Honors Analysis II

Math 396

University of Michigan

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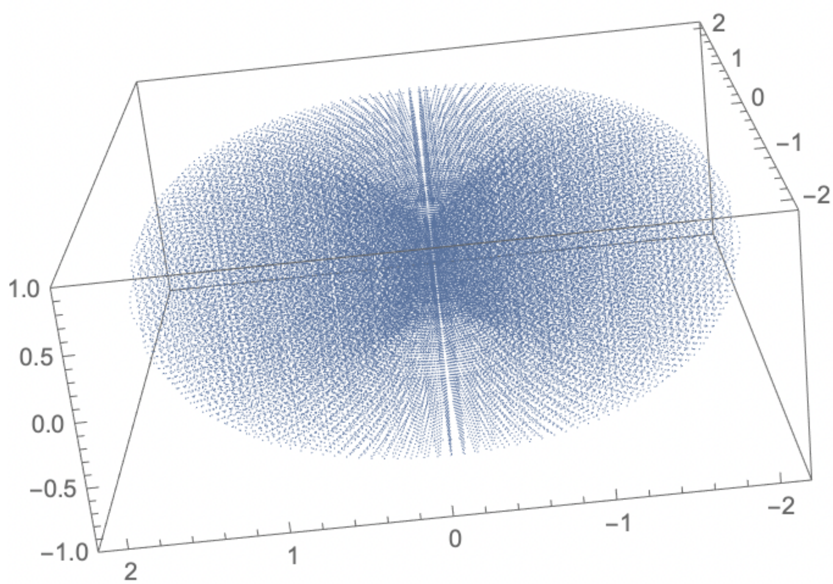

Contents

INTRODUCTION & MOTIVATION

Textbooks:

(i) Munkres, Analysis on Manifolds

(ii) Spivak, Calculus on Manifolds.

(iii) (Possibly) Fourier Analysis, an Introduction.

Content:

Manifolds are k-dimensional objects embedded in ambient n-dimensional space. We will be interested in integration over manifolds. Next, we will study differential forms are generalizations of functions and vector fields. We will then integrate differential forms on manifolds which will lead us to the celebrated **Stokes Theorem**. Stokes Theorem describes the relationship between the integral over a manifold and its boundary. We will study many classical examples.

EUCLIDEAN DIFFERENTIABLE MANIFOLDS

Motivation

Informally, a **topological manifold** is a topological space that is **homeomorphic** to Euclidean space. This means a manifold looks locally like $\mathbb{R}^n$.

For example, $\mathbb{S}^1$ is a manifold because when we "zoom into" the circle it looks like a line. Also $\mathbb{S}^1 \times \mathbb{S}^1$ is a manifold because donuts look locally like a plane (see front cover).

We want to do analysis on these manifolds, so we need to add more structure. A **differentiable manifold** is a special type of topological manifold that is "smooth."

Proposition. (Volume of a Parallelepiped)
If $v_1, \ldots, v_n \in \mathbb{R}^n$ are linearly independent. The volume of the parallelepiped generated by $v_1, \ldots, v_n$ is

$$\pm \det \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix}$$

We want to determine the k-dimensional volume of a parallelepiped determined by k vectors in $\mathbb{R}^n$. The vectors will determine a non-square matrix so we cannot use the determinant.

Definition. (Volume of Parallelepiped)
Let $k \leq n$, Let $M(n,k)$ be the space of $n \times k$ matrices. Define $V : M(n,k) \to [0,\infty)$ by

$$V(X) = \sqrt{\det(X^T X)}$$

Suppose $x_1, \ldots, x_k \in \mathbb{R}^n$ are linearly independent. We define the **k-dimensional volume** of the parallelepiped generated by $x_1, \ldots, x_k$ by $V(X)$ where

$$X = \begin{bmatrix} | & & | \\ x_1 & \cdots & x_k \\ | & & | \end{bmatrix}$$

This is well defined because $X^T X$ is a positive definite matrix.

Examples:

(i) $k = n$ (should agree with previous proposition)

$$V(X) = \sqrt{\det(X^T X)} = \sqrt{\det(X) \cdot \det(X)} = |\det(X)|.$$

(ii) $k = 1$ (should agree with length of vector)

$$\sqrt{v^T v} = \|v\|.$$

(iii) $k = 2$ and $n = 3$ (should agree with cross product of generators)

$$X = \begin{bmatrix} | & | \\ a & b \\ | & | \end{bmatrix} \implies X^T X = \begin{bmatrix} a^T a & a^T b \\ b^T a & b^T b \end{bmatrix} \implies \det(X^T X) = \det \begin{bmatrix} \|a\|^2 & a \cdot b \\ a \cdot b & \|b\|^2 \end{bmatrix}$$

$$= \|a\|^2 \|b\|^2 - (a \cdot b)^2 = \|a\|^2 \|b\|^2 \sin \theta$$

So $\det(X^T X) = \|a \times b\|^2$.

This implies an interesting fact about the determinant of $X^T X$.

Definition. (Ascending k-tuple)
Let $k \leq n$.

(a) An **ascending k-tuple** from the set $[n]$ is $I = (i_1, \ldots, i_k)$ satisfying $1 \leq i_1 \leq \cdots \leq i_k$.

(b) Denote by $\mathrm{ASC}_{k,n}$ the set of all ascending k-tuples from $[n]$.
So $|\mathrm{ASC}_{k,n}| = \binom{n}{k}$.

Theorem. (Cauchy-Binet Identity)
Let $k \leq n$. If $A \in M(k, n)$ and $B \in M(n, k)$, then

$$\det(AB) = \sum_{\mathrm{ASC}_{k,n}} \det(A^I) \det(B_I)$$

where for $I = (i_1, \ldots, i_k)$, A^I is the $k \times k$ submatrix of A containing the columns $i_1, \ldots, i_k$ and B_I, is the $k \times k$ submatrix of A containing the rows $i_1, \ldots, i_k$.

Corollary. For $k \leq n, X \in M(n, k)$

$$V(X)^2 = \det(X^T X) = \sum_{\mathrm{ASC}_{k,n}} (\det X_I)^2$$

This generalizes the Pythagorean Theorem.

Check directly for a 2×3 matrix.

Proof.
We will prove for $k = 2$ and n arbitrary.

$$\det(AB) = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ \vdots & \vdots \\ b_{n1} & b_{n2} \end{bmatrix}$$

So,

$$\begin{aligned}
\det(AB) &= \det \begin{bmatrix} \sum_{i\in[n]} a_{1i}b_{i1} & \sum_{i\in[n]} a_{1i}b_{i2} \\ \sum_{j\in[n]} a_{2j}b_{j1} & \sum_{j\in[n]} a_{2j}b_{j2} \end{bmatrix} && \text{matrix product} \\
&= \sum_{i\in[n]} \sum_{j\in[n]} \det \begin{bmatrix} a_{1i}b_{i1} & a_{1i}b_{i2} \\ a_{2j}b_{j1} & a_{2j}b_{j2} \end{bmatrix} && \text{det is multilinear} \\
&= \sum_{i\in[n]} \sum_{j\in[n]} a_{1i}a_{2j} \det \begin{bmatrix} b_{i1} & b_{i2} \\ b_{j1} & b_{j2} \end{bmatrix} && \text{det is multilinear} \\
&= \sum_{i\in[n]} \sum_{j\in[n]} \delta_{ij} \cdot a_{1i}a_{2j} \det \begin{bmatrix} b_{i1} & b_{i2} \\ b_{j1} & b_{j2} \end{bmatrix} && \text{det is alternating} \\
&= \sum_{i\in[n]} \sum_{i<j} a_{1i}a_{2j} \det \begin{bmatrix} b_{i1} & b_{i2} \\ b_{j1} & b_{j2} \end{bmatrix} + \sum_{i\in[n]} \sum_{i>j} a_{1i}a_{2j} \det \begin{bmatrix} b_{i1} & b_{i2} \\ b_{j1} & b_{j2} \end{bmatrix} && \text{expansion of sum} \\
&= \sum_{i\in[n]} \sum_{i<j} a_{1i}a_{2j} \det \begin{bmatrix} b_{i1} & b_{i2} \\ b_{j1} & b_{j2} \end{bmatrix} + \sum_{j\in[n]} \sum_{j>i} a_{1j}a_{2i} \det \begin{bmatrix} b_{j1} & b_{j2} \\ b_{i1} & b_{i2} \end{bmatrix} && \text{permute } i \text{ and } j \\
&= \sum_{i\in[n]} \sum_{i<j} a_{1i}a_{2j} \det \begin{bmatrix} b_{i1} & b_{i2} \\ b_{j1} & b_{j2} \end{bmatrix} - \sum_{j\in[n]} \sum_{j>i} a_{1j}a_{2i} \det \begin{bmatrix} b_{i1} & b_{i2} \\ b_{j1} & b_{j2} \end{bmatrix} && \text{det is alternating} \\
&= \sum_{i\in[n]} \sum_{i<j} (a_{1i}a_{2j} - a_{1j}a_{2i}) \det \begin{bmatrix} b_{i1} & b_{i2} \\ b_{j1} & b_{j2} \end{bmatrix} && \text{factor} \\
&= \sum_{(i,j)\in \mathrm{ASC}_{2,n}} \det \begin{bmatrix} a_{i1} & a_{i2} \\ a_{j1} & a_{j2} \end{bmatrix} \det \begin{bmatrix} b_{i1} & b_{i2} \\ b_{j1} & b_{j2} \end{bmatrix} && \text{definition of det.}
\end{aligned}$$

□

Parametrized Manifolds

Almost always we will use n to denote the dimension of the ambient space and k the dimension of the subspace.

