2024年05月16日 星期四10-11

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1. $\iiint_\Omega f(x,y,z)\,dv \xrightarrow{\text{投影法}} \iint_{D_{xy}} dxdy \int_{\varphi_1(x,y)}^{\varphi_2(x,y)} f(x,y,z)\,dz$

截面法 $\int_{z_1}^{z_2} dz \iint_{D_z} f(x,y,z)\,dxdy$

$\iiint_\Omega g(z)\,dv \xrightarrow{\text{用}z\text{=常数}} \int_{z_1}^{z_2} g(z)\cdot\sigma(z)\,dz$

常用截面法的区域：

(1) $z=\sqrt{x^2+y^2} \to x^2+y^2=z^2$

截面面积 $\sigma(z)=\pi z^2$.

截面区域：D_z：$x^2+y^2\le z^2$.

$z=2-\sqrt{x^2+y^2}$.

$\sigma(z)=\pi(2-z)^2$

D_z：$x^2+y^2\le(2-z)^2$

(2) $z=x^2+y^2$.

$\sigma(z)=\pi z$

D_z：$x^2+y^2\le z$

$z=1-x^2-y^2$

$\sigma(z)=\pi(1-z)$

D_z：$x^2+y^2\le 1-z$.

(3) $z=\sqrt{1-x^2-y^2}$.

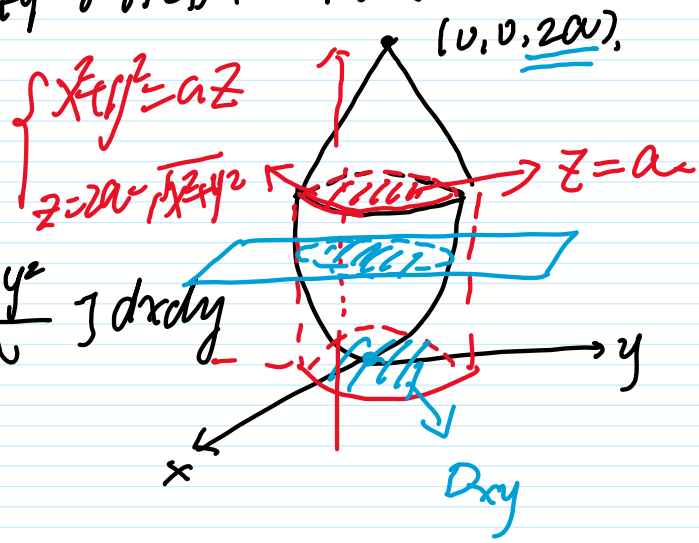
$\sigma(z)=\pi(1-z^2)$

D_z：$x^2+y^2\le 1-z^2$.

例. $x^2+y^2=az$ 与 $z=2a-\sqrt{x^2+y^2}$ 所围成的体积.

(0,0,2a).

例. $x^2+y^2=az$ 与 $z=2a-\sqrt{x^2+y^2}$ 所围成的立体.



解: 法一:

$V = V_{上曲} - V_{下曲}$

$= \iint_D \left[(2a-\sqrt{x^2+y^2}) - \frac{x^2+y^2}{a}\right] dxdy$

$D: x^2+y^2 \le a^2.$

法二: $V = \iiint_\Omega 1 \, dv.$

$= \iint_{D_{xy}} dxdy \int_{\frac{x^2+y^2}{a}}^{2a-\sqrt{x^2+y^2}} 1 \, dz$

法三: $V = \iiint_\Omega 1 \, dv = \int_0^{2a} 1 \cdot G(z) \, dz$

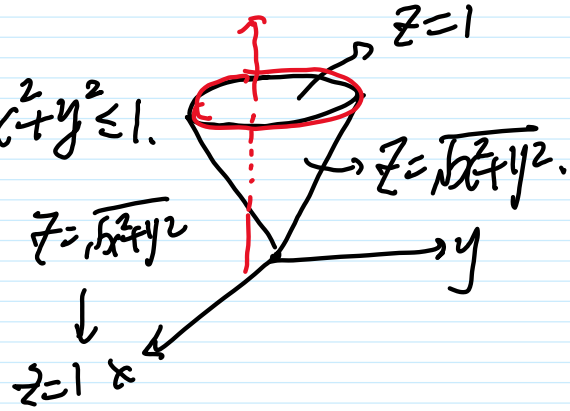
$= \int_0^a \pi a z \, dz + \int_a^{2a} \pi (2a-z)^2 dz = \cdots$

例. $I = \iiint_\Omega (x^2+y^2) \, dv.$ $\Omega: z=\sqrt{x^2+y^2}$ 与 $z=1$ 所围成的

解: 法一: 投影法

① Ω 在 xOy 面上的 $D_{xy}: x^2+y^2 \le 1.$

② 平行于 z 轴且穿过 Ω 内部的直线与 Ω 的边界曲面交点不多于两个.

