Epsilon-Regularity for Non-Collapsed Limits of Ricci Flows

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Outline

1. Overview of the Ricci Flow
2. Metric Flows and $\mathbb{F}$-Convergence
3. Statement of Results

Overview of the Ricci Flow

Definition (Ricci flow)

Let M^n be a closed manifold and $g(t)$ be a smooth 1-parameter family of Riemannian metrics. We say that $g(t)$ evolves by the **Ricci flow** if:

$$\frac{\partial}{\partial t} g_{ij} = -2Rc_{ij}.$$

1. **Harmonic coordinates**: $\frac{\partial}{\partial t} g_{ij} = \Delta(g_{ij})$ + lower order.
2. **Idea**: "Improve" the metric using Ricci flow. Better geometry gives an easier to understand topology.
3. **Notable Results:** $n = 2$ *(Hamilton, Chow, Chen, Lu, Tian)*, $n = 3$ *(Hamilton, Perelman)*, general n *(Böhm, Wilking)*.

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1. (Evolution of curvature): $\frac{\partial}{\partial t} Rm = \Delta Rm + Rm * Rm$.
2. Expect the formation of singularities ($\lim_{t \nearrow T} \sup_M |Rm| \nearrow \infty$).

Singularity Analysis

Take $\lambda_i \to \infty$, and (x_i, t_i) with $t_i \to T$ and $|Rm(x_i, t_i)| \nearrow \infty$. Study the blowup sequence:

$$(M^n, \lambda_i^2 g(t_i + \lambda_i^{-2} s)_{s \in [-t_i \lambda_i^2, 0]}, x_i).$$

Aim: analyze possible limits and continue the flow past singularities via surgery constructions.

1. $n = 2, 3$ are well understood *(Hamilton, Chow, Perelman)*
2. $n = 4$ largely open.

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Definition (Gradient Shrinking Ricci Soliton)

Let (M, g) be a Riemannian manifold and $f \in C^\infty(M)$. We say that (M, g, f) is a **gradient shrinking Ricci soliton** (GSRS) if

$$Rc + \nabla^2 f = \frac{1}{2} g.$$

1. Any GSRS gives rise to a self-similar solution to the Ricci flow via $g(t) = (1 - t)\varphi^*(t)g$ where $\varphi^*(t)$ is the flow generated by $(\frac{1}{1-t})\nabla f$.
2. GSRS model "most" singularities of the Ricci flow.
3. **Examples:**
   1. Ricci flat cones over $\mathbb{S}^n/\Gamma$ $(f = \frac{1}{4}r^2)$,
   2. Gaussian shrinker: $(\mathbb{R}^n, g_{\mathbb{R}^n}, \frac{1}{4}|x|^2)$,
   3. Einstein manifolds $(f \equiv c)$, e.g. $\mathbb{S}^n$,
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The Picture in Dimension 4 *(Bamler 2021)*

1. **First bubble:** Every type-(1) ($\lambda_i \approx (T - t_i)^{-1}$) blowup of a closed 4-dimensional Ricci flow converges to a gradient shrinking Ricci soliton with finitely many orbifold singularities.
2. **Deeper bubbles:** singularities are "mostly modelled" on one of:
   1. generalized cylinders $\mathbb{S}^k \times \mathbb{R}^{4-k}$,
   2. Cones over some smooth link $\mathbb{S}^3/\Gamma$,

Corollary of Epsilon-Regularity (F.-Hallgren 2024)

If $(M^4, g(t))$ is a solution to the Kähler Ricci flow, then every "bubble" at an orbifold point contributes a definite amount of L^2-curvature.

Hence if $(M^4, g(t))$ satisfies a finite energy assumption, the "bubble tree" is finite.

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The Case of Ricci Bounded from Below

Epsilon-Regularity for Ricci Limit Spaces *(Cheeger, Colding, Tian 2002)*

1. If (M_i^4, g_i) satisfies $|Rc_i| \leq 3$, $Vol(B_1(p_i)) \geq \nu$. Then $(M_i, g_i, p_i) \xrightarrow{GH} (X, d, p^*)$.
2. (X, d, p^*) has singularities that are modeled on Ricci flat cones $(C(\mathbb{S}^3/\Gamma), x^*)$.
3. If $d_{GH}(B_1(p), B_1(x^*)) < \epsilon_i$, then there exists $r_i^2 : B_1 \to \mathbb{R}^+$ such that

$$\begin{cases} \Delta r_i^2 = 8, \\ |r_i^2 - \pi_r| < \epsilon_i', \\ \int_{B_1(p)} (|\nabla r_i^2| - 2r_i)^2 + |\nabla^2 r_i^2 - 2g|^2 + |Rc(\nabla r_i^2, \nabla r_i^2)| \, dg < \epsilon_i', \\ \sup |\nabla r_i^2| \leq c(n). \end{cases}$$

