Ricci Flow and the urvature Operator of Kind

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Outline

Hamilton's Ricci Flow

Definition: Ricci Flow (Hamilton '82)

Let (M, g_0) be a Riemannian manifold. The equation

$$\frac{\partial}{\partial t} g_{ij} = \quad 2R_{ij} + (\frac{2\int_M R d\mu}{nVol(M)} g_{ij}), \ g(0) = g$$

is known as the (normalized) Ricci flow. pair $(M, g(t))$ closed manifold and $g(t)$ is a solution to the Ricci flow is compact Ricci flow.

Short Time Existence

Theorem (Hamilton '82)

For any closed Riemannian manifold (M, g_0), there is a un
short time solution $g(t), t \in [0, \delta)$ to the Ricci flow equat
$g(0) = g_0$.

Remark

The Ricci tensor is not elliptic:

$$\sigma D(\quad 2Rc)_{\zeta}(\nu) = |\zeta|^2 \nu_{ij} + \zeta_i \zeta_j Tr(\nu) \quad \zeta_i \zeta_k \nu_{kj}$$

That is, the Ricci flow is not a strictly parabolic flow.

Short Time Existence

Theorem (DeTurck '83)

Let (M, g_0) be a closed Riemannian manifold. There is a flow

$$\frac{\partial}{\partial t} g_{ij} = \quad 2R_{ij} + \mathcal{L}_X g_{ij}, \; g(0) = g_0$$

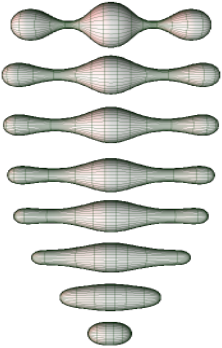
and a family of diffeomorphisms $\varphi(t) : M \to M$, such tha
unique smooth short time solution to (1) with $g(0) = g_0$,
solves the Ricci flow equation with initial date g_0.

Intuition for the Ricci Flow

Ricci Flow as a Heat Equation

In geodesic normal coordinates centered at a point p, we

$$\frac{\partial}{\partial t} g_{ij} = \quad 2R_{ij} = 3\Delta_{\mathrm{Euc}}(g_{ij}).$$



Idea of the Ricci

1. Ricci Flow sho similarly to the for the metric.
2. Ricci flow shou apriori badly b
3. Use this fact t the topology o

Singularity Formation in Ricci Flow

Singularity Formation

For a compact Ricci flow $(M, g(t))$ with $R(0) > 0$, $\exists\ T \in$

$$\lim_{t \nearrow T} |Rm|_{g(t)} = \infty.$$

Modelling the Singularity

For $K_i := |Rm(x_i, t_i)| \nearrow \infty$, $t_i \nearrow T$, we aim to study the

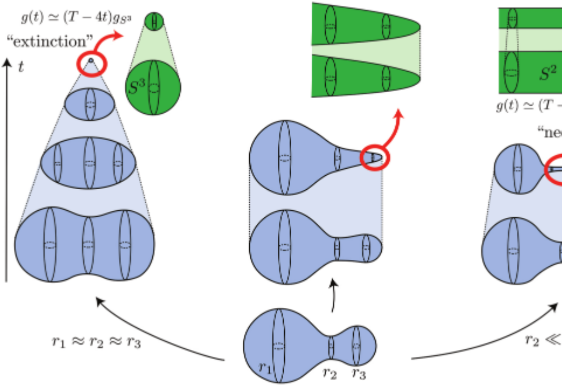
$$(M, g_i(t), x_i), \quad g_i(t) = K_i g(t_i + K_i^{\ 1} t).$$

If, in the C^∞ pointed sense of Cheeger and Gromov, we h

$$(M, g_i(t), x_i) \to (M_\infty, g_\infty(t), x_\infty)$$

we call $(M_\infty, g_\infty(t))$ a singularity model for the flow.

Intuitive Solutions of the Ricci Flow



The Ricci Flow on Surfaces

The Uniformization Theorem (Poincaré)

Every closed Riemann surface contains a metric in its conf
is locally isometric to one of the 3 model geometries: S^2,

Remarks

1. This is equivalent to the existence of a metric of con
curvature in the conformal class.
2. In dimension 2, Ricci flow is equivalent to the scalar

$$\frac{\partial}{\partial t} u = \Delta_{g_0} log(u) \quad R_{g_0},$$

where $u \in C^{\infty}(M)$. In particular, Ricci flow preserves
class of the metric.

