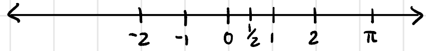
MATRICES AND VECTORS

$\mathbb{R}$ refers to all real numbers



$\in$ is used to indicate membership

ex. $0 \in \mathbb{R}$ → "0 is a real number"

$1 \in \mathbb{R}$ → "1 is a real number"

$-2 \in \mathbb{R}$ → "-2 is a real number"

'hello' $\notin \mathbb{R}$ → "hello is not a real number"

Numbers are also called "**Scalars**"

Scalars inside a vector are called "**coordinates**"

What is a vector?

- A vector in $\mathbb{R}^n$ is a list of n scalars organized vertically into a list

Examples: These are all vectors in $\mathbb{R}^n$

*n refers to the # of scalars in a vector

list of 2 scalars (1 and 2)

$$v = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad w = \begin{bmatrix} 3 \\ -5 \\ -5 \end{bmatrix} \quad x = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad b = \begin{bmatrix} -7/9 \\ 0 \\ \pi \\ 1 \end{bmatrix}$$

Using $\in$ to indicate membership

↳ $v \in \mathbb{R}^2$ $\quad w \in \mathbb{R}^3$ $\quad x, b \in \mathbb{R}^4$ $\quad v \notin \mathbb{R}^5$

m x n matrix → array of scalars with m rows and n columns

$$A = \begin{bmatrix} 13 & 4 & -9 & 1 & 0 & 0 \\ -27 & -2 & 2 & 16 & -21 & -2 \\ -13 & -12 & -2 & 2 & 3 & 19 \\ -5 & 18 & -22 & 2 & -1 & 1 \end{bmatrix}$$

rows → ↓ columns

Ex. This matrix has 4 rows and 6 columns: $A \in \mathbb{R}^{4\times6}$

"A belongs to $\mathbb{R}^{4\times6}$"

(i,j) entry of a matrix is the scalar in the i^{th} row and j^{th} column

Example:

$$A = \begin{bmatrix} 11 & 5 & 0 \\ -6 & -3 & -2 \\ 2 & 1 & 4 \\ -1 & -2 & 3 \\ -2 & -11 & 3 \end{bmatrix}$$

The (4,2) entry of this A is $a_{42} = -2$

Basic arithmetic

Vector arithmetic

Scalar-vector products work coordinatewise

*Take any vector, multiply by any scalar

$$2 \cdot \begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \cdot 3 \\ 2 \cdot 0 \\ 2 \cdot 4 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 8 \end{bmatrix}$$

⇑

$c \cdot v$ for $c \in \mathbb{R}$

scalar · vector

c we used to scale is a scalar

Vector addition also works coordinatewise

*must have same # of coordinates

$$\begin{bmatrix} 2 \\ 3 \\ 7 \end{bmatrix} + \begin{bmatrix} 0 \\ 6 \\ 1 \end{bmatrix} = \begin{bmatrix} 2+0 \\ 3+6 \\ 7+1 \end{bmatrix} = \begin{bmatrix} 2 \\ 9 \\ 8 \end{bmatrix}$$

⇑

$v + w$ for $\underbrace{v, w}_{\text{Both vectors have the same \# of coordinates}} \in \mathbb{R}^n$

<u>Scalar-matrix products</u> work componentwise

$$8\cdot\begin{bmatrix}3&4\\6&0\\2&1\end{bmatrix}=\begin{bmatrix}8\cdot3&8\cdot4\\8\cdot6&8\cdot0\\8\cdot2&8\cdot1\end{bmatrix}=\begin{bmatrix}24&32\\48&0\\16&8\end{bmatrix}$$

so $c\cdot A$ $\quad c\in\mathbb{R}$

we can take any Matrix A and scale by Scalar c

<u>Matrix addition</u> also works component wise

* Matrices must have same # of rows and columns

$$\begin{bmatrix}0&7\\4&2\\9&1\end{bmatrix}+\begin{bmatrix}1&0\\1&2\\8&2\end{bmatrix}=\begin{bmatrix}0+1&7+0\\4+1&2+2\\9+8&1+2\end{bmatrix}=\begin{bmatrix}1&7\\5&4\\17&3\end{bmatrix}$$

