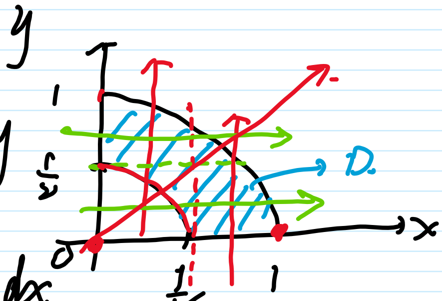
1. 直角坐标系下的计算

$$\iint_D f(x,y)\,dxdy = \int_a^b dx\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\,dy = \int_c^d dy\int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\,dx.$$

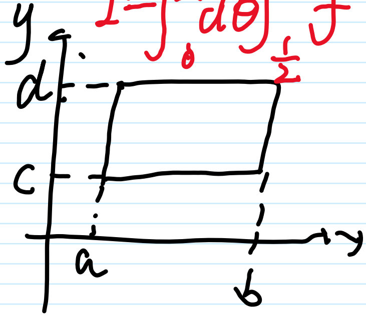
$I=\iint_D f(x,y)\,dxdy$. D: $\frac{1}{4}\le x^2+y^2\le 1$, $x\ge 0$ 且 $y\ge 0$.



X-型 $I=\int_0^{\frac{1}{2}} dx\int_{\sqrt{\frac{1}{4}-x^2}}^{\sqrt{1-x^2}} f(x,y)\,dy+\int_{\frac{1}{2}}^1 dx\int_0^{\sqrt{1-x^2}} f(x,y)\,dy$

Y-型 $=\int_0^{\frac{1}{2}} dy\int_{\sqrt{\frac{1}{4}-y^2}}^{\sqrt{1-y^2}} f(x,y)\,dx+\int_{\frac{1}{2}}^1 dy\int_0^{\sqrt{1-y^2}} f(x,y)\,dx$

$I=\int_0^{\frac{\pi}{2}} d\theta\int_{\frac{1}{2}}^1 f\cdot\rho\,d\rho$



推论1: D: $a\le x\le b$, $c\le y\le d$.

$$\iint_D f(x,y)\,dxdy = \int_a^b dx\int_c^d f(x,y)\,dy$$

$$= \int_c^d dy\int_a^b f(x,y)\,dx$$

推论2. D: $a\le x\le b$, $c\le y\le d$. 且 $f(x,y)=g(x)\cdot h(y)$.

$$\iint_D f(x,y)\,dxdy = \int_a^b dx\int_c^d f(x,y)\,dy$$

$$=\int_a^b\left[\int_c^d g(x)\cdot h(y)\,dy\right]dx$$

$$=\int_a^b\left[g(x)\underbrace{\int_c^d h(y)\,dy}_{\text{常数}}\right]dx.$$

$$=\left[\int_a^b g(x)\,dx\right]\left[\int_c^d h(y)\,dy\right]$$

$=[\int_a g(x)dx][\int_c h(y)dy]$

例: $$\left[\int_a^b f(x)\cdot g(x)dx\right]^2 \le \left[\int_a^b f^2(x)dx\right]\left[\int_a^b g^2(x)dx\right].$$

$(\vec{\alpha},\vec{\beta})^2 \le (\vec{\alpha},\vec{\alpha})(\vec{\beta},\vec{\beta})$ ✓

证明: ①法: $h(x) = f(x) + t\cdot g(x)$. $\Delta \le 0$.

$$\int_a^b h^2(x)dx \ge 0.$$

②法: 左 $=\left[\int_a^b f(x)g(x)dx\right]\left[\int_a^b f(y)\cdot g(y)dy\right]$

$=\iint_D [f(x)g(x)\cdot f(y)g(y)]dxdy$ 其中 D: $a\le x\le b$, $a\le y\le b$.

