CS 70	Discrete Mathematics and Probability Theory	
Spring 2025	Rao	HW 06

1 Counting, Counting, and More Counting

Note 10

The only way to learn counting is to practice, practice, practice, so here is your chance to do so. Although there are many subparts, each subpart is fairly short, so this problem should not take any longer than a normal CS70 homework problem. You do not need to show work, and **Leave your answers as an expression** (rather than trying to evaluate it to get a specific number).

(a) How many ways are there to arrange n 1s and k 0s into a sequence?

(b) How many 19-digit ternary (0,1,2) bitstrings are there such that no two adjacent digits are equal?

(c) A bridge hand is obtained by selecting 13 cards from a standard 52-card deck. The order of the cards in a bridge hand is irrelevant.

   (i) How many different 13-card bridge hands are there?

   (ii) How many different 13-card bridge hands are there that contain no aces?

   (iii) How many different 13-card bridge hands are there that contain all four aces?

   (iv) How many different 13-card bridge hands are there that contain exactly 4 spades?

(d) Two identical decks of 52 cards are mixed together, yielding a stack of 104 cards. How many different ways are there to order this stack of 104 cards?

(e) How many 99-bit strings are there that contain more ones than zeros?

(f) An anagram of ALABAMA is any re-ordering of the letters of ALABAMA, i.e., any string made up of the letters A, L, A, B, A, M, and A, in any order. The anagram does not have to be an English word.

   (i) How many different anagrams of ALABAMA are there?

   (ii) How many different anagrams of MONTANA are there?

(g) How many different anagrams of ABCDEF are there if:

   (i) C is the left neighbor of E

   (ii) C is on the left of E (and not necessarily E's neighbor)

(h) We have 8 balls, numbered 1 through 8, and 25 bins. How many different ways are there to distribute these 8 balls among the 25 bins? Assume the bins are distinguishable (e.g., numbered 1 through 25).

(i) How many different ways are there to throw 8 identical balls into 25 bins? Assume the bins are distinguishable (e.g., numbered 1 through 25).

(j) We throw 8 identical balls into 6 bins. How many different ways are there to distribute these 8 balls among the 6 bins such that no bin is empty? Assume the bins are distinguishable (e.g., numbered 1 through 6).

(k) There are exactly 20 students currently enrolled in a class. How many different ways are there to pair up the 20 students, so that each student is paired with one other student? Solve this in at least 2 different ways. **Your final answer must consist of two different expressions.**

(l) How many solutions does $x_0 + x_1 + \cdots + x_k = n$ have, if each x must be a non-negative integer?

(m) How many solutions does $x_0 + x_1 = n$ have, if each x must be a *strictly positive* integer?

(n) How many solutions does $x_0 + x_1 + \cdots + x_k = n$ have, if each x must be a *strictly positive* integer?

Solution:

(a) $\binom{n+k}{k}$

(b) There are 3 options for the first digit. For each of the next digits, they only have 2 options because they cannot be equal to the previous digit. Thus, $3 \cdot 2^{18}$

(c) (i) We have to choose 13 cards out of 52 cards, so this is just $\binom{52}{13}$.

(ii) We now have to choose 13 cards out of 48 non-ace cards. So this is $\binom{48}{13}$.

(iii) We now require the four aces to be present. So we have to choose the remaining 9 cards in our hand from the 48 non-ace cards, and this is $\binom{48}{9}$.

(iv) We need our hand to contain 4 out of the 13 spade cards, and 9 out of the 39 non-spade cards, and these choices can be made separately. Hence, there are $\binom{13}{4}\binom{39}{9}$ ways to make up the hand.

(d) If we consider the 104! rearrangements of 2 identical decks, since each card appears twice, we would have overcounted each distinct rearrangement. Consider any distinct rearrangement of the 2 identical decks of 52 cards and see how many times this appears among the rearrangement of 104 cards where each card is treated as different. For each identical pair (such as the two Ace of spades), there are two ways they could be permuted among each other (since $2! = 2$). This holds for each of the 52 pairs of identical cards. So the number 104! overcounts the actual number of rearrangements of 2 identical decks by a factor of 2^{52}. Hence, the actual number of rearrangements of 2 identical decks is $\frac{104!}{2^{52}}$.

(e) **Answer 1:** There are $\binom{99}{k}$ 99-bit strings with k ones and $99-k$ zeros. We need $k > 99-k$, i.e. $k \geq 50$. So the total number of such strings is $\sum_{k=50}^{99} \binom{99}{k}$.

This expression can however be simplified. Since $\binom{99}{k} = \binom{99}{99-k}$, we have

$$\sum_{k=50}^{99}\binom{99}{k} = \sum_{k=50}^{99}\binom{99}{99-k} = \sum_{l=0}^{49}\binom{99}{l}$$

by substituting $l = 99-k$.

Now $\sum_{k=50}^{99}\binom{99}{k} + \sum_{l=0}^{49}\binom{99}{l} = \sum_{m=0}^{99}\binom{99}{m} = 2^{99}$. Hence, $\sum_{k=50}^{99}\binom{99}{k} = \frac{1}{2}\cdot 2^{99} = 2^{98}$.

