$$\iint_D f(x,y)\,d\sigma = \iint_D f(x,y)\,dxdy$$

例. $\displaystyle\lim_{r\to 0}\frac{1}{\pi r^2}\iint_D e^{x^2-y^2}\cos(x+y)\,dxdy = \underline{\quad 1 \quad}$.

其中 $D: x^2+y^2\le r^2$　　$D': x^2+(y-1)^2\le r^2$.

解: $\displaystyle\iint_D e^{x^2-y^2}\cos(x+y)\,dxdy = f(\xi,\eta)\cdot\sigma$.　　$(\xi,\eta)\in D$, $(\xi,\eta)\to(0,0)$ $\xleftarrow{r\to 0}$ $(0,0)$

$= e^{\xi^2-\eta^2}\cos(\xi+\eta)\cdot\pi r^2$.

原式 $=\underline{\displaystyle\lim_{r\to 0} e^{\xi^2-\eta^2}\cos(\xi+\eta)} = \displaystyle\lim_{(\xi,\eta)\to(0,0)} e^{\xi^2-\eta^2}\cos(\xi+\eta) = 1$

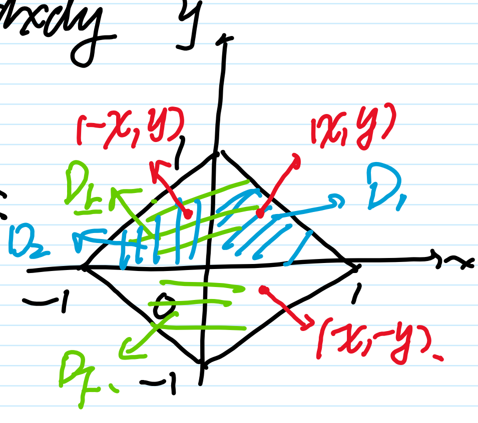
原式 $=\displaystyle\lim_{r\to 0}\frac{1}{\pi r^2}\iint_{D'} f(x,y)\,dxdy = \lim_{r\to 0} e^{\xi'^2-\eta'^2}\cos(\xi'+\eta')$.

$(\xi',\eta')\in D'$, $(\xi',\eta')\to(0,1)$, $r\to 0$: $(0,1)$

$=\displaystyle\lim_{(\xi',\eta')\to(0,1)} f(\xi',\eta')$

$= e^{-1}\cdot\cos 1$.

例. $\displaystyle I=\iint_D (x^2+x^3y^4)\,dxdy = 4\iint_{D_1} x^2\,dxdy$



$D: |x|+|y|\le 1$.　D_1: D的Ⅰ象限部分

证明: D关于x轴对称: $D=D_上+D_下$.

$f(x,y)=\underline{x^2+x^3y^4}$

$$f(x,y)=x^2+x^3y^4$$

$$f(x,-y)=x^2+x^3(-y)^4=f(x,y)$$

$$I=2\iint_{D_上}(x^2+x^3y^4)\,dxdy.$$

$D_上$关于y轴对称：　$D_上=D_1+D_2$

$$f(-x,y)=(-x)^2+(-x)^3y^4=x^2-x^3y^4.$$

线性.

$$I=2\iint_{D_上}x^2\,dxdy+2\iint_{D_上}x^3y^4\,dxdy.$$

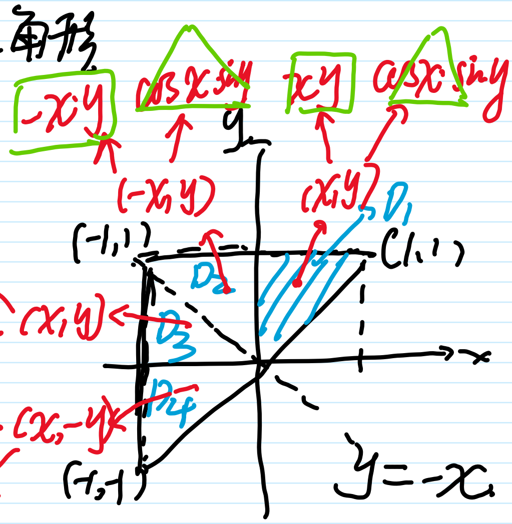
$$=2\cdot 2\iint_{D_1}x^2\,dxdy+2\cdot 0=4\iint_{D_1}x^2\,dxdy.$$

