Syllabus

Physics-I [BS-PH 201]

. Mechanics

Problems including constraints & friction. Basic ideas of vector calculus and partial differential equations. Potential energy function F = -grad V, equipotential surfaces and meaning of gradient. Conservative and non-conservative forces. Conservation laws of energy & momentum. Non-inertial frames of reference. Harmonic oscillator; Damped harmonic motion forced oscillations and resonance. Motion of a rigid body in a plane and in 3D. Angular velocity vector. Moment of inertia.

2. Optics

- Distinction between interference and diffraction, Fraunhofer and Fresnel diffraction, Fraunhofer diffraction at single slit, double slit, and multiple slits (only the expressions for maximum, & intensity and qualitative discussion of fringes); diffraction grating(resolution formulac only), characteristics of diffraction grating and its applications.
- Polarisation: Introduction, polarisation by reflection, polarisation by double reflection, scattering of light, circular and elliptical polarisation, optical activity.
- Lasers: Principles and working of laser : population inversion, pumping, various modes, threshold population inversion with examples .

3. Electromagnetism and Dielectric Magnetic Properties of Materials

- Maxwell's equations. Polarisation, permeability and dielectric constant, polar and non-polar dielecrrics, internal fields in a solid, Clausius-Mossotti equation (expression only), applications of dielectrics.
- Magnetisation, permeability and susceptibility, classification of magnetic materials, ferromagnetism, magnetic domains and hysteresis, applications.

4. Quantum Mechanics

- Introduction to quantum physics, black body radiation, explanation using the photon concept, Compton effect, de Broglie hypothesis, wave-particle duality, verification of matter waves, uncertainty principle, Schrodinger wave equation, particle in box, quantum harmonic oscillator, hydrogen atom.

5. Statistical Mechanics

- Macrostate, Microstate, Density of states, Qualitative treatment of Maxwell Boltzmann, Fermi-Diraç and Bose-Einstein statistics.

Chemistry-I [BS-CH 201]

i) Atomic and molecular structure

- Schrodinger equation. Particle in a box solutions and their applications for simple sample. Molecular orbitals of diatomic molecules (e.g.H2). Energy level diagrams of diatomic. Pi-molecular orbitals of butadiene and benzene and aromaticity. Crystal field theory and the energy level diagrams for transition metal ions and their magnetic properties. Band structure of solids and the role of doping on band structures.

ii) Spectroscopic techniques and applications

- Principles of spectroscopy and selection rules. Electronic spectroscopy. Fluorescence and it applications in medicine. Vibrational and rotational spectroscopy of diatomic molecules Applications. Nuclear magnetic resonance and magnetic resonance imaging, surfac characterisation techniques. Diffraction and scattering.

Mathematics – II B [BS-M 202]

Multivariate Calculus (Integration):

- Multiple Integration: Double integrals (Cartesian), change of order of integration in double integrals, change of variables (Cartesian to Polar), Applications: Areas and volumes, Center of mass and Gravity (constant and variable densities); Triple integrals (Cartesian), Orthogonal curvilinear coordinates, Simple applications involving cubes, sphere and rectangular parallelepipeds; Scalar line integrals, vector line integrals, scalar surface integrals, vector surface integrals, Theorems of Green, Gauss and Stokes.

First order ordinary differential equations:

- Exact, linear and Bernoulli's equations, Equations not of first degree: equations solvable for p, equations solvable for y, equations solvable for x and Clairaut's type.

Ordinary differential equations of higher orders:

- Second order linear differential equations with constant coefficients, Use of Doperators, Second order linear differential equations with variable coefficients, method of variation of parameters, Cauchy-Euler equation; Power series solutions; Legendre polynomials, Bessel functions of the first kind and their properties.

Complex Variable – Differentiation

- Differentiation of complex functions, Cauchy-Riemann equations, Analytic functions, Harmonic functions, determination of harmonic conjugate, elementary analytic functions (exponential, trigonometric, logarithmic) and their properties; Conformal mappings, Mobius transformations and their properties.

Complex Variable – Integration

- Contour integrals, Cauchy-Goursat theorem (without proof), Cauchy integral formula (without proof), Liouville's theorem and Maximum-Modulus theorem (without proof); Taylor's series, Zeros of analytic functions, Singularities, Laurent's series; Residues, Cauchy residue theorem (without proof), Evaluation of definite integral involving sine and cosine, Evaluation of certain improper integrals using the Bromwich contour.

Programming for Problem Solving [ES-CS 201]

Unit 1: Introduction to Programming

- Introduction to components of a computer system (disks, memory, processor, where a program is stored and executed, operating system, compilers etc.)
- Idea of Algorithm: steps to solve logical and numerical problems. Representation of Algorithm: Flowchart/Pseudocode with examples.
- From algorithms to programs; source code, variables (with data types) variables and memory locations, Syntax and Logical Errors in compilation, object and executable code

Unit 2: Arithmetic expressions and precedence

Unit 3: Conditional Branching and Loops

- Writing and evaluation of conditionals and consequent branching
- Iteration and loops

Unit 4: Arrays

- Arrays (1-D, 2-D), Character arrays and Strings

Unit 5: Basic Algorithms

- Searching, Basic Sorting Algorithms (Bubble, Insertion and Selection), Finding roots of equations, notion of order of complexity through example programs (no formal definition required)

Unit 6: Function

- Functions (including using built in libraries), Parameter passing in functions, call by value, Passing arrays to functions: idea of call by reference

Unit 7: Recursion

- Recursion as a different way of solving problems. Example programs, such as Finding Factorial, Fibonacci series, Ackerman function etc. Quick sort or Merge sort.

Unit 8: Structure

- Structures, Defining structures and Array of Structures

Unit 9: Pointers

- Idea of pointers, Defining pointers, Use of Pointers in self-referential structures, notion of linked list (no implementation)

Unit 10: File handling (only if time is available, otherwise should be done as part of the lab)

English [HM-HU 201]

1. Vocabulary Building

1.1 The concept of Word Formation
1.2 Root words from foreign languages and their use in English
1.3 Acquaintance with prefixes and suffixes from foreign languages in English to form derivatives.
1.4 Synonyms, antonyms, and standard abbreviations.

2. Basic Writing Skills

2.1 Sentence Structures
2.2 Use of phrases and clauses in sentences
2.3 Importance of proper punctuation
2.4 Creating coherence
2.5 Organizing principles of paragraphs in documents
2.6 Techniques for writing precisely

3. Identifying Common Errors in Writing

3.1 Subject-verb agreement
3.2 Noun-pronoun agreement
3.3 Misplaced modifiers
3.4 Articles
3.5 Prepositions
3.6 Redundancies
3.7 Clichés

4. Nature and Style of sensible Writing

4.1 Describing
4.2 Defining
4.3 Classifying
4.4 Providing examples or evidence
4.5 Writing introduction and conclusion

5. Writing Practices

5.1 Comprehension
5.2 Précis Writing
5.3 Essay Writing

6. Oral Communication

(This unit involves interactive practice sessions in Language Lab)

- Listening Comprehension
- Pronunciation, Intonation, Stress and Rhythm
- Common Everyday Situations: Conversations and Dialogues
- Communication at Workplace
- Interviews
- Formal Presentations

Contents

PHYSICS-I

NOTE:

MAKAUT course structure and syllabus of 1st year has been changed from 2018. Present syllabus of **PHYSICS – I** has been completely redesigned and restructured with selected topics from 1st year & 2nd year PHYSICS. Few new topics have been introduced also. Taking special care of this matter we are providing the relevant university solutions of both years along with some model questions & answers for newly introduced topics, so that students can get an idea about university questions patterns.

MECHANICS

Chapter at a Glance

Newton's Laws of Motion

Newton's First Law of Motion: An object remains at rest (if originally at rest) or moves in a straight line with constant velocity if the net force on it is zero.

Newton's Second Law of Motion: A particle with a force acting on it has an acceleration proportional to the magnitude of the force and in the direction of that force.

Newton's Third Law of Motion: The forces of action and reaction between interacting bodies are equal in magnitude and opposite in direction. - or - For every action there is an equal and opposite reaction.

Preliminary concepts of SHM

Simple Harmonic Motion (SHM) is a periodic motion of a body, in which acceleration of the body (or force on the body) is directly proportional to its displacement from a fixed point and is always directed towards the fixed point.

Energy of a SHM

- P.E. $= \int_0^y F dx = \int_0^y \text{mass} \times \text{acceleration}\, dx = \int_0^y m\omega^2 x dx = \dfrac{m\omega^2 x^2}{2} = \dfrac{m\omega^2 a^2 \sin^2\omega t}{2}$
- K.E. $= \dfrac{mv^2}{2} = \dfrac{ma^2\omega^2\cos^2\omega t}{2}$
- Total energy P.E + K.E $= E = \dfrac{ma^2\omega^2}{2}$
- Average kinetic energy, $\langle E_k \rangle = \dfrac{1}{T}\left[\dfrac{1}{2}m\int_0^T \left(\dfrac{dx}{dt}\right)^2 dt\right] = \dfrac{1}{4}m\omega^2 a^2$
- Average potential energy, $\langle E_p \rangle = \dfrac{1}{T}\left[\dfrac{1}{2}s\int_0^T x^2 dt\right] = \dfrac{1}{4}m\omega^2 a^2$

Damped harmonic motion

In reality when a pendulum is allowed to swing in air, frictional force due to air will come into play and as a result it's amplitudes decreases down continuously with time. This nature of pendulum's motion is deviating from free vibration as discussed in chapter one. So free vibration is a idealistic case but not a realistic one. *In presence of some dissipative force if a body executes a vibrational motion and it's amplitude decays down continuously with time then the motion of the body is known as damped vibration.*

Examples:

(i) The motion of a coil in a suspended coil galvanometer

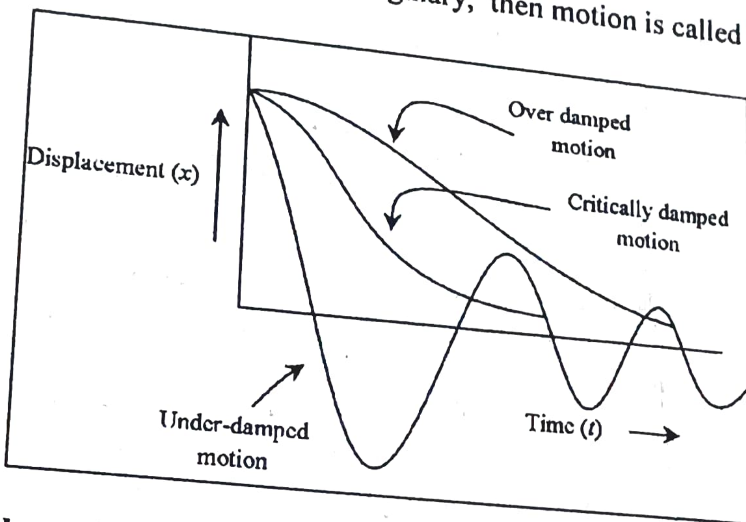
(ii) The motion of a mass-spring system etc...

- **Differential Equation of a Damped Harmonic Motion is** $m\frac{d^2x}{dt^2} = -kx - L\frac{dx}{dt}$
  where L is resistive constant, k is restoring force constant.
  If we adjust the above differential equation as
  $\frac{d^2x}{dt^2} + \frac{L}{m}\frac{dx}{dt} + \frac{k}{m}x = 0$ or, $\frac{d^2x}{dt^2} + 2K\frac{dx}{dt} + \omega^2 x = 0$ $\left[\text{where } K = \frac{L}{m} \text{ and } \omega^2 = \frac{k}{m}\right]$
- Solution of the differential equation of a damped vibration as
  $x = A_1 e^{\left(-K+\sqrt{K^2-\omega^2}\right)t} + A_2 e^{\left(-K-\sqrt{K^2-\omega^2}\right)t}$

Case 1: If $K > \omega$, i.e., $\sqrt{K^2-\omega^2}$ motion is called as *over damped* or *dead beat*

Case 2: If $K = \omega$, the motion will be critical

Case 3: When $K < \omega$ i.e., $\sqrt{K^2-\omega^2}$ is imaginary, then motion is called damped oscillatory motion



- **Logarithmic decrement can** be represented as

$$\lambda = \log_e \frac{C}{C_1} = \log_e \frac{C_1}{C_2} = \log_e \frac{C_2}{C_3} = \text{constant.}$$

- Relaxation time means the time taken to fall the amplitude $\frac{1}{e}$ times (37%) of its initial amplitude.
- When damping constant (K) is greater than restoring constant (ω) the motion is damped or dead beat.
- For critically damped motion damping constant is **equal to restoring constant.**
- When damping constant (K) is less than vestoring constant (ω) the is damped oscillatory.
- Damped oscillatory motion of frequency is

$f' = \frac{p}{2\pi} = \frac{\sqrt{\omega^2 - K^2}}{2\pi}$ Time period for under damped motion $T = \frac{2\pi}{\sqrt{\omega^2 - k^2}}$

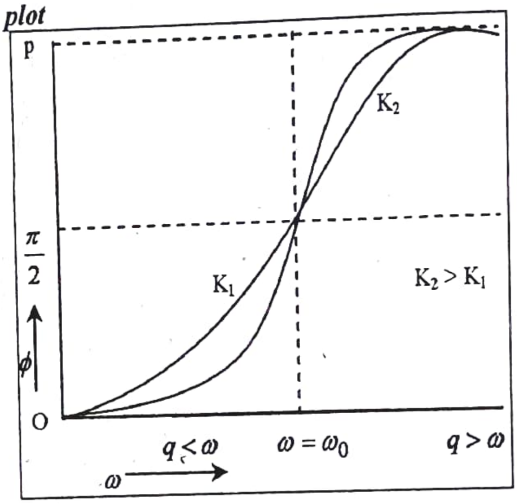
- When a body is set into vibrating condition by applying strong periodic force having a frequency different from the natural frequency of the body, such vibration of the body is termed as forced vibration.

Example: When a vibrating tuning fork is pressed by the stem of it on a table a forced vibration is produced.

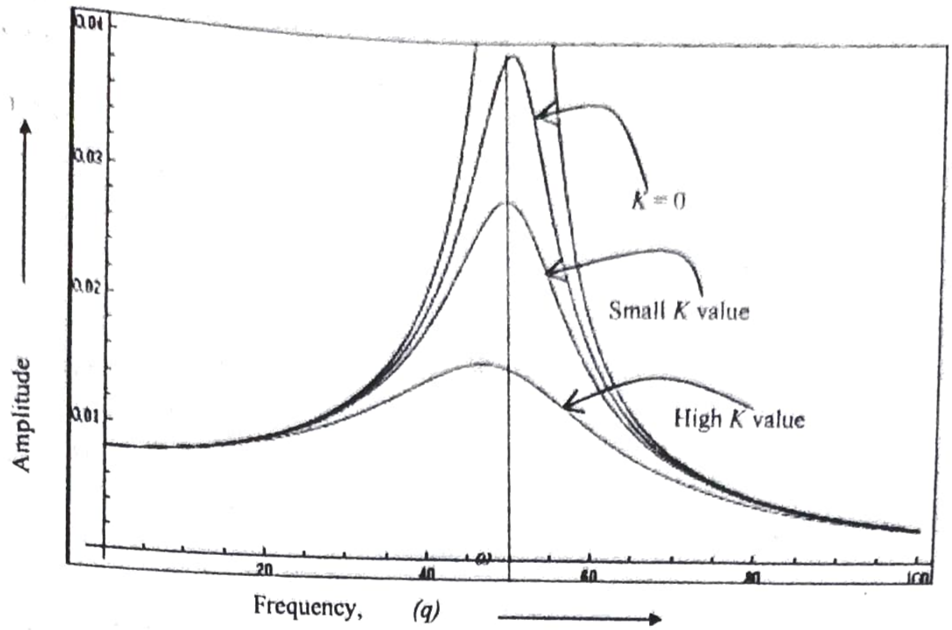
- Differential Equation of Forced Vibration is $m\frac{d^2x}{dt^2}=-kx-L\frac{dx}{dt}+F_0e^{iqt}$ where q is the frequency of the applied external force.
- General Solution of the differential Equation of Forced Vibration is sum of the complimentary function and particular integral at the start

$$\text{i.e., } x = Ae^{i(qt-\phi)}+Ce^{-Kt}\sin\left(\sqrt{\left(\omega^2-K^2\right)}\,t+\delta\right)$$

- After a lapse of time when the 2nd part becomes negligible, we can write the sustained forced vibration solution as $x=\frac{f_0}{\sqrt{4K^2q^2+\left(\omega^2-q^2\right)^2}}e^{i(qt-\varphi)}$, which is represented as **steady state motion**
- **Amplitude** is $A=\frac{f_0}{\sqrt{4K^2q^2+\left(\omega^2-q^2\right)^2}}$ and that of **phase** is $\phi=\tan^{-1}\left(\frac{2qK}{\omega^2-q^2}\right)$ of the seady state displacement of the particle under forced vibration.
- ***Phase vs. frequency plot***



- The phenomenon in which a body vibrates with its natural frequency under the influence of external periodic force is called ***resonance vibration***.
- The value of amplitude under ***amplitude resonance*** condition is $A=\frac{f_0}{2K\sqrt{\omega^2-K^2}}$. For ***weak damping*** (i.e., K very small) the amplitude becomes $A_{max}=\frac{f_0}{2K\omega}$.
- The amplitude resonance always occurs at or near $q=\omega$ provided the damping is not too large.



- Average power supplied by the driving force to the oscillator is equal to average loss of power of the oscillator i.e. = mKq^2A^2
- The sharpness of resonance may be defined (In terms of amplitude resonance) as the rate of fall in a amplitude (or decrement in amplitude) with the change of the frequency of the applied periodic force on each side of resonant frequency.
- We can define also sharpness of resonance in terms of velocity resonance as when the amplitude of velocity of the forced vibration under the action of impressed periodic force is maximum, then the phenomenon is known as velocity resonance with the applied force.
- The ***quality factor*** Q value is defined as 2π times the ratio of the energy stored in the system to the energy lost per period.

$$Q = 2\pi \frac{\text{Average energy stored}}{\text{Energy dissipated per cycle}} = \frac{2\pi E_{av}}{T \times P_{av}}$$

- Electrical analogue of forced vibration equation is $L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{c} = E_0 e^{i\omega t}$
- In the analogy with electrical circuit we may call $Z = R_m + i\left(qm - \frac{k}{q}\right)$ as complex **mechanical impedance** and $X_m = \left(qm - \frac{k}{q}\right)$ as **mechanical reactance.**
- At current resonance in the electrical circuit the reactance vanishes i.e., $\left(\omega L - \frac{1}{\omega c}\right) = 0$ i.e.,

$$\omega^2 = \frac{1}{LC} \rightarrow \qquad \omega = \frac{1}{\sqrt{LC}}$$

Multiple Choice Type Questions

1.1. The value of a for which $\vec{A} = \hat{i}2ax + \hat{j}2y + \hat{k}4z$ is solenoidal is equal to [WBUT 2018(ODD)]

a) 2 b) 3 c) –3 d) 1

Answer: (c)

1.2. Moment of inertia of solid sphere of Mass M and radius R is [WBUT 2018(ODD)]

a) $\frac{5}{2}MR^2$ b) $\frac{1}{2}MR^2$ c) $\frac{2}{3}MR^2$ d) $\frac{2}{5}MR^2$

Answer: (d)

1.3. In a conservative field [WBUT 2018(ODD)]

a) work done is zero
b) line integral is independent of path
c) curl is not equal to zero
d) divergence is zero

Answer: (b)

1.4. The moment of inertia of a thin uniform rod of mass M and length L about an axis perpendicular to the rod, through its centre is I. The moment of inertia of the rod about an axis perpendicular to the rod through its end point is [WBUT 2019(EVEN)]

a) $\frac{I}{4}$ b) $\frac{I}{2}$ c) $2I$ d) $4I$

Answer: (d)

1.5. The relaxation time is defined as the time during which the amplitude of a damped oscillator [WBUT 2019(EVEN)]

a) grows to e times the initial value
b) decays to $1/e$ times the initial value
c) grows to e^2 times the initial value
d) decays to $1/e^2$ times the initial value

Answer: (b)

1.6. For a particle executing S.H.M. the phase difference between displacement and velocity is [WBUT 2019(ODD)]

a) 0 b) $\pi/2$ c) $\pi/4$ d) π

Answer: (b)

1.7. If the vector $\vec{A}$ is solenoidal, then [WBUT 2019(ODD)]

a) $\vec{\nabla}\cdot\vec{A}=0$ b) $\vec{\nabla}\times\vec{A}=0$ c) $2(\vec{\nabla}\times\vec{A})=1$ d) $\vec{\nabla}|\vec{A}|=0$

Answer: (a)

1.8. Rigid body has constraints classified in which of the following group? [WBUT 2007(ODD), 2015(ODD)]

a) Rheonomic and Holonomic
b) Rheonomic and Non-holonomic
c) Scleronomic and Holonomic
d) Scleronomic and Non-holonomic

Answer: (c)

1.9. A system with time independent potential is in an energy state E. The wave function of this state is [WBUT 2011(ODD)]

a) independent of time
b) an exponentially decaying function of time
c) a periodic function of time with time period proportion to E
d) a periodic function of time with time period inversely proportional to E.

