Math 566

Algebraic Combinatorics

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Chapter 1
Algebraic Graph Theory

Notes from the Syllabus:

- No Exams.
- There will be five or six problem sets.
- Office Hours: Tuesdays and Friday at 1pm.
- There will be almost no calculus or analysis.
- You should expect this class to make frequent use of linear algebra.
- You need to solve all but two problems correctly to get 100%.
- There will be very little partial credit given.
- You can either bring in a printed copy of the homework to class.

1.1 Linear Algebraic Preliminaries

Definition 1.1.1 (Characteristic Polynomial)

Let M be a $p \times p$ matrix in $\mathbb{C}$, the **monic characteristic polynomial** is

$$\det(tI - M) = \prod_{k=1}^{p} (t - \lambda_k)$$

where $\lambda_1, ..., \lambda_k \in \mathbb{C}$ are the p eigenvalues with multiplicity.

Lemma 1.1.2 (Eigenvalues of Matrix Polynomial)

If $f(t) \in \mathbb{C}[t]$ then $f(M)$ has eigenvalues $f(\lambda_1), ..., f(\lambda_k)$.

Proof.

M is diagonalizable, so conjugation commutes with taking powers and therefore computation of a polynomial. The statement is trivial up to general nonsense consideration.

Diagonalizable matrices are dense in the set of matrices. A matrix is only diagonalizable if there are multiple equal eigenvalues. Thus this is a subvariety within the set of matrices (obtained by imposing an algebraic condition of equal eigenvalues) which has dimension less than the set of matrices.

Now we can take a limit within the set of diagonalizable matrices and the limit converges to the general matrix. □

This statement can be extended to more general functions (those with converging power series).

Lemma 1.1.3 (Trace of Matrix)

The trace of a matrix M is the sum of its eigenvalues.

Proof.
The coefficient of t^{p-1} in $\det(tI - M)$ is

$$-\operatorname{tr}(M) = \sum_{i=1}^{p} -\lambda_k.$$

□

Combining this fact with Lemma 1.1.2 gives us that

$$\operatorname{tr}\left(M^\ell\right) = \sum_{k=1}^{p} \lambda_k^\ell.$$

Thus it is easy to compute the sum of powers of eigenvalues of M. This leads to an algorithm of finding roots of polynomials from power sums.
We can recover the multi-set $\left\{\lambda_1, ..., \lambda_p\right\}$ from the traces $\operatorname{tr}(M), \operatorname{tr}\left(M^2\right), ...$

Theorem 1.1.4 (Multiset Recovery)

Let $\alpha_1, ..., \alpha_r, \beta_1, ..., \beta_s \in \mathbb{C}$ such that $\forall \ell \in \mathbb{Z}^+$

$$\sum_{i=1}^{r} \alpha_i^\ell = \sum_{i=1}^{s} \beta_i^\ell \quad (\star)$$

then $r = s$ and the β's are permutations of the α's.

If the values where over the reals this would be easy since we could just observe the asymptotic behavior and remove each leading term. This cannot be applied to complex numbers since some may have the same modulus

Proof.
We will use the method of generating functions in a noncombinatorial sense. First multiply $(\star)$ by t^ℓ and sum giving,

$$\sum_{i=1}^{r} \frac{\alpha_i t}{1 - \alpha_i t} = \sum_{i=1}^{s} \frac{\beta_i t}{1 - \beta_i t}.$$

We use that $\sum_{k=1}^{\infty} = \frac{x}{1-x}$. Pick $\gamma \in \mathbb{C}$, and multiply by $1 - \gamma t$ and set $t = \frac{1}{\gamma}$. Take the limit $t \to \frac{1}{\gamma}$. Each term will become one or zero depending on whether the corresponding α_k or β_k equals γ or not.

After that, LHS will be the number of α's equal to γ and the RHS will be the number of β's equal to γ. This shows the multisets are equal. □

And what if I put a little more text

1.2 Enumerating Walks with Eigenvalues

We assume all basic vocabulary of basic graph theory. All graphs will be assumed finite. We allow loops & multiple edges.

Lemma 1.2.1 (Walks on a Graph)

Let G be a graph on the vertex set $[p]$. Then put $M = A(G)$, the **adjacency matrix** of G. So M_{ij} is the number of edges from i to j. Then M is symmetric, with entries in $\mathbb{Z}_{\geq 0}$. The number of walks of length ℓ from i to j is $\left(M^{\ell}\right)_{ij}$.

Proof.

Follows from matrix multiplication (also works for directed graphs). □

This suggests that there must be a way of counting walks using eigenvalues of the graph. But we will have to restrict to closed paths. A walk can move anywhere in the graph, a path cannot repeat vertices. A **marked closed walk** is one that starts and begins at the same vertex.