The Ricci Flow on Surfaces

Theorem (Hamilton '88+ Chow '91)

For any closed (M^2, g_0), the solution to the normalized R
all time and converges to a metric of constant sectional c

Remarks

1. Hamilton '88: Case of $\chi(M) \leq 0$ and $\chi(M) > 0$ with
2. Chow '91: Case of $\chi(M) > 0$ with arbitrary initial m
3. Hamilton and Chow's proof required uniformization t
4. (Chen, Lu+ Tian '06): Give a Ricci flow proof of the theorem.

The Ricci Flow on 3-Manifolds

Thurston's Geometrization Conjecture

Every closed prime 3-manifold has a canonical geometric s
when cut along tori, each tori is locally isometric to one o
geometries.

Remarks

1. Implies, as a corollary, the Poincaré conjecture:

   Every closed simply connected 3-manifold is homeo

2. proof of Thurston's conjecture was the long term g
   Ricci flow programme.
3. Resolved by Perelman in 2003 using Ricci flow.

The Ricci Flow on 3-Manifolds

Theorem (Hamilton '82)

For any closed (M^3, g_0) with $Rc > 0$, the solution to the
flow exists for all time and converges to a metric of const
sectional curvature. Hence, $M^3 \cong S^3/\ $ for some $\ \in$ Iso(

Theorem (Perelman '03)

Let $(M_\infty, g_\infty(t))$ be a singularity model of a 3-dimensiona
flow. Then there are sequences $\lambda_i, \beta_i \in \mathbb{R}$, points $x_i \in M_$
$t_i \to \ \infty$ such that the rescaled solutions

$$(M_\infty, g_i(t), x_i),\ g_i(t) = \lambda_i g(t_i + \beta_i t)$$

converge to the standard solution on either $S^3/\ $ or $(S^2$

Ricci Flow in Higher Dimensions

Remarks on $n \geq 4$

1. Singularities of the Ricci flow for dimensions $n \geq 4$ ar more complicated.
2. Understanding singularity formation in dimension 4 is contemporary area of research.
3. To attain any general results, one needs to impose st assumptions on the initial metric.

The urvature Operator

Definition: Curvature Operator

Let (M, g) be a Riemannian manifold. The curvature tens
bundle map on the space of 2-forms:

$$Rm : \wedge^2 T^*M \to \wedge^2 T^*M$$
$$e^i \wedge e^j \to R_{ijkl} e^k \wedge e^l$$

known as the curvature operator.

Definition: Positive Curvature Operator

We say Rm is positive ($Rm > 0$) if $Rm|_p$ has strictly posit
for each $p \in M$. Note that

$$Rm > 0 \Rightarrow sec > 0.$$

The urvature Operator

Classical Problem

Which manifolds admit metrics with positive curvature op

Examples

1. If $n = 2$, Gauss Bonnet implies $M = S^2$ or $M = \mathbb{RP}^2$
2. If $M^n = S^n/$ with the standard round metric, then
3. Some non-examples:

$$Rm_{S^2 \times S^1} = \begin{pmatrix} 0 & & \\ & 0 & \\ & & 1 \end{pmatrix}, \quad Rm \ _{\mathbb{P}^2} = \begin{pmatrix} 0 & & \\ & 0 & \\ & & 1 \end{pmatrix}$$

The urvature Operator

Space Form Conjecture

Let (M, g) is a closed Riemannian manifold with $Rm > 0$
diffeomorphic to a spherical spherical space form. That is
some $\in \mathrm{Iso}(S^n)$.

The urvature Operator

Hamilton's Conjecture

Let (M, g) be a closed Riemannian manifold with $Rm > 0$
solution to the normalized Ricci flow exists for all time an
metric of constant positive sectional curvature. Hence, M