Answer 2 (Symmetry): Since the answer from above looked so simple, there must have been a more elegant way to arrive at it. Since 99 is odd, no 99-bit string can have the same number of zeros and ones. Let A be the set of 99-bit strings with more ones than zeros, and B be the set of 99-bit strings with more zeros than ones. Now take any 99-bit string x with more ones than zeros i.e. $x \in A$. If all the bits of x are flipped, then you get a string y with more zeros than ones, and so $y \in B$. This operation of bit flips creates a one-to-one and onto function (called a bijection) between A and B. Hence, it must be that $|A| = |B|$. Every 99-bit string is either in A or in B, and since there are 2^{99} 99-bit strings, we get $|A| = |B| = \frac{1}{2}\cdot 2^{99}$. The answer we sought was $|A| = 2^{98}$.

(f) ALABAMA: The number of ways of rearranging 7 distinct letters and is 7!. In this 7 letter word, the letter A is repeated 4 times while the other letters appear once. Hence, the number 7! overcounts the number of different anagrams by a factor of 4! (which is the number of ways of permuting the 4 A's among themselves). Aka, we only want 1/4! out of the total rearrangements. Hence, there are $\frac{7!}{4!}$ anagrams.

MONTANA: In this 7 letter word, the letter A and N are each repeated 2 times while the other letters appear once. Hence, the number 7! overcounts the number of different anagrams by a factor of $2! \times 2!$ (one factor of 2! for the number of ways of permuting the 2 A's among themselves and another factor of 2! for the number of ways of permuting the 2 N's among themselves). Hence, there are $\frac{7!}{(2!)^2}$ different anagrams.

(g) (i) Suppose we consider CE to be a new letter X; with this replacement, the question is just to count the number of rearrangements of 5 distinct letters, which is 5!.

(ii) Symmetry: Let A be the set of all the rearranging of ABCDEF with C on the left side of E, and B be the set of all the rearranging of ABCDEF with C on the right side of E. $|A \cup B| = 6!, |A \cap B| = 0$. There is a bijection between A and B by construct a operation of exchange the position of C and E. Thus $|A| = |B| = \frac{6!}{2}$.

(h) Each ball has a choice of which bin it should go to. So each ball has 25 choices and the 8 balls can make their choices separately. Hence, there are 25^8 ways.

(i) Since there is no restriction on how many balls a bin needs to have, this is just the problem of throwing k identical balls into n distinguishable bins, which can be done in $\binom{n+k-1}{k}$ ways. Here $k = 8$ and $n = 25$, so there are $\binom{32}{8}$ ways.

(j) **Answer 1:** Since each bin is required to be non-empty, let's throw one ball into each bin at the outset. Now we have 2 identical balls left which we want to throw into 6 distinguishable bins. There are 2 cases to consider:

Case 1: The 2 balls land in the same bin. This gives 6 ways.

Case 2: The 2 balls land in different bins. This gives $\binom{6}{2}$ ways of choosing 2 out of the 6 bins for the balls to land in. Note that it is *not* 6×5 since the balls are identical and so there is no order on them.

Summing up the number of ways from both cases, we get $6 + \binom{6}{2}$ ways.

Answer 2: Since each bin is required to be non-empty, let's throw one ball into each bin at the outset. Now we have 2 identical balls left which we want to throw into 6 distinguishable bins. From class (see note 10), we already saw that the number of ways to put k identical balls into n distinguishable bins is $\binom{n+k-1}{k}$. Taking $k = 2$ and $n = 6$, we get $\binom{7}{2}$ ways to do this.

EXERCISE: Can you give an expression for the number of ways to put k identical balls into n distinguishable bins such that no bin is empty?

(k) **Answer 1:** Let's number the students from 1 to 20. Student 1 has 19 choices for her partner. Let i be the smallest index among students who have not yet been assigned partners. Then no matter what the value of i is (in particular, i could be 2 or 3), student i has 17 choices for her partner. The next smallest indexed student who doesn't have a partner now has 15 choices for her partner. Continuing in this way, the number of pairings is $19 \times 17 \times 15 \times \cdots \times 1 = \prod_{i=1}^{10}(2i-1)$.

Answer 2: Arrange the students numbered 1 to 20 in a line. There are 20! such arrangements. We pair up the students at positions $2i-1$ and $2i$ for i ranging from 1 to 10. You should be able to see that the 20! permutations of the students doesn't miss any possible pairing. However, it counts every different pairing multiple times. Fix any particular pairing of students. In this pairing, the first pair had freedom of 10 positions in any permutation that generated it, the second pair had a freedom of 9 positions in any permutation that generated it, and so on. There is also the freedom for the elements within each pair i.e. in any student pair (x, y), student x could have appeared in position $2i-1$ and student y could have appeared in position $2i$ and also vice versa. This gives 2 ways for each of the 10 pairs. Thus, in total, these freedoms cause $10! \times 2^{10}$ of the 20! permutations to give rise to this particular pairing. This holds for each of the different pairings. Hence, 20! overcounts the number of different pairings by a factor of $10! \times 2^{10}$. Hence, there are $\frac{20!}{10! \cdot 2^{10}}$ pairings.