> **Definition**. (Parametrized Manifold)
> Let $k \le n$ and $A \subseteq \mathbb{R}^k$ be open. Let $\alpha : A \subseteq \mathbb{R}^k \to \mathbb{R}^n$ be a C^1 map. Put $Y = \alpha(A)$.
> The pair $Y_\alpha = (Y, \alpha)$ is called a **parametrized manifold** of dimension k.

Examples:

(a) $\alpha : (0, 3\pi) \subseteq \mathbb{R} \to \mathbb{R}^2$ given by $\alpha(t) = (2\cos t, 2\sin t)$. Think of this manifold not as a circle but the trajectory of a particle that moves around the circle 1.5 times.

(b) $\alpha : (0,\pi) \times (0,\pi) \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ given by $\alpha(\theta, \phi) = (2\cos\theta\sin\phi, 2\sin\theta\sin\phi, 2\cos\phi)$. This is the portion of $\mathbb{S}^2$ in the positive x quadrant.

(c) Let $\Omega \subseteq \mathbb{R}^n$ be open. Let $h : \Omega \to \mathbb{R}$ be a C^1 function. Put $\alpha : \Omega \to \mathbb{R}^{n+1}$ with $\alpha(x) = (x, h(x))$. Then (G_h, α) is a parametrized manifold.

We want to compute the k-dimensional volume of parametrized manifolds, and in general compute integrals over them. We now define reasonable notions of length, area, and volume.

Take a rectangle in A with vertex at p and lengths $\Delta x_1, \Delta x_2$. Then it should be that the volume of this rectangle in the image is $\alpha(p + (\Delta x_i)e_i) - \alpha(p) \approx \frac{\partial \alpha}{\partial x_i}\Delta x_i$. So the volume in the image should be approximately the volume of the parallelepiped determined by $\frac{\partial \alpha}{\partial x_1}(p)\Delta x_1, \dots, \frac{\partial \alpha}{\partial x_k}(p)\Delta x_k$ which is equal to $V(D\alpha(p))\Delta x_1 \Delta x_2 \cdots \Delta x_k$. Where

$$D\alpha = \begin{bmatrix} | & & | \\ \frac{\partial \alpha}{\partial x_1} & \cdots & \frac{\partial \alpha}{\partial x_k} \\ | & & | \end{bmatrix}.$$

This motivates the following definition

Definition. (Volume of Parametrized Manifold)
Let $k \leq n$, $A \subseteq \mathbb{R}^k$ be open, $\alpha : A \to \mathbb{R}^n$ be C^1. Set $Y = \alpha(A)$ and $Y_\alpha = (Y, \alpha)$.
Define the **volume** of Y_α as

$$v(Y_\alpha) = \int_A V(D\alpha)$$

For a continuous function $f : Y \to \mathbb{R}$, define the **integral** of f over Y_α as

$$\int_{Y_\alpha} f dV = \int_A (f \circ \alpha) V(D\alpha)$$

if the RHS exists[a].

[a]Here we are using the concept of a Pullback.

Examples:

(1) $\alpha : (0, 3\pi) \subseteq \mathbb{R} \to \mathbb{R}^2$ given by $\alpha(t) = (2\cos t, 2\sin t)$.

$$D\alpha = \begin{bmatrix} -2\sin t \\ 2\cos t \end{bmatrix} \implies V(D\alpha) = \sqrt{4} = 2 \implies v(Y_\alpha) = \int_0^{3\pi} 2 = 6\pi$$

(2) For $k = 2, n = 3$ and $\alpha : A \subseteq \mathbb{R}^2 \to \mathbb{R}^3$.

$$D_\alpha = \begin{bmatrix} | & | \\ \frac{\partial \alpha}{\partial x} & \frac{\partial \alpha}{\partial y} \\ | & | \end{bmatrix} \implies V(D\alpha) = \left\| \frac{\partial \alpha}{\partial x} \times \frac{\partial \alpha}{\partial y} \right\| \implies v(Y_\alpha) = \int_A \left\| \frac{\partial \alpha}{\partial x} \times \frac{\partial \alpha}{\partial y} \right\|$$

More generally,

$$\int_{Y_\alpha} f dV = \int_A (f \circ \alpha) \left\| \frac{\partial \alpha}{\partial x} \times \frac{\partial \alpha}{\partial y} \right\|$$

(3) $\alpha : (0, \pi) \times (0, \pi) \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ given by $\alpha(\theta, \phi) = (2\cos\theta \sin\phi, 2\sin\theta\sin\phi, 2\cos\phi)$.
Check that $V(D\alpha) = 4\sin\phi$.

(4) Let $\alpha : \Omega \to \mathbb{R}^{n+1}$ be given by $\alpha(x) = (x, g(x))$ for C^1 g.
Check that

$$v(D\alpha) = \sqrt{1 + \sum_{i \in [n]} \left(\frac{\partial g}{\partial x_i} \right)^2}$$

Now we show that integrals over parametrized manifolds are invariant under reparametrization.

For a parametrized manifold to exist there is one α the following theorem says any β diffeomorphic to α will agree on integrals. It does not say anything about two "randomly" chosen maps which define the same parametrized manifold.

Theorem. (Reparametrization Invariance)
Let $A, B \subseteq \mathbb{R}^k$ be open. Let $g : A \to B$ be a diffeomorphism. Let $\beta : B \to \mathbb{R}^n$ be a C^1 map. Let $\alpha = \beta \circ g : A \to \mathbb{R}^n$. Put $Y = \beta(B) = \alpha(A)$. For a continuous function $f : Y \to \mathbb{R}$, f is integrable on $Y_\alpha \iff$ f is integrable on Y_β. If so,

$$\int_{Y_\alpha} f dV = \int_{Y_\beta} f dV.$$

Proof.
We need to show

$$\int_A (f \circ \alpha) V(D\alpha) = \int_B (f \circ \beta) V(D\beta) \qquad (\star)$$

This amounts to change of variables in $\mathbb{R}^k$.

$$\int_A (f \circ \alpha) V(D\alpha) = \int_B f(\beta(y)) V(D\beta(y)) = \int_A f(\beta(g(x))) V(D\beta(g(x))) \cdot |\det Dg(x)| .$$

By the Chain rule

$$D\alpha(x) = D\beta(g(x)) Dg(x)$$

$$\implies V(D\alpha(x))^2 = \det(D\alpha(x)^T D\alpha(x)) = \det\left([D\beta(g(x))Dg(x)]^T D\beta(g(x))Dg(x)\right)$$

$$= \det\left(Dg(x)^T D\beta(g(x))^T D\beta(g(x)) Dg(x)\right) = \det(Dg(x))^2 V(D\beta \circ g(x))^2$$

The last step follows from the multiplicativity of the determinant and commutativity[1].
Taking square roots gives $(\star)$. $\square$

Manifolds Without Boundary

Definition. (Homeomorphism)
Let X and Y be topological spaces (such as subsets of Euclidean spaces). A map $f : X \to Y$ is called a **homeomorphism** provided that f is bijective, continuous, and f^{-1} is continuous (equivalently f is an open map). If there is a homeomorphism between X and Y we say that they are **homeomorphic**.

Examples[2]:

(a) $(0, 1)$ and the unit square minus the point $(0, 1)$ are homeomorphic.

(b) $f(x) = (\cos x, \sin x)$ with $f : [0, 2\pi) \to \mathbb{S}^1$ is a continuous bijective map. However $[0, 2\pi)$ and $\mathbb{S}^1$ are *not* homeomorphic because f^{-1} is not continuous (this makes sense because their fundamental groups are different).

[1] Get used to this proof. It's techniques will show up often.
[2] Algebraic Topology is the study of classifying topological spaces invariant under homeomorphism

Recall the definition of the **subspace topology**.

Definition. (Differentiable Manifold)
Let $k \leq n$. Let $M \subseteq \mathbb{R}^n$. We call M a **differentiable k-manifold without boundary** in $\mathbb{R}^n$ provided that $\forall p \in M$, there is

(i) a set $\mathcal{V} \subseteq M$, containing p, that is open in M. (open containment)
(ii) a set $\mathcal{U} \subseteq \mathbb{R}^k$, that is open in $\mathbb{R}^k$, (local homeomorphism)
(iii) a diffeomorphism $\alpha : \mathcal{U} \to \mathcal{V}$ such that rank $D\alpha(x) = k$, (rank condition) $\forall x \in \mathcal{U}$.

If α is C^r we say M is of class C^r. If α is C^∞ then we say M is **smooth**.
The **manifold** is the set M together with its coordinate patches (atlas). A manifold without the rank condition is called a **topological manifold**.

Terminology: We call the map $\alpha : \mathcal{U} \subseteq \mathbb{R}^k \to \mathcal{V} \subseteq M$ a **coordinate patch** (**coordinate system**) on M about p. The map $\varphi = \alpha^{-1} : \mathcal{V} \subseteq M \to \mathcal{U} \subseteq \mathbb{R}$ is called a **coordinate chart**. The collection of coordinate charts $(\varphi_\lambda, \mathcal{V}_\lambda)$ such that $\bigcup_\lambda \mathcal{V}_\lambda = M$ is called an **atlas**.

Intuition: Intuitively the rank condition assures the linear independence of the columns of

$$D\alpha = \begin{bmatrix} | & & | \\ \frac{\partial \alpha}{\partial x_1} & \cdots & \frac{\partial \alpha}{\partial x_k} \\ | & & | \end{bmatrix} \qquad \text{where} \qquad \frac{\partial \alpha}{\partial x_i} = \lim_{\varepsilon \to 0} \frac{\alpha(x + \varepsilon e_i) - \alpha(x)}{\varepsilon}$$

when it exists is the **tangent vector** to M at $\alpha(x)$. So, rank condition means that there is a k-dimensional tangent "plane" to M at every point.

