第六节 几何应用

空间曲线：

$$\begin{cases} x=\varphi(t) \\ y=\psi(t) \\ z=\omega(t) \end{cases}$$

或 $\begin{cases} F(x,y,z)=0 \\ G(x,y,z)=0 \end{cases}$

空间直线：

$$\begin{cases} x=x_0+lt \\ y=y_0+mt \\ z=z_0+nt \end{cases}$$

方向向量 $\vec{s}=(l,m,n)$ ✓

点向式：$\boxed{\dfrac{x-x_0}{l}=\dfrac{y-y_0}{m}=\dfrac{z-z_0}{n}}$

空间曲面：$F(x,y,z)=0$

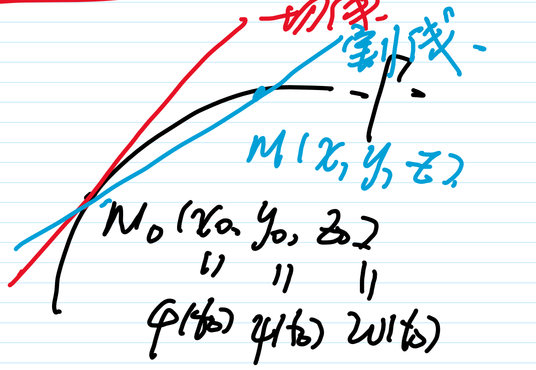
或 $z=f(x,y)$

空间平面：$Ax+By+Cz+D=0$

法向量 $\vec{n}=(A,B,C)$ ✓

点法式 $\boxed{A(x-x_0)+B(y-y_0)+C(z-z_0)=0}$

一、空间曲线的切线与法平面



空间曲线

$$\Gamma: \begin{cases} x=\varphi(t) \\ y=\psi(t) \\ z=\omega(t) \end{cases}$$

割线方向向量：$\overrightarrow{M_0M}=(x-x_0,\ y-y_0,\ z-z_0)\to\vec{0}$

$(M\to M_0)$

$$\frac{1}{t-t_0}\overrightarrow{M_0M}=\left(\frac{x-x_0}{t-t_0},\frac{y-y_0}{t-t_0},\frac{z-z_0}{t-t_0}\right)\to\vec{T}.$$

$(t\to t_0)$

$(M\to M_0)$.

定义：切向量 $\vec{T}$：即 切线的方向向量

$$\boxed{\vec{T}=\lim_{t\to t_0}\frac{1}{t-t_0}\overrightarrow{M_0M}=(\varphi'(t_0),\psi'(t_0),\omega'(t_0)).}$$

切线方程：$\dfrac{x-x_0}{\varphi'(t_0)}=\dfrac{y-y_0}{\psi'(t_0)}=\dfrac{z-z_0}{\omega'(t_0)}$

法平面方程：$\varphi'(t_0)(x-x_0)+\psi'(t_0)(y-y_0)+\omega'(t_0)(z-z_0)=0$

参数形式

一般形式：

$$\Gamma:\begin{cases}F(x,y,z)=0\\G(x,y,z)=0\end{cases}\longrightarrow\begin{cases}x=x\\y=y(x)\\z=z(x)\end{cases}$$

在 $M_0(x_0,y_0,z_0)$ 的切线和法平面方程

$\vec{T}=(1,y'(x),z'(x))$

$$J=\frac{\partial(F,G)}{\partial(y,z)}=\begin{vmatrix}F_y&F_z\\G_y&G_z\end{vmatrix}$$

$$y'(x)=-\frac{1}{J}\frac{\partial(F,G)}{\partial(x,z)}=-\frac{1}{J}\begin{vmatrix}F_x&F_z\\G_x&G_z\end{vmatrix}$$

$$z'(x)=-\frac{1}{J}\frac{\partial(F,G)}{\partial(y,x)}=-\frac{1}{J}\begin{vmatrix}F_y&F_x\\G_y&G_x\end{vmatrix}.$$

