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二重极限的计算：（一元函数极限的计算方法）
四则、复合、重要极限、等价无穷小替换、
泰勒公式、夹逼定理、无穷小×有界量等

例 $\lim\limits_{(x,y)\to(0,0)}\left(x\sin\frac{1}{y}+y\sin\frac{1}{x}\right)=0+0=0$

$\lim\limits_{x\to0^+}x^x=1$

$x^x=e^{x\ln x}\to e^0\ (x\to0^+)$

例 $\lim\limits_{(x,y)\to(0,0)}(x^2+y^2)^{(x^2+y^2)}=\underline{\ 1\ }$

换元. $t=x^2+y^2$

$\lim\limits_{(x,y)\to(0,0)}(x^2+y^2)=0$

连续 $\Leftrightarrow \lim\limits_{(x,y)\to(x_0,y_0)}f(x,y)=f(x_0,y_0)$

特值法. $\lim\limits_{(x,y)\to(0,0)}(x^2+0^2)^0=\lim\limits_{x\to0}1=1$

例. $\lim\limits_{(x,y)\to(0,0)}(x^2+y^2)^{xy}=\underline{\ 1\ }$ $y=0$

解: $(x^2+y^2)^{xy}=e^{xy\ln(x^2+y^2)}$ $\lim\limits_{(x,y)\to(0,0)}xy\ln(x^2+y^2)=0$

夹逼定理

$0\le\left|xy\ln(x^2+y^2)\right|\le\frac{(x^2+y^2)}{2}\ln(x^2+y^2)$

$\downarrow$ $\downarrow (x,y)\to(0,0)$

$0\longrightarrow 0 \longleftarrow 0$

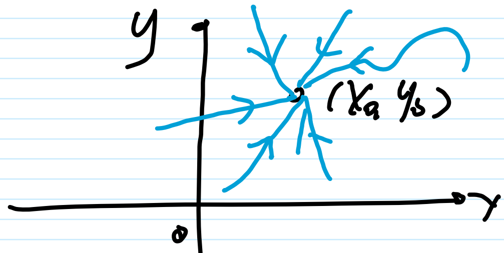
例. $\lim\limits_{(x,y)\to(0,0)}\frac{\sin(xy)}{x}$

解: 原式 $=\lim\frac{\sin(\boxed{xy})}{\boxed{xy}}\cdot y=1\cdot0=0.$

解1 原式 $= \lim\limits_{(x,y)\to(0,0)} \dfrac{\sin(xy)}{\boxed{xy}} \cdot y = 1\cdot 0 = 0.$

原式 $= \lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x} = 0.$

例 $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x^2+y^2}$ 不存在



解： $\lim\limits_{\substack{(x,y)\to(0,0)\\ y=0}} \dfrac{x\cdot 0}{x^2+0^2} = \lim\limits_{x\to 0} 0 = 0$

$\lim\limits_{\substack{(x,y)\to(0,0)\\ x=0}} \dfrac{0\cdot y}{0^2+y^2} = 0.$

$\lim\limits_{\substack{(x,y)\to(0,0)\\ y=x}} \dfrac{x\cdot x}{x^2+x^2} = \lim\limits_{x\to 0} \dfrac{x^2}{2x^2} = \dfrac{1}{2}.$

$\lim\limits_{\substack{(x,y)\to(x_0,y_0)\\ \varphi_1(x,y)=0}} f(x,y) \neq \lim\limits_{\substack{(x,y)\to(x_0,y_0)\\ \varphi_2(x,y)=0}} f(x,y) \Rightarrow$ 极限不存在

$\lim\limits_{\substack{(x,y)\to(0,0)\\ y=kx}} \dfrac{xy}{x^2+y^2} = \lim\limits_{x\to 0} \dfrac{kx^2}{(1+k^2)x^2} = \dfrac{k}{1+k^2}$