Proposition 1.2.2 (Marked Closed Walks)

The number of marked closed walks of length ℓ on G is $\sum_{k=1}^{p} \lambda_k^{\ell}$.

Proof.

The LHS equals $\operatorname{tr}\left(M^{\ell}\right)$ be observation and Lemma 1.1.3 gives the equality with the RHS. □

Example: Let $G = K_p$ (the complete graph).

Let J be the $p \times p$ matrix with all entries equal to 1. Then $J - I = A(G)$. Then $\operatorname{rank}(J) = 1$. So the eigenvalues of J are $0, ..., 0$ ($p - 1$ times) and p. Now $J - I$ is applying the polynomial $t - 1$ to J. Therefore by Lemma 1.1.2, its eigenvalues are $-1, ..., -1$ ($p - 1$ times) and $p - 1$. Thus the number of marked closed walks in G is

$$(p-1)^{\ell} + (p-1)(-1)^{\ell}.$$

Restatement: The number of marked closed walks of length ℓ in G is the number of $(\ell + 1)$-letter words in a p-symbol alphabet such that consecutive letters are distinct and first letter equals the last.

Homework 1.1

Show that the number of walks of length ℓ between two distinct vertices in K_p differs by one from the number of closed walks of length ℓ starting and ending at a given vertex. Use a simple combinatorial argument.

We can also compute eigenvalues via counting walks. Using combinatorics for linear algebra!

Example: Suppose $G = K_{m,n}$, the complete bipartite graph. Let's count the marked closed walks in G. This number is

$$\left(\sqrt{mn}\right)^{\ell} + \left(-\sqrt{mn}\right)^{\ell} + 0^{\ell} + \cdots + 0^{\ell} = \begin{cases} 0 \text{ if } \ell \bmod 2 = 0 \\ 2(mn)^{\frac{\ell}{2}} \text{ if } \ell \bmod 2 = 1. \end{cases}$$

Now we conclude from Theorem 1.1.4 that $K_{m,n}$ has eigenvalues $-\sqrt{mn}, \sqrt{mn}, 0, ..., 0$. No linear algebra involved!

Homework 1.2

Prove that the **diameter** (the largest minimal pairwise distance of a connected graph is strictly less than the number of distinct eigenvalues.

Inequalities for Maximal Eigenvalues

Proposition 1.2.6 (Max)

Let G be a graph on $[p]$. Denote $\lambda_{\max} = \max_i |\lambda_i|$. The Perron-Frobenius Theorem ensures that the largest eigenvalue is positive (hence equal to $\lambda_{\max}$).

$$\lambda_{\max} \leq \max_{v \in G} \deg(v)$$

Perron-Frobenius holds because a matrix with nonnegative entries can be though of in probabilistic terms (so each row adds to one). Now we have a Markov Chain and taking powers means considering all possible walks in the state-space. As the matrix is raised to higher powers the largest eigenvalue will dominate. The steady state must be positive (since we are taking powers of nonnegative entries) and have eigenvalue one since it is "steady."

Proof.

Informal: As you count your walks your number of choices at each step is at most $\operatorname{maxdeg}(G) = \left(\max_v \deg(v)\right)^{\ell}$. So the inequality must be in this direction since otherwise Lemma 1.2.1 cannot hold.

Formal: For any vector set $x = (x_i)$,

$$\max_j \left|(A(G)x)_j\right| = \max_j \left|\sum_{\text{edge } ij} x_i\right| \leq \text{maxdeg}(G) \cdot \max_k |x_k|.$$

Now assume that x is an eigenvector of $A(G)$, with eigenvalue λ. Then this inequality becomes $|\lambda| \cdot \max_k |x_k| \leq \text{maxdeg(G)} \cdot \max_k |x_k| \Rightarrow |\lambda| \leq \text{maxdeg(G)}$. This is just mathematical legalese (the argument is philosophical). $\square$

Homework 1.3

Prove that

$$\lambda_{\text{max}} \geq \frac{1}{p} \sum_{v \in G} \deg(v).$$

This is like a kind of convexity or Jensson Inequality.

Hint: Use that for symmetric real M,

$$\lambda_{\text{max}} = \max_{|x| = 1} x^T M x$$

Corollary: The number of closed walks grows exponentially in ℓ. With rate at least equal to the average degree.

1.3 Eigenvalues of Special Graphs

We now study the eigenvalues of graphs which take certain specified forms.

Block Anti-Diagonal Matrices

Now let's discuss in more detail the bipartite graphs. Whose matrices take the block form $\begin{bmatrix} 0 & * \\ * & 0 \end{bmatrix}$. We will use implicitly the concept of singular values.

