ST116 Problem Sheet 3

Deadline: Thursday 27 October 2016, 11 am.
Please submit answers to **all** questions in the assignment box on the first floor landing of the atrium.
Make sure to attach a cover sheet to the assignment!
Only one of the questions will be marked. The mark counts towards the module mark.

Let A and B be subsets of the universal set U. For Questions 1 and 2 you may use the statements below without further proof. However, you are expected to be able to prove these.

- $A \cap A = A$;
- $A \cap B \subseteq A$;
- $A \backslash B = A \cap B^c$;
- the laws listed in Section 3.5 of the workbook.

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1. **Set-theoretic proofs:** Let A, B and C be subsets of the universal set U.
   (a) Using Venn diagrams explain why $(A\backslash B)^c = A^c \cup B$.
   (b) Prove the following statement.

   $$\text{If } C \subseteq A, \text{ then } A\backslash(B\backslash C) = (A\backslash B) \cup C.$$

   (Hint: you may use the statement in (a) without further proof.)
   (c) A student claims that the following statement is true for all sets A, B and C.

   $$\text{If } A \subseteq B \cup C, \text{ then } A \subseteq B \text{ or } A \subseteq C.$$

   (i) Provide a counter-example, that is, provide example sets A, B and C such that $A \subseteq B \cup C$, but neither $A \subseteq B$ nor $A \subseteq C$.
   (ii) Below is the student's proof. Explain what went wrong in the proof.

   Proof: Let $a \in A$. As $A \subseteq B \cup C$ it follows that $a \in B$ or $a \in C$.
   Case 1: $a \in B$. Hence $A \subseteq B$.
   Case 2: $a \in C$. Hence $A \subseteq C$.
   It follows that $A \subseteq B$ or $A \subseteq C$.

2. **Power sets:** Let A, B and C be subsets of the universal set U.

   (a) Write out the sets $\mathcal{P}(\emptyset)$ and $\mathcal{P}\Big(\mathcal{P}(\emptyset)\Big)$.

   (b) Use the statement in Question 3 (a) of Problem Sheet 2 to show that

   $$\text{if } A \subseteq B \text{ and } A \subseteq C, \text{ then } A \subseteq B \cap C.$$

   (c) What is the relationship between $\mathcal{P}(A \cap B)$ and $\mathcal{P}(A) \cap \mathcal{P}(B)$? Provide a proof.

   (d) Find a set C such that $\mathcal{P}(A \backslash B) = \mathcal{P}(A) \cap \mathcal{P}(C)$. Justify your answer.

3. **Images of sets under a function:**

   (a) Let $g : \mathbb{R} \to \mathbb{R}$ be given by $g(x) = x^2 + 1$ and let $A = (-1, 3]$.
   Derive $g(A)$, that is, the image of A under g.

   (b) Let $A = (-1, 3], B = [-1, 4]$ and $C = (1, 4)$. For the function g in (a) express (without proof) the following images as intervals: $g(B), g(C)$ and $g(A \cap C)$.

   (c) Let $f : X \to Y$ be a function. Consider sets $A \subseteq B \subseteq X$. What can you say about the relationship between $f(A)$ and $f(B)$? Justify your answer.

   (d) Let $f : X \to Y$ be a function. Consider sets $A \subseteq X$ and $C \subseteq X$. Show that $f(A \cap C) \subseteq f(A) \cap f(C)$.

   (e) Find an example of a function $f : X \to Y$ and sets $A \subseteq X$ and $C \subseteq X$ such that $f(A \cap C) \neq f(A) \cap f(C)$.
   Note: you may provide the images $f(A)$, $f(C)$ and $f(A \cap C)$ without formal derivation, but you must explain why your example has the required properties.

4. **Logic:** Raymond Smullyan, a mathematician and logician, is the author of some wonderful books containing logical puzzles. These often feature island inhabitants, some of which always tell the truth and the others always lie. Suppose Jack and Jill are two such inhabitants.

   (a) Consider the statements P and Q, where P is the statement "Jack is telling the truth" and Q is the statement "Jill is telling the truth". Express the following statements in terms of P and Q using appropriate logical connectives.

      (i) None of the two inhabitants is lying.

      (ii) Exactly one of the two inhabitants is lying.

      (iii) Both are lying.

      (iv) At most one of the two inhabitants is lying.

   (b) Jack says "Jill is lying". Jill replies "We are both lying". Who, if anyone, is lying? Explain why.