与 k 有关
路径

$\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x^2+y^2} = \lim\limits_{x\to 0} \dfrac{kx^3}{x^2+k^2x^4}$

$$\lim_{\substack{(x,y)\to(0,0)\\ y=kx^2}} \frac{[illegible]}{x^2+y^2} = \lim_{x\to 0} \frac{[illegible]}{x^2+k^2x^4}$$

$$= \lim_{x\to 0} \frac{kx}{1+k^2x^2} = \frac{0}{1+0} = 0$$

例. $\lim\limits_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}$ 不存在 取 $y=kx^2$.

$$\lim_{(x,y)\to(0,0)} \frac{x^3y}{x^4+y^2} = 0$$

解:

$$\lim_{\substack{(x,y)\to(0,0)\\ y=0}} \frac{x^3\, 0}{x^4+0^2} = 0$$

$$\lim_{\substack{(x,y)\to(0,0)\\ y=x}} \frac{x^4}{x^4+x^2} = \lim_{x\to 0} \frac{x^2}{x^2+1} = 0.$$

$$\lim_{\substack{(x,y)\to(0,0)\\ y=x^2}} \frac{x^5}{x^4+x^4} = \lim_{x\to 0} \frac{x}{2} = 0$$

$$0 \le \left|\frac{x^3y}{x^4+y^2}\right| \le \frac{|x|^3|y|}{2x^2|y|} = \frac{|x|}{2}$$

$$\downarrow \qquad\qquad\qquad\qquad\qquad \downarrow$$

$$0 \qquad \to 0 \leftarrow \qquad 0$$

例. $\lim\limits_{(x,y)\to(0,0)} \frac{(|x|+|y|)^3}{x^2+y^2} = 0$

例. $x=1$

解：极坐标换元 $\begin{cases} x=\rho\cos\theta \\ y=\rho\sin\theta \end{cases}$

$\downarrow$ $\rho\cos\theta=1$ $\Downarrow$ $\rho=\frac{1}{\cos\theta}$.

原式 $=\lim\limits_{\substack{\rho\to 0^+ \\ \theta任意}} \frac{\rho^3(|\cos\theta|^3+|\sin\theta|^3)}{\rho^2} = \lim\limits_{\rho\to 0^+} \underset{无穷小}{\rho}\,\underset{有界}{(|\cos\theta|^3+|\sin\theta|^3)} = 0$.

例. $f(x,y)=\begin{cases} \frac{xy}{x^2+y^2}, & (x,y)\neq(0,0) \\ 0, & (x,y)=(0,0) \end{cases}$

在 $(0,0)$ 处 不连续. 在除 $(0,0)$ 外 R^2 上都连续

$\lim\limits_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} \neq 0 = f(0,0)$

$\lim\limits_{(x,y)\to(x_0,y_0)} \frac{xy}{x^2+y^2} = \frac{x_0y_0}{x_0^2+y_0^2}$ $[(x_0,y_0)\neq(0,0)]$

例 $f(x,y)=\begin{cases} \frac{\sin(xy)}{x(y^2+1)}, & x\neq 0 \\ \boxed{0, \quad x=0} \end{cases}$

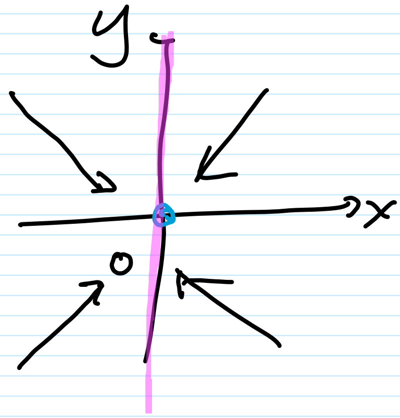
在 $(0,0)$ 不连续 $\begin{cases} 0, & (x,y)=(0,0) \\ 1, & x=0 且 y\neq 0 \end{cases}$

在 $(0,0)$ 处 连续. $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)=f(x_0,y_0)$.

