Imperial College London

Coursework

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Department of Mathematics

# MATH70129 - Portfolio Management

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**Abstract.** In this project, we investigate portfolio construction methods and replicate the comparison done in "Optimal Versus Naive Diversification: How Inefficient is the 1/N Portfolio Strategy?" by DeMiguel, Garlappi, and Uppal [1]. We evaluate various strategies, including the equally weighted 1/N portfolio, sample-based mean-variance portfolio, other moment estimators, and shrinkage estimation.

The analysis employs a rolling-window estimation approach to assess the out-of-sample performance of these strategies using multiple metrics, including Sharpe ratio. Our analysis spans multiple empirical datasets, including S&P 500 sector portfolios (1981–2002), industry portfolios (1963–2004), and factor-based portfolios (1963–2004).

# I Introduction

In this project, we analyse and replicate the paper: Optimal Versus Naive Diversification: How Inefficient is the 1/N Portfolio Strategy?[1] The paper compares different portfolio strategies using three metrics: out-of-sample Sharpe ratio, certainty equivalent returns, and portfolio turnover, which we define in Section 2. In Section 3, we present the empirical results applied to each dataset, and in Section 4, we seek to explain the results obtained in Section 3 with analytical results.

Introduced in 1952 by Markowitz [2], the mean-variance framework is a model that compares the expected return of a portfolio with its variance, constructing a capital allocation line that represents the optimal allocation between risky assets. While the mean-variance model with sample mean and covariance estimators gives the best in-sample performance, it completely ignores estimation errors and often produces extreme portfolio weights, performing poorly out-of-sample. We explore models using different moment estimators, including shrinkage estimation, and estimators computed from constrained optimisation. We then compare the results under the aforementioned metrics to obtain a comprehensive overview on the strategies considered.

# II Theory

## II.I Portfolio optimisation methods

Suppose we have a matrix of returns of different strategies $X = (X_1, ..., X_N)$ over time, such that $\mu \in \mathbb{R}^N$ is the expected return and $\Sigma \in \mathcal{M}^{N,N}$ is the covariance matrix. An investor has an initial wealth of $W$ and creates a portfolio in these risky strategies, i.e. they want to choose a weight vector $w \in \mathbb{R}^N$ corresponding to the proportion invested in each strategy.

**Equally Weighted 1/N (1/N)** One way to allocate to each strategies is to assign the same weight to each strategies, i.e.

$$w = \begin{bmatrix} \frac{1}{N} \\ \frac{1}{N} \\ \vdots \\ \frac{1}{N} \end{bmatrix} \tag{1}$$

Given that each of these strategies already invests in a wide range of assets, this could further diversify away the idiosyncratic risks.

**Mean-Variance (mv)** By fixing a risk aversion constant $\gamma$, one can find the portfolio that gives the maximum expected return for that risk aversion level, i.e. the investor solves

$$\max_x x^\intercal \mu - \frac{\gamma}{2} x^\intercal \Sigma x$$

This maximization has the solution $x = \frac{1}{\gamma}\Sigma^{-1}\mu$ and the relative portfolio weights are $w = \frac{x}{|\mathbf{1}_N^\intercal x|}$ ($\mathbf{1}_N$ is the vector of size $N$ with only ones).

**Minimum Variance (min)** The minimum variance portfolio corresponds to the portfolio obtained by minimising the variance, i.e. $\min_w w^\intercal \Sigma w$ under the constraint $\mathbf{1}_N^\intercal w = 1$. The analytical solution is $w = \frac{\Sigma^{-1}\mathbf{1}_N}{\mathbf{1}_N^\intercal \Sigma^{-1}\mathbf{1}_N}$.

**Bayes-Stein Shrinkage (bs)** Portfolio optimisation techniques often rely on estimating expected returns, which can be noisy due to market volatility or small sample sizes. The Bayes-Stein portfolio accounts for this issue by "shrinking" the sample mean, $\hat{\mu}$, toward the mean of the minimum-variance portfolio, $\mu^{\min}$, by using the following estimator:

$$\hat{\mu}_t^{\mathrm{bs}} = (1 - \hat{\phi}_t)\hat{\mu}_t + \hat{\phi}_t \hat{\mu}_t^{\min},$$

$$\hat{\phi}_t = \frac{N+2}{(N+2) + M(\hat{\mu}_t - \mu_t^{\min})^\top \hat{\Sigma}^{-1} (\hat{\mu}_t - \mu_t^{\min})},$$

where $0 < \hat{\phi}_t < 1$, $\hat{\Sigma}_t = \frac{1}{M-N-2} \sum_{s=t-M+1}^{t} (R_s - \hat{\mu}_t)(R_s - \hat{\mu}_t)^\top$, and $\hat{\mu}_t^{\min} \equiv \hat{\mu}^\top \hat{w}_t^{\min}$.