Answer: (d)

1.10. The degrees of freedom for a system of N particles with K constraint relation is given by [WBUT 2012(EVEN)]

a) $N-K$
b) $N-3K$
c) $3N-K$
d) $3(N-K)$

Answer: (c)

1.11. If the constraint relations can be made independent of velocity, then the constraints are called [WBUT 2013(EVEN)]

a) sclerenomic
b) bilateral
c) holonomic
d) conservative

Answer: (c)

1.12. A rigid body whose equation of constraint is given by $\left|\vec{r}_i - \vec{r}_j\right|^2 = \text{constant}$ is governed by [WBUT 2014(EVEN)]

a) holonomic, rheonomic, dissipative and bilateral constraints
b) non-holonomic, conservative and schleronomic constraints
c) holonomic, schleronomic and conservative constraints
d) non-holonomic, rheonomic and unilateral constraints

Answer: (c)

1.13. The constraint associated with the motion of a pendulum is [WBUT 2014(EVEN)]

a) Dissipative
b) Holonomic
c) Rheonomic
d) Non-Holonomic

Answer: (b)

1.14. The degree of freedom of particle constrained to move on a surface of sphere is [WBUT 2016(ODD)]

a) 1
b) 2
c) 3
d) 4

Answer: (b)

1.15. In case of a rigid body, the inertia forces of action and reaction between any two particles are [MODEL QUESTION]

a) generalized forces
b) pseudo forces
c) forces of constraint
d) none of the above

Answer: (c)

1.16. For a conservative system, the potential function [MODEL QUESTION]

a) dependent on position and velocity
b) purely dependent on velocity
c) purely dependent on coordinates
d) purely dependent on momentum

Answer: (c)

1.17. A particle of mass m is executing oscillations about the origin on the x-axis. Its potential energy is $V(x)=k|x|^3$ where k is appositive constant. If the amplitude of oscillation is 'a', then its time period T is [MODEL QUESTION]

a) proportional to $\frac{1}{\sqrt{a}}$
b) independent of a
c) proportional to $\sqrt{a}$
d) proportional to $a^{3/2}$

Answer: (c)

1.18. If a vector field can be expressed as the gradient of a scalar, then the vector field is called [MODEL QUESTION]

a) Solenoidal
b) Irrotaional
c) Lamellar
d) None of these

Answer: (c)

1.19. The vector $\vec{A}$ is solenoidal if [MODEL QUESTION]

a) $\vec{\nabla}\times\vec{A}=0$
b) $\vec{\nabla}.\vec{A}=0$
c) $\vec{A}=0$
d) None of these

Answer: (b)

1.20. The vector field is irrotational if [MODEL QUESTION]

a) $\vec{\nabla}\times\vec{A}=0$
b) $\vec{\nabla}.\vec{A}=0$
c) $\vec{\nabla}\times\vec{A}=1$
d) $\vec{\nabla}.\vec{A}=1$

Answer: (a)

1.21. A vector field A is conservative if [MODEL QUESTION]

a) $\vec{A}=\vec{\nabla}.\varphi$
b) $\vec{A}=\vec{\nabla}\times\varphi$
c) $\vec{A}=-\vec{\nabla}\varphi$
d) $\vec{A}=\nabla^2\varphi$

Answer: (c)

1.22. The line integral of the vector field around a closed path can always be written as [MODEL QUESTION]

a) $\iint_S\left(\vec{\nabla}.\vec{A}\right)dS$
b) $\oiint_S\left(\vec{\nabla}.\vec{A}\right).\vec{dS}$
c) $\iint_S\left(\vec{\nabla}.\vec{A}\right).\vec{dS}$
d) $\oiint\left(\vec{\nabla}.\vec{A}\right).\vec{dS}$

Answer: (c)

1.23. For a conservative field [MODEL QUESTION]

a) $\oint\vec{E}.\vec{dl}=0$
b) $\oint\vec{E}.\vec{dl}=1$
c) $\oint\vec{E}.\vec{dl}=El$
d) $\oint\vec{E}.dl=Edl\cos\theta$

Answer: (a)

1.24. Gauss's divergence theorem permits change from [MODEL QUESTION]

a) volume integral to surface integral and vice versa
b) volume to line integral and vice versa
c) surface integral to line integral and vice versa
d) none of these

Answer: (a)

1.25. Gravitational force field is [MODEL QUESTION]

a) central force field
b) non-central force field
c) non-conservative force field
d) repulsive force field
e) conservative force field

Answer: (a), (e)

1.26. Which of the following has coulomb as the unit? [MODEL QUESTION]

a) $\oint \vec{H}.\vec{dl}$
b) $\oint \vec{E}.\vec{dl}$
c) $\iint \vec{D}.\vec{ds}$
d) $\iint \vec{E}.\vec{ds}$

Answer: (c)

1.27. Maxwell's equation involving $\frac{\partial \vec{B}}{\partial t}$ is obtained from [MODEL QUESTION]

a) Ampere's law
b) Gauss's law
c) Biot and Savart's law
d) Faraday's law

Answer: (d)

1.28. Velocity of plane electromagnetic wave in vacuum is given by [MODEL QUESTION]

a) $c=\frac{1}{\sqrt{\mu_0\varepsilon_0}}$
b) $c=\left(\frac{\mu_0}{\varepsilon_0}\right)^{1/2}$
c) $c=\sqrt{\mu_0\varepsilon_0}$
d) $c=\left(\frac{\varepsilon_0}{\mu_0}\right)^{1/2}$

Answer: (a)

1.29. A force $\vec{F}=-k\left(\hat{i}y+\hat{j}x\right)$, where k is a positive constant acts on a particle moving in the xy plane. Starting from the origin, the particle is taken along the positive x-axis to the point (a,a). The total work done by the force $\vec{F}$ on the particle is, [MODEL QUESTION]

a) $-2ka^2$
b) $2ka^2$
c) $-ka^2$
d) ka^2

Answer: (c)

1.30. Which relation between restoring force and potential energy is correct? [MODEL QUESTION]

a) Potential energy = ½ restoring force × displacement^2
b) Potential energy = ½ restoring force × displacement
c) Potential energy = restoring force × displacement
d) Potential energy = restoring force × displacement^2

Answer: (a)

1.31. S.I unit of spring constant is [MODEL QUESTION]

a) N m^{-1} b) N m^{-2} c) N m d) none of these

Answer: (a)

1.32. For a particle executing S.H.M the phase difference between displacement and velocity is [MODEL QUESTION]

a) π b) $\pi/2$ c) 0 d) $-\pi/2$

Answer: (b)

1.33. The relaxation time τ for a mechanical oscillator is related to damping constant L is [MODEL QUESTION]

a) $\tau = \frac{L}{2m}$ b) $\tau = \frac{2m}{L}$ c) $\tau = 2mL$ d) $\tau = \frac{L}{m}$

Answer: (b)

1.34. What is the phase difference between driving force and velocity of forced oscillator? [MODEL QUESTION]

a) ϕ b) $\phi + \frac{\pi}{2}$ c) $\phi - \frac{\pi}{2}$ d) $\frac{\pi}{2} - \phi$

Answer: (a)

1.35. Example of weakly damped harmonic oscillator is [WBUT 2007(ODD), 2011(EVEN), 2013(ODD)]

a) Dead-bead galvanometer
b) Tangent galvanometer
c) Ballistic galvanometer
d) Discharge of a charged capacitor through a resistance

Answer: (c)

1.36. Motion of a system in critical damped condition is [WBUT 2008(ODD)]

a) oscillatory b) damped oscillatory
c) harmonic d) non-oscillatory

Answer: (d)

1.37. If a is the force constant of an oscillating body of mass m, the Q-factor is where C = relaxation time. [WBUT 2009(ODD)]

a) $Q = C\sqrt{am}$ b) $Q = C\sqrt{m/a}$ c) $Q = C\sqrt{a/m}$

Answer: (c)

1.38. If the equation of motion of an oscillator is given by $\ddot{x} + \frac{\gamma^2}{4}x + \gamma\dot{x} = 0$ then the motion is [WBUT 2011(ODD), 2016(ODD)]

a) simple harmonic without damping
b) a critically damped simple harmonic
c) an over damped simple harmonic
d) an under damped simple harmonic

Answer: (b)

1.39. If the damping force on an one-dimensional harmonic oscillator of natural frequency ω_0, is $2bmv$, where m is the mass and v is the instantaneous velocity of the oscillator then the frequency of oscillation (when $b << \omega_0$) is

[WBUT 2011(EVEN)]

a) ω_0
b) b
c) $\omega_0\left(1-\frac{b^2}{2\omega_0^2}\right)$
d) $\omega_0\left(1+\frac{b^2}{2\omega_0^2}\right)$

Answer: (c)

1.40. The relaxation time (τ) of a damped harmonic oscillator with damping constant (K) is

[WBUT 2012(ODD), 2015(ODD)]

a) $\tau = 1/K$
b) $\tau = 1/2K$
c) $\tau = K$
d) $\tau = 2K$

Answer: (a) or (b)

1.41. If 'α' is the force constant of an oscillating body of mass 'm' the Q-factor is

[WBUT 2012(ODD)]

a) $Q \propto \sqrt{\alpha m}$
b) $Q \propto \sqrt{\frac{m}{\alpha}}$
c) $Q \propto \sqrt{\frac{\alpha}{m}}$
d) $Q \propto \frac{\alpha}{m}$

Answer: (a)

1.42. The relaxation time is the time in which the amplitude of the damped oscillator

[WBUT 2014(EVEN)]

a) grows to e times the initial value
b) decays to 1/e times the initial value
c) grows to e^2 times the initial value
d) decays to $1/e^2$ times the initial value

Answer: (b)

1.43. For large values of damping constant the Q-factor

[WBUT 2014(ODD), 2015(EVEN)]

a) increase
b) decreases
c) remains same
d) becomes zero

Answer: (b)

1.44. The quality factor Q for an L-C-R circuit is

[WBUT 2010(ODD), 2015(ODD)]

a) $\frac{\omega R}{L}$
b) $\frac{\omega L}{R}$
c) $\frac{\omega}{LR}$
d) $\frac{R}{\omega L}$

Answer: (b)

1.45. An external force $F = F_0 e^{i\omega t}$ is applied to a slightly damped oscillator of natural frequency ω_0 then in steady state it will oscillate with a frequency

[WBUT 2010(ODD)]

a) ω_0
b) ω
c) $\omega_0 - \omega$
d) $\sqrt{\omega_2^0 - \omega^2}$

Answer: (b)

1.46. An oscillator with natural frequency ω_0 experiences a damping force and also a periodic force $A\cos(\Omega t)$. Amplitude of displacement velocity of the resulting steady state forced oscillation is studied with slow variation of Ω, when $\Omega = \omega_0$ [WBUT 2011(ODD)]

a) the displacement amplitude shows a maximum
b) the velocity amplitude shows a maximum
c) amplitudes of both displacement and velocity show maxima
d) amplitude of displacement shows minimum while the amplitude of velocity shows maximum

Answer: (b)

1.47. The resonant frequency of an electrical oscillator is given by [WBUT 2012(ODD)]

a) $\nu = 2\pi\sqrt{LC}$ b) $\nu = \frac{1}{2\pi\sqrt{LC}}$ c) $\nu = \frac{2\pi}{\sqrt{LC}}$ d) $\nu = 2\pi\sqrt{\frac{L}{C}}$

Answer: (b)

1.48. For larger value of damping constant k the resonance curve will be [WBUT 2012(ODD)]

a) Unchanged b) Flatter c) Sharper d) None of these

Answer: (b)

1.49. The amount of power supplied to a system is equal to the rate of dissipation of energy in [WBUT 2014(ODD)]

a) forced vibration b) damped vibration
c) simple harmonic motion d) oscillatory motion

Answer: (a)

Short Answer Type Questions

2.1. An oscillator executing SHM has zero displacement at time $t = 0$. If the displacement are 1mm and 1.5mm at instants 0.1 and 0.2 seconds, calculate the frequency and amplitude of oscillation. [WBUT 2018(ODD)]

Answer:

Using the formula, $x = a\sin\omega t$

we have $a\sin(0.1\omega) = 0.1$ and $a\sin(0.2\omega) = 0.15$

or, $0.15 = 0.1\frac{\sin(0.2\omega)}{\sin(0.1\omega)} \rightarrow 1.5 = \frac{2\sin(0.1\omega)\cos(0.1\omega)}{\sin(0.1\omega)}$

or, $\cos(0.1\omega) = 0.75 \rightarrow \omega = 0.723\,\text{rad}$ or 2.3π rad

So, Frequency $\nu = \frac{\omega}{2\pi} = \frac{0.723}{2\pi} = 1.15\,\text{Hz}$

and amplitude $a = \frac{0.1}{\sin(0.1\omega)} = \frac{0.1}{\sin 41.4} = 0.151\,\text{cm}$

2.2. a) Find $\vec{\nabla}\cdot\vec{F}$ where $\vec{F}=\vec{\nabla}\left(x^3+y^3+z^3-3xyz\right)$. **[WBUT 2019(EVEN)]**

Answer:

Here, $\vec{F}=\vec{\nabla}\left(x^3+y^3+z^3-3xyz\right)$

$$=\hat{i}\frac{\partial}{\partial x}\left(x^3+y^3+z^3-3xyz\right)+\hat{j}\frac{\partial}{\partial y}\left(x^3+y^3+z^3-3xyz\right)+\hat{k}\frac{\partial}{\partial z}\left(x^3+y^3+z^3-3xyz\right)$$

$$=\left(3x^2-3yz\right)\hat{i}+\left(3y^2-3xz\right)\hat{j}+\left(3z^2-3xy\right)\hat{k}$$

Now $\vec{\nabla}\cdot\vec{F}=\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\cdot\left[\left(3x^2-3yz\right)\hat{i}+\left(3y^2-3xz\right)\hat{j}+\left(3z^2-3xy\right)\hat{k}\right]$

or, $\vec{\nabla}\cdot\vec{F}=6x+6y+6z$

b) Show that the vector field $\vec{F}=\dfrac{x\hat{i}+y\hat{j}}{\sqrt{\left(x^2+y^2\right)}}$ is a "source" field. **[WBUT 2019(EVEN)]**

Answer:

For a source field, $\vec{\nabla}\cdot\vec{F}$ should be positive and for a sink field, $\vec{\nabla}\cdot\vec{F}$ should be negative.

Here, $\vec{F}=\dfrac{\hat{i}x+\hat{j}y}{\sqrt{x^2+y^2}}$

$$\therefore\quad \vec{\nabla}\cdot\vec{F}=\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\cdot\left(\frac{\hat{i}x+\hat{j}y}{\sqrt{x^2+y^2}}\right)=\frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2+y^2}}\right)+\frac{\partial}{\partial y}\left(\frac{y}{\sqrt{x^2+y^2}}\right)$$

$$=-\frac{x^2}{\left(x^2+y^2\right)^{3/2}}+\frac{1}{\left(x^2+y^2\right)^{1/2}}-\frac{y^2}{\left(x^2+y^2\right)^{3/2}}+\frac{1}{\left(x^2+y^2\right)^{1/2}}$$

$$=-\frac{\left(x^2+y^2\right)}{\left(x^2+y^2\right)^{3/2}}+\frac{2}{\left(x^2+y^3\right)^{1/2}}=-\frac{1}{\left(x^2+y^2\right)^{1/2}}+\frac{2}{\left(x^2+y^2\right)^{1/2}}=\frac{1}{\sqrt{x^2+y^2}}$$

$\therefore$ $\vec{\nabla}\cdot\vec{F}$ is +ve, so $\vec{F}$ is a source field except at the origin (0,0)

2.3. At an instant of time displacement of a particle is 12 cm, velocity is 5 cm/sec and when its displacement is 5 cm, the velocity is 12 cm/sec. Calculate the amplitude, frequency and time period. **[WBUT 2019(ODD)]**

Answer:

Velocity of the particle, $v=\dfrac{dy}{dt}=\omega\sqrt{a^2-y^2}$

In the first case:

$$v_1=\omega\sqrt{a^2-y_1^2}$$

where, $v_1 = 5\text{ cm s}^{-1}$ and $y_1 = 12\text{ cm}$

or, $5 = \omega\sqrt{a^2 - 144}$ .... (1)

in the second case:

$$v_2 = \omega\sqrt{a^2 - y_2^2}$$

where, $v_2 = 12\text{ cm s}^{-1}$ and $y_2 = 5\text{ cm}$

$\therefore$ $12 = \omega\sqrt{a^2 - 25}$ .... (2)

Dividing (1) by (2) we have, $\frac{5}{12} = \frac{\sqrt{a^2 - 144}}{\sqrt{a^2 - 25}}$

$$\frac{25}{144} = \frac{a^2 - 144}{a^2 - 25} \Rightarrow 25a^2 - 25^2 = 144a^2 - 144^2$$

$$\Rightarrow 119a^2 = 144^2 - 25^2$$

$$\Rightarrow 119a^2 = (144 - 25)(144 + 25)$$

$$\Rightarrow a^2 = 169 \Rightarrow a = 13\text{ cm}$$

$$5 = \omega\sqrt{13^2 - 144} \Rightarrow 5 = \omega\sqrt{25}$$

$\therefore$ $\omega = 1\text{ rad s}^{-1}$

So, frequency, $\gamma = \frac{\omega}{2\pi} = \frac{1}{2\pi}\text{ Hz}$

The time period, $T = \frac{1}{\gamma} = 2\pi\text{ s}.$

2.4. If r is the magnitude of the position vector $\vec{r}$ then prove that $\nabla^2(\log_e r) = \frac{1}{r^2}$.

[WBUT 2007(ODD)]

Answer:

In the given problem $\nabla^2(\log_e r)$ can be represented as $\nabla^2 f(r)$, where $f(r)$ is a scalar. We will first find out the value of $\nabla^2 f(r)$

Now,

$\nabla^2 = \nabla.\nabla = \nabla.\frac{d}{dr}\hat{r}$ Hence $\nabla^2 f(r) = \vec{\nabla}.\left\{\frac{d}{dr}(\hat{r}f(r))\right\} = \vec{\nabla}.\frac{df(r)}{dr}\hat{r}$

$$\therefore \nabla^2 f(r) = \vec{\nabla}.\frac{df(r)}{dr}\hat{r} = \vec{\nabla}.\frac{df(r)}{dr}\frac{\vec{r}}{r} = \vec{\nabla}.(\frac{1}{r}\frac{df(r)}{dr})\vec{r}$$

comparing standard formula $\vec{\nabla}.(\phi\,\vec{r}) = \phi\,\vec{\nabla}.\vec{r} + \vec{\nabla}\phi.\vec{r}$ with the above equation we have scalar function $\phi = \frac{1}{r}\frac{df(r)}{dr}$ and $\vec{r} = \vec{r}$. So we can write

$$\nabla^2 f(r) = \frac{1}{r}\frac{df(r)}{dr}\nabla.\vec{r} + \nabla\frac{1}{r}\frac{df(r)}{dr}.\vec{r}$$

$$= \frac{3}{r}\frac{df(r)}{dr} + \frac{d}{dr}\hat{r}\left[\frac{1}{r}\frac{df(r)}{dr}\right].\vec{r} \qquad [\because \nabla.\vec{r} = 3]$$

$$\nabla^2 f(r) = \frac{3}{r}\frac{df(r)}{dr} - \frac{1}{r^2}\frac{df(r)}{dr}\hat{r}.\vec{r} + \frac{1}{r}\frac{d^2 f(r)}{dr^2}\hat{r}.\vec{r}$$

$$\nabla^2 f(r) = \frac{3}{r}\frac{df(r)}{dr} - \frac{1}{r^2}\frac{df(r)}{dr}\frac{\vec{r}}{r}.\vec{r} + \frac{1}{r}\frac{d^2 f(r)}{dr^2}\frac{\vec{r}}{r}.\vec{r}$$

$$\nabla^2 f(r) = \frac{3}{r}\frac{df(r)}{dr} - \frac{1}{r}\frac{df(r)}{dr} + \frac{1}{r}\frac{d^2 f(r)}{dr^2}$$

$$\nabla^2 f(r) = \frac{2}{r}\frac{df(r)}{dr} + \frac{1}{r}\frac{d^2 f(r)}{dr^2}$$

Now substituting $\log_e r$ in the above identity we have,

$$\nabla^2(\log_e r) = \frac{2}{r}\frac{d}{dr}(\log_e r) + \frac{d^2}{dr^2}(\log_e r)$$

$$\nabla^2(\log_e r) = \frac{2}{r}\times\frac{1}{r} + \left(-\frac{1}{r^2}\right)$$

$$\therefore \nabla^2(\log_e r) = \frac{1}{r^2}$$

2.5. Find if the work done in moving an object in the field

$$\vec{F} = (2xy + z^3)\,\hat{i} + x^2\hat{j} + 3xz^2\hat{k}$$

from point (1, – 2, 1) to (3, 1, 4) is independent of the path chosen. **[WBUT 2007(ODD), 2015(EVEN)]**

Answer:

$$\text{Work done} = \int_{1,-2,1}^{3,1,4} \vec{F}.d\vec{r} = \int_1^3 (2xy + z^3)dx + \int_{-2}^{1} x^2 dy + \int_1^4 3xz^2 dz$$

$$\text{Work done} = \int_{1,-2,1}^{3,1,4} \vec{F}.d\vec{r} = \int_{1,-2,1}^{3,1,4} d(x^2y + xz^3) = \left[x^2y + xz^3\right]_{1,-2,1}^{3,1,4} = 202$$

If we calculate the curl of the given field we have

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy + z^3 & x^2 & 3xz^2 \end{vmatrix} = \hat{i}(0-0) + \hat{j}(3z^2 - 3z^2) + \hat{k}(2x - 2x) = 0 \quad \text{.... (1)}$$

Since curl of the force field $\vec{F} = 0$ so the force field is conservative.