$A+B$ $\quad A,B\in\mathbb{R}^{m\times n}$

↑ we can add any matrix A and B as long as they have the same # of rows and columns

WARNING

Only vectors and matrices with the same shape can be summed

$$\cancel{\begin{bmatrix}*\\*\end{bmatrix}+\begin{bmatrix}*\\*\\*\end{bmatrix}} \qquad \cancel{\begin{bmatrix}*&*&*\\*&*&*\end{bmatrix}+\begin{bmatrix}*&*\\*&*\\*&*\end{bmatrix}}$$

Be careful when writing $v+w$ or $A+B$

↑ you are assuming that they have the same shape

Basic operations

<u>Transposition:</u> The <u>transpose A^T</u> is formed by interchanging the rows and columns of A

$$A=\underbrace{\begin{bmatrix}10&17&19\\1&-10&11\\-15&-3&2\\9&18&-17\end{bmatrix}}_{4\times3} \qquad A^T=\underbrace{\begin{bmatrix}10&1&-15&9\\17&-10&-3&18\\19&11&2&-17\end{bmatrix}}_{3\times4}$$

* 1st row becomes 1st column, etc.

The (i,j) entry of A^T is $\underline{a_{ji}}$

*When you transpose, you change the size of your matrix

A^T is $n\times m$ if A is $m\times n$

Vertical space is saved by writing vectors as transposes of $\mathbf{1\times n}$ matrices

$$v=\begin{bmatrix}9\\-3\\4\\7\end{bmatrix}=\begin{bmatrix}9&-3&4&7\end{bmatrix}^T$$

← Called "**Horizontal Notation**"

SAME THING

Transposition is a linear operation which means...

$$(c_1 \cdot A_1 + c_2 \cdot A_2)^T = c_1 \cdot A_1^T + c_2 A_2^T$$ } useful when working w/ symbolic expressions

Multiplying two matrices by 2 scalars, adding, then transposing

You can distribute transpose sign, then scale, then add together
You can distribute transposes

Transposition is also an involution } involuted operation: If you do the thing twice, you get back the original thing

↳ $(A^T)^T = A$

Symmetric matrix: Matrix S is symmetric if $S^T = S$

$$S = \begin{bmatrix} -2 & 17 & -9 & -1 \\ 17 & 14 & -5 & 30 \\ -9 & -5 & 26 & 7 \\ -1 & 30 & 7 & -22 \end{bmatrix} = S^T = \begin{bmatrix} -2 & 17 & -9 & -1 \\ 17 & 14 & -5 & 30 \\ -9 & -5 & 26 & 7 \\ -1 & 30 & 7 & -22 \end{bmatrix}$$

"Diagonal" of S

"Symmetry across the diagonal"

Diagonal: All of the entries in the (i, i) position

Trace of an $n \times m$ matrix is the sum of its diagonal

$$\text{trace} \begin{bmatrix} -12 & 5 & 3 & 8 \\ -5 & -19 & -6 & 5 \\ 18 & -8 & 14 & 0 \\ 12 & 10 & -7 & 6 \end{bmatrix} = (-12) + (-19) + (14) + (6) = -11$$

* Must be a square *

trace(A) is a scalar

Trace is a linear operation → $\text{trace}(c_1 \cdot A_1 + c_2 \cdot A_2) = c_1 \cdot \text{trace}(A_1) + c_2 \cdot \text{trace}(A_2)$

↳ can be distributed

Example:

trace(A) = -3   trace(B) = 5

then $\text{trace}(6 \cdot A - B) = 6 \cdot \text{trace}(A) - \text{trace}(B) = 6 \cdot (-3) - 5 = -23$