**Value-Weighted (vw)** This portfolio simply invests all its capital into the market portfolio $MKT$, without estimation or rebalancing throughout the entire period.

**Shortsale-Constrained Portfolios (mv-c, min-c, g-min-c)** The mv-c and min-c portfolios correspond to the the mean-variance and minimum variance portfolios with no shorting allowed. The g-min-c portfolio is obtained by adding the constraint

$$w > a\mathbf{1}_N, \ a \in [0, 1/N]$$

to the minimum variance optimisation. This strategy attempts to generalise the **min-c** portfolio by adjusting the threshold, with $a = 0$ being the original **min-c** portfolio and $a = \frac{1}{N}$ corresponding to **1/N**. $a = \frac{1}{2N}$ was chosen arbitrarily to be the middle ground.

**Mixture of Equally Weighted and Minimum Variance (ew-min)** This portfolio is a linear combination of the equally weighted (1/N) and minimum variance portfolios,

$$\hat{w}^{ew-min} = c\frac{1}{N}\mathbf{1}_N + d\hat{\Sigma}^{-1}\mathbf{1}_N, \ \ \mathbf{1}_N^T \hat{w}^{ew-min} = 1$$

## II.II Evaluation methodology

The methodology for evaluating the performance used in the paper uses a rolling-sample approach with window size M. Suppose we have a dataset containing T monthly returns. At each step $1 \leq i \leq T - M$, we use month $i$ to $M + i - 1$ to estimate our portfolio weights and calculate the out-of-sample return in month $M + i$ using these weights. This gives a vector $r_k$ of monthly returns for the strategy $k$. We use $M = 120$, consistent with the estimation window used in the original paper.

In order to compare the relative performance between strategies, we will use three metrics to evaluate portfolio performance.
For each strategy $k$, let us denote by $\widehat{\mu}_k$ the out-of-sample mean and $\widehat{\sigma}_k$ the out-of-sample standard deviation.

$$\widehat{\mu}_k = \frac{1}{T-M} \sum_{i=1}^{T-M} r_{k,i}$$

$$\widehat{\sigma}_k = \sqrt{\frac{1}{T-M-1} \sum_{i=1}^{T-M} (r_{k,i} - \widehat{\mu}_k)^2}$$

For strategies that do not require moment estimation (1/N and value-weighted), we simply use the mean and standard deviation over the whole time period $T$.

Then we calculate the following metrics :

i. Sharpe ratio :

$$\widehat{SR_k} = \frac{\widehat{\mu}_k}{\widehat{\sigma}_k}$$

We then test the hypothesis

$$H_0 : \widehat{SR_i} = \widehat{SR_n}, \ \ H_1 : \widehat{SR_i} > \widehat{SR_n}$$

where $SR_n$ is the Sharpe ratio of the 1/N strategy. The one-sided p-value is then calculated using the following test statistic:

$$\hat{z} = \frac{\hat{\sigma}_n \hat{\mu}_i - \hat{\sigma}_i \hat{\mu}_n}{\sqrt{\hat{\nu}}} \xrightarrow{H_0} N(0,1)$$

with

$$\hat{\nu} = \frac{1}{T-M}(2\hat{\sigma}_i^2\hat{\sigma}_n^2 - 2\hat{\sigma}_i\hat{\sigma}_n\hat{\sigma}_{i,n} + \frac{1}{2}\hat{\mu}_i^2\hat{\sigma}_n^2 + \frac{1}{2}\hat{\mu}_n^2\hat{\sigma}_i^2 - \frac{\hat{\mu}_i\hat{\mu}_n}{\hat{\sigma}_i\hat{\sigma}_n}\hat{\sigma}_{i,n}^2)$$

where $\hat{\mu}_n$ and $\hat{\sigma}_n$ are the moment estimators of the $1/N$ strategy returns, and $\hat{\sigma}_{i,n}^2$ is the sample covariance.

ii. Certainty-equivalent (CEQ) return :

$$\widehat{CEQ}_k = \widehat{\mu}_k - \frac{\gamma}{2}\widehat{\sigma}_k^2$$

We take risk aversion $\gamma = 1$ and similarly test the hypothesis

$$H_0 : \widehat{CEQ_i} = \widehat{CEQ_n}, \ \ H_1 : \widehat{CEQ_i} > \widehat{CEQ_n}$$