4. Proof uses "good slices" coming from the approximates r_i^2.

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Conjugate Heat Kernel Measures

Setup: $(M, g(t)_{t\in(-T,0]})$ a closed RF, $(x_0, t_0) \in M \times (T, 0]$, $s \le t_0$.

Definition (Conjugate Heat Kernels)

The **conjugate heat operator** is defined as

$$\Box^* = -\frac{\partial}{\partial t} - \Delta + R.$$

If $K(x_0, t_0, y, s)$ is a **conjugate heat kernel**, that is:

$$\begin{cases} \Box^*_{y,s} K(x_0, t_0, y, s) = 0, \\ \lim_{s \nearrow t_0} K(x_0, t_0, y, s) = \delta_{x_0}. \end{cases}$$

Then the family $d\nu_{x_0,t_0;s} = K(x_0, t_0, -, s)dg_s$ are probability measures, known as the **conjugate heat kernel measures** based at (x_0, t_0).

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Metric Flows and $\mathbb{F}$-Convergence

Definition (Metric Flow)

A **metric flow** $\mathcal{X}$ is a 1-parameter family of complete separable metric spaces $(\mathcal{X}_t, d_t)$ together with probability measures $d\nu_{x;s}$, $x \in \mathcal{X}_t, s \leq t$ that satisfy the following:

1. For every $x \in \mathcal{X}_t$, $\lim_{s \nearrow t} d\nu_{x;s} = \delta_x$,
2. Lipschitz estimates improve for "heat flows" $u_t(y) = \int_{\mathcal{X}_s} u_s d\nu_{y;s}$,
3. The measures are compatible: $d\nu_{x;t_1} = \int_{\mathcal{X}_{t_2}} d\nu_{y;t_1} d\nu_{x;t_2}(y)$.

1. Properties (1) – (3) characterize conjugate heat kernels on a (super) Ricci flow background.
2. Ricci flows $(M, g(t)_{t \in (-T,0]})$, together with the conjugate heat kernel measures $d\nu_{x_0,t_0;s}$, are metric flows.

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Metric Flows and $\mathbb{F}$-Convergence

Theorem *(Bamler 2020)*

The space $\mathbb{F}_I$ of metric flow pairs over I can be equipped with a distance function $d_{\mathbb{F}}$ such that $(\mathbb{F}_I, d_{\mathbb{F}})$ is a complete metric space.

If $(M^i, g_i(t)_{t\in(-T,0]}, d\nu_{(x^i,0);s})$ is a sequence of closed RFs, then there is a metric flow pair $(\mathcal{X}^\infty, d\nu_{x^\infty;s})$ such that

$$(M^i, g_i(t)_{t\in(-T,0]}, d\nu_{(x^i,0);s}) \xrightarrow{d_{\mathbb{F}}} (\mathcal{X}^\infty, d\nu_{x^\infty;s}).$$

1. $\mathbb{F}$-convergence $\approx GW^1$-convergence for metric measure spaces on a.e. time slice,
2. Compactness uses good properties of conjugate heat kernels $d\nu_{x,t;s}$ coupled to Ricci flow.

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Limits that are Ricci Flat Cones

Definition (Integral Almost Properties)

Let $(M, g(t)_{t\in(-T,0]})$ be a RF, $(x_0, t_0) \in M \times (-T, 0]$. Let $\delta > 0$, and $r > 0$ be such that $[t_0 - \delta^{-1}r^2, t_0 - \delta r^2] \subset (-T, 0]$. Let $d\nu_{x_0,t_0;s} = (4\pi\tau)^{-n/2}e^{-f}dg_s$ and $W = \mathcal{N}_{x_0,t_0}(r^2)$. We say that (x_0, t_0) is (δ, r)-selfsimilar if:

$$\begin{cases} \int_{t_0-\delta^{-1}r^2}^{t_0-\delta r^2} \int_M \tau|\nabla^2 f + Rc - \frac{1}{2\tau}g|^2 \, d\nu_t dt < \delta, \\ \sup_{[t_0-\delta^{-1}r^2, t_0-\delta r^2]} \int_M |\tau(2\Delta f - |\nabla f|^2 + R) + f - n - W| d\nu_t < \delta, \\ \inf_{M\times[t_0-\delta^{-1}r^2, t_0-\delta r^2]} R \geq -\delta r^{-2}, \end{cases}$$

and (δ, r)-static if:

$$\begin{cases} \int_{t_0-\delta^{-1}r^2}^{t_0-\delta r^2} \int_M r^2|Rc|^2 \, d\nu_t dt < \delta, \\ \sup_{[t_0-\delta^{-1}r^2, t_0-\delta r^2]} \int_M R < \delta, \\ \inf_{M\times[t_0-\delta^{-1}r^2, t_0-\delta r^2]} R \geq -\delta r^{-2}. \end{cases}$$

Limits that are Ricci Flat Cones

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Limits that are Ricci Flat Cones

Proposition *(Bamler 2021)*

Let $(M_i^n, g_i(t), d\nu_{x^i,0;s})$ be a sequence of closed RFs, $(x^i, 0)$ be (δ_i, r)-selfsimilar and (δ_i, r)-static for $\delta_i \to 0$, $W_i = \mathcal{N}_{x^i,0}(r^2) \geq -Y$. Then

$$(M_i, g_i(t), d\nu_{x^i,0;s}) \xrightarrow{d_{\mathbb{F}}} (C(\mathbb{S}^n/\Gamma), d\nu_{x^*;s})$$

where $C(\mathbb{S}^n/\Gamma)$ is a Ricci flat cone with link $\mathbb{S}^n/\Gamma$ and vertex x^*.

Moreover, if $d\nu_{x^i,0;s} = (4\pi\tau)^{-n/2} e^{-f_i} dg_t$, then there is a sequence of embeddings $\varphi_i : V_i \subset C(\mathbb{S}^n/\Gamma) \backslash \{x^*\} \to M_i$ such that

$$4\tau(f_i \circ \varphi_i - W_i) \to d^2(x^*, -)$$

locally smoothly on $C(\mathbb{S}^n/\Gamma) \backslash \{x^*\}$.

Statement of Results

Theorem *(F.- Hallgren 2024)*

Let $Y, \sigma > 0$. Then there exists $\bar{\epsilon} = \bar{\epsilon}(\sigma, Y)$ such that the following statement holds for all $\epsilon \leq \bar{\epsilon}$. Let $(M^4, g(t)_{t\in(-T,0]})$ be a closed Ricci flow, and (x_0, t_0) be a point that is both (ϵ, r)-selfsimilar and (ϵ, r)-static for some scale $r > 0$. Suppose that $\mathcal{N}_{x_0,t_0}(r^2) \geq -Y$, and that

$$\int_{-2r^2}^{-r^2} \int_{(P^*)_t^-(x_0,t_0;\epsilon^{-1}r)} r^{2-n}|Rm|^2 dg_t dt < \epsilon.$$

Then one of the following holds:

1. The curvature scale satisfies $r_{Rm} \geq \epsilon r$.
2. We have $d_{\mathbb{F}}((M, g(t), d\nu_{x_0,t_0;s}), C(L_{p,q})) < r\sigma$ where $L(p,q)$, $q^2 \equiv -1 \pmod p$ is an exceptional Lens space.

Remark: (2) does not occur in the Kähler setting.

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Idea of Proof

Slicing Argument *(Cheeger, Colding, Tian 2002)*

1. **Argue by contradiction:** $\exists\, \sigma, Y > 0$, $(M^i, g_i, d\nu_{x_i,0;s})$ satisfying the conditions of the proposition for $\epsilon_i \to 0$ s.t. conclusion (a) and (b) fail.
2. **Pass to a limit:** Static cone $(C(\mathbb{S}^3/\Gamma), d\nu_{x^*;s})$, suffices to show $\mathbb{S}^3/\Gamma \cong L(p,q)$,
3. **Slice:** Consider level sets of approximate distance functions

$$\tilde{q}_i := 4\tau(f_i \circ \varphi_i + W_i) \xrightarrow{C^\infty_{loc}(\mathcal{R})} d^2(x^*, -)$$

$$\tilde{\Sigma}_{i,r} := q_i^{-1}(r) \xrightarrow{C^\infty_{loc}(\mathcal{R})} \mathbb{S}^3/\Gamma,$$

4. **Idea:** Small L^2-curvature will contradict some properties of the Chern-Simon invariant associated to the first Pontryagin form.

