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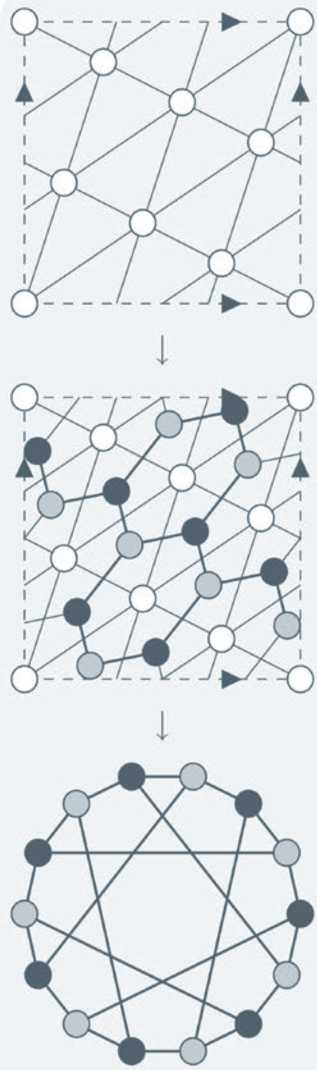
Fundamentals of Graph Theory

Allan Bickle



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Preface

Graph theory is a fascinating and inviting branch of mathematics. It is full of results, both simple and profound. Natural visual representations of many problems invite exploration by new students and professional mathematicians. The goal of this textbook is to present the fundamentals of graph theory to a wide range of readers.

Unlike many other subjects in mathematics, graph theory is quite new. While results on graphs date back to Euler's solution of the Konigsberg Bridge Problem in 1736, graph theory was first studied as a coherent theory by Julius Petersen in 1891. It remained very small until an explosion of research in the 1960s, and it has grown dramatically since then.

One consequence of this history is that teaching of this subject is much less standard than in older subjects like calculus and linear algebra. Any graph theory text must begin with many definitions in the first chapter. After that, there are a number of common topics that can be covered in almost any order. My goal in ordering these topics was to present the Four Color Theorem, the most significant theorem in the history of graph theory, early enough in the book that any class should have no difficulty in reaching it before the end. This theorem ties together vertex coloring and planarity, so I used it as a bridge between the two topics. It comes after a thorough discussion of coloring, and is used to motivate the idea of planarity.

Connectivity and degeneracy are both important in understanding vertex coloring, so they are covered before it, along with basic topics, such as trees and Eulerian graphs. The topics of isomorphisms and degree sequence characterizations are often covered near the beginning in other textbooks, but they tend to cause students problems. They have been moved back, allowing students to learn more graph theory before tackling them. This also facilitates a somewhat deeper examination of these topics. Other common topics, such as Hamiltonian graphs, matchings and domination, are covered later in the book.

Amongst many possible optional topics, I choose topics for which I have a new perspective, topics I find interesting, and topics that have not appeared in textbooks before. These topics include cores, rigid graphs, generalized vertex colorings, and Nordhaus-Gaddum Theorems.

Distinctive Features. There are many other graph theory textbooks, some very good in parts, yet I find all of them unsatisfactory in some ways. This text has a number of distinct features that make it different, and hopefully better than other texts.

(1) **Current Notation.** The fact that graph theory is relatively new means that there have been many changes in notation and terminology. Every effort has been made to use the most current notation, and when there are multiple notations in use, to use the most natural and instructive notation.

(2) **New Results.** New results are continually proved in graph theory, some settling old conjectures or open problems. This text includes a number of significant recent results, including the proof of Alspach's Conjecture, the proof (for large orders) of the 1-factorization and Hamiltonian decomposition conjectures, progress on the chromatic number of unit distance graphs, and the discovery of a quasipolynomial algorithm for the graph isomorpism problem.

(3) **New proofs.** Theorems including Menger's Theorem, Brooks' Theorem, Turan's Theorem, the Perfect Graph Theorem, Tutte's 1-Factor Theorem, Vizing's Theorem, Nash-Williams' Theorem, and the Nordhaus-Gaddum Theorem form the backbone of a first course in graph theory. However, the original proofs of theorems are often not the best possible. Over time, new proofs of classic results are discovered. Unfortunately, many textbooks recycle the same older proofs of these theorems. I have endeavored to include the shortest, most elegant, most intuitive proofs of these classic theorems, including some that have never before appeared in textbooks. Some alternative proofs of theorems are explored in the Exercises, illustrating different proof techniques.

(4) **Motivation by Applications.** The text begins with a section full of practical problems that can be solved using graph theory. There are also separate sections on applications of trees, vertex coloring, and Hamiltonian graphs. Other major topics are introduced with practical applications that motivate their development. This should encourage students who are not sold on mathematics for its own sake, while also interesting pure mathematicians.

(5) **New Approaches.** Many classic topics are presented in new ways. Some of my approaches are (I believe) better, while other are simply different, and offer new ways to think about old topics. Vertex coloring is approached using degeneracy, which simplifies the proofs of many theorems and provides bounds that are superior to most of the better known theorems presented in other textbooks. Chordal graphs and k-trees are presented as generalizations of trees, which facilitates comparisons between these classes and basic results about them. List coloring, vertex arboricity, and other generalizations of vertex coloring are presented in a chapter that illustrates how thinking about a familiar concept in different ways leads to generalizations and new research questions. Block designs, Ramsey numbers, and Nordhaus-Gaddum Theorems

are presented as decompositions, which helps to illustrate connections between these topics. There are many other new approaches scattered throughout this text.

(6) **Applying Theorems.** Many textbooks present theorems, but do not clearly explain how to use them in practice. I include many examples illustrating how to use basic theorems to solve standard problems. For example, several examples show how to apply the numerous bounds on chromatic number to verify the chromatic number of moderate-sized graphs. There is also a detailed discussion of how to find Hamiltonian cycles (or prove none exist) in sparce graphs.

(7) **Brute Force Solutions.** Whether you like it or hate it, it is a fact that many graph theory problems are solved by brute force (case-checking), that is, generating or checking a large number of graphs to verify some fact. Even when there is an elegant solution to a problem, in practice it is often found by checking many or all relevant graphs, observing a pattern, and then conjecturing and proving a theorem. This is a skill that all graph theorists need, but it is entirely ignored by other textbooks. I include several long examples illustrating this process, and some exercises require brute force solutions.

(8) **Internet Resources.** This textbook does not ignore the existence of the internet. It references appropriate internet resources to help students expand their learning. Such resources include the Online Encyclopedia of Integer Sequences (OEIS), which enumerates many graph classes and other graph theory problems.

(9) **Related Terms.** The internet has made it much easier to look up research papers. But one of the biggest problems in mathematical research is not knowing whether something has already been done and given a different name. Without knowing the name of an existing concept, it is very difficult to search for it. Many sections in this text have a list of related terms immediately before the Exercises. A reader looking to extend or modify concepts in this book should first search for these related terms to see what has already been done.

(10) **Homework.** The text has more than 1200 homework exercises, far more than most books. Most of the Exercises are new. There is a wide range of difficultly in the Exercises, allowing the text to be used at many different levels of difficulty. Especially difficult exercises are marked (+). These are generally not good homework problems for undergraduates, but may make good projects or presentation problems. There are also many types of questions, including simple conceptual questions, applications, evaluating parameters, proofs, case-checking problems, and exploration of related concepts. My goal in writing the Exercises was to ask the sort of questions that one would naturally ask when learning a topic.

(11) **Appendices.** There are some "meta-topics" that recur throughout the book. Rather than awkwardly halt a section to discuss them, they are contained in appendices at the end of the book, so they can be covered at any point in a course, or left for background reading. One appendix discusses proofs,

including common proof techniques and how they are applied in graph theory. Another discusses counting techniques and identities, which have many applications in graph theory. Graph theory contains many problems and algorithms, so there is an appendix on the increasingly important topic of computational complexity. Other appendices cover bounds, extremal graphs, and graph characterizations, which allows a deeper look at these common topics in graph theory.

Using This Book. This book is designed to be used at different levels of difficulty. There are almost no formal mathematical prerequisites, only *mathematical maturity*, that is, exposure to enough mathematics to be able to grasp the concepts presented. A previous course in discrete math would be beneficial. However, the essentials on proofs, counting techniques, and computational complexity are presented in the appendices.

A basic knowledge of matrices, including matrix multiplication, is essential. Somewhat more advanced concepts from linear algebra (rank and determinants) are used in the proof of the Perfect Graph Theorem. The concept of groups is alluded to when discussing automorphisms, Tait coloring, and Steiner triple systems. Other areas of math are occasionally mentioned as asides, but these can easily be avoided when readers are unfamiliar with them.

The following are four distinct levels of difficulty at which this book can be used.

(1) **No Proofs.** This would be a class for undergraduates, perhaps in computer science or math education rather than pure math. The students would not be expected to do proofs. The instructor could present proofs, but often an informal illustration of a theorem would be more useful than a formal proof. Such a class would focus on applications, algorithms, and more practical problems, such as evaluating parameters and finding structures like Eulerian trails and Hamiltonian cycles.

(2) **Introduction to Proofs.** This would be a class for undergraduates who have little or no experience writing proofs. It should begin by covering the appendix on proofs. Proofs should be presented formally, though informal illustrations are often good motivations for formal proofs. Since careful proofs take time, more difficult material may need to be omitted. Many easier proofs should be given as exercises.

(3) **Experienced Undergraduates.** This would be a class for math majors who have already had at least one proof-based class, such as modern algebra or number theory. Some proofs may be presented informally, assuming that students are capable of filling in the details themselves. Some more challenging exercises should be assigned.

(4) **Beginning Graduate Students.** This would be an introductory class for graduate students in math. Applications and the first chapter would be omitted as background or summarized quickly. Most of Chapters 2–9 would be presented, including the most difficult proofs. Difficult exercises would be assigned.

Each of these levels requires a careful selection of which topics to cover, and which to omit. Some professional mathematicians never take a course in graph theory and may be unsure which topics to cover or how different topics are related. The following table (Guide for Using this Book, pp. xiv–xv) describes importance (Imp.) of each topic on a scale of 1 to 5, with 5 being the most important. Less important topics are still good, interesting mathematics, but are less well known and are less likely to come up in other books and papers.

The guide also describes how sections depend on each other. Essential dependence (ED) means that an earlier section or chapter must be covered or well understood for a later section to make sense. Inessential dependence (ID) means that some concepts or theorems from the earlier section are used, but they can be quickly explained or omitted when covering the later section.

The guide also includes a recommendation of how many lectures to devote to each section for each of the four levels of difficultly (L1–L4) described above. Recommendations for a 3-credit (36 lecture) course exclude the parenthesized numbers. Recommendations for a 4-credit or two quarter (48 lecture) course include the parenthesized numbers.

Acknowledgments. I dedicate this book to Allen Schwenk, my doctoral advisor, mentor, and friend. I am grateful to him for patiently answering my many questions and providing support for my career. Many of his insights on graph theory are scattered throughout this book.

This book would not be possible without the mathematicians who developed graph theory and the earlier textbooks on this subject. I have been particularly influenced by the texts by West [**2001**] and Chartand and Lesniak [**2005**]. I am also grateful to the reviewers and editors who provided constructive suggestions for improving this text.

Feedback. Comments, corrections, and constructive criticism are welcome. Please inform me of any errors. You can contact me through my website: `allanbickle.wordpress.com`.

Allan Bickle
Penn State Altoona

Guide for Using this Book

§§	Title	Imp.	ED	ID	L1	L2	L3	L4
1.1	Graphs as Models	3			1	.5	.5	0
1.2	Representations of Graphs	5		1.1	1	1	1	.5
1.3	Graph Parameters	5	1.2		.5	.5	.5	0
1.4	Common Graph Classes	5	1.2	1.3	1	.5	.5	0
1.5	Graph Operations	5	1.4	1.3	1	1	1	0
1.6	Distance	4	1.4	1.5	1	1	1	.5
1.7	Bipartite Graphs	5	1.6	1.5	.5	1	.5	(.5)
1.8	Generalizations of Graphs	3	1.2	1.6, 1.7	1	.5	.5	0
2.1	Trees	5	1.6	1.5	1	1.5	1.5	1
2.2	Tree Algorithms	3	2.1		1	(1)	1	(.5)
2.3	Connectivity	5	1.6	1.3, 1.5, 2.1	1.5	2	1.5	1
2.4	Menger's Theorem	5	1.8, 2.3		2	2	2	2
3.1	Eulerian Graphs	5	C1	2.2	1.5	1.5	1	1
3.2	Graph Isomorphism	4	C1		2	2	2	2
3.3	Degree Sequences	3	1.3, 1.4	1.5	(1)	(1)	(2)	(2)
3.4	Degeneracy	4	C1	2.1, 2.4, 3.3	1	1	1.5	2
4.1	Applications of Coloring	2	C1		1	(.5)	1	0
4.2	Coloring Bounds	5	2.3, 3.4		3	3	2	2
4.3	Coloring and Operations	4	4.2	2.3	1	(1)	1	1
4.4	Extremal k-Chromatic Graphs	5	4.2	4.3	1	1	1	1
4.5	Perfect Graphs	4	4.2	3.4	1	(2)	(2)	2
5.1	Four Color Theorem	5	4.2	5.2	1	1	1	1
5.2	Planar Graphs	5	C1	2.1	2	2	2	2

Guide for Using this Book (continued)

§§	Title	Imp.	ED	ID	L1	L2	L3	L4
5.3	Kuratowski's Theorem	4	5.2	2.3, 5.1	1	1	2	2
5.4	Dual Graphs and Geometry	3	5.2	4.2, 5.3	2	2	2	2
5.5	Genus of Graphs	3	4.2, 5.2	5.3	(2)	0	(2)	1
6.1	Finding Hamiltonian Cycles	5	2.3, 2.4	1.3, 3.1	2	2	1.5	1.5
6.2	Hamiltonian Applications	3	6.1		1	.5	0	0
6.3	Hamiltonian Planar Graphs	3	5.2, 6.1		(.5)	.5	.5	.5
6.4	Tournaments	3	1.8, 6.1		0	(.5)	(.5)	0
7.1	Bipartite Matchings	5	C1	2.4	2	2	2	2
7.2	Tutte's 1-Factor Theorem	3	7.1	3.1	(1)	(2)	(1.5)	2
7.3	Edge Coloring	4	7.1	4.2	1	(1.5)	1	2
7.4	Tait Coloring	3	7.3	5.1, 5.2, 6.3	(1)	(1)	1	1
7.5	Domination	3	2.1	3.4, 7.1	(2)	(1.5)	(2)	(1)
8.1	List Coloring	3	4.2, 5.2		0	0	0	1
8.2	Vertex Arboricity	2	2.1, 4.2	6.3	0	0	0	(.5)
8.3	Grundy Numbers	1	4.2		0	0	0	(.5)
8.4	Distance and Sets	1	4.2		0	0	0	(1)
9.1	Decomposing Complete Graphs	3	C1	5.2, 5.5, 6.1	(2)	0	(2)	(3)
9.2	General Decompositions	2	2.1	3.4	0	0	0	(1)
9.3	Ramsey Numbers	3	C1	2.1, 3.4, 4.2	0	0	0	2
9.4	Nordhaus-Gaddum Theorems	2	3.4, 4.2		0	0	0	(2)
10.1	Proofs	5			0	3	0	0
10.2	Counting Techniques	5		10.1	(2.5)	2	2	0

Chapter 1

Basics of Graphs

1.1. Graphs as Models

Social networks have become increasingly prominent in modern life. On the internet, a social network has many people as members. These people are friends with some people and not with others. This is a virtual version of real-life friendships. Instead of friendship, the relationship between people could be acquaintance or it could be biological relation.

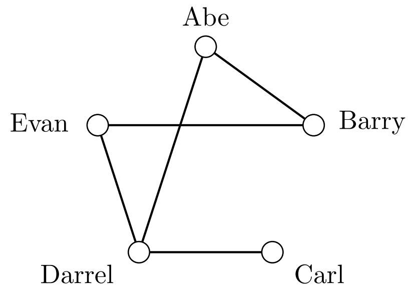
In other industries, more distinct relationships are possible. In the entertainment industry, we can ask whether two people have ever collaborated on some project. In academia, a natural relationship to consider is whether two people have ever coauthored a paper.

There are many questions we can ask about social networks. Who has the most friends? What is the largest number of people who are all friends? Given two people, what is the smallest number of people required to "connect" them?

Rather than try to solve equivalent problems separately in different contexts, we should use a mathematical model that can describe all of these situations. Then we can solve problems once in this abstract setting and apply the results to many different real-world problems.

We can model a social network by drawing a dot to represent each person and a line between two dots when two people are friends.

Example. Abe is friends with Barry and Darrel. Barry is friends with Abe and Evan. Carl is friends with Darrel. Darrel is friends with Abe, Carl, and Evan. Evan is friends with Barry and Darrel. These friendships are illustrated in the figure below.



The same model works just as well for other relationships between people, and many other real-world problems.

Definition 1.1. A **graph** G is a mathematical object consisting of a finite nonempty set of objects called **vertices** $V(G)$ (the **vertex set**), and a set of **edges** $E(G)$ (the **edge set**). An **edge** is two-element subset of the vertex set.

We commonly use G and H for graphs; u, v, $w, \ldots,$ for vertices; and e and f for edges. An edge $e = \{u, v\}$ will typically be written uv or vu, dropping the inconvenient braces.

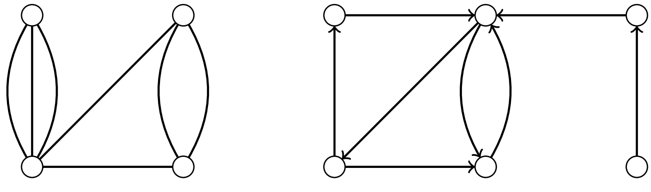
The name "graph" should not be confused with the graph of a function, an unrelated mathematical concept. The term "network", which is common in computer science and technology, would probably be a more intuitive name. However, "graph" is now standard in mathematics. The terms "vertex" and "edge" come from geometry, as they can be used to represent the geometric objects with the same names.

Several variations on the concept of a graph are possible, allowing multiple edges or directed edges. If we allow multiple edges between vertices, the edges must be in a **multiset**, which allows multiple copies of the same object. For directed edges, we replace unordered pairs with ordered pairs of vertices.

Definition 1.2. A **multigraph** G is a mathematical object consisting of a finite nonempty set of objects called vertices $V(G)$ and a multiset $E(G)$ of pairs of vertices.

A **directed graph** (digraph) D is a mathematical object consisting of a finite nonempty set of objects called vertices $V(D)$ and a set $E(D)$ of ordered pairs of distinct vertices called **directed edges**.

A **directed multigraph** replaces the set $E(D)$ with a multiset.



Examples of a multigraph and a digraph are shown above. Every graph is also a multigraph. A multigraph is allowed, but not required, to have multiple edges between pairs of vertices.

Each of these mathematical objects can be used to model many real-world situations. Which one is chosen depends on whether multiple or directed edges make sense in the context of the problem.

Graphs can model transportation networks.

Example. A network of roads can be modeled using graph theory. Vertices represent intersections, and edges represent road segments. Often, a graph will be sufficient to model this situation. However, some areas have one-way streets, which should be represented by directed edges. Thus a digraph is the appropriate model in this situation. Sometimes there may be more than one road between the same pair of intersections. In this case, we would use a multigraph with multiple edges between some vertices.

Example. Airplane flights can be modeled with vertices representing airports. If edges represent flights, they are directed and likely multiple. Edges could alternatively represent the existence of a regular flight between two airports. In this case, the edges are not multiple.

Graphs also model communication and information networks.

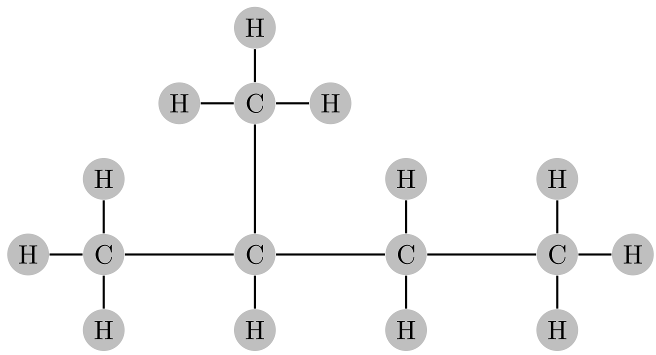
Example. A computer network has several computers connected by cables. A graph modeling this situation has vertices represent computers and edges represent cables.

Example. The **web graph** is a digraph that models the internet. Vertices represent webpages. A directed edge represents when one website links to another. The web graph is very large and growing. It changes as sites are added or deleted, and as links are added or deleted. Many of the other graphs modeling real-world situations also change over time.

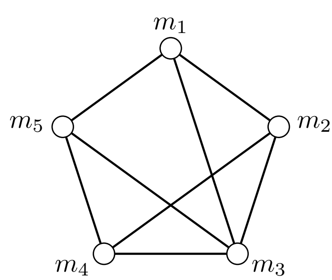
Example. A **citation graph** has vertices representing documents such as academic papers, patents, or legal opinions. A directed edge goes to the cited document from the document that cites it. Citations point toward documents further back in time. However, it is possible for two academic papers to both cite each other if they were written by the same author or if their authors corresponded while they were being written.

Graphs can model topics as diverse as chemical molecules and scheduling meetings.

Example. A chemical molecule has several atoms that are held together by chemical bonds. A graph models this situation with vertices for atoms and edges for chemical bonds. Vertex and edge labels may be necessary to identify different atoms and different types of bonds. A graph representing isopentane, C_5H_{12}, is shown below.



Example. Eight businessmen must have five meetings. The sets of people in meetings are $m_1 = \{1, 2, 3, 4\}$, $m_2 = \{1, 5, 6\}$, $m_3 = \{2, 5, 7\}$, $m_4 = \{6, 7, 8\}$, and $m_5 = \{3, 4, 7, 8\}$. We draw a graph where vertices represent meetings and there is an edge between vertices when the sets have a nonempty intersection. We would like to schedule the meetings in as few time slots as possible. Observation shows that meetings 1 and 4 can be scheduled in one slot, 2 and 5 in another, and 3 in a third. Meetings 1, 2, and 3 all need separate slots, so three slots are required. The graph in this example is called an **intersection graph**, which has vertices represent sets and edges between vertices when the sets have a nonempty intersection.



The situations that can be modeled with graph theory are seemingly endless. You can even construct a graph whose vertices represent other graphs! Many more situations that can be modeled with graph theory are explored in the Exercises.

1.2. Representations of Graphs

There are several terms associated with a graph containing an edge $e = uv$.

Definition 1.3. If $e = uv$ and $f = uw$ are edges of a graph, then we write $u \leftrightarrow v$, and we say u and v are **adjacent**, u and v are **neighbors**, u is **adjacent to** v, e **joins** u and v, u is **joined to** v, e and v are **incident**, and e and f are **adjacent**. If u and v are not adjacent, we write $u \not\leftrightarrow v$, and we say u and v are **nonadjacent.**

We can describe a graph by simply listing the vertex and edge sets. While this description is accurate, it tends not to be particularly helpful for solving problems or making new discoveries. There are alternative ways of describing graphs.

Definition 1.4. An **adjacency list** of a graph is a list of each vertex in a separate row followed by a list of the vertices that it is adjacent to. An **adjacency matrix**

of a graph G with vertex set $\{v_1, \ldots, v_n\}$ is an $n \times n$ matrix with i, j entry 1 when $v_i v_j \in E(G)$, and 0 otherwise.

The adjacency matrix of a graph is symmetric and has 0's on the diagonal.

Example. Let G be the graph with vertex set $V(G) = \{1, 2, 3, 4, 5\}$ and edge set $E(G) = \{12, 13, 23, 24, 34, 45\}$. The adjacency list and adjacency matrix are given below.

Vertex	Adjacent to
1	2, 3
2	1, 3, 4
3	1, 2, 4
4	2, 3, 5
5	4

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

For sparse graphs (those with relatively few edges), an adjacency list uses less space than an adjacency matrix. For dense graphs (those with many edges), the space usage is comparable. Each has advantages and disadvantages for computational efficiency. However, the adjacency matrix has more theoretical uses, since the theory of linear algebra can be applied. For example, in Section 1.6, it is used to count walks in graphs.

It is often beneficial to represent a graph using a drawing. Indeed, much of the appeal of graph theory comes from fact that graphs can be drawn and analyzed visually.

Definition 1.5. A **drawing of a graph** is a diagram with a small circle (either open or solid) in the plane representing each vertex and a curve (often a straight line) joining two circles representing each edge.

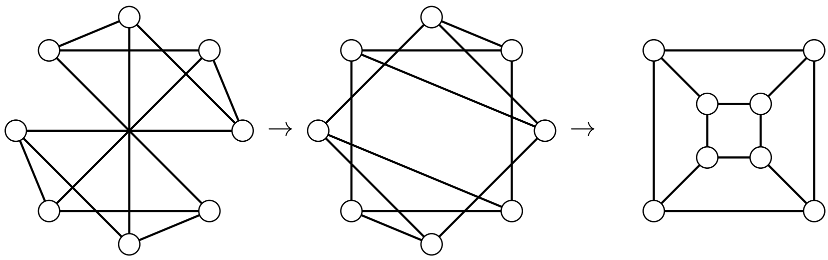
The names of the vertices can be written near them in the drawing. Less commonly, the names of edges can also.

But how should a graph be drawn? At a minimum, vertices must have distinct locations, and edges must not intersect vertices to which they are not incident. Edges may cross, but usually they should not cross more than once. But there are still (infinitely) many ways to draw a graph. We may prefer a drawing that has other desirable properties, such as no edge crossings (if possible), reflective or rotational symmetry, and no long edges. There is no one best drawing; different drawings may emphasize different properties of a graph. The field of **graph drawing** studies how best to visually represent graphs.

One way to satisfy the essential conditions above is to place all the vertices on a circle and make all the edges straight lines that are chords of the circle. Some computer programs use this method when not given any information about where to place the vertices. This method tends to lead to many edge crossings and long edges, however.

Another possibility is to start with one vertex, draw its neighbors close to it, try to draw their neighbors close to them, and so on. No matter what method is used, it usually takes several attempts to find a desirable drawing. An undesirable drawing can be improved by moving one or more vertices to another location and repeating this step until a better drawing is found.

Example. Suppose we draw the graph below left, with its vertices initially around a circle. We notice that the graph contains several 4-cycles. We swap two pairs of vertices to unravel two 4-cycles, obtaining the middle drawing. Finally, we move one 4-cycle inside the other so that no edges cross, obtaining the drawing at right. We obtain a symmetric, visually pleasing drawing.



The fact that a graph can be drawn in many different ways raises the issue that two graphs that appear to be different may be essentially the same. One way this may be is that the vertex sets are different, but the vertices can be renamed so that the graphs are the same. Equivalently, the graphs can be drawn identically except for the vertex names. Another possibility is that the vertices may have the same names, but the graphs may be drawn differently.

We say that two such graphs are **isomorphic**. Loosely speaking, this means that the vertices can be renamed or moved so that the graphs are the same. It is usually not too hard to tell whether or not two graphs are the same, but finding a method that always works is a difficult problem. We consider this problem in depth in Section 3.2, where we also state a precise definition of **isomorphism**.

It is possible to draw the diagram of a graph without naming its vertices. We call this an **unlabeled graph**. An unlabeled graph can be viewed as representing all possible labeled graphs with the same structure. Two unlabeled graphs are isomorphic if it is possible to label their vertices so that they are the same. If two unlabeled graphs G and H are isomorphic, we write $G = H$.

We will refer to a graph with named vertices as a **labeled graph**. Without additional context, "graph" will mean an unlabeled graph. Practical applications usually involve labeled graphs, while theoretical problems more commonly involve unlabeled graphs. When we work with a labeled graph that does not derive from a specific application, we simply need n distinct names for the vertices, where n is the number of vertices in the graph. The natural choice is to use the set $[n] = \{1, \ldots, n\}$ as the vertex set.

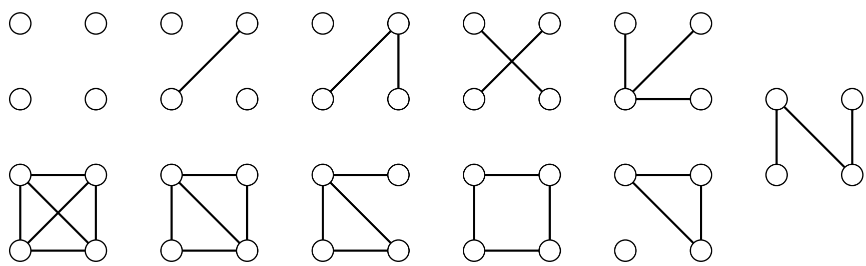
Example. Suppose we want to count the number of labeled graphs with vertex set $[n]$. There are $\binom{n}{2} = \frac{n(n-1)}{2}$ pairs of vertices, each of which may be joined by an edge or not. There are two possibilities for each pair, and the choices are independent. Thus there are $2^{\binom{n}{2}}$ labeled graphs with vertex set $[n]$.

Basic counting techniques, including those just used, are discussed in Section 2 of the Appendix.

There are $n!$ ways to permute the vertices of a labeled graph. Thus there are at least $\frac{2^{\binom{n}{2}}}{n!}$ unlabeled graphs. This is not exact since the number of labeled graphs corresponding to each unlabeled graph varies. Using generating functions, it is possible to generate the sequence of the number of unlabeled graphs with n vertices: 1, 2, 4, 11, 34, 156, 1044, 12346, ... (sequence A000088 in the Online Encyclopedia of Integer Sequences (OEIS)). This sequence is asymptotic to $\frac{2^{\binom{n}{2}}}{n!}$.

Example. To find all unlabeled graphs with four vertices, we could just start drawing graphs. But then we could not be certain that we had found all of them. This is one of many problems in graph theory that require a careful "case-checking" argument to be certain that all solutions have been found. We need to break down the possibilities using the properties of the graphs.

With four vertices, the number of edges must be between 0 and 6. There is only one graph when the number of edges is 0, 1, 5, or 6. When there are two edges, they can either be adjacent or not. When there are four edges, the two missing edges can either be adjacent or not. When there are three edges, they could all be incident with the same vertex. When two are adjacent, the third can either be adjacent to both of their other ends, or only one. Thus there are 11 graphs with four vertices (graphed below), and we can be confident we have them all.



1.3. Graph Parameters

We need simple notation for the number of vertices and edges of a graph.

Definition 1.6. The **order** $n(G) = |V(G)|$ of a graph G is the number of vertices of G. The **size** $m(G) = |E(G)|$ of a graph G is the number of edges of G.

When the context is clear (typically when there is a single graph under discussion), we will simplify the notation for order and size to n and m, respectively. We will only use n and m to represent order and size (thus the complete graph K_n has order n), but caution is still warranted. For example, the statement "$n(K_n - v) = n - 1$" should be revised to "$n(K_r - v) = r - 1$" or "the order of $K_n - v$ is $n - 1$."

We note in passing that graph theory is a relatively young subject, and its notation is still evolving. Most terminology and notation have changed from past texts and papers. Some notation is still not standardized (for example, the notations p, $|G|$, n, $n(G)$, and $|V(G)|$ are all common for order). Other terms are now standard but have varied in the past. To avoid unnecessary confusion, this text will usually not mention alternate terminology or notation, but readers are urged to use caution when reading other sources.

Order and size are examples of functions that are defined on all graphs.

Definition 1.7. A **graph parameter** $f(G)$ is a function that is from some or all graphs to the real numbers.

Most parameters of interest have integer values. Thinking of parameters as functions is not especially beneficial in graph theory. For one thing, parameters of interest are almost never one-to-one, so inverse functions usually do not exist.

Some parameters are defined on the vertices of a particular graph.

Definition 1.8. The **degree** of a vertex v, written $d_G(v)$ or $d(v)$ when the graph in question is clear, is the number of edges incident with v. A vertex with degree 0 is an **isolated vertex**. A vertex of degree 1 is a **leaf**. An **even vertex** has even degree; an **odd vertex** has odd degree. The **neighborhood** of a vertex v, $N(v)$, is the set of neighbors of v.

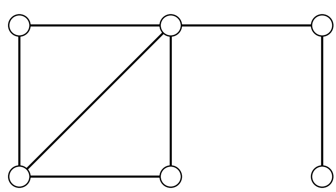
Note that the degree of a vertex is the number of vertices in its row of the adjacency list, and the number of 1's in its row (or column) of the adjacency matrix.

Example. Airlines usually do not fly directly between two smaller cities. Instead, they fly from a smaller city airport to a hub airport, which has flights to many other smaller airports and other hub airports. A hub airport can be identified by having a relatively large degree in a graph representing flights.

We can also consider the degrees of an entire graph.

Definition 1.9. The **degree sequence** of a graph G is the list of its degrees, usually written in nonincreasing order. Its **minimum degree** is $\delta(G)$. Its **maximum degree** is $\Delta(G)$. It is **regular** if every vertex has the same degree (**k-regular** if the common degree is k). A 3-regular graph is a **cubic graph**.

Example. The following graph G has degree sequence 4, 3, 2, 2, 2, 1, $\Delta(G) = 4$, and $\delta(G) = 1$.



There are many relationships among graph parameters. The following theorem is basic enough that it is called the First Theorem of Graph Theory.

Theorem 1.10 (First Theorem of Graph Theory). *If G is a graph, then* $\sum d(v_i) = 2m$.

Proof. Consider the set of all vertex-edge incidences (the "ends" of edges) in a graph. Partitioning the set by vertices shows its cardinality is the sum of the degrees. Partitioning the set by edges shows each edge appears twice, so its cardinality is $2m$. Thus $\sum d(v_i) = 2m$. □

This theorem is also known as the **degree sum formula**. The proof uses the technique known as *counting two ways*, which partitions a set two different ways and counts both ways to obtain an identity. This technique is explored in detail in Section 2 of the Appendix on counting techniques.

Corollary 1.11. *Every graph has an even number of vertices of odd degree.*

Proof. Assume to the contrary that a graph has an odd number of odd vertices. Then its degree sum must be odd, which contradicts the First Theorem. □

Both this corollary and the First Theorem are sometimes called the **Handshaking Lemma**. The results are interpreted involving people at a party shaking hands. With vertices representing people and edges representing handshakes, it follows that an even number of people shake an odd number of hands.

Several other observations follow immediately from the First Theorem. The average degree of a graph is $\frac{2m}{n}$. Hence $\delta(G) \le \frac{2m}{n} \le \Delta(G)$. Also, the size of a k-regular graph with order n is $m = \frac{nk}{2}$.

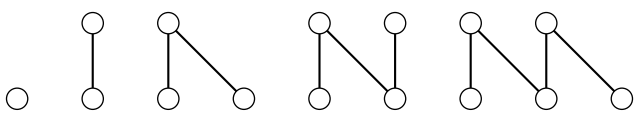
1.4. Common Graph Classes

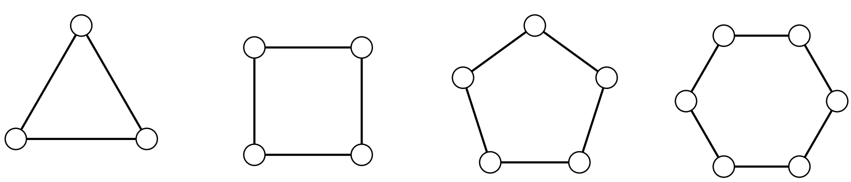
Definition 1.12. A **graph class** is a set of graphs. We denote a graph class using blackboard bold ($\mathbb{G}$ or $\mathbb{H}$) if it does not have its own notation.

We now introduce several important graph classes.

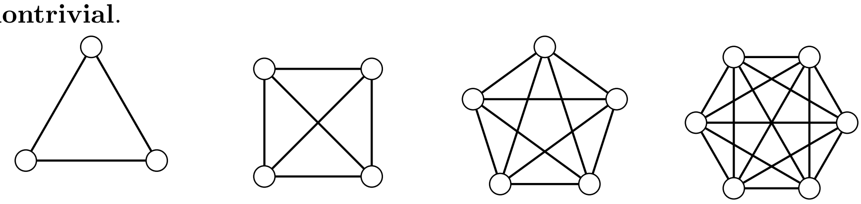
Definition 1.13. A **path** P_n is a graph whose vertices can be numbered $v_1, v_2, \ldots,$ v_n so that its edges are $v_1v_2, \ldots, v_{n-1}v_n$. A **cycle** C_n (or n-cycle) is a graph whose vertices can be numbered $v_1, v_2, \ldots, v_n$ so that its edges are $v_1v_2, \ldots, v_{n-1}v_n$, and v_nv_1. An **even cycle** has n even, and an **odd cycle** has n odd.

Small paths and cycles are illustrated in the following figures.





Definition 1.14. A **complete graph** K_n has order n and every pair of vertices is adjacent. An **empty graph** $\overline{K}_n$ has order n and no edges. The (unique) graph with one vertex K_1 is called the **trivial graph**. A graph with more than one vertex is **nontrivial**.



The complete graph K_n has size $\binom{n}{2} = \frac{n(n-1)}{2}$ since every pair of vertices produces an edge. Complete and empty graphs are opposite extremes for the size of a graph with n vertices.

These graph classes are interesting in their own right. When we study an unfamiliar parameter or property of graphs, we usually apply it to these classes first, since they are familiar and easy to define. However, these classes are also important because they are contained in other graphs.

Many mathematical objects contain subobjects. Sets contain subsets. Groups contain subgroups. Vector spaces contain subspaces. The same is true for graphs.

Definition 1.15. A graph H is a **subgraph** of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ so that the edges in $E(H)$ use only vertices in $V(H)$. We write $H \subseteq G$ and say G **contains** H. An **induced subgraph** is a subgraph $G[S]$ with vertex $S \subseteq V(G)$ and all edges with both ends in S.

Equivalently, an induced subgraph can be obtained by deleting a set of vertices and all edges incident with them.

Definition 1.16. A **clique** is a complete subgraph, or the set of vertices inducing a complete subgraph. An **independent set** of vertices is a set that induces an empty graph.

Both of these concepts are used heavily in graph coloring and elsewhere in graph theory.

Graphs that do not contain certain subgraphs are also of interest.

Definition 1.17. A graph G is H**-free** if it does not contain any induced subgraph isomorphic to H.

The graph K_3 is called a **triangle**. A K_3-free graph is called **triangle-free**.

Paths are relevant as subgraphs when discussing connectivity and distance (Section 1.6). Cycles are relevant as subgraphs in Eulerian and Hamiltonian graphs, planar graphs, and more.

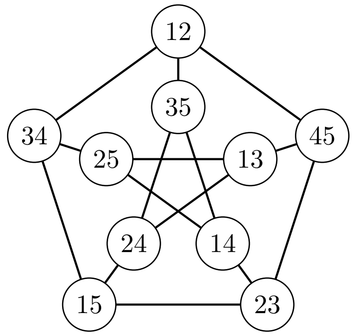
Definition 1.18. A graph, subgraph, or structure of a graph is **maximal** if no larger one contains it. It is **maximum** if it is as large as possible. The definitions of **minimal** and **minimum** are similar.

Lemma 1.19. *If a graph G has $\delta(G) \geq 2$, then it contains a cycle.*

Proof. Let P be a maximal path in G with vertices $v_1, \ldots, v_n$. Now v_n has degree at least 2, so it is adjacent to some vertex $u \neq v_{n-1}$. If u is not on the path, it could be extended, contradicting its maximality. Thus u must be on the path, forming a cycle. □

The distinction between *maximal* and *maximum* is important. Any maximum structure is maximal, but the converse may not be true. A maximal path may not be maximum. The same is true for cliques and independent sets.

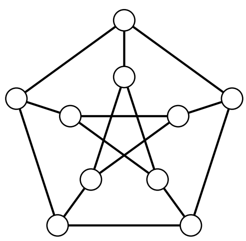
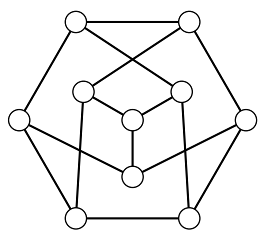
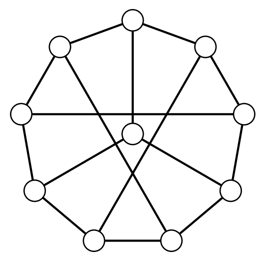
Definition 1.20. The **Petersen graph** has vertices that represent the 2-element subsets of $[5] = \{1, 2, 3, 4, 5\}$. Edges join vertices whose subsets are disjoint.



The Petersen graph is a particularly important example in graph theory. It has several surprising properties that will be revealed throughout this text. In fact, there is an entire book about it (Holton/Sheehan [**1993**])!

There are $\binom{5}{2} = 10$ 2-element subsets of [5], so the Petersen graph has order 10. There are three elements of [5] not contained in a given 2-element subset, so each vertex is adjacent to three others. Thus the Petersen graph is cubic, so it has size $\frac{10\cdot 3}{2} = 15$. There cannot be three disjoint 2-element subsets of [5], so the Petersen graph is triangle-free.

It can be drawn to emphasize two 5-cycles with vertices 12, 34, 51, 23, 45 and 13, 52, 41, 35, 24, respectively, with edges matching vertices on each cycle. Other drawings of the Petersen graph emphasize other interesting properties.

Proposition 1.21. *Two nonadjacent vertices of the Petersen graph have exactly one common neighbor. Thus the Petersen graph has no* 4*-cycle.*

Proof. Nonadjacent vertices have sets with one element in common. Thus there are two elements of [5] in neither of them, which produces one vertex adjacent to both. A 4-cycle would require nonadjacent vertices with two common neighbors. □

Definition 1.22. The **girth** of a graph is the length of its shortest cycle.

Thus the Petersen graph has girth 5.

Counting different types of graphs or subgraphs is a common problem in graph theory. The following result uses the counting technique known as counting by bijection (Appendix, Section 2).

Proposition 1.23. *The Petersen graph contains* 15 8*-cycles.*

Proof. Any 8-cycle omits two vertices. They must be adjacent, or else they would have a common neighbor that could not be on the cycle. Deleting two adjacent vertices from the Petersen graph results in a graph that clearly contains a single 8-cycle. The Petersen graph has 15 edges and, hence, 15 pairs of adjacent vertices, so it has 15 8-cycles. □

There are several classes of graphs that generalize the Petersen graph, which are explored in the Exercises.

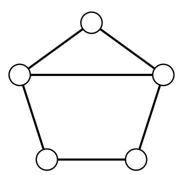
1.5. Graph Operations

Almost any type of mathematical object has corresponding mathematical operations to manipulate the objects and produce new objects. Numbers can be negated, added, or divided. Functions can be multiplied or composed. Sets can be complemented or intersected. Thus it is natural that analogous operations exist for graphs.

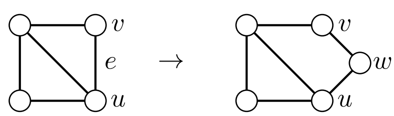
A **unary operation** produces a new graph when given a single graph.

Definition 1.24. The graph $G-e$ or $G-X$ is obtained by deleting edge e or edge set X from $E(G)$. The graph $G-v$ or $G-S$ is obtained by deleting vertex v or vertex set S and all incident edges from G. The graph $G+e$ is obtained by adding e to the edges of G.

When G is an unlabeled graph, $G-e$ will only be defined when a particular edge is specified or when deleting any edge produces the same graph. For example, $C_n - e = P_n$ no matter which edge is deleted. The same will hold for vertices. When e is unspecified, $G+e$ will only be defined when any choice yields the same graph. The graph C_5 satisfies this condition, so C_5+e is defined.



Definition 1.25. A **subdivision of an edge** $e = uv$, deletes e and adds vertex w and edges uw and wv. A graph H is a **subdivision of a graph** G if it can be obtained by some number (perhaps zero) of subdivisions of edges of G.



Loosely speaking, a subdivision can be thought of as inserting a vertex on an edge. Subdivisions are important when we are mainly interested in whether there is a path between vertices of a graph, regardless of length. This includes connectivity (Section 2.3) and planarity (Section 5.2). Note that a subdivision of an edge increases the order and size by one.

The preceding operations can be considered "local" operations that leave most of the graph unchanged. Unary operations can also change the entire graph.

Definition 1.26. The **complement** $\overline{G}$ of a graph G has the same vertex set and edge $uv \in V\left(\overline{G}\right)$ if and only if $uv \notin V(G)$. A **decomposition** of G is a set of nonempty subgraphs whose edge sets partition $E(G)$. The subgraphs are said to **decompose** G.

Note that an empty graph is the complement of a complete graph of the same order (and vice versa), explaining the notation $\overline{K}_n$ for empty graphs. A graph G of order n and its complement decompose K_n.

Example. The complement of C_4 is $2K_2$, so $\{C_4, 2K_2\}$ is a decomposition of K_4.

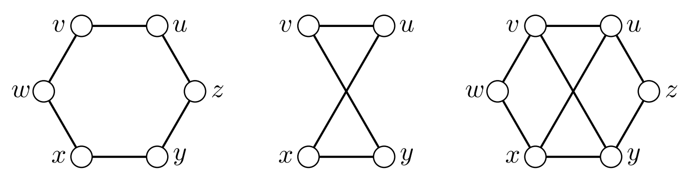
A unary operation may produce a graph with a completely different vertex set. An example of this is the line graph (Section 2.4).

A **binary graph operation** uses two graphs to produce a new graph. We consider several binary operations: the union, join, and Cartesian product of graphs.

Definition 1.27. The **union of graphs** G and H, $G \cup H$, is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. The union is a **disjoint union** if the vertex sets of the graphs are disjoint. A disjoint union of k copies of G is denoted kG.

When we work with unlabeled graphs, we consider a union disjoint.

Example. The union of the two graphs at left is the graph at right.



Empty graphs, complete graphs, and cycles are all regular. In fact, 0-regular graphs are exactly the empty graphs, which can also be denoted nK_1. Any 1-regular graph must be composed of disjoint copies of K_2, that is kK_2. Next we characterize the structure of 2-regular graphs.

Proposition 1.28. *Any 2-regular graph is a disjoint union of cycles.*

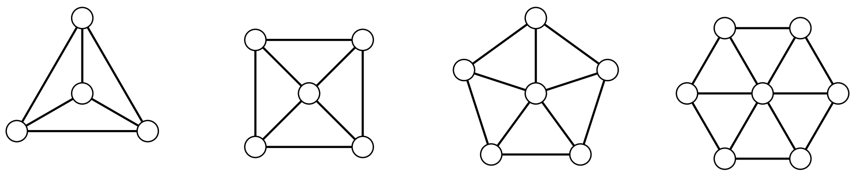
Proof. We use strong induction on the number of cycles of a 2-regular graph G. By Lemma 1.19, G must contain a cycle. If G is a single cycle, we are done. Assume the result holds for graphs with fewer than k cycles. Let G contain k cycles, one of which is C. All of the vertices of C have degree 2 in G. Deleting these vertices produces a 2-regular graph H with fewer than k cycles. Thus H must be a disjoint union of cycles, and hence so is G. $\square$

Strong induction is a common proof technique in graph theory. Section 1 of the Appendix reviews common proof techniques.

The structure of k-regular graphs is much more complicated when $k \geq 3$. This problem is considered further in later sections.

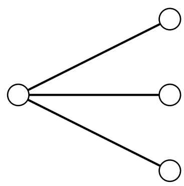
Definition 1.29. The **join of graphs** G and H, $G + H$, is obtained from the disjoint union $G \cup H$ by adding the edges $\{uv : u \in V(G), v \in V(H)\}$.

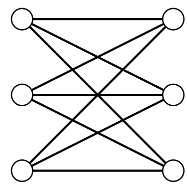
The **wheel** W_n, $n \geq 4$, is the join of a cycle and a single vertex, $W_n = C_{n-1} + K_1$.



Definition 1.30. The **complete bipartite graph** $K_{r,s}$ is the join of two empty graphs, $K_{r,s} = \overline{K}_r + \overline{K}_s$. A **star** is the complete bipartite graph $K_{1,s}$. The star $K_{1,3}$ is known a **claw** when it is an induced subgraph of another graph.

General (noncomplete) bipartite graphs will be defined in Section 1.7. The graphs $K_{1,3}$ and $K_{3,3}$ are shown below.

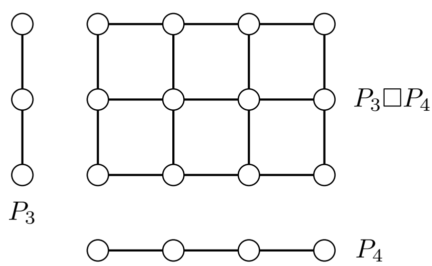




Definition 1.31. The **Cartesian product** of G and H, $G \square H$, has vertex set $V(G) \times V(H)$ and (u, v) adjacent to (u', v') if $u = u'$ and $vv' \in E(H)$ or $v = v'$ and $uu' \in E(G)$.

The notation $G \square H$ symbolizes the fact that $K_2 \square K_2 = C_4$. Note that $G \square H$ has a copy of G for each vertex of H, and vice versa. Thus it decomposes into copies of G and H.

Example. The Cartesian product of P_3 and P_4 is shown below.

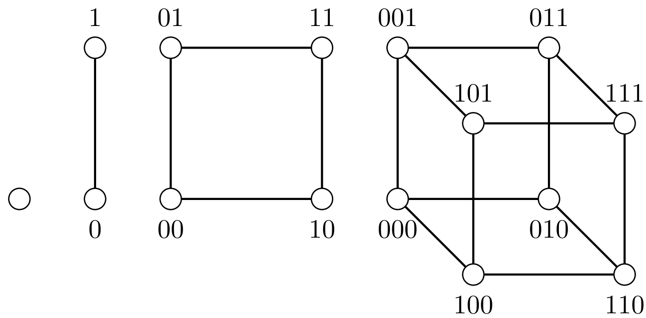


Definition 1.32. The **grid** $G_{r,s}$ is the Cartesian product of two paths, $G_{r,s} = P_r \square P_s$.

The definitions of the union, join, and Cartesian product all extend in the natural way to more than two graphs. (Note that they are all associative and commutative operations.) This can be used to define another important class of graphs.

Definition 1.33. The **hypercube** Q_k is defined recursively by $Q_1 = K_2$ and $Q_k = Q_{k-1} \square K_2$.

Note that $Q_2 = C_4$ and Q_3 represents the usual three-dimensional cube. The hypercube Q_k can also be defined as a graph with vertices representing k-digit bitstrings that are adjacent when they differ in a single digit. Note that Q_k is k-regular and has order $n(Q_k) = 2^k$. Thus $m(Q_k) = \frac{k \cdot 2^k}{2} = k \cdot 2^{k-1}$.



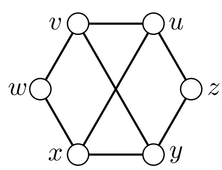
1.6. Distance

We now consider how we can travel around a graph via adjacent edges.

Definition 1.34. A **walk** is a list $v_0, e_1, v_1, \ldots, e_k, v_k$ of vertices and edges such that $e_i = v_{i-1}v_i$. A $u-v$ walk has $u = v_0$ and $v = v_k$. These are its **endpoints**. A walk is **closed** if its endpoints are the same. A $u-v$ **path** is a path with endpoints u and v. Its other vertices are **internal vertices**. The **length** of a walk, path, or cycle is the number of edges it contains.

Note that a path is a walk with no repeated vertices. The length of P_n is $n-1$.

Example. In the graph below, u, z, y, v, u, x, y, v is a $u-v$ walk of length 7, and u, x, y, v is a $u-v$ path of length 3.



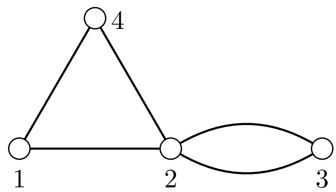
The adjacency matrix can be used to count walks and some cycles.

Theorem 1.35. *Let G be a multigraph with vertices $v_1, \ldots, v_n$ and adjacency matrix $A(G)$. Then the i, j entry of A^k is the number of $v_i - v_j$ walks of length k in G.*

Proof. We use induction on k. For $k = 1$, the result is immediate since a walk of length 1 contains a single edge. Assume for some k that the i, j entry of A^k, $a_{ij}^{(k)}$, is the number of $v_i - v_j$ walks of length k in G. Now $A^{k+1} = A^k A$, so $a_{ij}^{(k+1)} = \sum_{t=1}^{n} a_{it}^{(k)} a_{tj}$. Now every $v_i - v_j$ walk of length $k+1$ contains a $v_i - v_t$ walk of length k and an edge from v_t to v_j. Thus the summation counts the number of $v_i - v_j$ walks of length $k+1$. □

Example. In the movie Good Will Hunting, a four-part problem is presented. For the graph below, the first two parts ask for

1. the adjacency matrix A,
2. the matrix giving the number of 3-step walks.



The adjacency matrix A is below left. The matrix of 3-step walks is just A^3.

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 2 & 1 \\ 0 & 2 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}, A^3 = \begin{bmatrix} 2 & 7 & 2 & 3 \\ 7 & 2 & 12 & 7 \\ 2 & 12 & 0 & 2 \\ 3 & 7 & 2 & 2 \end{bmatrix}$$

The movie portrays the problem as being so difficult that only a genius like Will Hunting can solve it. But the first two parts are easy. The last two parts, which involve generating functions, are harder, but not that hard.

