# Honors Analysis II

# Math 396

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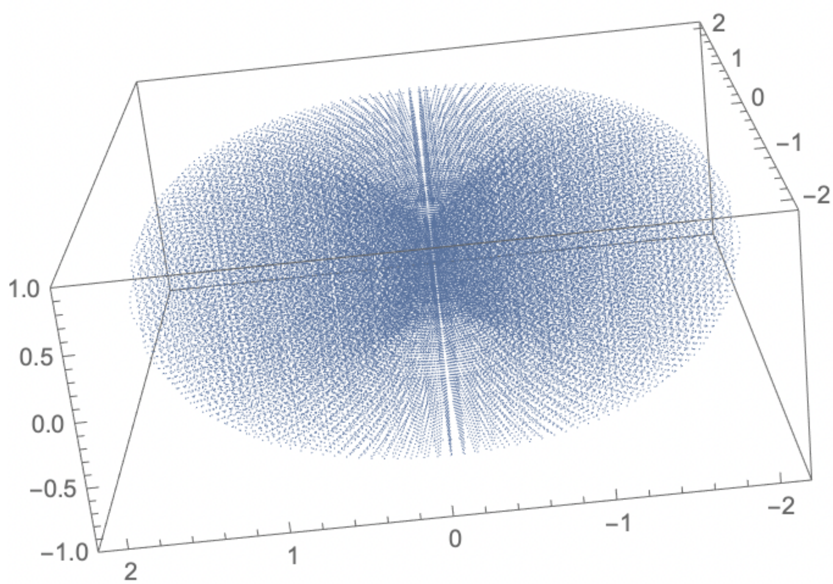

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# INTRODUCTION & MOTIVATION

Textbooks:

(i) Munkres, Analysis on Manifolds

(ii) Spivak, Calculus on Manifolds.

(iii) (Possibly) Fourier Analysis, an Introduction.

Content:

Manifolds are $k$-dimensional objects embedded in ambient $n$-dimensional space. We will be interested in integration over manifolds. Next, we will study differential forms which are generalizations of functions and vector fields. We will then integrate differential forms on manifolds which will lead us to the celebrated **Stokes Theorem**. Stokes Theorem describes the relationship between the integral over a manifold and its boundary. We will study many classical examples.

# EUCLIDEAN DIFFERENTIABLE MANIFOLDS

## Motivation

Informally, a **topological manifold** is a topological space that is **homeomorphic** to Euclidean space. This means a manifold, locally, looks like $\mathbb{R}^k$.

For example, $\mathbb{S}^1$ is a manifold because when we "zoom into" the circle it looks like a line. Also $\mathbb{S}^1 \times \mathbb{S}^1$ is a manifold because donuts look locally like a plane (see front cover).

We want to do calculus on manifolds, so we need to add more structure. A **differentiable manifold** is a special type of topological manifold that is "smooth."

**Proposition**. (Volume of a Parallelepiped)
If $v_1, \ldots, v_n \in \mathbb{R}^n$ are linearly independent. The volume of the parallelepiped generated by $v_1, \ldots, v_n$ is the absolute value of

$$\det \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix}$$

We want to determine the $k$-dimensional volume of a parallelepiped determined by $k$ vectors in $\mathbb{R}^n$. Since there are only $k$-vectors, appending them into a matrix will form a non-square matrix—we cannot use the determinant.

**Definition**. (Volume of Parallelepiped)
Let $k \leq n$, Let $M(n,k)$ be the space of $n \times k$ matrices. Define $V : M(n,k) \to [0,\infty)$ by

$$V(X) = \sqrt{\det(X^T X)}$$

Suppose $x_1, \ldots, x_k \in \mathbb{R}^n$ are linearly independent. We define the **k-dimensional volume** of the parallelepiped generated by $x_1, \ldots, x_k$ to be $V(X)$ where

$$X = \begin{bmatrix} | & & | \\ x_1 & \cdots & x_k \\ | & & | \end{bmatrix}$$

This is well defined because $X^T X$ is a positive definite matrix (it has positive determinant).

Examples:

(i) $k = n$ (should agree with previous proposition)

$$V(X) = \sqrt{\det(X^T X)} = \sqrt{\det(X) \cdot \det(X)} = |\det(X)|.$$

(ii) $k = 1$ (should agree with length of vector)

$$\sqrt{v^T v} = \|v\|.$$

(iii) $k = 2$ and $n = 3$ (should agree with cross product of the rectangular sides)

$$X = \begin{bmatrix} | & | \\ a & b \\ | & | \end{bmatrix} \implies X^T X = \begin{bmatrix} a^T a & a^T b \\ b^T a & b^T b \end{bmatrix} \implies \det(X^T X) = \det \begin{bmatrix} \|a\|^2 & a \cdot b \\ a \cdot b & \|b\|^2 \end{bmatrix}$$

$$= \|a\|^2 \|b\|^2 - (a \cdot b)^2 = \|a\|^2 \|b\|^2 \sin \theta$$

So $\det(X^T X) = \|a \times b\|^2$.

Generalizing this, we obtain an interesting fact about the determinant of the product of matrices.

**Definition**. (Ascending $k$-tuple)
Let $k \leq n$.

(a) An **ascending $k$-tuple** from the set $[n]$ is $I = (i_1, \ldots, i_k)$ satisfying $1 \leq i_1 \leq \cdots \leq i_k$.

(b) Denote by $\text{ASC}_{k,n}$ the set of all ascending $k$-tuples from $[n]$.
So $|\text{ASC}_{k,n}| = \binom{n}{k}$ and $\text{ASC}_{k,n} \cong \binom{[n]}{k}$.

**Theorem**. (Cauchy-Binet Identity)
Let $k \leq n$. If $A \in M(k, n)$ and $B \in M(n, k)$, then

$$\det(AB) = \sum_{\text{ASC}_{k,n}} \det(A^I) \det(B_I)$$

where for $I = (i_1, \ldots, i_k)$, $A^I$ is the $k \times k$ submatrix of $A$ containing the columns $i_1, \ldots, i_k$ and $B_I$, is the $k \times k$ submatrix of $A$ containing the rows $i_1, \ldots, i_k$.

**Corollary**. For $k \leq n, X \in M(n, k)$

$$V(X)^2 = \det(X^T X) = \sum_{\text{ASC}_{k,n}} (\det X_I)^2$$

The corollary generalizes the Pythagorean Theorem.

Check directly for a $2 \times 3$ matrix.

*Proof.*
We will prove for $k = 2$ and $n$ arbitrary.

$$\det(AB) = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ \vdots & \vdots \\ b_{n1} & b_{n2} \end{bmatrix}$$

So,

$$\begin{aligned}
\det(AB) &= \det \begin{bmatrix} \sum_{i\in[n]} a_{1i}b_{i1} & \sum_{i\in[n]} a_{1i}b_{i2} \\ \sum_{j\in[n]} a_{2j}b_{j1} & \sum_{j\in[n]} a_{2j}b_{j2} \end{bmatrix} && \text{matrix product} \\
&= \sum_{i\in[n]}\sum_{j\in[n]} \det \begin{bmatrix} a_{1i}b_{i1} & a_{1i}b_{i2} \\ a_{2j}b_{j1} & a_{2j}b_{j2} \end{bmatrix} && \text{det is multilinear} \\
&= \sum_{i\in[n]}\sum_{j\in[n]} a_{1i}a_{2j} \det \begin{bmatrix} b_{i1} & b_{i2} \\ b_{j1} & b_{j2} \end{bmatrix} && \text{det is multilinear} \\
&= \sum_{i\in[n]}\sum_{j\in[n]} \delta_{ij} \cdot a_{1i}a_{2j} \det \begin{bmatrix} b_{i1} & b_{i2} \\ b_{j1} & b_{j2} \end{bmatrix} && \text{det is alternating} \\
&= \sum_{i\in[n]}\sum_{i<j} a_{1i}a_{2j} \det \begin{bmatrix} b_{i1} & b_{i2} \\ b_{j1} & b_{j2} \end{bmatrix} + \sum_{i\in[n]}\sum_{i>j} a_{1i}a_{2j} \det \begin{bmatrix} b_{i1} & b_{i2} \\ b_{j1} & b_{j2} \end{bmatrix} && \text{expansion of sum} \\
&= \sum_{i\in[n]}\sum_{i<j} a_{1i}a_{2j} \det \begin{bmatrix} b_{i1} & b_{i2} \\ b_{j1} & b_{j2} \end{bmatrix} + \sum_{j\in[n]}\sum_{j>i} a_{1j}a_{2i} \det \begin{bmatrix} b_{j1} & b_{j2} \\ b_{i1} & b_{i2} \end{bmatrix} && \text{permute } i \text{ and } j \\
&= \sum_{i\in[n]}\sum_{i<j} a_{1i}a_{2j} \det \begin{bmatrix} b_{i1} & b_{i2} \\ b_{j1} & b_{j2} \end{bmatrix} - \sum_{j\in[n]}\sum_{j>i} a_{1j}a_{2i} \det \begin{bmatrix} b_{i1} & b_{i2} \\ b_{j1} & b_{j2} \end{bmatrix} && \text{det is alternating} \\
&= \sum_{i\in[n]}\sum_{i<j} (a_{1i}a_{2j} - a_{1j}a_{2i}) \det \begin{bmatrix} b_{i1} & b_{i2} \\ b_{j1} & b_{j2} \end{bmatrix} && \text{factor} \\
&= \sum_{(i,j)\in \mathrm{ASC}_{2,n}} \det \begin{bmatrix} a_{i1} & a_{i2} \\ a_{j1} & a_{j2} \end{bmatrix} \det \begin{bmatrix} b_{i1} & b_{i2} \\ b_{j1} & b_{j2} \end{bmatrix} && \text{definition of det.}
\end{aligned}$$

$\square$

# Parametrized Manifolds

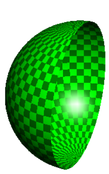
We will almost always use $n$ to denote the dimension of the ambient space and $k$ the subspace. We now turn to study manifolds given by a single patch, called parametrized manifolds.

**Definition**. (Parametrized Manifold)
Let $k \leq n$ and $A \subseteq \mathbb{R}^k$ be open. Let $\alpha : A \subseteq \mathbb{R}^k \to \mathbb{R}^n$ be a $C^1$ map. Put $Y = \alpha(A)$.
The pair $Y_\alpha = (Y, \alpha)$ is called a **parametrized manifold** of dimension $k$.

Examples:

(a) $\alpha : (0, 3\pi) \subseteq \mathbb{R} \to \mathbb{R}^2$ given by $\alpha(t) = (2\cos t, 2\sin t)$. Think of this manifold not as a circle but the trajectory of a particle that moves around the circle 1.5 times.

(b) $\alpha : (0, \pi) \times (0, \pi) \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ given by $\alpha(\theta, \phi) = (2\cos\theta\sin\phi, 2\sin\theta\sin\phi, 2\cos\phi)$. This is the portion of $\mathbb{S}^2$ in the positive $x$ quadrant.



(c) Let $\Omega \subseteq \mathbb{R}^n$ be open. Let $h : \Omega \to \mathbb{R}$ be a $C^1$ function. Put $\alpha : \Omega \to \mathbb{R}^{n+1}$ with $\alpha(x) = (x, h(x))$. Then $(G_h, \alpha)$ is a parametrized manifold.

We want to compute the $k$-dimensional volume of parametrized manifolds, and in general compute integrals over them. We now define reasonable notions of length, area, and volume.

Take a rectangle in $A$ with vertex at $p$ and lengths $\Delta x_1, \Delta x_2$. Then it should be that the volume of this rectangle in the image is $\alpha(p + (\Delta x_i)e_i) - \alpha(p) \approx \frac{\partial \alpha}{\partial x_i}\Delta x_i$. So the volume in the image should be approximately the volume of the parallelepiped determined by $\frac{\partial \alpha}{\partial x_1}(p)\Delta x_1, \ldots, \frac{\partial \alpha}{\partial x_k}(p)\Delta x_k$ which is equal to $V(D\alpha(p))\Delta x_1 \Delta x_2 \cdots \Delta x_k$. Where

$$D\alpha = \begin{bmatrix} | & & | \\ \frac{\partial \alpha}{\partial x_1} & \cdots & \frac{\partial \alpha}{\partial x_k} \\ | & & | \end{bmatrix}.$$

This motivates the following definition

**Definition**. (Volume of Parametrized Manifold)
Let $k \leq n$, $A \subseteq \mathbb{R}^k$ be open, $\alpha : A \to \mathbb{R}^n$ be $C^1$. Set $Y = \alpha(A)$ and $Y_\alpha = (Y, \alpha)$.
Define the **volume** of $Y_\alpha$ as

$$v(Y_\alpha) = \int_A V(D\alpha)$$

For a continuous function $f : Y \to \mathbb{R}$, define the **integral** of $f$ over $Y_\alpha$ as

$$\int_{Y_\alpha} f dV = \int_A (f \circ \alpha) V(D\alpha)$$

if the RHS exists[a].

[a]Here we are using the concept of a Pullback.

Examples:

(1) $\alpha : (0, 3\pi) \subseteq \mathbb{R} \to \mathbb{R}^2$ given by $\alpha(t) = (2\cos t, 2\sin t)$.

$$D\alpha = \begin{bmatrix} -2\sin t \\ 2\cos t \end{bmatrix} \implies V(D\alpha) = \sqrt{4} = 2 \implies v(Y_\alpha) = \int_0^{3\pi} 2 = 6\pi$$

(2) For $k = 2, n = 3$ and $\alpha : A \subseteq \mathbb{R}^2 \to \mathbb{R}^3$.

$$D_\alpha = \begin{bmatrix} | & | \\ \frac{\partial \alpha}{\partial x} & \frac{\partial \alpha}{\partial y} \\ | & | \end{bmatrix} \implies V(D\alpha) = \left\| \frac{\partial \alpha}{\partial x} \times \frac{\partial \alpha}{\partial y} \right\| \implies v(Y_\alpha) = \int_A \left\| \frac{\partial \alpha}{\partial x} \times \frac{\partial \alpha}{\partial y} \right\|$$

More generally,

$$\int_{Y_\alpha} f dV = \int_A (f \circ \alpha) \left\| \frac{\partial \alpha}{\partial x} \times \frac{\partial \alpha}{\partial y} \right\|$$

(3) $\alpha : (0, \pi) \times (0, \pi) \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ given by $\alpha(\theta, \phi) = (2\cos\theta \sin\phi, 2\sin\theta\sin\phi, 2\cos\phi)$.
Check that $V(D\alpha) = 4\sin\phi$.

(4) Let $\alpha : \Omega \to \mathbb{R}^{n+1}$ be given by $\alpha(x) = (x, g(x))$ for $C^1$ $g$.
Check that

$$v(D\alpha) = \sqrt{1 + \sum_{i\in[n]} \left(\frac{\partial g}{\partial x_i}\right)^2}$$

Now we show that integrals over parametrized manifolds are invariant under reparametrization.

For a parametrized manifold to exist there is one $\alpha$ the following theorem says any $\beta$ diffeomorphic to $\alpha$ will agree on integrals. It does not say anything about two "randomly" chosen maps which define the same parametrized manifold.

**Theorem**. (Reparametrization Invariance)
Let $A, B \subseteq \mathbb{R}^k$ be open. Let $g : A \to B$ be a diffeomorphism. Let $\beta : B \to \mathbb{R}^n$ be a $C^1$ map. Let $\alpha = \beta \circ g : A \to \mathbb{R}^n$. Put $Y = \beta(B) = \alpha(A)$. Diagrammatically,

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[dr, bend right, "\alpha"'] & B \arrow[d, "\beta"] \\ & Y \end{tikzcd}$$

For a continuous function $f : Y \to \mathbb{R}$, $f$ is integrable on $Y_\alpha \iff$ f is integrable on $Y_\beta$. If so,

$$\int_{Y_\alpha} f dV = \int_{Y_\beta} f dV.$$

*Proof.*
We need to show

$$\int_A (f \circ \alpha) V(D\alpha) = \int_B (f \circ \beta) V(D\beta) \qquad (\star)$$

This amounts to change of variables in $\mathbb{R}^k$.

$$\int_A (f \circ \alpha) V(D\alpha) = \int_B f(\beta(y)) V(D\beta(y)) = \int_A f(\beta(g(x))) V(D\beta(g(x))) \cdot |\det Dg(x)| \,.$$

By the Chain rule

$$D\alpha(x) = D\beta(g(x)) Dg(x)$$

$$\implies V(D\alpha(x))^2 = \det(D\alpha(x)^T D\alpha(x)) = \det\Big([D\beta(g(x))Dg(x)]^T D\beta(g(x))Dg(x)\Big)$$

$$= \det\Big(Dg(x)^T D\beta(g(x))^T D\beta(g(x))Dg(x)\Big) = \det(Dg(x))^2 V(D\beta(g(x)))^2$$

The last step follows from the multiplicativity of the determinant and commutativity[1].
Taking square roots gives $(\star)$. $\square$

[1]Get used to this proof. It's techniques will show up often.

