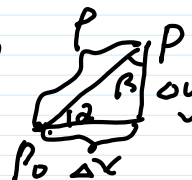


$$1. \frac{\partial f(x, y)}{\partial \vec{r}} \Big|_{(x_0, y_0)} = \lim_{\rho \rightarrow 0^+} \frac{f(x_0 + \rho \cos \alpha, y_0 + \rho \cos \beta) - f(x_0, y_0)}{\rho}$$


其中 $(\cos \alpha, \cos \beta) = \frac{1}{|\vec{r}|} \vec{r}$.

2. f 可微 \Rightarrow 方向导数存在

$$\left| \frac{\partial f}{\partial \vec{r}} \right|_{(x_0, y_0)} = f_x(x_0, y_0) \cdot \cos \alpha + f_y(x_0, y_0) \cdot \cos \beta$$

例. ① $u = x^2y + yz - z$ 在 $P_0(1, -1, 0)$ 沿

② 1. P_0 为起点 1.5 $P_1(2, 0, -1)$ 为终点何量的方向导数
 u 在 P_0 沿何方向的方向导数最大, 且求值.

解: $u_x(1, -1, 0) = 2xy \Big|_{(1, -1, 0)} = -2$
 $u_y(1, -1, 0) = x^2 + z \Big|_{(1, -1, 0)} = 1$
 $u_z(1, -1, 0) = y - 1 \Big|_{(1, -1, 0)} = -2$

$\nabla u(1, -1, 0) = (-2, 1, -2)$
 $\left| \frac{\partial u}{\partial \vec{r}} \right|_{\max} = \sqrt{(-2)^2 + 1^2 + (-2)^2} = 3$

$\vec{r} = \vec{P_0 P_1} = (1, 1, -1)$ $\frac{1}{|\vec{r}|} \vec{r} = \frac{1}{\sqrt{3}} (1, 1, -1)$

$\cos \alpha = \frac{\sqrt{3}}{3}$ $\cos \beta = \frac{\sqrt{3}}{3}$ $\cos \gamma = -\frac{\sqrt{3}}{3}$

$\frac{\partial u}{\partial \vec{r}} \Big|_{(1, -1, 0)} = (-2) \times \frac{\sqrt{3}}{3} + 1 \times \frac{\sqrt{3}}{3} + (-2) \times (-\frac{\sqrt{3}}{3}) = \frac{\sqrt{3}}{3}$

$\frac{\partial f}{\partial \vec{r}} = f_x \cdot \cos \alpha + f_y \cdot \cos \beta$

记作: $\text{grad } f = (f_x, f_y) \cdot (\cos \alpha, \cos \beta)$

$$\text{定义: } \text{grad} f = (\underline{f_x}, \underline{f_y}) \cdot (\cos \alpha, \cos \beta)$$

$$= \nabla f = \nabla f \cdot \frac{1}{|\vec{r}|} \vec{r}$$

$$= (f_x, f_y)$$

$$\vec{\alpha} \cdot \vec{\beta} = |\vec{\alpha}| \cdot |\vec{\beta}| \cdot \cos \theta$$

$$\frac{\partial f}{\partial t} = |\nabla f| \cdot 1 \cdot \cos \theta$$

$$\theta = \angle \nabla f, \vec{r}$$

$$\theta \in [0, \pi]$$

1) $\theta = 0, \cos \theta = 1$, 方向导数最大 $|\nabla f| = \sqrt{f_x^2 + f_y^2}$.

与梯度同向, 方向导数沿梯度方向最大

函数沿梯度方向增加最快.

函数沿
梯度(2)
其反方向

变化最快

$\theta = \pi, \cos \theta = -1$, 方向导数最小. $-|\nabla f| = -\sqrt{f_x^2 + f_y^2}$

与梯度反向, 方向导数沿负梯度方向最小.