Definition 1.36. The **trace** of a matrix A, $\operatorname{tr}(A)$, is the sum of the entries on its forward diagonal.

Corollary 1.37. *The number of triangles in a graph G with adjacency matrix A is $\frac{1}{6}\operatorname{tr}\left(A^3\right)$.*

Proof. A triangle is a closed walk of length 3. These walks are counted on the forward diagonal of A^3. Each triangle can be counted six ways, since there are three starting vertices and two directions. Thus $\frac{1}{6}\operatorname{tr}\left(A^3\right)$ is the number of triangles in G. □

Paths are important subgraphs because they are used to determine whether a graph is all one piece or not.

Definition 1.38. A graph is **connected** if it contains a $u-v$ path for all vertices u and v. A graph is **disconnected** if it is not connected. A **component** of a graph is a maximal connected subgraph.

Any disconnected graph is a disjoint union of its components. For example, the components of $C_4 \cup P_3$ are C_4 and P_3.

Many graph parameters can be determined by calculating them over each component and then taking the maximum, minimum, or sum. For example, the size of a graph is the sum of the sizes of its components, and the maximum degree of a graph is the maximum of the maximum degrees of its components.

When a graph is connected, we would like to know how far apart its vertices are.

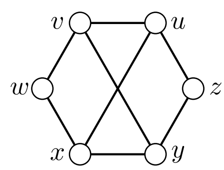
Definition 1.39. The **distance** between vertices u and v, $d(u,v)$, is the length of the shortest $u-v$ path. If u and v are in distinct components, we say $d(u,v) = \infty$. A $u-v$ **geodesic** is a $u-v$ path of length $d(u,v)$. The **diameter** of a graph G, $\operatorname{diam}(G)$, is the maximum length of a geodesic in G. The **eccentricity** of a vertex

$e(v)$ is the maximum length of a geodesic starting at v. The **radius** of a graph G, $\operatorname{rad}(G)$, is the minimum eccentricity of its vertices. The **center** of a graph is the subgraph induced by the vertices of minimum eccentricity.

Example. The social network that represents collaboration in Hollywood is called the **Hollywood graph**. The **Bacon number** of an actor is the distance between that actor and Kevin Bacon in the Hollywood graph. Bacon's own Bacon number is 0. Tom Hanks's Bacon number is 1, since they appeared together in Apollo 13. Matt Damon's Bacon number is 2, since Damon has not worked with Bacon, but appeared with Hanks in Saving Private Ryan. In the game "**Six Degrees of Kevin Bacon**", the goal is to find a geodesic path from some actor to Bacon. It has become a common party game and internet meme.

Example. The **academic collaboration graph** has an edge between two academics who are coauthors of a paper. The **Erdos number** of a mathematician (or other academic) is the distance between him and Paul Erdos in the academic collaboration graph. Erdos (1913–1996) was an iconic mathematician who solved many problems in graph theory, number theory, and other areas. Erdos is famous due to his eccentric lifestyle and the record number of papers he published, most with coauthors. This author's Erdos number is 2, as I coauthored a paper with Allen Schwenk, who coauthored with Erdos.

Example. In the following graph, $d(u,v) = 1$, the radius is 2, the diameter is 3, and the center is C_4, induced by u, v, y, and x.



Example. The hypercube Q_k has diameter k, since changing all k coordinates requires k edges. The wheel $W_{n+1} = C_n + K_1$ has radius 1 and diameter 2. The center is the vertex corresponding to K_1.

The definition of a path resembles the definition of a walk. However, a path cannot repeat vertices or edges.

Lemma 1.40. *Every $u - v$ walk contains a $u - v$ path.*

Proof. Assume to the contrary that there is a $u - v$ walk that does not contain a $u - v$ path, and let W be such a walk with minimum length. Then W must repeat some vertex w. Deleting all the vertices and edges on the walk between the occurrences of w produces a shorter $u - v$ walk. Either it is a $u - v$ path or it is a smaller counterexample. Either way, there is a contradiction. $\square$

Distance satisfies the triangle inequality $d(u,w) \le d(u,v) + d(v,w)$ since concatenating the $u-v$ and $v-w$ paths produces a $u-w$ walk, which contains a $u-w$ path. Along with the obvious properties $d(u,v) \ge 0$ with equality exactly when $u = v$, and $d(u,v) = d(v,u)$, this shows that graphs with distance are a metric space.

Proposition 1.41. *For every connected graph,* $\operatorname{rad}(G) \leq \operatorname{diam}(G) \leq 2\operatorname{rad}(G)$.

Proof. The first inequality follows from the definitions. Let u and v be vertices with $d(u,v) = \operatorname{diam}(G)$, and let w be a vertex with $e(w) = \operatorname{rad}(G)$. By the triangle inequality,

$$\operatorname{diam}(G) = d(u,v) \leq d(u,w) + d(w,v) \leq 2e(v) = 2\operatorname{rad}(G). \qquad \square$$

Given a result like $\operatorname{diam}(G) \leq 2\operatorname{rad}(G)$, it is natural to ask whether this bound is as good as possible. If it were the case that $\operatorname{diam}(G) \leq 2\operatorname{rad}(G) - 1$, we say that the bound could be strengthened. We say that a bound in a theorem is **attained** if there is a graph for which it is an equality (preventing it from being strengthened). A bound is **sharp** if there are infinitely many graphs for which it is an equality. Note that for paths of odd order, $\operatorname{rad}(P_{2k+1}) = k$ and $\operatorname{diam}(P_{2k+1}) = 2k$. This infinite class shows that the bound is sharp. Graphs that make a bound an equality are called **extremal graphs** for the bound. For some bounds, it is possible to characterize the extremal graphs, thereby improving the bound. Graph theory bounds, sharpness, and extremal graphs are explored in detail in Section 4 of the Appendix.

If a graph has a large diameter, it has relatively few edges, so it follows that its complement has many edges and a small diameter.

Proposition 1.42. *If* $\operatorname{diam}(G) \geq 4$, *then* $\operatorname{diam}\left(\overline{G}\right) \leq 2$.

Proof. Let $\operatorname{diam}(G) \geq 4$, and let x and y be vertices with $d(x,y) = 4$. If u and v are nonadjacent in G, then they are adjacent in $\overline{G}$. If u and v are adjacent in G, then x and y cannot both be in the neighborhoods of u and v, since that would imply an $x - y$ path of length at most 3. Thus u and v are both nonadjacent to either x or y. Then $d(u,v) = 2$ in $\overline{G}$. $\square$

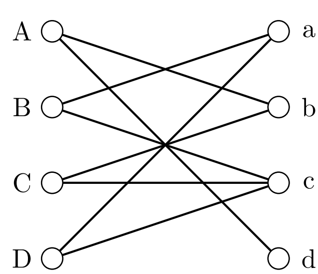
1.7. Bipartite Graphs

Some graphs have vertex sets that are divided into two distinct types of objects.

Example. A small college offers four math courses in the summer and has four math teachers who can each teach one class. We model this using a graph with four vertices representing courses and another four representing classes. An edge joins a course to a qualified teacher. Note that it would make no sense for an edge to join two teachers or two classes.

The four teachers and the classes they are qualified to teach are given in the table. The situation is modeled with a graph at right.

Teacher	Classes
Alice	biomath, diff eq
Betty	algebra, calculus
Cindy	biomath, calculus
Dot	algebra, calculus



The natural question is whether it is possible to assign each teacher a course she is qualified to teach. Some thought shows that it is. Since only Alice can teach diff eq, Cindy must teach biomath, and Betty and Dot can choose between algebra and calculus. Thus there is a way to match each teacher with a class.

Definition 1.43. A graph is **bipartite** if the vertex set can be partitioned into two independent sets (called **partite sets**). A **matching** is a set of edges, none of which are adjacent. A **perfect matching** includes an endpoint of every vertex.

Bipartite graphs are also called 2-colorable since the two partite sets can be colored so that no adjacent vertices have the same color. A complete bipartite graph has all possible edges between partite sets. Paths, even cycles, grids, and hypercubes are other classes of bipartite graphs. Matchings in bipartite graphs are explored in Section 7.1.

The concept of distance is essential to the characterization of bipartite graphs.

Theorem 1.44. (Konig [**1936**]) *A graph is bipartite if and only if it contains no odd cycles.*

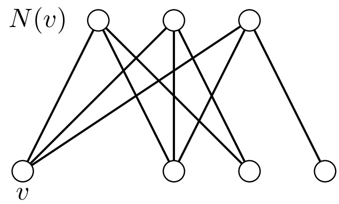
Proof. ($\Rightarrow$) Let G be a bipartite graph. Every walk alternates between the two partite sets, so it can only return to the first set after an even number of steps. Thus any cycle has even length.

($\Leftarrow$) Let G be a graph with no odd cycles. We assume G is connected, as this argument can be repeated for each component. Let u be a vertex, let U be the set of all vertices with even distance from u, and let W be the set of all vertices with odd distance from u. This is a partition of the vertex set. Suppose to the contrary that there is an edge between vertices x and y in the same partite set. Then the $u - x$ and $u - y$ geodesics have the same parity. These paths may have vertices in common, but there must be a last vertex v they have in common. Then the $v - x$ and $v - y$ geodesics and the edge xy can be combined to form an odd cycle. This is a contradiction, so U and W are independent sets. $\square$

Note that this is a constructive proof, as it provides a simple way to find the partite sets of a bipartite graph or show that it is not bipartite. This theorem is a **forbidden subgraph characterization**, where the class of graphs is characterized by a class of subgraphs that they do not contain. Such characterizations tell us about the structure of the graph class under consideration and may lead to practical algorithms for testing whether a graph is in the class. These and other graph characterizations are explored in Section 5 of the Appendix.

Bipartite graphs are useful in solving many problems. They are essential in the following result on triangle-free graphs.

Theorem 1.45 (Mantel [**1907**]). *The maximum number of edges in a triangle-free graph of order n is $\left\lfloor \frac{n^2}{4} \right\rfloor$, and the unique extremal graph is $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$.*



Proof. Let G be a graph with order n containing a vertex v with maximum degree d. None of v's neighbors are adjacent, so each edge is incident with a vertex not in $N(v)$. Then $m(G) \leq \sum_{u \notin N(v)} d(u) \leq d(n-d) = m(K_{d,n-d})$ since we sum over $n-d$ vertices with degree at most d. To maximize the size of $K_{d,n-d}$, we note that moving a vertex from the partite set of size d to the other one adds $d-1$ edges and subtracts $n-d$ edges. The net gain of $2d-n-1$ is positive when $d > \frac{n+1}{2}$ and negative when $d < \frac{n+1}{2}$. Then the size is maximized when d is $\lfloor \frac{n}{2} \rfloor$ or $\lceil \frac{n}{2} \rceil$, so the maximum is $\left\lfloor \frac{n^2}{4} \right\rfloor$.

Equality requires that G is bipartite with $n-d$ vertices of degree d, and these numbers are as close as possible. Thus $G = K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$. □

It is interesting that the number of edges required to guarantee a triangle is the same number required to guarantee any odd cycle.

1.8. Generalizations of Graphs

There are many models related to, but different from, graphs as we have defined them. Recall the definition of a graph.

Definition 1.46. A **graph** G is a mathematical object consisting of a finite nonempty set of objects called **vertices** $V(G)$ (the **vertex set**), and a set of **edges** $E(G)$ (the **edge set**). An edge is two-element subset of the vertex set.

Consider how changing a few words in the definition of a graph results in different models. What if we remove "finite" or "nonempty"?

Definition 1.47. The **null graph** has an empty vertex and edge set.

There isn't much to do with the null graph. It is an exception to some theorems whose hypotheses it would vacuously satisfy. It is sometimes convenient to consider in counting formulas, but the complications outweigh the benefits, so we do not consider it a graph. These issues are considered further in Frank Harary and Ronald Read's [**1974**] tongue-in-cheek paper, *Is the Null Graph a Pointless Concept?*

Definition 1.48. An **infinite graph** has an infinite vertex set. A **finite graph** is not infinite.

There are many substantial results on infinite graphs, but it is better to have a firm understanding of finite graphs before considering them. Some results on finite graphs extend naturally to infinite graphs, but others do not. All graphs in this book will be finite unless explicitly stated.

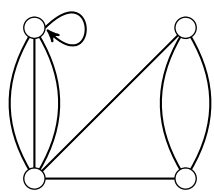
What if the edges are not 2-element subsets?

Definition 1.49. A **hypergraph** is a mathematical object consisting of a **vertex set** and an **edge set**. An **edge** is nonempty subset of the vertex set.

Thus an edge can have any cardinality. Hypergraphs are a large subject, but as with infinite graphs, it is better to study graphs first. Also, hypergraphs cannot be represented with drawings as nicely as graphs, so the subject is less visually appealing. See Berge [**1973**] for more on hypergraphs.

If we allow multiple edges between vertices, the edges must be in a **multiset**, which allows multiple copies of the same object.

Definition 1.50. A **multigraph** G is a mathematical object consisting of a finite nonempty set of objects called **vertices** $V(G)$ and a multiset $E(G)$ of pairs of vertices. A **loop** is an edge with both ends the same vertex. **Multiple edges** are edges having the same ends.



We have seen that multigraphs are natural models in some situations. In some topics in graph theory, multiple edges are irrelevant. Often multiple edges can be replaced by positive integer labels on the edges of a graph. Drawings of graphs become crowded and less useful when many edges are present.

Digraphs are the variation on graphs that we will consider the most.

Definition 1.51. A **directed graph** (digraph) D is a mathematical object consisting of a finite nonempty set of objects called vertices $V(D)$ and a set $E(D)$ of ordered pairs of distinct vertices called **directed edges**. A **directed multigraph** replaces the set $E(D)$ with a multiset.

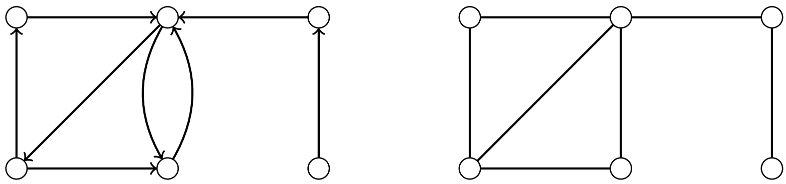
If $e = uv$ is a directed edge of a digraph, then we write $u \to v$, and we say u is the **tail** of e, v is the **head** of e, u is **adjacent to** v, v is **adjacent from** u, u and v are **neighbors**, e **joins** u and v, u is **joined to** v, and e and v are **incident**.

In a drawing of a digraph, a directed edge $e = uv$ is typically drawn as an arrow with its tail at u and its head at v. When a digraph has edges in both directions between two vertices, you can either draw two arrows or one two-headed arrow between them.

There is a natural relationship between graphs and digraphs.

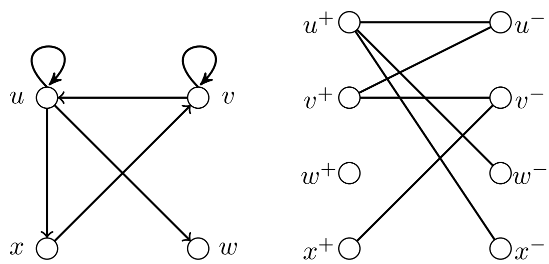
Definition 1.52. The **underlying graph** of a digraph D is obtained by treating each edge of D as an unordered pair. An **orientation** of a graph is a digraph formed by replacing each edge with a directed edge. Such a digraph is **asymmetric**. A digraph D is **symmetric** if whenever uv is an edge of D, so is vu.

Each digraph has a unique underlying graph, but most graphs have many orientations. Graphs correspond to symmetric loopless digraphs since any pair of symmetric edges can be replaced with an undirected edge. A digraph and its underlying graph are illustrated below.



There is also a natural correspondence between digraphs and bipartite graphs.

Definition 1.53. The **split** of a digraph D is a bipartite graph G with partite sets V^- and V^+ copying $V(D)$. Each vertex $v \in V(D)$ is duplicated to $v^- \in V^-$ and $v^+ \in V^+$. Each edge $uv \in E(D)$ corresponds to $u^+v^- \in E(G)$.



Many definitions of terms for digraphs are the same as for graphs, or follow naturally by replacing edges with directed edges. Examples include order and size. However, degree requires distinguishing the directions of the edges.

Definition 1.54. Let v be a vertex in a digraph. The **outdegree** $d^+(v)$ is the number of edges from v. The **indegree** $d^-(v)$ is the number of edges to v.

Example. In the web graph, pages that are more commonly linked tend to be better sources of information. Since the web graph is a digraph, these are pages with high indegree. This observation is the basic idea behind PageRank, a significant algorithm used by Google to provide quality search results. This process can be manipulated by sites that provide links solely to influence search engines, so PageRank weights the links differently depending on the quality of the site.

Proposition 1.55. *In a digraph D, $\sum_v d^+(v) = \sum_v d^-(v) = m(D)$.*

This result is analogous to the First Theorem of Graph Theory. It follows since each each edge has a head and a tail.

Directed walks, paths, and cycles are analogous to their undirected versions except that edges must have a consistent direction. The head of one edge leads to the tail of the next. There are two versions of a digraph being connected.

Definition 1.56. A digraph is **connected** (or **weakly connected**) if its underlying graph is connected. A digraph is **strong** (or **strongly connected**) if there are $u-v$ and $v-u$ paths for all vertices u and v. A **strong component** of a digraph is a maximal strong subgraph.

Proposition 1.57. *A digraph is strong if and only if it contains a closed spanning walk.*

Proof. ($\Leftarrow$) Assume a digraph D contains a closed spanning walk. Then any vertices u and v are on the walk, so D contains a $u - v$ walk and a $v - u$ walk following the spanning walk. Then D contains $u - v$ and $v - u$ paths (analogous to Lemma 1.40), so it is strong.

($\Rightarrow$) Let D be a strong digraph, and let W be a walk with the maximum number of vertices. Suppose to the contrary that v is a vertex not on W, and let u be on W. Then D contains $u - v$ and $v - u$ paths. Inserting them into W at u produces a walk with more vertices. $\square$

Many results on digraphs are analogous to those for graphs; however, some others that are not will be explored throughout this text.

Related Terms: call graph, module dependency graph, niche overlap graph, protein interaction graph, influence graph, incidence matrix, theta graph, Kneser graph, odd graph, Durer graph, dodecahedron, Desargues graph, Heawood graph, odd girth, double wheel, fan graph, tensor product, biregular graph, graph median, peripheral vertex, graph periphery, boundary vertex of a graph, interior vertex, location number, detour distance, detour diameter, Wiener index, functional digraph, converse of a digraph.

Exercises

Section 1.1:

(1) A telephone company tracks telephone calls between different phone numbers. How can you model this situation using graph theory? Are the edges directed or multiple?

(2) It is considered good practice to break computer programs into modules, which are smaller subprograms. Modules take input and produce output. A module can call another module by sending it data. How can you model this situation using graph theory? Are the edges directed or multiple? What condition should a **module dependency graph** satisfy in a well-written program?

(3) When several species share an environment, they may compete for food. How can you model this situation using graph theory? Are the edges directed or multiple?

(4) In biology, some proteins interact chemically with each other, while others do not. How can you model this situation using graph theory? Are the edges directed or multiple?

(5) A map has different countries, some of which share borders. The countries are colored with different pastel colors.
   (a) How can you model this situation using graph theory? Are the edges directed or multiple?
   (b) What is a natural question to ask regarding the coloring of the regions?

(6) The game of *Risk* has a map with different regions, some of which share borders. Armies can only attack adjacent regions. How can you model this situation using graph theory? Are the edges directed or multiple?

(7) Some people in society may influence others. That is, their ideas or actions affect the actions of some others. How can you model this situation using graph theory? Are the edges directed or multiple?

(8) A criminal network has a hierarchy with some members giving orders to others. How can you model this situation using graph theory? Are the edges directed or multiple?

(9) Some college courses have prerequisites. How can you model this situation using graph theory? Are the edges directed or multiple?

(10) Email messages can be sent over a network. How can you model this situation using graph theory? Are the edges directed or multiple?

(11) A number of traffic lanes approach a road intersection. Some pairs of lanes cannot safely have traffic flow simultaneously. How can you model this situation using graph theory? What is a natural question to ask about such a graph?

(12) Give an example of a real world situation that is not an example or exercise in this section that can be modeled using graph theory. Are the edges directed or multiple?

Section 1.2:

(1) Aaron knows Clay, Aaron knows Fred, Blake knows Dave, Blake knows Ethan, Clay knows Fred, Dave knows Ethan, and Ethan knows Fred. Find the adjacency list, the adjacency matrix, and a drawing of the graph modeling this situation.

(2) An airline has flights (both ways) between Detroit and Grand Rapids, Detroit and Fort Wayne, Chicago and Indianapolis, Chicago and Green Bay, Chicago and Kalamazoo, Detroit and Cincinnati, Detroit and Chicago, Detroit and Milwaukee, Chicago and the Quad Cities, and Detroit and Indianapolis. Find the adjacency list, the adjacency matrix, and a drawing of the graph modeling this situation.

(3) There are cables connecting the following seven computers: A and C, B and F, B and G, C and D, C and F, and E and G. Find the adjacency list, the adjacency matrix, and a drawing of the graph modeling this situation.

(4) Find an adjacency matrix and drawing of a graph representing propane (C_3H_8) given that no hydrogen atoms can be adjacent to each other, and each carbon atom has bonds with four other atoms.

(5) Find the intersection graph of the sets $\{1, 3, 5\}$, $\{2, 5, 6\}$, $\{3, 4, 7\}$, $\{3, 7, 8\}$, $\{4, 5, 7\}$, $\{5, 6, 8\}$.

(6) Find the intersection graph of the sets $\{1, 2, 5, 6\}$, $\{1, 7, 9\}$, $\{2, 3, 7, 8\}$, $\{3, 5, 10\}$, $\{4, 6, 8\}$, $\{5, 6, 9, 10\}$.

(7) Sue wants to prove that six statements are logically equivalent. She has proved that $A \Rightarrow D$, $F \Rightarrow A$, $C \Rightarrow E$, $D \Rightarrow B$, $E \Rightarrow A$, and $B \Rightarrow F$. Has she proved they are all equivalent? If not, what else must she prove?

(8) Sally wants to prove that six statements are logically equivalent. She has proved that $A \Rightarrow C$, $E \Rightarrow B$, $C \Rightarrow F$, $B \Rightarrow D$, $E \Rightarrow A$, and $F \Rightarrow E$. Has she proved they are all equivalent? If not, what else must she prove?

(9) For the graph with the adjacency list below left, find the adjacency matrix and a drawing of it.

1	3, 4
2	3, 5
3	1, 2, 5
4	1
5	2, 3

1	2, 4, 6
2	1, 3, 5
3	2, 4, 6
4	1, 3, 5
5	2, 4, 6
6	1, 3, 5

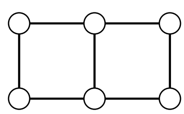
(10) For the graph with the adjacency list above right, find the adjacency matrix and a drawing of it.

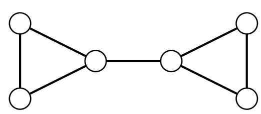
(11) For the graph with the adjacency matrix below left, find the adjacency list and a drawing of it.

$$\begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

(12) For the graph with the adjacency matrix above right, find the adjacency list and a drawing of it.

(13) For the graph drawn below left, find its adjacency matrix.





(14) For the graph drawn above right, find its adjacency matrix.

(15) The **incidence matrix** of a graph has rows that represent its vertices and columns that represent its edges. A 1 in the i, j position indicates vertex v_i and edge e_j are incident. Otherwise, the entry is 0. Find the incidence matrix for the graph in Exercise 13.

(16) Find the incidence matrix for the graph in Exercise 14.

Section 1.3:

(1) Draw all graphs of order 5. Use a case-checking argument to be sure you have found them all.

(2) Draw all graphs of order 6 and size 7. Use a case-checking argument to be sure you have found them all.

(3) How many edges does a graph have if its degree sequence is 4, 3, 3, 2, 2? Draw such a graph.

(4) How many edges does a graph have if its degree sequence is 5, 2, 2, 2, 2, 1? Draw such a graph.

(5) For the graph in Exercise 14 of Section 1.2, find the minimum degree, maximum degree, and average degree.

(6) Show that the number of labeled graphs of order n with all even degrees is $2^{\binom{n-1}{2}}$. (*Hint*: Find a bijection to graphs with vertex set $[n-1] = \{1, \dots, n-1\}$.)

Section 1.4:

(1) How many copies of P_n are contained in K_n?

(2) How many copies of C_n are contained in K_n?

(3) Let G be a connected graph without P_4 or C_4 as an induced subgraph. Show that G has a vertex adjacent to all other vertices. (*Hint*: Consider a vertex of maximum degree.)

(4) Use a counting argument involving complete graphs to show that $\binom{n}{2} = \binom{k}{2} + n\,(n-k) + \binom{n-k}{2}$.

(5) Find the size of the largest clique and largest independent set in the graph in Exercise 12 of Section 1.2.

(6) Find the largest size of an independent set in the Petersen graph. How many sets of this size are there? (*Hint*: Consider the labeling that gives rise to it.)

(7) Use a bijection to find the number of 6-cycles in the Petersen graph.

(8) Use a counting argument to find the number of 9-cycles in the Petersen graph.

(9) (a) Show that every edge is contained in four 5-cycles of the Petersen graph.
(b) Use part (a) to find the number of 5-cycles of the Petersen graph.

(10) Show that the Petersen graph does not have a 7-cycle by:
(a) supposing it does, and considering the vertices three edges around the 7-cycle from a given vertex;
(b) deleting three vertices from the Petersen graph;
(c) using the drawing that emphasizes two 5-cycles;
(d) supposing it does, and considering the edges incident with exactly one vertex on the 7-cycle.

(11) Find a decomposition of the Petersen graph into copies of
(a) P_4;
(b) a connected graph of size 5.

(12) The **theta graph** $\theta_{i,j,k}$ is formed from paths of length i, j, and k by identifying their endpoints.
(a) Determine the number and lengths of the cycles in $\theta_{i,j,k}$.
(b) Determine all theta graphs contained in the Petersen graph.

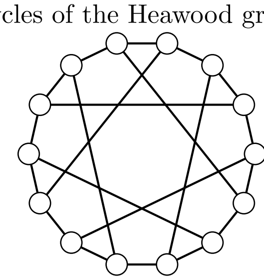
(13) (Kneser [**1955**]) The **Kneser graph** $KG_{r,k}$ has vertices representing the k-element subsets of $[r]$ and edges between disjoint subsets. The Petersen graph is $KG_{5,2}$.

(a) Draw $KG_{6,2}$.
(b) Determine the size of $KG_{r,k}$.
(c) Determine for what r and k Kneser graphs are connected.
(d) Determine for what r and k Kneser graphs contain a triangle.

(14) + The **odd graph** O_k is the Kneser graph $KG_{2k-1,k-1}$. The Petersen graph is O_3.
(a) Show that there is a path of length $2s$ between two vertices whose sets have $k-s$ common elements.
(b) Show that O_4 has girth 6.

(15) The **generalized Petersen graph** $P(r,k)$, $1 \le \frac{r}{2} < k$, has vertices $u_1, \ldots, u_r$ and $v_1, \ldots, v_r$ and edges $u_i u_{i+1}$, $u_i v_i$, and $v_i v_{i+k}$ (addition mod r). The Petersen graph is $P(5,2)$.
(a) Draw $P(6,2)$, the **Durer graph**.
(b) Draw $P(10,2)$, the **dodecahedron**.
(c) Draw $P(10,3)$, the **Desargues graph**.
(d) Determine for what r and k generalized Petersen graphs contain a triangle.
(e) Determine for what r and k generalized Petersen graphs contain a 4-cycle.
(f) Show that any generalized Petersen graph has girth at most 8.

(16) (a) Show that there is a unique cubic graph of order 14 and girth 6. This is called the **Heawood graph**.
(b) Find the number of 6-cycles of the Heawood graph.



Section 1.5:

(1) Which of the following graphs are well-defined? Draw those that are.
(a) $P_5 - e$
(b) $K_{3,3} - e$
(c) $K_5 - e$
(d) $W_6 - e$

(2) Which of the following graphs are well-defined? Draw those that are.
(a) $C_6 + e$
(b) $K_{3,3} + e$
(c) $P_4 + e$
(d) $PG + e$, where PG is the Petersen graph

(3) Which of the following graphs are well-defined? Draw those that are.
(a) $P_4 - v$
(b) $K_{3,4} - v$
(c) $K_6 - v$
(d) $K_{3,3} - v$

(4) Identify the complements of the following graphs (without using complementation).
(a) C_5
(b) C_6
(c) P_4
(d) Q_3

(5) A **double wheel** is the graph $C_{n-2} + 2K_1$. Find the size and number of triangles of double wheels.

(6) A **fan graph** is $P_{n-1} + K_1$. Find the size of all fan graphs. How are they related to wheels?

(7) Show that every graph of order 4 can be expressed using the classes complete graphs, paths, and cycles, and the operations complement, disjoint union, and join.

(8) Find three graphs of order 5, none of which are complements of another, that cannot be expressed using the classes complete graphs, paths, and cycles, and the operations complement, disjoint union, and join.

(9) Draw a Venn diagram representing the graph classes P_n, C_n, K_n, $\overline{K}_n$, $K_{r,s}$, Q_k. Identify all graphs in the intersections of classes.

(10) Use strong induction on order to prove Proposition 1.28.

(11) Explain how to count 2-regular graphs in terms of solutions to a linear equation.

(12) Explain how to count $n-3$-regular graphs of order n.

(13) Show that $\overline{G+H} = \overline{G} \cup \overline{H}$.

(14) Identify the graph $\overline{K}_{r,s}$ as another graph without using complementation.

(15) Determine $m\,(G+H)$ in terms of the orders and sizes of G and H.

(16) Determine $m\,(K_{r,s})$ in terms of r and s.

(17) Determine $m\,(G \Box H)$ in terms of the orders and sizes of G and H.

(18) Show that $m\,(Q_k) = k \cdot 2^{k-1}$ using induction.

(19) Find the number of copies of Q_j contained in Q_k.

(20) (a) Find the number of 6-cycles in Q_3.
(b) Find the number of 6-cycles in Q_k.

(21) Find the number of copies of $K_{2,3}$ in Q_k.

(22) Show that a k-regular graph with girth 4 has at least $2k$ vertices. Which such graphs have exactly $2k$ vertices?

(23) When does $K_{r,s}$ decompose into two copies of the same graph?

(24) If G and $\overline{G}$ are both regular, what can be concluded about the order of G?

(25) Show that a graph G with maximum degree d is an induced subgraph of some d-regular graph. (*Hint*: Consider a subgraph of $G \Box Q_k$ for some k.)

(26) (Imrich/Klavzar [**2000**]) Show that

$$(K_1 \cup K_2 \cup C_4)\Box(K_1 \cup Q_3) = (K_1 \cup C_4 \cup Q_4)\Box(K_1 \cup K_2).$$

(*Note*: This is a disconnected graph that can be expressed as a Cartesian product in two different ways.)

(27) Find the number of 4-cycles in $K_{r,s}$.

(28) Find the number of 6-cycles in $K_{r,s}$.

(29) Find the number of $2r$-cycles in $K_{r,r}$.

(30) The **complete multipartite graph** $K_{n_1,\ldots,n_k}$ is a join of multiple empty graphs, $K_{n_1,\ldots,n_k} = \overline{K}_{n_1} + \cdots + \overline{K}_{n_k}$.
 (a) Determine the size $m\left(K_{r,s,t}\right)$ in terms of r, s, and t.
 (b) Draw the **octahedron** $K_{2,2,2}$ without any edges crossing.

(31) Define an operation to undo a subdivision in a graph. Is this operation well-defined for any vertex of degree 2?

(32) The **Mobius ladder** M_n is formed from C_n (n even) by adding edges between opposite vertices.
 (a) Show that M_n can be formed from $C_{\frac{n}{2}} \square K_2$ by deleting and replacing two edges.
 (b) Find the number of 4-cycles of M_n.

(33) The **tensor product** of G and H, $G \times H$, has vertex set $V\left(G\right) \times V\left(H\right)$ and vertices (u, u') and (v, v') are adjacent in $G \times H$ if and only if $uv \in E\left(G\right)$ and $u'v' \in E\left(H\right)$.
 (a) Draw $P_4 \times P_4$.
 (b) Determine $m\left(G \times H\right)$ in terms of the orders and sizes of G and H.
 (c) Show that $K_2 \times P(5, 2) = P(10, 3)$. (The notation refers to generalized Petersen graphs.)

(34) The **triangular grid** T_l has vertices (i, j, k) such that i, j, k are nonnegative integers and $i + j + k = l$. Two vertices are adjacent when the total of the absolute differences in corresponding coordinates is 2.
 (a) Draw T_4.
 (b) Determine $m\left(T_l\right)$.
 (c) Determine the number of triangles of T_l.

Section 1.6:

(1) Let G be a graph with adjacency matrix A. Explain how to find the vertex degrees of G from A^2.

(2) Show that the number of 5-cycles in a triangle-free graph G with adjacency matrix A is $\frac{1}{10}\operatorname{tr}\left(A^5\right)$.

(3) The **odd girth** of a graph is the length of the shortest odd cycle of a graph. Find a formula for the number of k-cycles in a graph with odd girth k.

(4) + Explain how to use the adjacency matrix of a graph to count the number of 4-cycles. (You must exclude walks of length 4 that are not cycles.)

(5) Show that a graph is connected if any only if, for every partition of its vertices into two nonempty sets, there is an edge joining vertices in different sets.

(6) Show that two paths of maximum length in a connected graph G must have a common vertex. Must they have a common edge?

(7) If G has r components and H has s components, how many components does $G \square H$ have?

(8) Find the minimum number of edges that will guarantee that a graph with order n is connected. Characterize the extremal graphs.

(9) Show that if $d(u) + d(v) \geq n - 1$ for every two nonadjacent vertices u and v of a graph G with order n, then $\operatorname{diam}(G) \leq 2$.

(10) The previous problem implies that if $\delta(G) \geq \frac{n-1}{2}$, then G is connected. Show that this bound is sharp by characterizing the extremal graphs.

(11) Let u and v be adjacent vertices in a connected graph G. Show that
(a) $|e(u) - e(v)| \leq 1$.
(b) $|d(u,w) - d(v,w)| \leq 1$ for any vertex w of G.

(12) Find the diameter and radius of
(a) C_n.
(b) $K_{r,s}$.
(c) P_n.
(d) $G_{r,s}$.

(13) Show that every $u - v$ walk contains a $u - v$ path by
(a) induction on the length of a $u - v$ walk.
(b) considering a shortest $u - v$ walk contained in a $u - v$ walk W.

(14) Show that every walk of odd length contains an odd cycle.

(15) Show that if $\operatorname{diam}(G) \geq 3$, then $\operatorname{diam}\left(\overline{G}\right) \leq 3$. Find an infinite class of graphs with $\operatorname{diam}(G) = 3$ and $\operatorname{diam}\left(\overline{G}\right) = 3$.

(16) Find all pairs of integers (a, b) such that there is a graph G with $\operatorname{diam}(G) = a$ and $\operatorname{diam}\left(\overline{G}\right) = b$. Give an example for each case.

(17) For integers r and d with $0 \leq r \leq d \leq 2r$, construct a graph with radius r and diameter d.

(18) Let G be graph. Show that if $\operatorname{rad}(G) \geq 3$, then $\operatorname{rad}\left(\overline{G}\right) \leq 2$.

(19) + Let G have diameter d and $\Delta(G) = k$. Show that $n \leq 1 + \frac{\left((k-1)^d - 1\right)k}{k-2}$.

(20) The **Wiener index** of a graph $W(G) = \sum_{u,v \in G} d(u,v)$ is the sum of distances between all pairs of vertices in a graph. Calculate the Wiener index for all
(a) stars.
(b) paths.
(c) cycles.
(d) wheels.

(21) Find the center of
(a) P_n.
(b) $K_{r,s}$.

(22) Show that any graph G can be the center of a graph. (*Hint*: Add four vertices and some edges to G.)

(23) + The **detour distance** $D(u,v)$ between u and v is the length of the longest $u-v$ path. Show that detour distance satisfies the triangle inequality $D(u,w) \leq D(u,v) + D(v,w)$.

(24) The **detour diameter** is the length of the longest path in a graph. Calculate this for
(a) wheels.
(b) $K_{r,s}$.
(c) $G_{r,s}$.

Section 1.7:

(1) Several employees of a company are assigned to work on a project requiring several distinct skills (accounting, engineering, programming, etc.), but each employee is only qualified for some of the tasks. How can this situation be modeled using graph theory? What is a natural question to ask in this situation?

(2) A social network has a number of groups that its users can join. How can this situation be modeled using graph theory?

(3) Show that if G is a nonempty regular bipartite graph with partite sets U and W, then $|U| = |W|$.

(4) Every edge of G joins an odd vertex and an even vertex. Show that G is bipartite with even size.

(5) Let G have adjacency matrix A and order n, and let r be the odd integer that is either n or $n-1$. Show that G is bipartite if and only if $\operatorname{tr}(A^r) = 0$.

(6) Show that a graph G is bipartite if and only if every subgraph H of G has an independent set containing at least half the vertices of H.

(7) Show that $G \square H$ is bipartite if and only if G and H are bipartite.

(8) Let G be a connected graph not containing P_4 or C_3 as an induced subgraph. Show that G is a complete bipartite graph.

(9) + Show that K_n decomposes into k bipartite graphs if and only if $n \leq 2^k$.

(10) Show that a cubic graph decomposes into claws if and only if it is bipartite.

(11) Show that every graph has a bipartite subgraph with at least $\frac{m}{2}$ edges. (*Hint*: Start with a partition and swap vertices.)

(12) Find the largest number of edges in a bipartite subgraph of the Petersen graph.

(13) (Barefoot et al. [**1995**]) Show that a triangle-free graph G with $n \geq 3$ is maximal triangle-free if and only if $\operatorname{diam}(G) = 2$.

(14) Show that there is a triangle-free nonbipartite graph with size $\left\lfloor \frac{(n-1)^2}{4} \right\rfloor + 1$. (*Note*: Barefoot et al. [**1995**] showed that this is the maximum size of such graphs.)

Section 1.8:

(1) The null graph:
(a) Determine the order, size, adjacency matrix, complement, degree sequence, maximum and minimum degrees of the null graph if they exist.

(b) Does the null graph satisfy the First Theorem of Graph Theory?
(c) Let G be a graph, and let N be the null graph. Determine $G + N$ and $G \square N$.
(d) Determine whether the null graph is connected.

(2) For infinite graphs:
(a) Explain how there are two different types of infinite paths.
(b) Determine whether Lemma 1.19 and Proposition 1.28 hold for infinite graphs.

(3) For the multigraph with the adjacency list below left, find the adjacency matrix and a drawing of it.

1	2, 2, 4
2	1, 1, 3, 5, 5
3	2, 3
4	1, 5, 5
5	2, 2, 4, 4, 5

$$\begin{bmatrix} 0 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 3 & 0 \\ 2 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 2 \\ 1 & 3 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \end{bmatrix}$$

(4) For the multigraph with the adjacency matrix above right, find the adjacency list and a drawing of it.

(5) Do multigraphs satisfy the First Theorem of Graph Theory? Explain carefully.

(6) Draw a Venn diagram with sets representing graphs, the null graph, infinite graphs, hypergraphs, multigraphs, and digraphs.

(7) Let f be a function from a finite set to itself. A **functional digraph** has an edge from x to $f(x)$ for each x. Describe the structure of a functional digraph.

(8) Give an example of a real world situation that is not an example or exercise in this chapter that can be modeled using a digraph.

(9) For the digraph with the adjacency list below left, find the adjacency matrix and a drawing of it and its underlying graph.

1	2
2	3
3	1, 4
4	5
5	3

$$\begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(10) For the digraph with the adjacency matrix above right, find the adjacency list and a drawing of it and its underlying graph.

(11) Find the split of the digraph with vertex set $\{1, 2, 3, 4\}$ and edge set $\{12, 23, 34, 41, 31, 44\}$.

(12) What digraph is Q_3 the split of if vertices with all corresponding bits different are identified (e.g., 101 is identified with 010)?

(13) Determine all digraphs of order 4 and size 3.

(14) Determine all digraphs of order 4 and size 4.

(15) Show that if D is a digraph with minimum outdegree at least 1, then D contains a cycle.

(16) Prove or disprove: If D is a digraph with minimum indegree and minimum outdegree at least 1, then every vertex of D is on a cycle.

(17) The **converse of a digraph** is obtained by reversing the direction of every arc. Show that a digraph is strong if and only if its converse if strong.

(18) (a) Show that the strong components of a graph have no common vertices.
(b) Form a digraph D' whose vertices represent the strong components of a digraph D with an edge between them if there is an edge between vertices in their strong components in D. Show that D' contains no cycles.
(c) Show that in every digraph, some strong component has no entering edges and some strong component has no exiting edges.

Chapter 2

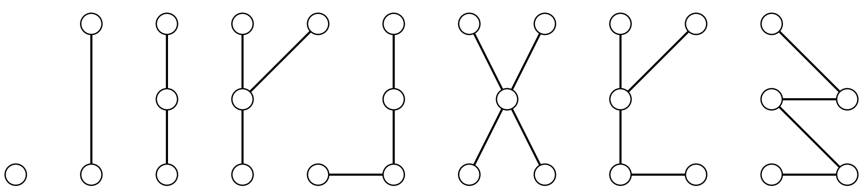
Trees and Connectivity

Trees are minimally connected graphs. They have many theoretical uses and practical applications, such as finding the distance between two vertices. Disconnecting a connected graph requires deleting at least one vertex or edge. Connectivity measures how much must be deleted from a graph to disconnect it. This has a natural relationship with the number of disjoint paths between vertices, which is the essence of Menger's Theorem.

2.1. Trees

Definition 2.1. A graph is a **forest** (or **acyclic graph**) if it does not contain a cycle. A **tree** is a connected forest.

Thus a graph is a forest if and only if every component is a tree; hence, we seek to characterize the structure of trees. All trees of orders 1–5 are shown below.



All paths and stars are trees. Indeed, they are opposite extremes with respect to many parameters such as diameter and maximum degree. Trees can be thought of as an extremal graph class. They are both minimal connected graphs and maximal acyclic graphs.

Recall that a leaf is a vertex with degree 1.

Proposition 2.2. *Every nontrivial tree contains at least two leaves.*

Proof. Let T be a nontrivial tree. Let P be a maximal path between vertices u and v in T. Then v cannot be adjacent to more than one vertex on the path, since this would create a cycle. It cannot be adjacent to another vertex not on P, since then it could be extended to a longer path. The same holds for u, so u and v have degree 1. $\square$

This leads to a natural way to construct any tree.

Theorem 2.3. *A graph is a tree if and only if it can be constructed from K_1 by repeatedly applying the operation of adding a new vertex adjacent to one existing vertex.*

Proof. ($\Leftarrow$) K_1 is a tree, and the operation keeps the graph connected and acyclic. Thus any graph produced this way must be a tree.

($\Rightarrow$) We use induction on n. The result is obvious when $n = 1$. Assume that any tree of order $n-1$ can be constructed from K_1 using the operation, and let T have order $n > 1$. Then T has a leaf v, so $T - v$ is a tree with order $n-1$. Thus $T - v$ can be constructed using the operation, so T can also. $\square$

This is an example of an operation characterization of a class of graphs.

Definition 2.4. An **operation characterization** of a class of graphs describes them recursively using one or more starting graphs and one or more operations that can be applied to any graph in the class to yield another graph in the class.

An operation characterization produces all graphs in a class, and only graphs in that class. It would be even more desirable to produce each graph exactly once, but for more complicated classes such as trees, that is too much to hope for.

Operation characterizations are useful for proofs by induction, since if a property holds for all initial graphs and is preserved by the operation(s), then it holds for all graphs in the class.

Corollary 2.5. *A tree with order n has size $n-1$. A forest with k components has size $n-k$.*

Proof. The operation adds an edge for each new vertex, and $m(K_1) = 0$. Let F be a forest with components F_i with orders n_i and sizes m_i. Then $\sum_{i=1}^{k} m_i = \sum_{i=1}^{k} (n_i - 1) = n - k$. $\square$

In the Exercises, you are asked to show that any two of the three properties connected, acyclic, and $m = n - 1$ imply that a graph is a tree. There is another characterization of trees.

Proposition 2.6. *A graph T is a tree if and only if for any two vertices u and v, T contains a unique $u - v$ path.*

Proof. ($\Rightarrow$) (contrapositive) Suppose that for some u and v in T, there is not a unique $u - v$ path. Either there is no $u - v$ path and T is disconnected or there are two $u - v$ paths, and some of their edges induce a cycle. Either way, T is not a tree.

($\Leftarrow$) (contrapositive) If T is not a tree, then either T is disconnected (and there is no $u - v$ path for some u and v in T) or T contains a cycle (and there are two $u - v$ paths for u and v on it). $\square$

A graph with large enough minimum degree can be forced to contain any tree of a given size.

Theorem 2.7. *If $\delta(G) \geq k$, then G contains all trees of size k.*

Proof. We use induction on k. When $k = 0$, every graph contains K_1, the only tree with size 0.

Assume the result holds for all trees with $k-1$ edges, $k \geq 1$. Let T be a tree with size k containing leaf v with neighbor u. By assumption, G contains $T - v$, which has size $k-1$. Now $d_G(u) \geq k$, so u is adjacent to at least one vertex in G that is not in $T - v$. Thus G also contains T. □

We can characterize the degree sequences of trees.

Proposition 2.8. *A list of $n > 1$ positive integers is the degree sequence of some tree T if and only if they sum to $2n-2$.*

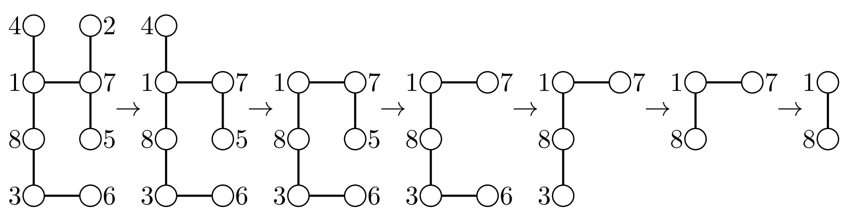
Proof. ($\Rightarrow$) A tree has size $n-1$, so its degree sum is $2(n-1) = 2n-2$.

($\Leftarrow$) We use induction on n. If $n = 2$, the sum is two and $T = K_2$. Assume the result holds for trees of order $n-1$. Suppose we have a list of $n > 2$ positive integers that sum to $2n-2$. Then the smallest integer is 1 and the largest is $d > 1$. Delete the 1 and replace d with $d-1$. Then we have a list of $n-1$ positive integers with sum $2(n-1)-2$, so there is some tree T' with these degrees. Construct T by adding a leaf adjacent to the vertex of degree $d-1$ in T'. □

The problem of counting the number of labeled trees of order n is not too difficult. Our strategy is to find a bijection between labeled trees and a certain type of sequence. When we count the sequences, we will find a formula for the number of labeled trees.

Algorithm 2.9 (**Prufer coding**—Prufer [**1918**]). *Find the leaf with the smallest label, record the label of its neighbor, and delete it. Repeat this step until there are only two vertices remaining.*

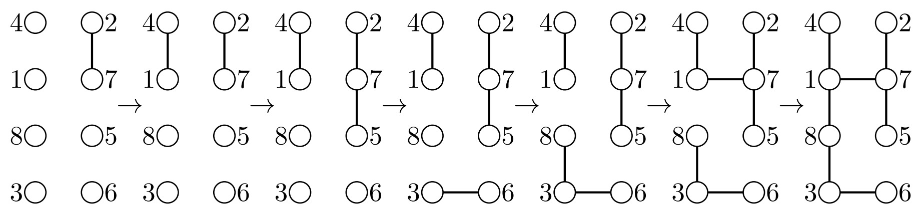
Example. The algorithm produces the Prufer code $(7, 1, 7, 3, 8, 1)$ for the tree below.



The algorithm produces a sequence of length $n-2$ with elements from $[n]$. We call this a **Prufer code**. There are n^{n-2} Prufer codes. This establishes a function from labeled trees to Prufer codes. To show that it is a bijection, we need an inverse function from codes to trees.

Algorithm 2.10 (**Prufer uncoding**). *Start with a Prufer code S and a list $L = 1, 2, \ldots, n$. Find the smallest element x of L not in S, and add an edge between* *vertex x and the first element of S. Then delete both elements. Repeat this step until S is empty and two elements of L remain. Then add an edge between those two vertices.*

Example. Consider the code $(7,1,7,3,8,1)$ produced in the previous example. The smallest number from $[8]$ not appearing is 2. Thus the first iteration adds edge 27. The remaining iterations are shown below. In each step, the edge added is the edge that was deleted in the corresponding step in the previous example.



717381	17381	7381	381	81	1	
12345678	1345678	135678	13678	1378	178	18

Theorem 2.11 (**Cayley's Formula**—Borchardt [**1860**]). *There are n^{n-2} labeled trees of order n.*

Proof (Prufer [**1918**]). This is obvious when $n = 1$. For $n \geq 2$, we show that there is a bijection between Prufer codes and labeled trees. For $n = 2$, there is one labeled tree and one (empty) code. Assume the bijection holds for labeled trees with order $n - 1 \geq 2$. That is, if $S = P(T)$ is the Prufer code of a tree, and $T = P'(S)$ is the tree formed by Prufer uncoding of S, then $P'(P(T)) = T$.

Let T be a tree with vertex set $[n]$, whose least leaf i has neighbor j. Then $S = P(T)$ begins with j and has smallest missing element i. By induction, $P'(P(T - i)) = T - i$. Applying the uncoding algorithm to S adds edge ij, and then adds the edges of $T - i$. Thus $P'(P(T)) = T$. $\square$

We can modify the process used to prove this theorem to find the number of trees with a given degree sequence.

Corollary 2.12 (Cayley [**1889**]). *Given positive integers $d_1, \ldots, d_n$ summing to $2n - 2$, there are $\frac{(n-2)!}{\Pi(d_i-1)!}$ labeled trees with each vertex i having degree d_i.*

Proof. Each nonleaf v will have $d(v) - 1$ of its neighbors deleted when constructing its Prufer code, so its label will be recorded that many times. Thus we must count codes of length $n - 2$ containing $d_i - 1$ copies of i for each i. There are $(n-2)!$ ways to permute $n - 2$ distinct numbers. Now $d_i - 1$ copies of i can be permuted in $(d_i - 1)!$ ways, so we must divide by this number for each i. $\square$

Enumerating unlabeled trees is a much more difficult problem. When n is not too large, it can be done by brute force.

Example. We can use brute force to find all trees of order 7. We organize them by maximum degree, and then by the number of vertices with this degree.

$\Delta = 6$. The star $K_{1,6}$ is the only possibility.

$\Delta = 5$. Start with $K_{1,5}$ and add one vertex adjacent to a leaf.

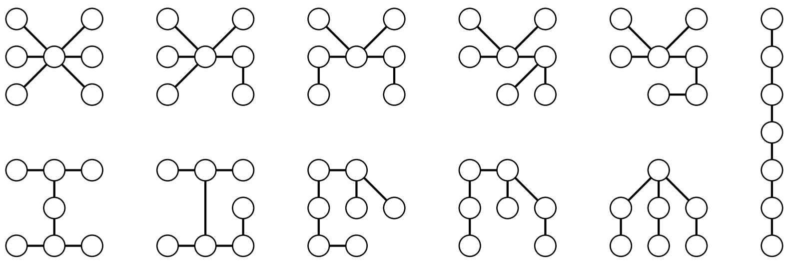
$\Delta = 4$. Start with $K_{1,4}$ and add two leaves. They can be adjacent to two distinct leaves, both adjacent to one leaf, or form a path from one leaf.

$\Delta = 3$ (two degree 3 vertices). There must be one degree 2 vertex, which can either be between the two vertices of degree 3 or not.

$\Delta = 3$ (one degree 3 vertex). There must be three paths that meet at a vertex. Their lengths must sum to 6 and must all be at least 1. The possibilities are $(1, 1, 4)$, $(1, 2, 3)$, and $(2, 2, 2)$.

$\Delta = 2$. The path P_7 is the only possibility.

Thus there are eleven possibilities, and we can be confident we have found them all.



Let t_n be the number of unlabeled trees with order n. There are $n!$ ways to label the vertices of each tree, some of which may produce the same labeled graph. Thus $t_n \cdot n! \geq n^{n-2}$, so $t_n \geq \frac{n^{n-2}}{n!}$. Using Stirling's approximation, $n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$, we see an approximate lower bound for t_n is

$$\frac{n^{n-2}}{n!} \approx \frac{\frac{n^n}{n^2}}{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n} = \frac{e^n}{\sqrt{2\pi} n^{2.5}}.$$

The first few terms of t_n are 1, 1, 1, 2, 3, 6, 11, 23, 47, 106, 235, ... (OEIS A000055). There is no simple formula for t_n, but there is an asymptotic formula that can be found using generating functions. It says that $t_n \sim C\frac{\alpha^n}{n^{2.5}}$, where $C \approx 0.534949606$ and $\alpha \approx 2.95576528565$ (Otter [**1948**]).

Related Terms: cactus, caterpillar, lobster, spider, broom, double broom, binary tree, Matrix Tree Theorem.

2.2. Tree Algorithms

Trees are also important as subgraphs of connected graphs.