Examples:

(a) Let M be $\mathbb{S}^1 \subseteq \mathbb{R}^2$ (the unit circle). For every $p \in M \setminus \{(-1,0)\}$, put $V = M \setminus \{(-1,0)\}$, $\mathcal{U} = (-\pi, \pi) \subset \mathbb{R}$, and $\alpha(t) = (\cos t, \sin t)$. α is clearly C^∞, onto, 1-1, continuous inverse, and the rank of $D\alpha(t)$ is 1 $\forall t$.

For the point $p = (-1,0)$, put $V = M \setminus \{(1,0)\}$, $\mathcal{U} = (0, 2\pi) \subset \mathbb{R}$, and $\alpha(t) = (\cos t, \sin t)$. α is clearly C^∞, onto, 1-1, continuous inverse, and the rank of $D\alpha(p)$ is 1.

So $\mathbb{S}^1$ is a differentiable manifold. We showed this by considering a covering of $\mathbb{S}^1$ whose constituents are homeomorphic to $\mathbb{R}$.

(b) Let M be $\mathbb{S}^1 \subseteq \mathbb{R}^2$ (the unit circle). For every p in the upper half of M, put $\alpha_1 : (-1,1) \to V_1$ given by $\alpha_1(t) = (t, \sqrt{1-t^2})$. Do the same with the lower half of M. Then do the same with the right and left hand sides of M but with $\alpha_3 : (-1,1) \to V_3$ given by $\alpha_3(t) = (-\sqrt{1-t^2}, t)$.

(c) Let $M = \mathbb{R}^n$. Then M is a smooth n-manifold without boundary ($\alpha = \mathrm{Id}$).

(d) Finite dimensional vector space W. Let $v_1, \ldots, v_k$ be a basis of W.
Then,

$$W = \left\{ \sum_{i \in [k]} c_i v_i : c_1, \ldots, c_k \in \mathbb{R} \right\}.$$

Let $\alpha : \mathbb{R}^k \to W$ such that

$$\alpha(x) = \sum_{i \in [k]} x_i v_i.$$

Then

$$D\alpha(x) = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_k \\ | & & | \end{bmatrix}$$

has rank k.

(e) Translates and dilates of a manifold (any diffeomorphism). If $M \subseteq \mathbb{R}^n$ and $p \in \mathbb{R}^n$ such that M is a manifold then $N = M + p_0$ is a manifold. The translation map is continuous and has rank 0. $N = rM$ is also a manifold.

(f) Spheres. $\mathbb{S}^{n-1}\{x \in \mathbb{R}^n : \|x\| = 1\}$ is a smooth manifold without boundary of dimension $n-1$. Consider all $2n$ half spheres of $\mathbb{S}^{n-1}$ and consider the patch

$$\alpha_1(x_1, \ldots, x_{n-1}) = \left(x_1, \ldots, x_{n-1}, \sqrt{1 - \sum_{i \in [n]} x_i^2}\right).$$

(g) Open subsets of a manifold (**submanifold**). The restriction of C^r maps are C^r. Therefore, open sets in $\mathbb{R}^n$ are differentiable manifolds without boundary. Any open sets in $\mathbb{S}^{n-1}$ are differentiable manifolds without boundary. $\mathrm{GL}(n, \mathbb{R})$ the set of $n \times n$ invertible manifolds is an n^2-manifold without boundary, this is an open subset of $\mathbb{R}^{n^2}$.

(h) **Product manifold**. For $i \in [\ell]$, M_i an k_i-manifold without boundary in $\mathbb{R}^{n_i}$. Then

$$M = \prod_{i \in [\ell]} M_i$$

is a manifold of dimension $\sum_{i \in [\ell]} k_i$.

The coordinate patches are the products of coordinate patches. $T^n = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$ is an n-torus which is a smooth n-manifold without boundary in $\mathbb{R}^{2n}$. So $\mathbb{S}^1 \times \mathbb{S}^1$ is a 4-manifold but we can clearly embed it in $\mathbb{R}^3$ because we all have seen 3-dimensional donuts coated in sprinkles (this is called the em edibility question). This is because we can realize the torus as a quotient manifold.

(i) Singletons or discrete sets are by definition 0-dimensional manifolds.

(j) Quotient manifold.

Non-Examples:

(a) $\alpha : (0, \pi) \to \mathbb{R}^2$ given by $\alpha(t) = \sin(2t) \begin{bmatrix} |\cos t| \\ \sin t \end{bmatrix}$. Then α is 1-1 and onto but the inverse is not continuous.

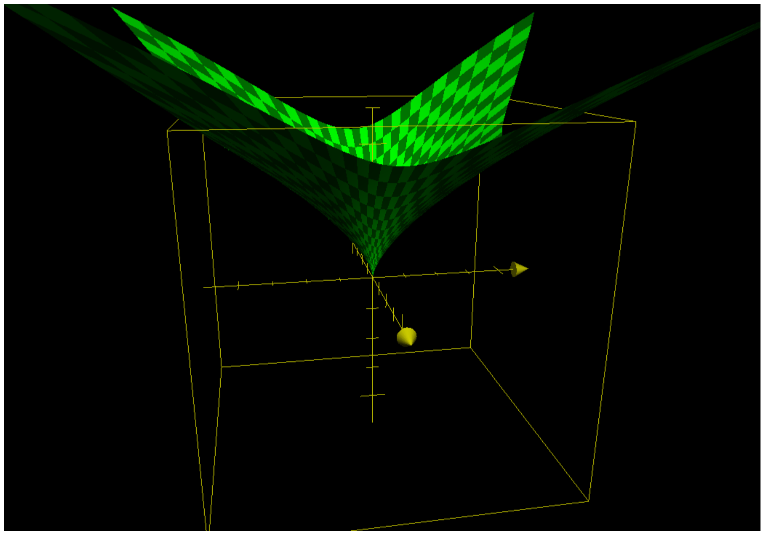

Figure 1: Not a manifold.

Why is the cross not a manifold.

(b) $\alpha : \mathbb{R}^2 \to \mathbb{R}^3$ given by $\alpha(x, y) = (x(x^2 + y^2), y(x^2 + y^2), x^2 + y^2)$. Put $M = \alpha(\mathbb{R}^2)$. α is C^∞, a homeomorphism (check!), but

$$D\alpha(0, 0) = \vec{0}_{3\times 2}$$

so rank $D\alpha(0, 0) \neq 2$.

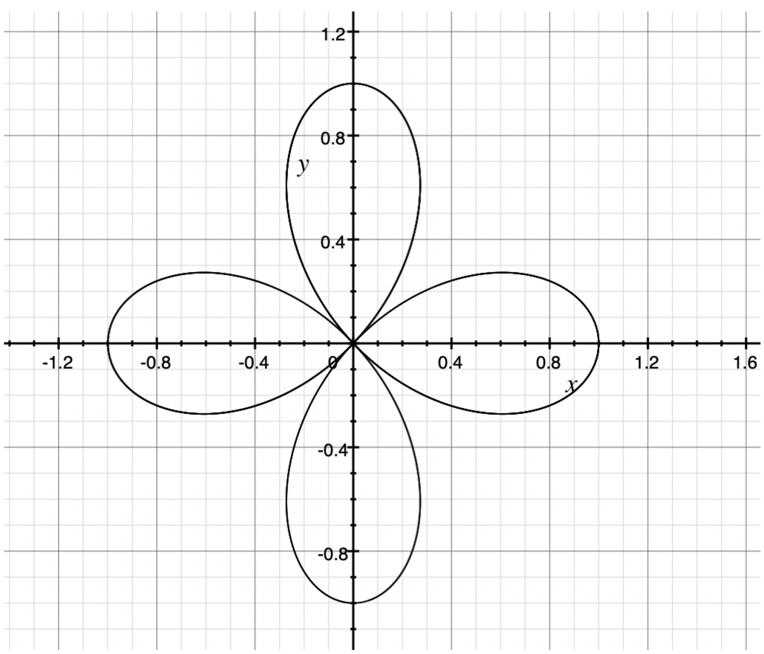

Figure 2: Not a manifold.

At all other point $D\alpha$ has rank two. So M is not a manifold. The surface looks like a parabolic funnel. The set does not have a two dimensional tangent plane at the origin.

(c) Put $\alpha(t) = (t, |t|)$ and $M = \alpha(\mathbb{R})$. This α does not give rise to a (differentiable) manifold. Put $\beta(t) = (t^3, t^2|t|)$. Note that

$$f(x) = t^2|t| = \begin{cases} t^3 & t \geq 0 \\ -t^3 & t < 0 \end{cases}$$

is C^1.

Since

$$f'(x) = \begin{cases} 3t^2 & t > 0 \\ 0 & t = 0 \\ -3t^2 & t < 0 \end{cases}$$

But the rank condition still fails because $\operatorname{rank} D\beta(0) = \operatorname{rank} \vec{0} \neq 1$.

Moral of the story: if you try to be clever, the rank condition will kick in and you will fail.

Is the topologist's sine curve a manifold?
What topology is generated by using the euclidean topology on $\mathbb{R}$ and then considering a space filling curve.

Definition. (Continuous Differentiability)
Let $S \subseteq \mathbb{R}^\ell$. A function $f : S \to \mathbb{R}^m$ is said to be C^r **on** S provided that f extends to a C^r function on an open set in $\mathbb{R}^2$ containing S. There is an open $\Omega \subseteq \mathbb{R}^\ell$ with $\Omega \supseteq S$ and $\tilde{f} : \Omega \to \mathbb{R}^m$, such that $\tilde{f}$ is C^r and $\tilde{f} \restriction S = f$.

Example:

(a) Let $f : S \to \mathbb{R}$ where $S = \operatorname{Span}(\{e_1 + e_2\})$ and $f(x, y) = xy$ then f is C^∞ on S.

