

# Math 566

Algebraic Combinatorics

Harrison Centner

University of Michigan

February 20, 2024

# Contents

<b>1 Algebraic Graph Theory .....</b>	<b>3</b>
1.1 Linear Algebraic Preliminaries .....	3
1.2 Enumerating Walks with Eigenvalues .....	5
1.3 Eigenvalues of Special Graphs .....	7
 <b>2 Tilings, Trees, and Networks .....</b>	 <b>15</b>
2.1 Domino Tilings .....	15
2.2 Spanning Trees .....	18
 <b>3 The Diamond Lemma .....</b>	 <b>21</b>
3.1 Statement and Preliminaries .....	21
3.2 Applications to Terminating Games .....	22
3.3 Applications to Non-Terminating Games .....	24
3.4 Wilson's Algorithm .....	26
 <b>4 Electrical Networks .....</b>	 <b>29</b>
4.1 Flows and Edge Weighted Graphs .....	29
4.2 Resistor Networks and Kirchhoff's Laws .....	30
4.3 Effective Conductance .....	34

# Chapter 1

## Algebraic Graph Theory

Notes from the Syllabus:

- No Exams.
- There will be five or six problem sets.
- Office Hours: Tuesdays and Friday at 1pm.
- There will be almost no calculus or analysis.
- You should expect this class to make frequent use of linear algebra.
- You need to solve all but two problems correctly to get 100%.
- There will be very little partial credit given.
- You can either bring in a printed copy of the homework to class.

### 1.1 Linear Algebraic Preliminaries

#### Definition 1.1.1 (Characteristic Polynomial)

Let  $M$  be a  $p \times p$  matrix in  $\mathbb{C}$ , the **monic characteristic polynomial** is

$$\det(tI - M) = \prod_{k=1}^p (t - \lambda_k)$$

where  $\lambda_1, \dots, \lambda_k \in \mathbb{C}$  are the  $p$  eigenvalues with multiplicity.

#### Lemma 1.1.2 (Eigenvalues of Matrix Polynomial)

If  $f(t) \in \mathbb{C}[t]$  then  $f(M)$  has eigenvalues  $f(\lambda_1), \dots, f(\lambda_k)$ .

*Proof.*

$M$  is diagonalizable, so conjugation commutes with taking powers and therefore computation of a polynomial. The statement is trivial up to general nonsense consideration.

Diagonalizable matrices are dense in the set of matrices. A matrix is only diagonalizable if there are multiple equal eigenvalues. Thus this is a subvariety within the set of matrices (obtained by imposing an algebraic condition of equal eigenvalues) which has dimension less than the set of matrices.

Now we can take a limit within the set of diagonalizable matrices and the limit converges to the general matrix.  $\square$

This statement can be extended to more general functions (those with converging power series).

**Lemma 1.1.3 (Trace of Matrix)**

The trace of a matrix  $M$  is the sum of its eigenvalues.

*Proof.*

The coefficient of  $t^{p-1}$  in  $\det(tI - M)$  is

$$-\operatorname{tr}(M) = \sum_{i=1}^p -\lambda_i.$$

□

Combining this fact with [Lemma 1.1.2](#) gives us that

$$\operatorname{tr}(M^\ell) = \sum_{k=1}^p \lambda_k^\ell.$$

Thus it is easy to compute the sum of powers of eigenvalues of  $M$ . This leads to an algorithm of finding roots of polynomials from power sums.

We can recover the multi-set  $\{\lambda_1, \dots, \lambda_p\}$  from the traces  $\operatorname{tr}(M), \operatorname{tr}(M^2), \dots$

**Theorem 1.1.4 (Multiset Recovery)**

Let  $\alpha_1, \dots, \alpha_r, \beta_1, \dots, \beta_s \in \mathbb{C}$  such that  $\forall \ell \in \mathbb{Z}^+$

$$\sum_{i=1}^r \alpha_i^\ell = \sum_{i=1}^s \beta_i^\ell \quad (\star)$$

then  $r = s$  and the  $\beta$ 's are permutations of the  $\alpha$ 's.

If the values were over the reals this would be easy since we could just observe the asymptotic behavior and remove each leading term. This cannot be applied to complex numbers since some may have the same modulus

*Proof.*

We will use the method of generating functions in a noncombinatorial sense. First multiply  $(\star)$  by  $t^\ell$  and sum giving,

$$\sum_{i=1}^r \frac{\alpha_i t}{1 - \alpha_i t} = \sum_{i=1}^s \frac{\beta_i t}{1 - \beta_i t}.$$

We use that  $\sum_{k=1}^\infty x^k = \frac{x}{1-x}$ . Pick  $\gamma \in \mathbb{C}$ , and multiply by  $1 - \gamma t$  and set  $t = \frac{1}{\gamma}$ . Take the limit  $t \rightarrow \frac{1}{\gamma}$ . Each term will become one or zero depending on whether the corresponding  $\alpha_k$  or  $\beta_k$  equals  $\gamma$  or not.

After that, LHS will be the number of  $\alpha$ 's equal to  $\gamma$  and the RHS will be the number of  $\beta$ 's equal to  $\gamma$ . This shows the multisets are equal.  $\square$

And what if I put a little more text

## 1.2 Enumerating Walks with Eigenvalues

We assume all basic vocabulary of basic graph theory. All graphs will be assumed finite. We allow loops & multiple edges.

### Lemma 1.2.1 (Walks on a Graph)

Let  $G$  be a graph on the vertex set  $[p]$ . Then put  $M = A(G)$ , the **adjacency matrix** of  $G$ . So  $M_{ij}$  is the number of edges from  $i$  to  $j$ . Then  $M$  is symmetric, with entries in  $\mathbb{Z}_{\geq 0}$ . The number of walks of length  $\ell$  from  $i$  to  $j$  is  $(M^\ell)_{ij}$ .

*Proof.*

Follows from matrix multiplication (also works for directed graphs).  $\square$

This suggests that there must be a way of counting walks using eigenvalues of the graph. But we will have to restrict to closed paths. A walk can move anywhere in the graph, a path cannot repeat vertices. A **marked closed walk** is one that starts and begins at the same vertex.

### Proposition 1.2.2 (Marked Closed Walks)

The number of marked closed walks of length  $\ell$  on  $G$  is  $\sum_{k=1}^p \lambda_k^\ell$ .

*Proof.*

The LHS equals  $\text{tr}(M^\ell)$  by observation and [Lemma 1.1.3](#) gives the equality with the RHS.  $\square$

Example: Let  $G = K_p$  (the complete graph).

Let  $J$  be the  $p \times p$  matrix with all entries equal to 1. Then  $J - I = A(G)$ . Then  $\text{rank}(J) = 1$ . So the eigenvalues of  $J$  are  $0, \dots, 0$  ( $p - 1$  times) and  $p$ . Now  $J - I$  is applying the polynomial  $t - 1$  to  $J$ . Therefore by [Lemma 1.1.2](#), its eigenvalues are  $-1, \dots, -1$  ( $p - 1$  times) and  $p - 1$ . Thus the number of marked closed walks in  $G$  is

$$(p - 1)^\ell + (p - 1)(-1)^\ell.$$

Restatement: The number of marked closed walks of length  $\ell$  in  $G$  is the number of  $(\ell + 1)$ -letter words in a  $p$ -symbol alphabet such that consecutive letters are distinct and first letter equals the last.

### Homework 1.1

Show that the number of walks of length  $\ell$  between two distinct vertices in  $K_p$  differs by one from the number of closed walks of length  $\ell$  starting and ending at a given vertex. Use a simple combinatorial argument.

We can also compute eigenvalues via counting walks. Using combinatorics for linear algebra!