Lemma 1.3.2 (Eigenvalues of Block Anti-Diagonal Matrix)

For a real matrix B, $M = \begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix}$ has nonzero eigenvalues $\pm\sqrt{\mu_i}$ where $\mu_1, \mu_2, \ldots$ are the eigenvalues of $B^T B$ counted with multiplicity. Note that $B^T B$ is real, symmetric, and most importantly positive semidefinite. Since $\langle B^T B x, x \rangle = \langle Bx, Bx \rangle \geq 0$ and the signature of a quadratic form is given by the signs of eigenvalues. Note that the $\mu_1, \mu_2, \ldots$ are the singular values of B.

The proof is going to be the worst kind of proof in mathematics.

For a $p \times p$ matrix X we will denote $F_X(t) := \det\big(tI_p - X\big)$.

Proof.

Suppose B is $n \times m$. Then

$$\begin{bmatrix} tI_n & -B \\ -B^T & tI_m \end{bmatrix} \begin{bmatrix} I_n & B \\ 0 & tI_m \end{bmatrix} = \begin{bmatrix} tI_n & 0 \\ -B^T & t^2 I_m - B^T B \end{bmatrix}.$$

Taking determinants gives

$$F_M(t) \cdot t^m = t^n \cdot F_{B^T B}(t^2).$$

Now we are after the eigenvalues which is precisely when $F_M(t)$ vanishes. Now we know that t^2 is an eigenvalue of $B^T B$ which means that t is a singular value of B. There exists a proof from the book of this fact but it is much longer. $\square$

Example: Let $G = K_{m,n}$. Let B be the $m \times n$ matrix of all ones. Then $B^T B$ is an $m \times m$ matrix of all n's. This is a rank one matrix so it has one nonzero eigenvalue and it is $\mathrm{tr}(A(G)) = nm$. Thus the eigenvalues of $A(G)$ are $\pm\sqrt{mn}, 0, ..., 0$.

Homework 1.4

Let G be the graph obtained by removing n disjoint edges from $K_{n,n}$. Find the eigenvalues of G.

Example: Let G be a $2n$-cycle, thus G is bipartite. Denote $A(G) = M_{2n}$ Then this matrix has the form $\begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix}$. Then $B^T B = 2I_n + M_n$ with the appropriate labeling. This is intuitive because we can return to our original vertex in two ways: counter clockwise or clockwise. [Why?]
This implies that the eigenvalues of G are $\pm\sqrt{2 + \lambda_i}$ where λ_i are the eigenvalues of M_n. Now compute the eigenvalues of M_n for small n (say 4). So the eigenvalues of 2^r-cycle are $\sqrt{2 \pm \sqrt{2 \pm \cdots}}$.

Circulant Graphs

We are talking about matrices which are act upon the set of basis vectors as a cyclic group.

Definition 1.3.5 (Circulant Matrix)

A square matrix is called **circulant** provided that

$$C = \begin{bmatrix} s_0 & s_1 & \cdots & s_{p-1} \\ s_{p-1} & s_0 & \cdots & s_{p-2} \\ \vdots & \vdots & \ddots & \vdots \\ s_1 & s_2 & \cdots & s_0 \end{bmatrix}$$

.

Lemma 1.3.6 (Eigenvalues of Circulant Matrices)

$$\lambda_k = \sum_{j=0}^{p-1} s_j e^{\frac{2\pi i}{p} jk}$$

for $k = 0, ..., p - 1$.

Remark: Let $s(x) = \sum_{j=0}^{p-1} s_j x^j$, then the eigenvalues of C are the values of $s(x)$ at the p-th roots of unity.

Proof.
Let T be the circulant matrix such that $e_1^T T = [0, 1, 0, ..., 0]$. T acts on the basis $e_1, ..., e_p$ by cyclically shifting. So T^k will again be a circulant matrix. Then $C = s(T)$. Thus finding the eigenvalues of T will give us the eigenvalues of C. The eigenvalues of C are the p-th roots of unity. By observation, $\det(tI_p - T) = t^p - 1$. Also $T^p = I_p$ so then T itself is a p-th root of unity and so it must be that the eigenvalues of T are p-th roots of unity.
The claim follows. □

Definition 1.3.7 (Circulant Graph)

A graph G is **circulant** if $A(G)$ is circulant for some labeling of G.
Corollary: The eigenvalues of the p-cycle are $2\cos\left(\frac{2\pi k}{p}\right)$ for $k = 0, ..., p - 1$.

Proof.
Let G be a p-cycle. So $A(G) = T + T^{-1} = T + T^{p-1}$. Nowa, Lemma 1.1.2 implies that the eigenvalues of G are

$$e^{\frac{2\pi i}{p}k} + e^{\frac{2\pi i}{p}k(p-1)} = e^{\frac{2\pi i}{p}k} + e^{-\frac{2\pi i}{p}k} = 2 \cdot \Re\left(e^{\frac{2\pi i}{p}k}\right) = 2\cos\left(\frac{2\pi k}{p}\right).$$

□

Recall: The eigenvalues of a $2n$-cycle are $\pm\sqrt{2 + \mu_i}$ where μ_i is an eigenvalue of the n-cycle. Note that this is consistent with the more general corollary above via the cosine double angle identity.