Lemma. (Local $C^r \implies C^r$)
Let $S \subseteq \mathbb{R}^\ell$ and $f : S \to \mathbb{R}^m$. Suppose that $\forall x \in S$ f is locally C^r near x (i.e. $\exists S_x$ open in S such that $x \in S_x$ and f is C^r on S_x), then f is C^r on S.

Proof.
We did this in the 395 homework. □

Lemma. (Coordinate Charts are C^r)
Let M be a differentiable k-manifold without boundary in $\mathbb{R}^n$ of class C^r. Let $\alpha : \mathcal{U} \subseteq \mathbb{R}^k \to V \subseteq M \subseteq \mathbb{R}^m$ be a coordinate path on M. Then $\alpha^{-1} : V \to \mathcal{U}$ is C^r on V.

Proof.
It suffices to prove locally. Choose $p_0 \in V$ with $x_0 = \alpha^{-1}(p_0)$.

Since $\operatorname{rank} D\alpha(x_0) = k$ (and row rank equals column rank) there are k linearly independent rows. Without loss of generality we assume that the first k rows of $D\alpha(x_0)$ are linearly independent. Let $\pi : \mathbb{R}^n \to \mathbb{R}^k$ be the projection map onto $\mathbb{R}^k$ (the indices of the k independent rows).

Note π is C^∞ and

$$D\pi = \begin{bmatrix} 1 & & 0 & 0 & \cdots & 0 \\ & \ddots & & 0 & \cdots & 0 \\ 0 & & 1 & 0 & \cdots & 0 \end{bmatrix}$$

Define $g = \pi \circ \alpha$. Then g is C^r and the chain rule gives us

$$Dg(x_0) = D\pi(p_0) D\alpha(x_0)$$

$$\begin{bmatrix} 1 & & 0 & 0 & \cdots & 0 \\ & \ddots & & 0 & \cdots & 0 \\ 0 & & 1 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \\ \heartsuit & \heartsuit & \heartsuit \\ \heartsuit & \heartsuit & \heartsuit \end{bmatrix} = \begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$$

which is invertible (by rank condition).

By the Inverse Function Theorem, g is a diffeomorphism locally near x_0 and g^{-1} is C^r near $\pi(p_0)$.

Note that $\alpha^{-1} = \pi \circ g^{-1}$, so α^{-1} is C^r[3]. □

> **Theorem.** (Coordinate Patches Overlap Differentiably)
> Let M be a differentiable k-manifold without boundary in $\mathbb{R}^n$ of class C^r. Let α_1, α_2 be coordinate patches from $\mathcal{U}_1, \mathcal{U}_2$ to $\mathcal{V}_1, \mathcal{V}_2$ respectively with $\mathcal{V}_1 \cap \mathcal{V}_2 \neq \varnothing$.
> The map $\alpha_2^{-1} \circ \alpha_1 : \mathcal{W}_1 \to \mathcal{W}_2$ is C^r where $W_i = \alpha_i^{-1}(\mathcal{V}_1 \cap \mathcal{V}_2)$ are open in $\mathbb{R}^k$.

Proof.
Easy. The lemma above tells us that α_2 is C^2 and composition of C^2 maps is C^r by the Chain Rule. The map $\alpha_2^{-1} \circ \alpha_1$ is called a **transition map**[4]. □

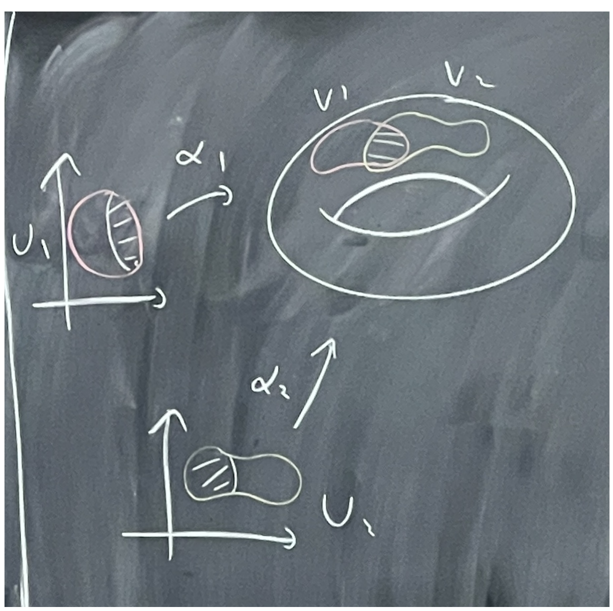

Figure 3: Overlapping coordinate patches.

[3] This step needs more thinking!

[4] In more abstract manifold theory we take the existence of transition maps as the definition of a differentiable manifold.

Manifolds With Boundary

Someone should make a hat with a donut on the top!

Notation: $\mathbb{H}^k = \{x \in \mathbb{R}^k : x_k \geq 0\}$ and $\mathbb{H}^k_+ = \{x \in \mathbb{R}^k : x_k > 0\}$.

Lemma. (Differentiability on Boundary)
Let $\mathcal{U} \subseteq \mathbb{H}^k$ be open in $\mathbb{H}^k$ but not in $\mathbb{R}^k$. Suppose $\alpha : \mathcal{U} \to \mathbb{R}^n$ is C^r. Let $\tilde{\alpha} : \tilde{\mathcal{U}} \to \mathbb{R}^n$ be a C^r extension of α, then $\forall x \in \mathcal{U}$, $D\tilde{\alpha}(x)$ depends only on α. As a consequence, $D\alpha(x)$ is well defined.

Proof.
Note that

$$\frac{\partial \tilde{\alpha}}{\partial x_i} = \lim_{\varepsilon \to 0} \frac{\tilde{\alpha}(x + \varepsilon e_i) - \tilde{\alpha}(x)}{\varepsilon}$$

exists by the assumption that α is C^r.

Since the limit exists, it is unique and equal for every path (so we can always approach from within $\mathbb{H}^k$. By taking $\varepsilon > 0$ we see that

$$\frac{\partial \alpha}{\partial x_i} = \lim_{\varepsilon \downarrow 0} \frac{\alpha(x + \varepsilon e_i) - \alpha(x)}{\varepsilon}$$

Definition. (Differentiable Manifold with Boundary)
A differentiable k-manifold (**with boundary**) in $\mathbb{R}^n$ of class C^r is a set $M \subseteq \mathbb{R}^n$ such that $\forall p \in M$, $\exists \alpha : \mathcal{U} \to \mathcal{V}$ where

(1) $\mathcal{U}$ is open in either $\mathbb{R}^k$ or $\mathbb{H}^k$,
(2) $\mathcal{V}$ is open in M,
(3) α is a C^r homeomorphism, and

$$\operatorname{rank} D\alpha(x) = k$$

for all $x \in \mathcal{U}$.

Examples:

(a) $\mathbb{S}^1 \cap \mathbb{H}^k_+$ has manifold structure (without boundary). Consider $\alpha(t) = (\cos t, \sin t)$.

(b) $\mathbb{S}^1 \cap \mathbb{H}^k$ has manifold structure (with boundary).
For $p \in M \setminus \{(-1, 0)\}$, $\alpha : [0, \pi) \subseteq \mathbb{H}^1 \to M \setminus \{(-1, 0)\}$ given by $\alpha(t) = (\cos t, \sin t)$ is a coordinate patch.
For $p \in M \setminus \{(1, 0)\}$, $\alpha : [0, \pi) \subseteq \mathbb{H}^1 \to M \setminus \{(-1, 0)\}$ given by $\alpha(t) = (\cos(\pi - t), \sin(\pi - t))$ is a coordinate patch.
So this is a manifold with boundary.

(c) The convex hull of $\mathbb{S}^1$ (considered as a subset of $\mathbb{R}^2$) is a manifold with boundary.

(d) The portion of the unit disk that lies in the closed first quadrant does not have a differentiable manifold structure.

Lemma. (Coordinate Charts are C^r)
Let M be a differentiable k-manifold in $\mathbb{R}^n$ of class C^r. Let $\alpha : \mathcal{U} \subseteq \mathbb{R}^k \to V \subseteq M \subseteq \mathbb{R}^m$ be a coordinate path on M. Then $\alpha^{-1} : V \to \mathcal{U}$ is C^r on V.

Theorem. (Coordinate Patches Overlap Differentiably)
Let M be a differentiable k-manifold in $\mathbb{R}^n$ of class C^r. Let α_1, α_2 be coordinate patches from $\mathcal{U}_1, \mathcal{U}_2$ to $\mathcal{V}_1, \mathcal{V}_2$ respectively with $\mathcal{V}_1 \cap \mathcal{V}_2 \neq \varnothing$.
The map $\alpha_2^{-1} \circ \alpha_1 : \mathcal{W}_1 \to \mathcal{W}_2$ is C^r where $W_i = \alpha_i^{-1}(\mathcal{V}_1 \cap \mathcal{V}_2)$ are open in $\mathbb{R}^k$ or $\mathbb{H}^k$.

Definition. (Interior and Boundary of Manifold)
Let M be a k-manifold in $\mathbb{R}^n$. Take $p \in M$.

(a) p is called an **interior point** of M if there is a coordinate patch $\alpha : \mathcal{U} \to \mathcal{V}$ on M about p such that $\mathcal{U}$ is open in $\mathbb{R}^n$.
(b) p is called an **boundary point** of M if p is not an interior point. The set of boundary points of M is denoted by ∂M.

We want a condition to characterize boundary points.

Lemma. (Restrictions of Coordinate Maps)
Let M be a manifold and $\alpha : \mathcal{U} \to \mathcal{V}$ a coordinate patch. If $\mathcal{U}_0 \subseteq \mathcal{U}$ is open in $\mathcal{U}$, then $\alpha \restriction \mathcal{U}_0 : \mathcal{U}_0 \to \alpha(\mathcal{U}_0)$ is also a coordinate patch.