The urvature Operator Under Ricci Flow

The Lie lgebra Structure of $\wedge^2 T^*$

Let (M, g) be a Riemannian manifold. For each $p \in M$, t
$\wedge^2 T_p^* M \cong \mathfrak{s}\ (n)$ as a Lie algebra, with bracket defined by

$$[U, V]_{ij} = g^{kl}(U_{ik} V_{lj} \quad V_{ik} U_{lj})$$

Fix a basis φ of $\wedge^2 T_p^* M$. The Lie algebra square of Rm

$$Rm \quad : \wedge^2 T^* M \to \wedge^2 T^* M$$

$$(Rm \quad)\ _{\beta} = C^{\gamma\delta} C_{\beta}^{\epsilon\zeta} Rm_{\gamma\epsilon} Rm_{\delta\zeta},$$

where $C^{\gamma\delta}$ are the structure constants for the bracket in t

The urvature Operator Under Ricci Flow

The Evolution Equation for Rm

Let $(M, g(t))$ be a compact Ricci flow. The Riemann cur
satisfies the reaction diffusion equation

$$\frac{\partial}{\partial t} R_{ijkl} = \Delta R_{ijkl} + 2(R_{pijq} R_{qklp} + R_{pilq} R_{qkjp} \quad R_{pijq} R_{qlk}$$

The reaction terms can be grouped so that the curvature

$$\frac{\partial}{\partial t} Rm = \Delta Rm + Rm^2 + Rm \quad .$$

The urvature Operator Under Ricci Flow

Hamilton's ODE→PDE Maximum Principle

Let (M, g) be a Riemannian manifold, $\pi : E \to M$ be a H
bundle with compatible connection, and $K \subset E$ be a close
which is invariant under parallel translation. For a section
consider the non-linear PDE

$$\frac{\partial}{\partial t} e(t) = \Delta e + f(e(t)).$$

Suppose that the subset K is preserved by the ODE

$$\frac{d}{dt} e = f(e(t)).$$

Then the same is true for solutions to (2).

The urvature Operator Under Ricci Flow

The ODE→PDE Maximum Principle for Rm

Let $(M, g(t))$ be a compact Ricci flow. Recall that the cu satisfies

$$\frac{\partial}{\partial t} Rm = \Delta Rm + Rm^2 + Rm \quad .$$

Let $K \subset S^2(\wedge^2 T^* M)$ be a closed, convex subset invarian translation. Suppose that solutions to

$$\frac{d}{dt} \quad = \quad ^2 +$$

which begin in K, remain in K. Then the same is true for Ricci flow.

The urvature Operator Under Ricci Flow

Theorem (Huisken '85)

Let $(M, g(t))$ be a compact Ricci flow such that $R(0) >$
$\exists \delta > 0$ s.t. the estimate

$$|\tilde{Rm}| \leq R^{1 \quad \delta}$$

holds $\forall t \in [0, T)$. Then the solution to the normalized Ri
all time and converges to a metric of constant positive se

The urvature Operator Under Ricci Flow

Definition: Pinching Sets

subset $K \subset S^2(\wedge^2 T^*M)$ is called a pinching set if it is invariant under parallel translation, preserved by the ODE

$$\frac{d}{dt} = {}^2 +$$

and satisfies the pinching estimate

$$|\tilde{\ }| \leq | \ |^{1 \ \delta}$$

for some $\delta > 0$ and all $\in K$. n open subset $U \subset S^2(\wedge$ the pinching condition if every compact subset $K \subset U$ is pinching set.

In particular, the normalized Ricci flow evolves a manifold $R > 0$ into one of constant positive sectional curvature if satisfies the pinching condition.

The urvature Operator Under Ricci Flow

The ODE in Dimension 3

The Lie algebra square Rm is the adjoint matrix and the
the system

$$\frac{d}{dt}\lambda_i = \lambda_i^2 + \lambda_j\lambda_k,$$

where $\lambda_1, \lambda_2, \lambda_3$ are the eigenvalues of Rm.

Hamilton studied this system to prove his conjecture for r

The urvature Operator Under Ricci Flow

The ODE in Dimension 4

There is a splitting $\bigwedge^2 = \bigwedge^2_+ \oplus \bigwedge^2$ by eigenspaces of the
operator so that

$$Rm = \begin{pmatrix} & B \\ B^t & C \end{pmatrix}, Rm \quad = \begin{pmatrix} & B \\ (B \quad)^t & C \end{pmatrix}$$

where $, B, C$ are 3×3 matrices and is the adjoint m
then reduces to a system of 3×3 matrix ODE e.g.

$$\frac{d}{dt} \quad = \quad {}^2 + BB^t + \quad .$$

Hamilton studied this system to prove his conjecture for

The urvature Operator Under Ricci Flow

Theorem (Böhm+Wilking '08)

Let (M, g) be a closed Riemannian manifold with 2 positi
operator. Then the solution to the normalized Ricci flow e
all time and converges to a metric of constant positive se
Hence, $M \cong S^n /$ is diffeomorphic to a spherical space fo

Remark

We say a linear operator $\in \text{End}(\mathbb{R}^n)$ is m positive if the
m eigenvalues of is positive.

