Epsilon-Regularity for Non-Collapsed Limi Flows

Harry Fluck
(in collaboration with Max Hallgren)

ornell University

October 27th 2024

Outline

Overview of the Ricci Flow

Definition (Ricci flow)

Let M^n be a closed manifold and $g(t)$ be a smooth 1-par
Riemannian metrics. We say that $g(t)$ evolves by the **Ric**

$$\frac{\partial}{\partial t} g_{ij} = \quad 2Rc_{ij}.$$

1. **Harmonic coordinates**: $\frac{\partial}{\partial t} g_{ij} = \Delta(g_{ij}) +$ lower orde
2. **Idea**: "Improve" the metric using Ricci flow. Better g
   easier to understand topology.
3. **Notable Results:** $n = 2$ *(Hamilton, Chow, Chen, Lu*
   (Hamilton, Perelman), general n *(öhm, Wilking)*.

Overview of the Ricci Flow

Definition (Ricci flow)

Let M^n be a closed manifold and $g(t)$ be a smooth 1-par
Riemannian metrics. We say that $g(t)$ evolves by the **Ric**

$$\frac{\partial}{\partial t} g_{ij} = \quad 2Rc_{ij}.$$

1. **Harmonic coordinates**: $\frac{\partial}{\partial t} g_{ij} = \Delta(g_{ij}) +$ lower orde
2. **Idea**: "Improve" the metric using Ricci flow. Better g
easier to understand topology.
3. **Notable Results:** $n = 2$ *(Hamilton, Chow, Chen, Lu*
(Hamilton, Perelman), general n *(öhm, Wilking)*.

Overview of the Ricci Flow

Definition (Ricci flow)

Let M^n be a closed manifold and $g(t)$ be a smooth 1-par
Riemannian metrics. We say that $g(t)$ evolves by the **Ric**

$$\frac{\partial}{\partial t} g_{ij} = \quad 2Rc_{ij}.$$

1. **Harmonic coordinates**: $\frac{\partial}{\partial t} g_{ij} = \Delta(g_{ij}) +$ lower orde
2. **Idea**: "Improve" the metric using Ricci flow. Better g
easier to understand topology.
3. **Notable Results:** $n = 2$ *(Hamilton, Chow, Chen, Lu*
(Hamilton, Perelman), general n *(öhm, Wilking)*.

Overview of the Ricci Flow

Definition (Ricci flow)

Let M^n be a closed manifold and $g(t)$ be a smooth 1-par
Riemannian metrics. We say that $g(t)$ evolves by the **Ric**

$$\frac{\partial}{\partial t} g_{ij} = \quad 2Rc_{ij}.$$

1. **Harmonic coordinates**: $\frac{\partial}{\partial t} g_{ij} = \Delta(g_{ij}) +$ lower orde
2. **Idea**: "Improve" the metric using Ricci flow. Better g
easier to understand topology.
3. **Notable Results:** $n = 2$ *(Hamilton, Chow, Chen, Lu*
(Hamilton, Perelman), general n *(öhm, Wilking)*.

Overview of the Ricci Flow

1. (Evolution of curvature): $\frac{\partial}{\partial t} Rm = \Delta Rm + Rm * Rm$
2. Expect the formation of singularities ($\lim_{t \nearrow T} \sup_M |$

Singularity nalysis

Take $\lambda_i \to \infty$, and (x_i, t_i) with $t_i \to T$ and $|Rm(x_i, t_i)| \nearrow$
blowup sequence:

$$(M^n, \lambda_i^2 g(t_i + \lambda_i^{\ 2} s)_{s \in [\ t_i \lambda_i^2, 0]}, x_i).$$

im: analyze possible limits and continue the flow past si
surgery constructions.

1. $n = 2, 3$ are well understood *(Hamilton, Chow, Perel*
2. $n = 4$ largely open.

Overview of the Ricci Flow

1. (Evolution of curvature): $\frac{\partial}{\partial t} Rm = \Delta Rm + Rm * Rm$
2. Expect the formation of singularities ($\lim_{t \nearrow T} \sup_M |R$

Singularity nalysis

Take $\lambda_i \to \infty$, and (x_i, t_i) with $t_i \to T$ and $|Rm(x_i, t_i)| \nearrow$
blowup sequence:

$$(M^n, \lambda_i^2 g(t_i + \lambda_i^{\ 2} s)_{s \in [\ t_i \lambda_i^2, 0]}, x_i).$$

im: analyze possible limits and continue the flow past si
surgery constructions.

1. $n = 2, 3$ are well understood *(Hamilton, Chow, Perel*
2. $n = 4$ largely open.

Overview of the Ricci Flow

1. (Evolution of curvature): $\frac{\partial}{\partial t} Rm = \Delta Rm + Rm * Rm$
2. Expect the formation of singularities ($\lim_{t \nearrow T} \sup_M |$

Singularity nalysis

Take $\lambda_i \to \infty$, and (x_i, t_i) with $t_i \to T$ and $|Rm(x_i, t_i)| \nearrow$ blowup sequence:

$$(M^n, \lambda_i^2 g(t_i + \lambda_i \ ^2 s)_{s \in [\ \ t_i \lambda_i^2, 0]}, x_i).$$

im: analyze possible limits and continue the flow past si surgery constructions.

1. $n = 2, 3$ are well understood *(Hamilton, Chow, Perel*
2. $n = 4$ largely open.

Overview of the Ricci Flow

1. (Evolution of curvature): $\frac{\partial}{\partial t} Rm = \Delta Rm + Rm * Rm$
2. Expect the formation of singularities ($\lim_{t \nearrow T} \sup_M |$

Singularity nalysis

Take $\lambda_i \to \infty$, and (x_i, t_i) with $t_i \to T$ and $|Rm(x_i, t_i)| \nearrow$
blowup sequence:

$$(M^n, \lambda_i^2 g(t_i + \lambda_i{}^{2} s)_{s \in [\ t_i \lambda_i^2, 0]}, x_i).$$

im: analyze possible limits and continue the flow past si
surgery constructions.

1. $n = 2, 3$ are well understood *(Hamilton, Chow, Pereli*
2. $n = 4$ largely open.

