平面曲线积分与路径无关.

定理: (1). D单连通

(2) P, Q 偏导连续

则 (i) $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y} \Longrightarrow$ (ii) $\int_{L_1} P\,dx+Q\,dy = \int_{L_2} P\,dx+Q\,dy$

其中 L_1与L_2起点、终点相同.

P, Q在 L_1 与 L_2 所围成区域 D' 上偏导连续.

(iii) $\oint_L P\,dx+Q\,dy = 0$.

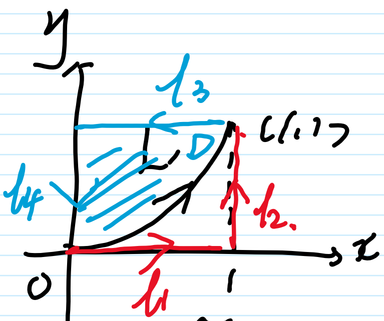
(iv). $du = P\,dx+Q\,dy$ $u(x,y)$为一个原函数

且 $u(s,t) = \int_{(x_0,y_0)}^{(s,t)} P(x,y)\,dx + Q(x,y)\,dy$

记 $I_2 =$

例 $I = \int_L (2x\cos y - y^2 \cdot \sin x + y)\,dx + (2y\cos x - x^2 \cdot \sin y)\,dy$.

其中 $L: y=x^2$. 从 $(0,0) \to (1,1)$.



解. (1)法:

$I = \int_0^1 (2x\cos x^2 - x^4 \cdot \sin x) \cdot 1\,dx + \int_0^1 (2x^2\cos x - x^2 \sin x^2) \cdot 2x\,dx = \cdots$ 计算繁琐.

$\int_0 (2x \cdots$

法二: $\frac{\partial Q}{\partial x} = -2y\sin x - 2x\sin y$

$\frac{\partial P}{\partial y} = -2x\sin y - 2y\sin x$.

与路径无关

$\frac{\partial Q}{\partial x} + \frac{\partial P}{\partial y}$

$I = \int_{L_1} Pdx + Qdy + \int_{L_2} Pdx + Qdy$.

$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = -1$.

格林公式,

$= \int_0^1 P(x,0)\,dx + \int_0^1 Q(1,y)\,dy$

$= \int_0^1 2x\,dx + \int_0^1 (2y\cdot\cos 1 - 1^2\sin y)\,dy$

$= 1^2 - 0^2 + \cos 1\cdot(1^2 - 0^2) + \cos 1 - \cos 0 = \cdots$

构造封闭曲线 $L' = L + L_3 + L_4$. 正向

由格林公式 $\oint_{L'} Pdx + Qdy = + \iint_D (-1)\,dxdy = -S_D$.

$I + I_{L_1} + I_{L_2}$.

例: 求一个原函数 $u(x,y)$.

使得 $du = (\underbrace{2x\cos y - y^2\sin x}_{P})dx + (\underbrace{2y\cos x - x^2\sin y}_{Q})dy$

解: 法: 偏积分法

$P = \frac{\partial u}{\partial x}$　　　$Q = \frac{\partial u}{\partial y}$

$u(x,y) = \int (2x\cdot\cos y - y^2\sin x)\,dx$

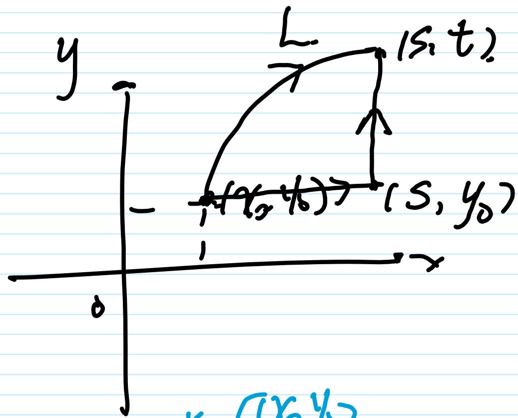
$= x^2\cos y + y^2\cos x + \boxed{C(y)}$

$$\frac{\partial u}{\partial y} = -x^2\sin y + 2y\cos x + c'(y) = 2y\cos x - x^2\sin y$$

$$c'(y) = 0. \Rightarrow c(y) = C.$$

综上. $u(x,y) = x^2\cos y + y^2\cos x + C.$

②法: 线积分法.