Homework 1.5

Find the eigenvalues of the graph obtained by removing n disjoint edges from K_{2n}. Note that this graph is the 1-skeleton of the n-dimensional **cross polytope** (i.e. the **hyperoctahedron**).

The hyperoctahedron is the n-dimensional polytope which is dual to the n-cube. Thus it is the convex hull of the set $\{\pm e_i : i \in [n]\}$. The only non-adjacent vertices are the vertices at e_i and $-e_i$. Counting walks in the hyperoctahedron counts walks on the faces of the hypercube.

Cartesian Products of Graphs

Cartesian products is a general categorical construction.

Definition 1.3.10 (Product of Graphs)

Let G, H be graphs. Then $G \times H$ is the **Cartesian product** of G and H with vertex set $V_{G\times H} = V_G \times V_H$ and edges of two kinds $(g,h)(g',h)$ for edge $gg' \in E_G$. and $(g,h)(g,h')$ for edge $hh' \in E_H$.

Example:

- Let $G = \bullet - \bullet - \bullet$ and $H = \bullet - \bullet - \bullet - \bullet$. Then $G \times H$ is the 3×4 grid graph.
- The n-cube graph (skeleton of the n-cube) is isomorphic to $\underbrace{K_2 \times \cdots \times K_2}_{n \text{ times}}$.

Proposition 1.3.11 (Eigenvalues of Direct Product)

Suppose G have eigenvalues $\lambda_1, \lambda_2, \ldots$ and H has eigenvalues $\mu_1, \mu_2, \ldots$ then $G \times H$ has eigenvalues $\lambda_i + \mu_j$ for all i, j.

We will give two proofs. The first is highbrow. The second is elementary.

Proof.

Take two vectors u, v in the space of formal linear combinations of the vertices of G and H (i.e. functions on the vertex space of the graphs). Put $u = \sum \alpha_g g$ and $v = \sum \beta_h h$. Define the tensor product of two vectors

$$u \otimes v \stackrel{\text{def}}{=} \sum_g \sum_h \alpha_g \beta_h (g,h).$$

Then we claim that the matrix

$$A(G \times H)(u \otimes v) = A(G)u \otimes v + u \otimes A(H)v.$$

This is true for the basis vectors of the tensor space $V_G^* \otimes V_H^*$ thus it holds for any u and v by linearity. Now if u and v are eigenvectors with eigenvalues λ and μ for G and H respectively, then, we have

$$A(G \times H)(u \otimes v) = \lambda u \otimes + u \otimes \mu v = (\lambda + \mu)(u \otimes v).$$

So $u \otimes v$ is an eigenvector for $G \times H$ with eigenvalue $\lambda + \mu$. □

Proof.

A marked closed walk of length ℓ in $G \times H$ is a **shuffle** (i.e. interleaving) of marked closed walks of length k and $\ell - k$ in G and H, respectively. Hence, the number of such walks is

$$\sum_{k=0}^{\ell} \binom{\ell}{k} \left(\sum_i \lambda_i^k \right) \left(\sum_j \mu_j^{\ell-k} \right) = \sum_i \sum_j \sum_k \binom{\ell}{k} \lambda_i^k \mu_j^{\ell-k} = \sum_i \sum_j (\lambda_i + \mu_j)^\ell.$$

Where the first equality follows from nontrivial summation rearrangement and the second follows from the binomial theorem. Now by Theorem 1.1.4 we have found the eigenvalues. □

The first proof is better because there is no computation. Also the second proof does not give us the eigenvectors.

Homework 1.6

Find the number of marked closed walks of length ℓ in the 3×3 grid graph. Note that this graph is bipartite so we should get zero for odd ℓ.

Homework 1.7

Find the eigenvalues of the 1-skeleton of an octagonal prism. Note $G = C_2 \times C_8$.

Eigenvalues of the Cube

Let $Q_n = (K_2)^n$ be the 1-skeleton of the n-cube.

Proposition 1.3.15 (Eigenvalues of Cube Graph)

The eigenvalues of Q_n are

$$\left\{ \left(n - 2k; \binom{n}{k} \right) : k = 0, ..., n \right\}.$$

Corollary: The number of marked closed walks in Q_n of length ℓ is

$$\sum_{k=0}^{\ell} \binom{n}{k} (n - 2k)^\ell.$$

Also this number must be divisible by 2^n and zero for odd ℓ since Q_n is bipartite.