Example: Suppose  $G = K_{m,n}$ , the complete bipartite graph. Let's count the marked closed walks in  $G$ . This number is

$$(\sqrt{mn})^\ell + (-\sqrt{mn})^\ell + 0^\ell + \dots + 0^\ell = \begin{cases} 0 & \text{if } \ell \bmod 2 = 0 \\ 2(mn)^{\frac{\ell}{2}} & \text{if } \ell \bmod 2 = 1. \end{cases}$$

Now we conclude from Theorem 1.1.4 that  $K_{m,n}$  has eigenvalues  $-\sqrt{mn}, \sqrt{mn}, 0, \dots, 0$ . No linear algebra involved!

### Homework 1.2

Prove that the **diameter** (the largest minimal pairwise distance of a connected graph) is strictly less than the number of distinct eigenvalues.

## *Inequalities for Maximal Eigenvalues*

### Proposition 1.2.6 (Max)

Let  $G$  be a graph on  $[p]$ . Denote  $\lambda_{\max} = \max_i |\lambda_i|$ . The Perron-Frobenius Theorem ensures that the largest eigenvalue is positive (hence equal to  $\lambda_{\max}$ ).

$$\lambda_{\max} \leq \max_{v \in G} \deg(v)$$

Perron-Frobenius holds because a matrix with nonnegative entries can be thought of in probabilistic terms (so each row adds to one). Now we have a Markov Chain and taking powers means considering all possible walks in the state-space. As the matrix is raised to higher powers the largest eigenvalue will dominate. The steady state must be positive (since we are taking powers of nonnegative entries) and have eigenvalue one since it is “steady.”

*Proof.*

**Informal:** As you count your walks your number of choices at each step is at most  $\max \deg(G) = (\max_v \deg(v))^\ell$ . So the inequality must be in this direction since otherwise Lemma 1.2.1 cannot hold.

**Formal:** For any vector set  $x = (x_i)$ ,

$$\max_j |(A(G)x)_j| = \max_j \left| \sum_{\text{edge } ij} x_i \right| \leq \maxdeg(G) \cdot \max_k |x_k|.$$

Now assume that  $x$  is an eigenvector of  $A(G)$ , with eigenvalue  $\lambda$ . Then this inequality becomes  $|\lambda| \cdot \max_k |x_k| \leq \maxdeg(G) \cdot \max_k |x_k| \Rightarrow |\lambda| \leq \maxdeg(G)$ . This is just mathematical legalese (the argument is philosophical).  $\square$

### Homework 1.3

Prove that

$$\lambda_{\max} \geq \frac{1}{p} \sum_{v \in G} \deg(v).$$

This is like a kind of convexity or Jensen's Inequality.

*Hint:* Use that for symmetric real  $M$ ,

$$\lambda_{\max} = \max_{|x|=1} x^T M x$$

**Corollary:** The number of closed walks grows exponentially in  $\ell$ . With rate at least equal to the average degree.

## 1.3 Eigenvalues of Special Graphs

We now study the eigenvalues of graphs which take certain specified forms.

### Block Anti-Diagonal Matrices

Now let's discuss in more detail the bipartite graphs. Whose matrices take the block form  $\begin{bmatrix} 0 & * \\ * & 0 \end{bmatrix}$ . We will use implicitly the concept of singular values.

#### Lemma 1.3.2 (Eigenvalues of Block Anti-Diagonal Matrix)

For a real matrix  $B$ ,  $M = \begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix}$  has nonzero eigenvalues  $\pm\sqrt{\mu_i}$  where  $\mu_1, \mu_2, \dots$  are the eigenvalues of  $B^T B$  counted with multiplicity. Note that  $B^T B$  is real, symmetric, and most importantly positive semidefinite. Since  $\langle B^T Bx, x \rangle = \langle Bx, Bx \rangle \geq 0$  and the signature of a quadratic form is given by the signs of eigenvalues. Note that the  $\mu_1, \mu_2, \dots$  are the singular values of  $B$ .

The proof is going to be the worst kind of proof in mathematics.

For a  $p \times p$  matrix  $X$  we will denote  $F_X(t) := \det(tI_p - X)$ .

*Proof.*

Suppose  $B$  is  $n \times m$ . Then

$$\begin{bmatrix} tI_n & -B \\ -B^T & tI_m \end{bmatrix} \begin{bmatrix} I_n & B \\ 0 & tI_m \end{bmatrix} = \begin{bmatrix} tI_n & 0 \\ -B^T & t^2 I_m - B^T B \end{bmatrix}.$$

Taking determinants gives

$$F_M(t) \cdot t^m = t^n \cdot F_{B^T B}(t^2).$$

Now we are after the eigenvalues which is precisely when  $F_M(t)$  vanishes. Now we know that  $t^2$  is an eigenvalue of  $B^T B$  which means that  $t$  is a singular value of  $B$ . There exists a proof from the book of this fact but it is much longer.  $\square$

Example: Let  $G = K_{m,n}$ . Let  $B$  be the  $m \times n$  matrix of all ones. Then  $B^T B$  is an  $n \times n$  matrix of all ones. This is a rank one matrix so it has one nonzero eigenvalue and it is  $\text{tr}(A(G)) = nm$ . Thus the eigenvalues of  $A(G)$  are  $\pm\sqrt{mn}, 0, \dots, 0$ .

#### Homework 1.4

Let  $G$  be the graph obtained by removing  $n$  disjoint edges from  $K_{n,n}$ . Find the eigenvalues of  $G$ .

Example: Let  $G$  be a  $2n$ -cycle, thus  $G$  is bipartite. Denote  $A(G) = M_{2n}$ . Then this matrix has the form  $\begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix}$ . Then  $B^T B = 2I_n + M_n$  with the appropriate labeling. This is intuitive because we can return to our original vertex in two ways: counter clockwise or clockwise. [Why?]

This implies that the eigenvalues of  $G$  are  $\pm\sqrt{2 + \lambda_i}$  where  $\lambda_i$  are the eigenvalues of  $M_n$ . Now compute the eigenvalues of  $M_n$  for small  $n$  (say 4). So the eigenvalues of  $2^n$ -cycle are  $\sqrt{2 \pm \sqrt{2 \pm \dots}}$ .

### Circulant Graphs

We are talking about matrices which act upon the set of basis vectors as a cyclic group.

#### Definition 1.3.5 (Circulant Matrix)

A square matrix is called **circulant** provided that

$$C = \begin{bmatrix} s_0 & s_1 & \cdots & s_{p-1} \\ s_{p-1} & s_0 & \cdots & s_{p-2} \\ \vdots & \vdots & \ddots & \vdots \\ s_1 & s_2 & \cdots & s_0 \end{bmatrix}$$



**Lemma 1.3.6 (Eigenvalues of Circulant Matrices)**

$$\lambda_k = \sum_{j=0}^{p-1} s_j e^{\frac{2\pi i}{p} jk}$$

for  $k = 0, \dots, p-1$ .

Remark: Let  $s(x) = \sum_{j=0}^{p-1} s_j x^j$ , then the eigenvalues of  $C$  are the values of  $s(x)$  at the  $p$ -th roots of unity.

*Proof.*

Let  $T$  be the circulant matrix such that  $e_1^T T = [0, 1, 0, \dots, 0]$ .  $T$  acts on the basis  $e_1, \dots, e_p$  by cyclically shifting. So  $T^k$  will again be a circulant matrix. Then  $C = s(T)$ . Thus finding the eigenvalues of  $T$  will give us the eigenvalues of  $C$ . The eigenvalues of  $C$  are the  $p$ -th roots of unity. By observation,  $\det(tI_p - T) = t^p - 1$ . Also  $T^p = I_p$  so then  $T$  itself is a  $p$ -th root of unity and so it must be that the eigenvalues of  $T$  are  $p$ -th roots of unity.

