1. $z=f(x,y)$

$f_{xx}$, $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$, $f''_{11}$

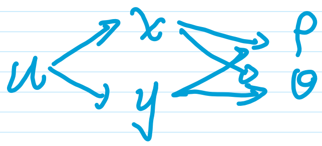
例. 证明: $\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2=\left(\frac{\partial u}{\partial \rho}\right)^2+\frac{1}{\rho^2}\left(\frac{\partial u}{\partial \theta}\right)^2$

$u=u(x,y)$ ✓

$$\begin{cases} x=\rho\cos\theta \\ y=\rho\sin\theta \end{cases}$$

$\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$

$u=u(\rho\cos\theta,\rho\sin\theta)$ ✓




$\frac{\partial u}{\partial \rho}$. $\frac{\partial u}{\partial \theta}$

$u''_{11}=\frac{\partial^2 u(s,t)}{\partial s^2}$

$u''_{22}=\frac{\partial^2 u(s,t)}{\partial t^2}$

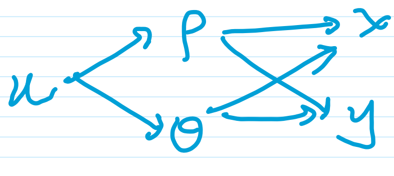
证明: $\frac{\partial u}{\partial \rho}=u'_1\cdot\cos\theta+u'_2\cdot\sin\theta.$

$\frac{\partial u}{\partial \theta}=u'_1\cdot\rho\cdot(-\sin\theta)+u'_2\cdot\rho\cdot\cos\theta.$

右 $=\left(\frac{\partial u}{\partial \rho}\right)^2+\left(\frac{\partial u}{\partial \theta}\right)^2\frac{1}{\rho^2}=u''_{11}+u''_{22}=$ 左.

例: $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{\partial^2 u}{\partial \rho^2}+\frac{1}{\rho}\frac{\partial u}{\partial \rho}+\frac{1}{\rho^2}\frac{\partial^2 u}{\partial \theta^2}.$

解: $u=u(\rho\cos\theta,\rho\sin\theta)$




$u=u(\rho(x,y),\theta(x,y))$

$\frac{\partial u}{\partial \rho}=u'_1\cos\theta+u'_2\sin\theta$

$$\frac{\partial u}{\partial \rho} = u_1'\cos\theta + u_2'\sin\theta$$

$u_1' = u_1'(\rho\cos\theta, \rho\sin\theta)$

$$\frac{\partial u}{\partial \theta} = -u_1'\rho\sin\theta + u_2'\rho\cos\theta$$

$$\frac{\partial^2 u}{\partial \rho^2} = \frac{\partial}{\partial \rho}\left(\frac{\partial u}{\partial \rho}\right) = \frac{\partial}{\partial \rho}(u_1'\cos\theta + u_2'\sin\theta) = \cos\theta\frac{\partial u_1'}{\partial \rho} + \sin\theta\frac{\partial u_2'}{\partial \rho}$$

$$= \cos\theta(u_{11}''\cos\theta + u_{12}''\sin\theta) + \cos\theta(u_{21}''\cos\theta + u_{22}''\sin\theta)$$

$u_{12}'' = u_{21}''$

$= \cdots$

$$\frac{\partial^2 u}{\partial \theta^2} = \frac{\partial}{\partial \theta}(-u_1'\rho\sin\theta + u_2'\rho\cos\theta)$$

$u \to u_1'$

$u \to u_2'$

$$= \rho\left[\frac{\partial u_2'}{\partial \theta}\cdot\cos\theta + u_2'\cdot(-\sin\theta) - \frac{\partial u_1'}{\partial \theta}\sin\theta - u_1'\cdot\cos\theta\right]$$

$$= \rho\cos\theta(-u_{21}''\rho\sin\theta + u_{22}''\rho\cos\theta) - \rho\sin\theta(u_{11}''\rho\sin\theta + u_{12}''\rho\cos\theta)$$

$$- \rho(\underbrace{u_1'\cos\theta + u_2'\sin\theta}_{\frac{\partial u}{\partial \rho}})$$