# Manifolds Without Boundary

> **Definition**. (Homeomorphism)
> Let $X$ and $Y$ be topological spaces (such as subsets of Euclidean spaces). A map $f : X \to Y$ is called a **homeomorphism** provided that $f$ is bijective, continuous, and $f^{-1}$ is continuous (equivalently $f$ is an open map). If there is a homeomorphism between $X$ and $Y$ we say that they are **homeomorphic**.

Examples:[2]

(a) $(0, 1)$ and the unit square minus the point $(0, 1)$ are homeomorphic.

(b) $f(x) = (\cos x, \sin x)$ with $f : [0, 2\pi) \to \mathbb{S}^1$ is a continuous bijective map. However $[0, 2\pi)$ and $\mathbb{S}^1$ are *not* homeomorphic because $f^{-1}$ is not continuous (this makes sense because their fundamental groups are different).

Recall the definition of the **subspace topology**.

> **Definition**. (Differentiable Manifold)
> Let $k \le n$. Let $M \subseteq \mathbb{R}^n$. We call $M$ a **differentiable $k$-manifold without boundary** in $\mathbb{R}^n$ provided that $\forall p \in M$, there is
> (i) a set $\mathcal{U} \subseteq \mathbb{R}^k$, that is open in $\mathbb{R}^k$, (local homeomorphism)
> (ii) a set $\mathcal{V} \subseteq M$, containing $p$, that is open in $M$, and (open containment)
> (iii) a diffeomorphism $\alpha : \mathcal{U} \to \mathcal{V}$ with rank $D\alpha(x) = k$, $\forall x \in \mathcal{U}$. (rank condition)
>
> If $\alpha$ is $C^r$ we say $M$ is of class $C^r$. If $\alpha$ is $C^\infty$ then we say $M$ is **smooth**.
> The **manifold** is the set $M$ together with its coordinate patches (atlas). A manifold without the rank condition is called a **topological manifold**.

Terminology: We call the map $\alpha : \mathcal{U} \subseteq \mathbb{R}^k \to \mathcal{V} \subseteq M$ a **coordinate patch** (**coordinate system**) on $M$ about $p$. The map $\varphi = \alpha^{-1} : \mathcal{V} \subseteq M \to \mathcal{U} \subseteq \mathbb{R}$ is called a **coordinate chart**. The collection of coordinate charts $(\varphi_\lambda, \mathcal{V}_\lambda)$ such that $\bigcup_\lambda \mathcal{V}_\lambda = M$ is called an **atlas**.

Intuition: the rank condition assures the linear independence of the columns of

$$D\alpha = \begin{bmatrix} | & & | \\ \frac{\partial \alpha}{\partial x_1} & \cdots & \frac{\partial \alpha}{\partial x_k} \\ | & & | \end{bmatrix} \qquad \text{where} \qquad \frac{\partial \alpha}{\partial x_i} = \lim_{\varepsilon \to 0} \frac{\alpha(x + \varepsilon e_i) - \alpha(x)}{\varepsilon}$$

when it exists is the **tangent vector** to $M$ at $\alpha(x)$. So, rank condition means that there is a $k$-dimensional tangent "plane" to $M$ at every point.

Examples:

(a) Let $M$ be $\mathbb{S}^1 \subseteq \mathbb{R}^2$ (the unit circle). For every $p \in M \setminus \{(-1, 0)\}$, put $V = M \setminus \{(-1, 0)\}$, $\mathcal{U} = (-\pi, \pi) \subset \mathbb{R}$, and $\alpha(t) = (\cos t, \sin t)$. $\alpha$ is clearly $C^\infty$, onto, 1-1, continuous inverse, and the rank of $D\alpha(t)$ is 1 $\forall t$.

[2] Algebraic Topology is the study of classifying topological spaces invariant under homeomorphism.

For the point $p = (-1, 0)$, put $V = M \setminus \{(1, 0)\}$, $\mathcal{U} = (0, 2\pi) \subset \mathbb{R}$, and $\alpha(t) = (\cos t, \sin t)$. $\alpha$ is clearly $C^\infty$, onto, 1-1, continuous inverse, and the rank of $D\alpha(p)$ is 1.

So $\mathbb{S}^1$ is a differentiable manifold. We showed this by considering a covering of $\mathbb{S}^1$ whose constituents are homeomorphic to $\mathbb{R}$.

(b) Let $M$ be $\mathbb{S}^1 \subseteq \mathbb{R}^2$ (the unit circle). For every $p$ in the upper half of $M$, put $\alpha_1 : (-1, 1) \to V_1$ given by $\alpha_1(t) = (t, \sqrt{1 - t^2})$. Do the same with the lower half of $M$. Then do the same with the right and left hand sides of $M$ but with $\alpha_3 : (-1, 1) \to V_3$ given by $\alpha_3(t) = (-\sqrt{1 - t^2}, t)$.

(c) Let $M = \mathbb{R}^n$. Then $M$ is a smooth $n$-manifold without boundary ($\alpha = \text{Id}$).

(d) Finite dimensional vector space $W$. Let $v_1, \ldots, v_k$ be a basis of $W$.
Then,

$$W = \left\{ \sum_{i \in [k]} c_i v_i : c_1, \ldots, c_k \in \mathbb{R} \right\}.$$

Let $\alpha : \mathbb{R}^k \to W$ such that

$$\alpha(x) = \sum_{i \in [k]} x_i v_i.$$

Then

$$D\alpha(x) = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_k \\ | & & | \end{bmatrix}$$

has rank $k$.

(e) Translates and dilates of a manifold (any diffeomorphism). If $M \subseteq \mathbb{R}^n$ and $p \in \mathbb{R}^n$ such that $M$ is a manifold then $N = M + p_0$ is a manifold. The translation map is continuous and has rank 0. $N = rM$ is also a manifold.

(f) Spheres. $\mathbb{S}^{n-1}\{x \in \mathbb{R}^n : \|x\| = 1\}$ is a smooth manifold without boundary of dimension $n - 1$. Consider all $2n$ half spheres of $\mathbb{S}^{n-1}$ and consider the patch

$$\alpha_1(x_1, \ldots, x_{n-1}) = \left( x_1, \ldots, x_{n-1}, \sqrt{1 - \sum_{i \in [n]} x_i^2} \right).$$

(g) Open subsets of a manifold (**submanifold**). The restriction of $C^r$ maps are $C^r$. Therefore, open sets in $\mathbb{R}^n$ are differentiable manifolds without boundary. Any open sets in $\mathbb{S}^{n-1}$ are differentiable manifolds without boundary. $\text{GL}(n, \mathbb{R})$ the set of $n \times n$ invertible manifolds is an $n^2$-manifold without boundary, this is an open subset of $\mathbb{R}^{n^2}$.

(h) **Product manifold**. For $i \in [\ell]$, $M_i$ an $k_i$-manifold without boundary in $\mathbb{R}^{n_i}$. Then

$$M = \prod_{i \in [\ell]} M_i$$

is a manifold of dimension $\sum_{i \in [\ell]} k_i$.

The coordinate patches are the products of coordinate patches. $T^n = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$ is an $n$-torus which is a smooth $n$-manifold without boundary in $\mathbb{R}^{2n}$. So $\mathbb{S}^1 \times \mathbb{S}^1$ is a 4-manifold but we can clearly embed it in $\mathbb{R}^3$ because we all have seen 3-dimensional donuts coated in sprinkles (this is called the edibility[3] question). This is because we can realize the torus as a quotient manifold.

(i) Singletons or discrete sets are by definition 0-dimensional manifolds.

(j) Quotient manifold (not covered).

Non-Examples:

(a) $\alpha : (0, \pi) \to \mathbb{R}^2$ given by $\alpha(t) = \sin(2t) \begin{bmatrix} |\cos t| \\ \sin t \end{bmatrix}$. Then $\alpha$ is 1-1 and onto but the inverse is not continuous.

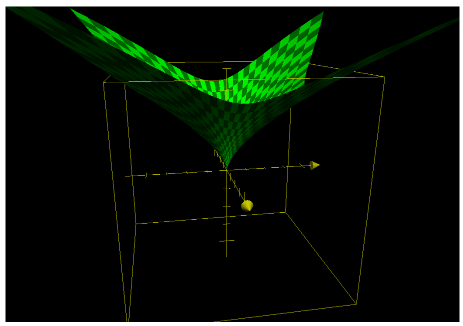


Figure 1: Not a manifold.

Why is the cross not a manifold.

(b) $\alpha : \mathbb{R}^2 \to \mathbb{R}^3$ given by $\alpha(x, y) = (x(x^2 + y^2), y(x^2 + y^2), x^2 + y^2)$. Put $M = \alpha(\mathbb{R}^2)$. $\alpha$ is $C^\infty$, a homeomorphism (check!), but

$$D\alpha(0, 0) = \vec{0}_{3\times 2}$$

so rank $D\alpha(0, 0) \neq 2$.

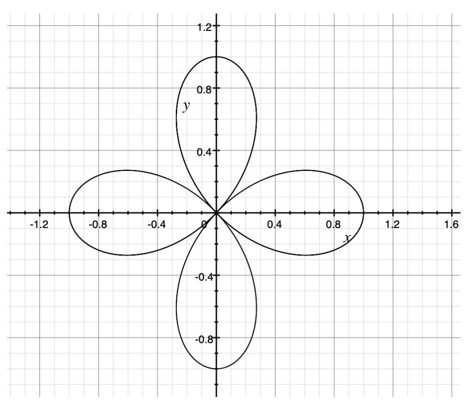



Figure 2: Not a manifold.

At all other point $D\alpha$ has rank two. So $M$ is not a manifold. The surface looks like a parabolic funnel. The set does not have a two dimensional tangent plane at the origin.

[3]It is really called the embedibility question, but I guess AutoCorrect was hungry.

(c) Put $\alpha(t) = (t, |t|)$ and $M = \alpha(\mathbb{R})$. This $\alpha$ does not give rise to a (differentiable) manifold. Put $\beta(t) = (t^3, t^2|t|)$. Note that

$$f(x) = t^2|t| = \begin{cases} t^3 & t \geq 0 \\ -t^3 & t < 0 \end{cases}$$

is $C^1$.

Since

$$f'(x) = \begin{cases} 3t^2 & t > 0 \\ 0 & t = 0 \\ -3t^2 & t < 0 \end{cases}$$

But the rank condition still fails because $\operatorname{rank} D\beta(0) = \operatorname{rank} \vec{0} \neq 1$.

*Moral of the story*: if you try to be clever, the rank condition will kick in and you will fail.

Is the topologist's sine curve a manifold?
What topology is generated by using the euclidean topology on $\mathbb{R}$ and then considering a space filling curve.

Now we generalize the notion of $C^r$ to maps with differing dimension and codimension.

**Definition**. (Continuous Differentiability)
Let $S \subseteq \mathbb{R}^\ell$. A function $f : S \to \mathbb{R}^m$ is said to be $C^r$ **on** $S$ provided that $f$ extends to a $C^r$ function on an open set in $\mathbb{R}^\ell$ containing $S$. There is an open $\Omega \subseteq \mathbb{R}^\ell$ with $\Omega \supseteq S$ and $\tilde{f} : \Omega \to \mathbb{R}^m$, such that $\tilde{f}$ is $C^r$ and $\tilde{f} \restriction S = f$.

Example:

(a) Let $f : S \to \mathbb{R}$ where $S = \operatorname{Span}(\{e_1 + e_2\})$ and $f(x, y) = xy$ then $f$ is $C^\infty$ on $S$.

**Lemma**. (Local $C^r \implies C^r$)
Let $S \subseteq \mathbb{R}^\ell$ and $f : S \to \mathbb{R}^m$. Suppose that $\forall x \in S$ f is locally $C^r$ near $x$ (i.e. $\exists S_x$ open in $S$ such that $x \in S_x$ and $f$ is $C^r$ on $S_x$), then $f$ is $C^r$ on $S$.

*Proof.*
We did this in the 395 homework using Partitions of Unity. □

**Lemma**. (Coordinate Charts are $C^r$)
Let $M$ be a differentiable $k$-manifold without boundary in $\mathbb{R}^n$ of class $C^r$. Let $\alpha : \mathcal{U} \subseteq \mathbb{R}^k \to \mathcal{V} \subseteq M$ be a coordinate patch. Then $\alpha^{-1} : \mathcal{V} \to \mathcal{U}$ is $C^r$ on $\mathcal{V}$ and is a coordinate chart.

*Proof.*
It suffices to prove locally. Choose $p_0 \in V$ with $x_0 = \alpha^{-1}(p_0)$.

Since $\operatorname{rank} D\alpha(x_0) = k$ (and row rank equals column rank) there are $k$ linearly independent rows. Without loss of generality we assume that the first $k$ rows of $D\alpha(x_0)$ are linearly independent. Let $\pi : \mathbb{R}^n \to \mathbb{R}^k$ be the projection map onto $\mathbb{R}^k$ (the indices of the $k$ independent rows).

Note $\pi$ is $C^\infty$ and $D\pi = \left[\begin{array}{c|c} \quad I_k \quad & \vec{0}_{k\times(n-k)} \end{array}\right]$.

Define $g = \pi \circ \alpha$. Then $g$ is $C^r$ and the chain rule gives us

$$Dg(x_0) = D\pi(p_0)D\alpha(x_0) = \left[\begin{array}{c|c} \quad I_k \quad & \vec{0}_{k\times(n-k)} \end{array}\right] \begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \\ \heartsuit & \heartsuit & \heartsuit \\ \heartsuit & \heartsuit & \heartsuit \end{bmatrix} = \begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$$

which is invertible (by rank condition).

By the Inverse Function Theorem, $g$ is a diffeomorphism locally near $x_0$ and $g^{-1}$ is $C^r$ near $\pi(p_0)$.

Note that $\alpha^{-1} = \pi \circ g^{-1}$, so $\alpha^{-1}$ is $C^r$.[4] □

**Theorem**. (Coordinate Patches Overlap Differentiably)
Let $M$ be a differentiable $k$-manifold without boundary in $\mathbb{R}^n$ of class $C^r$. Let $\alpha_1, \alpha_2$ be coordinate patches from $\mathcal{U}_1, \mathcal{U}_2$ to $\mathcal{V}_1, \mathcal{V}_2$ respectively with $\mathcal{V}_1 \cap \mathcal{V}_2 \neq \varnothing$.
The map $\alpha_2^{-1} \circ \alpha_1 : \mathcal{W}_1 \to \mathcal{W}_2$ is $C^r$ where $W_i = \alpha_i^{-1}(\mathcal{V}_1 \cap \mathcal{V}_2)$ are open in $\mathbb{R}^k$.

*Proof.*
Easy. The lemma above tells us that $\alpha_2$ is $C^r$ and composition of $C^r$ maps is $C^r$ by the Chain Rule. The map $\alpha_2^{-1} \circ \alpha_1$ is called a **transition map**[5]. □

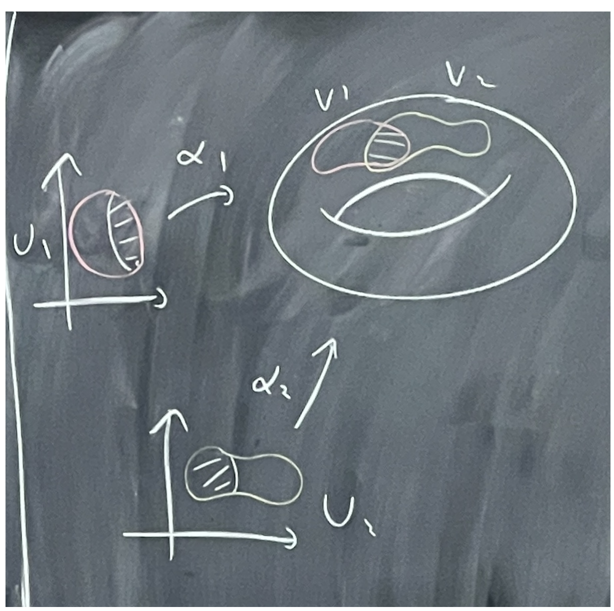



Figure 3: Overlapping coordinate patches.

[4] This step needs more thinking!

[5] In more abstract manifold theory we take the existence of transition maps as the definition of a differentiable manifold.

# Manifolds With Boundary

Someone should make a hat with a donut on the top!

Notation: $\mathbb{H}^k = \{x \in \mathbb{R}^k : x_k \geq 0\}$ and $\mathbb{H}^k_+ = \{x \in \mathbb{R}^k : x_k > 0\}$.

**Lemma**. (Differentiability on Boundary)
Let $\mathcal{U} \subseteq \mathbb{H}^k$ be open in $\mathbb{H}^k$ but not in $\mathbb{R}^k$. Suppose $\alpha : \mathcal{U} \to \mathbb{R}^n$ is $C^r$. Let $\tilde{\alpha} : \tilde{\mathcal{U}} \to \mathbb{R}^n$ be a $C^r$ extension of $\alpha$ where $\tilde{\mathcal{U}} \supset \mathcal{U}$ is open in $\mathbb{R}^k$, then $\forall x \in \mathcal{U}$, $D\tilde{\alpha}(x)$ depends only on $\alpha$. As a consequence, $D\alpha(x)$ is well defined.