Definition 2.13. A **spanning tree** of a connected graph G is a tree that is a subgraph of G with the same order. A **unicyclic graph** is a connected graph with exactly one cycle.

Every connected graph contains a spanning tree. This can be seen by deleting edges on cycles until no more remain, resulting in a tree. Any unicyclic graph can be formed by adding an edge to some tree. A connected graph that is not a tree has multiple spanning trees, which can be obtained from each other by swapping edges. The proof of the following lemma is an exercise.

Lemma 2.14. *If T and T' are spanning trees of a connected graph G and e is an edge in T and not in T', then there is an edge e' in T' and not in T such that $T - e + e'$ and $T' + e - e'$ are spanning trees of G.*

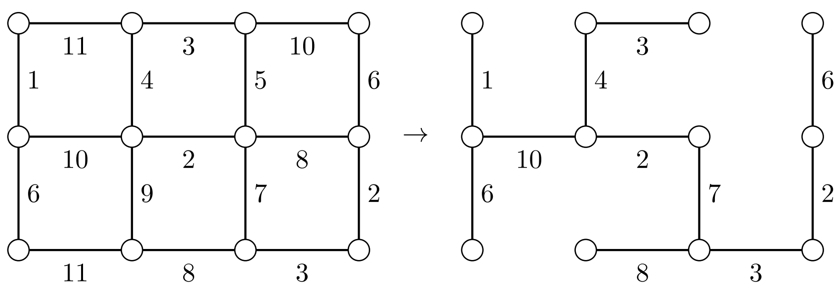
Suppose that there are several isolated villages that we wish to connect together with electric lines. The costs to connect different pairs of villages vary depending on the distance, terrain, and value of the land between the villages. We wish to find the minimum cost to connect the villages with electric lines. We can model this situation with a weighted graph.

Definition 2.15. A **weighted graph** is a graph for which each edge e has a nonnegative number $w(e)$ called a **weight** associated with it. (If an edge does exist, we may say its weight is ∞.) A **minimum spanning tree** is a spanning tree of a graph with minimum total edge weight.

There is a simple algorithm to find a minimum spanning tree.

Algorithm 2.16 (**Kruskal's Algorithm**). *Start with an empty graph with the same vertex set as G. While it is disconnected, add an edge with minimum weight that does not complete a cycle.*

Example. Consider the weighted graph below left. Begin by adding the edge with weight 1. Both edges weighted 2 can be added (in either order). Both edges with 3's and the edge with 4 can be added. However, the edge with weight 5 cannot be added, since it would create a cycle. Both edges with 6 and the one with 7 can be added. Only one edge with 8 can be added; the other would create a cycle. We eventually obtain the spanning tree at right.



Kruskal's Algorithm is an example of a **greedy algorithm**, that is, it makes the optimal choice at each step. Greedy algorithms often do not produce the best solution overall, but Kruskal's algorithm does.

Theorem 2.17 (Kruskal [**1956**]). *Kruskal's Algorithm produces a minimum spanning tree in a connected weighted graph.*

Proof. Kruskal's Algorithm must produce a spanning tree T. Let T' be a minimum spanning tree. If $T \neq T'$, let e be the first edge chosen for T that is not in T'. Adding e to T' creates a unicyclic graph, which contains an edge e' not in T. Consider the spanning tree $T' + e - e'$. All the edges chosen before e are contained in both trees, so $w(e) \le w(e')$ since the algorithm chooses e. Thus $T' + e - e'$ has total weight at most that of T' and agrees on a longer list of initial edges. Repeating this argument eventually shows that T is a minimum spanning tree. $\square$

Kruskal's Algorithm is not the only way to produce a minimum spanning tree. **Prim's Algorithm** (Jarnik [**1930**], Prim [**1957**], and Dijkstra [**1959**]) starts with a single vertex and iteratively adds the smallest weight edge that uses a new vertex. This algorithm is explored in the Exercises. When the edges are presorted by weight, these algorithms have similar running times.

Popular map software not only displays a map including a starting point and destination but also finds the shortest route between two points. We may wish to minimize geographical distance, travel time, or monetary cost. How can we do this using graph theory?

Definition 2.18. The **distance** $d(u, z)$ in a weighted graph is the minimum sum of weights on any $u - z$ path.

We wish to find the minimum distance from u to any vertex z. We note that if v is on the shortest $u - z$ path, then the shortest $u - v$ path must be contained in it.

Algorithm 2.19 (**Dijkstra's Algorithm**—Dijkstra [**1959**])**.** *Start with a weighted graph and starting vertex u. Create a set S of vertices whose shortest distance to u is known. Initially, $S = \{u\}$. Maintain a list of tentative distances $t(x)$ from u to each x, which may be decreased if a shorter path is found. Initially, set $t(u) = 0$, $t(x) = w(ux)$ for $x \in N(u)$, and $t(x) = \infty$ otherwise.*

While there are vertices not in S with finite tentative distances, iterate the following steps:

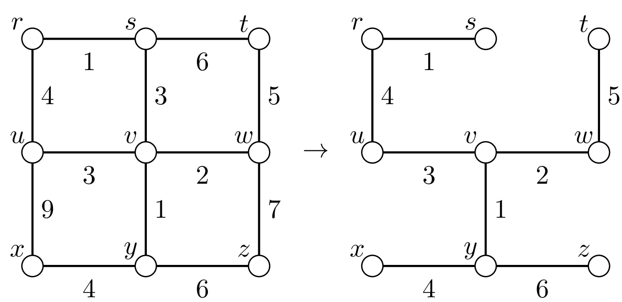
(1) *Select the vertex $v \notin S$ with smallest tentative distance $t(v)$.*

(2) *Add v to S.*

(3) *Update the tentative distances for each edge vy with $y \notin S$; set $t(y) = \min(t(y), t(v) + w(vy))$.*

The final value of $t(x)$ is $d(u, x)$.

Example. For the graph below left, shortest paths starting at u are found to v, r, y, s, w, x, t, z in order. These paths have lengths 3, 4, 4, 5, 5, 8, 10, 10, respectively. The union of these paths produces the spanning tree at right. Note that this is not a minimum weight spanning tree, since it includes edge ur rather than vs.



The complexity of Dijkstra's Algorithm is $\mathcal{O}\left(n^2\right)$, meaning that for a graph of order n, at most cn^2 operations are needed for some constant c. See Section 3 of the Appendix to learn about the computational complexity of algorithms.

Theorem 2.20 (Dijkstra [**1959**]). *Given a graph containing vertex u, Dijkstra's Algorithm computes $d\left(u, z\right)$ for every vertex z.*

Proof. We use induction on $r = |S|$, the number of visited vertices. When $r = 1$, the algorithm gives $d\left(u, u\right) = 0$, which is clearly true. Assume that the algorithm works for the first $r - 1$ vertices. Let $v \notin S$ be the vertex with smallest tentative distance at this step.

Suppose there is a shorter path to v than that produced by the algorithm. If it uses an unvisited vertex x, then x would be selected by the algorithm before v, a contradiction. If the path used all visited vertices except v, then its length cannot be less than the tentative distance.

Thus the algorithm produces a shortest path to v, and to every vertex. □

Distance in (unweighted) graphs is equivalent to distance in a weighted graph where every edge has weight 1. In this case, Dijkstra's Algorithm is known as **Breadth First Search**. The eccentricity of u is the longest distance from it to another vertex. The diameter of a graph can be computed by running Breadth First Search starting at each vertex.

When a graph is connected, the union of the paths found by Dijkstra's Algorithm or Breadth First Search form a spanning tree. The vertex that the algorithm starts with is important.

Definition 2.21. A **rooted tree** is a tree with one distinguished vertex called the **root**.

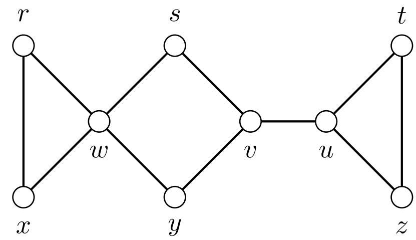
Rooted trees are common data structures in computer science, as they can be used to store information that can be retrieved quickly. They are also useful in some theoretical applications, as they can be used to construct larger trees by adding an edge adjacent to the root.

2.3. Connectivity

In a graph representing an electric grid, a vertex would represent a substation and an edge would represent an transmission line. If a substation or transmission line is disabled, it could knock out electricity to some of the grid. Vertices and edges of a graph whose deletion disconnect it deserve special attention.

Definition 2.22. Let G be a graph. A **cut-vertex** v is a vertex such that $G - v$ has more components than G. A **bridge** is an edge e such that $G - e$ has more components than G. A **block** of a graph is a maximal subgraph with no cut-vertex.

Example. In the following graph, u, v, and w are cut-vertices, and uv is a bridge. The blocks are the subgraphs induced by vertices $\{r, w, x\}$, $\{s, v, w, y\}$, $\{u, v\}$, and $\{t, u, z\}$.



Proposition 2.23. *An edge e of a graph G is a bridge if and only if e is not contained on any cycle.*

Proof. ($\Rightarrow$) Let $e = uv$ be a bridge, and let w and x be vertices in the components of u and v in $G - e$, respectively. If e is on a cycle, then there is a walk from w to u to v (along the cycle) to x and, hence, there is a $w - x$ path.

($\Leftarrow$) (contrapositive) Let $e = uv$ be an edge of G that is not a bridge. Then $G - e$ is connected, and so has a $u - v$ path. Together with e, this forms a cycle in G. $\square$

Any cut-vertex of a graph is contained in at least two blocks. Any two blocks have at most one cut-vertex in common. If two blocks had two vertices in common, their union would have no cut-vertex and, hence, be a larger block. Thus the blocks of a graph decompose it. The blocks of a graph are useful since many graph parameters can be found if they are known for each block. For example, the size of a graph is the sum of the sizes of its blocks.

Definition 2.24. The **block-cutvertex graph** of a graph G is a bipartite graph with one partite set representing the cut-vertices of G and the other partite set representing the blocks of G. There is an edge between two vertices if the given cut-vertex is contained in the given block.

Example. The block-cutvertex graph of the graph is the preceding example is P_7.

The block-cutvertex graph of a graph provides useful information about its structure.

Proposition 2.25. *Let G be a connected graph with a cut-vertex. Then the block-cutvertex graph H of G is a tree with all leaves in the same partite set. Further, G has at least two blocks that only contain one cut-vertex (**end-blocks**).*

Proof. If the H contained a cycle, then the subgraph of G induced by the blocks on that cycle would have no cut-vertex, so it would not contain separate blocks. Thus H is a tree. The blocks and cut-vertices of G are partite sets of H, so all leaves of H are in the same partite set. Any nontrivial tree contains two leaves, so G has at least two end-blocks. $\square$

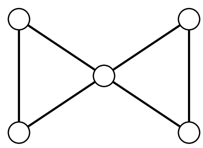
When a graph has no bridge or cut-vertex, it is useful to know how many vertices or edges must be deleted before it becomes disconnected.

Definition 2.26. A **vertex cut** of a graph G is a set S of vertices so that $G - S$ has more components than G. The **connectivity** $\kappa(G)$ of a connected noncomplete graph is the smallest size of a vertex cut. We define $\kappa(K_n) = n-1$, and $\kappa(G) = 0$ when G is disconnected. A graph is **k-connected** if $\kappa(G) \geq k$.

For S, $T \subseteq V(G)$, let $[S,T]$ be the set of edges with one end in S and the other in T. An **edge cut** of a multigraph G is a set $X = [S,T]$, of edges so that $G-X$ has more components than G. The **edge connectivity** $\kappa'(G)$ of a connected graph is the smallest size of an edge cut. A disconnected graph has $\kappa'(G) = 0$. A graph is **k-edge-connected** if $\kappa'(G) \geq k$.

Often we can express an edge cut as $X = [S, \overline{S}]$, where $\overline{S} = V(G) - S$. These definitions generalize cut-vertices and bridges. A connected graph G with a cut-vertex has $\kappa(G) = 1$. A connected graph G with a bridge has $\kappa'(G) = 1$.

Example. The "bowtie" $G = K_1 + 2K_2$ is connected with a cut-vertex, so $\kappa(G) = 1$. It has no bridge, but deleting two edges of a triangle disconnects the graph, so $\kappa'(G) = 2$.



Example. An edge cut of K_n must separate k vertices, $1 \leq k < n$, from the other $n-k$. Thus at least $k(n-k)$ edges must be deleted. The minimum value of this expression is $n-1$ when $k = 1$, so $\kappa'(K_n) = n-1$.

Definition 2.27. A **trivial edge cut** has all edges incident with a common vertex. A **bond** is a minimal nonempty edge cut.

Proposition 2.28. *If G is a connected graph with bond X, then $G-X$ has exactly two components.*

Proof (contrapositive). Assume X is an edge cut and $G-X$ has more than two components. Then X contains an edge e between two of them. Adding back e connects two components, resulting in a disconnected graph with fewer components than $G-X$. Thus X is not a bond. □

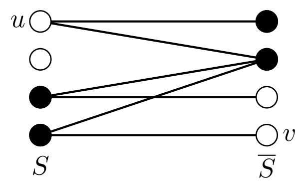
There is a natural relationship between the connectivity, edge connectivity, and minimum degree.

Theorem 2.29 (**Whitney's Theorem**—Whitney [**1932**]). *For any graph G, $\kappa(G) \leq \kappa'(G) \leq \delta(G)$.*

Proof. The edges incident with a vertex of minimum degree form a trivial edge cut, so $\kappa'(G) \leq \delta(G)$.

Consider a minimum edge cut $X = [S, \overline{S}]$. If all edges are present between S and $\overline{S}$, then $\kappa'(G) = |S|\,|\overline{S}| \geq n-1 \geq \kappa(G)$.

Else let $u \in S$ and $v \in \overline{S}$ be nonadjacent. Let U contain all neighbors of u in $\overline{S}$ and all vertices of $S - \{u\}$ with neighbors in $\overline{S}$ (colored black below). Any $u-v$ path contains a vertex of U, so U is a vertex cut. Now the edges from u to $\overline{S}$ and one edge from each vertex of $U \cap S$ to $\overline{S}$ yield $|U|$ distinct edges of X. Thus $\kappa'(G) \geq |U| \geq \kappa(G)$. □



Note that we may be tempted to just delete S in the previous proof. However, this may not disconnect the graph, e.g., if $G = P_n$ and S is a leaf.

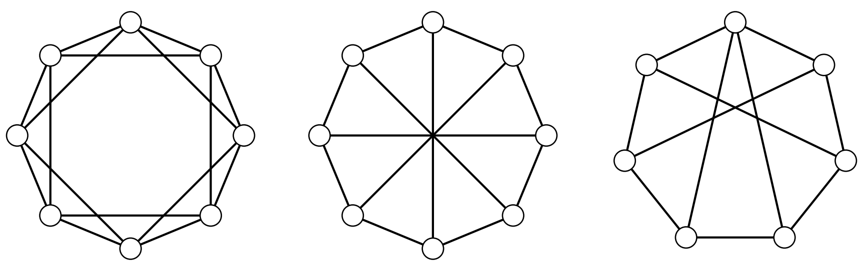
Since $\kappa(G) \le \delta(G)$, a k-connected graph must have size at least $\left\lceil \frac{kn}{2} \right\rceil$. We now describe a class of graphs that achieve this bound.

Definition 2.30. The kth **power** G^k of a graph G adds all edges between pairs of vertices with distance at most k. The graphs G^2 and G^3 are the **square** and **cube** of G.

Given $2 \le k < n$, the **Harary graph** $H_{k,n}$ is $(C_n)^{k/2}$ when k is even. (Harary graphs are named after Frank Harary, an early promoter of graph theory and author of several books on the subject.)

When n is even and k is odd, add edges between the pairs of most distant vertices of $(C_n)^{k/2}$.

When n and k are both odd, construct $H_{k,n}$ from $H_{k-1,n}$ by numbering the vertices consecutively around the circle and adding edges $i \leftrightarrow i + \frac{n-1}{2}$ for $0 \le i \le \frac{n-1}{2}$.



The graphs $H_{4,8}$, $H_{3,8}$, and $H_{3,7}$ appear above. Note that $H_{3,n}$ for n even is the Mobius ladder M_n. We prove in one case that $\kappa(H_{k,n}) = k$; the others are left to the Exercises. To prove that $\kappa(G) = k$, you must find a vertex cut of size k and show that no smaller vertex cut is possible.

Proposition 2.31 (Harary [**1962**])**.** *If k is even, $\kappa(H_{k,n}) = k$.*

Proof. Any set of k neighbors of a vertex form a vertex cut. Let $|S| < k$ and $u, v \in V(G) - S$. There are two $u - v$ paths around C_n. One of these two paths must have fewer than $\frac{k}{2}$ internal vertices in S. Since each vertex is adjacent to the next $\frac{k}{2}$ vertices, deleting S leaves a path in one of the two directions. $\square$

For various classes of graphs, more can be said about the inequalities in Theorem 2.29.

Theorem 2.32. *If G is cubic, then $\kappa(G) = \kappa'(G)$.*

Proof. If $\kappa(G) = 0$, then G is disconnected, so $\kappa'(G) = 0$.

If $\kappa(G) = 1$, then G has a cut-vertex v. Then v must be adjacent to only one vertex of some component of $G - v$, so G contains a bridge.

If $\kappa(G) = 2$, let u and v be vertices of a minimum vertex cut. Each is adjacent to only one vertex in some component of $G-u-v$. If this can be the same component for both, the corresponding edges form a 2-edge cut. If not, then $u \not\leftrightarrow v$, and the unique edges to different components still form an edge cut.

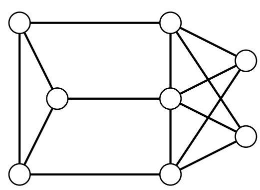
If $\kappa(G) = 3$, then $\kappa'(G) = 3$, since $\kappa(G) \le \kappa'(G) \le \delta(G) = 3$. □

Theorem 2.33 (Plesnik [**1975**]). *If G has diameter 2, then $\kappa'(G) = \delta(G)$.*

Proof (Bickle/Schwenk [**2019**]). Let $[S, \overline{S}]$ be a minimum edge cut. Now S and $\overline{S}$ cannot both have vertices u and v that are not incident with $[S, \overline{S}]$, for then $\operatorname{diam}(G) \ge d(u, v) \ge 3$. Say S has every vertex incident with $[S, \overline{S}]$. Thus $|S| \le \left|[S, \overline{S}]\right| = \kappa'(G) \le \delta(G)$. Each vertex in S is incident with at most $|S| - 1$ edges in $G[S]$, and so at least $\delta(G) - |S| + 1$ edges in $[S, \overline{S}]$. Thus

$$\kappa'(G) = \left|[S, \overline{S}]\right| \ge |S| (\delta(G) - |S| + 1).$$

This last expression attains its minimum value of $\delta(G)$ when $|S| = 1$ or $|S| = \delta(G)$. In both cases we have $\kappa'(G) \ge \delta(G)$, so $\kappa'(G) = \delta(G)$. □



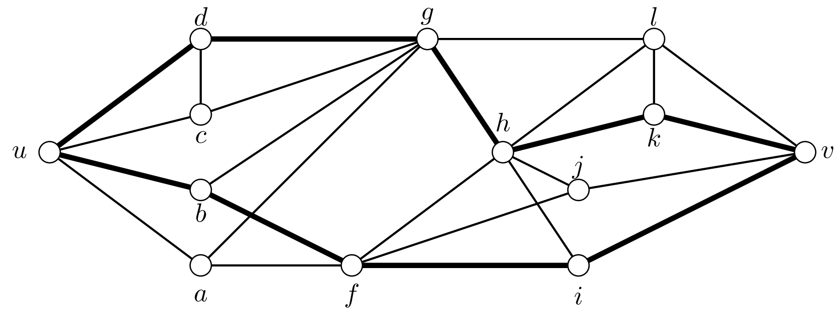
The graph above has diameter 2 and a nontrivial minimum edge cut. It follows from the proof that if a graph G has diameter 2, then either the edge cut is trivial or one of the components of $G - X$ is $K_{\delta(G)}$.

Related Terms: k-connectivity, connection number, cyclic connectivity, circulant graph.

2.4. Menger's Theorem

2.4.1. Menger's Theorem (vertices). By definition, a graph is connected if and only if there is a path between any pair of vertices. This implies that a graph has connectivity 0 if and only if there is no path between some pair of vertices. In the Exercises, you are asked to show that a graph has connectivity 1 (has a cut-vertex) if and only if any path between some pair of vertices goes through the cut-vertex. In this section, we will generalize these observations to characterize graphs with connectivity k.

Example. In the following graph, $S = \{f, g\}$ is a vertex cut that separates vertices u and v. There are two $u - v$ paths, shown in bold, that have no common internal vertices. There cannot be more than two such paths, since each vertex in S must be on one of them.



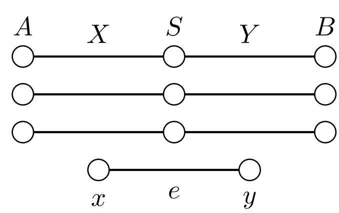
Menger's Theorem reveals a simple relationship between vertex cuts and disjoint paths between vertices. We begin with a lemma that contains the essence of this theorem.

Definition 2.34. Let D be a digraph. For sets of vertices A and B of D, an AB**-path** is a path starting in A, ending in B, and containing no other vertices of either. (A single vertex of $A \cap B$ is considered an AB-path.) An AB**-cut** is a set S of vertices of D such that $D - S$ contains no AB-path. An AB**-connector** is a subgraph of D with each component an AB-path (an empty graph is also an AB-connector).

Lemma 2.35. *Let D be a finite digraph, let A and B be sets of vertices of D, and let s be the minimum number of vertices forming an AB-cut. Then there is an AB-connector C consisting of s paths.*

Proof (Goring [**2000**]). We use induction on the number of edges. If D is empty, set $C = A \cap B$. Assume the result holds for digraphs with fewer than m edges. Let D be a digraph with size m containing edge $e = xy$. Now $D' = D - e$ has size $m - 1$, so D' has an AB-cut S with $|S| \leq s$. If $|S| = s$, then the induction hypothesis implies that D' has an AB-connector, so D does also.

Thus we assume $|S| < s$. Then $P = S \cup \{x\}$ and $Q = S \cup \{y\}$ are AB-cuts of D. Thus $|P| = |Q| = |S| + 1 = s$. An AP-cut (as well as a QB-cut) of D' is an AB-cut of D. Consequently, D' has an AP-connector X containing P and a QB-connector Y containing Q. Then $C = (X \cup Y) + e$ consists of s paths, since $X \cap Y = S$. $\square$



Definition 2.36. Let D be a digraph. For vertices u and v of D, let a uv**-cut** be a set S of vertices of D such that $D - S$ contains no $u - v$ path. A set of $u - v$ paths are **independent** if no pair of them have any common internal vertex.

Theorem 2.37 (**Menger's Theorem**—Menger [**1927**])**.** *Let u and v be nonadjacent vertices of a digraph D. Then the minimum size of a uv-cut equals the maximum number of independent $u - v$ paths.*

Proof. A uv-cut must contain a distinct internal vertex from each $u-v$ path. Thus the minimum size of a uv-cut is at least the maximum number of independent $u-v$ paths.

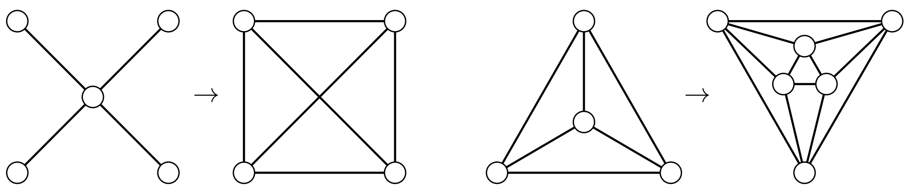
To show equality, let $A = N(u)$ and $B = N(v)$, and apply the lemma. Adding the edges incident with u and v produces the required number of $u-v$ paths. $\square$

The statement and proof of Menger's Theorem work just as well for undirected graphs. Menger's Theorem is an example of a min-max theorem, where the minimum value of one parameter is equal to the maximum value of another parameter.

2.4.2. Line Graphs. A result analogous to Menger's Theorem holds for edge cuts and paths with no common edges. To prove it, we will employ an operation that transforms edges into vertices.

Definition 2.38. The **line graph** $L(G)$ of a graph G is the graph whose vertices represent the edges of G, with vertices of $L(G)$ adjacent when the corresponding edges of G are adjacent. The **line digraph** of a digraph is defined similarly.

Example. The line graph of $K_{1,4}$ is K_4, $L(K_{1,4}) = K_4$. We have $L(K_4) = K_{2,2,2}$, the octahedron.



The structure of line graphs can be characterized.

Theorem 2.39 (Krausz [**1943**]). *For a graph G, there is a graph H with $L(H) = G$ if and only if G decomposes into cliques so that each vertex of G appears in at most two cliques.*

Proof. ($\Rightarrow$) Each vertex of H with degree at least 2 produces a clique in G corresponding to the edges that all meet at that vertex. These cliques decompose G. Each vertex of G belongs to the two cliques generated by the ends of the corresponding edge of H.

($\Leftarrow$) Let S_i be the sets of vertices of the cliques that decompose G. We will define H so that $L(H) = G$. Isolated vertices of G correspond to K_2-components in H, so we assume $\delta(G) \geq 1$.

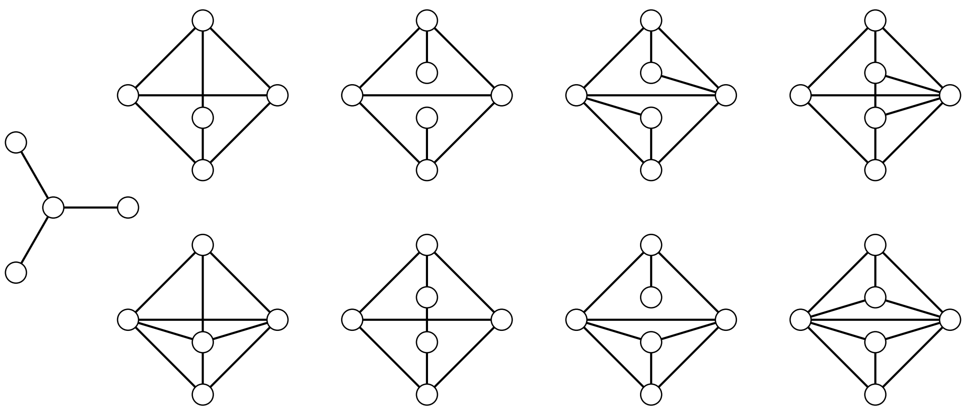
Let the vertices of H be the S_i and the vertices of G that appear in only one S_i (if any). Make two vertices of H adjacent if the corresponding vertex sets overlap in G. All vertices of G are contained in exactly two of the sets defining $V(H)$, and none occur in the same two sets. Thus the edges of H correspond to vertices of G. Adjacent vertices in G are in the same clique, so the corresponding edges in H are incident with a common vertex. Thus $L(H) = G$. $\square$

This immediately implies that any line graph is claw-free, since three cliques cannot overlap on one vertex of a line graph.

Example. The Petersen graph is not a line graph, since it contains a claw. For $k \geq 3$, the same is true for Q_k. However, $Q_2 = C_4 = L(C_4)$ and $Q_1 = K_2 = L(P_3)$.

The previous theorem can be used to prove the following forbidden induced subgraph characterization of line graphs.

Theorem 2.40 (Beineke [**1968**])**.** *A graph G is the line graph of some simple graph if and only if G does not contain any of the following graphs as an induced subgraph.*

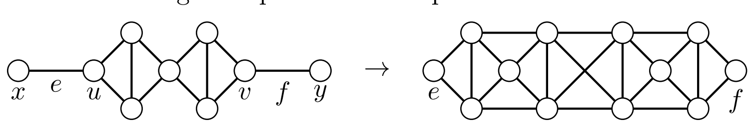


2.4.3. Extensions of Menger's Theorem. We now return to the problem of edge cuts and edge-disjoint paths. For edge cuts, multiple edges are relevant, so we consider multigraphs.

Definition 2.41. Let D be a directed multigraph. For vertices u and v of D, let a uv-**edge cut** be a set X of edges of D such that $D - X$ contains no $u - v$ path. A set of $u - v$ paths are **edge independent** if no pair of them have any common edge.

Theorem 2.42 (Elias/Feinstein/Shannon [**1956**], Ford/Fulkerson [**1956**])**.** *Let u and v be distinct vertices of a directed multigraph D. Then the minimum size of a uv-edge cut equals the maximum number of edge independent $u - v$ paths.*

Proof (West [**2001**]). Construct D' by adding vertices x and y and edges $e = xu$ and $f = vy$ to D. This does not change the quantities we are interested in, and each path can be seen as going from xu to vy. Now X is a uv-edge cut of D' exactly when the corresponding vertices form an ef-cut of $L(D')$. Also, edge independent $u - v$ paths in D correspond to independent $e - f$ paths in $L(D')$. Since $u \neq v$, $e \not\leftrightarrow f$ in $L(D')$. Since the minimum size of an ef-cut equals the maximum number of independent $e - f$ paths in $L(D')$, the minimum size of a uv-edge cut equals the maximum number of edge independent $u - v$ paths. $\square$



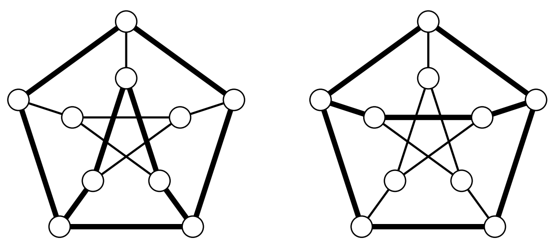
Menger's Theorem suggests a relationship between connectivity and independent paths. However, connectivity is a global property. That is, it involves all possible vertex cuts. Thus we should consider all possible pairs of vertices and the paths between them to determine connectivity.

Theorem 2.43. *The connectivity of a graph or digraph G is the maximum k such that the number of independent $u-v$ paths is at least k for all distinct vertices u and v. The edge connectivity of a graph or digraph G is the maximum k such that the number of edge independent $u-v$ paths is at least k for all distinct vertices u and v.*

Proof. For edge connectivity, this follows immediately from Theorem 2.42 and the definition of edge connectivity.

For connectivity, Menger's Theorem justifies the result when $u \not\leftrightarrow v$. When $u \leftrightarrow v$, let $\kappa(G) = k$. Then $G - uv$ is $k-1$-connected (see Exercise (19) of Section 2.3), so by Menger's Theorem there are at least $k-1$ independent $u-v$ paths in $G - uv$. Since uv is a $u-v$ path, there are at least k independent $u-v$ paths in G. Thus the connectivity is the maximum k such that the number of independent $u-v$ paths is at least k for all distinct vertices u and v. □

Example. Recall that any pair of vertices of the Petersen graph are either adjacent or have a common neighbor. In each case, there are three independent paths between the pairs of vertices. Thus the Petersen graph has connectivity 3. Since the Petersen graph is cubic, its edge connectivity must be 3 by Theorem 2.29.



2.4.4. Menger's Theorem and Cycles.

Menger's Theorem can be stated in the special case of 2-connected graphs, in which case there are additional characterizations.

Corollary 2.44 (Whitney [**1932**]). *The following are equivalent for a graph G:*

(1) *G is 2-connected.*

(2) *For any pair of vertices of G, there are two independent paths between them.*

(3) *Any pair of vertices of G lie on a common cycle.*

Proof. ($1 \Leftrightarrow 2$) This is just Menger's Theorem.

($2 \Leftrightarrow 3$) Two independent $u-v$ paths form a cycle, and a cycle containing vertices u and v can be split into two independent paths. □

Several other characterizations of 2-connected graphs are explored in the Exercises. The characterization of 2-connected graphs involving vertices on cycles is different than those we have seen before. It can be generalized to larger values of k.

Lemma 2.45. *If G is a k-connected graph, the graph formed by adding a vertex adjacent to k vertices of G is k-connected.*

Proof. Let S be a vertex cut of the graph G' formed by adding vertex u to G as described. If $u \in S$, then $S - \{u\}$ separates G, so $|S| \geq k+1$. If $u \notin S$ and $N(u) \subseteq S$, then $|S| \geq k$. Else u and $N(u) - S$ are in a single component of $G' - S$, so S must be a vertex cut of G, and $|S| \geq k$. □

Lemma 2.46 (Dirac [**1960**])**.** *If G is a k-connected graph and v, $v_1, \ldots, v_k$ are distinct vertices of G, then there are independent $v - v_i$ paths for $1 \leq i \leq k$.*

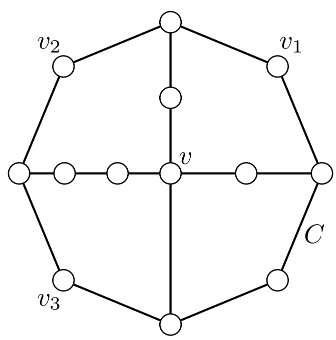
Proof. Construct G' by adding to G vertex u adjacent to $v_1, \ldots, v_k$. Then G' is also k-connected by Lemma 2.45. Menger's Theorem says that there are k independent $u - v$ paths, which must go through $v_1, \ldots, v_k$. Deleting edges uv_i from these paths proves the result. □

Theorem 2.47 (Dirac [**1960**])**.** *If G is a k-connected graph, $k \geq 2$, then any k vertices of G are on a cycle.*

Proof (West [**2001**]). We use induction on k. For $k = 2$, Corollary 2.44 is the result.

Assume that G is k-connected, $k > 2$, and let S be a set of k vertices, including v. Since G is $k-1$-connected, the induction hypothesis implies that the vertices of $S - \{v\}$ lie on a cycle C. If $n(C) = k-1$, then by Lemma 2.46 there are independent paths from v to all vertices of C, so the cycle can be expanded to include v.

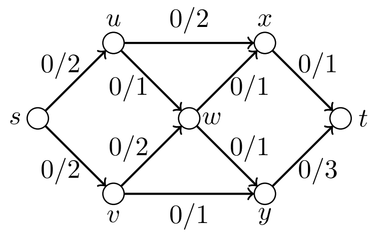
Assume that $n(C) \geq k$. Let $v_1, \ldots, v_{k-1}$ be the vertices of $S - \{v\}$ in order on C, and let V_i be the vertices starting with v_i up to but not including v_{i+1}. The sets V_i partition $V(C)$ into $k-1$ disjoint sets. Since there are k independent paths from v to $V(C)$, two enter $V(C)$ in some V_i. Then the cycle can be extended to include v using these two paths. □

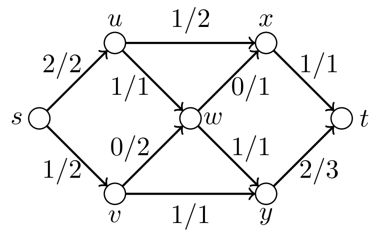


2.4.5. Max-flow Min-cut Theorem. Suppose there is a network of pipes where a liquid can flow in only one direction. Each pipe has a capacity for how much can flow in a given time. There is also a source that the liquid flows out of, and a sink that it flows into. This can be modeled with a digraph where edges represent pipes, vertices represent junctions, and each edge is assigned a nonnegative integer. The natural question to ask is what is the largest amount of liquid that can flow through the network in a given time.

Definition 2.48. A **network** is a digraph with a nonnegative capacity $c(e)$ on each edge e and distinguished **source** and **sink** vertices s and t. A **flow** f is a nonnegative function on the edges satisfying $0 \leq f(e) \leq c(e)$ for each edge and the flow into and out of each vertex other than s and t are equal. The **value** of a flow is the amount from the source to the sink. A **source-sink cut** $[S, T]$ consists of edges from set S to T, with $s \in S$ and $t \in T$. The **capacity** of a cut is the total of the capacities of the edges on $[S, T]$.

Example. Below left there is a network with no flow. Below right we see a flow with value 3; $\{vy, wy, xt\}$ is a cut with capacity 3.





It is not surprising that there is a min-max relationship between flows and capacities. The following theorem is true for all real capacities, but our proof will be restricted to rational capacities.

Theorem 2.49 (**Max-flow Min-cut Theorem**—Ford/Fulkerson [**1956**]). *In any network, the maximum value of a flow equals the minimum value of a source-sink cut.*

Proof. Suppose first that all capacities are integers. Let G be a network, and let D be a directed multigraph formed by replacing an edge with capacity r with r edges.

We claim that a maximum flow with value k in G corresponds to k edge independent paths in D. Certainly, k edge independent paths can be merged into a flow, as paths using j copies of an edge in D implies a flow having value j on that edge in G. Conversely, a flow with integer values can be split into edge independent paths in D by splitting off one $s-t$ path at a time.

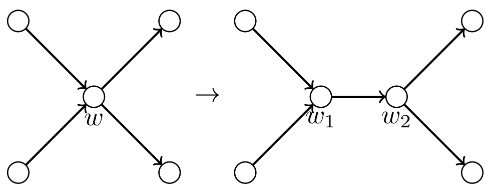
Now Menger's Theorem (edge version) says that the maximum number of edge independent $s-t$ paths equals the minimum size of an st-edge cut in D. Equivalently, the maximum value of a flow equals the minimum value of a source-sink cut in G.

If all capacities are rational, multiply all capacities by their least common denominator and apply the previous case. □

The Ford-Fulkerson Algorithm (see West [**2001**]) provides a practical way to find a maximum flow in a network. The basic idea is to find an $s-t$ path with unused capacity on all edges and increase the flow on this path. This may require decreasing a positive flow on the reverse directions of some edges. The Ford-Fulkerson Algorithm can be implemented in $\mathcal{O}\left(n^3\right)$ time.

By Menger's Theorem, this can be used to calculate edge connectivity. Select a vertex u of a digraph. Some minimum edge cut must separate u from another vertex. Running the Ford-Fulkerson Algorithm (with all capacities 1) between u and the other $n-1$ vertices shows that edge connectivity can be computed in $\mathcal{O}\left(n^4\right)$ time. Other algorithms with better efficiency also exist.

Connectivity can be computed similarly. Consider a pair of nonadjacent vertices u and v in a digraph D. Form a digraph D' by replacing each vertex w by two vertices w_1 and w_2 and an edge w_1w_2 with the edges to w going to w_1 and the edges from w coming from w_2. This converts the problem of finding a uv-cut of D to finding a uv-edge cut of D'. The Ford-Fulkerson Algorithm can be run on D (with all capacities 1) for all pairs of nonadjacent vertices of D. The minimum of the maximum values of the flows must be the connectivity of D.



Related Terms: total graph, odd triangle, even triangle, ear, closed ear, ear decomposition, closed ear decomposition, algebraic connectivity, Cheeger constant, strength of a graph, structural cohesion, augmenting path, maximum flow problem, baseball elimination problem, transportation network, consistent rounding of a matrix, Edmunds-Karp Algorithm, Dinic's Blocking Flow Algorithm, Integral Flow Theorem, Approximate Max-flow Min-cut Theorem, maximum cut.

Exercises

Section 2.1:

(1) Characterize the trees that have exactly two leaves.

(2) Characterize the trees of order n that have exactly $n-1$ leaves.

(3) Draw all trees of order 6.

(4) Draw all trees of order 8 (organize them by maximum degree).

(5) Draw all trees of order 10 with no vertices of degree 2. (*Note*: This problem appears in the movie *Good Will Hunting*, where it is claimed that it took a team of MIT professors two years to solve it.)

(6) Determine how many forests of order 6 there are.

(7) Show that a graph is a tree if and only if it can be constructed from K_1 by repeatedly applying the operation of adding an edge between any two vertices of two trees.

(8) Find an operation characterization of complete graphs.

(9) Find an operation characterization of cycles.

(10) Find an operation characterization of stars.

(11) Find a forbidden subgraph characterization for trees that are paths.

(12) Find a forbidden subgraph characterization for trees that are stars.

(13) Show that a tree T with order $n \geq 3$ is a star if and only if it has diameter 2.

(14) A **double star** is a tree with two nonleaves. Show that a tree is a double star if and only if it has diameter 3.

(15) Count the number of double stars with order n.

(16) Describe the structure of the complement of a double star, and show that it has diameter 3.

(17) Characterize the trees of order n with $\Delta(T) = n - 2$.

(18) Count the number of trees of order $n \geq 5$ with $\Delta(T) = n - 3$.

(19) Count the number of trees of order $n \geq 7$ with $\Delta(T) = n - 4$.

(20) Show that any two of the three properties—connected, acyclic, and $m = n - 1$—imply the third, and hence characterize trees.

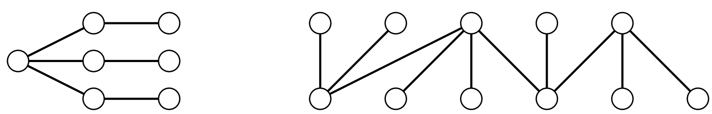
(21) A **spider** is a tree with a single vertex with degree 3, and all others with smaller degree. Explain how to count the number of spiders with order $n \geq 4$.

(22) Show that a tree is a spider if and only if it has exactly three leaves.

(23) Let T be a tree with order n, k leaves, and $\Delta(T) = k$. Show that G is the union of k paths with a common endpoint.

(24) A **broom** is a tree formed by identifying the center of a star with the end of a path. Find the number of brooms of order $n \geq 3$.

(25) (Harary/Schwenk [**1971**]) A **caterpillar** is a tree that is K_1, K_2, or has the property that deleting all its leaves results in a path. Show that a tree is a caterpillar if and only if it does not contain the tree shown below left.



(26) (Harary/Schwenk [**1972**]) Show that a tree is a caterpillar if and only if it can be drawn so that its vertices are on two parallel lines and its edges are straight lines that don't cross (see above right).

(27) + (Harary/Schwenk [**1973**]) Show that the number of caterpillars of order $n \geq 3$ is $2^{n-4} + 2^{\lfloor \frac{n-4}{2} \rfloor}$. (*Hint*: Label the edges of a caterpillar with $1, 2, \ldots, n-1$ so that edges with consecutive labels are adjacent.)

(28) Draw a Venn diagram representing paths, stars, double stars, brooms, and caterpillars. If there is a single tree in the intersection of two classes, identify it.

(29) + (Jordan [**1869**]) Show that the center of a tree is either a single vertex or two adjacent vertices.

(30) Let T be a tree. Show that the vertices of T all have odd degree if and only if for every edge, both components of $T - e$ have odd order.

(31) Show that every tree with $k \geq 2$ leaves decomposes into $k - 1$ paths.

(32) Find all trees of order 6 such that decompose K_6.

(33) Let n_i be the number of vertices with degree i in a tree T for $1 \le i \le k$. Show that $n_1 = n_3 + 2n_4 + \cdots + (k-2)\, n_k + 2$.

(34) (Bose et al. [**2008**]) Let D be a list of n positive integers. Show that if $\sum D = 2n - 2$, then for any $l, k \in D$, D can be realized as a tree in which a vertex of degree l is adjacent to a vertex of degree k, unless $n > 2$ and $l = k = 1$.

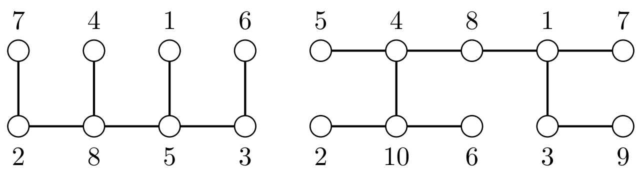
(35) Let $l\,(G)$ be the length of the longest path of G. Show that $\delta\,(G) \le l\,(G)$, and find the extremal graphs for this bound.

(36) (Bondy/Murty [**1976**]) A **saturated hydrocarbon** is a molecule containing k carbon atoms and l hydrogen atoms by adding bonds between atoms so that each carbon atom is in four bonds and each hydrogen atom is in one bond. Assume that no sequence of bonds forms a cycle. Prove that $l = 2k + 2$.

(37) Propane is a saturated hydrocarbon with formula C_3H_8. Sketch a graph representing propane.

(38) Butane and isobutane are saturated hydrocarbons with formula C_4H_{10}. Sketch graphs representing them.

(39) Find the Prufer code of the tree below left.



(40) Find the Prufer code of the tree above right.

(41) Find the tree with Prufer code
(a) $(4, 7, 6, 4, 1, 1)$.
(b) $(9, 7, 9, 9, 3, 4, 1, 10)$.

(42) Characterize trees with Prufer codes containing
(a) a single number.
(b) exactly two numbers.
(c) all different numbers.

(43) Count the distinct vertex labelings of the trees of the following orders and verify that they sum to the number given in Cayley's Formula.
(a) order 4
(b) order 5
(c) + order 6

(44) Show that $K_n - e$ has $(n-2)\, n^{n-3}$ spanning trees. (*Hint*: Count the number of times each edge is used by the spanning trees.)

(45) + (Aigner/Ziegler [**1998**]) An alternative proof of Cayley's Formula.
(a) Let T_n be the number of labeled trees of order n. Count the number of ways to choose a tree on n vertices, choose a single vertex (the root), and label the edges with distinct numbers.
(b) Starting with n isolated vertices, each of which is initially considered the root of a tree, count the number of ways to add $n - 1$ directed edges so

that each edge added can be from any vertex and to any root of a tree in the forest. (Once an edge points to a vertex, it is no longer a root.)

(c) Use parts (a) and (b) to prove $T_n = n^{n-2}$.

(46) + Show that there is a one-to-one function from the set of unlabeled trees of order n to the set of oriented $2n-2$-cycles with $n-1$ edges in each direction. Use this to find an exponential upper bound for the number of unlabeled trees of order n.

Section 2.2:

(1) A graph is formed by adding k edges to a tree. How many cycles can the graph contain if
 (a) $k = 2$?
 (b) $k = 3$?

(2) Prove or disprove: There is a graph with exactly two spanning trees.

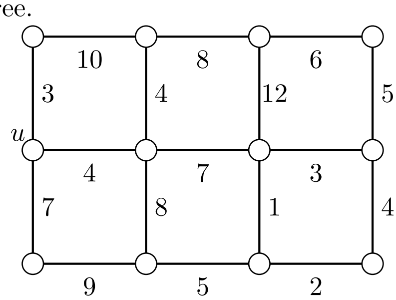
(3) Let G be a connected graph with order n. Prove that G has exactly one cycle if and only if G has size n.

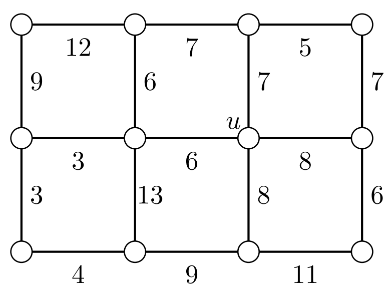
(4) Let T and T' be spanning trees of a connected graph G, and let e be an edge in T and not in T'. Show that there is an edge e' in T' and not in T such that
 (a) $T - e + e'$ is a spanning tree of G.
 (b) $T' + e - e'$ is a spanning tree of G.

(5) A **binary tree** is a rooted tree with each vertex adjacent to at most two vertices further from the root, which are distinguished as the **left child** and **right child**.
 (a) Find a recurrence relation for the number of **full binary trees**, in which every nonleaf has exactly two children.
 (b) + Find an explicit formula for part (a).

(6) An ancestral **family tree** has vertices representing a person and all of his biological ancestors, with edges between each biological parent and child. Under what assumptions is this a tree in the graph theory sense? Are these assumptions realistic?

(7) For the graph below left, use Kruskal's Algorithm to find a minimum spanning tree.





(8) For the graph above right, use Kruskal's Algorithm to find a minimum spanning tree.

(9) For the graph in Exercise 7, use Prim's Algorithm to find a minimum spanning tree.

(10) For the graph in Exercise 8, use Prim's Algorithm to find a minimum spanning tree.

(11) + Prove that Prim's Algorithm produces a minimum spanning tree.

(12) Show that if a graph has distinct edge weights, it has a unique minimum spanning tree.

(13) Let the **weight of a cycle** in K_n be the sum of its edge weights. Show that all cycles have even weight if and only if the subgraph formed by the edges with odd weight is a spanning complete bipartite graph.

(14) Show that the total weight on every cycle of K_n is even if and only if the total weight on every triangle is even.

(15) For the graph in Exercise 7, use Dijkstra's Algorithm to find the distance starting at vertex u.

(16) For the graph in Exercise 8, use Dijkstra's Algorithm to find the distance starting at vertex u.

(17) For the graph in Exercise 2 of Section 2.3, use Breadth First Search to find the distance starting at vertex u.

(18) For the first graph in Section 2.3, use Breadth First Search to find the distance starting at vertex u.

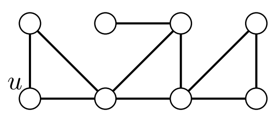
(19) Explain how to find the girth of a graph using a modification of Breadth First Search.

(20) Develop an algorithm to determine whether a graph is bipartite.

Section 2.3:

(1) What do cut-vertices and bridges represent in
   (a) a road intersection graph?
   (b) a computer network graph?

(2) Find the cut-vertices, bridges, and block-cutvertex graph of the graph below.



(3) Show that a vertex v of a graph G is a cut-vertex if and only if there are vertices u and w in G such that every $u - w$ path contains v.

(4) Show that an edge e of a graph G is a bridge if and only if G contains two vertices u and w such that every $u - w$ path contains e.

(5) Show that a k-regular bipartite graph with $k \geq 2$ does not contain a bridge.

(6) Show that a graph with all even degrees does not contain a bridge.

(7) Let $e = uv$ be a bridge of a graph G. Characterize when v is a cut-vertex.

(8) Show that a cubic graph has a cut-vertex if and only if it has a bridge.

(9) Show that if v is a cut-vertex of a graph G, then v is not a cut-vertex of $\overline{G}$.

(10) Show that if a graph with diameter 2 has a cut-vertex, then its complement has an isolated vertex.

(11) Show that a graph is a forest if any only if every edge is a bridge.

(12) Prove or disprove: A connected nontrivial graph is a tree if and only if every nonleaf is a cut-vertex.

(13) Show that a nontrivial connected graph has at least two vertices that are not cut-vertices.

(14) Show that an edge e of a connected graph G is a bridge if and only if e belongs to every spanning tree of G.

(15) A graph G has blocks $B_1, \ldots, B_k$. Show that $n(G) = \sum n(B_i) - k + 1$.

(16) A **cactus** is a connected graph in which every block is an edge or a cycle. Show that a cactus G has $m(G) \leq \frac{3(n-1)}{2}$. Describe the extremal graphs.

(17) + (Harary/Norman [**1953**]) Show that the center of a graph is always contained in a single block.

(18) Find a formula for the number of spanning trees of a graph G in terms of this for its blocks.

(19) Find the connectivity and edge connectivity of the following graphs. Determine whether every minimum edge cut must be trivial.
(a) trees
(b) $G_{r,s}$
(c) theta graphs

(20) Find the connectivity and edge connectivity of the following graphs. Determine whether every minimum edge cut must be trivial.
(a) $K_{r,s}$
(b) W_n
(c) the triangular grid T_l

(21) Find the connectivity of Q_k. (*Hint*: Use the recursive definition of Q_k.)

(22) Characterize the graphs with connectivity $\kappa(G) = n - 2$.

(23) Find a 4-regular graph with order 14 that does not contain any 3-connected subgraph.

(24) Determine the connectivity and edge connectivity of $G + K_1$ in terms of the connectivity and other parameters of G.

(25) Let G be a graph of order $n \geq 2$, and let k be an integer such that $1 \leq k \leq n - 1$. Show that if $\delta(G) \geq \left\lceil \frac{n+k-2}{2} \right\rceil$, then G is k-connected. Show that this bound is sharp.

(26) For any integers k, l, and m with $0 < k \leq l \leq m$, construct a graph with $\kappa(G) = k$, $\kappa'(G) = l$, $\delta(G) = m$.

(27) (a) For all $k \geq 1$, find the size of P_n^2.
(b) Let T be a tree with order n. Determine sharp bounds on the size of T^2.

(28) If G is a graph with diameter k and order n, what graph is G^k?

(29) (Harary [**1962**]) Show that the Harary graph $H_{k,n}$ has connectivity k when
(a) n is even and k is odd.
(b) n and k are both odd.

(30) Find the diameter and radius of $H_{k,n}$.

(31) Show that if $\Delta(G) \leq 3$, then $\kappa(G) = \kappa'(G)$.

(32) Prove or disprove: If G is a graph with one vertex of degree 4 and all others with degree 3, then $\kappa(G) = \kappa'(G)$.

(33) Show that if $\delta(G) \geq \lfloor \frac{n}{2} \rfloor$, then $\kappa'(G) = \delta(G)$. Show that this bound is sharp.

(34) Show that $\kappa'(G)$ and $\delta(G)$ can be arbitrarily far apart when G is a graph with diameter 3.

(35) + (Bickle/Schwenk [**2019**]) Let $\mathbb{G}$ be the class of graphs containing $K_{\frac{n}{2}} \Box K_2$, $n \geq 4$, and those graphs that can be constructed as follows. Let H_1 be a graph with order $d > 1$ and $\delta(H_1) \geq d - r - 1$, and let H_2 be a graph with order r. Add a perfect matching between K_d and H_1 and join all the vertices of H_1 and H_2. Show that a graph has diameter 2 and contains a nontrivial minimum edge cut if and only if it is in $\mathbb{G}$.