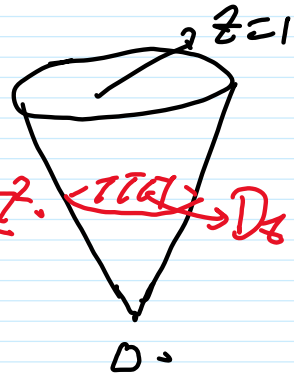


$I = \iint_{D_{xy}} dxdy \int_{\sqrt{x^2+y^2}}^{1} (x^2+y^2) \, dz$

$= \iint_{D_{xy}} (x^2+y^2)(1-\sqrt{x^2+y^2}) \, dxdy$ 极坐标

$$= \iint_{D_{xy}} (x^2+y^2)\cdot\left(1-\sqrt{x^2+y^2}\right) dxdy \overset{\text{极坐标}}{=} \cdots$$

②法：截面法.



用平行于xoy面去截

$D_z: x^2+y^2 \le z^2$

$$I = \int_0^1 dz \iint_{D_z} (x^2+y^2)\,dxdy$$

$$= \int_0^1 \left[\int_0^{2\pi} d\theta \int_0^z \rho^2 \rho\, d\rho\right] dz = \cdots$$

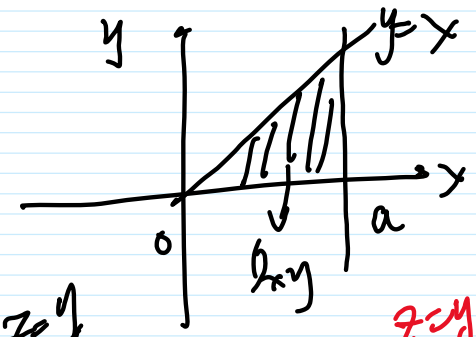
例：$I = \int_0^a dx \int_0^x dy \int_0^y f(z)\,dz$ 化为定积分

解：①法：

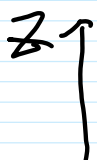
Ω：$\overbrace{\boxed{0\le x\le a,\ 0\le y\le x,}}^{D_{xy}}$ $0\le z\le y$

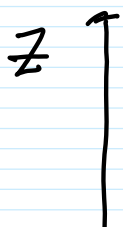
$$I = \iiint_\Omega f(z)\,dv \overset{\text{截面法}}{=} \int_0^a f(z)\,\underbrace{\sigma(z)}_{=\frac{1}{2}(a-z)^2}\,dz$$

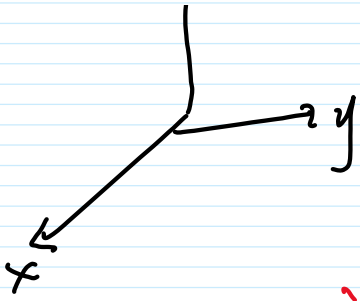
投影法 $\iint_{D_{xy}} dxdy \int_0^y f(z)\,dz$

