Math 566

Algebraic Combinatorics

Harrison Centner

University of Michigan

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Contents

Chapter 1
Algebraic Graph Theory

Notes from the Syllabus:
- No Exams.
- There will be five or six problem sets.
- Office Hours: Tuesdays and Friday at 1pm.
- There will be almost no calculus or analysis.
- You should expect this class to make frequent use of linear algebra.
- You need to solve all but two problems correctly to get 100%.
- There will be very little partial credit given.
- You can either bring in a printed copy of the homework to class.

1.1 Linear Algebraic Preliminaries

Definition 1.1.1 (Characteristic Polynomial)

Let M be a $p \times p$ matrix in $\mathbb{C}$, the **monic characteristic polynomial** is

$$\det(tI - M) = \prod_{k=1}^{p} (t - \lambda_k)$$

where $\lambda_1, ..., \lambda_k \in \mathbb{C}$ are the p eigenvalues with multiplicity.

Lemma 1.1.2 (Eigenvalues of Matrix Polynomial)

If $f(t) \in \mathbb{C}[t]$ then $f(M)$ has eigenvalues $f(\lambda_1), ..., f(\lambda_k)$.

Proof.
M is diagonalizable, so conjugation commutes with taking powers and therefore computation of a polynomial. The statement is trivial up to general nonsense consideration.

Diagonalizable matrices are dense in the set of matrices. A matrix is only diagonalizable if there are multiple equal eigenvalues. Thus this is a subvariety within the set of matrices (obtained by imposing an algebraic condition of equal eigenvalues) which has dimension less than the set of matrices.

Now we can take a limit within the set of diagonalizable matrices and the limit converges to the general matrix. $\square$

This statement can be extended to more general functions (those with converging power series).

Lemma 1.1.3 (Trace of Matrix)

The trace of a matrix M is the sum of its eigenvalues.

Proof.
The coefficient of t^{p-1} in $\det(tI - M)$ is

$$-\operatorname{tr}(M) = \sum_{i=1}^{p} -\lambda_k.$$

□

Combining this fact with Lemma 1.1.2 gives us that

$$\operatorname{tr}(M^\ell) = \sum_{k=1}^{p} \lambda_k^\ell.$$

Thus it is easy to compute the sum of powers of eigenvalues of M. This leads to an algorithm of finding roots of polynomials from power sums.
We can recover the multi-set $\{\lambda_1, ..., \lambda_p\}$ from the traces $\operatorname{tr}(M), \operatorname{tr}(M^2), ...$

Theorem 1.1.4 (Multiset Recovery)

Let $\alpha_1, ..., \alpha_r, \beta_1, ..., \beta_s \in \mathbb{C}$ such that $\forall \ell \in \mathbb{Z}^+$

$$\sum_{i=1}^{r} \alpha_i^\ell = \sum_{i=1}^{s} \beta_i^\ell \quad (\star)$$

then $r = s$ and the β's are permutations of the α's.

If the values where over the reals this would be easy since we could just observe the asymptotic behavior and remove each leading term. This cannot be applied to complex numbers since some may have the same modulus

Proof.
We will use the method of generating functions in a noncombinatorial sense. First multiply $(\star)$ by t^ℓ and sum giving,

$$\sum_{i=1}^{r} \frac{\alpha_i t}{1 - \alpha_i t} = \sum_{i=1}^{s} \frac{\beta_i t}{1 - \beta_i t}.$$

We use that $\sum_{k=1}^{\infty} = \frac{x}{1-x}$. Pick $\gamma \in \mathbb{C}$, and multiply by $1 - \gamma t$ and set $t = \frac{1}{\gamma}$. Take the limit $t \to \frac{1}{\gamma}$. Each term will become one or zero depending on whether the corresponding α_k or β_k equals γ or not.

After that, LHS will be the number of α's equal to γ and the RHS will be the number of β's equal to γ. This shows the multisets are equal. □

And what if I put a little more text

1.2 Enumerating Walks with Eigenvalues

We assume all basic vocabulary of basic graph theory. All graphs will be assumed finite. We allow loops & multiple edges.

Lemma 1.2.1 (Walks on a Graph)

Let G be a graph on the vertex set $[p]$. Then put $M = A(G)$, the **adjacency matrix** of G. So M_{ij} is the number of edges from i to j. Then M is symmetric, with entries in $\mathbb{Z}_{\geq 0}$. The number of walks of length ℓ from i to j is $\left(M^{\ell}\right)_{ij}$.

Proof.
Follows from matrix multiplication (also works for directed graphs). □

This suggests that there must be a way of counting walks using eigenvalues of the graph. But we will have to restrict to closed paths. A walk can move anywhere in the graph, a path cannot repeat vertices. A **marked closed walk** is one that starts and begins at the same vertex.

Proposition 1.2.2 (Marked Closed Walks)

The number of marked closed walks of length ℓ on G is $\sum_{k=1}^{p} \lambda_k^{\ell}$.

Proof.
The LHS equals $\operatorname{tr}\left(M^{\ell}\right)$ be observation and Lemma 1.1.3 gives the equality with the RHS. □

Example: Let $G = K_p$ (the complete graph).
Let J be the $p \times p$ matrix with all entries equal to 1. Then $J - I = A(G)$. Then $\operatorname{rank}(J) = 1$. So the eigenvalues of J are $0, ..., 0$ ($p - 1$ times) and p. Now $J - I$ is applying the polynomial $t - 1$ to J. Therefore by Lemma 1.1.2, its eigenvalues are $-1, ..., -1$ ($p - 1$ times) and $p - 1$. Thus the number of marked closed walks in G is

$$(p-1)^{\ell} + (p-1)(-1)^{\ell}.$$

Restatement: The number of marked closed walks of length ℓ in G is the number of $(\ell + 1)$-letter words in a p-symbol alphabet such that consecutive letters are distinct and first letter equals the last.

Homework 1.1

Show that the number of walks of length ℓ between two distinct vertices in K_p differs by one from the number of closed walks of length ℓ starting and ending at a given vertex. Use a simple combinatorial argument.

We can also compute eigenvalues via counting walks. Using combinatorics for linear algebra!

Example: Suppose $G = K_{m,n}$, the complete bipartite graph. Let's count the marked closed walks in G. This number is

$$\left(\sqrt{mn}\right)^{\ell} + \left(-\sqrt{mn}\right)^{\ell} + 0^{\ell} + \cdots + 0^{\ell} = \begin{cases} 0 \text{ if } \ell \bmod 2 = 0 \\ 2(mn)^{\frac{\ell}{2}} \text{ if } \ell \bmod 2 = 1. \end{cases}$$

Now we conclude from Theorem 1.1.4 that $K_{m,n}$ has eigenvalues $-\sqrt{mn}, \sqrt{mn}, 0, ..., 0$. No linear algebra involved!

Homework 1.2

Prove that the **diameter** (the largest minimal pairwise distance of a connected graph is strictly less than the number of distinct eigenvalues.

Inequalities for Maximal Eigenvalues

Proposition 1.2.6 (Max)

Let G be a graph on $[p]$. Denote $\lambda_{\max} = \max_i |\lambda_i|$. The Perron-Frobenius Theorem ensures that the largest eigenvalue is positive (hence equal to $\lambda_{\max}$).

$$\lambda_{\max} \leq \max_{v \in G} \deg(v)$$

Perron-Frobenius holds because a matrix with nonnegative entries can be though of in probabilistic terms (so each row adds to one). Now we have a Markov Chain and taking powers means considering all possible walks in the state-space. As the matrix is raised to higher powers the largest eigenvalue will dominate. The steady state must be positive (since we are taking powers of nonnegative entries) and have eigenvalue one since it is "steady."

Proof.

Informal: As you count your walks your number of choices at each step is at most $\operatorname{maxdeg}(G) = \left(\max_v \deg(v)\right)^{\ell}$. So the inequality must be in this direction since otherwise Lemma 1.2.1 cannot hold.

Formal: For any vector set $x = (x_i)$,

$$\max_j |(A(G)x)_j| = \max_j \left| \sum_{\text{edge } ij} x_i \right| \leq \operatorname{maxdeg}(G) \cdot \max_k |x_k|.$$

Now assume that x is an eigenvector of $A(G)$, with eigenvalue λ. Then this inequality becomes $|\lambda| \cdot \max_k |x_k| \leq \operatorname{maxdeg}(\mathrm{G}) \cdot \max_k |x_k| \Rightarrow |\lambda| \leq \operatorname{maxdeg}(\mathrm{G})$. This is just mathematical legalese (the argument is philosophical). □

Homework 1.3

Prove that

$$\lambda_{\max} \geq \frac{1}{p} \sum_{v \in G} \deg(v).$$

This is like a kind of convexity or Jensson Inequality.

Hint: Use that for symmetric real M,

$$\lambda_{\max} = \max_{|x|=1} x^T M x$$

Corollary: The number of closed walks grows exponentially in ℓ. With rate at least equal to the average degree.

1.3 Eigenvalues of Special Graphs

We now study the eigenvalues of graphs which take certain specified forms.

Block Anti-Diagonal Matrices

Now let's discuss in more detail the bipartite graphs. Whose matrices take the block form $\begin{bmatrix} 0 & * \\ * & 0 \end{bmatrix}$. We will use implicitly the concept of singular values.

Lemma 1.3.2 (Eigenvalues of Block Anti-Diagonal Matrix)

For a real matrix B, $M = \begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix}$ has nonzero eigenvalues $\pm\sqrt{\mu_i}$ where $\mu_1, \mu_2, \ldots$ are the eigenvalues of $B^T B$ counted with multiplicity. Note that $B^T B$ is real, symmetric, and most importantly positive semidefinite. Since $\langle B^T Bx, x \rangle = \langle Bx, Bx \rangle \geq 0$ and the signature of a quadratic form is given by the signs of eigenvalues. Note that the $\mu_1, \mu_2, \ldots$ are the singular values of B.

The proof is going to be the worst kind of proof in mathematics.

For a $p \times p$ matrix X we will denote $F_X(t) := \det\left(tI_p - X\right)$.

Proof.

Suppose B is $n \times m$. Then

$$\begin{bmatrix} tI_n & -B \\ -B^T & tI_m \end{bmatrix} \begin{bmatrix} I_n & B \\ 0 & tI_m \end{bmatrix} = \begin{bmatrix} tI_n & 0 \\ -B^T & t^2 I_m - B^T B \end{bmatrix}.$$

Taking determinants gives

$$F_M(t) \cdot t^m = t^n \cdot F_{B^T B}(t^2).$$

Now we are after the eigenvalues which is precisely when $F_M(t)$ vanishes. Now we know that t^2 is an eigenvalue of $B^T B$ which means that t is a singular value of B. There exists a proof from the book of this fact but it is much longer. □

Example: Let $G = K_{m,n}$. Let B be the $m \times n$ matrix of all ones. Then $B^T B$ is an $m \times m$ matrix of all n's. This is a rank one matrix so it has one nonzero eigenvalue and it is $\operatorname{tr}(A(G)) = nm$. Thus the eigenvalues of $A(G)$ are $\pm\sqrt{mn}, 0, ..., 0$.

Homework 1.4

Let G be the graph obtained by removing n disjoint edges from $K_{n,n}$. Find the eigenvalues of G.

Example: Let G be a $2n$-cycle, thus G is bipartite. Denote $A(G) = M_{2n}$ Then this matrix has the form $\begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix}$. Then $B^T B = 2I_n + M_n$ with the appropriate labeling. This is intuitive because we can return to our original vertex in two ways: counter clockwise or clockwise. [Why?]
This implies that the eigenvalues of G are $\pm\sqrt{2+\lambda_i}$ where λ_i are the eigenvalues of M_n. Now compute the eigenvalues of M_n for small n (say 4). So the eigenvalues of 2^r-cycle are $\sqrt{2 \pm \sqrt{2 \pm \cdots}}$.

Circulant Graphs

We are talking about matrices which are act upon the set of basis vectors as a cyclic group.

Definition 1.3.5 (Circulant Matrix)

A square matrix is called **circulant** provided that

$$C = \begin{bmatrix} s_0 & s_1 & \cdots & s_{p-1} \\ s_{p-1} & s_0 & \cdots & s_{p-2} \\ \vdots & \vdots & \ddots & \vdots \\ s_1 & s_2 & \cdots & s_0 \end{bmatrix}$$

.

Lemma 1.3.6 (Eigenvalues of Circulant Matrices)

$$\lambda_k = \sum_{j=0}^{p-1} s_j e^{\frac{2\pi i}{p} jk}$$

for $k = 0, ..., p-1$.

Remark: Let $s(x) = \sum_{j=0}^{p-1} s_j x^j$, then the eigenvalues of C are the values of $s(x)$ at the p-th roots of unity.

Proof.
Let T be the circulant matrix such that $e_1^T T = [0, 1, 0, ..., 0]$. T acts on the basis $e_1, ..., e_p$ by cyclically shifting. So T^k will again be a circulant matrix. Then $C = s(T)$. Thus finding the eigenvalues of T will give us the eigenvalues of C. The eigenvalues of C are the p-th roots of unity. By observation, $\det(tI_p - T) = t^p - 1$. Also $T^p = I_p$ so then T itself is a p-th root of unity and so it must be that the eigenvalues of T are p-th roots of unity.
The claim follows. □

Definition 1.3.7 (Circulant Graph)

A graph G is **circulant** if $A(G)$ is circulant for some labeling of G.
Corollary: The eigenvalues of the p-cycle are $2\cos\left(\frac{2\pi k}{p}\right)$ for $k = 0, ..., p-1$.

Proof.
Let G be a p-cycle. So $A(G) = T + T^{-1} = T + T^{p-1}$. Nowa, Lemma 1.1.2 implies that the eigenvalues of G are

$$e^{\frac{2\pi i}{p}k} + e^{\frac{2\pi i}{p}k(p-1)} = e^{\frac{2\pi i}{p}k} + e^{-\frac{2\pi i}{p}k} = 2 \cdot \Re\left(e^{\frac{2\pi i}{p}k}\right) = 2\cos\left(\frac{2\pi k}{p}\right).$$

□

Recall: The eigenvalues of a $2n$-cycle are $\pm\sqrt{2 + \mu_i}$ where μ_i is an eigenvalue of the n-cycle. Note that this is consistent with the more general corollary above via the cosine double angle identity.

Homework 1.5

Find the eigenvalues of the graph obtained by removing n disjoint edges from K_{2n}. Note that this graph is the 1-skeleton of the n-dimensional **cross polytope** (i.e. the **hyperoctahedron**).

The hyperoctahedron is the n-dimensional polytope which is dual to the n-cube. Thus it is the convex hull of the set $\{\pm e_i : i \in [n]\}$. The only non-adjacent vertices are the vertices at e_i and $-e_i$. Counting walks in the hyperoctahedron counts walks on the faces of the hypercube.

Cartesian Products of Graphs

Cartesian products is a general categorical construction.

Definition 1.3.10 (Product of Graphs)

Let G, H be graphs. Then $G \times H$ is the **Cartesian product** of G and H with vertex set $V_{G\times H} = V_G \times V_H$ and edges of two kinds $(g, h)(g', h)$ for edge $gg' \in E_G$. and $(g, h)(g, h')$ for edge $hh' \in E_H$.

Example:

- Let $G = \bullet - \bullet - \bullet$ and $H = \bullet - \bullet - \bullet - \bullet$. Then $G \times H$ is the 3×4 grid graph.
- The n-cube graph (skeleton of the n-cube) is isomorphic to $\underbrace{K_2 \times \cdots \times K_2}_{n \text{ times}}$.

Proposition 1.3.11 (Eigenvalues of Direct Product)

Suppose G have eigenvalues $\lambda_1, \lambda_2, \ldots$ and H has eigenvalues $\mu_1, \mu_2, \ldots$ then $G \times H$ has eigenvalues $\lambda_i + \mu_j$ for all i, j.

We will give two proofs. The first is highbrow. The second is elementary.

Proof.

Take two vectors u, v in the space of formal linear combinations of the vertices of G and H (i.e. functions on the vertex space of the graphs). Put $u = \sum \alpha_g g$ and $v = \sum \beta_h h$. Define the tensor product of two vectors

$$u \otimes v \stackrel{\text{def}}{=} \sum_g \sum_h \alpha_g \beta_h (g, h).$$

Then we claim that the matrix

$$A(G \times H)(u \otimes v) = A(G)u \otimes v + u \otimes A(H)v.$$

This is true for the basis vectors of the tensor space $V_G^* \otimes V_H^*$ thus it holds for any u and v by linearity. Now if u and v are eigenvectors with eigenvalues λ and μ for G and H respectively, then, we have

$$A(G \times H)(u \otimes v) = \lambda u \otimes + u \otimes \mu v = (\lambda + \mu)(u \otimes v).$$

So $u \otimes v$ is an eigenvector for $G \times H$ with eigenvalue $\lambda + \mu$. □

Proof.