Overview of the Ricci Flow

1. (Evolution of curvature): $\frac{\partial}{\partial t} Rm = \Delta Rm + Rm * Rm$
2. Expect the formation of singularities ($\lim_{t \nearrow T} \sup_M |$

Singularity nalysis

Take $\lambda_i \to \infty$, and (x_i, t_i) with $t_i \to T$ and $|Rm(x_i, t_i)| \nearrow$
blowup sequence:

$$(M^n, \lambda_i^2 g(t_i + \lambda_i\ ^2 s)_{s \in [\ \ t_i \lambda_i^2, 0]}, x_i).$$

im: analyze possible limits and continue the flow past si
surgery constructions.

1. $n = 2, 3$ are well understood *(Hamilton, Chow, Perel*
2. $n = 4$ largely open.

Overview of the Ricci Flow

1. (Evolution of curvature): $\frac{\partial}{\partial t} Rm = \Delta Rm + Rm * Rm$
2. Expect the formation of singularities ($\lim_{t \nearrow T} \sup_M |$

Singularity nalysis

Take $\lambda_i \to \infty$, and (x_i, t_i) with $t_i \to T$ and $|Rm(x_i, t_i)| \nearrow$
blowup sequence:

$$(M^n, \lambda_i^2 g(t_i + \lambda_i{}^2 s)_{s \in [\ t_i \lambda_i^2, 0]}, x_i).$$

im: analyze possible limits and continue the flow past si
surgery constructions.

1. $n = 2, 3$ are well understood *(Hamilton, Chow, Perel*
2. $n = 4$ largely open.

Overview of the Ricci Flow

Definition (Gradient Shrinking Ricci Soliton)

Let (M, g) be a Riemannian manifold and $f \in C^\infty(M)$. W
(M, g, f) is a **gradient shrinking Ricci soliton** (GSRS) i

$$Rc + \nabla^2 f = \frac{1}{2} g.$$

1. ny GSRS gives rise to a self-similar solution to the
   $g(t) = (1 \quad t)\varphi^*(t)g$ where $\varphi^*(t)$ is the flow genera
2. GSRS model "most" singularities of the Ricci flow.
3. Examples:
   1. Ricci flat cones over $\mathbb{S}^n/ \quad (f = \frac{1}{}r^2)$,
   2. Gaussian shrinker: $(\mathbb{R}^n, g_{\mathbb{R}^n}, \frac{1}{}|x|^2)$,
   3. Einstein manifolds $(f \equiv c)$, e.g. $\mathbb{S}^n$,
   4. Products e.g. cylinders $\mathbb{S}^n \times \mathbb{R}^{n \quad k}$.

Overview of the Ricci Flow

Definition (Gradient Shrinking Ricci Soliton)

Let (M, g) be a Riemannian manifold and $f \in C^\infty(M)$. W
(M, g, f) is a **gradient shrinking Ricci soliton** (GSRS) i

$$Rc + \nabla^2 f = \frac{1}{2} g.$$

1. ny GSRS gives rise to a self-similar solution to the
   $g(t) = (1 \quad t)\varphi^*(t)g$ where $\varphi^*(t)$ is the flow genera
2. GSRS model "most" singularities of the Ricci flow.
3. Examples:
   1. Ricci flat cones over $\mathbb{S}^n/ \quad (f = \frac{1}{}r^2)$,
   2. Gaussian shrinker: $(\mathbb{R}^n, g_{\mathbb{R}^n}, \frac{1}{}|x|^2)$,
   3. Einstein manifolds $(f \equiv c)$, e.g. $\mathbb{S}^n$,
   4. Products e.g. cylinders $\mathbb{S}^n \times \mathbb{R}^{n \quad k}$.

Overview of the Ricci Flow

Definition (Gradient Shrinking Ricci Soliton)

Let (M, g) be a Riemannian manifold and $f \in C^\infty(M)$. W
(M, g, f) is a **gradient shrinking Ricci soliton** (GSRS) i

$$Rc + \nabla^2 f = \frac{1}{2} g.$$

1. ny GSRS gives rise to a self-similar solution to the
   $g(t) = (1 \quad t)\varphi^*(t)g$ where $\varphi^*(t)$ is the flow genera
2. GSRS model "most" singularities of the Ricci flow.
3. Examples:
   1. Ricci flat cones over $\mathbb{S}^n/ \quad (f = \frac{1}{}r^2)$,
   2. Gaussian shrinker: $(\mathbb{R}^n, g_{\mathbb{R}^n}, \frac{1}{}|x|^2)$,
   3. Einstein manifolds $(f \equiv c)$, e.g. $\mathbb{S}^n$,
   4. Products e.g. cylinders $\mathbb{S}^n \times \mathbb{R}^{n \quad k}$.

Overview of the Ricci Flow

Definition (Gradient Shrinking Ricci Soliton)

Let (M, g) be a Riemannian manifold and $f \in C^\infty(M)$. V
(M, g, f) is a **gradient shrinking Ricci soliton** (GSRS) i

$$Rc + \nabla^2 f = \frac{1}{2} g.$$

1. ny GSRS gives rise to a self-similar solution to the
   $g(t) = (1 \quad t)\varphi^*(t)g$ where $\varphi^*(t)$ is the flow genera
2. GSRS model "most" singularities of the Ricci flow.
3. **Examples:**
   1. Ricci flat cones over $\mathbb{S}^n/ \quad (f = \frac{1}{\ } r^2)$,
   2. Gaussian shrinker: $(\mathbb{R}^n, g_{\mathbb{R}^n}, \frac{1}{\ }|x|^2)$,
   3. Einstein manifolds $(f \equiv c)$, e.g. $\mathbb{S}^n$,
   4. Products e.g. cylinders $\mathbb{S}^n \times \mathbb{R}^{n \quad k}$.

Overview of the Ricci Flow

The Picture in Dimension 4 *(amler 2021)*

1. **First bubble:** Every type-(1) ($\lambda_i \approx (T \quad t_i) \quad ^1$) blow
   4-dimensional Ricci flow converges to a gradient shri
   with finitely many orbifold singularities.
2. **Deeper bubbles:** singularities are "mostly modelled"
   1. generalized cylinders $\mathbb{S}^k \times \mathbb{R} \quad ^k$,
   2. ones over some smooth link $\mathbb{S}^3 / \quad$,

Corollary of Epsilon-Regularity (F.-Hallgren 2024)

If $(M^4, g(t))$ is a solution to the Kähler Ricci flow, then e
an orbifold point contributes a definite amount of L^2-curv

Hence if $(M^4, g(t))$ satisfies a finite energy assumption, th
finite.