Let $v = (\mu_i, \mu_n, \sigma_i^2, \sigma_n^2)$ and $u = (1, -1, -\frac{\gamma}{2}, \frac{\gamma}{2})$
Then under $H_0$,

$$\sqrt{T}(\widehat{CEQ_i} - \widehat{CEQ_n}) \xrightarrow{H_0} N(0, u\Theta u^T)$$

where

$$\Theta = \begin{pmatrix} \sigma_i^2 & \sigma_{i,n} & 0 & 0 \\ \sigma_{i,n} & \sigma_n^2 & 0 & 0 \\ 0 & 0 & 2\sigma_i^4 & 2\sigma_{i,n}^2 \\ 0 & 0 & 2\sigma_{i,n}^2 & 2\sigma_n^4 \end{pmatrix}$$

iii. Turnover :

$$Turnover = \frac{1}{T-M}\sum_{t=1}^{T-M}\sum_{j=1}^{N}|\hat{w}_{j,t+1} - \hat{w}_{j,t^+}|$$

where $\hat{w}_{j,t^+}$ and $\hat{w}_{j,t+1}$ are the portfolio weights for strategy $j$ right before and after balancing at time $t+1$.

We will also consider these metrics for the in-sample mean-variance method as a benchmark.

As in the paper, we have considered the following datasets, where $N$ represents the total number of strategies.

| **Dataset and source** | **N** | **Time period** | **Abbreviation** |
|---|---|---|---|
| Ten sector portfolios of the S&P 500 and MKT factor, *Source: Roberto Wessels* | 10 +1 | 01/1981–12/2002 | S&P Sectors |
| Ten industry portfolios and MKT factor, *Source: Ken French's Web site* | 10 +1 | 07/1963–11/2004 | Industry |
| SMB and HML portfolios and MKT factor, *Source: Ken French's Web site* | 2 +1 | 07/1963–11/2004 | MKT/SMB/HML |
| Twenty size- and book-to-market portfolios and MKT factor, *Source: Ken French's Web site* | 20 +1 | 07/1963–11/2004 | FF-1-factor |
| Twenty size- and book-to-market portfolios and MKT, SMB, HML, and UMD factors, *Source: Ken French's Web site* | 20 +4 | 07/1963–11/2004 | FF-4-factor |

*For each dataset portfolios, we consider monthly excess returns over the 90-day nominal US T-bill.*
*MKT: US equity market portfolio, SMB: small-minus-big portfolio,*
*HML: high-minus-low portfolio, UMD: momentum portfolio.*

# III Empirical Results

| | S&P Sectors N=11 | Industry N=11 | MKT/SMB/HML N=3 | FF-1 N=21 | FF-4 N=24 |
|---|---|---|---|---|---|
| 1/N | 0.1525 | 0.1251 | 0.2085 | 0.1428 | 0.1475 |
| mv (in-sample) | 0.3248 | 0.2097 | 0.2589 | 0.4518 | 0.4578 |
| mv | 0.0643 | -0.0512 | 0.2096 | -0.0479 | 0.1980 |
| | (0.9141) | (0.9914) | (0.6791) | (0.9983) | (0.3354) |
| min | 0.0961 | 0.1588 | 0.2556 | 0.2532 | 0.0349 |
| | (0.9280) | (0.2853) | (0.2640) | (0.0265) | (0.9725) |
| bs | 0.0704 | 0.0530 | 0.2506 | 0.0531 | -0.1042 |
| | (0.9272) | (0.8962) | (0.3444) | (0.9285) | (0.9999) |
| vw | 0.1138 | 0.1053 | 0.1053 | 0.1053 | 0.1053 |
| | (1.0000) | (0.9648) | (0.9998) | (0.9935) | (0.9953) |
| mv-c | 0.1653 | 0.0965 | 0.2345 | 0.2053 | 0.2589 |
| | (0.6912) | (0.8976) | (0.4972) | (0.0044) | (0.0390) |
| min-c | 0.1236 | 0.1632 | 0.2169 | 0.1704 | 0.2551 |
| | (0.9543) | (0.1541) | (0.7833) | (0.2911) | (0.0435) |
| g-min-c | 0.1597 | 0.1515 | 0.2224 | 0.1667 | 0.2360 |
| | (0.8533) | (0.1734) | (0.7235) | (0.2800) | (0.0041) |
| ew-min | 0.0411 | 0.0284 | 0.0401 | 0.1170 | 0.0989 |
| | (0.9733) | (0.9667) | (0.9972) | (0.7170) | (0.9913) |

