Ricci Flow and the Curvature Operator of the Second Kind

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Outline

Hamilton's Ricci Flow

Definition: Ricci Flow (Hamilton '82)

Let (M, g_0) be a Riemannian manifold. The equation

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij} + (\frac{2\int_M R d\mu}{nVol(M)} g_{ij}), \ g(0) = g_0$$

is known as the (normalized) Ricci flow. A pair $(M, g(t))$ where M is a closed manifold and $g(t)$ is a solution to the Ricci flow is known as a compact Ricci flow.

Short Time Existence

Theorem (Hamilton '82)

For any closed Riemannian manifold (M, g_0), there is a unique smooth short time solution $g(t), t \in [0, \delta)$ to the Ricci flow equation with $g(0) = g_0$.

Remark

The Ricci tensor is not elliptic:

$$\sigma D(-2Rc)_\zeta(\nu) = |\zeta|^2 \nu_{ij} + \zeta_i \zeta_j Tr(\nu) - \zeta_i \zeta_k \nu_{kj} - \zeta_j \zeta_k \nu_{ki}.$$

That is, the Ricci flow is not a strictly parabolic flow.

Short Time Existence

Theorem (DeTurck '83)

Let (M, g_0) be a closed Riemannian manifold. There is a strictly parabolic flow

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij} + \mathcal{L}_X g_{ij}, \ g(0) = g_0 \tag{1}$$

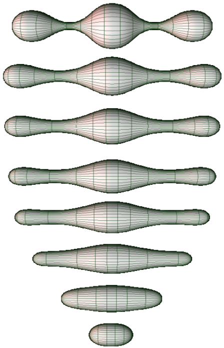
and a family of diffeomorphisms $\varphi(t) : M \to M$, such that if $g(t)$ is the unique smooth short time solution to (1) with $g(0) = g_0$, then $\varphi^* g(t)$ solves the Ricci flow equation with initial date g_0.

Intuition for the Ricci Flow

Ricci Flow as a Heat Equation

In geodesic normal coordinates centered at a point p, we have

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij} = 3\Delta_{\text{Euc}}(g_{ij}).$$



Idea of the Ricci Flow

1. Ricci Flow should behave similarly to the heat equation for the metric.
2. Ricci flow should "improve" apriori badly behaved metrics.
3. Use this fact to gain insight into the topology of the manifold.

Singularity Formation in Ricci Flow

Singularity Formation

For a compact Ricci flow $(M, g(t))$ with $R(0) > 0$, $\exists\ T \in (0, \infty)$ s.t.

$$\lim_{t \nearrow T} |Rm|_{g(t)} = \infty.$$

Modelling the Singularity

For $K_i := |Rm(x_i, t_i)| \nearrow \infty$, $t_i \nearrow T$, we aim to study the limit of:

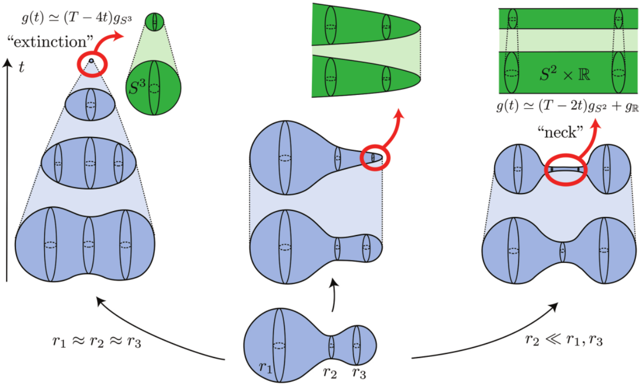
$$(M, g_i(t), x_i), \;\; g_i(t) = K_i g(t_i + K_i^{-1} t).$$

If, in the C^∞ pointed sense of Cheeger and Gromov, we have

$$(M, g_i(t), x_i) \to (M_\infty, g_\infty(t), x_\infty)$$

we call $(M_\infty, g_\infty(t))$ a singularity model for the flow.

Intuitive Solutions of the Ricci Flow



The Ricci Flow on Surfaces

The Uniformization Theorem (Poincaré)

Every closed Riemann surface contains a metric in its conformal class which is locally isometric to one of the 3 model geometries: $S^2, \mathbb{R}^2$, or $\mathbb{H}^2$.