A marked closed walk of length ℓ in $G \times H$ is a **shuffle** (i.e. interleaving) of marked closed walks of length k and $\ell - k$ in G and H, respectively. Hence, the number of such walks is

$$\sum_{k=0}^{\ell} \binom{\ell}{k} \left(\sum_i \lambda_i^k \right) \left(\sum_j \mu_j^{\ell-k} \right) = \sum_i \sum_j \sum_k \binom{\ell}{k} \lambda_i^k \mu_j^{\ell-k} = \sum_i \sum_j (\lambda_i + \mu_j)^{\ell}.$$

Where the first equality follows from nontrivial summation rearrangement and the second follows from the binomial theorem. Now by Theorem 1.1.4 we have found the eigenvalues. □

The first proof is better because there is no computation. Also the second proof does not give us the eigenvectors.

Homework 1.6

Find the number of marked closed walks of length ℓ in the 3×3 grid graph. Note that this graph is bipartite so we should get zero for odd ℓ.

Homework 1.7

Find the eigenvalues of the 1-skeleton of an octagonal prism. Note $G = C_2 \times C_8$.

Eigenvalues of the Cube

Let $Q_n = (K_2)^n$ be the 1-skeleton of the n-cube.

Proposition 1.3.15 (Eigenvalues of Cube Graph)

The eigenvalues of Q_n are

$$\left\{ \left(n - 2k; \binom{n}{k} \right) : k = 0, ..., n \right\}.$$

Corollary: The number of marked closed walks in Q_n of length ℓ is

$$\sum_{k=0}^{\ell} \binom{n}{k} (n - 2k)^{\ell}.$$

Also this number must be divisible by 2^n and zero for odd ℓ since Q_n is bipartite.

Proof.

The eigenvalues of K_2 are ± 1. Now recalling Proposition 1.3.11, we see that the eigenvalues of Q_n are all possible sums of the form $\underbrace{\pm 1 \pm 1 \pm 1 \cdots \pm 1}_{n \text{ times}}$. $\square$

Homework 1.8

Find a direct proof of the formula using generating functions.

Now we have enough background to consider **random walks** in the cube. We will consider Brownian motion within a graph. When G is a **regular** graph of degree d on p vertices, a **simple** random walk on G proceeds by choosing (uniformly at random) an adjacent vertex and moving into it.

Instead of regularity, we will assume the stronger condition that any two vertices are related by an automorphism of G, meaning the group of automorphisms of G acts transitively on G.

Assuming that the random walk originates at v,

$$\begin{aligned}\mathbb{P}[\text{after } \ell \text{ steps we return to } v] &= d^{-\ell} \cdot \#\{\text{closed walks of length } \ell \text{ starting at } v\} \\ &= \frac{1}{d^\ell p} \cdot \sum_{i=1}^{p} \lambda_i^\ell .\end{aligned}$$

Example: For Q_n we put $d = n$ and $p = 2^n$, the probability of return after ℓ steps is

$$\mathbb{P}[\text{return after } \ell \text{ steps}] = \frac{1}{n^\ell 2^n} \sum_{k=0}^{n} \binom{n}{k} (n-2k)^\ell .$$

Homework 1.9

Find the analogous probability of return for the discrete torus with nm vertices (the product of an n-cycle and an m-cycle). Alternatively, this is the quotient of the grid graph by its “boundary” graph.

Note that the discrete torus satisfies the automorphism transitivity assumption.

Eigenvalues of a Chain Graph

Let G be the n-cycle with one edge removed. This graph is not regular.

$$A \overset{\text{def}}{=} A(G) = \begin{bmatrix} 0 & 1 & & & \\ 1 & 0 & 1 & & \\ & 1 & 0 & 1 & \\ & & & \ddots & \\ & & & & \end{bmatrix}$$

Proposition 1.3.19 (Eigenvalues of Chain Graph)

The eigenvalues of A_n are $2 \cdot \cos\left(\frac{\pi k}{n+1}\right)$.

Proof.

The characteristic polynomial of A_n is $T_{n(t)} = \det(tI_n - A_n)$. It turns out this determinant is closely related to Chebyshev Polynomials.

Consider a more general determinant (of Toeplitz Matrices)

$$h_n(a,b) = \begin{bmatrix} a+b & -ab & & \\ -1 & a+b & -ab & \\ & -1 & a+b & -ab \\ & & & \ddots \\ & & & \end{bmatrix}.$$

Exploring small values we see that

$$\begin{aligned} h_1(a,b) &= a+b \\ h_2(a,b) &= a^2+ab+b^2 \\ h_3(a,b) &= (a+b)^3 - 2ab(a+b) \\ &= a^3+a^2b+ab^2+b^3. \end{aligned}$$

This suggests the pattern that

$$h_{n(a,b)} = \sum_{k=0}^{n} a^{n-k}b^k = \frac{a^{n+1}-b^{n+1}}{a-b}.$$

We will use a recurrence to prove this formula. We expand the determinant in the last row

$$h_n(a,b) = (a+b)h_{n-1}(a,b) - ab \cdot h_{n-2}(a,b).$$

The proof follows from induction on n. All that must be done is to plug the formula into the recurrence and check that it holds. This trick of expanding the determinant into a recurrence is common for (tri)-diagonal matrices.

Now set $b := \frac{1}{a}$. Then

$$T_n(a+a^{-1}) = h_n(a,a^{-1}) = \frac{a^{n+1}-a^{-(n+1)}}{a-a^{-1}}.$$

This vanishes when $a^{2(n+1)} = 1$ but $a^2 \neq 1$. That is, $a = e^{\frac{2\pi ik}{2(n+1)}} = e^{\frac{\pi ik}{n+1}}$ for $k \in [n]$ since 1 and $n+1$ are forbidden by our vanishing rules.

Summing $t = a + a^{-1} = 2 \cdot \cos\left(\frac{\pi k}{n+1}\right)$. $\square$

Chapter 2

Tilings, Trees, and Networks

2.1 Domino Tilings

Definition 2.1.1 (Domino Tiling)

A **domino tiling** is a tiling of the plane $\mathbb{Z} \times \mathbb{Z}$ by 2×1 rectangles. A tiling is **cryptomorphic** to a perfect matching in on $\mathbb{Z} \times \mathbb{Z}$.

This may seem like a recreational mathematics question, but it arises in applications. The most common is statistical mechanics. Consider the center of unit squares as locations in a crystal (think NaH). So this is related to dimer models and the ising model. This problem was solved in the late 1950s by physicists.

The Permanent-Determinant Method

The method we will consider was invented by Peter W. Kastelen (1960). We will consider an $n \times m$ board and suppose n is even. Let G be the bipartite graph which is the product of the m-chain and the n-chain. Put

$$M := A(G) = \begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix}.$$

Where B is an $\frac{m}{2} \times \frac{m}{2}$ matrix. Choosing a perfect matching in G amounts to choosing exactly one nonzero entry from each row and column in B. This should perk your ears and make you think of a determinant.

Definition 2.1.3 (Permanent)

The permanent of an $n \times n$ matrix A is

$$\operatorname{per}(B) = \sum_{\sigma \in S_n} \prod_{i \in [n]} a_{i,\sigma(i)}.$$

This is notoriously difficult computationally (should be exponentially hard).

Let $T(m, n)$ be the number of perfect tilings in G. Perfect matchings in G are in bijection with nonvanishing terms in $\det(B) = \operatorname{per}(B)$.

The key trick is to turn these permanents into determinants. Let $\tilde{B}$ be the matrix obtained from B by replacing the 1s that correspond with <u>vertical</u> tiles with i.

Lemma 2.1.4 (Determinant Permanent Equivalence)

Kastelen proved that

$$\det\left(\tilde{B}\right) = \pm\ \mathrm{per}(B) = \pm T(m,n).$$

The homework problem below is due to Bill Thurston. "This is a problem for a very very smart ten year old."

Homework 2.1

Show that any two tilings of a rectangular board can be connected by a sequence of "flips."

One way of showing this is by connecting every tiling to a canonical tiling.

We will use this result to show Lemma 2.1.4.

Proof.

In a vertical 2×2 block we have the submatrix of those four entries looks like $\begin{bmatrix} i & * \\ * & i \end{bmatrix}$. A flip then takes it to $\begin{bmatrix} * & 1 \\ 1 & * \end{bmatrix}$. So the flip replaces a product $\cdots 1 \cdot 1 \cdots$ with $\cdots i \cdot i \cdots$ and these products are equal up to exactly one transposition. Thus the term in the permanent and the term in the determinant are equal. The horizontal tiling is a canonical tiling which gives a real term in the determinant equal to ± 1. Now using Homework 2.1 we are done. □

Now we are prepared to enumerate the number of domino tilings in G. Put

$$\widetilde{M} := \begin{bmatrix} 0 & \tilde{B} \\ \tilde{B}^T & 0 \end{bmatrix} \Rightarrow \det\left(\widetilde{M}\right) = \pm \det\left(\tilde{B}\right)\det\left(\tilde{B}^T\right) = \pm(T(m,n))^2$$

by Lemma 2.1.4.

Recall that A_n is the adjacency matrix of an n-chain. Then

$$\widetilde{M} = I_m \otimes A_n + iA_m \otimes I_n.$$

By Proposition 1.3.11, this implies that the eigenvalues of $\widetilde{M}$ are sums of eigenvalues of A_n and A_m. Recall that the eigenvalues of A_n are $2 \cdot \cos\left(\frac{\pi k}{n+1}\right)$ for $k \in [n]$. We now conclude that,

$$\begin{aligned} \det\left(\widetilde{M}\right) &= \prod_{j \in [n]} \prod_{k \in [m]} \left(2 \cdot \cos\left(\frac{\pi j}{n+1}\right) + 2i \cdot \cos\left(\frac{\pi k}{m+1}\right)\right) \\ &= \pm \prod_{j \in \left[\frac{n}{2}\right]} \prod_{k \in [m]} \left(4\cos^2\left(\frac{\pi j}{n+1}\right) + 4\cos^2\left(\frac{\pi k}{m+1}\right)\right) \end{aligned}$$

The first equality follows since n is even, $n+1$ is odd and so the cos values from the RHS come in pairs which cancel out each others imaginary parts.
Since $T(m,n) = \sqrt{|\det(\widetilde{M})|}$, we obtain: a theorem by P. Kasteleyn (1961), M. Fisher and N.Temperley (1961).

Theorem 2.1.6 (Enumeration of Domino Tilings)

$T(m,n)$ has two formulas. If m is odd then,

$$T(m,n) = \prod_{j\in\left[\frac{n}{2}\right]} 2\cos\left(\frac{\pi j}{n+1}\right) \prod_{k\in\left[\frac{m-1}{2}\right]} \left(4\cos^2\left(\frac{\pi j}{n+1}\right) + 4\cos^2\left(\frac{\pi k}{m+1}\right)\right)$$

If m is even then, we get

$$T(m,n) = \prod_{j\in\left[\frac{n}{2}\right]} \prod_{k\in\left[\frac{m}{2}\right]} \left(4\cos^2\left(\frac{\pi j}{n+1}\right) + 4\cos^2\left(\frac{\pi k}{m+1}\right)\right)$$

<u>Note</u>: $T(8,8) = 12,988,816 = 3604^2$. It is a difficulty theorem to prove, but for any square board whose length is divisible by four, the number of tilings is a perfect square. And otherwise if it is an square board that can be tiled, then it will be twice a perfect square.
But the physicists don't care about this they want to know the asymptotics.
Let $m = n$. Let's take the log

$$\begin{aligned}
\frac{\log T(n,n)}{n^2} &= \frac{1}{n^2} \sum_{j\in\left[\frac{n}{2}\right]} \sum_{k\in\left[\frac{n}{2}\right]} \log\left(4\cos^2\left(\frac{\pi j}{n+1}\right) + 4\cos^2\left(\frac{\pi k}{m+1}\right)\right) \\
&\sim \frac{1}{\pi^2} \sum\sum \left(\frac{\pi}{n+1}\right)^2 \log\left(4\cos^2\left(\frac{\pi j}{n+1}\right) + 4\cos^2\left(\frac{\pi k}{m+1}\right)\right) \\
&\sim \frac{1}{\pi^2} \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \left(\frac{\pi}{n+1}\right)^2 \log\left(4\cos^2\left(\frac{\pi j}{n+1}\right) + 4\cos^2\left(\frac{\pi k}{m+1}\right)\right) dxdy \\
&= \frac{K}{\pi}
\end{aligned}$$

So now we have a Riemann sum (we are multiplying the area of the square $4\cos^2 x + 4\cos^2 y$ evaluated at the point $\left(\frac{\pi j}{n+1}, \frac{\pi k}{n+1}\right)$.
Where K is the Catalan Constant

$$K = 1 - \frac{1}{9} + \frac{1}{25} - \frac{1}{49} + \cdots = \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{t}{\sin(t)} dt$$

So $T(n,n) \sim 1.34^{n^2}$. To the physicists it is obvious that the complexity is c^{n^2} for some $c \in \mathbb{R}^+$. This is because placing two dominoes are independent in expectation.

2.2 Spanning Trees

The following theorem is due to H.N.V. Temperley (1974).

Theorem 2.2.1 (Tilings & Spanning Trees)

The number of domino tilings of the $(2k-1) \times (2\ell-1)$ rectangle without a corner box is equal to the number of spanning trees of the $k \times \ell$ grid graph.

Homework 2.2

Prove that this rule produces a tree. Something connected and acyclic. (A cycle requires an odd number of squares)

Proof.

for each domino which covers a node with two odd coordinates, draw a segment in the direction it is pointing. This will create a spanning tree by Homework 2.2.

For the other direction, direct each edge in the spanning tree to the node in the missing corner. Then draw dominoes in the dual graph. The remaining area uncovered by the dominoes splits into a rooted forest in the dual graph of the original board. Each tree in the forest grows from (is rooted) somewhere on the boundary of the dual graph. This gives us a unique domino placement for the uncovered areas. Since if the uncovered area contained a cycle, then the original tree would not be spanning (or maybe not a tree). □

Homework 2.3

Prove that if we remove a different box from the boundary, we get the same count (so long as a tiling is possible). You should build this bijection via a composition of two bijections.

Corollary 2.2.4 (Asymptotics of Spanning Trees)

The number of spanning trees in an $n \times n$ grid is $\sim e^{\frac{4K}{\pi}n^2} \approx 2.24^{n^2}$.

We can generalize to arbitrary planar graphs. Suppose P is a polygon on $\mathbb{R}^2$. Subdivide P into smaller polygons (not necessarily convex) to get a graph G. This description gets rid of annoying planar graphs (those have vertices of degree one, unconnected, etc.). Obtain a new graph H by placing a vertex inside each face of G and the midpoint of each edge of G. For each face F, connect each vertex in the F to those on the midpoints of the edges of F.

Homework 2.3

Fix a vertex v of P (hence of G and H). Show that the number of spanning trees on G is equal to the number of perfect matchings on $H \setminus \{v\}$.

Example: If G is the graph on the pentagon labeled cyclically with $[5]$ and adding the edge $\{1,3\}$ then there are $4+4+3=11$ spanning trees. Then H is isomorphic to the 4×3 grid graph which also has 11 perfect matchings.

Chapter 3

The Diamond Lemma

3.1 Statement and Preliminaries

The Diamond Lemma is a general mathematical argument. There aren't that many: induction, pigeon-hold principle, and switching summations.

Definition 3.1.1 (One Player Game)

A **one-player game** is defined by the set of positions S, for each position $s \in S$, there is a set of positions $s' \neq s$ into which a player can move. We will use notation $s \rightsquigarrow s'$. If this set is empty, then s is called **terminal**.

Definition 3.1.2 (Play Sequence)

A **play sequence** is a sequence of the form $s \rightsquigarrow s' \rightsquigarrow s'' \rightsquigarrow \cdots$ (either finite or infinite). A **terminating game** is one without an infinite play sequence. A game is **confluent** if its outcome is determined by the initial position.

Lemma 3.1.3 (Diamond Condition)

Suppose that a game is terminating and satisfies $(\diamondsuit)$ $\forall s \in S$ and any $s \rightsquigarrow s'$ and $s \rightsquigarrow s''$, there exists a $t \in S$ such that $s' \rightsquigarrow \cdots \rightsquigarrow t$ and $s'' \rightsquigarrow \cdots \rightsquigarrow t$. Then the game is confluent.

Note that if a game is confluent, then the game satisfies $(\diamondsuit)$.

Proof.

