近似计算

一元函数： $\Delta y = \overbrace{f'(x)\cdot\Delta x}^{dy} + o(\Delta x)$ $\qquad \Delta y \approx dy$

$$f(x_0+\Delta x) - f(x_0) \approx f'(x_0)\cdot\Delta x$$

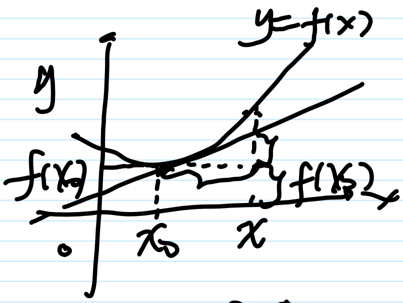
$$f(x_0+\Delta x) \approx f(x_0) + f'(x_0)\Delta x.$$

$x = x_0+\Delta x$

$$\underbrace{f(x)}_{\text{曲线上的值}} \approx \underbrace{f(x_0) + f'(x_0)(x-x_0)}_{\text{切线上的值}}$$

求 $y=f(x)$ 在 $(x_0, f(x_0))$ 的切线方程.

$y - f(x_0) = f'(x_0)(x-x_0)$



二元函数： $\Delta u \approx du.$

$$f(x_0+\Delta x, y_0+\Delta y) - f(x_0,y_0) \approx f_x(x_0,y_0)\cdot\Delta x + f_y(x_0,y_0)\Delta y$$

$x = x_0+\Delta x$

$y = y_0+\Delta y$

$$\underbrace{f(x,y)}_{\text{曲面上的值}} \approx \underline{f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)}$$

平面 $z = f(x_0,y_0)+f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)$

曲面 $z=f(x,y)$ 在 $(x_0, y_0, f(x_0,y_0))$ 处的切平面

第四节 复合函数求导

抽象复合函数

例. $f(x,y) = e^{xy}\cdot\cos(x+y)$. $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$

$\underline{z = f(u,v). \quad u = xy, \quad v = x+y.}$ $\quad \frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$, $\frac{\partial^2 z}{\partial x^2}$, ...

一元复合函数： $y = f(u)$, $u = \varphi(x)$. $\quad y \xrightarrow{f(u)} u \xrightarrow{\varphi(x)} x$

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因　中　自

$$y=f[\varphi(x)],\quad \frac{dy}{dx}=f'(u)\cdot\varphi'(x)$$

多元复合函数 "沿线相乘、分线相加"

1、中间变量是一元函数

定理: ① $z=f(u,v)$. 在(u,v)偏导连续

② $u=\varphi(x)$, $v=\psi(x)$ 在x可导.

则复合函数 $z=f[\varphi(x), \psi(x)]$ 在x处可导.

$$\begin{tikzcd} & u \arrow[dr, "\frac{d\varphi}{dx}"] & \\ z \arrow[ur, "\frac{\partial f}{\partial u}"] \arrow[dr, "\frac{\partial f}{\partial v}"'] & & x \\ & v \arrow[ur, "\frac{d\psi}{dx}"'] & \end{tikzcd}$$

因　中　自

全导数

且 $$\frac{dz}{dx}=\frac{\partial f}{\partial u}\cdot\frac{d\varphi}{dx}+\frac{\partial f}{\partial v}\cdot\frac{d\psi}{dx}$$

全微分 $$dz=\frac{\partial f}{\partial u}\cdot du+\frac{\partial f}{\partial v}dv.$$

$$\begin{tikzcd} & u \arrow[dr] & \\ z \arrow[ur] \arrow[r] \arrow[dr] & v \arrow[r] & x \\ & w \arrow[ur] & \end{tikzcd}$$

例. $z=f(u,v,w)$, $u=u(x)$. $v=v(x)$, $w=w(x)$

$$\frac{dz}{dx}=\frac{\partial f}{\partial u}\cdot\frac{du}{dx}+\frac{\partial f}{\partial v}\cdot\frac{dv}{dx}+\frac{\partial f}{\partial w}\cdot\frac{dw}{dx}.$$

例 $z=x^{\boxed{x}}$. 则 $\frac{dz}{dx}=$ ______.

