Honors Analysis II

Math 396

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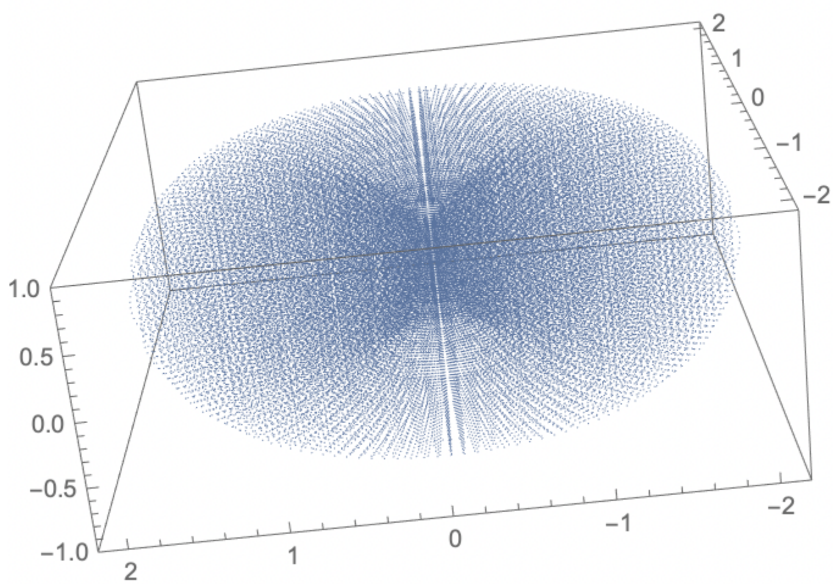

Contents

INTRODUCTION & MOTIVATION

Textbooks:

(i) Munkres, Analysis on Manifolds

(ii) Spivak, Calculus on Manifolds.

(iii) (Possibly) Fourier Analysis, an Introduction.

Content:

Manifolds are k-dimensional objects embedded in ambient n-dimensional space. We will bee interested in integration over manifolds. Next, we will study differential forms are generalizations of functions and vector fields. We will then integrate differential forms on manifolds which will lead us to the celebrated **Stokes Theorem**. Stokes Theorem describes the relationship between the integral over a manifold and its boundary. We will study many classical examples.

DIFFERENTIABLE MANIFOLDS

Motivation

Informally, a **topological manifold** is a topological space that is **homeomorphic** to Euclidean space. This means a manifold looks locally like $\mathbb{R}^n$.

For example, S^1 is a manifold because when we "zoom into" the circle it looks like a line. Also $S^1 \times S^1$ is a manifold because donuts look locally like a plane (see front cover).

We want to do analysis on these manifolds, so we need to add more structure. A **differentiable manifold** is a special type of topological manifold that is "smooth."

Proposition. (Volume of a Parallelepiped)
If $v_1, \dots, v_n \in \mathbb{R}^n$ are linearly independent. The volume of the parallelepiped generated by $v_1, \dots, v_n$ is

$$\pm \det \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix}$$

We want to determine the k-dimensional volume of a parallelepiped determined by k vectors in $\mathbb{R}^n$. The vectors will determine a non-square matrix so we cannot use the determinant.

Definition. (Volume of Parallelepiped)
Let $k \leq n$, Let $M(n,k)$ be the space of $n \times k$ matrices. Define $V : M(n,k) \to [0, \infty)$ by

$$V(X) = \sqrt{\det(X^T X)}$$

Suppose $x_1, \dots, x_k \in \mathbb{R}^n$ are linearly independent. We define the **k-dimensional volume** of the parallelepiped generated by $x_1, \dots, x_k$ by $V(X)$ where

$$X = \begin{bmatrix} | & & | \\ x_1 & \cdots & x_k \\ | & & | \end{bmatrix}$$

This is well defined because $X^T X$ is a positive definite matrix.

Examples:

(i) $k = n$ (should agree with previous proposition)

$$V(X) = \sqrt{\det(X^T X)} = \sqrt{\det(X) \cdot \det(X)} = |\det(X)|.$$

(ii) $k = 1$ (should agree with length of vector)

$$\sqrt{v^T v} = \|v\|.$$

(iii) $k = 2$ and $n = 3$ (should agree with cross product of generators)

$$X = \begin{bmatrix} | & | \\ a & b \\ | & | \end{bmatrix} \implies X^T X = \begin{bmatrix} a^T a & a^T b \\ b^T a & b^T b \end{bmatrix} \implies \det(X^T X) = \det \begin{bmatrix} \|a\|^2 & a \cdot b \\ a \cdot b & \|b\|^2 \end{bmatrix}$$

$$= \|a\|^2 \|b\|^2 - (a \cdot b)^2 = \|a\|^2 \|b\|^2 \sin\theta$$

So $\det(X^T X) = \|a \times b\|^2$.