Color each position s red if there exists two different terminal positions reachable from s. Color each position green otherwise. We want to show that there are no red positions. We know that every terminal position is green. Suppose s is red. Then make moves into red positions while we can. Since the game is terminating, at some point we must choose a terminal position. Thus we will have $s \rightsquigarrow s$‘ for all $s \rightsquigarrow s'$. Since s is red, there must be distinct $s \rightsquigarrow s'$ $s \rightsquigarrow s''$. But since s', s'' are green then they have unique terminal end points t_1, t_2. But now invoking $(\diamondsuit)$ shows that $t_1 = t_2$ so s', s'' cannot be green. Contradiction! $\square$

Typically, we will use the Diamond Lemma to show something is well defined.

3.2 Applications to Terminating Games

Definition 3.2.1 (Chip Firing)

Let $G = (V, E)$ be a finite directed simple graph. Let t be a **sink**, reachable from any vertex in G. A **chip configuration** is a placement of a nonnegative integer x_v at each $v \in V$. A move in the chip firing game occurs by choosing a $v \neq t$ satisfying $x_v \geq \text{outdegree}(v)$, then "fire" by sending one chip along each outward edge from v.

So after a move,

$$x_v := x_v - \text{outdeg}(v) \quad x_w := x_w + 1$$

$\forall w$ incident to v.

$3 \to 3 \to 3 \to 0$

Theorem 3.2.2 (Chip Firing is Confluent)

Chip Firing is confluent.

Proof.

We employ the Diamond Lemma. Suppose there is a position with $v, v' \in V$ which can both be fired. The order which we fire v and v' does not matter, so the fires at v and v' commute $\Rightarrow$ ($\Diamond$). $\square$

Thus any game will have the same number of moves.

Definition 3.2.3 (Young Diagram)

A **Young Diagram** is a sequence of unit boxes arranged in tabular format which monotonically decrease in length. Young Diagrams of size n are in bijection with integer partitions of n.

Definition 3.2.4 (Skew Shape)

A **Skew Diagram** is obtained from a Young Diagram via removing a left aligned sub-(Young Diagram). Alternatively a Skew Shape can be defined as a convex region of $\mathbb{Z} \times \mathbb{Z}$ where convexity is understood as (having empty squares when top left and bottom right corners are missing).

Think of Young Diagram as chocolate bars and Skew Shapes as bitten Chocolate bars and Cores as nuts (or maybe bolts that would break your teeth).

Definition 3.2.5 (Cores)

The positions of the game are Young Diagrams and the moves consist of removing dominoes from the southeast. The 2-**core** of a Young Tableaux is the remainder after the removal game has ended. Cores are uniquely determined, because taking non-overlapping bites commutes (and two overlapping bites can result in the same aftermath if the whole 2×2 square that the dominoes occupy is removed).

Fix $p \in \mathbb{Z}_{\geq 0}$. Modify the game by only allowing removals of border strips of length p. The border is the set of Young Diagram is the set of unit boxes which have no southeast neighbor.

Homework 2.4

Show that the generalized Core game is confluent. So the p-core of a Young Diagram is well defined.

"A curious ten year-old would be impressed by this ... of course a regular ten year-old would not care."

We did a quick grammar lesson on when to use "well defined" and "well-defined."

Now it is natural to ask how to describe the uniquely determined output from a game that satisfies the Diamond Lemma.

Note that tableaux is the plural of tableau.

Definition 3.2.7 (Standard Young Tableaux)

The **Standard Young Tableau** of a skew shape is given by filling it with numbers in $[n]$ so that the numbers increase along the rows and columns.

These Young Tableaux play important roles in the representation of linear groups.

The following game was introduced by M.-P. Schutzenberger (~1970).

Definition 3.2.8 (Jeu De Taquin or The Teasing Game)

Let the positions of the game be Standard Young Tableaux. Let the moves be "slides."

Homework 2.5

Show that Jeu De Taquin is confluent. This problem is not easy.

Tutte was involved in solving the Enigma code. The Tutte Polynomial is an invariant of graphs. The simplest definition involves a recursive definition involving edge deletion and contraction. This recursion depends on an arbitrary choice (but it is still well defined). Look it up on Wikipedia.

Homework 2.6

Show the Tutte Polynomial is well defined.

3.3 Applications to Non-Terminating Games

A one-player game is confluent if, for any initial position, either

- the game never terminates (no matter how it's played)
- it always arrives at the canonical terminal position.

The old version of the Diamond Lemma (Lemma 3.1.3) is no longer true. Consider the game on the states $\mathbb{Z}^+ \cup \{\infty\}$, with arrows from $n \to n+1$ and $n \to \infty$ for all $n \in \mathbb{Z}^+$. The diamond condition is satisfied, yet the game is not confluent (in the old sense).

Homework 2.7

Lemma 3.3.2 (The Diamond Lemmaa)

Suppose that $\forall s \in S$ and any $s \rightsquigarrow s'$ and $s \rightsquigarrow s''$, $\exists t \in S$ and paths $s' \rightsquigarrow \cdots \rightsquigarrow t$ and $s'' \rightsquigarrow \cdots \rightsquigarrow t$ with the same number of steps. Then, the game is confluent. Moreover, if it terminates (for a given initial position), then it arrives at a canonical terminal position in a fixed (predetermined) number of steps.

Meaning, the poset is graded.

Loop-Erased Walks

This definition is due to G. Lawler (1980) at UChicago. Unfortunately, Greg is a probabalist not a combinatorialist, so a better name would be cycle-erased walks.

Definition 3.3.3 (Loop-Erased Walk)

Let G be a connected graph. Let π be a (finite) walk in G. The **loop-erasure** of π denoted $\mathrm{LE}(\pi)$ is a walk defined recursively by

if π doesn't visit any vertex more than once, *then* $\mathrm{LE}(\pi) = \pi$.

else Remove the first cycle that π makes to get π' and set $\mathrm{LE}(\pi) = \pi'$.

A very good leetcode problem would be to find the loop erased walk of a walk.

Markov Chains are an elementary concept, but running them with stacks is a very little known technology. This will make the Markov process run deterministically.

Definition 3.3.4 (Running a Markov Chain with Stacks)

Suppose M is a **Finite Markov Chain** (stationary i.e., transition probabilities do not depend on time). For each state s, decide in advance (by making appropriate random choices) where the chain will move from s after visiting s for the n-th time. This creates a **stack** $u(s) = (u_1, u_2, ...)$ at s.

Assumption: A Markov Chain arrives, with probability 1, at a (possibly non-unique) absorbing state. The stack at such a state is garbage (or simple empty).

Definition 3.3.5 (Cycle Popping)

Consider a set of partially depleted stacks. Their tops determine a directed graph with edges $s \to s'$ colored by $\mathbb{Z}^+$, according to the number of visits to the s that have occurred. The out degree of every vertex is one. Removing the tops from the stacks lying on a directed cycle in this graph is called **cycle-popping**.

Observation: Loop erasure can be viewed as a sequence of cycle poppings. [Why?]

Lemma 3.3.6 (Cycle Popping Game is Confluent)

The Cycle Popping Game is confluent.

Proof.

Follows immediately from Lemma 3.3.2. ☐

3.4 Wilson's Algorithm

Definition 3.4.1 (Wilson's Algorithm)

The algorithm proceeds as follows:

Input: connected loopless graph G.

Output: random spanning tree T in G.

Fix a vertex r in G (a *root*).
Put $T := \{r\}$.
while T doesn't span G **do**
  pick a vertex $v \notin T$;
  from v run a simple random walk π **until** π hits T;
  append $\mathrm{LE}(\pi)$ to T;
return T;

This is a very involved process for choosing a spanning tree. The following result is from D.B. Wilson (1996) while Prof. Fomin was his instructor at MIT.

Theorem 3.4.2 (Wilson's Theorem)

Wilson's Algorithm outputs a uniformly distributed random spanning tree.

By uniformly, we mean for any spanning tree the probability that Wilson's Algorithm produces that spanning tree is the same.

There is a huge subject in Theoretical Computer Science called generation of combinatorial objects. Monte Carlo algorithms heavily depend on these generation procedures. For example choosing a k-element subset from an n element set. Volume II of Knuth's *Art of Computer Programming* concerns this for a major portion.

Proof.

Make r (later, all of T) into an absorbing state. Run the algorithm with stacks. We know that loop erasure amounts to cycle popping. The algorithm terminates with probability 1. The output is the (uniquely defined by the stacks) tree lying underneath all the pop-able cycles.

For simplicity, assume that v is chosen using some deterministic rule. For any spanning tree T of G and any "heap" H of colored cycles, we could possibly remove them, let $P(T, H)$ be $\mathbb{P}[\text{we end up with } H\&T]$. Then,

$$P(T,H) = \left(\prod_{v\in H} \frac{1}{\deg(v)}\right)\left(\prod_{v\in T\setminus\{r\}} \frac{1}{\deg(v)}\right)$$

Since for each chip there is exactly one direction we want it to point. So we just need to remove enough chips until that direction is uncovered.

Now the key observation is that the expression above does not depend on T! This implies that $\mathbb{P}[T]$ (Wilson's Algorithm outputs T) and

$$\mathbb{P}[T] = \sum_H P(T, H).$$

$\therefore$ The we don't depend on T. $\square$

The following exercise was proved five years before Wilson's result.

Homework 2.9

Suppose G has vertices a, b. Consider two random experiments.

1. Run a simple random walk π from a until it hits b and output $\mathrm{LE}(\pi)$.
2. Generate a uniformly random spanning tree T in G and output the unique path in T connecting a and b.

Prove that the outputs of (1) and (2) are identically distributed.

Corollary 3.4.4 (Wilson's Corollary)

The distribution in the exercise above remains the same if we swap a and b.

This corollary is surprising because the loop erasure of a path $a \to b$ is not the same as the loop erasure of the same path $b \to a$.

Example: in the four cycle the probability of connecting two adjacent vertices uniformly at random is $\frac{3}{4}$ the short way around and $\frac{1}{4}$ the long way around. Now finding the same probability according to (2) is the same at looking at all four spanning trees and then noting that only one contains the long way around.

Chapter 4

Electrical Networks

4.1 Flows and Edge Weighted Graphs

From now on, G is a connected loopless graph. Designate two vertices $a, b \in G$ as boundary vertices and the rest are interior vertices. We will run some commodity (that is infinitely divisible) through our wires/pipes.

Definition 4.1.1 (Flow)

A **flow** f in G is a function that assigns a real number $f(e,u,v)$ to every edge e connecting $u-v$, so that
1. $f(e,u,v) = -f(e,v,u)$
2. for each interior vertex u,

$$\sum_{u \overset{e}{-} v} f(e,u,v) = 0.$$

Thus $\exists!$ number $|f|$ called the **total flow** from a to b such that

$$\sum_{u \overset{e}{-} v} f(e,u,v) = \begin{cases} |f| \text{ if } u = a, \\ -|f| \text{ if } u = b, \\ 0. \end{cases}$$

Note that total flow may be negative.

Definition 4.1.2 (Weighted Graph)

A **weighted graph** or **network** is a graph G with a function on its edges w. Assign to each edge $e \in G$ a positive weight in $\mathbb{R}$ (which we secretly call **conductance**) and denote

$$w(u) = \sum_{u \overset{e}{-} v} w(e).$$

We now study electrical networks. We could give edges complex weights which would give us the theory of solenoid networks.

Definition 4.1.3 (Weighted Random Walk)

The **weighted randoom walk** on (G, w) chooses each edge e incident to a current vertex u with probability $\frac{w(e)}{w(u)}$, then moves to v. Thus the transition probability is given by

$$\mathbb{P}[u \rightsquigarrow v] = \sum_{u \overset{e}{-} v} \frac{w(e)}{w(u)}.$$

The following homework is optional.

Homework 3.0

Generalize Wilson's Theorem to weighted graphs. So the probability of a particular spanning tree will be

$$o(1) \cdot \sum_{e \in T} w(e).$$

4.2 Resistor Networks and Kirchhoff's Laws

Think of (G, w) as a resistor network and $w(e)$ is the conductance of edge $e = \frac{1}{\text{resistance}(e)} \Omega^{-1}$ (where $\Omega =$ Ohms). We are talking about direct current here. The first statement is the conservation of energy.

Proposition 4.2.1 (Kirchhoff's First Law)

$\exists!$ number $|f|$ called the **total flow** from a to b such that

$$\sum_{u \overset{e}{-} v} f(e, u, v) = \begin{cases} |f| \text{ if } u = a, \\ -|f| \text{ if } u = b, \\ 0. \end{cases}$$

Proposition 4.2.2 (Kirchhoff's Second Law)

For each ~~closed walk~~ cycle in G,

$$u_0 \overset{e_1}{-} u_1 \overset{e_2}{-} \cdots - u_{k-1} \overset{e_k}{-} u_k = u_0$$

we have

$$\sum_{i=1}^{k} \frac{f(e_i, u_{i-1}, u_i)}{w(e_i)} = 0.$$

Each of the terms is the product of resistance and _ so this gives us potential energy. We want this axiom because you cannot have perpetual motion machines and your electric field should be conservative.

One observes that Kirchhoff's equations are linear in the variables $f(e, u, v)$ (assuming that $|f|$ is given) and almost all equations are homogeneous. Therefore,

Theorem 4.2.3 (Uniqueness of Flow)

Kirchhoff's Equations have a unique solution (for a given $|f|$).

To a physicist this is patently obvious since only the initial condition can have an effect, since there is no randomness involved. This is predicated on the assumption that the "real world objectively exists" which might be an illusion.

Definition 4.2.4 (Potentials)

Let f be a function satisfying Proposition 4.2.2. Define a **potential** $p = p_f$ on the vertices of G:

1. Assign arbitrary values $p(a)$ to the boundary (choose a value for ground)
2. For any walk $a = u_0 \overset{e_1}{-} u_1 \overset{e_2}{-} \cdots - u_{k-1} \overset{e_k}{-} u_k = u$, set

$$p(u) \overset{\text{def}}{=} p(a) + \sum_{i=1}^{k} \frac{f(e_i, u_{i-1}, u_i)}{w(e_i)}.$$

Remark that $p(u)$ is unambiguously defined, thanks to Proposition 4.2.2. Since concatenating two paths in opposite directions will give two paths whose potentials sum to zero.

Now we will rewrite the Kirchhoff equations in terms of potentials.

Proposition 4.2.5 (Ohm's Law)

If f satisfies Proposition 4.2.2 then $p = p_f$ satisfies, for every edge $u \overset{e}{-} v$ the equation

$$\frac{f(e, u, v)}{w(e)} = p(v) - p(u).$$

Proof.

Immediate from rewriting sums. □

We can start with an arbitrary function p and define f via Proposition 4.2.5. Then Proposition 4.2.2 is automatic. [Why?] It basically follows because the sums will telescope. This is a discrete Fundamental Theorem of Calculus.

Proving Proposition 4.2.1 is not automatic however (not even true). Specifically, for internal vertices Proposition 4.2.1 is equivalent to

$$0 = \sum_{u \overset{e}{-} e} v)w(e)(p(v) - p(u)).$$

This not true when picking p willy-nilly. Remember that $p(u)$ is fixed because we are looking at all edges out of u. So

$$\iff \sum_{u \overset{e}{-} v} w(e)p(v) = p(u)w(u)$$

$$\iff p(u) = \sum_{u \overset{e}{-} v} \frac{w(e)}{w(u)} p(v)$$

This is saying that the value at every vertex is equal to the arithmetic mean of the surrounding edges. Thus the potential is a **discrete harmonic function** meaning its value at each internal vertex is the weighted average of its values neighboring vertices. For a **conservative flow** (i.e. $|f| = 0$), the potential is harmonic everywhere.

Lemma 4.2.6 (The Maximum Principle)

A (real) discrete harmonic function on the vertices of a connected loopless graph G attains its extrema at the boundary vertices a & b. In the absence of boundary, (the flow is conservative) the function must be constant.

Proof.

Immediate. □

Now we can prove Theorem 4.2.3.

Proof.

Write the equations in terms of potential. If $\exists$ two distinct solutions, then their difference is harmonic everywhere (including the boundary). Hence this difference is constant and consequently zero. This implies that f is unique. □

Now we will prove existence constructively, say for $|f| = 1$.