*Proof.*
Note that

$$\frac{\partial \tilde{\alpha}}{\partial x_i} = \lim_{\varepsilon \to 0} \frac{\tilde{\alpha}(x + \varepsilon e_i) - \tilde{\alpha}(x)}{\varepsilon}$$

exists by the assumption that $\alpha$ is $C^r$.

Since the limit exists, it is unique and equal for every path (so we can always approach from within $\mathbb{H}^k$. By taking $\varepsilon > 0$ we see that

$$\frac{\partial \alpha}{\partial x_i} = \lim_{\varepsilon \downarrow 0} \frac{\alpha(x + \varepsilon e_i) - \alpha(x)}{\varepsilon}$$

$\square$

**Definition**. (Differentiable Manifold with Boundary)
A differentiable $k$-manifold (**with boundary**) in $\mathbb{R}^n$ of class $C^r$ is a set $M \subseteq \mathbb{R}^n$ such that $\forall p \in M$, $\exists \alpha : \mathcal{U} \to \mathcal{V}$ where

(1) $\mathcal{U}$ is open in either $\mathbb{R}^k$ or $\mathbb{H}^k$,
(2) $\mathcal{V}$ is open in $M$,
(3) $\alpha$ is a $C^r$ homeomorphism, and $\operatorname{rank} D\alpha(x) = k$ for all $x \in \mathcal{U}$.

Note that any manifold without boundary is necessarily a manifold with boundary.

Examples:

(a) $\mathbb{S}^1 \cap \mathbb{H}^k_+$ has manifold structure (without boundary). Consider $\alpha(t) = (\cos t, \sin t)$.

(b) $\mathbb{S}^1 \cap \mathbb{H}^k$ has manifold structure (with boundary).
For $p \in M \setminus \{(-1, 0)\}$, $\alpha : [0, \pi) \subseteq \mathbb{H}^1 \to M \setminus \{(-1, 0)\}$ given by $\alpha(t) = (\cos t, \sin t)$ is a coordinate patch.
For $p \in M \setminus \{(1, 0)\}$, $\alpha : [0, \pi) \subseteq \mathbb{H}^1 \to M \setminus \{(-1, 0)\}$ given by $\alpha(t) = (\cos(\pi - t), \sin(\pi - t))$ is a coordinate patch.
So this is a manifold with boundary.

(c) The convex hull of $\mathbb{S}^1$ (considered as a subset of $\mathbb{R}^2$) is a manifold with boundary (this is the closed unit disk about the origin).

(d) The portion of the closed unit disk that lies in the closed first quadrant does not have a differentiable manifold structure.

**Lemma**. (Coordinate Charts are $C^r$)
Let $M$ be a differentiable $k$-manifold in $\mathbb{R}^n$ of class $C^r$. Let $\alpha : \mathcal{U} \subseteq \mathbb{R}^k \to \mathcal{V} \subseteq M \subseteq \mathbb{R}^m$ be a coordinate path on $M$. Then $\alpha^{-1} : \mathcal{V} \to \mathcal{U}$ is $C^r$ on $\mathcal{V}$.

**Theorem**. (Transition Maps are Differentiable)
Let $M$ be a differentiable $k$-manifold in $\mathbb{R}^n$ of class $C^r$. Let $\alpha_1, \alpha_2$ be coordinate patches from $\mathcal{U}_1, \mathcal{U}_2$ to $\mathcal{V}_1, \mathcal{V}_2$ respectively with $\mathcal{V}_1 \cap \mathcal{V}_2 \neq \varnothing$.
The map $\alpha_2^{-1} \circ \alpha_1 : \mathcal{W}_1 \to \mathcal{W}_2$ is $C^r$ where $W_i = \alpha_i^{-1}(\mathcal{V}_1 \cap \mathcal{V}_2)$ are open in $\mathbb{R}^k$ or $\mathbb{H}^k$.

**Definition**. (Interior and Boundary of Manifold)
Let $M$ be a $k$-manifold in $\mathbb{R}^n$. Take $p \in M$.

(a) $p$ is called an **interior point** of $M$ if there is a coordinate patch $\alpha : \mathcal{U} \to \mathcal{V}$ on $M$ about $p$ such that $\mathcal{U}$ is open in $\mathbb{R}^n$.
(b) $p$ is called an **boundary point** of $M$ if $p$ is not an interior point. The set of boundary points of $M$ is denoted by $\partial M$.

We want a condition to characterize boundary points.

**Lemma**. (Restrictions of Coordinate Patches)
Let $M$ be a manifold and $\alpha : \mathcal{U} \to \mathcal{V}$ a coordinate patch. If $\mathcal{U}_0 \subseteq \mathcal{U}$ is open in $\mathcal{U}$, then $\alpha \restriction \mathcal{U}_0 : \mathcal{U}_0 \to \alpha(\mathcal{U}_0)$ is also a coordinate patch.

*Proof.*
Easy. Restrictions of diffeomorphisms are diffeomorphisms onto their image. $\square$

**Definition**. (Conditions for Boundary and Interior)
Let $M$ be a $k$-manifold in $\mathbb{R}^k$ and $\alpha : \mathcal{U} \to \mathcal{V}$ a coordinate patch on $M$ about $p$.

(1) $\mathcal{U}$ is open in $\mathbb{R}^k \implies p$ is an interior point of $M$.
(2) $\mathcal{U}$ is open in $\mathbb{H}^k$ and $p = \alpha(x_0)$ for some $x_0 \in \mathbb{H}^k_+ \implies p$ is an interior point of $M$.
(3) $\mathcal{U}$ is open in $\mathbb{H}^k$ and $p = \alpha(x_0)$ for some $x_0 \in \mathbb{R}^{k-1} \times \{0\} \implies p$ is a boundary point.

*Proof.*
(1) is clear by definition. (2) Put $\mathcal{U}_0 \coloneqq \mathcal{U} \cap \mathbb{H}^k_+$, which is open in $\mathbb{R}^k$. Now restrict $\alpha$ to $\mathcal{U}_0$ which witnesses that $p$ is an interior point.
(3)[6] Suppose, for the sake of contradiction, $p$ is an interior point. Then, $\exists \beta : \mathcal{U}' \to \mathcal{V}'$ with $\mathcal{U}'$ open in $\mathbb{R}^k$. Consider $\mathcal{U} \cap \mathcal{U}'$ the transition map $\gamma = \alpha^{-1} \circ \beta : \mathcal{W}_1 \to \mathcal{W}_2$ is $C^r$, a homeomorphism, and $D\gamma(x)$ has rank $k$ for all $x \in \mathcal{W}_1$.

So $\gamma : \mathcal{W}_1 \subseteq \mathbb{R}^k \to \mathbb{R}^k$ so $\gamma$ should be an *open map*. Therefore $\mathcal{W}_2 = \gamma(\mathcal{W}_1)$ is open in $\mathbb{R}^k$. Contradiction! Since $x_0 \in \mathcal{W}_2$ and $x_0 \in \mathbb{R}^{k-1} \times \{0\}$. $\square$

Example:

(i) $\partial(\mathbb{S}^1 \cap \mathbb{H}^K_+) = \{(1,0), (-1,0)\}$.

[6] You should be able to do this.

(ii) $\partial \mathbb{H}^k = \mathbb{R}^{k-1} \times \{0\}$.

Here's a cute theorem!

> **Theorem**. (Boundary Manifold)
> Let $M$ be a $k$-manifold of class $C^r$ in $\mathbb{R}^n$. If $\partial M \neq \varnothing$, then $\partial M$ is $(k-1)$-manifold without boundary of class $C^r$ in $\mathbb{R}^n$.

*Proof.*
Read the book. Use the boundary coordinate patches and project them onto $\mathbb{R}^{k-1}$. □

Here is a workhorse theorem:

> **Theorem**. (Condition for Level Set Manifold)
> Let $\mathcal{O} \subseteq \mathbb{R}^n$ be open and $f : \mathcal{O} \to \mathbb{R}$ be $C^r$. Define $N \coloneqq \{x \in \mathcal{O} : f(x) \geq 0\}$ and $M \coloneqq \{x \in \mathcal{O} : f(x) = 0\}$. We say that $M$ is a **level set** of $f$. Suppose $M \neq \varnothing$ and $\operatorname{rank} Df(x) = 1$ for all $x \in M$. Then, $N$ is a $C^r$ $n$-manifold in $\mathbb{R}^n$ and $M = \partial N$.

*Proof.*
Suppose $p \in N$ and $f(p) > 0$. Let $M_0 = \{x \in \mathcal{O} : f(x) > 0\}$, which is open in $\mathbb{R}^n$. Put $\alpha : \mathcal{U} \subseteq \mathbb{R}^n \to \mathcal{U} \subseteq N$, $\alpha = \text{Id}$. Then $\alpha$ is a coordinate patch about $p$.

Suppose $p \in N$ and $f(p) = 0$ (i.e. $p \in M$). Since $\operatorname{rank} Df(p) = 1$, at least one of $\frac{\partial f}{\partial x_i}(p) \neq 0$ for $i \in [n]$. Without loss of generality, we may assume $\frac{\partial f}{\partial x_n}(p) \neq 0$. Define $F : \mathcal{O} \to \mathbb{R}^n$, $F(x) = (x_1, \ldots, x_{n-1}, f(x))$. $F$ is $C^r$ and

$$DF = \left[\begin{array}{ccc|c} & & & 0 \\ & I_{n-1} & & \vdots \\ & & & 0 \\ \hline \dfrac{\partial f}{\partial x_1} & \cdots & \dfrac{\partial f}{\partial x_{n-1}} & \dfrac{\partial f}{\partial x_n} \end{array}\right] \implies \det DF(p) = \frac{\partial f}{\partial x_n}(p) \neq 0$$

The Inverse Function Theorem guarantees that $F$ is a diffeomorphism locally near $p$. Meaning, there exists open $A, B \subseteq \mathbb{R}^n$ with $p \in (A)$ such that $F : A \to B$ is a $C^r$ diffeomorphism and $F(A)$ is identically zero. Let $\mathcal{U} = B \cap \mathbb{H}^n$, $\mathcal{V} = A \cap N$, $\alpha = F^{-1} : \mathcal{U} \to \mathcal{V}$. $\alpha$ is a coordinate patch. Hence, $N$ is a $C^r$ $n$-manifold. This computation also shows us that $M = \partial N$. □

<u>Example</u>:
Let $f : \mathbb{R}^n \to \mathbb{R}$ be given by $f(x) = a^2 - \sum_{i \in [n]} x_i^2$. Then $N = B_a^n(0)$ or $\mathbb{B}^n(a)$ and $M = \mathbb{S}^{n-1}(a)$. $Df(x) = -2\vec{x}^T$ is not the zero vector for $x \in \mathbb{S}^{n-1}(a)$. Thus, $\mathbb{B}^n(a)$ is a smooth $n$-manifold in $\mathbb{R}^n$ of class $C^\infty$ and $\partial \mathbb{B}^n(a) = \partial \mathbb{S}^{n-1}(a)$.

# Integration of Scalar Functions on Manifolds

Later we will integrate vector fields and differential forms over manifolds. For now, we will just be integrating scalar valued functions over a manifold. For simplicity of presentation, we will only consider integration over **compact manifolds**, meaning a closed and bounded subset of $\mathbb{R}^n$ which has manifold structure.

Suppose $f : M \to \mathbb{R}$ where $M$ is a manifold with boundary. Suppose $\operatorname{supp} f$ is contained in a single coordinate patch.

**Definition**. (One Patch Integral over Manifold)
Let $M$ be a compact $k$-manifold in $\mathbb{R}^n$. Let $f : M \to \mathbb{R}$ be continuous. Suppose there is a coordinate patch $\alpha : \mathcal{U} \to \mathcal{V}$ such that $\operatorname{supp} f \subseteq \mathcal{V}$. Note that since $\alpha^{-1}(\operatorname{supp} f)$ is compact in $\mathbb{R}^k$, we may choose $\mathcal{U}$ to be bounded.
Define

$$\int_M f\, dV = \int_{\operatorname{Int}\mathcal{U}} (f \circ \alpha) \cdot V(D\alpha)$$

Note that $\operatorname{Int}\mathcal{U} = \mathcal{U}$ if $\mathcal{U}$ is open in $\mathbb{R}^k$ and $\operatorname{Int}\mathcal{U} = \mathcal{U} \cap \mathbb{H}^k_+$ if $\mathcal{U}$ is open in $\mathbb{H}^k$.
**Lemma**. The RHS is ordinary integrable.
**Lemma**. $\int_M f\, dV$ does not depend on the choice of $\alpha$.

Check that the integral is patch-independent and the integral is well defined (recall theorem 13.5 of Munkres).

Example: Suppose $M = \{(x, y) : (x, y) \in \mathbb{S}^1(3),\ x \leq 0 \vee y \geq 0\}$. Put

$$f(x, y) = \begin{cases} y & y \geq 0 \\ 0 & y < 0 \end{cases}$$

Then $\operatorname{supp} f = \mathbb{S}^1(3) \cap \mathbb{H}^2$. We can find one coordinate patch "to rule them all."
Put $\alpha : [0, \frac{3\pi}{2}) \subseteq \mathbb{H}^1 \to M \setminus \{(0, -3)\}$, $\alpha(t) = (3\cos t, 3\sin t)$. We have,

$$\int_M f\, dV = \int_0^{\frac{3\pi}{2}} \alpha \circ f(3\cos t, 3\sin t) \cdot 3 = \int_0^{\pi} 9 \sin t = 18.$$

Recall the definition of a Partition of Unity subordinate to $\mathscr{A}$.

**Lemma**. (Partition of Unity on a Manifold)
Let $M$ be a compact $k$-manifold in $\mathbb{R}^n$. Given a covering of $M$ by coordinate patches, there is a finite collection of $C^\infty$ $\phi_i : \mathbb{R}^n \to \mathbb{R}$ $i \in [\ell]$ such that
(i) $\phi_i(x) \geq 0$, $\forall x \in \mathbb{R}^n$, $\forall i \in [\ell]$.
(ii) $\sum_{i \in [\ell]} \phi_i(p) = 1$, $\forall p \in M$
(iii) $\forall i \in [\ell]$, there is a coordinate patch $\alpha_i : \mathcal{U}_i \to \mathcal{V}_i$ such that $\operatorname{supp} \phi_i \cap M \subseteq \mathcal{V}_i$.

*Proof.*
Read the book. □

**Definition**. (Integral over Manifold)
Let $M$ be a compact $k$-manifold in $\mathbb{R}^n$ and $f : M \to \mathbb{R}$ continuous.
Define

$$\int_M f\, dV = \sum_{i \in [\ell]} \int_M (\phi_i \cdot f) = \sum_{i \in [\ell]} \int_{\mathcal{U}_i} ((\phi_i \cdot f) \circ \alpha) \cdot V(D\alpha)$$

for a partition of unity $\{\phi_i\}_{i \in [\ell]}$ of $M$.

We need to check:

(a) If $\operatorname{supp} f$ lies in one coordinate patch, then the two definitions agree.

(b) $\int_M f\, dV$ is independent of the choice of partition of unity on $M$.

(c) $\int_M (\alpha f + \beta g)\, dV = \alpha \int_M f\, dV + \beta \int_M g\, dV$ and monotonicity in the domain.

Now we need a practical way to compute the integral over a manifold. We will extend the notion of measure zero on a manifold.

**Definition**. (Measure Zero Sets in a Manifold)
Let $M \subseteq \mathbb{R}^n$ be a compact $k$-manifold. $D \subseteq M$ is said to have **measure zero in** $M$ provided that $D$ can be covered by at most countably many coordinate patches $\alpha_i : \mathcal{U}_i \to \mathcal{V}_i$ such that

$$\bigcup_{i \in \mathbb{N}} \alpha_i^{-1}(D \cap \mathcal{V}_i)$$

has measure zero in $\mathbb{R}^k$.

Example:

$M = \mathbb{S}^2(a) \subseteq \mathbb{R}^3$ and $D = \mathbb{S}^1(a) \times \{0\}$. Let $\alpha$ be the stereographic projection from the north pole. Then, $\alpha^{-1}(D)$ is a circle in $\mathbb{R}^2$.

**Theorem**. (Measure Zero Sets Do Not Affect Integrals)
Let $M \subseteq \mathbb{R}^n$ be a compact $k$-manifold and $f : M \to \mathbb{R}$ continuous. Suppose $\alpha_i : A_i \to M_i$ for $i \in [\ell]$ are coordinate patches such that $M_1, \ldots, M_N$ are disjoint and

$$M = \left( \bigcup_{i \in [\ell]} M_i \right) \cup K$$

where $K$ is of measure zero in $M$.

Then,

$$\int_M f\, dV = \sum_{i \in [\ell]} \int_{M_i} f\, dV$$

*Proof.*
Since both sides of the equation are linear in $f$, it is enough to show

$$\int_M f\, dV = \sum_{i \in [N]} \int_{A_i} (f \circ \alpha_i) \cdot V(D\alpha)$$

coordinate patch. Hence WLOG we may assume $\operatorname{supp} f$ lies in one coordinate patch.