The claim follows. □

**Definition 1.3.7 (Circulant Graph)**

A graph  $G$  is **circulant** if  $A(G)$  is circulant for some labeling of  $G$ .

**Corollary**: The eigenvalues of the  $p$ -cycle are  $2 \cos\left(\frac{2\pi k}{p}\right)$  for  $k = 0, \dots, p-1$ .

*Proof.*

Let  $G$  be a  $p$ -cycle. So  $A(G) = T + T^{-1} = T + T^{p-1}$ . Now, Lemma 1.1.2 implies that the eigenvalues of  $G$  are

$$e^{\frac{2\pi i}{p} k} + e^{\frac{2\pi i}{p} k(p-1)} = e^{\frac{2\pi i}{p} k} + e^{-\frac{2\pi i}{p} k} = 2 \cdot \Re\left(e^{\frac{2\pi i}{p} k}\right) = 2 \cos\left(\frac{2\pi k}{p}\right).$$

□

Recall: The eigenvalues of a  $2n$ -cycle are  $\pm\sqrt{2 + \mu_i}$  where  $\mu_i$  is an eigenvalue of the  $n$ -cycle. Note that this is consistent with the more general corollary above via the cosine double angle identity.

**Homework 1.5**

Find the eigenvalues of the graph obtained by removing  $n$  disjoint edges from  $K_{2n}$ . Note that this graph is the 1-skeleton of the  $n$ -dimensional **cross polytope** (i.e. the **hyperoctahedron**).

The hyperoctahedron is the  $n$ -dimensional polytope which is dual to the  $n$ -cube. Thus it is the convex hull of the set  $\{\pm e_i : i \in [n]\}$ . The only non-adjacent vertices are the vertices at  $e_i$  and  $-e_i$ . Counting walks in the hyperoctahedron counts walks on the faces of the hypercube.

### Cartesian Products of Graphs

Cartesian products is a general categorical construction.

#### Definition 1.3.10 (Product of Graphs)

Let  $G, H$  be graphs. Then  $G \times H$  is the **Cartesian product** of  $G$  and  $H$  with vertex set  $V_{G \times H} = V_G \times V_H$  and edges of two kinds  $(g, h)(g', h)$  for edge  $gg' \in E_G$ . and  $(g, h)(g, h')$  for edge  $hh' \in E_H$ .

Example:

- Let  $G = \bullet - \bullet - \bullet$  and  $H = \bullet - \bullet - \bullet - \bullet$ . Then  $G \times H$  is the  $3 \times 4$  grid graph.
- The  $n$ -cube graph (skeleton of the  $n$ -cube) is isomorphic to  $\underbrace{K_2 \times \cdots \times K_2}_{n \text{ times}}$ .

#### Proposition 1.3.11 (Eigenvalues of Direct Product)

Suppose  $G$  have eigenvalues  $\lambda_1, \lambda_2, \dots$  and  $H$  has eigenvalues  $\mu_1, \mu_2, \dots$  then  $G \times H$  has eigenvalues  $\lambda_i + \mu_j$  for all  $i, j$ .

We will give two proofs. The first is highbrow. The second is elementary.

*Proof.*

Take two vectors  $u, v$  in the space of formal linear combinations of the vertices of  $G$  and  $H$  (i.e. functions on the vertex space of the graphs). Put  $u = \sum \alpha_g g$  and  $v = \sum \beta_h h$ . Define the tensor product of two vectors

$$u \otimes v \stackrel{\text{def}}{=} \sum_g \sum_h \alpha_g \beta_h (g, h).$$

Then we claim that the matrix

$$A(G \times H)(u \otimes v) = A(G)u \otimes v + u \otimes A(H)v.$$

This is true for the basis vectors of the tensor space  $V_G^* \otimes V_H^*$  thus it holds for any  $u$  and  $v$  by linearity. Now if  $u$  and  $v$  are eigenvectors with eigenvalues  $\lambda$  and  $\mu$  for  $G$  and  $H$  respectively, then, we have

$$A(G \times H)(u \otimes v) = \lambda u \otimes v + u \otimes \mu v = (\lambda + \mu)(u \otimes v).$$

So  $u \otimes v$  is an eigenvector for  $G \times H$  with eigenvalue  $\lambda + \mu$ . □

*Proof.*

A marked closed walk of length  $\ell$  in  $G \times H$  is a **shuffle** (i.e. interleaving) of marked closed walks of length  $k$  and  $\ell - k$  in  $G$  and  $H$ , respectively. Hence, the number of such walks is

$$\sum_{k=0}^{\ell} \binom{\ell}{k} \left( \sum_i \lambda_i^k \right) \left( \sum_j \mu_j^{\ell-k} \right) = \sum_i \sum_j \sum_k \binom{\ell}{k} \lambda_i^k \mu_j^{\ell-k} = \sum_i \sum_j (\lambda_i + \mu_j)^{\ell}.$$

Where the first equality follows from nontrivial summation rearrangement and the second follows from the binomial theorem. Now by [Theorem 1.1.4](#) we have found the eigenvalues. □

The first proof is better because there is no computation. Also the second proof does not give us the eigenvectors.

### Homework 1.6

Find the number of marked closed walks of length  $\ell$  in the  $3 \times 3$  grid graph. Note that this graph is bipartite so we should get zero for odd  $\ell$ .

### Homework 1.7

Find the eigenvalues of the 1-skeleton of an octagonal prism. Note  $G = C_2 \times C_8$ .

## Eigenvalues of the Cube

Let  $Q_n = (K_2)^n$  be the 1-skeleton of the  $n$ -cube.

### Proposition 1.3.15 (Eigenvalues of Cube Graph)

The eigenvalues of  $Q_n$  are

$$\left\{ \binom{n-2k}{k} : k = 0, \dots, n \right\}.$$

**Corollary:** The number of marked closed walks in  $Q_n$  of length  $\ell$  is

$$\sum_{k=0}^{\ell} \binom{n}{k} (n-2k)^{\ell}.$$

Also this number must be divisible by  $2^n$  and zero for odd  $\ell$  since  $Q_n$  is bipartite.

*Proof.*

The eigenvalues of  $K_2$  are  $\pm 1$ . Now recalling [Proposition 1.3.11](#), we see that the eigenvalues of  $Q_n$  are all possible sums of the form  $\underbrace{\pm 1 \pm 1 \pm 1 \cdots \pm 1}_{n \text{ times}}$ .  $\square$

### Homework 1.8

Find a direct proof of the formula using generating functions.

Now we have enough background to consider **random walks** in the cube. We will consider Brownian motion within a graph. When  $G$  is a **regular** graph of degree  $d$  on  $p$  vertices, a **simple** random walk on  $G$  proceeds by choosing (uniformly at random) an adjacent vertex and moving into it.

Instead of regularity, we will assume the stronger condition that any two vertices are related by an automorphism of  $G$ , meaning the group of automorphisms of  $G$  acts transitively on  $G$ .

Assuming that the random walk originates at  $v$ ,

$$\begin{aligned} \mathbb{P}[\text{after } \ell \text{ steps we return to } v] &= d^{-\ell} \cdot \#\{\text{closed walks of length } \ell \text{ starting at } v\} \\ &= \frac{1}{d^\ell p} \cdot \sum_{i=1}^p \lambda_i^\ell. \end{aligned}$$

Example: For  $Q_n$  we put  $d = n$  and  $p = 2^n$ , the probability of return after  $\ell$  steps is

$$\mathbb{P}[\text{return after } \ell \text{ steps}] = \frac{1}{n^\ell 2^n} \sum_{k=0}^n \binom{n}{k} (n - 2k)^\ell.$$

### Homework 1.9

Find the analogous probability of return for the discrete torus with  $nm$  vertices (the product of an  $n$ -cycle and an  $m$ -cycle). Alternatively, this is the quotient of the grid graph by its “boundary” graph.