Remarks

1. This is equivalent to the existence of a metric of constant sectional curvature in the conformal class.
2. In dimension 2, Ricci flow is equivalent to the scalar PDE:

$$\frac{\partial}{\partial t} u = \Delta_{g_0} log(u) - R_{g_0},$$

where $u \in C^\infty(M)$. In particular, Ricci flow preserves the conformal class of the metric.

The Ricci Flow on Surfaces

Theorem (Hamilton '88+ Chow '91)

For any closed (M^2, g_0), the solution to the normalized Ricci flow exists for all time and converges to a metric of constant sectional curvature.

Remarks

1. Hamilton '88: Case of $\chi(M) \leq 0$ and $\chi(M) > 0$ with $sec_{g_0} \geq 0$.
2. Chow '91: Case of $\chi(M) > 0$ with arbitrary initial metric.
3. Hamilton and Chow's proof required uniformization theorem.
4. (Chen, Lu+ Tian '06): Give a Ricci flow proof of the uniformization theorem.

The Ricci Flow on 3-Manifolds

Thurston's Geometrization Conjecture

Every closed prime 3-manifold has a canonical geometric structure so that when cut along tori, each tori is locally isometric to one of 8 model geometries.

Remarks

1. Implies, as a corollary, the Poincaré conjecture:

   Every closed simply connected 3-manifold is homeomorphic to S^3.

2. A proof of Thurston's conjecture was the long term goal of Hamilton's Ricci flow programme.
3. Resolved by Perelman in 2003 using Ricci flow.

The Ricci Flow on 3-Manifolds

Theorem (Hamilton '82)

For any closed (M^3, g_0) with $Rc > 0$, the solution to the normalized Ricci flow exists for all time and converges to a metric of constant positive sectional curvature. Hence, $M^3 \cong S^3/\Gamma$ for some $\Gamma \in \mathrm{Iso}(S^3)$.

Theorem (Perelman '03)

Let $(M_\infty, g_\infty(t))$ be a singularity model of a 3-dimensional compact Ricci flow. Then there are sequences $\lambda_i, \beta_i \in \mathbb{R}$, points $x_i \in M_\infty$, and times $t_i \to -\infty$ such that the rescaled solutions

$$(M_\infty, g_i(t), x_i),\ g_i(t) = \lambda_i g(t_i + \beta_i t)$$

converge to the standard solution on either S^3/Γ or $(S^2 \times \mathbb{R})/\Gamma$.

Ricci Flow in Higher Dimensions

Remarks on $n \geq 4$

1. Singularities of the Ricci flow for dimensions $n \geq 4$ are significantly more complicated.
2. Understanding singularity formation in dimension 4 is a large contemporary area of research.
3. To attain any general results, one needs to impose strong curvature assumptions on the initial metric.

The Curvature Operator

Definition: Curvature Operator

Let (M, g) be a Riemannian manifold. The curvature tensor R_{ijkl} defines a bundle map on the space of 2-forms:

$$
\begin{aligned}
Rm : \wedge^2 T^* M &\to \wedge^2 T^* M \\
e^i \wedge e^j &\to R_{ijkl} e^k \wedge e^l
\end{aligned}
$$

known as the curvature operator.

Definition: Positive Curvature Operator

We say Rm is positive ($Rm > 0$) if $Rm|_p$ has strictly positive eigenvalues for each $p \in M$. Note that

$$Rm > 0 \Rightarrow sec > 0.$$

The Curvature Operator

A Classical Problem

Which manifolds admit metrics with positive curvature operator?

Examples

1. If $n = 2$, Gauss Bonnet implies $M = S^2$ or $M = \mathbb{RP}^2$.
2. If $M^n = S^n/\Gamma$ with the standard round metric, then $Rm = \mathrm{Id} > 0$.
3. Some non-examples:

$$Rm_{S^2 \times S^1} = \begin{pmatrix} 0 & & \\ & 0 & \\ & & 1 \end{pmatrix}, \quad Rm_{\mathbb{CP}^2} = \begin{pmatrix} 0 & & & & & \\ & 0 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix}.$$

A Space Form Conjecture

Let (M, g) is a closed Riemannian manifold with $Rm > 0$. Then M is diffeomorphic to a spherical spherical space form. That is, $M \cong S^n/\Gamma$ for some $\Gamma \in \mathrm{Iso}(S^n)$.