Proof.
Easy. Restrictions of diffeomorphisms are diffeomorphisms onto their image. $\square$

Definition. (Conditions for Boundary and Interior)
Let M be a k-manifold in $\mathbb{R}^k$ and $\alpha : \mathcal{U} \to \mathcal{V}$ a coordinate patch on M about p.

(1) $\mathcal{U}$ is open in $\mathbb{R}^k \implies p$ is an interior point of M.
(2) $\mathcal{U}$ is open in $\mathbb{H}^k$ and $p = \alpha(x_0)$ for some $x_0 \in \mathbb{H}^k_+ \implies p$ is an interior point of M.
(3) $\mathcal{U}$ is open in $\mathbb{H}^k$ and $p = \alpha(x_0)$ for some $x_0 \in \mathbb{R}^{k-1} \times \{0\} \implies p$ is a boundary point.

Proof.
(1) is clear by definition. (2) Put $\mathcal{U}_0 \coloneqq \mathcal{U} \cap \mathbb{H}^k_+$, which is open in $\mathbb{R}^k$. Now restrict α to $\mathcal{U}_0$ which witnesses that p is an interior point.
(3)[5] Suppose, for the sake of contradiction, p is an interior point. Then, $\exists \beta : \mathcal{U}' \to \mathcal{V}'$ with $\mathcal{U}'$ open in $\mathbb{R}^k$. Consider $\mathcal{U} \cap \mathcal{U}'$ the transition map $\gamma = \alpha^{-1} \circ \beta : \mathcal{W}_1 \to \mathcal{W}_2$ is C^r, a homeomorphism, and $D\gamma(x)$ has rank k for all $x \in \mathcal{W}_1$.

So $\gamma : \mathcal{W}_1 \subseteq \mathbb{R}^k \to \mathbb{R}^k$ so γ should be an *open map*. Therefore $\mathcal{W}_2 = \gamma(\mathcal{W}_1)$ is open in $\mathbb{R}^k$. Contradiction! Since $x_0 \in \mathcal{W}_2$ and $x_0 \in \mathbb{R}^{k-1} \times \{0\}$. $\square$

Example:

(i) $\partial(\mathbb{S}^1 \cap \mathbb{H}^K_+) = \{(1, 0), (-1, 0)\}$.

[5] You should be able to do this.

(ii) $\partial\mathbb{H}^k = \mathbb{R}^{k-1} \times \{0\}$.

Here's a cute theorem!

Theorem. (Boundary Manifold)
Let M be a k-manifold of class C^r in $\mathbb{R}^n$. If $\partial M \neq \varnothing$, then ∂M is $(k-1)$-manifold without boundary of class C^r in $\mathbb{R}^n$.

Proof.
Read the book. Use the boundary coordinate patches and project them onto $\mathbb{R}^{k-1}$. □

Here is a workhorse theorem:

Theorem. (Condition for Level Set Manifold)
Let $\mathcal{O} \subseteq \mathbb{R}^n$ be open and $f : \mathcal{O} \to \mathbb{R}$ be C^r. Define $N \coloneqq \{x \in \mathcal{O} : f(x) \geq 0\}$ and $M \coloneqq \{x \in \mathcal{O} : f(x) = 0\}$. We say that M is a **level set** of f. Suppose $M \neq \varnothing$ and $\operatorname{rank} Df(x) = 1$ for all $x \in M$. Then, N is a C^r n-manifold in $\mathbb{R}^n$ and $M = \partial N$.

Proof.
Suppose $p \in N$ and $f(p) > 0$. Let $\mathcal{U} = \{x \in \mathcal{O} : f(x)0\}$, which is open in $\mathbb{R}^n$. Put $\alpha : \mathcal{U} \subseteq \mathbb{R}^n \to \mathcal{U} \subseteq N \subseteq \mathbb{R}^n$, $\alpha = \mathrm{Id}$. Then α is a coordinate patch about p.

Suppose $p \in N$ and $f(p) = 0$ (i.e. $p \in M$). Since $\operatorname{rank} Df(p) = 1$, at least one of $\frac{\partial f}{\partial x_i}(p) \neq 0$ for $i \in [n]$. Without loss of generality, we may assume $\frac{\partial f}{\partial x_n}(p) \neq 0$. Define $F : \mathcal{O} \to \mathbb{R}^n$, $F(x) = (x_1, \ldots, x_{n-1}, f(x))$. F is C^r and

$$DF = \left[\begin{array}{ccc|c} & & & 0 \\ & I_{n-1} & & \vdots \\ & & & 0 \\ \hline \dfrac{\partial f}{\partial x_1} & \cdots & \dfrac{\partial f}{\partial x_{n-1}} & \dfrac{\partial f}{\partial x_n} \end{array}\right] \implies \det DF(p) = \frac{\partial f}{\partial x_n}(p) \neq 0$$

The Inverse Function Theorem guarantees that F is a diffeomorphism locally near p. Meaning, there exists open $A, B \subseteq \mathbb{R}^n$ with $p \in (A)$ such that $F : A \to B$ is a C^r diffeomorphism and $F(A)$ is identically zero. Let $\mathcal{U} = B \cap \mathbb{H}^n$, $\mathcal{V} = A \cap N$, $\alpha = F^{-1} : \mathcal{U} \to \mathcal{V}$. α is a coordinate patch. Hence, N is a C^r n-manifold. This computation also shows us that $M = \partial N$. □

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Example:
Let $f : \mathbb{R}^n \to \mathbb{R}$ be given by $f(x) = a^2 - \sum_{i\in[n]} x_i^2$. Then $N = B_a^n(0)$ or $\mathbb{B}^n(a)$ and $M = \mathbb{S}^{n-1}(a)$. $Df(x) = -2\vec{x}^T$ is not the zero vector for $x \in \mathbb{S}^{n-1}(a)$. Thus, $\mathbb{B}^n(a)$ is a smooth n-manifold in $\mathbb{R}^n$ of class C^∞ and $\partial\mathbb{B}^n(a) = \partial\mathbb{S}^{n-1}(a)$.

Integration of Scalar Functions on Manifolds

Later we will integrate vector fields and differential forms over manifolds. For now, we will just be integrating scalar valued functions over a manifold. For simplicity of presentation, we will only consider integration over **compact manifolds**, meaning a closed and bounded subset of $\mathbb{R}^n$ which has manifold structure.

Suppose $f : M \to \mathbb{R}$ where M is a manifold with boundary. Suppose $\operatorname{supp} f$ is contained in a single coordinate patch.

Definition. (One Patch Integral over Manifold)
Let M be a compact k-manifold in $\mathbb{R}^n$. Let $f : M \to \mathbb{R}$ be continuous. Suppose there is a coordinate patch $\alpha : \mathcal{U} \to \mathcal{V}$ such that $\operatorname{supp} f \subseteq \mathcal{V}$. Note that since $\alpha^{-1}(\operatorname{supp} f)$ is compact in $\mathbb{R}^k$, we may choose $\mathcal{U}$ to be bounded.
Define
$$\int_M f\, dV = \int_{\operatorname{Int}\mathcal{U}} (f \circ \alpha) \cdot V(D\alpha)$$
Note that $\operatorname{Int}\mathcal{U} = \mathcal{U}$ if $\mathcal{U}$ is open in $\mathbb{R}^k$ and $\operatorname{Int}\mathcal{U} = \mathcal{U} \cap \mathbb{H}^k_+$ if $\mathcal{U}$ is open in $\mathbb{H}^k$.
Lemma. The RHS is ordinary integrable.
Lemma. $\int_M f\, dV$ does not depend on the choice of α.

Check that the integral is patch-independent and the integral is well defined (recall theorem 13.5 of Munkres).

Example: Suppose $M = \{(x, y) : (x, y) \in \mathbb{S}^1(3),\ x \leq 0 \vee y \geq 0\}$. Put
$$f(x, y) = \begin{cases} y & y \geq 0 \\ 0 & y < 0 \end{cases}$$
Then $\operatorname{supp} f = \mathbb{S}^1(3) \cap \mathbb{H}^2$. We can find one coordinate patch "to rule them all."
Put $\alpha : [0, \frac{3\pi}{2}) \subseteq \mathbb{H}^1 \to M \setminus \{(0, -3)\}$, $\alpha(t) = (3\cos t, 3\sin t)$. We have,
$$\int_M f\, dV = \int_0^{\frac{3\pi}{2}} \alpha \circ f(3\cos t, 3\sin t) \cdot 3 = \int_0^{\pi} 9\sin t = 18.$$

Recall the definition of a Partition of Unity subordinate to $\mathscr{A}$.

Lemma. (Partition of Unity on a Manifold)
Let M be a compact k-manifold in $\mathbb{R}^n$. Given a covering of M by coordinate patches, there is a finite collection of C^∞ $\phi_i : \mathbb{R}^n \to \mathbb{R}$ $i \in [\ell]$ such that
(i) $\phi_i(x) \geq 0$, $\forall x \in \mathbb{R}^n$, $\forall i \in [\ell]$.
(ii) $\sum_{i \in [\ell]} \phi_i(p) = 1$, $\forall p \in M$
(iii) $\forall i \in [\ell]$, there is a coordinate patch $\alpha_i : \mathcal{U}_i \to \mathcal{V}_i$ such that $\operatorname{supp} \phi_i \cap M \subseteq \mathcal{V}_i$.

Proof.
Read the book. $\square$

Definition. (Integral over Manifold)
Let M be a compact k-manifold in $\mathbb{R}^n$ and $f : M \to \mathbb{R}$ continuous.
Define

$$\int_M f\,dV = \sum_{i\in[\ell]} \int_M (\phi_i \cdot f) = \sum_{i\in[\ell]} \int_{\mathcal{U}_i} ((\phi_i \cdot f)\circ\alpha)\cdot V(D\alpha)$$

for a partition of unity $\{\phi_i\}_{i\in[\ell]}$ of M.