Note that the discrete torus satisfies the automorphism transitivity assumption.

### *Eigenvalues of a Chain Graph*

Let  $G$  be the  $n$ -cycle with one edge removed. This graph is not regular.

$$A \stackrel{\text{def}}{=} A(G) = \begin{bmatrix} 0 & 1 & & \\ 1 & 0 & 1 & \\ & 1 & 0 & 1 \\ & & & \ddots \end{bmatrix}$$

**Proposition 1.3.19 (Eigenvalues of Chain Graph)**

The eigenvalues of  $A_n$  are  $2 \cdot \cos\left(\frac{\pi k}{n+1}\right)$ .

*Proof.*

The characteristic polynomial of  $A_n$  is  $T_{n(t)} = \det(tI_n - A_n)$ . It turns out this determinant is closely related to Chebyshev Polynomials.

Consider a more general determinant (of Toeplitz Matrices)

$$h_n(a, b) = \begin{bmatrix} a+b & -ab & & \\ -1 & a+b & -ab & \\ & -1 & a+b & -ab \\ & & \ddots & \ddots \end{bmatrix}.$$

Exploring small values we see that

$$\begin{aligned} h_1(a, b) &= a + b \\ h_2(a, b) &= a^2 + ab + b^2 \\ h_3(a, b) &= (a + b)^3 - 2ab(a + b) \\ &= a^3 + a^2b + ab^2 + b^3. \end{aligned}$$

This suggests the pattern that

$$h_n(a, b) = \sum_{k=0}^n a^{n-k} b^k = \frac{a^{n+1} - b^{n+1}}{a - b}.$$

We will use a recurrence to prove this formula. We expand the determinant in the last row

$$h_n(a, b) = (a + b)h_{n-1}(a, b) - ab \cdot h_{n-2}(a, b).$$

The proof follows from induction on  $n$ . All that must be done is to plug the formula into the recurrence and check that it holds. This trick of expanding the determinant into a recurrence is common for (tri)-diagonal matrices.

Now set  $b := \frac{1}{a}$ . Then

$$T_n(a + a^{-1}) = h_n(a, a^{-1}) = \frac{a^{n+1} - a^{-(n+1)}}{a - a^{-1}}.$$

This vanishes when  $a^{2(n+1)} = 1$  but  $a^2 \neq 1$ . That is,  $a = e^{\frac{2\pi i k}{2(n+1)}} = e^{\frac{\pi i k}{n+1}}$  for  $k \in [n]$  since 1 and  $n + 1$  are forbidden by our vanishing rules.

Summing  $t = a + a^{-1} = 2 \cdot \cos\left(\frac{\pi k}{n+1}\right)$ . □



# Chapter 2

## Tilings, Trees, and Networks

### 2.1 Domino Tilings

#### Definition 2.1.1 (Domino Tiling)

A **domino tiling** is a tiling of the plane  $\mathbb{Z} \times \mathbb{Z}$  by  $2 \times 1$  rectangles. A tiling is **cryptomorphic** to a perfect matching in on  $\mathbb{Z} \times \mathbb{Z}$ .

This may seem like a recreational mathematics question, but it arises in applications. The most common is statistical mechanics. Consider the center of unit squares as locations in a crystal (think NaH). So this is related to dimer models and the ising model. This problem was solved in the late 1950s by physicists.

#### *The Permanent-Determinant Method*

The method we will consider was invented by Peter W. Kastelen (1960). We will consider an  $n \times m$  board and suppose  $n$  is even. Let  $G$  be the bipartite graph which is the product of the  $m$ -chain and the  $n$ -chain. Put

$$M := A(G) = \begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix}.$$

Where  $B$  is an  $\frac{m}{2} \times \frac{m}{2}$  matrix. Choosing a perfect matching in  $G$  amounts to choosing exactly one nonzero entry from each row and column in  $B$ . This should perk your ears and make you think of a determinant.

#### Definition 2.1.3 (Permanent)

The permanent of an  $n \times n$  matrix  $A$  is

$$\text{per}(B) = \sum_{\sigma \in S_n} \prod_{i \in [n]} a_{i, \sigma(i)}.$$

This is notoriously difficult computationally (should be exponentially hard).

Let  $T(m, n)$  be the number of perfect tilings in  $G$ . Perfect matchings in  $G$  are in bijection with nonvanishing terms in  $\det(B) = \text{per}(B)$ .

The key trick is to turn these permanents into determinants. Let  $\tilde{B}$  be the matrix obtained from  $B$  by replacing the 1s that correspond with vertical tiles with  $i$ .

**Lemma 2.1.4 (Determinant Permanent Equivalence)**

Kastelen proved that

$$\det(\tilde{B}) = \pm \operatorname{per}(B) = \pm T(m, n).$$

The homework problem below is due to Bill Thurston. “This is a problem for a very very smart ten year old.”

**Homework 2.1**

Show that any two tilings of a rectangular board can be connected by a sequence of “flips.”

One way of showing this is by connecting every tiling to a canonical tiling. We will use this result to show [Lemma 2.1.4](#).

*Proof.*

In a vertical  $2 \times 2$  block we have the submatrix of those four entries looks like  $\begin{bmatrix} i & * \\ * & i \end{bmatrix}$ . A flip then takes it to  $\begin{bmatrix} * & 1 \\ 1 & * \end{bmatrix}$ . So the flip replaces a product  $\cdots 1 \cdot 1 \cdots$  with  $\cdots i \cdot i \cdots$  and these products are equal up to exactly one transposition. Thus the term in the permanent and the term in the determinant are equal. The horizontal tiling is a canonical tiling which gives a real term in the determinant equal to  $\pm 1$ . Now using [Homework 2.1](#) we are done.  $\square$

Now we are prepared to enumerate the number of domino tilings in  $G$ . Put

$$\tilde{M} := \begin{bmatrix} 0 & \tilde{B} \\ \tilde{B}^T & 0 \end{bmatrix} \Rightarrow \det(\tilde{M}) = \pm \det(\tilde{B}) \det(\tilde{B}^T) = \pm (T(m, n))^2$$

by [Lemma 2.1.4](#).

Recall that  $A_n$  is the adjacency matrix of an  $n$ -chain. Then

$$\tilde{M} = I_m \otimes A_n + iA_m \otimes I_n.$$

By [Proposition 1.3.11](#), this implies that the eigenvalues of  $\tilde{M}$  are sums of eigenvalues of  $A_n$  and  $A_m$ . Recall that the eigenvalues of  $A_n$  are  $2 \cdot \cos\left(\frac{\pi k}{n+1}\right)$  for  $k \in [n]$ . We now conclude that,

$$\begin{aligned} \det(\tilde{M}) &= \prod_{j \in [n]} \prod_{k \in [m]} \left( 2 \cdot \cos\left(\frac{\pi j}{n+1}\right) + 2i \cdot \cos\left(\frac{\pi k}{m+1}\right) \right) \\ &= \pm \prod_{j \in [\frac{n}{2}]} \prod_{k \in [m]} \left( 4 \cos^2\left(\frac{\pi j}{n+1}\right) + 4 \cos^2\left(\frac{\pi k}{m+1}\right) \right) \end{aligned}$$



The first equality follows since  $n$  is even,  $n + 1$  is odd and so the cos values from the RHS come in pairs which cancel out each others imaginary parts.