解. i) $\lim\limits_{(x,y)\to(0,0)} f(x,y)$

$= \lim\limits_{\substack{(x,y)\to(0,0)\\x\neq 0}} \frac{\sin(xy)}{x(y^2+1)} = \lim\limits_{(x,y)\to(0,0)} \frac{xy}{x(y^2+1)} \ldots$

$= 0 = f(0,0)$



(ii) $\lim\limits_{\substack{(x,y)\to(0,0)\\ x=0}} f(x,y)$

$= \lim\limits_{(x,y)\to(0,0)} 0 = 0$

综上 $\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0)$ 故连续

有界闭区域的连续函数，

① 有界、最值

② 介值 $m \le \mu \le M$ $\exists P_0 \in D$，使 $f(P_0)=\mu$

③ 一致连续：$\forall \varepsilon>0$，$\exists \delta=\delta(\varepsilon)$，当 $0<|P_1P_2|<\delta$，

有 $|f(P_1)-f(P_2)|<\varepsilon$ 成立

第二节 偏导数

一、偏导数概念 (二元函数)

x x_0

定义：$z=f(x,y)$ 在 $U(P_0)$ 有定义，当固定 y 为 y_0 时

y y_0 Δy

x 在 x_0 处有增量 Δx 时，函数也有相应的增量

$\Delta_y z = f(x_0, y_0+\Delta y) - f(x_0, y_0)$

y

关于 x 的偏增量 $\Delta_x z = f(x_0+\Delta x, y_0) - f(x_0, y_0)$

$\lim\limits_{\Delta y\to 0}\frac{\Delta_y z}{\Delta y}$

若 $\lim \frac{f(x_0+\Delta x, y_0) - f(x_0, y_0)}{}$ 存在

若 $\lim\limits_{\Delta x\to 0}\dfrac{f(x_0+\Delta x,y_0)-f(x_0,y_0)}{\Delta x}$ 存在.

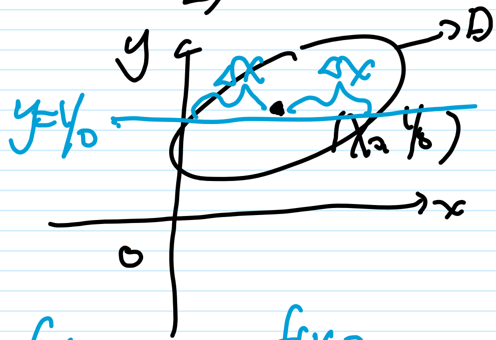
则称为 $z=f(x,y)$ 在 (x_0,y_0) 关于 x（y）的偏导数

记为 $\left.\dfrac{\partial z}{\partial x}\right|_{(x_0,y_0)}$ $\left.\dfrac{\partial f}{\partial x}\right|_{(x_0,y_0)}$. $z_x(x_0,y_0)$、$f_x(x_0,y_0)$

$\left.\dfrac{\partial f}{\partial y}\right|_{(x_0,y_0)}$ $f_y(x_0,y_0)$

否则、偏导不存在

$$\boxed{\begin{array}{c}\left.\dfrac{\partial f(x,y)}{\partial x}\right|_{(x_0,y_0)}\\ \| \\ \left.\dfrac{df(x,y_0)}{dx}\right|_{x=x_0}\end{array}} = \lim_{\Delta x\to 0}\frac{f(x_0+\Delta x,y_0)-f(x_0,y_0)}{\Delta x}$$



一元函数导数: $f'(x_0)=\lim\limits_{\Delta x\to 0}\dfrac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$

例: $y=2x$. $y'=2$. $y=ax$, $y'=a$

$f(x,y)=x\cdot y$.

$\dfrac{\partial f}{\partial x}$ 把y看成常数 y

$\dfrac{\partial f}{\partial y}$ 把x看成常数 x

$$\frac{\partial y}{} = \qquad x$$

$f(x,y) = x^y \quad (x>0 且 x \neq 1)$

$$\frac{\partial f}{\partial x} \overset{幂函数}{=\!=} y x^{y-1}$$

$$\frac{\partial f}{\partial y} \overset{指数函数}{=\!=} x^y \cdot \ln x.$$

定义：偏导(函)数

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, \boxed{y}) - f(x, \boxed{y})}{\Delta x}.$$

例 $f(x,y) = x + (y-1)\arctan\sqrt{\frac{x}{y}}$.