Then the equation to prove becomes

$$\int_{\operatorname{Int} \mathcal{U}} (f \circ \alpha) \cdot V(D\alpha) = \sum_{i \in [N]} \int_{M_i} f\, dV$$

Put $L = \alpha^{-1}(K \cap \mathcal{V})$. Put $\mathcal{W}_i = \alpha^{-1}(M_i \cap \mathcal{V})$, which is open in $\mathbb{R}^k$ or $\mathbb{H}^k$. Try to prove that $L$ is measure zero in $\mathbb{R}^k$ (HW), you should use that $C^1$ maps take measure zero sets to measure zero sets.

$$\int_{\operatorname{Int}\mathcal{U}}(f\circ\alpha)\cdot V(D\alpha)=\int_{\operatorname{Int}\mathcal{U}\setminus L}(f\circ\alpha)\cdot V(D\alpha)=\sum_{i\in[N]}\int_{\mathcal{W}_i}(f\circ\alpha)\cdot v(D\alpha)$$
$$=\sum_{i\in[N]}\int_{\alpha_i^{-1}(M_i\cap\mathcal{V})}(f\circ\alpha_i)\cdot v(D\alpha_i)$$

The last equality follows from change of variables and the fact that $\operatorname{supp}\alpha_i$ lies almost entirely in $A_i$.

□

$M=\mathbb{S}^2(a)\subseteq\mathbb{R}^3$. Let's compute $v(M)$. $K=\{(x,y,z)\in M: y=0, x\geq 0\}$ (half the meridian). Let $\alpha:(0,2\pi)\times(0,\pi)\to M\setminus K$ be given by $\alpha(\theta,\phi)=(a\sin\phi\cos\theta, a\sin\phi\sin\theta, a\cos\phi)$.

$$v(M)=\int_{M\setminus K}1\,dV=\int_{(0,2\pi)\times(0,\pi)}1\cdot V(D\alpha)=\int_0^{2\pi}\int_0^{\pi}a^2\sin\phi\,d\phi d\theta=a^2(2)(2\pi)=4\pi a^2$$

Let's compute again with another method: **Cavalieri's Principle**. $\alpha:(-a,a)\times(0,2\pi)\to M\setminus K$. $\alpha(z,\theta)=(\sqrt{a^2-z^2}\cos\theta,\sqrt{a^2-z^2}\sin\theta,z)$. Check that $\alpha$ is a coordinate patch and $V(D\alpha)=a$.

$$v(M)=\int_{M\setminus K}1\,dV=\int_{-a}^{a}\int_0^{2\pi}a\,d\theta dz=4\pi a^2$$

A similar computation will give you $v\left(\mathbb{S}^1(a)\right)=2\pi a$. What about the surface area of $\mathbb{S}^k(\alpha)$?

# DIFFERENTIAL FORMS

## Tensors & Alternating Tensors

**Definition**. (Tensor Product)
Let $E, F, T, H$ be vector spaces and $\phi : E \times F \to H$ a bilinear map. There is a bilinear map $\otimes$ called the **tensor product** which is unique, up to isomorphism, and obeys the **universal property** that for every bilinear $\phi : E \times F \to H$, $\exists! f : T \to H$ which is linear and makes

$$\begin{tikzcd} E \times F \arrow[r, "\phi"] \arrow[d, "\otimes"'] & H \\ T \arrow[ur, "f"'] & \end{tikzcd}$$

commutes.

For $f \in \mathcal{L}^k(V)$ and $g \in \mathcal{L}^\ell(V)$ we define

$$(f \otimes g)(v_1, \ldots, v_{k+\ell}) = f(v_1, \ldots, v_k) \cdot g(v_{k+1}, \ldots, v_{k+\ell})$$

Tensors generalize vectors and matrices.

**Definition**. (Tensor)
Suppose $V$ is a vector space of dimension $n$ with basis $\{b_i : i \in [n]\}$. A $k$-**tensor** is a function $f : V^K \to \mathbb{R}$ that is **multilinear**.
We write $\mathcal{L}^k(V)$ as the set of $k$-tensors on $V$. $\mathcal{L}^k(V)$ forms a vector space with basis $\{\phi_{\sigma(1)} \otimes \cdots \otimes \phi_{\sigma(n)} : \sigma \in [n]^k\}$ where $\{\phi_i : i \in [n]\}$ is the standard basis for $V^*$.
We call $k$ the **order** of the tensor. $g \otimes h$ has order $\ell + m$ if $\mathcal{L}^\ell(V)$ and $h \in \mathcal{L}^m(V)$

Example:

(a) $\mathcal{L}^1(V) = V^*$ is the dual space of $V$.

(b) $\mathcal{L}^2(V) = \left\{ f : V^2 \to \mathbb{R} : f \mapsto (f(a_i, a_j))_{i,j \in [n]} \right\}$ is isomorphic to $\mathrm{Hom}(V, V)$.

(c) Let $V = \mathbb{R}^n$ let $f \in \mathcal{L}^2(\mathbb{R}^n)$. Then for $x, y \in \mathbb{R}^n$,

$$f(x,y)) = \left( \sum_{i,j \in [n]} c_{ij} (\phi_i \otimes \phi_j) \right)(x,y) = \sum_{i,j \in [n]} c_{ij} \cdot \phi_i(x) \cdot \phi_j(y) = \sum_{i,j \in [n]} c_{ij} x_i y_j = x^T C y$$

since $\phi_i(e_j) = \delta_{ij}$.

**Definition**. (Alternating Tensor)
$f \in \mathcal{L}^k(V)$ is said to be **alternating** provided that for every $i \in [n-1]$,

$$f(\cdots, v_i, v_{i+1}, \cdots) = -f(\cdots, v_{i+1}, v_i, \cdots)$$

We write $\Lambda^k(V)$ (or $\Lambda_k(V)$ $\mathcal{A}^k(V)$) as the set of alternating $k$-tensors on $V$.
$\Lambda^k(V)$ forms a vector subspace of dimension $\binom{n}{k}$ with basis

$$\left\{ \bigwedge_{j\in[k]} \phi_{\sigma(j)} \mid \sigma \in \mathrm{ASC}_{k,n} \right\}$$

Where $\{\phi_i : i \in [n]\}$ is the standard basis for $V^*$ and the $\bigwedge$ denotes the wedge product.
Note that the space $\Lambda^n(V)$ is one dimensional. When $k > n$, $\Lambda^{\dim(V)}(V) = \{0\}$.

Example:

(a)
$$\begin{aligned} f \in \Lambda^k(V)) &\iff f(x,y) = x^T C y \ \wedge \ f(x,y) = -f(y,x) \\ &\iff x^T C y = -y^T C x = (y^T C x)^T = -x^T C^t y \\ &\iff x^T (C + C^T) y = 0, \ \forall x, y \in \mathbb{R}^n \\ &\iff f(x,y) = x^T C y \ \wedge \ C = -C^T \end{aligned}$$

So $\Lambda^k(V)$ is isomorphic so the set of **skew-symmetric** matrices (note the diagonal must be zero).

(b) $\Lambda^2(\mathbb{R}^3)$ has basis $\{\omega_{12}, \omega_{23}, \omega_{13}\}$ where $\omega_{ij}(x,y) = x_i y_j - x_j y_i = (\phi_i \otimes \phi_j - \phi_j \otimes \phi_i)(x,y)$ .

Check that for $\omega \in \Lambda^k(V)$ and $\sigma \in S_k$,

$$\omega(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) = \mathrm{sgn}(\sigma) \cdot \omega(v_1, \ldots, v_k)$$

Note that for $\omega \in \Lambda^k(V)$ then

$$\omega(\cdots, v, \ldots, v, \cdots) = 0$$

**Definition**. (Alternization)
Define $\mathrm{Alt} : \mathcal{L}^k(V) \to \Lambda^k(V)$ by

$$\mathrm{Alt}(f)(v_1, \ldots, v_k) = \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma) \cdot f(v_{\sigma(1)}, \ldots, v_{\sigma(k)})$$

**Lemma**.
(1) Alt is a linear map and $f \in \mathcal{L}^k(V) \implies \mathrm{Alt}(f) \in \Lambda^k(V)$.
(2) If $\omega \in \Lambda^k(V)$, then $\mathrm{Alt}(\omega) = k! \cdot \omega$.
(3) $f \in \mathcal{L}^k(V) \implies \mathrm{Alt}(\mathrm{Alt}(f)) \in k!\,\mathrm{Alt}(f)$.
(4) $f \in \mathcal{L}^k(V), g \in \mathcal{L}^\ell(V) \implies \mathrm{Alt}(f \otimes g) = (-1)^{k+\ell}\,\mathrm{Alt}(g \otimes f)$.

*Proof.*
We have

(1) It suffices to show $\tau \in S_k$,

$$\mathrm{Alt}(f)(v_{\tau(1)}, \dots, v_{\tau(k)}) = \mathrm{sgn}(\tau) \cdot \omega(v_1, \dots, v_k)$$

for all $\tau \in S_k$.

Fix $\tau \in S_k$ then

$$\begin{aligned}
&\mathrm{Alt}(f)(v_{\tau(1)}, \dots, v_{\tau(k)}) \\
&= \sum_{\sigma \in S_k} (\mathrm{sgn}\,\sigma) f(v_{\sigma\circ\tau(1)}, \dots, v_{\sigma\circ\tau(k)}) && \text{definition of Alt} \\
&= \sum_{\pi \in S_k} (\mathrm{sgn}\,\sigma) f(v_{\pi(1)}, \dots, v_{\pi(k)}) && \text{with } \pi = \sigma \circ \tau \\
&= \sum_{\pi \in S_k} (\mathrm{sgn}\,\pi)(\mathrm{sgn}\,\tau) f(v_{\pi(1)}, \dots, v_{\pi(k)}) && \text{transposition properties} \\
&= (\mathrm{sgn}\,\tau) \sum_{\pi \in S_k} (\mathrm{sgn}\,\pi) f(v_{\pi(1)}, \dots, v_{\pi(k)}) && \text{distributivity} \\
&= (\mathrm{sgn}\,\tau)\,\mathrm{Alt}(f)) && \text{definition of Alt}
\end{aligned}$$

(2) Easy

(3) $(1) \wedge (2) \implies (3)$.

(4) Let $\pi \in S_{k+1}$ for $\pi(i) = \ell + i, \pi(j+1) = j, i \in [k], j \in [\ell]$

$$\begin{aligned}
&\mathrm{Alt}(f \otimes g)(v_1, \dots, v_{k+\ell}) \\
&= \sum_{\sigma \in S_{k+\ell}} (\mathrm{sgn}\,\sigma) f(v_{\sigma(1)}, \dots, v_{\sigma(k+\ell)}) && \text{definition of Alt} \\
&= \sum_{\sigma \in S_{k+\ell}} (\mathrm{sgn}\,\sigma) f(v_{\sigma(1)}, \dots, v_{\sigma(k)}) g(v_{\sigma(k+1)}, \dots, v_{\sigma(k+\ell)}) && \text{definition of Alt} \\
&= \sum_{\sigma \in S_{k+\ell}} (\mathrm{sgn}\,\tau \circ \pi) f(v_{\tau\circ\pi(1)}, \dots, v_{\sigma(k)}) g(v_{\tau\circ\pi(k+1)}, \dots, v_{\tau\circ\pi(k+\ell)}) && \sigma \coloneqq \tau \circ \pi \\
&= \sum_{\sigma \in S_{k+\ell}} (\mathrm{sgn}\,\tau \circ \pi) f(v_{\tau(\ell+1)}, \dots, v_{\tau(k+\ell)}) g(v_{\tau(1)}, \dots, v_{\tau(\ell)}) && \text{definition of } \pi \\
&= (\mathrm{sgn}\,\pi)\,\mathrm{Alt}(g \otimes f) && \text{definition of Alt}
\end{aligned}$$

# The Wedge Product

**Definition**. (Wedge Product)
For $\omega \in \Lambda^k(V), \eta \in \Lambda^\ell(V)$ define the **wedge product**

$$\omega \wedge \eta = \frac{1}{k!\ell!} \mathrm{Alt}(\omega \otimes \eta)$$

**Lemma**.
(1) If $\omega$ and $\eta$ are of the same order, then $(\omega \wedge \eta) \wedge \theta = \omega \wedge \theta + \eta \wedge \theta$.
(2) $\wedge : \Lambda^k(V) \times \Lambda^\ell(V) \to \Lambda^{k+\ell}(V)$ is a bilinear map.
(3) $\omega \wedge \eta = (-1)^{k \cdot \ell}(\eta \wedge \omega)$.

*Proof.*
Done in IBL □

**Lemma**. (Associativity of Wedge Product)

(1) If $f \in \mathcal{L}^k(V), g \in \mathcal{L}^\ell(V)$, and $\operatorname{Alt}(f) = 0$, then $\operatorname{Alt}(f \otimes g) = 0$.
(2) If $f \in \mathcal{L}^k(V)$ and $\theta \in \Lambda^m(V)$ then $\operatorname{Alt}(f) \wedge \theta = \frac{1}{m!} \operatorname{Alt}(f \otimes \theta)$.
(3) If $\omega \in \Lambda^k(V), \eta \in \Lambda^\ell(V), \theta \in \Lambda^m(V)$, then

$$(\omega \wedge \eta) \wedge \theta = \omega \wedge (\eta \wedge \theta) = \frac{1}{k!\ell!m!} \operatorname{Alt}(\omega \otimes \eta \otimes \theta)$$

*Proof.*
**(1)** This is the difficult part. Note that

$$\operatorname{Alt}(f) = 0 \iff \sum_{\pi \in S_k} f(w_{\pi(1)}, \ldots, w_{\pi(k)} = 0$$

For each $I \in [k+\ell]^\ell$. Let $G_I$ be the set of permutations $\sigma \in S_{k+\ell}$ satisfying $\sigma(k+j) = I(j)$ for each $j \in [\ell]$.

For example $G_I = \{(14352), (41352)\}$ for $I = (3, 5, 2)$.

We have

$$\begin{aligned} k!\ell! \operatorname{Alt}(f \otimes g)(v_1, \ldots, v_{k+\ell}) &= \sum_{\sigma \in S_{k+\ell}} f(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) \cdot g(v_{I(1)}, \ldots, v_{I(\ell)}) \\ &= \sum_{I \in [k+\ell]^\ell} \sum_{\sigma \in G_I} f(v_1, \ldots, v_{k+\ell}) \cdot g(v_{k+1}, \ldots, v_{k+\ell}) \\ &= \sum_{I \in [k+\ell]^\ell} \left[ \sum_{\sigma \in G_I} f(v_1, \ldots, v_{k+\ell}) \right] \cdot g(v_{k+1}, \ldots, v_{k+\ell}) \end{aligned}$$

Now fix $I$. Note that if $\sigma, \tau \in G_I$, then $\{\sigma(j) : j \in [k]\} = \{\tau(j) : j \in [k]\}$.

Denote $\{v_{\sigma(j)} : j \in [k]\} = \{w_j : j \in [k]\}$. Then,

$$\begin{aligned} &= \sum_{\sigma \in G_I} f(v_1, \ldots, v_{k+\ell}) \\ &= \pm \sum_{\pi \in S_k} (\operatorname{sgn} \pi) f(w_{\pi(1)}, \ldots, w_{\pi(k)}) \\ &= \pm \operatorname{Alt} f(w_1, \ldots, w_k) \\ &= 0 \end{aligned}$$

**(2)** Put $F = \operatorname{Alt}(f) - k!f$. Then, $\operatorname{Alt}(F) = 0$. Use (1) with $f \coloneqq F$. Then

$$\begin{aligned} &\implies \\ &\implies \operatorname{Alt}(F \otimes \theta) = 0 \\ &\implies \operatorname{Alt}(\operatorname{Alt}(f) \otimes \theta) = k! \operatorname{Alt}(f \otimes \theta) \\ &\implies k!m! \operatorname{Alt}(f) \wedge \theta && \text{definition of } \wedge \end{aligned}$$

**(3)** In two parts,

$$\begin{aligned}(\omega \wedge \eta) \wedge \theta &= \frac{1}{k!\ell!} \operatorname{Alt}(\omega \otimes \eta) \wedge \theta \\ &= \frac{1}{k!\ell!m!} \operatorname{Alt}((\omega \otimes \eta) \otimes \theta) \\ &= \frac{1}{k!\ell!m!} \operatorname{Alt}(\omega \otimes \eta \otimes \theta) \qquad \otimes \text{ is associative}\end{aligned}$$

For part 2,

$$\begin{aligned}\theta \wedge (\omega \wedge \eta) &= (-1)(\omega \wedge \eta) \wedge \theta \\ &= \frac{(-1)^{k(\ell+m)}}{k!\ell!m!} \operatorname{Alt}(\eta \otimes \theta \otimes \omega) \\ &= \frac{1}{k!\ell!m!} \operatorname{Alt}(\omega \otimes \eta \otimes \theta)\end{aligned}$$

# Fields & Forms on Euclidean Space

**Definition**. (Fields)
Let $A \subseteq \mathbb{R}^n$ be open. We define a

(i) a **scalar field** on $A$ is a function $f : A \to \mathbb{R}$.
(ii) a **vector field** on $A$ is a function $F : A \to \mathbb{R}^n$ (note the dimension).
(iii) a $k$**-tensor field** is a function $F : A \to \mathcal{L}^k(\mathbb{R}^n)$.
(iv) a (**differential**) $k$**-form** is a function $F : A \to \Lambda^k(\mathbb{R}^n)$.