(36) (Bickle/Schwenk [**2019**]) Show that, if $G \in \mathbb{G}$ as defined in the previous problem, it has between $d = \delta(G)$ and $\max\{n - d, 3d - 1\}$ trivial edge cuts.

(37) Show that if $X = [S, \overline{S}]$ is an edge cut, then $|X| = \left|\sum_{v \in S} d(v)\right| - 2m(G[S])$.

(38) + Show that if $X = [S, \overline{S}]$ is an edge cut and $|X| < \delta(G)$, then $|S| > \delta(G)$.

Section 2.4:

(1) Find the line graph of the following graphs.
   (a) $K_1 + (K_1 \cup K_2)$
   (b) P_n
   (c) $K_{r,s}$
   (d) double stars with vertices of degrees $r > 1$ and $s > 1$

(2) Show that the following graphs from Theorem 2.40 are not line graphs.
   (a) $K_2 + 2K_2$
   (b) $K_2 + P_3$
   (c) W_5
   (d) K_4 with a subdivided edge

(3) Determine which graphs in the following classes are line graphs. For those that are, find a graph with the given line graph.
   (a) K_n
   (b) W_n
   (c) trees
   (d) $K_{r,s}$

(4) Determine which graphs in the following classes are line graphs. For those that are, find a graph with the given line graph.
   (a) $G_{r,s}$
   (b) C_n
   (c) the triangular grid T_l
   (d) bipartite graphs

(5) Find the complement of the line graph of K_5. What graph is this?

(6) Show that the complement of the Kneser graph $KG_{r,2}$ is the line graph of K_r.

(7) Show that vertices form a clique in $L(G)$ if and only if the corresponding edges in G have a common endpoint or form a triangle.

(8) Find a formula for the number of triangles of $L(G)$ in terms of properties of G.

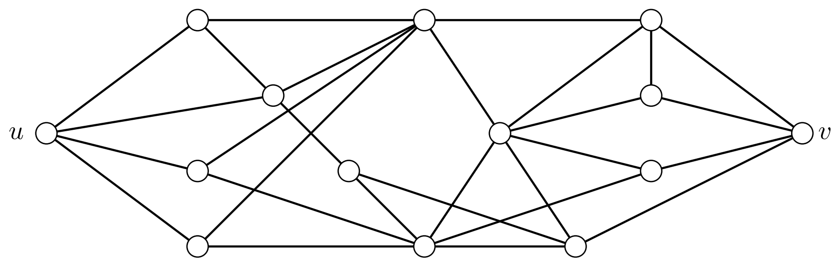
(9) (Ray-Chaudhuri [**1967**]) Let G be a graph without isolated vertices. Show that if $L(G)$ is connected and regular, then either G is regular or G is a bipartite graph with vertices in the same partite set having the same degree.

(10) Let G be a k-edge-connected graph. Show that $L(G)$ is k-connected and is $2k-2$-edge-connected.

(11) Let G be a graph.
 (a) Show that the size of $L(G)$ is $\sum_v \binom{d(v)}{2}$.
 (b) Determine for which graphs $L(G) = G$.
 (c) (van Rooij/Wilf [**1965**]) For all graphs, determine the limit of the sequence of graphs formed by iterating the line graph operation G, $L(G)$, $L(L(G)), \ldots$.

(12) (van Rooij/Wilf [**1965**]) An **odd triangle** T of G has $|N(v) \cap V(T)|$ odd for some $v \in V(G)$. Show that if G is a line graph, then G is claw-free and no **double triangle** (induced $K_4 - e$) of G has two odd triangles. (*Note*: The converse holds.)

(13) In the graph below, find a minimum uv-cut and the same number of independent paths.



(14) In the graph above, find a minimum uv-edge cut and the same number of edge independent paths.

(15) Use Menger's Theorem to find the connectivity of the graphs in the following classes.
 (a) $K_{r,s}$
 (b) W_n
 (c) Q_n

(16) Use Menger's Theorem to find the edge connectivity of the graphs in the following classes.
 (a) K_n
 (b) $G_{r,s}$
 (c) $K_1 + 2K_r$

(17) Prove Lemma 2.35, given Menger's Theorem.

(18) Let G be a graph with order n and $k = \kappa(G)$. Show that $\operatorname{diam}(G) \leq \frac{n-2}{k} + 1$. Show that this bound is sharp.

(19) Show that if G is k-connected, then $G - e$ is $k-1$-connected.

(20) Show that if G is k-edge connected, then $G - e$ is $k-1$-edge connected.

(21) Show that if G is a 2-connected graph containing an odd cycle, then any vertex v of G is on an odd cycle.

(22) Show that every 3-connected graph contains a subdivision of K_4.

(23) Let G be k-connected, where $k \geq 3$, and let it contain vertices $v_1, \ldots, v_k$. Show that G has a cycle containing $v_1, \ldots, v_{k-1}$ but not v_k and independent $v_i - v_k$ paths, where $1 \leq i \leq k-1$.

(24) Show that Theorem 2.47 is sharp by finding for each k a k-connected graph with $k+1$ vertices not on a cycle.

(25) If G is 2-connected graph with vertices u and v and P is a $u-v$ path of G, must there exist a $u-v$ path P' so that P and P' are independent?

(26) Show that a graph is 2-connected if and only if $\delta(G) \geq 1$ and every pair of edges in G lies on a common cycle.

(27) Show that if G is 2-connected, then the graph obtained by subdividing an edge of G is 2-connected.

(28) Show that if G is 2-edge connected, then the graph obtained by subdividing an edge of G is 2-edge connected.

(29) (Robbins [**1939**]) Show that a graph has a strong orientation if and only if it is 2-edge-connected.

(30) (Chartrand/Lesniak [**1986**]) Let v be a vertex of a 2-connected graph G. Show that v has a neighbor u so that $G - v - u$ is connected.

(31) (Bickle [**2013**]) Show that a graph G is connected with $\delta(G) \geq 2$ if and only if it is contained in the set S whose members can be constructed by the following rules.
  (a) All cycles are in S.
  (b) Given one or two graphs in S, the result of joining the ends of a (possibly trivial) path to it or them is in S.

(32) + (Whitney [**1932**]) An **ear** of a graph is a maximal path whose internal vertices have degree 2 (an edge can be an ear). An **ear decomposition** is a decomposition of a graph into a cycle and some ears. Show that a graph is 2-connected if and only if it has an ear decomposition. Further, any cycle in a 2-connected graph can be the initial cycle in some ear decomposition.

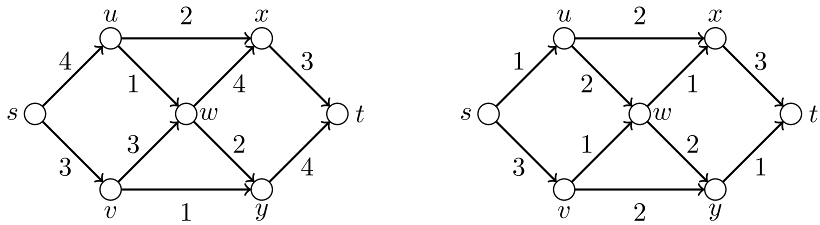
(33) (Dirac [**1967**]) Show that a minimally 2-connected graph G has $\delta(G) = 2$, and $m(G) \leq 2n - 4$ for $n \geq 4$, with equality only for $K_{2,n-2}$. (*Hint*: Use an ear decomposition.)

(34) + (West [**2001**]) A **closed ear** is a cycle of a graph with all vertices except one having degree 2. A **closed ear decomposition** is a cycle and some ears and closed ears. Show that a graph is 2-edge connected if and only if it has a closed ear decomposition. Further, any cycle in a 2-edge connected graph can be the initial cycle in some closed ear decomposition.

(35) Explain how network flows can be used to model traffic in a road intersection network.

(36) Explain how network flows can be used to model current in an electrical network.

(37) Find a maximum flow and minimum cut for the network below left.



(38) Find a maximum flow and minimum cut for the network above right.

(39) Suppose a flow network has multiple sources and multiple sinks. Explain how to modify this network to be able to apply the Max-flow Min-cut Theorem.

(40) Use the Max-flow Min-cut Theorem to prove Menger's Theorem.

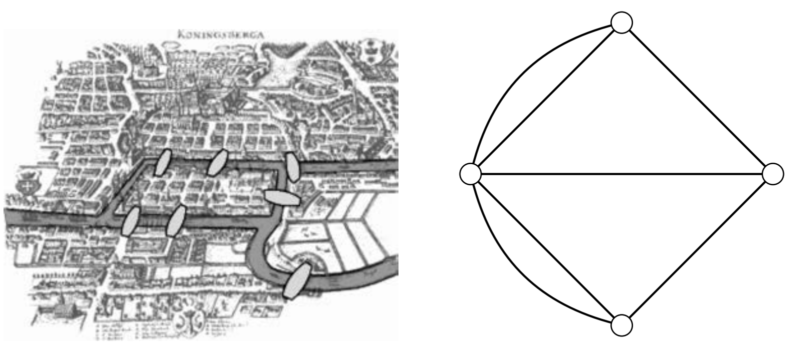
Chapter 3

Structure and Degrees

Structural graph theory is a loosely related group of problems and ideas concerning how graphs are constructed or structured. Common topics include subgraphs, graph characterizations, graph operations, and symmetry. Degree sequences are particularly important to structural graph theory. This chapter considers the problems of characterizing graphs with a walk that does not repeat edges, determining whether two graphs are isomorphic, characterizing degree sequences of graph classes, and constructing graphs whose degrees satisfy certain conditions.

3.1. Eulerian Graphs

In 1736, citizens of the Prussian town of Konigsberg had a problem. The city straddled the Pregel river, and there were seven bridges between the four landmasses. The citizens wanted to take a Sunday walk, crossing each bridge exactly once, and returning to where they started. But no one could find a way to do so. The problem came to the attention of famous Swiss mathematician Leonhard Euler, who was in St. Petersberg at the time.



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Euler (pronounced "Oiler") is perhaps the most prolific mathematician in history, making huge contributions to many areas of mathematics. Euler saw that the problem could be modeled with a point representing each landmass and a curve representing each bridge. In modern terms, the result was a multigraph. This problem is seen as the beginning of the subject of graph theory.

Euler saw that any walk would enter and exit a landmass each time it was encountered. Thus such a walk would require that an even number of bridges incident with each landmass. But in Konigsberg, each landmass had an odd number of bridges incident with it, so what the citizens wanted was impossible. Thus the problem was solved. Euler also considered the harder question of what conditions would guarantee that such a walk was possible. Some definitions are necessary.

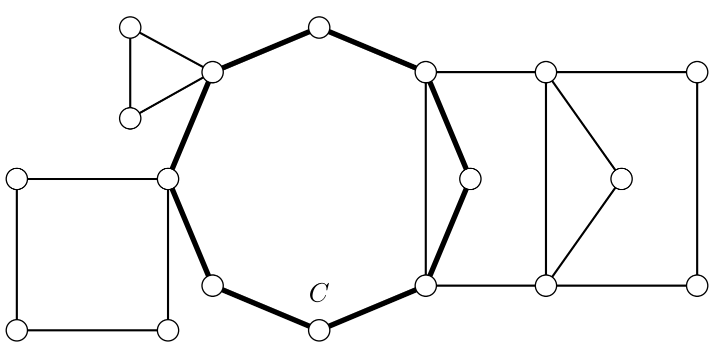
Definition 3.1. A $u - v$ **trail** is a $u - v$ walk that does not repeat any edges. A **closed trail** (or **circuit**) has $u = v$; an **open trail** has $u \neq v$. An **Eulerian circuit** contains all the edges of a graph. An **Eulerian trail** contains all the edges of a graph but is not closed. A graph with an Eulerian circuit is an **Eulerian graph**.

Now we can characterize Eulerian graphs. Since a circuit cannot cross between different components, we restrict ourselves to connected graphs.

Theorem 3.2. *A connected multigraph G is Eulerian if and only if it has all even degrees.*

Proof. ($\Rightarrow$) (Euler [**1736**]) If G has an Eulerian circuit, then each time it goes through a vertex, it uses two edges. The first and last edge also pair up at the first vertex. Thus every vertex has even degree.

($\Leftarrow$) (Hierholzer [**1873**]) Assume G is connected with all even vertex degrees. We use strong induction on the size m. If $m = 0$, G has a trivial circuit. Assume that all connected graphs with even vertex degrees and size less than m have an Eulerian circuit. Let G have size $m > 0$. Then $\delta(G) \geq 2$, so by Lemma 1.19 (which works for multigraphs), G contains a cycle C. Deleting the edges of C results in a (possibly disconnected) graph G' with all even degrees. Now each component of G' has size less than m, so applying the induction hypothesis shows that each has an Eulerian circuit. Now an Eulerian circuit for G can be constructed by splicing together segments of C and the circuits of each component of G'. $\square$



Euler's paper was not clear on the harder direction of the theorem. The first clear proof of this direction was given by Hierholzer in 1873. He used a somewhat different approach (see Exercise 11 of Section 3.1). The proof that we have used has the advantage of implying an algorithm for how to construct an Eulerian circuit, not just proving that one exists. This algorithm can be implemented in $\mathcal{O}(m)$ time. Note that strong induction is necessary since we do not know in general the length of cycle C.

Konigsberg was later absorbed by the Soviet Union and had its name changed to Kaliningrad. After the breakup of the Soviet Union, Kaliningrad became part of an *oblast* (region or province) on the Baltic Sea between Poland and Lithuania disconnected from the rest of Russia. Of the original seven bridges, two were removed, and three were replaced, leaving two of the original seven remaining.

The proof of the previous theorem immediately implies another result.

Corollary 3.3. *A multigraph can be decomposed into cycles if and only if all vertices have even degree.*

Proof. ($\Rightarrow$) If G decomposes into cycles, then every cycle containing vertex v contributes two to its degree. Thus its degree is even.

($\Leftarrow$) If all vertices have even degree, then each component is Eulerian. The proof of Theorem 3.2 implies that each nontrivial component contains a cycle, and the induction implies that they can be decomposed into cycles. □

When a graph is not Eulerian, the next question is whether it has an Eulerian trail. We will prove a more general result.

Theorem 3.4. *The minimum number of open trials that decompose a connected non-Eulerian multigraph with exactly $2k$ odd vertices is k.*

Proof. Let G be a connected graph with $2k > 0$ odd vertices. An open trail adds an even degree to every non–end vertex and odd degree to both ends. Thus there must be at least k open trails to account for the $2k$ odd vertices. Form a multigraph H by adding k edges between odd vertices of G so that H has all even degrees. Then H has an Eulerian circuit. Deleting the k edges results in k open trails that decompose G. □

Note that the proof modifies the graph to employ Theorem 3.2. This technique saves work compared to proving the result directly (see Exercise 12 of Section 3.1). When $k = 1$, this theorem implies that a connected graph has an Eulerian trail if and only if it has exactly two odd vertices, in which case they are the ends of the trail.

Example. The multigraph representing the Konigsberg Bridge Problem has four odd vertices, so it decomposes into two odd trails. Any theta graph, such as $K_{2,3}$, has exactly two odd vertices, so it has an Eulerian trail. The Petersen graph has ten odd vertices, so it decomposes into five open trails. Since K_5 is 4-regular, it is Eulerian.

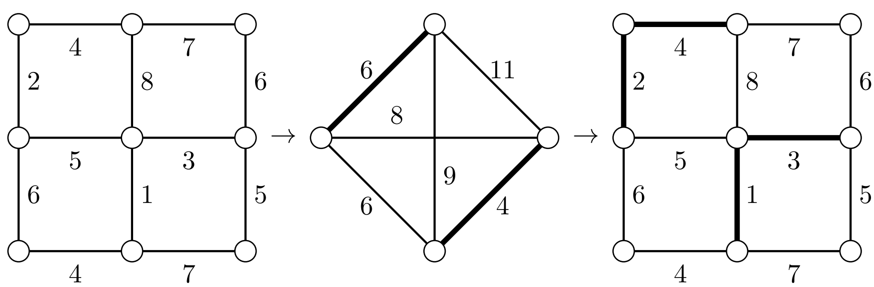
We have seen in the Konigsberg Bridge Problem that Eulerian graphs are relevant to graphs representing road networks. A more timely example is a postman who must deliver to all the houses in a neighborhood. A neighborhood road network graph is unlikely to be Eulerian, since there are many 3-way intersections. But we would still like to find the shortest route that covers every edge.

We use a weighted graph, and we want to find a closed walk of minimum length that uses all edges. This is called the **Chinese Postman Problem**. It is named in honor of Mei-ko Kwan [**1962**] who proposed it. (The postman is not required to be Chinese!)

If the graph G is Eulerian, there is nothing to solve. When there is an odd vertex, the postman must repeat an edge incident with it, and must continue to do so along a path until he reaches another odd vertex. Thus to return to the starting point, he must repeat paths joining pairs of odd vertices. Equivalently, we could duplicate edges along these paths so that in the new graph, each edge is used exactly once, making it Eulerian. Since each edge of G must be used once, we seek the minimum total weight of edges to add that will make the graph Eulerian. We must add k paths (which may intersect) between the $2k$ odd vertices of the graph.

We can use Dijkstra's Algorithm to find the shortest paths between each of the $2k$ odd vertices. Construct a complete graph K_{2k} whose vertices represent the odd vertices of G, and whose edges have weights that are the distances between the corresponding vertices of G. Thus we must find a perfect matching in K_{2k} with minimum total weight, which can be done efficiently. This solution is due to Edmonds and Johnson [**1973**]. The solution has complexity $\mathcal{O}\left(n^3\right)$.

Example. We solve the Chinese Postman Problem for the weighted graph below left. There are four odd vertices. The middle graph is K_4 on those vertices, with each edge labeled with the distance between the corresponding pair of vertices. There are three possible matchings, with total weights $8 + 9 = 17$, $6 + 11 = 17$, and $6 + 4 = 10$. At right, we see the corresponding edges that must be repeated in bold.



For digraphs, the problem of finding an Eulerian circuit is quite similar to that for undirected graphs.

Theorem 3.5. *A connected digraph D is Eulerian if and only if $od\,(v) = id\,(v)$ for every vertex v of D.*

You are asked to prove this in the Exercises, along with generalizations of Corollary 3.3 and Theorem 3.4.

Related Terms: de Bruijn graph, extendible vertex.

3.2. Graph Isomorphism

3.2.1. The Isomorphism Problem. Suppose a chemist notices that two molecules have similar properties. He suspects that they may actually be the same molecule. How can he tell?

We briefly introduced the idea of isomorphism between graphs in Section 1.2. Informally, two graphs are isomorphic if they have the same structure, despite having different vertex names or drawings. We now make this idea precise.

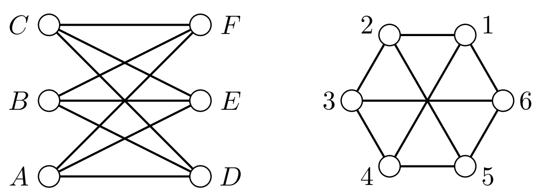
Definition 3.6. An **isomorphism** from graph G to graph H is a bijection $\phi : V(G) \to V(H)$ such that $uv \in E(G)$ if and only if $\phi(u)\phi(v) \in E(H)$. We say G is **isomorphic to** H and write $G \cong H$ for labeled graphs or $G = H$ for unlabeled graphs. Two graphs are **nonisomorphic** if they are not isomorphic.

It is a straightforward exercise to show that isomorphism of graphs is an equivalence relation. This implies that the relation partitions the collection of all graphs into equivalence classes. An unlabeled graph can be thought of as representing the isomorphism class of all graphs with the same structure.

How can we determine whether two graphs are isomorphic? The most naive answer is to check all bijections between the vertex sets of G and H. Either one is an isomorphism or none are. This is a theoretical answer but not a practical answer. For a graph of order n, there are $n!$ permutations of its vertices. Since the factorial function grows very quickly, the brute force method is not practical for even fairly small graphs.

A somewhat better approach is to pair up one vertex, then try to match up their neighborhoods, and so on until an isomorphism is found or this attempt fails.

Example. Consider the following graphs.

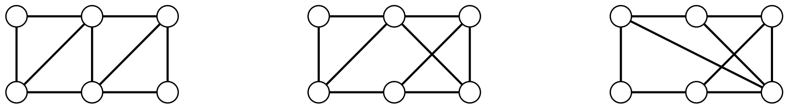


We try to construct an isomorphism by mapping vertex A to vertex 1. Then vertices D, E, and F must map onto $\{2, 4, 6\}$, so map D to 2, E to 4, and F to 6. Then an isomorphism can be completed by mapping B to 3 and C to 5. The graphs are isomorphic; both are $K_{3,3}$.

Definition 3.7. A **graph invariant** is a property of a graph that is preserved by isomorphism.

To show that two graphs are not isomorphic, a much better approach than checking all possible isomorphisms is to find a graph invariant that varies on them. What properties are invariants? It is immediate that the order and size are invariant since a bijection matches up vertices and edges. Similarly, the degree sequence is preserved since every vertex is mapped to a vertex with the same number of neighbors.

Example. Consider the following graphs. All have order 6 and size 9. However, the graph on the left has degree sequence 443322, while the others have degree sequence 433332. The graph in the middle has adjacent vertices with degrees 2 and 4, while the graph on the right does not. Thus all three are nonisomorphic.

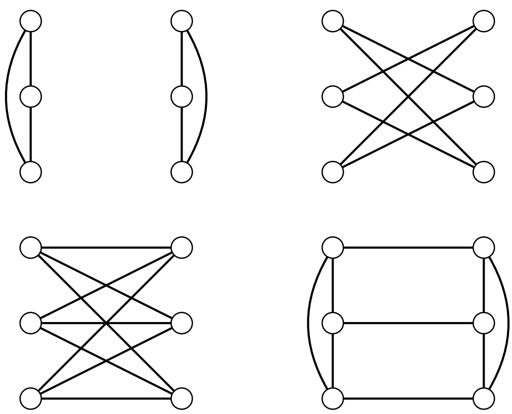


An isomorphism matches both edges and nonedges. This proves the following observation.

Proposition 3.8. *Let G and H be graphs. Then $G \cong H$ if and only if $\overline{G} \cong \overline{H}$.*

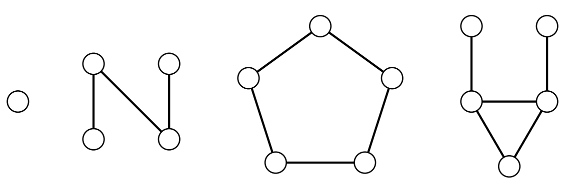
When a graph is dense, it can be easier to analyze its complement.

Example. Consider cubic graphs of order 6. The complement of a 3-regular graph of order 6 is a 2-regular graph of order 6. There are two such graphs, $2C_3$ and C_6. Thus there are two cubic graphs of order 6, $K_{3,3}$ and $K_3 \square K_2$.



Definition 3.9. A graph G is **self-complementary** if $G = \overline{G}$.

Example. The only self-complementary graphs with order at most 5 are K_1, P_4, and C_5, and a graph with order 5 is called the **bull graph**. This can be shown by examining all small graphs.



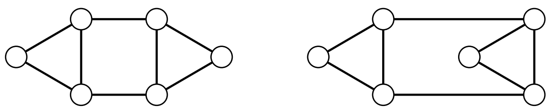
A self-complementary graph of order n has size $m = \frac{1}{2}\binom{n}{2} = \frac{n(n-1)}{4}$, which must be an integer. Then $4|n$ or $4|n-1$, so $n \equiv 0$ or $n \equiv 1 \bmod 4$. The first few values of the sequence of the number of self-complementary graphs of order n is 1, 0, 0, 1, 2, 0, 0, 10, 36, 0, 0, 720, 5600, ... (OEIS A000171). The structure of self-complementary graphs is explored in the Exercises.

Proposition 3.10. *The existence of a particular subgraph, and the number of occurrences of that subgraph, is a graph invariant.*

Proof. A subgraph is determined by its vertices and edges. Since corresponding vertices and edges are matched by an isomorphism, the existence of a subgraph is invariant. Similarly, each copy of a subgraph must match with a distinct copy in the other graph, so the number of occurrences of a subgraph is invariant. □

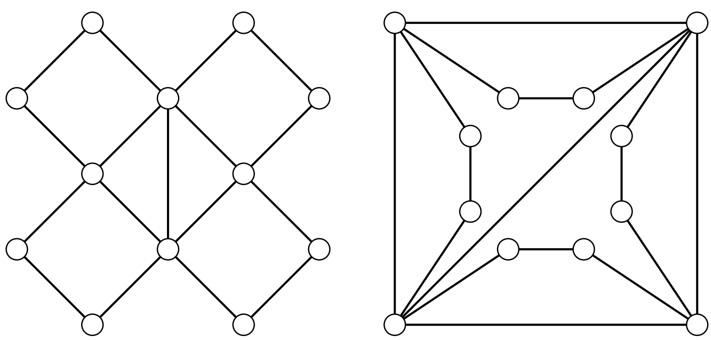
A little thought shows that graph properties that are invariant include maximum and minimum degree, number of components, diameter and radius, bipartiteness, girth, connectivity, and edge connectivity. What is not preserved? A property that depends on the particular drawing of a graph may not be preserved. For example, whether there is an edge crossing in the drawing is not preserved. The length of the regions in a plane drawing (see Section 5.2) is also not preserved.

Example. Two drawings of the same graph are shown below. The drawing at left has regions with lengths 3, 3, 4, and 6, while the drawing at right has region lengths 3, 3, 5, and 5.



When two graphs are isomorphic, isomorphic subgraphs can be used to help find the isomorphism.

Example. Consider the two graphs below. Both contain a single copy of $K_4 - e$. Mapping one of them onto the other quickly produces an isomorphism.



A good strategy to test a pair of graphs for isomorphism starts with their degree sequences. If they are the same, and the graphs are not regular, examine the subgraphs induced by edges between vertices of particular degrees. If these subgraphs are nonisomorphic, neither are the larger graphs. If they are are isomorphic, this provides information on how to find an isomorphism for the larger graphs.

Example. In the graphs above, the subgraphs induced by the vertices of degree 5 are both K_2. The subgraphs induced by the vertices of degrees 4 and 5 are $K_4 - e$. The subgraphs induced by the vertices of degrees 4 and 2 are $2P_5$.

However, this is not helpful when the graphs are regular. Since vertex degrees tell us how many vertices are distance 1 away from a vertex, it may be helpful to determine how many vertices are distance 2 from each vertex. More generally, we could compute the number of walks of distance 2 between each pair of vertices. This can be done simply by squaring the adjacency matrix (Theorem 1.35). Then we can consider the vectors counting the number of distance 2 walks starting at each vertex. Since we don't know the isomorphism (if it exists), sort the numbers in each vector in nonincreasing order.

Example. Consider the graphs with matrices shown below. Inspection quickly reveals that both are cubic graphs of order 6. However, it is not immediately clear whether they are isomorphic.

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

Now consider squaring the matrices. In the graph at left, each vertex has distance 2 walk vector $(3, 3, 3, 0, 0, 0)$. In the graph at right, each vertex has distance 2 walk vector $(3, 2, 2, 1, 1, 0)$. No matter how the vertices are permuted, the rows of the two matrices are different, so the corresponding graphs are nonisomorphic. In fact, they are $K_{3,3}$ and $K_3 \square K_2$ again.

$$\begin{bmatrix} 3 & 3 & 3 & 0 & 0 & 0 \\ 3 & 3 & 3 & 0 & 0 & 0 \\ 3 & 3 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 3 & 3 \\ 0 & 0 & 0 & 3 & 3 & 3 \\ 0 & 0 & 0 & 3 & 3 & 3 \end{bmatrix} \qquad \begin{bmatrix} 3 & 1 & 1 & 0 & 2 & 2 \\ 1 & 3 & 1 & 2 & 0 & 2 \\ 1 & 1 & 3 & 2 & 2 & 0 \\ 0 & 2 & 2 & 3 & 1 & 1 \\ 2 & 0 & 2 & 1 & 3 & 1 \\ 2 & 2 & 0 & 1 & 1 & 3 \end{bmatrix}$$

If the squares of the adjacency matrix cannot be distinguished, higher powers of it could be analyzed. This will distinguish many, but not all regular graphs.

Definition 3.11. A graph is **strongly regular** if there are integers λ and μ such that every two adjacent vertices have λ common neighbors and every two nonadjacent vertices have μ common neighbors. A strongly regular graph with order n and degree k is denoted $\operatorname{srg}(n, k, \lambda, \mu)$.

Example. Complete graphs are $\operatorname{srg}(n, n-1, n-2, 0)$. The Petersen graph is an $\operatorname{srg}(10, 3, 0, 1)$.

There are nonisomorphic strongly regular graphs with the same parameters. They must have the same distance 2 walk vectors, so this technique will not distinguish them.

Thus the graph isomorphism problem is both easy and hard in different senses. Two graphs are usually easy to tell apart. Two randomly chosen graphs will almost certainly have different orders; two randomly chosen graphs with the same order will almost certainly have different sizes, etc. But some pairs of strongly regular graphs exist which have many properties in common. Yet even these can be distinguished with some work. In knot theory, there are pairs of knots that are not known to be the same or different for some time; this is not the case in graph theory.

The difficulty in the graph isomorphism problem is that there is no algorithm that is known to work efficiently for any pair of graphs. For several decades, the best known algorithms had complexity essentially $\mathcal{O}\left(2^{\sqrt{n}}\right)$. In 2017, Laszlo Babai [**2017**] announced a quasipolynomial time algorithm (that has complexity $2^{\mathcal{O}((\log n)^c)}$ for some $c > 0$). The algorithm uses group theory.

The graph isomorphism problem is in class NP, but is not known to belong to either class P or NP-complete. It has been used to define its own complexity class, GI. The graph isomorphism problem can be solved in polynomial time for some special classes of graphs, including trees.

3.2.2. Applications of Isomorphisms. Isomorphism and graph invariants can be used to classify all graphs of a certain type. We characterized 2-regular graphs in Proposition 1.28. Cubic graphs are much more difficult to classify.

Example. Suppose we want to find all cubic graphs of order 8. Note that if we just start drawing graphs, we cannot be sure we have found them all, and we may draw the same graph more than once without realizing it. Instead, we use a case-checking argument based on graph invariants.

Start by asking whether a cubic graph of order 8 can be disconnected. If so, each component must have order 4, and so must be K_4. Thus we find $G_1 = 2K_4$.

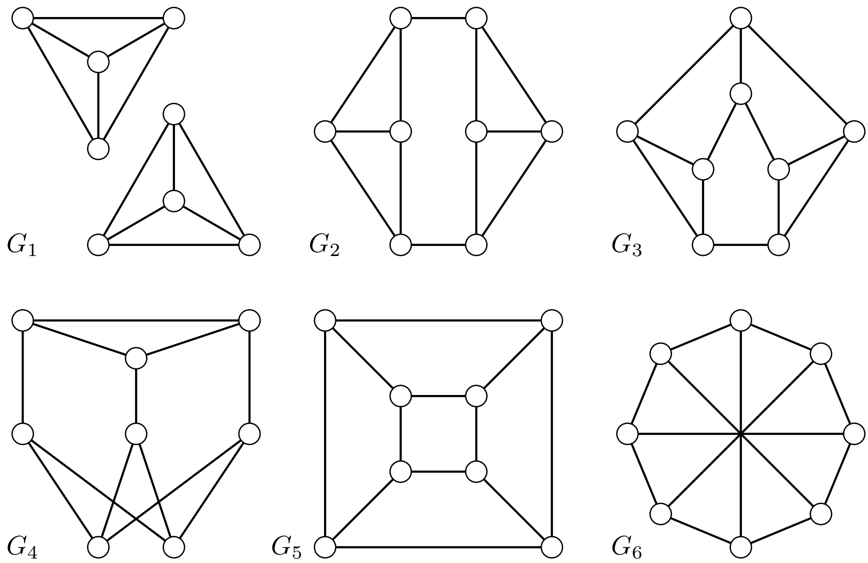
It is easily checked that a cubic graph of order 8 cannot have a cut-vertex or bridge. If it has a 2–edge cut, the components when it is deleted both have degree sequence 3, 3, 2, 2, so they must be $K_4 - e$. Thus we have a unique graph G_2.

The remaining graphs must be 3-connected. If there is an edge cut of three nonadjacent edges, then the orders of the components when they are deleted must be odd since else they would have an odd number of odd vertices. One must be 2-regular with order 3, so K_3. The other must have degree sequence 3, 3, 2, 2, 2. The vertices of degree 3 are either adjacent or not, leading to $C_5 + e$ and $K_{2,3}$. Thus we have found G_3 and G_4.

Any triangle leads to one of the edge cuts in the previous two cases, so we consider triangle-free graphs. If a graph is bipartite, each partite set has four vertices, so each vertex is not adjacent to one in the other set. There are four mutually nonadjacent nonedges, so there is one possible graph G_5. Since the 3-cube is cubic and bipartite with order 8, $G_5 = Q_3$.

If there is a triangle-free nonbipartite graph, suppose it has a 5-cycle. The other three vertices have degree sum 9, and five edges join to the 5-cycle. There must be two edges between the three vertices, so they induce P_3. The two edges from an end of the path join to nonadjacent vertices in the 5-cycle (else there would be a triangle). This leaves only one option. The final graph G_6 can be shown to be a Mobius ladder.

Note that if a graph contains a 7-cycle, the final vertex is adjacent to three vertices on it, creating a smaller odd cycle. The six cubic graphs of order 8 are shown below.



Many of the techniques used in the previous example can be applied to cubic graphs in general. These ideas are explored further in the Exercises.

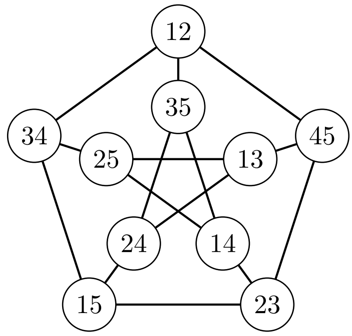
Isomorphisms can be used to describe the symmetries of a graph.

Definition 3.12. An **automorphism** of G is an isomorphism from G to G. A graph is **vertex-transitive** if for every pair of vertices u and v, there is an automorphism that maps u to v. A graph is **edge-transitive** if for every pair of edges e and f, there is an automorphism that maps e to f.

In a vertex-transitive graph, every vertex resembles every other, so a statement for all vertices can be proved by just looking at one.

Example. A nontrivial path has two automorphisms. The identity leaves it in place. It can also be reflected so that the ends are interchanged. Thus a nontrivial path has two automorphisms, and only P_2 is vertex-transitive. Both P_2 and P_3 are edge-transitive.

Example. The Petersen graph is defined using 2-element subsets of [5]. Permuting these elements produces $5! = 120$ automorphisms. It follows that the Petersen graph is both vertex-transitive and edge-transitive. The 5-cycle 12-34-51-23-45 must be mapped to a 5-cycle with each two numbers on two nonadjacent vertices. This implies that any automorphism of the Petersen graph comes from permuting elements of [5].



The automorphisms of a graph have implications for graph labelings.

Definition 3.13. A **labeling** $f(v) : V(G) \to \mathbb{R}$ of the vertices of a graph G is an assignment of numbers (**labels**) to each vertex. Two labelings are **distinct** if they don't produce the same edge set.

Proposition 3.14. *Let* $|\mathrm{Aut}(G)|$ *be the number of automorphisms of a graph* G *with order* n*. Then the number of distinct labelings of* G *from* $[n]$ *is* $\frac{n!}{|\mathrm{Aut}(G)|}$.

Proof. Any graph has $n!$ labelings from $[n]$ without regard to whether they are distinct. Each automorphism of a labeled graph G produces the same labeling. Thus, dividing $n!$ by $|\mathrm{Aut}(G)|$ produces the number of distinct labelings. □

This result can be used to find either the number of labelings or the number of automorphisms given the other.

Example. The path P_6 has two automorphisms, so it has $\frac{6!}{2} = 360$ distinct labelings from [6]. The star $K_{1,5}$ has six distinct labelings from [6], as once the label for the center is chosen, there is nothing else to choose. Thus $K_{1,5}$ has $\frac{6!}{6} = 120$ automorphisms.

Related Terms: graph homomorphism, covering graph, automorphism group, symmetric graph, asymmetric graph, arc-transitive, distance-transitive.

3.3. Degree Sequences

We have seen in previous sections that the degree sequence of a graph is important when considering isomorphisms and other problems. In this section, we consider what sequences can be degree sequences of graphs, graph classes, or unique graphs.

Definition 3.15. The **degree sequence** of a graph G is the list of its degrees $d_1, \ldots, d_n$, usually written in nonincreasing order, $d_1 \geq \cdots \geq d_n$. A **graphic sequence** is a sequence of nonnegative integers that is the degree sequence of some graph. A graph with degree sequence S **realizes** S.

When there is no danger of confusion, we may omit the commas from the list of degrees.

Some requirements are immediate. The maximum degree is at most $n-1$. By the First Theorem of Graph Theory, the degree sum $\sum d_i$ must be even.

Example. Determine whether the following sequences are graphic.

(a) 53321: No, the first term is too large.

(b) 222211: Yes, P_6 realizes this sequence.

(c) 3322221: No, the sum is odd.

(d) 3311: This satisfies the two conditions we listed. However, a vertex of degree 3 must be adjacent to all other vertices. If we delete this vertex, the remaining graph must have degree sequence 200. But this is impossible, so the sequence is not graphic.

The last example can be generalized. We show that a sequence is graphic if and only if a particular shorter sequence is graphic. This makes the problem easier to solve.

Theorem 3.16 (**Havel-Hakimi Theorem**—Havel [**1955**], Hakimi [**1962**]). *The sequence S: $d_1 \geq \cdots \geq d_n$ is graphic if and only if the sequence S_1 formed by deleting d_1 and subtracting* 1 *from the d_1 next largest terms is graphic.*

Proof. ($\Leftarrow$) Suppose that there is a graph G_1 with degree sequence S_1 formed as stated in the theorem. Form a graph G by adding one vertex adjacent to the vertices whose degrees were reduced. Then G realizes S, which is graphic.

($\Rightarrow$) Let G realize S with $d(v_i) = d_i$. We want v_1 to be adjacent to vertices with degrees $d_2, \ldots, d_{d_1+1}$. If it is not, we will modify the graph so that it is.

Assume that there are vertices w and x with $d(w) > d(x)$ so that $v_1 \leftrightarrow x$ and $v_1 \not\leftrightarrow w$. Then there is some vertex y that is adjacent to w and not x, since $d(w) > d(x)$. Delete edges v_1x and yw and add edges v_1w and yx. This keeps the degrees the same and increases the sum of degrees of neighbors of v_1. This operation increases the number of neighbors of v_1 with the desired degrees, so repeating it eventually produces a graph G' with the desired property.

Thus, deleting vertex v_1 produces a graph realizing S_1. $\square$

The Havel-Hakimi Theorem immediately implies an algorithm for determining whether a sequence is graphic.

Algorithm 3.17. *Given a nonincreasing sequence S, iteratively delete the first element Δ and subtract* 1 *from the Δ largest remaining elements. Stop when the sequence is all* 0*'s* (*in which case S is graphic*) *or contains a negative number* (*in which case S is not graphic*).

Note that we will often need to reorder the sequences to keep them nonincreasing. We can shortcut the end conditions if an earlier sequence produced by the algorithm is recognized as graphic or not.

Example. To determine whether 887664432 is graphic, we apply the algorithm. We find the following.

S: 887664432

S_1: 76553321

S_2: 5442210

S_3: 331100

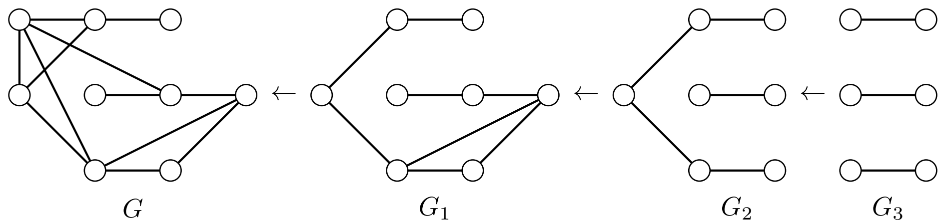
S_4: 20000

S_5: 0,0,-1,-1

Thus S is not graphic. We could have concluded this from S_3 (which resembles example (d) above) or certainly from S_4.

Example. To determine whether 443333211 is graphic, we apply the algorithm. We see that $3K_2$ realizes S_3, so S is graphic.

$$S\text{: } 443333211 \to S_1\text{: } 33222211 \to S_2\text{: } 2221111 \to S_3\text{: } 111111$$



We reverse the process to find a graph realizing S. Add new vertices adjacent to those whose degrees were reduced. Starting with $G_3 = 3K_2$, we get $G_2 = P_5 \cup K_2$, G_1, and G, as shown above.

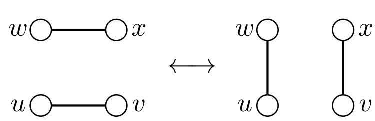
This can be generalized to an algorithm to construct a graph realizing a graphic sequence.

Algorithm 3.18. *Given nonincreasing degree sequences $S_0, \ldots, S_k$ produced by Algorithm 3.17, find a graph G_k realizing S_k. Iterate adding a vertex whose degree is the first term of S_i, adjacent to the vertices of G_{i+1} whose degrees were reduced in S_{i+1}. Stop when G_0 is produced.*

Note that the graph produced by this algorithm is often not unique, since there is sometimes a choice of which vertices the newly added vertex is adjacent to. Also, some graphs, such as $2P_3$, cannot be produced using this algorithm.

When several graphs have the same degree sequence, we may wonder how all of them can be produced. We use an operation contained in the proof of the Havel-Hakimi Theorem that preserves degree sequences.

Definition 3.19. Given a graph containing edges uv and wx and not containing uw and vx, a **2-switch** deletes uv and wx and adds uw and vx.



Theorem 3.20 (Berge [**1973**]). *Two graphs G and H have the same nonincreasing degree sequence if and only if there is a sequence of 2-switches that transforms G into H.*

Proof. ($\Leftarrow$) Every 2-switch preserves vertex degrees and, hence, degree sequences.

($\Rightarrow$) (West [**2001**]) Assume G and H have the same vertex set and $d_G(v) = d_H(v)$ for every vertex v. We use induction on n. For $1 \leq n \leq 3$, each degree sequence corresponds to only one graph. Assume the result holds for graphs with order $n - 1 \geq 3$, and let G and H have order n.

Let u be a vertex with $d(u) = \Delta(G) = \Delta$. Let S be a set of vertices with the Δ largest degrees other than Δ. As in the proof of the Havel-Hakimi Theorem, some sequence of 2-switches transforms G into a graph G' with $N_{G'}(u) = S$, and some sequence of 2-switches transforms H into a graph H' with $N_{H'}(u) = S$.

Now $N_{G'}(u) = N_{H'}(u)$, so deleting u produces graphs $G' - u$ and $H' - u$ with equal degrees on corresponding vertices. By the induction hypothesis, some sequence of 2-switches transforms $G' - u$ into $H' - u$. Since they don't involve u, which has the same neighbors in G' and H', the sequence transforms G' into H'. Thus there is a sequence of 2-switches that transforms G into H (via G' and H'). $\square$

The numbers of distinct degree sequences of graphs with order n begin 1, 2, 4, 11, 31, 102, 342, 1213, 4361, … (OEIS A004251).

The Havel-Hakimi Theorem provides a recursive test of whether a sequence is graphic. There are other explicit characterizations of degree sequences. Loosely speaking, the problem with the nongraphic sequences that we have seen is that the large terms are too large and the small terms too small for there to be enough edges between vertices of large and small degree.

Theorem 3.21 (**Erdos-Gallai Theorem**—Erdos/Gallai [**1960**]). *The sequence $S\colon d_1 \geq \cdots \geq d_n$ of nonnegative integers is graphic if and only if $\sum d_i$ is even and for each k with $1 \leq k \leq n-1$,*

$$\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{n} \min(k, d_i).$$

Proof. ($\Rightarrow$) Consider the vertices with the k largest degrees in a graph G realizing S. The graph induced by them has degree sum at most $k(k-1)$ (if it is complete). Each of the other $n-k$ vertices can be adjacent to at most k of those vertices and also at most d_i.

($\Leftarrow$) (Tripathi/Venugoplan/West [**2010**]) Let a **subrealization** of a nonincreasing sequence $S : d_1, \ldots, d_n$ be a graph with vertices $v_1, \ldots, v_n$ such that $d(v_i) \leq d_i$ for $1 \leq i \leq n$. Given a sequence $d_1, \ldots, d_n$ with an even sum that satisfies the inequalities, we construct a realization through successive subrealizations. The initial subrealization is $\overline{K}_n$.

In a subrealization, the critical index r is the largest index such that $d(v_i) = d_i$ for $1 \leq i < r$. Initially, $r = 1$ unless the list is all 0, in which case the process is complete. While $r \leq n$, we obtain a new subrealization with smaller deficiency $d_r - d(v_r)$ at vertex v_r while not changing the degree of any vertex v_i with $i < r$ (the degree list increases lexicographically). The process can only stop when the subrealization is a realization of D.

Let $T = \{v_{r+1}, \ldots, v_n\}$. We maintain the condition that T is an independent set, which certainly holds initially.

Case 0. $v_r \not\leftrightarrow v_i$ for some vertex v_i such that $d(v_i) < d_i$. Add the edge $v_r v_i$.

Case 1. $v_r \nleftrightarrow v_i$ for some i with $i < r$. Since $d(v_i) = d_i \geq d_r > d(v_r)$, there exists $u \in N(v_i) - (N(v_r) \cup \{v_r\})$. If $d_r - d(v_r) \geq 2$, then replace uv_i with $\{uv_r, v_iv_r\}$. If $d_r - d(v_r) = 1$, then since $\sum d_i - \sum d(v_i)$ is even, there is an index k with $k > r$ such that $d(v_k) < d_k$. Case 0 applies unless $v_r \leftrightarrow v_k$; replace $\{v_rv_k, uv_i\}$ with $\{uv_r, v_iv_r\}$.

Case 2. $v_1, \ldots, v_{r-1} \in N(v_r)$, and $d(v_k) \neq \min(r, d_k)$ for some k with $k > r$. In a subrealization, $d(v_k) \leq d_k$. Since T is independent, $d(v_k) \leq r$. Hence $d(v_k) < \min(r, d_k)$, and Case 0 applies unless $v_k \leftrightarrow v_r$. Since $d(v_k) < r$, there exists i with $i < r$ such that $v_k \nleftrightarrow v_i$. Since $d(v_i) > d(v_r)$, there exists $u \in N(v_i) - (N(v_r) \cup \{v_r\})$. Replace uv_i with $\{uv_r, v_iv_k\}$.

Case 3. $v_1, \ldots, v_{r-1} \in N(v_r)$, and $v_i \nleftrightarrow v_j$ for some i and j with $i < j < r$. Case 1 applies unless $v_iv_j \in N(v_r)$. Since $d(v_i) \geq d(v_j) \geq d(v_r)$, there exist $u \in N(v_i) - (N(v_r) \cup \{v_r\})$ and $w \in N(v_j) - (N(v_r) \cup \{v_r\})$ (possibly $u = w$). Since $u, w \notin N(v_r)$, Case 1 applies unless $u, w \in T$. Replace $\{uv_i, wv_j\}$ with $\{v_iv_j, uv_r\}$.

If none of these cases applies, then $v_1, \ldots, v_r$ are pairwise adjacent, and $d(v_k) = \min(r, d_k)$ for $k > r$. Since T is independent,

$$\sum_{i=1}^{r} d(v_i) \leq r(r-1) + \sum_{k=r+1}^{n} \min(r, d_k).$$

Now $\sum_{i=1}^{r} d_i$ is bounded by the right side. Hence, we have already eliminated the deficiency at vertex r. Increase r by 1 and continue. $\square$

We can also characterize the degree sequences of classes of graphs. In Proposition 2.8, the degree sequences of trees are characterized. In the next section, this is generalized to characterize the degree sequences of maximal k-degenerate graphs. These characterizations have the following form, where $\mathbb{G}$ is some graph class and $\mathbb{S}$ is some specified set of degree sequences.

There exists $G \in \mathbb{G}$ realizing S if and only if $S \in \mathbb{S}$.

Note that this allows the existence of another graph realizing S that is not in $\mathbb{G}$. That is, $\mathbb{G}$ may not be closed under 2-switches. For some classes, it is possible for the reverse direction to have the following stronger form.

If $S \in \mathbb{S}$, then for any graph G realizing S, $G \in \mathbb{G}$.

Example. The star $K_{1,n-1}$ has degree sequence $n-1, 1, \ldots, 1$. No 2-switch can be performed on a star. Thus we can say that a graph is a star if and only if it has a degree sequence of the form $n-1, 1, \ldots, 1$.

The degree sequences in the preceding example each correspond to only one graph.

Definition 3.22. A degree sequence is **unigraphic** if there is exactly one graph with this sequence.

Example. The sequence 3333 is unigraphic, as K_4 is the only graph with this sequence. The sequence 333333 is not unigraphic, as $K_{3,3}$ and $K_3 \square K_2$ both realize it.

We have already learned a fair amount about regular graphs. The opposite extreme would be a graph with all distinct degrees. However, this is impossible.

Proposition 3.23. *No nontrivial graph has all vertex degrees distinct.*

Proof. A graph with $n \geq 2$ vertices has n possible degrees, $0, 1, \ldots, n-1$. However, a vertex with degree $n-1$ is adjacent to all other vertices, which implies that none may have degree 0. Thus, at most $n-1$ of the numbers $0, 1, \ldots, n-1$ can be degrees of a graph, so by the Pigeonhole Principle, at least two vertices have the same degree. $\square$

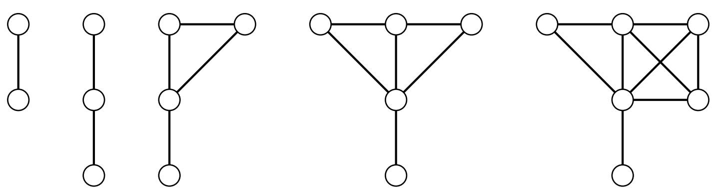
It is possible for a graph to have exactly two vertices with equal degrees. The structure of such graphs can be described exactly.

Proposition 3.24. *For $n \geq 2$, there is exactly one connected graph of order n with degrees 1 through $n-1$ (with $\lfloor \frac{n}{2} \rfloor$ repeated).*

Proof. We use induction on n. The graphs K_2 and P_3 uniquely satisfy the theorem when $n = 2$ or 3.

Assume the result holds for graphs with order less than n, and let G be connected with order n and degrees 1 to $n-1$ with $\lfloor \frac{n}{2} \rfloor$ repeated. The vertex v with degree $n-1$ is adjacent to all others, so $G = H + K_1$ for some unique graph H with degrees 0 through $n-3$ with $\lfloor \frac{n}{2} \rfloor - 1 = \lfloor \frac{n-2}{2} \rfloor$ repeated. Now H has an isolated vertex u, and $H - u$ satisfies the conditions for order $n-2$, so it is also unique. Thus G exists and is unique. $\square$

Definition 3.25. The **irregular graph** I_n is the unique connected graph of order n with degrees 1 through $n-1$ with $\lfloor \frac{n}{2} \rfloor$ repeated.



Note that the proof implies that $I_n = (I_{n-2} \cup K_1) + K_1$ for $n \geq 4$. Irregular graphs are explored further in the Exercises.

Related Terms: degree set, graphical partition, degree distribution, bipartite realization problem, bigraphic, Gale-Ryser Theorem, highly irregular graph.

3.4. Degeneracy

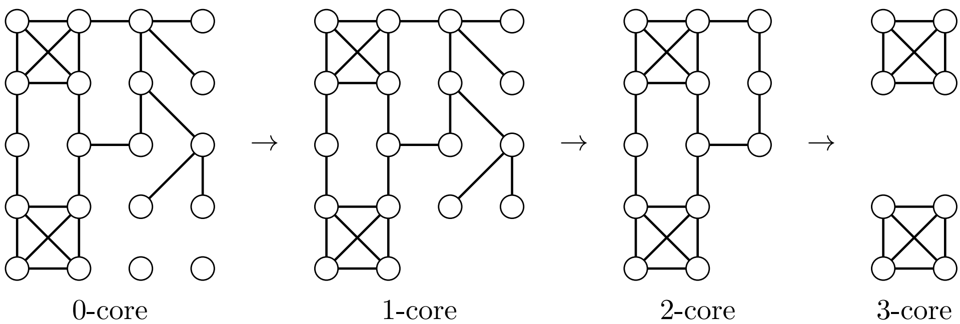
One question that can be asked about a social network is whether there is a group of people who all know many other people in the group. In graph theory terms, we are asking whether there is a subgraph with some given minimum degree.

3.4.1. Cores of Graphs.

Definition 3.26. The **k-core of a graph** G, $C_k(G)$, is the maximal induced subgraph $H \subseteq G$ such that $\delta(G) \geq k$, if it exists. If $\delta(G) \geq k$, we say G is a **k-core**. A graph is **k-core-free** if it does not contain a k-core. The **core number** of a vertex, $C(v)$, is the largest value for k such that $v \in C_k(G)$. The **maximum core** of G is induced by the vertices with maximum core number.

It is straightforward to show that the k-core is well defined. The cores of a graph are nested; if $k > j$, then $C_k(G) \subseteq C_j(G)$.

Example. The graph below left is its own 0-core. Its other cores are shown.



