

$$dx = \cos \alpha \cdot ds \quad dy = \cos \beta \cdot ds$$

$$\int_L p dx + Q dy = \int_L (p \cdot \cos \alpha + Q \cdot \cos \beta) ds$$

$$\int_\Gamma p dx + Q dy + R dz = \int_\Gamma (p \cos \alpha + Q \cos \beta + R \cos \gamma) ds$$

$$\cos \alpha = \frac{dx}{ds} = \pm \frac{x'(t) dt}{\sqrt{x'^2 + y'^2} dt}$$

$$\cos \beta = \frac{dy}{ds} = \pm \frac{y'(t) dt}{\sqrt{x'^2 + y'^2} dt}$$

有向 L:  $t: \alpha \rightarrow \beta$  若  $\alpha \leq \beta$ , 则取 "+"  
若  $\alpha \geq \beta$ , 则取 "-"

$$\int x dy - y dx$$

例: 将  $\int_L (-y) dx + x dy$

$$L: y = \sqrt{a^2 - x^2} \quad (a > 0)$$

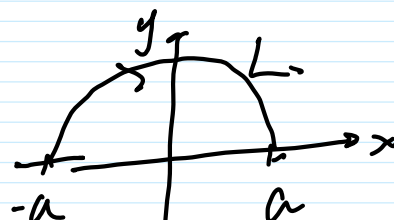
化为 I 型曲线积分

解: 由法 L:  $\begin{cases} x = a \cos t \\ y = a \sin t \end{cases}$

$$t: \pi \rightarrow 0$$

逆时针

$$(-a, 0) \rightarrow (a, 0)$$



$$\cos \alpha = - \frac{x'(t)}{\sqrt{x'^2 + y'^2}} = - \frac{-a \sin t}{a} = \sin t = \frac{y}{a}$$

$$\cos \beta = - \frac{y'(t)}{a} = - \frac{a \cos t}{a} = - \cos t = - \frac{x}{a}$$

$$I = \int_L [-y \cdot \sin t + x \cdot (-\cos t)] ds$$

$$= \int_L (-y \cdot \frac{y}{a} - x \cdot \frac{x}{a}) ds = \int_L \frac{-x^2 - y^2}{a} ds$$

$$= \int_L \frac{-a^2}{a} ds = -a \int_L 1 ds = -a \cdot \pi a = -\pi a^2.$$

572:  $L: y = \sqrt{a^2 - x^2}$ .  $x: -a \rightarrow a$   
 $\overline{xy} \leq \overline{yz}$

$$\cos \alpha = + \frac{1}{\sqrt{1+y'^2}} = \dots$$

$$\cos \beta = + \frac{y'}{\sqrt{1+y'^2}} = \dots$$

第三讲 格林公式.

1. 单连通区域: 无洞

复连通区域: 有洞



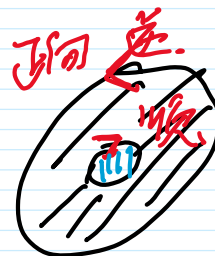
$x^2 + y^2 \leq 1$  单.



$1 \leq x^2 + y^2 \leq 4$  复.

$0 < x^2 + y^2 \leq 1$  复.

2. L 正向



3. 定理 (格林公式).

(1) D 单连通

(2) L 正向

(3) P, Q 偏导连续.

$$\left| \frac{\partial}{\partial x} \quad \frac{\partial}{\partial y} \right|$$

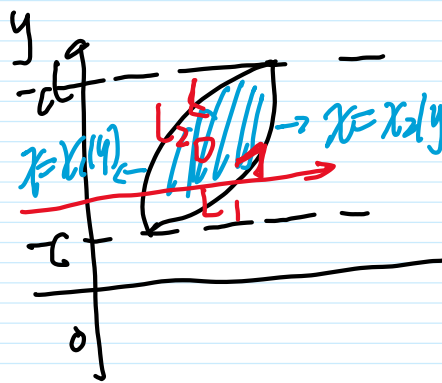
P Q

③  $P, Q$  偏导连续.  $\begin{vmatrix} \frac{\partial x}{\partial P} & \frac{\partial y}{\partial Q} \\ P & Q \end{vmatrix}$

$$\Rightarrow \oint_L \underbrace{P}_{\text{复杂}} dx + \underbrace{Q}_{\text{简单}} dy = \iint_D \left( \underbrace{\frac{\partial Q}{\partial x}}_{\text{简单}} - \frac{\partial P}{\partial y} \right) dx dy.$$

证明  $\oint_L Q dy = \iint_D \frac{\partial Q}{\partial x} dx dy.$

$$\iint_D \frac{\partial Q}{\partial x} dx dy = \int_a^b dx \int_{y_1(x)}^{y_2(x)} \frac{\partial Q}{\partial x} dy$$



$$= \int_c^d dy \int_{x_1(y)}^{x_2(y)} \frac{\partial Q}{\partial x} dx$$

$$= \int_c^d \left[ \int_{x_1(y)}^{x_2(y)} dQ(x, y) \right] dy$$

$$= \int_c^d [Q(x_2(y), y) - Q(x_1(y), y)] dy$$

$L = L_1 + L_2$  其中  $L_1: x = x_2(y), y: c \rightarrow d$   
 $L_2: x = x_1(y), y: d \rightarrow c$

$$\oint_L Q dy = \int_{L_1} Q(x_2(y), y) dy + \int_{L_2} Q(x_1(y), y) dy$$

$$= \int_c^d Q(x_2(y), y) \cdot 1 dy + \int_d^c Q(x_1(y), y) \cdot 1 dy$$

$$= \int_c^d [Q(x_2(y), y) - Q(x_1(y), y)] dy = \text{右}.$$

推广1.  $\square$  单连通.

