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1. $f_x(x,y)=\lim\limits_{\Delta x\to 0}\dfrac{f(x+\Delta x,y)-f(x,y)}{\Delta x}$

$f_y(x,y)=\lim\limits_{\Delta y\to 0}\dfrac{f(x,y+\Delta y)-f(x,y)}{\Delta y}$

$\dfrac{d^2}{dx^2}(\cdots\cdots)$

$\dfrac{d^2y}{dx^2}$

2. 高阶偏导

$\dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial x}\right)=\lim\limits_{\Delta x\to 0}\dfrac{f_x(x+\Delta x,y)-f_x(x,y)}{\Delta x}=\dfrac{\partial^2 f}{\partial x^2}=f_{xx}$

二阶混合偏导

$\dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial x}\right)=\lim\limits_{\Delta y\to 0}\dfrac{f_x(x,y+\Delta y)-f_x(x,y)}{\Delta y}=\dfrac{\partial^2 f}{\partial x\partial y}=f_{xy}$

$\dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial y}\right)=\dfrac{\partial^2 f}{\partial y\partial x}=f_{yx}$.

$\dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial y}\right)=\dfrac{\partial^2 f}{\partial y^2}=f_{yy}$

类似：$\dfrac{\partial^3 f}{\partial x^3}=f_{xxx}=\dfrac{\partial}{\partial x}\left(\dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial x}\right)\right)$

$\dfrac{\partial^3 f}{\partial x^2\partial y}=f_{xxy}=\dfrac{\partial}{\partial y}\left(\dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial x}\right)\right)$

例：$z=e^{x+2y}$ 求所有二阶偏导.及 $\dfrac{\partial^3 z}{\partial x^2\partial y}$. $\dfrac{\partial^3 z}{\partial y\partial x^2}$

解：$\dfrac{\partial z}{\partial x}=e^{x+2y}\cdot 1$，$\dfrac{\partial z}{\partial y}=e^{x+2y}\cdot 2$

$\dfrac{\partial^2 z}{\partial x^2}=\dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial x}\right)=\dfrac{\partial}{\partial x}\left(e^{x+2y}\right)=e^{x+2y}$

$\dfrac{\partial^2 z}{\partial x\partial y}=\dfrac{\partial}{\partial y}\left(\dfrac{\partial z}{\partial x}\right)=\dfrac{\partial}{\partial y}\left(e^{x+2y}\right)=2e^{x+2y}$

$x+2y$

$$\frac{\partial^2 z}{\partial x\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}(\ldots)$$

$$\frac{\partial^2 z}{\partial y\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(2e^{x+2y}\right) = 2e^{x+2y}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left(2e^{x+2y}\right) = 4e^{x+2y}$$

$$\frac{\partial^3 z}{\partial x^2\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial^2 z}{\partial x^2}\right) = \frac{\partial}{\partial y}\left(e^{x+2y}\right) = 2e^{x+2y}$$

$$\frac{\partial^3 z}{\partial y\partial x^2} = \frac{\partial^2}{\partial x^2}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2}{\partial x^2}\left(2e^{x+2y}\right) = 2e^{x+2y}$$

$$\frac{\partial^3 z}{\partial x\partial y\partial x} = \frac{\partial}{\partial x}\left(\underbrace{\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)}\right) = \frac{\partial}{\partial x}\left(\frac{\partial^2 z}{\partial y\partial x}\right) = \frac{\partial}{\partial x}\left(2e^{x+2y}\right)$$

$$\frac{\partial^3 z}{\partial y^2\partial x} = \frac{\partial^3 z}{\partial x\partial y^2} = \frac{\partial^3 z}{\partial y\partial x\partial y}$$

例：

$$f(x,y) = \begin{cases} xy\,\dfrac{x^2-y^2}{x^2+y^2}, & (x,y)\neq(0,0) \\ 0, & (x,y)=(0,0) \end{cases}$$