$$\vec{T}=\left(1,\ -\frac{1}{J}\begin{vmatrix}F_x&F_z\\G_x&G_z\end{vmatrix},\ -\frac{1}{J}\begin{vmatrix}F_y&F_x\\G_y&G_x\end{vmatrix}\right).$$

$$\boxed{\vec{T}=J\vec{T}=\left(\begin{vmatrix}F_y&F_z\\G_y&G_z\end{vmatrix},\ -\begin{vmatrix}F_x&F_z\\G_x&G_z\end{vmatrix},\ \begin{vmatrix}F_x&F_y\\G_x&G_y\end{vmatrix}\right)}$$

$$\begin{vmatrix}\vec{i}&\vec{j}&\vec{k}\end{vmatrix}=\ \ldots$$

$$\vec{T}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ F_x & F_y & F_z\\ G_x & G_y & G_z\end{vmatrix}=\vec{\alpha}\times\vec{\beta}$$

$\vec{\alpha}=(F_x, F_y, F_z)$

$\vec{\beta}=(G_x, G_y, G_z)$

例. $\begin{cases} y=\sqrt{2x} \\ z=1-x \end{cases}$ 在 $(2,2,-1)$ 处的切线方程

解: $\vec{T}=\left(1, \sqrt{2}\cdot\frac{1}{2\sqrt{x}}, -1\right)\Big|_{(2,2,-1)}$

$=\left(1, \frac{1}{2}, -1\right)$. $\vec{T}=\underline{(2,1,-2)}$

切线方程: $\frac{x-2}{1}=\frac{y-2}{\frac{1}{2}}=\frac{z+1}{-1}$.

或. $\frac{x-2}{2}=\frac{y-2}{1}=\frac{z+1}{-2}$.

例. $\begin{cases} x^2+y^2+z^2=6 \\ x+y+z=0 \end{cases}$ 在 $(1,1,-2)$ 处的法平面方程.

解: $F(x,y,z)=x^2+y^2+z^2-6$

$G(x,y,z)=x+y+z$.

$F_x=2x$, $F_y=2y$, $F_z=2z$

$\vec{\alpha}=(2x, 2y, 2z)\Big|_{(1,1,-2)}=(2,2,-4)=2\underline{(1,1,-2)}$

$G_x=1.\quad G_y=1.\quad G_z=1$

$$\vec{T_2}=(1,1,1)\Big|_{(1,1,-2)}=(1,1,1)$$

$$\vec{T}=\vec{T_1}\times\vec{T_2}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 1 & 1 & -2\\ 1 & 1 & 1\end{vmatrix}$$

$$=\left(\begin{vmatrix}1 & -2\\ 1 & 1\end{vmatrix},\ -\begin{vmatrix}1 & -2\\ 1 & 1\end{vmatrix},\ \begin{vmatrix}1 & 1\\ 1 & 1\end{vmatrix}\right)$$

$$=(3,\ -3,\ 0).=3(1,-1,0)$$

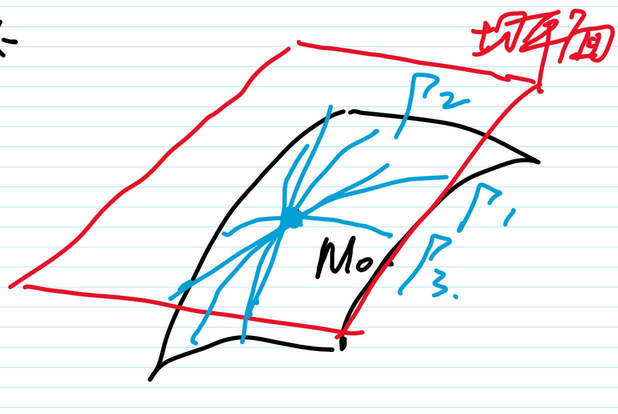
法平面方程: $1\cdot(x-1)+(-1)\cdot(y-1)+0\cdot(z+2)=0.$