推论1.

① 单连通.

②  $P, Q$  偏导连续.

$$\Rightarrow \oint_L P dx + Q dy = \pm \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy$$

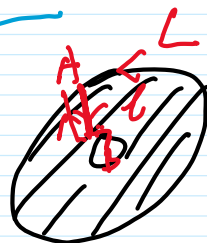
L 正向, 取“+” ; L 负向, 取“-”.

推论2.

①  $D$  复连通.

②  $L, l$  正向

③  $P, Q$  偏导连续



$$\Rightarrow \oint_{L+l} P dx + Q dy = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy$$

$$\oint_{L + \overline{AB} + l + \overline{BA}} P dx + Q dy = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy$$

||

$$\oint_{L+l} P dx + Q dy + \underbrace{\oint_{\overline{AB} + \overline{BA}} P dx + Q dy}_0$$

证明:

$$\iint_D 1 dx dy = \sigma.$$

当  $Q=x, P=0$  时

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1.$$

或  $Q=\frac{x}{2}, P=-\frac{y}{2}$  时

$$\text{例 } Q = \frac{x}{2}, P = -\frac{y}{2}$$

$$\oint_C \frac{x}{2} dy - \frac{y}{2} dx = \iint_D 1 dxdy = \sigma.$$

$$\text{例: } L: \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (a>0, b>0) \text{ 所围成面积}$$

$$\text{解: } S = \frac{1}{2} \oint_L x dy - y dx.$$

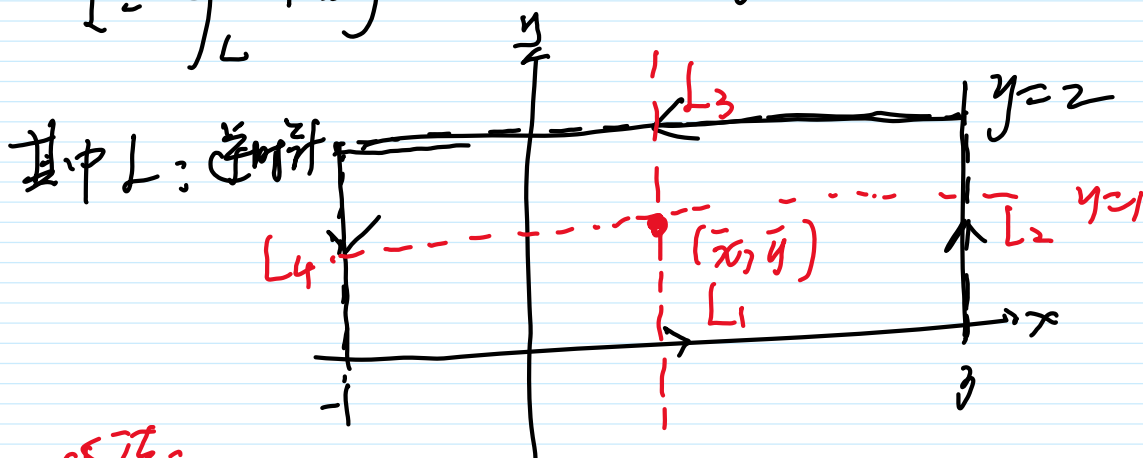
其中  $L$ : 逆时针.

$$L: \begin{cases} x = a \cos t \\ y = b \sin t, \end{cases} \quad t: 0 \rightarrow 2\pi$$

$$S = \frac{1}{2} \int_0^{2\pi} [a \cos t \cdot b \cos t - b \sin t \cdot (-a \sin t)] dt$$

$$= \frac{1}{2} \int_0^{2\pi} ab dt = \pi ab$$

$$\text{例: } I = \oint_L 4xy dx + 3x^2 dy$$



解: 顺时针:

$$L = L_1 + L_2 + L_3 + L_4.$$

$$L_1: y=0, \quad x: -1 \rightarrow 3 \quad (dy=0)$$

$$I = \int_{-1}^3 4xy dx + 3x^2 dy = \int_{-1}^3 4xy dx$$

$$I_{L_1} = \int_{L_1} 4xy \, dx + \underline{3x^2} \, dy = \int_{L_1} 4xy \, dx$$