So we can write $\vec{F}=-\vec{\nabla}\phi$

So, Work done $=\int_{1,-2,1}^{3,1,4}\vec{F}.d\vec{r}=\int_{1,-2,1}^{3,1,4}-\vec{\nabla}\phi=\left[-\phi\right]_{1,-2,1}^{3,1,4}=\phi(\text{initial})-\phi(\text{final})$

i.e., the work done is independent of path and it is a state function.

2.6. Given $\phi=\phi(r)$ and $\vec{\nabla}=\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}$, find $\nabla^2\phi=\vec{\nabla}\cdot\left(\vec{\nabla}\phi\right)$ without using directly the explicit form of ∇^2 in the spherical polar coordinate. [WBUT 2008(ODD)]

Answer:

$\nabla^2=\vec{\nabla}.\vec{\nabla}=\vec{\nabla}.\frac{d}{dr}\hat{r}$ Hence $\nabla^2\phi(r)=\vec{\nabla}.\left(\vec{\nabla}\phi(r)\right)$

Now $\nabla^2\phi(r)=\vec{\nabla}\cdot\frac{d\phi(r)}{dr}\hat{r}=\vec{\nabla}\cdot\frac{d\phi(r)}{dr}\frac{\vec{r}}{r}=\vec{\nabla}\cdot(\frac{1}{r}\frac{d\phi(r)}{dr})\vec{r}$

using vector identity $\vec{\nabla}\cdot\left(\alpha\vec{A}\right)=\alpha\vec{\nabla}\cdot\vec{A}+\vec{A}\cdot\left(\vec{\nabla}\alpha\right)$ we have

$$\nabla^2\phi(r)=\frac{1}{r}\frac{d\phi(r)}{dr}\vec{\nabla}\cdot\vec{r}+\vec{\nabla}\left[\frac{1}{r}\frac{d\phi(r)}{dr}\right]\cdot\vec{r}$$

$$=\frac{3}{r}\frac{d\phi(r)}{dr}+\frac{d}{dr}\hat{r}\left[\frac{1}{r}\frac{d\phi(r)}{dr}\right]\cdot\vec{r}\qquad\left[\because\quad\vec{\nabla}\cdot\vec{r}=3\right]$$

$$\nabla^2\phi(r)=\frac{3}{r}\frac{d\phi(r)}{dr}-\frac{1}{r^2}\frac{d\phi(r)}{dr}\hat{r}.\vec{r}+\frac{1}{r}\frac{d^2\phi(r)}{dr^2}\hat{r}.\vec{r}$$

$$\nabla^2\phi(r)=\frac{3}{r}\frac{d\phi(r)}{dr}-\frac{1}{r^2}\frac{d\phi(r)}{dr}\frac{\vec{r}}{r}\cdot\vec{r}+\frac{1}{r}\frac{d^2\phi(r)}{dr^2}\frac{\vec{r}}{r}\cdot\vec{r}$$

$$\nabla^2\phi(r)=\frac{3}{r}\frac{d\phi(r)}{dr}-\frac{1}{r}\frac{d\phi(r)}{dr}+\frac{d^2\phi(r)}{dr^2}$$

$$\nabla^2\phi(r)=\frac{2}{r}\frac{d\phi(r)}{dr}+\frac{d^2\phi(r)}{dr^2}$$

Which is the required expression.

2.7. Given $\vec{F}=f(r)\vec{r}$, show that $\vec{\nabla}\times\vec{F}=0$ and hence show that $\oint_C\vec{F}\cdot d\vec{r}=0$, where C is a simple closed curve. [WBUT 2008(ODD)]

Answer:

Given that, $\vec{F}=f(r)\vec{r}$,

So

$$\vec{\nabla}\times\vec{F}=\vec{\nabla}\times\left[f(r)\vec{r}\right]$$

$$=\left[\frac{\partial}{\partial x}\hat{i}+\frac{\partial}{\partial y}\hat{j}+\frac{\partial}{\partial z}\hat{k}\right]\times f(r)\left(x\hat{i}+y\hat{j}+z\hat{k}\right)$$

$$=\left[\frac{\partial}{\partial x}\hat{i}+\frac{\partial}{\partial y}\hat{j}+\frac{\partial}{\partial z}\hat{k}\right]\times\left\{f(r)x\hat{i}+f(r)y\hat{j}+f(r)z\hat{k}\right\}$$

$$=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ f(r)x & f(r)y & f(r)zk\end{vmatrix}$$

$$=\left[z\frac{\partial f(r)}{\partial y}-y\frac{\partial f(r)}{\partial z}\right]\hat{i}-\left[z\frac{\partial f(r)}{\partial x}-x\frac{\partial f(r)}{\partial z}\right]\hat{j}+\left[y\frac{\partial f(r)}{\partial x}-x\frac{\partial f(r)}{\partial y}\right]\hat{k}$$

$$=\left[z\frac{df(r)}{\partial r}\frac{\partial r}{\partial y}-y\frac{\partial f(r)}{\partial r}\frac{\partial r}{\partial z}\right]\hat{i}-\left[z\frac{df(r)}{dr}\frac{\partial r}{\partial x}-x\frac{df(r)}{\partial r}\frac{\partial r}{\partial z}\right]\hat{j}+\left[y\frac{df(r)}{dx}\frac{\partial r}{\partial x}-x\frac{df(r)}{\partial r}\frac{\partial r}{\partial y}\right]\hat{k}$$

$$=\left[zf'(r)\frac{y}{r}-yf'(r)\frac{z}{r}\right]\hat{i}-\left[zf'(r)\frac{x}{r}-xf'(r)\frac{z}{r}\right]\hat{j}+\left[yf'(r)\frac{x}{r}-xf'(r)\frac{y}{r}\right]\hat{k}$$

$$=\frac{f'(r)}{r}\left[(yz-yz)\hat{i}-\hat{j}(xz-xz)+(xy-xy)\hat{k}\right]=0 \text{ Hence proved}$$

Now $\oint_c \vec{F}\cdot d\vec{r}=\int_s(\nabla\times\vec{F})\cdot d\vec{S}$ (using stokes law)

As we have found that $\nabla\times\vec{F}=0$ and as $d\vec{S}\neq 0$, So $\oint_c \vec{F}\cdot d\vec{r}=0$ Proved.

2.8. Prove that $\bar{A}\times\bar{B}\times\bar{C}=\bar{B}\times(\bar{A}\cdot\bar{C})-\bar{C}\times(\bar{A}\cdot\bar{B})$.

[WBUT 2009(ODD)]

Answer:

Let $\vec{A}=A_1\hat{i}+A_2\hat{j}+A_3\hat{k}$ and $\vec{B}=B_1\hat{i}+B_2\hat{j}+B_3\hat{k}$ and $\vec{C}=C_1\hat{i}+C_2\hat{j}+C_3\hat{k}$

Hence, $(\vec{B}\times\vec{C})=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ B_1 & B_2 & B_3\\ C_1 & C_2 & C_3\end{vmatrix}$

$$=\hat{i}\{B_2C_3-C_2B_3\}+\hat{j}\{B_3C_1-C_3B_1\}+\hat{k}\{B_1C_2-C_1B_2\}$$

$$\vec{A}\times(\vec{B}\times\vec{C})=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ A_1 & A_2 & A_3\\ \{B_2C_3-C_2B_3\} & \{B_3C_1-C_3B_1\} & \{B_1C_2-C_1B_2\}\end{vmatrix}$$

So, the $\hat{i}^{th}$ component $[A_2B_1C_2-A_2B_2C_1-A_3B_3C_1+A_3B_1C_3]$

Adding & subtracting $A_1B_1C_1$, collecting positive terms and negative terms and arranging, we get the $\hat{i}^{th}$ component

$= [A_2B_1C_2 + A_3B_1C_3 + A_1B_1C_1] - [A_2B_2C_1 + A_3B_3C_1 + A_1B_1C_1]$

$= [B_1(A_1C_1 + A_2C_2 + A_3C_3) - C_1(A_1B_1 + A_2B_2 + A_3B_3)][B_1(\vec{A}.\vec{C}) - C_1(\vec{A}.\vec{B})]$

similarly $\hat{j}^{th}$ component $[B_2(\vec{A}.\vec{C}) - \vec{C}_2(\vec{A}.\vec{B})]$

and $\hat{k}^{th}$ component $[B_3(\vec{A}.\vec{C}) - \vec{C}_3(\vec{A}.\vec{B})]$

So adding all above three terms we have $\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}.\vec{C}) - \vec{C}(\vec{A}.\vec{B})$

2.9. Prove that curl grad $\phi = 0$. **[WBUT 2009(ODD)]**

Answer:

$$\vec{\nabla}\times\vec{\nabla}\phi = \vec{\nabla}\times\left(\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right)$$

$$\vec{\nabla}\times\vec{\nabla}\phi = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial y}\right)\right]\hat{i} + \left[\frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial x}\right) - \frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial z}\right)\right]\hat{j} + \left[\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial x}\right)\right]\hat{k}$$

If the scalar point function ϕ possesses continuous second order partial derivatives then,

$\frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial z}\right) = \frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial y}\right)$ and so on. So, $\vec{\nabla}\times\vec{\nabla}\phi = 0$.

2.10. A vibrator of mass 10 gm is acted on by a restoring force of 5 dyne/cm and a damping force 2 dyne-sec/cm. Find whether the motion is overdamped or oscillatory. If at $t = 0$ the vibrator was at position $x = 0$, when a velocity 1 cm/sec is imparted to it then calculate the maximum deviation along positive x-axis. **[WBUT 2010(ODD)]**

Answer:

We know $2K = \frac{L}{m}$ [where K is the damping constant]

$\therefore$ Damping constant $K = \frac{L}{2m}$ $\left[\because m\frac{d^2x}{dt^2} = -L\frac{dx}{dt} - kx\right]$ and $\omega = \sqrt{\frac{k}{m}}$

$\therefore \omega = \sqrt{\frac{5}{10}} = \sqrt{0.5} = 0.707 s^{-1}$ and $K = \frac{2}{2\times10} = 0.1 s^{-1}$

$\because\ \omega > K$, hence the motion is oscillatory at $t=0;\ x=0 \rightarrow \frac{dx}{dt}=v_0$

Amplitude $A=\frac{v_0}{\omega_0}=\frac{v_0}{\sqrt{\omega^2-K^2}}=\frac{1}{\sqrt{0.5-0.01}}=1.145$ cm.

2.11. If a damped harmonic oscillator, with natural frequency ω_o and damping force $2mbV$ (V = velocity, m = mass, b = constant) is driven by a periodic force $Fe^{i\omega t}$ then the steady state displacement is given by $x=\frac{F}{m}\frac{e^{i\omega t}}{\sqrt{(\omega_0^2-\omega^2)+i2b\omega}}$. Use this fact to show that at the velocity resonance the phase of the periodic force and the velocity is the same.

[WBUT 2011(EVEN)]

Answer:

Let us assume $(\omega_0^2-\omega^2)=Z\cos\phi$ and $2b\omega=Z\sin\phi$ .... (1)

So we can write displacement $x=\frac{F}{m}\frac{e^{i\omega t}}{Ze^{i\phi}}=Ae^{i(\omega t-\phi)}$ where $A=\frac{F}{Zm}$

Now velocity $v=\frac{dx}{dt}=Ai\omega e^{i(\omega t-\phi)}=A\omega e^{i\left(\omega t-\phi+\frac{\pi}{2}\right)}=v_o e^{i\left(\omega t-\phi+\frac{\pi}{2}\right)}$ .... (2)

At velocity resonance amplitude of velocity, i.e., $v_o=\max$ i.e., $A\omega=\max$ for $\omega=\omega_0$.

From (1) we can write $\tan\phi=\frac{2b\omega}{(\omega_0^2-\omega^2)}$. At velocity resonance i.e., $\omega=\omega_0$ phase angle $\phi=\tan^{-1}\left(\frac{2b\omega}{0}\right)=\tan^{-1}(\infty)\rightarrow\phi=\frac{\pi}{2}$. Putting $\phi=\frac{\pi}{2}$ in (2) we have velocity $v=v_o e^{i\left(\omega t-\frac{\pi}{2}+\frac{\pi}{2}\right)}=v_o e^{i\omega t}$, which is in same phase of periodic force $Fe^{i\omega t}$

2.12. If an oscillator's motion is described by $x=e^{\beta t}\sin\Omega t$ where t is time, then calculate its logarithmic decrement.

[WBUT 2011(ODD)]

Answer:

Given that $x=e^{-\beta t}\sin\Omega t$

As the amplitudes decreases down exponentially so, amplitudes will be

$A_1=A_0e^{\frac{-\beta T^*}{4}}$

$A_2=A_0\beta\left(\frac{T^*}{4}+\frac{T^*}{2}\right)$ and $A_3=A_0\beta\left(\frac{3T^*}{4}+\frac{T^*}{2}\right)$ .... etc

Where T^* is the effective time period of a damped harmonic motion.

By the definition of logarithmic decrement

$\log d = \lambda = \frac{\beta T^*}{2} \cdot \frac{2\pi}{\Omega} = \frac{\pi\beta}{\Omega}$ [$\because$ in the given problem $\omega = \Omega$]

2.13. State Stokes theorem in vector calculus. Find the unit vectors perpendicular to $x^2 + y^2 - z^2 = 100$ at the point (1, 2, 3). **[WBUT 2012(EVEN)]**

Answer:

Stoke's Theorem: It states that the surface integral of the curl of a vector field taken over any surfaces is equal to the line integral for $\vec{A}$ round the boundary C of the surface i.e.

$\int_S (\vec{\nabla} \times \vec{A}) \cdot \vec{ds} = \oint_C \vec{A} \cdot \vec{dl}$

Perpendicular vector to the surface $x^2 + y^2 - z^2 = 100$ at $(1,2,3)$ can be calculated as

$\vec{\nabla}(x^2 + y^2 - z^2 - 100) = [2x\hat{i} + 2y\hat{j} - 2z\hat{k}]_{(1,2,3)} = 2\hat{i} + 4\hat{j} - 6\hat{k}$

So unit perpendicular vector on the surface will be $\hat{n} = \pm\frac{2\hat{i} + 4\hat{j} - 6\hat{k}}{\sqrt{4+16+36}} = \pm\frac{\hat{i} + 2\hat{j} - 3\hat{k}}{\sqrt{14}}$

2.14. A fluid motion is given by $\vec{V} = (y+z)\hat{i} + (z+x)\hat{j} + (x+y)\hat{k}$. Show that the motion is irrotational. **[WBUT 2012(ODD)]**

Answer:

For a vector $\vec{V}$ to be irrotational ***curl*** $\vec{V}$ should be zero (null vector)

$$\text{So } \vec{\nabla} \times \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (y+z) & (z+x) & x+y \end{vmatrix} = \hat{i}(1-1) + \hat{j}(1-1) + \hat{k}(1-1) = 0$$

Hence $\vec{V}$ is irrotational.

2.15. a) If the vectors A and B be irrotational, then show that the vector $A \times B$ is solenoidal. **[WBUT 2013(EVEN)]**

b) Prove that $i \times (j \times k) = j \times (k \times i) = k \times (i \times j) = 0$.

Answer:

a) $\hat{i} \times (\hat{j} \times \hat{k}) = \hat{i} \times \hat{i} = 0$; $\hat{j} \times (\hat{k} \times \hat{i}) = \hat{j} \times \hat{j} = 0$ and $\hat{k} \times (\hat{i} \times \hat{j}) = \hat{k} \times \hat{k} = 0$ again by interchanging $\hat{i}$, $\hat{j}$ and $\hat{k}$ we have $i \times (j \times k) = j \times (k \times i) = k \times (i \times j)$. Hence proved.

b) By the given condition we have $\vec{\nabla} \times \vec{A} = 0$ and $\vec{\nabla} \times \vec{B} = 0$.

Now $\vec{\nabla} \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\vec{\nabla} \times \vec{A}) - \vec{A} \cdot (\vec{\nabla} \times \vec{B}) = 0$ Hence proved.

2.16. a) Show that the fluid motion given by the vector $\vec{V}=(y+z)\hat{i}+(z+x)\hat{j}+(x+y)\hat{k}$ is solenoidal and irrotational.

b) Using Gauss' divergence theorem prove $\int_S \vec{r}.d\vec{S}=3V$, where V is the volume bounded by surface S.

Answer: [WBUT 2014(EVEN)]

a) $\vec{\nabla}\cdot\vec{V}=\left(\frac{\partial}{\partial x}\hat{i}+\frac{\partial}{\partial x}\hat{j}+\frac{\partial}{\partial z}\hat{k}\right)\cdot\left[(y+z)\hat{i}+(z+x)\hat{j}+(x+y)\hat{k}\right]$

or, $\vec{\nabla}\cdot\vec{V}=\left(\frac{\partial}{\partial x}(y+z)+\frac{\partial}{\partial x}(z+x)+\frac{\partial}{\partial z}(x+y)\right)=0$ Hence $\vec{V}$ is solenoidal.

Again, $\vec{\nabla}\times\vec{V}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ (y+z) & (z+x) & (x+y)\end{vmatrix}=(1-1)\hat{i}+(1-1)\hat{j}+(1-1)\hat{k}=0$

Hence $\vec{V}$ is irrotational.

b) Using Gauss' divergence theorem we can write $\iint_S \vec{r}\cdot\hat{n}dS=\int_V \vec{\nabla}\cdot\vec{r}dV$

Again $\vec{\nabla}\cdot\vec{r}=\left(\frac{\partial}{\partial x}\hat{i}+\frac{\partial}{\partial y}\hat{j}+\frac{\partial}{dz}\hat{k}\right)\cdot\left(x\hat{i}+y\hat{j}+x\hat{k}\right)=(1+1+1)=3$

Hence $\int_V \vec{\nabla}\cdot\vec{r}dV=\int_V 3dV=3V$

2.17. Show that for a system of forced vibration, the ratio of two energy as follows

$$\frac{\text{average potential energy}}{\text{average kinetic energy}}=\frac{\omega^2}{\omega'^2}$$

[WBUT 2014(EVEN)]

Answer:

Displacement due to a force vibration under a force $F=F_0\cos\omega t$ is $x=A\cos(\omega t-\phi)$

$$E=K\cdot E+P\cdot E=\frac{1}{2}mv^2+\frac{1}{2}Sx^2$$

$$E=\frac{1}{2}mA^2\omega^2\sin^2(\omega t-\phi)+\frac{1}{2}SA^2\cos^2(\omega t-\phi)$$

Clearly E is not constant and varies with time t.