Let π be the random walk on G (no loops) which starts at a, stops at b, and chooses an edge from $u \overset{e}{-} v$ with probability $\frac{w(e)}{w(u)}$. Denote by $\pi(t)$ the (random) vertex occupied by π at time t. Thus $\pi(0) = a$. Define the Bernoulli random variables

$$\xi_t(e, u, v) = \begin{cases} 1 \text{ if } \pi(t) = u \text{ and proceeds along } e, \\ 0 \text{ otherwise.} \end{cases}$$

This is a fairly unlikely event. Note that

$$\sum_t \mathbb{E}[\xi_t(e,u,v)] = \frac{w(e)}{w(u)} \sum_t \mathbb{P}[\pi(t) = u] < \infty.$$

The first equality follows since we need to be at u then move. So this says that we visit each vertex u on average a finite number of times. This claim follows since the probability of returning to any vertex decays exponentially. The probability of returning is less than one. This follows since we stop if we get to b and the graph is connected, so there is a nonzero probability of hitting b before returning.

Theorem 4.2.7 (Existence for Kirchhoff's Equations)

The function f defined by

$$f(e,u,v) = \sum_t \mathbb{E}[\xi_t(e,u,v)] - \sum_t \mathbb{E}[\xi_t(e,v,u)]$$

satisfies Proposition 4.2.1 and Proposition 4.2.2 with $|f| = 1$.

"You might be able to think about how you can do this without proof, but I certainly doubt that you can."

Proof.
For a vertex u,

$$\begin{aligned}
\sum_{u \overset{e}{-} v} f(e,u,v) &= \sum_{u \overset{e}{-} v} \left(\sum_t \mathbb{E}[\xi_t(e,u,v)] - \sum_t \mathbb{E}[\xi_t(e,v,u)] \right) \\
&= \mathbb{E}[\#\{\text{departures from } u\}] - \mathbb{E}[\#\{\text{arrivals at } u\}] \\
&= \begin{cases} 1 & \text{if } u = a \\ -1 & \text{if } u = b \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

This shows that we satisfy Proposition 4.2.1.
Now we will use linearity of expectation. To show Proposition 4.2.2 is satisfied.

$$\begin{aligned}
\sum_{i=1}^k \frac{f(e_i,u_{i-1},u_i)}{w(e_i)} &= \sum_t \left(\sum_{i=1}^k \frac{\mathbb{E}[\xi_t(e_i,u_{i-1},u_i)]}{w(e_i)} - \sum_{i=1}^k \frac{\mathbb{E}[\xi_t(e_i,u_i,u_{i-1})]}{w(e_i)} \right) \\
&= \sum_t \left(\sum_{i=1}^k \frac{\mathbb{P}[\pi(t) = u_{i-1}]}{w(u_{i-1})} - \sum_{i=1}^k \frac{\mathbb{P}[\pi(t) = u_i]}{w(u_i)} \right) \\
&= 0
\end{aligned}$$

Since we are in a cycle, shifting sub scripts by one gives the same sum. □

The following theorem is due to Kirchhoff (1847).

Theorem 4.2.8 (Electric Networks & Spanning Trees)

Suppose G contains an edge $a \overset{e_0}{-} b$ that directly connects a and b. Run a unit current from a to b (i.e. solve Kirchhoff equations with $|f| = 1$). Then

$$f(e_0, a, b) = \mathbb{P}[\text{random spanning tree } T \subseteq G \text{ contains } e_0].$$

Where $\mathbb{P}[T] = \prod_{e \in T} w(e)$.

The key insight of Kirchhoff was that there is a meaning to the polynomials obtained from taking products of edge weights. So you discover that the solution is generating functions for all trees in the denominator and those containing e_0 in the numerator.

"The first proof has a right to be ugly."

Proof.

Recall that a random spanning tree T can be generated by Wilson's Algorithm (with edge weights). We can choose to run the algorithm with b as the root and with a as the first starting vertex.

$$\mathbb{P}[e_0 \in T] = \mathbb{P}[e_0 \in \mathrm{LE}(\pi)] = \mathbb{P}[e_0 \in \pi]$$

On the other hand,

$$\begin{aligned} f(e_0, a, b) &= \sum_t \mathbb{E}[\xi_t(e_0, a, b)] - \sum_t \mathbb{E}[\xi_t(e_0, b, a)] \\ &= \sum_t \mathbb{E}[\xi_t(e_0, a, b)] - 0 \\ &= \mathbb{P}[\pi \text{ uses } e_0 \text{ to arrive at } b] \\ &= \mathbb{P}[e_0 \in \pi]. \end{aligned}$$

$\square$

This is our happy ending of loop erasure and electrical networks!

Moral of the Story: So you can view electric networks as random processes of random walks, or even better a result of the theory of random spanning trees.

4.3 Effective Conductance

Next Step: We can think of our electric network as one huge resistor. It makes sense to believe that we can replace the whole graph with a single resistor and the outside world will not notice. We know the answer of that since we know the proportion of flow that will go through e_0. This will tell us the **effective resistance** of G. This will allow us to remove our special edge e_0.

Homework 3.1

Suppose $G \subseteq H$ are electric networks and that G only attaches to H at the boundary of G. Show that replacing G by a single edge e with $w(e) = w(G, a, b)$ will not be felt outside of G.

This is some sort of functoriality property when you think about it properly.

Corollary 4.3.2 (Potential and Flow)

The **potential** $p = p_f$ is uniquely determined (up to constant shift) by:
1. p is discrete harmonic at all interior vertices, and
2. at the boundary $p(b) - p(a) = \frac{|f|}{w(G,a,b)}$.

Proof.
Suppose we have two solutions p and p', then $\Delta p = p - p'$ is harmonic in the interior and satisfies $\Delta p(b) - \Delta p(a) = 0$. By Lemma 4.2.6, Δp is constant. So $p' = p + c$ for some constant c. □

Homework 3.2

Let $p = p_f$ be the potential associated to the unit flow from a to b. For a vertex $u \in G$, let π_u be the (weighted) random walk in G that starts at u. Prove that

$$p(u) = p(a) + \frac{\mathbb{P}(\pi_u \text{ visits } b \text{ before } a)}{w(G, a, b)}.$$

Corollary 4.3.4 (Probabilistic Potential and Flow)

The **potential** p of any flow in G satisfies:

$$\frac{p(u) - p(a)}{p(b) - p(a)} = \mathbb{P}(\pi_u \text{ visits } b \text{ before } a).$$

Kirchhoff showed the following Theorem in 1847. For simplicity assume unit conductance. But instead of enumeration (“the number of”) you will just need to replace with the generating function (“edges counted with weights”).

Theorem 4.3.5 (Effective Conductance)

$$w(G,a,b)$$
$$= \frac{\#\{\text{spanning trees in } G\}}{\#\{\text{spanning forests in } G \text{ with only two trees, one with } a \text{ and one with } b\}}$$

Equivalently this is the number of spanning trees in G divided by the number of spanning trees in G_{ab} which is the graph obtained by gluing together a and b.

Example: Consider a line graph with n edges (series). There is only one spanning tree in G. And there are n spanning trees in G_{ab}. The conductance is $\frac{1}{n}$.
Example: Consider a two vertices connected by n edges (parallel). Then there are n spanning trees in G and 1 spanning tree in G_{ab}. The conductance is n.

Proof.
Add an extra edge $a \overset{e}{-} b$ of unit weight to G to form a network H. Run a unit current in H. Then since $w(e)$ is one,

$$f(e,a,b) = \frac{f(e,a,b)}{w(e)} = \frac{f(e_G,a,b)}{w(G,a,b)} = \frac{1-f(e,a,b)}{w(G,a,b)}.$$

By Ohm's Law, since both are the difference in potential. We conclude that

$$f(e,a,b) = \frac{1}{1+w(G,a,b)}.$$

We did this since we have a formula for the flow through the extra edge given by Theorem 4.2.8. Taking reciprocals completes the proof.
INSERT PICTURE ☐

Homework 3.3

Find the effective conductance between antipodal vertices of K_2^n, the graph formed by the edges of the n-dimensional cube. Each edge is a unit resistor.

Homework 3.4

Let G be a graph with n vertices and m edges, and suppose that every edge of G belongs to the same number of spanning trees (e.g. the automorphism group of G acts transitively on edges of G). Find the effective conductance between two adjacent vertices in G. Use your result to find the effective conductance between two adjacent vertices of the 1-skeleton of the dodecahedron.

4.4 Squaring the Square

Is it possible to partition a square into pairwise non-congruent squares. The following Theorem is from L. Brooks, C. Smith, A. Stone, and W. Tutte in 1938 while a team of undergraduates at Cambridge. They suggested the problem to themselves and solved it affirmatively. The smallest solution has 21 squares. Amazingly, their method uses electric networks in a substantial way.

Make every horizontal bar an ideal conductor (infinite conductance or zero resistance). Now make the potential equal to height. The square tiles are all unit resistors. The boundary is the bottom and the top.

INSERT PICTURE

Now by Ohm's law, the flow from one vertex to another is the difference in height of the vertices. Harmonicity, or equivalently the in-flow out-flow conditions, then the rectangles will stack correctly (there will be no overhanging rectangles). So harmonicity follows from the fact that the tiles are squares.

Now imagine we are just given a electric network and we choose potentials for the vertices. We use symmetry and the in-flow/out-flow conditions to find the possible potentials.

$$x = \frac{2x+3y}{3}$$
$$x = 3y$$

INSERT PICTURE

This gives a symmetric tiling.

If we want asymmetry then we should introduce it into the graph. Let's study an example from Z. Moro'n. This is the simplest graph with no inner vertex having degree two (since this forces identical squares).

INSERT PICTURE

Example: This graph has 9 edges. Give the bottom vertex zero. Now you have four equations in five unknowns. Give an assignment to one arbitrarily and solve the system of linear equations. Now scale to get integers.

This is the smallest tiling of a rectangle by squares (32×33).

Homework 3.5

Find a different tiling of a rectangle by 9 distinct squares.

Note: The electric network must be planar (for the construction to be reversed).

Now wee riot and say we want to square the square.

The smallest example of a tiling of a square by distinct squares was found by a dutch ~~mathematician~~ programmer A.J.W. Duijvestijn ("precisely if was found by the computer that Duijvestijn programmed") on 03/22/1978. The only way we know it is the smallest is from an exhaustive search of all planar graphs with 21 edges.

Chapter 5

Spanning Trees

5.1 More Linear Algebraic Preliminaries

Theorem 5.1.1 (Laplace Expansion)

Suppose $X = \left(x_{ij}\right)$ is a matrix. Let S be a subset of column indices. Denote by $X[S] = \left(x_{ij}\right)_{j \in S}$. Let X be an $n \times (n+m)$ matrix and let Y be an $m \times (n+m)$ matrix. Then

$$\det \begin{bmatrix} X \\ Y \end{bmatrix} = \sum_{S \in \binom{[n+m]}{n}} \operatorname{sgn}(S) \det(X[S]) \det\left(Y\left[\overline{S}\right]\right)$$

where $S \subseteq [n+m]$, $S = [n+m] \setminus S$, and $\operatorname{sgn}(S) = (-1)^{|K|}$ where $K = \left\{\overline{s} < s \mid s \in S, \overline{s} \in \overline{S}\right\}$.

Proof.
You just need to lock in and think about the determinant. ☐

Homework 3.6

Let A, B be matrices of sizes $n \times m$ and $m \times n$, respectively, with $n \leq m$. Then,

$$\det \begin{bmatrix} 0_n & A \\ B & I_m \end{bmatrix} = (-1)^n \sum_{S \in \binom{[m]}{n}} \det(A[S]) \det(B^T[S]).$$

This is like a dot product. Up to global sign, this is the same at the determinant. This is an easy exercise, just do the naive cut and it will come out.

All you need to do here is split along the first n horizontal rows and the remaining rows and use Laplace Expansion.

Binet and Cauchy announced the same theorem at the same conference and both gave lectures on the same day.

Sometimes the matrix $A[S]$ is called the **maximal minor**.

Theorem 5.1.3 (Binet-Cauchy Formula)

Let A, B be matrices of sizes $n \times m$ and $m \times n$, respectively, with $n \leq m$. Then,

$$\det(AB) = \sum_{S \in \binom{[m]}{n}} \det(A[S]) \det(B^T[S]).$$

Note that if we swap n and m while keeping $n \leq m$ then the determinant will be zero (follows a volume argument).

Proof.
We will use the following identity that

$$\begin{bmatrix} -I_n & A \\ 0 & I_m \end{bmatrix} \begin{bmatrix} 0 & A \\ B & I_m \end{bmatrix} = \begin{bmatrix} AB & 0 \\ B & I_m \end{bmatrix}$$
$$\Rightarrow (-1)^n (-1)^n \sum \det(A[S]) \det(B^T[S]) = \det(AB)$$

$\square$

This is a horrible proof because this is a trick that is not illuminating. Maybe we will discuss better proofs later on, but these require more theory. This identity has deep meaning in the representation theory of semi-simple Lie Groups.

Definition 5.1.4 (Cofactors)

Suppose $Z = (z_{ij})$ is a square matrix. Then

$$Z_{ab} \stackrel{\text{def}}{=} (-1)^{a+b} \det (z_{ij})_{i \neq a, j \neq b}.$$

Also cofactor expansion holds

$$\det Z = \sum_j z_{aj} Z_{aj} = \sum_i z_{ib} Z_{ib}.$$

Everyone knows the statement above, but you might not know the following lemma.

Lemma 5.1.5 (Cofactor Sums)

Let $Z = (z_{ij})$ be square. Suppose the row sums in Z are all zero (equivalently $Z[1 \cdots 1]^T = [0 \cdots 0]^T$). Then all cofactors with a given row are equal. So $Z_{i1} = Z_{i2} = \cdots = Z_{in}$.

Proof.

Let Z' be Z except add one to the $(i, 1)$ entry. Now add all columns of Z' to column k, then expand with respect to that column.

$$\det(Z') = Z_{ik}$$

But the LHS doesn't depend on k, so all Z_{ik} are equal for $k \in [n]$. $\square$

5.2 A Return to Combinatorics

Definition 5.2.1 (Incidence Matrix)

Suppose D is a directed loopless graph. The **incidence matrix** $M = (m_{i,e})$ is obtained from D where i is a vertex and e is an oriented edge. (The rows are labeled by vertices and columns by directed edges). Then if $i \xrightarrow{e} k$ occurs in D then the e column in M will be all zero except a one at the k-th spot and a -1 at the i-th spot.

Example: $1 \to 2 \to 3$. Then an incidence matrix of this graph is $\begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix}$.

Lemma 5.2.2 (Tree Determinants are Bounded)

Let T be a tree and M_0 an incidence matrix of an orientation of T, with one of its rows removed. Then $\det(M_0) = \pm 1$.

Proof.
Designate the removed vertex (with corresponds to the removed row) as the root. Now the rows of M_0 are labelled by non-root vertices and the columns by (directed) edges. Every non-vanishing term in the determinant gives a bijection between non-root vertices and edges, such that each edge is sent to one of its endpoints. But there is only one such bijection (every vertex is assigned to it edge closer to the root).
So only one term is non-vanishing and this term is ± 1. $\square$

Now let's talk about minors of an incidence matrix.

Corollary 5.2.3 (Minors)

Let G be an unoriented graph. Let M_0 be an incidence matrix of an orientation of G, with one row removed. Then the maximal minors of G

$$\det(M_0[S]) = \begin{cases} \pm 1 & \text{if the edges in } S \text{ form a spanning tree in } G, \\ 0 & \text{otherwise.} \end{cases}$$

Proof.

The first case (the det is nonzero) follows from Lemma 5.2.2. In the second case, the subgraph of G that only uses the edges in S is disconnected. Take the connected component I not containing v. Now if we add the rows of $M_0[S]$ labelled by I will sum to zero (by the nature of having a cycle). So the rows are degenerate (have a linear dependence) and the determinant vanishes. □

Definition 5.2.4 (Discrete Laplacian)

Let G be a loopless graph on $[n]$. The **Laplacian** of G is the $n \times n$ matrix $L = L(G) = \left(\ell_{ij}\right)$ defined by

$$\ell_{ij} = \begin{cases} \deg(i) & \text{if } i = j, \\ -k & \text{if } i \text{ and } j \text{ are connected by } k \text{ edges} \end{cases}$$

Example: Let G be the four cycle. Then

$$L(G) = \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}.$$

Remark: L gives the system of equations satisfied by harmonic functions on G. Furthermore, each row/column of $L(G)$ sums to zero.

Therefore, all cofactors of $L(G)$ are equal to each other.

Lemma 5.2.5 (Laplacian is Given by Incidence Matrix)

Let M be an incidence matrix of any orientation of G, then

$$L(G) = MM^T.$$

Proof.

I was not paying attention.

$$\sum_e M_{ue} M_{ve} = \begin{cases} -1 \cdot (\text{edges between } u \text{ and } v) & \text{if } u \neq v, \\ \deg(u) & \text{if } u = v. \end{cases}$$

□

This is the discrete analogue of the identity $\Delta = \nabla^2$ in elliptic PDEs. Remember the Laplacian is the linear operator whose kernel is harmonic functions.