$$u(s,t) = \int_{(x_0,y_0)}^{(s,t)} p\,dx + Q\,dy$$

$$= \int_{x_0}^{s} p(x, y_0)\,dx + \int_{y_0}^{t} Q(s, y)\,dy$$

差 $\underbrace{\int_{(0,0)}^{(x_0,y_0)} p\,dx + Q\,dy}_{常数}$

$$u(s,t) = \int_{(0,0)}^{(s,t)} p\,dx + Q\,dy$$

$$= \int_0^s p(x, 0)\,dx + \int_0^t Q(s,y)\,dy$$

$$= \int_0^s 2x\,dx + \int_0^t (2y\cos s - s^2\sin y)\,dy$$

$$= s^2 + \cos s\,(t^2 - 0^2) + s^2(\cos t - \cos 0)$$

$$= t^2\cos s + s^2\cos t$$

$$u(x,y) = x^2\cos y + y^2\cos x$$

例. $I = \int_L p\,dx + Q\,dy$ P, Q 同上.

$$= \int_A^B P\,dx + Q\,dy.$$

L_1: $(1,1) \to (2,2)$. $\quad I_{L_1} = u(2,2) - u(1,1)$

L_2: $(1,1) \to (3,-4)$ $\quad I_{L_2} = u(3,-4) - u(1,1)$.

定理: $\quad du = P\,dx + Q\,dy$

$$\Rightarrow \int_L P\,dx + Q\,dy = \underline{\int_A^B P\,dx + Q\,dy = u(B) - u(A)}.$$

例. $I = \int_L P\,dx + Q\,dy$. P, Q 带参数与路径无关

① 求参数, 原函数. ② L: $(x_1, y_1) \to (x_2, y_2)$. 计算 I.

解: $\quad du = P\,dx + Q\,dy$ 求 P, Q 中的参数

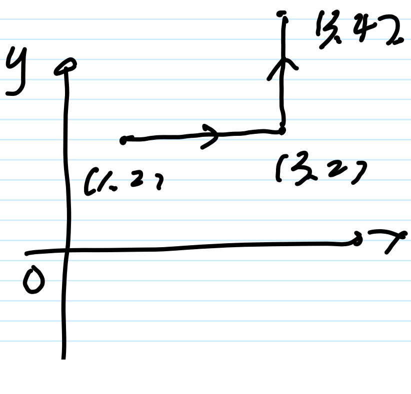
$P = \frac{\partial u}{\partial x}$ $\quad Q = \frac{\partial u}{\partial y}$ $\quad \underline{\frac{\partial^2 u}{\partial x \partial y}} = \frac{\partial^2 u}{\partial y \partial x}$. $\quad \boxed{\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}}$

$\Downarrow$

参数

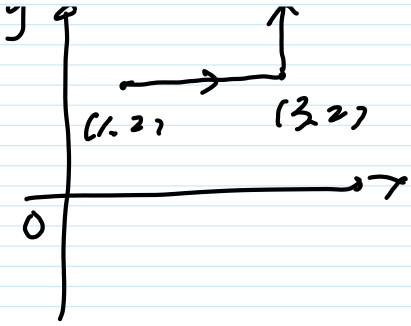
例. $I = \int_{(1,2)}^{(3,4)} (6xy^2 - y^3)\,dx + (6x^2y - 3xy^2)\,dy$

解: $\quad \frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$.



①法: $\quad (1,2) \to (3,2) \to (3,4)$

解: $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$.