Example:

(i) $F : \mathbb{R}^2 \to \mathbb{R}^2$ given by $F(x, y) = xe_1 + ye_1$ is a vector field that describes radial growth.
(ii) $F : \mathbb{R}^2\{\vec{0}\} \to \mathbb{R}^2$ given by $F(x, y) = \frac{xe_2 - ye_1}{\sqrt{x^2+y^2}}$ is a vector field that describes counterclockwise rotation at unit speed.
(iii) $F : \mathbb{R}^3 \to \Lambda^2(\mathbb{R}^2)$ given by

$$\omega(x, y, z) = xy(\phi_1 \wedge \phi_2) + xz(\phi_1 \wedge \phi_3) + yz(\phi_2 \wedge \phi_3) \quad \longleftrightarrow \quad \begin{bmatrix} 0 & xy & xz \\ -xy & 0 & yz \\ -xz & -yz & 0 \end{bmatrix}$$

Generally, a $k$-form on $\mathbb{R}^n$ can be interpreted as an $\underbrace{n \times n \times \cdots \times n}_{k \text{ times}}$ array-valued function.

**Definition**. (Zero Form)
A 0**-form** is a scalar field.

Let $F : A \subseteq \mathbb{R}^n \to \mathcal{L}^k\left(\mathbb{R}^k\right)$, a $k$-tensor field on $A$, then for all $x \in A$ and $v_1, \ldots, v_k \in \mathbb{R}^n$

$$F(x)(v_1, \ldots, v_k) \in \mathbb{R}$$

**Definition**. (Smooth Tensor Fields)
A $k$-tensor field $F$ on $A$ (open in $\mathbb{R}^n$) is said to be $C^r$ the function $A \times \mathbb{R}^n \times \cdots \times \mathbb{R}^n \to \mathbb{R}$ given by $(x, v_1, \ldots, v_k) \mapsto F(x)(v_1, \ldots, v_k)$ is $C^r$.

(a) $F : \mathbb{R}^3 \to \Lambda^2(\mathbb{R}^2)$ given by

$$\omega(x, y, z) = xy(\phi_1 \wedge \phi_2) + xz(\phi_1 \wedge \phi_3) + yz(\phi_2 \wedge \phi_3)$$

$$\omega(x, y, z)(v, w) = xy(v_1w_2 - v_2w_1) + xz(v_1w_3 - v_3w_1) + yz(v_2w_3 - v_3w_2)$$

is $C^\infty$ since it is a polynomial.

Remark: It is enough to check the coefficients (of the tensor space basis) are $C^r$ functions.
Meaning, $\omega : A \to \Lambda^k(\mathbb{R}^n)$ we can write

$$\omega(x) = \sum_{I \in \mathrm{ASC}_{k,n}} \omega_I(x) \bigwedge_{j \in [k]} \phi_{I(j)}$$

and $\omega_I$ is smooth for each $I \in \mathrm{ASC}_{k,n}$.

Notation: For open $A \subseteq \mathbb{R}^n$, $\Omega^k(A)$ will denote the set of smooth $k$-forms on $A$. $\Omega^0(A) = C^\infty(A)$. If $k > n$, then $\Omega^k(A) = \{0\}$. Note that $\Omega^k(A)$ is a vector space.

We will take the convention that all forms are smooth on their domain and worry about continuity in an *ad hoc* manner. This is more fun than always worrying about continuity.

**Definition**. (Differential of a 0-Form)
Let $A \subseteq \mathbb{R}^n$ be open and $f \in \Omega^0(A)$. Define $df \in \Omega^1(A)$ by

$$df(x) = \sum_{i \in [n]} D_i f(x) \phi_i$$

Example:

(a) Consider $f(x,y,z) = xyz \implies df(x,y,z) = yz\phi_1 + xz\phi_2 + xy\phi_3$. Then

$$df(x)(v) = \sum_{i \in [n]} D_i f(x) v_i = Df(x) \cdot v$$

which is the directional derivative.

(b) The projection function $\pi_i : \mathbb{R}^n \to \mathbb{R}$ given by $\pi_i(x) = x_i \implies d\pi_i = \phi_i$.

Notation: Whenever we see $\phi_i$ it is often more convenient to write $\phi_i = d\pi_i$ as $dx_i$. This is a formal notation and has no meaning. So $dx_i$ is the 1-form satisfying $dx_i(e_j) = \delta_{ij}$. The standard basis of $\Lambda^k(\mathbb{R}^n)$ is the set

$$\{dx_{i_1} \wedge \cdots \wedge dx_{i_k} : 1 \leq i_1 \leq \cdots \leq i_k \leq n\}$$

(a) We can rewrite $f(x,y,z) = xyz \implies df(x,y,z) = yz\,dx + xz\,dy + xy\,dz$.

**Propsotition**. (Properties of Differentials)
Differentials obey the following properties:

(i) $d(fg) = g\,df + f\,dg$
(ii) $d : \Omega^k(A) \to \Omega^{k+1}(A)$ is linear.
(iii)

**Definition**. (Differential of a Form)
Let $A \subseteq \mathbb{R}^n$ be open and $\omega \in \Lambda^k(A)$. We write

$$\omega = \sum_{I \in \mathrm{ASC}_{k,n}} \omega_I \left( \bigwedge_{j \in [k]} dx_{I(j)} \right) \in \Omega^k(A)$$

Define $d\omega \in \Omega^{k+1}(A)$ the **differential** (or **exterior derivative**) of $\omega$ by

$$d\omega = \sum_{I \in \mathrm{ASC}_{k,n}} (d\omega_I) \wedge \left( \bigwedge_{j \in [k]} dx_{I(j)} \right) \in \Omega^k(A)$$

Example:

(a) Consider

$$\begin{aligned}\omega &= xy\,dx + 3\,dy - yz\,dz \in \Omega^1(\mathbb{R}^3)\\ &\Downarrow\\ d\omega &= d(xy)\wedge dx + d(3)\wedge dy + d(-yz)\wedge dz\\ &= (y\,dx + x\,dy + 0\,dz)\wedge dx + 0\wedge\ dy - (z\,dy + y\,dz)\wedge dz\\ &= -(x\,dx\wedge dy + z\,dx\wedge dz)\end{aligned}$$

(b) Consider

$$\begin{aligned}\omega &= (x+z)\,dx\wedge dy - y\,dx\wedge dz + (x^2+y^2)\,dy\wedge dz \in \Omega^2(\mathbb{R}^3)\\ &\Downarrow\\ d\omega &= (dx+dz)\wedge dx\wedge dy - dy\wedge dx\wedge dz + (dx + 2y\,dy)\wedge dy\wedge dz\\ &= dx\wedge dy\wedge dz\end{aligned}$$

Understanding the differential more carefully we have

$$\begin{aligned}\omega &= \sum_{i\in[n]} \omega_i\,dx_i \in \Omega^1(\mathbb{R}^n)\\ &\Downarrow\\ d\omega &= \sum_{i,j\in[n]} ((D_j\omega_i)\,dx_j)\wedge dx_i\\ &= \sum_{1\le i<j\le n} ((D_j\omega_i - D_i\omega_j)\,dx_i\wedge dx_j)\end{aligned}$$

So this is equivalent to a matrix whose diagonal is zero and $ij$-entry is $(D_j\omega_i - D_i\omega_j)$. There are $n^2$ possible partial derivatives of order 2, but we are only choosing $\frac{n(n-1)}{2}$. We will discuss the importance of this choice to follow. Differential forms vastly generalize the notation of gradient, divergence, and curl one encounters in Calc III.

**Definition**. (Gradient, Divergence, & Curl) We define the following

(1) For $f:\mathbb{R}^n\to\mathbb{R}$, the **gradient** of $f$ is

$$\nabla f = \sum_{i\in[n]}(D_i f)e_i.$$

(2) For $f:\mathbb{R}^n\to\mathbb{R}^n$ the **divergence** of $F$ is

$$\operatorname{Div} F = \nabla\cdot F = \sum_{i\in[n]} D_iF_i$$

(3) For $R:\mathbb{R}^3\to\mathbb{R}^3$ the **curl** of $F$ can be written formally

$$\operatorname{curl} F = \det\begin{bmatrix} e_1 & e_2 & e_3\\ D_1 & D_2 & D_3\\ F_1 & F_2 & F_3\end{bmatrix}$$

So grad maps smooth scalar fields to vector fields. Similarly, $d:\Omega^0(\mathbb{R}^n)\to\Omega^1(\mathbb{R}^n)$. There is a bijection $\alpha_1$ between vector fields on $\mathbb{R}^n$ and $\Omega^1(\mathbb{R}^n)$ given by $\alpha_1(f) = \alpha_1\left(\sum_{i\in[n]} F_i e_i\right) = \sum_{i\in[n]} F_i \, dx_i$. Also $\alpha_0(f) = f$ maps smooth scalar fields to $\Omega^0(\mathbb{R}^n)$. $\alpha_0, \alpha_1$ are isomorphisms (of vector spaces). Furthermore, the diagram commutes, (i.e. $d \circ \alpha_0 = \alpha_1 \circ \operatorname{grad}$ or $\operatorname{grad} = \alpha_1^{-1} \circ d \circ \alpha_0$). We say that the gradient operator is equivalent to $d$ modulo conjugation of $\alpha_1^{-1}$ and $\alpha_0$.

For $\omega \in \Omega^{n-1}(\mathbb{R}^n)$,

$$\begin{aligned}
\omega &= \sum_{i\in[n]} \omega_i \left( \bigwedge_{j\in[n]\setminus\{i\}} dx_j \right) \\
&\Downarrow \\
d\omega &= \sum_{i\in[n]} (D_i\omega_i)\, dx_i \wedge \left( \bigwedge_{j\in[n]\setminus\{i\}} dx_j \right) \\
&= \sum_{i\in[n]} (-1)^{i-1} (D_i\omega_i) \left( \bigwedge_{j\in[n]} dx_j \right)
\end{aligned}$$

$$\begin{bmatrix} \mathrm{Vec}(\mathbb{R}^n) \\ \mathrm{Div}\downarrow \\ C^\infty(\mathbb{R}^n, \mathbb{R}) \end{bmatrix}$$

Put $\beta_n(f) = f \bigwedge_{i\in[n]} dx_i$ and

$$\beta_{n-1}(F) == \sum_{i\in[n]} (-1)^{i-1} (F_i) \left( \bigwedge_{j\in[n]\setminus\{i\}} dx_j \right)$$

For $\omega \in \Omega^1(\mathbb{R}^3)$,

$$\begin{aligned}
\omega &= \omega_1\, dx + \omega_2\, dy + \omega_3\, dz \\
&\Downarrow \\
d\omega &= (D_1\omega_2 - D_2\omega_1)\, dx \wedge dy + (D_1\omega_3 - D_3\omega_1)\, dx \wedge dz + (D_2\omega_3 - D_3\omega_2)\, dy \wedge dz
\end{aligned}$$

$$\begin{tikzcd}
V \arrow[r, "\mathrm{Id}"] \arrow[rrr, bend left, "T"] & V \arrow[r, "T"] & W \arrow[r, "\mathrm{Id}\, W"] \arrow[d, "i^{-1}\beta"] & W \arrow[d, "i^{-1}\beta'"] \\
F^n \arrow[u, "i\alpha'"] \arrow[r, "f_\alpha \,\mathrm{Id}\, \alpha'"'] \arrow[rrr, bend right, "f_{\alpha'} T \beta'"'] & F^n \arrow[u, "i\alpha"] \arrow[r, "f_\beta T \alpha"'] & F^m \arrow[r, "f\beta' \,\mathrm{Id}\, \beta"'] & F^m
\end{tikzcd}$$

**Proposition**. (Wedge Product of Dual Basis)

$$\bigwedge_{j=1}^{k} dx_{i_j} = \det B_I$$

where $B_I$ is the $k \times k$ matrix obtained from the rows of $i_1, \ldots, i_k$ of

$$B = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} \in \mathbb{R}^{n \times k}$$

**Proposition**. (Properties of Differential)

(a) $d\,f = \sum_{i=1}^{n} (D_i f) d\,x_i$ for all $f \in \Omega^0(\mathbb{R}^n)$.
(b) $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$
(c) $d(d\omega) = 0$ for all $\omega \in \Lambda^k(V)$.

For $k \in [n]$, $d$ is the **only** linear transformation from $\Omega^k(\mathbb{R}^n)$ to $\Omega^{k+\ell}(\mathbb{R}^n)$ satisfying these properties.

*Proof.*

(a)

(b)

(c) By the linearity $d$, it is enough to consider $\omega = f\,dx_I$.

$$\begin{aligned} d\omega &= df \wedge dx_I \\ &= \left( \sum_{i \in [n]} (D_i f) dx_i \right) \wedge dx_I \\ &= \sum_{i \in [n]} ((D_i f) dx_i) \wedge dx_i \wedge dx_I \\ &= \Downarrow \\ d(d\omega) &= \qquad\qquad = \sum_{i \in [n]} (d(D_i f) dx_i) \wedge dx_i \wedge dx_I \\ &= \sum_{i \in [n]} \left( d \sum_{j \in [n]} (D_j D_i f\, dx_j \right) dx_i \wedge dx_i \wedge dx_I \\ &= \left[ \sum_{i,j \in [n]} (D_j D_i f)\, dx_i \wedge d_j \right] \wedge dx_I \\ &= 0 \end{aligned}$$

<u>Note</u>: In principle these properties are all we need to compute $d\omega$. So $f \in \Omega^0(\mathbb{R}^n)$ we interpret

$$f\,d\omega = f \wedge d\omega$$

Example:

$$\begin{aligned}
\omega = f\,dz \wedge dy &= f \wedge dz \wedge dy \\
&\Downarrow (2) \\
d\omega &= df \wedge dz \wedge dy + (-1)^0 f \wedge d(dz \wedge dy) \\
&\Downarrow \\
d\omega &= df \wedge dz \wedge dy \\
&= (D_1 f\,dx + D_2 f\,dy + D_3 f\,dz) \wedge dz \wedge dy
\end{aligned}$$

Note: $d^2 = 0$ implies, when $n = 3$, $\text{curl}(\text{grad}(f)) = 0,\ \text{Div}(\text{curl}(F)) = 0 \qquad \forall F, f.$

# Fields & Forms on Manifolds

We want to integrate differential forms over manifolds (and potatoes). To do so we must define tangent spaces.

Suppose $\gamma : (a, b) \subseteq \mathbb{R} :: \mathbb{R}^3$ is a local parametrization of a 1-manifold in $\mathbb{R}^3$. so

$$\gamma(t) = x(t)e_1 + y(t)e_2 + z(t)e_3 \implies DD\gamma(t) = \begin{bmatrix} x'(t) \\ y'(t) \\ z'(t) \end{bmatrix}$$

So $D\gamma(t_0)$ is a tangent vector to the curve at $p = \gamma(t_0)$. So the tangent line to the curve at $p$ is $\{(D\gamma(t_0))\, v_1 \mid v_1 \in \mathbb{R}\}$ (we consider this to be a vector space with $p$ as the origin).

Suppose $\alpha : D \subseteq \mathbb{R}^2 :: \mathbb{R}^3$ is a local parametrization of a 2-manifold in $\mathbb{R}^3$. For $v \in \mathbb{R}^2$, consider $\gamma(t) = \alpha(x_0 + tv) : \mathbb{R} \to \mathbb{R}^3$ and the image of $\gamma$ is a curve on the surface. Then $\gamma'(0) = (D\alpha(x_0))\, v$ is tangent to the embedded curve, hence tangent to the surface at $p = \alpha(x_0)$. The tangent place is $\{D\alpha(x_0)v \mid v \in \mathbb{R}^2\}$ (again up to translation).

**Definition**. (Tangent Space of a Manifold)
Let $M$ be a (smooth) $k$-manifold in $\mathbb{R}^n$. Let $p \in M$ and let $\alpha : \mathcal{U} \to \mathcal{V}$ be a coordinate patch about $p$. Define the **tangent space** to $M$ at $p$ as

$$T_p(M) = \left\{ (p, D\alpha(x_0)v) : v \in \mathbb{R}^k \right\}$$

where $x_0 = \alpha^{-1}(p)$.

Remark: The definition does not depend on the choice of coordinate patch. Meaning,

$$\left\{ D\alpha(x_0)v : v \in \mathbb{R}^k \right\} = \left\{ D\beta(x_0)v : v \in \mathbb{R}^k \right\}.$$

This holds since the transition map $\gamma \coloneqq \beta^{-1} \circ \alpha$ is smooth and so $\beta \circ \gamma$ is a diffeomorphism.