Overview of the Ricci Flow

The Picture in Dimension 4 *(amler 2021)*

1. **First bubble:** Every type-(1) ($\lambda_i \approx (T \quad t_i) \ ^{1}$) blow
4-dimensional Ricci flow converges to a gradient shri
with finitely many orbifold singularities.
2. **Deeper bubbles:** singularities are "mostly modelled"
   1. generalized cylinders $\mathbb{S}^k \times \mathbb{R} \quad ^{k}$,
   2. ones over some smooth link $\mathbb{S}^3 / \quad$,

Corollary of Epsilon-Regularity (F.-Hallgren 2024)

If $(M^4, g(t))$ is a solution to the Kähler Ricci flow, then e
an orbifold point contributes a definite amount of L^2-curv

Hence if $(M^4, g(t))$ satisfies a finite energy assumption, th
finite.

Overview of the Ricci Flow

The Picture in Dimension 4 *(amler 2021)*

1. **First bubble:** Every type-(1) ($\lambda_i \approx (T \quad t_i) \quad ^1$) blow 4-dimensional Ricci flow converges to a gradient shrin with finitely many orbifold singularities.
2. **Deeper bubbles:** singularities are "mostly modelled"
   1. generalized cylinders $\mathbb{S}^k \times \mathbb{R} \quad ^k$,
   2. ones over some smooth link $\mathbb{S}^3/ \quad$,

Corollary of Epsilon-Regularity (F.-Hallgren 2024)

If $(M^4, g(t))$ is a solution to the Kähler Ricci flow, then e an orbifold point contributes a definite amount of L^2-curv

Hence if $(M^4, g(t))$ satisfies a finite energy assumption, th finite.

Overview of the Ricci Flow

The Picture in Dimension 4 *(amler 2021)*

1. **First bubble:** Every type-(1) ($\lambda_i \approx (T \quad t_i) \quad ^1$) blow
4-dimensional Ricci flow converges to a gradient shrin
with finitely many orbifold singularities.
2. **Deeper bubbles:** singularities are "mostly modelled"
    1. generalized cylinders $\mathbb{S}^k \times \mathbb{R} \quad ^k$,
    2. ones over some smooth link $\mathbb{S}^3 /$,

Corollary of Epsilon-Regularity (F.-Hallgren 2024)

If $(M^4, g(t))$ is a solution to the Kähler Ricci flow, then e
an orbifold point contributes a definite amount of L^2-curv

Hence if $(M^4, g(t))$ satisfies a finite energy assumption, th
finite.

The Case of Ricci Bounded from Below

Epsilon-Regularity for Ricci Limit Spaces *(Cheeger, C*
2002)

1. If (M_i^4, g_i) satisfies $|Rc_i| \leq 3$, $Vol(B_1(p_i)) \geq \nu$. Then
   $(M_i, g_i, p_i) \overset{GH}{\to} (X, d, p^*)$.
2. (X, d, p^*) has singularities that are modeled on Ricci
   $(C(\mathbb{S}^3/\quad), x^*)$.
3. If $d_{GH}(B_1(p), B_1(x^*)) < \epsilon_i$, then there exists $r_i^2 : B_1$

   $$\begin{cases} \Delta r_i^2 = 8, \\ |r_i^2 \quad \pi_r| < \epsilon_i', \\ \int_{B_1(p)} (|\nabla r_i^2| \quad 2r_i)^2 + |\nabla^2 r_i^2 \quad 2g|^2 + |Rc(\nabla r_i^2, \nabla \\ \sup |\nabla r_i^2| \leq c(n). \end{cases}$$

4. Proof uses "good slices" coming from the approximat

The Case of Ricci Bounded from Below

Epsilon-Regularity for Ricci Limit Spaces *(Cheeger, C*
2002)

1. If (M_i^4, g_i) satisfies $|Rc_i| \leq 3$, $Vol(B_1(p_i)) \geq \nu$. Then
   $(M_i, g_i, p_i) \overset{GH}{\to} (X, d, p^*)$.
2. (X, d, p^*) has singularities that are modeled on Ricci
   $(C(\mathbb{S}^3/\ \ ), x^*)$.
3. If $d_{GH}(B_1(p), B_1(x^*)) < \epsilon_i$, then there exists $r_i^2 : B_1$

$$\begin{cases} \Delta r_i^2 = 8, \\ |r_i^2 \quad \pi_r| < \epsilon_i', \\ \int_{B_1(p)} (|\nabla r_i^2| \quad 2r_i)^2 + |\nabla^2 r_i^2 \quad 2g|^2 + |Rc(\nabla r_i^2, \nabla \\ \sup |\nabla r_i^2| \leq c(n). \end{cases}$$

4. Proof uses "good slices" coming from the approximat

The Case of Ricci Bounded from Below

Epsilon-Regularity for Ricci Limit Spaces *(Cheeger, C*
2002)

1. If (M_i^4, g_i) satisfies $|Rc_i| \leq 3$, $Vol(B_1(p_i)) \geq \nu$. Then
   $(M_i, g_i, p_i) \overset{GH}{\to} (X, d, p^*)$.
2. (X, d, p^*) has singularities that are modeled on Ricci
   $(C(\mathbb{S}^3/ \quad), x^*)$.
3. If $d_{GH}(B_1(p), B_1(x^*)) < \epsilon_i$, then there exists $r_i^2 : B_1$

$$\begin{cases} \Delta r_i^2 = 8, \\ |r_i^2 \quad \pi_r| < \epsilon_i', \\ \int_{B_1(p)} (|\nabla r_i^2| \quad 2r_i)^2 + |\nabla^2 r_i^2 \quad 2g|^2 + |Rc(\nabla r_i^2, \nabla \\ \sup |\nabla r_i^2| \leq c(n). \end{cases}$$

4. Proof uses "good slices" coming from the approximat

The Case of Ricci Bounded from Below

Epsilon-Regularity for Ricci Limit Spaces *(Cheeger, C*
2002)

1. If (M_i^4, g_i) satisfies $|Rc_i| \leq 3$, $Vol(B_1(p_i)) \geq \nu$. Ther
   $(M_i, g_i, p_i) \xrightarrow{GH} (X, d, p^*)$.
2. (X, d, p^*) has singularities that are modeled on Ricci
   $(C(\mathbb{S}^3/\ \ ), x^*)$.
3. If $d_{GH}(B_1(p), B_1(x^*)) < \epsilon_i$, then there exists $r_i^2 : B_1$

$$\begin{cases} \Delta r_i^2 = 8, \\ |r_i^2 \quad \pi_r| < \epsilon_i', \\ \int_{B_1(p)} (|\nabla r_i^2| \quad 2r_i)^2 + |\nabla^2 r_i^2 \quad 2g|^2 + |Rc(\nabla r_i^2, \\ \sup |\nabla r_i^2| \leq c(n). \end{cases}$$