The 2^{nd} urvature Operator

Definition: The 2^{nd} Curvature Operator

Let (M, g) be a Riemannian manifold. The curvature tens
bundle map on the space of symmetric 2-tensors:

$$\overline{Rm} : S^2(T^*M) \to S^2(T^*M)$$
$$e_i \odot e_j \to R_{kilj} e_k \odot e_l$$

Let $\pi : S^2(T^*M) \to S_0^2(T^*M)$ denote the projection ontc
traceless 2-forms. The operator

$$\mathring{Rm} = \pi \circ \overline{Rm}|_{S_0^2} : S_0^2(T^*M) \to S_0^2(T^*M$$

is known as the curvature operator of the second kind.

The 2^{nd} urvature Operator

Restricting to S_0^2 T^*)

1. There a splitting
$$S^2(T^*M) = S_0^2(T^*M) \oplus \mathbb{R}g$$
into $O(n)$-invariant subbundles.
2. For the round metric $g_{\mathbb{S}^n}$ on S^n,
$$\overline{Rm}_{g\ \ n} : S^2(T^*M) \to S^2(T^*M)$$
is not positive. In particular, $\overline{Rm}|_{\mathbb{R}g}$ has eigenvalue
3. Hence, we work with the restricted operator
$$\mathring{Rm} : S_0^2(T^*M) \to S_0^2(T^*M)$$

The 2^{nd} urvature Operator

Remarks

1. We have

$$\mathring{Rm} > 0 \Rightarrow sec > 0.$$

2. In general, the relationship between Rm and $\mathring{Rm}$ is u

$$\mathring{Rm}_{\mathbb{P}^2} = \begin{pmatrix} \frac{1}{2}\mathrm{Id} & & \\ & \mathrm{Id} & \\ & & \mathrm{Id} \end{pmatrix}, Rm_{\mathbb{P}^2} = \begin{pmatrix} 0 & & \\ & 0 & \\ & & 1 \end{pmatrix}$$

The 2^{nd} urvature Operator

Conjecture (Nishikawa '86)

Let (M, g) be a closed Riemannian manifold with $\mathring{Rm} > ($
$M \cong S^n/$ is diffeomorphic to a spherical space form.

Examples

1. If $n = 2$, Gauss Bonnet implies $M = S^2$ or $M = \mathbb{RP}^2$
2. If $M^n = S^n/$ with the standard round metric then I
3. Some non-examples:

$$\mathring{Rm}_{S^2 \times S^1} = \begin{pmatrix} \frac{1}{3} & & & & \\ & 0 & & & \\ & & 0 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}, \quad \mathring{Rm}_{\ \mathbb{P}^2} = \Big($$

The 2^{nd} urvature Operator

Theorem (Cao, Gursky, Tran '21)

Let (M^n, g) be a closed Riemannian manifold with 2-posi
curvature operator. Then M^n is diffeomorphic to a spheri

The 2^{nd} urvature Operator

Definition: Positive Isotropic Curvature

Riemannian manifold (M^n, g), $n \geq 4$ is said to have pos
curvature (PIC) if for all orthonormal 4-frames $\{e_1, e_2, e_3,$

$$R_{1331} + R_{1441} + R_{2332} + R_{2442} \quad 2R_{1234} >$$

We say (M^n, g), $n \geq 3$ is PICk if $M \times \mathbb{R}^k$ has PIC.

Remark

We have the following implications:

$$2 \text{ positive } Rm \Rightarrow \text{PIC1} \Rightarrow \begin{cases} \text{PIC} \\ Rc > 0 \end{cases}$$

The 2^{nd} urvature Operator

Theorem (Brendle '08)

Let (M^n, g_0), $n \geq 3$ be a closed, PIC1 Riemannian manifo
solution to the normalized Ricci flow exists for all time co
metric of constant positive sectional curvature.

Remark

Since Rm is 2 positive $\Rightarrow$ PIC1, Brendle's result can be se
strengthening of the theorem of Böhm and Wilking.