**Definition**. (Tangent Bundle)
Taking all tangent spaces over $M$ gives,

$$T(M) = \bigsqcup_{p \in M} T_p(M) \cong \bigcup_{p \in M} (p, T_p(M))$$

which is called the **tangent bundle** of $M$.

**Definition**. (Differential Forms on Manifolds)
We may define an $\ell$-form on $M$ as a function $\omega$ such that for all $p \in M$, $\omega(p) \in \Lambda^\ell(T_p(M))$. In other words, $\omega(p)(v_1, \ldots, v_\ell)$ makes sense for $v_1, \ldots, v_\ell \in T_p(M)$ but not necessarily for all $v_1, \ldots, v_\ell \in \mathbb{R}^n$.

Although it is not trivial, every form on a manifold can be locally extended to $\mathbb{R}^n$.

Note: If we have an $\ell$-form $\omega$ on $\mathbb{R}^n$, $\omega \restriction M$ is an $\ell$-form on $M$ since $\omega(p)(v_1, \ldots, v_\ell)$ makes sense for $v_1, \ldots, v_\ell \in T_p(M)$.

**Fact**. ($\ell$-forms Can Be Extended)
An $\ell$-form on $M$ can be extended to an $\ell$-form on an open set in $\mathbb{R}^n$ containing $M$.

*Proof.*
The proof is not trivial (and deep).[7] □

Convention:
Given this, we will only consider forms that are defined in an open neighborhood of $M$ in $\mathbb{R}^n$

Suppose $\omega \in \Omega^2(\mathbb{R}^2)$ and $\alpha$ is a coordinate patch. We need to define the dual transform $\alpha^*\omega \in \Omega^2(\mathbb{R}^2)$ (which is a pullback).

**Definition**. (Dual Transforms)
The **dual transform** of a linear map $T : V \to W$ is the map $T^* : \mathcal{L}^k(W) \to \mathcal{L}^k(V)$ given by the form for $f \in \mathcal{L}^k(W)$,

$$(T^* f)(v_1, \ldots, v_k) = f(Tv_1, \ldots, Tv_k)$$

**Lemma**. (Properties of Dual Transforms)
(i) $T^* \in \operatorname{Hom}\left(\mathcal{L}^k(W), \mathcal{L}^k(V)\right)$.
(ii) $T^*(f \otimes g) = T^* f \otimes T^* g$.[a]
(iii) $(S \circ T)^* = T^* \circ S^*$.
(iv) $f \in \Lambda^k(W) \implies T^* f \in \Lambda^k(V)$.
(v) $T^*(\omega \wedge \eta) = (T^*\omega) \wedge (T^*\eta)$.

[a] The dual operation and the tensor operation commute.

[7] see pg. 244-249 of Munkres

*Proof.*

(i) easy.

(ii) easy.

(iii) easy.

(iv) easy.

(v) See homework to prove $T^* \circ \operatorname{Alt} = \operatorname{Alt} \circ T^*$. Thus

$$T^*(\omega \wedge \eta) = T^*\left(\frac{1}{k!\ell!}\operatorname{Alt}(\omega \otimes \eta\right) = \frac{1}{k!\ell!}\operatorname{Alt}(T^*\omega \otimes T^*\eta) = (T^*\omega) \wedge (T^* \wedge \eta)$$

Recall: For a coordinate patch $\alpha : \mathcal{U} \subseteq \mathbb{R}^k \to \mathcal{V} \subseteq M \subseteq \mathbb{R}^n$. Then $D\alpha(x_0)$ (is an $n \times k$ matrix and) is a linear map from $\mathbb{R}^k$ centered at $x_0$ to the affine tangent space centered at $p$. Therefore this map is linear when $x_0$ is held constant.

**Definition**. (Dual of a Continuous Map)
Let $\alpha : A \subseteq \mathbb{R}^k \to B \subseteq \mathbb{R}^n$ be a smooth map for $A, B$ open sets. Define the **dual transform** of forms $\alpha^* : \Omega^\ell(B) \to \Omega^\ell(A)$ for $\ell = 0, 1, 2, \ldots$ by

$$(\alpha^* f)(x) = \begin{cases} f \circ \alpha(x) & \ell = 0 \\ f \circ \alpha(x)\Big(D\alpha(x)(v_1), \ldots, D\alpha(x)v_k\Big) & \ell > 0 \end{cases}$$

where $f \in \Omega^\ell(B),\ x \in A,\ v_1, \ldots, v_\ell \in \mathbb{R}^k$.

**Lemma**.

(a) Let $A \subseteq \mathbb{R}^k \to^\alpha B \subseteq \mathbb{R}^n \to^\beta C \subseteq \mathbb{R}^m$ $\beta^*(a\omega + b\eta) = a\beta^*\omega + b\beta^*\eta$.

(b) $(\beta \circ \alpha)^* = \alpha^* \circ \beta^*$.

(c) $\beta^*(\omega \wedge \eta) = (\beta^*\omega) \wedge (\beta^*\eta)$.

*Proof.*
Easy. $\square$

Example: Let $\alpha : \mathbb{R}^2 \to \mathbb{R}^3$ given by $\alpha(s,t) = (st, s + t + 1, t^3)$. Then,

$$D\alpha(s,t) = \begin{bmatrix} t & s \\ 1 & 1 \\ 0 & 3t^2 \end{bmatrix}$$

Let $\omega \in \Lambda^1(\mathbb{R}^3)$ given by $\omega(x,y,z) = yz\,dy + x\,dz$. We now compute $(\alpha^*\omega)(0,1)$. Note $\alpha(0,1) = (0,2,1)$ and

$$D\alpha(0,1) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 3 \end{bmatrix}$$

For $v = (v_1, v_2) \in \mathbb{R}^2$,

$$\begin{aligned}
(\alpha^*\omega)(0,1)v &= \omega(0,2,1)\Big(v_1e_1, v_1+v_2, 3v_2\Big) \\
&= (2\,dy + 0\,dz)\Big(v_1e_1 + (v_1+v_2)e_2 + 3v_2e_3\Big) \\
&= 2(v_1+v_2) \\
&\Downarrow \\
(\alpha^*\omega)(0,1) &= 2\,ds + 2\,dt.
\end{aligned}$$

Now let's consider a general point $(s,t)$.

$$\begin{aligned}
(\alpha^*\omega)(s,t)v &= \omega(st, s+t+1, t^3)\Big((tv_1+sv_2), (v_1+v_2), 3t^2v_2\Big) \\
&= \Big((s+t+1)t^3\,dy + st\,dz\Big)\Big((tv_1+sv_2), (v_1+v_2), 3t^2v_2\Big) \\
&= (s+t+1)t^3(v_1+v_2) + (st)(3t^2v_2) \\
&\Downarrow \\
(\alpha^*\omega)(s,t) &= (s+t+1)t^3(ds+dt) + (st)(3t^2)dt.
\end{aligned}$$

This strategy works in general, the coefficients in the end are the differentials of the coefficients ???

**Theorem**. (Computation of Dual Map)
Take $x = (x_1, \ldots, x_k) \in \mathbb{R}^k$ and $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$. Let $A \subseteq \mathbb{R}^k$ and $B \subseteq \mathbb{R}^n$ be open and $\alpha : A \to B$ a smooth map.
Take a 1-form, then

$$\alpha^*(dy_i) = d\alpha_i = \sum_{i=1}^{k} (D_jx_i)\ dx_j.$$

For a $k$-form we have. For $I \in \mathrm{ASC}_{k,n}$.

$$\alpha^*\left(\bigwedge_{j\in[k]} dy_{i_j}\right) = \left(\det\frac{\partial\alpha_I}{\partial x}\right)\left(\bigwedge_{j\in[k]} dy_{i_j}\right)$$

where

$$\frac{\partial\alpha_I}{\partial x} = \frac{\partial\left(\alpha_{i_1}, \ldots, \alpha_{i_k}\right)}{\partial(x_1, \ldots, x_k)}$$

*Proof.*
1-form case. Take $v \in \mathbb{R}^k$ then

$$\begin{aligned}
\alpha^*(dy_i)(x)(v) &= dy_i \circ \alpha(x)(D\alpha(x)v) \\
&= D\alpha(x)(v)e_1 \\
&= \sum_{j=1}^{k} D_j\alpha_i(x)v_j \\
&= \sum_{j=1}^{k} (D_j\alpha_i)\,dx_j.
\end{aligned}$$

$k$-form case.

$$
\begin{aligned}
\alpha^*\left(\bigwedge_{j\in[k]} dy_{i_j}\right) &= \bigwedge_{j\in[k]} \alpha^*\left(dy_{i_j}\right) && \text{dual distributes over } \wedge \\
&= \bigwedge_{j\in[k]} d\alpha_{i_j} \\
&= \bigwedge_{\ell=1}^{k}\left(\sum_{j=1}^{k}(D_j\alpha_{i_\ell})\,dx_j\right) && \text{1-form case} \\
&= \bigwedge_{\ell=1}^{k}\left(\sum_{b=1}^{k} C_{\ell b}\,dx_b\right) && \text{putting } C_{ab} = D_b\alpha_{i_a} \\
&= \left(\det(C_{ab})_{a,b\in[k]}\right) && \text{by homework}
\end{aligned}
$$

**Theorem**. (The Differential and Dual Commute)
Let $A \subseteq \mathbb{R}^k$ and $B \subseteq \mathbb{R}^n$ be open and $\alpha : A \to B$ smooth. Then,

$$\alpha^*(d\omega) = d\left(\alpha^*\omega\right)$$

$\forall \omega \in \Omega^\ell(B)$.

*Proof.*
$\ell = 0$ case: read the book.
$\ell > 0$ By linearity it is enough to consider $\omega = f\,dy_I$, for $I \in \mathrm{ASC}_{\ell,n}$.

$$\alpha^*(d\omega) = \alpha^*(df \wedge dy_I) = \alpha^*(df) \wedge \alpha^*(dy_I)$$

$$d(\alpha^*\omega) = d(\alpha^*(f \wedge dy_I)) = d(\alpha^* f \wedge \alpha^* dy_I) = d\alpha^* f \wedge \alpha^*(dy_I) = (-1)^0(\alpha^* f) \wedge d(\alpha^*(dy_I))$$

But $d(\alpha^*(dy_I)) = d(d\alpha_{i_1} \wedge \cdots \wedge d\alpha_{i_k})$ is zero since $d^2$ is zero.

By the 1-form case we have $\alpha^*(df) = d\left(\alpha^* f\right)$.

# INTEGRATION OF FORMS ON MANIFOLDS

## Integration on Parametrized Manifolds

**Definition**. (Integration over Parametrized Manifold)
Suppose $\eta \in \Omega^k(A)$ for $A \subseteq \mathbb{R}^k$ open. For $\eta = f\, dx_1 \wedge \cdots \wedge dx_k$, define

$$\int_A \eta = \int_A f$$

**Definition**. (Integral of a Form on a Parametrized Manifold)
Let $\alpha : A \subseteq \mathbb{R}^k \to \mathbb{R}^n$ be smooth. Let $Y = \alpha(A)$ and $B \subseteq \mathbb{R}^n$ be open and contain $Y$. Then $Y_\alpha$ is the manifold parametrized by $\alpha$. For $\omega \in \Omega^k(B)$,

$$\int_{Y_\alpha} \omega = \int_A \alpha^* \omega$$

which equals $\int_A f$ provided that $\omega = f\, dx_1 \wedge \cdots \wedge dx_k$,

Example: Let $\alpha : (0, \pi) \times (0, 1) \to \mathbb{R}^3$ be given by $\alpha(\theta, t) = (2\cos\theta, 2\sin\theta, t)$. Then $Y_\alpha$ looks like half the label of a pop can. Let $\omega = xz\, dy \wedge dz - yz\, dx \wedge dz$. This form is identified with the vector field $F(x, y, z) = (xz, yz, 0)$ by the musical isomorphism $\sharp$. This form acts outwards on the manifold and gains magnitude as $z$ increases.

$$\begin{aligned}
\int_{Y_\alpha} \omega &= \int_{Y_\alpha} \alpha^* \omega \\
&= \int_A \alpha^*(xz)\alpha^*(dy) \wedge \alpha^*(dz) - \alpha^*(yz)\alpha^*(dx) \wedge \alpha^*(dz) \\
&= \int_A (2\cos\theta)(t)(2\cos\theta\, d\theta) \wedge dt - (2\sin\theta)(t)(-2\sin\theta\, d\theta) \wedge dt \\
&= \int_A 2t\, d\theta \wedge dt \\
&= \int_0^\pi \int_0^1 2t\, d\theta dt \\
&= 2\pi
\end{aligned}$$

where $d\theta dt$ is a formal expression.

Now we do the same computation with $\beta(t, \theta) = (2\cos\theta, 2\sin\theta, t)$. Then

$$\begin{aligned}
\int_{Y_\alpha} \omega &= \int_A 2t\, dt \wedge d\theta \\
&= \int_A 2t\, d\theta \wedge dt \\
&= -2\pi
\end{aligned}$$

Even though $Y_\beta = Y_\alpha$ as a set, the integrals are equal only up to sign.

Now we do the same computation with $\gamma : (-2, 2) \times (0, 1) \to \mathbb{R}^3$ given by $\gamma(u, v) = (u, \sqrt{4 - u^2}, v)$. Then again $Y_\gamma \cong Y_\alpha$

$$\begin{aligned}\int_{Y_\gamma} \omega &= \int_{(-2,2)\times(0,1)} \frac{4u}{\sqrt{4-u^2}}\, du \wedge dv \\ &= \int_{-2}^{2}\int_{0}^{1} \frac{4u}{\sqrt{4-u^2}}\, dudv \\ &= -2\pi\end{aligned}$$

using substitution with $u = \sin(t)$.

**Theorem**. (Reparametrization Up To Sign)
Let $A, B \subseteq \mathbb{R}^k$ be open. If $g : A \to B$ is a diffeomorphism and $\det Dg$ does not change sign on $A$. Let $\beta : B \to \mathbb{R}^n$ be smooth and put $\alpha = \beta \circ g$. Let $O \subseteq \mathbb{R}^n$ be open and contain $Y = \alpha(A) = \beta(B)$. Then for every $\omega \in \Omega^k(O)$

$$\int_{Y_\alpha} \omega = \pm \int_{Y_\beta}$$

the minus sign is obtained only when $\det Dg < 0$ on $A$.

Remark: if $A$ is connected, the condition "$\det Dg$ does not change sign" is automatically satisfied.

*Proof.*

$$\begin{aligned}\int_{Y_\beta} &= \int_B \beta^*\omega && \beta^*\omega = f\, d_{y_1} \wedge \cdots \wedge d_{y_k} \\ &= \int_B f \\ &= \\ &= \int_{Y_\alpha} \omega \\ &= \int_A \alpha^*\omega \\ &= \int_A g^*\left(\beta^*\omega\right) && \alpha \coloneqq \beta \circ g \\ &= \int_A g^*\left(f\, d_{y_1} \wedge \cdots \wedge d_{y_k}\right) && \text{properties of pullback} \\ &= \int_A (f \circ g) \cdot (\det Dg)\;\; d_{y_1} \wedge \cdots \wedge d_{y_k} \\ &= \pm \int_B f && \text{change of variables}\end{aligned}$$

**Theorem**. (Computation of Integrals of Forms)
Suppose $A \subseteq \mathbb{R}^k$ is open and $\alpha : A \to \mathbb{R}^n$ is smooth. For $I \in \mathrm{ASC}_{k,n}$,

$$\int_{Y_\alpha} f\, dz_I = \int_A (f \circ \alpha) \cdot \left(\det \frac{\partial \alpha_I}{\partial x}\right)$$

*Proof.*
Do it!

Remark: If $A = (a, b) \subset \mathbb{R}$.

$$\int_{(a,b)} f\, dx = \int_{(a,b)} f = \int_a^b f(x)\, dx$$

where the $dx$ on the LHS has rigorous meaning and the $dx$ on the RHS has formal meaning. So the formal notation $dx$ in a one dimensional integral immediately makes formal sense.

This does not hold for any higher dimensions.

About the integral of a form question on the IBL.[8]

# Integration on Orientable Manifolds

In a parametrized manifold, the manifold is a one-patch manifold and the patch is given. For general differential manifolds we need several patches (this issue is easy to handle with a partition of unity). We also have a freedom of choice of parametrization about any given points.

A reasonable definition would be

$$\int_M \omega = \int_{\operatorname{Int} \mathcal{U}} \alpha^* \omega$$

for some patch $\alpha : \mathcal{U} \to M$ where $\mathcal{U}$ is open in $\mathbb{R}^k$ or $\mathbb{H}^k$.

Unfortunately this definition causes sign issues, which we seek to solve here.

