This document is a derivation of the formula used in Euler317.

We begin with a vector description of projectile motion:

$$\vec{r} = ut\cos\theta\hat{\mathbf{i}} + ut\sin\theta + \frac{1}{2}at^2 + h\hat{\mathbf{j}}$$

where u, θ, h, t are initial velocity, launch angle, height from which it is launched and time elapsed respectively. We find that:

$$x = ut\cos\theta \tag{1}$$

$$y = ut\sin\theta + \frac{1}{2}at^2 + h \tag{2}$$

These can be rearranged and substituted in for t.

$$\begin{aligned}
y &= u\left(\frac{x}{u\cos\theta}\right)\sin\theta + \frac{1}{2}a\left(\frac{x}{u\cos\theta}\right)^2 + h \\
y &= \frac{x\sin\theta}{\cos\theta} + \frac{ax^2}{2u^2\cos^2\theta} + h \\
y &= x\tan\theta - \frac{ax^2}{2u^2}\sec^2\theta + h
\end{aligned}$$

As acceleration is due to gravity, $a = g$, and to make it look nice let $u = v$. Hence the final equation is

$$y = x\tan\theta - \frac{gx^2}{2v^2}\sec^2\theta + h.$$

This is all well and good, but we must find an 'envelope' that encompasses all parabolas for values of θ. Wkipedia says that the envelope of the family of curves parametised by t is given as the set of points where

$$F(t, x, y) = \frac{\delta F}{\delta t}(t, x, y) = 0.$$

We have $F(\theta, x, y) = x\tan\theta - \frac{gx^2}{2v^2}\sec^2\theta + h - y$, hence:

$$\frac{\delta F}{\delta\theta}(\theta, x, y) = x\sec^2\theta + \frac{gx^2}{v^2}\sec^2\theta\tan\theta$$

by applying the chain rule. Solving $\frac{\delta F}{\delta\theta} = 0$:

$$\begin{aligned}
0 &= x\sec^2\theta + \frac{gx^2}{v^2}\sec^2\theta\tan\theta \\
0 &= \frac{x}{v^2}\sec^2\theta\left(v^2 - gx\tan\theta\right) \\
\therefore \tan\theta &= \frac{v^2}{gx}
\end{aligned}$$

Using the trig identity $\sec^2 x = 1 + \tan^2 x$, we find that $\sec^2 x = \frac{g^2x^2+v^4}{g^2x^2}$. Substituting into $F(\theta, x, y) = 0$:

$$\begin{aligned}
y &= \frac{v^2}{g} - \frac{gx^2}{2v^2}\left(\frac{g^2x^2 + v^4}{g^2x^2}\right) + h \\
y &= \frac{v^2}{g} - \frac{g^2x^2 + v^4}{2v^2g} + h \\
y &= \frac{v^2}{2g} - \frac{gx^2}{2v^2} + h \qquad (3)
\end{aligned}$$

To find the solid of revolution around the y-axis:

$$V = \pi \int_{b}^{a} [f(y)]^2 dy$$

Using equation 3:

$$\begin{aligned} y &= \frac{v^2}{2g} - \frac{gx^2}{2v^2} + h \\ y - \frac{v^2}{2g} - h &= -\frac{gx^2}{2v^2} \\ \frac{v^2}{g}(y - \frac{v^2}{2g} - h) &= x^2 \\ x &= \sqrt{\frac{v^2}{g}(y - \frac{v^2}{2g} - h)} \end{aligned}$$

The bounds on the integral will be 0 and $f(0)$, where $f(x)$ is equation 3.

Using Julia and plugging in $\{v, g, h\} = \{20, 9.81, 8\}$