Table 1: Sharpe ratios with one-sided p-values

The monthly Sharpe ratios and corresponding p-values of different strategies applied to the aforementioned datasets are displayed in Table 1. Note that a higher Sharpe ratio does not necessarily imply a lower p-value, as the test statistic also takes the correlation of the returns into account. As suggested in the original paper, the difference between the in-sample and out-of-sample mean-variance portfolios provides an approximation of the estimation error of the moments. This difference is immense, with the out-of-sample Sharpe ratio being negative for two out of five datasets considered (Industry and FF-1), despite having a 0.21 and 0.45 Sharpe in-sample. The mean-variance portfolio outperforms the 1/N strategy in the MKT/SMB/HML and FF-4 datasets, but the difference is marginal and statistically insignificant, consistent with the findings in the original paper. This suggests that using sample-based moment estimators under the mean-variance framework without considering estimation error is ineffective.

On the other hand, the minimum variance portfolio ignores the estimation of expected returns, and is effective in reducing extreme portfolio weights, from [-1388676%, 667892%] under **mv** to [-565%, 228%] under **min**. This resulted in the **min** portfolio achieving positive Sharpe ratios for all datasets, outperforming **1/N** in three out of five datasets, although the difference is only statistically significant in the FF-1 dataset.

While the Bayes-Stein shrinkage strategy is designed to reduce estimation error of the expected return, it underperforms **1/N** in all but one dataset - MKT/SMB/HML - and the difference only yields a p-value of 0.34. Also, it only offers marginal improvement compared to **mv**.

The market portfolio (**vw**) strategy underperforms **1/N** in the two market return time series considered (Dataset 2-5 have the same market return series as a factor portfolio). This is not surprising, as companies with lower market capitalisations tend to outperform larger companies, known as the small firm effect. However, there are advantages to the **vw** strategy, demonstrated in the later section on portfolio turnover.

Contrary to the results presented in the original paper, shortsale-constrained strategies seem to offer a decent improvement from the **1/N** strategy. In particular, the **mv-c** and **min-c** strategies both outperform **1/N** in four out of five datasets, achieving statistically significant improvements in two and one dataset respectively. **g-min-c** outperforms **1/N** in all datasets, again with one statistically significant p-value.

Finally, the mixture strategy performs noticeably worse than the equally weighted portfolio.

We observe unsurprising results for the CEQ returns in Table 2, with in-sample mean-variance exhibiting the highest CEQ returns. However, out of all the datasets and strategies, there is only one superior CEQ return that is statistically significant. This is achieved in the FF-1 dataset with the **mv-c** strategy, achieving a 0.0092 CEQ against 0.0062 for the **1/N** strategy with a p-value $< 10^{-4}$. This table also highlights the drawbacks of the **mv** and **bs** strategy's inability to adjust for estimation error. This is shown by the negative CEQ return attained by these strategies, in two and three out of five datasets respectively.

| | S&P Sectors N=11 | Industry N=11 | MKT/SMB/HML N=3 | FF-1 N=21 | FF-4 N=24 |
|---|---|---|---|---|---|
| 1/N | 0.0057 | 0.0044 | 0.0038 | 0.0062 | 0.0061 |
| mv (in-sample) | 0.0176 | 0.0101 | 0.0041 | 0.0249 | 0.0146 |
| mv | 0.0016 | -2.3682 | 0.0042 | -165.5758 | 0.0181 |
| | (0.7819) | (1.0000) | (0.3182) | (1.0000) | (0.1517) |
| min | 0.0028 | 0.0052 | 0.0040 | 0.0090 | 0.0001 |
| | (0.9496) | (0.2876) | (0.3111) | (0.0853) | (0.9963) |
| bs | 0.0025 | -1.1470 | 0.0042 | -42.0027 | -0.0388 |
| | (0.8363) | (1.0000) | (0.2410) | (1.0000) | (1.0000) |
| vw | 0.0042 | 0.0037 | 0.0037 | 0.0037 | 0.0037 |
| | (1.0000) | (0.8450) | (0.5148) | (0.9915) | (0.9914) |
| mv-c | 0.0062 | 0.0036 | 0.0041 | 0.0092 | 0.0052 |
| | (0.3612) | (0.7200) | (0.3215) | (0.0000) | (0.6713) |
| min-c | 0.0038 | 0.0054 | 0.0038 | 0.0069 | 0.0042 |
| | (0.9548) | (0.1540) | (0.5143) | (0.2158) | (0.8304) |
| g-min-c | 0.0053 | 0.0051 | 0.0038 | 0.0071 | 0.0058 |
| | (0.7184) | (0.1310) | (0.4162) | (0.0264) | (0.5927) |
| ew-min | -0.0087 | -0.0030 | 0.0004 | 0.0051 | 0.0027 |
| | (0.9293) | (0.9429) | (0.8591) | (0.5528) | (0.7264) |