4. Proof uses "good slices" coming from the approximat

Conjugate Heat Kernel Measures

Setup: $(M, g(t)_{t\in(\quad T,0]})$ a closed RF, $(x_0, t_0) \in M \times (T$

Definition (Conjugate Heat Kernels)

The **conjugate heat operator** is defined as

$$^{*} = \quad \frac{\partial}{\partial t} \quad \Delta + R.$$

If $K(x_0, t_0, y, s)$ is a **conjugate heat kernel**, that is:

$$\begin{aligned} {}^{*}_{y,s} K(x_0, t_0, y, s) &= 0, \\ \lim_{s \nearrow t_0} K(x_0, t_0, y, s) &= \ {}_{x_0}. \end{aligned}$$

Then the family $d\nu_{x_0,t_0;s} = K(x_0, t_0, \quad, s)dg_s$ are probabil
known as the **conjugate heat kernel measures** based at

Conjugate Heat Kernel Measures

Setup: $(M, g(t)_{t\in(\ \ T,0]})$ a closed RF, $(x_0, t_0) \in M \times (T$

Definition (Conjugate Heat Kernels)

The **conjugate heat operator** is defined as

$$^* = \ \ \frac{\partial}{\partial t} \quad \Delta + R.$$

If $K(x_0, t_0, y, s)$ is a **conjugate heat kernel**, that is:

$$\begin{aligned} {}^*_{y,s} K(x_0, t_0, y, s) &= 0, \\ \lim_{s\nearrow t_0} K(x_0, t_0, y, s) &= \ \ _{x_0}. \end{aligned}$$

Then the family $d\nu_{x_0,t_0;s} = K(x_0, t_0, \quad, s)dg_s$ are probabil
known as the **conjugate heat kernel measures** based at

Metric Flows and -Convergence

Definition (Metric Flow)

metric flow $\mathcal{X}$ is a 1-parameter family of complete sep
spaces $(\mathcal{X}_t, d_t)$ together with probability measures $d\nu_{x;s}$,
that satisfy the following:

1. For every $x \in \mathcal{X}_t$, $\lim_{s \nearrow t} d\nu_{x;s} = \ _x$,
2. Lipschitz estimates improve for "heat flows" $u_t(y) =$
3. The measures are compatible: $d\nu_{x;t_1} = \int_{\mathcal{X}_{t_2}} d\nu_{y;t_1} d\nu$

1. Properties (1) (3) characterize conjugate heat kern
   Ricci flow background.
2. Ricci flows $(M, g(t)_{t\in(\ T,0]})$, together with the conj
   measures $d\nu_{x_0,t_0;s}$, are metric flows.

Metric Flows and -Convergence

Definition (Metric Flow)

metric flow $\mathcal{X}$ is a 1-parameter family of complete sep
spaces $(\mathcal{X}_t, d_t)$ together with probability measures $d\nu_{x;s}$,
that satisfy the following:

1. For every $x \in \mathcal{X}_t$, $\lim_{s \nearrow t} d\nu_{x;s} = \ _x$,
2. Lipschitz estimates improve for "heat flows" $u_t(y) =$
3. The measures are compatible: $d\nu_{x;t_1} = \int_{\mathcal{X}_{t_2}} d\nu_{y;t_1} d\nu$

1. Properties (1) (3) characterize conjugate heat kern Ricci flow background.
2. Ricci flows $(M, g(t))_{t \in (\ T, 0]})$, together with the conju measures $d\nu_{x_0, t_0; s}$, are metric flows.

Metric Flows and -Convergence

Definition (Metric Flow)

metric flow $\mathcal{X}$ is a 1-parameter family of complete sep
spaces $(\mathcal{X}_t, d_t)$ together with probability measures $d\nu_{x;s}$,
that satisfy the following:

1. For every $x \in \mathcal{X}_t$, $\lim_{s \nearrow t} d\nu_{x;s} = \quad {}_x$,
2. Lipschitz estimates improve for "heat flows" $u_t(y) =$
3. The measures are compatible: $d\nu_{x;t_1} = \int_{\mathcal{X}_{t_2}} d\nu_{y;t_1} d\nu$

1. Properties (1) (3) characterize conjugate heat kern
   Ricci flow background.
2. Ricci flows $(M, g(t)_{t\in(\quad T,0]})$, together with the conju
   measures $d\nu_{x_0,t_0;s}$, are metric flows.

Metric Flows and -Convergence

Theorem (amler 2020)

The space $_I$ of metric flow pairs over I can be equipped
function d such that $(\ _I, d\ )$ is a complete metric space.

If $(M^i, g_i(t)_{t\in(\ T,0]}, d\nu_{(x^i,0);s})$ is a sequence of closed RF
metric flow pair $(\mathcal{X}^\infty, d\nu_{x^\infty;s})$ such that

$$(M^i, g_i(t)_{t\in(\ T,0]}, d\nu_{(x^i,0);s}) \xrightarrow{d} (\mathcal{X}^\infty, d\nu_{x^\infty}$$

1. -convergence $\approx$ GW^1-convergence for metric measu
time slice,
2. Compactness uses good properties of conjugate heat
coupled to Ricci flow.

Metric Flows and -Convergence

Theorem (amler 2020)

The space $_I$ of metric flow pairs over I can be equipped
function d such that $(_I, d)$ is a complete metric space.

If $(M^i, g_i(t)_{t\in(T,0]}, d\nu_{(x^i,0);s})$ is a sequence of closed RF
metric flow pair $(\mathcal{X}^\infty, d\nu_{x^\infty;s})$ such that

$$(M^i, g_i(t)_{t\in(T,0]}, d\nu_{(x^i,0);s}) \xrightarrow{d} (\mathcal{X}^\infty, d\nu_{x^\infty$$

1. -convergence $\approx GW^1$-convergence for metric measu
time slice,
2. Compactness uses good properties of conjugate heat
coupled to Ricci flow.

Metric Flows and -Convergence

Theorem (amler 2020)

The space $_I$ of metric flow pairs over I can be equipped function d such that $(_I, d)$ is a complete metric space.