Set L_0 to be the matrix L with some row v and column v removed. Then $L_0 = M_0 M_0^T$.

"What I'm saying actually has nothing to do with virtually anything."

The following theorem is due to Kirchhoff in 1847.

Theorem 5.2.6 (Matrix Tree)

The number of spanning trees in G is equal to any cofactor of $L(G)$.

There exists a weighted version of this theorem (instead the cofactor will compute the generating function for the spanning trees).

Example: Let G be the four cycle. Then

$$L(G) = \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix} \qquad L_{11} = 8 - 2 - 2 = 4.$$

Proof.

By Theorem 5.1.3,

$$\det(L_0) = \sum_S (\det M_0[S])^2 = \#\text{ spanning trees}$$

since each term $\det M_0[S]$ is ± 1. □

There many different proofs, but this is the canonical one.

5.3 A Return to Eigenvalues

Proposition 5.3.1 (Cofactors from Eigenvalues)

Let Z be an $n \times n$ matrix with zero row/column sums. Note from any $(n-1) \times (n-1)$ matrix can be made into such an $n \times n$ matrix. Suppose Z has eigenvalues $\lambda_1, ..., \lambda_n$. Then at least one $\lambda_i = 0$. WLOG $\lambda_n := 0$. Then $\forall i, j$,

$$Z_{ij} = \frac{1}{n} \prod_{k \in [n-1]} \lambda_k.$$

Proof.

From already know that all cofactors are equal to each other. Let's look at the (non-monic) characteristic polynomial

$$\begin{aligned} \det(Z - tI_n) &= \det(Z) - t\left(\sum_i Z_{ii}\right) + \cdots \\ &= 0 - t(n \cdot Z_{11}) + \cdots \end{aligned}$$

since $\det(Z) = 0$ and all cofactors are equal.

Also note that

$$\det(Z - tI_n) = \prod_i (\lambda_i - t)$$
$$= -t\left(\prod_{k\in[n-1]} \lambda_k\right) + \cdots$$

Comparing the linear term in the two expressions completes the proof. □

Corollary 5.3.2 (Spanning Trees from Eigenvalues)

Let G be a graph as before. Let $\lambda_1, ..., \lambda_n$ be the eigenvalues of $L(G)$ with $\lambda_n := 0$. Then the number of spanning trees in G is

$$\frac{1}{n} \prod_{i\in[n-1]} \lambda_i.$$

In general eigenvalues of $A(G)$ and $L(G)$ have little to do with each other. However, in the setting of regular graphs they are closely related.

Proposition 5.3.3 (Eigenvalues of Laplacians of Regular Graphs)

Suppose G is regular with degree d. Then $L(G) = dI_n - A(G)$. This is applying the polynomial $f(t) = d - t$ to $A(G)$. So now Lemma 1.1.2 gives the eigenvalues of $L(G)$ in terms of $A(G)$.

Example: Put $G := (K_2)^d$. Then $n = 2^d$. Recall the eigenvalues of G are $d - 2k$ with multiplicity $\binom{d}{k}$ with $k \in \{0, ..., d\}$. Therefore, the eigenvalues of $L(G)$ are $2k$ with multiplicity $\binom{d}{k}$. We are lucky that $\binom{d}{0} = 1$.
Thus, the number of spanning trees in $(K_2)^d$ is

$$\frac{1}{2^d} \prod_{k\in[d]} (2k)^{\binom{d}{k}} = 2^{2^d-1-d}\left(\prod_{k\in[d]} k^{\binom{d}{k}}\right)$$

Example: If $d = 3$ then

$$2^{8-1-3} \cdot 2^3 \cdot 3 = 16 \cdot 8 \cdot 3 = 384.$$

Homework 3.9

Count the spanning trees in Peterson's Graph.

Example: Put $G := K_n$ the complete graph. This leads to Cayley's Formula as an easy corollary of . It was first stated by Sylvester in 1857 with no proof. It was first proved by Borchardtt in 1860. All Cayley did was advertise the result (without a proof no less!) but he still got credit.

Definition 5.3.5 (Cayley's Formula)

The number of trees on a fixed n-element set of vertices is n^{n-2}.

Proof.

We need to count spanning trees in K_n. K_n is $(n-1)$-regular. The eigenvalues of K_n are $\{(-1; n-1), (n-1; 1)\}$. It follows that the eigenvalues of $L(K_n)$ are $\{(n; n-1), (0; 1)\}$. Thus the number of spanning trees is

$$\frac{1}{n} n^{n-1} = n^{n-2}.$$

□

Note that these are not trees on n vertices up to isomorphism, but labeled trees. Another way of counting this would be to determine the number of trees on n vertices up to isomorphism then count with the number of labelings on each of these trees.

For example if we were counting labeled trees on four vertices there is the line graph and the tree with a fork (triangular graph). For the triangle there are four choices for the center vertex and none after that. Then for the chain there are $\frac{k!}{2}$ labeling so there are $\frac{24}{6} + 4 = 16 = 4^{4-2}$ spanning trees on K_4 as expected.

For K_5. The chain graph has 60 labelings. There is the tree that looks like an airplane has $5 \cdot 4 \cdot 3 = 60$. Tree that is center and spokes which has only 5 labelings. So there are $125 = 5^{5-2}$ trees.

Now we wonder if there is a direct way. It turns out there is no simple argument. The first bijective proof is by Prufer, it is rather tricky.

William Tutte proved the following result in 1948.

Theorem 5.3.6 (Directed Matrix-Tree)

Let D be a directed loopless graph. An **arborescence** rooted at v is a tree whose edges are oriented away from the root. Denote by $\tau(D, v)$ the number of spanning arborescences in D rooted at v. Define $L(D) = \left(\ell_{ij}\right)$ by

$$\ell_{ij} = \begin{cases} \#\{\text{edges } j \to i\} & \text{if } i \neq j, \\ \text{indeg}(i) & \text{if } i = j. \end{cases}$$

So $L(D)$ has row sums $= 0$.

Then, $\tau(D, v)$ is equal to any cofactor of $L(D)$ in row v.

Proof.

Check any textbook. □

We will not rely on the proof, but it's good to know that this can be computed efficiently. Now we are "seemingly" switching gears but it will turn out to be closely related.

5.4 Eulerian Tours

Definition 5.4.1 (Eulerian Tour)

Let G be a graph (possibly directed). An **Eulerian Tour** is a closed walk that visits all vertices, traversing each edge exactly once.

This comes from the Konigsburg bridges problem.

Theorem 5.4.2 (Eulerian Graph Characterization)

An undirected graph is Eulerian ($\exists$ an Eulerian tour) iff the graph is connected and every vertex has even degree.

For a directed graph D, the following are equivalent

(a) D is Eulerian,

(b) D is connected (disregarding orientation) and balanced ($\text{outdeg}(v) = \text{indeg}(v)\ \forall v$), and

(c) D is strongly connected (there is an oriented path from any v to any u) and balanced.

Proof.

Immediately (a) $\Rightarrow$ (c) $\Rightarrow$ (b).

(SKETCH) (b) $\Rightarrow$ (a). Use induction on the number of edges. Make a closed walk. Now the remaining portions that you didn't cover all have Eulerian tours by the inductive assumption and you can stitch these together with your initial walk because the graph is connected.

The argument is the same as the unoriented version. $\square$

The following theorem is due to T. van Aardenne-Ehrenfest and N. de Bruijn based on the work of C.A.B. Smith and W. Tutte. Which is how we get BEST in alphabetical order.

de Bruijn lived to be 94 (died in 2012). "Apparently combinatorics is good for your health … although not everybody, Ramsey died in his 20s."

Theorem 5.4.3 (The BEST)

Denote by $\varepsilon(D)$ the number of Eulerian tours starting starting with a fixed edge e. That is the number of Eulerian tours up to cyclical shifts.

Let v be any fixed vertex then in an Eulerian graph,

$$\varepsilon(D) = \tau(D, v)\left(\prod_u (\mathrm{indeg}(u) - 1)!\right)$$

As a corollary, in an Eulerian graph $\tau(D, v)$ does not depend on v.

Proof.

Fix an edge e directed towards v. The number of Eulerian tours ending with e (must start at v) is $\varepsilon(D)$. For each vertex $u \neq v$, consider the edge of 1st arrival of a given Eulerian tour at u. The key observation is that these edges form a spanning arborescence. For each Eulerian tour, order $\mathrm{indeg}(u)$ edges pointing at a vertex u in the order in which they are traversed by the tour. The resulting collection of linear orderings has the following properties:

1. for $u \neq v$, the unique edge that belongs to T is the first in the ordering (at u).
2. for $u = v$, the edge e is the last one in the ordering.

So now we have a map that takes an Eulerian tour to a pair with one component as spanning arborescence T rooted at v and the other a collection of linear orderings satisfying (1) and (2).

For each T, The number of linear orderings satisfying (1) and (2) is $(\mathrm{indeg}(u) - 1)!$.

We still need to show that this map is injective, hence bijective. □

Homework 4.1

Finish the proof of the BEST theorem.

Now we study two applications.

Definition 5.4.5 (Postman Routes)

Let G be an undirected connected graph and let $D(G)$ be the Eulerian directed graph obtained by replacing each edge in G with a pair of oppositely oriented edges. Then $\tau(D(G))$ gives the number of spanning trees in G.

A **postman route** is an Eulerian tour in $D(G)$ with a distinguished starting edge.

Using Theorem 5.4.3 we can compute the number of postman routes. Let $G = (V, E)$ then the number of postman routes is $2\,|E| \cdot \varepsilon(D(G))$. This is equal to

$$2\,|E|\cdot|T(G)|\cdot\prod_u(\deg(u)-1)!$$

where $T(G)$ is the number of spanning trees on G.

Example: Put $G \coloneqq K_n$, the number of postman routes is

$$2\binom{n}{2}n^{n-2}((n-2)!)^n.$$

For $n = 4$, this gives 3072 postman routes.

Definition 5.4.6 (De Bruijn Sequences)

A binary de Bruijn sequence is a $(0, 1)$-sequence that is periodic of period 2^n where all binary strings of length n occur in the sequnce.

For $n = 3$ a de Bruijn sequence is 00011101. Then all binary triples occur contiguously. This is sometimes called a "rotating drum" for random numbers.

Definition 5.4.7 (De Bruijn Graph)

We let D_n be the **de Bruijn graph** with vertices $(a_1, ..., a_{n-1}) \in \{0, 1\}^{n-1}$ and edges $(a_1, ..., a_{n-1}) \longrightarrow (a_2, ..., a_{n-1}, a_n)$.

Now de Bruijn sequences of order n are cryptomorphic to Eulerian tours in D_n. D_n is connected and balanced, so Eulerian tours exist for every n.

Corollary 5.4.8 (Enumeration of de Bruijn Sequences)

The number of binary de Bruijn sequences of order n is equal to the value of any cofactor in $L(D_n)$.

Proof.
The LHS is $\varepsilon(D_n) = \tau(D, v)\prod_u(\mathrm{indeg}(u) - 1)! = \tau(D, v)\prod_u 1 = \tau(D, v)$. By the directed matrix tree theorem $\tau(D, v)$ is the RHS. $\square$

Let $A_n = A(D_n)$. Where $A_n = \left(a_{ij}\right)$ and

$$a_{ij} = \begin{cases} 1 & \exists i \to j, \\ 0 & \text{otherwise} \end{cases}$$

Then

$$L(D_n) = 2I_n - A_n^T.$$

So the eigenvalues of $L(D_n)$ are $2 - \lambda_i$ where λ_i is an eigenvalue of A_n.

$\forall u, v$ there is a unique walk of length $n - 1$ from u to v. This implies that $A^{n-1} = J$ where J is the matrix of all ones. Since J is rank one, there is only one nonzero eigenvalue. It has a 0 as an eigenvalue with multiplicity $2^{n-1}-1$. So A_n has eigenvalue 0 with multiplicity $2^{n-1}-1$. So two is an eigenvalue of $L(D_n)$ with multiplicity $2^{n-1}-1$. Since $L(D_n)$ is a Laplacian matrix, the last eigenvalue must be zero (the Laplacian is singular).

The following theorem is due to Flye St. Marie (1894) and N. de Bruijn (1946).

Theorem 5.4.9 (Enumeration of de Bruijn Sequences)

The number of de Bruijn sequences of order n up to cyclic shift is

$$\frac{1}{2^n}\prod(\text{nonzero eigenvalues of } L(D_n))$$

$$= \frac{2^{2^{n-1}}}{2^n} = 2^{2^{n-1}-n}$$

Thus, the number of de Bruijn sequences of order n is $2^{2^{n-1}} = \sqrt{2^{2^n}}$. Thus pairs of de Bruijn sequences are in bijection with boolean functions of n arguments and also the powerset of the powerset of $[n]$. Only in 2011 was a bijection found, but it was a very difficult construction.

We can generalize this result to an alphabet of m symbols instead of 2 symbols.

Homework 4.2

Find the number of m-ary de Bruijn sequences of order n.

Chapter 6

Partitions and Tableaux

We are now finished with the first part of the course Algebraic Graph Theory. This is a huge subject which is closely related to the study of symmetric functions which grew out of the representation theory of general linear groups and Schubert calculus.

6.1 Introduction

Definition 6.1.1 (Integer Partition)

A **partition** of $n \in \mathbb{Z}^+$ is a weakly decreasing (equivalently, unordered) finite sequence $\lambda = (\lambda_1, \lambda_2, ..., \lambda_\ell)$. of positive integers whose sum is n.
The set of all partitions of n is denoted $\mathrm{Par}(n)$. We also write $|\lambda| = n$ and $\ell = \ell(\lambda)$ for the length of λ.

Example: $\mathrm{Par}(4) = \{(4), (3, 1), (2, 2), (2, 1, 1), (1, 1, 1, 1)\}$.

Recall that to each partition there is an associated Young diagram. Taking the transpose of a Young Diagram gives the **conjugate** partition. Without Young Diagrams it would be difficult to prove that this is an involution.

We denote $p(n) = |\mathrm{Par}(n)|$ the number of partitions of n.

$p(k) = 1, 1, 2, 3, 5, 7, 11, 15, 22, 30, 42, ...$ for $k = 0, 1, 2, 3, 4, ...$

There is no formula for $p(n)$. The asymptotic behavior is known.

The following theorem is due to G.H. Hardy and S. Ramanujan in 1918.

Theorem 6.1.2 (Asymptotics of Partitions)

$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right)$$

This is the first example of a mathematical quantity that arises naturally that grows faster than any polynomial and slower than any exponential (i.e. has intermediate growth). This result is proved using Analytic Number Theory.

H. Rademacher in 1937 obtained a complete asymptotic expansion for $p(n)$. If you are interested you can look into George Andrews' book **The Theory of Partitions**.

Euler introduced the technology of generating functions via the e following theorem.

Theorem 6.1.3 (Generating Functions for Partitions)

$$\sum_{n=0}^{\infty} p(n)x^n = \prod_{\ell=1}^{\infty} \frac{1}{1-x^\ell}$$

$$\sum_{n=0}^{\infty} p_{\leq k}(n)x^n = \sum_{n=0}^{\infty} p^{\leq k}(n)x^n = \prod_{\ell\in[k]} \frac{1}{1-x^\ell}$$

$$\sum_{n=0}^{\infty} p_{\text{unique}}(n)x^n = \prod_{\ell=1}^{\infty} 1+x^\ell$$

$$\sum_{n=0}^{\infty} p_{\text{odd}}(n)x^n = \prod_{\ell=1}^{\infty} \frac{1}{1-x^{2\ell+1}}$$

Moreover,

$$\sum_{n=0}^{\infty} p_{\text{odd}}(n)x^n = \sum_{n=0}^{\infty} p_{\text{unique}}(n)x^n$$

$p_{\leq k}(n)$ is the number of partitions with parts less than or equal to k.
$p^{\leq k}(n)$ is the number of partitions with less than or equal to k parts.
$p_{\text{unique}}(n)$ is the number of partitions with distinct parts.
$p_{\text{odd}}(n)$ is the number of partitions with only odd parts.

Proof.
This is nothing but the multiplication principal for generating functions. And the fact that

$$\frac{1}{1-x^k} = \sum_{\ell=0}^{\infty} x^{k\cdot\ell}.$$

The second identity follows via taking conjugate partitions.
The last equality follows since

$$\prod_{\ell=1}^{\infty} 1+x^\ell = \prod_{\ell=1}^{\infty} \frac{1-x^{2\ell}}{1-x^\ell}$$

and the product telescopes away all the even powers. □

Nowadays this is easy but in the 1800s this was a highly unusual insight.