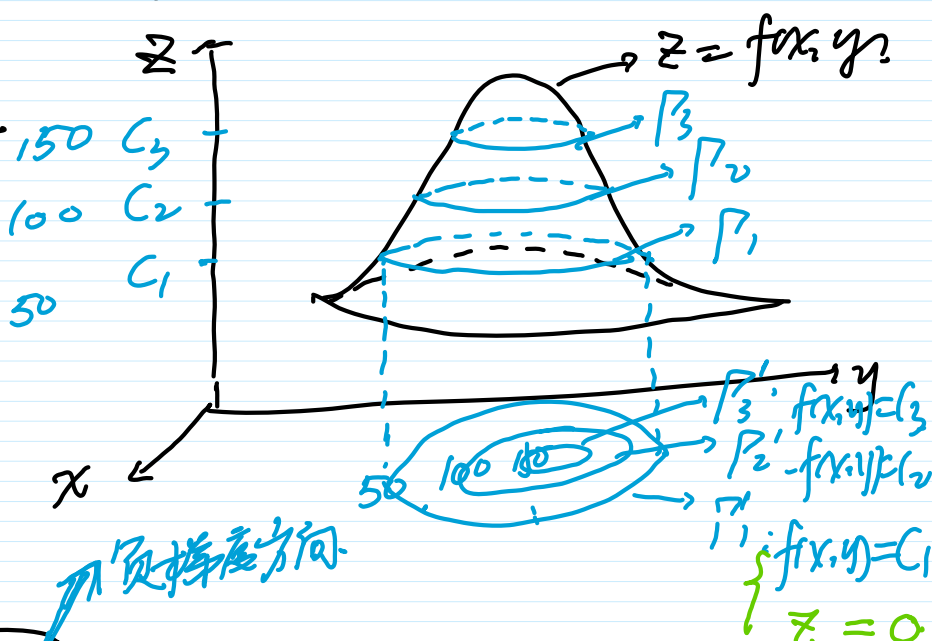
函数沿负梯度方向减少最快.

2) $\theta = \frac{\pi}{2}, \cos \theta = 0$, 方向导数为 0.

$\vec{r} \perp \nabla f$, 方向导数沿与梯度垂直为 0

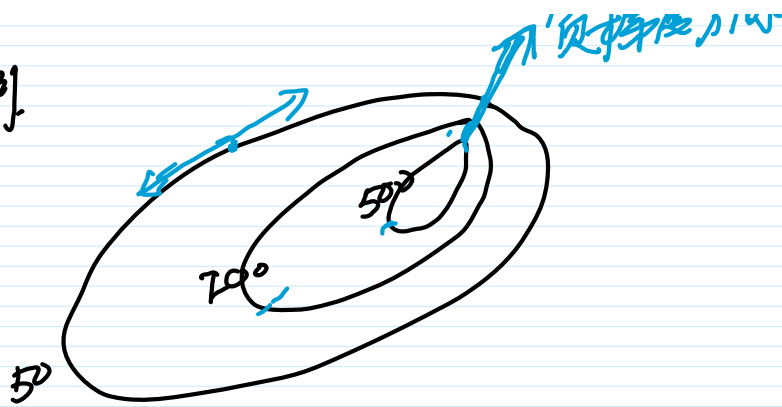
函数沿与梯度垂直的方向不变.

等值线: $z = f(x, y)$.



例

例



$$\begin{cases} f(x, y) = z \\ z = 0 \end{cases}$$

例. $z = f(x, y)$ 在 $M_0(x_0, y_0)$ 沿 x轴正半轴 的方向导数最大, 且为 25, 则 _____

解: ∇f 与 x 轴正半轴同方向. $\vec{i} = (1, 0)$.

设 $\nabla f = a\vec{i} = a(1, 0)$ 且 $a > 0$. $\nabla f = a(1, 2)$
 $= (a, 2a)$
 $a > 0$.

$$\nabla f = (25, 0) = (f_x(x_0, y_0), f_y(x_0, y_0))$$

$$|\nabla f| = 25 = \sqrt{a^2 + (2a)^2} = a \cdot \sqrt{5}.$$

例. $\vec{r} = (0, 2, 2)$ $\nabla f \cdot \vec{r} = 1$.

$$\text{则 } \frac{\partial f}{\partial r} \Big|_{(x_0, y_0)} = \frac{1}{\sqrt{0^2 + 2^2 + 2^2}} \cdot 1 = \frac{1}{2\sqrt{2}}.$$

解: $\frac{\partial f}{\partial r} \Big|_{(x_0, y_0)} = \nabla f \cdot \frac{1}{|\vec{r}|} \vec{r} = \frac{1}{|\vec{r}|} \nabla f \cdot \vec{r}$

第八节 极值.