解: ①法 $$\frac{dz}{dx}=(e^{\ln x^x})'=(e^{x\ln x})'$$

$$=x^x\cdot(x\ln x)'=x^x(\ln x+1)$$

$x^{\boxed{x}-1}$.

$= x^{x}\cdot(x\ln x)' \ \dots$

$= \boxed{x}^{x}\cdot\ln\boxed{x} + \boxed{x}\cdot x^{\boxed{x}-1}.$

回顾：$z = f(u,v) = u^{v}$. $u = x$, $v = x$.

$$\frac{dz}{dx} = \frac{\partial f}{\partial u}\cdot\frac{du}{dx} + \frac{\partial f}{\partial v}\cdot\frac{dv}{dx}$$

$$= v\cdot u^{v-1}\cdot 1 + u^{v}\cdot\ln u\cdot 1$$

$$= x\cdot x^{x-1} + x^{x}\cdot\ln x.$$

例：
$$z = f(u,v) = \boxed{\begin{cases} \dfrac{u^2 v}{u^2+v^2}, & (u,v)\neq(0,0) \\ 0, & (u,v)=(0,0) \end{cases}}$$

一阶偏导在(0,0)不连续

$u = x$, $v = x$. 则 $\left.\dfrac{dz}{dx}\right|_{x=0} = \underline{\ \dfrac{1}{2}\ }$.

解：
$$z = f(x,x) = \begin{cases} \dfrac{x^2\cdot x}{x^2+x^2}, & x\neq 0 \\ 0, & x=0 \end{cases}$$

$$= \begin{cases} \dfrac{x}{2}, & x\neq 0 \\ 0, & x=0 \end{cases}$$

故 $z = \dfrac{x}{2}$, $x\in\mathbb{R}$.

$$\left.\frac{dz}{dx}\right|_{x=0} \neq \left.\frac{\partial f}{\partial u}\right|_{(0,0)}\cdot\left.\frac{du}{dx}\right|_{x=0} + \left.\frac{\partial f}{\partial v}\right|_{(0,0)}\cdot\left.\frac{dv}{dx}\right|_{x=0}$$

$z = f(u,v)$, $u = \varphi(x)$, $v = \psi(x)$.

$$\frac{dz}{dx} = \frac{\partial f}{\partial u}\cdot\varphi' + \frac{\partial f}{\partial v}\cdot\psi' = f_u\cdot u' + f_v\cdot v'$$
$$= f_1'\cdot u' + f_2'\cdot v'$$

例 $z = f(\underset{u}{x}, \underset{v}{x^2})$. 则 $\dfrac{dz}{dx} = \dfrac{\partial f}{\partial u}\cdot 1 + \dfrac{\partial f}{\partial v}\cdot 2x$.

$f(u,v)$二阶偏导连续 $\Rightarrow f''_{12} = f''_{21}$　　$\dfrac{d^2z}{dx^2} =$ ____

$$\frac{dz}{dx} = f_1'\cdot\frac{dx}{dx} + f_2'\cdot\frac{dx^2}{dx} = f_1' + 2xf_2'.$$

$z = f(u,v,w)$.

$$z \to u, v, w \to x$$

例. $z = f(x, e^x, \sin x)$.

$$\frac{dz}{dx} = f_1'\cdot 1 + f_2'\cdot e^x + f_3'\cdot\cos x.$$

解: $\dfrac{d^2z}{dx^2} = \dfrac{d}{dx}\left(\dfrac{dz}{dx}\right) = \dfrac{d}{dx}(f_1' + 2xf_2')$

$$= \frac{df_1'}{dx} + \frac{d}{dx}(2xf_2')$$ 乘积求导

$$= \frac{df_1'}{dx} + 2f_2' + 2x\frac{df_2'}{dx}$$

$f_1'(u,v)$

$$f_1' = \frac{\partial f(u,v)}{\partial u} = \boxed{g}(u,v)$$

例: $f(u,v) = u+v$.
$f(u,v) = u\cdot e^v$.