Homework 4.3

Prove that $p_{\text{odd}}(n) = p_{\text{unique}}(n)$ by a bijective argument.

Homework 4.4

Show that the number of self-conjugate partitions of n is equal to the number of partitions of n into distinct odd parts.

Homework 4.5

Show that $\forall k \geq 0$ the number of partitions of n into parts not divisible by k is equal to the number of partitions of n such that the multiplicity of each part is less than k.
Do this algebraically.

6.2 Relevant Partial Orders

The Dominance Order

Definition 6.2.2 (Dominance Order)

Then **dominance order** is a partial order on $\mathrm{Par}(n)$. Where $\mu \leq \lambda$ provided that $\forall i$, $\mu_1 + \cdots + \mu_i \leq \lambda_1 + \cdots + \lambda_i$ with remaining parts (when $i > \ell(\mu), \ell(\lambda)$) assumed to be zero.

This poset has a maximum and minimum. $n = 6$ is the first dominance order that is not a linear order. $n = 7$ is the first dominance order that is not graded. Nonetheless, the dominance order is a lattice.

Homework 4.7

Show that the dominance order is self-dual under conjugation

$$\mu \leq \lambda \Longleftrightarrow \mu' \geq \lambda'.$$

It is enough to prove in only one direction (just switch μ and λ).

Homework 4.8

Show that the dominance order is a **lattice**. Given the previous problem, you only need to show every subset has a unique join.

The dominance order arises naturally in various applications. The Jordan Normal Form, since the sizes of Jordan blocks form a partition of n. It turns out that the subsets of matrices that share the Jordan partition then these subsets can be ordered by $\subseteq$-containment under closures then you recover the dominance order. It also plays an important role in representation theory.

Now we study dominance in $\mathbb{R}^n$. Take $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n) \in \mathbb{R}^n$. We can extend the dominance order to $\mathbb{R}^n$.

Homework 4.9

For $\lambda \in \mathbb{R}^n_{\geq 0}$ let P_λ be the convex hull of the S_n-orbit of λ (S_n acting on the coordinates of λ). Prove that for $\mu, \lambda \in \mathrm{Par}(n)$,

$$\mu \leq \lambda \iff P_\mu \subseteq P_\lambda.$$

This statement is true for non-negative reals not just partitions.

This dominance order plays a role in the famous theorem below.

Theorem 6.2.6 (Schur-Horn)

For every Hermitian matrix, the vector of eigenvalues dominates the vector of diagonal entries in the matrix. Conversely, every pair of vectors in which one dominates the other has a matrix with eigenvalues of the dominating vector and diagonal entries of the dominated vector.

The Lattices L(m,n)

Fix $m, n \in \mathbb{Z}^+$ and consider an $m \times n$ rectangle. Now we study $L(m,n)$ the set of partitions whose young diagrams fit in this rectangle. Explicitly,

$$L(m,n) = \left\{ \lambda \in \bigcup_{k=1}^{\infty} \mathrm{Par}(k) \mid \lambda_1 \leq n, \ell(\lambda) = \lambda_{1'} \leq m \right\}.$$

Now partially order $L(m,n)$ by $\subseteq$-containment of their young diagrams.

This lattice has a maximum (the $m \times n$ rectangle) and minimum (the empty partition) and is graded (the rank function is the order of the partition). $L(m,n)$ is a **distributive** lattice (meaning the $\wedge$ and $\vee$ operators are distributive). This follows since $\wedge$ is $\cap$ and $\vee$ is $\cup$ on the young diagrams.

$$|L(m,n)| = \binom{m+n}{m} = \binom{n+m}{n}$$

Since selecting an element of $L(m,n)$ is equivalent to choosing a lattice path on the $m \times n$ rectangle.

We can count the size of $L(m,n)$ at each rank and package them together in a generating function.

Definition 6.2.8 (q-binomial coefficient)

The q-binomial coefficient is a polynomial in a formal variable q defined by

$$\begin{bmatrix} m+n \\ m \end{bmatrix}_q \stackrel{\text{def}}{=} \sum_{\lambda \in L(m,n)} q^{|\lambda|} \stackrel{\text{def}}{=} \sum_{a=0}^{m \cdot n} g(m,n,a) q^a$$

where $g(m,n,a)$ is the **Gaussian coefficient** which gives the number of partitions in $L(m,n)$ of rank a.

There is no formula for these coefficients (hence why we give it a new name). But there is a formula for the generating function.

<u>Example</u>:

$$\begin{aligned}
\begin{bmatrix} 5 \\ 2 \end{bmatrix}_q &= 1 + q + 2q^2 + 2q^3 + 2q^4 + q^5 + q^5 \\
&= \underbrace{(1 + q + q^2 + q^3 + q^4)}_{[5]}(1 + q^2) \\
&= (1 + q + q^2 + q^3 + q^4)\frac{1 + q + q^2 + q^3}{1 + q} \\
&= \frac{[5] \cdot [4]}{[2] \cdot [1]} \\
&= \frac{[5] \cdot [4] \cdot [3] \cdot [2] \cdot [1]}{[2] \cdot [1] \cdot [3] \cdot [2] \cdot [1]} \\
&= \frac{[5]!}{[2]! \cdot [3]!} \\
&= \frac{[5]!}{[2]! \cdot [5-2]!}.
\end{aligned}$$

Which is exactly the q-analogue of the binomial coefficient.

$[k]_q = [k] = 1 + q + q^2 + \cdots + q^{k-1}$ is generally understood to be a perturbation or refinement of the positive integer k. Another interpretation, is that $g(m,n,\ell)$ is the number of sequences with m zeros and n ones that have ℓ inversions (meaning a one occurs before a zero). You can see this bijection by viewing zeros as steps north and ones as steps east.

Now we can return to the q-binomial coefficients via the "noncommutative binomial theory."

Definition 6.2.9 (Non-commutative Binomial Theorem)

Suppose x, y are variables that q-commute meaning that $yx = qxy$. Then,

$$(x+y)^n = \sum_{k \in [n]} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k y^{n-k}$$

Now we will study an important interpretation of this lattice.

Definition 6.2.10 (Grassmannian)

Let $\mathbb{F}$ be a field and $V \cong \mathbb{F}^{m+n}$ be a vector space over $\mathbb{F}$. Then the m-**Grassmannian** is the set of m-dimensional subspaces of V denoted $G(m, m+n, \mathbb{F})$.

Recall from linear algebra that

Theorem 6.2.11 (Representation of Subspaces)

Any m-dimensional subspace in $\mathbb{F}^{m+n}$ can be uniquely represented as the row span of an $m \times (m+n)$ matrix in reduced row echelon form.

The definition of reduced row echelon form gives us a bijection between RREF $m \times (m+n)$ matrices and young diagrams contained in an $m \times n$ rectangle.

So now m-subspaces of $\mathbb{F}^{m+n}$ are in bijection with pairs of $\lambda \in L(m,n)$ and filling of λ with elements of $\mathbb{F}$.

Corollary 6.2.12 (Dimension of Grassmannian)

Let $\mathbb{F}$ be a finite field of q elements. Then $|\mathbb{F}^k| = q$ and

$$|G(m, m_n, \mathbb{F}_q)| = \begin{bmatrix} m+n \\ m \end{bmatrix}_q = \sum_{\lambda \in L(m,n)} q^{|\lambda|}$$

Now we can directly calculate the cardinality of the Grassmannian.

$$\begin{aligned}
\#G(m, m+n, \mathbb{F}_q) &= \frac{\#\{\text{sequences of } m \text{ linearly independent vectors in } \mathbb{F}_q^{m+n}\}}{\#\{\text{ordered basis in } \mathbb{F}^m\}} \\
&= \frac{(q^{m+n}-1)(q^{m+n}-q)(q^{m+n}-q^2)\cdots(q^{m+n}-q^{m-1})}{(q^m-1)(q^m-q)\cdots(q^m-q^{m-1})} \\
&= \frac{(q^{m+n}-1)(q^{m+n-1}-1)(q^{m+n-2}-1)\cdots(q^{n+1}-1)}{(q^m-1)(q^{m-1}-1)\cdots(q-1)} \\
&= \frac{[m+n][m+n-1]\cdots[n+1]}{[m][m-1]\cdots[1]} \\
&= \frac{[m+n]!}{[n]![m]!}
\end{aligned}$$

This is the other definition of the q-binomial coefficient. When q is a prime power, we have

$$\begin{bmatrix} m+n \\ m \end{bmatrix}_q = \#G(m, m+n, \mathbb{F}_q) = \frac{[m+n]!}{[n]![m]!}$$

Now we have two polynomials in q on the LHS and the RHS. We don't *a priori* know the RHS is a polynomial, but we see that these take equal values on any prime number. So now we have two rational functions (but they're really polynomials) that agree on infinitely many points (the primes) so they must be the same polynomial.

Some textbooks define the q-binomial coefficient first using the LHS expression. This is difficult to then show that the RHS is a generating for the LHS.

So the q-binomial coefficient are to vector spaces as binomial coefficients are to sets.

Homework 5.1

Prove that

$$([m]![n]!) \cdot \left(\sum_{\lambda \in L(m,n)} q^{|\lambda|} \right) = [m+n]!$$

using a direct bijective argument.

View the RHS as a generating function for permutations in S_{m+n} with respect to number of inversions. So we want a bijection $S_m \times S_n \times L(m,n) \simeq S_{m+n}$ so that $\operatorname{inv}(u) + \operatorname{inv}(v) + |\lambda| = \operatorname{inv}(w)$ where $u \in S_m$, $v \in S_n$, and $w \in S_{m+n}$.

q-analogue of the Pascal Recurrence

Definition 6.2.15 (Recurrence for Gaussian Coefficients)

$$g(m,n,a) = g(m-1,n,a-n) + g(m,n-1,a)$$

We use the convention that $g(m,n,a) = 0$ if any of m, n, a are negative.

Proof.

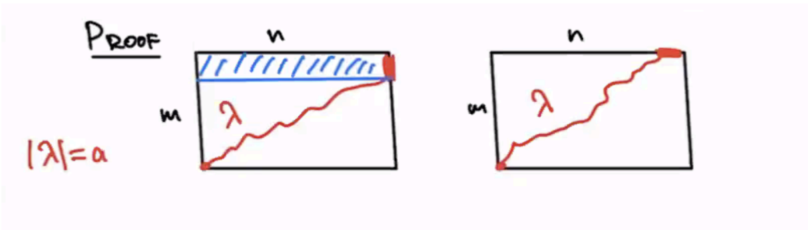

Figure 1: There are two ways to produce a Young Diagram.

The last step can either go north or east. The proof follows immediately from the addition principle. □

Now that we've proved this identity, let's pass to generating functions.

$$\begin{aligned}
\begin{bmatrix} m+n \\ m \end{bmatrix}_q &= \sum_{a\in[mn]} g(m,n,a)q^a \\
&= \sum_{a\in[mn]} g(m-1,n,a-n)q^a + \sum_{a\in[mn]} g(m,n-1,a)q^a \\
&= \sum_{a\in[mn]} g(m-1,n,a-n)q^a + \begin{bmatrix} m+n-1 \\ n-1 \end{bmatrix}_q \\
&= q^n \sum_{a\in[mn]} g(m-1,n,a-n)q^{a-n} + \begin{bmatrix} m+n-1 \\ n-1 \end{bmatrix}_q \\
&= q^n \begin{bmatrix} m+n-1 \\ m-1 \end{bmatrix}_q + \begin{bmatrix} m+n-1 \\ n-1 \end{bmatrix}_q
\end{aligned}$$

Changing notation gives,

Theorem 6.2.16 (q-Pascal Recurrence)

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_q + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q$$

Now we can construct the q-Pascal Triangle.

Remark: One can use the q-Pascal Recurrence and induction to prove the q-binomial coefficient identity. But this whole proof looks like a circus trick and doesn't explain the geometry of the situation. The combinatorial and geometric proofs are much more illuminating.

Homework 5.2

Find exponents $e(i,k,n,m)$ such that

$$\begin{bmatrix} n+m \\ k \end{bmatrix}_q = \sum_{i=0}^{k} q^{e(i,k,n,m)} \begin{bmatrix} n \\ i \end{bmatrix}_q \begin{bmatrix} m \\ k-i \end{bmatrix}_q.$$

Then deduce that

$$\begin{bmatrix} 2n \\ n \end{bmatrix}_q = \sum_{i=0}^{n} q^{i^2} \left(\begin{bmatrix} n \\ i \end{bmatrix}_q \right)^2$$

Homework 5.3

This identity is due to Cauchy.

$$\prod_{j \in [n]} (1 + yq^j) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q y^k q^{\frac{k(k+1)}{2}}$$

There are lots of other theorems, but we are just going to skip them.

6.3 Young Tableaux

Definition 6.3.1 (Young Lattice)

The **Young Lattice** $\mathbb{Y}$ is the set of all Young Diagrams (equiv., partitions) partially ordered by inclusion (resp., component-wise domination).

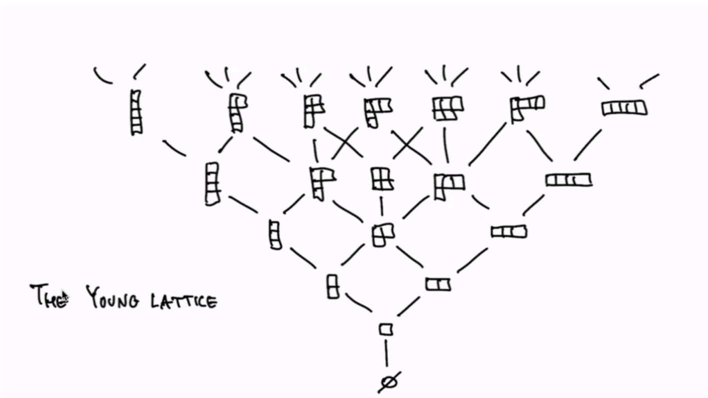

Figure 2: An initial segment of $\mathbb{Y}$.

It is easy to check that this is a lattice since union and intersection serve as join and meet, respectively.

Recall that a **Standard Young Tableau** is a filling of a Young Diagram with an initial segment of $\mathbb{Z}^+$ so that order is respected across all columns and rows.

Observe, that a Standard Young Tableau corresponds to a path in the Hasse Diagram of the Young Lattice.

Let $f^\lambda \stackrel{\text{def}}{=} \#\{\text{standard Young Tableaux of shape } \lambda\}$, then

$$f^\lambda = \sum_{\substack{|\lambda \setminus \mu| = 1 \\ \lambda, \mu \in \mathbb{Y}}} f^\mu$$

since we obtain a Young Diagram from Young Diagrams with one less box. So the identity above follows from the Addition Principle.

These numbers give the dimensions of irreducible representations of symmetric groups. They also give the degrees of Schubert varieties in Algebraic Geometry. So it is important we find a formula.

Definition 6.3.2 (Hooks & Hooklengths)

Let λ be a Young Diagram. Given a box b in λ, the **hook** $h \subseteq \lambda$ are the set of boxes directly south and directly east of b. The **hooklength** of b is

$$|h| = h_\lambda(b).$$

Now comes the much celebrated formula due to J.S. Frame, G. de B. Robinson, and R.M. Thrall in 1954. They discovered this in East Lansing and Ann Arbor a day apart independently.

Prof. Fomin is the Robert Thrall Collegiate professor.

Theorem 6.3.3 (Hooklength Formula)

For $|\lambda| = n$,

$$f^\lambda = \frac{n!}{\prod_{x \in \lambda} h_\lambda(x)}$$

There are no simple proofs. We will study the probabalistic one. This proof is due to Curtis Greene, A. Nijenhuis, and H. Wilf (Greene is a good friend of Prof. Fomin).

The proof is fairly intricate.

Proof.

Put $H(\lambda) \overset{\text{def}}{=} \prod_{x\in\lambda} h_{\lambda(x)}$. We want to show that $f^\lambda = \frac{n!}{H(\lambda)}$. Denote by del($\lambda$) the set of corner boxes of λ (so the boxes whose removal leaves a valid Young Diagram) or the "deletable" boxes.

$$f^\lambda = \sum_{x\in\ \mathrm{del}(\lambda)} f^{\lambda\setminus\{x\}}$$

Since we can put n in any of the deletable boxes. It turns out to be surprisingly hard that the formula satisfies the recurrence basically because these are sums of fractions and we are checking that is equal to another fraction. There is a proof like that by Don Knuth (check the *Art of Computer Programming*).