Cores were introduced by S. B. Seidman [**1983**] and have been studied extensively in (Bickle [**2010**], Bickle [**2013**]). Seidman briefly explores applications to social networks in his paper. Cores also have applications in computer science to network visualization (Alvarez-Hamelin et al. [**2006**], Gaertler/Patrignani [**2004**]) and in bioinformatics (Altaf-Ul-Amin et al. [**2003**], Bader/Hogue [**2003**], Wuchty/Almaas [**2005**]).

There is a simple algorithm for determining the k-core of a graph, called the k-Core Algorithm.

Algorithm 3.27 (**k-Core Algorithm**). *Input a graph G and an integer k. Iterate the step of deleting all vertices of degree less than k. Stop when there are no more such vertices. If a graph remains, it is the k-core. If no graph remains, G has no k-core.*

Theorem 3.28. *Applying the k-Core Algorithm to a graph G yields the k-core of G, if it exists. That is, a vertex v is in the k-core of G if and only if it is not deleted by the algorithm.*

Proof. ($\Rightarrow$) The vertices in the k-core all have at least k neighbors in the k-core. None of these vertices will be deleted in the first iteration. If none have been deleted after i iterations, none will be deleted by the next iteration. Thus none will ever be wrongly deleted.

($\Leftarrow$) The vertices not deleted by the algorithm all have degree at least k in the graph produced by the algorithm. Thus they are all in the k-core, so no vertices will be wrongly included. $\square$

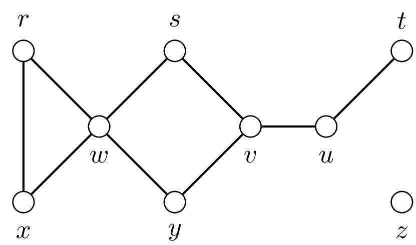
The k-Core Algorithm can be implemented in polynomial time. If an adjacency matrix is employed, it can be implemented in $O\left(n^2\right)$ time, while Batagelj and Zaversnik [**2011**] showed that using an edge list, it can be implemented in $O(m)$ time, which is better for sparse graphs.

We can define a sequence of vertices based on the order that they are deleted by the k-Core Algorithm. We may also wish to construct a graph by successively adding vertices of relatively small degree.

Definition 3.29. A **deletion sequence** of a graph G is a sequence of its vertices formed by iterating the operation of deleting a vertex of smallest degree and adding it to the sequence until no vertices remain. A **construction sequence** of a graph is the reversal of a corresponding deletion sequence. A graph is **k-degenerate** if its vertices can be successively deleted so that when deleted, each has degree at most k. The **degeneracy** $D(G)$ of a graph G is the smallest k such that it is k-degenerate.

The term k-degenerate was introduced in 1970 by Lick and White [**1970**]; the concept has been introduced under other names both before and since.

Example. A deletion sequence of the graph below is z, t, u, s, v, y, r, w, x. The graph has degeneracy 2.



As a corollary of Theorem 3.28, we have the following min-max relationship.

Corollary 3.30. *For any graph, its maximum core number is equal to its degeneracy.*

Proof. Let G be a graph with degeneracy D and maximum core number k. By Theorem 3.28, since G has a k-core, it is not $k-1$-degenerate, so $k \leq D$. Since G has no $k+1$-core, it is k-degenerate, so $k = D$. $\square$

Thus a graph G is k-degenerate if and only if G is $k+1$-core-free.

It is immediate from the definition of degeneracy that $\delta(G) \leq D(G) \leq \Delta(G)$. We can characterize the extremal graphs for the upper bound. For simplicity, we restrict the statement to connected graphs.

Proposition 3.31. *Let G be a connected graph. Then $D(G) = \Delta(G)$ if and only if G is regular.*

Proof. ($\Leftarrow$) If G is regular, then its maximum and minimum degrees are equal, so the result is obvious.

(⇒) Let $D(G) = \Delta(G) = k$. Then G has a subgraph H with $\delta(H) = \Delta(G) \geq \Delta(H)$, so H is k-regular. If H were not all of G, then since G is connected, some vertex of H would have a neighbor not in H, implying that $\Delta(G) > \Delta(H) = \delta(H) = \Delta(G)$. But this is not the case, so $G = H$, and G is regular. □

We also consider the extremal graphs for the lower bound $\delta(G) \leq D(G)$.

Definition 3.32. A graph G is **k-monocore** if $D(G) = \delta(G) = k$. A graph is **monocore** if it is k-monocore for some k.

Note that 0-monocore graphs are exactly empty graphs, and connected 1-monocore graphs are exactly nontrivial trees. Regular graphs are monocore. In fact, almost all the graph classes we have considered so far are monocore. Now we present an operation characterization of 2-monocore graphs.

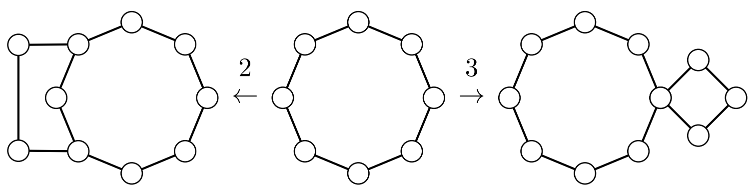
Definition 3.33. An **ear** of a graph is a maximal path whose internal vertices have degree 2.

Theorem 3.34 (Bickle [**2013**]). *The set of connected* 2*-monocore graphs is the set* S *of graphs that can be constructed using the following rules.*

(1) *All cycles are in* S.

(2) *Given one or two graphs in* S*, the graph* H *formed by identifying the ends of a path of length at least* 2 *with vertices of the graph or graphs is in* S.

(3) *Given a graph* G *in* S*, form* H *by taking a cycle and either identifying a vertex of the cycle with a vertex of* G *or adding an edge between one vertex in each.*



Proof. (⇐) We first show that if G is in S, then G is 2-monocore. Certainly cycles are 2-monocore. Let H be formed from G in S by applying rule (2). Then $\delta(H) = 2$, and since G is 3-core-free and internal vertices of the path have degree 2, H is also 3-core-free. Thus H is 2-monocore. The same argument works for adding a path between two graphs. Let H be formed from G in S by applying rule (3). Then $\delta(H) = 2$, and since G is 3-core-free and all but one vertex of the cycle have degree 2, H is also 3-core-free. Thus H is 2-monocore.

(⇒) We now use induction to show that if G is 2-monocore, it is in S. This clearly holds for all cycles, including C_3, so assume it holds for all 2-monocore graphs of order less than n. Let G be 2-monocore of order n and not a cycle. Then $\delta(G) = 2$, so let $d(v) = 2$. Then v is contained in P, an ear of length at least 2, or C, a cycle which has all but one vertex of degree 2.

Case 1. G has an ear P. If $G - P$ is disconnected, then the components of G are 2-monocore, and hence in S. Then G can be formed from them using rule (2), so G is in S. If $G - P$ is connected, then it is 2-monocore, and hence in S. Then G can be formed from $G - P$ using rule (2), so G is in S.

Case 2. We may assume that G has no such ear P. Then G has a cycle C with all but one vertex of degree 2, and one vertex u of degree more than 2. If u has degree at least 4 in G, then let H be formed by deleting all the vertices of C except u. Then H is 2-monocore, and G can be formed from it using rule (3). If $d(u) = 3$, then its neighbor not in the cycle has degree at least 3, so $G - C$ is 2-monocore, and G can be formed from it by using rule (3). $\square$

The edges not in the 2-core induce trees that may be attached to the 2-core.

Definition 3.35. The **1-shell** of a graph G is the subgraph of G induced by the edges not contained in the 2-core.

The 1-shell, if it exists, is a forest with no trivial components and at most one vertex per component contained in the 2-core. Other shells of a graph are defined in the Exercises.

3.4.2. Maximal k-Degenerate Graphs. We now consider the properties of maximal k-degenerate graphs, starting with their size.

Theorem 3.36. *The size of a maximal k-degenerate with order $n \geq k$ is $k \cdot n - \binom{k+1}{2}$.*

Proof. If G is k-degenerate, then its vertices can be successively deleted so that when deleted they have degree at most k. Since G is maximal, the degrees of the deleted vertices will be exactly k until the number of vertices remaining is at most k. After that, the $(n-j)$-th vertex deleted will have degree j. Thus the size m of G is

$$m = \sum_{i=0}^{k-1} i + \sum_{i=k}^{n-1} k = \frac{k(k-1)}{2} + k(n-k) = k \cdot n + \frac{k(k-1)}{2} - \frac{2k^2}{2} = k \cdot n - \binom{k+1}{2}.$$

$\square$

Thus for k-degenerate graphs, maximal and maximum are equivalent. Hence a k-degenerate graph is maximal if and only if it has size $k \cdot n - \binom{k+1}{2}$.

Corollary 3.37. *Every graph with order n and size $m \geq (k-1)n - \binom{k}{2} + 1$, $1 \leq k \leq n-1$, has a k-core.*

Plugging small positive values of k into $k \cdot n - \binom{k+1}{2}$ produces $n-1$, $2n-3$, $3n-6, \ldots$. We have seen that $n-1$ is the size of a tree. The other two quantities are sizes of several interesting graph classes.

The following basic properties of maximal k-degenerate graphs are presented without proof.

Theorem 3.38 (Lick/White [**1970**], Mitchem [**1977**]). *Let G be a maximal k-degenerate graph of order n, $1 \leq k \leq n-1$. Then*

(a) *G contains a $k+1$-clique, and, for $n \geq k+2$, G contains $K_{k+2} - e$ as a subgraph.*

(b) *For $n \geq k+2$, G has $\delta(G) = k$, and no two vertices of degree k are adjacent.*

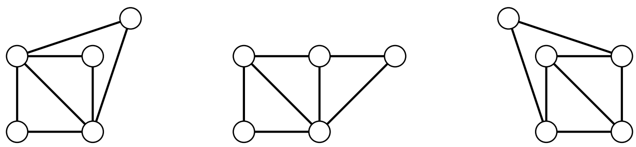
(c) *G has connectivity $\kappa(G) = k$.*

(d) *For any integer r, $1 \le r \le n$, G contains a maximal k-degenerate graph of order r as an induced subgraph. For $n \ge k+2$, if $d(v) = k$, then G is maximal k-degenerate if and only if $G - v$ is maximal k-degenerate.*

(e) *G is maximal 1-degenerate if and only if G is a tree.*

In fact, maximal k-degenerate graphs are one of several generalizations of trees. Note that part (b) implies that maximal k-degenerate graphs are monocore.

Example. The three maximal 2-degenerate graphs of order 5 are shown below.

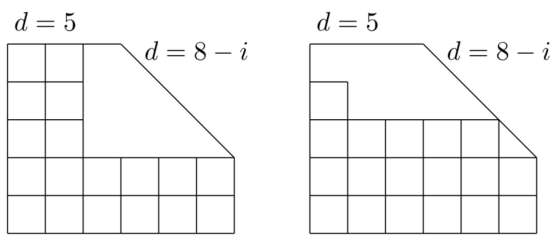


We can characterize the degree sequences of maximal k-degenerate graphs. A different characterization and proof was offered in Borowiecki et al. [**1995**].

Lemma 3.39. *Let G be maximal k-degenerate with order n and nonincreasing degree sequence $d_1, \ldots, d_n$. Then $d_i \le k + n - i$.*

Proof. Assume to the contrary that $d_i > k + n - i$ for some i. Let H be the graph formed by deleting the $n - i$ vertices of smallest degree. Then $\delta(H) > k$, so G has a $k+1$-core. □

Lemma 3.40 (Bickle [**2012**]). *Let $d_1, \ldots, d_n$ be nonincreasing sequence of integers with $\sum d_i = 2\left[k \cdot n - \binom{k+1}{2}\right]$, such that $k \le d_i \le \min\{n-1, k+n-i\}$. Then at most $k+1$ terms of the sequence achieve the upper bound.*



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Proof. Visualize the problem as stacking boxes in adjacent columns so that the height of the column i is d_i (see the examples above for $n = 6$ and $k = 2$). If all the terms other than d_n that achieve the upper bound are at the beginning of the sequence, then there are at most k, since $\sum d_i \ge k(n-1) + (n-k)k = 2k \cdot n - k(k+1)$. Filling the row at height $k+1$ would require $n - k - 1$ more boxes, which would have to be moved from at least two of the columns. Similarly, filling more rows requires disrupting at least as many columns. Thus there are at most $k+1$ terms that achieve the upper bound when all the columns that achieve the upper bound are at the beginning or end of the sequence.

Suppose there is a sequence that is a counterexample, and let it maximize the number of columns at the beginning or end that achieve the maximum. There must be a column somewhere in the middle that achieves the upper bound. Then some boxes can be moved to a column or row next to the the run of those at the beginning or end that to achieve the upper bound, producing a contradiction. □

Similar analysis shows that only k columns at the beginning and one at the end can achieve the upper bound exactly $k+1$ times, in which case the corresponding graph must be $K_k + \overline{K}_{n-k}$.

Theorem 3.41 (Bickle [**2012**]). *A nonincreasing sequence of integers $d_1, \ldots, d_n$ is the degree sequence of a maximal k-degenerate graph G if and only if $k \le d_i \le \min\{n-1, k+n-i\}$ and $\sum d_i = 2\left[k \cdot n - \binom{k+1}{2}\right]$ for $0 \le k \le n-1$.*

Proof. Let $d_1, \ldots, d_n$ be such a sequence.

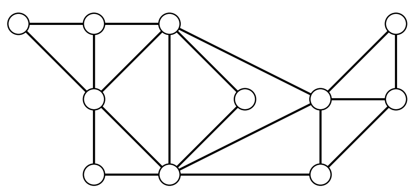
($\Rightarrow$) Certainly $\Delta(G) \le n-1$. The other three conditions have already been shown.

($\Leftarrow$) For $n = k+1$, the result holds for $G = K_{k+1}$. Assume the result holds for order r. Let $d_1, \ldots, d_{r+1}$ be a nonincreasing sequence that satisfies the given properties. Let $d_1', \ldots, d_r'$ be the sequence formed by deleting d_{r+1} and decreasing k other numbers greater than k by 1, including any that achieve the maximum. (There are at most k by the preceding lemma.) Then the new sequence satisfies all the hypotheses and has length r, so it is the degree sequence for some maximal k-degenerate graph H. Add vertex v_{r+1} to H, making it adjacent to the vertices with degrees that were decreased for the new sequence. Then the resulting graph G has the original degree sequence and is maximal k-degenerate. □

3.4.3. k-trees. One class of maximal k-degenerate graphs is particularly important.

Definition 3.42. A **k-tree** is a graph that can be formed by starting with K_{k+1} and iterating the operation of making a new vertex adjacent to all the vertices of a k-clique of the existing graph. The clique used to start the construction is called the **root** of the k-tree.

Example. The graph below is a 2-tree. Any triangle could be the root.



It is easy to see that a k-tree is maximal k-degenerate. The class of 1-trees is just the class of trees. However, k-trees and maximal k-degenerate graphs are not equivalent for $k \ge 2$. Any maximal k-degenerate graph contains a k-tree of some order, but this may not be the whole graph. One characterization of the maximal k-degenerate graphs that are k-trees involves chordal graphs (Section 4.5). Another characterization involves subdivisions.

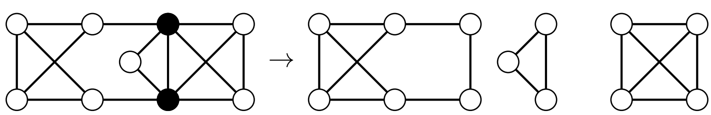
Theorem 3.43 (Bickle [**2012**]). *A maximal k-degenerate graph is a k-tree if and only if it contains no subdivision of K_{k+2}.*

Proof. ($\Rightarrow$) Let G be a k-tree. Certainly K_{k+1} contains no subdivision of K_{k+2}. Suppose G is a counterexample of minimum order with a vertex v of degree k. Then $G-v$ is a k-tree with no subdivision of K_{k+2}, so the subdivision in G contains v. But then v is not one of the $k+2$ vertices of degree $k+1$ in the subdivision, so it is on a path P between two such vertices. Let its neighbors on P be u and w. But since the neighbors of v form a clique, $uw \in G-v$, so P could avoid v, implying $G-v$ has a subdivision of K_{k+2}. This is a contradiction.

($\Leftarrow$) (contrapositive) Let G be maximal k-degenerate and not a k-tree. Since G is constructed beginning with a k-tree, for a given construction sequence there is a first vertex in the sequence that makes G not a k-tree. Let v be this vertex, and let H be the maximal k-degenerate subgraph induced by the vertices of the construction sequence up to v. Then $n(H) \geq k+3$, $d_H(v) = k$, v has nonadjacent neighbors u and w, and $H-v$ is a k-tree. Now there is a sequence of at least two $k+1$-cliques starting with one containing u and ending with one containing w, such that each pair of consecutive $k+1$-cliques in the sequence overlap on a k-clique. Then two of these cliques and a path through v produces a subdivision of K_{k+2}. □

Any k-clique of a k-tree can be the root used to construct it. Thus identifying two k-cliques of two k-trees produces another k-tree. This operation can be reversed to split a k-tree into two or more smaller k-trees. This operation can be generalized.

Definition 3.44. An **S-lobe** of a graph G is a subgraph of G induced by a cutset S and a component of $G-S$.



There is a simple condition that guarantees the existence of a subdivision of K_4 in a graph.

Theorem 3.45 (Dirac [**1964**]). *Every graph with at most one vertex with degree less than* 3 *contains a subdivision of* K_4.

Proof. We use induction on order n. The smallest order with a graph (K_4) satisfying the hypothesis is $n=4$, which certainly satisfies the conclusion. Assume the result holds for all graphs with order n', $4 \leq n' < n$, and G has order n. If G has more than one component or block, then some component or end-block satisfies the hypotheses, and by induction it contains a subdivision of K_4. Hence we may assume that G is 2-connected.

If G has a cutset $S = \{u, v\}$, consider a lobe H that does not contain any vertex with degree less than 3. If this lobe contains another cutset of size 2, consider the new set and smaller lobe. In this way, we may assume H has no cutset of size 2. Then u and v both have at least two neighbors in H (else replacing one with its neighbor would yield another cutset in H). If $uv \in H$, then H satisfies the induction hypothesis. If not, then there is a $u-v$ path outside H, which can be treated as a subdivided edge.

If G is 3-connected, then for any vertex v, $G - v$ is 2-connected. Thus $G - v$ contains a cycle C. Then Lemma 2.46 says that there are three independent paths between v and C. These paths and C produce a subdivision of K_4. □

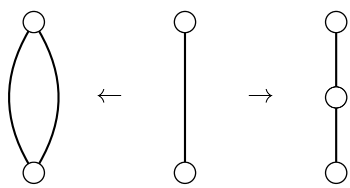
Corollary 3.46 (Dirac [**1964**]). *If G has $m > 2n-3$, then G contains a subdivision of K_4, and the graphs of size $2n-3$ that don't contain a subdivision of K_4 are exactly the 2-trees.*

Proof. Let G have $m > 2n - 3 = (3-1)\,n - \binom{3}{2}$. By Corollary 3.37, G contains a 3-core. By Theorem 3.45, it contains a subdivision of K_4. If a graph of size $2n-3$ has no 3-core, it is maximal 2-degenerate. By Theorem 3.43, exactly the 2-trees do not contain a subdivision of K_4. □

It is natural to ask what forces a graph to contain a subdivision of K_5. Minimum degree 4 does not suffice, as shown by $K_{2,2,2}$. However, $m > 3n - 6$ forces G to contain a subdivision of K_5. This was conjectured by Dirac and proved by Mader [**1998**].

Electrical circuits can be modeled using graphs, with edges representing wires, and vertices representing their intersections. Components of an electrical circuit can be combined in series (one after another) or in parallel (beside each other).

Definition 3.47. A **series-parallel multigraph** is a multigraph that can be constructed from K_2 by subdividing edges and duplicating edges.



A series-parallel multigraph can be viewed as having two distinguished vertices, a **source** and a **sink**, which are the initial two vertices when it is constructed from K_2. Current would flow from the source to the sink. Thus the edges of a series-parallel multigraph could all be directed from the source to the sink, but this is not required. Series-parallel circuits lead to easy computations of physical quantities such as resistance. Resistance sums on series circuits ($R = R_1 + R_2$), while for parallel circuits, reciprocals are summed ($\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$).

Theorem 3.48 (Duffin [**1965**]). *A multigraph G has no K_4-subdivision if and only if every block of G is series-parallel.*

Proof. ($\Leftarrow$) Let G be series-parallel, so it can be constructed as in the definition. There is no K_4-subdivision in K_2, and neither operation can create one. Thus G contains no K_4-subdivision.

($\Rightarrow$) (contrapositive) Let G have a block H that is not series-parallel, so $\delta(H) \geq 2$. If H has duplicate edges or a degree 2 vertex, then it can be constructed using one of the operations from a graph of smaller size. Thus we may assume that H is a graph (not multigraph) with $\delta(H) \geq 3$. But then Theorem 3.45 implies H has a K_4-subdivision. □

Graphs that are series-parallel have a natural relationship with 2-trees.

Theorem 3.49 (Wald/Colbourn [**1983**]). *A graph G contains no K_4-subdivision if and only if G is contained in a 2-tree.*

Proof. ($\Leftarrow$) Theorem 3.43 implies this.

($\Rightarrow$) Assume G contains no K_4-subdivision and, to the contrary, is not contained in a 2-tree. We may assume G is a minimal counterexample. Thus G is 2-connected. Now G is not 3-connected, since every 3-connected graph contains a K_4-subdivision. Thus $\kappa(G) = 2$, so G has a cutset $S = \{u, v\}$. We can add uv if it is not already present, since each S-lobe contains a $u - v$ path. Then each S-lobe of $G + uv$ is contained in a 2-tree, so identifying them on uv results in a graph that is contained in a 2-tree. $\square$

This implies that any series-parallel graph (not multigraph) has size at most $2n - 3$, and 2-trees are the extremal graphs.

Definition 3.50. The **treewidth** $\mathrm{tw}(G)$ of a graph G is the minimum value of k such that G is a subgraph of some k-tree. A **partial k-tree** is a graph with treewidth at most k.

Thus all series-parallel graphs have treewidth 2 (except paths). Forests are exactly the graphs with treewidth 1. Note that $D(G) \leq \mathrm{tw}(G)$. Treewidth is an important parameter in computer science, as many problems that are intractable for graphs in general are tractable for graphs with bounded treewidth. Treewidth was used by Neil Robertson and Paul Seymour in the proof of the Graph Minor Theorem (Theorem 5.30).

Degeneracy, cores, and k-trees are interesting in their own right. But they also have significant applications to graph coloring, which is the focus of the next chapter.

Related Terms: k-shell, k-collapsible graph, graph contraction, pathwidth, branchwidth, series composition, parallel composition, confluent graph.

Exercises

Section 3.1:

(1) Show that every $u - v$ walk contains a $u - v$ trail.

(2) Show that every $u - v$ trail contains a $u - v$ path.

(3) Determine which graphs in the following classes have an Eulerian circuit or Eulerian trail. For those with neither, determine the smallest number of trails into which they decompose.
   (a) C_n
   (b) K_n
   (c) W_n
   (d) Q_k

(4) Determine which graphs in the following classes have an Eulerian circuit or Eulerian trail. For those with neither, determine the smallest number of trails into which they decompose.

(a) $K_{r,s}$
(b) $G_{r,s}$
(c) the triangular grid T_l
(d) the Kneser graph $KG_{r,k}$

(5) Prove or disprove: Any subdivision of an Eulerian graph is Eulerian.

(6) State and prove when $\overline{G}$ is Eulerian in terms of basic parameters of a graph G.

(7) State and prove when the line graph $L(G)$ is Eulerian in terms of basic parameters of a graph G.

(8) State and prove when $G+H$ is Eulerian in terms of basic parameters of graphs G and H.

(9) State and prove when $G \square H$ is Eulerian in terms of basic parameters of graphs G and H.

(10) State and prove when the tensor product $G \times H$ is Eulerian in terms of basic parameters of graphs G and H.

(11) + Prove Theorem 3.2 using
(a) a trail of maximum length.
(b) minimum counterexample.

(12) + Prove Theorem 3.4 directly using induction on
(a) k.
(b) size.

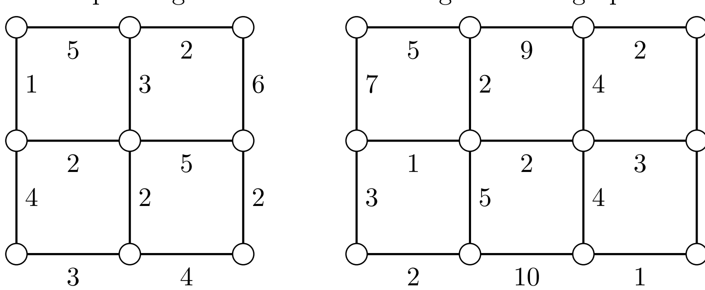
(13) + Prove Theorem 3.5.

(14) Prove Theorem 3.2 given Theorem 3.5.

(15) State and prove an analogue of Corollary 3.3 for digraphs.

(16) Show that if D is a connected digraph with $\sum_v |od(v) - id(v)| = 2k$, $k \geq 1$, then D can be decomposed into k directed open trails.

(17) Find a closed spanning walk of minimum length for the graph below left.



(18) Find a closed spanning walk of minimum length for the graph above right.

(19) Show that every nontrivial connected graph has a closed spanning walk that uses every edge exactly twice.

(20) Form a graph D_k whose vertices are bitstrings of length $k-1$. An edge joins vertex u to v if the last $k-2$ bits of u agree with the first $k-2$ bits of v. Label the edge with the last entry of v. This is called a **de Bruijn graph**.
(a) Draw D_4.

(b) Show that D_k is Eulerian.
(c) Show that the edge labels in any Eulerian circuit of D_k form a cyclic arrangement in which the 2^k substrings of length k are distinct.

(21) Two equivalent Eulerian circuits have the same sequence of edges, ignoring starting point and direction. How many nonequivalent Eulerian circuits are there in
(a) $K_{2,2k}$?
(b) the triangular grid T_2?

(22) + Eulerian graphs and cycle counting.
(a) (Toida [**1973**]) Let G be Eulerian with $e = uv$. Show that there is an odd number of $u - v$ trails in $G - e$ containing v only once (at the end). Show that there is an even number of $u - v$ trails that are not paths in $G - e$ containing v only once.
(b) (McKee [**1984**]) Let v have odd degree, and let $c(e)$ be the number of cycles containing e. Use $\sum_v c(e)$ to show that $c(e)$ is even for some edge e incident with v.
(c) Use (a) and (b) to show that a nontrivial connected graph is Eulerian if and only if every edge is contained in an odd number of cycles.

(23) Prove or disprove: Any two adjacent edges in an Eulerian graph are consecutive on some Eulerian circuit of it.

(24) + (Chartrand/Lesniak [**1986**]) Let G be an Eulerian graph with order at least 3. A vertex v is **extendible** if every trail beginning at v can be extended to form an Eulerian circuit. Show that
(a) (Ore [**1951**]) a vertex v is extendible if and only if $G - v$ is a forest.
(b) (Babler [**1953**]) if v is extendible, it has maximum degree.
(c) any vertex of G is extendible if and only if G is a cycle.
(d) if G is not a cycle, G has at most two extendible vertices.

Section 3.2:

(1) A computer chip manufacturer suspects a corporate rival of stealing its chip design. How can graph theory be used to model and help to answer this question?

(2) Show that graph isomorphism is an equivalence relation by verifying that it is
(a) reflexive.
(b) symmetric.
(c) transitive.

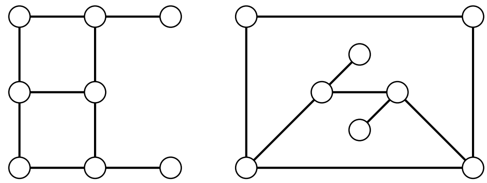
(3) Find two nonisomorphic graphs of smallest order with the same order and size.

(4) Find two nonisomorphic trees of smallest order with the same order.

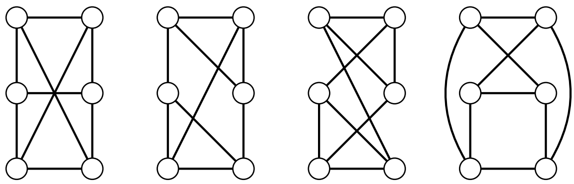
(5) Find two nonisomorphic graphs of smallest order with the same degree sequence.

(6) Find two nonisomorphic trees of smallest order with the same degree sequence.

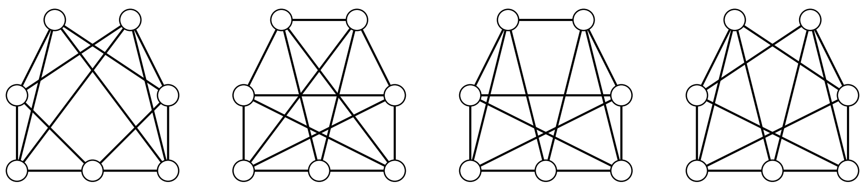
(7) Determine whether the following graphs are isomorphic or not.



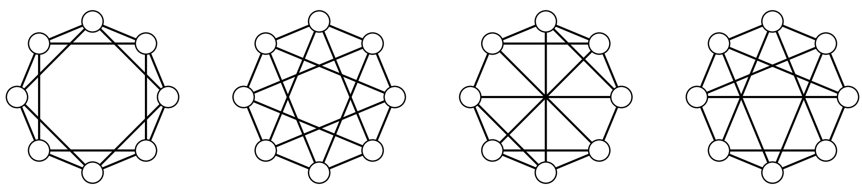
(8) Determine which pairs of graphs below are isomorphic.



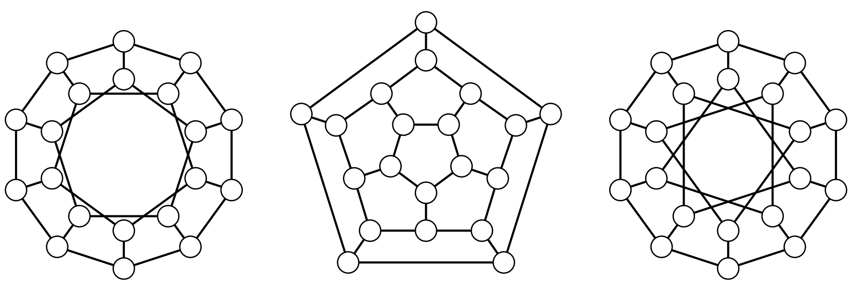
(9) Determine which pairs of graphs below are isomorphic.



(10) Determine which pairs of graphs below are isomorphic.



(11) Determine which pairs of graphs below are isomorphic.



(12) Given n graphs that are contained in r isomorphism classes, what is the minimum number of pairs that must be tested to determine all the isomorphism classes? What is the minimum of this quantity over all possible values of r?

(13) Show that $K_3 \square K_3$ and $K_3 \square K_3 - v$ are self-complementary.

(14) (Akiyama/Harary [**1981**]) Let G be a self-complementary graph. Show that G has a cut-vertex if and only if G has a leaf.

(15) Let G and H be self-complementary graphs, where G has even order n. Form a graph by joining each vertex of H to every vertex of G that has degree less than $\frac{n}{2}$. Show that this graph is self-complementary.

(16) Let G and H be self-complementary graphs, where G has even order n. Form a graph by joining each vertex of H to every vertex of G that has degree at least $\frac{n}{2}$. Show that this graph is self-complementary.

(17) Let G and H be graphs. We **substitute** H **for a vertex** v **of** G by deleting v, and add a copy of H with each of its vertices adjacent to each vertex in $N(v)$. Show that if G and H are self-complementary, the graph formed by substituting H for every vertex of G is self-complementary.

(18) Let G be a graph. Show that the graph formed by substituting G for the leaves of P_4 and $\overline{G}$ for the nonleaves of P_4 is self-complementary.

(19) Let G be a self-complementary graph with order $n \equiv 1 \bmod 4$. Show that G contains an odd number of vertices with degree $\frac{n-1}{2}$ (and hence at least 1).

(20) + (Ringel [**1963**], Sachs [**1962**]) Structure of self-complementary graphs.
 (a) Let G be a self-complementary graph, and let σ be a permutation of $V(G)$ that maps G to $\overline{G}$. Show that each cycle of the permutation has length a multiple of 4, except for at most one cycle of length 1.
 (b) Use part (a) to show that each self-complementary graph G with order $n \equiv 1 \bmod 4$ has a fixed point under any permutation σ mapping G to $\overline{G}$.

(21) + Let σ be a permutation of $V(G)$ so that each cycle of the permutation has length a multiple of 4, except for at most one cycle of length 1. Show that each cycle in the induced permutation of edges is even, and selecting alternate edges along each edge cycle produces a self-complementary graph.

(22) Ten self-complementary graphs of order 8.
 (a) Find two self-complementary graphs of order 8 using the operations in Exercises (15) and (16) above.
 (b) + Find five self-complementary graphs of order 8 containing two disjoint copies of $K_4 - e$.
 (c) + Find three self-complementary graphs of order 8 containing four triangles, each of which has one edge in common with a 4-cycle.

(23) Determine a necessary condition for the orders of complete graphs that decompose into three isomorphic graphs.

(24) Determine a necessary condition for the orders of complete graphs that decompose into four isomorphic graphs.

(25) Use Proposition 3.10 to explain carefully why girth is a graph invariant.

(26) Use Proposition 3.10 to explain carefully why diameter is a graph invariant.

(27) Let G be an $\operatorname{srg}(n, k, \lambda, \mu)$. Show that $\overline{G}$ is strongly regular, and find its parameters.

(28) + Show that an $\operatorname{srg}(n, k, \lambda, \mu)$ has $k(k - \lambda - 1) = \mu(n - k - 1)$.

(29) Determine which graphs in the following classes are strongly regular.
 (a) C_n
 (b) complete multipartite graphs
 (c) cubic graphs

(30) (a) Show that the line graphs $L(K_n)$ and $L(K_{r,r})$ are strongly regular.
(b) The **Shrikhande graph** has vertices (i, j), $0 \leq i, j \leq 4$. Vertex (i, j) is adjacent to $(i \pm 1, j)$, $(i, j \pm 1)$, $(i+1, j+1)$, and $(i-1, j-1)$ (all mod 4). Show that it is strongly regular.
(c) Show that $L(K_{4,4})$ and the Shrikhande graph are $\operatorname{srg}(16, 6, 2, 2)$, but they are nonisomorphic. (*Note*: This is the pair of nonisomorphic strongly regular graphs of smallest order.)

(31) Explain how to count disconnected cubic graphs if you know the numbers of connected cubic graphs of various orders. Find the number of disconnected cubic graphs of orders 12 and 14.

(32) Let G and H be cubic graphs with edges $e = uv$ and $f = wx$, respectively. Show that the graph formed by deleting e and f and adding edges uw and vx is also cubic. Can this operation always be reversed to separate a cubic graph with edge connectivity 2 into two cubic graphs?

(33) Let G be cubic with edge connectivity 2. Show that G has subgraphs G_1 and G_2 and vertices $u, v \in V(G_1)$, and $w, x \in V(G_2)$, with $uv \notin E(G_1)$ and $wx \notin E(G_2)$ so that there is a **ladder** ($P_k \square K_2$ with the neighboring vertices of degree 2 joined to u and v at one end and w and x at the other end) joining G_1 and G_2. Explain how such a cubic graph can be constructed from smaller cubic graphs.

(34) + Use the previous problems to show how any cubic graph with connectivity less than 3 can be constructed from smaller 3-connected cubic graphs.

(35) + Let G be a cubic graph with a minimal edge cut of three nonadjacent edges. Define an operation that can be used to construct G from smaller cubic graphs.

(36) Let G be a cubic graph with edges $e = uv$ and $f = wx$. We **add a handle** to G by deleting e and f and adding vertices y and z and edges uy, yv, wz, zx, and yz.
(a) Show that adding a handle produces another cubic graph.
(b) Draw a graph with vertices representing cubic graphs of order 4–8 and edges between two graphs when one can be constructed from the other by adding a handle.
(c) Find an infinite class of 2-connected cubic graphs that cannot be constructed by adding handles. (*Hint*: Generalize graph G_2 of order 8.)

(37) (Steinitz/Rademacher [**1934**]) Show that all 3-connected cubic graphs can be formed from K_4 by adding handles.

(38) For a multigraph containing edge $e = uv$, define adding a handle as before or as deleting e and adding vertices y and z and edges uy, vz, and two between y and z. Show that every cubic multigraph can be formed from the cubic multigraph with order 2 and size 3 by adding handles between one or two cubic multigraphs.

(39) Let G be a 3-connected cubic graph containing a triangle. Show that contracting the triangle to a single vertex results in another cubic graph. Define an operation to construct G from this smaller cubic graph.

(40) Cubic graphs of order 10.
(a) Find all disconnected cubic graphs of order 10.

(b) Are there any cubic graphs of order 10 that contain a bridge? If so, find them.
(c) Find all cubic graphs of order 10 with edge connectivity 2.
(d) Find all cubic graphs of order 10 with an edge cut with three nonadjacent edges.
(e) Find all bipartite cubic graphs of order 10.
(f) + Fill in the remaining cases to find all cubic graphs of order 10.

(41) Determine all graphs with degree sequence $3, 3, 2, 2, 2, 2$.

(42) Determine all graphs with degree sequence $3, 3, 3, 3, 2, 2$.

(43) Find the number of automorphisms and number of distinct labelings from $[n]$ of the following graphs. State if they are vertex-transitive or edge-transitive.
(a) C_n
(b) K_n
(c) W_n

(44) Find the number of automorphisms and number of distinct labelings from $[n]$ of the following graphs. State if they are vertex-transitive or edge-transitive.
(a) $K_{r,s}$
(b) $\overline{K}_n$
(c) $C_n \square K_2$

(45) A graph is **asymmetric** if the only automorphism is the identity. Find an asymmetric graph that
(a) has order 6.
(b) is a tree with order 7.
(c) + is cubic with order 12.
(d) + is 4-regular with order 10.

(46) Show that any copy of P_4 in the Petersen graph can be mapped to any other copy.

(47) Show that graphs G and $\overline{G}$ have the same number of automorphisms.

(48) + Show that the automorphisms of a graph form a group with the operation function composition. Verify that this operation is well defined, is associative, has an identity, and has inverses for each automorphism.

Section 3.3:

(1) Use the Havel-Hakimi Theorem to determine whether the following sequences are graphic. If so, construct a graph realizing the sequence.
(a) 3331
(b) 555333
(c) 666333111
(d) 77655432111

(2) Use the Havel-Hakimi Theorem to determine whether the following sequences are graphic. If so, construct a graph realizing the sequence.
(a) 6442211
(b) 44111111
(c) 9866443211
(d) 88555443222

(3) Draw a graph whose vertices represent cubic graphs of order 8 (Section 3.2) and that has an edge where a graph can be converted to another by a single 2-switch.

(4) Draw a graph whose vertices represent 2-regular graphs of order 12 and that has an edge where a graph can be converted to another by a single 2-switch.

(5) Explain why the statement of the Havel-Hakimi Theorem does not include a requirement that the sum of the degrees is even.

(6) (Wang/Kleitman [**1973**]) Show that the sequence S: $d_1 \geq \cdots \geq d_n$ is graphic if and only if the sequence S_1 formed by deleting d_k and subtracting 1 from the d_k largest remaining terms is graphic. (*Note*: This generalizes the Havel-Hakimi Theorem.)

(7) Show that any sequence of nonnegative integers with even sum, largest entry less than n, and largest and smallest entries differing by at most 1 is graphic.

(8) Use the Erdos-Gallai Theorem to determine whether the following sequences are graphic.
(a) 3311
(b) 22211
(c) 43221
(d) 544321

(9) Characterize the degree sequences of the following graph classes. State whether they are closed under 2-switches.
(a) paths
(b) 2-regular graphs
(c) wheels
(d) theta graphs

(10) Characterize the degree sequences of the following graph classes. State whether they are closed under 2-switches.
(a) $K_{r,s}$
(b) cubic graphs
(c) double stars
(d) triangular grids

(11) Determine whether the following sequences are unigraphic. If so, draw the graph.
(a) 21111
(b) 33211
(c) 322111

(12) Determine whether the following sequences are unigraphic. If so, draw the graph.
(a) 33332
(b) 533322
(c) $11 \cdots 11$

(13) Given the degree sequences of graphs G and H, describe the degree sequence of $G + H$.

(14) Given the degree sequences of graphs G and H, describe the degree sequence of $G\square H$.

(15) Show that the sequence $d_1, \ldots, d_n$ is graphic if and only if $n - d_1 - 1, \ldots, n - d_n - 1$ is graphic.

(16) (Gale [**1957**], Ryser [**1957**]) The **Gale-Ryser Theorem** states that a pair of sequences of nonnegative integers $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ with $a_1 \geq \cdots \geq a_n$ are the degrees of the partite sets of a bipartite graph if and only if $\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i$ and $\sum_{i=1}^{k} a_i \leq \sum_{i=1}^{n} \min(b_i, k)$ holds when $1 \leq k \leq n$. Prove the forward direction ($\Rightarrow$) of this theorem.

(17) Show that $d_1, \ldots, d_n$ is the degree sequence of some multigraph if and only if $\sum d_i$ is even.

(18) + (Hakimi [**1962**]) Let $d_1 \geq \cdots \geq d_n$ be nonnegative integers. Show that there is a loopless multigraph with degree sequence $d_1, \ldots, d_n$ if and only if $\sum d_i$ is even and $d_1 \leq d_2 + \cdots + d_n$.

(19) Show that a graph G has a unigraphic degree sequence if and only if $G + K_1$ has a unigraphic degree sequence.

(20) Show that a graph G has a unigraphic degree sequence if and only if $G \cup K_1$ has a unigraphic degree sequence.

(21) (a) For $n \geq 2$, show that there is exactly one disconnected graph of order n with degrees 0 through $n-2$ (with $\lfloor \frac{n-1}{2} \rfloor$ repeated).
(b) Show that the graphs in part (a) can be described as $\overline{I_n}$ and $I_{n-1} \cup K_1$.

(22) Determine the size of the irregular graph I_n.

(23) Show that the two vertices with equal degree in I_n are adjacent when n is even and nonadjacent when n is odd.

(24) Show that I_n contains $K_{\frac{n+2}{2}}$ when n is even and $K_{\frac{n+2}{2}} - e$ when n is odd. Show that I_{n+1} can be formed from I_n by adding a vertex joined to all vertices in a copy of $K_{\lceil \frac{n}{2} \rceil}$ contained in a clique of maximum cardinality.

Section 3.4:

(1) Show that the k-core of a graph is well defined (that is, unique).

(2) Explain how to modify the k-Core Algorithm to find
(a) the degeneracy of a graph.
(b) the core numbers of all vertices of a graph.

(3) Determine the degeneracy of the graphs in the following classes. State whether they are monocore.
(a) forests
(b) $K_{r,s}$, $r \leq s$
(c) W_n
(d) $G_{r,s}$

(4) Determine the degeneracy of the graphs in the following classes. State whether they are monocore.
(a) theta graphs
(b) double wheels

(c) triangular grids
(d) fan graphs

(5) (a) + Let G and H be graphs. Show that

$$C_k(G \square H) = \bigcup_{i+j=k} [C_i(G) \square C_j(H)].$$

(b) Let $v = (u, w) \in V(G \square H)$. Show that $C(v) = C(u) + C(w)$.
(c) Show that $D(G \square H) = D(G) + D(H)$.

(6) (a) + Let G and H be graphs. Show the following, where the minimum is unique over both i and j.

$$C_k(G+H) = \min_{i,j} \{C_i(G) + C_j(H) \mid i + |C_j(H)| \geq k \text{ and } j + |C_i(G)| \geq k\}$$

(b) Show that $D(G+H) = \max_{i,j} \min(i + |C_j(H)|, j + |C_i(G)|)$.

(7) Let G be a k-core with order n. Show that if $k+1 < n < 2k+2$, then $\operatorname{diam}(G) = 2$.

(8) + (Moon [**1965**]) Show that if G is a connected k-core with order $n \geq 2k+2$, then $\operatorname{diam}(G) \leq 3\left\lfloor \frac{n}{k+1} \right\rfloor - 3 + \min(n \bmod k+1, 2)$. Show that this bound is sharp.

(9) Show that if a graph is connected, then its 2-core is connected.

(10) Show that a vertex v of G is contained in the 2-core of G if and only if v is on a cycle or a path between vertices of distinct cycles.

(11) Show that a graph G is a 2-core if and only if every end-block of G is 2-connected.

(12) Show that every 2-core G has a unique decomposition into 2-connected blocks and trees so that if any tree T in the decomposition is nontrivial, each end-vertex of T is shared with a distinct 2-connected block, if T is trivial, it is a cut-vertex contained in at least two 2-connected blocks, and there are no two disjoint paths between two distinct blocks.

(13) Show that degeneracy and average degree are incomparable over all graphs (either can be larger). Is this true for monocore graphs?

(14) For $k > 0$, the **k-shell** of a graph G is the subgraph of G induced by the edges contained in the k-core and not contained in the $k+1$-core. Show that the size m of a k-shell with order n satisfies $\left\lceil \frac{k \cdot n}{2} \right\rceil \leq m \leq k \cdot n$. Characterize the extremal graphs for the lower bound.

(15) Show that the size m of a k-monocore graph G of order n satisfies $\left\lceil \frac{k \cdot n}{2} \right\rceil \leq m \leq k \cdot n - \binom{k+1}{2}$. Show that both bounds are sharp.

(16) (Bickle [**2014**]) Show that if a nonincreasing sequence of integers $d_1, \ldots, d_n$ is the degree sequence of some k-monocore graph G, $0 \leq k \leq n-1$, then $k \leq d_i \leq \min\{n-1, k+n-i\}$ and $\sum d_i = 2m$, where $\left\lceil \frac{k \cdot n}{2} \right\rceil \leq m \leq k \cdot n - \binom{k+1}{2}$. (*Note*: The converse also holds.)

(17) Let $d_1, \ldots, d_n$ be a nonincreasing sequence of integers that can be the degree sequence of a k-monocore graph G. Show that it must be the degree sequence of a k-monocore graph if $d_{k+2} = k$.

(18) Let $d_1, \ldots, d_n$ be a nonincreasing sequence of integers that can be the degree sequence of a 1-monocore graph G. Show that it must be the degree sequence of a 1-monocore graph if and only if $d_3 = 1$.

(19) (Bickle [**2018**]) A graph G is **k-collapsible** if it is k-monocore and has no proper induced k-core. Show that every k-monocore graph G contains a k-collapsible graph as an induced subgraph.

(20) (Bickle [**2018**]) Show that for $k \geq 1$, the size m of a k-collapsible graph G of order n satisfies $\left\lceil \frac{k \cdot n}{2} \right\rceil \leq m \leq (k-1) \cdot n - \binom{k}{2} + 1$.

(21) Let G be a maximal k-degenerate graph of order n, $1 \leq k \leq n-1$. Show that for $k \geq 2$, there are exactly 3 nonisomorphic maximal k-degenerate graphs of order $k+3$.

(22) Find all maximal 2-degenerate graphs of order 6.

(23) Let G be a maximal k-degenerate graph of order n, $1 \leq k \leq n-1$. Show that G has edge-connectivity $\kappa'(G) = k$, and for $k \geq 2$, an edge set is a minimum edge cut if and only if it is a trivial edge cut.

(24) (Franceschini/Luccio/Pagli [**2006**]) Let G be a maximal k-degenerate graph with $n \geq 2k$. Let S be a subset of vertices with $n_1 = |S| \leq \frac{n}{2}$, and let W be the edge cut separating S from $\overline{S}$. Show that if $n_1 \leq k$, then $|W| \geq n_1 k - \binom{n_1}{2}$, and if $n_1 > k$, then $|W| \geq \binom{k+1}{2}$.

(25) Let $t_1, \ldots, t_r$ be r positive integers which sum to t. Show that a maximal t-degenerate graph can be decomposed into r graphs with degeneracies at most $t_1, \ldots, t_r$, respectively. Show that a k-degenerate graph decomposes into k forests.

(26) (a) The **graph contraction** G/H is formed by contracting the subgraph H of G to a single vertex. Show that a maximal k-degenerate graph G of order $n \geq k$ can be decomposed into K_k and k trees of order $n-k+1$, which span G/K_k.
(b) Show that if k is odd, a maximal k-degenerate graph decomposes into k trees of order $n - \frac{k-1}{2}$.

(27) (a) (Bickle [**2012**]) Show that a maximal k-degenerate graph G with $n \geq k+2$ has $2 \leq \operatorname{diam}(G) \leq \frac{n-2}{k} + 1$.
(b) Show that if the upper bound is an equality, then G has exactly two vertices of degree k and every diameter path has them as its endpoints.

(28) + (Bickle [**2012**]) Show that a maximal k-degenerate graph G has $\operatorname{diam}(G) = \frac{n-2}{k} + 1$ if and only if G can be constructed by the following algorithm.
(1) Begin with either $K_k + 2K_1$ or a graph formed from any maximal k-degenerate graph of order $2k$ by adding two vertices of degree k with no common neighbors.
(2) Iterate the following operation. Let $v = v_0$ be vertex of degree k in G with neighbors $\{u_1, \ldots, u_k\}$. Successively add $k-1$ vertices $\{v_1, \ldots, v_{k-1}\}$ with degree k when added so that the neighbors of v_i are all in $\{u_1, \ldots, u_k, v_0, \ldots, v_{i-1}\}$. Then add a new vertex v' adjacent to $\{v_0, \ldots, v_{k-1}\}$.

(29) (Borowiecki et al. [**1995**]) Let G be maximal k-degenerate with $\Delta(G) = r$, let $n \geq k+1$, and let n_i be the number of vertices of degree i, $k \leq i \leq r$. Show

that

$$k \cdot n_k + (k-1)\, n_{k+1} + \ldots + 2n_{2k-2} + n_{2k-1}$$
$$= n_{2k+1} + 2n_{2k+2} + \ldots + (r-2k)\, n_r + k\,(k+1)\,.$$

(30) (Bickle [**2012**]) Show that for a nonincreasing degree sequence $d_1, \ldots, d_n$ of a maximal k-degenerate graph, $d_i \leq \frac{k(n-k-1)}{i} + k$. Hence

$$d_i \leq \min \left\{ n-1, \left\lfloor \frac{k\,(n-k-1)}{i} \right\rfloor + k, k+n-i \right\}.$$

Show that for each i, there is some maximal k-degenerate graph that attains this bound.

(31) + (Filakova et al. [**1997**]) Degree sequences of maximal k-degenerate graphs. Let G be maximal k-degenerate of order n.
  (a) Let $k \geq 2$ and $0 \leq s \leq k-2$. Show that if $n > \frac{k^2+(3+2s)k}{2(1+s)} - \frac{s}{2}$, then $\Delta\,(G) \geq 2k - s$.
  (b) Show that if $n \geq \binom{k+2}{2}$, then $\Delta(G) \geq 2k$.
  (c) Show that if $n \leq \frac{1+\sqrt{1+8k}}{2} + k$, then $\Delta(G) = n-1$.

(32) Let $d_1, \ldots, d_n$ be the nonincreasing degree sequence of a maximal k-degenerate graph G with $k \geq 2$ and $0 \leq s \leq k-2$. Use the previous problem to show that if $n > \frac{k^2+(3+2s)k}{2(1+s)} - \frac{s}{2}$, then $d_i \geq 2k+1-s-i$. (*Hint*: Show that if G has a vertex v of degree $n-1$, then $G-v$ is maximal $k-1$-degenerate.)

(33) Find all 2-trees of order at most 6.

(34) Find all 3-trees of order at most 7.

(35) Show that every maximal k-degenerate graph of order $n \geq k+2$ must contain an induced k-tree of order $k+2$. For all $k \geq 2$ and $n \geq k+3$, find a maximal k-degenerate graph that does not contain a larger induced k-tree.

(36) (Bickle [**2012**]) Show that every maximal k-degenerate graph G of order $n \geq k+1$ contains a unique k-tree of largest possible order containing a $k+1$-clique that can be used to begin the construction of G.

(37) + Show that every 3-core contains a subdivision of K_4 by showing that any 3-core contains either adjacent vertices with no common neighbors, $K_4 - e$, or a triangle whose vertices all have distinct neighbors.

(38) (Bose et al. [**2008**]) Let S be a list of four d's and $n-4$ 2's. Show that S is not the degree sequence of a 2-tree. (*Note*: Bose et al. [**2008**] characterized the degree sequences of 2-trees. No characterization is known for k-trees when $k > 2$.)

(39) (Caminiti/Fusco [**2007**]) Use a generalization of Prufer codes to show that there are at most $\binom{n}{k}^{n-k-1}$ labeled maximal k-degenerate graphs, with equality exactly when $k = 1$ or $n < \frac{k(k+1)}{k-1}$.

(40) + (Beineke/Pippert [**1969**], Moon [**1969**]) Let $N_k\,(n,d)$ be the number of k-trees with n labeled vertices in which the root (a distinguished k-clique) is joined to exactly d other vertices.
  (a) Show that $N_k\,(n,d) = \binom{n-k}{d} \sum_{t=1}^{n-d-k} N_k\,(n-d,t)\,(kd)^t$.