ADJECTIVES

Square matrices: An $m \times n$ matrix is square if $m=n$

Ex:

$$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \qquad \begin{bmatrix} * & * \\ * & * \end{bmatrix}$$

Diagonal: refers to collection of (i,i) entries

Ex.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \\ b_{41} & b_{42} & b_{43} \end{bmatrix}$$

Triangular matrices:

Upper triangle: If every entry below the diagonal is zero

Ex.

$$\begin{bmatrix} 1 & 3 & 5 \\ 0 & 0 & 1 \\ 0 & 0 & 9 \end{bmatrix} \qquad \begin{bmatrix} 0 & 7 & 4 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 2 & 8 \\ 0 & 6 & 0 & 4 \\ \underline{9} & 0 & 9 & 3 \end{bmatrix}$$

Not upper triangular

Equivalent to → $a_{ij} = 0$ for $i > j$

Lower triangle: If every entry above the diagonal is zero

Ex.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 \\ 9 & 4 & 2 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & 0 & 0 \\ 7 & 2 & 0 & \underline{5} \\ 3 & 4 & 1 & 0 \\ 9 & 8 & 0 & 0 \\ 0 & 3 & 1 & 2 \end{bmatrix}$$

← Not lower triangular

Equivalent to → $a_{ij} = 0$ for $i < j$

A matrix is diagonal if every nondiagonal entry is zero

Ex.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 7 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 4 & 0 \end{bmatrix}$$

Diagonal matrix

Diagonal matrix

Not a diagonal matrix

* upper triangle matrix

Diagonal matrices are both upper and lower triangle

Equivalent to $a_{ij} = 0$ for $i \neq j$

$\underline{\text{diag}(d_1, \ldots d_n)}$ is the $n \times n$ diagonal matrix with diagonal $d_1, \ldots d_n$

$$\text{diag}(4, 0, -7, 12, 5) \rightarrow \begin{bmatrix} 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -7 & 0 & 0 \\ 0 & 0 & 0 & 12 & 0 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 4 & & & & \\ & 0 & & & \\ & & -7 & & \\ & & & 12 & \\ & & & & 5 \end{bmatrix}$$

5x5 matrix

<u>Blank entries</u> are understood to equal zero

<u>Zero vector and zero matrix</u> are full of zeroes

$$\mathbf{0}_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \mathbf{0}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{0}_{3\times 2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \mathbf{0}_{4\times 4} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

If the context is clear, we omit subscripts and just write $\mathbf{0}$

<u>Identity matrices</u>: $n \times n$ identity matrix I_n has ones on diagonal and zeroes elsewhere

$$I_2 = \begin{array}{c} \begin{array}{cc} e_1 & e_2 \end{array} \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{array} \qquad I_3 = \begin{array}{c} \begin{array}{ccc} e_1 & e_2 & e_3 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array} \qquad I_4 = \begin{array}{c} \begin{array}{cccc} e_1 & e_2 & e_3 & e_4 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

<u>standard basis</u> $\{e_1, \ldots e_n\}$ of $\mathbb{R}^n$ are the columns of I_n

$n \leftarrow$ number of standard basis vectors

Rank One Matrices

A matrix $A \neq \mathbf{0}$ is rank one if every column is a multiple of one column

$$\begin{array}{cc} & \begin{array}{ccc} \text{Col}_1 & & -1\cdot\text{Col}_1 \end{array} \\ A = & \begin{bmatrix} 2 & 6 & -2 \\ 5 & 15 & -5 \\ 3 & 9 & -3 \\ 6 & 18 & -6 \end{bmatrix} \\ & \begin{array}{ccc} & 3\cdot\text{Col}_1 & \end{array} \end{array} \qquad \begin{array}{cc} & \begin{array}{ccc} & \text{Col}_2 & \neq c\cdot\text{Col}_2\text{, so} \end{array} \\ B = & \begin{bmatrix} 0 & 7 & -3 \\ 0 & 14 & 0 \\ 0 & 1 & 5 \\ 0 & 5 & 8 \end{bmatrix} \\ & \begin{array}{ccc} 0\cdot\text{Col}_2 & & \end{array} \end{array}$$