$$= \int_{-1}^3 4x \cdot 0 \, dx = 0$$

$$I_{L_2} \stackrel{dx=0}{=} \int_{L_2} Q \, dy = \int_0^2 Q(3, y) \cdot 1 \, dy$$

$$= \int_0^2 3 \cdot 3^2 \, dy = \dots$$

$$I_{L_3} \stackrel{dy=0}{=} \int_{L_3} P \, dx = \int_3^{-1} P(x, 2) \cdot 1 \, dx$$

$$= \int_3^{-1} 4x \cdot 2 \, dx = \dots$$

$$I_{L_4} = \int_{L_4} Q \, dy = \int_2^0 3 \cdot (-1)^2 \, dy = \dots$$

②法: 格林公式

$$I = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx \, dy$$

$$= \iint_D (6x - 4x) \, dx \, dy$$

$$= 2 \iint_D x \, dx \, dy \stackrel{\text{平均值}}{=} 2 \cdot \bar{x} \cdot 6 = 2 \times 1 \times 4 \times 2 = 16.$$

平均值法

$$\bar{x} = \frac{\iint_D x \, d\sigma}{\sigma}$$

$$\bar{x} = \frac{\iiint_V x \, dv}{V}$$

$$= 2 \int_{-1}^3 dx \int_0^2 x \, dy$$

$$= 2 \int_{-1}^3 2x \, dx = 2 \cdot (9 - 1) = 16.$$

$$v = \frac{u}{\sqrt{u}} = 2 \int_{-1}^3 2x \cdot dx = 2 \cdot \frac{1}{2} (1 - 1) = 0.$$

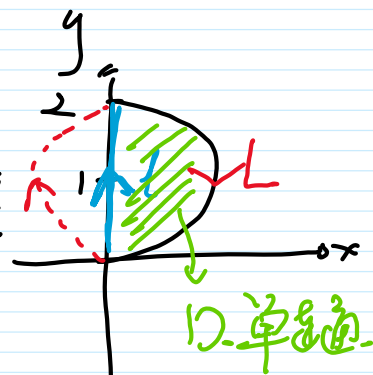
$$\bar{x} = \frac{\int_L x ds}{s}$$

例.  $I = \int_L (x^3 - e^x \cos y) dx + (e^x \sin y + 4x) dy$

其中  $L: x^2 + (y-1)^2 = 1$  右半圆. 从  $(0, 2) \rightarrow (0, 0)$ ?

解:  $L: \begin{cases} x = \cos t \\ y = 1 + \sin t \end{cases}$

$t: \frac{\pi}{2} \rightarrow -\frac{\pi}{2}$



用定积分计算非常麻烦. 格林公式

添加辅助线: 平行于坐标轴直线.

构造封闭曲线  $L' = L + l$ . 负向.

$$\oint_{L'} p dx + q dy = - \iint_D \left( \frac{\partial q}{\partial x} - \frac{\partial p}{\partial y} \right) dx dy = - \iint_D 4 dx dy$$

$$= -4 \cdot \frac{1}{2} \pi \cdot 1^2 = -2\pi.$$

$$\frac{\partial p}{\partial y} = \frac{\partial}{\partial y} (x^3 - e^x \cos y) = e^x \sin y$$

$$\frac{\partial q}{\partial x} = \frac{\partial}{\partial x} (e^x \sin y + 4x) = e^x \sin y + 4.$$

$$\oint_{L'} p dx + q dy = I + \int_l p dx + q dy = -2\pi.$$

注: 1.  $t=0$  时,  $0 \rightarrow 2$ .

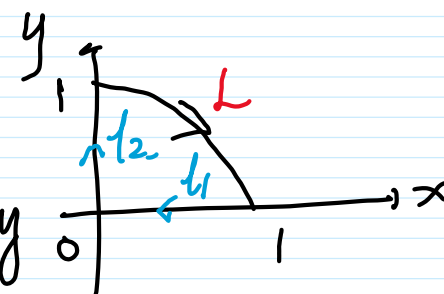
其中 1:  $x=0, y: 0 \rightarrow 2$ .

$$\begin{aligned} \int_L p dx + q dy &= \int_1 0 dy = \int_0^2 0 < 0, y > dy \\ &= \int_0^2 (e^{\sin y} + 4x0) dy \end{aligned}$$

例:  $I = \int_L x dy$        $L: y = \sqrt{1-x^2}$ .

$L: (0,1) \rightarrow (1,0)$

解: 构造  $L' = L + l_1 + l_2$ . 负向



$$\oint_{L'} x dy = - \iint_D \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ q & p \end{vmatrix} dx dy$$

$$= - \iint_D 1 dx dy = -\frac{\pi}{4}.$$

$$\oint_{L'} x dy = I + \int_{l_1} x dy + \int_{l_2} x dy$$

$$l_1: y=0, x: 1 \rightarrow 0, dy=0. \quad \int_{l_1} x dy = 0$$

$$l_2: x=0, y: 0 \rightarrow 1. \quad \int_{l_2} x dy = \int_0^1 0 dy = 0$$

$$I = -\frac{\pi}{4}.$$