即 $x-y=0$

切线方程: $$\frac{x-1}{1}=\frac{y-1}{-1}=\frac{z+2}{0}$$

或者 $\begin{cases}x+y=2.\\ z+2=0\end{cases}$

二. 空间曲面的切平面与法线



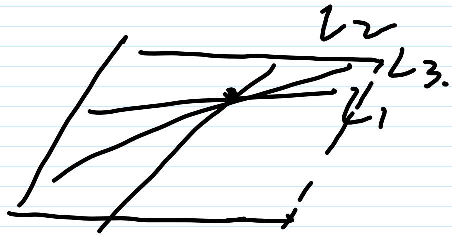
空间曲面: $F(x,y,z)=0$

$M_0(x_0,y_0,z_0)$

定义: 切平面是由切线共面构成的

Γ Γ Γ 三条曲线. 相交. 去记共面.

L_1, L_2, L_3 三条直线. 相交. 去证共面.



$L \perp L_1 \quad L \perp L_2$

$\left.\begin{array}{l} \Rightarrow L \perp L_1, L_2 \text{所确定平面} \\ L \perp L_3 \end{array}\right\} \Rightarrow L_1, L_2, L_3$ 共面

证切线共面 $\Longrightarrow$ 证 $L \perp$ 所有的切线.

$\Longleftrightarrow a \cdot \vec{T}_i = 0$.

设曲面内过 M_0 的任一条曲线 Γ_i: $\begin{cases} x = \varphi_i(t) \\ y = \psi_i(t) \\ z = \omega_i(t) \end{cases}$

$x_0 = \varphi_i(t_0)$

$y_0 = \psi_i(t_0)$

$z_0 = \omega_i(t_0)$

对 $F(\varphi_i(t), \psi_i(t), \omega_i(t)) = 0$ 对 t 求导

$$F_1' \cdot \varphi_i'(t) + F_2' \cdot \psi_i'(t) + F_3' \cdot \omega_i'(t) = 0.$$

$$\underbrace{(F_1', F_2', F_3')}_{= \vec{n}} \cdot \underbrace{(\varphi_i'(t), \psi_i'(t), \omega_i'(t))}_{T_i(t)} = 0$$

定义: $\boxed{\text{法向量 } \vec{n} = (F_x(x_0, y_0, z_0), F_y(x_0, y_0, z_0), F_z(x_0, y_0, z_0))}$

切平面方程：$F_x(P_0)(x-x_0)+F_y(P_0)\cdot(y-y_0)+F_z(P_0)(z-z_0)=0$

法线方程：$\dfrac{x-x_0}{F_x(P_0)}=\dfrac{y-y_0}{F_y(P_0)}=\dfrac{z-z_0}{F_z(P_0)}$

$z=f(x,y)$，$\vec{n}=(f_x,\ f_y,\ -1)|_{P_0}$

$\downarrow$ 或 $\vec{n}=(-f_x,\ -f_y,\ 1)|_{P_0}$

$f(x,y)-z=0$

或 $z-f(x,y)=0$

例 $z=4-x^2-y^2$．切平面平行于 $2x+2y+z-1=0$、

$4-x^2-y^2-z=0$ 或 $x^2+y^2+z-4=0$

$2x+2(-y)+z-1=0$

$\vec{n}_3=(2,-2,1)$

解：$\pi_1 \parallel \pi_2 \Leftrightarrow \vec{n}_1 \parallel \vec{n}_2$

设切点 (x_0,y_0,z_0)

法向量 $\vec{n}_1=(-2x,-2y,-1)|_{(x_0,y_0,z_0)}$

$=(-2x_0,-2y_0,-1)$

法向量 $\vec{n}_2=(2,2,1)$

由 $\vec{n}_1 \parallel \vec{n}_2$，$\Rightarrow$ $\dfrac{-2x_0}{2}=\dfrac{-2y_0}{2}=\dfrac{-1}{1} \Rightarrow x_0=1,\ y_0=1.$