Not a rank one matrix

Rank one matrix

MATRIX-VECTOR PRODUCTS

Linear combinations

A linear combination of $v_1, v_2, \dots v_m \in \mathbb{R}^n$ ("n" scalars) is any expression of the form:

$$c_1 \cdot v_1 + c_2 \cdot v_2 + \dots + c_m \cdot v_m$$

scalars (called "weights")

Example of a linear combination

$$\underset{c_1}{-3} \cdot \underset{v_1}{\begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}} + \underset{c_2}{6} \cdot \underset{v_2}{\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}} + \underset{c_3}{4} \cdot \underset{v_3}{\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}} + \underset{c_4}{2} \cdot \underset{v_4}{\begin{bmatrix} 6 \\ 0 \\ -3 \end{bmatrix}} = \underbrace{\begin{bmatrix} 24 \\ 7 \\ -16 \end{bmatrix}}_{\text{vector "b"}}$$

b is a linear combination of $v_1, v_2, v_3, v_4 \in \mathbb{R}^3$

Generic combinations of $v_1 = [2 \ 0 \ 3]^T$ and $v_2 = [-1 \ 0 \ 5]^T$

$$\hookrightarrow c_1 \cdot \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} + c_2 \cdot \begin{bmatrix} -1 \\ 0 \\ 5 \end{bmatrix} = \begin{bmatrix} 2c_1 - c_2 \\ 0 \\ 3c_1 + 5c_2 \end{bmatrix}$$

← No matter how many linear combinations, the value will always stay 0

Ex. $[2 \ 1 \ 3]^T$ cannot be a combination because the center is not 0

Linear combinations are important but clumsy ... so:

Matrix-vector products encode data into a single expression: Av

Matrix-vector product Av of $A \in \mathbb{R}^{m \times n}$ and $v \in \mathbb{R}^n$

* # of columns in A must equal the # of coordinates in v

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = c_1 \cdot \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{bmatrix} + c_2 \cdot \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \dots + c_n \cdot \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

n # of columns; n coordinates

→ Av is the combination of columns of A using the weights in v

# of coordinates of Av is "m" ($Av \in \mathbb{R}^m$)

Example: 4×3 matrix A can be multiplied by any vector $v \in \mathbb{R}^3$

$$\underbrace{\overset{A}{\begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & 4 \\ 2 & 1 & 0 \\ 4 & 5 & 1 \end{bmatrix}}}_{\text{3 columns}} \underbrace{\overset{v}{\begin{bmatrix} -2 \\ 3 \\ 5 \end{bmatrix}}}_{\text{3 Coordinates}} = -2\cdot\begin{bmatrix} 1 \\ 0 \\ 2 \\ 4 \end{bmatrix} + 3\cdot\begin{bmatrix} 0 \\ 3 \\ 1 \\ 5 \end{bmatrix} + 5\cdot\begin{bmatrix} 2 \\ 4 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ -4 \\ -8 \end{bmatrix} + \begin{bmatrix} 0 \\ 9 \\ 3 \\ 15 \end{bmatrix} + \begin{bmatrix} 10 \\ 20 \\ 0 \\ 5 \end{bmatrix} = \begin{bmatrix} 8 \\ 29 \\ -1 \\ 12 \end{bmatrix}$$