The Curvature Operator

Hamilton's Conjecture

Let (M, g) be a closed Riemannian manifold with $Rm > 0$. Then the solution to the normalized Ricci flow exists for all time and converges to a metric of constant positive sectional curvature. Hence, $M \cong S^n/\Gamma$.

The Curvature Operator Under Ricci Flow

The Lie Algebra Structure of $\wedge^2 T^* M$

Let (M, g) be a Riemannian manifold. For each $p \in M$, the fiber $\wedge^2 T_p^* M \cong \mathfrak{so}(n)$ as a Lie algebra, with bracket defined by:

$$[U, V]_{ij} = g^{kl}(U_{ik} V_{lj} - V_{ik} U_{lj})$$

Fix a basis φ^α of $\wedge^2 T_p^* M$. The Lie algebra square of Rm is the operator

$$\begin{aligned} Rm^{\#} : \wedge^2 T^* M \to \wedge^2 T^* M \\ (Rm^{\#})_{\alpha\beta} = C_\alpha^{\gamma\delta} C_\beta^{\epsilon\zeta} Rm_{\gamma\epsilon} Rm_{\delta\zeta}, \end{aligned}$$

where $C_\alpha^{\gamma\delta}$ are the structure constants for the bracket in the given basis.

The Curvature Operator Under Ricci Flow

The Evolution Equation for Rm

Let $(M, g(t))$ be a compact Ricci flow. The Riemann curvature tensor satisfies the reaction diffusion equation

$$\frac{\partial}{\partial t} R_{ijkl} = \Delta R_{ijkl} + 2(R_{pijq}R_{qklp} + R_{pilq}R_{qkjp} - R_{pijq}R_{qlkp} - R_{pikq}R_{qljp}).$$

The reaction terms can be grouped so that the curvature operator satisfies

$$\frac{\partial}{\partial t} Rm = \Delta Rm + Rm^2 + Rm^{\#}.$$

The Curvature Operator Under Ricci Flow

Hamilton's ODE→PDE Maximum Principle

Let (M, g) be a Riemannian manifold, $\pi : E \to M$ be a Hermitian vector bundle with compatible connection, and $K \subset E$ be a closed, convex subset which is invariant under parallel translation. For a section $f(e) \in \Gamma(E)$, consider the non-linear PDE

$$\frac{\partial}{\partial t} e(t) = \Delta e + f(e(t)). \tag{2}$$

Suppose that the subset K is preserved by the ODE

$$\frac{d}{dt} e = f(e(t)).$$

Then the same is true for solutions to (2).

The Curvature Operator Under Ricci Flow

The ODE→PDE Maximum Principle for Rm

Let $(M, g(t))$ be a compact Ricci flow. Recall that the curvature operator satisfies

$$\frac{\partial}{\partial t} Rm = \Delta Rm + Rm^2 + Rm^{\#}.$$

Let $K \subset S^2_B(\wedge^2 T^* M)$ be a closed, convex subset invariant under parallel translation. Suppose that solutions to

$$\frac{d}{dt} A = A^2 + A^{\#}$$

which begin in K, remain in K. Then the same is true for Rm under the Ricci flow.

The Curvature Operator Under Ricci Flow

Theorem (Huisken '85)

Let $(M, g(t))$ be a compact Ricci flow such that $R(0) > 0$. Suppose $\exists \delta > 0$ s.t. the estimate

$$|\tilde{Rm}| \leq R^{1-\delta}$$

holds $\forall t \in [0, T)$. Then the solution to the normalized Ricci flow exists for all time and converges to a metric of constant positive sectional curvature.

