计算物理第一次作业

祝茗 2024202020022

Bessel 函数

Bessel 方程为

$$x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+x\frac{\mathrm{d}x}{\mathrm{d}y}+(x^2-n^2)y=0$$

其中 n 被称为其对应 Bessel 方程的阶数。实际应用中最常见的情形为 n 是整数，对应解称为 n 阶 Bessel 函数 $J_n(x)$

Bessel 函数 $J_n(x)$ 之间有递推关系

$$J_{n+1}(x)-\frac{2n}{x}J_n(x)+J_{n-1}(x)=0 \tag{1}$$

和一个求和恒等式

$$[J_0(x)]^2+2\sum_{n=1}^{\infty}[J_n(x)]^2=1 \tag{2}$$

根据这两个关系，我们可以来绘制，当 $x=1,10$ 时 Bessel 函数 $J_n(x)$ 作为 n 的函数图像。

可以预见的是

$$|J_n(x)|_{n\to\infty}\to 0$$

考虑到数值计算的精度，当 n 大到一定程度 (例如 $n>M$ 时)，都小于计算机浮点数可以允许的精度

$$|J_n(x)|_{n\geq M}\leq eps$$

所以我们可以做一个假设

$$J_M(x)=0,\quad J_{M-1}=1$$

然后根据递推关系，递推地算出其余 $J_n(x)$ 的相对大小，由于是齐次的线性的关系式，最后根据求和恒等式，就可以得到需要的答案。

导入前置的包并设置基本的参数

In [37]:

```
import numpy as np
import matplotlib.pyplot as plt
from scipy.special import jn
```

```python
epsi = 1e-9
```

计算 Bessel 函数 $J_n(x)$

In [38]:
```python
def BesselJ_with_rescale(x: float | np.ndarray, M: int) -> np.ndarray:
    """计算 Bessel 函数 J_n(x) 的递推值，基于递推关系式。

    对 J_n(x) 进行归一化处理，防止数值溢出。

    Args:
        x (float): 自变量
        M (int): 返回的阶数

    Returns:
        np.ndarray: J_n(x) 的数组

    Example:
        >>> x = 1
        >>> Jn = BesselJ(x, 10)
        >>> plt.plot(x, Jn.T)
        >>> plt.show()
    """
    Jn = np.zeros(M+1)
    Jn[M] = 0  # 假设 J_M(x) = 0
    Jn[M-1] = 1  # 假设 J_{M-1}(x) = 1

    for n in range(M-2, -1, -1):
        Jn[n] = (2*(n+1)/x) * Jn[n+1] - Jn[n+2]

        # 整体缩放，防止递推过程中数值溢出
        if abs(Jn[n]) > 1/epsi:
            Jn *= epsi

    normalization_factor = np.sqrt(Jn[0]**2 + 2 * np.sum(Jn[1:]**2))
    Jn /= normalization_factor
    return Jn


def BesselJ_without_rescale(x: float | np.ndarray, M: int) -> np.ndarray:
    """计算 Bessel 函数 J_n(x) 的递推值，基于递推关系式。

    Args:
        x (float): 自变量
        M (int): 返回的阶数

    Returns:
        np.ndarray: J_n(x) 的数组

    Example:
        >>> x = np.linspace(0, 20, 100)
        >>> Jn = BesselJ(x, 10)
        >>> plt.plot(x, Jn.T)
        >>> plt.show()
    """
    Jn = np.zeros(M+1)
    Jn[M] = 0  # 假设 J_M(x) = 0
    Jn[M-1] = 1  # 假设 J_{M-1}(x) = 1
```