①法: $(1,2) \to (3,2) \to (3,4)$

$$I = \int_1^3 (6x\cdot 2^2 - 2^3)\,dx + \int_2^4 (6\cdot 3^2\cdot y - 3\cdot 3\cdot y^2)\,dy$$

②法: $\frac{\partial u}{\partial x} = 6xy^2 - y^3$

$$u = 3x^2y^2 - xy^3 + C(y)$$

$$\frac{\partial u}{\partial y} = 6x^2y - 3xy^2 + C'(y) = 6x^2y - 3xy^2$$

得 $C'(y) = 0$

故取 $C(y)=0$, 则一个原函数 $u = 3x^2y^2 - xy^3$

$$I = u(3,4) - u(1,2)$$

例. $I = \int_L \frac{y\,dx - x\,dy}{x^2+y^2}$

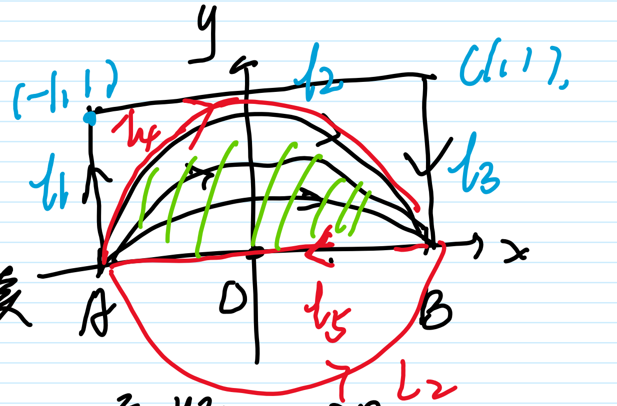
$-\pi$ $\parallel$ I_{L_2} ✗ π $\parallel$ I.

L_2与L所包围区域含$(0,0)$. P,Q无定义.

L_2: $A \to B$ 下半平面

其中 L: $A(-1,0) \to B(1,0)$ 任意一条不过原点的曲线(上半平面)

解: $P = \frac{y}{x^2+y^2}$ $Q = -\frac{x}{x^2+y^2}$



P, Q 在$(0,0)$无定义. 不满足偏导连续

$(x^2+y^2) - x\cdot 2x$ x^2-y^2 ∂P

$$\frac{\partial Q}{\partial x} = -\frac{1\cdot(x^2+y^2)-x\cdot 2x}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2} = \frac{\partial P}{\partial y}.$$

法二: $L = l_1+l_2+l_3$、有向折线

$$I = \int_0^1 -\frac{(-1)}{(-1)^2+y^2}\,dy + \int_{-1}^{1} \frac{1}{x^2+1^2}\,dx + \int_1^0 -\frac{1}{1^2+y^2}\,dy$$

$= \cdots$

$l_5: y = -\sqrt{1-x^2}$ $\quad t: \pi \to 2\pi$

法三: $L = l_4$. $l_4: y = \sqrt{1-x^2}$, $x: -1 \to 1$. $\begin{cases} x = \cos t \\ y = \sin t \end{cases}$ $t: \pi \to 0$

$$I = \int_{l_4} \frac{y\,dx - x\,dy}{x^2+y^2} = \int_{l_4} \underbrace{y}_{P'}\,dx \underbrace{-x}_{Q'}\,dy.$$

$\frac{\partial Q'}{\partial x} = -1$ $\quad \frac{\partial P'}{\partial y} = 1$

(i) ☆ 利用参数形式

$$= \int_{\pi}^{0} (-\sin t\cdot\sin t - \cos t\cdot\cos t)\,dt = \pi.$$

$\frac{\partial Q'}{\partial x} \ne \frac{\partial P'}{\partial y}$ 格林公式.