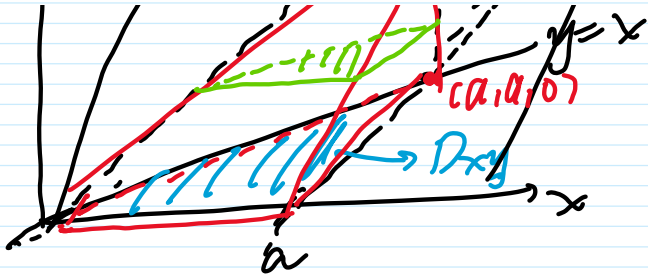


$$= \int_0^a dx \int_0^y dy \int_0^y f(z)\,dz$$







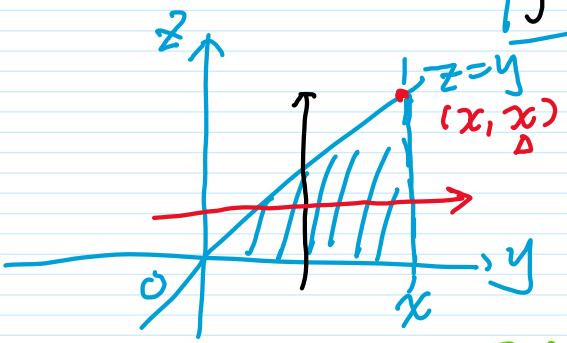


方法：交换积分次序

$$I=\int_0^a dx\int_0^x dy\int_0^y f(z)\,dz=\int dz\int dx\int f(z)\,dy$$

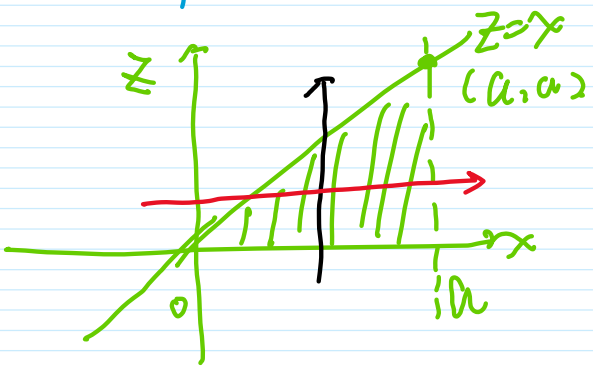
$(z\to y\to x)$，　$(y\to x\to z)$

$(y\to z\to x)$



$$\int_0^a dx\int_0^x dz\int_z^x f(z)\,dy$$

$$=\int_0^a dz\int_z^a dx\int_z^x f(z)\,dy$$



$$=\int_0^a dz\int_z^a f(z)(x-z)\,dx$$

$$=\int_0^a f(z)\frac{(a-z)^2-(z-z)^2}{2}\,dz$$

$$=\int_0^a f(z)\frac{(a-z)^2}{2}dz$$

2. 柱面坐标系下的计算 ⟶ 投影法 + 极坐标

$$\begin{cases}x=\rho\cos\theta\\ y=\rho\sin\theta\end{cases}\qquad dv=\rho\,d\rho\,d\theta\,dz.$$

$$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \\ z = z \end{cases}$$

$dv = \rho\, d\rho\, d\theta\, dz$

ρ = 常数　圆柱面

θ = 常数　半平面

z = 常数　平面

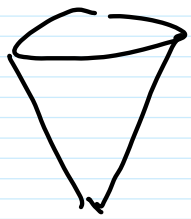
$$\iiint_{\Omega} f(x,y,z)\,dv = \iiint_{\Omega} f(\rho\cos\theta, \rho\sin\theta, z)\,\rho\, d\rho\, d\theta\, dz$$

$$= \int_{\alpha}^{\beta} d\theta \int_{\rho_1(\theta)}^{\rho_2(\theta)} \rho\, d\rho \int_{z_1(\rho,\theta)}^{z_2(\rho,\theta)} f(\rho\cos\theta, \rho\sin\theta, z)\, dz$$

例. $I = \iiint_{\Omega} (x^2+y^2)\,dv$　　Ω: $z = \sqrt{x^2+y^2}$ 与 $z=1$ 所围成

解. 投影法

D_{xy}: $x^2+y^2 \le 1$.



界线: $z = \sqrt{x^2+y^2}$

↓

$z = 1$.

$$I = \iint_{D_{xy}} dx\,dy \int_{\sqrt{x^2+y^2}}^{1} (x^2+y^2)\,dz$$

$$= \iint_{D_{xy}} (x^2+y^2)\left(1-\sqrt{x^2+y^2}\right) dx\,dy$$

极坐标

$$= \int_0^{2\pi} d\theta \int_0^1 \rho^2(1-\rho)\,\rho\, d\rho$$

极坐标 $= \int_0^{2\pi} d\theta \int_0^1 \rho^2 (1-\rho)$ $\rho d\rho$

⊗法:

D_{xy}: $x^2+y^2 \le 1$. $\begin{cases} \theta: 0 \to 2\pi \\ \rho: 0 \to 1 \end{cases}$

射线: $z=\sqrt{x^2+y^2} \xrightarrow{\text{柱}} z=\rho$

$\downarrow \qquad\qquad \downarrow$

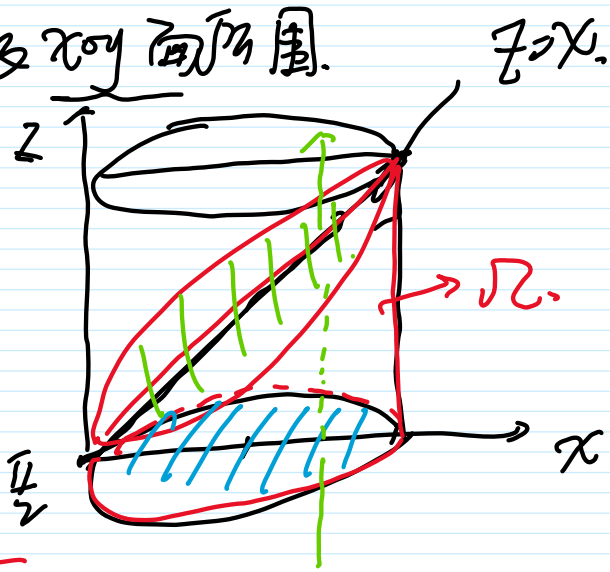
$z=1 \qquad\qquad z=1$

$$I=\int_0^{2\pi} d\theta \int_0^1 \rho\, d\rho \int_\rho^1 \rho^2 \, dz$$

$$=\int_0^{2\pi} d\theta \int_0^1 \rho \, \rho^2 (1-\rho) \, d\rho$$

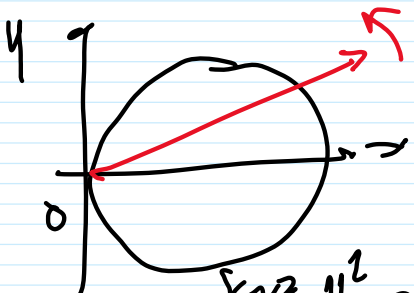
例. $I=\iiint_\Omega z\sqrt{x^2+y^2}\, dv$.