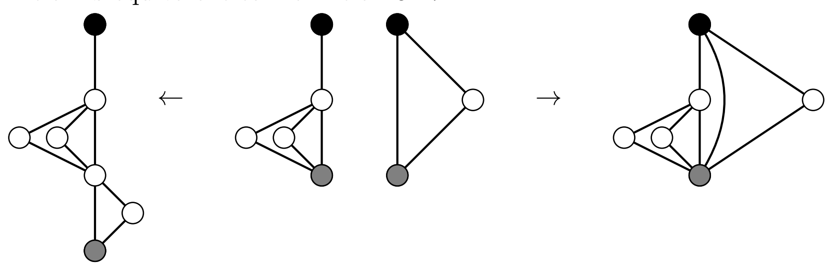
(b) Use induction to show that $N_k(n,d) = \binom{n-k-1}{d-1}(k(n-k))^{n-d-k}$.

(c) Show that the number of labeled rooted k-trees is $(k(n-k)+1)^{n-k-1}$.

(d) Show that the number of labeled (unrooted) k-trees is

$$\binom{n}{k}(k(n-k)+1)^{n-k-2}.$$

(41) There is an alternative definition of the class of series-parallel multigraphs, which starts with K_2 with its vertices identified as the source and sink. Two series-parallel graphs can be combined using the operations of **parallel composition** (identifying the sources and sinks of two series-parallel graphs) and **series composition** (identifying the source of one series-parallel graph with the sink of another). These operations are illustrated below. Show that this definition is equivalent to Definition 3.47.



(42) (Duffin [**1965**]) A graph is **confluent** if it does not have two cycles containing two common edges so that some orientations of the cycles contain one edge in the same direction and the other edge in opposite directions. Show that a graph is confluent if and only if it does not contain a subdivision of K_4.

(43) Determine the treewidth of the graphs in the following classes.
(a) K_n
(b) W_n
(c) $+\ G_{r,s}$

(44) Determine the treewidth of the graphs in the following classes.
(a) C_n
(b) $K_{r,s}$
(c) theta graphs

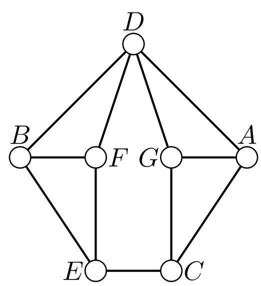
Chapter 4

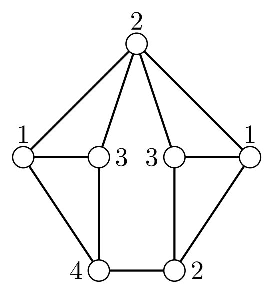
Vertex Coloring

4.1. Applications of Coloring

Example. Six math students will take seven summer math classes, denoted A–G. The students' schedules are listed below.

Student	Classes
Al	A, D, G
Bob	B, E
Carl	A, C, G
Dave	C, E
Edna	E, F
Frank	B, D, F



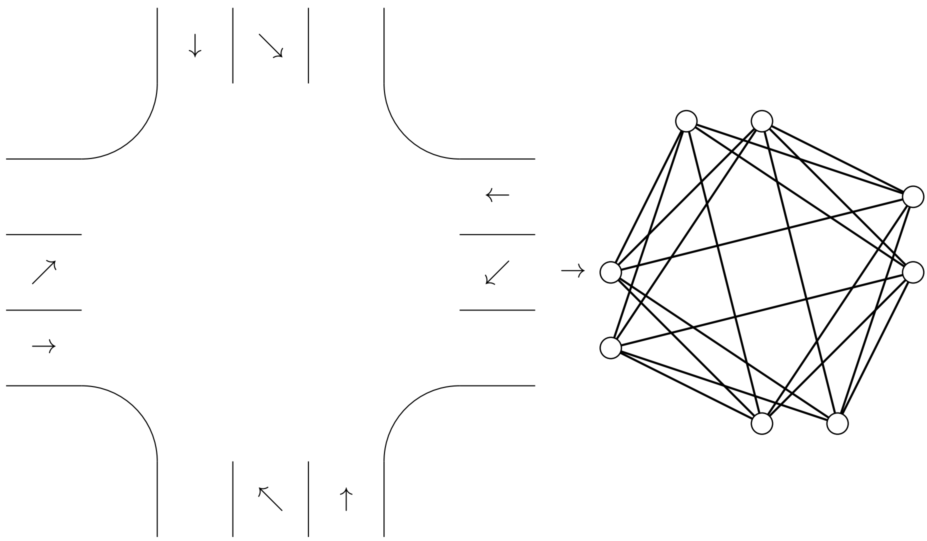


We model this situation with a graph. Each vertex represents a class. Put an edge between two classes when they have a common student. Thus Al's schedule imposes edges AD, AG, and DG. Thus we construct the graph above, known as the **Moser spindle**. A natural question to ask here is how few time slots we can schedule the classes in, and how to construct such a schedule.

If a vertex is in one slot, its neighbors must be in different slots. Number the slots $1, 2, 3, \ldots$. Vertices A, D, and G must be in three different slots, say 1, 2, and 3. If we try to schedule the rest of the classes in three slots, C must be in slot 2. Vertices B and F must use slots 1 and 3. However, we find it is not possible to schedule vertex E in slots 1, 2, or 3, so a fourth slot is needed.

We refer to the slots as **colors**, and the process we have just completed as **coloring**. The minimum number of colors that can be used on a graph is its **chromatic number**. These terms will be defined precisely in the next section. For now, we consider several very different situations that can be analyzed using these ideas.

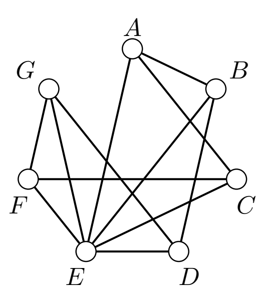
Example. Two roads meet at an intersection where there is a traffic light. Both roads have one lane going in each direction and a left-turn lane. Some pairs of lanes cannot safely have traffic flow simultaneously. We model this situation with a graph. Vertices represent each of the eight lanes entering the intersection. Add an edge between the two vertices whose traffic lanes conflict. We find the following graph (note that it is $\overline{2C_4}$).

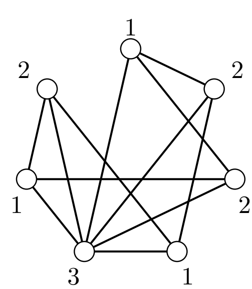


We want to program the traffic light so that there are as few traffic cycles as possible and determine which lanes go in which cycles. Thus we must partition the lanes into groups so that no lanes in a group conflict. Examining the graph, we see four lanes that all conflict, so we need at least four cycles. One option is to have each cycle be the two lanes that approach from the same direction. Another option is to pair the opposite lanes that go straight and the opposite left-turn lanes. This may be a better option if significantly more drivers go straight than turn left.

Example. An aquarium has a number of tanks displaying tropical fish. Some species of fish will attack other species if they are in the same tank. The conflicts are described in the following table, where an X indicates that the row species will attack the column species.

	A	B	C	D	E	F	G
A							
B	X				X		
C	X				X	X	
D		X			X		X
E	X					X	
F							
G					X	X	

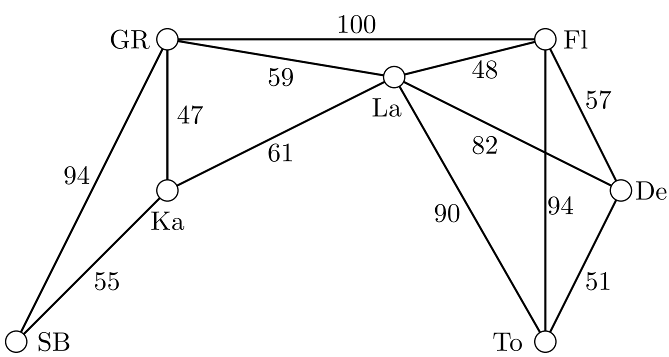




The graph above models the situation. It can be colored using three colors. There are three species that all conflict (e.g., A, B, and E), so three tanks is the minimum acceptable number.

Example. Television stations may not broadcast on the same channel if they are within some distance D of each other. We can model this situation with a graph, where the vertices represent television stations and the edges join stations within D miles. To minimize the number of channels used, we want to minimize the number of colors used in a coloring of this graph.

	De	Fl	GR	La	Ka	To	SB
Detroit		57	141	82	130	51	171
Flint			100	48	109	94	160
Grand Rapids				59	47	141	94
Lansing					61	90	114
Kalamazoo						113	55
Toledo							139
South Bend							



The table above lists distances between several cities. Suppose $D = 100$. The graph above models this situation. It can be colored using four colors.

These examples illustrate the fact that graph coloring has many applications. The rest of this chapter studies how to color graphs efficiently.

4.2. Coloring Bounds

Definition 4.1. A **vertex coloring** of a graph assigns one color to each vertex. A **proper vertex coloring** requires that adjacent vertices are colored differently.

While the colors could be actual colors (red, green, blue, . . .), we will typically use natural numbers $1, 2, \ldots, k$.

Definition 4.2. A k**-coloring** of a graph is a proper vertex coloring using **colors** $1, \ldots, k$ (not necessarily all of them). A graph is k**-colorable** if it has a k-coloring. The **chromatic number** $\chi(G)$ is the minimum number of colors used in any k-coloring of a graph G. A graph with $\chi(G) = k$ is said to be k**-chromatic**. A **minimum coloring** of a graph is one using $\chi(G)$ colors. A **color class** is all vertices with the same color in some coloring of the graph.

A k-coloring G can be thought of as a function $f : V(G) \to [k]$. However, using function notation is only occasionally beneficial when discussing graph coloring.

To determine the chromatic number of a graph, it is useful to have bounds that are easier to calculate. It is immediate that

$$1 \leq \chi(G) \leq n.$$

The extremal graph for the lower bound is the empty graph $\overline{K_n}$, since only a graph with no edges can be colored with one color. The extremal graph for the upper bound is K_n, since only a graph with all possible edges requires a different color on each vertex.

To determine the chromatic number exactly, we will need better bounds. The following observation is useful.

Proposition 4.3. *If $H \subseteq G$, then $\chi(H) \leq \chi(G)$.*

Proof. A coloring of G with $\chi(G)$ colors can be restricted to H. □

Definition 4.4. The **clique number** $\omega(G)$ of a graph G is the size of the largest clique of G.

Corollary 4.5. *For any graph G, $\chi(G) \geq \omega(G)$.*

Definition 4.6. The **independence number** $\alpha(G)$ of a graph G is the size of the largest independent set of G.

The independence number and clique number are complementary parameters, since $\omega\left(\overline{G}\right) = \alpha(G)$ and vice versa. The notation suggests that the beginning (α) of a graph is empty and the end (ω) is complete. For small graphs, independence number and clique number can be determined by inspection. For larger graphs, a systematic argument or exhaustive search is required.

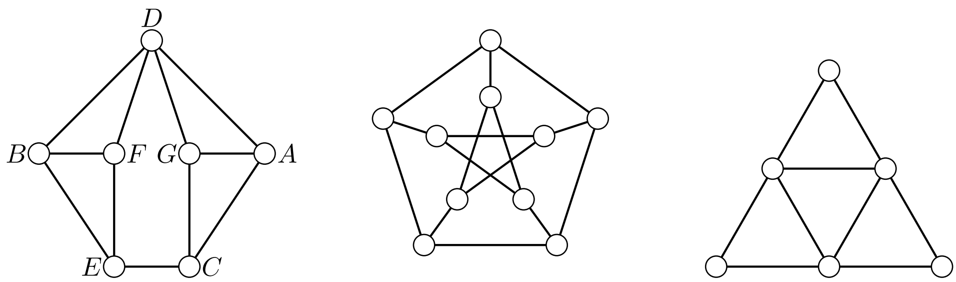
Any color class in a proper vertex coloring is an independent set. A k-coloring partitions the vertex set into k color classes. The chromatic number is the smallest number of independent sets into which $V(G)$ can be partitioned.

Proposition 4.7. *For any graph G, $\chi(G) \geq \frac{n}{\alpha(G)}$.*

Proof. Let $k = \chi(G)$, so G has color classes $V_1, \ldots, V_k$ for some k-coloring. Then $n = \sum_{i=1}^{k} |V_i| \leq k \cdot \alpha(G)$. Thus $\chi(G) \geq \frac{n}{\alpha(G)}$. □

The two basic lower bounds on the chromatic number are $\omega(G)$ and $\frac{n}{\alpha(G)}$. Which one is better depends on the graph.

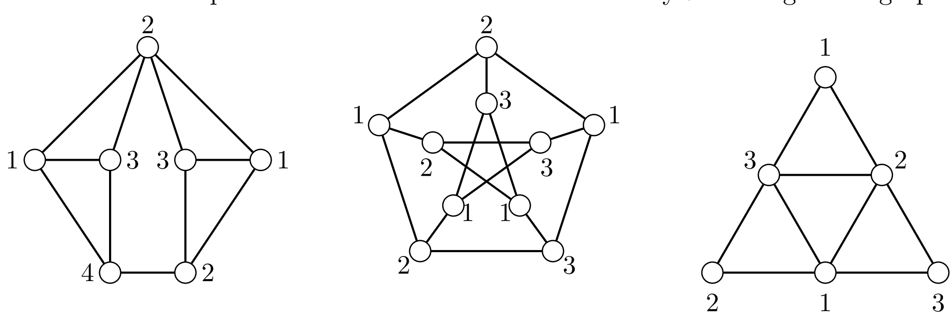
Example. Find ω, α, and χ for the following graphs.



The Moser spindle has $\omega(G) = 3$, with $\{A, D, G\}$ being one of four triangles. It has $\alpha(G) = 2$, with $\{C, D\}$ one of many independent sets of size 2. Corollary 4.5 implies $\chi(G) \geq 3$, and Proposition 4.7 implies $\chi(G) \geq \frac{7}{2} = 3.5$. Since the chromatic number must be an integer, $\chi(G) \geq 4$. A 4-coloring is shown below, so $\chi(G) = 4$.

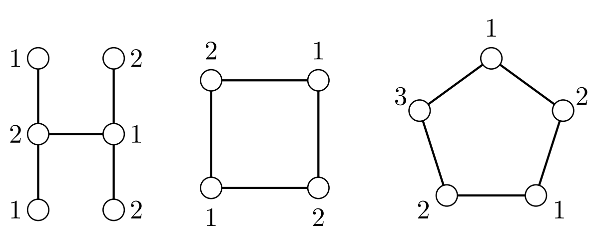
The Petersen graph has $\omega(G) = 2$ and $\alpha(G) = 4$, with the four vertices colored 1 below being one maximum independent set. The latter implies that $\chi(G) \geq \frac{10}{4} = 2.5$. A 3-coloring is shown below, so $\chi(G) = 3$.

The graph at right has $\omega(G) = 3$ and $\alpha(G) = 3$, with the three vertices of degree 2 being the unique maximum independent set. These imply lower bounds of 3 and $\frac{6}{3} = 2$ for the chromatic number. A 3-coloring is shown below. Note that the maximum independent set cannot be a color class in any 3-coloring of this graph.



Any nonempty bipartite graph requires exactly two colors, and the color classes are the partite sets. In fact, a graph is 2-colorable if and only if it is bipartite. Thus there is a good characterization of 2-colorable graphs (Theorem 1.44); a graph is 2-colorable if and only if it contains no odd cycle.

Example. Trees are bipartite, so they are 2-colorable. Even cycles have $\chi(C_{2k}) = 2$. Odd cycles have $\chi(C_{2k+1}) = 3$.



We have good characterizations of graphs with chromatic number 1 or 2. Unfortunately, there is no good characterization of graphs with $\chi(G) = k$ when $k \geq 3$. In fact, determining $\chi(G)$ when $k \geq 3$ is an NP-complete problem.

The class of NP-complete problems could all be solved in polynomial time if any can be solved in polynomial time. The fact that these problems have been studied extensively without anyone finding a polynomial time solution for any of them suggests (but does not prove) that no such algorithm exists. See Section 3 of the Appendix for more on computational complexity and NP-complete problems.

Determining α and ω are also NP-complete problems. They are essentially equivalent due to complementation. A naive algorithm would check all 2^n vertex subsets of a graph. A better algorithm (Robson [**1986**]) runs in $\mathcal{O}(1.2108^n)$ time, but no polynomial algorithm is known.

To show that $\chi(G) = k$, we must show

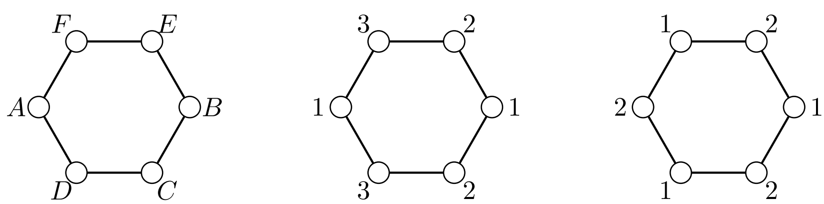
(1) $\chi(G) \geq k$. Use a lower bound, or find a contradiction to show that $\chi(G) < k$ is impossible.

(2) $\chi(G) \leq k$. Find a k-coloring, or use an upper bound (discussed below).

How can we find a k-coloring? Trial and error may work for small graphs, but larger graphs may require a more systematic approach.

Algorithm 4.8 (**Greedy Coloring**). *Given some vertex order, color each vertex with the smallest color that has not already been used on an adjacent vertex.*

Example. Color the vertices of the graph below left in order A–F. The 3-coloring produced is in the center. However, the coloring at right uses only two colors.

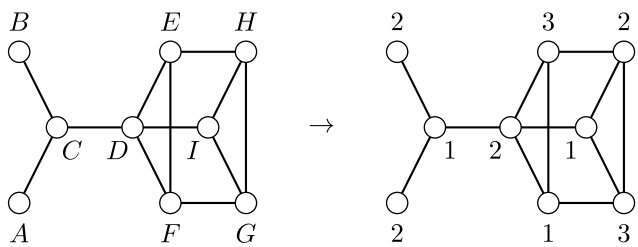


Greedy coloring must produce a proper coloring, but as with many greedy algorithms, it is not guaranteed to produce an optimal solution. How good the coloring is depends on the vertex order used. Some vertex order must produce a minimum coloring, but checking all $n!$ vertex orders is not practical. We need a vertex order that is likely to produce a good coloring, namely a construction sequence (Section 3.4).

Theorem 4.9 (**The Degeneracy Bound**). *For any graph G, $\chi(G) \leq 1 + D(G)$.*

Proof. Greedily color a construction sequence of G. Each vertex has at most $D(G)$ neighbors when colored, so at most $1 + D(G)$ colors are needed. $\square$

Example. A deletion sequence of the graph below left is the vertices A through I in alphabetical order. Greedy coloring using the corresponding construction sequence produces the 3-coloring below right. This is a minimum coloring, one better than the 4-coloring guaranteed by the Degeneracy Bound. Note that beginning the construction sequence with HGI requires four colors.



Since $D(G) \leq \Delta(G)$, an immediate corollary to the Degeneracy Bound is $\chi(G) \leq 1 + \Delta(G)$. The maximum degree bound is more famous, but is often much worse. For nontrivial trees, the Degeneracy Bound gives two—the correct value, while the maximum degree bound may be arbitrarily large. A single vertex of large degree will determine the maximum degree bound, while only a core of many large degree vertices determines the Degeneracy Bound. Several other common upper bounds for $\chi(G)$ are worse than the Degeneracy Bound, and can be proved as corollaries of it (see the Exercises). Nonetheless, the Degeneracy Bound still fails to give good results for some graphs, such as $K_{r,r}$.

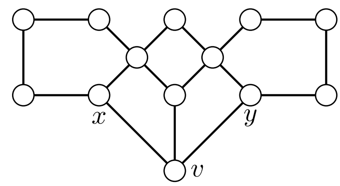
The Degeneracy Bound has been observed many times in various forms. The quantity $1 + D(G)$ has been called the **coloring number** (Erdos/Hajnal [**1966**]) and the **Szekeres-Wilf number** (Szekeres and Wilf [**1968**]). Unfortunately, the Degeneracy Bound is often presented in the confusing form $\chi(G) \leq 1 + \max_{H \subseteq G} \delta(H)$, which seems to imply that all 2^n induced subgraphs of G must be checked. In fact, only one subgraph (the maximum core) must be checked, which can be done in $O(m)$ time.

Among connected graphs, the Degeneracy Bound equals the maximum degree bound only for regular graphs (Proposition 3.31). The next theorem shows which regular graphs equal the maximum degree bound. We start with a lemma.

Lemma 4.10 (Lovasz [**1975**])**.** *Given $r \geq 3$, if G is an r-regular 2-connected noncomplete graph, then G has a vertex v with two nonadjacent neighbors x and y such that $G - x - y$ is connected.*

Proof. If G is 3-connected, let v be any vertex, and let x and y be two nonadjacent neighbors of v, which must exist since G is noncomplete.

If $\kappa(G) = 2$, let $\{u, v\}$ be any 2-vertex-cut of G. Then $\kappa(G - v) = 1$, so $G - v$ has at least two end-blocks and v has neighbors in all of them. Let x, y be two such neighbors. They must be nonadjacent, and $G - x - y$ is connected since blocks have no cut-vertices and $r \geq 3$. $\square$



Theorem 4.11 (**Brooks' Theorem**—Brooks [**1941**]). *If G is connected, then $\chi(G) = 1 + \Delta(G)$ if and only if G is complete or an odd cycle.*

Proof. ($\Leftarrow$) Equality certainly holds for cliques and odd cycles.

($\Rightarrow$) Let G satisfy the hypotheses. Then by Proposition 3.31, G is r-regular. The result certainly holds for $r \leq 2$, so we may assume $r \geq 3$. If G had a cut-vertex, each block could be colored with fewer than $r + 1$ colors to agree on that vertex, so we may assume G is 2-connected and, to the contrary, not a clique.

By the lemma, we can establish a deletion sequence for G starting with some vertex v and ending with its nonadjacent neighbors x and y so that all vertices but v have at most $r - 1$ neighbors when deleted. Reversing this yields a construction sequence, and coloring greedily gives x and y the same color, so G needs at most r colors. □

Thus the extremal graphs for $\chi(G) \leq 1 + \Delta(G)$ are complete graphs and odd cycles. For the Degeneracy Bound, the extremal graphs include these, and also trees, fans, maximal k-degenerate graphs, irregular graphs, and many more. No complete characterization of the extremal graphs for this bound is known.

There is another approach to graph coloring that is sometimes useful.

Algorithm 4.12. *Find a maximum independent set S of a graph, and color it with a single color. For $G - S$, repeat this step until all vertices are colored.*

This approach may not be optimal, as some graphs have no minimum coloring with any color class that is a maximum independent set. Finding a maximum independent set is not easy in general, so replacing "maximum" with "maximal" yields a faster algorithm. This algorithm does not translate directly into a bound, since most graphs have several maximum independent sets, and which one is chosen may change the number of colors used. Nonetheless, this algorithm may still yield decent results. One bound based on this approach follows.

Theorem 4.13 (Brigham/Dutton [**1985**]). *For any graph G,*

$$\chi(G) \leq \frac{\omega(G) + n + 1 - \alpha(G)}{2}.$$

Proof. We use induction on n. Note the result is true for empty graphs, which have $\chi = n$, $\omega = 1$, and $\alpha = n$. Thus it holds for $n = 1$. Assume the result holds for graphs with fewer than n vertices, and let G be a nonempty graph with $n > 1$. Let S be a maximum independent set of G and $H = G - S$.

If H is complete, then $\chi(G) = \omega(G) = \frac{\omega(G)+\omega(G)}{2} \leq \frac{\omega(G)+n+1-\alpha(G)}{2}$. If H is not complete, then $\alpha(H) \geq 2$, so

$$\begin{aligned}\chi(G) &\leq \chi(H) + 1 \\ &\leq \frac{\omega(H) + n - \alpha(G) + 1 - \alpha(H)}{2} + 1 \\ &\leq \frac{\omega(G) + n - \alpha(G) + 1 - 2}{2} + 1 \\ &= \frac{\omega(G) + n - \alpha(G) + 1}{2}.\end{aligned}$$ □

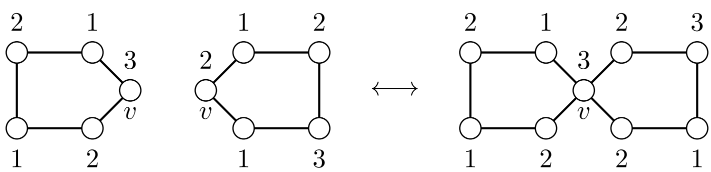
This bound is superior to the Degeneracy Bound for some classes, such as complete bipartite graphs, and it is inferior for many others. As noted before, it will not always be easy to calculate.

4.3. Coloring and Operations

Consider coloring a disjoint union of two graphs G and H. Since there are no edges between the two graphs, they can both be colored separately, and the colorings combined. Thus $\chi(G \cup H) = \max\{\chi(G), \chi(H)\}$. Similarly, the components G_i of a disconnected graph G can be colored separately, so $\chi(G) = \max \chi(G_i)$. When a noncomplete graph has connectivity one, it decomposes into multiple blocks.

Proposition 4.14. *Let G be a graph with blocks B_i. Then $\chi(G) = \max \chi(B_i)$.*

Proof. Each block can be colored separately. The colorings of separate blocks may not agree on the cut-vertices. However, the graph can be assembled by adding one block at a time and permuting the colors on the added block so that the cut-vertex it shares with the rest of the graph has the same color. □



When the join of two graphs is colored, each graph can be colored separately. No colors may be repeated between the two graphs, since the join has edges between each pair of vertices in distinct graphs.

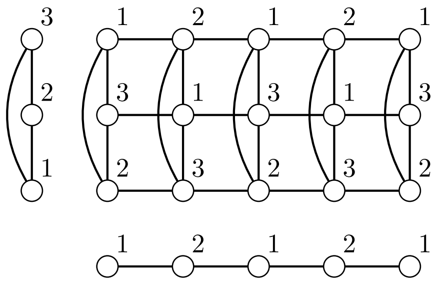
Proposition 4.15. *Let G and H be graphs. Then $\chi(G + H) = \chi(G) + \chi(H)$.*

This result is also useful when a graph can be represented as a join of two graphs. Any vertex of degree $n - 1$ in a graph must be colored differently than all other vertices.

Example. Wheels are defined as joins, so $\chi(W_n) = 3$ when n is odd, and $\chi(W_n) = 4$ when n is even.

Proposition 4.16 (Sabidussi [**1957**], Vizing [**1963**])**.** *Let G and H be graphs. Then $\chi(G \square H) = \max\{\chi(G), \chi(H)\}$.*

Proof. Since G and H are both subgraphs of $G\Box H$, certainly $\chi(G\Box H) \geq \max\{\chi(G), \chi(H)\}$. Assume $j = \chi(G) \leq \chi(H) = k$, and consider minimum colorings $g : V(G) \to [j]$ and $h : V(H) \to [k]$. Define a coloring of $G\Box H$ by $f((u_i, v_j)) = g(u_i) + h(v_j) \mod k$. Now adjacent vertices in each copy of G or H receive distinct colors since the same number is added to them, so f is a proper k-coloring of $G\Box H$. $\Box$

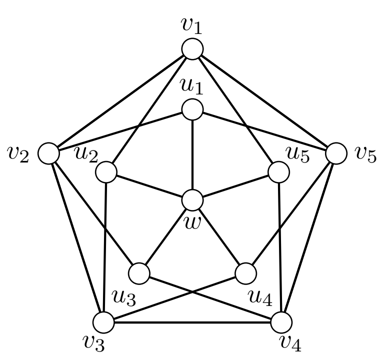


The use of arithmetic properties of numbers in this proof illustrates why numbers are used for colors.

What is it about a graph that forces it to have large chromatic number? A large clique is the most obvious answer but not the only possibility. Note that $\chi(C_5) = 3$, despite C_5 being triangle-free. Next we consider a construction that produces triangle-free graphs with arbitrarily large chromatic number.

Definition 4.17. Let G be a graph. The **Mycielskian** $M(G)$ is formed by adding a vertex u_i for each $v_i \in V(G)$ with u_i adjacent to the neighbors of v_i. Finally, a vertex w is adjacent to all u_i.

Example. Note $M(K_2) = C_5$; $M(C_5)$ is a graph known as the **Grotzch graph**. The Grotzch graph is the smallest triangle-free 4-chromatic graph.



Theorem 4.18 (Mycielski [**1955**]). *Let G be a graph with $\chi(G) = k$. Then $\chi(M(G)) = k + 1$. If G is triangle-free, so is $M(G)$.*

Proof. Let G be a graph, and let $M(G)$ be constructed as in Definition 4.17. If G is triangle-free, then no vertex u_i has adjacent neighbors, nor does w. Thus $M(G)$ is triangle-free also.

Let $\chi(G) = k$. Construct a $(k+1)$-coloring of $M(G)$ by assigning the color of v_i to u_i, which has the same neighbors. Assign color $k+1$ to w. Thus $\chi(M(G)) \leq k+1$.

Suppose to the contrary that $M(G)$ has a k-coloring. Color w with k (permute the colors if necessary). Then none of the u_i's require color k. Whatever colors they receive can be repeated on the corresponding v_i's, since they have the same neighbors. This produces a $(k-1)$-coloring of G, a contradiction. $\square$

Applying the Mycielskian repeatedly produces a sequence of triangle-free graphs with arbitrarily large chromatic numbers. Surprisingly, even longer cycles are not required to exist in k-chromatic graphs.

Theorem 4.19 (Erdos [**1959**])**.** *For any $k \geq 2$ and $g \geq 3$, there is a k-chromatic graph with girth g.*

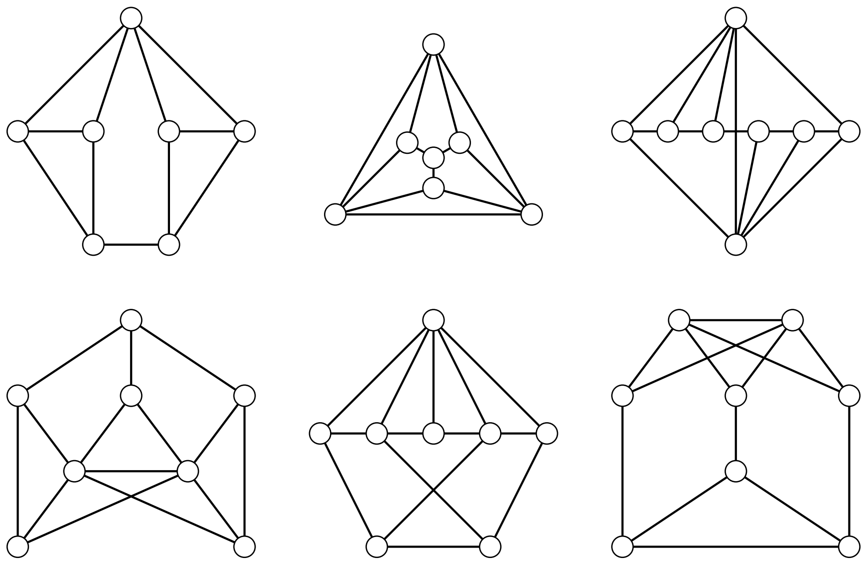
The proof of this theorem uses a probabilistic argument that does not provide explicit constructions of such graphs. Explicit constructions were later found for some larger girths.

4.4. Extremal k-chromatic Graphs

By examining extremal k-chromatic graphs, we may hope to better understand what makes a graph need k colors.

Definition 4.20. A graph is **critically k-chromatic** (**k-critical** when the context is clear) when $\chi(G) = k$ and $\chi(H) < k$ for any proper subgraph $H \subset G$. A **color-critical** graph is k-critical for some k.

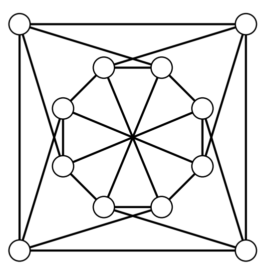
The only 1-critical graph is K_1. The only 2-critical graph is K_2. The 3-critical graphs are odd cycles, since a graph is 2-colorable if and only if it has no odd cycle. There is no known characterization of k-critical graphs for $k > 3$. Instead, we will determine some properties of these graphs. It follows from Proposition 4.14 that they must be 2-connected. In the Exercises, you are asked to show that joins and Mycielskians of color-critical graphs are color-critical. Thus wheels with even order are 4-critical. Thus the smallest nontrivial case is 4-critical graphs with no vertex of degree $n-1$. All such graphs of orders 7 and 8 are shown below (Toft [**1974**]).



Corollary 4.21. *If a graph G is k-critical, then $\delta(G) \geq k-1$.*

Proof. If G had no $(k-1)$-core, it would be $(k-1)$-colorable by the Degeneracy Bound. If G had a vertex v not in the $(k-1)$-core, $G - v$ could be $(k-1)$-colored; hence so could G. □

Note however that a critically k-chromatic graph need not have minimum degree $k-1$. For example, the **Chvatal graph** (Chvatal [**1970**]) is 4-critical and 4-regular.



It follows from Corollary 4.21 that if G has a 2-core, then $\chi(G) = \chi(C_2(G))$. This is because the rest of the graph (the 1-shell) is a forest, and requires at most two colors. Similarly, we find that if G has a 3-core which is not bipartite, then $\chi(G) = \chi(C_3(G))$. Thus the problem of optimally coloring a graph can be readily reduced to coloring its 3-core. This may lead to an improvement in the bound $\chi(G) \geq \frac{n}{\alpha(G)}$, since the vertices not in the 3-core may have lower degrees.

The following theorem strengthens Corollary 4.21.

Theorem 4.22 (Dirac [**1953**]). *If a graph G is k-critical, $k \geq 2$, then it is $(k-1)$-edge-connected.*

Proof (Chartrand/Lesniak [**2005**]). The result certainly holds when k is 2 or 3, so assume $k \geq 4$. Assume to the contrary that G is not $(k-1)$-edge-connected. Let

V_1 and V_2 partition $V(G)$ so that there are fewer than $k-1$ edges between them. Since G is k-critical, $G_1 = G[V_1]$ and $G_2 = G[V_2]$ are $(k-1)$-colorable. Color G_1 and G_2 with the same set of $k-1$ colors. Since G is k-critical, there must be some edge incident with vertices colored the same.

Let $U_1, \ldots, U_t$ be the color classes of G_1 so that for each i, $1 \le i \le t \le k-2$, there is at least one edge joining U_i and G_2. For each i, $1 \le i \le t$, there are $k_i > 0$ edges joining U_i and G_2, so $\sum k_i \le k-2$.

If some vertex of U_1 is adjacent to a vertex of G_2 with the same color, permute the colors so that the color of U_1 is replaced with one of at least $k-1-k_1 > 0$ nonconflicting colors. Repeat the same process for the color of U_i, leaving the earlier colors fixed. This produces a $(k-1)$-coloring of G, a contradiction. □

How large must the size of a k-critical graph of order n be? Certainly Corollary 4.21 implies that $m \ge \frac{n(k-1)}{2}$. But this bound is not sharp, since a k-critical graph ($k \ge 4$) is not regular unless $n = k$. The following is a better bound.

Theorem 4.23 (Gallai [**1963**]). *If G is a k-critical graph, $k \ge 3$,*

$$m \ge \frac{n}{2}\left(k-1+\frac{k-3}{k^2-3}\right).$$

The maximal k-chromatic graphs can be characterized. If we fix a k-coloring of a graph G and add all possible edges between differently colored vertices, we obtain a complete k-partite graph $K_{n_1,\ldots,n_k} = \overline{K}_{n_1} + \cdots + \overline{K}_{n_k}$. Thus the complete k-partite graphs are the maximal k-chromatic graphs. Next consider finding the maximum size of a k-chromatic graph.

Definition 4.24. The **Turan graph** $T_{n,k}$ is the complete k-partite graph of order n with all parts having cardinalities as close to equal as possible.

For example, $T_{7,3} = K_{3,2,2}$.

Lemma 4.25. *The Turan graph is the unique graph with maximum size among all complete k-partite graphs of order n.*

Proof. Consider a complete k-partite graph with partite sets that differ by more than 1 in size. Move a vertex from the largest set (size r) to the smallest set (size s). The vertex gains $r-1$ neighbors in its old set and loses s neighbors in its new class. Thus $r-1-s > 0$ edges are added, so the size increases. Thus maximizing the size requires partite sets as close in size as possible. □

Theorem 4.26 (**Turan's Theorem**—Turan [**1941**]). *Among all K_{k+1}-free graphs with order n, $T_{n,k}$ is the unique graph with maximum size.*

Proof. We show that if G is K_{k+1}-free, then there is a k-partite graph H with equal order and larger size, unless $H = G$. Use induction on k. When $k = 1$, G and H are empty. Let $k > 1$, and let G be a K_{k+1}-free graph with order n containing vertex v with $d(v) = \Delta(G) = \Delta$. Let G' be induced by the neighbors of v. Since G is K_{k+1}-free, G' is K_k-free. By induction, there is a $(k-1)$-partite graph H' with

$m(H') > m(G')$ unless $H' = G'$. Let $H = H' + (n - \Delta) K_1$. So H is k-partite, and

$$m(G) \leq m(G') + \Delta(n - \Delta) < m(H') + \Delta(n - \Delta) = m(H)$$

unless $H' = G'$ and $H = G$. By Lemma 4.25, $T_{n,k}$ is the unique graph with maximum size. □

You are asked to find the size of the Turan graph in the Exercises. Turan's Theorem is considered the foundation of extremal graph theory.

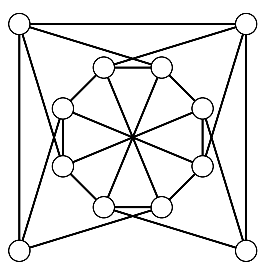
A complete k-partite graph has only one partition into k color classes. Thus it has only one way to be colored, provided that we don't care which particular colors are used.

Definition 4.27. A graph is **uniquely k-colorable** if any k-coloring produces the same vertex partition. A graph is **uniquely colorable** if any minimum coloring produces the same vertex partition.

Adding edges consistent with a minimum coloring of a graph limits the possible minimum colorings, until eventually the graph is uniquely colorable. Thus uniquely k-colorable graphs are a larger class containing maximal k-chromatic graphs.

One way to produce a uniquely colorable graph is to have many overlapping cliques. Trees are uniquely colorable, as every edge corresponds to a 2-clique. If G is uniquely k-colorable and a vertex v of degree $k-1$ is added so that it is adjacent to vertices in all but one color class, the new graph is uniquely k-colorable. Thus k-trees are uniquely colorable. Adding a vertex with degree less than $k-1$ yields more than one possible color for this vertex, so a uniquely k-colorable graph G has $\delta(G) \geq k-1$.

Surprisingly, there are also triangle-free uniquely 3-colorable graphs. The example below has one edge deleted from the Chvatal graph (Harary/Hedetniemi/Robinson [**1969**]).



The number of k-colorings of a graph (considered as partitions) can be determined using standard counting techniques. If the particular colors are important, this number can be multiplied by $k!$.

Related Terms: vertex-color-critical, chromatic polynomial.

4.5. Perfect Graphs

4.5.1. Introduction.

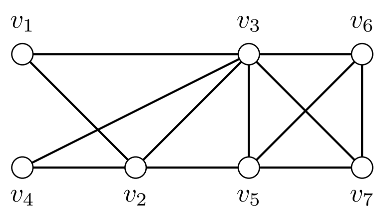
Example. A university student center has a number of identical rooms that are available for group meetings and events. They cannot be reserved ahead of time but are available when not already in use. Two natural questions are what is the minimum number of rooms needed and how should they be assigned?

Note that this problem differs from the scheduling problem in Section 4.1 because we have no control over when the meetings take place, only to which rooms they are assigned. There is a natural solution to this problem. Simply assign each meeting to the smallest-numbered room that is available when it starts. We will need k rooms if and only if k rooms are in use at once at some time.

Thus in this special situation, the chromatic number is equal to the clique number, and both can be easily determined.

Definition 4.28. An **intersection graph** has vertices representing sets and edges between sets that have nonempty intersections. An **interval graph** in an intersection graph with sets that are intervals.

Example. Suppose that meetings occur during the intervals $I_1 = [0,3]$, $I_2 = [2,6]$, $I_3 = [3,8]$, $I_4 = [4,5]$, $I_5 = [6,9]$, $I_6 = [7,10]$, $I_7 = [8,11]$. The interval graph is shown below. Its chromatic number is 4, since it contains K_4.



Any graph can be an intersection graph, but interval graphs are more restrictive. In the Exercises, you are asked to show that C_4 is not an interval graph. Note also that any induced subgraph of an interval graph can be formed by deleting some of the intervals, so it is also an interval graph.

Proposition 4.29. *If G is an interval graph, then $\chi(G) = \omega(G)$.*

Proof. Use a greedy coloring, and order the vertices by the left endpoint of the interval. For each vertex, a new color is only needed when all the existing colors are in use on intervals containing the left end of the next interval. Thus G can be colored with $\omega(G)$ colors. □

The properties shown above for interval graphs hold for a larger class of graphs.

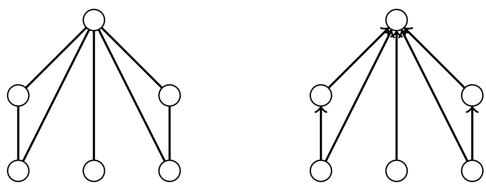
Definition 4.30. A graph G is **perfect** if for every induced subgraph H, $\chi(H) = \omega(H)$. A graph class $\mathbb{G}$ is **hereditary** if every induced subgraph of a graph in $\mathbb{G}$ is also in $\mathbb{G}$.

Perfect graphs can be viewed as the extremal graphs for the bound $\chi(G) \geq \omega(G)$. The condition on induced subgraphs is needed to avoid trivial examples.

Nonempty bipartite graphs have $\chi(G) = \omega(G) = 2$, and bipartite graphs are hereditary. Thus bipartite graphs are perfect. Bipartite graphs can have all their edges oriented from one set to the other, so they are contained in the following graph class.

Definition 4.31. A **transitive orientation** of a graph G is an orientation D such that when uv and vw are edges in D, G contains edge uw, oriented that way in D. A **comparability graph** is one with a transitive orientation.

Example. The comparability graph below left has the transitive orientation below right.



Proposition 4.32 (Berge [**1960**]). *Comparability graphs are perfect.*

Proof. Every induced subdigraph of a transitive digraph is transitive. Let D be a transitive orientation of a comparability graph G. Certainly D has no cycle. Color G by assigning each vertex v the number of vertices in the longest path of D ending at v in a proper coloring. If $uv \in E(D)$, then any path ending at u could be extended to v, so they must have different colors. By transitivity, the vertices of a path in D form a clique in G, so G can be colored with $\omega(G)$ colors. □

4.5.2. Chordal Graphs. What graphs are not perfect? Any odd cycle (except K_3) has $\omega(C_{2k+1}) = 2$ and $\chi(C_{2k+1}) = 3$, and so is not perfect. It is easily checked that any complement of an odd cycle (except K_3) is also not perfect. Thus any graph containing an odd cycle or its complement (except K_3) as an induced subgraph is not perfect. Thus when a perfect graph contains an odd cycle (except K_3), it must also have an edge not on the cycle joining two vertices of the cycle.

Definition 4.33. A **chord** of a cycle is an edge not on the cycle joining two vertices of the cycle. A graph is **chordal** if any cycle (except K_3) has a chord. A **simplicial vertex** is a vertex whose neighbors induce a clique.

Equivalently, a chordal graph has no induced cycle (except K_3). Any induced subgraph of a chordal graph is chordal.

Trees are (vacuously) chordal. They have the property that any minimal cutset is a single vertex. This observation can be generalized to characterize chordal graphs.

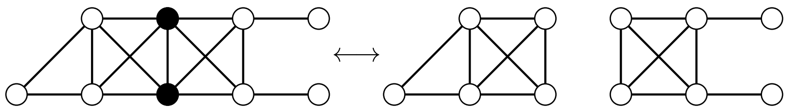
Lemma 4.34. *Any minimal cutset of a chordal graph induces a clique.*

Proof. Let S be a minimal cutset of a chordal graph G, and let X and Y be components of $G - S$. If $|S| = 1$, the result is obvious. Assume $|S| \geq 2$, and let $u, v \in S$. Then there are $u - v$ paths of minimum length through X and Y. Combining them forms a cycle whose only chord can be uv. Thus every possible edge in $G[S]$ is present, so it is a clique. □

Theorem 4.35 (Hajnal/Suranyi [**1958**], Dirac [**1961**]). *A graph is chordal if and only if it is complete or can be formed by identifying cliques of two smaller chordal graphs.*

Proof. ($\Leftarrow$) Complete graphs are chordal, and the operation cannot create a chordless cycle, so any graph formed this way is chordal.

($\Rightarrow$) Let G be chordal and not complete, and let S be a minimal cutset of G, which induces a clique. Let G_1 be an S-lobe of G, and let G_2 be the union of all other S-lobes. Both graphs are smaller than G and are chordal since they are induced subgraphs. Thus G can be formed by identifying the vertices of S in G_1 and G_2. □



Lemma 4.36. *Every chordal graph is either complete or has at least two nonadjacent simplicial vertices.*

Proof. We use induction on order n. The result is obvious for $n = 1$. Assume the result is true for graphs with fewer than n vertices. Let G be chordal with order n, let S be a minimal cutset, and let G_1 and G_2 be smaller chordal graphs that overlap on S. Then they are either complete or have at least two nonadjacent simplicial vertices. In a complete graph, every vertex is simplicial. Now $G[S]$ cannot contain all of G_1 (if it is complete) and cannot contain nonadjacent vertices, so at least one simplicial vertex of G_1 is also simplicial in G. Thus G has at least two nonadjacent simplicial vertices. □

Simplicial vertices with degree 1 are leaves. The previous lemma is analogous to the fact that nontrivial trees have at least two leaves. Along with maximal k-degenerate graphs and k-trees, connected chordal graphs are another generalization of trees. All k-trees are chordal. In the Exercises, you are asked to show that a graph is a k-tree if and only if it is maximal k-degenerate and chordal with $n \geq k+1$.

The simplicial vertices of a chordal graph can be successively deleted until only a single vertex remains. Conversely, a chordal graph can be constructed starting with K_1 and then adding simplicial vertices. This makes it easy to find minimum colorings of chordal graphs and show that they are perfect.

Theorem 4.37 (Voloshin [**1982**]). *A graph G is chordal if and only if $\omega(H) = 1 + D(H)$ for all induced subgraphs H in G.*

Proof. ($\Rightarrow$) We use induction on order n. The result holds when $n = 1$. Assume it holds for graphs with order less than n, and let G be chordal with order n. Let v be a simplicial vertex, and let $H = G - v$. Then H has $\omega(H) = 1 + D(H)$. Now v's neighborhood is a clique, so $d(v) \leq \omega(H)$. Thus, adding v to H can leave both ω and D unchanged or increase them both by one, so $\omega(G) = 1 + D(G)$.

($\Leftarrow$) (contrapositive) Let G be not chordal. Then G contains a cycle C_n, $n \geq 4$, as an induced subgraph. Then $\omega(C_n) = 2 < 3 = 1 + D(C_n)$. □

Corollary 4.38. *Chordal graphs are perfect, and any construction sequence produces a minimum coloring.*

Proof. For any graph G, $\omega(G) \leq \chi(G) \leq 1 + D(G)$, and coloring G with a construction sequence uses at most $1 + D(G)$ colors. By the previous theorem, these are all equalities. $\square$

Other vertex orders for chordal graphs also produce minimum colorings. Many classes of perfect graphs are related. In the exercises, you are asked to show that interval graphs are chordal, and their complements are comparability graphs.

4.5.3. The Perfect Graph Theorem. Early in the study of perfect graphs, Claude Berge noticed that many classes of perfect graphs also had perfect complements. In 1961, he conjectured that a graph is perfect if and only if its complement is perfect. This conjecture was proved in 1972 by Lazlo Lovasz. He actually proved a somewhat stronger result, which follows. The proof uses some ideas from linear algebra.

Theorem 4.39 (Lovasz [**1972B**]). *A graph G is perfect if and only if for every induced subgraph H, $\alpha(H)\,\omega(H) \geq n$.*

Proof. ($\Rightarrow$) Assume that G is perfect. Then, for every induced subgraph H, $\chi(H) = \omega(H)$. Then $\alpha(H)\,\omega(H) = \alpha(H)\,\chi(H) \geq \alpha(H)\,\frac{n}{\alpha(H)} = n$.

($\Leftarrow$) (Gasparian [**1996**]) (contrapositive) Assume that G is not perfect. Let H be a minimally imperfect subgraph of G, and let $n = n(H)$, $\alpha = \alpha(H)$ and $\omega = \omega(H)$. Then H satisfies $\omega = \chi(H - v)$ for every vertex $v \in V(H)$ and $\omega = \omega(H - S)$ for every independent set $S \subseteq V(H)$.

Let A_0 be a maximum independent set of H. Fix an ω-coloring of each of the α graphs $H - s$ for $s \in A_0$, let $A_1, \ldots, A_{\alpha\omega}$ be the independent sets occurring as a color class in one of these colorings and let $\mathbb{A} = \{A_0, A_1, \ldots, A_{\alpha\omega}\}$. Let A be the corresponding independent set versus a vertex incidence matrix. Define $\mathbb{B} = \{B_0, B_1, \ldots, B_{\alpha\omega}\}$ where B_i is an ω-clique of $H - A_i$. Let B be the corresponding clique versus a vertex incidence matrix.

Let $S_1, \ldots, S_\omega$ be any ω-coloring of $H - v$. Since any ω-clique C of H has at most one vertex in each S_i, C intersects all S_i's if $v \notin C$ and all but one if $v \in C$. Since C has at most one vertex in A_0, every ω-clique of H intersects all but one of the independent sets in $\mathbb{A}$.

In particular, it follows that $AB^T = J - I$. (J is a matrix of all 1's, and I is an identity matrix.) Now $\operatorname{rank}(J - I) \leq \min\left\{\operatorname{rank}(A), \operatorname{rank}\left(B^T\right)\right\}$. Since $J - I$ is nonsingular, A and B have at least as many columns as rows, so $n \geq \alpha\omega + 1$. $\square$

Corollary 4.40 (**Perfect Graph Theorem**—Lovasz [**1972A, 1972B**]). *A graph G is perfect if and only if $\overline{G}$ is perfect.*

Proof. Let G be perfect, and let H be any induced subgraph. Since $\alpha(H) = \omega\left(\overline{H}\right)$ and $\omega(H) = \alpha\left(\overline{H}\right)$, $\alpha\left(\overline{H}\right)\omega\left(\overline{H}\right) = \alpha(H)\,\omega(H) \geq n$. Thus $\overline{G}$ is perfect. $\square$

In addition to the conjecture that led to the Perfect Graph Theorem, in 1961 Claude Berge made a stronger conjecture (Berge [**1961**]). It eventually became the following theorem.

Theorem 4.41 (**Strong Perfect Graph Theorem**—Chudnovsky/Robertson et al. [**2006**]). *A graph is perfect if and only if it contains no induced C_{2k+1} or $\overline{C_{2k+1}}$, $k > 1$.*

This conjecture was the topic of much research over the next 50 years but only partial results were proved. Finally, Neal Robertson, Paul Seymour, Robin Thomas, and Maria Chudnovsky attacked the problem from January 2000 to May 2002, when they found a solution. Their work ran to about 150 pages.

Their general approach was to show that all graphs with no induced C_{2k+1} or $\overline{C_{2k+1}}$, $k > 1$, belong to few basic classes or have certain features corresponding to decompositions. Those classes (bipartite graphs and their complements, line graphs of bipartite graphs and their complements, and **double split graphs**) are easily shown to be perfect (see Theorem 7.18). They showed that no graph with the specified features could be a minimum counterexample, proving the theorem (Seymour [**2006**]). Their approach also resulted in a polynomial time algorithm to determine whether a graph is perfect (Chudnovsky/Cornuejols et al. [**2005**]).

Related Terms: stable set, duplicating a vertex, vertex multiplication, intersection representation, subtree representation, clique tree, *o*-triangulated graph, parity graph, Meyniel graph, weakly chordal graph, strongly chordal graph, strongly perfect graph, perfect order, odd hole, odd antihole, Berge graph, circular-arc graph, circle graph, cograph, split graph, asteroidal triple, skew partition, perfectly orderable graph, threshold graph.

Exercises

Section 4.1:

(1) Several math students will take summer math classes. The students' schedules are listed below left. Draw a graph representing this situation, and find the smallest number of slots that they can be scheduled in.