求 $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, $\left.\frac{\partial f}{\partial x}\right|_{(5,1)}$, $\left.\frac{\partial f}{\partial y}\right|_{(5,1)}$ "," $\to \frac{\partial}{\partial x}$

解 $\frac{\partial f}{\partial x} \overset{y是常数}{=\!=} (x)' + \left[(y-1)\arctan\sqrt{\frac{x}{y}}\right]'$

"先导再代"

$$= 1 + (y-1)\frac{1}{1+\frac{x}{y}}\left(\sqrt{\frac{x}{y}}\right)'$$

(连续)

$$= 1 + (y-1)\frac{1}{1+\frac{x}{y}} \cdot \frac{1}{\sqrt{y}} \cdot \frac{1}{2\sqrt{x}}$$

"," $\to \frac{\partial}{\partial y}$

$$\frac{\partial f}{\partial y} = 0 + (y-1)'\arctan\sqrt{\frac{x}{y}} + (y-1)\left(\arctan\sqrt{\frac{x}{y}}\right)'$$

$$= \arctan\sqrt{\frac{x}{y}} + (y-1)\frac{1}{1+\frac{x}{y}}\sqrt{x}\cdot\left(-\frac{1}{2}\right)y^{-\frac{3}{2}}.$$

$$=\arctan\sqrt{\frac{x}{y}}+(y-1)\frac{1}{1+\frac{x}{y}}\cdot\sqrt{x}\cdot(-\tfrac{1}{2})y$$

代入(5,1)分别得到 $f_x(5,1)$, $f_y(5,1)$.

$$\left.\frac{\partial f(x,y)}{\partial x}\right|_{(5,1)}=\left.\frac{d f(x,1)}{dx}\right|_{x=5}$$ “先代后导” 常

$$=\left.\frac{dx}{dx}\right|_{x=5}=1$$

$$\left.\frac{\partial f(x,y)}{\partial y}\right|_{(5,1)}=\left.\frac{d f(5,y)}{dy}\right|_{y=1}$$ ✓

例
$$f(x,y)=\begin{cases}\dfrac{xy}{x^2+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$$

则 $f_x=\begin{cases}\dfrac{y(y^2-x^2)}{(x^2+y^2)^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$ $f_y=\begin{cases}\dfrac{x(x^2-y^2)}{(y^2+x^2)^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$

解：当 $(x,y)\neq(0,0)$

$$f_x=\frac{y(x^2+y^2)-(xy)\cdot 2x}{(x^2+y^2)^2}=\frac{y(y^2-x^2)}{(x^2+y^2)^2}$$

当 $(x,y)=(0,0)$

$$f_x(0,0)=\lim_{\Delta x\to0}\frac{f(0+\Delta x,0)-f(0,0)}{\Delta x}$$

$$=\lim_{\Delta x\to0}\frac{\frac{\Delta x\cdot 0}{(\Delta x)^2+0^2}-0}{\Delta x}=0$$

$$\lim\frac{y(y^2-x^2)}{\ldots}=\lim_{(x,y)\to(0,0)}\frac{y^3}{y^4}\text{不存在}$$

$$\lim_{(x,y)\to(0,0)} \frac{y(y^2-x^2)}{(x^2+y^2)^2} = \lim_{\substack{(x,y)\to(0,0)\\ x=0}} \frac{y^3}{y^4} \text{ 不存在}$$

$$\lim_{(x,y)\to(0,0)} \frac{y(y^2-x^2)}{(x^2+y^2)^2} \neq 0 = f_x(0,0)$$

$\Rightarrow f_x(x,y)$ 在 $(0,0)$ 不连续.

若 $f(x,y)=f(y,x)$, $f_x(x,y)=\varphi(x,y)$

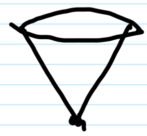
则 $f_y(x,y)=\varphi(y,x)$

在 $(0,0)$ 处偏导存在,但不连续 重极限

一元函数极限

一元函数; 可导 $\Rightarrow$ 连续.

