

1. $f(x) = f(x+2l)$. $T = 2l$. $l = \frac{T}{2}$ $\int_0^T f(x) dx = \int_a^{a+T} f(x) dx$

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l})$

$x \in (-\infty, +\infty)$ 且 x 非间断点

其中 $a_n = \frac{1}{l} \int_{-l}^l f(x) \cos \frac{n\pi x}{l} dx$ ($n=0, 1, 2, \dots$)

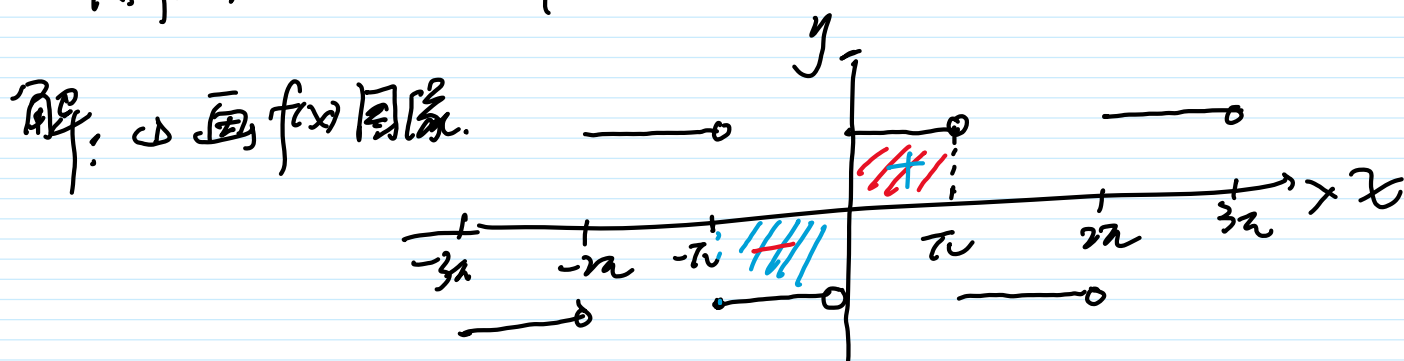
$b_n = \frac{1}{l} \int_{-l}^l f(x) \sin \frac{n\pi x}{l} dx$ ($n=1, 2, \dots$)

$= \frac{1}{l} \int_0^{2l} f(x) \sin \frac{n\pi x}{l} dx$

2. $S(x) = \begin{cases} f(x) & , x \text{ 连续点} \\ \frac{f(x^-) + f(x^+)}{2} & , x \text{ 间断点} \end{cases}$

例 $f(x) = f(x+2\pi)$ $f(x) = \begin{cases} -1, & -\pi \leq x < 0 \\ 1, & 0 \leq x < \pi \end{cases}$
 $T = 2\pi$.

将 $f(x)$ 展成傅里叶级数。



② 间断点 $x = k\pi$ ($k \in \mathbb{Z}$)

③ $f(x) = \dots$, $x \in \mathbb{R}$ 且 $x \neq k\pi$ ($k \in \mathbb{Z}$)

(3) $f(x) = \dots$, $x \in \mathbb{R}$

$$T = \pi. \quad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \underbrace{f(x)}_{\sim \text{奇}} \cos nx \, dx$$

$$= \frac{1}{\pi} \left(\int_{-\pi}^0 (-1) \cos nx \, dx + \int_0^{\pi} \cos nx \, dx \right)$$

$$= 0. \quad (n = 0, 1, 2, \dots)$$

$$b_n = \frac{2}{\pi} \int_0^{\pi} 1 \cdot \sin nx \, dx$$

$$= -\frac{2}{n\pi} \int_0^{\pi} d \cos nx = -\frac{2}{n\pi} [\cos n\pi - 1]$$

故. $f(x) = \sum_{n=1}^{\infty} -\frac{2}{n\pi} (\cos n\pi - 1) \sin nx, \quad (x \neq k\pi, k \in \mathbb{Z})$

$$\cos n\pi = (-1)^n. \quad b_n = \frac{2}{n\pi} (1 - (-1)^n)$$

$$= \begin{cases} 0 & , n = 2k \\ \frac{4}{(2k-1)\pi} & , n = 2k-1 \end{cases}$$

$$f(x) = \sum_{n=1}^{\infty} \frac{4}{(2n-1)\pi} \sin(2n-1)x, \quad x \neq k\pi.$$

定义: 正弦级数. $\sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$,

余弦级数. $\frac{a_0}{2} \cdot 1 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}$.