右 $=\left[\int_a^b f^2(x)dx\right]\left[\int_a^b g^2(y)dy\right]$

$=\iint_D [f^2(x)\cdot g^2(y)]dxdy$

$=\left[\int_a^b f^2(y)dy\right]\left[\int_a^b g^2(x)dx\right]$

$=\iint_D [f^2(y)\cdot g^2(x)]dxdy$

$=\iint_D \frac{f^2(x)g^2(y)+f^2(y)g^2(x)}{2}dxdy.$

D 关于 $y=x$ 对称

$\Downarrow$

$\iint_D f(x,y)\,d\sigma$

$=\iint_D f(y,x)\,d\sigma$

$=\iint_D \frac{f(x,y)+f(y,x)}{2}d\sigma.$

$\because \frac{f^2(x)g^2(y)+f^2(y)g^2(x)}{2} \ge f(x)g(x)f(y)g(y)$ $\therefore$ 右 $\ge$ 左.

2. 极坐标系的计算

2. 极坐标系的计算

$$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases}$$

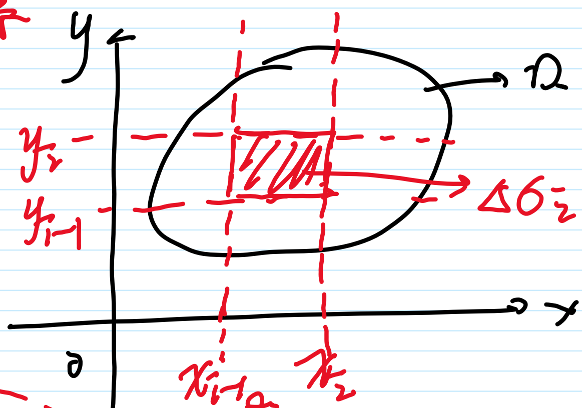
$$\iint_D f(x,y)\,\underbrace{d\sigma}_{\text{面积元素}} = \iint_D f(x,y)\,\underbrace{dxdy}_{\text{面积元素}} = \iint_D f(\rho\cos\theta,\rho\sin\theta)\,d\sigma$$

$x=$常数

$y=$常数

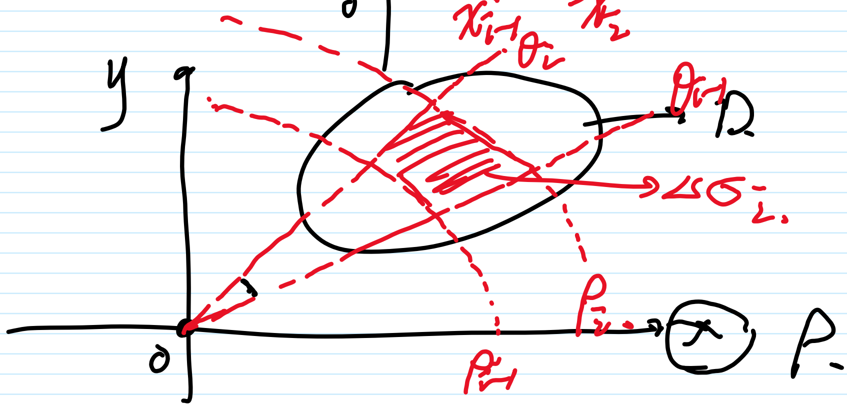
$\Delta\sigma_i = \Delta x_i \cdot \Delta y_i$.

$d\sigma = dx\,dy$.



$\theta=$常数、射线.

$\rho=$常数 圆



$$\Delta\sigma_i = (\pi\rho_i^2 - \pi\rho_{i-1}^2)\cdot\frac{\theta_i-\theta_{i-1}}{2\pi}$$

$$= \frac{(\rho_i+\rho_{i-1})\cdot\Delta\rho_i}{2}\cdot\Delta\theta_i$$

$$= \frac{(2\rho_{i-1}+\Delta\rho_i)}{2}\cdot\Delta\rho_i\cdot\Delta\theta_i$$

$$= \rho_i\Delta\rho_i\cdot\Delta\theta_i + \frac{1}{2}(\Delta\rho_i)^2\Delta\theta_i$$

$$\Delta\sigma_i \approx \rho_i\,\Delta\rho_i\,\Delta\theta_i$$

$$d\sigma = \boxed{\rho}\, d\rho\, d\theta.$$

$$\iint_D f(x,y)\,dx\,dy = \iint_D f(\rho\cos\theta,\rho\sin\theta)\cdot \boxed{\rho}\, d\rho\, d\theta$$