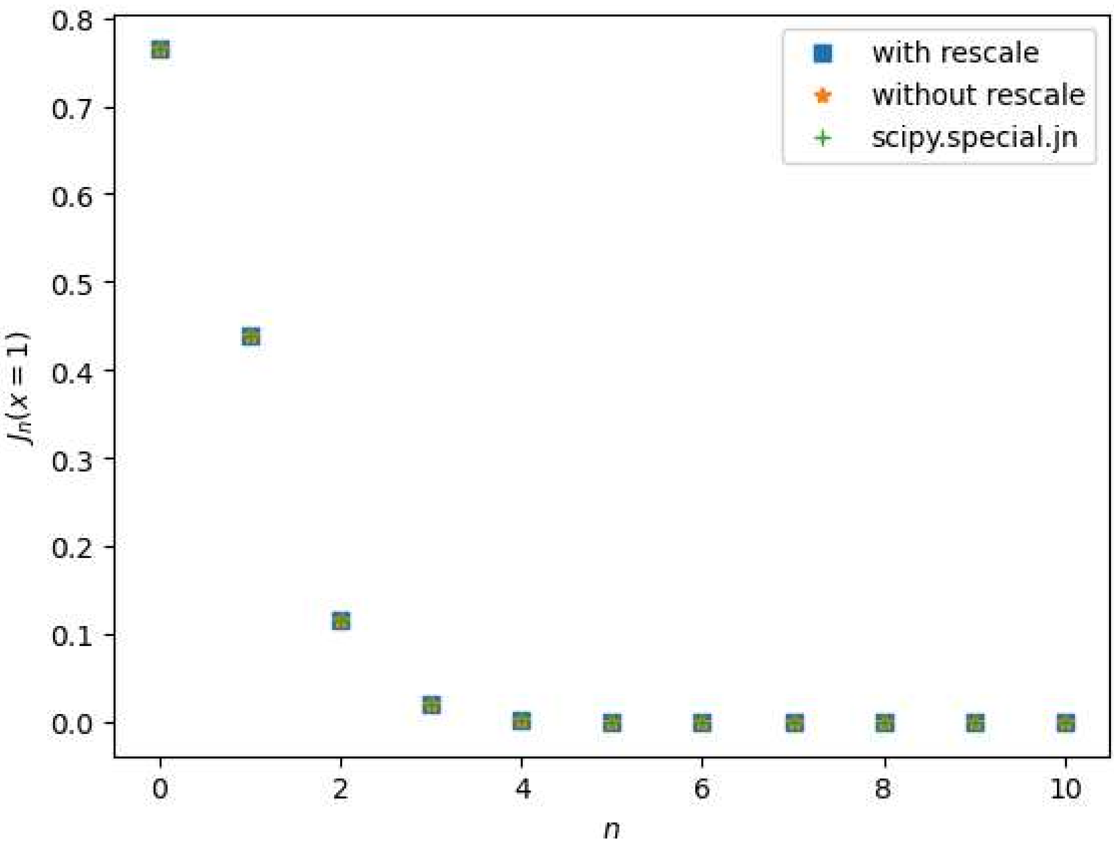
```
    for n in range(M-2, -1, -1):
        Jn[n] = (2*(n+1)/x) * Jn[n+1] - Jn[n+2]

        # 不进行数值溢出处理
        # if abs(Jn[n]) > 1/epsi:
        #     Jn *= epsi

    normalization_factor = np.sqrt(Jn[0]**2 + 2 * np.sum(Jn[1:]**2))
    Jn /= normalization_factor
    return Jn
```

函数效果直观比较

$x = 1, M = 20$

In [73]:

```
x = 1
M = 20

J1 = BesselJ_with_rescale(x, M)
J2 = BesselJ_without_rescale(x, M)
J3 = np.array([jn(n, 1) for n in range(M+1)])

# 前 m 个点的比较
m = 11
J1 = J1[:m]
J2 = J2[:m]
J3 = J3[:m]

fig = plt.figure()

plt.xlabel(f'$n$')
plt.ylabel(f'$J_n(x=1)$')

plt.plot(J1, 's', label='with rescale')
plt.plot(J2, '*', label='without rescale')
plt.plot(J3, '+', label='scipy.special.jn')
plt.legend()
plt.show()
```