极值
 最值
 条件极值 \rightarrow 最值

一元函数

$$y=f(x).$$

极值点: $\begin{cases} \text{驻点 } f'(x)=0 & \text{例: } y=x^3, x=0 \\ \text{不可导点} & \text{例: } y=|x|, x=0 \end{cases}$

$$f'(x_0)=0, f''(x_0) \neq 0. \begin{cases} f''(x_0) > 0 \Rightarrow \text{极小值点} & \text{例 } y=x^2 \\ f''(x_0) < 0 \Rightarrow \text{极大值点} & \text{例 } y=-x^2 \end{cases}$$

定理: $z=f(x,y)$ 在 $U(p_0)$ 各阶偏导连续.

泰勒公式: 令 $x-x_0=h, y-y_0=k$.

$$f(x,y) = f(x_0,y_0) + \frac{1}{1!} [f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)] + \frac{1}{2!} [f_{xx}(x_0,y_0)\underbrace{(x-x_0)^2}_{h^2} + 2f_{xy}(x_0,y_0)\underbrace{(x-x_0)(y-y_0)}_{hk} + f_{yy}(x_0,y_0)\underbrace{(y-y_0)^2}_{k^2}] + R.$$

$$f(x_0+h, y_0+k) = f(x_0, y_0) + \frac{1}{1!} (h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y}) f(x_0, y_0) + \frac{1}{2!} (h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y})^2 f(x_0, y_0) + \dots + \frac{1}{n!} (h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y})^n f(x_0, y_0) + R.$$

定义: $z=f(x,y)$ 在 $U(p_0)$ 有定义. $p_0(x_0,y_0)$. $p(x,y) \in U(p_0)$

$\forall p \in U(p_0)$. 都有 $f(p) \geq f(p_0)$ (或 $f(p) \leq f(p_0)$)

称 $p_0(x_0,y_0)$ 为极小(大)值点.

$f(p_0)$ 为极小(大)值

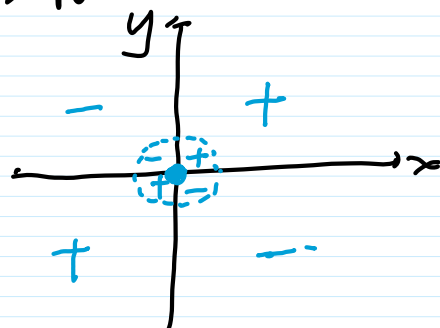
例: $z=f(x,y)=x^2+y^2$. $(0,0)$ 是极小值点.

例. $z = f(x, y) = x^2 + y^2$. $(0, 0)$ 是极小值点.

例. $z = f(x, y) = xy$. $(0, 0)$ 是否极值点.

解. $f(0, 0) = 0$

但 $(0, 0)$ 不是极值点.



例. $\lim_{(x,y) \rightarrow (0,0)} \frac{f(x,y) - xy}{(x^2+y^2)^2} = 1$. 且 $f(x,y)$ 连续.

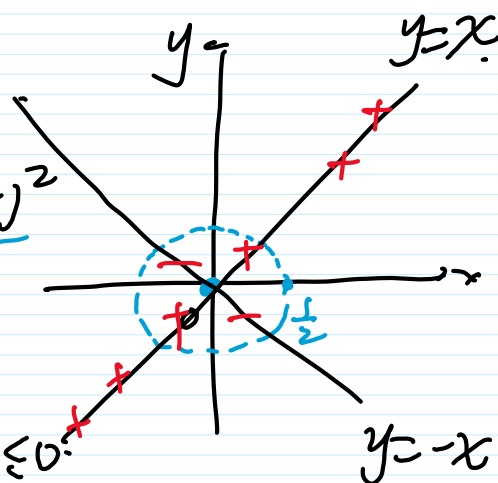
判断 $(0, 0)$ 不是 极值点.