Now average values of $K\cdot E$ and $P\cdot E$ are given by $\langle K\cdot E\rangle=\frac{1}{4}mA^2\omega^2$

$\langle P\cdot E\rangle=\frac{1}{2}SA^2=\frac{1}{4}mA^2\omega_0^2.$ $\left[\because \langle\sin^2(\omega t-\phi)\rangle=\langle\cos^2(\omega t-\phi)\rangle=\frac{1}{2}\right]$

$\therefore \quad \dfrac{\langle P.E\rangle}{\langle K.E\rangle} = \dfrac{\omega_0^2}{\omega^2}$

Using the formula $2d\sin\theta = n\lambda$ we have $2d\sin 8^\circ 35' = 0.842\times10^{-8}$

$d = \dfrac{1\times0.9\times10^{-8}}{2\sin 8^\circ 6'} = \dfrac{0.9\times10^{-8}}{2\times0.1409} = 3.19\times10^{-8}cm = 3.19\,\mathring{A}$

Using $\sin\theta = \dfrac{3\lambda}{2d} = \dfrac{3\times0.9}{6.38} \rightarrow \theta = 25.03^\circ$ the required angle.

2.18. Difference between velocity resonance and amplitude resonance. **[WBUT 2014(EVEN)]**

Answer:

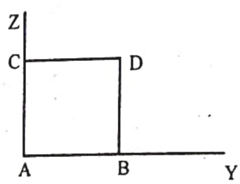
Velocity Resonance	Amplitude Resonance
(1) In velocity resonance, the velocity amplitude of the forced oscillator is the maximum at a particular frequency of the applied force.	(1) In amplitude resonance the amplitude of forced oscillator is the maximum for a particular frequency of the applied force.
(2) Velocity resonance occurs at frequency $\omega = q$ where ω is the natural frequency of the oscillator.	(2) Amplitude resonance occurs at resonant frequency $q = \sqrt{\omega^2 - 2K^2}$ where K is the damping constant.
(3) At applied frequency $q = 0$, the velocity amplitude is zero.	(3) At applied frequency $q = 0$, the amplitude of the forced oscillator is $\dfrac{f_0}{k}$, where f_0 is the force amplitude and k is the force constant.
(4) At velocity resonance the phase of the forced oscillator w.r.t that of the applied force is zero.	(4) At amplitude resonance the phase of the forced oscillator w.r.t that of the applied force $\dfrac{\pi}{2}$

2.19. State Stoke's theorem, and verify it for $\vec{A} = (x+3yz)\hat{j} + xy\hat{k}$ for the square surface ABCD on y-z plane. **[WBUT 2015(EVEN)]**

Answer:

It states that the surface integral of the curl of a vector field taken over any surfaces is equal to the line integral for $\vec{A}$ round the boundary C of the surface i.e.

$\int_S (\vec{\nabla}\times\vec{A})\cdot\vec{ds} = \oint_C \vec{A}\cdot\vec{dl}$



$\oint_C \vec{A}\cdot\vec{dl} = \oint_C \left[(x+3yz)\hat{j} + xy\hat{k}\right]\cdot\left[dx\hat{i} + dy\hat{j} + dz\hat{k}\right] = \oint (x+3yz)dy + xydz$

...ince the plane is on yz plane so $x=0$ so the line integral becomes $\oint_c \vec{A}\cdot\vec{dl}=\oint 3yzdy$.

Path integral along individual line is stated as follows

$\int_{AB} 3yzdy = 0$ (along this path $z=0$)

$\int_{DC} 3yzdy = \int_1^0 3ydy = -\dfrac{3}{2}$

$\int_{BD} 3yzdy =$ (along this path dy=0) $\int_{CA} 3yzdy = 0$ (along this path $dy=0$)

So $\oint_c \vec{A}\cdot\vec{dl} = \oint 3yzdy = -\dfrac{3}{2}$ .... (1)

The unit vector of the surface is along x axis i.e. $\hat{i}$. So the surface integral will be $\int_s (\vec{\nabla}\times\vec{A})\cdot\vec{ds}$

$$\vec{\nabla}\times\vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 0 & (x+3yz) & xy \end{vmatrix} = \hat{i}(x-3y)+x\hat{j}+\hat{k}$$

$$\int_s (\vec{\nabla}\times\vec{A})\cdot\vec{ds} = \iint_s \left[\hat{i}(x-3y)+x\hat{j}+\hat{k}\right].\hat{i}dzdy = \int_0^1\int_0^1 (x-3y)dzdy = -\frac{3}{2} \;\ldots.\; (2)$$

Comparing (1) and (2) it is found that stoke's law is verified.

2.20. Prove that $\nabla^2 f(r) = \dfrac{d^2 f(r)}{dr^2} + \dfrac{2}{r}\dfrac{df(r)}{dr}$.

Answer: **[WBUT 2015(EVEN)]**

$\nabla^2 = \nabla.\nabla = \nabla.\dfrac{d}{dr}\hat{r}$ Hence $\nabla^2 f(r) = \vec{\nabla}.\left\{\dfrac{d}{dr}(\hat{r}f(r))\right\} = \vec{\nabla}.\dfrac{df(r)}{dr}\hat{r}$

$\therefore \nabla^2 f(r) = \vec{\nabla}.\dfrac{df(r)}{dr}\hat{r} = \vec{\nabla}.\dfrac{df(r)}{dr}\dfrac{\vec{r}}{r} = \vec{\nabla}.(\dfrac{1}{r}\dfrac{df(r)}{dr})\vec{r}$

comparing standard formula $\vec{\nabla}.(\phi\,\vec{r}) = \phi\,\vec{\nabla}.\vec{r} + \vec{\nabla}\phi.\vec{r}$ with the above equation we have scalar function $\phi = \dfrac{1}{r}\dfrac{df(r)}{dr}$ and $\vec{r} = \vec{r}$. So we can write

$$\nabla^2 f(r) = \frac{1}{r}\frac{df(r)}{dr}\nabla.\vec{r} + \nabla\frac{1}{r}\frac{df(r)}{dr}.\vec{r} \qquad [\because \; \nabla.\vec{r}=3]$$

$$= \frac{3}{r}\frac{df(r)}{dr} + \frac{d}{dr}\hat{r}\left[\frac{1}{r}\frac{df(r)}{dr}\right].\vec{r}$$

$$\nabla^2 f(r) = \frac{3}{r}\frac{df(r)}{dr} - \frac{1}{r^2}\frac{df(r)}{dr}\hat{r}.\vec{r} + \frac{1}{r}\frac{d^2 f(r)}{dr^2}\hat{r}.\vec{r}$$

$$\nabla^2 f(r) = \frac{3}{r}\frac{df(r)}{dr} - \frac{1}{r^2}\frac{df(r)}{dr}\frac{\vec{r}}{r}.\vec{r} + \frac{1}{r}\frac{d^2 f(r)}{dr^2}\frac{\vec{r}}{r}.\vec{r}$$

$$\nabla^2 f(r) = \frac{3}{r}\frac{df(r)}{dr} - \frac{1}{r}\frac{df(r)}{dr} + \frac{1}{r}\frac{d^2 f(r)}{dr^2}$$

$$\nabla^2 f(r) = \frac{2}{r}\frac{df(r)}{dr} + \frac{1}{r}\frac{d^2 f(r)}{dr^2}$$

2.21. A mechanical oscillator has initial energy $E_0 = 50$ joule, having a damping coefficient $b = 1s^{-1}$. Calculate the relaxation time. **[WBUT 2015(EVEN)]**

Answer:

Relaxation time, $\tau = \frac{1}{2b} = \frac{1}{2} = 0.5s$

2.22. What is decay constant (or relaxation time)? In damped harmonic motion, calculate the time in which the energy of the system falls to e^{-1} times of its initial value. **[WBUT 2015(EVEN)]**

Answer:

In damped harmonic motion, how fast amplitude of the vibration of the body decay with time by the factor e^{-kt} depends on the damping coefficient 'K'. The time at which amplitude Ce^{-kt} decreases to e^{-1} i.e., 1/e of its original value C is called **relaxation time** or **decay constant**. Let us take that time t be τ. Putting τ in place of t we have $K\tau = 1$ or $\tau = 1/K$, So the time τ when new amplitude becomes 1/e i.e., 0.368 of its initial value. This τ is called the **relaxation time** or **decay constant**.

The displacement of damped harmonic motion is $x = C e^{-Kt}\sin(pt + \delta)$ where K is the damping constant and other symbols have their usual meaning.

We can have the average energy $\langle E \rangle = \frac{1}{2}mC^2\omega^2 e^{-2Kt} = E_0 e^{-2Kt}$. When $2Kt = 1$, i.e., $t = \frac{1}{2K}$ $\langle E \rangle = E_0 e^{-1}$. We define relaxation time t as τ such that $\tau = \frac{1}{2K}$. So relaxation time is defined as the time taken to fall the energy of the damped oscillatory motion to $\frac{1}{e}$ (nearly 37%) of its initial energy.

Long Answer Type Questions

3.1. a) Establish the differential equation of damped harmonic motion and solve the equation for light damping. **[WBUT 2018(ODD)]**

Answer:

1[st] Part: *Refer to Question No. 3.4(a).*

2[nd] Part: *Refer to Question No. 3.4(b) (1[st] Part & Case III).*

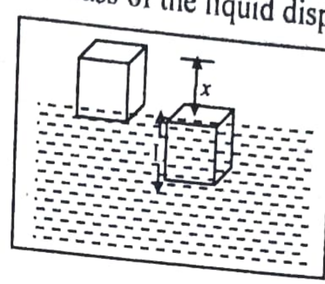
b) A cubical box of L cm side and density ρ is floating on water of density $\sigma(\rho<\sigma)$. The block is slightly depressed and released. Shows that it will execute S.H.M. also determine the time period of oscillation. **[WBUT 2018(ODD)]**

Answer:

As the cubical block of mass M is floating on the water of density ρ creates a static equilibrium. The weight of the block is balanced by the weight of liquid (water) it displaces.

Let we depress the block by a distance x (by dipping it further in the water), the buoyant force on the cubical block increases by $\rho L^2 gx$ as $L^2 \rho x$ is the mass of the liquid displaced by further dipping.

Neglecting viscous force we can write restoring force



$$F=-\rho L^2 gx \quad \Rightarrow M\frac{d^2x}{dt^2}=-\rho L^2 g\,x$$

$$\therefore \frac{d^2x}{dt^2}=-\frac{\rho L^2 g}{M}x$$

We observe that acceleration is directly proportional to displacement hence, the motion of the block is simple harmonic.

Comparing this equation with the general equation of S.H.M $f=-\omega^2 x$ we have

$$\omega^2=\frac{\rho L^2 g}{M} \quad \rightarrow \omega=\sqrt{\frac{\rho L^2 g}{M}}$$

Now mass of the block is $L^3 d$

$$\therefore\ \omega=\sqrt{\frac{\rho L^2 g}{L^3 d}}=\sqrt{\frac{\rho g}{Ld}} \qquad \Rightarrow \text{Time period, } T=2\pi\sqrt{\frac{Ld}{\rho g}}$$

c) The motion of a particle in S.H.M. is given by $y=a\sin\omega t$. If it has speed u when the displacement is y_1, and a speed v when the displacement is y_2, show that the amplitude of the motion is a $\left[\frac{v^2y_1^2-u^2y_2^2}{v^2-u^2}\right]^{1/2}$. **[WBUT 2018(ODD)]**

Answer:

$y = a\sin\omega t$ then, $v = \frac{dy}{dt} = \omega a\cos\omega t$

$\Rightarrow \sin\omega t = \frac{y}{a}$ and $\cos\omega t = \frac{v}{a\omega}$

$\therefore \frac{y^2}{a^2} + \frac{v^2}{a^2\omega^2} = 1$ According to the given problem we have

$\frac{y_1^2}{a^2} + \frac{u^2}{a^2\omega^2} = 1$ ....(i) and, $\frac{y_2^2}{a^2} + \frac{v^2}{a^2\omega^2} = 1$ ......(ii)

From (i) we have $\frac{y_1^2}{a^2} - 1 = \frac{u^2}{a^2\omega^2} \Rightarrow \frac{y_1^2 - a^2}{a^2} = \frac{u^2}{a^2\omega^2} \Rightarrow \omega^2 = \frac{u^2}{y_1^2 - a^2}$ ......(iii)

From (ii) we have $\frac{y_2^2}{a^2} - 1 = \frac{v^2}{a^2\omega^2} \Rightarrow \frac{y_2^2 - a^2}{a^2} = \frac{v^2}{a^2\omega^2} \Rightarrow \omega^2 = \frac{v^2}{y_2^2 - a^2}$ ......(iv)

Comparing (iii) and (iv) we have $\frac{u^2}{y_1^2 - a^2} = \frac{v^2}{y_2^2 - a^2}$ or, $\frac{y_2^2 - a^2}{y_1^2 - a^2} = \frac{v^2}{u^2}$

or, $u^2y_2^2 - u^2a^2 = v^2y_1^2 - v^2a^2 \Rightarrow a^2 = \frac{v^2y_1^2 - u^2y_2^2}{v^2 - u^2}$

So, amplitude $a = \left[\frac{v^2y_1^2 - u^2y_2^2}{v^2 - u^2}\right]^{1/2}$ *proved*

3.2. a) Write down the differential equation of a discharging series LCR circuit. Identify the natural frequency of the circuit. Under what condition will this circuit show an oscillatory decay? **[WBUT 2019(EVEN)]**

Answer:

Potential difference across the resistor is $V_R = IR$ and that of across the inductor L is $V_L = -L\frac{dI}{dt}$.

$\therefore$ In the series circuit we can write, $\frac{q}{C} + IR + L\frac{dI}{dt} = 0$

[$\therefore$ Circuit is open no energy is being supplied from the battery so, right hand side is zero.]

$$\because I = \frac{dq}{dt}, \quad \text{so,} \quad R\frac{dq}{dt} + L\frac{d^2q}{dt^2} + \frac{q}{C} = 0$$

or, $$\frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{q}{LC} = 0$$

or, $$\frac{d^2q}{dt^2} + 2K\frac{dq}{dt} + \omega^2 q = 0 \quad \text{where } 2K = \frac{R}{L} \text{ and } \frac{1}{LC} = \omega^2 \quad \dots (1)$$

We know the differential equation of a mechanical oscillator is given by

$$\frac{d^2x}{dt^2}+2K\frac{dx}{dt}+\omega^2x=0 \quad \dots (2)$$

Eqn. (1) and (2) are identical in nature, so, the solution of the equation will be,

$$q=A_1e^{\left(-K+\sqrt{K^2-\omega^2}\right)t}+A_2e^{\left(-K-\sqrt{K^2-\omega^2}\right)t}$$

So natural frequency of the series *L-C-R* circuit is $\omega=\frac{1}{\sqrt{LC}}$

For small damping i.e. $K<\omega$ i.e., $\frac{R}{2L}<\frac{1}{\sqrt{LC}}$ we know the solution will be, $q=Ce^{-kt}(\sin pt+\phi)$ which is ***oscillatory in nature***

where $p=\sqrt{\omega^2-k^2}$ and $\phi=\tan^{-1}\frac{\sqrt{\omega^2-k^2}}{k}=\frac{\sqrt{\frac{1}{LC}-\frac{R^2}{4L^2}}}{\frac{R}{2L}}$

Therefore, amplitude, $C=\frac{q\omega}{\sqrt{\omega^2-k^2}}=\frac{q\sqrt{\frac{1}{LC}}}{\sqrt{\frac{1}{LC}-\frac{R^2}{4L^2}}}$

$\therefore$ Solution becomes, $q=Ce^{-\frac{Rt}{2L}}\left[\sin\left\{\sqrt{\frac{1}{LC}-\frac{R^2}{4L^2}}\right\}t-\phi\right]$

Time period of oscillation, $T=\frac{2\pi}{p}=\frac{2\pi}{\sqrt{\omega^2-k^2}}=\frac{2\pi}{\sqrt{\frac{1}{LC}-\frac{R^2}{4L^2}}}$

or, frequency, $n=\frac{1}{T}=\frac{1}{2\pi}\sqrt{\frac{1}{LC}-\frac{R^2}{4L^2}}$

Special Case: when in the series circuit resistance is small i.e., R can be neglected in the frequency term. So, frequency, $n=\frac{1}{2\pi\sqrt{LC}}$

b) Calculate the damped frequency of oscillation and relaxation time of an LCR circuit with $L=3$ H, $C=0.05\,\mu$F and $R=100\,\Omega$.

Answer: **[WBUT 2019(EVEN)]**

We know the angular frequency of oscillation for an LCR circuit $\omega'=\left(\frac{1}{LC}-\frac{R^2}{4L^2}\right)^{\frac{1}{2}}$.

Substituting L = 3H; C = 5μF = 0.05 × 10^{-6}F and R = 100Ω

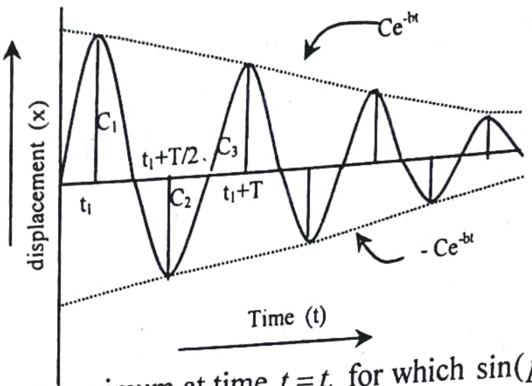
We have, $\omega' = \left[\dfrac{1}{3\times0.05\times10^{-6}} - \dfrac{(100)^2}{4\times9}\right]^{\frac{1}{2}}$ or, $\omega' = 0.816\times10^3$ rad/sec.

So, frequency $f' = \dfrac{\omega'}{2\pi} = \dfrac{0.816\times10^3}{2\times3.414} = 129.9Hz \approx 130Hz$

c) Derive an expression with illustration for logarithmic decrement. **[WBUT 2019(EVEN)]**

Answer:

The displacement of a damped harmonic oscillator $x = Ce^{-bt}\sin(pt+\delta)$ where C is the initial amplitude at $t=0$ as shown below.



Next displacement becomes maximum at time $t = t_1$ for which $\sin(pt+\delta)$ is ±1. Let at time $t = t_1$ amplitude $C_1 = Ce^{-bt_1}$. If $T = \dfrac{2\pi}{\omega}$ is the time period of oscillation, then half time period later the amplitude C_2 is given by $C_2 = Ce^{-b\left(t_1+\frac{T}{2}\right)} = C_1e^{-b\frac{T}{2}}$

Thus $\dfrac{C_1}{C_2} = e^{K\frac{T}{2}} = d(\text{say})$ i.e., $\log_e d = K\dfrac{T}{2} = \lambda(\text{say})$ ...(1)

The quantity d, which is the ratio of two successive amplitudes (on opposite side of the mean position) is called the decrement and $\log_e d = \lambda$ is called *logarithmic decrement* of the damped oscillation. If C_1 and C_{n+1} are the amplitudes for the first and $(n+1)$th half oscillation then

$$\frac{C_1}{C_{n+1}} = \frac{C_1}{C_2}\cdot\frac{C_2}{C_3}\cdots\cdots\frac{C_n}{C_{n+1}} = (d)^n$$

So from equation (1) we have $\lambda = \log_e d = \dfrac{1}{n}\log_e\dfrac{C_1}{C_{n+1}}$

Thus λ can be measured by noting C_1 and C_{n+1}, i.e., measurement of logarithmic decrement is possible in terms of successive amplitudes.

3.3. a) Define curl of a vector point function with example. Give its physical significance. **[WBUT 2019(ODD)]**

Answer:

One important property that characterizes a vector field is its rotation. Let us consider a turbulent fluid field in which $\vec{v}$ is the velocity at any point.

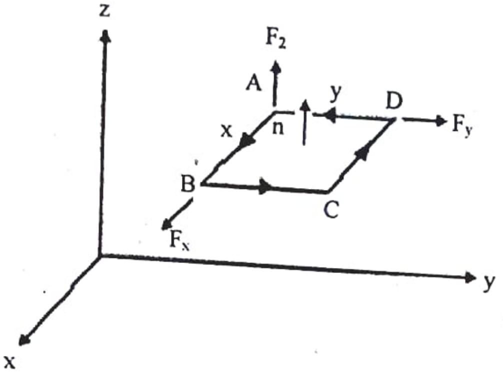
An appropriate characterization of the behaviour of the fluid is its rotation. The strength of the rotation can be measured by the circulation R, which is defined as $R = \oint_{C(S)} \vec{v} \cdot d\vec{l}$.

The larger the value of R, the stronger the rotation of the fluid in the area S which is bounded by the closed contour C.