(ii) 格林公式:

构造一条封闭曲线 $L' = l_4 + l_5$. 负向

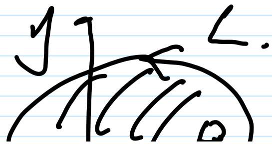
$$\oint_{L'} P'dx + Q'dy = -\iint_D \left(\frac{\partial Q'}{\partial x} - \frac{\partial P'}{\partial y}\right)dxdy = 2\cdot\frac{1}{2}\pi\cdot 1^2 = \pi.$$

$$I + \int_{l_5} P'dx + Q'dy = I + \int_1^{-1} 0\,dx = I$$

例 $$I = \oint_L \frac{y\,dx - x\,dy}{x^2+y^2}$$

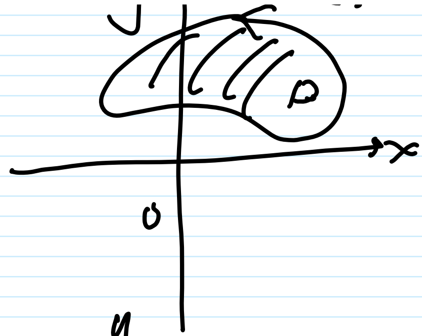
L: 一条不过原点的封闭曲线. 方向: 逆时针.

解: (1) L 不包含原点



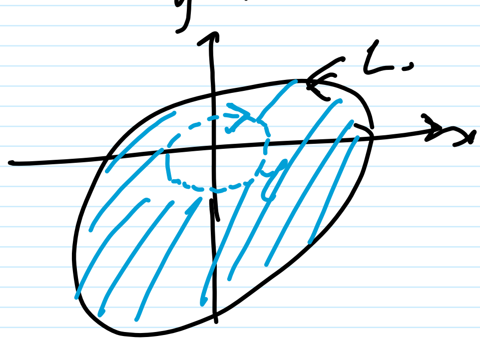
$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y} \Rightarrow I = 0.$$

解: $\dfrac{\partial Q}{\partial x}=\dfrac{\partial P}{\partial y} \Rightarrow I=0.$



P, Q 在 D 偏导连续. 故 $I=0$

②. L 包含原点



P, Q 在 D 上 有无定义点、不满足偏导连续.

特殊情况: L: $x^2+y^2=R^2$. 逆时针

$$I=\frac{1}{R^2}\oint_L y\,dx-x\,dy$$

$$=\frac{1}{R^2}\iint_D(-1-1)\,dxdy=-\frac{2}{R^2}\iint_D 1\,dxdy$$

$$=-\frac{2}{R^2}\cdot\pi R^2=\underline{-2\pi}.$$

构造封闭曲线 l: 顺时针

L 与 l 构造一个复连通区域 D', P, Q 在 D' 偏导连续.

L. l: 正向

$$\oint_{L+l}P\,dx+Q\,dy=\iint_{D'}\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dxdy=0.$$

$$\|$$

$$\oint_L P\,dx+Q\,dy+\oint_l P\,dx+Q\,dy=0$$

$$\oint_L Pdx+Qdy + \oint_l Pdx+Qdy = 0$$

前提：$\dfrac{\partial Q}{\partial x}=\dfrac{\partial P}{\partial y}$

$$\boxed{\begin{aligned} I &= \oint_L Pdx+Qdy = -\oint_l Pdx+Qdy \\ &= \oint_{l^-} Pdx+Qdy \end{aligned}}$$

L 与 l^- 均为逆时针.

不妨设 l：$x^2+y^2=\varepsilon^2$（$\varepsilon>0$ 足够小） 逆时针.

得出 $I=-2\pi$.

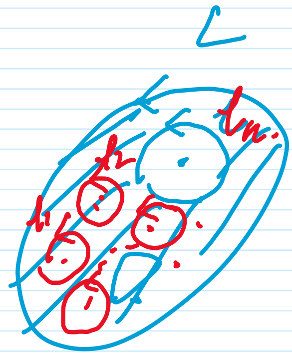
定理：① L、l 方向同为逆或顺时针

推广：L 外，$l_1, l_2, \cdots l_n$ 内.

② L，l 围成区域 D 复连通.