The 2^{nd} urvature Operator

Theorem (Cao, Gursky, Tran '21)

Let (M^n, g) be a closed Riemannian manifold with 2-posi
curvature operator. Then M^n is diffeomorphic to a spheri

Idea of the Proof

Show algebraically that 2 positive $\mathring{R}m$ implies positive PIC
the convergence result of Brendle.

The 2^{nd} urvature Operator

Definition: $k + \epsilon)$ non-negative

linear operator $\in End(\mathbb{R}^n)$ is $(k + \epsilon)$ non-negative wh
$1 \leq k \leq n$ if

$$\lambda_1 + ... + \lambda_k + \epsilon\lambda_{k+1} \geq 0$$

for any eigenvalues $\lambda_1, ..., \lambda_{k+1}$ of .

Note that setting $\epsilon = 0$ gives k non-negative and setting
non-negative.

The 2^{nd} urvature Operator

Theorem (Li '22)

The assumption of Cao, Gursky, and Tran can be weakene
That is, if (M, g) is a closed manifold such that $\mathring{R}m$ is 3
is diffeomorphic to a spherical space form.

For $n = 3$, the assumption can be weakened further to $3\frac{1}{3}$

For $n = 4$, the assumption can be weakened even further
$\mathring{R}m$.

Conjecture (Li '22)

If (M^n, g) is a closed manifold with $(n + \frac{n\ \ 2}{n})$ positive se
operator, then M^n is diffeomorphic to a spherical space fo

The 2^{nd} Curvature Operator and Ricci Flow

1. The evolution of $\mathring{R}m$ under the Ricci flow has not be
2. This is natural to investigate c.f. the work of Hamilt
3. Fundamentally, positivity conditions of $\mathring{R}m$ would nee by Ricci flow for this to provide anything useful.

Statement of Results

Main Theorem: Fluck and Li '23

Let $(M^3, g(t)), t \in [0, T)$ be a 3 dimensional compact Ri
$g(0)$ has non-negative second curvature operator for so
Then $g(t)$ has non-negative second curvature operator

Statement of Results

Theorem: Fluck and Li '23

Let (M^3, g) be a 3-dimensional Riemannian manifold. Th
$\mathring{R}m$ are given by a, b, c and

$$\lambda_{\pm} = \frac{a+b+c}{3} \pm \frac{\sqrt{2}}{3}\sqrt{3(a^2+b^2+c^2) \quad (a+$$

where $a \leq b \leq c$ denote the eigenvalues of Rm. Note we

$$\lambda \quad \leq a \leq b \leq c \leq \lambda_{+}.$$

Statement of Results

Proof

1. Recall that the Weyl tensor vanishes in dimension 3 s

$$Rm = S \owedge g := (Rc \quad \frac{R}{4} g) \owedge g$$

and the eigenvalues of S are

$$\frac{1}{2}(a+b \quad c) \leq \frac{1}{2}(a+c \quad b) \leq \frac{1}{2}(b+c$$

2. Thus, the problem reduces to a general algebraic one second curvature operator of $\owedge$ Id, where has kn

Statement of Results

lgebraic Lemma: Fluck and Li '23

Let V be a finite dimensional real vector space and $\in S$
eigenvalues of the algebraic second curvature operator of
$\bigcirc\!\!\!\!\wedge\ \mathrm{Id} \in S^2(\wedge^2 V)$ are given by

$$\begin{cases} \mu_i + \mu_j \text{ with multiplicity } n_i n_j \text{ where } 1 \leq i < \\ 2\mu_i \text{ with multiplicity } n_i \quad 1 \text{ where } 1 \leq i \leq \\ \text{the k-1 non-zero solutions of } \sum_{i=1}^{k} \frac{n_i \mu_i}{2\mu_i \quad \lambda} = \end{cases}$$

where μ_i for $1 \leq i \leq k$ are the eigenvalues of with mult

Statement of Results

Proof

1. pply the lemma to $S \owedge \mathrm{Id}$ where S has eigenvalues

$$\mu_1 = \frac{1}{2}(a + b \quad c), \mu_2 = \frac{1}{2}(a + c \quad b), \mu_3 = \frac{1}{2}$$