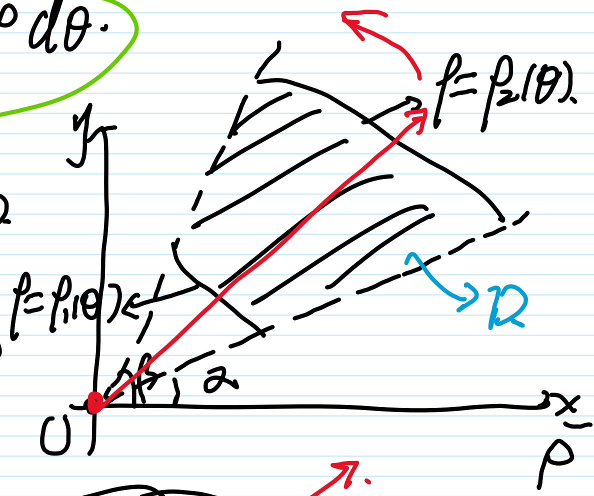
$$=\int d\theta \int f(\rho\cos\theta,\rho\sin\theta)\,\rho\, d\rho$$

$$=\int d\rho\int f(\rho\cos\theta,\rho\sin\theta)\,\rho\, d\theta.$$

穿线法: D: $\alpha \le \theta \le \beta \quad \rho_1(\theta) \le \rho \le \rho_2(\theta)$

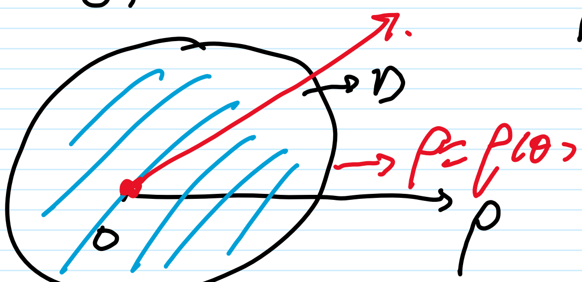
(i)

$$I=\int_\alpha^\beta d\theta\int_{\rho_1(\theta)}^{\rho_2(\theta)} f(\rho\cos\theta,\rho\sin\theta)\,\rho\, d\rho.$$



(ii)

$$I=\int_0^{2\pi} d\theta\int_0^{\rho(\theta)} f(\rho\cos\theta,\rho\sin\theta)\,\rho\, d\rho$$



例. $I=\iint_D 1\, d\sigma$:

D': $x^2+y^2 \le 2Ry$.

D: $x^2+y^2 \le 2Rx$.

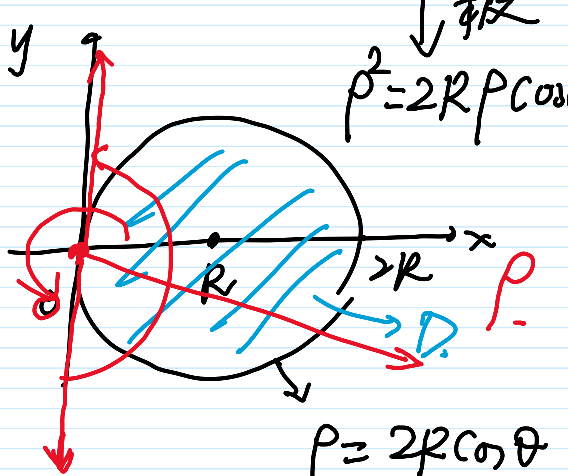
$x^2+y^2=2Rx$

$\downarrow$ 极

$\rho^2=2R\rho\cos\theta$

解, ①法

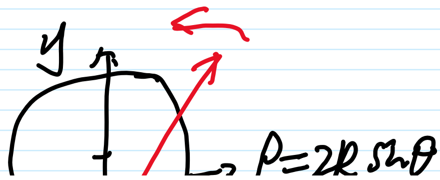
$$I=\int_0^{2R} dx\int_{-\sqrt{2Rx-x^2}}^{\sqrt{2Rx-x^2}} 1\, dy$$