The curl of a vector field $\vec{F}$ about a point is defined as the circulation per unit surface as the contour surface shrinks to zero. Mathematically it is represented as

$$curl\,\vec{F} \cdot \hat{n} = \lim_{\Delta S \to 0} \frac{\oint_C \vec{F} \cdot d\vec{l}}{\Delta S}$$

Above equation gives the component of the curl which is in the direction of unit normal to ΔS. In the example illustrated in figure below the rotation of the paddle wheel is in the plane of the page and the axis of the wheel as well as that of rotation is perpendicular to the plane containing the wheel. The paddle wheel can be oriented along three independent directions. Each of these three orientations indicate a measure of the circulation in three orthogonal planes



Determination of the z-component of the curl of a vector field $\vec{F}$

b) Find the unit vector perpendicular to both $2\hat{i}+\hat{j}+3\hat{k}$ and $\hat{i}+7\hat{j}-5\hat{k}$. What is the angle between them?

Answer: **[WBUT 2019(ODD)]**

$\vec{A} = 2\hat{i}+\hat{j}+3\hat{k}$ and $\vec{B} = \hat{i}+7\hat{j}-5\hat{k}$ (say)

$\vec{C} = \vec{A} \times \vec{B}$

where, $\vec{C}$ is perpendicular to both $\vec{A}$ and $\vec{B}$

$$\therefore \quad \vec{C}=\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & 3 \\ 1 & 7 & -5 \end{vmatrix}=\hat{i}\{-5-21\}+\hat{j}\{3+10\}+\hat{k}\{14-2\}=-26\hat{i}+13\hat{j}+12\hat{k}$$

So, unit vector of $\vec{C}$ is $\hat{n}=\pm\dfrac{-26\hat{i}+13\hat{j}+12\hat{k}}{\sqrt{(26)^2+(13)^2+(12)^2}}=\pm\dfrac{-26\hat{i}+13\hat{j}+12\hat{k}}{\sqrt{989}}$

Using $\vec{A}\cdot\vec{B}=|\vec{A}||\vec{B}|\cos\theta$, we have,

$$(2\hat{i}+\hat{j}+3\hat{k})\cdot(\hat{i}+7\hat{j}-5\hat{k})=|(2\hat{i}+\hat{j}+3\hat{k})||(\hat{i}+7\hat{j}-5\hat{k})|\cos\theta$$

$$\Rightarrow \quad 2+7-15=\sqrt{14}\sqrt{75}\cos\theta$$

$$\Rightarrow \quad \cos\theta=\frac{-6}{\sqrt{14\times 75}}=\frac{-6}{5\sqrt{42}} \Rightarrow \theta=\cos^{-1}\left[-\frac{6}{5\sqrt{42}}\right]$$

which the required angle between the vectors.

c) A rigid body is rotating with a constant angular velocity $\vec{\omega}$ about a fixed axis, if $\vec{v}$ is the velocity of a point of the body, prove that $\vec{\nabla}\times\vec{v}=2\vec{\omega}$ [WBUT 2019(ODD)]

Answer:

Let $\vec{V}$ be the linear velocity of a particle having position vector $\vec{r}$ on the rigid body and $\vec{w}$ its angular velocity. If x, y and z are the coordinates of the point having position vector $\vec{r}$, then

$$\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$$

If w_x, w_y and w_z are the x, y and z components of $\vec{w}$, then

$$\vec{w}=w_x\hat{i}+w_y\hat{j}+w_z\hat{k}$$

$$\text{Now,} \quad \vec{V}=\vec{w}\times\vec{r}=\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ w_x & w_y & w_z \\ x & y & z \end{vmatrix}$$

$$\vec{V}=\hat{i}(w_yz-w_zy)+\hat{j}(w_zx-w_xz)+\hat{k}(w_xy-w_yx)$$

$$\therefore Curl\,\vec{V}=\vec{\nabla}\times\vec{V}=\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ (w_yz-w_zy) & (w_zx-w_xz) & (w_xy-w_yx) \end{vmatrix}$$

$$=\hat{i}\left[\frac{\partial}{\partial y}\left(w_x y-w_y x\right)-\frac{\partial}{\partial z}\left(w_z x-w_x z\right)\right]$$

$$+\hat{j}\left[\frac{\partial}{\partial z}\left(w_y z-w_z y\right)-\frac{\partial}{\partial x}\left(w_x y-w_y x\right)\right]+\hat{k}\left[\frac{\partial}{\partial x}\left(w_z y-w_x z\right)-\frac{\partial}{\partial y}\left(w_y z-w_z y\right)\right]$$

As the angular velocity $\vec{w}$ for all the particles of a rigid body rotating about an axis passing through it is the same $\vec{w}$ is a constant. Hence all the derivatives $\frac{\partial}{\partial x},\frac{\partial}{\partial y}$ and $\frac{\partial}{\partial z}$ for the components of $w\left(w_x, w_y \text{ and } w_z\right)$ are zero. Also $\left(\frac{\partial y}{\partial x},\frac{\partial z}{\partial x}\right)$; $\left(\frac{\partial x}{\partial y},\frac{\partial z}{\partial y}\right)$; $\left(\frac{\partial x}{\partial z},\frac{\partial y}{\partial z}\right)$ are all zero.

Hence $Curl\,\vec{V}=\hat{i}\left(w_x+w_x\right)+\hat{j}\left(w_y+w_y\right)+\hat{k}\left(w_z+w_z\right)=2\left(w_x\hat{i}+w_y\hat{j}+w_z\hat{k}\right)=2\vec{w}$

Hence when a body is in rotation the curl of its linear velocity at any point is twice its angular velocity (both in direction as well as magnitude).

d) Calculate the angle between r and $\frac{dr}{dt}$ where r be a vector of constant magnitude.

[WBUT 2019(ODD)]

Answer:

$$\vec{r}\cdot\vec{r}=|\vec{r}|^2=\text{Constant}$$

So, $\frac{d}{dt}(\vec{r}\cdot\vec{r})=0 \Rightarrow \frac{d\vec{r}}{dt}\cdot\vec{r}+\vec{r}\cdot\frac{d\vec{r}}{dt}=0$

Hence, $2\vec{r}\cdot\frac{d\vec{r}}{dt}=0$

So, $\vec{r}$ is perpendicular to $\frac{d\vec{r}}{dt}$. Proved

3.4. a) Establish the differentiation equation of damped harmonic motion.

[WBUT 2009(ODD), 2013(ODD)]

b) Solve the equation for light damping and prove that the amplitude of vibration decreases exponentially with time.

[WBUT 2009(ODD)]

OR,

Obtain the necessary solution for the damped oscillatory motion for different cases.

[WBUT 2013(EVEN)]

Answer:

a) Equation of motion of a body of mass m, subject to friction is of the form

$$m\frac{d^2x}{dt^2}=-kx-L\frac{dx}{dt}$$

Where k and L are constants. L is often known as resistive force constant.

The above said expression can be written as $\frac{d^2x}{dt^2}+2K\frac{dx}{dt}+\omega^2 x=0$

Where we define L/m = 2K and k/m = ω^2

2K represents the ***frictional force per unit mass per unit velocity***. Hence $K=\frac{L}{2m}$ is called ***damping constant*** and $\omega=\sqrt{\frac{k}{m}}$ as ***restoring constant***.

b) Trying $x=Ae^{bt}$ as a trial solution we have, $\frac{dx}{dt}=Abe^{bt}$; $\frac{d^2x}{dt^2}=Ab^2e^{bt}$

So that the value satisfies the above equation, $b^2+2Kb+\omega^2=0$

Which is a quadratic equation of b having two roots.

$$\therefore b=-K\pm\sqrt{K^2-\omega^2}$$

i.e., the general solution is $x=A_1e^{\left(-K+\sqrt{K^2-\omega^2}\right)t}+A_2e^{\left(-K-\sqrt{K^2-\omega^2}\right)t}$

The actual solution depends upon the relative magnitudes of K and ω.

Case I: If $K>\omega$ i.e., $\sqrt{K^2-\omega^2}$ is real (but less than K). Hence x consists in this case of two terms, both dying off exponentially and there is no oscillation.

This type of motion is called as *over damped* or *dead beat*.

Case II: If K = ω, becomes $x=A_1e^{-Kt}+A_2e^{Kt}$. But According to the theory of differential equation solution will be $x=e^{-Kt}\{P+Qt\}$ where P and Q are two constant.

Now above equation represents one form of the solution. It is clear from that as t increases the factor $(P+Qt)$ increases but the e^{-Kt} decreases. Thus, the displacement 'x' first increases due to the factor $(P+Qt)$ but at the same time reversal occurs due to the term e^{-Kt} and displacement approaches zero at t increases. Further, in this case exponent is e^{-Kt} where as in the first case, it was more that e^{-Kt}. Hence in this case the particle tends to move to equilibrium much more rapidly than in case (I) shown in figure given below.

Such motion is called *critically damped* motion. In this case also there is no oscillation.

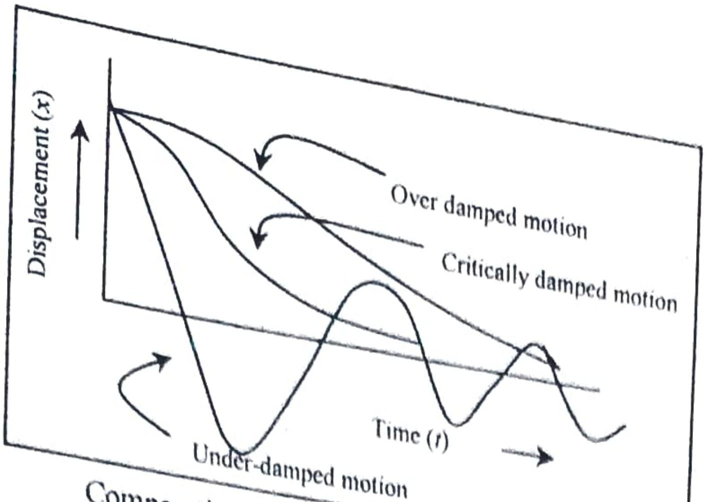
Case III: When K< ω i.e., $\sqrt{K^2-\omega^2}$ is imaginary, we may write it as $\sqrt{K^2-\omega^2}=ip$

or, $\omega^2-K^2=p^2$

i.e., $p=\sqrt{\omega^2-K^2}$ which is a real quantity so that the solution becomes,

$x=A_1\left\{e^{(-K+ip)t}\right\}+A_2\left\{e^{(-K-ip)t}\right\}=e^{-Kt}\{A_1e^{ipt}+A_2e^{-ipt}\}$

This may be put in the form $x=Ce^{-Kt}\sin(pt+\delta)$



Comparative displacement of three vibration

We have seen in damped vibration, the displacement of the body undergoing damped harmonic motion can be written as, $x = Ce^{-Kt}\sin(pt+\delta)$

The amplitude of the particle is Ce^{-Kt} i.e., as times goes on amplitude of the damped vibration falls off exponentially.

3.5. a) Starting from the equation of motion and after solving it show that, for a forced oscillator in the steady state, the displacement amplitude at low frequencies $(\omega \to 0)$, the velocity amplitude at velocity resonance $(\omega = \omega_0)$ are independent of the frequency of the driving force. **[WBUT 2010(ODD)]**

Answer:

Let a particle of mass m executing damped harmonic motion. At the same time ***external periodic*** force (driving) is acting on it. Equation of such forced vibration can be written as $m\dfrac{d^2x}{dt^2} = -kx - L\dfrac{dx}{dt} + F_0e^{i\omega t}$

where q is the frequency of the driving force .... (1)

Solution for Complementary part:

For the solution of the complementary part of the differential equation $m\dfrac{d^2x}{dt^2} = -kx - L\dfrac{dx}{dt} + F_0e^{i\omega t}$ we will choose the differential equation $m\dfrac{d^2x}{dt^2} + L\dfrac{dx}{dt} + kx = 0$. Taking $x = Ae^{bt}$ be the trial solution and solving the differential equation for oscillatory motion

Solution is $x = Ce^{-Kt}\sin(pt+\delta)$

Solution for particular integral:

Let $x = x_0e^{i\omega t}$ be the trial solution of the above equation

Now, $\dfrac{d^2x}{dt^2} = -\omega^2x_0e^{i\omega t}$; $\dfrac{dx}{dt} = i\omega x_0e^{i\omega t}$

On substituting these values in (1) we have,

$$\therefore \quad -m\omega^2 x_0 e^{i\omega t} = -k x_0 e^{i\omega t} - Li\omega x_0 e^{i\omega t} + F_0 e^{i\omega t}$$

$$or, \quad x_0[-m\omega^2 + i\omega L + k]e^{i\omega t} = F_0 e^{i\omega t}$$

$$\Rightarrow \quad x_0 = \frac{F_0}{[i\omega L + (k - m\omega^2)]} = \frac{F_0}{[(k - m\omega^2) + i\omega L]}$$

$$\Rightarrow \quad x_0 = \frac{F_0/m}{\left[\left(\frac{k}{m} - \omega^2\right) + i\omega\frac{L}{m}\right]}$$

$$\therefore \quad x_0 = \frac{F_0/m}{\left[\left(\omega_0^2 - \omega^2\right) + 2i\omega K\right]} \qquad \left[\text{on substitutung } \frac{L}{m} = 2K \ \& \ \frac{k}{m} = \omega_0^2\right]$$

Let $(\omega_0^2 - \omega^2) = Z\cos\phi$ & $2\omega K = Z\sin\phi$

$\therefore Z^2 = 4\omega^2 K^2 + (\omega_0^2 - \omega^2)^2$ [squaring and adding $Z\cos\phi$ and $Z\sin\phi$]

$$\Rightarrow \quad Z = \sqrt{4\omega^2 K^2 + (\omega_0^2 - \omega^2)^2}$$

$\Rightarrow \tan\phi = \dfrac{2\omega K}{(\omega_0^2 - \omega^2)}$ [dividing $Z\sin\phi$ and $Z\cos\phi$]

$$\therefore \quad x_0 = \frac{F_0/m}{Z\{\cos\phi + i\sin\phi\}} = \frac{F_0/m}{Ze^{i\phi}} = \frac{F_0/m}{Z}e^{-i\phi}$$

$\therefore$ Solution of the differential equation is $x = x_0 e^{iqt}$.

On substituting x_0 in the above solution we have, $\therefore x = \dfrac{F_0/m}{Z}e^{-i\phi}e^{i\omega t} = \dfrac{F_0/m}{Z}e^{i(\omega t - \phi)}$

So general solution of differential will be
x = Particular Integral (P.F) + Complementary Function (C.F)

$$\therefore \quad x = \frac{F_0/m^{e^{i(\omega t-\phi)}}}{\sqrt{4\omega^2K^2 + (\omega_0^2 - \omega^2)}} + Ce^{-Kt}\sin(pt+\delta) = \frac{f_0 e^{i(\omega t - \phi)}}{\sqrt{4\omega^2K^2 + (\omega_0^2 - \omega^2)}} + Ce^{-kt}\sin(pt+\delta)$$

defining $\dfrac{F_0}{m} \Rightarrow$ external force per unit mass i.e., f_0

So the solution of the differential equation is

$$x = Ae^{i(\omega t - \phi)} + Ce^{-Kt}\sin\left(\sqrt{(\omega_0^2 - K^2)}\,t + \delta\right) \text{where, amplitude } A = \frac{f_0}{\sqrt{4K^2\omega^2 + (\omega_0^2 - \omega^2)^2}}$$

...d vibration will be obtained from as $\varphi = \tan^{-1}\left(\frac{2\omega K}{\omega_0^2 - \omega^2}\right)$

Further we can write the general solution as

$$x = Ae^{i(\omega t-\phi)} + Ce^{-Kt}\sin\left(\sqrt{\left(\omega_0^2 - K^2\right)}\,t + \delta\right)$$

First term of the solution for x is a particular integral and the second part of the solution (complementary function) for x represents natural vibration set up in the damped system by the harmonic force at the start. These vibration become negligible very soon as the amplitude diminishes exponentially with time. If damping is very small, the natural vibration will persists for a longer time. After a lapse of time when the 2nd part becomes negligible, we can write the sustained forced vibration solution as $x = A\,e^{i(\omega t-\phi)}$, which is represented as steady state motion.

When the driving frequency is low i.e., $q < \omega_0$ then amplitude

$$A = \frac{f_0}{\left[\left(\omega_0^2 - \omega^2\right)^2 + 4K^2\omega^2\right]^{1/2}} = \frac{f_0}{\left\{\omega_0^4\left[1 - \frac{\omega^2}{\omega_0^2}\right]^2 + \omega_0^4\left[\frac{4K\omega^2}{\omega_0^4}\right]\right\}^{1/2}}$$

Because, $q/\omega_0 << 1$ $\quad \therefore A = \frac{f_0}{\omega_0^2} = Const$

This shows amplitude of the vibration is independent frequency external force.

The velocity of the oscillator in the steady state may be written as,

$$v = \frac{dx}{dt} = -A\omega\sin(\omega t - \phi) = A\omega\cos(\omega t - \phi + \frac{\pi}{2})$$

So velocity is represented as $v = v_0\cos(\omega t - \delta)$

$$\therefore \quad v = \frac{f_0 q}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + 4K^2\omega^2}}\cos(qt - \delta)$$

For velocity resonance i.e., velocity amplitude has to be maximum. So from expression we have $\frac{f_0}{\sqrt{\frac{\left(\omega_0^2 - \omega^2\right)^2}{\omega^2} + 4K^2}}$ has to maximum.

We know velocity resonance occur at $\omega_0 = \omega$. So at velocity resonance velocity amplitude $\boxed{v_{0\max} = \frac{f_0}{2mK}}$ which is independent of the frequency q. **Hence proved.**

b) Explain the terms logarithmic decrement and quality factor of a damped oscillatory system. How are they related?

[WBUT 2010(ODD)]

Answer:
The displacement of a damped harmonic oscillator $x = Ce^{-Kt}\sin(pt+\delta)$ becomes maximum at time $t = t_1$ for which $\sin(pt+\delta)$ is ± 1. Let at time $t = t_1$ amplitude $C_1 = Ce^{-Kt_1}$.

If $T = \frac{2\pi}{\omega}$ is the time period of oscillation, then half time period later the amplitude is given by $C_2 = Ce^{-K\left(t_1+\frac{T}{2}\right)} = C_1 e^{-K\frac{T}{2}}$

Thus $\frac{C_1}{C_2} = e^{K\frac{T}{2}} = d(say)$ i.e., $\log_e d = K\frac{T}{2} = \lambda(say)$

The quantity d which is the ratio of two successive amplitudes (on opposite side of the mean position) is called the decrement and $\log_e d = \lambda$ is called *logarithmic decrement* of the damped oscillation.

2π times the ratio of average energy stored in one period and average energy lost in one period.

Quality factor is represented as $Q = \frac{\omega}{2K}$. Now from logarithmic decrement formula $K\frac{T}{2} = \lambda$ where λ is logarithmic decrement Hence the required relation is $Q = \frac{\omega T}{4\lambda}$.

3.6. Find the displacement as a function of time of a particle of mass m which is subjected to overdamped harmonic motion with natural frequency ω_o and damping force γV (V being the instantaneous velocity), given that the displacement is zero initially and the initial velocity is V_0. Sketch the displacement as a function of time.
[WBUT 2011(EVEN)]

Answer:
The differential equation of a particle undergoing damped motion is $\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2 x = 0$ where the symbols have their usual meaning. In order to find the solution of this differential equation let us assume $x = Ae^{bt}$ be the trial solution. So calculating $\frac{d^2x}{dt^2}$ and $\frac{dx}{dt}$ and substituting in the differential equation we have

$b^2 + \gamma b + \omega_0^2 = 0$.

The roots of this auxiliary equation is $\therefore b = -\frac{\gamma}{2} \pm \sqrt{\frac{\gamma^2}{4} - \omega_0^2}$

So the general solution of the 2nd order above differential equation is

$$x = x_1 + x_2 = A_1 \exp\left(-\frac{\gamma}{2} + \sqrt{\frac{\gamma^2}{4} - \omega_0^2}\right)t + A_2 \exp\left(-\frac{\gamma}{2} - \sqrt{\frac{\gamma^2}{4} - \omega_0^2}\right)t$$

For over damped harmonic motion $\frac{\gamma}{2} > \omega_0$

Say $\sqrt{\frac{\gamma^2}{4} - \omega_0^2} = q$. So the above solution will be

$$x = x_1 + x_2 = A_1 \exp\left(-\frac{\gamma}{2} + q\right)t + A_2 \exp\left(-\frac{\gamma}{2} - q\right)t \quad \text{.... (1)}$$

And velocity

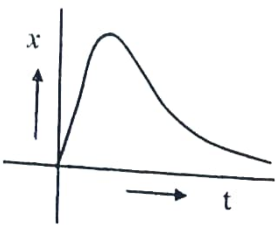
$$v = \frac{dx}{dt} = \left(-\frac{\gamma}{2} + q\right)A_1 \exp\left(-\frac{\gamma}{2} + q\right)t + \left(-\frac{\gamma}{2} - q\right)A_2 \exp\left(-\frac{\gamma}{2} - q\right)t \quad \text{.... (2)}$$

At $t=0$, $x=0$ So, putting this condition in (1) we have $A_1 + A_2 = 0 \rightarrow A_1 = -A_2$.