$$I'=\iint|\sin x-y^{101}|\,dxdy$$

例：
$$I=\iint_D(xy+\cos x\sin y)\,dxdy=2\iint_{D_1}\cos x\sin y\,dxdy$$

D：(1,1)，(-1,1)，(-1,-1)为顶点的三角形

D_1：I象限部分



证明：　$D=D_1+D_2+D_3+D_4$

$-xy$　(x,y)　$\boxed{\cos x\sin y}$

其中 D_1+D_2关于y轴对称

$-xy$　$(x,-y)$

D_3+D_4关于x轴对称

$\boxed{-\cos x\sin y}$

$$I=\iint_{D_1+D_2}(xy+\cos x\sin y)\,dxdy+\iint_{D_3+D_4}(xy+\cos x\sin y)\,dxdy.$$

$$=\iint_{D_1+D_2}xy\,dxdy+\iint_{D_3+D_4}xy\,dxdy+\iint_{D_1+D_2}\cos x\sin y\,dxdy+\iint_{D_3+D_4}\cos x\sin y\,d\sigma$$

$$=0+0+2\iint_{D}\cos x\sin y\,d\sigma+0$$

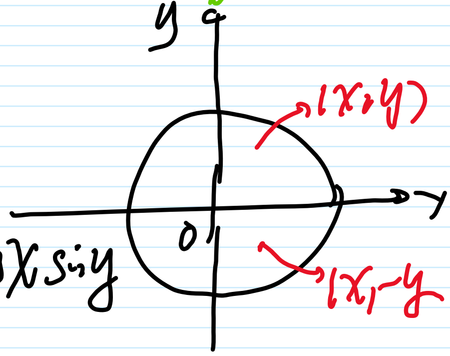
$$= 0 + 0 + 2\iint_{D_1}\cos x\cdot\sin y\,d\sigma + 0$$

$$= 2\iint_{D_1}\cos x\sin y\,dxdy$$

$$I' = \iint_D[\sin(x-y)+x^2]\,dxdy = \iint_D x^2\,dxdy = \iint_D y^2\,dxdy = \frac{1}{2}\iint_D(x^2+y^2)\,dxdy$$

例. $\iint_D \sin(x-y)\,dxdy = \underline{\quad 0 \quad}$.

其中, $D: x^2+y^2\le 1$.



解, ①法 D关于x轴对称

$f(x,y) = \sin(x-y) = \sin x\cos y - \cos x\sin y$

$f(x,-y) = \sin(x+y) = \sin x\cos y + \cos x\sin y$

②法: $\boxed{D\text{关于 } y=x \text{ 对称} \Rightarrow \iint_D f(x,y)\,dxdy = \iint_D f(y,x)\,dxdy}$