$\underbrace{f(-x) = -f(x)}_{\text{奇}} \leftarrow \sin \frac{n\pi x}{l} \leftarrow \text{正弦级数}$

$\underbrace{f(x) = f(x)}_{\text{偶}} \leftarrow \cos \frac{n\pi x}{l} \leftarrow \text{余弦级数}$

$$f(-x) = f(x) \text{ 偶} \leftarrow \cos \frac{n\pi x}{l} \leftarrow \text{余弦级数}$$

推论: $f(x) = f(x+2l), f(-x) = -f(x).$

$$\Rightarrow a_n = 0, (n=0, 1, 2, \dots)$$

$$b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} dx.$$

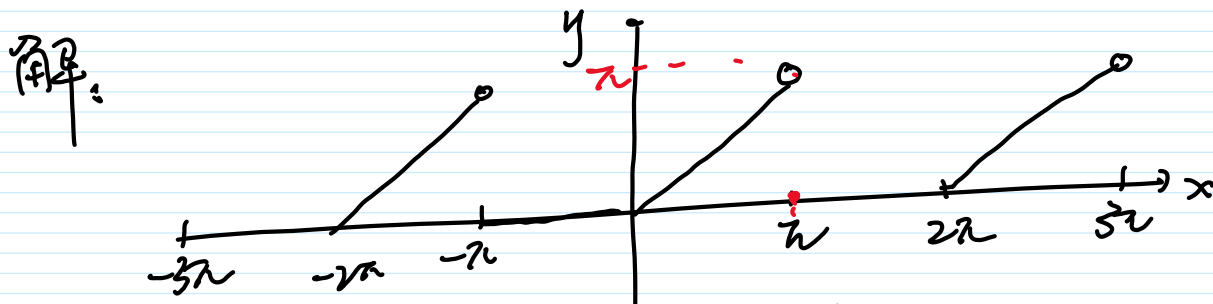
② $f(x) = f(x+2l), f(-x) = f(x)$

$$\Rightarrow b_n = 0, (n=1, 2, \dots)$$

$$a_n = \frac{2}{l} \int_0^l f(x) \cos \frac{n\pi x}{l} dx.$$

例. $f(x) = f(x+2\pi), f(x) = \begin{cases} 0, & -\pi \leq x < 0 \\ x, & 0 \leq x < \pi. \end{cases}$

则 $f(x) = \dots, x \in (2k+1)\pi, k \in \mathbb{Z}$



间断点 $x = (2k+1)\pi, k \in \mathbb{Z}$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx$$

$$= \frac{1}{\pi} \int_0^{\pi} x \cos nx dx$$

$$= \frac{1}{\pi} \int_0^{\pi} x \cos nx dx$$

$$= \frac{1}{\pi} \cdot \frac{1}{n} \int_0^{\pi} x d \sin nx.$$

$$= \frac{1}{n\pi} \left(\underbrace{x \sin nx} \Big|_0^{\pi} - \int_0^{\pi} \sin nx dx \right)$$

$$= -\frac{1}{n\pi} \cdot \left(-\frac{1}{n}\right) \int_0^{\pi} d \cos nx.$$

$$= \frac{1}{n^2\pi} (\cos n\pi - 1) \quad (n=1, 2, \dots)$$

$$a_0 = \frac{1}{\pi} \int_0^{\pi} x \cdot 1 dx = \frac{\pi}{2}.$$

$$b_n = \frac{1}{\pi} \int_0^{\pi} x \sin nx dx = \frac{(-1)^{n+1}}{n} \quad (n=1, 2, \dots)$$

综上. $f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty} \left[\frac{(-1)^n - 1}{n^2\pi} \cos nx + \frac{(-1)^{n+1}}{n} \sin nx \right].$
 $(x \neq (2k+1)\pi, k \in \mathbb{Z})$

例 $\sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n^2} = \frac{\pi^2}{4}.$ $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}.$

当 $x=0$ 时 $x=0$ 是驻点

$$0 = f(0) = \frac{\pi}{4} + \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2\pi}$$

$$\frac{1}{\pi} \cdot \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n^2} = \frac{\pi}{4}$$

$$\sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n^2} = \frac{2}{1^2} + \frac{0}{2^2} + \frac{2}{3^2} + \frac{0}{4^2} + \dots$$

$$= \sum_{n=1}^{\infty} \frac{2}{(2n-1)^2}$$

$$S = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots$$

$$\frac{\pi^2}{8} = \sum_{n=1}^{\infty} \frac{1}{(2n)^2} = \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \dots$$

$$S - \frac{\pi^2}{8} = \sum_{n=1}^{\infty} \frac{1}{(2n)^2} = \frac{1}{4} \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{4} S.$$