The Curvature Operator Under Ricci Flow

Definition: Pinching Sets

A subset $K \subset S^2_B(\wedge^2 T^*M)$ is called a pinching set if it is closed, convex, invariant under parallel translation, preserved by the ODE

$$\frac{d}{dt}A = A^2 + A^{\#}$$

and satisfies the pinching estimate

$$|\tilde{A}| \leq |A|^{1-\delta}$$

for some $\delta > 0$ and all $A \in K$. An open subset $U \subset S^2_B(\wedge^2 T^*M)$ satisfies the pinching condition if every compact subset $K \subset U$ is contained in a pinching set.
In particular, the normalized Ricci flow evolves a manifold (M, g) with $R > 0$ into one of constant positive sectional curvature if $Rm \in U$ where U satisfies the pinching condition.

The Curvature Operator Under Ricci Flow

The ODE in Dimension 3

The Lie algebra square $Rm^{\#}$ is the adjoint matrix and the ODE reduces to the system

$$\frac{d}{dt}\lambda_i = \lambda_i^2 + \lambda_j\lambda_k,$$

where $\lambda_1, \lambda_2, \lambda_3$ are the eigenvalues of Rm.

Hamilton studied this system to prove his conjecture for $n = 3$.

The Curvature Operator Under Ricci Flow

The ODE in Dimension 4

There is a splitting $\bigwedge^2 = \bigwedge^2_+ \oplus \bigwedge^2_-$ by eigenspaces of the Hodge star operator so that

$$Rm = \begin{pmatrix} A & B \\ B^t & C \end{pmatrix}, Rm^\# = \begin{pmatrix} A^\# & B \\ (B^\#)^t & C^\# \end{pmatrix}$$

where A, B, C are 3×3 matrices and $A^\#$ is the adjoint matrix. The ODE then reduces to a system of 3×3 matrix ODE e.g.

$$\frac{d}{dt}A = A^2 + BB^t + A^\#.$$

Hamilton studied this system to prove his conjecture for $n = 4$.

The Curvature Operator Under Ricci Flow

Theorem (Böhm+Wilking '08)

Let (M, g) be a closed Riemannian manifold with 2 positive curvature operator. Then the solution to the normalized Ricci flow equation exists for all time and converges to a metric of constant positive sectional curvature. Hence, $M \cong S^n/\Gamma$ is diffeomorphic to a spherical space form.

Remark

We say a linear operator $A \in \mathrm{End}(\mathbb{R}^n)$ is m positive if the sum of the first m eigenvalues of A is positive.

The 2^{nd} Curvature Operator

Definition: The 2^{nd} Curvature Operator

Let (M, g) be a Riemannian manifold. The curvature tensor R_{ijkl} defines a bundle map on the space of symmetric 2-tensors:

$$\overline{Rm} : S^2(T^*M) \to S^2(T^*M)$$
$$e_i \odot e_j \to R_{kilj} e_k \odot e_l$$

Let $\pi : S^2(T^*M) \to S_0^2(T^*M)$ denote the projection onto the subbundle of traceless 2-forms. The operator

$$\mathring{Rm} = \pi \circ \overline{Rm}|_{S_0^2} : S_0^2(T^*M) \to S_0^2(T^*M)$$

is known as the curvature operator of the second kind.

The 2^{nd} Curvature Operator

Restricting to $S_0^2(T^*M)$

1. There a splitting
$$S^2(T^*M) = S_0^2(T^*M) \oplus \mathbb{R}g$$
into $O(n)$-invariant subbundles.
2. For the round metric $g_{\mathbb{S}^n}$ on S^n,
$$\overline{Rm}_{g_{\mathbb{S}^n}} : S^2(T^*M) \to S^2(T^*M)$$
is not positive. In particular, $\overline{Rm}|_{\mathbb{R}g}$ has eigenvalue $-(n-1)$.
3. Hence, we work with the restricted operator
$$\mathring{Rm} : S_0^2(T^*M) \to S_0^2(T^*M)$$

The 2^{nd} Curvature Operator

Remarks

1. We have

$$\mathring{Rm} > 0 \Rightarrow sec > 0.$$

2. In general, the relationship between Rm and $\mathring{Rm}$ is unclear e.g.

$$\mathring{Rm}_{\mathbb{CP}^2} = \begin{pmatrix} -\frac{1}{2}\mathrm{Id} & & \\ & \mathrm{Id} & \\ & & \mathrm{Id} \end{pmatrix}, Rm_{\mathbb{CP}^2} = \begin{pmatrix} 0 & & & & & \\ & 0 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix}.$$

The 2^{nd} Curvature Operator

Conjecture (Nishikawa '86)

Let (M, g) be a closed Riemannian manifold with $\mathring{Rm} > 0$. Then $M \cong S^n/\Gamma$ is diffeomorphic to a spherical space form.