$$\iint_D \sin(x-y)\,dxdy = \iint_D \sin(y-x)\,dxdy$$

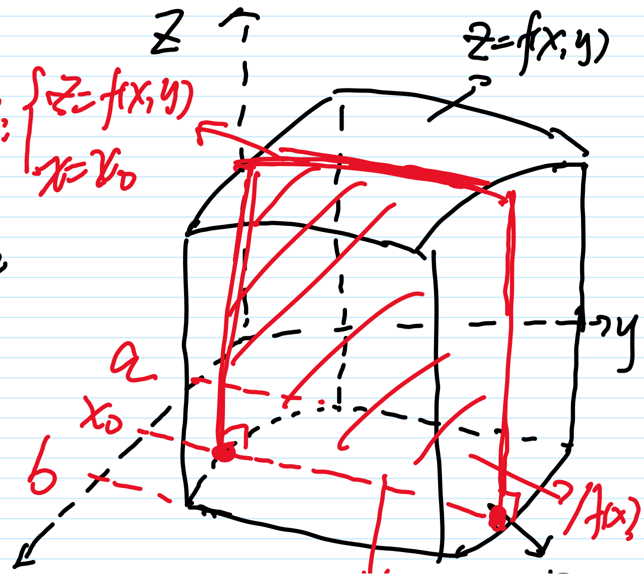
$$= -\iint_D \sin(x-y)\,dxdy.$$

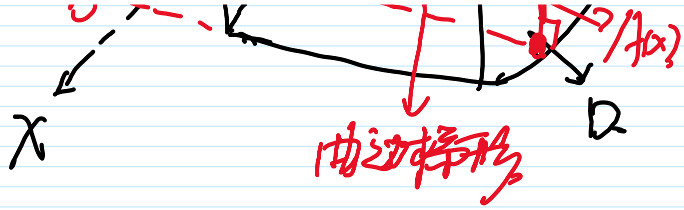
第二节 二重积分的计算

$$V_{\text{曲顶柱体}} = \iint_D f(x,y)\,dxdy$$

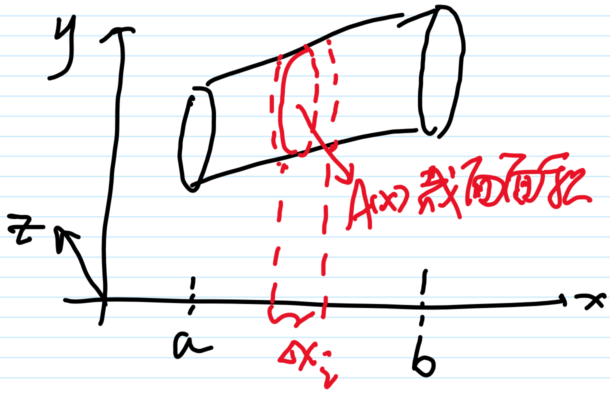
$$= \lim_{\lambda\to 0}\sum_{i=1}^{n} f(\xi_i,\eta_i)\Delta\sigma_i$$

$\Gamma: \begin{cases} z=f(x,y) \\ x=x_0 \end{cases}$



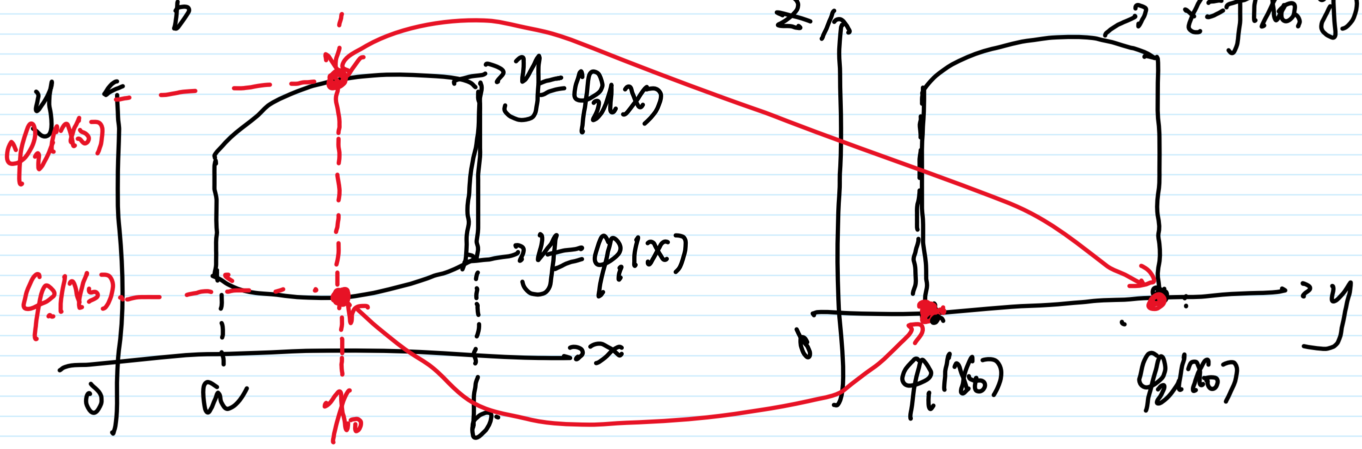


$$V_{旋} = \int_a^b \pi f^2(x)\,dx$$



$$V = \int_a^b A(x)\,dx$$

$$V_{曲顶} = \iint_D f(x,y)\,dxdy = \int_a^b A(x)\,dx$$



$$A(x_0) = \int_{\varphi_1(x_0)}^{\varphi_2(x_0)} f(x_0,y)\,dy$$

$$A(x) = \int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\,dy$$

$$V_{曲顶柱体} = \iint_D f(x,y)\,dxdy$$

$$= \int_a^b \left[\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\,dy \right] dx$$

$$= \int_a^b \left[\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\,dy \right] dx$$

$$= \int_a^b dx \int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\,dy$$ 二次积分