Since  $T(m, n) = \sqrt{|\det(\tilde{M})|}$ , we obtain: a theorem by P. Kasteleyn (1961), M. Fisher and N. Temperley (1961).

### Theorem 2.1.6 (Enumeration of Domino Tilings)

$T(m, n)$  has two formulas. If  $m$  is odd then,

$$T(m, n) = \prod_{j \in [\frac{n}{2}]} 2 \cos\left(\frac{\pi j}{n+1}\right) \prod_{k \in [\frac{m-1}{2}]} \left(4 \cos^2\left(\frac{\pi j}{n+1}\right) + 4 \cos^2\left(\frac{\pi k}{m+1}\right)\right)$$

If  $m$  is even then, we get

$$T(m, n) = \prod_{j \in [\frac{n}{2}]} \prod_{k \in [\frac{m}{2}]} \left(4 \cos^2\left(\frac{\pi j}{n+1}\right) + 4 \cos^2\left(\frac{\pi k}{m+1}\right)\right)$$

Note:  $T(8, 8) = 12,988,816 = 3604^2$ . It is a difficult theorem to prove, but for any square board whose length is divisible by four, the number of tilings is a perfect square. And otherwise if it is an square board that can be tiled, then it will be twice a perfect square.

But the physicists don't care about this they want to know the asymptotics.

Let  $m = n$ . Let's take the log

$$\begin{aligned} \frac{\log T(n, n)}{n^2} &= \frac{1}{n^2} \sum_{j \in [\frac{n}{2}]} \sum_{k \in [\frac{n}{2}]} \log\left(4 \cos^2\left(\frac{\pi j}{n+1}\right) + 4 \cos^2\left(\frac{\pi k}{m+1}\right)\right) \\ &\sim \frac{1}{\pi^2} \sum \sum \left(\frac{\pi}{n+1}\right)^2 \log\left(4 \cos^2\left(\frac{\pi j}{n+1}\right) + 4 \cos^2\left(\frac{\pi k}{m+1}\right)\right) \\ &\sim \frac{1}{\pi^2} \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \left(\frac{\pi}{n+1}\right)^2 \log\left(4 \cos^2\left(\frac{\pi j}{n+1}\right) + 4 \cos^2\left(\frac{\pi k}{m+1}\right)\right) dx dy \\ &= \frac{K}{\pi} \end{aligned}$$

So now we have a Riemann sum (we are multiplying the area of the square  $4 \cos^2 x + 4 \cos^2 y$  evaluated at the point  $(\frac{\pi j}{n+1}, \frac{\pi k}{n+1})$ ).

Where  $K$  is the Catalan Constant

$$K = 1 - \frac{1}{9} + \frac{1}{25} - \frac{1}{49} + \dots = \frac{1}{2} \int_0^{\frac{\pi}{2}} \frac{t}{\sin(t)} dt$$

So  $T(n, n) \sim 1.34^{n^2}$ . To the physicists it is obvious that the complexity is  $c^{n^2}$  for some  $c \in \mathbb{R}^+$ . This is because placing two dominoes are independent in expectation.

## 2.2 Spanning Trees

The following theorem is due to H.N.V. Temperley (1974).

### Theorem 2.2.1 (Tilings & Spanning Trees)

The number of domino tilings of the  $(2k - 1) \times (2\ell - 1)$  rectangle without a corner box is equal to the number of spanning trees of the  $k \times \ell$  grid graph.

### Homework 2.2

Prove that this rule produces a tree. Something connected and acyclic. (A cycle requires an odd number of squares)

*Proof.*

for each domino which covers a node with two odd coordinates, draw a segment in the direction it is pointing. This will create a spanning tree by [Homework 2.2](#). For the other direction, direct each edge in the spanning tree to the node in the missing corner. Then draw dominoes in the dual graph. The remaining area uncovered by the dominoes splits into a rooted forest in the dual graph of the original board. Each tree in the forest grows from (is rooted) somewhere on the boundary of the dual graph. This gives us a unique domino placement for the uncovered areas. Since if the uncovered area contained a cycle, then the original tree would not be spanning (or maybe not a tree).  $\square$

### Homework 2.3

Prove that if we remove a different box from the boundary, we get the same count (so long as a tiling is possible). You should build this bijection via a composition of two bijections.

### Corollary 2.2.4 (Asymptotics of Spanning Trees)

The number of spanning trees in an  $n \times n$  grid is  $\sim e^{\frac{4K}{\pi}n^2} \approx 2.24^{n^2}$ .

We can generalize to arbitrary planar graphs. Suppose  $P$  is a polygon on  $\mathbb{R}^2$ . Subdivide  $P$  into smaller polygons (not necessarily convex) to get a graph  $G$ . This description gets rid of annoying planar graphs (those have vertices of degree one, unconnected, etc.). Obtain a new graph  $H$  by placing a vertex inside each face of  $G$  and the midpoint of each edge of  $G$ . For each face  $F$ , connect each vertex in the  $F$  to those on the midpoints of the edges of  $F$ .

**Homework 2.3**

Fix a vertex  $v$  of  $P$  (hence of  $G$  and  $H$ ). Show that the number of spanning trees on  $G$  is equal to the number of perfect matchings on  $H \setminus \{v\}$ .

Example: If  $G$  is the graph on the pentagon labeled cyclically with  $[5]$  and adding the edge  $\{1, 3\}$  then there are  $4 + 4 + 3 = 11$  spanning trees. Then  $H$  is isomorphic to the  $4 \times 3$  grid graph which also has 11 perfect matchings.



# Chapter 3

## The Diamond Lemma

### 3.1 Statement and Preliminaries

The Diamond Lemma is a general mathematical argument. There aren't that many: induction, pigeon-hole principle, and switching summations.

#### Definition 3.1.1 (One Player Game)

A **one-player game** is defined by the set of positions  $S$ , for each position  $s \in S$ , there is a set of positions  $s' \neq s$  into which a player can move. We will use notation  $s \rightsquigarrow s'$ . If this set is empty, then  $s$  is called **terminal**.

#### Definition 3.1.2 (Play Sequence)

A **play sequence** is a sequence of the form  $s \rightsquigarrow s' \rightsquigarrow s'' \rightsquigarrow \dots$  (either finite or infinite). A **terminating game** is one without an infinite play sequence. A game is **confluent** if its outcome is determined by the initial position.

#### Lemma 3.1.3 (Diamond Condition)

Suppose that a game is terminating and satisfies  $(\Diamond) \forall s \in S$  and any  $s \rightsquigarrow s'$  and  $s \rightsquigarrow s''$ , there exists a  $t \in S$  such that  $s' \rightsquigarrow \dots \rightsquigarrow t$  and  $s'' \rightsquigarrow \dots \rightsquigarrow t$ . Then the game is confluent.

Note that if a game is confluent, then the game satisfies  $(\Diamond)$ .

*Proof.*

Color each position  $s$  **red** if there exists two different terminal positions reachable from  $s$ . Color each position **green** otherwise. We want to show that there are no **red** positions. We know that every terminal position is **green**. Suppose  $s$  is red. Then make moves into **red** positions while we can. Since the game is terminating, at some point we must choose a terminal position. Thus we will have  $s \rightsquigarrow s'$  for all  $s \rightsquigarrow s'$ . Since  $s$  is red, there must be distinct  $s \rightsquigarrow s' \rightsquigarrow s''$ . But since  $s', s''$  are green then they have unique terminal end points  $t_1, t_2$ . But now invoking  $(\Diamond)$  shows that  $t_1 = t_2$  so  $s', s''$  cannot be green. Contradiction!  $\square$

Typically, we will use the Diamond Lemma to show something is well defined.

