} \frac{1}{n^2}e^{in\theta} \implies \frac{\pi}{2} - \frac{4}{\pi}\sum_{n\in\mathbb{Z}^+\setminus(2\mathbb{Z})} \frac{\cos(n\theta)}{n^2}$$

This gives us

$$\sum_{n\in\mathbb{Z}^+\setminus(2\mathbb{Z})} \frac{1}{n^2} = \frac{\pi^2}{8}$$

Lemma. (Coefficients of Conjugate Function)

$$\hat{\bar{f}} = \overline{\hat{f}(-n)}$$

Proof.

$$\hat{\bar{f}}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \overline{f(\theta)}e^{-in\theta}\,d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} \overline{f(\theta)e^{in\theta}}\,d\theta = \overline{\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-in\theta}\,d\theta} = \overline{\hat{f}(-n)}(n)$$

□

Convolutions

Definition. (Convolution)
For integrable functions f and g on the circle, the **convolution** of f and g is

$$(f \star g)(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)g(x-y)\,dy$$

$\forall x \in [-\pi, \pi]$.

Note: For every fixed $x \in \mathbb{R}$, the function $F(y) = f(y)g(x-y)$ is integrable over $[-\pi, \pi]$. Since the discontinuities of F will be the union of the discontinuities of f and g.

Some properties follow immediately from the definition:

(i) $(f \star g)(x + 2\pi) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\overline{g(x + 2\pi - y)}\,dy = (f \star g)(x)$. So $(f \star g)$ is an integrable function on the circle.

(ii) If $g(y) = 1$, then $(f \star g)(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\,dy$ which is the average of f on [-π, π].

(iii) For general g, we think of the convolution $f \star g$ is thought of as a "weighted average" of f where the weight is determined by $g(x)$.

Example: Let

$$f(x) = \begin{cases} 1 & x \in [0, \pi] \\ 0 & [-\pi, 0] \end{cases}$$

and extend f periodically.

Then

$$(f \star f)(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)f(x-y)\,dy = \frac{1}{2\pi}\int_{0}^{\pi} f(x-y)\,dy = \frac{1}{2\pi}\int_{x}^{x-\pi} f(s)\,(-ds)$$
$$= \frac{1}{2\pi}\int_{x-\pi}^{x} f(s)\,ds = \begin{cases} \frac{1}{2\pi}x & x \in [0, \pi] \\ -\frac{1}{2\pi}x & x \in [-\pi, 0] \end{cases}$$

which we extend periodically.

Observation: $f \notin C^0$ but $(f \star f) \in C^0$. This will be true in general.

Definition. (Dirichlet Kernel)

$$S_N(f)(\theta) := \sum_{n=-N}^{N} \hat{f}(n)e^{in\theta} = \sum_{n=-N}^{N}\left[\frac{1}{2\pi}f(\phi)e^{-in\varphi}d\varphi\right]e^{in\theta} = \frac{1}{2\pi}f(\varphi)\left(\sum_{n=-N}^{N} e^{in(\theta-\varphi)}\right)d\varphi$$

Recalling that the **Dirichlet Kernel** is $D_N(x) = \sum_{n=-N}^{N} e^{inx}$, shows that

$$S_N(f) = f \star D_N$$

Note: If f is integrable on the circle and $x \in \mathbb{R}$,

$$\int_{-\pi}^{\pi} F(y)\,dy = \int_{-\pi}^{\pi} F(x-y)\,dy$$

This follows by a change of variables $s = x - y$ and noting that since f is periodic the integral is unchanged.

This shows that

$$(f \star g)(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)g(x-y)\,dy = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-y)g(y)\,dy == (g \star f)(x).$$

Proposition. (Properties of Convolution)
Let f, g, h be integrable functions on $\mathbb{T}$ and $c \in \mathbb{C}$, then

(i) $f \star (g + h) = f \star g + f \star h$
(ii) $(cf) \star g = c(f \star g) = f \star (cg)$
(iii) $f \star g = g \star f$
(iv) $f \star g$ is at least continuous and if $f \in C^r$ and $g \in C^k$ then $f \star g$ is C^m where $m = \max(r, k)$.
(v) $(f \star g) \star h = f \star (g \star h)$
(vi) $f \hat{\star} g(n) = \hat{f}(n) \cdot \hat{g}(n)$

Proof.
(1)-(3) is clear.