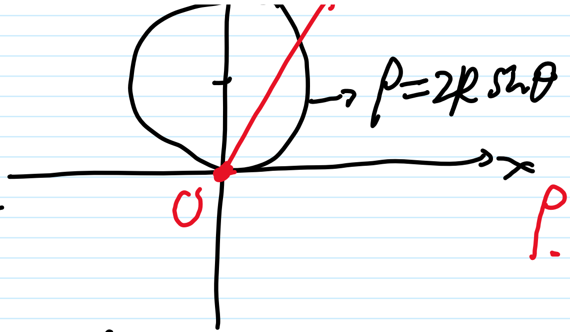
②法:

$$I=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta\int_0^{2R\cos\theta} 1\,\rho\, d\rho.$$

$$I'=\iint_{D'} 1\, d\sigma=\int_0^{\pi} d\theta\int_0^{2R\sin\theta} 1\,\rho\, d\rho.$$



$I = \iint_{D'} 1\, d\sigma = \int_0 d\theta \int_0 1 \rho\, d\rho.$

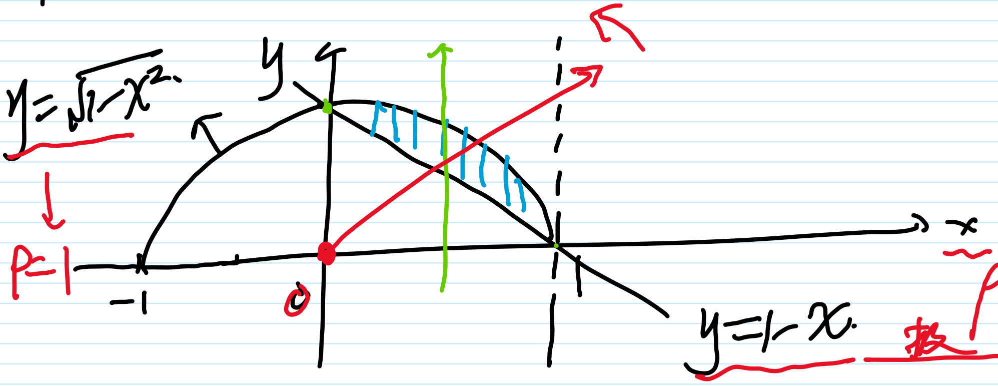


$x^2+y^2=2Ry \xrightarrow{\text{极}} \rho^2 = 2R\rho\sin\theta$

极坐标计算：
$\begin{cases} ① & D \text{ 与圆有关.} \\ ② & f(x,y) \text{ 含有 } \underbrace{x^2+y^2}_{\rho^2} \text{ 或 } \underbrace{\dfrac{y}{x}}_{\tan\theta} \text{ 形式} \end{cases}$

例：$\displaystyle\int_0^1 dx \int_{1-x}^{\sqrt{1-x^2}} f(x,y)\,dy = \int_0^{\frac{\pi}{2}} d\theta \int_{\frac{1}{\cos\theta+\sin\theta}}^{1} f(\rho\cos\theta, \rho\sin\theta)\rho\, d\rho$（极坐标）