$g \to w \begin{cases} \to x \\ \to y \end{cases}$

例. $z = f(xy, x^2+y^2) + g(xy^2)$, 则 $\frac{\partial^2 z}{\partial x^2}$

解. $$\frac{\partial z}{\partial x} = f_1'\cdot y + f_2'\cdot 2x + g'\cdot y^2.$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}(f_1'y + f_2'\cdot 2x + g'\cdot y^2)$$

$$= y\frac{\partial f_1'}{\partial x} + 2\left(\frac{\partial f_2'}{\partial x}\cdot x + f_2'\cdot 1\right) + y^2\cdot\frac{\partial g'}{\partial x}$$

$$= y(f_{11}''\cdot y + f_{12}''\cdot 2x) + 2x(f_{21}''y + f_{22}''\cdot 2x)$$

$$= y\left(f''_{11}\cdot y + f''_{12}\cdot 2x\right) + 2x\left(f''_{21} y + f''_{22}\cdot 2x\right)$$

$$+2f_2' + y^2\cdot(g''\cdot y^2)$$

特值法：$f(x,y) = ax+by-a-b+1$

例 $z=f(x,y)$ $\boxed{f(1,1)=1.}$ $f_x(1,1)=a$、$f_y(1,1)=b$

$\varphi(x) = f\{x, f[x, f(x,x)]\}$ 则 $\varphi'(1) = \_\_\_\_$

$\varphi'(1) = (a+b[a+b(a+b)])b + a$

解：$\varphi'(x) = f_1'\cdot 1 + f_2'\cdot \frac{d}{dx} f[x, f(x,x)]$

$$= f_1' + f_2'\cdot\left(f_1'\cdot 1 + f_2'\cdot\frac{d f(x,x)}{dx}\right)$$

$$= f_1' + f_2'\left(f_1' + f_2'(f_1'\cdot 1 + f_2'\cdot 1)\right).$$

故 $\varphi'(1) = a+b[a+b(a+b)]$

$f_1'\{x, f[x, f(x,x)]\}$ $\xrightarrow{x=1}$ $f_1'\{1, f(1, f(1,1))\} = f_1'(1,1).$

$f_1'(x,x)$ $\xrightarrow{x=1}$ $= f_1'(1,1)$

例. $u=u(x,y)$ 二阶偏导连续 $u_{xy}=u_{yx}$

$u_{xx}=u_{yy}$. $u(x,2x)=x$, $u_x(x,2x)=x^2$.

求 $u_{xx}(x,2x)$. $u_{xy}(x,2x)$

例 $u(x,y) = -x+y$

解： ... $=x$ $u(x,2x)=x$

解： $u(x,y)|_{y=2x}=x$ 　　$u(x,2x)=x$

$u_x(x,y)|_{y=2x}=x^2=\frac{\partial u(s,t)}{\partial s}\Big|_{(x,2x)}$

例 $u(x,y)=\frac{1}{3}x^3+y$

对 $u(x,2x)=x$ 两边关于 $x$ 求导

$u_1'\cdot 1+u_2'\cdot 2=1$ ✓

$\underbrace{u_1'(x,2x)}_{=u_x(x,2x)}+2\underbrace{u_2'(x,2x)}_{=u_y(x,2x)}=1.$

对 $u_2'=\frac{1-x^2}{2}$ 两边关于 $x$ 求导

$\underbrace{u_{21}''}_{=u_{yx}(x,2x)=u_{xy}(x,2x)}\cdot 1+\underbrace{u_{22}''}_{=u_{yy}(x,2x)=u_{xx}(x,2x)}\cdot 2.=-x$ 　　$u_{12}''+2u_{11}''=-x$ ①

对 $u_x(x,2x)=x^2$ 两边求导

$u_{11}''\cdot 1+u_{12}''\cdot 2.=2x$ ②

故得 $u_{11}''=u_{xx}(x,2x)$ 　　$u_{12}''=u_{xy}(x,2x)$

2.