**Definition**. (Sign of Coordinate Overlap)
Let $M \subseteq \mathbb{R}^n$ be a $k$-manifold. Let $\alpha_1 : \mathcal{U}_1 \to \mathcal{V}_1$ and $\alpha_2 : \mathcal{U}_2 \to \mathcal{V}_2$ be two coordinate matches with non-empty overlap (i.e. $\mathcal{V}_1 \cap \mathcal{V}_2 \neq \varnothing$). Recall the transition map $\alpha_2^{-1} \circ \alpha_1$ is a diffeomorphism (sufficiently restricted). We say that $\alpha_1$ and $\alpha_2$ **overlap positively** if

$$\det\left(\alpha_2^{-1} \circ \alpha_1\right) > 0$$

everywhere it is defined.
We define **overlap negatively** similarly (these are the only cases).

**Definition**. (Orientable Manifold)
Let $M \subseteq \mathbb{R}^n$ be a $k$-manifold in $\mathbb{R}^n$. We say $M$ is **orientable** if it can be covered by coordinate patches which pairwise overlap positively. Otherwise, we say $M$ is **non-orientable**.

Example: Consider $\mathbb{S}^1 \subseteq \mathbb{R}^2$. For $i \in [4]$, let $\alpha_i : (-1, 1) \to \mathbb{R}^2$ be given by

$$\begin{aligned}
\alpha_1(t) &= \left(t, \sqrt{1 - t^2}\right) \\
\alpha_2(t) &= \left(t, -\sqrt{1 - t^2}\right) \\
\alpha_3(t) &= \left(\sqrt{1 - t^2}, t\right) \\
\alpha_4(t) &= \left(-\sqrt{1 - t^2}, t\right)
\end{aligned}$$

[8] "I was hoping this was a trick question, but it took you all 40 minutes to figure it out."

The overlap of $\alpha_1$ and $\alpha_3$ is in the first quadrant of the coordinate plane. Thus $\alpha_1, \alpha_3$ overlap negatively. This occurs because the "trajectory" of these maps is opposite.

$$\left(\alpha_3^{-1} \circ \alpha_1\right)(t) = \sqrt{1-t^2}$$

for $0 < t < 1$.

Thus,

$$D\left(\alpha_3^{-1} \circ \alpha_1\right)(t) = \frac{-t}{\sqrt{1-t^2}} < 0$$

$\forall t \in (0,1)$.

Instead, for $i \in [4]$, put $\beta_i : (-1,1) \to \mathbb{R}^2$

$$\begin{aligned} \beta_1(t) &= \alpha_1(t) \\ \beta_2(t) &= \alpha_1(-t) \\ \beta_3(t) &= \alpha_1(-t) \\ \beta_4(t) &= \alpha_1(t) \end{aligned}$$

Then all transition maps overlap positively. Thus, $\mathbb{S}^1$ is orientable.

Example: Let $\alpha : (0,\pi) \times (0,1) \to \mathbb{R}^3$ be given by $\alpha(\theta, t) = (2\cos\theta, 2\sin\theta, t)$. $Y_\alpha$ looks like half the label of a pop can. Since $Y_\alpha$ can be covered by a single patch, $Y_\alpha$ is orientable.

Let $\gamma : (-2,2) \times (0,1) \to \mathbb{R}^3$ be given by $\gamma(u,v) = (u, \sqrt{4-u^2}, v)$. Then again $Y_\gamma \cong Y_\alpha$

Check that $\alpha, \gamma$ overlap negatively.

Example: The Möbius band is non-orientable and so is the Klein Bottle. The Klein bottle is a 2-manifold in $\mathbb{R}^4$. The Klein Bottle contains the Möbius strip as a subspace.

Do we have a way of showing the Möbius band as a set is a manifold.

**Definition**. (Orientation of a Manifold)
Let $M \subseteq \mathbb{R}^n$ be an orientable $k$-manifold. An **orientation** of $M$ is a "maximal collection" of positively overlapping coordinate patches.
An **oriented manifold** is an orientable manifold together with a specific orientation.

There are simpler descriptions of orientation of manifolds in $\mathbb{R}^n$ for three cases $k = 1, (n-1), n$ that correspond nicely with intuition.

**Definition**. (One Dimensional Orientation of a Manifold)
Let $M \subseteq \mathbb{R}^n$ be a 1-manifold. For $p \in M$, let $\alpha : \mathcal{U} \to \mathcal{V}$ be a patch about $p$ belonging to the orientation. Define $T(p) := \left(p, \frac{D\alpha(t_0)}{\|D\alpha(t_0)\|}\right)$ where $t_0 = \alpha^{-1}(p)$ for some $\alpha$ in the orientation of $M$. We call $T : M \to T(M)$ the **unit tangent field** corresponding to the orientation of $M$.
Remark: $T(p)$ is well defined. Thus for $k = 1$ we just need to give our curve a direction.
Let $\mathbb{L} := \mathbb{R}^1 \setminus \mathbb{H}^1_+$ then we can remedy the case for $k$-manifolds with boundary by allowing patches from $\mathbb{R}, \mathbb{H}$, or $\mathbb{L}$.
**Fact**: Every 1-manifold is orientable.

**Definition**. (Orientation of a Hyper-Manifold)
Let $M \subseteq \mathbb{R}^n$ be an oriented $(n-1)$-manifold in $\mathbb{R}^n$. Take $p \in M$ and a $\alpha$ a coordinate patch about $p$ belonging to the orientation. Put $x_0 = \alpha^{-1}(p)$.
Let $N(p)$ be the unit vector such that

(i) $N(p)$ is perpendicular (or normal) to $T_p(M)$, meaning

$$N(p) \cdot \frac{\partial \alpha}{\partial x_i}(x_0) = 0$$

$\forall i \in [n-1]$.

(ii)

$$\det \begin{bmatrix} N(p) & \Big| D\alpha(x_0) \end{bmatrix} > 0$$

Remark: $N : M \to \mathbb{R}^n$ is well defined and smooth.
$N$ is called the **unit normal field** corresponding to the orientation of $M$.

Example: Let $\alpha : (-\pi, \pi) \times (0,1) \to \mathbb{R}^3$ be given by $\alpha(\theta, t) = (2\cos\theta, 2\sin\theta, t)$.

$$D\alpha(\theta, t) = \begin{bmatrix} -2\sin\theta & 0 \\ 2\cos\theta & 0 \\ 0 & 1 \end{bmatrix} \qquad N(\alpha(\theta,t)) = \pm \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix}$$

We choose the positive vector.

Check that

$$N(\gamma(u,v)) = -\frac{1}{2} \begin{bmatrix} u \\ \sqrt{4-u^2} \\ 0 \end{bmatrix}$$

**Definition**. (Orientation of an $n$-Manifold)
Let $M \subseteq \mathbb{R}^n$ be an $n$-manifold. Define the **natural orientation** of $M$ to be the collection of coordinate patches $\alpha : \mathcal{U} \to \mathcal{V}$ satisfying $\det D\alpha > 0$.

**Definition**. (Reverse Orientation)
Define $r : \mathbb{R}^k \to \mathbb{R}^k$ by $r(x_1, \ldots, x_k) = (-x_1, x_2, \ldots, x_k)$.
Note $r : \mathbb{H}^1 \to \mathbb{L}^1$ and for $k > 1$ we have $r : \mathbb{H}^k \to \mathbb{H}^k$.

**Lemma**.
Let $M$ be an oriented manifold in $\mathbb{R}^n$, for a patch $\alpha : \mathcal{U} \to \mathcal{V}$ belonging to the orientation $\mathscr{A}$, let $\beta = \alpha \circ r : r(\mathcal{U}) \to \mathcal{V}$. Then $\beta$ is a coordinate patch and overlaps with $\alpha$ negatively. So $\beta$ does not belong to the orientation. If $\alpha_1, \alpha_2 \in \mathscr{A}$, then $r(\alpha_1), r(\alpha_2)$ overlap positively. The collection $r(\mathscr{A})$ is called the **reverse** (or **opposite**) **orientation**.

This shows that every manifold has an even number of orientations. For a connected manifold, we will show that $M$ admits at most 2 orientations. A manifold with $n$ connected components admits at most $2^n$ orientations. Furthermore, if a manifold admits an orientation, then it admits the maximal number.

# Induced Orientation on Boundary Manifolds

An orientation on a manifold induces (by restriction) an orientation on the boundary manifold. We now turn to study this relationship. Later we will see how to induce this orientation without using restriction.

**Theorem**. (Induced Orientation on $\partial M$)
Let $k > 1$, if $M \subseteq \mathbb{R}^n$ is an oriented $k$-manifold with non-empty boundary, then $\partial M$ is orientable. The proof will construct an orientation on $\partial M$.

*Proof.*
Let $p \in \partial M$, there is a patch (about $p$) given by $\alpha : \mathcal{U} \to \mathcal{V}$ that belongs to the orientation of $M$. Note that $\mathcal{U} \subseteq \mathbb{H}^k$. Let $b : \mathbb{R}^{k-1} \to \mathbb{R}^k$ be given by $b(x_1, \ldots, x_{k-1}) = (x_1, \ldots, x_{k-1}, 0)$. Put $\alpha_0 \coloneqq \alpha \circ b : \mathcal{U}_0 \to \mathcal{V} \cap \partial M$. Recall that $\alpha_0$ is a coordinate patch on $\partial M$ about $p$.

Let $\alpha, \beta$ be two coordinate patches of $M$ about $p$, in the orientation of $M$. So, $\det D\left(\beta^{-1} \circ \alpha\right) > 0$.

We will show that $\det D\left(\beta_0^{k-1} \circ \alpha_0\right) > 0$. Put $g \coloneqq \beta^{-1} \circ \alpha$ and $h \coloneqq \beta_0^{-1} \circ \alpha_0$.

$h(x_1, \ldots, x_{k-1}) = \Big(g_1(x_1, \ldots, x_{k-1}, 0), \ldots, g_{k-1}(x_1, \ldots, x_{k-1}, 0)\Big)$. Let $\mathcal{W}_1 = \alpha^{-1} \circ \beta(\mathcal{U}_2)$ and $\mathcal{W}_2 = \beta^{-1} \circ \alpha(\mathcal{U}_1)$. For all $x \in \mathcal{W}_1$, $g)k(x) \geq 0$ and $\forall x_0 \in \mathcal{W}_1 \cap \partial \mathbb{H}^k$, $g_k(x) = 0$. Let $x_0 \in \mathcal{W}_1 \cap \partial \mathbb{H}^k$,

$$\frac{\partial g_k}{\partial x_i} = \lim_{t \to 0^+} \frac{g_k(x_0 + te_i) - g_k(x_0)}{t} = \lim_{t \to 0^+} \frac{g_k(x_0 + te_i)}{t} \geq 0$$

$\forall i \in [k]$ and equality holds $\forall i \in [k-1]$.

$$Dg(x_0) = \frac{\partial(g_1, \ldots, g_k)}{\partial(x_1, \ldots, x_k)}(x_0) = \left[\begin{array}{c|c} \dfrac{\partial(g_1, \ldots, g_{k-1})}{\partial(x_1, \ldots, x_{k-1})} & \begin{matrix} * \\ \vdots \\ * \end{matrix} \\ \hline 0 \ \cdots \ 0 & \dfrac{\partial g_k}{\partial x_k}(x_0) \end{array}\right]$$

$$\implies \det Dg(b(q)) = (Dh(q)) \left(\frac{\partial g_k}{\partial x_k}(b(q))\right)$$

Since $\frac{\partial g_k}{\partial x_k}(b(q)) > 0$, we find $\det Dg(b(q)) > 0$. Thus $\alpha_0, \beta_0$ overlap positively.

$\square$

<u>Notation</u>: Given an orientation of $M$, we will call the orientation on $\partial M$ obtained in the way above the **restriction orientation**.[9]

**Definition**. (Induced Orientation)
Let $M \subseteq \mathbb{R}^n$, be an oriented $k$-manifold with $\partial M \neq \varnothing$. The **induced orientation** of $\partial M$ is
(i) the restriction orientation if $k$ is even, and
(ii) the reverse orientation of the restriction orientation if $k$ is odd.

[9] No one calls it this.

Example: Let $M = \mathbb{S}^2 \cap \mathbb{H}^3$. Then $\partial M = \mathbb{S}^1 \subseteq \mathbb{R}^3$. Let the orientation of $M$ contain the patch $\alpha : (-1,1) \times [0,1) \to \mathbb{R}^3$ given by $\alpha(u,v) = \left(u, \sqrt{1-u^2-v^2}, v\right)$. Te restriction of $\alpha$ on $\partial M$ is $\alpha_0 : (-1,1) \to \mathbb{R}^3$ given by $\alpha_0 = \alpha(u,0) = \left(u, \sqrt{1-u^2}, 0\right)$.

Then $T(\alpha_0(u)) = \frac{D\alpha_0(u)}{\|D\alpha_0(u)\|} = (\sqrt{1-u^2}, -u, 0)$. In particular, $T(0,1,0) = T(\alpha_0(0)) = (1,0,0)$.

$$N(\alpha(u,v)) = \frac{\frac{\partial \alpha}{\partial u} \times \frac{\partial \alpha}{\partial v}}{\left\|\frac{\partial \alpha}{\partial u} \times \frac{\partial \alpha}{\partial v}\right\|} = (-u, -\sqrt{1-u^2-v^2}, -v) = -\alpha(u,v)$$

In particular, $N(0,1,0) = N(\alpha(0,0)) = (0,-1,0)$.

Let $W(p)$ be the unit vector tangent to $M$ at $p$ that points "into" $M$. This $\left(N(p), T(p), W(p)\right)$ system form the **right-handed frame** in $\mathbb{R}^3$. Even if you do the reverse orientation, you still get the right-handed frame.

Example: Put $M = \mathbb{B}^3$. Then $\partial \mathbb{B}^3 = \mathbb{S}^2$. Since $M$ is a 3-dimensional manifold in $\mathbb{R}^3$, it has a natural orientation. For a patch $\alpha : \mathcal{U} \subseteq \mathbb{H}^3 \to \mathcal{V}$ in the orientation, the natural orientation guarantees that

$$0 < \det D\alpha = \det \begin{bmatrix} \frac{\partial \alpha}{\partial x_1} & \frac{\partial \alpha}{\partial x_2} & \frac{\partial \alpha}{\partial x_3} \end{bmatrix}.$$

Now consider the induced orientation on $\partial M = \mathbb{S}^2$. This amounts to asking where does the Normal vector point?

Recall that we have

$$\det \begin{bmatrix} N & \frac{\partial \beta}{\partial x_1} & \frac{\partial \beta}{\partial x_2} \end{bmatrix} > 0$$

for some patch $\beta$.

Using the restriction of $\alpha$ (and since we use the opposite of the restricted orientation), we find that $N$ should satisfy

$$\det \begin{bmatrix} -N & \frac{\partial \alpha}{\partial x_1} & \frac{\partial \alpha}{\partial x_2} \end{bmatrix} > 0.$$

Since

$$0 < \det D\alpha = \det \begin{bmatrix} \frac{\partial \alpha}{\partial x_3} & \frac{\partial \alpha}{\partial x_1} & \frac{\partial \alpha}{\partial x_2} \end{bmatrix}$$

We obtain that $-N$ and $\frac{\partial \alpha}{\partial x_3}$ point the same side of the tangent plane. Thus $N$ points "outward."

# Integration of forms on Oriented Manifolds

We proceed just as before. We begin with the single patch case.

**Definition**. (Integration of form over Single Patch Oriented Manifold)
Let $M \subseteq \mathbb{R}^n$ be a compact, oriented $k$-manifold. Let $\omega$ be a $k$-form on and open $B \subseteq \mathbb{R}^n$ containing $M$. Let $C = M \cap \operatorname{supp}(\omega)$ be compact. Suppose there is a coordinate patch in the orientation $\alpha : \mathcal{U} \to \mathcal{V}$ such that $V \supseteq C$ and $\mathcal{U}$ is bounded. Define

$$\int_M \omega = \int_{\operatorname{Int} \mathcal{U}} \alpha^* \omega = \int_{\operatorname{Int} \mathcal{U}} f$$

where $\alpha^* \omega = f \, dx_1 \wedge \cdots \wedge dx_k$.

We can integrate over $\mathrm{Int}\,\mathcal{U}$ because we are at most removing $\partial\mathbb{H}^k$ which is measure zero.

We need to check that the RHS is ordinary integrable, satisfies properties like linearity, and does not depend on the choice of coordinate patch. These concerns are satisfied by the Change of Variables Theorem since

$$\int_{\mathrm{Int}\,\mathcal{U}_1} f \circ \alpha_1 = \pm \int_{\mathrm{Int}\,\mathcal{U}_2} f \circ \alpha_2$$

But since $\alpha_1$ and $\alpha_2$ overlap positively these equations are equal.

Note that we often write $-M$ to indicate the oriented manifold with the reverse orientation. Then,

$$\int_{-M} \omega = -\int_M \omega$$

**Definition**. (Integration of form over Oriented Manifold)
Let $M \subseteq \mathbb{R}^n$ be a compact, oriented $k$-manifold. Let $\omega$ be a $k$-form on and open $B \subseteq \mathbb{R}^n$ containing $M$. Let $\{\phi_i\}_{i\in[\ell]}$ be a partition of unity on $M$ dominated by the orientation. Define,

$$\int_M \omega = \sum_{i\in[\ell]} \int_M \phi_i \cdot \omega$$

One can check this is well-defined, satisfies linearity, and sign flip for orientation reversal.