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Idea of Proof

Definition (Secondary Invariant Associated to p_1)

Let $\pi : E \to M^3$ be a vector bundle with connection 1-form A, and curvature 2-form Ω. Define:

$$\hat{p}_1(E) := \int_M \mathrm{Tr}(dA \wedge A + \frac{2}{3} A \wedge A \wedge A) \pmod{\mathbb{Z}}$$

1. **Boundary formula:** If $M^3 = \partial N^4$ then:

$$\hat{p}_1(TN|_M) = \int_N p_1(N) = -\frac{1}{4\pi^2} \int_N \mathrm{Tr}(\Omega^2).$$

2. $\hat{p}_1(T(\mathbb{S}^3/\Gamma)) = 0 \iff \mathbb{S}^3/\Gamma \cong L(p,q),\ q^2 \equiv -1 \pmod{p}$,
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Setup: For $\tilde{q}_i = 4\tau(f_i \circ \varphi_i + W_i)$, $\tilde{\Sigma}_{i,r} := \tilde{q}_i^{-1}(r)$ we have

$$\begin{cases} \tilde{q}_i \xrightarrow{C^\infty_{loc}(\mathcal{R}\backslash x^*)} d^2(x^*, -), \\ \tilde{\Sigma}_{i,r} \xrightarrow{C^\infty_{loc}(\mathcal{R}\backslash x^*)} \mathbb{S}^3/\Gamma, \\ \hat{p}_1(TM^i|_{\tilde{\Sigma}_{i,r}}) \to \hat{p}_1(\mathbb{S}^3/\Gamma). \end{cases}$$

Assumption: $\varphi_i^*(\tilde{\Sigma}_{i,r}) = \partial S_{i,r}$ for some $S_{i,r} \subset P^*(x_0, t_0; \epsilon_i^{-1} r)$.
Since

$$\int_{-2r^2}^{-r^2} \int_{(P^*)_t^-(x_0, t_0; \epsilon_i^{-1} r)} r^{2-n} |Rm|^2 dg_t dt < \epsilon_i,$$

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Justifying the Assumption

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1. This is not an issue in the bounded Ricci curvature setting.
    1. The approximate distance functions r_i^2 are constructed by solving elliptic equations on a ball.
2. The conjugate heat kernels $d\nu_{x_i,t_i;s}$ are constructed *globally* on M, and may be badly behaved "far away" from (x_i, t_i).

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Lemma *(F.- Hallgren 2024)*

Let $\epsilon, Y > 0$. There exists $A(Y)$, $\Lambda(Y)$, $\delta(Y, \epsilon) > 0$ such that the following holds: If $(M^4, g(t)_{t\in(-T,0]})$ is a closed RF, (x_0, t_0) is (δ, r)-static and (δ, r)-selfsimilar, $\mathcal{N}_{x_0,t_0}(\delta r^2) \geq -Y$. Then there is a connected component Σ_A of $q^{-1}(A)$ such that:

1. Σ_A is ϵ-close in the C^∞_{loc}-topology to $\mathbb{S}^3/\Gamma$.
2. Σ_A separates M into two connected components M'_A, M''_A.
3. $\partial M'_A = \Sigma_A$ and $M'_A \subset P^*(x_0, t_0; \Lambda r)$.

Justifying the Assumption

Lemma (Gaussian Isoperimetric Inequality)

Set $\Phi(x) := \int_{-\infty}^{x} (4\pi)^{-\frac{1}{2}} e^{-x^2}\, dx$ and $\eta = \Phi' \circ \Phi^{-1}$. Then for any $\mathcal{A} \subset M$ open with smooth boundary, we have

$$\eta(\nu_{x,0;t}(\mathcal{A})) \leq \sqrt{-t} \int_{\partial\mathcal{A}} K(x,0;-,t)\, d\mathcal{H}^{n-1}.$$

1. Applying to the annulus $\mathcal{A} := \varphi(B(x^*,\epsilon^{-1}) \setminus \overline{B(x^*,\epsilon)})$, gives that

$$\nu_{x,0;t}(\mathcal{A}) \approx \begin{cases} 1 \\ 0 \end{cases}$$

if x is "far away" from $\varphi(\partial B(x^*,\epsilon^{-1}))$.

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Thank You for Listening!