P，Q 在 D 上 偏导连续

③ $\dfrac{\partial Q}{\partial x}=\dfrac{\partial P}{\partial y}$



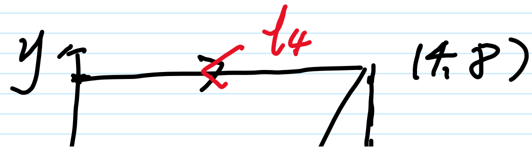
$$\Rightarrow \oint_L Pdx+Qdy = \oint_l Pdx+Qdy$$

$$\oint_L Pdx+Qdy = \sum_{i=1}^{n} \oint_{l_i} Pdx+Qdy$$

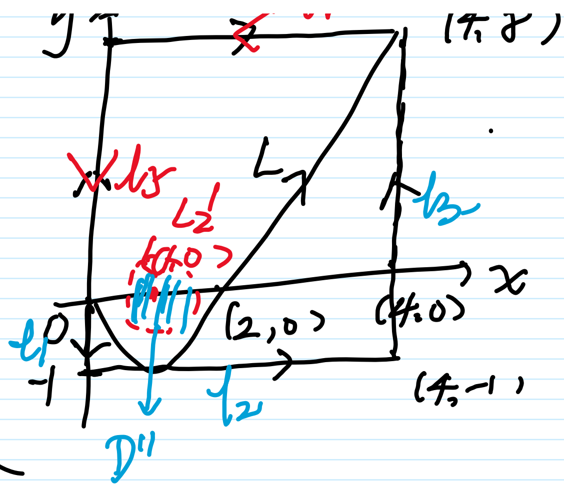
例 $I=\displaystyle\int_L \frac{-y\,dx+(x-1)dy}{(x-1)^2+y^2}$.

其中 L：$y=x^2-2x$. 从 $(0,0)\to(4,8)$.

解：



解: $P=\frac{-y}{(x-1)^2+y^2}$ $Q=\frac{x-1}{(x-1)^2+y^2}$



在(1,0)无定义.

①法: $\frac{\partial Q}{\partial x}=\frac{\partial P}{\partial y}$ ⇒ 与路径无关

$$I=\int_{l_1+l_2+l_3}Pdx+Qdy$$

$$=\int_0^{-1}\frac{0-1}{(0-1)^2+y^2}dy+\int_0^4\frac{-(-1)}{(x-1)^2+(-1)^2}dx+\int_{-1}^{8}\frac{4-1}{(4-1)^2+y^2}dy$$

$=\cdots$

②法: 构造一封闭曲线 $L_1'=L+l_4+l_5$ 方向: 逆时针

构造另一封闭曲线 L_2': $(x-1)^2+y^2=\varepsilon^2$ ($\varepsilon>0$ 足够小) 方向: 逆时针

L_1'与L_2'构造一个复连通区域D'. P,Q在D'偏导连续

$$\frac{\partial Q}{\partial x}=\frac{\partial P}{\partial y}$$

故 $\oint_{L_1'}Pdx+Qdy=\oint_{L_2'}Pdx+Qdy$. 参数形式

$$=\frac{1}{\varepsilon^2}\oint_{L_2'}\underbrace{-y}_{P'}dx+\underbrace{(x-1)}_{Q'}dy$$

$\frac{\partial Q'}{\partial x}=1$ $\frac{\partial P'}{\partial y}=-1$

格林公式: L_2': D''.

$$=\frac{1}{\varepsilon^2}\iint_{D''}(1-(-1))dxdy$$

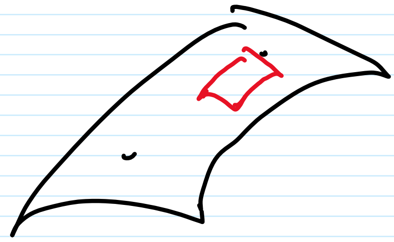
$$=\frac{1}{\varepsilon^2}\cdot2\cdot\pi\varepsilon^2=2\pi.$$