## 3.2 Applications to Terminating Games

### Definition 3.2.1 (Chip Firing)

Let  $G = (V, E)$  be a finite directed simple graph. Let  $t$  be a **sink**, reachable from any vertex in  $G$ . A **chip configuration** is a placement of a nonnegative integer  $x_v$  at each  $v \in V$ . A move in the chip firing game occurs by choosing a  $v \neq t$  satisfying  $x_v \geq \text{outdegree}(v)$ , then “fire” by sending one chip along each outward edge from  $v$ .

So after a move,

$$x_v := x_v - \text{outdeg}(v) \quad x_w := x_w + 1$$

$\forall w$  incident to  $v$ .

$3 \rightarrow 3 \rightarrow 3 \rightarrow 0$

### Theorem 3.2.2 (Chip Firing is Confluent)

Chip Firing is confluent.

*Proof.*

We employ the Diamond Lemma. Suppose there is a position with  $v, v' \in V$  which can both be fired. The order which we fire  $v$  and  $v'$  does not matter, so the fires at  $v$  and  $v'$  commute  $\Rightarrow (\diamond)$ .  $\square$

Thus any game will have the same number of moves.

### Definition 3.2.3 (Young Diagram)

A **Young Diagram** is a sequence of unit boxes arranged in tabular format which monotonically decrease in length. Young Diagrams of size  $n$  are in bijection with integer partitions of  $n$ .

### Definition 3.2.4 (Skew Shape)

A **Skew Diagram** is obtained from a Young Diagram via removing a left aligned sub-(Young Diagram). Alternatively a Skew Shape can be defined as a convex region of  $\mathbb{Z} \times \mathbb{Z}$  where convexity is understood as (having empty squares when top left and bottom right corners are missing).

Think of Young Diagram as chocolate bars and Skew Shapes as bitten Chocolate bars and Cores as nuts (or maybe bolts that would break your teeth).

### Definition 3.2.5 (Cores)

The positions of the game are Young Diagrams and the moves consist of removing dominoes from the southeast. The **2-core** of a Young Tableaux is the remainder after the removal game has ended. Cores are uniquely determined, because taking non-overlapping bites commutes (and two overlapping bites can result in the same aftermath if the whole  $2 \times 2$  square that the dominoes occupy is removed).

Fix  $p \in \mathbb{Z}_{\geq 0}$ . Modify the game by only allowing removals of border strips of length  $p$ . The border of a Young Diagram is the set of unit boxes which have no southeast neighbor.

### Homework 2.4

Show that the generalized Core game is confluent. So the  $p$ -core of a Young Diagram is well defined.

“A curious ten year-old would be impressed by this ... of course a regular ten year-old would not care.”

We did a quick grammar lesson on when to use “well defined” and “well-defined.”

Now it is natural to ask how to describe the uniquely determined output from a game that satisfies the Diamond Lemma.

Note that tableaux is the plural of tableau.

### Definition 3.2.7 (Standard Young Tableaux)

The **Standard Young Tableau** of a skew shape is given by filling it with numbers in  $[n]$  so that the numbers increase along the rows and columns.

These Young Tableaux play important roles in the representation of linear groups.

The following game was introduced by M.-P. Schutzenberger (~1970).

### Definition 3.2.8 (Jeu De Taquin or The Teasing Game)

Let the positions of the game be Standard Young Tableaux. Let the moves be “slides.”

### Homework 2.5

Show that Jeu De Taquin is confluent. This problem is not easy.

Tutte was involved in solving the Enigma code. The Tutte Polynomial is an invariant of graphs. The simplest definition involves a recursive definition in-

volving edge deletion and contraction. This recursion depends on an arbitrary choice (but it is still well defined). Look it up on Wikipedia.

### Homework 2.6

Show the Tutte Polynomial is well defined.

## 3.3 Applications to Non-Terminating Games

A one-player game is confluent if, for any initial position, either

- the game never terminates (no matter how it's played)
- it always arrives at the canonical terminal position.

The old version of the Diamond Lemma ([Lemma 3.1.3](#)) is no longer true. Consider the game on the states  $\mathbb{Z}^+ \cup \{\infty\}$ , with arrows from  $n \rightarrow n+1$  and  $n \rightarrow \infty$  for all  $n \in \mathbb{Z}^+$ . The diamond condition is satisfied, yet the game is not confluent (in the old sense).

### Homework 2.7

#### Lemma 3.3.2 (The Diamond Lemmaa)

Suppose that  $\forall s \in S$  and any  $s \rightsquigarrow s'$  and  $s \rightsquigarrow s''$ ,  $\exists t \in S$  and paths  $s' \rightsquigarrow \dots \rightsquigarrow t$  and  $s'' \rightsquigarrow \dots \rightsquigarrow t$  with the same number of steps. Then, the game is confluent. Moreover, if it terminates (for a given initial position), then it arrives at a canonical terminal position in a fixed (predetermined) number of steps.

Meaning, the poset is graded.

### *Loop-Erased Walks*

This definition is due to G. Lawler (1980) at UChicago. Unfortunately, Greg is a probabalist not a combinatorialist, so a better name would be cycle-erased walks.

#### Definition 3.3.3 (Loop-Erased Walk)

Let  $G$  be a connected graph. Let  $\pi$  be a (finite) walk in  $G$ . The **loop-erasure** of  $\pi$  denoted  $\text{LE}(\pi)$  is a walk defined recursively by  
*if*  $\pi$  doesn't visit any vertex more than once, *then*  $\text{LE}(\pi) = \pi$ .  
*else* Remove the first cycle that  $\pi$  makes to get  $\pi'$  and set  $\text{LE}(\pi) = \pi'$ .

A very good leetcode problem would be to find the loop erased walk of a walk.



Markov Chains are an elementary concept, but running them with stacks is a very little known technology. This will make the Markov process run deterministically.

**Definition 3.3.4 (Running a Markov Chain with Stacks)**

Suppose  $M$  is a **Finite Markov Chain** (stationary i.e., transition probabilities do not depend on time). For each state  $s$ , decide in advance (by making appropriate random choices) where the chain will move from  $s$  after visiting  $s$  for the  $n$ -th time. This creates a **stack**  $u(s) = (u_1, u_2, \dots)$  at  $s$ .

Assumption: A Markov Chain arrives, with probability 1, at a (possibly non-unique) absorbing state. The stack at such a state is garbage (or simple empty).

**Definition 3.3.5 (Cycle Popping)**

Consider a set of partially depleted stacks. Their tops determine a directed graph with edges  $s \rightarrow s'$  colored by  $\mathbb{Z}^+$ , according to the number of visits to the  $s$  that have occurred. The out degree of every vertex is one. Removing the tops from the stacks lying on a directed cycle in this graph is called **cycle-popping**.

Observation: Loop erasure can be viewed as a sequence of cycle poppings. [Why?]

**Lemma 3.3.6 (Cycle Popping Game is Confluent)**

The Cycle Popping Game is confluent.

*Proof.*

Follows immediately from Lemma 3.3.2. □

### 3.4 Wilson’s Algorithm

#### Definition 3.4.1 (Wilson’s Algorithm)

The algorithm proceeds as follows:

Input: connected loopless graph  $G$ .

Output: random spanning tree  $T$  in  $G$ .

Fix a vertex  $r$  in  $G$  (a *root*).

Put  $T := \{r\}$ .

**while**  $T$  doesn’t span  $G$  **do**

    pick a vertex  $v \notin T$ ;

    from  $v$  run a simple random walk  $\pi$  **until**  $\pi$  hits  $T$ ;

    append  $\text{LE}(\pi)$  to  $T$ ;

**return**  $T$ ;

This is a very involved process for choosing a spanning tree. The following result is from D.B. Wilson (1996) while Prof. Fomin was his instructor at MIT.