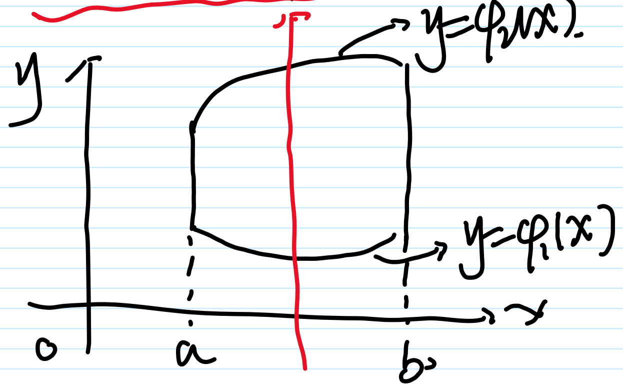
$$* \quad \left(\int_a^b dx\right)\left(\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\,dy\right)$$

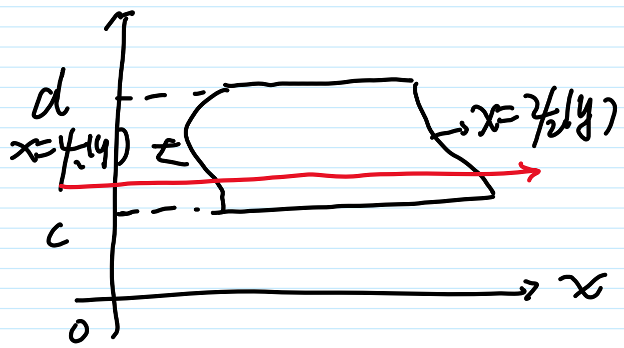
1、直角坐标系下的计算.

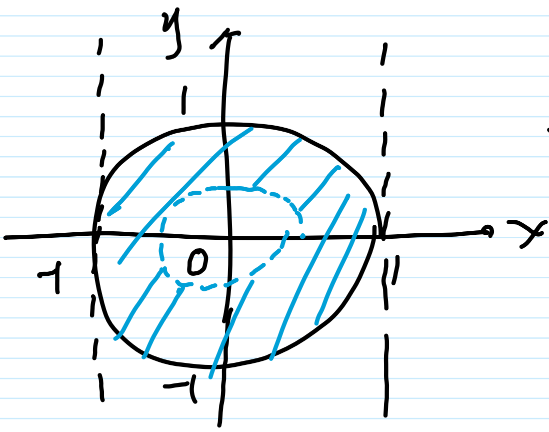
定义： X-型区域　　　　Y-型区域.

D: $a \le x \le b$, $\varphi_1(x) \le y \le \varphi_2(x)$　　　　D: $c \le y \le d$, $\psi_1(y) \le x \le \psi_2(y)$.

穿线法







X-型: $-1 \le x \le 1$, $-\sqrt{1-x^2} \le y \le \sqrt{1-x^2}$

Y-型: $-1 \le y \le 1$. $-\sqrt{1-y^2} \le x \le \sqrt{1-y^2}$.

可加性.

定理：

(1). X-型区域： $a \le x \le b$, $\varphi_1(x) \le y \le \varphi_2(x)$

$$\iint_D f(x,y)\,dx\,dy = \int_a^b dx \int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\,dy.$$ (先y后x).