(iv)

$$\begin{aligned}
f \hat{\star} g(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} (f \star g)(\theta) e^{-in\theta}\, d\theta \\
&= \frac{1}{(2\pi)^2}\int_{-\pi}^{\pi} \left[\int_{-\pi}^{\pi} f(\varphi) g(\theta-\varphi)\, d\varphi\right] e^{-in\theta}\, d\theta \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} (f \star g)(\theta) e^{-in\theta}\, d\theta \\
&= \frac{1}{(2\pi)^2}\int_{-\pi}^{\pi} \left[\int_{-\pi}^{\pi} f(\varphi) g(\theta-\varphi) e^{-in\theta}\, d\theta\right] d\varphi \\
&= \frac{1}{(2\pi)^2}\int_{-\pi}^{\pi} f(\varphi) e^{-in\varphi} \left[\int_{-\pi}^{\pi} g(\theta-\varphi) e^{-in(\theta-\varphi)}\, d\theta\right] d\varphi \\
&= \frac{1}{(2\pi)^2}\int_{-\pi}^{\pi} f(\varphi) e^{-in\varphi} \left[\int_{-\pi}^{\pi} g(\theta) e^{-in(\theta)}\, d\theta\right] d\varphi \\
&= \frac{1}{(2\pi)^2}\int_{-\pi}^{\pi} f(\varphi) e^{-in\varphi} \left[2\pi \hat{g}(n)\right] d\varphi \\
&= \hat{f}(n) \cdot \hat{g}(n)
\end{aligned}$$

(v) is similar assuming (iv)

Lemma. (Uniformity of Convolutions)
Let f be an integrable function on the circle. Let $B > 0$ such that $|f(x)| \le B,\ \forall x$. Suppose $\forall k \in \mathbb{N}$ there exists a continuous functions f_k on the circle which together satsify
(a) $|f_k(x)| \le B,\ \forall x\ \forall k$
(b) $\lim_{k\to\infty} \|f - f_k\| = 0$[a]

[a] where $\|\cdot\|$ is the L^1 semi norm induced by the standard inner product on the space of integrable functions on $\mathbb{T}$.

HW 10 Q1 $\implies$ 2 and same proof gives 1. See the proof of lemma 1.5 on p. 285. The idea is to approximate f by step functions which can be further approximated by smooth bump functions.

Case 1. Suppose g is continuous. If $f \equiv 0$ up to a measure zero set, then $(f \star g) \equiv 0$, so $f \star g$ is continuous. Assume f is not zero almost everywhere. Put $C := \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(y)|\, dy > 0$. Since g is continuous on a compact set, it is uniformly continuous. Let $\varepsilon > 0$ be arbitrary, put $\varepsilon' = \frac{\varepsilon}{C}$. $\exists \delta$ such that $|g(x_1 - y) - g(x_2 - y)| \le \varepsilon'\ \forall |s-t| \le \delta$. This gives

$$\begin{aligned}
|(f \star g)(x_1) - (f \star g)(x_2)| &= \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\left(g(x_1 - y) - g(x_2 - y)\right) dy\right| \\
&\le \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(y)|\, |g(x_1 - y) - g(x_2 - y)|\, dy \\
&\le \frac{\varepsilon'}{2\pi}\int_{-\pi}^{\pi} |f(y)|\, dy \\
&= \varepsilon
\end{aligned}$$

Case 2. Suppose g is integrable.[13]

Since f, g are Riemann integrable, they are bounded by a $M, B > 0$ respectively. Let $\{g_k\}_{k\in\mathbb{N}}$ get the set of continuous functions on the circle satisfying the lemma assumption. By case 1, $f \star g_k$ are continuous. We want to show that $(f \star g_k)(x) \to (f \star g)(x)$ uniformly so that $f \star g$ is continuous.

$$\begin{aligned}
|(f \star g_k)(x) - (f \star g_k)(x)| &= \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-y)\,(g_k(y) - g(y))\ dy\right| \\
&\leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x-y)|\,|g_k(y) - g(y)|\ dy \\
&\leq \frac{M}{2\pi}\int_{-\pi}^{\pi} |g_k(y) - g(y)|\ dy \\
&= M\|g_k - g\|
\end{aligned}$$

Since the last expression is independent of x, $(f \star g_k) \to (f \star g)$ uniformly. $\square$

In $\mathbb{R}^n$, all norms are equivalent, but in functions spaces norms are not equivalent. Oftentimes arguments in analysis, end up being density arguments in function spaces.