求 $f_{xy}(0,0)$，$f_{yx}(0,0)$

$f_{xy}(x_0,y_0)=f_{yx}(x_0,y_0)$

$(x_0,y_0)\neq(0,0)$

解：

$$f_{xy}(0,0) = \frac{\partial}{\partial y}\left(\boxed{\frac{\partial f}{\partial x}}\right)\Bigg|_{(0,0)}$$

$$= \lim_{\Delta y\to 0}\frac{f_x(0,0+\Delta y)-f_x(0,0)}{\Delta y}$$

首先求 $f_x(x,y)$

当 $(x,y)\neq(0,0)$.

$$f_x(x,y)=\frac{\partial}{\partial x}\left(xy\cdot\frac{x^2-y^2}{x^2+y^2}\right)$$

y常数

$$=y\cdot\frac{x^2-y^2}{x^2+y^2}+(xy)\cdot\frac{2x(x^2+y^2)-(x^2-y^2)\cdot 2x}{(x^2+y^2)^2}$$

例. $f(x,y)=\begin{cases}\dfrac{xy}{x^2+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$

$$=\frac{y(x^4+4x^2y^2-y^4)}{(x^2+y^2)^2}$$

当 $(x,y)=(0,0)$

$$f_x(0,0)=\lim_{\Delta x\to 0}\frac{f(0+\Delta x,0)-f(0,0)}{\Delta x}=0$$

综上. $f_x(x,y)=\begin{cases}\dfrac{y(x^4+4x^2y^2-y^4)}{(x^2+y^2)^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$

$$\text{故 } f_{xy}(0,0)=\lim_{\Delta y\to 0}\frac{\dfrac{\Delta y(0^4+4\cdot 0^2\cdot(\Delta y)^2-(\Delta y)^4)}{(0^2+(\Delta y)^2)^2}-0}{\Delta y}$$

$=-1.$

$$\text{同理 } f_{yx}(0,0)=\lim_{\Delta x\to 0}\frac{f_y(0+\Delta x,0)-f_y(0,0)}{\Delta x}=1.$$

$$f_{xy}(0,0)\neq f_{yx}(0,0)$$

定理、 二阶偏导连续 $\Rightarrow \dfrac{\partial^2 f}{\partial x\partial y}=\dfrac{\partial^2 f}{\partial y\partial x}.$

推广: 三阶偏导连续 $\Rightarrow \dfrac{\partial^3 f}{\partial x^2\partial y}=\dfrac{\partial^3 f}{\partial y\partial x^2}=\dfrac{\partial^3 f}{\partial x\partial y\partial x}$

$$\frac{\partial^3 f}{\partial y^2\partial x}=\frac{\partial^3 f}{\partial x\partial y^2}=\frac{\partial^3 f}{\partial y\partial x\partial y}$$

$\frac{\partial^2 z}{\partial y^2 \partial x} = \frac{\partial^2 z}{\partial x \partial y^2} = \frac{\partial^2 z}{\partial y \partial x \partial y}$

例 $\frac{\partial^2 z}{\partial x \partial y} = x+y$. $z(x,0)=x$, $z(0,y)=y^2$.

$\frac{df(y)}{dy} = \tan y$

则二元函数 $z=$ ______

偏积分

$\frac{\partial z}{\partial x} = \int (x+y)\,dy$.

解: $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\boxed{\frac{\partial z}{\partial x}}\right) = x+y$ $= xy + \frac{y^2}{2} + \varphi(x)$

$$\frac{\partial z}{\partial x} = x\cdot y + \frac{1}{2}y^2 + \varphi(x).$$

$$z(x,y) = \int [xy + \frac{1}{2}y^2 + \varphi(x)]\,dx$$

$$= \frac{1}{2}x^2 y + \frac{1}{2}xy^2 + \boxed{\Phi(x) + \psi(y)}.$$

$\Phi'(x) = \varphi(x)$　　常数

$z(x,0) = \Phi(x) + \psi(0) = x$,
$z(0,y) = \Phi(0) + \psi(y) = y^2$.