Proof.

The eigenvalues of K_2 are ± 1. Now recalling Proposition 1.3.11, we see that the eigenvalues of Q_n are all possible sums of the form $\underbrace{\pm 1 \pm 1 \pm 1 \cdots \pm 1}_{n \text{ times}}$. $\square$

Homework 1.8

Find a direct proof of the formula using generating functions.

Now we have enough background to consider **random walks** in the cube. We will consider Brownian motion within a graph. When G is a **regular** graph of degree d on p vertices, a **simple** random walk on G proceeds by choosing (uniformly at random) an adjacent vertex and moving into it.

Instead of regularity, we will assume the stronger condition that any two vertices are related by an automorphism of G, meaning the group of automorphisms of G acts transitively on G.

Assuming that the random walk originates at v,

$$\begin{aligned}\mathbb{P}\{\text{after } \ell \text{ steps we return to } v\} &= d^{-\ell} \cdot \#\{\text{closed walks of length } \ell \text{ starting at } v\} \\ &= \frac{1}{d^\ell p} \cdot \sum_{i=1}^{p} \lambda_i^\ell.\end{aligned}$$

Example: For Q_n we put $d = n$ and $p = 2^n$, the probability of return after ℓ steps is

$$\mathbb{P}\{\text{return after } \ell \text{ steps}\} = \frac{1}{n^\ell 2^n} \sum_{k=0}^{n} \binom{n}{k} (n-2k)^\ell.$$

Homework 1.9

Find the analogous probability of return for the discrete torus with nm vertices (the product of an n-cycle and an m-cycle). Alternatively, this is the quotient of the grid graph by its “boundary” graph.

Note that the discrete torus satisfies the automorphism transitivity assumption.

Eigenvalues of a Chain Graph

Let G be the n-cycle with one edge removed. This graph is not regular.

$$A \overset{\text{def}}{=} A(G) = \begin{bmatrix} 0 & 1 & & \\ 1 & 0 & 1 & \\ & 1 & 0 & 1 \\ & & & \ddots \end{bmatrix}$$

Proposition 1.3.19 (Eigenvalues of Chain Graph)

The eigenvalues of A_n are $2 \cdot \cos\left(\frac{\pi k}{n+1}\right)$.

Proof.

The characteristic polynomial of A_n is $T_{n(t)} = \det(tI_n - A_n)$. It turns out this determinant is closely related to Chebyshev Polynomials.

Consider a more general determinant (of Toeplitz Matrices)

$$h_n(a,b) = \begin{bmatrix} a+b & -ab & & \\ -1 & a+b & -ab & \\ & -1 & a+b & -ab \\ & & & \ddots \end{bmatrix}.$$

Exploring small values we see that

$$\begin{aligned} h_1(a,b) &= a+b \\ h_2(a,b) &= a^2+ab+b^2 \\ h_3(a,b) &= (a+b)^3 - 2ab(a+b) \\ &= a^3+a^2b+ab^2+b^3. \end{aligned}$$

This suggests the pattern that

$$h_{n(a,b)} = \sum_{k=0}^{n} a^{n-k}b^k = \frac{a^{n+1}-b^{n+1}}{a-b}.$$

We will use a recurrence to prove this formula. We expand the determinant in the last row

$$h_n(a,b) = (a+b)h_{n-1}(a,b) - ab \cdot h_{n-2}(a,b).$$

The proof follows from induction on n. All that must be done is to plug the formula into the recurrence and check that it holds. This trick of expanding the determinant into a recurrence is common for (tri)-diagonal matrices.

Now set $b := \frac{1}{a}$. Then

$$T_n(a+a^{-1}) = h_n(a,a^{-1}) = \frac{a^{n+1}-a^{-(n+1)}}{a-a^{-1}}.$$

This vanishes when $a^{2(n+1)} = 1$ but $a^2 \neq 1$. That is, $a = e^{\frac{2\pi ik}{2(n+1)}} = e^{\frac{\pi ik}{n+1}}$ for $k \in [n]$ since 1 and $n+1$ are forbidden by our vanishing rules.

Summing $t = a + a^{-1} = 2 \cdot \cos\left(\frac{\pi k}{n+1}\right)$. $\square$

Chapter 2

Tilings, Trees, and Networks

2.1 Domino Tilings

Definition 2.1.1 (Domino Tiling)

A **domino tiling** is a tiling of the plane $\mathbb{Z} \times \mathbb{Z}$ by 2×1 rectangles. A tiling is **cryptomorphic** to a perfect matching in on $\mathbb{Z} \times \mathbb{Z}$.

This may seem like a recreational mathematics question, but it arises in applications. The most common is statistical mechanics. Consider the center of unit squares as locations in a crystal (think NaH). So this is related to dimer models and the ising model. This problem was solved in the late 1950s by physicists.