Student	Classes
Al	A, C, D
Bob	E, F
Carl	A, B
Dave	B, D, E
Edna	E, F

Student	Classes
Alice	C, D, F
Bee	E, H
Cindy	A, B, E
Doris	D, E, G
Ed	A, D, G

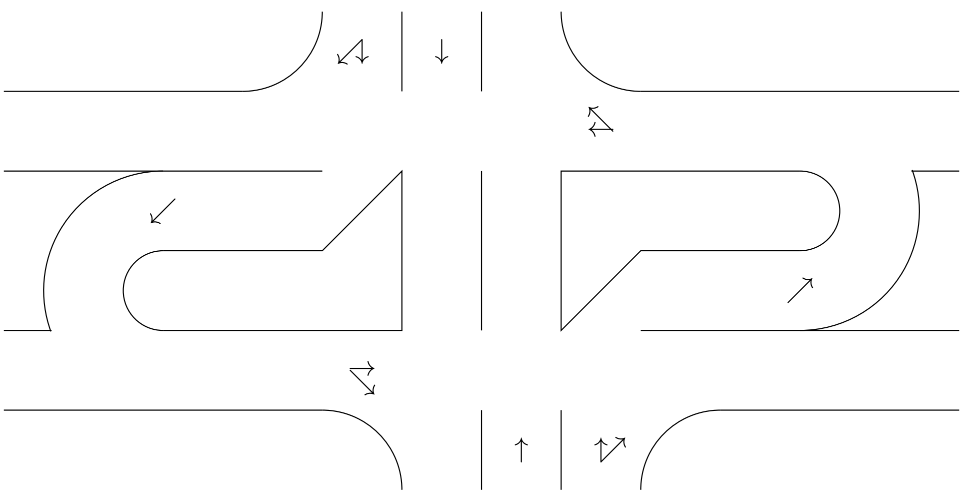
(2) Several math students will take summer math classes. The students' schedules are listed above right. Draw a graph representing this situation, and find the smallest number of slots that they can be scheduled in.

(3) Eight businessmen must have five meetings. The sets of people in meetings are $m_1 = \{A, B, C, D\}$, $m_2 = \{A, E, F\}$, $m_3 = \{B, E, G\}$, $m_4 = \{F, G, H\}$, and $m_5 = \{C, D, G, H\}$. Draw a graph representing this situation, and find the smallest number of slots that they can be scheduled in.

(4) Seven businessmen must have five meetings. The sets of people in meetings are $m_1 = \{C, E, F\}$, $m_2 = \{A, B, D\}$, $m_3 = \{D, F, G\}$, $m_4 = \{A, C, D\}$, and $m_5 = \{B, G\}$. Draw a graph representing this situation, and find the smallest number of slots that they can be scheduled in.

(5) A road ends at another road, producing a three-way intersection with a traffic light. Both roads have left-turn lanes, so there are five lanes entering the intersection. Draw a graph representing this situation and find the smallest number of traffic cycles the light can use. Describe the possibilities for each cycle.

(6) A **Michigan left** is an intersection where left turns on a divided road are performed by driving straight through the intersection, then making a U-turn to the opposite direction, followed by a right turn. Draw a graph representing this situation and find the smallest number of traffic cycles the light can use. What are the advantages and disadvantages of such an intersection compared to a standard 4-way intersection?



(7) An aquarium has a number of tanks displaying tropical fish. Some species of fish will attack other species if they are in the same tank. The conflicts are described in the table below left, where an X indicates that the row species will attack the column species. Draw a graph representing this situation, and find the smallest number of tanks that the fish can be held in.

	A	B	C	D	E	F	G
A				X			
B			X	X			
C						X	
D					X	X	X
E						X	
F		X					
G						X	

	A	B	C	D	E	F	G
A			X		X		
B	X						
C				X	X		X
D	X				X		
E						X	X
F				X			
G		X					

(8) A zoo has a number of environments displaying various species. Some species will attack other species if they are together. The conflicts are described in the table above right, where an X indicates that the row species will attack the column species. Draw a graph representing this situation, and find the smallest number of environments that the animals can be held in.

(9) A daycare finds that some children fight when they are together. It assigns them to separate play rooms to keep the peace. The table below left describes the conflicts. Draw a graph representing this situation, and find the smallest number of rooms that the children require.

	A	B	C	D	E	F	G
A		X	X				
B	X			X	X		
C	X					X	
D		X				X	
E		X					X
F			X	X			X
G					X	X	

	A	B	C	D	E	F	G
A		X			X		X
B	X		X	X		X	
C		X			X		X
D		X			X		X
E	X		X	X		X	
F		X			X		X
G	X		X	X		X	

(10) A prison finds that some prisoners fight when they are together. It assigns them to separate areas to keep the peace. The table above right describes the conflicts. Draw a graph representing this situation, and find the smallest number of areas that the prisoners require.

(11) Some volatile chemicals react when mixed together, so it is too dangerous to ship them in the same package. The table below lists several chemicals, with an X for pairs that react with each other. Draw a graph representing this situation and find the smallest number of packages that the chemicals can be shipped in.

	A	B	C	D	E	F	G
A		X	X	X			
B	X				X		
C	X			X		X	
D	X		X		X	X	X
E		X		X			X
F			X	X			X
G				X	X	X	

	Am	Au	CC	Da	Ho	SA	Wa
Amarillo		489	654	366	600	512	430
Austin			193	194	163	78	100
Corpus Christie				388	213	143	294
Dallas					238	272	96
Houston						195	188
San Antonio							178
Waco							

(12) The table above lists distances between several cities. Suppose that television stations must be at least 200 miles apart to share the same channel. Determine

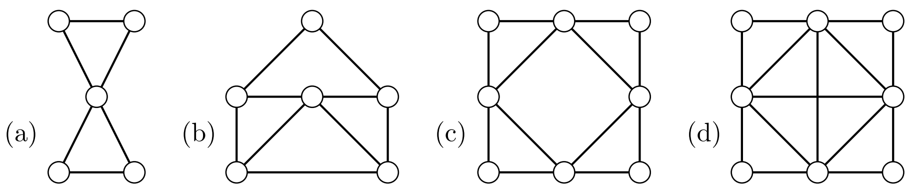
the minimum number of channels that must be used for one station in each city.

Section 4.2:

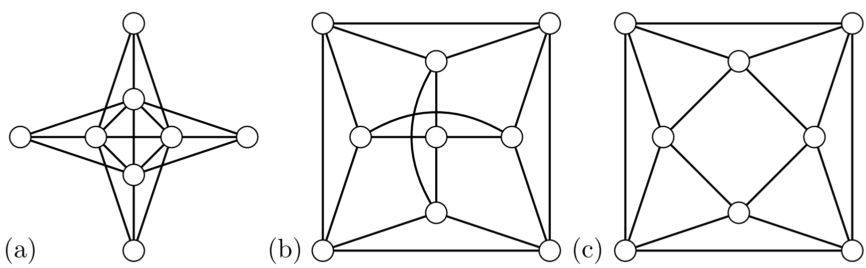
(1) Determine the clique number, independence number, and chromatic number of the graphs in the following classes.
   (a) Q_r
   (b) double wheels
   (c) triangular grids
   (d) irregular graphs I_n

(2) Determine the clique number, independence number, and chromatic number of the graphs in the following classes.
   (a) $G_{r,s}$
   (b) fans
   (c) theta graphs $\theta_{i,j,k}$
   (d) Mobius ladders M_n

(3) Determine the clique number, independence number, and chromatic number of the following graphs.



(4) Determine the clique number, independence number, and chromatic number of the following graphs.



(5) Determine the clique number, independence number, and chromatic number of the six cubic graphs of order 8.

(6) Recall that the Kneser graph $KG_{r,k}$ has vertices representing the k-element subsets of $[r]$ and edges between disjoint subsets.
   (a) Show that any independent set of $KG_{r,2}$ of size more than 3 can be extended to a maximum independent set. Determine $\alpha(KG_{r,2})$.
   (b) Show that $\chi(KG_{r,2}) = r-2$. (*Note*: Lovasz [**1978**] showed that $\chi(KG_{r,k}) = r - 2k + 2$ using topology.)

(7) Recall that the generalized Petersen graph $P(r, k)$, $1 \leq \frac{r}{2} < k$, has vertices $u_1, \ldots, u_r$ and $v_1, \ldots, v_r$ and edges $u_i u_{i+1}$, $u_i v_i$, and $v_i v_{i+k}$ (addition mod r).

Determine the clique number, independence number, and chromatic number of $P(r,k)$.

(8) Determine the clique number, independence number, and chromatic number of the Harary graph $H_{k,n}$.

(9) Characterize all graphs G with $\chi(G) = n-1$.

(10) Characterize all graphs G with $\chi(G) = n-2$.

(11) If $\chi(G) = \frac{n}{\alpha(G)}$, what does this imply about colorings of G?

(12) Find a graph G of smallest order for which both basic lower bounds on $\chi(G)$ (ω and $\frac{n}{\alpha}$) fail to be exact.

(13) Given a minimum coloring of a graph, show that for each color i, there is some vertex colored i that is adjacent to vertices with all other colors.

(14) Prove or disprove: The independence number of a bipartite graph is the size of its largest partite set.

(15) Show that for any graph G, $\frac{n}{1+\delta(G)} \leq \alpha(G) \leq n - \delta(G)$.

(16) Show that for an r-regular graph G, $r \geq 1$, $\frac{n}{1+r} \leq \alpha(G) \leq \frac{n}{2}$. Characterize the extremal graphs for both bounds.

(17) Show that for any graph, there is a vertex order for which greedy coloring produces a minimum coloring.

(18) Find a sequence of trees to show that greedy coloring can require an arbitrarily large number of colors on them.

(19) For the following graphs, determine what proportion of their vertex orders produces a minimum coloring using greedy coloring.
(a) $K_{3,3}$
(b) C_6
(c) the triangular grid T_2

(20) For the following graphs, determine what proportion of their vertex orders produces a minimum coloring using greedy coloring.
(a) C_5
(b) P_5
(c) Q_3

(21) Find a sequence of trees T_i to show that Algorithm 4.12 may be forced to produce colorings using an arbitrarily large number of colors. (*Hint*: $T_1 = K_2$, T_2 has six vertices.)

(22) Find a sequence of trees T_i to show that Algorithm 4.12 may produce either a minimum coloring or a colorings using an arbitrarily large number of colors, depending on the choice of a maximum independent set. (*Hint*: $T_1 = K_2$, T_2 has five vertices.)

(23) (a) Show that $\chi(G) \leq n + 1 - \alpha(G)$.
(b) Determine the extremal graphs for the bound in part (a).
(c) Show that $1 + D(G) \leq n + 1 - \alpha(G)$. (Thus the bound in part (a) is no better than the Degeneracy Bound.)

(24) (a) Let $l(G)$ be the length of the longest path of G. Use Theorem 2.7 to show that $D(G) \leq l(G)$, and hence $\chi(G) \leq 1 + l(G)$. (Thus the bound is no better than the Degeneracy Bound.)
(b) Determine the extremal graphs for the bound $D(G) \leq l(G)$.

(25) Show that if a graph G has $\chi(G) = k$, then G has an orientation whose longest (directed) path has length $k - 1$.

(26) (a) (Welsh/Powell [**1967**]) Let G be a graph with degree sequence $d_1 \geq \cdots \geq d_n$. Show that $\chi(G) \leq 1 + \max_i \min\{d_i, i-1\}$.
(b) Show that the bound in part (a) is never superior to the Degeneracy Bound.

(27) Let l be the length of the longest odd cycle in a nonbipartite graph G.
(a) Show that l may be less than $D(G)$.
(b) + (Erdos/Hajnal [**1966**]) Show that if G is 2-connected, $D(G) \leq l$. (*Hint*: Consider the end of a path of maximum length in the $D(G)$-core. This implies that $\chi(G) \leq 1 + l$, and the Degeneracy Bound is superior to this bound.)
(c) + (Kenkre/Vishwanathan [**2007**]) Show that $D(G) = l$ if and only if $G = K_{l+1}$, l odd.

(28) Let G be a graph in which every two odd cycles share a vertex. Show that $\chi(G) \leq 5$.

(29) (a) (Reed [**1998**]) Use induction on n (not Theorem 4.13) to show that $\chi(G) \leq \frac{n+\omega(G)}{2}$.
(b) Show that the bound in part (a) is never superior to the Degeneracy Bound for bipartite graphs.
(c) Find a graph for which the bound in part (a) is superior to the Degeneracy Bound.

(30) (Schiermeyer [**2007**]) Show that any graph G with $\alpha(G) + \omega(G) = n + 1$, or $C_5 + K_{\omega-2} \subseteq G$ and $C_5 + K_{n-\omega-3} \subseteq \overline{G}$ is an extremal graph for Theorem 4.13.

(31) **Reed's Conjecture** (Reed [**1998**]) is that $\chi(G) \leq \left\lceil \frac{\omega(G)+1+\Delta(G)}{2} \right\rceil$. Find an infinite class of graphs other than cliques and odd cycles that make this an equality.

(32) + (Rabern [**2013**]) An alternate proof of Brooks' Theorem.
(a) Show that if a cubic graph contains $K_4 - e$, it is 3-colorable.
(b) Let G be a cubic graph containing a cycle C with $x, y \in N(C)$. Let $H = G - C$ if $x \leftrightarrow y$ or $H = G - C + xy$ if $x \not\leftrightarrow y$. Show that a 3-coloring of H can be extended to a 3-coloring of G.
(c) Let $\Delta = \Delta(G) \geq 4$, and let G be a minimum counterexample. Show that some color must be used on every K_Δ of $G - v$ for some vertex v. Use this to derive a contradiction and complete the proof of Brooks' Theorem.

Section 4.3:

(1) State and prove results analogous to Proposition 4.15 for ω and α for a disjoint union and the join.

(2) State and prove a result analogous to Proposition 4.14 for ω over the blocks of a graph.

(3) Let G be a graph with blocks $B_1, \ldots, B_k$. Show that $\sum \alpha(B_i) - k + 1 \leq \alpha(G) \leq \sum \alpha(B_i)$ and that both bounds are sharp.

(4) (Berge [**1973**]) Show that a graph G is k-colorable if and only if $\alpha(G \square K_k) \geq n$.

(5) Show that for a graph G containing edge e, $\chi(G) - 1 \leq \chi(G - e) \leq \chi(G)$.

(6) Show that for a graph G containing vertex v, $\chi(G) - 1 \leq \chi(G - v) \leq \chi(G)$.

(7) Let H be a graph formed by subdividing a single edge of a nonempty graph G. State and prove a result relating $\chi(G)$ and $\chi(H)$.

(8) Let H be a graph that is a subdivision of a nonempty graph G. State and prove a result relating $\chi(G)$ and $\chi(H)$.

(9) (a) Let G and H be graphs. Show that for the tensor product, $\chi(G \times H) \leq \min\{\chi(G), \chi(H)\}$.
    (b) Show that when $1 \leq \min\{\chi(G), \chi(H)\} \leq 3$, the bound in part (a) is an equality. (*Note*: Hedetniemi [**1966**] conjectured that it is always an equality.)

(10) Show that for a graph G, $\chi(G) \leq \chi\left(G^2\right) \leq n$, and both bounds are sharp.

(11) If G has r triangles, how many does $M(G)$ have?

(12) If G has r 4-cycles, how many does $M(G)$ have?

(13) Let $G_0 = K_2$ and $G_{k+1} = M(G_k)$. Find the order of G_k.

(14) Let $G_0 = K_2$ and $G_{k+1} = M(G_k)$. Find the size of G_k.

Section 4.4:

(1) Find the minimum size of a k-chromatic graph with order n.

(2) Find the minimum size of a connected k-chromatic graph with order n.

(3) Show that $G + H$ is k-critical if and only if G and H are k-critical.

(4) Show that if G is k-critical, then $M(G)$ is k-critical.

(5) Show that if $\chi(G - u - v) = \chi(G) - 2$ for all vertices u and v of G, then G is complete.

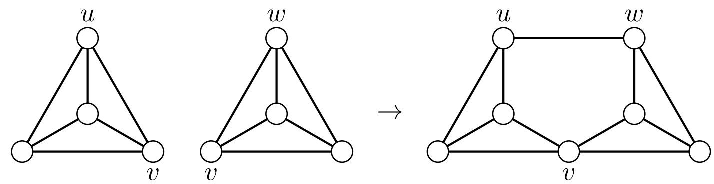
(6) (Niessen/Kind [**2000**]) Let G be a claw-free graph. Show that the subgraph induced by the union of any two color classes in a coloring of G consists of paths and even cycles. Use this to show that if G has a k-coloring, it has a k-coloring where the color classes differ in size by at most 1.

(7) Show that there is no k-critical graph with order $k + 1$.

(8) Show that a k-critical graph G cannot have a cutset containing two adjacent vertices.

(9) (a) (Hajos [**1961**]) Let G and H be k-critical graphs sharing only v with $uv \in E(G)$ and $vw \in E(H)$. The **Hajos construction** is $G \cup H - uv - vw + uw$. Show that it is k-critical. (*Note*: The Moser spindle is produced when $G = H = K_4$.)

(b) For all $n \geq 6$, construct a 4-critical graph with order n.



(10) Construct an infinite class of k-critical graphs with connectivity 2.

(11) Find the maximum size of P_3-free graphs and characterize the extremal graphs.

(12) Find the maximum size of $2K_2$-free graphs and characterize the extremal graphs.

(13) Find the maximum size of P_4-free graphs and characterize the extremal graphs.

(14) Find the maximum size of $K_{1,3}$-free graphs and characterize the extremal graphs.

(15) The Turan graph $T_{n,k}$ has b partite sets of size $a+1$ and has $k-b$ sets of size a, where $a = \left\lfloor \frac{n}{k} \right\rfloor$ and $b = n - ka$.
(a) Show that $m(T_{n,k}) = \left(1 - \frac{1}{k}\right)\frac{n^2}{2} - \frac{b(k-b)}{2k}$.
(b) Part (a) implies $m(T_{n,k}) \leq \left\lfloor \left(1 - \frac{1}{k}\right)\frac{n^2}{2} \right\rfloor$. Determine when this is a strict inequality.

(16) Find the size of the Turan graph $T_{n,k}$ by considering its complement.

(17) Characterize uniquely 2-colorable graphs.

(18) Determine which graphs in the following classes are uniquely colorable.
(a) K_n
(b) C_n
(c) W_n

(19) Show that if G is k-critical and uniquely k-colorable, then $G = K_k$.

(20) Prove or disprove: If G is $(k+1)$-critical, then $G - e$ is uniquely k-colorable for any edge e.

(21) Show that G and H are uniquely colorable graphs if and only if $G + H$ is uniquely colorable.

(22) Prove or disprove: A maximal k-degenerate graph is uniquely colorable if and only if it is a k-tree.

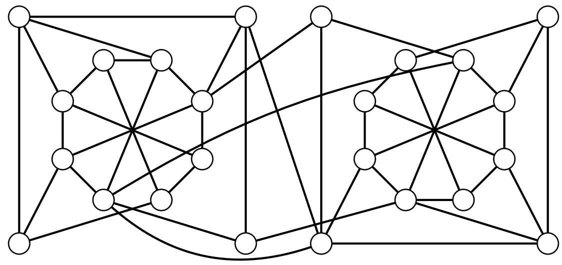
(23) Let G be a uniquely k-colorable graph with $d(v) = k-1$ for some vertex v. Show that $G - v$ is uniquely colorable.

(24) (a) (Harary et al. [**1969**]) Show that in a uniquely colorable graph, any two color classes induce a connected graph.
(b) Show that a uniquely k-colorable graph G has $\kappa(G) \geq k-1$.
(c) (Shaoji [**1990**]) Show that a uniquely k-colorable graph has

$$m \geq (k-1)\,n - \frac{k\,(k-1)}{2}.$$

(25) Verify that the Chvatal graph is 4-critical and that the graph formed by deleting one edge from the outer 4-cycle is uniquely 3-colorable.

(26) + (Akbari/Mirrokni/Sadjad [**2001**]) Show that the following graph is uniquely 3-colorable, triangle-free, and has $m = 2n - 3$.



(27) Find the number of k-colorings of a tree with order n.

(28) Find the number of k-colorings of a cycle with order n.

(29) (a) Find the number of maximum independent sets of the Petersen graph.
(b) Find the number of distinct 3-colorings of the Petersen graph.

(30) Find the number of independent sets of size k in P_n.

Section 4.5:

(1) A computer stores variables in memory locations called registers. When two values are never used at the same time, they can be stored in the same register. Explain how to model this situation using the ideas of this section.

(2) A hotel takes reservations for a number of identical rooms. When two people have overlapping stays, they cannot use the same room. Explain how to model this situation using the ideas of this section.

(3) Find the intersection graph of the sets of prime numbers, squares, cubes, even numbers, Fibonacci numbers, and multiples of 6. Label each edge with a number the two sets have in common.

(4) Show that a graph G with m edges is the intersection graph of a family of sets with at most m total elements.

(5) Find the interval graph for the intervals $[0, 3]$, $[1, 4]$, $[2, 6]$, $[5, 8]$, $[6, 9]$, $[8, 9]$, and determine its chromatic number.

(6) Find the interval graph for the intervals $[3, 5]$, $[1, 3]$, $[6, 9]$, $[0, 2]$, $[4, 7]$, $[2, 4]$, $[3, 6]$, $[7, 8]$, and determine its chromatic number.

(7) Determine which graphs in the following classes are interval graphs.
(a) P_n
(b) $K_{1,n-1}$
(c) C_n

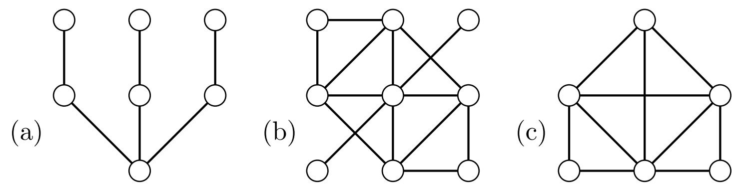
(8) Determine which graphs in the following classes are chordal graphs.
(a) W_n
(b) bipartite graphs
(c) triangular grids

(9) Determine which graphs in the following classes are perfect graphs.
(a) theta graphs
(b) complete multipartite graphs

(c) cactuses

(10) Determine which graphs in the following classes are comparability graphs.
(a) C_n
(b) W_n
(c) complete multipartite graphs

(11) Determine whether the graphs below are interval graphs.



(12) Determine whether the graphs above are chordal graphs.

(13) Determine ω and χ for the complements of the following graphs.
(a) C_{2k+1}, $k > 1$
(b) the Petersen graph
(c) Q_r

(14) Determine ω and χ for the complements of the following graphs.
(a) P_n
(b) Mobius ladders
(c) $G_{r,s}$

(15) Show that every interval graph is chordal.

(16) Show that if G is an interval graph, then $\overline{G}$ is a comparability graph.

(17) Show that a tree is an interval graph if and only if it is a caterpillar. (*Hint*: Use a forbidden subgraph.)

(18) + (Lakkerkerker/Boland [**1962**]) Three vertices form an **asteroidal triple** in a graph G if, for each two, there exists a path containing those two but no neighbor of the third. Show that a graph is an interval graph if and only if it is chordal and has no asteroidal triple.

(19) (Dirac [**1961**]) A **simplicial elimination ordering** is formed by successively deleting a simplicial vertex of a graph until none remain. Show that a graph has a simplicial elimination ordering if and only if it is chordal.

(20) Show that coloring a chordal graph using the reverse of a simplicial elimination ordering produces a minimum coloring.

(21) Show that a simplicial elimination ordering of a chordal graph need not be a deletion sequence, and vice versa.

(22) Characterize chordal graphs with exactly two simplicial vertices.

(23) Show that the graph induced by the simplicial vertices of a chordal graph is a disjoint union of cliques.

(24) (Fulkerson/Gross [**1965**]) Show that a chordal graph G has at most n maximal cliques, with equality only for empty graphs.

(25) Prove or disprove: A chordal graph is uniquely colorable if and only if it is a k-tree.

(26) Let G be a chordal graph not containing K_{k+2}. Show that $m \leq k \cdot n - \binom{k+1}{2}$ and that the extremal graphs are the k-trees.

(27) (Bickle [**2012**]) Show that a graph G is a k-tree if and only if G is maximal k-degenerate and G is chordal with $n \geq k+1$.

(28) + (Zeng/Yin [**2015**]) Show that a graph with $n \geq k+1$ is a k-tree if and only if it contains no subdivision of K_{k+2}, it is chordal, and it is k-connected.

(29) Prove or disprove: The complement of a chordal graph is chordal.

(30) Characterize graphs that are both bipartite and chordal.

(31) (a) Show that if $T_1, \ldots, T_k$ are pairwise intersecting subtrees of a tree T, then there is a vertex belonging to all of them. (*Hint*: Prove the contrapositive, and mark an edge for each vertex in T.)
(b) + Show that a graph is chordal if and only if it is the intersection graph of a family of subtrees of some tree.

(32) (Habib/Stacho [**2012**]) Let G be a connected chordal graph. Two maximal cliques C and C' of G form a **separating pair** if $C \cap C'$ is nonempty, and every path in G from a vertex of $C - C'$ to a vertex of $C' - C$ contains a vertex of $C \cap C'$.
(a) + Show that a set S is a minimal cutset of G if and only if G has a separating pair C and C' such that $S = C \cap C'$.
(b) Form a graph whose vertices represent maximal cliques of G with edges between those that form separating pairs. Show that this graph is a tree (a **clique tree**).

(33) A **split graph** is a graph whose vertex set can be partitioned into a clique and an independent set. (The partition may not be unique.)
(a) Show that the complement of a split graph is a split graph.
(b) Find a graph of smallest order that is not a split graph.
(c) Characterize the split graphs that are trees.
(d) Show that a split graph is chordal.

(34) + (Hammer/Simeone [**1981**]) Properties of split graphs.
(a) Show that G is a split graph if and only if G has no induced $2K_2$, C_4, or C_5.
(b) Let the degree sequence of a graph G be $d_1 \geq \cdots \geq d_n$, and let m be the largest value of i such that $d_i \geq i-1$. Show that G is a split graph if and only if $\sum_{i=1}^{m} d_i = m(m-1) + \sum_{i=m+1}^{n} d_i$.

(35) (Seinsche [**1974**]) A **cograph** (**complement reducible graph**) can be constructed from a single vertex using the operations of complementation and disjoint union.
(a) Show that a graph is a cograph if and only if it is P_4-free.
(b) Show that cographs are perfect.

(36) + (Christen/Selkow [**1979**]) A **perfect order** is a vertex order such that greedy coloring produces a minimum coloring on every subgraph. Show that a graph is a cograph if and only if every vertex order is a perfect order.

(37) Find a graph G of smallest order such that $\chi(G) = \omega(G)$ and G is not perfect.

(38) Let G be a minimal imperfect graph.

(a) Show that $n = \alpha(G)\,\omega(G) + 1$.
(b) Show that for every vertex v, $G - v$ has a partition into $\omega(G)$ independent sets of size $\alpha(G)$, and it has a partition into $\alpha(G)$ cliques of size $\omega(G)$.

(39) (Fulkerson [**1971**]) Given a graph G containing vertex v, a **replication** of v adds a new vertex with the same neighborhood as v. The **Replication Lemma** is that if G is perfect, then replicating a vertex produces a perfect graph. Prove this using the Perfect Graph Theorem.

(40) In a transitive digraph D, a **chain** is a directed path and an **antichain** is an independent set. **Dilworth's Theorem** (Dilworth [**1950**]) says that the maximum size of an antichain equals the minimum number of chains that partition the vertices of D. Use Proposition 4.32 to prove Dilworth's Theorem.

Chapter 5

Planarity

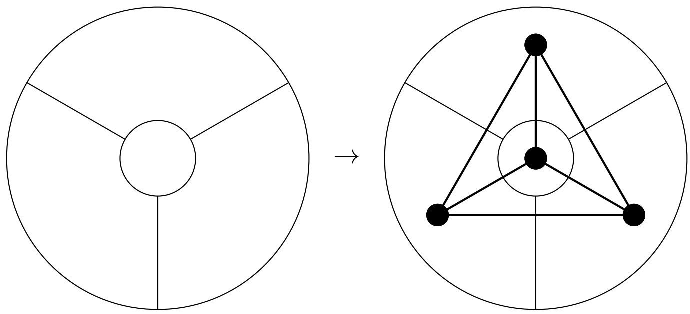
5.1. The Four Color Theorem

A map may emphasize different regions, such as states or countries, by coloring them differently. Using a different color for each region may be expensive, and using many shades would mean that some of them appear almost identical. Thus a publisher might ask what is the smallest number of colors that any map can be colored with so that regions that share a boundary are colored differently. We assume that all regions are contiguous and sharing a single point does not count as sharing a boundary.

This seems to be a new kind of coloring, where we color regions of a plane rather than vertices of a graph. However, we can model this type of coloring using graph coloring.

Definition 5.1. A **map coloring** (**region coloring**) is a coloring of regions of a map with regions sharing a boundary colored differently. The **dual of a map** is a graph with vertices representing regions of a map and edges between regions that share boundaries.

Thus, map coloring can be converted into graph coloring, and many of the techniques that we have seen before will be useful. The distinct feature of the dual of a map is that it can be drawn in the plane without any edges crossing. This is called a **planar graph**. Thus, we are interested in how many colors are necessary to color a planar graph. It is immediate that four may be necessary, since K_4 is planar.



Definition 5.2. The **Four Color Problem** asks whether every planar graph can be colored with four colors. The **Four Color Conjecture** asserts that this is true.

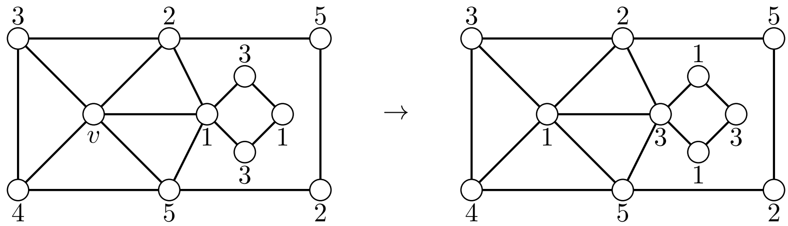
This question was first asked by Francis Guthrie, an English college graduate, in 1852. He mentioned the question to his brother Frederick, who in turn proposed it to his professor, Augustus DeMorgan. DeMorgan, who is famous for DeMorgan's laws in logic, stated that he believed it to be a new problem. He wrote about the problem to William Rowan Hamilton, and the problem spread within the mathematical community. Despite the simple statement of the problem, it defied an easy solution.

In 1879, Alfred Kempe announced a solution to the problem. For over a decade, the mathematical community believed that the problem was solved. However, in 1890, Percy Heawood found an error in Kempe's proof. While Kempe's proof was flawed, it did contain some good ideas, and Heawood was able to adapt it to prove that all planar graphs are 5-colorable. He also addressed the problem of coloring graphs on other topological surfaces (Section 5.5).

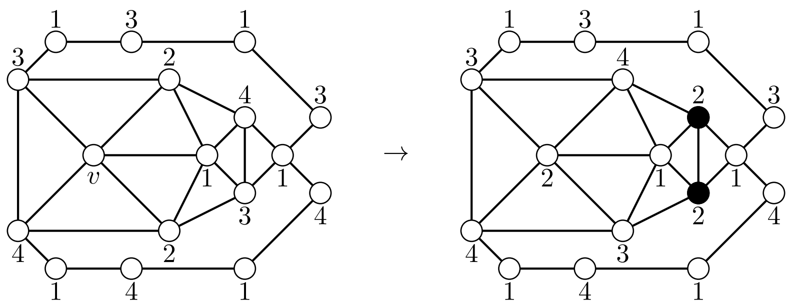
The key to Heawood's proof is the fact that all planar graphs are 5-degenerate (Corollary 5.14). Thus the Degeneracy Bound immediately implies that any planar graph is 6-colorable. Heawood improved this using Kempe's ideas.

Theorem 5.3 (**Five Color Theorem**—Heawood [**1890**]). *Every planar graph is 5-colorable.*

Proof. We find a 5-coloring, using the fact that any planar graph is 5-degenerate. Using a construction sequence, we can color each vertex of G easily unless a vertex has degree 5 and all five neighbors use different colors. Assume this is true for vertex v with neighbors $u_1, \ldots, u_5$ colored $1, \ldots, 5$ consecutively around it. Let G_{ij} be the subgraph of $G - v$ induced by vertices colored i and j. The subgraphs G_{13} and G_{25} cannot both be connected, since $u_1 - u_3$ and $u_2 - u_5$ paths in $G - v$ would contain a common vertex, as G is planar. Say G_{13} is disconnected. Switch the colors on the component of G_{13} containing u_1, so u_1 has color 3. Then v can be colored 1 without conflict. Thus G is 5-colorable. $\square$



The u_2-u_5 path is called a **Kempe chain** after Alfred Kempe, who introduced the idea. Given the argument in the Five Color Theorem, we may assume that a vertex v with degree 5 has two nonconsecutive neighbors with the same color. Say $u_1, \ldots, u_5$ are colored 1, 2, 3, 4, 2 consecutively around v. Color 1 could be eliminated from u_1 unless G_{13} contains a $u_1 - u_3$ path and G_{14} contains a $u_1 - u_4$ path. Kempe argued that in this case, 2 could be eliminated by switching 2 and 4 in the component of G_{24} containing u_2 and switching 2 and 3 in the component of G_{23} containing u_5. The flaw in Kempe's argument is that the $u_1 - u_3$ and $u_1 - u_4$ paths can intertwine, so that performing both switches leads to adjacent vertices colored the same.

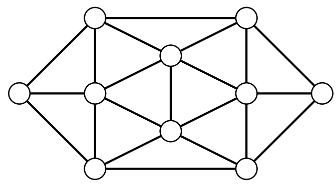


In 1880, Peter Tait offered another incorrect solution to the problem. He correctly showed that the Four Color Problem is equivalent to a problem involving coloring the edges of a graph (Section 7.4). However, he incorrectly believed that he had solved this problem. Problems in his argument were noticed by Julius Petersen and others.

With no simple solution apparent, mathematicians began to study the structure of planar graphs more closely. They sought a proof by minimum counterexample.

Definition 5.4. A **configuration** of a maximal planar graph is a cycle (the **ring**) and everything inside it. A set of configurations is **unavoidable** if some planar graph must contain one of them. A configuration is **reducible** if it cannot appear in any minimum counterexample to the Four Color Conjecture.

Thus to prove the Four Color Conjecture, the goal was to show that any planar graph has an unavoidable set of reducible configurations. It is not difficult to show that a vertex of degree 3 or 4 is reducible, but reducing a vertex of degree 5 was a much more difficult problem. Larger configurations had to be considered. In the first nontrivial result of this type, George Birkhoff (Birkhoff [**1913**]) showed that the **Birkhoff diamond** (see below) is reducible. A process called discharging was used to prove the existence of certain configurations.



The work of many mathematicians gradually expanded the number of reducible configurations and showed that the Four Color Conjecture is true for graphs of larger and larger order. Eventually the focus shifted toward planar graphs with a small ring.

In 1976, Kenneth Appel and Wolfgang Haken, with assistance from John Koch (Appel/Haken [**1977**]), announced a proof of the Four Color Conjecture. Their proof, which ran to about 140 pages, found an unavoidable set of 1936 reducible configurations with ring size at most 14 using 487 discharging rules. Their proof used over 1000 hours of computer time to check many cases and also produced 400 pages of output that had to be hand-checked.

Theorem 5.5 (**Four Color Theorem**—Appel/Haken [**1977**]). *Every planar graph is* 4*-colorable.*

The proof generated controversy. Older mathematicians objected to the use of a computer in a proof, while younger mathematicians objected that the large output was very difficult to check. The older mathematicians complained that no human has actually checked the entire proof. While it is true that computers can make mistakes, even when programmed correctly, humans can also make mistakes. A computer program can always be rerun, or run on a different computer, to double-check it. Over time, proofs using computers have become widely accepted by mathematicians.

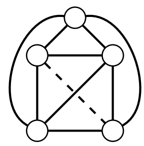
The lingering doubts about this proof led Neal Robertson, Paul Seymour, David Sanders, and Robin Thomas (Robertson et al. [**1996**]) to find a simplified proof using the same general techniques. It found an unavoidable set of 633 reducible configurations using 32 discharging rules. It also uses computer case-checking. They also found an $\mathcal{O}\left(n^2\right)$ algorithm to find a 4-coloring of a planar graph.

The Four Color Theorem is the most famous theorem in graph theory. Attempts to prove it led to the development of graph theory as a distinct subject. The history of this problem is surveyed in Chartrand/Zhang [**2009**]. To better understand the results and techniques used to prove it, we must study planar graphs more closely.

5.2. Planar Graphs

Example. Suppose a king has five sons, and wishes to divide his kingdom into five pieces, one for each prince, so that each prince's territory borders all the other territories. Is this possible? This is known as the **problem of the five princes**.

In graph theory terms, we are asking whether it is possible to draw K_5 in the plane with no edge crossings. We can safely start with a 5-cycle, since there is essentially only one way to draw this with no crossings. Inside the cycle, we can draw two chords without crossings, but no more. We can do the same outside the cycle, but there is nowhere to place the fifth chord without creating a crossing. Thus K_5 cannot be drawn in the plane with no crossings, so the king's wish is impossible.



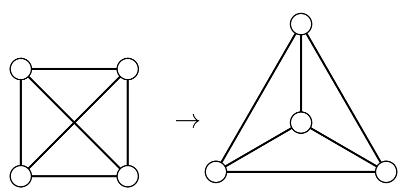
The preceding argument is somewhat informal. In what follows, we need the following theorem.

Theorem 5.6 (**Jordan Curve Theorem**). *Any simple closed curve in the plane divides the plane into two regions—one bounded* (*the interior*) *and one unbounded* (*the exterior*).

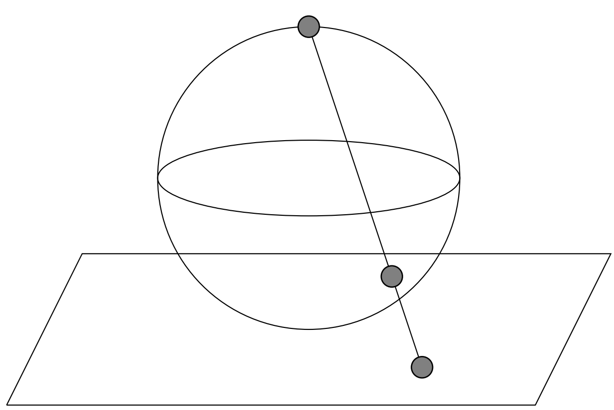
This seemingly obvious result is surprisingly hard to prove precisely. We will take it for granted.

Definition 5.7. A **plane drawing** of a graph is a drawing in the plane that has no crossings. A graph is **planar** if it has a plane drawing. A **nonplanar** graph is not planar. The **regions** of a plane drawing are the maximal pieces of the plane surrounded by edges and vertices. The infinite region is the **exterior region**. The **boundary** of a region is the subgraph induced by the edges that touch it. The **length** of a region is the length of a walk around it.

Not every drawing of a planar graph is a plane drawing, as shown below.



We could draw graphs on another topological surface, such as a sphere. Drawings on a plane or sphere are essentially equivalent. This can be proved via the process of stereographic projection, where a sphere missing a single point (a punctured sphere) is continuously mapped to a plane.



This also shows that any region of a plane drawing could be the exterior region, since a sphere could be punctured in any region. Thus which region is drawn as the exterior is a matter of convenience, not a fundamental property of a graph.

There is a basic relationship between the number of vertices, edges, and regions of a planar graph.

Theorem 5.8 (**Euler's Polyhedron Formula**). *For a connected planar graph with order n, size m, and r regions, $n - m + r = 2$.*

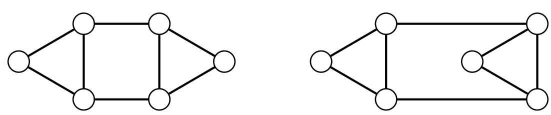
Proof. We use induction on m. For a connected graph G, m is minimum when $m = n - 1$ and G is a tree. Then $r = 1$, so $n - (n - 1) + 1 = 2$, as desired.

Assume the formula holds for graphs with size less than m, and let G be a connected planar graph with size $m > n - 1$. Then G contains a cycle. Let e be an edge of a cycle. Now $G - e$ is connected and planar with size $m - 1$. It has $r - 1$ regions, since the two regions bordering e merge into one when e is deleted. Then $n - m + r = n - (m - 1) + (r - 1) = 2$, so the formula holds for G. □

As the name implies, this identity was originally observed by Leonhard Euler in 1750 in the context of polyhedra in three-dimensional space. Any polyhedron has vertices, edges, and faces, and so can be modeled by a graph. Euler did not correctly prove the identity, however. It was later proved by Adrien-Marie Legendre. Indeed, the identity is difficult to prove solely in the context of polyhedra. This proof illustrates the surprising fact that sometimes it is easier to prove a more general result than a more special result.

This result implies that any plane drawing of a graph has the same number of regions, since n and m certainly are invariant. However, the lengths of the regions can vary between drawings.

Example. The graphs below are isomorphic. The graph on the left has region lengths 3, 3, 4, and 6. The graph on the right has region lengths 3, 3, 5, and 5.



Theorem 5.9. *The size of a planar graph with $n \geq 3$ satisfies $m \leq 3n - 6$. If it is triangle-free, then $m \leq 2n - 4$.*

Proof. Let G be a planar graph with order n, size m, and r regions with lengths r_i. Each region uses at least three edges, and each edge is used twice in region boundaries, so $3r \leq \sum r_i = 2m$. Now $n - m + r = 2$, so $6 = 3n - 3m + 3r \leq 3n - 3m + 2m = 3n - m$. Thus $m \leq 3n - 6$.

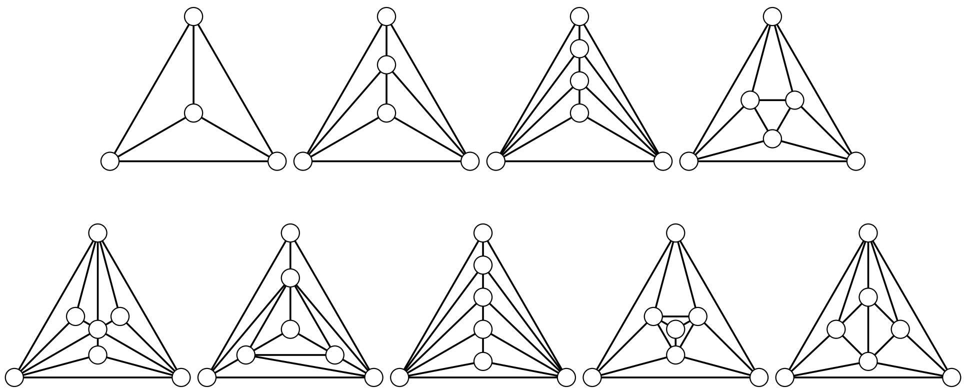
If G is triangle-free, then $4r \leq 2m$. Then $8 = 4n - 4m + 4r \leq 4n - 4m + 2m = 4n - 2m$, so $m \leq 2n - 4$. □

Example. We see that K_5 has $n = 5$ and $m = 10$. Since $10 > 9 = 3 \cdot 5 - 6$, K_5 is nonplanar.

Now $K_{3,3}$ is triangle-free, with $n = 6$ and $m = 9$. Since $9 > 8 = 2 \cdot 6 - 4$, $K_{3,3}$ is nonplanar.

Definition 5.10. A graph is **maximal planar** if no edge can be added without making it nonplanar. A plane drawing of a graph is a **triangulation** if every region is a triangle.

All maximal planar graphs with $4 \le n \le 7$ are shown below. The graphs in the first row are K_4, $P_3 + K_2$, $P_4 + K_2$, and $K_{2,2,2}$. The number of maximal planar graphs of order n (starting at $n = 4$) is 1, 1, 2, 5, 14, 50, 233, 1249, 7595, 49566,... [OEIS A000109].



Corollary 5.11. *The following are equivalent for a planar graph G.*

(1) *G is maximal planar.*

(2) *G has $m = 3n - 6$.*

(3) *A plane drawing of G is a triangulation.*

Proof. ($1 \Leftrightarrow 3$) A plane drawing of G is a triangulation if and only if there is no region with length longer than 3, since otherwise a chord could be added.

($2 \Leftrightarrow 3$) Following the proof of Theorem 5.9, every region is a triangle if and only if $3r = 2m$ if and only if $m = 3n - 6$. □

Corollary 5.12. *Let G be a maximal planar graph with $n \ge 4$, and n_i vertices of degree i. Then*

$$3n_3 + 2n_4 + n_5 = 12 + n_7 + 2n_8 + 3n_9 + \cdots .$$

Proof. We have $\sum n_i = n$, and by the First Theorem of Graph Theory, $\sum i \cdot n_i = 2m$. Then $2m = 6n - 12$, so $\sum i \cdot n_i = 6 \sum n_i - 12$, so $12 = \sum (6 - i)\, n_i$. The result follows. □

Proposition 5.13. *Any maximal planar graph G with $n \ge 4$ is 3-connected.*

Proof. This is immediate for K_4. If $n \ge 5$, G is not complete. Assume to the contrary that S is a cutset with $|S| \le 2$. Then S cannot induce a cycle, so there is a region of G containing vertices of distinct components of $G - S$, which could be made adjacent. This is a contradiction. □

This implies that any maximal planar graph G with $n \geq 4$ has $\delta(G) \geq 3$. Deleting any vertex of degree 3 from a maximal planar graph produces another maximal planar graph. Reversing this, adding a (degree 3) vertex adjacent to the vertices of any region of a maximal planar graph produces another maximal planar graph.

Corollary 5.12 is a condition that the degree sequence of a maximal planar graph must satisfy. Perhaps surprisingly, there is no known characterization of the degree sequences of either planar or maximal planar graphs. We do know the following.

Corollary 5.14. *Any planar graph is* 5*-degenerate. In particular, any planar graph with* $n \geq 4$ *has at least four vertices with degree at most* 5*.*

Proof. For a given planar graph, consider a maximal planar graph containing it. We see $3n_3 + 2n_4 + n_5 \geq 12$, so $n_3 + n_4 + n_5$ is minimized when $n_3 = 4$. Any subgraph of a planar graph is planar, so any planar graph is 5-degenerate. □

This result immediately implies that planar graphs are 6-colorable, and it provides the starting point for the Five Color Theorem and Four Color Theorem. Proving the Four Color Theorem requires knowing more about the structure of maximal planar graphs. A technique called **discharging** is helpful.

Proposition 5.15 (Wernicke [**1904**])**.** *Every maximal planar graph* G *with* $\delta(G) = 5$ *has a degree* 5 *vertex adjacent to a degree* 5 *or* 6 *vertex.*

Proof. Let G be a maximal planar graph with $\delta(G) = 5$. Assign each vertex v a number (**charge**) of $6 - d(v)$. Then Corollary 5.12 implies that the sum of the charges of G is 12. Assume to the contrary that G has no degree 5 vertex adjacent to a degree 5 or 6 vertex. For each degree 5 vertex, move $\frac{1}{5}$ of its charge to each neighbor, so its new charge is 0. Then any vertex u with $d(u) > 6$ has no consecutive degree 5 or 6 neighbors, so its new charge is at most $6 - d(u) + \frac{1}{2}d(u) \cdot \frac{1}{5} = 6 - \frac{9}{10}d(u) < 0$. Then every vertex has nonpositive charge, so the total charge is negative, a contradiction. □

This and other discharging rules are used to generate sets of unavoidable configurations in maximal planar graphs. Showing that each configuration in some unavoidable set is reducible is key to proving the Four Color Theorem.

When a graph is nonplanar, there are various ways to measure how close it is to being planar.

Definition 5.16. The **crossing number** $cr(G)$ of a graph G is the minimum number of crossings in all possible drawings of G.

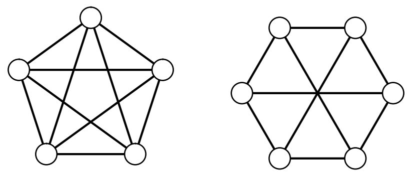
A graph G is planar if and only if $\mathrm{cr}(G) = 0$. Showing $\mathrm{cr}(G) = k$ requires a drawing with k crossings and an argument to show that a drawing with fewer than k crossings is impossible. This latter requirement tends to be difficult, and exact formulas for the crossing number of many common graph classes remain unknown. The crossing number is explored in the exercises.

Related Terms: rectilinear crossing number, maximum rectilinear crossing number, apex graph, nearly planar graph, Apollonian network.

5.3. Kuratowski's Theorem

We have seen that K_5 and $K_{3,3}$ are nonplanar. Then any subdivision of these graphs is also nonplanar. (Otherwise a plane drawing of such a subdivision could be used to find a plane drawing of K_5 or $K_{3,3}$.) Then any graph containing one of these graphs is nonplanar.

Lemma 5.17. *If a graph G is planar, then it does not contain a subdivision of K_5 or $K_{3,3}$.*



We will show that the converse of this result is also true. This requires two lemmas to restrict the type of graphs that we must consider.

Lemma 5.18. *A graph is planar if and only if each of its blocks is planar.*

Proof. ($\Rightarrow$) Any subgraph of a planar graph is planar.

($\Leftarrow$) Start with a plane drawing of one of the blocks B. A block that shares a common vertex v with B can be drawn inside one of the regions bordering v so that v is on its exterior region. Repeating this operation produces a plane drawing of G. □

This implies that any minimal nonplanar graph is 2-connected. This can be improved.

Lemma 5.19. *If G is a nonplanar graph of minimum size not containing a subdivision of K_5 or $K_{3,3}$, then G is 3-connected.*

Proof. Assume the hypothesis and, to the contrary, that G has a minimum vertex cut $S = \{x, y\}$. Let G_i be the S-lobes of G. Each G_i is either planar or contains a subdivision of K_5 or $K_{3,3}$. If $xy \in G$, then xy is in each G_i. If each G_i is planar, then G is also planar, since plane drawings of each G_i with xy on the exterior region can be combined into a plane drawing of G. If some G_i is nonplanar, then it contains a subdivision of K_5 or $K_{3,3}$, and hence so does G.

Thus $xy \notin G$. Let $H_i = G_i + xy$. As before, all H_i cannot be planar, so one contains a subdivision of K_5 or $K_{3,3}$. But replacing xy with an $x - y$ path through another H_j produces a subdivision of K_5 or $K_{3,3}$ in G. This is a contradiction. □

We note that the graph discussed in the preceding lemma does not exist; this argument will be part of a proof by minimum counterexample.

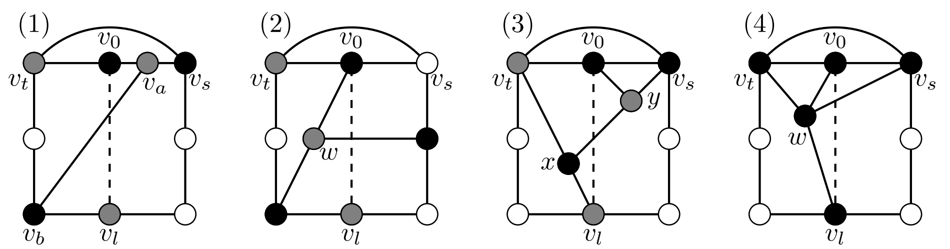
Theorem 5.20 (**Kuratowski's Theorem**—Kuratowski [**1930**]). *A graph G is planar if and only if it does not contain a subdivision of K_5 or $K_{3,3}$.*

Proof. ($\Rightarrow$) This is Lemma 5.17.

(⇐) (Dirac/Schuster [**1954**], Chartrand/Zhang [**2009**]) Let G be a nonplanar graph of minimum size not containing a subdivision of K_5 or $K_{3,3}$, which by Lemma 5.19 is 3-connected. Let $e = uv \in E(G)$, so $H = G - e$ is planar and 2-connected. By Theorem 2.47, u and v are on a cycle. For a given plane drawing of H, let C be a cycle $u = v_0, v_1, \ldots, v_l = v, \ldots, v_k = u$ containing u and v with the maximum number of regions in its interior. Let the interior subgraph be induced by the edges inside C, and the exterior subgraph be induced by the edges outside C. (Both exist, since otherwise e could be added to make G planar.)

There is no $v_i - v_j$ path with $0 \le i \le j \le l$ or $l \le i \le j \le k$ in the exterior, since otherwise there is a cycle containing u and v with more interior regions. Then there must be a $v_s - v_t$ path P, $0 < s < l < t < k$, in the exterior. Then no internal vertex of P is connected to any vertex of C by a path in the exterior.

Let H_1 be the component of the exterior subgraph that contains P. Now H_1 cannot be moved to the interior of C while maintaining planarity. This and the fact that G is nonplanar implies the interior subgraph must contain one of the following (see figures below):



(1) A $v_a - v_b$ path with $0 < a < s$ and $l < b < t$ with only v_a and v_b on C.

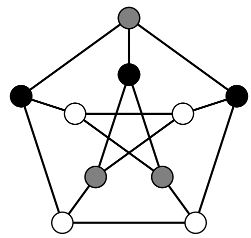
(2) A vertex w not on C connected to C by three independent paths with one of them ending at one of v_0, v_s, v_l, v_t. The other two ends must be positioned so neither uv nor P can be added inside without violating planarity.

(3) Two vertices x and y not on C connected by a path, with independent paths from x to two consecutive vertices in $\{v_0, v_s, v_l, v_t\}$ and independent paths from y to the other two.

(4) A vertex w not on C connected to v_0, v_s, v_l, v_t by four independent paths.

The first three cases produce a subdivision of $K_{3,3}$ in G; the fourth produces a subdivision of K_5 in G. This is a contradiction. □

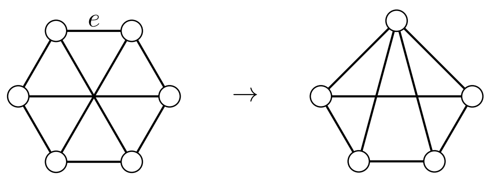
Example. The Petersen graph is nonplanar, since it contains a subdivision of $K_{3,3}$. This is shown below with the vertices of the two partite sets colored black and gray. It does not contain a subdivision of K_5, since it has no vertices of degree more than 3.