$z_0=4-x_0^2-y_0^2=2.$

切平面方程：$2(x-1)+2(y-1)+1\cdot(z-2)=0$

$z=f(x,y)$ 的切平面方程 $(x_0, y_0, f(x_0,y_0))$

$$\vec{n}=(f_x(x_0,y_0),\ f_y(x_0,y_0),\ -1)$$

$$f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)+(-1)\cdot(z-f(x_0,y_0))=0$$

$$z=f(x_0,y_0)+\underbrace{f_x(x_0,y_0)\cdot(\overset{dx}{x-x_0})+f_y(x_0,y_0)(\overset{dy}{y-y_0})}_{df(x,y)|_{(x_0,y_0)}}$$

例

直线 $\begin{cases} x+y+b=0 \\ x+ay-z-3=0 \end{cases}$ 在平面 π 上

且 π 与 $z=x^2+y^2$ 相切于 $(1,-2,5)$，则 $a=$_ $b=$__.

解：$\vec{n}=(2x, 2y, -1)|_{(1,-2,5)}=(2,-4,-1)$.

平面 π：$2\times(x-1)+(-4)(y+2)+(-1)\cdot(z-5)=0$

$$2x-4y-z-5=0.$$

(2)法：$\begin{cases} x+y+b=0 \\ x+ay-z-3=0 \\ 2x-4y-z-5=0 \end{cases}$ $\quad r(A)=r(A\vdots\eta)<3$.

$r(A)=r(A\vdots\eta)=2$.

$$(A\vdots\eta)=\left(\begin{array}{ccc|c} 1 & 1 & 0 & -b \\ 1 & a & -1 & 3 \\ 2 & -4 & -1 & 5 \end{array}\right)$$

$a=-5$. $b=-2$.

②法：平面束方程.

设π：$\lambda(x+y+6)+\mu(x+ay-z-3)=0$

π：$2x-4y-z-5=0$

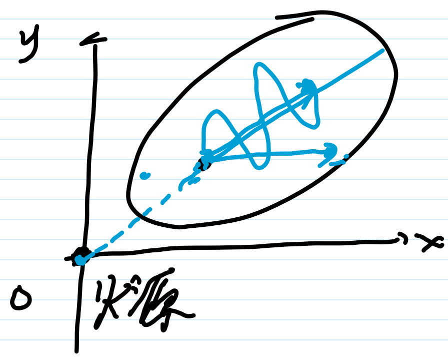
第七节 方向导数与梯度

实数　　向量

偏导数：偏增量与自变量增量之间关系

方向导数：全增量与自变量增量之间关系

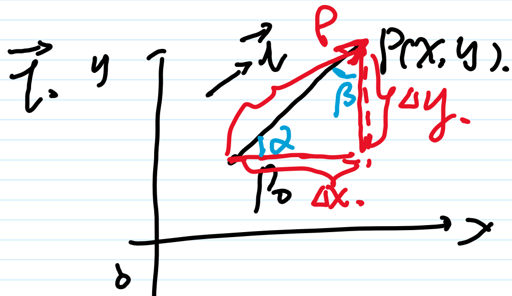
引例



定义：$z=f(x,y)$ 在 $U(P_0)$ 有定义.

沿 $\vec{l}$ 运动到 $P(x,y)$.

全增量 $\Delta z=f(x,y)-f(x_0,y_0)$



$\Delta x=\rho\cdot\cos\alpha$.

$\Delta y=\rho\cdot\cos\beta$.

距离 $\rho=|\overrightarrow{P_0P}|=\sqrt{(x-x_0)^2+(y-y_0)^2}$

若 $\lim\limits_{\rho\to0^+}\dfrac{f(x,y)-f(x_0,y_0)}{\rho}$ 存在. 则称为

$z=f(x,y)$ 在 $P_0(x_0,y_0)$ 沿 $\vec{l}$ 的方向导数 记为 $\left.\dfrac{\partial f}{\partial \vec{l}}\right|_{(x_0,y_0)}$

$z=f(x,y)$ 在 $P_0(x_0,y_0)$ 沿 $\vec{l}$ 的方向导数 记为 $\frac{\partial f}{\partial \vec{l}}\Big|_{(x_0,y_0)}$