Ω: $x^2+y^2=2x$. $z=x$ 及 xoy 面所围.



解: Ω 投影到 xoy 面上.

D_{xy}: $x^2+y^2 \le 2x$



θ: $-\frac{\pi}{2} \to \frac{\pi}{2}$

ρ: $0 \to 2\cos\theta$.

$x^2+y^2=2x \xrightarrow{\text{极}} \rho^2=2\rho\cos\theta$

射线：

$$z=0 \xrightarrow{\text{柱}} z=0$$

$$\downarrow \qquad\qquad \downarrow$$

$$z=x \qquad\qquad z=\rho\cos\theta.$$

$$I=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \int_0^{2\cos\theta} \rho\, d\rho \int_0^{\rho\cos\theta} z\cdot\rho \quad dz$$

$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \int_0^{2\cos\theta} \rho\cdot\frac{\rho^2\cos^2\theta-0^2}{2} \quad \rho d\rho$$

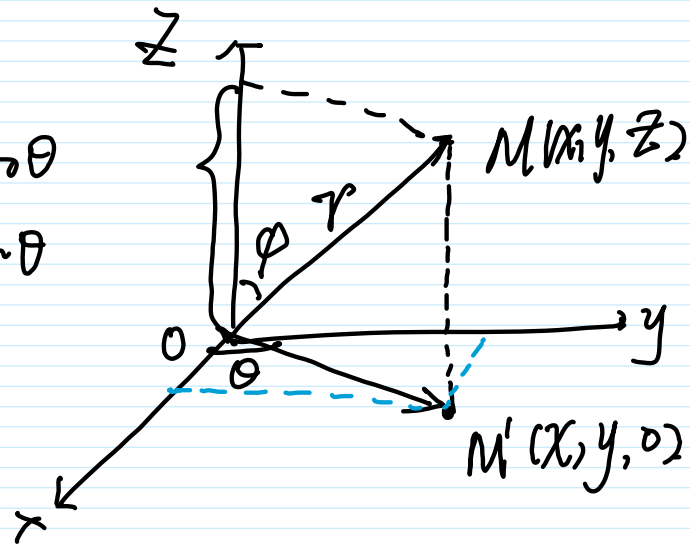
$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos^2\theta}{2}\cdot\frac{2^5\cdot\cos^5\theta}{5} \quad d\theta$$

$$=\frac{32}{5}\int_0^{\frac{\pi}{2}}\cos^7\theta\, d\theta=\frac{32}{5}\cdot 1\cdot\frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7}$$

$$\int_0^{\frac{\pi}{2}}\cos^8\theta\, d\theta=\frac{\pi}{2}\quad\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{7}{8}$$

3 球面坐标系下的计算

$$\begin{cases} x=\boxed{r\sin\varphi}\cdot\cos\theta \\ y=\boxed{r\sin\varphi}\cdot\sin\theta \\ z=r\cos\varphi. \end{cases}$$



$$dv=r^2\sin\varphi\, d\theta\, d\varphi\, dz$$

r=常数　球面

φ=常数　圆锥面

φ = 常数　锥面

θ = 常数　半平面

例：$I_1=\iiint_{\Omega_1} f(x,y,z)\,dv$，$\Omega_1$：$x^2+y^2+z^2\le R^2$.

解：$I_1=\int_0^{2\pi} d\theta \int_0^{\pi} \sin\varphi\, d\varphi \int_0^R f(r\sin\varphi\cos\theta, r\sin\varphi\sin\theta, r\cos\varphi)\, r^2 dr$

Ω_2：$x^2+y^2+z^2\le R^2$ 且 $z\ge 0$.

$I_2=\int_0^{2\pi} d\theta \int_0^{\frac{\pi}{2}} \sin\varphi\, d\varphi \int_0^R f(\cdots)\, r^2 dr$.

Ω_3：$x^2+y^2+z^2\le R^2$ 且

$x\ge 0$、$y\ge 0$、$z\ge 0$

$I_3=\int_0^{\frac{\pi}{2}} d\theta \int_0^{\frac{\pi}{2}} \sin\varphi\, d\varphi \int_0^R f(\cdots)\, r^2 dr$.