$\Phi(x) + \psi(y)$
$+ \boxed{\psi(0) + \Phi(0)} = x + y^2$.
$\Phi(0)+\psi(0) = 0$.

综上 $z(x,y) = \frac{x^2y + xy^2}{2} + x + y^2$.

第三节　全微分

一元函数微分：　不连续 $\Rightarrow$ 不可导 / 不可微

$y=f(x)$ 可微 $\Leftrightarrow$ $\Delta y = f(x+\Delta x) - f(x) = f'(x)\cdot\Delta x + o(\Delta x)$.

可微 $\Leftrightarrow$ 可导 $\rightarrow$ 连续

微分. $dy = f'(x)\cdot dx$.

连续 ⟸ 可导 (微分. $dy = f'(x)\,dx$)

二元函数微分

定义: $z=f(x,y)$. Δx、Δy

全增量 $\Delta z = f(x+\Delta x, y+\Delta y) - f(x,y)$

若 $\boxed{\Delta z = A(x,y)\,\Delta x + B(x,y)\cdot\Delta y + o\left(\sqrt{(\Delta x)^2+(\Delta y)^2}\right)}$ 其中 $A(x,y)$. $B(x,y)$ 与 Δx、Δy 无关

则 $z=f(x,y)$ 在 (x,y) 可微. 记为

全微分 $dz = A(x,y)\,\Delta x + B(x,y)\cdot\Delta y$

$dz = A(x,y)\,dx + B(x,y)\cdot dy$

定理 (可微的必要条件). 偏导存在. 但不可微

可微 $\Longrightarrow$ 偏导存在

且 $\boxed{dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}\cdot dy.}$

$dz\big|_{(x_0,y_0)} = f_x(x_0,y_0)\,dx + f_y(x_0,y_0)\,dy$

例 $z = x+2y$. 则 $dz = 1\cdot dx + 2\,dy$.

证明: 条件: $f(x+\Delta x, y+\Delta y) - f(x,y) = A(x,y)\cdot\Delta x + B(x,y)\,\Delta y + o\left(\sqrt{(\Delta x)^2+(\Delta y)^2}\right)$

去证: $f_x(x,y) = \lim\limits_{\Delta x\to 0}\frac{f(x+\Delta x, y+0) - f(x,y)}{\Delta x}$ 存在

$= \lim\limits_{\Delta x\to 0}\frac{A(x,y)\cdot\Delta x + B(x,y)\cdot 0 + o\left(\sqrt{(\Delta x)^2+0^2}\right)}{\Delta x}$

$= A(x,y) + \lim\limits_{\Delta x\to 0}\frac{o(|\Delta x|)}{\Delta x} = A(x,y)$

$$\lim_{\Delta x\to 0}\underbrace{\frac{o(|\Delta x|)}{|\Delta x|}}_{\text{趋向0}}\cdot\underbrace{\frac{|\Delta x|}{\Delta x}}_{\text{有界}}=0$$ 同理 $f_y(x,y)=B(x,y)$

例. $z=\sqrt{x^2+y^2}$. 在$(0,0)$处是否可微.

解: 在$(0,0)$连续. $f_x(0,0)$. $f_y(0,0)$ 不存在

偏导不存在 $\Longrightarrow$ 不可微

例: $f(x,y)=\begin{cases}\dfrac{xy}{x^2+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$

在$(0,0)$处不连续. $f_x(0,0)=f_y(0,0)=0$.

讨论在$(0,0)$是否可微. 不可微.

$\sin x$ $(x\to 0)$ $x^2=o(\sin x)$

x^2 $(x\to 0)$

$\lim_{x\to 0}\frac{x^2}{\sin x}=0$

解: 若可微.