Anyhow, we still need to show

$$\frac{n!}{H(\lambda)} = \sum_{x\in\ \mathrm{del}(\lambda)} \frac{(n-1)!}{H(\lambda\setminus\{x\})}$$

$$\iff \frac{n}{H(\lambda)} = \sum_{x\in\ \mathrm{del}(\lambda)} \frac{1}{H(\lambda\setminus\{x\})}$$

Strategy: find a Markov chain that starts at a box $u \in \lambda$ and terminates at a (random) $x \in \mathrm{del}(\lambda)$. Denote by $P(u,v)$ the probability that the process stops at v provided it started at u. So now $\forall u$

$$1 = \sum_{v\in\ \mathrm{del}(\lambda)} P(u,v).$$

Suppose: our Markov chain satisfies $\forall v \in \mathrm{del}(\lambda)$,

$$\sum_{u\in\lambda} P(u,v) = \frac{H(\lambda)}{H(\lambda\setminus\{v\})}.$$

Then we would get

$$\sum_{\substack{u\in\lambda\\ v\in\ \mathrm{del}(\lambda)}} n = P(u,v) = \sum_{v\in\ \mathrm{del}(\lambda)} \frac{H(\lambda)}{H(\lambda\setminus\{v\})}.$$

which would suffice.

Therefore, all that is left to do is find a Markov chain that establishes the identity above.

Removing a box from del(λ) only decrements the hooklengths of the other boxes in that same column and row. Therefore,

$$\frac{H(\lambda)}{H(\lambda\setminus\{v\})} = \prod_{x\in C(v)} \frac{h_{\lambda(x)}}{h_{\lambda(x)}-1}$$

Where $C(v)$ are the boxes in the same column and row as v.

Now we consider the **hook walk** (which is a probabilistic algorithm). Take as input $\lambda \in \mathbb{Y}$ and a box $u \in \lambda$. The output is going to be a box $v \in \mathrm{del}(\lambda)$.

```
v := u
while (not (v in del($lambda$)):
  choose v' in hook(v) without {v}
  uniformly at random a new box v' in the hooklength;
  v := v'
return v
}
```

Claim: the Hook Walk satisfies the identity. This is the hard part.

Lemma 1: Create a Cauchy matrix with the last column and row value being zero and follow a lattice path through it. Now take the sum over all lattice paths over products of the entries in the lattice path. This simplifies to the reciprocal of product of all row values and column values.

One can prove this by double induction on the height and the width (I think you could induct on the sum of the width and the height).

Lemma 2: If we allow jumps of arbitrary lengths vertically or horizontally, then (allowing arbitrary starting points)

$$\sum_{\substack{\text{trajectories} \\ \pi}} \prod_{x=(i,j)\in\pi} \frac{1}{a_i + b_j} = \prod_i \left(1 + \frac{1}{a_i}\right) \prod_j \left(1 + \frac{1}{b_j}\right)$$

Now the claim follows from Lemma 1 using the multiplication principle for generating functions.

The key observation *_*

$\square$

Example: Let λ be a $2 \times k$ rectangle. Then the top row hooklengths are $k+1, \ldots, 2$ and the bottom row's are $k, \ldots, 1$. Thus,

$$f^\lambda = \frac{(2k)!}{(k!)(k+1)!} = \frac{1}{k+1}\binom{2k}{k} = C_n$$

This is not a surprise because we can view the filling as a ballot sequence of length $2k$ (or a Dyck path). So Standard Young Tableaux are cryptomorphic to Dyck paths. So the numbers f^λ can be viewed as a huge generalization of the Catalan numbers.

Homework 5.4

Find the number of shortest lattice paths in $\mathbb{Z}^d$ connecting $(0, \ldots, 0)$ and $(k, \ldots, k)$ while staying inside the cone $\{x_1 \geq x_2 \geq \cdots \geq x_d \geq 0\}$. This is sometimes called the vial chamber.

Around the Hooklength Formula.

Homework 5.5

Suppose λ is a Young diagram with $|\lambda| = n$. Choose $u \in \lambda$ uniformly at random. Run the hook walk from u. Place the entry n in the resulting corner box $v \in \mathrm{del}(\lambda)$. Then remove v from λ, reset n accordingly and iterate. Prove that the resulting random Standard Young Tableaux is uniformly distributed.

Shifted Shapes & Tableaux.

Definition 6.3.6 (Shifted Shape)

Given a strict partition $\lambda = (\lambda_1 > \lambda_2 > \cdots)$ corresponds uniquely to a shifted shape (Young diagram) λ^*. In which row k starts k boxes indented to the right. Now we can introduce the concept of a standard shifted tableaux

Now define a shifted hooklength to be number of boxes to the East plus the number of boxes directly South and below along the diagonal. We write $h_{\lambda^*}(x)$.

This theorem is due to Bob Thrall. This was proved two years before the canonical hooklength formula.

Theorem 6.3.7 (Enumeration of Standard Shifted Shapes)

The number of standard shifted tableaux of shifted shape λ^* is

$$\frac{n!}{\prod h_{\lambda^*}(x)}$$

Definition 6.3.8 (Tree Tableaux)

This is the number of ways to place labels from [n] on a rooted tree with n vertices, so that they increase away from the root. Now define a tree hook-length of a node to be the number of nodes less than or equal to that node (ordered by descendency). The number of such increasing trees is

$$\frac{n!}{\prod h(t)}$$

Homework 5.6

Prove that the number of standard tree tableaux is given by the analogue of the hooklength formula.

Chapter 7

A Result from Representation Theory

Now lets talk about the significance of the numbers f^λ.

1. Enumerative geometry or Schubert Calculus. If you have two lines in $\mathbb{R}^3$ you can pierce them both by a third line, you have two degrees of freedom (a hyperboloid). Now having a line pierce three lines there is a single degree of freedom. Having a line pierce a collection of four lines is zero dimensional, is exactly two. The degree of the corresponding algebraic variety is two. This is the number of standard tableaux of shape 2×2.

   Affine lines in three space are the same as two two dimensional subspaces in four space. One of Hilbert's problems was to put Schubert Calculus onto rigorous foundations which was given by the modern theory of cohomology of algebraic varieties.

   The moduli space of line orientations is four dimensional.
2. Representation theory of finite dimensional (reducible) the symmetric group. Some representations decompose into irreducible elements. After choosing an appropriate basis you get block form matrices. Recall the result from Frobenius that there is the regular representation (left action of the group on the group algebra) which acts by permutation. This representation contains all irreducible representations with multiplicity equal to their dimension. The number of irreducible representations equinumerable to conjugacy classes. Thus in S_n the Young Diagrams correspond to irreducible representations of S_n. f^λ gives the dimensions of irreducible representations of S_n.

Burnside Frobenius-Young discovered the next result via representation theory.

Theorem 7.1.1 (Frobenius-Young Identity)

$$|S_n| = n! = \sum_{|\lambda| = n} {f^{\lambda}}^2$$

It took decades to find an elementary proof of this result. The first is linear algebraic. The second is a bijective proof from permutations to pairs of Standard Young Tableaux.

7.2 The Linear Algebraic Proof

Let $\mathbb{Y}$ be thee set of all Young diagrams. Let $\mathbb{R}\mathbb{Y}$ be the vector space formally spanned by $\mathbb{Y}$ over $\mathbb{R}$. Alternatively, real valued functions on $\mathbb{R}$ which have finite support.

Let U, D be the "up" and "down" operators in $\mathbb{R}\mathbb{Y}$. So U adds a box in all possible ways. For example $U(1 \times 2 - 2 \times 1) = 1 \times 3 - 3 \times 1$.

Explicitly,

$$U(\lambda) = \sum_{\mu \gtrdot \lambda} \mu \qquad\qquad D(\lambda) = \sum_{\mu \lessdot \lambda} \mu$$

where $\lessdot$ is the covering relation in $\mathbb{Y}$.

<u>Observe</u> that <u>Theorem 7.1.1</u> is equivalent to $|D^n U^n(\emptyset)| = n! \cdot \emptyset$

<u>Observe further</u> that $DU - UD = \mathrm{id}$. This is the Heisenberg Relation. When adding and removing a two distinct boxes the moves commute. When adding and removing the same box the moves do not commute. The number of ways a box can be added is one larger than the number of ways a box can be removed.

<u>Remark</u>: In a finite dimensional vector space V, $ST - TS = \mathrm{id}$ never holds for $S, T \in \mathrm{End}(V)$. Since $\mathrm{tr}(ST) = \mathrm{tr}(TS)$ and trace is linear.

<u>Example</u>: Take $V = \mathbb{R}[x]$ and define $U, D \in \mathrm{End}(V)$ by

$$(Uf)(x) = xf(x) \qquad\qquad (Df)(x) = \frac{d}{dx} f(x).$$

Then,

$$(DU - UD)f = (xf(x))' - xf'(x) = f(x)$$
$$\implies DU - UD = \mathrm{id}$$

This is not a coincidence. There is a different interpretation of $\mathbb{Y}$ as a graded ring of symmetric functions with two operators (multiplication by *** and differentiation) which gives an interesting relation.

"Homework 5 is due 04/16. One day after taxes, for those of you who pay them."

Recall by the Heisenberg identity that $UD + \mathrm{id} = DU$.

Lemma 7.2.1 (Young Interchange)

Let f be a formal power series in one variable. Then,

$$Df(UD)U = (UD + \mathrm{id})f(UD + \mathrm{id})$$

Corollary.

$$D^n U^n = \prod_{k \in [n]} (UD + k \cdot \mathrm{id})$$

Proof.
For a sanity check if $f = \text{id}$, then,

$$DUDU = (UD + \text{id})^2 = (DU)^2.$$

By linearity it is enough to check the case when f is a monomial, say x^k.

$$D(UD)^k U = (DU)^{k+1} = DU(DU)^k = (UD + \text{id})f(UD + \text{id}).$$

□

Corollary.

Proof.
Induction on n. When $n = 1$ this follows from the Heisenberg Identity.

$$\begin{aligned} D^{n+1}U^{n+1} &= DD^nU^nU \\ &= D\left(\prod_{k\in[n]}(UD + k\cdot\text{id})\right)U \\ &= (UD+\text{id})\left(\prod_{k\in[n]}(UD + (k+1)\cdot\text{id})\right) \end{aligned}$$

The second inequality follows from the inductive assumption and the second follows from Lemma 7.2.1. □

Now we can prove the Frobenius-Young Identity.

Proof.

$$\begin{aligned} \sum_{\lambda\in\ \text{Par}(n)} \left(f^\lambda\right)^2 &= \text{coefficient of } \varnothing \text{ in } D^nU^n(\varnothing) \\ &= \text{coefficient of } \varnothing \text{ in } \prod_{k\in[n]}(UD + k\cdot\text{id})(\varnothing) \\ &= n! \end{aligned}$$

Since we cannot grab any UD term because $UD(\varnothing)$ kills $\varnothing$ (there is no way to remove a box from the empty partition). □

This general approach finds great utility in poset theory.

Theorem 7.2.2 (Shifted Analogue of Frobenius-Young)

Let g^λ be the number of shifted Standard Young Tableaux of shape λ^*.

$$\sum_{|\lambda^*|\ = n} 2^{2-\ell(\lambda)}\left(g^\lambda\right)^2 = n!$$

Homework 5.7

Prove the Shifted Analogue of the Frobenius-Young Identity.

Prove it by using Up and Down operator type mathematics.

Hasse walks in $\mathbb{Y}$.

Homework 5.8

Consider $\mathbb{Y}$ as an undirected graph with edges given by the covering relation (its Hasse Diagram). Show that the total number of closed walks of length $2m$ which begin and end at $\varnothing$ is

$$\prod_{k\in[m]}(2k-1).$$

There are no closed walks of odd length.

This can be proved directly from the Heisenberg identity. Interpret the operators correctly.

7.3 Tableaux & Involutions

We will be following exactly the pioneering work of Donald Knuth. Just as we do in Pascal's triangle, it is natural to ask what is the sum of a row in $\mathbb{Y}$.

Schensted proved the following result.

Theorem 7.3.1 (Enumeration of Involutions)

Standard young tableaux on n boxes are equinumerable to involutions in S_n.

$$\sum_{\lambda\in \operatorname{Par}(n)} f^\lambda = \#\{\tau\in S_n : \tau^2 = \mathrm{id}\}$$

Involutions of S_n are given by partial matchings of $[n]$ since every cycle is either of length one or two.

Put $y_n := \#\{\tau\in S_n : \tau^2=\mathrm{id}\}$.

Theorem 7.3.2 (Recursive Formula)

The sequence (y_n) satisfy $y_0 = y_1 = 1$ and $y_{n+1} = y_n + ny_{n-1}$.

Proof.

Easy (think in terms of partial matchings). "Proof by intimidation." □

There is no formula for these numbers, but we can get their generating function. The right thing to use the exponential generating function. For labelled objects, exponential generating functions works better since we want to kill the action of the symmetric group.

Proposition 7.3.3 (Generating Function for Involutions)

The egf for (y_n) is

$$y(x) = \sum_{n=0} y_n \frac{x^n}{n!} = e^{x+\frac{x^2}{2}}$$

This is a special case of the number of permutations given a restriction on cycle length.

One can generalize this result via Polya's Theory of Counting, which enumerates finite sets under group action. It heavily uses the Burnside Lemma which counts elements fixed by a group action. Some algebraic combinatorics courses heavily focused on Polya Theory, it somehow became less important. This theory was used heavily to enumerate chemical compounds.

Proof.
Use Theorem 7.3.2.
Note that

$$y'(x) = \sum_{n\in\mathbb{Z}^+} \frac{y_n}{(n-1)!} x^{n-1} = \sum_{n=0}^{\infty} \frac{y_{n+1}}{(n)!} x^n.$$

Differentiation corresponds to a shift for exponential generating functions.
This gives us

$$\begin{aligned} \sum_{n\in\mathbb{Z}^+} \frac{y_{n+1}}{n!} x^n &= \sum_{n\in\mathbb{Z}^+} \frac{y_n}{n!} x^n + \sum_{n\in\mathbb{Z}^+} \frac{y_{n-1}}{(n-1)!} x^n \\ y' - 1 &= y - 1 + xy \end{aligned}$$

This gives an ODE that y satisfies and we can solve

$$\begin{aligned} y' &= y + xy = y(1+x) \\ \frac{y'}{y} &= (\log y)' = 1 + x \\ y &= e^{x+\frac{x^2}{2}} \end{aligned}$$

□

Homework Optional

Let $X \sim N(0,1)$ be a random variable with a standard Gaussian distribution. So its pdf is

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}.$$

Prove that

$$\mathbb{E}[(X+1)^n] = y_n.$$

One can also express these as powers of Hermite polynomials.

Now we proof Theorem 7.3.1 via the Heisenberg relation.

Proof.

Denote $Y_n := \sum_{\lambda \in \operatorname{Par}(n)} \lambda \in \mathbb{R}\mathbb{Y}$. So $Y_0 = \emptyset, Y_1 = 1 \times 1, Y_2 = 1 \times 2 + 2 \times 1$.

Observe that we can count paths from the n-th level to $\emptyset$ by studying the coefficient of $\emptyset$ in

$$D^n Y_n = \left(\sum_{\lambda \in \operatorname{Par}(n)} f^\lambda \right) \emptyset.$$

We want to show that $D^{n+1} Y_{n+1} = D^n Y_n + n D^{n-1} Y_{n-1}$ $(**)$.

Lemma: $D^n U = n D^{n-1} + U D^n$. This is like the Leibniz Rule (with an added commutator). We use induction. For $n = 1$ this gives exactly the Heisenberg Relation. Now

$$\begin{aligned} D^n U &= D D^{n-1} U \\ &= D((n-1)D^{n-2} + U D^{n-1}) \\ &= (n-1)D^{n-1} + U D^n + D^{n-1} \\ &= n D^{n-1} + U D^n. \end{aligned}$$

The second equality follows from the inductive assumption and the third FROM ???

Lemma: $D Y_{n+1} = U Y_{n-1} + Y_n$. The coefficient of λ on the LHS is #add(λ) (the number of boxes we can add to λ) and the coefficient of λ in $U Y_{n-1}$ is # del(λ) = # add(λ) $- 1$.

Now we can prove the desired identity $(**)$.

$$\begin{aligned} D^{n+1} Y_{n+1} &= D^n (U Y_{n-1} + Y_n) \\ &= n D^{n-1} Y_{n-1} + U D^n Y_{n-1} + D^n Y_n \\ &= n D^{n-1} Y_{n-1} + 0 \cdot \emptyset + D^n Y_n \end{aligned}$$

The first and second equalities follow from the lemmas and the third follows since D^n kills Y_{n-1}. □

This proof was only found in the 1980s.

7.4 The Schensted Insertion

This is a beautiful bijection between pairs of SYT of the same shape consisting of n-boxes $(P, Q) \longleftrightarrow \sigma$ and permutations in S_n. What will be cool is that $(P, Q) \longleftrightarrow \sigma \iff (Q, P) \longleftrightarrow \sigma^{-1}$. This will prove in one shot that sums of SYT give the number of involutions.