Table 2: CEQ returns with one-sided p-values

| | S&P Sectors N=11 | Industry N=11 | MKT/SMB/HML N=3 | FF-1 N=21 | FF-4 N=24 |
|---|---|---|---|---|---|
| 1/N | 0.0311 | 0.0222 | 0.0238 | 0.0165 | 0.0188 |
| mv (in-sample) | - | - | - | - | - |
| mv | 25.2108 | 3987.0870 | 1.0108 | 11471.1935 | 648.0970 |
| min | 2.2773 | 6.9104 | 0.9229 | 17.5321 | 4.8125 |
| bs | 10.1132 | 2660.2901 | 0.9197 | 4784.8193 | 2673.4524 |
| vw | 0 | 0 | 0 | 0 | 0 |
| mv-c | 0.8316 | 0.7544 | 0.8263 | 0.4413 | 1.1794 |
| min-c | 0.8930 | 0.8898 | 0.9466 | 0.6616 | 1.1692 |
| g-min-c | 0.9632 | 1.0274 | 0.9546 | 0.8893 | 1.5057 |
| ew-min | 113.0363 | 69.0205 | 5.7070 | 347.4025 | 89.4525 |

Table 3: Turnover of 1/N and relative turnover of other strategies

We observe that unconstrained strategies have significantly higher turnover in Table 3. This could be attributed to their high sensitivity to the input data used in the estimation process, with the turnover of **mv** being over 10000 times higher than that of **1/N** in the FF-1 dataset.

On the other hand, the constrained strategies achieve even lower turnovers than the **1/N** strategy. One of the reasons could be that these portfolios assign a lower weight to volatile assets/factors, which are the assets that require the most rebalancing, while **1/N** does not take this into account.

We also note that while the market portfolio (**vw**) did not achieve statistically significant values by the previous two metrics, it has a turnover of 0 and does not require rebalancing. In practice, the lack of transaction costs could be a significant advantage.

# IV Analytical Results

The paper also analyses the factors that contribute to the difference in empirical performance analytically. We would like to understand the conditions under which the sample-based **mv** strategy outperforms the **1/N** strategy. We first define the expected loss.

**Definition 1.** For a particular weight estimator

$$\hat{w} = \hat{w}(r_1, ..., r_M),$$

we define the expected loss

$$L(w^*, \hat{w}) = U(w^*) - E[U(\hat{w})]$$

where $U$ is the mean variance utility

$$U(w) = w^T \mu - \frac{\gamma}{2} w^T \Sigma w$$

and $w^*$ is the optimial weight.

Then, under the assumption that the returns distribution is jointly normal, we can compute the number of months needed for the sample-based **mv** strategy to outperform the **1/N** strategy, under the following three cases.

**Proposition 2.** *Suppose $S_*^2 = \mu^T \Sigma^{-1} \mu$ and $S_{1/N}^2 = \frac{(\mathbf{1}_N^T \mu)^2}{\mathbf{1}_N^T \Sigma \mathbf{1}_N}$ are the squared Sharpe ratios of the **mv** and **1/N** strategy respectively. Then, under each of these cases, **mv** outperforms **1/N** with respect to expected loss when:*

1. *$\mu$ unknown, $\Sigma$ known:*

$$S_*^2 - S_{1/N}^2 - \frac{N}{M} > 0$$

2. *$\mu$ known, $\Sigma$ unknown:*

$$kS_*^2 - S_{1/N}^2 > 0$$

*where*

$$k = (\frac{M}{M-N-2})(2 - \frac{M(M-2)}{(M-N-1)(M-N-4)})$$

3. *$\mu$ unknown, $\Sigma$ unknown:*

$$kS_*^2 - S_{1/N}^2 - h > 0$$

*where*

$$h = \frac{NM(M-2)}{(M-N-1)(M-N-2)(M-N-4)}$$

Using these conditions, we plot the boundary $M$ against $N$ where **mv** outperforms **1/N** (Figure 1) for $N$ ranging from 10 to 100. The values of $S_*$ and $S_{1/N}$ are chosen to match our empirical results in the previous section, for the S&P Sectors and FF-1/FF-4 datasets.