We need to check:

(a) If $\operatorname{supp} f$ lies in one coordinate patch, then the two definitions agree.

(b) $\int_M f\,dV$ is independent of the choice of partition of unity on M.

(c) $\int_M(\alpha f + \beta g)\,dV = \alpha\int_M f\,dV + \beta\int_M g\,dV$ and monotonicity in the domain.

Now we need a practical way to compute the integral over a manifold. We will extend the notion of measure zero on a manifold.

Definition. (Measure Zero Sets in a Manifold)
Let $M \subseteq \mathbb{R}^n$ be a compact k-manifold. $D \subseteq M$ is said to have **measure zero in** M provided that D can be covered by at most countably many coordinate patches $\alpha_i : \mathcal{U}_i \to \mathcal{V}_i$ such that

$$\bigcup_{i\in\mathbb{N}} \alpha_i^{-1}(D \cap \mathcal{V}_i)$$

has measure zero in $\mathbb{R}^k$.

Example:

$M = \mathbb{S}^2(a) \subseteq \mathbb{R}^3$ and $D = \mathbb{S}^1(a) \times \{0\}$. Let α be the stereographic projection from the north pole. Then, $\alpha^{-1}(D)$ is a circle in $\mathbb{R}^2$.

Theorem. (Measure Zero Sets Do Not Affect Integrals)
Let $M \subseteq \mathbb{R}^n$ be a compact k-manifold and $f : M \to \mathbb{R}$ continuous. Suppose $\alpha_i : A_i \to M_i$ for $i \in [\ell]$ are coordinate patches such that $M_1, \ldots, M_N$ are disjoint and

$$M = \left(\bigcup_{i\in[\ell]} M_i\right) \cup K$$

where K is of measure zero in M.

Then,

$$\int_M f\,dV = \sum_{i\in[\ell]} \int_{A_i} f\,dV$$

Proof.
Since both sides of the equation are linear in f, it is enough to show

$$\int_M f\,dV = \sum_{i\in[N]} \int_{A_i} (f\circ\alpha_i)\cdot V(D\alpha)$$

coordinate patch. Hence WLOG we may assume $\operatorname{supp} f$ lies in one coordinate patch.

Then the equation to prove becomes

$$\int_{\operatorname{Int}\mathcal{U}} (f \circ \alpha) \cdot V(D\alpha) = \sum_{i \in [N]} \int_{M_i} f \, dV$$

Put $L = \alpha^{-1}(K \cap \mathcal{V})$. Put $\mathcal{W}_i = \alpha^{-1}(M_i \cap \mathcal{V})$, which is open in $\mathbb{R}^k$ or $\mathbb{H}^k$. Try to prove that L is measure zero in $\mathbb{R}^k$ (HW), you should use that C^1 maps take measure zero sets to measure zero sets.

$$\int_{\operatorname{Int}\mathcal{U}} (f \circ \alpha) \cdot V(D\alpha) = \int_{\operatorname{Int}\mathcal{U} \setminus L} (f \circ \alpha) \cdot V(D\alpha) = \sum_{i \in [N]} \int_{\mathcal{W}_i} (f \circ \alpha) \cdot v(D\alpha)$$
$$= \sum_{i \in [N]} \int_{\alpha_i^{-1}(M_i \cap \mathcal{V})} (f \circ \alpha_i) \cdot v(D\alpha_i)$$

The last equality follows from change of variables and the fact that $\operatorname{supp} \alpha_i$ lies almost entirely in A_i.

$\square$

$M = \mathbb{S}^2(a) \subseteq \mathbb{R}^3$. Let's compute $v(M)$. $K = \{(x, y, z) \in M : y = 0, x \geq 0\}$ (half the meridian). Let $\alpha : (0, 2\pi) \times (0, \pi) \to M \setminus K$ be given by $\alpha(\theta, \phi) = (a \sin\phi \cos\theta, a \sin\phi \sin\theta, a \cos\phi)$.

$$v(M) = \int_{M \setminus K} 1 \, dV = \int_{(0,2\pi)\times(0,\pi)} 1 \cdot V(D\alpha) = \int_0^{2\pi} \int_0^{\pi} a^2 \sin\phi \, d\phi d\theta = a^2(2)(2\pi) = 4\pi a^2$$

Let's compute again with another method: **Cavalieri's Principle**. $\alpha : (-a, a) \times (0, 2\pi) \to M \setminus K$. $\alpha(z, \theta) = (\sqrt{a^2 - z^2} \cos\theta, \sqrt{a^2 - z^2} \sin\theta, z)$. Check that α is a coordinate patch and $V(D\alpha) = a$.

$$v(M) = \int_{M \setminus K} 1 \, dV = \int_{-a}^{a} \int_0^{2\pi} a \, d\theta dz = 4\pi a^2$$

A similar computation will give you $v(\mathbb{S}^1(a)) = 2\pi a$. What about the surface area of $\mathbb{S}^k(\alpha)$?

DIFFERENTIAL FORMS

Tensors & Alternating Tensors

Tensors generalize vectors and matrices.

Definition. (Tensor)
Suppose V is a vector space of dimension n with basis $\{b_i : i \in [n]\}$. A k**-tensor** is a function $f : V^K \to \mathbb{R}$ that is **multilinear**.
We write $\mathcal{L}^k(V)$ as the set of k-tensors on V. $\mathcal{L}^k(V)$ forms a vector space with basis $\{\phi_{\sigma(1)} \otimes \cdots \otimes \phi_{\sigma(n)} : \sigma \in [n]^k\}$ where $\{\phi_i : i \in [n]\}$ is the standard basis for V^*.
We call k the **order** of the tensor. $g \otimes h$ has order $\ell + m$ if $\mathcal{L}^\ell(V)$ and $h \in \mathcal{L}^m(V)$

Example:

(a) $\mathcal{L}^1(V) = V^*$ is the dual space of V.

(b) $\mathcal{L}^2(V) = \left\{f : V^2 \to \mathbb{R} : f \mapsto (f(a_i, a_j))_{i,j\in[n]}\right\}$ is isomorphic to $\mathrm{Hom}(V, V)$.

(c) Let $V = \mathbb{R}^n$ let $f \in \mathcal{L}^2(\mathbb{R}^n)$. Then for $x, y \in \mathbb{R}^n$,

$$f(x,y)) = \left(\sum_{i,j\in[n]} c_{ij}\left(\phi_i \otimes \phi_j\right)\right)(x,y) = \sum_{i,j\in[n]} c_{ij} \cdot \phi_i(x) \cdot \phi_j(y) = \sum_{i,j\in[n]} c_{ij} x_i y_j = x^T C y$$

since $\phi_i(e_j) = \delta_{ij}$.

Definition. (Alternating Tensor)
$f \in \mathcal{L}^k(V)$ is said to be **alternating** provided that for every $i \in [n-1]$,

$$f(\cdots, v_i, v_{i+1}, \cdots) = -f(\cdots, v_{i+1}, v_i, \cdots)$$

We write $\Lambda^k(V)$ (or $\Lambda_k(V)$ $\mathcal{A}^k(V)$) as the set of alternating k-tensors on V.
$\Lambda^k(V)$ forms a vector subspace of dimension $\binom{n}{k}$ with basis

$$\left\{\bigwedge_{j\in[k]} \phi_{\sigma(j)} \mid \sigma \in \mathrm{ASC}_{k,n}\right\}$$

Where $\{\phi_i : i \in [n]\}$ is the standard basis for V^* and the $\bigwedge$ denotes the wedge product.
Note that the space $\Lambda^n(V)$ is one dimensional. When $k > n$, $\Lambda^{\dim(V)}(V) = \{0\}$.

Example:

(a)
$$\begin{aligned}
f \in \Lambda^k(V)) &\iff f(x,y) = x^T C y \ \wedge \ f(x,y) = -f(y,x) \\
&\iff x^T C y = -y^T C x = (y^T C x)^T = -x^T C^t y \\
&\iff x^T (C + C^T) y = 0, \ \forall x, y \in \mathbb{R}^n \\
&\iff f(x,y) = x^T C y \ \wedge \ C = -C^T
\end{aligned}$$

So $\Lambda^k(V)$ is isomorphic so the set of **skew-symmetric** matrices (meaning the diagonal must be zero).

(b) $\Lambda^2(\mathbb{R}^3)$ has basis $\{\omega_{12}, \omega_{23}, \omega_{13}\}$ where $\omega_{ij}(x,y) = x_i y_j - x_j y_i = (\phi_i \otimes \phi_j - \phi_j \otimes \phi_i)(x,y)$.

Check that for $\omega \in \Lambda^k(V)$ and $\sigma \in S_k$,

$$\omega(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) = \operatorname{sgn}(\sigma) \cdot \omega(v_1, \ldots, v_k)$$

Note that for $\omega \in \Lambda^k(V)$ then

$$\omega(\cdots, v, \ldots, v, \cdots) = 0$$

Definition. (Alternization)
Define $\operatorname{Alt} : \mathcal{L}^k(V) \to \Lambda^k(V)$ by

$$\operatorname{Alt}(f)(v_1, \ldots, v_k) = \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) \cdot f(v_{\sigma(1)}, \ldots, v_{\sigma(k)})$$

Lemma.
(1) Alt is a linear map and $f \in \mathcal{L}^k(V) \implies \operatorname{Alt}(f) \in \Lambda^k(V)$.
(2) If $\omega \in \Lambda^k(V)$, then $\operatorname{Alt}(\omega) = k! \cdot \omega$.
(3) $f \in \mathcal{L}^k(V) \implies \operatorname{Alt}(\operatorname{Alt}(f)) \in k! \operatorname{Alt}(f)$.
(4) $f \in \mathcal{L}^k(V), g \in \mathcal{L}^\ell(V) \implies \operatorname{Alt}(f \otimes g) = (-1)^{k+\ell} \operatorname{Alt}(g \otimes f)$.