At $t=0$, $v=v_0$, on putting these conditions in (2) we have,

$$v_0 = \left(-\frac{\gamma}{2} + q\right)A_1 + \left(-\frac{\gamma}{2} - q\right)A_2$$

or, $v_0 = -\frac{\gamma}{2}(A_1 + A_2) + q(A_1 - A_2) \rightarrow v_0 = 2qA_1 \quad [\because A_1 = -A_2]$



Hence the required solution is $x = \frac{v_0}{2q} e^{-\frac{\gamma}{2}t}\left[e^{qt} - e^{-qt}\right]$

or, $x = \frac{v_0}{2q} e^{-\frac{\gamma}{2}t}\, 2\sinh(qt) \rightarrow x = \frac{v_0}{2} e^{-\frac{\gamma}{2}t} \sinh(qt)$

Nature of the graph displacement vs. time is shown in the figure.

3.7. a) If the same oscillator is driven by an external force $F = F_0 \cos\Omega t$, then find the phase difference between the displacement and the driving in the steady state.
b) A series *L-C-R* circuit is driven b a.c. source of frequency Ω. Write down the differential equation the charge across the capacitor should satisfy. Identify the parameters of forced damped oscillator in terms of *L-C-R*. Find the frequency at which the current in this circuit in maximum. You have to deduce the necessary formula from the differential equation. (You may use your results of part (b) of this question).

[WBUT 2011(ODD)]

Answer:

a) Differential equation of a forced vibration is $m\ddot{x} + \gamma\dot{x} + kx = F_0 \cos\Omega t$

So, $\ddot{x} + 2b\dot{x} + \omega_0^2 x = f_0 \cos\Omega t$ for undamped motion $b < 2\omega_0$ & displacement $x_2 = Ae^{-bt/2}\cos(\omega t - \delta)$ this is the complimentary function which dies out with time.

For particular integral solution of the differential equation we have
Let $x = A\cos(\Omega t - \phi)$ be the trial solution of the differential equation.
So, $\dot{x} = -A\Omega\sin(\Omega t - \phi)$ and $\ddot{x} = -A\Omega^2\cos(\Omega t - \phi)$
On substituting the values in (1) we have

$$-A\Omega^2\cos(\Omega t - \phi) - 2b\Omega\sin(\Omega t - \phi) + \omega_0^2 A\cos(\Omega t - \phi) = f_0\cos(\overline{\Omega t - \phi} + \phi)$$

$$\text{or, } (A\Omega^2 - \omega_0^2 A)\cos(\Omega t - \phi) - 2bA\Omega\sin(\Omega t - \phi) = f_0\cos(\Omega t - \phi)\cos\phi - f_0\sin(\Omega t - \ldots$$

Comparing the coefficients of $\cos(\Omega t - \phi)$ and $\sin(\Omega t - \phi)$ we have

$$(-A\Omega^2 + \omega_0^2 A) = f_0\cos\phi \quad \text{.... (1)} \quad \text{and} \quad 2bA\Omega = f_0\sin\phi \quad \text{.... (2)}$$

On squaring (1) and (2) we get the $(A\omega_0^2 - \Omega^2 A)^2 + 4b^2\Omega^2 A^2 = f_0^2$

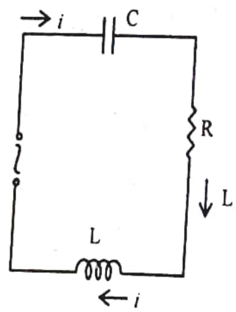
So, Amplitude of the solution $A = \dfrac{f_0}{\sqrt{(\omega_0^2 - \Omega^2)^2 + 4b^2\Omega^2}}$

Dividing (2) by (1) we have $\tan\phi = \dfrac{2b\Omega}{(\omega_0^2 - \Omega^2)}$

$$\boxed{\phi = \tan^{-1}\frac{\gamma\Omega}{m(\omega_0^2 - \Omega^2)}}$$

This is the phase, i.e., the angle by which the driving force leads the displacement a... depends on γ, ω_0 and Ω at steady state.

b) For a LCR circuit driven by oscillating emf $E_o\sin\Omega t$ as shown in the circuit. So differntial equation of the system will be, $L\dfrac{di}{dt} + \dfrac{q}{c} + iR = E_o\sin\Omega t$



If the current i is replaced by $\dfrac{dq}{dt}$, where q be the charge flows through the circuit w... have $L\dfrac{d^2q}{dt^2} + R\dfrac{dq}{dt} + \dfrac{q}{c} = E_o\sin\Omega t$

This equation is identical in form with the equation of motion of a mechanical oscillator under the action of external periodic force $F_0 \cos\Omega t$ as shown in answer (b) i.e., $m\ddot{x} + \gamma\dot{x} + kx = F_0 \cos\Omega t$

Thus we can say that mass m corresponds to self inductance L, mechanical resistance γ corresponds to electrical resistance R, the reciprocal of force constant or stiffness constant k corresponds to electrical capacitance c, displacement x corresponds to charge q, velocity v corresponds to current $i\left(=\frac{dq}{dt}\right)$ and force F_0 corresponds to E_0

From $L\frac{di}{dt} + \frac{q}{c} + iR = E_o \sin\Omega t$ we can write by differntiating both sides w.r.t t

$$L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{1}{c}\frac{dq}{dt} = E_o\Omega\cos\Omega t$$

At steady state the alternating current will be of the same ferquency Ω as the applied voltage. So the most general solution for current may be written as $i = i_o \sin(\Omega t - \phi)$.

So, $\frac{di}{dt} = \Omega i_0 \cos(\Omega t - \phi) \rightarrow \frac{d^2i}{dt^2} = -\Omega^2 i_0 \sin(\Omega t - \phi)$

So, on substituting these values in the above differential equation we have,

$$-L\Omega^2 i_0 \sin(\Omega t - \phi) + R\Omega i_0 \cos(\Omega t - \phi) + \frac{i_0 c}{c}\sin(\Omega t - \phi) = \Omega E_0 \cos\Omega t$$

$$i_0\left(\frac{1}{c} - L\Omega^2\right)(\sin\Omega t\cos\phi - \cos\Omega t\sin\phi) + R\Omega i_0(\cos\Omega t\cos\phi + \sin\Omega t\sin\phi) = \Omega E_0 \cos\Omega t$$

For all values of t this relation holds good. For $t = \frac{2\pi}{\omega}$

$$i_0\left(\frac{1}{c} - L\Omega^2\right)\cos\phi + R\Omega i_0 \sin\phi = 0$$

$$\tan\phi = \frac{\left(\Omega L - \frac{1}{\Omega c}\right)}{R}$$

So, $\sin\phi = \dfrac{\left(\Omega L - \frac{1}{\Omega c}\right)}{\sqrt{R^2 + \left(\Omega L - \frac{1}{\Omega c}\right)^2}}$ and $\cos\phi = \dfrac{R}{\sqrt{R^2 + \left(\Omega L - \frac{1}{\Omega c}\right)^2}}$

For $t = 0$ $\quad -i_0\left(\frac{1}{c} - L\Omega^2\right)\sin\phi + R\Omega i_0 \cos\phi = \Omega E_0$

On Substituting the values of $\sin\phi$ and $\cos\phi$ we have

$$i_0\left[R\Omega\frac{R}{\sqrt{R^2+\left(\Omega L-\frac{1}{\Omega c}\right)^2}}-\left(\frac{1}{c}-L\Omega^2\right)\frac{\left(\Omega L-\frac{1}{\Omega c}\right)}{\sqrt{R^2+\left(\Omega L-\frac{1}{\Omega c}\right)^2}}\right]=\Omega E_0$$

So, $i_0=\dfrac{E_0}{\sqrt{R^2+\left(\Omega L-\frac{1}{\Omega c}\right)^2}}$ which is the amplitude of the solution.

So, the equation for the instantaneous current becomes,

$$i=\frac{E_0}{\sqrt{R^2+\left(\Omega L-\frac{1}{\Omega c}\right)^2}}\sin(\Omega t-\phi)$$

In order to make current i maximum, denominator of the amplitude has to be minimum. Resistance R can not be zero but minimum value of the square term i.e, $\left(\Omega L-\frac{1}{\Omega c}\right)$ can be zero.

For current maximum $\left(\Omega L-\frac{1}{\Omega c}\right)=0$ i.e., $\Omega^2=\frac{1}{LC}\rightarrow\ \Omega=\frac{1}{\sqrt{LC}}$

which is the required frequency.

3.8. a) Given $\vec{v}=\vec{w}\times\vec{r}$, where $\vec{r}$ is the position vector and $\vec{w}$ is a constant angular velocity vector, then find out $\vec{\nabla}\times\vec{v}$. **[WBUT 2011(ODD)]**

Answer:

$$\nabla\times\vec{v}=\vec{\nabla}\times(\vec{\omega}\times\vec{r})=\vec{\nabla}\times\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\ \omega_1&\omega_2&\omega_3\\ x&y&z\end{vmatrix}$$

$$=\vec{\nabla}\times\left[(\omega_2z-\omega_3y)\hat{i}+(\omega_3x-\omega_1z)\hat{j}+(\omega_1y-\omega_2x)\hat{k}\right]$$

$$\begin{vmatrix}i&j&k\\ \frac{\partial}{\partial x}&\frac{\partial}{\partial y}&\frac{\partial}{\partial z}\\ (\omega_2z-\omega_3y)&(\omega_3x-\omega_1z)&(\omega_1y-\omega_2x)\end{vmatrix}$$

$$=\hat{i}(\omega_1+\omega_1)+\hat{j}(\omega_2+\omega_2)+\hat{k}(\omega_3+\omega_3)=2\vec{\omega}$$

b) Use the expression of gradient in the spherical coordinate system to find the normal to the surface $r\sin\theta = 1$.

Answer: **[WBUT 2011(ODD)]**

In polar coordinate system laplacian operator can be written as

$$\vec{\nabla} = \frac{\partial}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\hat{\phi}$$

So, $\vec{\nabla}S = \frac{\partial}{\partial r}\{r\sin\theta\}\hat{r} + \frac{1}{r}\frac{\partial}{\partial \theta}\{r\sin\theta\}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}(r\sin\theta)\hat{\phi}$

$\boxed{\vec{\nabla}S = \sin\theta\,\hat{r} + \cos\theta\,\hat{\theta}}$ which is also the unit normal vector.

c) If $\vec{A}$ is a constant vector, show that $\vec{\nabla}\left(\vec{r}\cdot\vec{A}\right) = \vec{A}$, where $\vec{r}$ is the position vector.

Answer: **[WBUT 2011(ODD)]**

$$\vec{\nabla}(\vec{r}.\vec{A}) = \vec{\nabla}(xA_1 + yA_2 + zA_3)$$

$$= \frac{\partial}{\partial x}(xA_1 + yA_2 + zA_3)\hat{i} + \frac{\partial}{\partial y}(xA_1 + yA_2 + zA_3)\hat{j} + \frac{\partial}{\partial z}(xA_1 + yA_2 + zA_3)\hat{k}$$

$= A_1\hat{i} + A_2\hat{j} + A_3\hat{k} = \vec{A}$ Hence Proved.

3.9. a) Show that $\nabla^2\left(\frac{1}{r}\right) = 0$ when $r \neq 0$. **[WBUT 2011(ODD)]**

Answer:

$$\nabla^2\left(\frac{1}{r}\right) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\frac{1}{\sqrt{x^2+y^2+z^2}}$$

For $\frac{\partial^2}{\partial x^2}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) = \frac{\partial}{\partial x}\left[-x\left(x^2+y^2+z^2\right)^{-\frac{3}{2}}\right] = -\left(x^2+y^2+z^2\right)^{-\frac{3}{2}} + 3x^2\left(x^2+y^2+z^2\right)^{-\frac{5}{2}}$

So, $\frac{\partial^2}{\partial y^2}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) = -\left(x^2+y^2+z^2\right)^{-\frac{3}{2}} + 3y^2\left(x^2+y^2+z^2\right)^{-\frac{5}{2}}$

and $\frac{\partial^2}{\partial z^2}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) = -\left(x^2+y^2+z^2\right)^{-\frac{3}{2}} + 3z^2\left(x^2+y^2+z^2\right)^{-\frac{5}{2}}$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\frac{1}{\sqrt{x^2+y^2+z^2}} = -3\left(x^2+y^2+z^2\right)^{-\frac{3}{2}} + 3\left(x^2+y^2+z^2\right)\left(x^2+y^2+z^2\right)^{-\frac{5}{2}} = 0$$

b) Find the angle between the normal to the *x–y* plane and the normal to surface $x^2 + y^2 = z^2$ **[WBUT 2011(ODD)]**

Answer:

$$\hat{n}_1 = \nabla\left(x^2 + y^2 - z^2\right) = \frac{2x\hat{i} + 2y\hat{j} - 2z\hat{k}}{\sqrt{4x^2 + 4y^2 - 4z^2}} = \frac{x\hat{i} + y\hat{j} - z\hat{k}}{\sqrt{x^2 + y^2 - z^2}}$$

$$\hat{n}_2 = \hat{k} \text{ so, } \hat{n}_1 \cdot \hat{n}_2 = \frac{-z}{\sqrt{x^2 + y^2 - z^2}} = |\hat{n}_1||\hat{n}_2|\cos\theta$$

$$\Rightarrow \quad \cos\theta = \frac{-z}{\sqrt{x^2 + y^2 - z^2}} \Rightarrow \theta = \cos^{-1}\left(\frac{-z}{\sqrt{x^2 + y^2 - z^2}}\right)$$

3.10. Find the frequency of osculation of an under damped harmonic oscillator of mass m, in terms of its natural frequency ω_0 and damping constant γ (where $-v\gamma$ is the damping force, v being the velocity). **[WBUT 2011(ODD)]**

Answer:

According to the question, the equation of a damped harmonic oscillator is

$$\ddot{x} + \frac{\gamma}{m}\dot{x} + \frac{k}{m}x = 0$$

$$\Rightarrow \quad \ddot{x} + b\dot{x} + \omega_0^2 x = 0$$

Let $x = Ae^{\alpha t}$ be the trial solution of the above differential equation

$$\therefore \quad \dot{x} = A\alpha e^{\alpha t}$$

$$\Rightarrow \quad \ddot{x} = A\alpha^2 e^{\alpha t}$$

$$\left(\alpha^2 + b\alpha + \omega_0^2\right) Ae^{\alpha t} = 0$$

$$\Rightarrow \quad Ae^{\alpha t} \neq 0; \ \alpha^2 + b\alpha + \omega_0 = 0$$

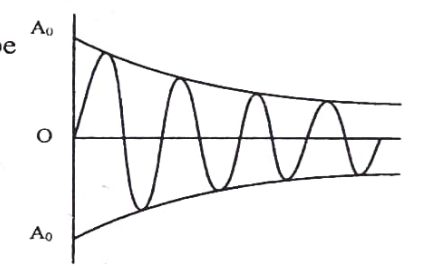
$$\alpha = \frac{-b \pm \sqrt{b^2 - 4\omega_0^2}}{2}$$

For oscillatory condition we have $b < 2\omega_0$

For oscillatory motion $\sqrt{\left(b^2 - 4\omega_0^2\right)}$ is imaginary.

So, let, $\sqrt{\left(\frac{b^2}{4} - \omega_0^2\right)} = i\omega^*$, Where, $\omega^* = \left(\omega_0^2 - \frac{b^2}{4}\right)^{\frac{1}{2}}$ is real positive,

So the solution of the above differential equation will be



$$x = e^{-bt/2}\left[A_1 \exp\left(i\omega^* t\right) + A_2 \exp\left(-i\omega^* t\right)\right]$$

$$x = e^{-bt/2}\left[A_1 \cos\omega^* t + iA_1 \sin\omega^* t + A_2 \cos\omega^* t - iA_2 \sin\omega^* t\right]$$

$$x = e^{-bt/2}\left[\left(A_1 + A_2\right)\cos\omega^* t + i\left(A_1 - A_2\right)\sin\omega^* t\right]$$

Let, $A_1 + A_2 = A\cos\delta$ and $i\left(A_1 + A_2\right) = A\sin\delta$

$x = e^{-bt/2}[A\cos\delta\cos\omega^{*}t + A\sin\delta\sin\omega^{*}t]$

$x = Ae^{-bt/2}\cos\left(\omega^{*}t - \delta\right)$

which is an oscillatory motion with frequency of oscillation.

$$\omega^{*} = \left(\omega_0^2 - \frac{b^2}{4}\right)^{\frac{1}{2}} \Rightarrow \boxed{\omega^{*} = \sqrt{\left(\omega_0^2 - \frac{\gamma^2}{4m}\right)}}$$

frequency is proportional to natural frequency ω_0 of the system.

3.11. a) If $\hat{a}$ and $\hat{b}$ are unit vectors and θ is the angle between them, show that $2\sin\frac{\theta}{2} = |\hat{a} - \hat{b}|$.

[WBUT 2012(EVEN)]

Answer:

$$\left(\hat{a} - \hat{b}\right).\left(\hat{a} - \hat{b}\right) = |\hat{a}|^2 + |\hat{b}|^2 - 2\hat{a}.\hat{b}$$

or, $|\hat{a} - \hat{b}|^2 = 1 + 1 - 2\cos\theta = 2(1 - \cos\theta) = 4\sin^2\theta/2$

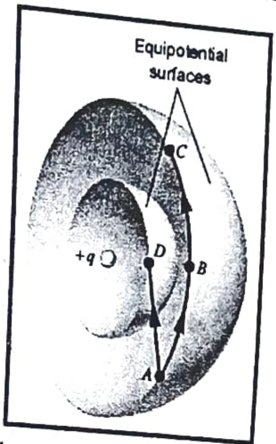
or, $|\hat{a} - \hat{b}| = 2\sin\theta/2$ Hence proved.

b) Show that the electric field is always perpendicular to the equipotential surface.

[WBUT 2012(EVEN)]

Answer:

An *equipotential surface* is a locus of points that are all at the same electric potential.



Thus an equipotential surface is a surface on which the electric potential is the same everywhere. The equipotential surfaces surrounding an isolated point charge are concentric spheres as shown in the figure.

If a small positive test charge q_0 moves along the outer equipotential surface along the path from A to B the work W_{AB} done by the electric field is given by $V_B - V_A = -W_{AB}/q_0$

But $V_B = V_A$ so that $W_{AB} = 0$ and the electric field does not work on the test charge. This is possible only if the electric force acts in a direction that is *perpendicular* to the displacement of the test charge. Hence Proved.

c) Show the $\vec{A} = \left(6xy + z^3\right)\hat{i} + \left(3x^2 - z\right)\hat{j} + \left(3xz^2 - y\right)\hat{k}$ is irrotational. Find ϕ such that $\vec{A} = \vec{\nabla}\phi$.

[WBUT 2012(EVEN)]

Answer:

Condition for the vector $A=(6xy+z^3)\hat{i}+(3x^2-z)\hat{j}+(3xz^2-y)\hat{k}$ is to be conser

if the curl $\vec{A}$ i.e., $\vec{\nabla}\times A=0$

$$\vec{\nabla}\times\vec{F}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ (6xy+z^3) & (3x^2-z) & (3xz^2-y)\end{vmatrix}$$

$$=\hat{i}\{-1+1\}+\hat{j}\{3z^2-3z^2\}+\hat{k}\{6x-6x\}=0$$

So the $\vec{A}$ will be conservative.