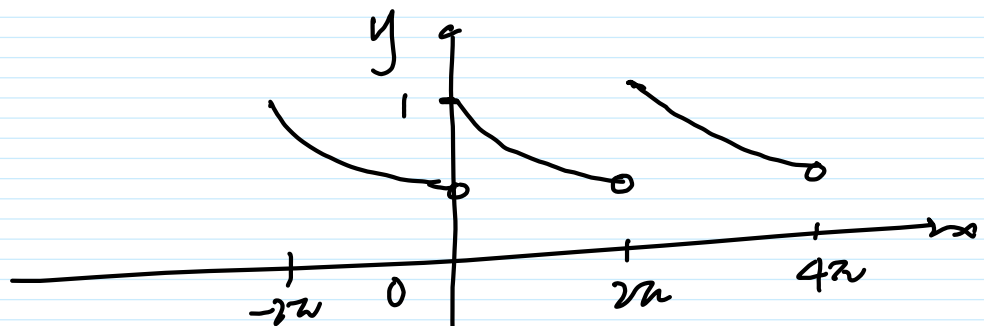
另法: 代入 $x = \pi$: $x = \pi$ 是间断点.

$$\frac{\pi+0}{2} = \frac{f(\pi^-) + f(\pi^+)}{2} = S(\pi) = \frac{\pi}{4} + \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n^2 \pi}$$

例 $f(x) = f(x + 2\pi)$ $f(x) = e^{-x}$ $[0, 2\pi)$

则 $f(x) =$ _____, $x \neq 2k\pi, k \in \mathbb{Z}$.

解:



间断点: $x = 2k\pi, k \in \mathbb{Z}$.

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$$a_n = \frac{1}{\pi} \int_0^{2\pi} e^{-x} \cos nx \, dx.$$

$$= \frac{1}{n\pi} \int_0^{2\pi} e^{-x} d \sin nx$$

$$= \frac{1}{n\pi} \left(e^{-x} \sin nx \Big|_0^{2\pi} + \int_0^{2\pi} \sin nx \cdot e^{-x} \, dx \right)$$

$$= \frac{1}{n\pi} (e^{-x} \sin nx \Big|_0^{2\pi})$$

$$a_n = \frac{1}{n\pi} \int_0^{2\pi} e^{-x} \sin nx dx, \quad (n=1, 2, \dots)$$

$$b_n = \frac{1}{n} \int_0^{2\pi} e^{-x} \cos nx dx = n a_n.$$

$$a_n = \frac{1}{n\pi} \cdot \left(-\frac{1}{n}\right) \int_0^{2\pi} e^{-x} d \cos nx$$

$$= -\frac{1}{n^2\pi} \left(e^{-x} \cos nx \Big|_0^{2\pi} + \int_0^{2\pi} \cos nx \cdot e^{-x} dx \right)$$

$$a_n = -\frac{e^{-2\pi} - 1}{n^2\pi} - \frac{1}{n^2} \cdot \frac{1}{\pi} \int_0^{2\pi} e^{-x} \cos nx dx.$$

$$a_n = \frac{1 - e^{-2\pi}}{(1+n^2)\pi} \quad (n=1, 2, \dots)$$

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} e^{-x} dx = \frac{1 - e^{-2\pi}}{\pi}$$

综上, $f(x) = \frac{1 - e^{-2\pi}}{\pi} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad x \in (0, 2\pi).$

例 $\sum_{n=1}^{\infty} \frac{1}{1+n^2} = \underline{\hspace{2cm}}.$

$$e^{-x} = \frac{1 - e^{-2\pi}}{2\pi} + \sum_{n=1}^{\infty} \left(\frac{1 - e^{-2\pi}}{(1+n^2)\pi} \cos nx + \frac{1 - e^{-2\pi}}{(1+n^2)\pi} \sin nx \right), \quad x \in (0, 2\pi)$$

三. 非周期函数 有延拓后展开成傅里叶级数.

例 $f(x) = e^{-x}, \quad x \in (0, 2\pi). \quad f(x) = \underline{\hspace{2cm}},$

注意: 周期函数才展开.

定义: 周期函数与拓.

$f(x)$ 定义 $[-1, 1]$ 或 $[0, 2\pi]$. 构造周期函数.

↓ 构造

$$F(x) = F(x+2\pi) \quad T=2\pi. \quad F(x) = f(x) \quad x \in [-1, 1] \text{ 或 } [0, 2\pi].$$

↓ 展开.

① 画 $f(x)$ 的图.