If $(M^i, g_i(t)_{t\in(T,0]}, d\nu_{(x^i,0);s})$ is a sequence of closed RF metric flow pair $(\mathcal{X}^\infty, d\nu_{x^\infty;s})$ such that

$$(M^i, g_i(t)_{t\in(T,0]}, d\nu_{(x^i,0);s}) \xrightarrow{d} (\mathcal{X}^\infty, d\nu_{x^\infty$$

1. -convergence $\approx GW^1$-convergence for metric measu time slice,
2. Compactness uses good properties of conjugate heat coupled to Ricci flow.

Limits that are Ricci Flat Cones

Definition (Integral lmost Properties)

Let $(M, g(t)_{t\in(\; T,0]})$ be a RF, $(x_0, t_0) \in M \times (\; T, 0]$. L
$r > 0$ be such that $[t_0 \quad {}^{1}r^2, t_0 \quad r^2] \subset (\; T, 0]$. Let
$d\nu_{x_0,t_0;s} = (4\pi\tau)^{\; n/2} e^{\; f} dg_s$ and $W = \mathcal{N}_{x_0,t_0}(r^2)$. We sa
$(\; , r)$-selfsimilar if:

$$\begin{cases} \int_{t_0 \quad {}^{1}r^2}^{t_0 \quad r^2} \int_M \tau |\nabla^2 f + Rc \quad \frac{1}{2\tau} g|^2 \, d\nu_t dt < \; , \\ \sup_{[t_0 \quad {}^{1}r^2, t_0 \quad r^2]} \int_M |\tau(2\Delta f \quad |\nabla f|^2 + R) + f \quad n \\ \inf_{M\times[t_0 \quad {}^{1}r^2, t_0 \quad r^2]} R \geq \quad r^{\; 2}, \end{cases}$$

and $(\; , r)$-static if:

$$\begin{cases} \int_{t_0 \quad {}^{1}r^2}^{t_0 \quad r^2} \int_M r^2 |Rc|^2 \, d\nu_t dt < \; , \\ \sup_{[t_0 \quad {}^{1}r^2, t_0 \quad r^2]} \int_M R < \; , \\ \inf_{M\times[t_0 \quad {}^{1}r^2, t_0 \quad r^2]} R \geq \quad r^{\; 2}. \end{cases}$$

Limits that are Ricci Flat Cones

Definition (Integral lmost Properties)

Let $(M, g(t)_{t\in(\ \ T,0]})$ be a RF, $(x_0, t_0) \in M \times (\ \ T, 0]$. L
$r > 0$ be such that $[t_0 \quad {}^{1}r^2, t_0 \quad r^2] \subset (\ \ T, 0]$. Let
$d\nu_{x_0,t_0;s} = (4\pi\tau)^{\ \ n/2} e^{\ \ f} dg_s$ and $W = \mathcal{N}_{x_0,t_0}(r^2)$. We say
$(\ \ , r)$-selfsimilar if:

$$\begin{cases} \int_{t_0 \quad {}^{1}r^2}^{t_0 \quad r^2} \int_M \tau |\nabla^2 f + Rc \quad \frac{1}{2\tau} g|^2 \, d\nu_t dt < \ \ , \\ \sup_{[t_0 \quad {}^{1}r^2, t_0 \quad r^2]} \int_M |\tau(2\Delta f \quad |\nabla f|^2 + R) + f \quad n \\ \inf_{M\times[t_0 \quad {}^{1}r^2, t_0 \quad r^2]} R \geq \quad r^{\ \ 2}, \end{cases}$$

and $(\ \ , r)$-static if:

$$\begin{cases} \int_{t_0 \quad {}^{1}r^2}^{t_0 \quad r^2} \int_M r^2 |Rc|^2 \, d\nu_t dt < \ \ , \\ \sup_{[t_0 \quad {}^{1}r^2, t_0 \quad r^2]} \int_M R < \ \ , \\ \inf_{M\times[t_0 \quad {}^{1}r^2, t_0 \quad r^2]} R \geq \quad r^{\ \ 2}. \end{cases}$$

Limits that are Ricci Flat Cones

Proposition *(amler 2021)*

Let $(M_i^n, g_i(t), d\nu_{x^i,0;s})$ be a sequence of closed RFs, $(x^i,$
$(\ _i, r)$-selfsimilar and $(\ _i, r)$-static for $\ _i \to 0$, $W_i = \mathcal{N}_{x^i,0}$

$$(M_i, g_i(t), d\nu_{x^i,0;s}) \xrightarrow{d} (C(\mathbb{S}^n/ \ ), d\nu_{x^*;s}$$

where $C(\mathbb{S}^n/ \ )$ is a Ricci flat cone with link $\mathbb{S}^n/$ and ver

Moreover, if $d\nu_{x^i,0;s} = (4\pi\tau) \ ^{n/2} e \ ^{f_i} dg_t$, then there is a
embeddings $\varphi_i : V_i \subset C(\mathbb{S}^n/ \ )\backslash\{x^*\} \to M_i$ such that

$$4\tau(f_i \circ \varphi_i \quad W_i) \to d^2(x^*, \quad)$$

locally smoothly on $C(\mathbb{S}^n/ \ )\backslash\{x^*\}$.

Statement of Results

Theorem *(F.- Hallgren 2024)*

Let $Y, \sigma > 0$. Then there exists $\bar{\epsilon} = \bar{\epsilon}(\sigma, Y)$ such that the
statement holds for all $\epsilon \leq \bar{\epsilon}$. Let $(M^4, g(t)_{t\in(\ \ T,0]})$ be a
and (x_0, t_0) be a point that is both (ϵ, r)-selfsimilar and (
some scale $r > 0$. Suppose that $\mathcal{N}_{x_0,t_0}(r^2) \geq \quad Y$, and th

$$\int_{2r^2}^{r^2} \int_{(P^*)_t \ (x_0,t_0;\epsilon \ \ ^1 r)} r^2 \ \ ^n |Rm|^2 dg_t dt <$$

Then one of the following holds:

1. The curvature scale satisfies $r_{Rm} \geq \epsilon r$.
2. We have $d \ ((M, g(t), d\nu_{x_0,t_0;s}), C(L_{p,q})) < r\sigma$ where
$q^2 \equiv \quad 1 \,(\mathrm{mod}\ p)$ is an exceptional Lens space.

Remark: (2) does not occur in the Kähler setting.