Examples

1. If $n = 2$, Gauss Bonnet implies $M = S^2$ or $M = \mathbb{RP}^2$.
2. If $M^n = S^n/\Gamma$ with the standard round metric then $\mathring{Rm} = \mathrm{Id}$.
3. Some non-examples:

$$\mathring{Rm}_{S^2 \times S^1} = \begin{pmatrix} -\frac{1}{3} & & & & \\ & 0 & & & \\ & & 0 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}, \quad \mathring{Rm}_{\mathbb{CP}^2} = \begin{pmatrix} -\frac{1}{2}\mathrm{Id} & & \\ & \mathrm{Id} & \\ & & \mathrm{Id} \end{pmatrix}.$$

The 2^{nd} Curvature Operator

Theorem (Cao, Gursky, Tran '21)

Let (M^n, g) be a closed Riemannian manifold with 2-positive second curvature operator. Then M^n is diffeomorphic to a spherical space form.

The 2^{nd} Curvature Operator

Definition: Positive Isotropic Curvature

A Riemannian manifold $(M^n, g), n \geq 4$ is said to have positive isotropic curvature (PIC) if for all orthonormal 4-frames $\{e_1, e_2, e_3, e_4\}$:

$$R_{1331} + R_{1441} + R_{2332} + R_{2442} - 2R_{1234} > 0.$$

We say (M^n, g), $n \geq 3$ is PICk if $M \times \mathbb{R}^k$ has PIC.

Remark

We have the following implications:

$$2 \text{ positive } Rm \Rightarrow \text{PIC1} \Rightarrow \begin{cases} \text{PIC} \\ Rc > 0 \end{cases}$$

The 2^{nd} Curvature Operator

Theorem (Brendle '08)

Let (M^n, g_0), $n \geq 3$ be a closed, PIC1 Riemannian manifold. Then the solution to the normalized Ricci flow exists for all time converges to a metric of constant positive sectional curvature.

Remark

Since Rm is 2 positive $\Rightarrow$ PIC1, Brendle's result can be seen as a strengthening of the theorem of Böhm and Wilking.

The 2^{nd} Curvature Operator

Theorem (Cao, Gursky, Tran '21)

Let (M^n, g) be a closed Riemannian manifold with 2-positive second curvature operator. Then M^n is diffeomorphic to a spherical space form.

Idea of the Proof

Show algebraically that 2 positive $\mathring{Rm}$ implies positive PIC1, then exploit the convergence result of Brendle.

The 2^{nd} Curvature Operator

Definition: $(k+\epsilon)$ non-negative

A linear operator $A \in End(\mathbb{R}^n)$ is $(k+\epsilon)$ non-negative where $\epsilon \in [0,1]$ and $1 \leq k \leq n$ if

$$\lambda_1 + ... + \lambda_k + \epsilon\lambda_{k+1} \geq 0$$

for any eigenvalues $\lambda_1, ..., \lambda_{k+1}$ of A.

Note that setting $\epsilon = 0$ gives k non-negative and setting $\epsilon = 1$ gives $k+1$ non-negative.

The 2^{nd} Curvature Operator

Theorem (Li '22)

The assumption of Cao, Gursky, and Tran can be weakened to 3 positive. That is, if (M, g) is a closed manifold such that $\mathring{Rm}$ is 3 positive. Then M is diffeomorphic to a spherical space form.

For $n = 3$, the assumption can be weakened further to $3\frac{1}{3}$ positive $\mathring{Rm}$.

For $n = 4$, the assumption can be weakened even further to $4\frac{1}{2}$ positive $\mathring{Rm}$.

