Representations

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Contents

1 Representations

1.1 Definitions

- For any group G, an n dimensional **matrix representation** of a group G is a homomorphism
$$\rho : G \to GL_n(\mathbb{C}).$$
- A representation is **faithful** if it is injective.
- Define $c_P : GL_n(\mathbb{C}) \to GL_n(\mathbb{C})$ by $c_P(M) = P^{-1}MP$. Two n dimensional representations ρ_1 and ρ_2 are said to be **equivalent** if $\rho_2 = c_P \circ \rho_1$ for some P.
- An n dimensional **representation** of a group G is (ρ, V) where $\rho : G \to GL(V)$ is a homomorphism and V is an n dimensional complex vector space.
- For any n dimensional complex vector space V with a basis $\mathcal{B}$, there is an isomorphism $\phi^{\mathcal{B}} : GL(V) \to GL_n(\mathbb{C})$. Therefore, for every representation $\rho : G \to GL(V)$, an **associated matrix representation** is $\rho^{\mathcal{B}} = \phi^{\mathcal{B}} \circ \rho : G \to GL(V) \to GL_n(\mathbb{C})$.
- Two matrix representations are equivalent iff they are both associated to the same representation.

1.2 Basic representations

- For any group G, the trivial representation of dimension n is $\rho : g \mapsto I_V$, where I_V is the identity map on V.
- The **permutation representation** of $H \subset S_n$ is constructed by considering the action of H on a set of n basis vectors for V, and extending this to a linear map in $GL(V)$.
- The **regular representation** of G is found by considering the permutation action of G on itself.
- If $f : H \to G$ is a homomorphism, and $\rho : G \to GL(V)$, then $\rho \circ f$ is a representation of H on V. (the composition of homomorphisms is a homomorphism)
- The sign representation of a group is $\rho : g \mapsto \text{sgn}(\sigma_g)$ where σ_g is the permutation corresponding to g.

1.3 G-Linear maps

- For two representations (ρ_1, V) and (ρ_2, W) of G, a G-linear map is a linear map $f : V \to W$ such that for all $g \in G$,
$$f \circ \rho_1(g) = \rho_2(g) \circ f.$$

- Two representations of G are said to be isomorphic if there is a G-linear map between them which is also an isomorphism.
- Isomorphism of representations is the same thing as equivalence of matrix representations in the sense that if two representations are isomorphic, then their associated matrix representations will be equivalent.

1.4 Subrepresentations

- A subrepresentation of (ρ, V) is a representation $(\rho_{|W}, W)$, where W is a subspace of V.
- If $f : V \to W$ is a G-linear map between two representations, then $\mathrm{Ker}(f)$ is a subrepresentation of V and $\mathrm{Im}(f)$ is a subrepresentation of W.
- Applying the above, we see that for a given ρ, if W is an eigenspace of V for every $\rho(g)$, then W is a subrepresentation of V.

1.5 Maschke's Theorem

- For any two representations (ρ_V, V) and (ρ_W, W), there is a natural representation on $V \oplus W$: $\rho_{V\oplus W}(g) : (x, y) \mapsto (\rho_V(g)(x), \rho_W(g)(y))$.
- If two bases $\{a_1, \ldots, a_n\}$ and $\{b_1, \ldots, b_n\}$ are chosen for V and W, then a natural basis for $V \oplus W$ is $\{(a_1, 0), \ldots, (a_n, 0), (0, b_1), \ldots, (0, b_n)\}$. Then if $\rho_V(g)$ and $\rho_W(g)$ have associated matrices M and N, then

$$\rho_{V\oplus W}(g) = \begin{pmatrix} M & 0 \\ 0 & N \end{pmatrix}.$$

- If (ρ, V) is a representation with subrepresentations W and U, where $W \cup W = \{0\}$ and $\dim U + \dim W = \dim V$, then $\rho_{V\oplus W}$ is isomorphic to ρ.
- If $W \subset V$ is a subspace of V, and $f : V \to W : x \mapsto x \forall x \in W$ is a linear map, then
- Maschke's theorem

2 questions

- Why do we sometimes treat a vector space as a representation? (i.e. there are statements like 'V is a 3 dimensional representation of G') Can we not have different homomorphisms from the same group to the vector space? i.e. C_2 to $GL(C^2)$ could map σ to

  $$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

  Are these not different?