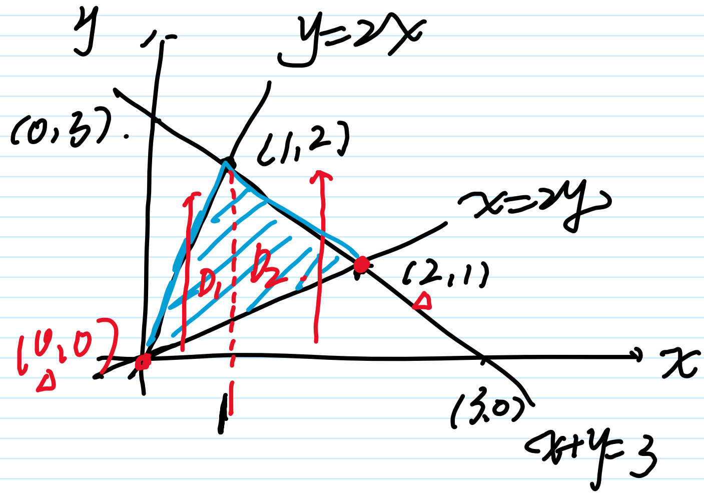
$$\iint_D f(x,y)\,dxdy = \int_a dx \int_{\varphi_1(x)} f(x,y)\,dy.$$ (先y后x).

(2) Y-型区域: $c \le y \le d$, $\psi_1(y) \le x \le \psi_2(y)$

$$\iint_D f(x,y)\,dxdy = \int_c^d dy \int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\,dx.$$ (先x后y).

例. D: $y=2x$, $x=2y$ 与 $x+y=3$ 所围成的

求 D 的面积.



解: $\sigma = \iint_D 1\,dxdy$

旧法: X-型区域. (↑)

$$I = \iint_D 1\,dxdy$$

$$= \int_0^2 dx \int_{\varphi_1(x)}^{\varphi_2(x)} 1\,dy$$

$$\varphi_1(x) = \frac{x}{2}, \qquad \varphi_2(x) = \begin{cases} 2x, & 0 \le x \le 1 \\ 3-x, & 1 \le x \le 2. \end{cases}$$

$$I = \int_0^2 [\varphi_2(x) - \varphi_1(x)]\,dx = \int_0^2 [\varphi_2(x) - \frac{x}{2}]\,dx$$

可加性 $= \int_0^1 (2x - \frac{x}{2})\,dx + \int_1^2 [(3-x) - \frac{x}{2}]\,dx = \cdots$

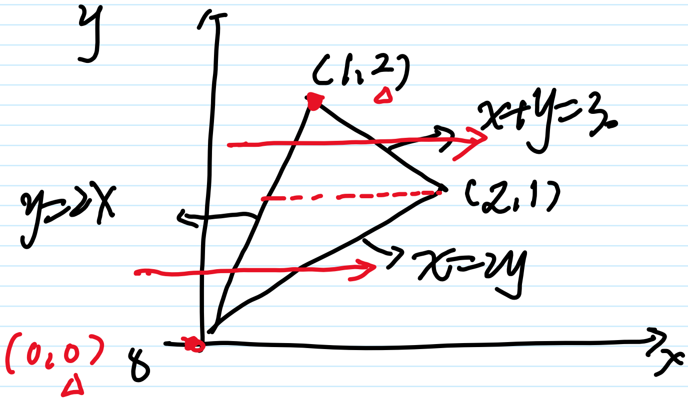
$$I = \iint_{D_1} 1\,dx + \iint_{D_2} 1\,dx.$$

$$= \int_0^1 dx \int_{\frac{x}{2}}^{2x} 1\,dy + \int_1^2 dx \int_{\frac{1}{2}x}^{3-x} 1 \cdot dy = \cdots$$

$\int_0 dx \int_{\frac{x}{2}} \quad dy + \int_1 dx \int_{\frac{1}{2}x} \quad dy$

② 法：Y-型区域（⟶）

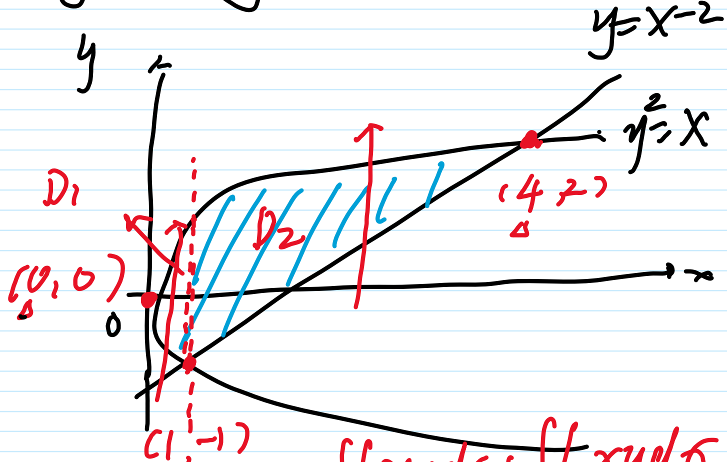
$$I = \int_0^1 dy \int_{\frac{y}{2}}^{2y} 1\, dx + \int_1^2 dy \int_{\frac{y}{2}}^{3-y} 1\, dx$$



例. $I = \iint\limits_D xy\, dxdy$. D: $y^2 = x$ 与 $y = x-2$ 所围成.