$$=\frac{1}{\varepsilon^2}\cdot 2\cdot \pi\varepsilon^2=2\pi.$$

$$\oint_{L'} Pdx+Qdy = I + \int_{l_4} Pdx+Qdy + \int_{l_5} Pdx+Qdy.$$

第四节 对面积的曲面积分

引例：曲面形状物体的质量. $\rho = f(x,y,z)$



$$M=\lim_{\lambda\to 0}\sum_{i=1}^{n} f(\xi_i,\eta_i,\zeta_i)\cdot \Delta S_i$$

定义：$f(x,y,z)$ 在 Σ 上有界.

① 任意分割 ② 任意取点

若 $\lim_{\lambda\to 0}\sum_{i=1}^{n} f(\xi_i,\eta_i,\zeta_i)\Delta S_i$ 存在 则称为

$f(x,y,z)$ 在 Σ 上对面积的曲面积分（I型）

记为 $\iint_{\Sigma} f(x,y,z)\,dS$ 曲面面积元素

当 $f(x,y,z)=1$ 时 $\iint_{\Sigma} 1\,dS = S$（曲面面积）

性质：① 积分中值定理

$$\iint_{\Sigma} f(x,y,z)\,dS = f(\xi,\eta,\zeta)\cdot S$$

② 对称性 —— 类似于三重积分

Σ

③对称性 —— 类似于三重积分

(i) Σ 关于 xoy 面对称 其中 Σ_1: 上半部分

$$\Rightarrow \iint_{\Sigma} f\,dS = \begin{cases} 2\iint_{\Sigma_1} f\,dS, & f(x,y,z)=f(x,y,-z) \\ 0, & -f(x,y,z)=f(x,y,-z). \end{cases}$$

(ii) Σ 具有轮换对称性

$$\iint_{\Sigma} f(x,y,z)\,dS = \iint_{\Sigma} f(y,z,x)\,dS = \iint_{\Sigma} f(z,x,y)\,dS$$

例. Σ: $x^2+y^2+z^2=R^2$. 具有轮换对称性

$$\iint_{\Sigma} x^2\,dS = \iint_{\Sigma} y^2\,dS = \iint_{\Sigma} z^2\,dS$$

$$= \frac{1}{3}\iint_{\Sigma}(x^2+y^2+z^2)\,dS = \frac{R^2}{3}\iint_{\Sigma} 1\,dS = \frac{R^2}{3}\cdot 4\pi R^2.$$

$$\iint_{\Sigma}(ax+by+cz)^2\,dS \xlongequal{\text{对称性}} \iint_{\Sigma}(a^2x^2+b^2y^2+c^2z^2)\,dS$$

$$\xlongequal{\text{轮换对称性}} \frac{a^2+b^2+c^2}{3}\iint_{\Sigma}(x^2+y^2+z^2)\,dS = \frac{a^2+b^2+c^2}{3}R^2\cdot 4\pi R^2.$$

例: $\iiint\limits_{\Omega} x^2 dv \qquad \Omega: x^2+y^2+z^2 \le R^2.$

$= \frac{1}{3}\iiint\limits_{\Omega}(x^2+y^2+z^2)dv \cancel{=} \frac{1}{3}R^2\iiint\limits_{\Omega}1dv = \frac{R^2}{3}\cdot\frac{4}{3}\pi R^3.$

$= \frac{1}{3}\int_0^{2\pi}d\theta\int_0^{\pi}\sin\varphi d\varphi\int_0^{R} r^2 \quad r^2dr = \cdots$

I型曲面积分的计算 ⟶ 二重积分

曲面 $z=z(x,y)$. $(x,y)\in D_{xy}$.

$$dS = \sqrt{1+z_x^2+z_y^2}\,dxdy.$$

故 $\iint\limits_{\Sigma} f(x,y,z)dS$

$= \iint\limits_{D_{xy}} f(x,y,z(x,y))\cdot\sqrt{1+z_x^2+z_y^2}\,dxdy$