Conjecture (Li '22)

If (M^n, g) is a closed manifold with $(n + \frac{n-2}{n})$ positive second curvature operator, then M^n is diffeomorphic to a spherical space form.

The 2^{nd} Curvature Operator and Ricci Flow

The 2^{nd} Curvature Operator and Ricci Flow

1. The evolution of $\mathring{Rm}$ under the Ricci flow has not been studied.
2. This is natural to investigate c.f. the work of Hamilton et.al.
3. Fundamentally, positivity conditions of $\mathring{Rm}$ would need to be preserved by Ricci flow for this to provide anything useful.

Statement of Results

Main Theorem: Fluck and Li '23

Let $(M^3, g(t)), t \in [0, T)$ be a 3 dimensional compact Ricci flow such that $g(0)$ has α non-negative second curvature operator for some $\alpha \in [1, 5]$. Then $g(t)$ has α non-negative second curvature operator for all $t \in [0, T)$.

Statement of Results

Theorem: Fluck and Li '23

Let (M^3, g) be a 3-dimensional Riemannian manifold. The eigenvalues of $\mathring{Rm}$ are given by a, b, c and

$$\lambda_{\pm} = \frac{a+b+c}{3} \pm \frac{\sqrt{2}}{3}\sqrt{3(a^2+b^2+c^2)-(a+b+c)^2}$$

where $a \leq b \leq c$ denote the eigenvalues of Rm. Note we have the ordering

$$\lambda_- \leq a \leq b \leq c \leq \lambda_+.$$

Statement of Results

Proof

1. Recall that the Weyl tensor vanishes in dimension 3 so that:

$$Rm = S \owedge g := (Rc - \frac{R}{4}g) \owedge g$$

and the eigenvalues of S are

$$\frac{1}{2}(a+b-c) \leq \frac{1}{2}(a+c-b) \leq \frac{1}{2}(b+c-a).$$

2. Thus, the problem reduces to a general algebraic one of studying the second curvature operator of $A \owedge \mathrm{Id}$, where A has known eigenvalues.

Statement of Results

Algebraic Lemma: Fluck and Li '23

Let V be a finite dimensional real vector space and $A \in S^2(V)$. Then the eigenvalues of the algebraic second curvature operator of $A \owedge \mathrm{Id} \in S^2(\wedge^2 V)$ are given by

$$\begin{cases} \mu_i + \mu_j \text{ with multiplicity } n_i n_j \text{ where } 1 \leq i < j \leq k \\ 2\mu_i \text{ with multiplicity } n_i - 1 \text{ where } 1 \leq i \leq k \\ \text{the k-1 non-zero solutions of } \sum_{i=1}^{k} \frac{n_i \mu_i}{2\mu_i - \lambda} = \frac{n}{2} \end{cases}$$

where μ_i for $1 \leq i \leq k$ are the eigenvalues of A with multiplicity n_i.

Statement of Results

Proof

1. Apply the lemma to $S \owedge \mathrm{Id}$ where S has eigenvalues
$$\mu_1 = \frac{1}{2}(a+b-c), \mu_2 = \frac{1}{2}(a+c-b), \mu_3 = \frac{1}{2}(b+c-a).$$
2. Indeed, $\{\mu_i + \mu_j\}_{1 \leq i < j \leq 3} = \{a, b, c\}$ and one verifies that the 2 non-zero solutions of
$$\sum_{i=1}^{k} \frac{n_i \mu_i}{2\mu_i - \lambda} = \frac{3}{2}$$
are
$$\lambda_\pm = \frac{a+b+c}{3} \pm \frac{\sqrt{2}}{3}\sqrt{3(a^2+b^2+c^2)-(a+b+c)^2}.$$

Statement of Results

Corollary: Fluck and Li '23

Let (M^3, g) be a complete Riemannian manifold such that $\mathring{Rm}$ is $(3 + \delta)$ non-negative for some $\delta \in [0, \frac{1}{3}]$. Then there exists $\epsilon > 0$ such that

$$Rc \geq \epsilon R.$$

Consequently, any such manifold is either flat or a spherical space form.