Definition. (Good Kernels)
A set of integrable functions $\{K_n\}_{n\in\mathbb{N}}$ on $\mathbb{T}$ is called a **family of good kernels** provided that $\forall n \in \mathbb{N}$,

(i) $\frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(x)\,dx = 1,$ (unit norm)
(ii) $\exists M > 0$ such that $\|K_n\|_1 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |K_n(x)|\,dx \leq M,$ (bounded absolutely)
(iii) $\forall \delta > 0,$ (concentrated)

$$\lim_{n\to\infty} \frac{1}{2\pi}\int_{\delta\leq|x|\leq\pi}^{\pi} |K_n(x)|\,dx = 0$$

Note: Frejér Kernels are good kernels. Poisson Kernels are a continuum family of good kernels. Dirichlet Kernels are not good kernels.

Why do we care?

$$(f \star K_n)(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)K_n(x-y)\,dy$$

which concentrates at $y = x$.

Theorem. (Convolutions with Good Kernels)
Let $\{K_n\}_{n\in\mathbb{N}}$ be a family of good kernels and f integrable on $\mathbb{T}$. Then,

$$\lim_{n\to\infty}(f \star K_n)(x_0) = f(x_0)$$

if f is continuous at x_0.

If f is continuous on $\mathbb{T}$, the convergence is uniform.

Proof.
Let f be continuous at x_0. Let $\varepsilon > 0$ be arbitrary and define $\varepsilon' = \frac{\varepsilon}{2M} > 0$. Then $\exists\delta > 0$ such that $|f(s) - f(x_0)| \leq \varepsilon',\ \forall s \in B_\delta(x_0)$.

[13] Here we use a "density argument."

$$\begin{aligned}
|(f \star K_n)(x_0) - f(x_0)| &= \left|(f \star K_n)(x_0) - \frac{f(x_0)}{2\pi}\int_{-\pi}^{\pi} K_n(y)\,dy\right| \\
&= \left|\int_{-\pi}^{\pi} \left(f(x_0 - y) - f(x_0)\right) K_n(y)\,dy\right| \\
&\le \int_{-\pi}^{\pi} |f(x_0 - y) - f(x_0)|\,|K_n(y)|\,dy \\
&\le \frac{1}{2\pi}\int_{|y|\le\delta} \quad + \frac{1}{2\pi}\int_{\delta\le|y|\le\pi} \\
&\le \frac{1}{2\pi}\int_{|y|\le\delta} \varepsilon'|K_n(y)|\,dy + \frac{1}{2M}\int_{\delta\le|y|\le\pi} 2B|K_n(y)|\,dy \\
&\le \frac{1}{2\pi}\int_{-\pi}^{\pi} \varepsilon'|K_n(y)|\,dy + \frac{B}{\pi}\int_{\delta\le|y|\le\pi} |K_n(y)|\,dy \\
&\le \frac{1}{2\pi}\varepsilon' M + \frac{1}{2\pi}\int_{\delta\le|y|\le\pi} |K_n(y)|\,dy \\
&\le \frac{\varepsilon}{2} + \frac{1}{2\pi}\int_{\delta\le|y|\le\pi}
\end{aligned}$$

Now choose n large and we are done. Thus $\exists N \in \mathbb{N}$ such that $|(f \star K_n)(x_0) - f(x_0)| \le \varepsilon$. Now we resolve the further issue of uniformity.

Suppose f is continuous on $\mathbb{T}$. Then the δ can be chosen independent of x_0 and N can be chosen independent of x_0 so we obtain that the convergence is uniform in x.

Good kernels approximate the identity operation for convolutions.

The Dirac delta function $\delta(x)$ is not a function (but a generalized function[14]) which has the property that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\delta(x)\,dy = f(0).$$

Now we turn to study Frejér Kernels and Poisson Kernels.

Recall that if

(i) $\sum_{N\in\mathbb{Z}} \left|\hat{f}(n)\right| < \infty$, then $S_N(f)(\theta) \to f(\theta)$ uniformly in θ as $N \to \infty$.

(ii) If f is C^2 on the circle, then $\sum_{N\in\mathbb{Z}} \left|\hat{f}(n)\right| < \infty$, and thus (i) holds.

What if f is not C^2, but just continuous?

There is a continuous functions f such that $S_N(f)(\theta)$ diverges at uncountably many points. [15]

It turns out that f need only continuous for the Cesaro sum to be convergent.