Craig Schensted made these discoveries in 1961 in Ann Arbor. He was an experimental physicist. He worked at Willow Run which was a military research lab. This was the main airfield base for bombers in WWII. His wife was a theoretical physicist and she told him about the SYT identity which she learned in quantum mechanics. Schensted also made lots of board games. "People are into video games nowadays ... and not intellectual ones." Later he went to live in a cabin in the woods in Main with no electricity. "There were some other Ann Arbor mathematicians with the same pattern of behavior, but this was totally benign."

We describe the correspondence as an algorithm.

Input: $\sigma = \sigma_1\sigma_2 \cdots \sigma_n \in S_n$.
Output: SYT $P = P(\sigma), Q = Q(\sigma)$. Often called the P-tableau and the Q-tableau also called the "insertion tableaux" and the "recording tableaux".

Proposition 7.4.1 (Schensted's Algorithm)

$P := \varnothing; Q := \varnothing;$
for j from 1 to n:
  $k := \sigma_j$; $i := 1$; $\beta := \top$;
  while $\beta = \top$ **do**:
    if all entries in row i of P are $\leq k$:
      then
        append k at the end of row i of P;
        place j in the corresponding box of Q;
        $\beta := \perp$;
      else
        replace the leftmost entry $\ell > k$ in row i of P by k;
        $k := \ell$; $i := i + 1$;
return (P, Q)

Definition 7.4.2 (Schensted's Algorithm Gives a Bijection)

Pairs of SYT of the same shape consisting of n boxes are bijectively mapped to permutations by Schensted's Algorithm.

It is not that hard to show the numbers increasing moving down columns.

Schensted's proof analyzed this algorithm in scrupulous detail, but it leaves totally unclear why the algorithm works.

Theorem 7.4.3 (Schensted's Inverse Rule)

If $(P, Q) \longleftrightarrow \sigma$, then $(Q, P) \longleftrightarrow \sigma^{-1}$.
Corollary: $(P, P) \longleftrightarrow \tau$ for involution τ. Thus SYT, $\cong$ involutions in S_n.

This algorithm has at least two very different interpretations. These various incarnations make some of these properties more transparent.

We will now discuss these other interpretations of the Schensted's Correspondence.

If you want to build a bijective proof for which you have an algebraic proof, one way of doing so is to combinatorialize the algebraic argument. Suppose you have a polynomial in two different ways and want to show that these expressions are the same combinatorially.

Let's bijectivize the identity $DU = UD + I$ for $D, U \in \operatorname{End}(\mathbb{R}\mathbb{Y})$. Both the LHS and RHS can be restricted to a graded component of this vector space, so they are just matrices. Consider the (λ, μ)-matrix element of LHS/RHS. This is $e_\lambda DU e_\mu = e_\lambda (UD + I) e_\mu$. This is thee λ coefficient of $DU(\mu)$. If $\lambda \neq \mu$, then either the LHS and RHS are empty or both one (since there is a unique way to add a box and remove a box to get λ from μ). If $\lambda = \mu$, then the LHS is $\text{add}(\mu) \longleftrightarrow \text{del}(\mu) \cup \{\bullet\}$ (where $\bullet$ means no action). To bijectivize, move each deletable box one step south then move west until you hit μ (or in line with column one of μ) and you've landed on an addable box. In other words, to each inner corner in row i associate the outer corner in row $i + 1$ and associate $\bullet$ to row 1.

We will now use this to bijectivize Theorem 7.1.1. The proof relied on showing $D^n U^n\ \varnothing = n!\ \varnothing$.

$$D \cdots D U \cdots U$$

Keep swapping the inner D and inner U. Every D will annihilate some U. For if not then we will end up with $U^k D^k$ which is zero. So we just need to know which D annihilates which U. There are $n!$ ways of assigning each D to a U. $\square$

Growth Diagrams

Represent the SYT as a string of shapes that grows horizontally.

$$\varnothing \to 1 \times 1 \to 2 \times 1 \to \cdots$$

INSERT PICTURE

Propagation rules:

$$\begin{Bmatrix} \lambda & \subseteq & \nu \\ \cup\!\mid & & \cup\!\mid \\ \rho & \subseteq & \mu \end{Bmatrix}$$

1. if $\lambda = \nu = \mu$, then $\rho = \lambda$
2. if $\lambda \subset \nu \supset \mu$,

$$\begin{cases} \lambda \neq \mu \Longrightarrow \rho = \lambda \cap \mu \\ \lambda = \mu \Longrightarrow \rho = \varphi(\nu) \end{cases}$$

where φ is the bijectivization of the $\operatorname{add}(\mu) \longleftrightarrow \operatorname{del}(u) \cup \{\bullet\}$ identity.

Now put green $\boxtimes$'s where λ, ρ, μ are all the same and ν is different.

Observe that the green $\boxtimes$'s form a permutation pattern. The green $\boxtimes$'s are the locations where U and D annihilate each other. Movements around these $\boxtimes$ are commuting D, U, id.

In the scenario pictures above we get the permutation 412563.

Proposition 7.4.5 (Claims)

This yields a bijection between pairs (P,Q) and permutations.
Moreover, $(P, Q) \longleftrightarrow \sigma$ then $(Q, P) \longleftrightarrow \sigma^{-1}$.

This follows since we can grow the diagram from thee southwest and the only ambiguity of what to fill occurs at the locations with $\boxtimes$ boxes (so we know what to do: add a box in the first row).

Swapping (P, Q) to get (Q, P) corresponds to transposing the growth diagram which inverts the permutation.

<u>Back to Schensted</u>: The columns of the growth diagram are precisely the intermediate P-tableaux in the Schensted algorithm.

Therefore, one possible way to think about the Schensted algorithm is to parallelize it (since the local propagation rules commute).

Jeu De Taquin Equivalence

The Jeu De Taquin slides can be used to construct an equivalence relation on skew SYT. The slides can also be inverted.

Definition 7.4.7 (Jeu De Taquin Equivalence)

Two tableaux are **JDT-equivalent** provided that one can be obtained from the other by a sequence of slides (or their inverses).

Observe:

1. each JDT-equivalence class contains at least one straight-shape SYT, meaning that the skew portion of the partition is empty.
2. any skew-tableau is JDT-equivalent to a "permutation tableau." This is a skew-tableau where this is only one number in each column and each row. Also constructed by reading the rows of the tableau read left to right, bottom up. We call this the **reading word** of the original skew tableau T.
3. The Schensted correspondence can be emulated using JDT. The final tableaux obtained from rectification of a permutation tableau is the P-tableau given by the Schensted algorithm.

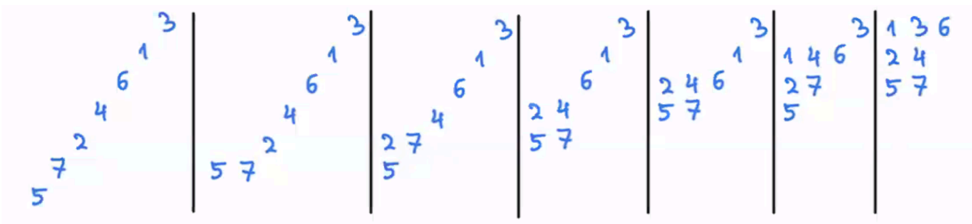

Figure 3: JDT rectification produces intermediate tableau from Schensted's algorithm.

Knuth Equivalence

This is due to D. Knuth's paper in 1970. We want to know which permutations are JDT-equivalent.

The non-increasing and non-decreasing permutations come in pairs of JDT-equivalence.

Definition 7.4.9 (Knuth Equivalence)

Knuth equivalence is the transitive closure of the JDT-equivalences. Explicitly, we can swap two consecutive entries x_1, x_2 if there is a witness on the left or right which is intermediate in value to x_1, x_2.

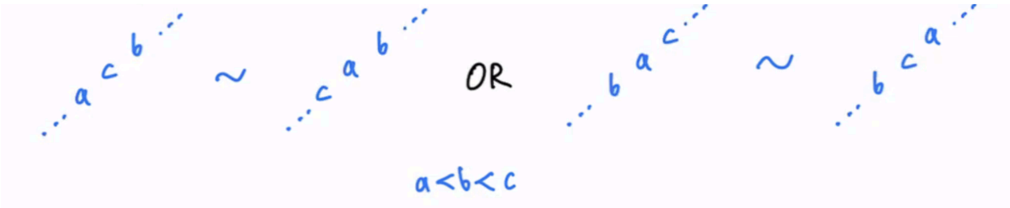

Figure 4: Knuth equivalence experiments.

It turns out that JDT-equivalence classes are equal to Knuth equivalence classes.

Lemma 7.4.10 (Preservation of Reading Word)

JDT slides preserve the Knuth equivalence class of the reading word of a tableau.

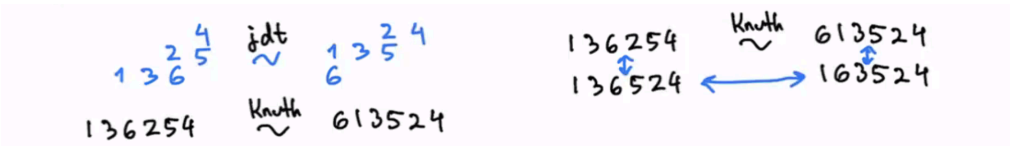

Figure 5: JDT slides can be given by Knuth moves.

We will give the idea of the proof because the actual proof is a bit technical.

Claim: The Knuth class of the reading word of a tableau remains invariant throughout all steps of the sliding process.

One has to show that moving a number laterally gives a Knuth move and moving horizontally does not change the Knuth word.

Homework 6.1

Complete the proof.

Corollary 7.4.12 (Preservation of Reading Word)

We obtain the following:

- Two permutations are Knuth equivalent if and only if the corresponding "permutation tableaux" are JDT-equivalent.
- Any permutation σ is Knuth equivalent to the reading word of its insertion tableau.

Moral of the Story: Knuth equivalence is a projection of JDT.

Greene's Invariants

This is based on the work of Curtis Greene from 1974. He was a professor at Haverford college and wrote several papers with Prof. Fomin.
Let σ be a permutation. Let

$$I_k(\sigma) \stackrel{\text{def}}{=} \max \# \text{ elements in a union of } k \text{ increasing subsequences of } \sigma$$

So I_1 gives the longest increasing subsequence. Take $\sigma = 236145$. Then $I_1(\sigma) = 4$ (2345), $I_2(\sigma) = 6$ (236 145). We cannot increase k further.
Define

$$D_\ell(\sigma) \stackrel{\text{def}}{=} \max \# \text{ elements in a union of } \ell \text{ decreasing subsequences of } \sigma$$

Then $D_1(\sigma) = 2$ (65), $D_2(\sigma) = 4$ (65 31). $D_4(\sigma) = 5$ (31 21 5). $D_5(\sigma) = 6$ (31 21 64). We cannot increase ℓ further.

Proposition 7.4.14 (Greene Invariants)

Let T be a straight shaped tableau, ν the shape of T, and σ the reading word of T. Then the Greene invariants of σ are given by

$$I_k(\sigma) = \nu_1 + \cdots + \nu_k$$
$$D_\ell(\sigma) = \nu_1' + \cdots + \nu_\ell'.$$

Proof.
By taking sequences given by the rows and columns of the tableau we get

$$I_k(\sigma) \geq \nu_1 + \cdots + \nu_k$$
$$D_\ell(\sigma) \geq \nu_1' + \cdots + \nu_\ell'.$$

The increasing subsequences can intersect the decreasing subsequences at most once. Therefore,

$$I_{k(\sigma)} + D_{\ell(\sigma)} \leq n + k\ell$$

The reverse direction holds because. □

Many permutations are not reaading words for example 126345. To reconstruct the SYT just break the permutation at all the descents.
In general Greene's invariants are difficult to compute.

Homework 6.2

Greene's invariants are preserved under Knuth Switches.

This can be done by swapping the tails of increasing subsequences of those using the swapped numbers.

From this follows the famous (in narrow circles) the following theorem.

Theorem 7.4.16 (Greene's Theorem)

Let $\lambda = |P(w)| = |Q(w)|$ be the common shape of the insertion tableau.

$$I_k(w) = \lambda_1 + \cdots + \lambda_k$$
$$D_\ell(w) = \lambda_{1'} + \cdots + \lambda_{\ell'}$$

Therefore,

$$\lambda_k = I_k(w) - I_{k-1}(w)$$
$$\lambda_{\ell'} = D_\ell(w) - D_{\ell-1}(w)$$

In particular, the formulas above define conjugate partitions. Very surprising!

Consequently the numbers $I_k(w)$ define the numbers $D_\ell(w)$.

Proof.

Let r_w be the reading word of $P(w)$ (read the tableau by rows bottom up). Now $w \sim r_w$ under Knuth equivalence due to JDT rectification of $P(w)$ into the maximal skew tableau placement of w.

It follows that w and r_w have the same Greene invariant. But

$$I_k(r_w) = I_k(w) = \lambda_1 + \cdots + \lambda_k$$
$$D_\ell(r_w) = D_\ell(w) = \lambda_{1'} + \cdots + \lambda_{\ell'}$$

$\square$

Schensted already noticed that the first row and column give the extrema for increasing/decreasing subsequences. Now all of this can be taught in an undergrad class.

The following theorem is normally stated for sequences of real numbers. But you can just replace equal entries by adding ε. It is a deep, special case of Theorem 7.4.16.

Corollary 7.4.17 (Erdos-Szekeres, 1935)

A permutation of length $ab + 1$ contains either an increasing subsequence of length $a + 1$ or a decreasing subsequence of length $b + 1$.

Proof.

Suppose the statement does not hold for a permutation w. Then $P(w)$ must be contained in $L(a,b)$ but this is not possible since there are $ab+1$ boxes. $\square$

This is in turn a special case of a special case of the following theorem.

Theorem 7.4.18 (C. Greene-D. Kleitman, 1972)

Let Π be a finite poset. Define $I_k(\Pi)$ as the maximal number in a union of k chains and $D_\ell(\Pi)$ as the maximal number in a union of k antichains. Define $\lambda_k = I_k(\Pi) - I_{k-1}(\Pi)$ and $\mu_\ell = D_\ell(\Pi) - D_{\ell-1}(\Pi)$. Then λ and μ are both partitions and they are conjugate to each other.

This theorem implies Dilworth's Theorem.

This is a very difficult theorem and we will not prove it. For any permutation we can create a Hasse Diagram in which a covers b if $b < a$ and a occurs after b. Now increasing (resp. decreasing) corresponds to chains (resp. antichains) in Π. In this poset if you take the reversed permutation then then the chains become antichains. In general Posets do not have the property that their incomparability graph is a poset.

This can be proved using standard techniques of linear programming: flows in networks, Unimodality, and matrices.

Theorem 7.4.19 (Growth Correspondence)

Let Π be a poset and attach a maximal element p. Now compute the Greene-Kleitman shape for Π. Then $\lambda(\Pi) \subset \lambda(\Pi \cup \{p\})$. Thus any linear extension of Π yields a SYT.

There are two natural ways to grow a permutation. The firsts is via order ideals moving from left to right adding the elements as they occur in the word. The second is given by always choosing the next largest number. Both of these growth processes give a SYT, and they are the P-tableau and the Q-tableau. Moreover, taking the intersections of order ideals in both methods gives the growth diagram of the Schensted algorithm via the Greene-Kleitman SYT.

The following theorem is due to Knuth in 1970.

Theorem 7.4.20 (Fundamental Theorem of Knuth Equivalence)

Each Knuth equivalence class contains exactly one reading word of a straight shape tableau.

Example:

$$[54123] = \{51243, 15243, 15423, 12543, 54123\}$$

We know that every equivalence class contains one reading word. It suffices to show that no two reading words of straight shape tableau are Knuth-equivalent.

Proof.

Suppose $r_P \sim r_{P'}$. Then both of these have the same Greene Invariants, consequently, $P = P'$. Now removing the largest entry n from both reading words does not break their Knuth equivalence since n is never a witness for a switch. Thus removing all entries greater than K from P and P' (and their reading words) results in Knuth-equivalent reading words (and so they have the same shape as tableaux). But now if the shape are the same after each removal then the SYT are the same. □

So the Knuth equivalence classes consist of all permutations that have a particular insertion tableaux.

The following theorem is due to Schutzenberger in 1976.

Theorem 7.4.21 (Fundamental Theorem of JDT)

Each JDT equivalence class contains exactly one straight shape tableau.

Since JDT slides acts as Knuth switches on the reading word. As a corollary JDT rectification is unique. This makes a rather difficult homework problem from earlier quite easy.