一阶全微分形式不变性：基于复合函数链式法则

一元函数：

一元函数：

$y=f(u)$ $u$自 $dy=f'(u)du$

$y=f(u)$ $u=\varphi(x)$ $u$中. $dy=[f[\varphi(x)]]'dx$

$=f'(u)\varphi'(x)dx=f'(u)du$

二元函数

$z=f(u,v)$ $u,v$自. $dz=\frac{\partial f}{\partial u}du+\frac{\partial f}{\partial v}dv.$

$z=f(u,v)$. $u=\varphi(x,y)$ $v=\psi(x,y)$ $u,v$中.

$dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy.$ $z=f[\varphi(x,y),\psi(x,y)]$

$=(f_1'\cdot\frac{\partial\varphi}{\partial x}+f_2'\cdot\frac{\partial\psi}{\partial x})dx+(f_1'\cdot\frac{\partial\varphi}{\partial y}+f_2'\cdot\frac{\partial\psi}{\partial y})dy$

$=f_1'(\frac{\partial\varphi}{\partial x}dx+\frac{\partial\varphi}{\partial y}dy)+f_2'(\frac{\partial\psi}{\partial x}dx+\frac{\partial\psi}{\partial y}dy)$

$=\frac{\partial f}{\partial u}du+\frac{\partial f}{\partial v}dv.$

例. $z=\frac{\cos(x^2y)}{e^{2x+y}}$ 则 $dz=$____________.

解.

$dz=d\frac{\boxed{\cos(x^2y)}}{\boxed{e^{2x+y}}}$

$d\frac{u}{v}=\frac{v\,du-u\,dv}{v^2}$

$(\frac{u}{v})'=\frac{u'v-u\cdot v'}{v^2}.$

$=\frac{e^{2x+y}d\cos\boxed{(x^2y)}-\cos(x^2y)\,d\,e^{\boxed{2x+y}}}{(e^{2x+y})^2}$

$$= \frac{e^{2x+y}\cdot a\cos(xy)\cdots}{(e^{2x+y})^2}$$

$$= \frac{e^{2x+y}\cdot(-\sin(x^2y))d(x^2y) - \cos(x^2y)\cdot e^{2x+y}d(2x+y)}{(e^{2x+y})^2}$$

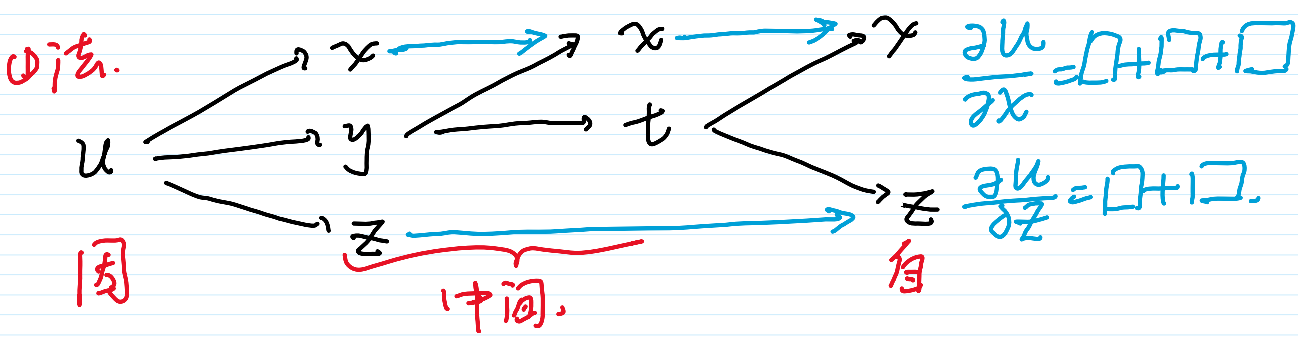
$$= \frac{-\sin(x^2y)[y\cdot 2x\,dx + x^2\,dy] - \cos(x^2y)\cdot(2dx+dy)}{e^{2x+y}}$$

$$= \left[\ \frac{\partial z}{\partial x}\ \right]dx + \left[\ \frac{\partial z}{\partial y}\ \right]dy$$