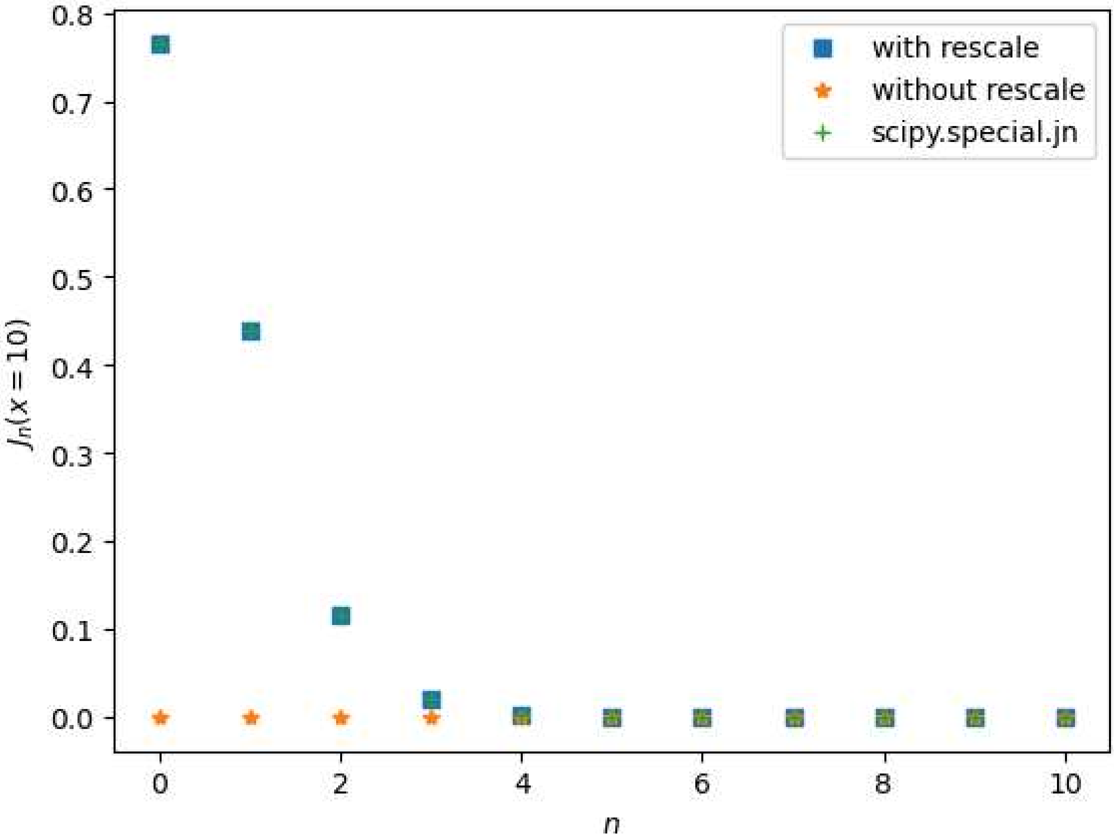


M 较小的时候，是否在递推过程中整体放缩影响不大。

$x = 1, M = 90$

In [75]:
```
x = 1
M = 90

J1 = BesselJ_with_rescale(x, M)
J2 = BesselJ_without_rescale(x, M)
J3 = np.array([jn(n, 1) for n in range(M+1)])

# 前 m 个点的比较
m = 11
J1 = J1[:m]
J2 = J2[:m]
J3 = J3[:m]

fig = plt.figure()

plt.xlabel(f'$n$')
plt.ylabel(f'$J_n(x=10)$')

plt.plot(J1, 's', label='with rescale')
plt.plot(J2, '*', label='without rescale')
plt.plot(J3, '+', label='scipy.special.jn')
plt.legend()
plt.show()
```

```
/tmp/ipykernel_13122/2824819278.py:62: RuntimeWarning: overflow encountered in sc
alar power
  normalization_factor = np.sqrt(Jn[0]**2 + 2 * np.sum(Jn[1:]**2))
/tmp/ipykernel_13122/2824819278.py:62: RuntimeWarning: overflow encountered in sq
uare
  normalization_factor = np.sqrt(Jn[0]**2 + 2 * np.sum(Jn[1:]**2))
```