Statement of Results

Theorem *(F.- Hallgren 2024)*

Let $Y, \sigma > 0$. Then there exists $\overline{\epsilon} = \overline{\epsilon}(\sigma, Y)$ such that the
statement holds for all $\epsilon \leq \overline{\epsilon}$. Let $(M^4, g(t)_{t\in(\quad T,0]})$ be a
and (x_0, t_0) be a point that is both (ϵ, r)-selfsimilar and (
some scale $r > 0$. Suppose that $\mathcal{N}_{x_0,t_0}(r^2) \geq \quad Y$, and th

$$\int_{2r^2}^{r^2} \int_{(P^*)_t \; (x_0,t_0;\epsilon \;\; ^1 r)} r^{2 \quad n} |Rm|^2 dg_t dt <$$

Then one of the following holds:

1. The curvature scale satisfies $r_{Rm} \geq \epsilon r$.
2. We have $d\;((M, g(t), d\nu_{x_0,t_0;s}), C(L_{p,q})) < r\sigma$ where
$q^2 \equiv \quad 1\,(\text{mod } p)$ is an exceptional Lens space.

Remark: (2) does not occur in the Kähler setting.

Idea of Proof

Slicing rgument *(Cheeger, Colding, Tian 2002)*

1. **rgue by contradiction:** $\exists\, \sigma, Y > 0$, $(M^i, g_i, d\nu_{x_i,0}$ conditions of the proposition for $\epsilon_i \to 0$ s.t. conclusio
2. **Pass to a limit:** Static cone $(C(\mathbb{S}^3/\ ), d\nu_{x^*;s})$, suffi $\mathbb{S}^3/\ \cong L(p,q)$,
3. **Slice:** Consider level sets of approximate distance fu

$$\tilde{q}_i := 4\tau(f_i \circ \varphi_i + W_i) \xrightarrow{C^\infty_{loc}(\mathcal{R})} d^2(x^*,$$

$$\tilde{\Sigma}_{i,r} := q_i^{\ \ 1}(r) \xrightarrow{C^\infty_{loc}(\mathcal{R})} \mathbb{S}^3/\ ,$$

4. **Idea:** Small L^2-curvature will contradict some proper Chern-Simon invariant associated to the first Pontrya

Idea of Proof

Slicing rgument *(Cheeger, Colding, Tian 2002)*

1. **rgue by contradiction:** $\exists\,\sigma, Y > 0$, $(M^i, g_i, d\nu_{x_i,0}$
conditions of the proposition for $\epsilon_i \to 0$ s.t. conclusio
2. **Pass to a limit:** Static cone $(C(\mathbb{S}^3/\ \ ), d\nu_{x^*;s})$, suffi
$\mathbb{S}^3/\ \ \cong L(p,q)$,
3. **Slice:** Consider level sets of approximate distance fu

$$\tilde{q}_i := 4\tau(f_i \circ \varphi_i + W_i) \xrightarrow{C^\infty_{loc}(\mathcal{R})} d^2(x^*,$$

$$\tilde{\Sigma}_{i,r} := q_i^{\ \ 1}(r) \xrightarrow{C^\infty_{loc}(\mathcal{R})} \mathbb{S}^3/\ \ ,$$

4. **Idea:** Small L^2-curvature will contradict some proper
Chern-Simon invariant associated to the first Pontrya

Idea of Proof

Slicing rgument *(Cheeger, Colding, Tian 2002)*

1. **rgue by contradiction:** $\exists\,\sigma, Y > 0$, $(M^i, g_i, d\nu_{x_i,0}$ conditions of the proposition for $\epsilon_i \to 0$ s.t. conclusio
2. **Pass to a limit:** Static cone $(C(\mathbb{S}^3/\ \ ), d\nu_{x^*;s})$, suffi $\mathbb{S}^3/\ \ \cong L(p,q)$,
3. **Slice:** Consider level sets of approximate distance fu

$$\tilde{q}_i := 4\tau(f_i \circ \varphi_i + W_i) \overset{C^\infty_{loc}(\mathcal{R})}{\longrightarrow} d^2(x^*,$$

$$\tilde{\Sigma}_{i,r} := q_i^{\ 1}(r) \overset{C^\infty_{loc}(\mathcal{R})}{\longrightarrow} \mathbb{S}^3/\ \ ,$$

4. **Idea:** Small L^2-curvature will contradict some proper Chern-Simon invariant associated to the first Pontrya

Idea of Proof

Slicing rgument *(Cheeger, Colding, Tian 2002)*

1. **rgue by contradiction:** $\exists\, \sigma, Y > 0$, $(M^i, g_i, d\nu_{x_i,0}$
   conditions of the proposition for $\epsilon_i \to 0$ s.t. conclusio
2. **Pass to a limit:** Static cone $(C(\mathbb{S}^3/\ \ ), d\nu_{x^*;s})$, suffi
   $\mathbb{S}^3/\ \ \cong L(p,q)$,
3. **Slice:** Consider level sets of approximate distance fu

$$\tilde{q}_i := 4\tau(f_i \circ \varphi_i + W_i) \overset{C^\infty_{loc}(\mathcal{R})}{\longrightarrow} d^2(x^*,$$

$$\tilde{\Sigma}_{i,r} := q_i\ ^1(r) \overset{C^\infty_{loc}(\mathcal{R})}{\longrightarrow} \mathbb{S}^3/\ \ ,$$

4. **Idea:** Small L^2-curvature will contradict some proper
   Chern-Simon invariant associated to the first Pontrya

Idea of Proof

Slicing rgument *(Cheeger, Colding, Tian 2002)*

1. **rgue by contradiction:** $\exists\,\sigma, Y > 0$, $(M^i, g_i, d\nu_{x_i,0}$
conditions of the proposition for $\epsilon_i \to 0$ s.t. conclusio
2. **Pass to a limit:** Static cone $(C(\mathbb{S}^3/\ \ ), d\nu_{x^*;s})$, suffi
$\mathbb{S}^3/\ \ \cong L(p,q)$,
3. **Slice:** Consider level sets of approximate distance fu

$$\tilde{q}_i := 4\tau(f_i \circ \varphi_i + W_i) \overset{C^\infty_{loc}(\mathcal{R})}{\longrightarrow} d^2(x^*,$$

$$\tilde{\Sigma}_{i,r} := q_i^{\ \ 1}(r) \overset{C^\infty_{loc}(\mathcal{R})}{\longrightarrow} \mathbb{S}^3/\ \ ,$$

4. **Idea:** Small L^2-curvature will contradict some proper
Chern-Simon invariant associated to the first Pontrya

Idea of Proof

Definition (Secondary Invariant ssociated to $_1$)