解. $f(0, 0) = 0$

① 法. 中等值法: 设 $f(x, y) = \boxed{xy} + (x^2 + y^2)^2$

$$y = x \quad f(x, x) = x^2 + 4x^4 \geq 0$$

$$y = -x \quad f(x, -x) = -x^2 + 4x^4 = x^2(4x^2 - 1) \leq 0$$



② 法: $f(x, y) = xy + \boxed{(x^2 + y^2)^2} + o((x^2 + y^2)^2)$ $\lim_{x \rightarrow x_0} f(x) = A$

$f(x) = A + o(x)$

定理1 极值必要条件: 可微的极值点必为驻点.

$z = f(x, y)$ 可微且取得极值

$$\Rightarrow \nabla f = (f_x, f_y) = \vec{0}$$

定义: 驻点. $\nabla f = \vec{0}$.

证明: $\frac{\partial f(x_0, y_0)}{\partial x} = \frac{df(x, y_0)}{dx} \Big|_{x=x_0} = 0$

记 $g(x) = f(x, y_0)$

$\forall (x, y) \in U(P_0)$

不妨设 (x_0, y_0) 是极大值点

$f(x, y) \geq f(x_0, y_0)$

$f(x, y_0) \geq f(x_0, y_0)$

$\exists p \quad g(x) \geq g(x_0)$

$x = x_0$ 是 $g(x)$ 可导的极值点 $\Rightarrow g'(x_0) = 0$

定理: 极值的充分条件.

$\nabla f(x_0, y_0) = \vec{0}$. 二阶偏导连续

$A = f_{xx}(x_0, y_0) \quad B = f_{xy}(x_0, y_0) \quad C = f_{yy}(x_0, y_0)$

则有

(i) $AC - B^2 > 0$. 取得极值 $\begin{cases} A > 0 \Rightarrow \text{极小值点} \quad f = x^2 + y^2 \\ A < 0 \Rightarrow \text{极大值点} \quad f = -x^2 - y^2 \end{cases}$

(ii) $AC - B^2 < 0$ 不是极值点 例: $f(x, y) = xy \quad (0, 0)$

(iii) $AC - B^2 = 0$ 无法判断.

$$f(x_0+h, y_0+k) = f(x_0, y_0) + \frac{1}{2!} (A \cdot h^2 + 2Bh \cdot k + C \cdot k^2) + R$$

$$A h^2 + 2B h \cdot k + C \cdot k^2 = (h, k) \begin{pmatrix} A & B \\ B & C \end{pmatrix} \begin{pmatrix} h \\ k \end{pmatrix}$$

黑塞矩阵.

$A > 0, AC - B^2 > 0$

$\begin{pmatrix} A & B \\ B & C \end{pmatrix}$ 正定矩阵.

$$\underline{A > 0, AC - B^2 > 0} \quad \begin{pmatrix} A & B \\ B & C \end{pmatrix} \text{ 正定阵,}$$

$$\Rightarrow Ah^2 + 2Bh \cdot k + C \cdot k^2 \geq 0$$

$$f(x_0+h, y_0+k) - f(x_0, y_0) = \frac{1}{2!} C \quad \geq 0$$

$\Rightarrow (x_0, y_0)$ 极小值点

$$A < 0, AC - B^2 > 0 \quad -\begin{pmatrix} A & B \\ B & C \end{pmatrix} = \begin{pmatrix} -A & -B \\ -B & -C \end{pmatrix} \text{ 正定阵.}$$

$$\Rightarrow -(Ah^2 + 2Bh \cdot k + C \cdot k^2) \geq 0$$

$$\text{即 } Ah^2 + 2Bh \cdot k + C \cdot k^2 \leq 0$$

$$f(x_0+h, y_0+k) - f(x_0, y_0) \leq 0 \Rightarrow (x_0, y_0) \text{ 极大值点}$$