方向导数：定义

$$\frac{\partial f}{\partial \vec{l}}\Big|_{(x_0,y_0)}=\lim_{\rho\to 0^+}\frac{f(x_0+\rho\cos\alpha,\ y_0+\rho\cos\beta)-f(x_0,y_0)}{\rho}$$

$$(\cos\alpha,\cos\beta)=\frac{1}{|\vec{l}|}\vec{l}$$

例：$\vec{l}=(2,-3)$.
$$\frac{1}{|\vec{l}|}\vec{l}=\frac{1}{\sqrt{2^2+(-3)^2}}(2,-3)$$
$$=\left(\frac{2}{\sqrt{13}},-\frac{3}{\sqrt{13}}\right)$$

$$\cos\alpha=\frac{2}{\sqrt{13}}\quad \cos\beta=-\frac{3}{\sqrt{13}}.$$

方向导数与偏导数之间关系.

(1) 沿 x 轴正半轴的方向导数，$\vec{l}=(1,0)$.

$$\frac{\partial f}{\partial x^+}\Big|_{(x_0,y_0)}=\lim_{\rho\to 0^+}\frac{f(x_0+\rho\cdot 1,\ y_0+\rho\cdot 0)-f(x_0,y_0)}{\rho}$$
$$=\lim_{\rho\to 0^+}\frac{f(x_0+\rho,\ y_0)-f(x_0,y_0)}{\rho}$$
$$\overset{\rho=\Delta x}{=}\lim_{\Delta x\to 0^+}\frac{f(x_0+\Delta x,y_0)-f(x_0,y_0)}{\Delta x}$$

$f_x(x_0,y_0)$ 存在 $\Longrightarrow \frac{\partial f}{\partial x^+}\Big|_{(x_0,y_0)}=f_x(x_0,y_0)$.

(2) 沿 x 轴负半轴 $\vec{l}=(-1,0)$

$$\cdots\ \frac{f(x_0-\rho,\ y_0)-f(x_0,y_0)}{\ }$$

(2) 沿 x 轴负方向 $\vec{l} = -\vec{i} = (-1, 0)$

$$\left.\frac{\partial f}{\partial x^-}\right|_{(x_0,y_0)} = \lim_{\rho\to 0^+}\frac{f(x_0-\rho, y_0)-f(x_0,y_0)}{\rho}$$

$$\overset{\Delta x = -\rho}{=\!=}\lim_{\Delta x\to 0^-}\frac{f(x_0+\Delta x, y_0)-f(x_0,y_0)}{-\Delta x}$$

$f_x(x_0,y_0)$ 存在 $\Rightarrow \left.\frac{\partial f}{\partial x^-}\right|_{(x_0,y_0)} = -f_x(x_0,y_0)$.

(3)

(4). $f_y(x_0,y_0)$ 存在 $\Rightarrow \begin{cases} \left.\frac{\partial f}{\partial y^+}\right|_{(x_0,y_0)} = f_y(x_0,y_0) \\ \left.\frac{\partial f}{\partial y^-}\right|_{(x_0,y_0)} = -f_y(x_0,y_0). \end{cases}$

推广：$u = f(x,y,z)$，$\vec{l}$.

$$\left.\frac{\partial u}{\partial \vec{l}}\right|_{(x_0,y_0,z_0)} = \lim_{\rho\to 0^+}\frac{u(x_0+\rho\cos\alpha, y_0+\rho\cos\beta, z_0+\rho\cos\gamma)-u(x_0,y_0,z_0)}{\rho}.$$

其中 $(\cos\alpha, \cos\beta, \cos\gamma) = \frac{1}{|\vec{l}|}\vec{l}$.

例. $z = \sqrt{x^2+y^2}$. 在 $(0,0)$ 连续. 偏导不存在. 不可微.

讨论 $(0,0)$ 沿各个方向的方向导数是否存在.