Proof.
We have

(1) It suffices to show $\tau \in S_k$,

$$\operatorname{Alt}(f)(v_{\tau(1)}, \ldots, v_{\tau(k)}) = \operatorname{sgn}(\tau) \cdot \omega(v_1, \ldots, v_k)$$

for all $\tau \in S_k$.

Fix $\tau \in S_k$ then

$$\begin{aligned}
&\operatorname{Alt}(f)(v_{\tau(1)}, \ldots, v_{\tau(k)}) \\
&= \sum_{\sigma \in S_k} (\operatorname{sgn}\sigma) f(v_{\sigma\circ\tau(1)}, \ldots, v_{\sigma\circ\tau(k)}) && \text{definition of Alt} \\
&= \sum_{\pi \in S_k} (\operatorname{sgn}\sigma) f(v_{\pi(1)}, \ldots, v_{\pi(k)}) && \text{with } \pi = \sigma \circ \tau \\
&= \sum_{\pi \in S_k} (\operatorname{sgn}\pi)(\operatorname{sgn}\tau) f(v_{\pi(1)}, \ldots, v_{\pi(k)}) && \text{transposition properties} \\
&= (\operatorname{sgn}\tau) \sum_{\pi \in S_k} (\operatorname{sgn}\pi) f(v_{\pi(1)}, \ldots, v_{\pi(k)}) && \text{distributivity} \\
&= (\operatorname{sgn}\tau) \operatorname{Alt}(f)) && \text{definition of Alt}
\end{aligned}$$

(2) Easy

(3) $(1) \land (2) \implies (3)$.

(4) Let $\pi \in S_{k+1}$ for $\pi(i) = \ell + i, \pi(j+1) = j, i \in [k], j \in [\ell]$

$$
\begin{aligned}
&\operatorname{Alt}(f \otimes g)(v_1, \dots, v_{k+\ell}) \\
&= \sum_{\sigma \in S_{k+\ell}} (\operatorname{sgn} \sigma) f(v_{\sigma(1)}, \dots, v_{\sigma(k+\ell)}) && \text{definition of Alt} \\
&= \sum_{\sigma \in S_{k+\ell}} (\operatorname{sgn} \sigma) f(v_{\sigma(1)}, \dots, v_{\sigma(k)}) g(v_{\sigma(k+1)}, \dots, v_{\sigma(k+\ell)}) && \text{definition of Alt} \\
&= \sum_{\sigma \in S_{k+\ell}} (\operatorname{sgn} \tau \circ \pi) f(v_{\tau\circ\pi(1)}, \dots, v_{\sigma(k)}) g(v_{\tau\circ\pi(k+1)}, \dots, v_{\tau\circ\pi(k+\ell)}) && \sigma \coloneqq \tau \circ \pi \\
&= \sum_{\sigma \in S_{k+\ell}} (\operatorname{sgn} \tau \circ \pi) f(v_{\tau(\ell+1)}, \dots, v_{\tau(k+\ell)}) g(v_{\tau(1)}, \dots, v_{\tau(\ell)}) && \text{definition of } \pi \\
&= (\operatorname{sgn} \pi) \operatorname{Alt}(g \otimes f) && \text{definition of Alt}
\end{aligned}
$$

The Wedge Product

Definition. (Wedge Product)
For $\omega \in \Lambda^k(V), \eta \in \Lambda^\ell(V)$ define the **wedge product**

$$\omega \wedge \eta = \frac{1}{k!\ell!} \operatorname{Alt}(\omega \otimes \eta)$$

Lemma.
(1) If ω and η are of the same order, then $(\omega \wedge \eta) \wedge \theta = \omega \wedge \theta + \eta \wedge \theta$.
(2) $\wedge : \Lambda^k(V) \times \Lambda^\ell(V) \to \Lambda^{k+\ell}(V)$ is a bilinear map.
(3) $\omega \wedge \eta = (-1)^{k \cdot \ell}(\eta \wedge \omega)$.

Proof.
Done in IBL $\square$

Lemma. (Associativity of Wedge Product)
(1) If $f \in \mathcal{L}^k(V), g \in \mathcal{L}^\ell(V)$, and $\operatorname{Alt}(f) = 0$, then $\operatorname{Alt}(f \otimes g) = 0$.
(2) If $f \in \mathcal{L}^k(V)$ and $\theta \in \Lambda^m(V)$ then $\operatorname{Alt}(f) \wedge \theta = \frac{1}{m!} \operatorname{Alt}(f \otimes \theta)$.
(3) If $\omega \in \Lambda^k(V), \eta \in \Lambda^\ell(V), \theta \in \Lambda^m(V)$, then

$$(\omega \wedge \eta) \wedge \theta = \omega \wedge (\eta \wedge \theta) = \frac{1}{k!\ell!m!} \operatorname{Alt}(\omega \otimes \eta \otimes \theta)$$

Proof.
(1) This is the difficult part. Note that

$$\operatorname{Alt}(f) = 0 \iff \sum_{\pi \in S_k} f(w_{\pi(1)}, \dots, w_{\pi(k)} = 0$$

For each $I \in [k+\ell]^\ell$. Let G_I be the set of permutations $\sigma \in S_{k+\ell}$ satisfying $\sigma(k+j) = I(j)$ for each $j \in [\ell]$.

For example $G_I = \{(14352), (41352)\}$ for $I = (3, 5, 2)$.

We have

$$\begin{aligned}
k!\ell! \operatorname{Alt}(f \otimes g)(v_1, \ldots, v_{k+\ell}) &= \sum_{\sigma \in S_{k+\ell}} f(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) \cdot g(v_{I(1)}, \ldots, v_{I(\ell)}) \\
&= \sum_{I \in [k+\ell]^\ell} \sum_{\sigma \in G_I} f(v_1, \ldots, v_{k+\ell}) \cdot g(v_{k+1}, \ldots, v_{k+\ell}) \\
&= \sum_{I \in [k+\ell]^\ell} \left[\sum_{\sigma \in G_I} f(v_1, \ldots, v_{k+\ell}) \right] \cdot g(v_{k+1}, \ldots, v_{k+\ell})
\end{aligned}$$

Now fix I. Note that if $\sigma, \tau \in G_I$, then $\{\sigma(j) : j \in [k]\} = \{\tau(j) : j \in [k]\}$.

Denote $\{v_{\sigma(j)} : j \in [k]\} = \{w_j : j \in [k]\}$. Then,

$$\begin{aligned}
&= \sum_{\sigma \in G_I} f(v_1, \ldots, v_{k+\ell}) \\
&= \pm \sum_{\pi \in S_k} (\operatorname{sgn} \pi) f(w_{\pi(1)}, \ldots, w_{\pi(k)}) \\
&= \pm \operatorname{Alt} f(w_1, \ldots, w_k) \\
&= 0
\end{aligned}$$

(2) Put $F = \operatorname{Alt}(f) - k!f$. Then, $\operatorname{Alt}(F) = 0$. Use (1) with $f \coloneqq F$. Then

$$\begin{aligned}
&\implies \\
&\implies \operatorname{Alt}(F \otimes \theta) = 0 \\
&\implies \operatorname{Alt}(\operatorname{Alt}(f) \otimes \theta) = k! \operatorname{Alt}(f \otimes \theta) \\
&\implies k!m! \operatorname{Alt}(f) \wedge \theta && \text{definition of } \wedge
\end{aligned}$$

(3) In two parts,

$$\begin{aligned}
(\omega \wedge \eta) \wedge \theta &= \frac{1}{k!\ell!} \operatorname{Alt}(\omega \otimes \eta) \wedge \theta \\
&= \frac{1}{k!\ell!m!} \operatorname{Alt}((\omega \otimes \eta) \otimes \theta) \\
&= \frac{1}{k!\ell!m!} \operatorname{Alt}(\omega \otimes \eta \otimes \theta) && \otimes \text{ is associative}
\end{aligned}$$

For part 2,

$$\begin{aligned}
\theta \wedge (\omega \wedge \eta) &= (-1)(\omega \wedge \eta) \wedge \theta \\
&= \frac{(-1)^{k(\ell+m)}}{k!\ell!m!} \operatorname{Alt}(\eta \otimes \theta \otimes \omega) \\
&= \frac{1}{k!\ell!m!} \operatorname{Alt}(\omega \otimes \eta \otimes \theta)
\end{aligned}$$

Fields & Forms on Euclidean Space

Definition. (Fields)
Let $A \subseteq \mathbb{R}^n$ be open. We define a
(i) a **scalar field** on A is a function $f : A \to \mathbb{R}$.
(ii) a **vector field** on A is a function $F : A \to \mathbb{R}^n$ (note the dimension).
(iii) a k**-tensor field** is a function $F : A \to \mathcal{L}^k(\mathbb{R}^n)$.
(iv) a (**differential**) k**-form** is a function $F : A \to \Lambda^k(\mathbb{R}^n)$.

Example:

(i) $F : \mathbb{R}^2 \to \mathbb{R}^2$ given by $F(x,y) = xe_1 + ye_1$ is a vector field that describes radial growth.

(ii) $F : \mathbb{R}^2\{\vec{0}\} \to \mathbb{R}^2$ given by $F(x,y) = \frac{xe_2 - ye_1}{\sqrt{x^2+y^2}}$ is a vector field that describes counterclockwise rotation at unit speed.

(iii) $F : \mathbb{R}^3 \to \Lambda^2(\mathbb{R}^2)$ given by

$$\omega(x,y,z) = xy(\phi_1 \wedge \phi_2) + xz(\phi_1 \wedge \phi_3) + yz(\phi_2 \wedge \phi_3) \quad \longleftrightarrow \quad \begin{bmatrix} 0 & xy & xz \\ -xy & 0 & yz \\ -xz & -yz & 0 \end{bmatrix}$$

Generally, a k-form on $\mathbb{R}^n$ can be interpreted as an $\underbrace{n \times n \times \cdots \times n}_{k \text{ times}}$ array-valued function.