This implies an interesting fact about the determinant of $X^T X$.

Definition. (Ascending k-tuple)
Let $k \leq n$.

(a) An **ascending k-tuple** from the set $[n]$ is $I = (i_1, \ldots, i_k)$ satisfying $1 \leq i_1 \leq \cdots \leq i_k$.

(b) Denote by $\mathrm{ASC}_{k,n}$ the set of all ascending k-tuples from $[n]$.

So $|\mathrm{ASC}_{k,n}| = \binom{n}{k}$.

Theorem. (Cauchy-Binet Identity)
Let $k \leq n$. If $A \in M(k,n)$ and $B \in M(n,k)$, then

$$\det(AB) = \sum_{\mathrm{ASC}_{k,n}} \det(A^I) \det(B_I)$$

where for $I = (i_1, \ldots, i_k)$, A^I is the $k \times k$ submatrix of A containing the columns $i_1, \ldots, i_k$ and B_I, is the $k \times k$ submatrix of A containing the rows $i_1, \ldots, i_k$.

Corollary. For $k \leq n, X \in M(n,k)$

$$V(X)^2 = \det(X^T X) = \sum_{\mathrm{ASC}_{k,n}} (\det X_I)^2$$

This generalizes the Pythagorean Theorem.

Check directly for a 2×3 matrix.

Proof.
We will prove for $k = 2$ and n arbitrary.

$$\det(AB) = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ \vdots & \vdots \\ b+n1 & b_{n2} \end{bmatrix}$$

So,

$$\begin{aligned}
\det(AB) &= \det \begin{bmatrix} \sum_{i\in[n]} a_{1i}b_{i1} & \sum_{i\in[n]} a_{1i}b_{i2} \\ \sum_{j\in[n]} a_{2j}b_{j1} & \sum_{j\in[n]} a_{2j}b_{j2} \end{bmatrix} && \text{matrix product} \\
&= \sum_{i\in[n]}\sum_{j\in[n]} \det \begin{bmatrix} a_{1i}b_{i1} & a_{1i}b_{i2} \\ a_{2j}b_{j1} & a_{2j}b_{j2} \end{bmatrix} && \text{det is multilinear} \\
&= \sum_{i\in[n]}\sum_{j\in[n]} a_{1i}a_{2j} \det \begin{bmatrix} b_{i1} & b_{i2} \\ b_{j1} & b_{j2} \end{bmatrix} && \text{det is multilinear} \\
&= \sum_{i\in[n]}\sum_{j\in[n]} \delta_{ij} \cdot a_{1i}a_{2j} \det \begin{bmatrix} b_{i1} & b_{i2} \\ b_{j1} & b_{j2} \end{bmatrix} && \text{det is alternating} \\
&= \sum_{i\in[n]}\sum_{i<j} a_{1i}a_{2j} \det \begin{bmatrix} b_{i1} & b_{i2} \\ b_{j1} & b_{j2} \end{bmatrix} + \sum_{i\in[n]}\sum_{i>j} a_{1i}a_{2j} \det \begin{bmatrix} b_{i1} & b_{i2} \\ b_{j1} & b_{j2} \end{bmatrix} && \text{expansion of sum} \\
&= \sum_{i\in[n]}\sum_{i<j} a_{1i}a_{2j} \det \begin{bmatrix} b_{i1} & b_{i2} \\ b_{j1} & b_{j2} \end{bmatrix} + \sum_{j\in[n]}\sum_{j>i} a_{1j}a_{2i} \det \begin{bmatrix} b_{j1} & b_{j2} \\ b_{i1} & b_{i2} \end{bmatrix} && \text{permute } i \text{ and } j \\
&= \sum_{i\in[n]}\sum_{i<j} a_{1i}a_{2j} \det \begin{bmatrix} b_{i1} & b_{i2} \\ b_{j1} & b_{j2} \end{bmatrix} - \sum_{j\in[n]}\sum_{j>i} a_{1j}a_{2i} \det \begin{bmatrix} b_{i1} & b_{i2} \\ b_{j1} & b_{j2} \end{bmatrix} && \text{det is alternating} \\
&= \sum_{i\in[n]}\sum_{i<j} (a_{1i}a_{2j} - a_{1j}a_{2i}) \det \begin{bmatrix} b_{i1} & b_{i2} \\ b_{j1} & b_{j2} \end{bmatrix} && \text{factor} \\
&= \sum_{(i,j)\in \mathrm{ASC}_{2,n}} \det \begin{bmatrix} a_{i1} & a_{i2} \\ a_{j1} & a_{j2} \end{bmatrix} \det \begin{bmatrix} b_{i1} & b_{i2} \\ b_{j1} & b_{j2} \end{bmatrix} && \text{definition of det.}
\end{aligned}$$

Parametrized Manifolds

Manifolds Without Boundary

Manifolds With Boundary

Integration on Manifolds