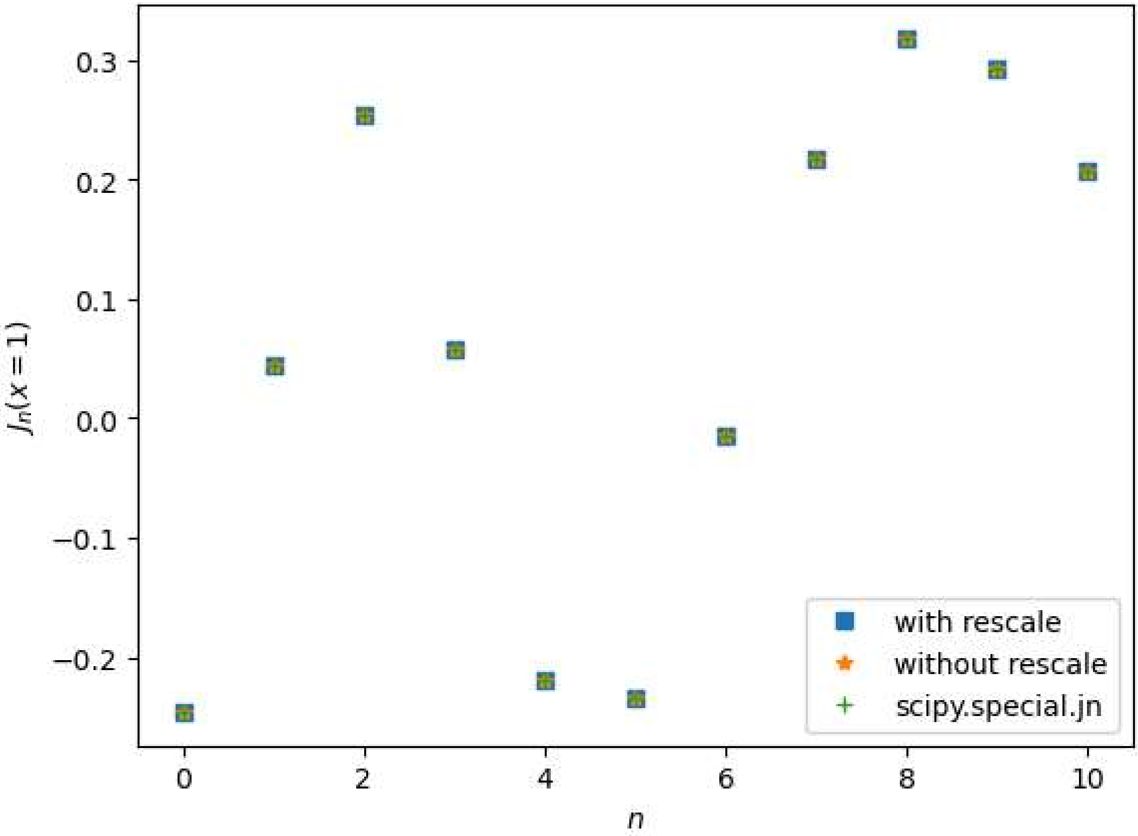
M 较大的时候，是否在递推过程中，如果不整体放缩，确实会对结果产生影响。

$x = 10, M = 20$

In [76]:

```python
x = 10
M = 20

J1 = BesselJ_with_rescale(x, M)
J2 = BesselJ_without_rescale(x, M)
J3 = np.array([jn(n, 10) for n in range(M+1)])

# 前 m 个点的比较
m = 11
J1 = J1[:m]
J2 = J2[:m]
J3 = J3[:m]

fig = plt.figure()

plt.xlabel(f'$n$')
plt.ylabel(f'$J_n(x=1)$')

plt.plot(J1, 's', label='with rescale')
plt.plot(J2, '*', label='without rescale')
plt.plot(J3, '+', label='scipy.special.jn')
```

```
plt.legend()
plt.show()
```