解：确定 D： $x=0$、$x=1$. $y=1-x$，$y=\sqrt{1-x^2}$



$y=1-x \xrightarrow{\text{极}} \rho\sin\theta = 1-\rho\cos\theta$

$\Leftrightarrow \rho = \dfrac{1}{\cos\theta+\sin\theta}$

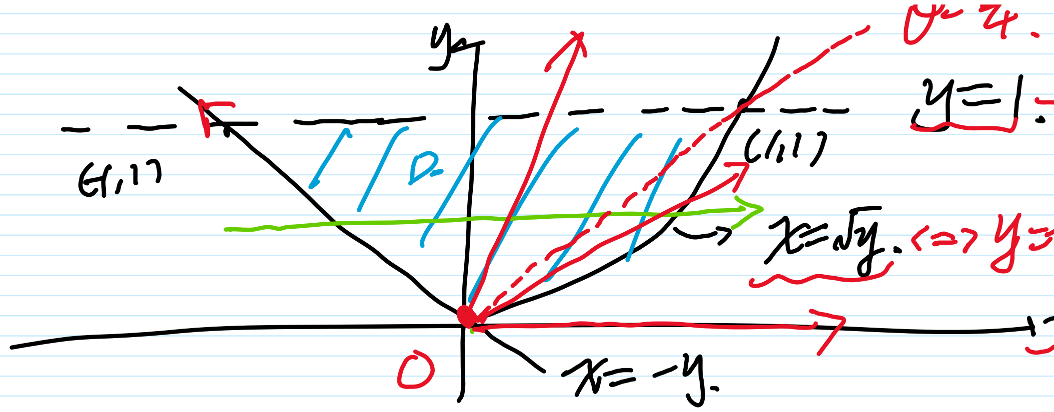
例 $\displaystyle\int_0^1 dy \int_{-y}^{\sqrt{y}} f(x,y)\,dx$

$\displaystyle= \int_0^{\frac{\pi}{4}} d\theta \int_0^{\frac{\sin\theta}{\cos^2\theta}} f\, \rho\, d\rho + \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} d\theta \int_0^{\frac{1}{\sin\theta}} f\, \rho\, d\rho.$ （极坐标）

解：

$y=1 \xrightarrow{\text{极}} \rho = \frac{1}{\dots}$

解:



$\theta=\frac{\pi}{4}$.

$y=1 \xrightarrow{换} \rho=\frac{1}{\sin\theta}$

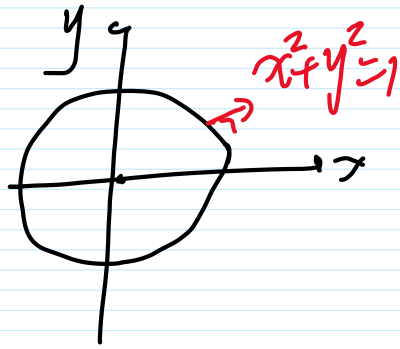
$x=\sqrt{y} \Leftrightarrow y=x^2 \xrightarrow{换} \rho\sin\theta=\rho^2\cos^2\theta$

ρ.

例. $I=\iint_D x^2\,dxdy$. D: $x^2+y^2\le 1$.

D: $0\le\theta\le 2\pi$, $0\le\rho\le 1$

解: $I=\int_0^{2\pi}d\theta\int_0^1 \rho^2\cos^2\theta\,\rho\,d\rho$



$=\left(\int_0^{2\pi}\cos^2\theta\,d\theta\right)\left(\int_0^1 \rho^3\,d\rho\right)=\cdots$

D关于 $y=x$ 对称

$\frac{1}{2}\iint_D 1\,dxdy=\frac{1}{2}\cdot\pi\cdot 1^2=\frac{\pi}{2}$

则 $I=\iint_D y^2\,dxdy=\frac{1}{2}\iint_D (\cancel{x^2+y^2})\,dxdy$.

$=\frac{1}{2}\left(\int_0^{2\pi}d\theta\right)\left(\int_0^1 \rho^2\cdot\rho\,d\rho\right)$

$=\frac{1}{2}\times 2\pi\times\frac{1}{4}=\frac{\pi}{4}$.

例: $I=\iint_D e^{-x^2-y^2}\,d\sigma$. D: $x^2+y^2\le a^2$ $(a>0)$

D_1: $x^2+y^2\le a^2$, $x\ge 0$ 且 $y\ge 0$

解: $I=\left(\int_0^{2\pi \; (\frac{\pi}{2})} d\theta\right)\left(\int_0^a e^{-\rho^2}\,\rho\,d\rho\right)$