**Theorem**. (Computation of Integral of a Form)
Let $M \subseteq \mathbb{R}^n$ be a compact, oriented $k$-manifold. Let $\omega$ be a $k$-form on and open $B \subseteq \mathbb{R}^n$ containing $M$. Suppose for each $i \in [\ell]$ there is a coordinate patch $\alpha_i : \mathcal{A}_i \to \mathcal{M}_i$ of $M$ in the orientation such that

$$M = \left(\bigcup_{i\in[\ell]} \mathcal{M}_i\right) \cup K$$

for measure zero $K \subseteq M$.

Then,

$$\int_M \omega = \sum_{i\in[\ell]} \int_{\mathcal{A}_i} \alpha_i^* \omega.$$

*Proof.*
Skip! □

Example:
Let $M = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1,\ 0 \le z \le 1\}$ with orientation so that $N\left(1, 0, \frac{1}{2}\right) = (-1, 0, 0)$. We compute $\int_M z\, dx \wedge dy$. Note that $N(x, y, z) = (-x, -y, 0)$ for all $(x, y, z) \in M$. Now we look for a nice patch. Put $\alpha : (-\pi, \pi) \times (0, 1) \to \mathbb{R}^3$ then $\alpha(\theta, t) = (\cos\theta, \sin\theta, t)$.

To find the orientation of $\alpha$, we compute the cross product

$$\frac{\partial \alpha}{\partial \theta} \times \frac{\partial \alpha}{\partial t} = \begin{bmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = (\cos\theta, \sin\theta, 0).$$

Whoops! Our sign was wrong, so we choose the opposite orientation. Putting $\gamma(\theta, t) = \alpha(-\theta, t)$ or $\gamma(\theta, t) = \alpha(t, \theta)$. We will use the later method. So $\beta : (0,1) \times (-\pi, \pi) \to \mathbb{R}^3$ with $\beta(\theta, t) = \alpha(t, \theta)$ is a coordinate patch in the orientation.

$$\begin{aligned}\int_M z\,dx \wedge dy &= \int_{(0,1)\times(-\pi,\pi)} \beta^*(z\,dx \wedge dy) \\ &= \int_{(0,1)\times(-\pi,\pi)} t\,d\beta_1 \wedge d\beta_2 \\ &= \int_0^1 \int_{-\pi}^{\pi} t \cdot 0 \\ &= 0\end{aligned}$$

**The exam is up to here.**

<u>Intuition</u>: We now consider the meaning of integrating a form over an $(n-1)$ manifold.

Let $\Omega \subseteq \mathbb{R}^2$ be open and $g : \Omega \to \mathbb{R}$ smooth. Put $M \coloneqq G_g$, the graph of $g$. Note that $M$ is an oriented manifold (it has a single patch). Fix $\alpha(x, y) = (x, y, g(x, y))$, so $\alpha$ is a patch that covers $M$. Consider the orientation of $M$ that contains $\alpha$. Since $3 - 1 = 2$, we can describe the orientation using the unit normal field.

$$\frac{\partial \alpha}{\partial x} \times \frac{\partial \alpha}{\partial y} = \begin{bmatrix} 1 \\ 0 \\ \frac{\partial g}{\partial x} \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ \frac{\partial g}{\partial y} \end{bmatrix} = \left( -\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, 1 \right)$$

So the unit normal vector, will be this vector but normalized. Thus $N$ points "upwards."

Let $x = u, y = v$, and $z = g(x, y)$ and we compute,

$$\begin{aligned}&\int_M F_1\,dy \wedge dz - F_2\,dx \wedge dz + F_3\,dx \wedge dy \\ &= \int_\Omega (F_1 \circ \alpha)\,dv \wedge \left( \frac{\partial g}{\partial u} du + \frac{\partial g}{\partial v} dv \right) - (F_2 \circ \alpha)\,u \wedge \left( \frac{\partial g}{\partial u} du + \frac{\partial g}{\partial v} dv \right) + (F_3) \circ \alpha)\,du \wedge dv \\ &= \int_\Omega \left( -(F_1 \circ \alpha) \frac{\partial g}{\partial u} - (F_2 \circ \alpha) \frac{\partial g}{\partial v} + (F_3) \circ \alpha) \right) du \wedge dv \\ &= \int_\Omega (F_1 \circ \alpha, F_3 \circ \alpha, F_2 \circ \alpha) \cdot \left( -\frac{\partial g}{\partial u}, \frac{\partial g}{\partial v}, 1 \right) (\star)\end{aligned}$$

Recall that for any $h : M \to \mathbb{R}$

$$\int_M h\,dV = \int_\Omega (h \circ \alpha)\,V(D\alpha)$$

For our case,

$$V(D\alpha) = \sqrt{\det\left(D\alpha^T D\alpha\right)} = \sqrt{1 + \left(\frac{\partial g}{\partial u}\right)^2 + \left(\frac{\partial g}{\partial v}\right)^2}$$

Multiplying $(\star)$ by $1 = \frac{\sqrt{1+\left(\frac{\partial g}{\partial u}\right)^2+\left(\frac{\partial g}{\partial v}\right)^2}}{\sqrt{1+\left(\frac{\partial g}{\partial u}\right)^2+\left(\frac{\partial g}{\partial v}\right)^2}}$, we obtain

$$\begin{aligned}&\int_M F_1\,dy\wedge dz - F_2\,dx\wedge dz + F_3\,dx\wedge dy\\&=\int_\Omega (F\cdot N)\ dV\end{aligned}$$

This result still holds if $M$ is an oriented 2-manifold in $\mathbb{R}^3$. More generally, this holds for an oriented $(n-1)$-manifold in $\mathbb{R}^n$. We can interpret this integral as a flux integral, we are finding the flux of the vector field $F$ across $M$.

Example: Let $M=\mathbb{S}^2(a)$. Orient $M$ such that $N(x,y,z)=\frac{(x,y,z)}{a}$ points outward. Compute

$$\int_{\mathbb{S}^2(a)} x\,dy\wedge dz - y\,dx\wedge dz + z\,dx\wedge dy = \int_{\mathbb{S}^2(a)}(x,y,z)\cdot\frac{(x,y,z)}{a}\,dV$$

$$=\int_{\mathbb{S}^2(a)}\frac{x^2+y^2+z^2}{a}4\pi a^2 = 4\pi a^3.$$

Intuition: Let $M$ be an oriented 1-manifold in $\mathbb{R}^n$ and suppose $\exists\alpha:(a,b)\to\mathbb{R}^n$, a coordinate patch in the orientation such that $M\setminus\alpha((a,b))$ has measure zero in $M$.

$$\begin{aligned}\int_M\sum_{i\in[n]}F_i\,dx_i &= \int_{(a,b)}\sum_{i\in[n]}(F_i\circ\alpha)\,(t)\alpha_i'(t)\,dt\,dx_i\\&=\int_{(a,b)}F(\alpha(t))\cdot D\alpha(t)\\&=\int_{(a,b)}F(\alpha(t))\cdot\frac{D\alpha(t)}{\|D\alpha(t)}\|D\alpha(t)\|\\&=\int_{(a,b)}(F\cdot T)(\alpha(t))\|D\alpha(t)\|D\alpha(t)\|\\&=\int_{(a,b)}(F\cdot T)(\alpha(t))\,dV\end{aligned}$$

because in the one manifold case $\|D\alpha(t)\|=V(D\alpha)$.

This can be interpreted as a work integral, meaning the total work done by $F$ along $M$. If the manifold is homeomorphic so $\mathbb{S}^1$, then this is called a "circulation" of the "velocity field" $F$ along $M$.

# REVIEW

Up until now we have learned 11 things.

(1) We computed the volume of a parallelepiped.

(2) Integration of a scalar function over a parametrized manifold.

(3) Differentiable manifolds without boundary.

(4) Differentiable manifolds with boundary.

(5) Integration of a scalar function over a manifold. We did so by defining measure zero sets on a manifold.

(6) Tensors.

(7) Differential forms (alternating tensor valued functions).

(8) Tangent spaces to a manifold.

(9) Dual transform.

(10) Integration of differential forms on parametrized manifolds.

(11) Orientations of a manifold.

(12) Integration of differential forms on orientable manifolds.

(13) The relationship of integrals of differential forms on an oriented manifold and its boundary.

(14) Special cases of Generalized Stoke's Theorem.

Note that 12-14 will not be covered on the Midterm.

Afterwards we will get to study Fourier Analysis … which "we are very excited to study".

# GENERALIZED STOKES' THEOREM

Recall the following theorem from Calculus

**Theorem**. (Fundamental Theorem of Calculus)

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

Interpreting this as an integration of a form over a manifold we obtain

$$\int_{\partial[a,b]} df = f(b) - f(a).$$

Let $I^k = [0,1]^k$ be the closed unit cube in $\mathbb{R}^k$. So, $\operatorname{Int} I^k = (0,1)^k$ and $\operatorname{Bd} I^k = I^k \setminus \operatorname{Int} I^k$. We will frequently be interested in the set $\operatorname{Int} I^{k-1} \times \{0\}$.

**Lemma**. ()
Let $k > 1$ and $\eta \in \Omega^{k-1}(A)$ for some open $A \subseteq \mathbb{R}^k$ which contains $I^k$. Suppose $\eta$ vanishes at all points of $\operatorname{Bd} I^k$, except possibly at points of $\operatorname{Int} I^{k-1} \times \{0\}$.
Then,

$$\int_{\operatorname{Int} I^k} d\eta = (-1)^k \int_{\operatorname{Int} I^{k-1}} b^*\eta$$

where $b : \mathbb{R}^{k-1} \hookrightarrow \mathbb{R}^k$.

*Proof.*
We can write

$$\eta = \sum_{j=1}^{k} f_j\, dx_1 \wedge \cdots \wedge \hat{dx_j} \wedge \cdots \wedge dx_k.$$

By the linearity of integrals, it is enough to check when $\eta$ is of the form $f\,dx_1 \wedge \cdots \wedge \hat{dx_j} \wedge \cdots \wedge dx_k$.

$$\begin{aligned} d\eta &= \left(\sum_{i\in[k]} (D_i f)\,dx_i\right) \wedge dx_1 \wedge \cdots \wedge \hat{dx_j} \wedge \cdots \wedge dx_k \\ &= (-1)^{j-1}(D_j f)\,dx_1 \wedge \cdots \wedge dx_j \wedge \cdots \wedge dx_k \end{aligned}$$

$$\begin{aligned}
\int_{\operatorname{Int} I^k} d\eta &= \int_{\operatorname{Int} I^k} (-1)^{j-1} D_j f \\
&= (-1)^{j-1} \int_{I^k} D_j f \\
&= (-1)^{j-1} \int_{v \in I^{k-1}} \int_{x_j \in [0,1]} D_j f(x_1, \ldots, x_k) && \text{where } v \coloneqq (x_1, \ldots, \hat{x}_j, \ldots, x_k) \\
&= (-1)^{j-1} \int_{v \in I^{k-1}} f(x_1, \ldots, 1, \ldots, x_k) - f(x_1, \ldots, 0, \ldots, x_k) && \text{1,0 in the } j\text{-th spot} \\
&= (-1)^{j-1} \int_{v \in I^{k-1}} f(x_1, \ldots, 0, \ldots, x_k) && \text{by assumption} \\
&= (-1)^{j-1} \int_{v \in I^{k-1}} f(x_1, \ldots, 0, \ldots, x_{k-1}, 0) \\
&= (-1)^{j-1} \int_{v \in I^{k-1}} (f \circ b)(x_1, \ldots, 0, \ldots, x_{k-1}) \\
&= (-1)^{j-1} \int_{\operatorname{Int} I^{k-1}} b^* f
\end{aligned}$$

Now we consider the RHS. We compute $b^*\eta = (f \circ b)\, db_1 \wedge \cdots \wedge \hat{db_j} \wedge \cdots \wedge db_k$. Is zero unless $j = k$. In which case $b^*\eta = (f \circ b)\, dx_1 \wedge \cdots \wedge dx_{k-1}$. Therefore,

$$(-1)^k \int_{\operatorname{Int} I^{k-1}} b^*\eta = \begin{cases} 0 & j < k \\ \int_{\operatorname{Int} I^{k-1}} f \circ b & j = k \end{cases}$$

□

This is half the proof for Generalized Stokes Theorem.

**Definition**. (Generalized Stokes' Theorem)
Let $k > 1$ and $M \subseteq \mathbb{R}^n$ be a compact, oriented $k$-manifold. Let $\omega$ be a $(k-1)$-form in an open neighborhood of $M$. Then,

$$\int_M d\omega = \int_{\partial M} \omega$$

where $\partial M$ is given the induced orientation.

In the case that $\partial M = \varnothing$, then the RHS is zero.

*Proof.*

Cover $M$ in patches in the following manner:

For $p \in M \setminus \partial M$. Let $\alpha : \mathcal{U} \to \mathcal{V}$ about $p$. Restrict, translate, and scale the domain of $\alpha$ to be $\operatorname{Int} I^k$. Note that $\alpha$ remains in the orientation of $M$.

For $p \in \partial M$, we do a similar maneuver but set the domain of $\alpha$ to be $\left(\operatorname{Int} I^k\right) \cup \left(\operatorname{Int} I^{k-1} \times \{0\}\right)$ which is open in $\mathbb{H}^k$, then $p \in \alpha\left(\operatorname{Int} I^{k-1} \times \{0\}\right)$.

"How do you do?"

Let $\{\phi_i\}_{i \in [\ell]}$ be a partition of unity dominated by the collection of patches constructed above.

$$\int_M d\omega = \sum_{i \in [\ell]} \int_M \phi_i d\omega$$

If we show the following identity,

$$\int_M d(\phi_i \omega) = \int_{\partial M} \phi_i \omega$$

$\forall i \in [\ell]$ Then,

$$\sum_{i\in[\ell]} \int_M \phi_i d(\phi_i \omega) = \int_{\partial M} \sum_{i\in[\ell]} \phi_i \omega = \int_{\partial M} \phi_i \omega.$$

Computing the LHS gives

$$\begin{aligned}
\sum_{i\in[\ell]} \int_M \phi_i d(\phi_i \omega) &= \sum_{i\in[\ell]} \int_M \Big( d\phi_i \wedge \omega + (-1)^0 \phi_i d\omega \Big) \\
&= \int_M \left( d \left( \sum_{i\in[\ell]} \phi_i \right) \wedge \omega + \sum_{i\in[\ell]} \phi_i d\omega \right) \\
&= \int_M \phi_i d\omega
\end{aligned}$$

By 2, it is enough to prove the theorem for $\omega$ such that $\operatorname{supp} \omega \cap M$ is contained in one coordinate patch constructed earlier.

Suppose $W = \operatorname{Int} I^k$ and $\alpha : W \to M$ is a coordinate patch.

$$\begin{aligned}
\int_M d\omega &= \int_{\operatorname{Int} I^k} \alpha^*(d\omega) \\
&= \int_{\operatorname{Int} I^k} d \circ \alpha^*(\omega) \\
&= \int_{\operatorname{Int} I^k} d \circ \alpha^*(\omega) \\
&= (-1)^k \int_{\operatorname{Int} I^k} b^* \circ \alpha^*(\omega) \\
&= 0
\end{aligned}$$

where the third equality follows from the Lemma. The last equality follows since $\operatorname{supp} \alpha^* \omega \subseteq \operatorname{Int} I^k$.

Similarly,

$$\int_{\partial M} \omega = 0$$

since $\operatorname{supp} \omega \cap \partial M = \varnothing$.

Now the last case. Suppose $W = \Big(\operatorname{Int} I^k\Big) \cup \Big(\operatorname{Int} I^k \times \{0\}\Big)$.

Then,

$$\begin{aligned}
\int_M d\omega &= \int_W \alpha^*(d\omega) \\
&= \int_{\operatorname{Int} I^k} \alpha^*(d\omega) \\
&= (-1)^k \int_{\operatorname{Int} I^{k-1}} b^* \alpha^*(\omega)
\end{aligned}$$

Put $\beta = \alpha \circ b : \operatorname{Int} I^{k-1} \to \partial M$. Then $\beta$ is a coordinate patch given by a restriction of $\alpha$. Therefore $\beta$ belongs to the orientation of $\partial M$ if $k$ is even. If $k$, is odd then we choose the reverse orientation of $\beta$.[10]

Thus,

$$\int_{\partial M} \omega = (-1)^k \int_{\operatorname{Int} I^{k-1}} \beta^* \omega = (-1)^k \int_{\operatorname{Int} I^{k-1}} b^* \alpha^*(\omega)$$

□

# Green's Theorem

**Theorem**. (Green's Theorem)
For a type 3 region $D \subseteq \mathbb{R}^2$ we have

$$\int_{\partial D} P\,dx + Q\,dy = \operatorname{Int}_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)$$

*Proof.*
Check that $d\omega = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \wedge dy$.

Then we apply Stokes' Theorem with $n = 2, k = 2$

# Stokes' Theorem

# Gauss' Theorem

[10]This is the sole reason for our definition of the induced orientation.