解：① 法：

$$I = \int_0^1 dx \int_{-\sqrt{x}}^{\sqrt{x}} xy\, dy + \int_1^4 dx \int_{x-2}^{\sqrt{x}} xy\, dy$$



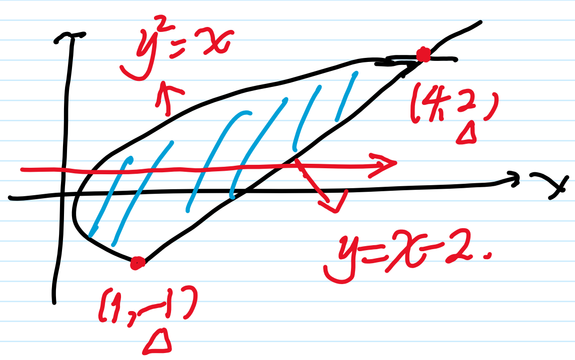
$$I = \iint\limits_{D_1} xy\, d\sigma + \iint\limits_{D_2} xy\, d\sigma = 0 + \iint\limits_{D_2} xy\, d\sigma.$$

$$= \int_0^1 \left[x \cdot \frac{y^2}{2} \Big|_{-\sqrt{x}}^{\sqrt{x}} \right] dx + \int_1^4 \left[x \cdot \frac{y^2}{2} \Big|_{x-2}^{\sqrt{x}} \right] dx = 0 + \int_1^4 \frac{x\left(x - (x-2)^2\right)}{2} dx$$

$= \cdots$

② 法：

$$I = \int_{-1}^2 dy \int_{y^2}^{y+2} xy\, dx.$$



$$= \int_{-1}^2 \left(y \cdot \frac{x^2}{2} \Big|_{y^2}^{y+2} \right) dy$$

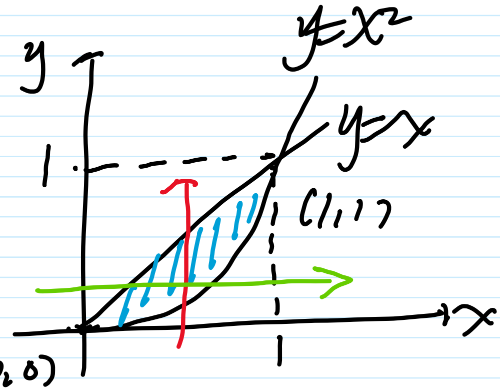
$= \cdots$

例. $I = \iint \frac{\sin x}{x} dxdy$ D: $y=x$ 和 $y=x^2$ 所围成.

例. $I=\iint\limits_{D}\frac{\sin x}{x}dxdy$. D: $y=x$ 和 $y=x^2$ 所围成.

解: ①法

$I=\int_0^1 dx\int_{x^2}^{x}\frac{\sin x}{x}dy$



$=\int_0^1\left[\frac{\sin x}{x}\cdot(x-x^2)\right]dx$

$=\int_0^1(\sin x - x\sin x)dx$. 分部积分 ...