Let $\pi : E \to M^3$ be a vector bundle with connection 1-for curvature 2-form Ω. Define:

$$\hat{p}_1(E) := \int_M \mathrm{Tr}(d \quad \wedge \quad + \frac{2}{3} \quad \wedge \quad \wedge \quad)\ (\mathrm{mo}$$

1. **Boundary formula:** If $M^3 = \partial N^4$ then:

$$\hat{p}_1(TN|_M) = \int_N p_1(N) = \quad \frac{1}{4\pi^2} \int_N \mathrm{Tr}($$

2. $\hat{p}_1(T(\mathbb{S}^3/\quad)) = 0 \iff \mathbb{S}^3/\quad \cong L(p,q),\ q^2 \equiv \quad 1\ (\mathrm{m}$
3. If M^4 is Kähler, we instead use a similar invariant ass

Idea of Proof

Definition (Secondary Invariant ssociated to $_1$)

Let $\pi : E \to M^3$ be a vector bundle with connection 1-for curvature 2-form Ω. Define:

$$\hat{p}_1(E) := \int_M \mathrm{Tr}(d \quad \wedge \quad + \frac{2}{3} \quad \wedge \quad \wedge \quad) \ (\mathrm{mo}$$

1. **Boundary formula:** If $M^3 = \partial N^4$ then:

$$\hat{p}_1(TN|_M) = \int_N p_1(N) = \quad \frac{1}{4\pi^2} \int_N \mathrm{Tr}($$

2. $\hat{p}_1(T(\mathbb{S}^3/ \quad)) = 0 \iff \mathbb{S}^3/ \quad \cong L(p,q),\ q^2 \equiv \quad 1\ (\mathrm{m}$
3. If M^4 is Kähler, we instead use a similar invariant ass

Idea of Proof

Definition (Secondary Invariant ssociated to $_1$)

Let $\pi : E \to M^3$ be a vector bundle with connection 1-for
curvature 2-form Ω. Define:

$$\hat{p}_1(E) := \int_M \mathrm{Tr}(d \quad \wedge \quad + \frac{2}{3} \quad \wedge \quad \wedge \quad) \text{ (mo}$$

1. **Boundary formula:** If $M^3 = \partial N^4$ then:

$$\hat{p}_1(TN|_M) = \int_N p_1(N) = \quad \frac{1}{4\pi^2} \int_N \mathrm{Tr}($$

2. $\hat{p}_1(T(\mathbb{S}^3/ \quad)) = 0 \iff \mathbb{S}^3/ \quad \cong L(p,q),\ q^2 \equiv \quad 1 \text{ (m}$
3. If M^4 is Kähler, we instead use a similar invariant ass

Idea of Proof

Definition (Secondary Invariant ssociated to $_1$)

Let $\pi : E \to M^3$ be a vector bundle with connection 1-for
curvature 2-form Ω. Define:

$$\hat{p}_1(E) := \int_M \mathrm{Tr}(d \quad \wedge \quad + \frac{2}{3} \quad \wedge \quad \wedge \quad) \text{ (mo}$$

1. **Boundary formula:** If $M^3 = \partial N^4$ then:

$$\hat{p}_1(TN|_M) = \int_N p_1(N) = \quad \frac{1}{4\pi^2} \int_N \mathrm{Tr}($$

2. $\hat{p}_1(T(\mathbb{S}^3/ \quad)) = 0 \iff \mathbb{S}^3/ \quad \cong L(p,q),\ q^2 \equiv \quad 1 \text{ (m}$
3. If M^4 is Kähler, we instead use a similar invariant ass

Idea of Proof

Setup: For $\tilde{q}_i = 4\tau(f_i \circ \varphi_i + W_i)$, $\tilde{\Sigma}_{i,r} := \tilde{q}_i^{\ -1}(r)$ we have

$$\begin{cases} \tilde{q}_i \xrightarrow{C^\infty_{loc}(\mathcal{R}\backslash x^*)} d^2(x^*, \ ), \\ \tilde{\Sigma}_{i,r} \xrightarrow{C^\infty_{loc}(\mathcal{R}\backslash x^*)} \mathbb{S}^3/ \ , \\ \hat{p}_1(TM^i|_{\tilde{\ }_{i\,r}}) \to \hat{p}_1(\mathbb{S}^3/ \ ). \end{cases}$$

ssumption: $\varphi_i^*(\tilde{\Sigma}_{i,r}) = \partial S_{i,r}$ for some $S_{i,r} \subset P^*(x_0, t_0;$
Since

$$\int_{2r^2}^{r^2} \int_{(P^*)_t\ (x_0,t_0;\epsilon_i^{\ 1} r)} r^2\ {}^n |Rm|^2 dg_t dt <$$

the boundary formula gives that $\hat{p}_1(TM^i|_{\tilde{\ }_{i\,r}}) \to 0$, so $\mathbb{S}^3$
exceptional Lens space.

Idea of Proof

Setup: For $\tilde{q}_i = 4\tau(f_i \circ \varphi_i + W_i)$, $\tilde{\Sigma}_{i,r} := \tilde{q}_i^{\ \ 1}(r)$ we hav

$$\begin{cases} \tilde{q}_i \xrightarrow{C^\infty_{loc}(\mathcal{R}\backslash x^*)} d^2(x^*, \quad), \\ \tilde{\Sigma}_{i,r} \xrightarrow{C^\infty_{loc}(\mathcal{R}\backslash x^*)} \mathbb{S}^3/ \quad , \\ \hat{p}_1(TM^i|_{\tilde{\ \ }_{i\ r}}) \to \hat{p}_1(\mathbb{S}^3/ \quad). \end{cases}$$

ssumption: $\varphi_i^*(\tilde{\Sigma}_{i,r}) = \partial S_{i,r}$ for some $S_{i,r} \subset P^*(x_0, t_0;$

Since

$$\int_{2r^2}^{r^2} \int_{(P^*)_t \ (x_0,t_0;\epsilon_i \ ^1 r)} r^2 \ \ ^n |Rm|^2 dg_t dt <$$

the boundary formula gives that $\hat{p}_1(TM^i|_{\tilde{\ \ }_{i\ r}}) \to 0$, so $\mathbb{S}^3$

exceptional Lens space.