<u>Recall</u>:

(a) Let $D_n(\theta)$ be the n-th Dirichlet kernel. Then

$$D_n(\theta) = \sum_{k=-n}^{n} e^{ik\theta} = \frac{\sin\left(\left(n + \frac{1}{2}\right)\theta\right)}{\sin\left(\frac{\theta}{2}\right)}$$

[14] which is an operator on the space of functions.

[15] Kolmogorov found one such example.

(b)

$$S_n(f)(\theta) \coloneqq \sum_{k=-n}^{n} \hat{f}(n)e^{ik\theta} = (f \star D_n)(\theta)$$

But $\{D_n\}_{n\in\mathbb{N}}$ is **not** a family of good kernels. It is complicated to show this, it is a problem in S&S.

Definition. (N-th Cesáro Mean)
We define the **N-th Cesáro Mean** of the Fourier series of f as

$$\sigma_N(f)(\theta) = \frac{1}{N}\sum_{k=0}^{N-1} S_k(f)(\theta) = \frac{1}{N}\sum_{k=0}^{N-1}(f \star D_n)(\theta) = \left(f \star \left(\frac{1}{N}\sum_{k=0}^{N-1} D_n\right)\right)(\theta)$$

So recalling that $F_N(\theta)$ is the N-th Fejer Kernel.

$$F_n(\theta) \coloneqq \frac{1}{N}\sum_{k=0}^{N-1} D_n(\theta) \implies \sigma_N(f) = f \star F_N = \frac{\sin^2\left(\frac{1}{2}N\theta\right)}{N\sin^2\left(\frac{\theta}{2}\right)}$$

Lemma. (Existence of Good Kernels)
$\{F_N\}_{N\in\mathbb{N}}$ is a family of good kernels.

Note:

$$\sigma_N(f)(\theta) = \frac{1}{N} + \sum_{n=0}^{N-1}\sum_{k=-n}^{n} e^{ik\theta}\hat{f}(k) = \frac{1}{N} + \sum_{k=-N+1}^{N-1}(N-|k|)e^{ik\theta}\hat{f}(k)$$

So, $\sigma_N(f)$ is a linear combination of $\hat{f}(k)$ for $k \in [-N, N] \cap \mathbb{Z}$.

Definition. (Trigonometric Polynomial)
A **trigonometric polynomial** is a function of the form

$$p(\theta) = \sum_{k=-N}^{N} c_k e^{ik\theta}$$

which can be written

$$p(\theta) = a_0 + \sum_{k=1}^{N}(a_k\cos(k\theta) + b_k\sin(k\theta))$$

for $a_k, b_k, c_k \in \mathbb{C}$.

Theorem. (Condition for Cesáro Summability)
If f is integrable on $\mathbb{T}$, then the Fourier series of f is Cesáro summable to f at every continuity point of f. Meaning,

$$\lim_{N\to\infty} \sigma_N(f)(\theta_0) = f(\theta_0)$$

provided that f is continuous at θ_0.

Moreover,
if f is continuous on the circle, then the Fourier series of f is Cesáro summable to f uniformly.

Corollary.
If f is integrable on $\mathbb{T}$, then $f \equiv 0$ almost everywhere (at every point of continuity).

Corollary.
If f is continuous on $\mathbb{T}$, then f can be uniformly approximated by trigonometric polynomials. So the trigonometric polynomials are dense in the space of continuous functions on $\mathbb{T}$ using the standard semi-norm.

Proof.
Lemma + convolutions with good kernels theorem.

The first corollary follows since $\sigma_N(f)(\theta)$ is a linear combination of $\hat{f}(k)$ for $k \in (-N, N) \cap \mathbb{Z}$ we find that $\sigma_N(f)(\theta) = 0,\ \forall N\ \forall \theta$. Thus,

$$\lim_{N\to\infty} \sigma_N(f)(\theta_0) = 0 = f(\theta_0)$$

where f is continuous at θ_0.

The second corollary follows by putting $p_N(\theta) = \sigma_N(f)(\theta),\ N \in \mathbb{N}$. Then p_N is a trig polynomial and

$$\lim_{N\to\infty} \mathit{Proof}._N(f) = f$$

uniformly by corollary 1.

□

Corollary. (Weierstrauss Approximation Theorem)
Let f be a continuous function on $[a, b]$. Then, $\exists$ a sequence $\{p_k\}_{k\in\mathbb{N}}$ of polynomials such that

$$\lim_{k\to\infty} p_k = f$$

uniformly.