#### Theorem 3.4.2 (Wilson’s Theorem)

Wilson’s Algorithm outputs a uniformly distributed random spanning tree.

By uniformly, we mean for any spanning tree the probability that Wilson’s Algorithm produces that spanning tree is the same.

There is a huge subject in Theoretical Computer Science called generation of combinatorial objects. Monte Carlo algorithms heavily depend on these generation procedures. For example choosing a  $k$ -element subset from an  $n$  element set. Volume II of Knuth’s *Art of Computer Programming* concerns this for a major portion.

*Proof.*

Make  $r$  (later, all of  $T$ ) into an absorbing state. Run the algorithm with stacks. We know that loop erasure amounts to cycle popping. The algorithm terminates with probability 1. The output is the (uniquely defined by the stacks) tree lying underneath all the pop-able cycles.

For simplicity, assume that  $v$  is chosen using some deterministic rule. For any spanning tree  $T$  of  $G$  and any “heap”  $H$  of colored cycles, we could possibly remove them, let  $P(T, H)$  be  $\mathbb{P}[\text{we end up with } H \& T]$ . Then,

$$P(T, H) = \left( \prod_{v \in H} \frac{1}{\deg(v)} \right) \left( \prod_{v \in T \setminus \{r\}} \frac{1}{\deg(v)} \right)$$

Since for each chip there is exactly one direction we want it to point. So we just need to remove enough chips until that direction is uncovered.

Now the key observation is that the expression above does not depend on  $T$ ! This implies that  $\mathbb{P}[T]$  (Wilson's Algorithm outputs  $T$ ) and

$$\mathbb{P}[T] = \sum_H P(T, H).$$

$\therefore$  The we don't depend on  $T$ . □

The following exercise was proved five years before Wilson's result.

### Homework 2.9

Suppose  $G$  has vertices  $a, b$ . Consider two random experiments.

1. Run a simple random walk  $\pi$  from  $a$  until it hits  $b$  and output  $\text{LE}(\pi)$ .
2. Generate a uniformly random spanning tree  $T$  in  $G$  and output the unique path in  $T$  connecting  $a$  and  $b$ .

Prove that the outputs of (1) and (2) are identically distributed.

### Corollary 3.4.4 (Wilson's Corollary)

The distribution in the exercise above remains the same if we swap  $a$  and  $b$ .

This corollary is surprising because the loop erasure of a path  $a \rightarrow b$  is not the same as the loop erasure of the same path  $b \rightarrow a$ .

Example: in the four cycle the probability of connecting two adjacent vertices uniformly at random is  $\frac{3}{4}$  the short way around and  $\frac{1}{4}$  the long way around. Now finding the same probability according to (2) is the same at looking at all four spanning trees and then noting that only one contains the long way around.



# Chapter 4

## Electrical Networks

### 4.1 Flows and Edge Weighted Graphs

From now on,  $G$  is a connected loopless graph. Designate two vertices  $a, b \in G$  as boundary vertices and the rest are interior vertices. We will run some commodity (that is infinitely divisible) through our wires/pipes.

#### Definition 4.1.1 (Flow)

A **flow**  $f$  in  $G$  is a function that assigns a real number  $f(e, u, v)$  to every edge  $e$  connecting  $u - v$ , so that

1.  $f(e, u, v) = -f(e, v, u)$
2. for each interior vertex  $u$ ,

$$\sum_{u \stackrel{e}{\sim} v} f(e, u, v) = 0.$$

Thus  $\exists!$  number  $|f|$  called the **total flow** from  $a$  to  $b$  such that

$$\sum_{u \stackrel{e}{\sim} v} f(e, u, v) = \begin{cases} |f| & \text{if } u = a, \\ -|f| & \text{if } u = b, \\ 0. & \end{cases}$$

Note that total flow may be negative.

#### Definition 4.1.2 (Weighted Graph)

A **weighted graph** or **network** is a graph  $G$  with a function on its edges  $w$ . Assign to each edge  $e \in G$  a positive weight in  $\mathbb{R}$  (which we secretly call **conductance**) and denote

$$w(u) = \sum_{u \stackrel{e}{\sim} v} w(e).$$

We now study electrical networks. We could give edges complex weights which would give us the theory of solenoid networks.

**Definition 4.1.3 (Weighted Random Walk)**

The **weighted random walk** on  $(G, w)$  chooses each edge  $e$  incident to a current vertex  $u$  with probability  $\frac{w(e)}{w(u)}$ , then moves to  $v$ . Thus the transition probability is given by

$$\mathbb{P}[u \rightsquigarrow v] = \sum_{u \stackrel{e}{\sim} v} \frac{w(e)}{w(u)}.$$

The following homework is optional.

**Homework 3.0**

Generalize Wilson's Theorem to weighted graphs. So the probability of a particular spanning tree will be

$$o(1) \cdot \sum_{e \in T} w(e).$$

**4.2 Resistor Networks and Kirchhoff's Laws**

Think of  $(G, w)$  as a resistor network and  $w(e)$  is the conductance of edge  $e = \frac{1}{\text{resistance}(e)} \Omega^{-1}$  (where  $\Omega = \text{Ohms}$ ). We are talking about direct current here.

The first statement is the conservation of energy.

**Proposition 4.2.1 (Kirchhoff's First Law)**

$\exists!$  number  $|f|$  called the **total flow** from  $a$  to  $b$  such that

$$\sum_{u \stackrel{e}{\sim} v} f(e, u, v) = \begin{cases} |f| & \text{if } u = a, \\ -|f| & \text{if } u = b, \\ 0. & \end{cases}$$

**Proposition 4.2.2 (Kirchhoff's Second Law)**

For each closed walk cycle in  $G$ ,

$$u_0 \stackrel{e_1}{\sim} u_1 \stackrel{e_2}{\sim} \dots \stackrel{e_k}{\sim} u_k = u_0$$

we have

$$\sum_{i=1}^k \frac{f(e_i, u_{i-1}, u_i)}{w(e_i)} = 0.$$

Each of the terms is the product of resistance and  $\_$  so this gives us potential energy. We want this axiom because you cannot have perpetual motion machines and your electric field should be conservative.

One observes that Kirchhoff's equations are linear in the variables  $f(e, u, v)$  (assuming that  $|f|$  is given) and almost all equations are homogeneous. Therefore,

### Theorem 4.2.3 (Uniqueness of Flow)

Kirchhoff's Equations have a unique solution (for a given  $|f|$ ).

To a physicist this is patently obvious since only the initial condition can have an effect, since there is no randomness involved. This is predicated on the assumption that the "real world objectively exists" which might be an illusion.

### Definition 4.2.4 (Potentials)

Let  $f$  be a function satisfying [Proposition 4.2.2](#). Define a **potential**  $p = p_f$  on the vertices of  $G$ :

1. Assign arbitrary values  $p(a)$  to the boundary (choose a value for ground)
2. For any walk  $a = u_0 \xrightarrow{e_1} u_1 \xrightarrow{e_2} \dots \xrightarrow{e_k} u_k = u$ , set

$$p(u) \stackrel{\text{def}}{=} p(a) + \sum_{i=1}^k \frac{f(e_i, u_{i-1}, u_i)}{w(e_i)}.$$

Remark that  $p(u)$  is unambiguously defined, thanks to [Proposition 4.2.2](#). Since concatenating two paths in opposite directions will give two paths whose potentials sum to zero.

Now we will rewrite the Kirchhoff equations in terms of potentials.