Answer 3: In the first step, pick a pair of students from the 20 students. There are $\binom{20}{2}$ ways to do this. In the second step, pick a pair of students from the remaining 18 students. There are $\binom{18}{2}$ ways to do this. Keep picking pairs like this, until in the tenth step, you pick a pair of students from the remaining 2 students. There are $\binom{2}{2}$ ways to do this. Multiplying all these, we get $\binom{20}{2}\binom{18}{2}\ldots\binom{2}{2}$. However, in any particular pairing of 20 students, this pairing could have been generated in 10! ways using the above procedure depending on which pairs in the pairing got picked in the first step, second step, ..., tenth step. Hence, we have to divide the above number by 10! to get the number of different pairings. Thus there are $\binom{20}{2}\binom{18}{2}\cdots\binom{2}{2}/10!$ different pairings of 20 students.

You may want to check for yourself that all three methods are producing the same integer, even though they are expressed very differently.

(l) $\binom{n+k}{k}$. This is just n indistinguishable balls into $k+1$ distinguishable bins (stars and bars). There is a bijection between a sequence of n ones and k plusses and a solution to the equation: x_0 is the number of ones before the first plus, x_1 is the number of ones between the first and second plus, etc. A key idea is that if a bijection exists between two sets they must be the same size, so counting the elements of one tells us how many the other has. Note that this is the exact same answer as part (a) — make sure you understand why!

(m) $n-1$. It's easiest just to enumerate the solutions here. x_0 can take values $1,2,\ldots,n-1$ and this uniquely fixes the value of x_1. So, we have $n-1$ ways to do this. But, this is just an example of the more general question below.

(n) $\binom{(n-(k+1))+k}{k} = \binom{n-1}{k}$. This is just $n-(k+1)$ indistinguishable balls into distinguishable $k+1$ bins. By subtracting 1 from all $k+1$ variables, and $k+1$ from the total required, we reduce it to problem with the same form as the previous problem. Once we have a solution to that we reverse the process, and adding 1 to all the non-negative variables gives us positive variables.

2 Fermat's Wristband

Note 7
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Let p be a prime number and let n be a positive integer. We have beads of n different colors, where any two beads of the same color are indistinguishable.

(a) We place p beads onto a string. How many different ways are there to construct such a sequence of p beads with up to n different colors?

(b) How many sequences of p beads on the string are there that use at least two colors?

(c) Now we tie the two ends of the string together, forming a wristband. Two wristbands are equivalent if the sequence of colors on one can be obtained by rotating the beads on the other. (For instance, if we have $n = 3$ colors, red (R), green (G), and blue (B), then the length $p = 5$ necklaces RGGBG, GGBGR, GBGRG, BGRGG, and GRGGB are all equivalent, because these are all rotated versions of each other.)

How many non-equivalent wristbands are there now? Again, the p beads must not all have the same color. (Your answer should be a simple function of n and p.)

[*Hint*: Think about the fact that rotating all the beads on the wristband to another position produces an identical wristband.]

(d) Use your answer to part (c) to prove Fermat's little theorem.

Solution:

(a) n^p. For each of the p beads, there are n possibilities for its colors. Therefore, by the first counting principle, there are n^p different sequences.

(b) $n^p - n$. You can have n sequences of a beads with only one color.

(c) Since p is prime, rotating any sequence by less than p spots will produce a new sequence. As in, there is no number x smaller than p such that rotating the beads by x would cause the pattern to look the same. This is because every other rotation of $x < p$ would only have the sequence and its rotated sequence being equivalent if the sequence was monochromatic (the sequence was just a repetition of one number). If we have a sequence $a_0, a_1, \ldots, a_{p-1}$ and rotate it by x to get $a_x, a_{x+1}, \ldots, a_{x-1}$, the two sequences would only be equal if $a_0 = a_x = a_{2x} = \ldots$, and thus each element would have to be the same. For example, if we had the sequence a_1, a_2, a_3, a_4, a_5, and rotated it by 2 to get a_3, a_4, a_5, a_1, a_2, we can analyze each position of the string. Looking at this first position, this implies that $a_1 = a_3$. Then, looking at the third position, this implies that $a_3 = a_5$, and then $a_5 = a_2$, and $a_2 = a_4$, thus they all have to be equal. This cannot happen in our count, because we are only considering wristbands for which there are at least 2 different colors.

So, every pattern which has more than one color of beads can be rotated to form $p - 1$ other patterns. So the total number of patterns equivalent with some bead sequence is p. Thus, the total number of non-equivalent patterns are $(n^p - n)/p$.

(d) $(n^p - n)/p$ must be an integer, because from the previous part, it is the number of ways to count something. Hence, $n^p - n$ has to be divisible by p, i.e., $n^p \equiv n \pmod{p}$, which is Fermat's Little Theorem.