例：$u=f(x,y,z)$，$y=\varphi(x,t)$，$t=\psi(x,z)$.

求 $\frac{\partial u}{\partial x}$，$\frac{\partial u}{\partial z}$，$du$.

解：①法.




$$u=f[x,\ \varphi[x,\ \psi(x,z)],\ z]$$

$$\frac{\partial u}{\partial x} = f_1'\cdot 1 + f_2'[\varphi_1'\cdot 1 + \varphi_2'\cdot(\psi_1'\cdot 1+\psi_2'\cdot 0)] + f_3'\cdot 0$$

$$\frac{\partial u}{\partial z} = f_1'\cdot 0 + f_2'[\varphi_1'\cdot 0 + \varphi_2'\cdot[\psi_1'\cdot 0 + \psi_1'\cdot 1]] + f_3'\cdot 1$$

②法：微分法.

$$du = df(x,y,z) = f_1'\cdot dx + f_2'\cdot dy + f_3'\cdot dz$$

$$du = df(x,y,z) = f_1' \cdot dx + f_2' \cdot dy + f_3' \cdot dz$$

$$dy = d\varphi(x,t) = \varphi_1' \cdot dx + \varphi_2' \cdot dt$$

$$dt = d\psi(x,z) = \psi_1' \, dx + \psi_2' \cdot dz$$

$$du = (\quad \frac{\partial u}{\partial x} \quad) dx + (\quad \frac{\partial u}{\partial z} \quad) dz$$

## 第五节 隐函数求导

例. $\sin y + y - e^x = 1$. 确定 $y = y(x)$.

则 $y' =$ ______.

解: 两边关于 $x$ 求导 $\quad \cos y \cdot y' + y' - e^x = 0$

$$y' = \frac{e^x}{1+\cos y}$$

定理: ① $F(x,y) = 0$

② 在 $U(P_0)$ 偏导连续. $F(x_0,y_0) = 0$

③ $F_y(x_0,y_0) \neq 0$. $F_x(x_0,y_0) \neq 0$ 偏导连续 $\Rightarrow F_y(x,y) \neq 0$.

$x = g(y)$

则可确定一隐函数 $y = f(x)$. 唯一.连续.可导.

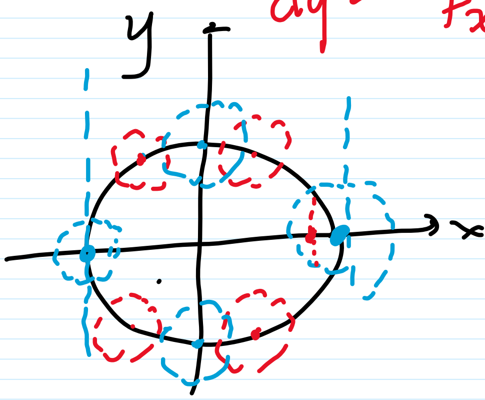
$y_0 = f(x_0)$. 且. $\boxed{\frac{dy}{dx} = -\frac{F_x(x,y)}{F_y(x,y)}}$ 公式法