The Permanent-Determinant Method

The method we will consider was invented by Peter W. Kastelen (1960). We will consider an $n \times m$ board and suppose n is even. Let G be the bipartite graph which is the product of the m-chain and the n-chain. Put

$$M := A(G) = \begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix}.$$

Where B is an $\frac{m}{2} \times \frac{m}{2}$ matrix. Choosing a perfect matching in G amounts to choosing exactly one nonzero entry from each row and column in B. This should perk your ears and make you think of a determinant.

Definition 2.1.3 (Permanent)

The permanent of an $n \times n$ matrix A is

$$\operatorname{per}(B) = \sum_{\sigma \in S_n} \prod_{i \in [n]} a_{i,\sigma(i)}.$$

This is notoriously difficult computationally (should be exponentially hard).

Let $T(m, n)$ be the number of perfect tilings in G. Perfect matchings in G are in bijection with nonvanishing terms in $\det(B) = \operatorname{per}(B)$.

The key trick is to turn these permanents into determinants. Let $\tilde{B}$ be the matrix obtained from B by replacing the 1s that correspond with <u>vertical</u> tiles with i.

Lemma 2.1.4 (Determinant Permanent Equivalence)

Kastelen proved that

$$\det\left(\tilde{B}\right) = \pm\ \mathrm{per}(B) = \pm T(m,n).$$

The homework problem below is due to Bill Thurston. "This is a problem for a very very smart ten year old."

Homework 2.1

Show that any two tilings of a rectangular board can be connected by a sequence of "flips."

One way of showing this is by connecting every tiling to a canonical tiling.

We will use this result to show Lemma 2.1.4.

Proof.

In a vertical 2×2 block we have the submatrix of those four entries looks like $\begin{bmatrix} i & * \\ * & i \end{bmatrix}$. A flip then takes it to $\begin{bmatrix} * & 1 \\ 1 & * \end{bmatrix}$. So the flip replaces a product $\cdots 1 \cdot 1 \cdots$ with $\cdots i \cdot i \cdots$ and these products are equal up to exactly one transposition. Thus the term in the permanent and the term in the determinant are equal. The horizontal tiling is a canonical tiling which gives a real term in the determinant equal to ± 1. Now using Homework 2.1 we are done. □

Now we are prepared to enumerate the number of domino tilings in G. Put

$$\widetilde{M} := \begin{bmatrix} 0 & \tilde{B} \\ \tilde{B}^T & 0 \end{bmatrix} \Rightarrow \det\left(\widetilde{M}\right) = \pm \det\left(\tilde{B}\right) \det\left(\tilde{B}^T\right) = \pm(T(m,n))^2$$

by Lemma 2.1.4.

Recall that A_n is the adjacency matrix of an n-chain. Then

$$\widetilde{M} = I_m \otimes A_n + iA_m \otimes I_n.$$

By Proposition 1.3.11, this implies that the eigenvalues of $\widetilde{M}$ are sums of eigenvalues of A_n and A_m. Recall that the eigenvalues of A_n are $2 \cdot \cos\left(\frac{\pi k}{n+1}\right)$ for $k \in [n]$. We now conclude that,

$$\begin{aligned} \det\left(\widetilde{M}\right) &= \prod_{j \in [n]} \prod_{k \in [m]} \left(2 \cdot \cos\left(\frac{\pi j}{n+1}\right) + 2i \cdot \cos\left(\frac{\pi k}{m+1}\right)\right) \\ &= \pm \prod_{j \in \left[\frac{n}{2}\right]} \prod_{k \in [m]} \left(4\cos^2\left(\frac{\pi j}{n+1}\right) + 4\cos^2\left(\frac{\pi k}{m+1}\right)\right) \end{aligned}$$

The first equality follows since n is even, $n+1$ is odd and so the cos values from the RHS come in pairs which cancel out each others imaginary parts.
Since $T(m,n) = \sqrt{|\det(\widetilde{M})|}$, we obtain: a theorem by P. Kasteleyn (1961), M. Fisher and N.Temperley (1961).