2. Indeed, $\{\mu_i + \mu_j\}_{1 \leq i < j \leq 3} = \{a, b, c\}$ and one verifies non-zero solutions of

$$\sum_{i=1}^{k} \frac{n_i \mu_i}{2\mu_i \quad \lambda} = \frac{3}{2}$$

are

$$\lambda_{\pm} = \frac{a+b+c}{3} \pm \frac{\sqrt{2}}{3}\sqrt{3(a^2+b^2+c^2) \quad (a}$$

Statement of Results

Corollary: Fluck and Li '23

Let (M^3, g) be a complete Riemannian manifold such tha
non-negative for some $\delta \in [0, \frac{1}{3}]$. Then there exists $\epsilon > 0$

$$Rc \geq \epsilon R.$$

Consequently, any such manifold is either flat or a spheric

Statement of Results

Remarks

1. It is already known due to Li that

$$\mathring{R}m \text{ is } 3\frac{1}{3} \text{ non-negative} \Rightarrow Rm \text{ has non-negative F}$$

2. There is already a classification of 3 manifolds with F
Hamilton (compact case) and Liu (complete non-con

$$M^3 = \begin{cases} \mathbb{R}^3, (N^2 \times \mathbb{R})/ & \text{if } M^3 \text{ is non-co} \\ S^3/\ , (S^2 \times \mathbb{R})/ & \text{if } M^3 \text{ is compa} \end{cases}$$

3. This result enhances this classification.

Statement of Results

Main Theorem: Fluck and Li '23

Let $(M^3, g(t)), t \in [0, T)$ be a 3 dimensional compact Ri
$g(0)$ has non-negative second curvature operator for so
Then $g(t)$ has non-negative second curvature operator

Statement of Results

Proof

1. By the lemma, the eigenvalues of $\mathring{Rm}$ are a, b, c and

$$\lambda_{\pm} = \frac{a+b+c}{3} \pm \frac{\sqrt{2}}{3}\sqrt{3(a^2+b^2+c^2) \quad (a$$

where $a \leq b \leq c$ are the eigenvalues of Rm.

2. By Hamilton's ODE$\rightarrow$PDE maximum principle, it suf
non-negativity is preserved by the system of ODE's

$$\begin{cases} \frac{da}{dt} & = \mathrm{a}^2 + bc \\ \frac{db}{dt} & = \mathrm{b}^2 + ac \\ \frac{dc}{dt} & = \mathrm{c}^2 + ab \end{cases}$$

coming from $\frac{dS}{dt} = S^2 + S$ in dimension 3.

Statement of Results

Proof

1. Define

$$f(\quad) = \begin{cases} \lambda \quad + (\quad 1)a, & \text{i} \\ \lambda \quad + (3 \quad)a + (\quad 2)(a+b), & \text{i} \\ \lambda \quad + \frac{R}{2} + (\quad 4)c, & \text{i} \\ \frac{R}{2} + \frac{R}{3}(\quad 4) + (5 \quad)\lambda_+ & \text{i} \end{cases}$$

so that

$$f(\quad) \geq 0 \iff \mathring{Rm} \text{ is } \quad \text{non-negativ}$$

2. It suffices to show that $f(\quad)$ is non-decreasing under

Statement of Results

Proof

1. Under the ODE we have
$$\frac{dR}{dt} = |Ric|^2 \geq 0,$$
and so w.l.o.g we may assume that $R(t) > 0$.
2. It thus suffices to show $f(\quad)/R$ is non-decreasing un
3. Direct calculation shows that each component is non
$$\frac{d}{dt}(\frac{a}{R}) = \frac{2}{S^2}(b^2(c \quad a) + c^2(b \quad a)) \geq$$

Open Problems

Conjecture: Preserving Positivity in rbitrary Dime

Let $(M^n, g(t)), t \in [0, T)$ be a compact Ricci flow. If $g(0$
non-negative second curvature operator for some $\in [1,$
$g(t)$ has non-negative second curvature operator for all

Open Problems

Space Form Conjecture (Li)

Let (M, g) be a closed Riemannian manifold with $(n + \frac{n}{n}$
second curvature operator. Then M is diffeomorphic to a
form. Moreover, this positivity condition is sharp all dimen

Remarks

1. Cases of dimension 3 and 4 have been resolved due t
   Sharpness of dimension 3 is due to Fluck and Li.
2. It is unknown whether $(n + \frac{n\ 2}{n})$ positive implies PIC
   non-Ricci flow approaches may not work.