$\frac{dx}{dy} = -\frac{F_y}{}$

$$\frac{dx}{dy}=-\frac{F_y}{F_x}.$$

例. $x^2+y^2-1=0.$

$\Rightarrow y=\pm\sqrt{1-x^2}.$




$$\frac{dy}{dx}=\frac{d\sqrt{1-x^2}}{dx}=\frac{-x}{\sqrt{1-x^2}}=-\frac{x}{y}$$

$$\frac{dy}{dx}=\frac{d\left(-\sqrt{1-x^2}\right)}{dx}=\frac{x}{\sqrt{1-x^2}}=-\frac{x}{y}$$

$F(x,y)=x^2+y^2-1.$ $F_x=2x$ $F_y=2y$ $\frac{dy}{dx}=-\frac{x}{y}$

直接法

对 $F(x,\ f(x))=0$ 两边关于 $x$ 求导

$F_1'\cdot 1+F_2'\cdot f'(x)=0$

$$f'(x)=\frac{dy}{dx}=-\frac{F_1'}{F_2'}=-\frac{F_x}{F_y}.$$

定理: ① $F(x,y,z)=0$

② $U(P_0)$ 下偏导连续 $F(x_0,y_0,z_0)=0$

③ $F_z(x_0,y_0,z_0)\neq 0.$

$$\frac{dy}{dx}=-\frac{F_x}{F_y}$$

则确定一隐函数 $z=f(x,y)$ 且.

公式法

$$\boxed{\frac{\partial z}{\partial x}=-\frac{F_x}{F_z}\qquad \frac{\partial z}{\partial y}=-\frac{F_y}{F_z}.}$$

$\frac{\partial z}{\partial x} \to F_z$ ... $F_z$.

对 $F(x, y, z(x,y))=0$ 两边关于 $x$、$(y)$ 求偏导

关于 $x$：$F_1' \cdot 1 + F_2' \cdot 0 + F_3' \frac{\partial z}{\partial x} = 0 \Rightarrow \cdots$

关于 $y$：$F_1' \cdot 0 + F_2' \cdot 1 + F_3' \cdot \frac{\partial z}{\partial y} = 0$

例. $x^3y - e^z = z$ 求 $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ 则 $dz=$ ______ $dz|_{(1,1)}=$ ______

解：①公式法：

设 $\underbrace{F(x,y,z)}_{\text{三元函数}} = x^3y - e^z - z = 0$

$F_x = 3x^2y$，$F_y = x^3$，$F_z = -e^z - 1$.

$\frac{\partial z}{\partial x} = -\frac{3x^2y}{-e^z-1} = \frac{3x^2y}{1+e^z}$，$\frac{\partial z}{\partial y} = \frac{x^3}{1+e^z}$

✗ $\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{3x^2y - e^z \cdot \frac{\partial z}{\partial x} - \frac{\partial z}{\partial x}}{-e^z - 1} \Rightarrow \frac{\partial z}{\partial x}$

②直接法：对 $x^3y - e^z = z$ 关于 $x$ 求偏导 是 $x, y$ 函数

$3x^2y - e^z \cdot \frac{\partial z}{\partial x} = \frac{\partial z}{\partial x} \Rightarrow \frac{\partial z}{\partial x} = \cdots$

③微分法：

对 $x^3y - e^z = z$ 两边微分

$d(x^3y - e^z) = dz$

$$d(x^3y-e^z)=dz$$

$$d(x^3y)-de^z=dz$$

$$3x^2y\,dx+x^3dy-e^z dz=dz$$

得 $$dz=\underbrace{\frac{3x^2y}{1+e^z}}_{=\frac{\partial z}{\partial x}}dx+\underbrace{\frac{x^3}{1+e^z}}_{=\frac{\partial z}{\partial y}}dy$$

$$dz\big|_{(1,1)}=\frac{3\times1^2\times1}{1+e^0}dx+\frac{1^3}{1+e^0}dy$$

$$=\frac{3}{2}dx+\frac{1}{2}dy$$

$$\frac{\partial^2 z(x,y)}{\partial x\partial y}=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial}{\partial y}\left(\frac{3x^2y}{1+e^z}\right)$$ → x, y的函数

$$=3x^2\frac{1\cdot(1+e^z)-y\cdot(0+e^z\frac{\partial z}{\partial y})}{(1+e^z)^2}$$