$=2\pi\cdot\left(-\frac{1}{2}\right)\int_0^a e^{-\rho^2}\,d(-\rho^2)$

$$= 2\pi\cdot\left(-\frac{1}{2}\right)\int_0^a e^{-\rho^2}\,\underbrace{d(-\rho^2)}$$

$$= -\pi\cdot e^{-\rho^2}\Big|_0^a = \pi(1-e^{-a^2}).$$

$$\boxed{\int_0^{+\infty} e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}.}\ \triangle$$

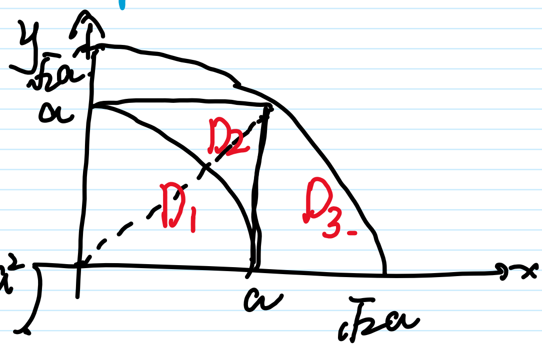
$$I_1 = \iint_{D_1} e^{-x^2-y^2}\,d\sigma = \underbrace{\frac{\pi}{4}(1-e^{-a^2})}$$

$$\Big\downarrow a\to+\infty$$

$$\frac{\pi}{4}.$$

$D_1 \subseteq D_2 \subseteq D_3$

$e^{-x^2-y^2} \geq 0.$



$$\iint_{D_1} e^{-x^2-y^2}\,d\sigma \leq \iint_{D_2} e^{-x^2-y^2}\,d\sigma \leq \iint_{D_3} e^{-x^2-y^2}\,d\sigma.$$

$$\frac{\pi}{4}(1-e^{-a^2}) \leq I_2 \leq \frac{\pi}{4}(1-e^{-2a^2})$$

$$\Big\downarrow a\to+\infty \qquad\qquad \Big\downarrow a\to+\infty$$

$$\frac{\pi}{4} \leq I_2 \leq \frac{\pi}{4}$$

$$\lim_{a\to+\infty} I_2 = \frac{\pi}{4}.$$

D_2: $0\leq x\leq a$, $0\leq y\leq a$

$$I_2 = \iint_{D_2}(e^{-x^2}\cdot e^{-y^2})\,dxdy$$

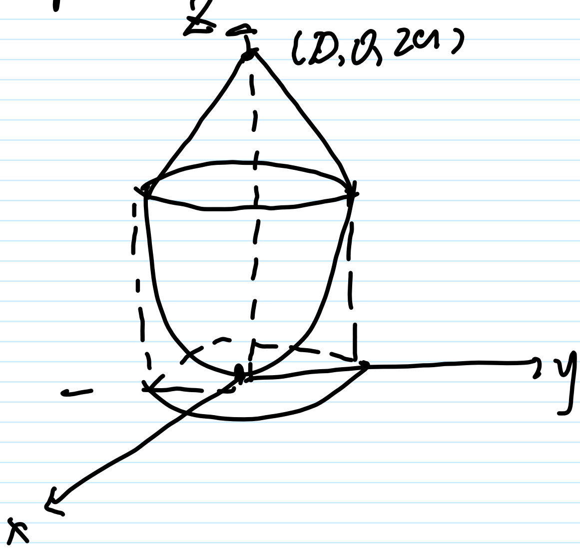
$$= \left(\int_0^a \underbrace{e^{-x^2}dx}\right)\left(\int_0^a \underbrace{e^{-y^2}dy}\right).$$

$$= \left(\int_0^a e^{-x^2} dx\right)\left(\int_0^a e^{-y^2} dy\right).$$

$$I_2 = \left(\int_0^a e^{-x^2} dx\right)^2$$

$$\lim_{a\to+\infty}\int_0^a e^{-x^2} dx = \lim_{a\to+\infty}\sqrt{I_2} = \sqrt{\frac{\pi}{4}} = \frac{\sqrt{\pi}}{2}$$

例. 求 $x^2+y^2=az$ 与 $z=2a-\sqrt{x^2+y^2}$ 所围成体积 $(a>0)$



解, $V = V_{上曲} - V_{下曲}$.