Let, $\vec{A}=\vec{\nabla}\phi$ [where ϕ is a scalar potential]

So, $\vec{A}=\frac{\partial\phi}{\partial x}\hat{i}+\frac{\partial\phi}{\partial y}\hat{j}+\frac{\partial\phi}{\partial z}\hat{k}$

So as per the condition $(6xy+z^3)\hat{i}+(3x^2-z)\hat{j}+(3xz^2-y)\hat{k}=\frac{\partial\phi}{\partial x}\hat{i}+\frac{\partial\phi}{\partial y}\hat{j}+\frac{\partial\phi}{\partial z}\hat{k}$

Now comparing the coefficients of $\hat{i},\hat{j},\hat{k}$ we have

$\frac{\partial\phi}{\partial x}=(6xy+z^3)\rightarrow\phi=3x^2y+z^3x+C_1(y,z)$ ... (1)

Similarly $\frac{\partial\phi}{\partial y}=(3x^2-z)\rightarrow\phi=3x^2y-zy+C_2(x,z)$ ... (2)

And $\frac{\partial\phi}{\partial z}=(3xz^2-y)\rightarrow\phi=z^3x-yz+C_3(x,y)$ ... (3)

From (1) and (2) and (3) we have the ***scalar potential*** as

$\phi=3x^2y-yz+z^3x+C(x,y,z)$ where C (x, y,z) is arbitrary constant.

d) Calculate the work done in moving a particle in a force field given by $\vec{F}=3xy\hat{i}-4z\hat{j}+8y\hat{k}$ along the curve $x=t^2+1, y=t^2, z=t^3$ from $t=0$ to $t=1$.

[WBUT 2012(EVEN)]

Answer:

Work done is $W=\int\vec{F}\cdot d\vec{r}=\int 3xydx-\int 4zdy+\int 8ydz$

As $x=t^2+1$ $\quad\therefore dx=2tdt$

$y=t^2$ $\quad\therefore dy=2tdt$

$z=t^3$ $\quad\therefore dz=3t^2dt$

$$W = \int \vec{F} \cdot d\vec{r} = \int_0^1 \left[3\left(t^2+1\right)t^2\right]2tdt - \int_0^1 \left[4t^3\right]2tdt + \int_0^1 \left[8t^2\right]3t^2dt$$

$$\text{or, } W = \int \vec{F} \cdot d\vec{r} = \int_0^1 6t^5dt + \int_0^1 6t^3dt - \int_0^1 8t^4dt + \int_0^1 24t^4dt$$

$$\text{or, } W = \int \vec{F} \cdot d\vec{r} = 1 + \frac{6}{4} - \frac{8}{5} + \frac{24}{5} = \frac{57}{10}$$

3.12. a) The forced harmonic oscillators have displacement amplitude at frequencies $\omega_1 = 400\,\text{sec}^{-1}$ and $\omega_2 = 600\,\text{sec}^{-1}$. Find the resonant frequency at which the displacement amplitude is maximum.
b) Differentiate between amplitude resonance and velocity resonance for a forced harmonic oscillator.
[WBUT 2012(EVEN)]

Answer:

a) **Information of this question is incomplete. However the problem is solved assuming** the frequencies $\omega_1 = 400$ sec^{-1} and $\omega_2 = 600$ sec^{-1} are equal.

Amplitude of the oscillator is given by

$A = \dfrac{f_0}{\left[\left(\omega^2 - q^2\right)^2 + 4K^2q^2\right]^{1/2}}$ where ω is the natural frequency and q frequency of the applied force.

At $\omega_1 = 400$ sec^{-1} and $\omega_2 = 600\,\text{sec}^{-1}$ A *are* equal. Hence

$$\left[\left(\omega^2-\omega_1^2\right)^2 + 4K^2\omega_1^2\right]^{1/2} = \left[\left(\omega^2-\omega_2^2\right)^2 + 4K^2\omega_2^2\right]^{1/2}$$

$$\text{or, } \left(\omega^2-\omega_1^2\right) + 4K^2\omega_1^2 = \left(\omega^2-\omega_2^2\right)^2 + 4K^2\omega_2^2$$

$$\text{or, } \left(\omega^2-\omega_1^2\right) - \left(\omega^2-\omega_2^2\right)^2 = 4K^2\omega_2^2 - 4K^2\omega_1^2$$

$$\text{or, } \left(2\omega^2-\omega_1^2-\omega_2^2\right) = 4K^2\left[\omega_2^2-\omega_1^2\right]$$

$$\text{or, } 4K^2 = \left(2\omega^2-\omega_1^2-\omega_2^2\right) \qquad \text{.... (1)}$$

Again we know, resonance frequency $q_{reso} = \sqrt{\omega^2 - 2K^2}$

$$\text{or, } q_{reso} = \sqrt{\omega^2 - \frac{\left(2\omega^2-\omega_1^2-\omega_2^2\right)}{2}} = \left(\frac{\omega_1^2+\omega_2^2}{2}\right)^{1/2}$$

$$\text{or, } q_{reso} = \left(\frac{(400)^2+(600)^2}{2}\right)^{1/2} = 510\,\text{sec}^{-1}$$

b) Amplitude resonance:

i) In amplitude resonance the amplitude of forced oscillator is the maximum for a particular frequency of the applied force.

ii) Amplitude resonance occurs at resonant frequency $q_{reso} = \omega\left\{1-\frac{2K^2}{\omega^2}\right\}^{1/2}$ where K be the damping constant

iii) At applied frequency $q = 0$, the amplitude of the forced oscillator is $\frac{F}{k}$, where F is the force amplitude and k the force constant.

iv) At amplitude resonance the phase of the forced oscillator w.r.t that of the applied force is $\frac{\pi}{2}$

Velocity resonance:

1. Amplitude of the velocity of the oscillating body at resonant condition is maximum.
2. Velocity resonance occurs at frequency $q = \omega$ where the natural frequency be ω.
3. At velocity resonance the phase of the forced oscillator w.r.t. that of the applied force is 0 (zero).

3.13. a) Show that the fractional change in the natural frequency of a damped simple harmonic oscillator is $\frac{1}{8Q^2}$, where Q is the quality factor of the oscillator. **[WBUT 2012(EVEN)]**

OR,

Find an expression for the fractional change in natural frequency of a damped harmonic oscillator in terms of the quality factor Q. **[WBUT 2014(EVEN)]**

b) A particle of mass 2 kg is subjected to an elastic force per unit displacement $0.03\,Nm^{-1}$ and frictional force per unit velocity $0.005\,\mathrm{Nm}^{-1}\mathrm{s}$. If it is displaced through 2 cm and then released, find whether the resulting motion is oscillatory or not. **[WBUT 2012(EVEN)]**

Answer:

a) Let ω' and ω be the angular frequencies of the damped and undamped oscillator respectively.

$\therefore \omega' = \sqrt{\omega^2 - k^2}$ where k is the damping constant.

$$\therefore \omega' = \sqrt{\omega^2 - \left(\frac{L}{2m}\right)^2} \quad \text{or, } \therefore \omega' = \omega\left[1-\left(\frac{L^2}{4m^2\omega^2}\right)\right]^{\frac{1}{2}} \text{ or, } \therefore \omega' = \omega\left[1-\left(\frac{L^2}{8m^2\omega^2}\right)\right]$$

Considering $\frac{L^2}{4m^2\omega^2}$ is very small for small value of L

We can write, $\frac{\omega'}{\omega} = 1 - \frac{L^2}{8m^2\omega^2}$

$$1 - \frac{\omega'}{\omega} = \frac{L^2}{8m^2\omega^2}$$

or, $$\frac{\omega - \omega'}{\omega} = \frac{L^2}{8m^2\omega^2} = \frac{1}{8Q^2} \left[\because Q = \frac{m\omega}{L}\right]$$

So, fractional change in angular frequency is equal to $\frac{1}{8Q^2}$

b) In the given Example restoring force per unit displacement is $k = 0.03$ N/m
Damping force or resisting force $L = 0.005\text{Nm}^{-1}\text{s}$ (force per unit velocity)

We know $\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{0.03}{2}} = 0.122\,s^{-1}$

Again $K = \sqrt{\frac{L}{2m}} = \sqrt{\frac{0.005}{4}} = 0.035\,\text{sec}$ ...(1)

Since the value of ω is higher than damping K, so motion is oscillatory. ...(2)

3.14. a) The equation for displacement of a point of a damped oscillator is given by $X = 5e^{-0.25t}\sin(\pi/2)t$ metre. Find the velocity of the oscillating point at $t = T/4$ and T, where T is the time period of oscillation.

[WBUT 2013(EVEN)]

Answer:

From the given displacement expression $X = 5\,e^{-0.25t}\sin\frac{\pi}{2}$ we have angular frequency $\omega = \frac{\pi}{2}$ and time period $T = \frac{2\pi}{\omega} = \frac{2\pi}{\frac{\pi}{2}} = 4s$

So velocity, $v = \frac{dX}{dt} = 5(-0.25)\,e^{-0.25t}\sin\frac{\pi}{2}t + 5\times\frac{\pi}{2}e^{-0.25t}\cos\frac{\pi}{2}t$ m/s

at $t = \frac{T}{4}$ i.e. $t = 1$s velocity will be $5(-0.25)\,e^{-0.25\times 1} = -0.974m/s$

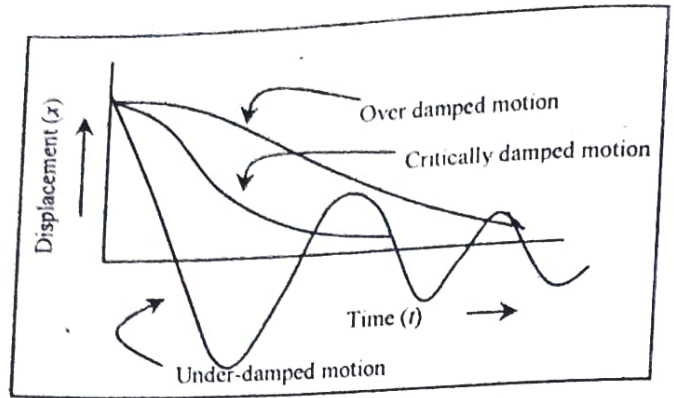
at time $t = T$ i.e. $t = 4$s velocity will be $5\times\frac{\pi}{2}\,e^{-0.25\times 4} = 2.89m/s$

Comparing the given equation with the damped vibration i.e., $x = C\,e^{-Kt}\sin(pt + \delta)$ we have amplitude $C = 5$m

b) Give a graphical comparison among underdamped, overdamped and critically damped harmonic motion.

[WBUT 2013(EVEN)]

Answer:



Comparative displacement of three vibration

3.15. What is resonance? Establish the condition of amplitude resonance and explain the sharpness of amplitude resonance. **[WBUT 2013(ODD)]**

Answer:

The phenomenon in which a body vibrates with its natural frequency under the influence of external periodic force is called resonance vibration.

Resonance can be classified into two categories. (a) Amplitude resonance (b) Velocity resonance.

Amplitude resonance: From the steady state solution $x = A\,e^{i(qt-\phi)}$ we note that the amplitude A of forced vibration depends on the frequency q of the driving force. For certain value of q the amplitude becomes maximum and we call it as amplitude vibration takes place between the driver and driven systems. For the amplitude A to be maximum

we must have $\dfrac{dA}{d\omega} = 0$ and $\dfrac{d^2A}{d\omega^2} < 0$

We have, $A = \dfrac{f_0}{\left[\left(\omega^2 - q^2\right)^2 + 4K^2q^2\right]^{1/2}}$.

$$\text{Hence, } \frac{dA}{d\omega} = f_0\left[\frac{-\frac{1}{2}\left\{2\left(\omega^2-q^2\right)\times(-2q)+4K^2.2q\right\}}{\left\{\left(\omega^2-q^2\right)^2+4K^2q^2\right\}^{3/2}}\right]$$

$$\frac{dA}{d\omega} = f_0q\left[\frac{\left\{2\left(\omega^2-q^2\right)-4K^2\right\}}{\left\{\left(\omega^2-q^2\right)^2+4K^2q^2\right\}^{3/2}}\right]$$

The above expression behaves $\dfrac{dA}{d\omega} = 0$. When the numerator of the above expression i.e.,

$2\left(\omega^2 - q^2\right) - 4K^2 = 0$

or, $\left(\omega^2-q^2\right)-2K^2=0$

So for maximum amplitude we have $\left(\omega^2-q^2\right)=2K^2$

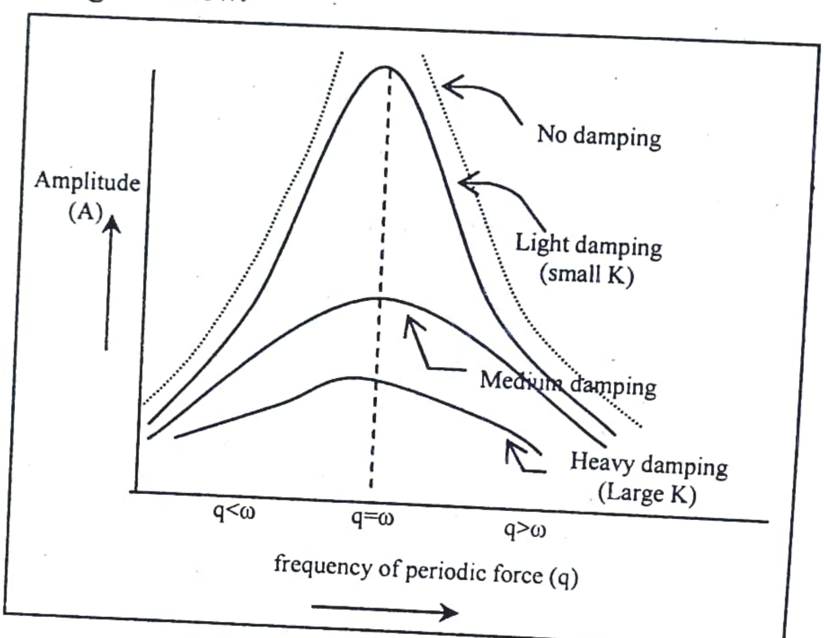
or, $q^2=\omega^2-2K^2$ i.e., $q=\sqrt{\omega^2-2K^2}$ this q is called resonance frequency or q_{reso} and for small K we have $q_{reso}=\omega\left\{1-\frac{2K^2}{\omega^2}\right\}^{1/2}=\omega-\frac{K^2}{\omega}$. It can be verified that for this q_{reso}, $\frac{d^2A}{d\omega^2}<0$ relation holds good.

This is the condition for amplitude resonance. The value of amplitude under this condition is

$$A=\frac{f_0}{\left[4K^2q^2+4K^2\right]^{1/2}}=\frac{f_0}{2K\sqrt{q^2+K^2}}=\frac{f_0}{2K\sqrt{\omega^2-K^2}}$$

However for ***weak damping*** (i.e., K very small) the amplitude becomes $A_{max}=\frac{f_0}{2K\omega}$.

The amplitude resonance always occurs at or near $q=\omega$ provided the damping is not too large. Only for heavy damping the amplitude resonance occurs at frequencies less than ω as shown in the figure below.



Graphical representation of amplitude with q

Velocity resonance: As we know that the sustained forced vibration is represented by $x=Ae^{i(qt-\phi)}$. Taking the real part of it we have $x=A\cos(qt-\phi)$

The velocity of the oscillator in the steady state may be written as,

$$v=\frac{dx}{dt}=-Aq\sin(qt-\phi)=Aq\cos(qt-\phi+\frac{\pi}{2})$$

So velocity is represented as $v=v_0\cos(qt-\delta)$

where amplitude $v_0=Aq=\frac{f_0q}{\sqrt{(\omega^2-q^2)^2+4K^2q^2}}$ and phase $\delta=\phi-\frac{\pi}{2}$

$$\therefore \quad v=\frac{f_0q}{\sqrt{(\omega^2-q^2)^2+4K^2q^2}}\cos(qt-\delta)$$

For velocity resonance i.e., velocity amplitude has to be maximum.

So from the above expression $\frac{f_0}{\sqrt{\frac{(\omega^2-q^2)^2}{q^2}+4K^2}}$ has to maximum i.e., denominator to be minimum for particular q i.e., $\frac{d}{d\omega}\left[\frac{(\omega^2-q^2)^2}{q^2}+4K^2\right]=0$

or, $\frac{d}{d\omega}\left[\frac{(\omega^2-q^2)^2}{q^2}\right]=0$ or, $2(\omega^2-q^2)\frac{(-2q)}{q^2}+(\omega^2-q^2)^2\left(\frac{-2}{q^3}\right)=0$

or, $(\omega^2-q^2)\left[-\frac{4}{q}-\frac{2(\omega^2-q^2)}{q^3}\right]=0$ or, $(\omega^2-q^2)\left[-\frac{2(\omega^2+q^2)}{q^3}\right]=0$

so $\omega^2=q^2$ which yields $\omega=q$

which is the condition for velocity resonance and at velocity resonance amplitude

$$v_{0\max}=\frac{f_0}{2mK}$$

So it is revealed that the angular frequency for the amplitude resonance is slightly smaller than that at velocity or energy resonance. The relation between the frequency of the driver and the amplitude of oscillation of the driver at different damping constants is explained by the curves drawn in the figure shown above.

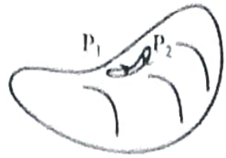
Note:

- Amplitude of the body at resonant condition is maximum.
- Smaller the value of the damping sharper will be the resonance and vice versa.
- At frequency of velocity resonance, velocity amplitude depends on amplitude of the damping force and driving force but independent of frequency.

3.16. a) Using Gauss's divergence theorem and Stokes theorem, prove that $div(curl\,A)=0$. **[WBUT 2013(ODD)]**

Answer:


(a)



(b)

Consider [figure (a)] the volume V enclosed by surface S. Apply the divergence theorem to function $\vec{F}(\vec{r})=\vec{\nabla}\times\vec{A}(\vec{r})$, giving $\int_V \vec{\nabla}\cdot\left(\vec{\nabla}\times\vec{A}(\vec{r})\right)d^3r=\int_{S\,of\,V}\left(\vec{\nabla}\times\vec{A}(\vec{r})\right)\cdot\hat{n}dA$

Now slice out and remove from the surface a tiny sliver [figure (b)]. Technically we have altered S, but this tiny alteration will not affect the value of the surface integral. The edge ε of the altered S is the edge of the sliver. Apply the circulation theorem to $\vec{F}(\vec{r})=\vec{A}(\vec{r})$, giving $\int_S\left(\vec{\nabla}\times\vec{A}(\vec{r})\right)\cdot\hat{n}dA=\int_{\varepsilon\,of\,S}\vec{A}(\vec{r})\cdot d\vec{l}$

But $\int_{\varepsilon\,of\,S}\vec{A}\cdot d\vec{l}=\int_{P_1\,of\,P_2}\vec{A}\cdot d\vec{l}+\int_{P_2\,of\,P_1}\vec{A}\cdot d\vec{l}=\int_{P_1\,of\,P_2}\vec{A}\cdot d\vec{l}-\int_{P_2\,of\,P_1}\vec{A}\cdot d\vec{l}=0$

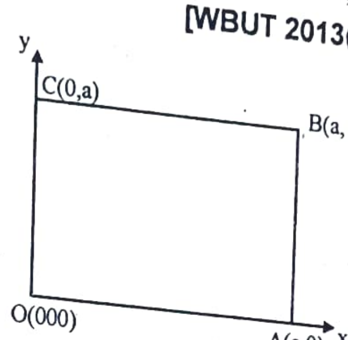
So for any volume V, $\quad\int_V\vec{\nabla}\cdot\left(\vec{\nabla}\times\vec{A}(\vec{r})\right)d^3r=0$

b) Verify Stokes theorem for $F=x^2i+xyj$, and S the area bounded by $x=0,\ y=0,\ x=a, y=a$, in the XY plane.

[WBUT 2013(ODD)]

Answer:

For face OABC, $\vec{ds}=|ds|\hat{n}=dx\,dy\hat{k}$.



$$\text{Now, }\nabla\times\vec{F}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ x^2 & xy & 0\end{vmatrix}=\hat{k}(y)$$

$$\therefore\ \int\left(\vec{\nabla}\times\vec{F}\right)\cdot d\vec{s}=\int_S\hat{k}(y).\,\hat{k}dxdy=\iint ydydx=\int_0^a ydy\int_0^a dx=\frac{1}{2}a^3 \qquad \ldots(1)$$

$$\oint_C\vec{F}.d\vec{r}=\int_{OA}\vec{F}.d\vec{r}+\int_{AB}\vec{F}.d\vec{r}+\int_{BC}\vec{F}.d\vec{r}+\int_{CO}\vec{F}.d\vec{r}$$

$$\text{or, }\oint_C\vec{F}.d\vec{r}=\int_{OA}\left(x^2dx+xydy\right)+\int_{AB}\left(x^2dx+xydy\right)+\int_{BC}\left(x^2dx+xydy\right)+\int_{CO}\left(x^2dx+xydy\right)$$

$$\text{or, }\oint_C\vec{F}.d\vec{r}=\frac{1}{2}a^3 \qquad \ldots(2)$$

From (1) and (2) Stokes theorem is verified.

c) Determine the constant a so that the vector $V=(x+3y)i+(y-2z)j+(x+az)k$ is solenoidal.