$f(0+\Delta x,0+\Delta y)-f(0,0)=f_x(0,0)\cdot\Delta x+f_y(0,0)\Delta y+o(\sqrt{(\Delta x)^2+(\Delta y)^2})$

$f(\Delta x,\Delta y)-0=0+0+o(\sqrt{(\Delta x)^2+(\Delta y)^2})$.

去证明: $\dfrac{\Delta x\cdot\Delta y}{(\Delta x)^2+(\Delta y)^2}=o(\sqrt{(\Delta x)^2+(\Delta y)^2})$

去证 $\lim_{(\Delta x,\Delta y)\to(0,0)}\dfrac{\Delta x\cdot\Delta y}{[(\Delta x)^2+(\Delta y)^2]^{\frac{3}{2}}}=0.$

但. 不存在. $(\Delta y=\Delta x)$

但，不存在。$(\Delta y=\Delta x)$

可微 $\Leftrightarrow f(x+\Delta x, y+\Delta y)-f(x,y)=f_x(x,y)\Delta x+f_y(x,y)\Delta y+o(\rho)$

其中 $\rho=\sqrt{(\Delta x)^2+(\Delta y)^2}$

$\Leftrightarrow o(\rho)=[f(x+\Delta x, y+\Delta y)-f(x,y)]-[f_x\cdot\Delta x+f_y\cdot\Delta y]$

☆ 可微 $\Leftrightarrow \boxed{\lim\limits_{(\Delta x,\Delta y)\to(0,0)}\dfrac{[f(x+\Delta x,y+\Delta y)-f(x,y)]-[f_x\cdot\Delta x+f_y\cdot\Delta y]}{\sqrt{(\Delta x)^2+(\Delta y)^2}}=0}$

推论：　可微 $\Rightarrow$ 连续；不连续 $\Rightarrow$ 不可微

证明：　去证 $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)=f(x_0,y_0)$

$\Leftrightarrow \lim\limits_{(x,y)\to(x_0,y_0)}[f(x,y)-f(x_0,y_0)]=0$　　$u=f(P)$

$x=x_0+\Delta x$
$y=y_0+\Delta y$

$\Leftrightarrow \lim\limits_{(\Delta x,\Delta y)\to(0,0)}\Delta z=0$　　$\lim\limits_{\Delta P\to 0}\Delta u=0$

条件：可微　$\lim\limits_{(\Delta x,\Delta y)\to(0,0)}[f_x\Delta x+f_y\Delta y+o(\sqrt{(\Delta x)^2+(\Delta y)^2})]=0$

（各项 $\to 0$）

例. $f(x,y)=\sqrt{|xy|}$ 在 $(0,0)$ 是否可微.

解：　连续。$f_y(0,0)=\lim\limits_{\Delta y\to 0}\dfrac{\sqrt{|0(0+\Delta y)|}-\sqrt{|0\cdot 0|}}{\Delta y}=0.$

同理 $f_x(0,0)=0$

$\Delta z=f(x_0+\Delta x, y_0+\Delta y)-f(x_0,y_0)$ ... $[f_x(0,0)\Delta x+f_y(0,0)\Delta y]$

$\Delta z = f(x_0+\Delta x, y_0+\Delta y) - f(x_0, y_0)$

$$\lim_{(\Delta x,\Delta y)\to(0,0)} \frac{[f(0+\Delta x, 0+\Delta y) - f(0,0)] - [f_x(0,0)\cdot\Delta x + f_y(0,0)\Delta y]}{\sqrt{(\Delta x)^2+(\Delta y)^2}}$$

$\Delta z \neq f(\Delta x, \Delta y)$

$$= \lim_{(\Delta x,\Delta y)\to(0,0)} \frac{(\sqrt{|\Delta x\Delta y|} - 0) - (0\cdot\Delta x + 0\,\Delta y)}{\sqrt{(\Delta x)^2+(\Delta y)^2}}$$

$$= \lim_{(\Delta x,\Delta y)\to(0,0)} \sqrt{\frac{|\Delta x\cdot\Delta y|}{(\Delta x)^2+(\Delta y)^2}}$$