② $f(x)$ 的间断点 在 $[-1, 1]$ 或 $[0, 2\pi]$

③ a_n, b_n .

$$a_n = \frac{1}{\pi} \int_{-1}^1 f(x) \cos \frac{n\pi x}{l} dx$$

$$b_n = \frac{1}{\pi} \int_{-1}^1 f(x) \sin \frac{n\pi x}{l} dx$$

得 $F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (\dots),$ x 非间断点

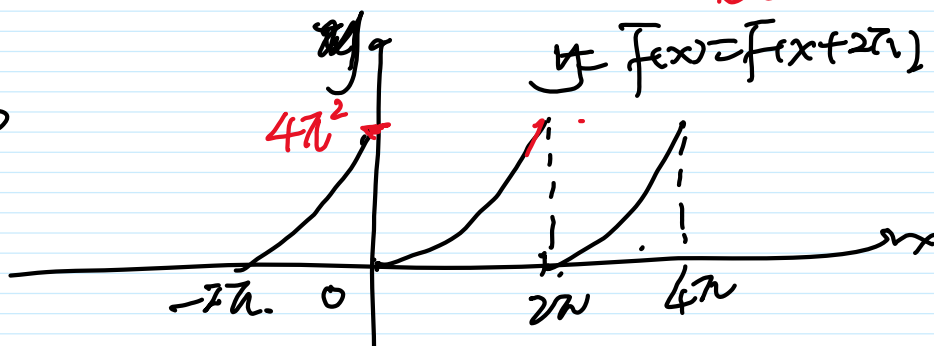
↓
 $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (\dots),$ $x \in [-1, 1] \text{ 或 } [0, 2\pi]$
 且 x 非 $F(x)$ 的间断点

例 $f(x) = x^2, [0, 2\pi]$ 则其傅里叶展开式为.

$f(x) =$ $x \in (0, 2\pi)$

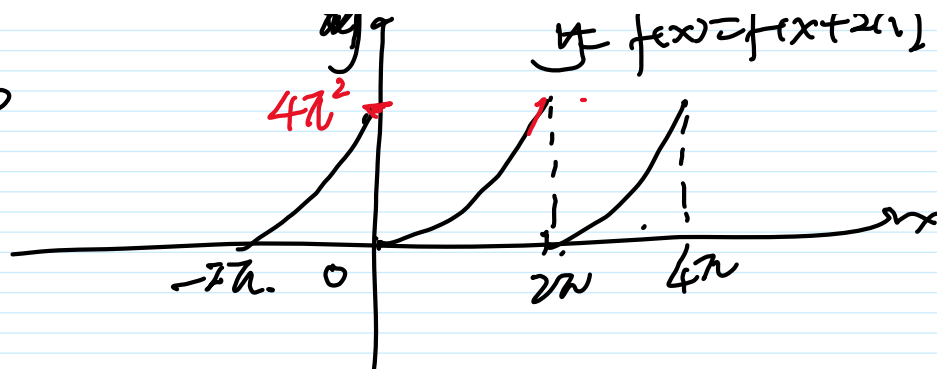
解: ① 周期函数与拓

$T = 2\pi$



解: ①. 周期及振幅

$$T = 2\pi$$



② $f(x)$ 的间断点 $x = 2k\pi \quad k \in \mathbb{Z}$

在 $[0, 2\pi]$ 上的间断点 $x=0$ 和 $x=2\pi$.

$$\textcircled{3}. \quad a_n = \frac{1}{2\pi} \int_0^{2\pi} x^2 \cos nx dx = \frac{4}{n^2} \quad (n=1, 2, \dots)$$

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} x^2 dx = \frac{8\pi^2}{3}$$

$$b_n = \frac{1}{2\pi} \int_0^{2\pi} x^2 \sin nx dx = -\frac{4\pi}{n} \quad (n=1, 2, \dots)$$

$$\text{得上 } x^2 = \frac{4\pi^2}{3} + \sum_{n=1}^{\infty} \left(\frac{4}{n^2} \cos nx - \frac{4\pi}{n} \sin nx \right)$$

$x \in (0, 2\pi)$

$$\text{例: } \sum_{n=1}^{\infty} \frac{1}{n^2} = \underline{\frac{\pi^2}{6}} \quad \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{8} - \frac{\pi^2}{24}$$

当 $x=0$ 时 $x=0$ 是间断点

$$\frac{4\pi^2}{2} = S(0) = \frac{4\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$= \frac{\pi^2}{12}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots$$

当 $x=\pi$ 时

$x=\pi$ 是连续点

$$\pi^2 = \frac{4}{3}\pi^2 + \sum_{n=1}^{\infty} \frac{4 \cdot (-1)^n}{n^2}$$

例. $\sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{1}{n} = \underline{\ln 2}$.

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} x^n.$$

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots$$