②法: $I=\int_0^1 dy\int_{y}^{\sqrt{y}}\frac{\sin x}{x}dx=$ ______

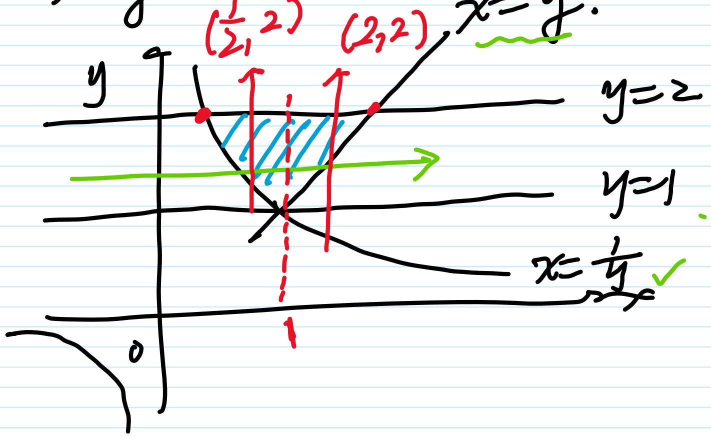
不可计算

交换积分次序:

例 $\int_1^2 dy\int_{\frac{1}{y}}^{y} f(x,y)dx=\int_{\square}^{\square}dx\int_{\square}^{\square}f(x,y)dy$ (交换次序)

解: 确定积分区域 D: $1\le y\le 2$, $\frac{1}{y}\le x\le y$

$y=1$. $y=2$. $x=\frac{1}{y}$ $x=y$



$I=\int_{\frac{1}{2}}^{1}dx\int_{\frac{1}{x}}^{2}f(x,y)dy$

$+\int_1^2 dx\int_x^2 f(x,y)dy$

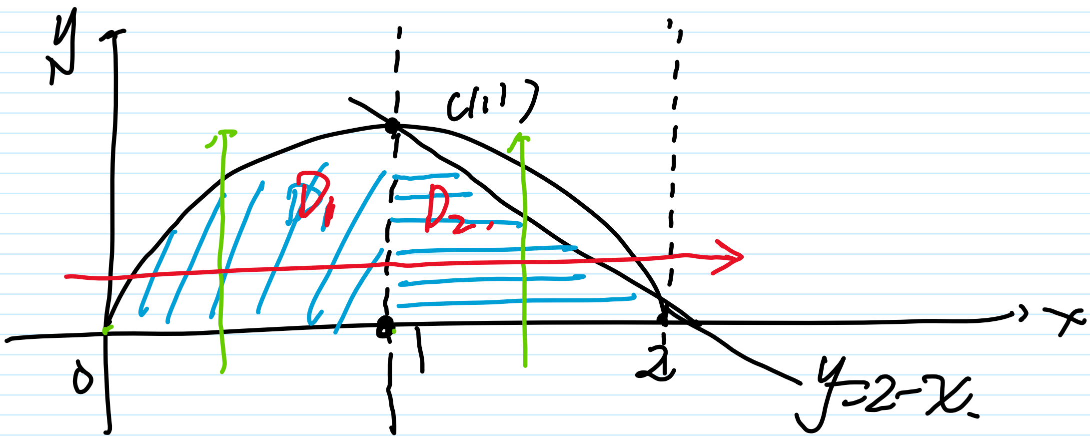
例. $I=\int_0^1 dx\int^{\sqrt{2x-x^2}}f(x,y)dy$

例. $I=\int_0^1 dx\int_0^{\sqrt{2x-x^2}} f(x,y)\,dy$

$+\underbrace{\int_1^2 dx\int_0^{2-x} f(x,y)\,dy}$ $=\int_{\square}^{\square} dy\int_{\square}^{\square} f(x,y)\,dx$ 交换次序.

解: $I=\iint\limits_D f(x,y)\,dxdy=\iint\limits_{D_1} f(x,y)\,dxdy+\iint\limits_{D_2} f(x,y)\,dxdy$

$D_1:\ 0\le x\le 1,\ 0\le y\le \sqrt{2x-x^2}.\quad D_2:\ 1\le x\le 2,\ 0\le y\le 2-x.$



$y=\sqrt{2x-x^2} \Leftrightarrow y^2=2x-x^2 \Leftrightarrow \underline{y^2+x^2-2x+1}=1$

$\Rightarrow x=1\pm\sqrt{1-y^2}$

$I=\int_0^1 dy\int_{1-\sqrt{1-y^2}}^{2-y} f(x,y)\,dx.$

$(x-a)^2+(y-b)^2=R^2.\qquad y=b\pm\sqrt{R^2-(x-a)^2}$

$x=a\pm\sqrt{R^2-(y-b)^2}.$

上半圆: $+$

左半圆: $-$