不存在
$(\Delta y = k\Delta x)$

故在 $(0,0)$ 不可微

推广 $u = f(x_1, x_2, \cdots, x_n)$ 可微

则 $du = \underbrace{\frac{\partial u}{\partial x_1}dx_1}_{\text{偏微分}} + \frac{\partial u}{\partial x_2}dx_2 + \cdots + \frac{\partial u}{\partial x_n}dx_n$

例: $z = x + 2y$

$dz = \underbrace{1\cdot dx}_{dg} + \underbrace{2\,dy}_{dh}$

$g(x) = x + 2a$

$h(y) = b + 2y$

$dg = g'(x)\,dx = 1\cdot dx$

$dh = h'(y)\,dy = 2\,dy$

定理 (可微的充分条件)

偏导连续 $\Rightarrow$ 可微

证明: 去证

$$\lim_{(\Delta x,\Delta y)\to(0,0)} \frac{[f(x+\Delta x, y+\Delta y) - f(x,y)] - [f_x\cdot\Delta x + f_y\cdot\Delta y]}{\sqrt{(\Delta x)^2+(\Delta y)^2}} = 0$$

$(\Delta x, \Delta y) \to (0,0)$ $\sqrt{(\Delta x)^2+(\Delta y)^2}$

拉格朗日中值定理 $f(x+\Delta x)-f(x)=f'(\xi)\cdot\Delta x$

$=f'(x+\theta\Delta x)\cdot\Delta x$

$$f(x+\Delta x,y+\Delta y)-f(x,y+\Delta y)+f(x,y+\Delta y)-f(x,y)$$

$$=f_x(x+\theta_1\Delta x,y+\Delta y)\cdot\Delta x+f_y(x,y+\theta_2\Delta y)\cdot\Delta y.$$

$$分子=[f_x(x+\theta_1\Delta x,y+\Delta y)-f_x(x,y)]\cdot\Delta x$$
$$+[f_y(x,y+\theta_2\Delta y)-f_y(x,y)]\cdot\Delta y.$$

$$\lim_{x\to x_0}f(x)=A \Leftrightarrow f(x)=A+\alpha$$

$$\lim_{x\to x_0}f(x)=f(x_0) \Leftrightarrow f(x)=f(x_0)+\alpha.$$

$$\lim_{(x,y)\to(x_0,y_0)}f(x,y)=f(x_0,y_0) \Leftrightarrow f(x,y)=f(x_0,y_0)+\alpha.$$

其中 $\lim_{(x,y)\to(x_0,y_0)}\alpha=0.$

$$f_x(x+\theta_1\Delta x,y+\Delta y)=f_x(x,y)+\alpha_1$$

$\theta_1\Delta x\to 0$, $\Delta y\to 0$, $\alpha_1\to 0$ $((\Delta x,\Delta y)\to(0,0))$

$$f_y(x,y+\theta_2\Delta y)=f_y(x,y)+\alpha_2.$$

$$\lim_{(\Delta x,\Delta y)\to(0,0)}\frac{\alpha_1\cdot\Delta x+\alpha_2\cdot\Delta y}{\sqrt{(\Delta x)^2+(\Delta y)^2}}=0. \Rightarrow 可微$$

例

$$f(x,y)=\begin{cases}(x^2+y^2)^2\sin\dfrac{1}{x^2+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$$

讨论在(0,0)是否连续、偏导存在、可微及偏导连续 ✗.