Idea of Proof

Setup: For $\tilde{q}_i = 4\tau(f_i \circ \varphi_i + W_i)$, $\tilde{\Sigma}_{i,r} := \tilde{q}_i\ ^{1}(r)$ we hav

$$\begin{cases} \tilde{q}_i \xrightarrow{C^\infty_{loc}(\mathcal{R}\backslash x^*)} d^2(x^*, \quad), \\ \tilde{\Sigma}_{i,r} \xrightarrow{C^\infty_{loc}(\mathcal{R}\backslash x^*)} \mathbb{S}^3/ \quad , \\ \hat{p}_1(TM^i|_{\tilde{\ }_{i\ r}}) \to \hat{p}_1(\mathbb{S}^3/ \quad). \end{cases}$$

ssumption: $\varphi_i^*(\tilde{\Sigma}_{i,r}) = \partial S_{i,r}$ for some $S_{i,r} \subset P^*(x_0, t_0;$
Since

$$\int_{2r^2}^{r^2} \int_{(P^*)_t\ (x_0,t_0;\epsilon_i\ ^1 r)} r^2\ \ ^n|Rm|^2 dg_t dt <$$

the boundary formula gives that $\hat{p}_1(TM^i|_{\tilde{\ }_{i\ r}}) \to 0$, so $\mathbb{S}^3$
exceptional Lens space.

Justifying the ssumption

ssumption: $\varphi_i^*(\tilde{\Sigma}_{i,r}) = \partial S_{i,r}$ for some $S_{i,r} \subset P^*(x_0, t_0;$

1. This is not an issue in the bounded Ricci curvature s
   1. The approximate distance functions r_i^2 are construct elliptic equations on a ball.
2. The conjugate heat kernels $d\nu_{x_i,t_i;s}$ are constructed and may be badly behaved "far away" from (x_i, t_i).

Justifying the ssumption

ssumption: $\varphi_i^*(\tilde{\Sigma}_{i,r}) = \partial S_{i,r}$ for some $S_{i,r} \subset P^*(x_0, t_0;$

1. This is not an issue in the bounded Ricci curvature s
   1. The approximate distance functions r_i^2 are construct elliptic equations on a ball.
2. The conjugate heat kernels $d\nu_{x_i,t_i;s}$ are constructed and may be badly behaved "far away" from (x_i, t_i).

Justifying the ssumption

ssumption: $\varphi_i^*(\tilde{\Sigma}_{i,r}) = \partial S_{i,r}$ for some $S_{i,r} \subset P^*(x_0, t_0;$

1. This is not an issue in the bounded Ricci curvature s
   1. The approximate distance functions r_i^2 are construct elliptic equations on a ball.
2. The conjugate heat kernels $d\nu_{x_i,t_i;s}$ are constructed g and may be badly behaved "far away" from (x_i, t_i).

Justifying the ssumption

Lemma *(F.- Hallgren 2024)*

Let $\epsilon, Y > 0$. There exists $(Y), \Lambda(Y), (Y,\epsilon) > 0$ such
following holds: If $(M^4, g(t)_{t\in(T,0]})$ is a closed RF, $(x_0,$
and $(, r)$-selfsimilar, $\mathcal{N}_{x_0,t_0}(r^2) \geq Y$. Then there is a
component Σ of $q^{1}( )$ such that:

1. Σ is ϵ-close in the C^∞_{loc}-topology to $\mathbb{S}^3/$.
2. Σ separates M into two connected components M'
3. $\partial M' = \Sigma$ and $M' \subset P^*(x_0, t_0; \Lambda r)$.

Justifying the ssumption

Lemma (Gaussian Isoperimetric Inequality)

Set $\Phi(x) := \int_{\infty}^{x} (4\pi) \ ^{\frac{1}{2}} e \ ^{x^2} dx$ and $\eta = \Phi' \circ \Phi \ ^{1}$. Then
open with smooth boundary, we have

$$\eta(\nu_{x,0;t}(\mathcal{A})) \leq \sqrt{\ \ t} \int_{\partial \mathcal{A}} K(x, 0; \ \ , t) d\mathcal{H}^n$$

1. pplying to the annulus $\mathcal{A} := \varphi(B(x^*, \epsilon \ ^{1}) \setminus \overline{B(x^*, \epsilon)}$

$$\nu_{x,0;t}(\mathcal{A}) \approx \begin{matrix} 1 \\ 0 \end{matrix}$$

if x is "far away" from $\varphi(\partial B(x^*, \epsilon \ ^{1}))$.

2. These facts will contradict continuity of $\nu_{x,0;t}(\mathcal{A})$ alc
curve.

Justifying the ssumption

Lemma (Gaussian Isoperimetric Inequality)

Set $\Phi(x) := \int_{\infty}^{x} (4\pi)^{\frac{1}{2}} e^{x^2} dx$ and $\eta = \Phi' \circ \Phi^{1}$. Then
open with smooth boundary, we have

$$\eta(\nu_{x,0;t}(\mathcal{A})) \leq \sqrt{ t} \int_{\partial \mathcal{A}} K(x, 0; , t) d\mathcal{H}^n$$

1. pplying to the annulus $\mathcal{A} := \varphi(B(x^*, \epsilon^{1}) \setminus \overline{B(x^*, \epsilon)}$

$$\nu_{x,0;t}(\mathcal{A}) \approx \begin{matrix} 1 \\ 0 \end{matrix}$$

   if x is "far away" from $\varphi(\partial B(x^*, \epsilon^{1}))$.

2. These facts will contradict continuity of $\nu_{x,0;t}(\mathcal{A})$ alo
   curve.

Justifying the ssumption

Lemma (Gaussian Isoperimetric Inequality)

Set $\Phi(x) := \int_{\infty}^{x}(4\pi)^{\ \frac{1}{2}}e^{\ x^2}\,dx$ and $\eta = \Phi' \circ \Phi^{\ 1}$. Then
open with smooth boundary, we have

$$\eta(\nu_{x,0;t}(\mathcal{A})) \leq \sqrt{\ \ t}\int_{\partial\mathcal{A}} K(x,0;\ \ ,t)\,d\mathcal{H}^n$$

1. pplying to the annulus $\mathcal{A} := \varphi(B(x^*,\epsilon^{\ 1})\backslash\overline{B(x^*,\epsilon)}$

$$\nu_{x,0;t}(\mathcal{A}) \approx \begin{matrix} 1 \\ 0 \end{matrix}$$

if x is "far away" from $\varphi(\partial B(x^*,\epsilon^{\ 1}))$.

2. These facts will contradict continuity of $\nu_{x,0;t}(\mathcal{A})$ alo
curve.

Thank You for istenin