[WBUT 2013(ODD)]

Answer:

For solenoidal $\vec{\nabla}\cdot\vec{V}=0$.

So, $\left(\hat{i}\dfrac{\partial}{\partial x}+\hat{j}\dfrac{\partial}{\partial y}+\hat{k}\dfrac{\partial}{\partial z}\right)\cdot\left((x+3y)i+(y-2z)j+(x+az)k\right)=0$

or, $1+1+a=0$ or, $a=-2$

3.17. a) Prove that $\vec{A}\times(\vec{B}\times\vec{C})=\vec{B}(\vec{A}.\vec{C})-\vec{C}(\vec{A}.\vec{B})$. **[WBUT 2014(EVEN)]**

Answer:

Let $\vec{A}=A_1\hat{i}+A_2\hat{j}+A_3\hat{k}$ and $\vec{B}=B_1\hat{i}+B_2\hat{j}+B_3\hat{k}$ and $\vec{C}=C_1\hat{i}+C_2\hat{j}+C_3\hat{k}$

Hence,

$$(\vec{B}\times\vec{C})=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ B_1 & B_2 & B_3\\ C_1 & C_2 & C_3\end{vmatrix}=\hat{i}\{B_2C_3-C_2B_3\}+\hat{j}\{B_3C_1-C_3B_1\}+\hat{k}\{B_1C_2-C_1B_2\}$$

$$\vec{A}\times(\vec{B}\times\vec{C})=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ A_1 & A_2 & A_3\\ \{B_2C_3-C_2B_3\} & \{B_3C_1-C_3B_1\} & \{B_1C_2-C_1B_2\}\end{vmatrix}$$

So, the $\hat{i}^{\text{th}}$ component $[A_2B_1C_2 - A_2B_2C_1 - A_3B_3C_1 + A_3B_1C_3]$

Adding & subtracting $A_1B_1C_1$, collecting positive terms and negative terms and arranging, we get the $\hat{i}^{\text{th}}$ component

$=[A_2B_1C_2+A_3B_1C_3+A_1B_1C_1]-[A_2B_2C_1+A_3B_3C_1+A_1B_1C_1]$

$=[B_1(A_1C_1+A_1C_2+A_3C_3)-C_1(A_1B_1+A_2B_2+A_3B_3)]$ $[B_1(\vec{A}.\vec{C})-\vec{C}_1(\vec{A}.\vec{B})]$

similarly $\hat{j}^{th}$ component $[B_2(\vec{A}.\vec{C})-\vec{C}_2(\vec{A}.\vec{B})]$

and $\hat{k}^{th}$ component $[B_3(\vec{A}.\vec{C})-\vec{C}_3(\vec{A}.\vec{B})]$

So adding all above three terms we have $\vec{A}\times(\vec{B}\times\vec{C})=\vec{B}(\vec{A}.\vec{C})-\vec{C}(\vec{A}.\vec{B})$

b) If A is a constant vector, show that $\vec{\nabla}(\vec{r}.\vec{A})=\vec{A}$, **where** $\vec{r}$ **is the position vector.** **[WBUT 2014(EVEN)]**

Answer:

$\vec{\nabla}(\vec{r}.\vec{A})=\vec{\nabla}(xA_1+yA_2+zA_3)$

$$=\frac{\partial}{\partial x}(xA_1+yA_2+zA_3)\hat{i}+\frac{\partial}{\partial y}(xA_1+yA_2+zA_3)\hat{j}+\frac{\partial}{\partial z}(xA_1+yA_2+zA_3)\hat{k}$$

$=A_1\hat{i}+A_2\hat{j}+A_3\hat{k}=\vec{A}$ Hence Proved.

c) Find $\vec{\nabla}.\vec{F}$ **and** $\vec{\nabla}\times\vec{F}$ **where** $\vec{F}=\vec{\nabla}(x^3+y^3+z^3+3xyz)$. **[WBUT 2014(EVEN)]**

Answer:

$$\vec{F}=\left(\frac{\partial}{\partial x}\hat{i}+\frac{\partial}{\partial x}\hat{j}+\frac{\partial}{\partial z}\hat{k}\right)(x^3+y^3+z^3+3xyz)$$

$$=\hat{i}\left(3x^2+3yz\right)+\hat{j}\left(3y^2+3xz\right)+\hat{k}\left(3z^2+3xy\right)$$

So, $$\vec{\nabla}\times\vec{F}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ \left(3x^2+3yz\right) & \left(3y^2+3xz\right) & \left(3z^2+3xy\right)\end{vmatrix}$$

$$=(3x-3x)\hat{i}+(3y-3y)\hat{j}+(3z-3z)\hat{k}=0$$

And $$\vec{\nabla}\cdot\vec{F}=\left(\frac{\partial}{\partial x}\hat{i}+\frac{\partial}{\partial x}\hat{j}+\frac{\partial}{\partial z}\hat{k}\right).\left\{\hat{i}\left(3x^2+3yz\right)+\hat{j}\left(3y^2+3xz\right)+\hat{k}\left(3z^2+3xy\right)\right\}$$

or, $\vec{\nabla}\cdot\vec{F}=6(x+y+z)$

d) The equation for displacement of a point of damped oscillator is given by $X=5e^{-0.5t}\sin(\pi/2)t$ m. Find the velocity of oscillating point at $t=T/5$ and T, where T is the time period of oscillation. **[WBUT 2014(EVEN)]**

Answer:

Comparing the given expression of displacement $x=5\,e^{-0.5t}\sin\frac{\pi}{2}t$

So velocity $v=\frac{dx}{dt}=5(-0.5)\,e^{-0.5t}\sin\frac{\pi}{2}t+5\times\frac{\pi}{2}e^{-0.5t}\cos\frac{\pi}{2}t$ m/s

Now with the general equation of displacement of a damped harmonic motion $x=Ce^{-Kt}\sin\left(\omega t+\delta\right)$

we have angular frequency $\omega=\frac{\pi}{2}$ and time period $T=\frac{2\pi}{\omega}=\frac{2\pi}{\frac{\pi}{2}}=4s$.

Now at $t=\frac{T}{5}$ i.e, $t=\frac{4}{5}=0.8$ s

velocity will be $v=5(-0.5)\,e^{-0.5\times0.8}\sin\frac{\pi}{2}0.8+5\times\frac{\pi}{2}e^{-0.5\times0.8}\cos\frac{\pi}{2}0.8=4.89\,m/s$

Now at $t=T$ i.e, $t=4$s

velocity will be $v=5(-0.5)\,e^{-0.5\times4}\sin\frac{\pi}{2}4+5\times\frac{\pi}{2}e^{-0.5\times4}\cos\frac{\pi}{2}4=1.06\,m/s$

3.18. a) What do you mean by Q-factor of a damped oscillator? Derive the relation between Q-factor and relaxation time.
b) If the natural angular frequency of a simple harmonic oscillator of mass 2g is 0.8 rad.s^{-1}. It undergoes critically damped motion when taken to a viscous medium. Find the damping force on the oscillator when its speed is 0.2 cm.s^{-1}. **[WBUT 2014(ODD)]**

Answer:

a) Q factor is often assigned to lightly damped oscillator, where Q is the ratio of average energy stored in the oscillator to the average energy dissipated per radian.

Mathematically we can write $Q = -\dfrac{\omega\langle E\rangle}{\langle P(t)\rangle}$

Ae we know displacement of a damped harmonic vibration is $x = Ce^{-Kt}\sin(pt+\delta)$ where C is the amplitude $p=\sqrt{\omega^2-K^2}$ and δ be the phase. Now time average energy stored of a damped vibration is $\langle E\rangle = \int_0^T\left(\frac{1}{2}m\left(\frac{dx}{dt}\right)^2+\frac{1}{2}kx^2\right)dt$

$$\text{or, } \langle E\rangle = \int_0^T\left(\frac{1}{2}m\left(\frac{dx}{dt}\right)^2+\frac{1}{2}kx^2\right)dt$$

$$\text{As } \frac{dx}{dt} = -CKe^{-Kt}\sin(pt+\delta)+Cpe^{-Kt}\cos(pt+\delta)$$

$$\text{So, } \langle E\rangle = \frac{1}{2}mC^2e^{-2Kt}\left(K^2\frac{1}{2}+p^2\frac{1}{2}+0+\omega^2\frac{1}{2}\right)$$

$$\langle E\rangle = \frac{1}{2}mC^2\omega^2e^{-2Kt} = E_0e^{-2Kt}$$

Average energy stored in the oscillator is $\langle E(t)\rangle$ which is equal to E_0e^{-2Kt}. So the average rate of energy loss is $\frac{d}{dt}\langle E(t)\rangle = \langle P(t)\rangle = \frac{d}{dt}\{E_0e^{-2Kt}\} = 2KE_0e^{-2Kt}$.

So, $Q = -\dfrac{\omega\langle E\rangle}{\langle P(t)\rangle}$ Hence $Q = \dfrac{\omega}{2K}$.

In damped harmonic motion, how fast amplitude of the vibration of the body decay with time by the factor e^{-Kt} that depends on the damping coefficient `K'. It is clear that the amplitude Ce^{-Kt} decreases to e^{-1} i.e., 1/e of its original value C in a time $t=\tau$. Putting τ in place of t we have $K\tau=1$ or $\tau=1/K$, then new amplitude becomes 1/e i.e., 0.368 of its initial value. This τ is called the **relaxation time.**

So the mathematical relation becomes $Q = \dfrac{\omega\tau}{2}$

b) Condition for critically damped condition is $\omega = b$, where ω is the natural frequency and b is the damping constant.
As per the given problem $\omega = b = 0.8$.
So, damping constant $L = 2b \times m = 2 \times 0.8 \times 0.2 = 0.32$ $dyne/cm\ s^{-1}$.
So, damping force is $L \times v = 0.32 \times 2 = 0.64$ dyne

3.19. a) In damped harmonic motion, calculate the time in which the energy falls to e^{-1} times of its initial value.
b) Draw the amplitude resonance curves for different values of damping.
[WBUT 2014(ODD)]

Answer:
a) In damped harmonic motion, how fast amplitude of the vibration of the body decay with time by the factor e^{-kt} that depends on the damping coefficient 'K'. It is clear that the amplitude Ce^{-kt} decreases to e^{-1} i.e., 1/e of its original value C in a time $t = \tau$. Putting τ in place of t we have $K\tau = 1$ or $\tau = 1/K$, then new amplitude becomes 1/e i.e., 0.368 of its initial value. This τ is called the **relaxation time** or **decay constant**. (Relaxation time can also be expressed in terms of energy as $\tau = \dfrac{1}{2K}$ is known as relaxation time. So relaxation time is defined as the time taken to fall the energy of the damped oscillatory motion to $\dfrac{1}{e}$ (nearly 37%) of its initial energy.
The Decay constant is defined in which the amplitude decays to 1/e of its initial value.

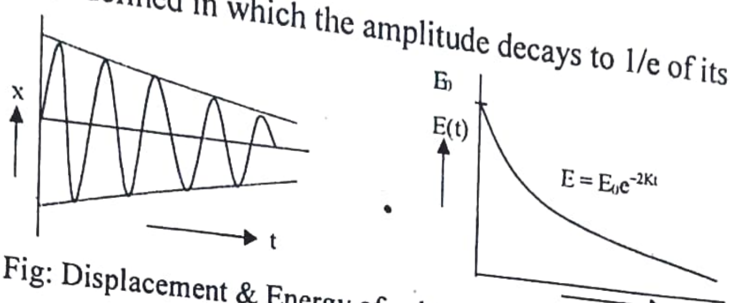
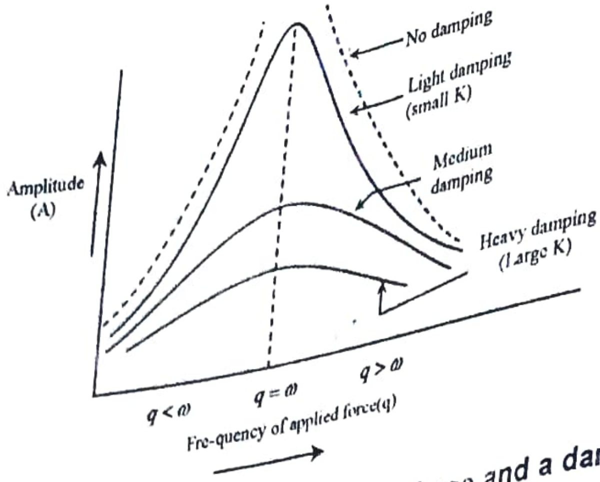


Fig: Displacement & Energy of a damped oscillatory motion

So for amplitude study relaxation time $\tau = \dfrac{1}{K}$ whereas for energy calculation relaxation time will be $\tau = \dfrac{1}{2K}$

b)



3.20. A particle is subjected to a harmonic restoring force and a damping force. Its equation of motion is given $\frac{d^2x}{dt^2}+2b\frac{dx}{dt}+\omega_0^2x=0$

Under the condition of small damping, find the expression for displacement as a function of time. **[WBUT 2014(ODD)]**

Answer:

The given differential equation is $\frac{d^2x}{dt^2}+2b\frac{dx}{dt}+\omega_0^2x=0$

Trying $x=Ae^{mt}$ as a trial solution we have, $\frac{dx}{dt}=Ame^{mt}$; $\frac{d^2x}{dt^2}=Am^2e^{mt}$

So that the value satisfies the above equation, $m^2+2bm+\omega_0^2=0$

Which is a quadratic equation of m having two roots.

$$\therefore\ m=-b\pm\sqrt{b^2-\omega_0^2}$$

i.e., the general solution is $x=A_1e^{\left(-b+\sqrt{b^2-\omega_0^2}\right)t}+A_2e^{\left(-b-\sqrt{b^2-\omega_0^2}\right)t}$

The actual solution depends upon the relative magnitudes of b and ω_0^2

For small damping i.e., when b< ω_0^2 i.e., $\sqrt{b^2-\omega_0^2}$ is imaginary, we may write it as

$\sqrt{b^2-\omega_0^2}=ip$

or, $\omega_0^2-b^2=p^2$

i.e., $p=\sqrt{\omega_0^2-b^2}$ which is a real quantity so that the solution becomes,

$x=A_1\left\{e^{(-b+ip)t}\right\}+A_2\left\{e^{(-b-ip)t}\right\}=e^{-bt}\{A_1e^{ipt}+A_2e^{-ipt}\}$

This may be put in the form $x=C\,e^{-bt}\sin(pt+\delta)$

...1. A particle of mass 'm' executes one dimensional motion under a restoring force proportional to displacement from equivalent position and a damping force proportional to its velocity (v).
a) Set up the equation of motion and solve it.
b) Find the condition for weakly damped, critically damped and over damped motion.
c) Show that for weakly damped motion, the logarithm of the ratio of successive amplitude on the same side of mean position is constant.
d) Show that the average energy E of vibration decays according to the law; $E = E_0 e^{-2\gamma t}$ (where the damping force is $-bv$ and $\gamma = b/2m$). **[WBUT 2015(EVEN)]**

Answer:
a) Equation of motion of a body of mass m, subject to damping force is of the form

$$m\frac{d^2x}{dt^2} = -kx - L\frac{dx}{dt} \quad \ldots(1)$$

Where k and L are constants.
L is often known as resistive force constant.
The above said expression can be written as

$$\frac{d^2x}{dt^2} + 2K\frac{dx}{dt} + \omega^2 x = 0 \quad \ldots(2)$$

Where we define $L/m = 2K$ and $k/m = \omega^2$

2K represents the ***frictional force per unit mass per unit velocity***. Hence $K = \dfrac{L}{2m}$ is called *damping constant* and $\omega = \sqrt{\dfrac{k}{m}}$ as *restoring constant.*

Let us choose $x = Ae^{bt}$ be a trial solution of the differential eqn.(2) we have,

$$\frac{dx}{dt} = Abe^{bt} \quad ; \quad \frac{d^2x}{dt^2} = Ab^2e^{bt}$$

So that the value satisfies the above equation

$$b^2 + 2Kb + \omega^2 = 0$$

Which is a quadratic equation of b having two roots.

$\therefore b = -K \pm \sqrt{K^2 - \omega^2}$ i.e., the general solution is

$$x = A_1 e^{\left(-K+\sqrt{K^2-\omega^2}\right)t} + A_2 e^{\left(-K-\sqrt{K^2-\omega^2}\right)t} \quad \ldots(3)$$

The actual solution depends upon the relative magnitudes of K and ω which is shown in the following cases.

b) ***Refer to Question No. 3.4(b) (Case I, II & III).***

c) The displacement of a damped harmonic oscillator $x = Ce^{-Kt}\sin(pt+\delta)$ becomes maximum at time $t = t_1$ for which $\sin(pt+\delta)$ is ± 1. Let at time $t = t_1$ amplitude $C_1 = Ce^{-Kt_1}$.

If $T = \frac{2\pi}{\omega}$ is the time period of oscillation, then amplitude on the same side of the mean position at time period T later be C_2 is given by $C_2 = Ce^{-K(t_1+T)} = C_1 e^{-KT}$. Similarly the other amplitudes at regular time period are C_3, C_4, C_5........

Now, $C_3 = Ce^{-K(t_1+2T)} = C_1 e^{-2KT} = C_2 e^{-KT}$

Thus $\frac{C_1}{C_2} = \frac{C_2}{C_3} = \frac{C_3}{C_4}$....... $= e^{KT} = d(say)$ i.e., $\log_e d = KT = \lambda(say)$

The quantity d which is the ratio of two successive amplitudes (on same side of the mean position) is called the decrement and $\log_e d = \lambda$ is called *logarithmic decrement* of the damped oscillation.

d) The total energy of a particle of mass m undergoing damped harmonic motion is

$$E = P.E + K.E = \frac{m\omega^2 x^2}{2} + \frac{m}{2}\left(\frac{dx}{dt}\right)^2$$

As we know from the solution of damped harmonic vibration as $x = Ce^{-Kt}\sin(pt+\delta)$

$$\text{or, } \frac{dx}{dt} = -CKe^{-Kt}\sin(pt+\delta) + Cpe^{-Kt}\cos(pt+\delta)$$

$$\frac{1}{2}m\left(\frac{dx}{dt}\right)^2 = \frac{1}{2}mC^2K^2e^{-2Kt}\sin^2(pt+\delta) + \frac{1}{2}mC^2p^2e^{-2Kt}\cos^2(pt+\delta) - \frac{1}{2}mC^2pK\sin 2(pt+\delta)$$

$$\text{Thus, } E = \frac{1}{2}m\omega^2C^2e^{-2Kt}\sin^2(pt+\delta) + \frac{1}{2}mK^2C^2e^{-2Kt}\sin^2(pt+\delta)$$
$$+\frac{1}{2}mp^2C^2e^{-2Kt}\cos^2(pt+\delta) - mC^2Kp\,e^{-2Kt}\sin 2(\omega t+\delta)$$

$$E = \frac{1}{2}mC^2e^{-2Kt}\begin{pmatrix} K^2\sin^2(pt+\delta) + p^2\cos^2(pt+\delta) \\ -Kp\,e^{-2Kt}\sin 2(\omega t+\delta) \\ +\omega^2\sin^2(pt+\delta) \end{pmatrix}$$

So by assuming e^{-2Kt} to be sensibly constant during the period T of the oscillation, the time average value of the total energy can be obtained by taking the averages of sine and cosine functions.

Now the average of a function $f(t)$ over T is defined by $\langle f(t)\rangle = \frac{1}{T}\int_0^T f(t)dt$

As we know $\langle\cos^2(\omega t+\delta)\rangle = \frac{1}{T}\int_0^T \cos^2(\omega t+\delta)dt = \frac{1}{2}$ and

$$\langle \sin^2(\omega t+\delta)\rangle = \frac{1}{T}\int_0^T \sin^2(\omega t+\delta)\,dt = \frac{1}{2}$$

so we have the time averaged energy of the oscillator is

$$\langle E\rangle = \frac{1}{2}mC^2e^{-2Kt}\left(K^2\frac{1}{2}+p^2\frac{1}{2}+0+\omega^2\frac{1}{2}\right)$$

$$\langle E\rangle = \frac{1}{2}mC^2\omega^2e^{-2Kt} = E_0e^{-2Kt} \qquad \ldots (1)$$

According to the given problem $L=b$ and $\gamma = \frac{b}{2m} = \frac{L}{2m} = K$

So the expression (1) can be written as $\langle E\rangle = E_0e^{-2\gamma t}$ Proved.