连续 $\not\Rightarrow$ 可导. 例 $y=|x|$ 

例. $z=\sqrt{x^2+y^2}$ 在 $(0,0)$ 连续但偏导不存在. 

解: $f_x(0,0)=\lim\limits_{\Delta x\to 0}\frac{\sqrt{(0+\Delta x)^2+0^2}-\sqrt{0^2+0^2}}{\Delta x}=\lim\limits_{\Delta x\to 0}\frac{|\Delta x|}{\Delta x}$ 不存在

例. $f(x,y)=\begin{cases}\frac{\sin(xy)-y}{xy^2}, & xy\neq 0\\ 0, & xy=0\end{cases}$

则 $f_y(1,0)=$ ____ $-\frac{1}{6}$ ____.

解: $\lim \frac{\sin(1\cdot\Delta y)-\Delta y}{1\cdot\Delta y^2} - 0$

解：$f_y(1,0)=\lim_{\Delta y\to 0}\dfrac{\dfrac{\sin(1\cdot\Delta y)-\Delta y}{1\cdot\Delta y^2}-0}{\Delta y}$

$=\lim_{\Delta y\to 0}\dfrac{\sin\Delta y-\Delta y}{\Delta y^3}=-\dfrac{1}{6}.$

例 $u(x,y,z)=\int_{xz}^{yz}e^{t^2}dt$. 求 $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial u}{\partial z}$.

$u=\int_{xz}^{yz}xyze^{t^2}dt.$

解：$\dfrac{\partial u}{\partial x}\ \underline{\underline{y,z常数}}\ -e^{(xz)^2}\cdot z.$

$\dfrac{\partial u}{\partial y}=e^{(yz)^2}\cdot z$

$\dfrac{\partial u}{\partial z}=e^{(yz)^2}\cdot y-e^{(xz)^2}\cdot x$

$\left[\int_{a(x)}^{b(x)}f(t)dt\right]'=f[b(x)]\cdot b'(x)-f[a(x)]\cdot a'(x).$

$\left[\int_{a(x)}^{b(x)}\underbrace{c(x)}_{常数}\cdot \underbrace{f(t)dt}_{积分变量}\right]'=\left[c(x)\cdot\int_{a(x)}^{b(x)}f(t)dt\right]'$

$=c'(x)\int_{a(x)}^{b(x)}f(t)dt+c(x)\cdot[\ \ ----\ \ ]$

$y'(x)=\dfrac{dy}{dx}$ 微商

例：$\dfrac{PV}{T}=C.$ （c常数）

则 $\dfrac{\partial p}{\partial V}\cdot\dfrac{\partial V}{\partial T}\cdot\dfrac{\partial T}{\partial p}=$ $\underline{-1}$.

解：$\dfrac{\partial p}{\partial V}=\dfrac{\partial}{\partial V}\left(\dfrac{CT}{V}\right)=-\dfrac{CT}{V^{\cancel{2}}}$

$\times$

$\dfrac{\partial V}{\partial T}=\dfrac{\partial}{\partial T}\left(\dfrac{CT}{p}\right)=\dfrac{\cancel{C}}{p}$

$\times$

$\dfrac{\partial T}{\partial p}=\dfrac{\partial}{\partial p}\left(\dfrac{pV}{C}\right)=\dfrac{V}{\cancel{C}}$

$= -1$

例 $\dfrac{d(\sin x^2)}{d(x^2)}=\dfrac{\cos x^2\, d(x^2)}{d(x^2)}=\cos x^2$

二、偏导数（多元函数）的几何意义：$\begin{cases} z=f(x,y) \\ y=y_0 \end{cases}$ 在 $(x_0,y_0,f(x_0,y_0))$ 切线斜率

定义

导数：切线斜率

$$\left.\frac{\partial f(x,y)}{\partial x}\right|_{(x_0,y_0)}=\left.\frac{d f(x,y_0)}{dx}\right|_{x=x_0}$$

空间曲线 $\begin{cases} z=f(x,y) \\ y=y_0. \end{cases}$