Statement of Results

Remarks

1. It is already known due to Li that

$$\mathring{Rm} \text{ is } 3\frac{1}{3} \text{ non-negative} \Rightarrow Rm \text{ has non-negative Ricci curvature.}$$

2. There is already a classification of 3 manifolds with $Rc \geq 0$ due to Hamilton (compact case) and Liu (complete non-compact):

$$M^3 = \begin{cases} \mathbb{R}^3, (N^2 \times \mathbb{R})/\Gamma & \text{if } M^3 \text{ is non-compact.} \\ S^3/\Gamma, (S^2 \times \mathbb{R})/\Gamma & \text{if } M^3 \text{ is compact.} \end{cases} .$$

3. This result enhances this classification.

Statement of Results

Main Theorem: Fluck and Li '23

Let $(M^3, g(t)), t \in [0, T)$ be a 3 dimensional compact Ricci flow such that $g(0)$ has α non-negative second curvature operator for some $\alpha \in [1, 5]$. Then $g(t)$ has α non-negative second curvature operator for all $t \in [0, T)$.

Statement of Results

Proof

1. By the lemma, the eigenvalues of $\mathring{Rm}$ are a, b, c and

$$\lambda_{\pm} = \frac{a+b+c}{3} \pm \frac{\sqrt{2}}{3}\sqrt{3(a^2+b^2+c^2)-(a+b+c)^2}$$

   where $a \leq b \leq c$ are the eigenvalues of Rm.

2. By Hamilton's ODE→PDE maximum principle, it suffices to show that α non-negativity is preserved by the system of ODE's:

$$\begin{cases} \frac{da}{dt} & = \mathrm{a}^2 + bc \\ \frac{db}{dt} & = \mathrm{b}^2 + ac \\ \frac{dc}{dt} & = \mathrm{c}^2 + ab \end{cases}$$

   coming from $\frac{dS}{dt} = S^2 + S^{\#}$ in dimension 3.

Statement of Results

Proof

1. Define

$$f(\alpha) = \begin{cases} \lambda_- + (\alpha - 1)a, & \text{if } \alpha \in [1, 2), \\ \lambda_- + (3 - \alpha)a + (\alpha - 2)(a + b), & \text{if } \alpha \in [2, 3), \\ \lambda_- + \frac{R}{2} + (\alpha - 4)c, & \text{if } \alpha \in [3, 4), \\ \frac{R}{2} + \frac{R}{3}(\alpha - 4) + (5 - \alpha)\lambda_+ & \text{if } \alpha \in [4, 5]. \end{cases}$$

   so that

$$f(\alpha) \geq 0 \iff \mathring{Rm} \text{ is } \alpha \text{ non-negative.}$$

2. It suffices to show that $f(\alpha)$ is non-decreasing under the ODE.

Statement of Results

Proof

1. Under the ODE we have
$$\frac{dR}{dt} = |Ric|^2 \geq 0,$$
and so w.l.o.g we may assume that $R(t) > 0$.
2. It thus suffices to show $f(\alpha)/R$ is non-decreasing under the ODE.
3. Direct calculation shows that each component is non-decreasing, e.g.
$$\frac{d}{dt}(\frac{a}{R}) = \frac{2}{S^2}(b^2(c-a) + c^2(b-a)) \geq 0.$$

Open Problems

Conjecture: Preserving α Positivity in Arbitrary Dimensions

Let $(M^n, g(t)), t \in [0, T)$ be a compact Ricci flow. If $g(0)$ has α non-negative second curvature operator for some $\alpha \in [1, \frac{(n+2)(n-1)}{2}]$, then $g(t)$ has α non-negative second curvature operator for all $t \in [0, T)$.

Open Problems

Space Form Conjecture (Li)

Let (M, g) be a closed Riemannian manifold with $(n + \frac{n-2}{n})$ positive second curvature operator. Then M is diffeomorphic to a spherical space form. Moreover, this positivity condition is sharp all dimensions.

Remarks

1. Cases of dimension 3 and 4 have been resolved due to work of Li. Sharpness of dimension 3 is due to Fluck and Li.
2. It is unknown whether $(n + \frac{n-2}{n})$ positive implies PIC1. Thus current non-Ricci flow approaches may not work.