解：$$\left.\frac{\partial f}{\partial \vec{l}}\right|_{(0,0)} = \lim_{\rho\to 0^+}\frac{f(0+\rho\cos\alpha, 0+\rho\cos\beta)-f(0,0)}{\rho}$$

$$= \lim_{\rho\to 0^+}\frac{\sqrt{\rho^2(\cos^2\alpha+\cos^2\beta)}-0}{\rho} = \lim_{\rho\to 0^+}\frac{\rho}{\rho} = 1.$$

$\frac{xy}{\sqrt{x^2+y^2}}$

例 $f = \frac{x^2 y}{\ldots}$ ... $f(0,0)$

例. $f(x,y)=\begin{cases}\dfrac{x^2y}{\sqrt{x^2+y^2}}\sin\dfrac{1}{x^2+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$ $\sqrt{x^2+y^2}$

? X ? X

讨论在(0,0)处连续性. 偏导存在. 可微性 及方向导数.

解: $f_x(0,0)=\lim\limits_{\Delta x\to 0}\dfrac{f(\Delta x,0)-f(0,0)}{\Delta x}=\lim\limits_{\Delta x\to 0}\dfrac{0-0}{\Delta x}=0$

$f_y(0,0)=0.$

$$\lim_{(\Delta x,\Delta y)\to(0,0)}\frac{[f(\Delta x,\Delta y)-f(0,0)]-[f_x(0,0)\cdot\Delta x+f_y(0,0)\Delta y]}{\sqrt{(\Delta x)^2+(\Delta y)^2}}$$

$$=\lim_{(\Delta x,\Delta y)\to(0,0)}\frac{(\Delta x)^2\cdot\Delta y}{(\Delta x)^2+(\Delta y)^2}\cdot\sin\frac{1}{(\Delta x)^2+(\Delta y)^2}=0$$

夹逼、极坐标变换

$$\lim_{\rho\to 0^+}\frac{f(0+\rho\cos\alpha,0+\rho\cos\beta)-f(0,0)}{\rho}$$

$$=\lim_{\rho\to 0^+}\frac{\dfrac{\rho^2\cos^2\alpha\cdot\rho\cos\beta}{\rho}\cdot\sin\dfrac{1}{\rho^2}}{\rho}$$

$$=\lim_{\rho\to 0^+}\rho\cos^2\alpha\cos\beta\sin\frac{1}{\rho^2}$$

$=0$

定理: 可微 ⇒ 方向导数存在且

$$\left.\frac{\partial f}{\partial \vec{l}}\right|_{(x_0,y_0)} = f_x(x_0,y_0)\cdot\cos\alpha + f_y(x_0,y_0)\cdot\cos\beta$$

其中$(\cos\alpha,\cos\beta) = \frac{\vec{l}}{|\vec{l}|}$

证明：$\left.\frac{\partial f}{\partial \vec{l}}\right|_{(x_0,y_0)} = \lim\limits_{\rho\to 0^+}\frac{\Delta z}{\rho}$

$$\overset{\text{可微}}{=\!=} \lim_{\rho\to 0^+}\frac{f_x(x_0,y_0)\cdot\Delta x + f_y(x_0,y_0)\cdot\Delta y + o(\sqrt{(\Delta x)^2+(\Delta y)^2})}{\rho}$$

$$\overset{\substack{\Delta x=\rho\cos\alpha\\ \Delta y=\rho\cos\beta}}{=\!=} \lim_{\rho\to 0^+}\left[\frac{f_x(x_0,y_0)\cdot\cancel{\rho}\cos\alpha + f_y(x_0,y_0)\cdot\cancel{\rho}\cos\beta}{\cancel{\rho}} + \frac{o(\rho)}{\rho}\right]$$

$$= f_x(x_0,y_0)\cdot\cos\alpha + f_y(x_0,y_0)\cdot\cos\beta$$

推广：$u=f(x,y,z)$ 可微

$$\left.\frac{\partial u}{\partial \vec{l}}\right|_{(x_0,y_0,z_0)} = f_x(P_0)\cdot\cos\alpha + f_y(P_0)\cos\beta + f_z(P_0)\cos\gamma.$$