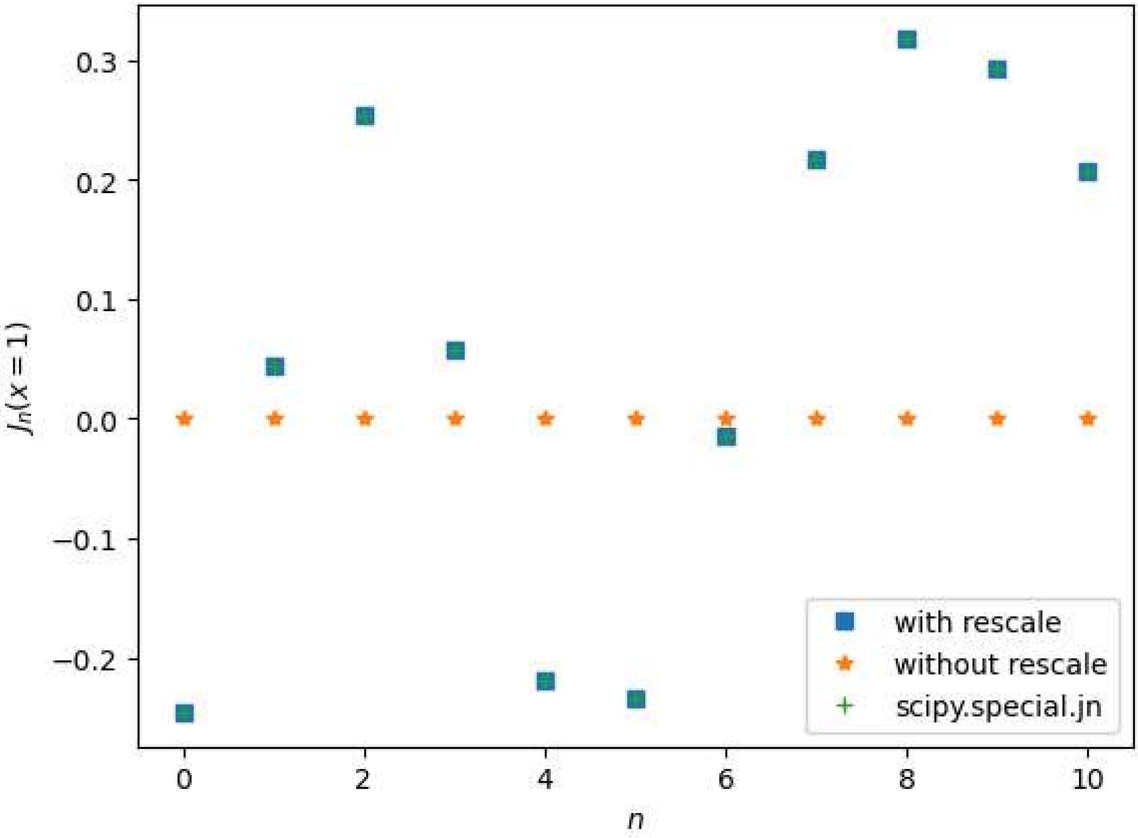


M 较小的时候，是否在递推过程中整体放缩影响不大。

$x = 10, M = 200$

In [81]:
```
x = 10
M = 200

J1 = BesselJ_with_rescale(x, M)
J2 = BesselJ_without_rescale(x, M)
J3 = np.array([jn(n, 10) for n in range(M+1)])

# 前 m 个点的比较
m = 11
J1 = J1[:m]
J2 = J2[:m]
J3 = J3[:m]

fig = plt.figure()

plt.xlabel(f'$n$')
plt.ylabel(f'$J_n(x=1)$')

plt.plot(J1, 's', label='with rescale')
plt.plot(J2, '*', label='without rescale')
plt.plot(J3, '+', label='scipy.special.jn')
plt.legend()
plt.show()
```

```
/tmp/ipykernel_13122/2824819278.py:62: RuntimeWarning: overflow encountered in sc
alar power
  normalization_factor = np.sqrt(Jn[0]**2 + 2 * np.sum(Jn[1:]**2))
/tmp/ipykernel_13122/2824819278.py:62: RuntimeWarning: overflow encountered in sq
uare
  normalization_factor = np.sqrt(Jn[0]**2 + 2 * np.sum(Jn[1:]**2))
```



M 较大的时候，是否在递推过程中，如果不整体放缩，确实会对结果产生影响。

函数效果数值比较

之后部分只使用函数 BesselJ_with_rescale

In [40]:
```python
import os
import pandas as pd

# 读取数据文件
data_x1 = np.loadtxt(os.path.join('source', 'bessel_x1.dat'))
data_x10 = np.loadtxt(os.path.join('source', 'bessel_x10.dat'))

# dateframe
df_x1 = pd.DataFrame(data_x1, columns=['n', 'J_n(x=1)'])
df_x10 = pd.DataFrame(data_x10, columns=['n', 'J_n(x=10)'])
```

