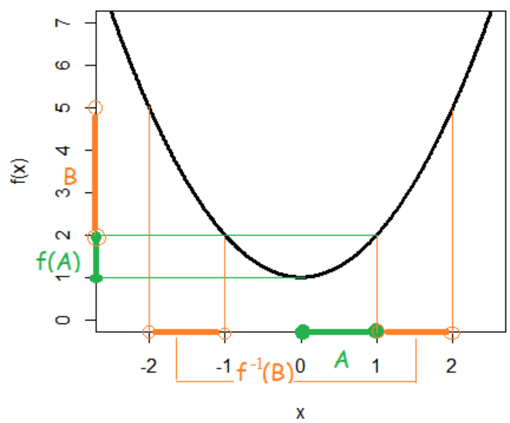
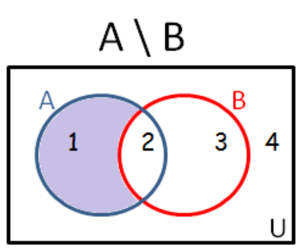
ST116 Problem Sheet 2 - Solutions

1. (a) They are sets.
   (b) $f(A) = [1, 2]$ and $f^{-1}(B) = (-2, -1) \cup (1, 2)$.



   (c) $A = f_{a,b}^{-1}([0, 1])$, $B = f_{a,b}([0, 1])$, $C = f_{a,b}(\mathbb{R})$.
   (d) $Z(1, 2) = -1$, $Z(5, 9) = -4$, $Z(7, 11) = -4$, hence
   $Z\Big(\Big\{(1, 2), (5, 9), (7, 11)\Big\}\Big) = \Big\{Z(i, j) \;\Big|\; (i, j) \in \{(1, 2), (5, 9), (7, 11)\}\Big\} = \{-1, -4\}$.
   $Z\Big(\Big\{8, 10\Big\}\Big) = \Big\{(12, 4), (12, 2)\Big\}$.
   (e) $A = Z(E)$, $B = Z^{-1}\Big(\{1, 3, 5, 8, 10\}\Big)$ or $B = Z^{-1}(\mathbb{R}^+)$, $C = Z^{-1}(\{-10, 10\})$,
   $D = Z\Big(\Big\{(i, k) \;\Big|\; i \in \{5, 7\}, j \in \{2, 4, 9, 11\}\Big\}\Big)$.
   (f) The image of a set A under the function f is the set containing the function values that f assigns to the elements of A. The pre-image of a set B under the function f is the set containing the elements in the domain of f that f maps onto an element of B.

2. The Venn diagram is the same in all sub-questions.



   (a) See Venn diagram above.
   (b) In the Venn diagram above, A is depicted as regions 1 and 2. The set B^c is given by the regions other than regions 2 and 3, and thus is depicted by regions 1 and 4. $A \cap B^c$ is depicted by the regions that are common to A and B^c and thus by region 1.

(c) The set $A \cap B$ is depicted by region 2, hence $(A \cap B)^c$ is depicted by the regions other than region 2 and thus by regions 1, 3 and 4. The set A is given by the regions 1 and 2. Finally, $A \cap (A \cap B)^c$ is illustrated by the regions that are common to A and $(A \cap B)^c$ and thus by region 1.

(d) B^c is depicted by regions 1 and 4. $A \cup B$ is represented by combining the regions depicting A (regions 1 and 2) and the regions depicting B (regions 2 and 3) and thus is given by regions 1, 2 and 3. Finally, $(A \cup B) \cap B^c$ is shown as the region common to $A \cup B$ and B^c and thus as region 1.

(e) $A \backslash B \;=\; A \cap B^c \;=\; A \cap (A \cap B)^c \;=\; (A \cup B) \cap B^c$.

3. (a) [**2 points**] As $A \subseteq B$, if $a \in A$, then $a \in B$. (*)
As $C \subseteq D$, if $a \in C$, then $a \in D$. (**)
Let $a \in A \cap C$, then $a \in A$ and $a \in C$. It follows from (*) that $a \in B$ and from (**) that $a \in D$. Hence $a \in B \cap D$. Therefore $A \cap C \subseteq B \cap D$.

(b) [**2 points**] Let $a \in A$. As $A \subseteq B$, it follows that $a \in B$. Since $B \subseteq C$, if $a \in B$, then $a \in C$. Taking both statements together, if $a \in A$, then $a \in C$. It follows that $A \subseteq C$.

(c) [**3 points**] If $A \subset B$, then $A \subseteq B$ and so we can deduce with (b) that $A \subseteq C$. So we only need to show that there is an element of C that does not belong to A. As $A \subset B$, there exists an element $b \in B$ such that $b \notin A$. As $B \subseteq C$, $b \in C$. Therefore $b \in C$ and $b \notin A$, hence $A \subset C$.

(d) [**3 points**] Let $a \in A$. As $A \subseteq B \cup C$ we have $a \in B \cup C$, that is $a \in B$ or $a \in C$.
Case 1: $a \in B$. Then $a \in A$ and $a \in B$ and so $a \in A \cap B$. However, this contradicts the assumption that $A \cap B = \emptyset$. Hence this case cannot occur.
Case 2: $a \in C$.
Since Case 2 is the only case that can occur, if $a \in A$, then $a \in C$. Hence $A \subseteq C$.

4. (a) $A \subseteq B$: the set A is a subset of B, that is, all elements of A belong to B.
$A \cup B$: the union of A and B is the set containing the elements that belong to A or belong to B.
$\mathcal{P}(A)$: the power set of A, that is, the set containing all subsets of A.

(b) The proposition specifies the relationship between the union of the power set of two sets and the power set of the union of these two sets.

(c) Let $A = \{1, 2\}$ and $B = \{2, 3\}$, then $\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$, $\mathcal{P}(B) = \{\emptyset, \{2\}, \{3\}, \{2, 3\}\}$ and $\mathcal{P}(A \cup B) = \mathcal{P}(\{1, 2, 3\}) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 3\}, \{1, 2, 3\}\}$. Hence $\mathcal{P}(A) \cup \mathcal{P}(B) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}\} \subseteq \mathcal{P}(A \cup B)$.

(d) Choose $A = \{1, 2\}$ and $B = \{2, 3\}$, then as shown above $\{1, 2, 3\} \in \mathcal{P}(A \cup B)$ but $\{1, 2, 3\} \notin \mathcal{P}(A) \cup \mathcal{P}(B)$.

(e) If $A \subseteq B$ or $B \subseteq A$, then $\mathcal{P}(A) \cup \mathcal{P}(B) = \mathcal{P}(A \cup B)$.
Proof:
Case 1: $A \subseteq B$. Then, by Proposition 4.4, $A \cup B = B$ and so $\mathcal{P}(A \cup B) = \mathcal{P}(B)$. Furthermore, if $C \subseteq A$, then, since $A \subseteq B$, $C \subseteq B$ (this was shown in Question 3 (b)). Hence if $C \in \mathcal{P}(A)$, then $C \in \mathcal{P}(B)$ and so $\mathcal{P}(A) \subseteq \mathcal{P}(B)$. By Proposition 4.4 it follows that $\mathcal{P}(A) \cup \mathcal{P}(B) = \mathcal{P}(B)$ and so $\mathcal{P}(A) \cup \mathcal{P}(B) = \mathcal{P}(A \cup B)$.
Case 2: $B \subseteq A$. We can make an analogous argument to Case 1 by reversing the roles of A and B.