$f_2'(u,v)$

$$f_2' = \frac{\partial f(u,v)}{\partial v} = \boxed{h}(u,v)$$

$$\frac{df(x,x^2)}{dx} = f_1'\cdot 1 + f_2'\cdot 2x$$

$$\frac{dg(u,v)}{dx} = g_1'\cdot 1 + g_2'\cdot 2x.$$

$$\frac{dh(u,v)}{dx} = h_1'\cdot 1 + h_2'\cdot 2x$$

$$\begin{cases} \dfrac{df_1'}{dx} = \dfrac{df_1'(u,v)}{dx} = f_{11}''\cdot 1 + f_{12}''\cdot 2x \\ \dfrac{df_2'}{dx} = \dfrac{df_2'(u,v)}{dx} = f_{21}''\cdot 1 + f_{22}''\cdot 2x \end{cases}$$

$$f_{11}'' = f_{11}''(u,v) = \frac{\partial^2 f(u,v)}{\partial u^2} \qquad f_{22}'' = \frac{\partial^2 f(u,v)}{\partial v^2}$$

$$f_{12}'' = \frac{\partial^2 f(u,v)}{\partial u \partial v} \qquad f_{21}'' = \frac{\partial^2 f(u,v)}{\partial v \partial u}$$

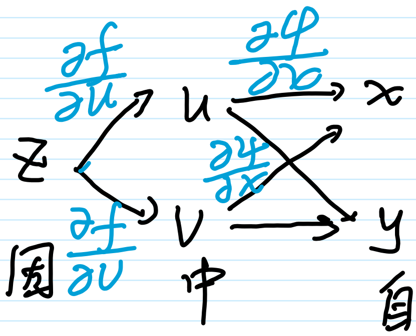
2、中间变量是多元函数

定理：① $z=f(u,v)$ 一阶偏导连续

② $u=\varphi(x,y)$, $v=\psi(x,y)$ 偏导存在

则 $z=f[\varphi(x,y),\psi(x,y)]$ 偏导存在，且

$$\begin{cases} \dfrac{\partial z}{\partial x} = \dfrac{\partial f}{\partial u}\cdot\dfrac{\partial \varphi}{\partial x} + \dfrac{\partial f}{\partial v}\cdot\dfrac{\partial \psi}{\partial x} \\ \dfrac{\partial z}{\partial y} = \dfrac{\partial f}{\partial u}\cdot\dfrac{\partial \varphi}{\partial y} + \dfrac{\partial f}{\partial v}\cdot\dfrac{\partial \psi}{\partial y} \end{cases}$$



$$\frac{\partial z}{\partial x} = f_1'\cdot\varphi_1' + f_2'\cdot\psi_1'$$

例 $z=e^{2x-y}\cos(xy)$，则 $\frac{\partial z}{\partial x}$，$\frac{\partial z}{\partial y}$。

解：$z=f(u,v)=e^{u}\cdot\cos v$. $u=2x-y$, $v=xy$

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$$\frac{\partial f}{\partial u}=e^{u}\cdot\cos v. \qquad \frac{\partial f}{\partial v}=e^{u}\cdot(-\sin v)$$

$$\frac{\partial u}{\partial x}=2 \qquad \frac{\partial v}{\partial x}=y$$

$$\frac{\partial u}{\partial y}=-1 \qquad \frac{\partial v}{\partial y}=x$$

$$\frac{\partial z}{\partial x}=e^{u}\cdot\cos v\cdot 2+e^{u}\cdot(-\sin v)\cdot y \overset{回代}{=} \cdots$$

$$\frac{\partial z}{\partial y}=e^{u}\cdot\cos v\cdot(-1)+e^{u}\cdot(-\sin v)\cdot x \overset{回代}{=} \cdots$$

$z=f(u,v)$ 二阶偏导连续.