### Proposition 4.2.5 (Ohm's Law)

If  $f$  satisfies [Proposition 4.2.2](#) then  $p = p_f$  satisfies, for every edge  $u \xrightarrow{e} v$  the equation

$$\frac{f(e, u, v)}{w(e)} = p(v) - p(u).$$

*Proof.*

Immediate from rewriting sums. □

We can start with an arbitrary function  $p$  and define  $f$  via [Proposition 4.2.5](#). Then [Proposition 4.2.2](#) is automatic. [Why?] It basically follows because the sums will telescope. This is a discrete Fundamental Theorem of Calculus.

Proving [Proposition 4.2.1](#) is not automatic however (not even true). Specifically, for internal vertices [Proposition 4.2.1](#) is equivalent to

$$0 = \sum_{u \stackrel{e}{\leftarrow} v} w(e) p(v) - p(u) w(u).$$

This not true when picking  $p$  willy-nilly. Remember that  $p(u)$  is fixed because we are looking at all edges out of  $u$ . So

$$\begin{aligned} &\Leftrightarrow \sum_{u \stackrel{e}{\leftarrow} v} w(e) p(v) = p(u) w(u) \\ &\Leftrightarrow p(u) = \sum_{u \stackrel{e}{\leftarrow} v} \frac{w(e)}{w(u)} p(v) \end{aligned}$$

This is saying that the value at every vertex is equal to the arithmetic mean of the surrounding edges. Thus the potential is a **discrete harmonic function** meaning its value at each internal vertex is the weighted average of its values neighboring vertices. For a **conservative flow** (i.e.  $|f| = 0$ ), the potential is harmonic everywhere.

**Lemma 4.2.6 (The Maximum Principle)**

A (real) discrete harmonic function on the vertices of a connected loopless graph  $G$  attains its extrema at the boundary vertices  $a$  &  $b$ . In the absence of boundary, (the flow is conservative) the function must be constant.

*Proof.*

Immediate. □

Now we can prove Theorem 4.2.3.

*Proof.*

Write the equations in terms of potential. If  $\exists$  two distinct solutions, then their difference is harmonic everywhere (including the boundary). Hence this difference is constant and consequently zero. This implies that  $f$  is unique. □

Now we will prove existence constructively, say for  $|f| = 1$ .

Let  $\pi$  be the random walk on  $G$  (no loops) which starts at  $a$ , stops at  $b$ , and chooses an edge from  $u \stackrel{e}{\leftarrow} v$  with probability  $\frac{w(e)}{w(u)}$ . Denote by  $\pi(t)$  the (random) vertex occupied by  $\pi$  at time  $t$ . Thus  $\pi(0) = a$ . Define the Bernoulli random variables

$$\xi_t(e, u, v) = \begin{cases} 1 & \text{if } \pi(t) = u \text{ and proceeds along } e, \\ 0 & \text{otherwise.} \end{cases}$$

This is a fairly unlikely event. Note that



$$\sum_t \mathbb{E}[\xi_t(e, u, v)] = \frac{w(e)}{w(u)} \sum_t \mathbb{P}[\pi(t) = u] < \infty.$$

The first equality follows since we need to be at  $u$  then move. So this says that we visit each vertex  $u$  on average a finite number of times. This claim follows since the probability of returning to any vertex decays exponentially. The probability of returning is less than one. This follows since we stop if we get to  $b$  and the graph is connected, so there is a nonzero probability of hitting  $b$  before returning.

### Theorem 4.2.7 (Existence for Kirchhoff's Equations)

The function  $f$  defined by

$$f(e, u, v) = \sum_t \mathbb{E}[\xi_t(e, u, v)] - \sum_t \mathbb{E}[\xi_t(e, v, u)]$$

satisfies [Proposition 4.2.1](#) and [Proposition 4.2.2](#) with  $|f| = 1$ .

“You might be able to think about how you can do this without proof, but I certainly doubt that you can.”

*Proof.*

For a vertex  $u$ ,

$$\begin{aligned} \sum_{u \stackrel{e}{\leftarrow} v} f(e, u, v) &= \sum_{u \stackrel{e}{\leftarrow} v} \left( \sum_t \mathbb{E}[\xi_t(e, u, v)] - \sum_t \mathbb{E}[\xi_t(e, v, u)] \right) \\ &= \mathbb{E}[\#\{\text{departures from } u\}] - \mathbb{E}[\#\{\text{arrivals at } u\}] \\ &= \begin{cases} 1 & \text{if } u = a \\ -1 & \text{if } u = b \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

This shows that we satisfy [Proposition 4.2.1](#).

Now we will use linearity of expectation. To show [Proposition 4.2.2](#) is satisfied.

$$\begin{aligned} \sum_{i=1}^k \frac{f(e_i, u_{i-1}, u_i)}{w(e_i)} &= \sum_t \left( \sum_{i=1}^k \frac{\mathbb{E}[\xi_t(e_i, u_{i-1}, u_i)]}{w(e_i)} - \sum_{i=1}^k \frac{\mathbb{E}[\xi_t(e_i, u_i, u_{i-1})]}{w(e_i)} \right) \\ &= \sum_t \left( \sum_{i=1}^k \frac{\mathbb{P}[\pi(t) = u_{i-1}]}{w(u_{i-1})} - \sum_{i=1}^k \frac{\mathbb{P}[\pi(t) = u_i]}{w(u_i)} \right) \\ &= 0 \end{aligned}$$

Since we are in a cycle, shifting sub scripts by one gives the same sum. □

The following theorem is due to Kirchhoff (1847).

**Theorem 4.2.8 (Electric Networks & Spanning Trees)**

Suppose  $G$  contains an edge  $a \stackrel{e_0}{\sim} b$  that directly connects  $a$  and  $b$ . Run a unit current from  $a$  to  $b$  (i.e. solve Kirchhoff equations with  $|f| = 1$ ). Then

$$f(e_0, a, b) = \mathbb{P}[\text{random spanning tree } T \subseteq G \text{ contains } e_0].$$

Where  $\mathbb{P}[T] = \prod_{e \in T} w(e)$ .

The key insight of Kirchhoff was that there is a meaning to the polynomials obtained from taking products of edge weights. So you discover that the solution is generating functions for all trees in the denominator and those containing  $e_0$  in the numerator.

“The first proof has a right to be ugly.”

*Proof.*

Recall that a random spanning tree  $T$  can be generated by Wilson’s Algorithm (with edge weights). We can choose to run the algorithm with  $b$  as the root and with  $a$  as the first starting vertex.

$$\mathbb{P}[e_0 \in T] = \mathbb{P}[e_0 \in \text{LE}(\pi)] = \mathbb{P}[e_0 \in \pi]$$

On the other hand,

$$\begin{aligned} f(e_0, a, b) &= \sum_t \mathbb{E}[\xi_t(e_0, a, b)] - \sum_t \mathbb{E}[\xi_t(e_0, b, a)] \\ &= \sum_t \mathbb{E}[\xi_t(e_0, a, b)] - 0 \\ &= \mathbb{P}[\pi \text{ uses } e_0 \text{ to arrive at } b] \\ &= \mathbb{P}[e_0 \in \pi]. \end{aligned}$$

□

This is our happy ending of loop erasure and electrical networks!

Moral of the Story: So you can view electric networks as random processes of random walks, or even better the theory of random spanning trees.

### 4.3 Effective Conductance

Next Step: We can think of our electric network as one huge resistor. It makes sense to believe that we can replace the whole graph with a single resistor and the outside world will not notice. We know the answer of that since we know the proportion of flow that will go through  $e_0$ . This will tell us the **effective resistance** of  $G$ . This will allow us to remove our special edge  $e_0$ .