Theorem 2.1.6 (Enumeration of Domino Tilings)

$T(m,n)$ has two formulas. If m is odd then,

$$T(m,n) = \prod_{j\in[\frac{n}{2}]} 2\cos\left(\frac{\pi j}{n+1}\right) \prod_{k\in[\frac{m-1}{2}]} \left(4\cos^2\left(\frac{\pi j}{n+1}\right) + 4\cos^2\left(\frac{\pi k}{m+1}\right)\right)$$

If m is even then, we get

$$T(m,n) = \prod_{j\in[\frac{n}{2}]} \prod_{k\in[\frac{m}{2}]} \left(4\cos^2\left(\frac{\pi j}{n+1}\right) + 4\cos^2\left(\frac{\pi k}{m+1}\right)\right)$$

<u>Note</u>: $T(8,8) = 12,988,816 = 3604^2$. It is a difficulty theorem to prove, but for any square board whose length is divisible by four, the number of tilings is a perfect square. And otherwise if it is an square board that can be tiled, then it will be twice a perfect square.
But the physicists don't care about this they want to know the asymptotics.
Let $m = n$. Let's take the log

$$\begin{aligned}
\frac{\log T(n,n)}{n^2} &= \frac{1}{n^2} \sum_{j\in[\frac{n}{2}]} \sum_{k\in[\frac{n}{2}]} \log\left(4\cos^2\left(\frac{\pi j}{n+1}\right) + 4\cos^2\left(\frac{\pi k}{m+1}\right)\right) \\
&\sim \frac{1}{\pi^2} \sum\sum \left(\frac{\pi}{n+1}\right)^2 \log\left(4\cos^2\left(\frac{\pi j}{n+1}\right) + 4\cos^2\left(\frac{\pi k}{m+1}\right)\right) \\
&\sim \frac{1}{\pi^2} \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \left(\frac{\pi}{n+1}\right)^2 \log\left(4\cos^2\left(\frac{\pi j}{n+1}\right) + 4\cos^2\left(\frac{\pi k}{m+1}\right)\right) dxdy \\
&= \frac{K}{\pi}
\end{aligned}$$

So now we have a Riemann sum (we are multiplying the area of the square $4\cos^2 x + 4\cos^2 y$ evaluated at the point $\left(\frac{\pi j}{n+1}, \frac{\pi k}{n+1}\right)$.
Where K is the Catalan Constant

$$K = 1 - \frac{1}{9} + \frac{1}{25} - \frac{1}{49} + \cdots = \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{t}{\sin(t)} dt$$

So $T(n,n) \sim 1.34^{n^2}$. To the physicists it is obvious that the complexity is c^{n^2} for some $c \in \mathbb{R}^+$. This is because placing two dominoes are independent in expectation.

2.2 Spanning Trees

The following theorem is due to H.N.V. Temperley (1974).

Theorem 2.2.1 (Tilings & Spanning Trees)

The number of domino tilings of the $(2k-1)\times(2\ell-1)$ rectangle without a corner box is equal to the number of spanning trees of the $k\times\ell$ grid graph.

Homework 2.2

Prove that this rule produces a tree. Something connected and acyclic. (A cycle requires an odd number of squares)

Proof.
for each domino which covers a node with two odd coordinates, draw a segment in the direction it is pointing. This will create a spanning tree by Homework 2.2.
For the other direction, direct each edge in the spanning tree to the node in the missing corner. Then draw dominoes in the dual graph. The remaining area uncovered by the dominoes splits into a rooted forest in the dual graph of the original board. Each tree in the forest grows from (is rooted) somewhere on the boundary of the dual graph. This gives us a unique domino placement for the uncovered areas. Since if the uncovered area contained a cycle, then the original tree would not be spanning (or maybe not a tree). □

Homework 2.3

Prove that if we remove a different box from the boundary, we get the same count (so long as a tiling is possible). You should build this bijection via a composition of two bijections.

Corollary 2.2.4 (Asymptotics of Spanning Trees)

The number of spanning trees in an $n\times n$ grid is $\sim e^{\frac{4K}{\pi}n^2}\approx 2.24^{n^2}$.

We can generalize to arbitrary planar graphs. Suppose P is a polygon on $\mathbb{R}^2$. Subdivide P into smaller polygons (not necessarily convex) to get a graph G. This description gets rid of annoying planar graphs (those have vertices of degree one, unconnected, etc.). Obtain a new graph H by placing a vertex inside each face of G and the midpoint of each edge of G. For each face F, connect each vertex in the F to those on the midpoints of the edges of F.

Homework 2.3

Fix a vertex v of P (hence of G and H). Show that the number of spanning trees on G is equal to the number of perfect matchings on $H \setminus \{v\}$.

Example: If G is the graph on the pentagon labeled cyclically with $[5]$ and adding the edge $\{1, 3\}$ then there are $4 + 4 + 3 = 11$ spanning trees. Then H is isomorphic to the 4×3 grid graph which also has 11 perfect matchings.

Chapter 3

The Diamond Lemma

3.1 Statement and Preliminaries

The Diamond Lemma is a general mathematical argument. There aren't that many: induction, pigeon-hold principle, and switching summations.

Definition 3.1.1 (One Player Game)

A **one-player game** is defined by the set of positions S, for each position $s \in S$, there is a set of positions $s' \neq s$ into which a player can move. We will use notation $s \rightsquigarrow s'$. If this set is empty, then s is called **terminal**.