例：$z=f(2x-y,\ xy)$，则 $\frac{\partial z}{\partial x}$，$\frac{\partial z}{\partial y}$ 及所有二阶偏导

解：
$$\frac{\partial z}{\partial x}=f_1'\cdot 2+f_2'\cdot y.$$
$$\frac{\partial z}{\partial y}=f_1'\cdot(-1)+f_2'\cdot x.$$

$$\frac{\partial f_1'(2x-y,xy)}{\partial x}$$

$$\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial}{\partial x}(2f_1'+yf_2')=2\boxed{\frac{\partial f_1'}{\partial x}}+y\frac{\partial f_2'}{\partial x}$$

$f\to f_1'$ $\qquad$ $f\to f_2'$

$$=2(2f_{11}''+yf_{12}'')+y(2f_{21}''+yf_{22}'')$$

$f_{12}''=f_{21}''$

$$=4f_{11}''+4yf_{12}''+y^2f_{22}''$$

$$\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial}{\partial y}(2f_1'+yf_2')$$

$$\frac{\partial f(2x-y,xy)}{\partial y}=-f_1'+xf_2'$$

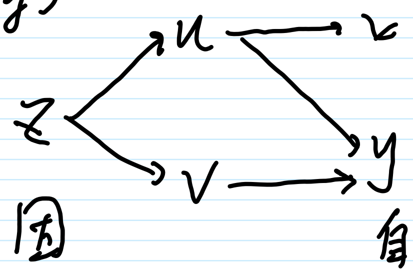
$$= 2\frac{\partial f_1'}{\partial y} + \boxed{1\cdot f_2'} + y\frac{\partial f_2'}{\partial y} \qquad = -f_1' + x f_2'$$

$$= 2(-f_{11}'' + x f_{12}'') + f_2' + y(-f_{21}'' + x f_{22}'')$$

$$f_{12}'' = f_{21}'' \quad = -2f_{11}'' + (2x - y) f_{12}'' + xy f_{22}'' + f_2'$$

3. 中间变量有一元函数又有多元函数

$z = f(u, v)$ $\quad u = \varphi(x, y)$, $\quad v = \psi(y)$



$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial u}\cdot\frac{\partial \varphi}{\partial x}$$

$$\frac{\partial z}{\partial y} = \frac{\partial f}{\partial u}\cdot\frac{\partial \varphi}{\partial y} + \frac{\partial f}{\partial v}\cdot\frac{d\psi}{dy}$$

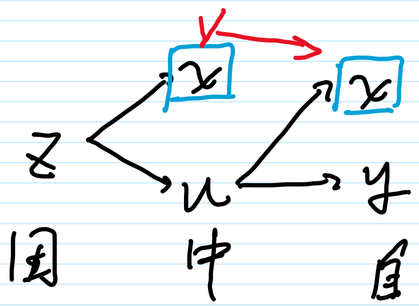
$$z = f[\varphi(x, y), \psi(y)]$$

$$\frac{\partial z}{\partial x} = f_1'\cdot\varphi_1' + f_2'\cdot 0$$

$$\frac{\partial z}{\partial y} = f_1'\cdot\varphi_2' + f_2'\cdot\psi'$$

4. 中间变量也是自变量

$z = f(\overset{v}{\boxed{x}}, u)$. $\quad u = \varphi(x, y)$ $\quad v = x$



$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial x}\cdot 1 + \frac{\partial f}{\partial u}\cdot\frac{\partial \varphi}{\partial x}.$$

$$z = f[x, \varphi(x, y)]$$

$$\frac{\partial z}{\partial x} = f_1' \cdot 1 + f_2' \cdot \varphi_1'$$

$$\frac{\partial z}{\partial y} = f_1' \cdot 0 + f_2' \cdot \varphi_2'$$