$\leftarrow Av \in \mathbb{R}^4$

4 coordinates = # of rows in A

Example: Converting linear combinations into matrix-vector products

$$-5\cdot\begin{bmatrix} 4 \\ 2 \\ 3 \\ 0 \\ 9 \end{bmatrix} + 3\cdot\begin{bmatrix} 1 \\ 6 \\ 4 \\ 2 \\ 0 \end{bmatrix} + 8\cdot\begin{bmatrix} 0 \\ 0 \\ 1 \\ 3 \\ 1 \end{bmatrix} + 5\cdot\begin{bmatrix} 1 \\ 0 \\ 1 \\ 7 \\ 5 \end{bmatrix} = \begin{bmatrix} 4 & 1 & 0 & 1 \\ 2 & 6 & 0 & 0 \\ 3 & 4 & 1 & 1 \\ 0 & 2 & 3 & 7 \\ 9 & 0 & 1 & 5 \end{bmatrix} \begin{bmatrix} -5 \\ 3 \\ 8 \\ 5 \end{bmatrix}$$

converted

$c_1 \cdot v_1 + \ldots + c_4 \cdot v_4$

↳ $Av \in \mathbb{R}^5$

Example: Consider this matrix-vector product

$$\overset{A}{\begin{bmatrix} | & | & | & | \\ Col_1 & Col_2 & Col_3 & Col_4 \\ | & | & | & | \end{bmatrix}} \overset{v}{\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}} = Col_1 - Col_4$$

$c_1 = 1$ $\quad \underbrace{c_2 = 0 \quad c_3 = 0}_{\text{irrelevant}}$ $\quad c_4 = -1$

* Because # of Coordinates in v is 4, # of Columns in A also must be 4

* warning: Av only makes sense if $A \in \mathbb{R}^{m\times n}$ and $v \in \mathbb{R}^n$

$$\cancel{\begin{bmatrix} * & * \\ * & * \\ * & * \\ * & * \end{bmatrix} \begin{bmatrix} * \\ * \\ * \end{bmatrix}} \qquad \cancel{\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \begin{bmatrix} * \\ * \end{bmatrix}}$$

Matrix-vector multiplication respects the linearity, identity, and zero rules

Linearity rule:
$A(c_1 \cdot v_1 + c_2 \cdot v_2) = c_1 \cdot Av_1 + c_2 \cdot Av_2$

Identity rule
$I_n v = v$

Zero rule
$A0_n = 0_m$

Think in terms of inputs and outputs : $Av = b$ inspires arrow notation

$A \in \mathbb{R}^{m \times n}$ "input" $v \in \mathbb{R}^n$ "output" $b \in \mathbb{R}^m$

Arrow notation

$\mathbb{R}^n \xrightarrow{A} \mathbb{R}^m$

* view matrix "A" as a process that accepts vectors from $\mathbb{R}^n$ and produces vectors from $\mathbb{R}^m$

Saying that:
A is mxn

Eigenvectors

If A is square, then both v and Av are vectors in $\mathbb{R}^n$

$$\underbrace{\begin{bmatrix} 3 & 3 & 3 \\ -8 & -2 & -4 \\ 2 & -4 & -2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}}_{w} = \underbrace{\begin{bmatrix} 0 \\ -6 \\ 6 \end{bmatrix}}_{Aw} \leftarrow Col_1 - Col_2$$

$Aw \neq \lambda \cdot w$

Not a scalar multiple of w

$$\underbrace{\begin{bmatrix} 3 & 3 & 3 \\ -8 & -2 & -4 \\ 2 & -4 & -2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}}_{v} = \underbrace{\begin{bmatrix} 0 \\ 2 \\ -2 \end{bmatrix}}_{Av} \leftarrow Col_2 - Col_3$$

*Notice a pattern

$Av = 2 \cdot v$

We call v an eigenvector of A with associated eigenvalue $\lambda = 2 \rightarrow v \in \mathcal{E}_A(2)$ but $w \notin \mathcal{E}_A(\lambda)$

No scalar λ makes $Aw = \lambda w$ work

Definition:

We call $v \neq 0$ an eigenvector of A with associated eigenvalue λ if:

Eigenvalue equation

$\boxed{Av = \lambda \cdot v}$ ← communicated as $\underbrace{v \in \mathcal{E}_A(\lambda)}_{\text{Declares that } Av = \lambda \cdot v}$