Definition 3.1.2 (Play Sequence)

A **play sequence** is a sequence of the form $s \rightsquigarrow s' \rightsquigarrow s'' \rightsquigarrow \cdots$ (either finite or infinite). A **terminating game** is one without an infinite play sequence. A game is **confluent** if its outcome is determined by the initial position.

Lemma 3.1.3 (Diamond Condition)

Suppose that a game is terminating and satisfies ($\diamondsuit$) $\forall s \in S$ and any $s \rightsquigarrow s'$ and $s \rightsquigarrow s''$, there exists a $t \in S$ such that $s' \rightsquigarrow \cdots \rightsquigarrow t$ and $s'' \rightsquigarrow \cdots \rightsquigarrow t$. Then the game is confluent.

Note that if a game is confluent, then the game satisfies ($\diamondsuit$).

Proof.
Color each position s red if there exists to different terminal positions reachable from s. Color each position green otherwise. We want to show that there are no red positions. We know that every terminal position is green. Suppose s is red. Then make moves into red positions while we can. Since the game is terminating, at some point we must choose a terminal position. Thus we will have $s \rightsquigarrow s$‘ for all $s \rightsquigarrow s'$. Since s is red, there must be distinct $s \rightsquigarrow s'$ $s \rightsquigarrow s''$. But since s', s'' are green then they have unique terminal end points t_1, t_2. But now invoking ($\diamondsuit$) shows that $t_1 = t_2$ so s', s'' cannot be green. Contradiction! $\square$

Typically, we will use the Diamond Lemma to show something is well defined.

3.2 Applications of the Diamond Lemma

Definition 3.2.1 (Chip Firing)

Let $G = (V, E)$ be a finite directed simple graph. Let t be a **sink**, reachable from any vertex in G. A **chip configuration** is a placement of a nonnegative integer x_v at each $v \in V$. A move in the chip firing game occurs by choosing a $v \neq t$ satisfying $x_v \geq \text{outdegree}(v)$, then "fire" by sending one chip along each outward edge from v.

So after a move,

$$x_v := x_v - \text{outdeg}(v) \quad x_w := x_w + 1$$

$\forall w$ incident to v.

$3 \to 3 \to 3 \to 0$

Theorem 3.2.2 (Chip Firing is Confluent)

Chip Firing is confluent.

Proof.

We employ the Diamond Lemma. Suppose there is a position with $v, v' \in V$ which can both be fired. The order which we fire v and v' does not matter, so the fires at v and v' commute $\Rightarrow (\Diamond)$. $\square$

Thus any game will have the same number of moves.

Definition 3.2.3 (Young Diagram)

A **Young Diagram** is a sequence of unit boxes arranged in tabular format which monotonically decrease in length. Young Diagrams of size n are in bijection with integer partitions of n.

Definition 3.2.4 (Skew Shape)

A **Skew Diagram** is obtained from a Young Diagram via removing a left aligned sub-(Young Diagram). Alternatively a Skew Shape can be defined as a convex region of $\mathbb{Z} \times \mathbb{Z}$ where convexity is understood as (having empty squares when top left and bottom right corners are missing).

Think of Young Diagram as chocolate bars and Skew Shapes as bitten Chocolate bars and Cores as nuts (or maybe bolts that would break your teeth).

Definition 3.2.5 (Cores)

The positions of the game are Young Diagrams and the moves consist of removing dominoes from the southeast. The 2-**core** of a Young Tableaux is the remainder after the removal game has ended. Cores are uniquely determined, because taking non-overlapping bites commutes (and two overlapping bites can result in the same aftermath if the whole 2×2 square that the dominoes occupy is removed).

Fix $p \in \mathbb{Z}_{\geq 0}$. Modify the game by only allowing removals of border strips of length p. The border is the set of Young Diagram is the set of unit boxes which have no southeast neighbor.

Homework 2.4

Show that the generalized Core game is confluent. So the p-core of a Young Diagram is well defined.

"A curious ten year-old would be impressed by this … of course a regular ten year-old would not care."

We did a quick grammar lesson on when to use "well defined" and "well-defined."

Now it is natural to ask how to describe the uniquely determined output from a game that satisfies the Diamond Lemma.

Note that tableaux is the plural of tableau.

Definition 3.2.7 (Standard Young Tableaux)

The **Standard Young Tableau** of a skew shape is given by filling it with numbers in $[n]$ so that the numbers increase along the rows and columns.

These Young Tableaux play important roles in the representation of linear groups.

The following game was introduced by M.-P. Schutzenberger (~1970).

Definition 3.2.8 (Jeu De Taquin or The Teasing Game)

Let the positions of the game be Standard Young Tableaux. Let the moves be "slides."

Homework 2.5

Show that Jeu De Taquin is confluent. This problem is not easy.