Proof.
Let $f : [a, b] \to \mathbb{R}$ be continuous. Extend f to be a continuous function g by setting

$$g(x) = \begin{cases} f(x) & x \in [a, b] \\ f(a(x - a + 1)) & x \in [a - 1, a] \\ f(-b(x - b - 1)) & x \in [b, b + 1] \end{cases}$$

Extend g to a periodic continuous function h on $\mathbb{R}$ by $h(x+b-a+2) = g(x)$. Now this is $(b-a+2)$-periodic. Scale by a linear map $x(\theta)$ the variable to define a 2π-periodic continuous function $\gamma(\theta) = h(x(\theta))$.

By the last corollary, γ is approximated uniformly by trig polynomials. Since $\sin(\theta)$ and $\cos(\theta)$ have MacLaurin series which converge uniformly for θ in any compact set.

You will complete the proof on homework. $\square$

We now study examples of Cesáro sums.

Example:

$$\sum_{k\in\mathbb{N}}(-1)^k k$$

Then $\sigma_{2m} = -\frac{1}{2}$ and $\sigma_{2m+1} = 1 - \frac{m+1}{2m}$. So $\lim_{N\to\infty}\sigma_N$ does not converge. So, the series is not Cesáro summable.

Now the Abel means is $A(r) = 1 - 2r + 3r^2 - 4r^3 + \cdots$. This is equal to $\frac{d}{dr}\frac{1}{1+r} = \frac{1}{(1+r)^2}$ (think of generating functions!). Taking

$$\lim_{r\uparrow 1} A(r) = \frac{1}{4}.$$

So the series is Abel summable.

Note: If a series is convergent $\implies$ Cesáro summable $\implies$ Abel summable but neither converse is true.

Definition. (Abel Means)
The **Abel means** of the Fourier series of f is

$$A_r(f)(\theta) = \sum_{n\in\mathbb{Z}} \hat{f}(n)e^{in\theta}r^{|n|}$$

$$r \in [0,1].$$

Note: If f is integrable on $\mathbb{T}$, then

$$\left|\hat{f}(n)\right| \leq \frac{1}{2\pi}\underbrace{\int_{-\pi}^{\pi}|f(\theta)|\,d\theta}_{\text{constant}}$$

$$\forall n.$$

Thus, $A_r(f)(\theta)$ is a convergent series for $r \in [0,1]$.

Note $A_r(f)(\theta) = (f \star P_r)(\theta)$ where

$$P_r(\theta) := \sum_{n\in\mathbb{Z}} e^{in\theta}r^{|n|} = \frac{1-r^2}{1-2r\cos\theta + r^2}.$$

Lemma. (Existence of Continuum of Good Kernels)
$\{P_r(\theta)\}_{r\in[0,1]}$ is a continuum family of good kernels.

Theorem. ()
If f is integrable on $\mathbb{T}$, then

$$\lim_{r\uparrow 1} A_r(f)(\theta_0) = f(\theta_0)$$

provided that f is continuous at θ_0.
Moreover, if f is continuous, then the convergence is uniform in θ_0.

The Heat Equation on the Disk

We will begin an application that will take a bit of time: the steady-state heat equation in the unit disk. Let $\mathcal{U}(x, y, t)$ be the temperature of a planar region at position (x, y) at time t. If the material of the region is relatively uniform (e.g. plastic) then the heat equation holds

$$\frac{\partial \mathcal{U}}{\partial t} = c\left(\frac{\partial^2 \mathcal{U}}{\partial x^2} + \frac{\partial^2 \mathcal{U}}{\partial y^2}\right)$$

where c is a constant that depends on the material. This follows from Newton's law of cooling (see [S&S] page 19). Steady-state thermal equilibrium occcurs when $\frac{\partial \mathcal{U}}{\partial t} = 0$, so the steady-state heat equation is Laplace's equation $\frac{\partial^2 \mathcal{U}}{\partial x^2} + \frac{\partial^2 \mathcal{U}}{\partial y^2} = 0$ or $\Delta \mathcal{U} = \mathrm{grad}(\mathcal{U}) \equiv 0$. Solutions of this equation are called **Harmonic functions** .

Let $D = \mathbb{B}^2$ and $C = \mathrm{Bd}\, D$. Let $g : C \to \mathbb{R}$ be a function. We will find a $\mathcal{V} : D \to \mathbb{R}$ such that

(i) $\mathcal{V}$ is C^2 and $\Delta \mathcal{V} = 0$.

(ii)
$$\lim_{r\uparrow 1} \mathcal{V}(r\cos\theta, r\sin\theta) = g(\cos\theta, r\sin\theta).$$