Definition. (Zero Form)
A 0**-form** is a scalar field.

Let $F : A \subseteq \mathbb{R}^n \to \mathcal{L}^k\left(\mathbb{R}^k\right)$, a k-tensor field on A, then for all $x \in A$ and $v_1, \ldots, v_k \in \mathbb{R}^n$

$$F(x)(v_1, \ldots, v_k) \in \mathbb{R}$$

Definition. (Smooth Tensor Fields)
A k-tensor field F on A (open in $\mathbb{R}^n$) is said to be C^r the function $A \times \mathbb{R}^n \times \cdots \times \mathbb{R}^n \to \mathbb{R}$ given by $(x, v_1, \ldots, v_k) \mapsto F(x)(v_1, \ldots, v_k)$ is C^r.

(a) $F : \mathbb{R}^3 \to \Lambda^2(\mathbb{R}^2)$ given by

$$\omega(x,y,z) = xy(\phi_1 \wedge \phi_2) + xz(\phi_1 \wedge \phi_3) + yz(\phi_2 \wedge \phi_3)$$

$$\omega(x,y,z)(v,w) = xy(v_1w_2 - v_2w_1) + xz(v_1w_3 - v_3w_1) + yz(v_2w_3 - v_3w_2)$$

is C^∞ since it is a polynomial.

Remark: It is enough to check the coefficients (of the tensor space basis) are C^r functions.
Meaning, $\omega : A \to \Lambda^k(\mathbb{R}^n)$ we can write

$$\omega(x) = \sum_{I \in \mathrm{ASC}_{k,n}} \omega_I(x) \bigwedge_{j \in [k]} \phi_{I(j)}$$

and ω_I is smooth for each $I \in \text{ASC}_{k,n}$.

Notation: For open $A \subseteq \mathbb{R}^n$, $\Omega^k(A)$ will denote the set of smooth k-forms on A. $\Omega^0(A) = C^\infty(A)$. If $k > n$, then $\Omega^k(A) = \{0\}$. Note that $\Omega^k(A)$ is a vector space.

We will take the convention that all forms are smooth on their domain and worry about continuity in an *ad hoc* manner. This is more fun than always worrying about continuity.

Definition. (Differential of a 0-Form)
Let $A \subseteq \mathbb{R}^n$ be open and $f \in \Omega^0(A)$. Define $df \in \Omega^1(A)$ by

$$df(x) = \sum_{i\in[n]} D_i f(x)\phi_i$$

Example:

(a) Consider $f(x,y,z) = xyz \implies df(x,y,z) = yz\phi_1 + xz\phi_2 + xy\phi_3$. Then

$$df(x)(v) = \sum_{i\in[n]} D_i f(x) v_i = Df(x)\cdot v$$

which is the directional derivative.

(b) The projection function $\pi_i : \mathbb{R}^n \to \mathbb{R}$ given by $\pi_i(x) = x_i \implies d\pi_i = \phi_i$.

Notation: Whenever we see ϕ_i it is often more convenient to write $\phi_i = d\pi_i$ as dx_i. This is a formal notation and has no meaning. So dx_i is the 1-form satisfying $dx_i(e_j) = \delta_{ij}$. The standard basis of $\Lambda^k(\mathbb{R}^n)$ is the set

$$\{dx_{i_1} \wedge \cdots \wedge dx_{i_k} : 1 \le i_1 \le \cdots \le i_k \le n\}$$

(a) We can rewrite $f(x,y,z) = xyz \implies df(x,y,z) = yz\,dx + xz\,dy + xy\,dz$.

Propsotition. (Properties of Differentials)
Differentials obey the following properties:
(i) $d(fg) = g\,df + f\,dg$
(ii) $d : \Omega^k(A) \to \Omega^{k+1}(A)$ is linear.
(iii)

Definition. (Differential of a Form)
Let $A \subseteq \mathbb{R}^n$ be open and $\omega \in \Lambda^k(A)$. We write

$$\omega = \sum_{I\in\text{ASC}_{k,n}} \omega_I \left(\bigwedge_{j\in[k]} dx_{I(j)}\right) \in \Omega^k(A)$$

Define $d\omega \in \Omega^{k+1}(A)$ by

$$d\omega = \sum_{I\in\text{ASC}_{k,n}} (d\omega_I) \wedge \left(\bigwedge_{j\in[k]} dx_{I(j)}\right) \in \Omega^k(A)$$

Example:

$$\begin{aligned} &\omega = xy\,dx + 3\,dy - yz\,dz \in \Omega^1(\mathbb{R}^3) \\ &\Downarrow \\ \text{(a)}\quad & d\omega = d(xy)\wedge dx + d(3)\wedge dy + d(-yz)\wedge dz \\ &\quad = (y\,dx + x\,dy + 0\,dz)\wedge dx + 0\wedge\ dy - (z\,dy + y\,dz)\wedge dz \\ &\quad = -(x\,dx\wedge dy + z\,dx\wedge dz) \end{aligned}$$

$$\begin{aligned} &\omega = (x+z)\,dx\wedge dy - y\,dx\wedge dz + (x^2+y^2)\,dy\wedge dz \in \Omega^2(\mathbb{R}^3) \\ \text{(b)}\quad &\Downarrow \\ &d\omega = (dx+dz)\wedge dx\wedge dy - dy\wedge dx\wedge dz + (dx + 2y\,dy)\wedge dy\wedge dz \\ &\quad = dx\wedge dy\wedge dz \end{aligned}$$

Understanding the differential more carefully we have

$$\begin{aligned} \omega &= \sum_{i\in[n]} \omega_i\,dx_i \in \Omega^1(\mathbb{R}^n) \\ &\Downarrow \\ d\omega &= \sum_{i,j\in[n]} ((Dj\omega_i)\,dx_j)\wedge dx_i \\ &= \sum_{1\le i<j\le n} ((Dj\omega_i - D_j\omega_i)\,dx_i\wedge dx_j) \end{aligned}$$

So this is equivalent to a matrix whose diagonal is zero and ij-entry is $(Dj\omega_i - D_j\omega_i)$. There are n^2 possible partial derivatives of order 2, but we are only choosing $\frac{n(n-1)}{2}$. We will discuss the importance of this choice to follow.

Differential forms vastly generalize the notation of gradient, divergence, and curl one encounters in Calc III.

Definition. (Gradient, Divergence, & Curl)

(1) For $f:\mathbb{R}^n\to\mathbb{R}$, the **gradient** of f is

$$\nabla f = \sum_{i\in[n]} (D_i f)e_i.$$

(2) For $f:\mathbb{R}^n\to\mathbb{R}^n$ the **divergence** of F is

$$\operatorname{Div} F = \nabla\cdot F = \sum_{i\in[n]} D_i F_i$$

(3) For $R:\mathbb{R}^3\to\mathbb{R}^3$ the **curl** of F can be written formally

$$\operatorname{curl} F = \det\begin{bmatrix} e_1 & e_2 & e_3 \\ D_1 & D_2 & D_3 \\ F_1 & F_2 & F_3 \end{bmatrix}$$

So grad maps smooth scalar fields to vector fields. Similarly, $d:\Omega^0(\mathbb{R}^n)\to\Omega^1(\mathbb{R}^n)$. There is a bijection α_1 between vector fields on $\mathbb{R}^n$ and $\Omega^1(\mathbb{R}^n)$ given by $\alpha_1(f) = \alpha_1\left(\sum_{i\in[n]} F_i e_i\right) = \sum_{i\in[n]} F_i\, dx_i$. Also $\alpha_0(f) = f$ maps smooth scalar fields to $\Omega^0(\mathbb{R}^n)$. α_0, α_1 are isomorphisms (of vector spaces). Furthermore, the diagram commutes, (i.e. $d \circ \alpha_0 = \alpha_1 \circ \text{grad}$ or $\text{grad} = \alpha_1^{-1} \circ d \circ \alpha_0$). We say that the gradient operator is equivalent to d modulo conjugation of α_1^{-1} and α_0.

For $\omega \in \Omega^{n-1}(\mathbb{R}^n)$,

$$\begin{aligned}
\omega &= \sum_{i\in[n]} \omega_i \left(\bigwedge_{j\in[n]\setminus\{i\}} dx_j \right) \\
&\Downarrow \\
d\omega &= \sum_{i\in[n]} (D_i\omega_i)\, dx_i \wedge \left(\bigwedge_{j\in[n]\setminus\{i\}} dx_j \right) \\
&= \sum_{i\in[n]} (-1)^{i-1} (D_i\omega_i) \left(\bigwedge_{j\in[n]} dx_j \right)
\end{aligned}$$

$$\begin{bmatrix} \mathrm{Vec}(\mathbb{R}^n) & \\ \mathrm{Div}\downarrow & \\ C^\infty(\mathbb{R}^n, \mathbb{R}) & \end{bmatrix}$$

Put $\beta_n(f) = f \bigwedge_{i\in[n]} dx_i$ and

$$\beta_{n-1}(F) == \sum_{i\in[n]} (-1)^{i-1} (F_i) \left(\bigwedge_{j\in[n]\setminus\{i\}} dx_j \right)$$

For $\omega \in \Omega^1(\mathbb{R}^3)$,

$$\begin{aligned}
\omega &= \omega_1\, dx + \omega_2\, dy + \omega_3\, dz \\
&\Downarrow \\
d\omega &= (D_1\omega_2 - D_2\omega_1)\, dx \wedge dy + (D_1\omega_3 - D_3\omega_1)\, dx \wedge dz + (D_2\omega_3 - D_3\omega_2)\, dy \wedge dz
\end{aligned}$$

Fields & Forms on Manifolds