In [41]:
```python
cal_J1 = BesselJ_with_rescale(x=1, M=13)
df_x1['cal_J_n(x=1)'] = cal_J1
df_x1['error'] = df_x1['cal_J_n(x=1)'] - df_x1['J_n(x=1)']
print(df_x1)
```

```
       n      J_n(x=1)  cal_J_n(x=1)         error
0    0.0  7.651977e-01  7.651977e-01 -4.203349e-11
1    1.0  4.400506e-01  4.400506e-01  4.493350e-11
2    2.0  1.149035e-01  1.149035e-01  3.190047e-11
3    3.0  1.956335e-02  1.956335e-02  2.668404e-12
4    4.0  2.476639e-03  2.476639e-03  1.099550e-13
5    5.0  2.497577e-04  2.497577e-04  1.123445e-14
6    6.0  2.093834e-05  2.093834e-05  2.389266e-15
7    7.0  1.502326e-06  1.502326e-06  4.368077e-16
8    8.0  9.422344e-08  9.422344e-08 -3.961428e-18
9    9.0  5.249250e-09  5.249250e-09 -1.904972e-19
10  10.0  2.630615e-10  2.630615e-10 -1.867515e-18
11  11.0  1.198007e-11  1.198003e-11 -3.661981e-17
12  12.0  4.999718e-13  4.991680e-13 -8.038662e-16
13  13.0  1.925617e-14  0.000000e+00 -1.925617e-14
```

In [42]:
```python
cal_J10 = BesselJ_with_rescale(x=10, M=34)
df_x10['cal_J_n(x=10)'] = cal_J10
df_x10['error'] = df_x10['cal_J_n(x=10)'] - df_x10['J_n(x=10)']
print(df_x10)
```

```
       n     J_n(x=10)  cal_J_n(x=10)         error
0    0.0 -2.459358e-01  -2.459358e-01  4.865169e-11
1    1.0  4.347275e-02   4.347275e-02 -1.138492e-12
2    2.0  2.546303e-01   2.546303e-01 -1.487943e-11
3    3.0  5.837938e-02   5.837938e-02 -4.813268e-12
4    4.0 -2.196027e-01  -2.196027e-01 -2.008560e-12
5    5.0 -2.340615e-01  -2.340615e-01  1.320641e-11
6    6.0 -1.445884e-02  -1.445884e-02 -4.785042e-12
7    7.0  2.167109e-01   2.167109e-01 -1.494843e-11
8    8.0  3.178541e-01   3.178541e-01  4.385720e-11
9    9.0  2.918557e-01   2.918557e-01 -3.487999e-11
10  10.0  2.074861e-01   2.074861e-01  3.335879e-11
11  11.0  1.231165e-01   1.231165e-01  1.597597e-12
12  12.0  6.337025e-02   6.337025e-02  1.559863e-13
13  13.0  2.897208e-02   2.897208e-02 -3.223248e-12
14  14.0  1.195716e-02   1.195716e-02 -5.364268e-13
15  15.0  4.507973e-03   4.507973e-03 -2.787484e-13
16  16.0  1.566756e-03   1.566756e-03 -2.998205e-13
17  17.0  5.056467e-04   5.056467e-04  1.932460e-14
18  18.0  1.524425e-04   1.524425e-04  4.552419e-14
19  19.0  4.314628e-05   4.314628e-05  4.562519e-15
20  20.0  1.151337e-05   1.151337e-05 -2.186612e-15
21  21.0  2.907199e-06   2.907199e-06 -3.089684e-16
22  22.0  6.968685e-07   6.968685e-07 -1.105346e-17
23  23.0  1.590220e-07   1.590220e-07 -1.966734e-17
24  24.0  3.463263e-08   3.463263e-08  5.837778e-19
25  25.0  7.214635e-09   7.214635e-09  4.694730e-19
26  26.0  1.440545e-09   1.440545e-09 -2.364181e-19
27  27.0  2.762005e-10   2.762005e-10  1.153141e-21
28  28.0  5.093755e-11   5.093755e-11 -2.735481e-20
29  29.0  9.049767e-12   9.049767e-12 -1.213401e-19
30  30.0  1.551096e-12   1.551095e-12 -6.798177e-19
31  31.0  2.568095e-13   2.568055e-13 -3.959366e-18
32  32.0  4.112271e-14   4.109885e-14 -2.386780e-17
33  33.0  6.375893e-15   6.227098e-15 -1.487946e-16
34  34.0  9.581766e-16   0.000000e+00 -9.581766e-16
```

可以看出符合的很好。