解: $$f_x(0,0)=\lim_{\Delta x\to 0}\frac{[(\Delta x)^2+0^2]\sin\frac{1}{(\Delta x)^2+0^2}-0}{\Delta x}=0$$

同理 $f_y(0,0)=0$.

$$\lim_{(\Delta x,\Delta y)\to(0,0)}\frac{\left[[(\Delta x)^2+(\Delta y)^2]^2\sin\frac{1}{(\Delta x)^2+(\Delta y)^2}-0\right]-[0\cdot\Delta x+0\cdot\Delta y]}{\sqrt{(\Delta x)^2+(\Delta y)^2}}$$

$$=\lim_{(\Delta x,\Delta y)\to(0,0)}\sqrt{(\Delta x)^2+(\Delta y)^2}\cdot\sin\frac{1}{(\Delta x)^2+(\Delta y)^2}=0$$

当 $(x,y)\neq(0,0)$ 时

$\dfrac{2x}{x^2+y^2}$ 不存在

$$f_x=\frac{\partial}{\partial x}\left((x^2+y^2)\sin\frac{1}{x^2+y^2}\right)$$

少平方

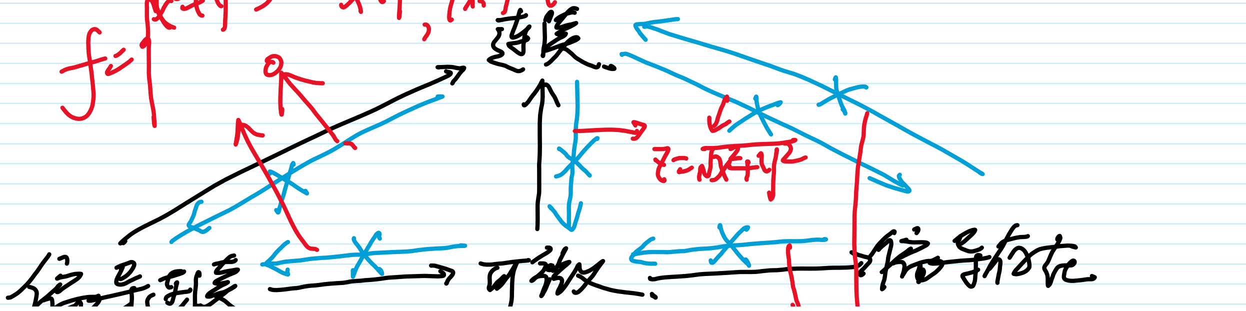
$$=2x\sin\frac{1}{x^2+y^2}+(x^2+y^2)\cdot\cos\frac{1}{x^2+y^2}\cdot(-1)\frac{2x}{(x^2+y^2)^2}$$

$\downarrow (x,y)\to(0,0)$ 　　　　　　$\downarrow (x,y)\to(0,0)$

0 　　　　　　不存在

$$\lim_{(x,y)\to(0,0)}f_x(x,y)\neq 0=f_x(0,0)\Rightarrow \text{偏导不连续}$$

$$f=\begin{cases}(x^2+y^2)\sin\frac{1}{x^2+y^2} & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$$



偏导连续 $\underset{\not\leftarrow}{\longrightarrow}$ 可微 $\underset{\not\leftarrow}{\longrightarrow}$ 偏导存在

$$f=\begin{cases}\dfrac{xy}{x^2+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$$

例. $(axy^3-y^2\cos x)dx+(1+by\sin x+3x^2y^2)dy$

是 $z=f(x,y)$ 的全微分. 则 $a=$ __2__, $b=$ __-2__.

解: (i) $dz=\dfrac{\partial z}{\partial x}dx+\dfrac{\partial z}{\partial y}dy.$

(ii) 法: 偏积分法

$$\int(axy^3-y^2\cos x)dx \;(+\varphi(y))=\int(1+by\sin x+3x^2y^2)dy \;(+\psi(x))$$

★ (iii) 法: 混合偏导相等

(i) $\dfrac{\partial}{\partial y}\left(\dfrac{\partial z}{\partial x}\right)=\dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial y}\right)$

$3axy^2-2y\cos x=0+by\cdot\cos x+6xy^2.$

$3a=6.\quad -2=b$