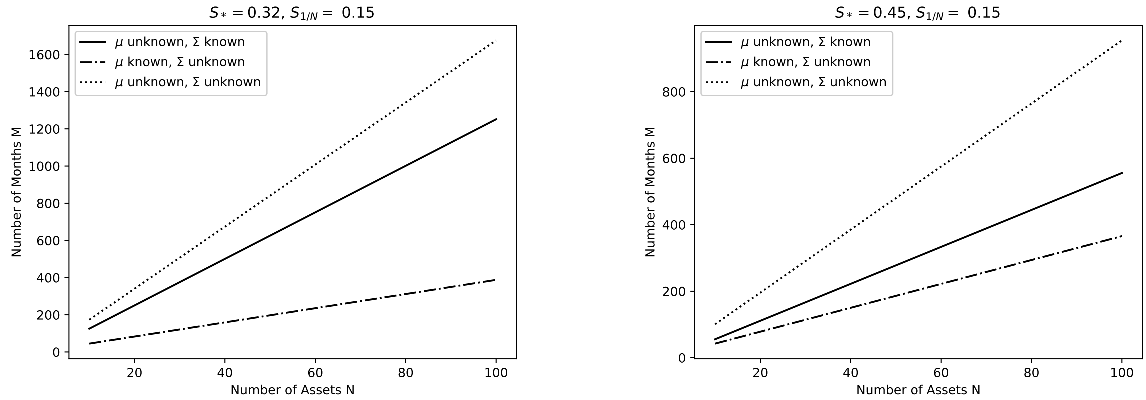


Figure 1: Number of Estimation Months Required Against Number of Assets

In this figure, we observe that when $\mu$ is known, the number of months required for estimation drastically decreases for both sets of parameters, by approximately 77% and 62% respectively. This partially explains the better performance of the minimum variance strategy compared to mean-variance, as it only estimates the covariance matrix.

It is also evident that the second set of parameters requires a much shorter estimation window. This supports our results in the previous section, where all of the statistically superior Sharpe ratios belong to the FF-1 and FF-4 datasets.

Although we only performed this analysis on the **mv** strategy, this provides us with insights into the reason behind the underperformance of the sample-based strategies. While we used $M = 120$ months in our empirical analysis, this is still far from the required estimation length shown in Figure 1. For $N = 11$ (S&P Sectors) and $N = 24$ (FF-4), the required lengths are $M = 180$ and $M = 240$ respectively. While increasing the estimation window should theoretically improve the **mv** performance, it is also unrealistic to assume that the expected returns and covariance structure will remain unchanged over such a long time period.

# V Conclusion

In this project, we have implemented portfolio construction methods on multiple datasets, evaluating the effectiveness of more sophisticated strategies compared to the straightforward 1/N strategy. We considered three metrics, Sharpe ratio, certainty equivalent returns, and portfolio turnover. By using test statistics from the original paper, we obtain p-values for hypothesis testing. While several strategies outperformed the equally weighted strategy by a statistically significant amount, it was only achieved in some datasets. None of the strategies considered outperformed the 1/N strategy in all datasets, under any metric. However, by examining the portfolio weights and certainty equivalent returns, it is clear that the constrained portfolios offer the best improvement in terms of the stability of portfolio weights. On the other hand, the turnover metric allowed us to view the portfolio construction process in a more practical aspect. While some strategies offer impressive returns in backtests, the transaction costs incurred during the process could offset any potential profits, and must be considered before a strategy is deployed in practice. We also presented the analytical results in the original paper, which demonstrates the relationship between the estimation window length and the performance of the mean-variance portfolio.

## References

[1] Victor DeMiguel, Lorenzo Garlappi, and Raman Uppal. Optimal versus naive diversification: How inefficient is the 1/n portfolio strategy? *The Review of Financial Studies, Vol. 22, Issue 5, pp. 1915-1953, 2009, hhm075[hhm075], Available at SSRN: https://ssrn.com/abstract=1376199 or http://dx.doi.org/hhm075*, 2009.

[2] Harry Markowitz. Portfolio selection. `https://doi.org/10.2307/2975974`, 1952.