例. $z=z(x,y)$ 由 $\varphi(x-az,y-bz)=0$ 确定隐函数

则 $a\cdot\frac{\partial z}{\partial x}+b\frac{\partial z}{\partial y}=$ ___1___

解: ①公式法:

$$F(x,y,z)=\varphi(x-az,y-bz)$$

$$F_x=\varphi_1'\cdot(1-0)+\varphi_2'\cdot(0-0)=\varphi_1'$$

$$F_y=\varphi_1'\cdot(0-0)+\varphi_2'\cdot(1-0)=\varphi_2'$$

$F_x = \cdots$

$F_y = \varphi_1' \cdot (0-0) + \varphi_2' \cdot (1-0) = \varphi_2'$

$F_z = \varphi_1' \cdot (0-a) + \varphi_2' (0-b) = -a\varphi_1' - b\varphi_2'$

故 $\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = \frac{\varphi_1'}{a\varphi_1'+b\varphi_2'}$ $\Rightarrow a\frac{\partial z}{\partial x}+b\frac{\partial z}{\partial y} = 1$

$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = \frac{\varphi_2'}{a\varphi_1'+b\varphi_2'}$.

③ 微分法：$d\varphi(\boxed{x-az}, \boxed{y-bz}) = 0$

$\varphi_1' \cdot d(x-az) + \varphi_2' \cdot d(y-bz) = 0$

$\varphi_1' dx - a\varphi_1' dz + \varphi_2' dy - b\varphi_2' dz = 0$

$dz = \frac{\varphi_1'}{a\varphi_1'+b\varphi_2'} dx + \frac{\varphi_2'}{a\varphi_1'+b\varphi_2'} dy.$

$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\varphi_1'}{a\varphi_1'+b\varphi_2'}\right)$

$\varphi_1' = \varphi_1'(x-az, y-bz)$

$\varphi_2' = \varphi_2'(x-az, y-bz)$

$= \frac{\frac{\partial \varphi_1'}{\partial x}\cdot(a\varphi_1'+b\varphi_2') - \varphi_1' \frac{\partial}{\partial x}(a\varphi_1'+b\varphi_2')}{(a\varphi_1'+b\varphi_2')^2}$

$\frac{\partial \varphi_1'}{\partial x} = \varphi_{11}'' \cdot (1 - a\cdot\frac{\partial z}{\partial x}) + \varphi_{12}''(0 - b\frac{\partial z}{\partial x})$

$\frac{\partial \varphi_2'}{\partial x} = \varphi_{21}''(1 - a\frac{\partial z}{\partial x}) + \varphi_{22}''(0 - b\frac{\partial z}{\partial x})$

$$\frac{\partial \varphi_2'}{\partial x} = \varphi_{21}''\left(1-u\frac{\partial z}{\partial x}\right)+\varphi_{22}''\left(0-u\frac{\partial z}{\partial x}\right)$$

方程组　　　　　　　　　　隐函数组

$$\begin{cases} F(x,y,u,v)=0 \\ G(x,y,u,v)=0 \end{cases} \longrightarrow \begin{cases} u=u(x,y) \\ v=v(x,y) \end{cases}$$

定理：① $F(x,y,u,v)=0$ 　$G(x,y,u,v)=0$

② $U(P_0)$ 偏导连续　$F(x_0,y_0,u_0,v_0)=0$
$G(x_0,y_0,u_0,v_0)=0$

③

$$J=\left.\frac{\partial(F,G)}{\partial(u,v)}\right|_{P_0}\neq 0.$$

其中 $\frac{\partial(F,G)}{\partial(u,v)}=\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}$

则确定一隐函数组 $\begin{cases} u=u(x,y) \\ v=v(x,y) \end{cases}$