There is an operation that undoes subdivisions.

Definition 5.21. Let G be a graph containing edge $e = uv$. A **contraction** of e, G/e, is formed from $G - u - v$ by adding a vertex w with neighborhood $N_G(u) \cup N_G(v)$. A **minor** H of a graph G is formed by some number of edge contractions or deletions on G.

Example. Contracting edge e of $K_{3,3}$ results in the graph W_5. Thus W_5 is a minor of $K_{3,3}$.



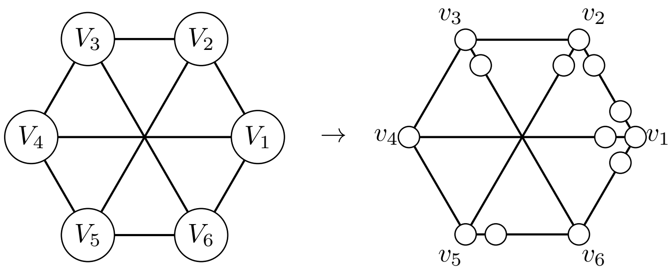
Note that contraction is not a perfect inverse for subdivision since it can be applied to an edge not incident with a vertex of degree 2. Vertices u and v may have a common neighbor, but the definition of contraction does not allow the formation of multiple edges. The existence of subdivisions and minors is equivalent for some graphs.

Lemma 5.22. *Let G and H be graphs with $\Delta(H) \leq 3$. Then G contains a subdivision of H if and only if G contains an H minor.*

Proof. ($\Rightarrow$) If G contains a subdivision of H, then performing edge contractions produces an H minor.

($\Leftarrow$) Assume G contains an H minor. Then H can be obtained by deleting edges (if necessary) to obtain a connected graph G', then contracting edges in G'. Denote the vertices of H by V_i, $1 \leq i \leq n$, where $G_i = G'[V_i]$ is a connected subgraph of G'. Each edge V_iV_j of H implies that there are adjacent vertices v_{ij} and v_{ji} of G_i and G_j.

If $d(V_i) = 3$, there is a vertex $v_i \in V_i$ connected to each v_{ij} by independent paths in V_i (possibly $v_i = v_{ij}$ for some or all i). If $d(V_i) = 2$, there is a path joining both vertices v_{ij} in V_i. Then the subgraph induced by the edges $v_{ij}v_{ji}$ and the paths in each V_i is a subdivision of H. $\square$



We can characterize planar graphs in terms of their minors.

Theorem 5.23 (**Wagner's Theorem**—Wagner [**1937**])**.** *A graph G is planar if and only if it does not have a K_5 or $K_{3,3}$ minor.*

Proof. ($\Leftarrow$) (contrapositive) If G is not planar, then by Kuratowski's Theorem, it contains a subdivision of K_5 or $K_{3,3}$. Performing edge contractions produces a K_5 or $K_{3,3}$ minor.

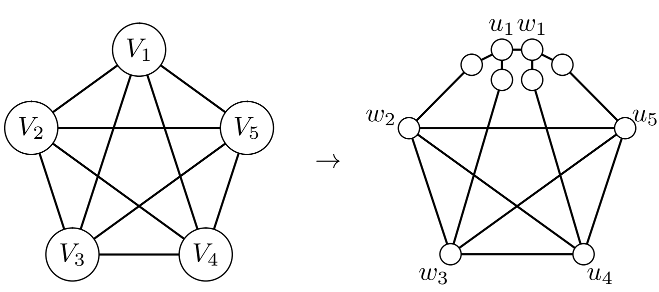
($\Rightarrow$) (contrapositive) Assume G has a K_5 or $K_{3,3}$ minor. If $K_{3,3}$ is a minor of G, then Lemma 5.22 implies G contains a subdivision of $K_{3,3}$. By Kuratowski's Theorem, G is nonplanar.

(Chartrand/Zhang [**2009**]) Assume $H = K_5$ is a minor of G. Then H can be obtained by deleting edges (if necessary) to obtain a connected graph G', then contracting edges in G'. Denote the vertices of H by V_i, $1 \le i \le 5$, where $G_i = G'[V_i]$ is a connected subgraph of G'. Each pair of distinct subgraphs G_i and G_j contains adjacent vertices v_{ij} and v_{ji}, $1 \le i, j \le 5$.

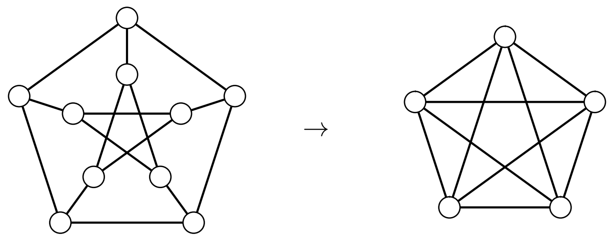
It may be that there is a vertex $v_1 \in V_1$ connected to v_{12}, v_{13}, v_{14}, v_{15} by independent paths (possibly $v_1 = v_{1i}$ for some or all i). If similar vertices $v_2, \ldots, v_5$ exist, then the subgraph induced by the edges $v_i v_j$ and the paths in each V_i is a subdivision of K_5.

Otherwise, v_i has not been defined for one or more i. For any such i (say $i = 1$), there are vertices u_1 and w_1 so that V_1 contains a $u_1 - w_1$ path and (possibly trivial) independent $u_1 - v_{12}$, $u_1 - v_{13}$, $w_1 - v_{14}$, and $w_1 - v_{15}$ paths.

Let w_2 be a vertex of G_2 connected to v_{21}, v_{24}, and v_{25} by (possibly trivial) independent paths. Similarly, w_3, u_4, and u_5 are defined for the sets $\{v_{31}, v_{34}, v_{35}\}$, $\{v_{41}, v_{42}, v_{43}\}$, and $\{v_{51}, v_{52}, v_{53}\}$. Now the edges $v_{ij} v_{ji}$ and the paths described for each G_i produce a subdivision of $K_{3,3}$ with partite sets $\{u_1, u_4, u_5\}$ and $\{w_1, w_2, w_3\}$. Thus G is nonplanar. $\square$



Example. The Petersen graph has no subdivision of K_5, but it has a K_5 minor, which can be seen by contracting the five "spokes" between two 5-cycles of the graph.



There are still other characterizations of planar graphs involving the "cycle space" of a graph (Mac Lane [**1937**]) and partial orders (Schnyder [**1989**]).

To show that a graph is nonplanar, we must show that it contains a subdivision of K_5 or $K_{3,3}$ or a K_5 or $K_{3,3}$ minor. To show that a graph is planar, we could show that it has no such subdivision or minor. However, such subdivisions can be arbitrarily large, so checking all subgraphs is not practical. Given the difficulty of other problems such as determining the chromatic number and independence number, it may be surprising that there are efficient algorithms to determine whether a graph is planar. In fact, this can be done in linear time. See (West [**2001**]) for discussion of such algorithms.

Planarity can also be shown by finding a plane drawing of a graph. When a given drawing has crossings, this involves moving one or more vertices at a time elsewhere on the graph to reduce the number of crossings. There is a game called Planarity (`planarity.net`) that requires the user to find a plane drawing of a graph. This game is good practice for this skill.

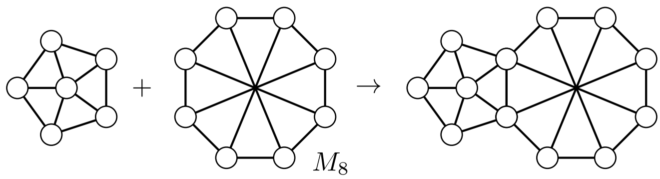
The motivation of Wagner's Theorem was to find a forbidden minor characterization of planar graphs. Wagner also examined the opposite problem of starting with a forbidden minor and characterizing the graphs that don't contain it. He solved this problem for K_5 and $K_{3,3}$. The corresponding graph classes necessarily contain all planar graphs, and also some others. The characterizations require a new operation.

Definition 5.24. A **k-sum** of G and H is formed by identifying copies of K_k in G and H and then possibly deleting some edges of the clique. Any k-sum is a **clique sum**.

Theorem 5.25 (Wagner [**1937**]). *Forbidden minor characterizations.*

(1) *A graph G does not contain a K_5 minor if and only if G can be contructed using k-sums, $0 \leq k \leq 3$, of planar graphs and M_8.*

(2) *A graph G does not contain a $K_{3,3}$ minor if and only if G can be contructed using k-sums, $0 \leq k \leq 2$, of planar graphs and K_5.*



Minors are also relevant to graph coloring. Recall that a graph with a large chromatic number need not contain a large clique, nor even a small cycle. Is there some other substructure that it must contain?

Conjecture 5.26. *Let G be a graph with $\chi(G) = k$.*

(a) (**Hajos' Conjecture**) *Then G contains a subdivision of K_k.*

(b) (**Hadwiger's Conjecture**—Hadwiger [**1943**]) *Then G contains a K_k minor.*

For $1 \leq k \leq 4$, these conjectures are equivalent, since $\Delta(K_4) = 3$. For $1 \leq k \leq 3$, they are obviously true.

Corollary 5.27 (Hadwiger [**1943**])**.** *Hadwiger's Conjecture is true for $k = 4$.*

Proof. A 4-chromatic graph G has a critical subgraph H with $\delta(H) \geq 3$. Then H contains a subdivision of K_4 by Theorem 3.45, so G does also. □

Hajos' Conjecture remains unresolved for $5 \leq k \leq 6$. For $k \geq 7$, it is false. A construction to show this is presented in the Exercises.

Hadwiger's Conjecture for $k = 5$ is equivalent to the Four Color Theorem. That is, they were both known to have the same truth value before either had been proved.

Theorem 5.28 (Wagner [**1937**])**.** *The Four Color Theorem is true if and only if every graph with $\chi(G) = 5$ contains a K_5 minor.*

Proof. ($\Rightarrow$) Assume the Four Color Theorem is true and (to prove the contrapositive) that G has no K_5 minor. Then Theorem 5.25 says that G can be contructed using clique-sums of size at most 3 of planar graphs and M_8. Now planar graphs and M_8 are 4-colorable, and cliques must be colored with distinct colors. Thus a clique-sum of two 4-colorable graphs is 4-colorable, so G is 4-colorable.

($\Leftarrow$) Assume Hadwiger's Conjecture is true for $k = 5$, and let G be a graph with $\chi(G) \geq 5$. Then G contains a K_5 minor, so G is not planar. □

Hadwiger's Conjecture for $k = 6$ was proved by Robertson, Seymour, and Thomas [**1993**] using the Four Color Theorem. For $k \geq 7$, the conjecture remains open. For $k = 7$, it is known that a 7-chromatic graph contains a K_7 or $K_{4,4}$ minor (Kawarabayashi/Toft [**2005**]). Hadwiger's Conjecture is considered one of the most important unsolved problems in graph theory.

Definition 5.29. A class of graphs is **minor-closed** if any minor of a graph in the class is also in the class.

Planar graphs are minor-closed, since contracting an edge in a plane drawing cannot produce a crossing. Wagner's Theorem says that planar graphs have only two forbidden minors.

Theorem 5.30 (**Graph Minor Theorem**—Robertson/Seymour [**2004**]). *Any minor-closed class of graphs has a finite set of forbidden minors.*

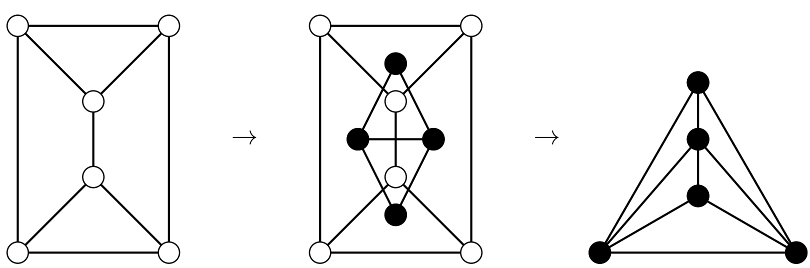
This is perhaps the deepest theorem in graph theory. It was proved in the series of papers totaling about 500 pages between 1983 and 2004 by Neal Robertson and Paul Seymour. It is an existence proof; it does not produce the set of forbidden minors for a given class. Some classes known to be minor-closed do not have complete characterizations of their forbidden minors. The Graph Minor Theorem implies that the class of all graphs having the property that all minors of graphs in the class can be $(k-1)$-colored has a finite set of forbidden minors; Hadwiger's Conjecture asserts that K_k is the only one.

5.4. Dual Graphs and Geometry

5.4.1. Dual Graphs. In the Four Color Problem, a map is modeled with a graph called a dual map. Something similar can be done for any planar graph.

Definition 5.31. The **dual of a plane drawing of a graph** G is a multigraph G^* with each vertex representing a region of the drawing, and each edge joining vertices representing regions that share an edge in the drawing. When all drawings of G have isomorphic duals, we call G^* the **dual graph** of G.

Example. To construct the dual of $G = K_3 \Box K_2$, put vertices inside its interior regions and draw the corresponding edges, so that each edge of G^* crossed exactly one edge of G. Drawing the edges incident with the vertex corresponding to the exterior region will require several curved edges. We draw G^* in a more aesthetically pleasing form, finding $G^* = P_3 + K_2$.



A dual of a drawing may not be a graph. It has a loop whenever G has a bridge. It has multiple edges whenever G has a minimal 2–edge cut. A graph and its dual necessarily have the same size. For a connected graph, $(G^*)^* = G$, so the dual graph is a dual operation (it is its own inverse). The sum of the region lengths of G, $\sum r_i = 2m$, is equivalent to the First Theorem of Graph Theory for the dual graph G^*.

In the definition, we were careful to specify the dual of a drawing, rather than the dual of a graph. We saw earlier that different drawings of a graph can have different region lengths, which must produce different duals. Some restrictions on the graph will guarantee a unique dual.

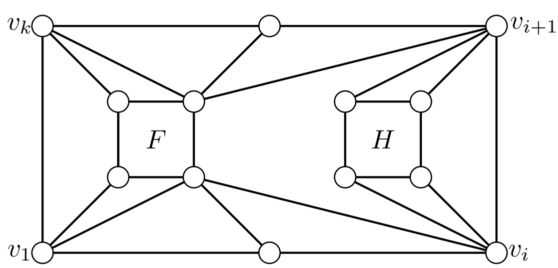
Lemma 5.32. *A cycle C of a planar graph G must bound a region in any drawing of G if and only if $G - C$ is connected.*

Proof. ($\Leftarrow$) Suppose $G - C$ is connected. Then in any plane drawing of G, all of $G - C$ is inside or outside (not both) of C. Thus C is the boundary of a region of G.

($\Rightarrow$) (contrapositive) Suppose $G-C$ is disconnected. Let F and H be subgraphs of G distinct from C with $F \cup H = G$ and $F \cap H = C$. Then G can be drawn with F inside C and H outside C, producing a plane drawing of G where C is not a region boundary. $\square$

Theorem 5.33 (Whitney [**1933**]). *If G is a 3-connected planar graph, then G has a unique plane drawing, up to rotation on a sphere (any plane drawing has the same regions).*

Proof. (contrapositive) Assume G has more than one distinct drawing, and let C be a cycle that is a region boundary in one drawing and not the other. By the lemma, $G - C$ is disconnected. Say F and H are components of $G - C$. Draw G with C on the outside, and let $v_1, \ldots, v_k$ be vertices of C adjacent to vertices of F. Then all vertices of C adjacent to vertices of H must be between some v_i and v_{i+1}. Then $\{v_i, v_{i+1}\}$ is a cutset of G. $\square$

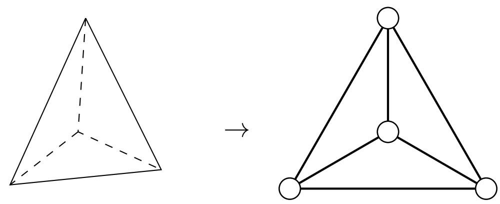


This implies that any 3-connected planar graph has a unique dual. The dual of any maximal planar graph with $n \geq 4$ is a 3-connected cubic planar graph. This partly explains our interest in cubic graphs elsewhere in this text.

5.4.2. Polyhedra. The main focus of this section is geometric applications of graph theory. In geometry, we consider figures in two or three-dimensional space. Polygons (two dimensions) have vertices at distinct points and straight line edges between some of them.

Definition 5.34. A **polyhedron** is a figure in three-dimensional space with points called vertices and straight-line edges connecting some vertices bounded by flat faces whose boundaries are polygons. A **regular polyhedron** has the same number of edges incident with each vertex and the same number of edges bounding each face.

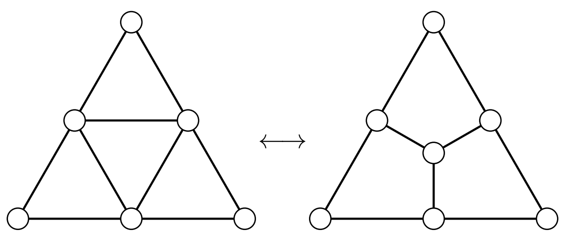
Any polygon or polyhedron can be modeled by a graph representing its vertices and edges. In fact, the names "vertex" and "edge" in graph theory come from this geometric terminology.



What can be said about a graph that represents a polyhedron? Certainly it must be planar. A vertex cut of size 2 would require the polyhedron to be "pinched", so it must be 3-connected. In fact, the converse is also true.

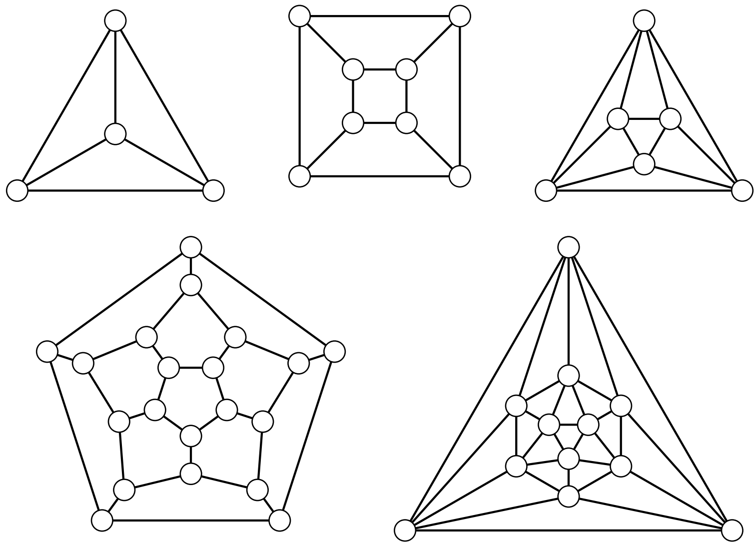
Theorem 5.35 (**Steinitz's Theorem**—Steinitz [**1922**]). *A graph represents a convex polyhedron if and only if it is planar and* 3*-connected.*

To prove the hard direction of this theorem, Steinitz observed that Euler's Polyhedron Formula implies that a planar 3-connected graph must contain either a degree 3 vertex or a triangular region. He showed that any planar 3-connected graph can be constructed from K_4 using an operation called a $\Delta - Y$ **transform** (see below). This corresponds to slicing off or adding a piece to a polyhedron. However, finding the appropriate sequence of $\Delta - Y$ transforms is not especially straightforward.



We can characterize regular polyhedra, which are also known as **Platonic solids**. In graph theory terms, the graph representing a polyhedron and its dual graph are both regular. Let G be a k-regular graph with order n, size m, and r regions, and let G^* be l-regular. Then $kn = 2m = lr$. Substituting into Euler's Formula, we see $\frac{2m}{k} - m + \frac{2m}{l} = 2$, so $m\left(\frac{2}{k} - 1 + \frac{2}{l}\right) = 2$. Since $m > 0$, $\frac{2}{k} - 1 + \frac{2}{l} > 0$. By Corollary 5.14, $3 \leq k, l \leq 5$. The five pairs of values of k and l that satisfy these inequalities are given in the table below. The corresponding values of n, m, and r are readily calculated. Each graph is 3-connected and has a unique realization. Each graph corresponds to a unique polyhedron.

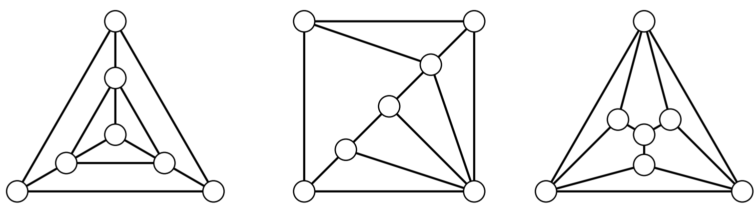
k	l	n	m	r	name	notation
3	3	4	6	4	tetrahedron	K_4
3	4	8	12	6	cube	Q_3
4	3	6	12	8	octahedron	$K_{2,2,2}$
3	5	20	30	12	dodecahedron	
5	3	12	30	20	icosahedron	



The cube and octahedron are dual graphs, as are the dodecahedron and icosahedron. The tetrahedron is its own dual.

Definition 5.36. A graph is **self-dual** if $G = G^*$.

The trivial graph is self-dual, as are all wheels. Three self-dual graphs of order 7 are shown below.

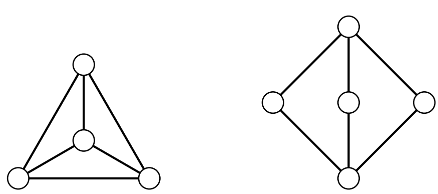


5.4.3. Outerplanar Graphs.

Some planar graphs can be drawn with all vertices on the outside.

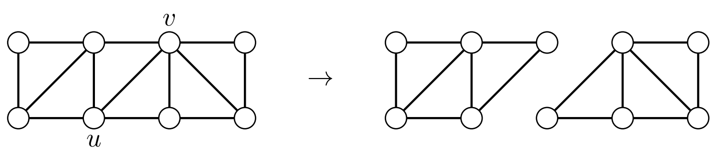
Definition 5.37. A graph is **outerplanar** if it has a plane drawing with all vertices on the exterior region.

The graphs K_4 and $K_{2,3}$ are not outerplanar, since any plane drawing of them must have one vertex in the interior.



Theorem 5.38. *Any outerplanar graph is 2-degenerate. In particular, any nontrivial outerplanar graph has at least two vertices of degree at most 2.*

Proof. The result is obvious when $2 \leq n \leq 4$, where K_4 is the only nonouterplanar graph. Assume the result holds for all maximal outerplanar graphs with order less than n, and let G be a maximal outerplanar graph of order n. Then all the vertices are on a spanning cycle C. Every other edge uv is a chord of C. The vertices of the two $u-v$ paths on C induce two maximal outerplanar graphs that only overlap on $\{u, v\}$. By induction, each of them has at least two vertices of degree at most 2, at least one of which is not u or v. Thus the result holds for G. Thus it holds for any nontrivial outerplanar graph. Since any subgraph of an outerplanar graph is outerplanar, any outerplanar graph is 2-degenerate. □



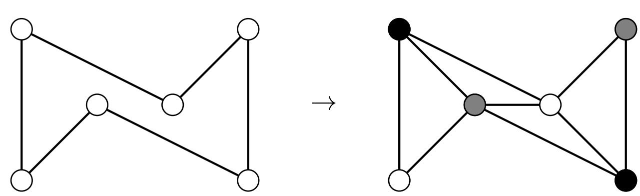
Corollary 5.39. *The size of a nontrivial outerplanar graph satisfies $m \leq 2n - 3$.*

This follows immediately from the bound on the size of k-degenerate graphs. Note that expressions of the form $k \cdot n - \binom{k+1}{2}$ occur repeatedly in graph theory. The size of a tree is $n-1$. A maximal outerplanar graph, which is a 2-tree, has size $2n-3$. A maximal planar graph has size $3n-6$. Some 3-trees are maximal planar, and Wagner [**1936**] showed that any maximal planar graph can be converted into a 3-tree via edge flips (Exercise 29).

Theorem 5.38 and the Degeneracy Bound immediately imply that any outerplanar graph is 3-colorable. This has a surprising geometric application. Imagine that an art gallery has a shape of an n-sided polygon. The gallery needs guards to prevent the art from being stolen. Guards must be positioned at vertices of the polygon so that every point of the interior is seen by some guard. How many guards are needed?

Theorem 5.40 (**Art Gallery Theorem**—Chvatal [**1975**]). *Any art gallery* (*an n-sided polygon*) *requires at most $\left\lfloor \frac{n}{3} \right\rfloor$ guards at its vertices to see all of its interior.*

Proof (Fisk [**1978**]). Add chords to the interior of the polygon so that it is split into triangles. Every vertex sees the interior of any triangle that contains it. The triangulated polygon can be represented by a maximal outerplanar graph, which is 3-colorable. In any 3-coloring, each triangle has each color on one of its three vertices. Thus each color class sees the interior of the polygon, and one of the classes has at most $\left\lfloor \frac{n}{3} \right\rfloor$ vertices. □

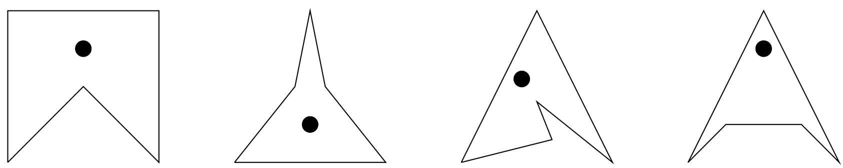


In the exercises, you are asked to show that this bound is sharp.

5.4.4. Straight Line Drawings. Kuratowski's Theorem guarantees that any graph not containing a subdivision of K_5 or $K_{3,3}$ has a plane drawing. In fact, it can be drawn in the plane using only straight lines for edges.

Lemma 5.41. *Any polygon with length at most* 5 *has a point in its interior that sees* (*has a straight line of sight to*) *all its vertices.*

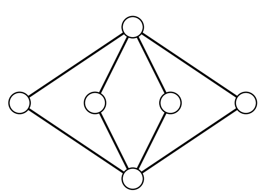
Proof. When the polygon is convex, any interior point works. Otherwise, there are several cases (see below) depending on the shape of the polygon. □



Theorem 5.42 (**Fary's Theorem**—Fary [**1948**], Wagner [**1936**]). *If a graph G is planar, then it has a drawing in the plane with straight lines for edges.*

Proof. Let G be planar, and let H be a maximal planar graph containing G. We use induction on order. The result is obvious when $1 \leq n(H) \leq 4$. Assume any maximal planar graph with order less than n has a straight line drawing. Let H have order $n \geq 4$. By Corollary 5.14, H has at least four vertices with degree at most 5, so at least one vertex v with $3 \leq d(v) \leq 5$ is not on the exterior region. If $H - v$ is not maximal planar, add one or two edges so that it is. Then this graph has order less than n, so it has a straight line drawing. Now deleting the added edges results in a region with length at most 5. By Lemma 5.41, there is a point in the interior of this region that sees all the vertices on its boundary. Add a vertex at this point, and add straight line edges to the vertices. Then H has a straight line drawing, so G does also. □

When a graph is maximal planar, every region is a triangle and, hence, convex. Otherwise, the last step in the proof of Fary's Theorem is to delete the edges added to make it maximal planar. This may result in regions that are not convex. For example, $K_{2,4}$ has no plane drawing where all interior regions are convex. Nonetheless, a stronger hypothesis can guarantee that all interior regions are convex.

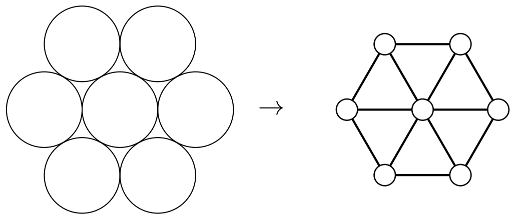


Theorem 5.43 (**Tutte's Spring Theorem**—Tutte [**1963**]). *Any* 3*-connected planar graph has a plane drawing with all interior regions convex.*

Tutte showed that if the exterior region is fixed and the interior edges are springs, then the equilibrium position of this system has every region convex. He proved this using linear algebra. Thomassen [**1980**] found a graph theoretic proof.

There is another way to produce planar graphs with straight line drawings.

Definition 5.44. A **coin graph** has vertices that are the centers of nonoverlapping circles and edges between centers of tangent circles. A **penny graph** is a coin graph with circles all having the same radius.



A coin graph must be planar. Surprisingly, the converse is also true.

Theorem 5.45 (**Circle Packing Theorem**—Koebe [**1936**])**.** *A graph is a coin graph if and only if it is planar.*

This theorem was originally proved using complex analysis. Fary's Theorem follows immediately from the Circle Packing Theorem, and Steinitz's Theorem can be proved using it. Penny graphs do not have a nice characterization, and it is an NP-complete problem to determine whether a graph is a penny graph.

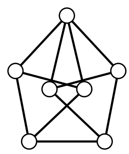
Since every planar graph has a straight edge drawing, we may seek additional properties. Heiko Harborth [**1987**] conjectured that every planar graph has a straight edge drawing with edges having integer length. Note that this is equivalent to every edge having rational length, since such a drawing could be scaled by the least common denominator of the edge lengths. This conjecture remains open, but the following more limited result is known.

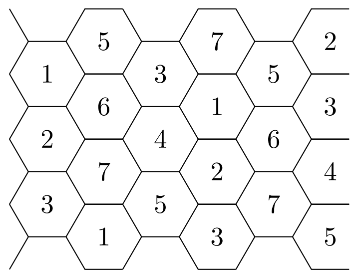
Theorem 5.46 (Geelen/Guo/McKinnon [**2008**])**.** *Let G be a planar graph that can be constructed so that, when added, each vertex has degree at most* 2*, or has degree* 3*, with two neighbors adjacent. Then G has a straight edge drawing with edges having integer length.*

The proof of this theorem uses number theory. The graphs in the hypothesis include cubic graphs and 3-trees, but not all 3-degenerate graphs.

Definition 5.47. A **unit distance graph** is a graph whose vertices are points in the plane and whose edges all have length 1. A **matchstick graph** is a unit distance graph with a plane drawing.

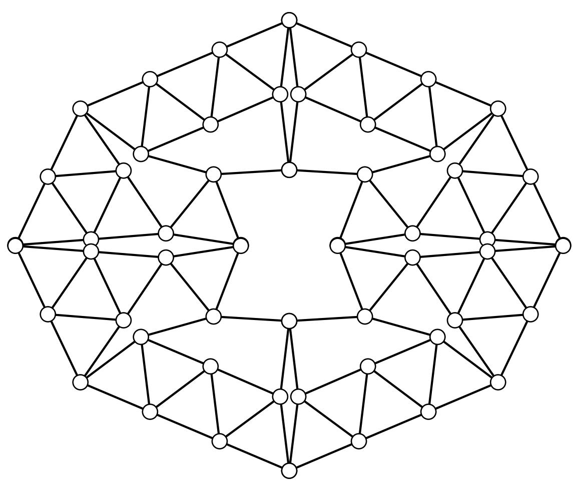
One natural question is how large the chromatic number of a unit distance graph can be. No wheel with even order (including K_4) is a unit distance graph. However, the Moser spindle is. There is a 7-coloring of the points of the plane using a tessellation by regular hexagons with diameter slightly less than 1. This shows that at most seven colors are required for any unit distance graph.





This problem was first discussed by Nelson in 1950 and first published by Hadwiger [**1961**]. For the next 57 years, 4 and 7 were the best known bounds for the maximum chromatic number of a unit distance graph. Then in 2018, biologist Aubrey de Grey [**2018**] found a 5-chromatic unit distance graph with order 1581. This graph contains many copies of the Moser spindle. Other researchers quickly went to work on the problem. Marijn Huele [**2018**] found several successively smaller examples of 5-chromatic unit distance graphs with order around 500.

Every penny graph is a matchstick graph. Penny graphs are 3-degenerate, but this is not true for all matchstick graphs. The **Harborth Graph** (Harborth [**1986**]) is a 4-regular matchstick graph. A precise description due to Gerbracht [**2011**] also shows that it is rigid.



5.4.5. Rigid Graphs. Consider a graph in the plane with edges that are line segments with fixed length that are hinged at the vertices (the angles at the vertices may vary).

Definition 5.48. A graph is **rigid** if when its vertices are placed in general position in the plane (fixing the lengths of the edges), there is no movement of the graph in the plane preserving the edge lengths that does not also preserve all distances between vertices. A graph is **flexible** if it is not rigid. A graph is a **Laman graph** if and only if $m = 2n - 3$ and each nontrivial subgraph with order n' has size $m' \leq 2n' - 3$.

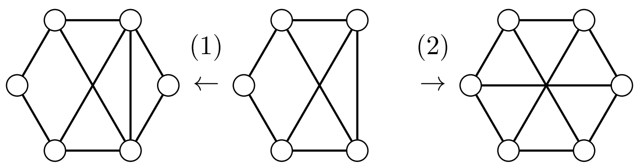
Any rigid graph must be 2-connected, since multiple components could be moved independently, and multiple blocks could be rotated at a cut-vertex. We will show that Laman graphs are exactly the rigid graphs. Laman graphs include all maximal 2-degenerate graphs, and also some with 3-cores.

Henneberg found an operation characterization of rigid graphs, which is proved using linear algebra.

Theorem 5.49 (Henneberg [**1911**]). *A graph is a minimal rigid graph if and only if it can be constructed by starting with K_2 and iterating the following two operations (**Henneberg operations**).*

(1) *Add a vertex of degree* 2*.*

(2) *Add a vertex of degree* 3 *adjacent to two vertices that are neighbors, and delete the edge between them.*



Laman proved a characterization of rigid graphs involving their sizes.

Theorem 5.50 (Laman [**1970**]). *A graph has a Henneberg construction if and only if it is a Laman graph.*

Proof. ($\Rightarrow$) Assume G has a Henneberg construction. Certainly K_2 is a Laman graph and both operations increase n by 1 and m by 2 in G. If a vertex is added to a subgraph of G, its size is increased by at most 2. Thus the operations preserve Laman graphs.

($\Leftarrow$) (Haas et al. [**2005**]) If $n = 2$, then K_2 is the only Laman graph. Assume that any Laman graph with order less than $n > 2$ has a Henneberg construction. If G is a Laman graph with order n, then $m = 2n - 3$, and it has a vertex of degree at most 3. If any vertex v has degree 0 or 1, then $G - v$ is not a Laman graph. If v has degree 2, then $G - v$ is a Laman graph since its size is $2(n-1) - 3$ and all its subgraphs are subgraphs of G.

If v has degree 3, let $N(v) = \{v_1, v_2, v_3\}$. Let $H = G - v$, which has order $n - 1$, but only $2(n-1) - 4$ edges. We must add one edge joining one of the three pairs of vertices in $N(v)$. Consider the rigid components of H: maximal subsets of some k vertices spanning $2k - 3$ edges. Now v_1, v_2, and v_3 cannot belong to the same rigid component; otherwise the size restriction would be violated in G on the subset consisting of this component and v. Two rigid components share at most one vertex; otherwise their union would be a larger Laman subgraph. Say v_1 and v_2 are not in a common rigid component. Then adding $e = v_1v_2$ doesn't violate the size restriction on any subset, and it converts H to a Laman graph H'. Then H' and hence G has a Henneberg construction. □

These ideas can be generalized, but there is no complete characterization of graphs that are rigid in three dimensions. Henneberg [**1911**] listed four operations that have been shown to produce all minimally rigid three-dimensional graphs starting from K_3, but it is unknown if they only produce rigid graphs.

The natural generalization of Laman graphs would be graphs with $m = 3n - 6$ and each subgraph with order $n' \geq 3$ having size $m' \leq 3n' - 6$. Each minimally rigid three-dimensional graph must satisfy these conditions, but there are graphs satisfying these conditions that are not rigid in three dimensions. See Tay/Whiteley [**1985**] for a survey of what is known on rigid graphs.

5.5. Genus of Graphs

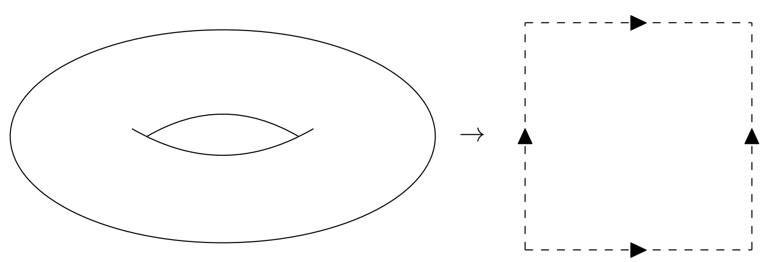
An electrical network can be modeled using a graph, with edges representing wires and vertices representing their intersections. A circuit board or computer chip has an electrical network on a flat surface (one or two-sided). If two wires cross, the network will short-circuit. Thus it is desirable for the network to be planar, so that it can be laid out with no crossings.

However, this may not be possible. In this case, we could build bridges to eliminate crossings. Alternatively, we could drill holes in the in the circuit board and run some wires through them to the other side. It turns out that these approaches are equivalent.

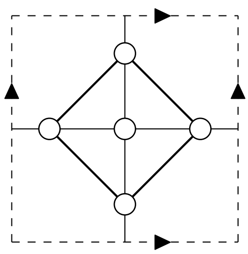
We have seen that drawing graphs in the plane and on a sphere is equivalent. However, there are other topological surfaces that are different. We could attach handles to a sphere or drill holes in it. Topologists consider two surfaces to be equivalent (homeomorphic) if one can be continuously deformed into another. That is, a surface can be stretched or shrunk but cannot be cut or pasted. In this way, we see that attaching a handle or drilling a hole amounts to the same operation.

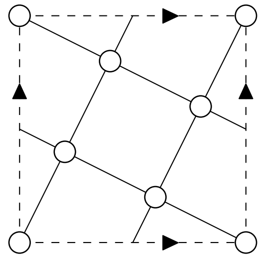
Definition 5.51. A **torus** is a surface with one handle (or hole). A graph that can be drawn on a torus with no crossings is **toroidal**.

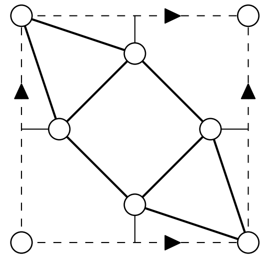
A torus can be drawn to look like a doughnut in space. However, this is not very helpful if we want to draw graphs on it. Instead, imagine cutting the torus twice, once in each direction, to obtain a rectangular figure. Opposite sides of this rectangle are identified, so that edges can leave one side of the rectangle and return on the other. The four corners of the rectangle are the same point.



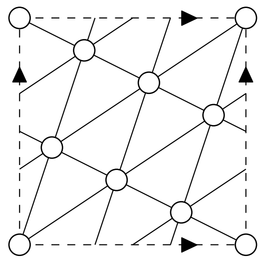
Any planar graph can be drawn without crossings on the torus, and many nonplanar graphs can as well. These include K_5 and $K_{3,3}$. We have seen that 3-connected planar graphs have an essentially unique drawing in the plane, but this is not the case for drawings on the torus. The lengths of regions may vary in different drawings. Three drawings of K_5 on the torus with different region lengths are shown below, and three more are requested in the Exercises.







Not just K_5, but K_6 and K_7 (see below) can be drawn on the torus. Larger complete graphs require surfaces with more handles.



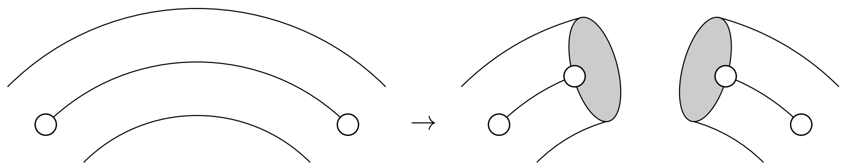
Definition 5.52. The **genus of a surface** is the number of handles (or holes) it has. The surface with k handles is denoted S_k. An **embedding** of a graph is a drawing with no crossings. The **genus of a graph** $\gamma(G)$ is the minimum genus so that it has an embedding on a surface with this genus. A region is a **2-cell** if any closed curve within that region can be continuously contracted to a point within that region. A **2-cell embedding** has every region a 2-cell.

Requiring a 2-cell embedding forbids any unused handles; a curve drawn around a handle could not be continuously contracted to a point within its region. A sphere has genus 0; a torus has genus 1. Euler's formula can be generalized to surfaces with genus other than 0.

Theorem 5.53 (Generalized Euler Identity—Lhuilier [**1812**]**).** *If G is a connected multigraph with r regions that embeds on a surface with genus γ, then $n - m + r = 2 - 2\gamma$.*

Proof. We use induction on γ. When $\gamma = 0$, we have Euler's Identity. Essentially the same argument as before shows that it holds for multigraphs, not just graphs.

Assume that the result holds for all graphs with genus less than $\gamma > 0$, and let G have genus γ, order n, size m, and r regions. We cut through one of the handles of S_γ and cap both holes this creates. Since the embedding is 2-cell, there are $l > 0$ edges that are cut through. To restore a 2-cell embedding, we add a vertex at the end of each cut edge, and a cycle of l edges connecting them on each side of the cut. Thus we have increased n by $2l$, m by $2l$, r by 2, and reduced γ by 1. The new graph H has $(n + 2l) - (m + 2l) + (r + 2) = 2 - 2(\gamma - 1)$, so $n - m + r = 2 - 2\gamma$ for G. □



Note that the generalized Euler Identity is proved by induction with Euler's Identity as the base case. Euler's Identity is proved using induction with the size of trees with order n as the base case. The size of trees with order n is proved using induction with a trivial graph as the base case. That's three levels of induction!

As with planar graphs, we have $3r \leq 2m$. Substituting, we see $n - m + \frac{2m}{3} = 2 - 2\gamma$. We obtain the following bounds.

Corollary 5.54. *If G is a graph with $n \geq 3$ and r regions that embeds on a surface with genus γ, then $m \leq 3\,(n - 2 + 2\gamma)$ and $\gamma\,(G) \geq \frac{m}{6} - \frac{n}{2} + 1$.*

The Generalized Euler Identity and its corollary show that some results for planar graphs generalize naturally to other surfaces. Others, however, do not. For example, a maximal toroidal graph need not be a triangulation (Exercise (14)).

We can substitute the size of K_n, $m = \frac{n(n-1)}{2}$, into the bound in Corollary 5.54. We find $\gamma\,(K_n) \geq \frac{n(n-1)}{12} - \frac{n}{2} + 1 = \frac{n^2-n-6n+12}{12} = \frac{(n-3)(n-4)}{12}$. It turns out that this is an equality.

Theorem 5.55 (**Ringel-Youngs Theorem**). *For $n \geq 3$, $\gamma\,(K_n) = \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil$.*

Proving that K_n embeds on the stated surface turns out to be a very difficult problem. The proof breaks into 12 cases depending on the order mod 12. The proof spanned a series of papers throughout the 1960s, primarily by Gerhard Ringel and J. W. T. Youngs. The proof was completed in (Ringel/Youngs [**1968**]) and published in the book Map Color Theorem (Ringel [**1974**]).

The Four Color Theorem famously states the maximum chromatic number of graphs that embed in the plane. This can be generalized for surfaces of higher genus. First we bound the degeneracy of graphs of a given genus.

Lemma 5.56. *Let G be a monocore graph with genus γ. Then $D\,(G) \leq 6 + \frac{12(\gamma-1)}{n}$.*

Proof. Recall that $\frac{2m}{n}$ is the average degree of a graph. If G embeds on S_γ, then $D\,(G) = \delta\,(G) \leq \frac{2m}{n} \leq \frac{6(n-2+2\gamma)}{n} = 6 + \frac{12(\gamma-1)}{n}$. $\square$

This confirms that $D(G) \leq 5$ if G is planar and confirms $D(G) \leq 4$ when $n < 12$. It also shows that if G is toroidal, $D(G) \leq 6$. Monocore graphs with genus larger than 1 can have large degeneracy with a small order (e.g., complete graphs), but sufficiently large order guarantees degeneracy at most 6.

Recall that the maximum core of a graph G is the maximal subgraph with minimum degree $D\,(G)$.

Theorem 5.57 (**Heawood's Theorem**—Heawood [**1890**]). *Let G be a graph that embeds on surface S_γ, $\gamma > 0$. Then*

$$\chi(G) \leq \left\lfloor \frac{7+\sqrt{1+48\gamma}}{2} \right\rfloor.$$

Proof. Let G be a graph that embeds on surface S_k, $k > 0$, with maximum core H with order n and size m. Let $k = \frac{7+\sqrt{1+48\gamma}}{2}$, so that $1+48\gamma = (2k-7)^2$ and $k = 7 + \frac{12(\gamma-1)}{k}$. Now $\chi(G) \leq 1 + D(H) \leq k$ if $n \leq k$, so suppose $n > k$. By Lemma 5.56,

$$\chi(G) \leq 1 + D(H) \leq 1 + 6 + \frac{12(\gamma-1)}{n} \leq 7 + \frac{12(\gamma-1)}{k} = k = \frac{7+\sqrt{1+48\gamma}}{2}.$$

□

Note that the proof does not work for the plane ($\gamma = 0$), since the final inequality fails in this case. This is despite the fact that the formula would give $\chi(G) \leq 4$ for planar graphs.

Percy Heawood initially believed he had shown that the bound above was sharp. However, it was soon pointed out that this was not the case. Curiously, proving the upper bound is very hard for planar graphs, while showing it is sharp is easy; however, for other surfaces, the upper bound is relatively easy, but showing that it is sharp is very hard. Not until the completion of the Ringel-Youngs Theorem was Heawood's Conjecture verified.

Theorem 5.58 (**Heawood Map Coloring Theorem**). *Let $\chi(S_\gamma)$ be the maximum chromatic number of graphs that embed on S_γ. For $\gamma > 0$,*

$$\chi(S_\gamma) = \left\lfloor \frac{7+\sqrt{1+48\gamma}}{2} \right\rfloor.$$

Proof (Chartrand/Zhang [**2004**]). Let $n = \left\lfloor \frac{7+\sqrt{1+48\gamma}}{2} \right\rfloor$. Certainly $\chi(K_n) = n = \left\lfloor \frac{7+\sqrt{1+48\gamma}}{2} \right\rfloor$.

Now solving $n \leq \frac{7+\sqrt{1+48\gamma}}{2}$ for γ, we find $2n-7 \leq \sqrt{1+48\gamma}$ and $\gamma \geq \frac{(n-3)(n-4)}{12}$. Since γ is an integer, $\gamma \geq \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil = \gamma(K_n)$, so K_n embeds on S_γ. □

A similar formula can also be shown to hold for nonorientable surfaces such as the projective plane and Klein bottle.

The class of graphs that embeds on any given surface is minor-closed, so the Graph Minor Theorem implies that there is a finite list of forbidden minors for each surface. For the plane, there are only two, K_5 and $K_{3,3}$. However, for the torus there are at least 16,000 forbidden minors, and at least 239,000 forbidden subdivisions, and no complete list is known (Chambers [**2002**]). Thus it is not practical to determine whether a graph is toroidal this way. We have seen that there are algorithms to determine whether a graph is planar in linear time. However, determining the genus of a graph is NP-complete.

A key text on the rich subject of topological graph theory is *Graphs of Groups on Surfaces* by Arthur White [**2001**].

Related Terms: splitting number, unit distance graph, Betti number, maximum genus, graph skewness, graph dimension.

Exercises

Section 5.1:

(1) Find the dual of the map of the countries of South America and find a minimum coloring of it.

(2) Find the dual of the map of the contiguous United States and find a minimum coloring of it.

(3) Find a map whose dual is $K_{2,2,2}$.

(4) Find a map whose dual is $C_5 \square K_2$.

(5) Write an algorithm to find a 5-coloring of a planar graph.

(6) Evaluate the following "proof" of the Four Color Theorem.
"Let G be a planar graph. If G has five mutually adjacent vertices, it would require five colors. But K_n is nonplanar for $n \geq 5$. Thus G does not contain K_5. Therefore G is 4-colorable."

(7) If regions that share a common point must be colored differently, how many colors may be required to color a map?

(8) How many colors may be required to color a map with one discontiguous region of two pieces and all other regions contiguous?

(9) If three-dimensional space is divided into contiguous solids, and solids sharing a boundary region with positive area must be colored differently, how many colors may be required?

(10) (Hedetniemi [**1969**]) Show that every planar graph decomposes into two bipartite graphs.

(11) Show that configurations with ring size 3 are reducible.

(12) + (Birkhoff [**1913**]) Show that configurations with ring size 4 are reducible.

Section 5.2:

(1) Three houses are to be connected to three utilities (gas, water, electricity). Can this be done without any lines crossing?

(2) Find plane drawings of W_5 with the exterior region having length 3 and length 5.

(3) Find plane drawings of $K_3 \square K_2$ with the exterior region having length 3 and length 4.

(4) Generalize Euler's Polyhedron Formula to allow disconnected graphs.

(5) Prove Euler's Polyhedron Formula using induction on the number of regions r.

(6) Prove Euler's Polyhedron Formula using induction on n.

(7) Let G be a planar graph with girth g. Show that $m \leq \frac{g}{g-2}(n-2)$.

(8) Use the bound in the previous problem to test the following graphs for planarity.
  (a) the Petersen graph
  (b) the Heawood graph, which has order 14 and girth 6.

(9) Show that all graphs in the following classes are maximal planar.
  (a) $P_{n-2} + K_2$
  (b) $C_{n-2} + 2K_1$
  (c) P_n^3

(10) Verify that the the maximal planar graphs of order 7 are those pictured in Section 5.2.

(11) Draw all 14 maximal planar graphs of order 8.

(12) Show that if G is a maximal planar graph with a three-vertex cutset, those vertices induce K_3. Find the smallest such graph with $\delta(G) > 3$.

(13) Show that if G is a maximal planar graph, $\operatorname{diam}(G) \leq \frac{n+1}{3}$, and this bound is sharp.

(14) Let xyz be a region of $K_{2,2,2}$. Form G by subdividing yz (new vertex a) and replacing the regions containing xy, xz, and yaz with $P_{k-2} + K_2$, $P_{k-2} + K_2$, and $P_k + \overline{K}_2$, respectively (the vertices with large degree are x, y, or z). Finally, add b adjacent to $\{x, y, a\}$ and c adjacent to $\{x, z, a, b\}$. Show that this graph has diameter 2 and $n = \frac{3}{2}\Delta(G)+1$. (*Note*: Seyffarth [**1989**] showed that a diameter 2 maximal planar graph with $\Delta(G) \geq 8$ has $n \leq \frac{3}{2}\Delta + 1$.)

(15) Show that for any planar graph, $\alpha(G) \geq \frac{n}{4}$. Show that this bound is sharp.

(16) (Caro/Roditty [**1985**]) Show that for any maximal planar graph with $n \geq 4$, $\alpha(G) \leq \frac{2n-4}{3}$. Characterize the extremal graphs.

(17) Show that any plane Eulerian graph has an Eulerian circuit that does not cross itself in the plane.

(18) + (Heawood [**1898**]) Show that any maximal planar graph G with $n \geq 3$ has $\chi(G) = 3$ if and only if G is Eulerian.

(19) (Chartrand/Geller [**1969**]) Show that every uniquely 4-colorable planar graph is maximal planar. (*Note*: Fowler [**1998**] showed every uniquely 4-colorable planar graph is a 3-tree.)

(20) Without using the Four Color Theorem, show that planar graphs with at most three triangles are 4-colorable. (*Note*: Grotzsch [**1959**] showed that triangle-free planar graphs are 3-colorable, and Grunbaum [**1963**] extended this to planar graphs with at most three triangles.)

(21) Prove Proposition 5.15 using degeneracy.

(22) Show that every maximal planar graph G with $\delta(G) = 5$ has a degree 5 vertex adjacent to two degree 5 or 6 vertices.

(23) Show that $\operatorname{cr}(G) \geq m - 3n + 6$ for any graph G and that $\operatorname{cr}(G) \geq m - 2n + 4$ if G is triangle-free.

(24) Let G be a graph with blocks B_i. Show that $\operatorname{cr}(G) = \sum \operatorname{cr}(B_i)$.

(25) Find the crossing number of the following graphs.
  (a) K_5

(b) K_6
(c) the Petersen graph

(26) Find the crossing number of the following graphs.
(a) $K_{3,3}$
(b) $K_{3,4}$
(c) $K_{4,4}$
(d) $K_{2,2,3}$

(27) For the Mobius ladder M_n, determine $\operatorname{cr}(M_n)$.

(28) (Harary/Kainen [**1993**]) Use the fact that P_n^3 is maximal planar to find $\operatorname{cr}\left(P_n^4\right)$.

(29) (Guy [**1960**]) Show that $\operatorname{cr}(K_n) \leq \frac{1}{4}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor$. (*Hint*: Position the vertices on two circles of a cylinder.)

(30) (Zarankiewicz [**1954**]) Show that $\operatorname{cr}(K_{r,s}) \leq \left\lfloor\frac{r}{2}\right\rfloor\left\lfloor\frac{r-1}{2}\right\rfloor\left\lfloor\frac{s}{2}\right\rfloor\left\lfloor\frac{s-1}{2}\right\rfloor$. (*Note*: The bounds in this problem and the previous problem are conjectured to be equalities.)

Section 5.3:

(1) Play the game Planarity (`planarity.net`) through level 5.

(2) Play the game Planarity (`planarity.net`) through level 8.

(3) Determine which graphs in the following classes are planar. Find a plane drawing or a forbidden substructure.
(a) W_n
(b) I_n
(c) C_n^2
(d) 2-trees

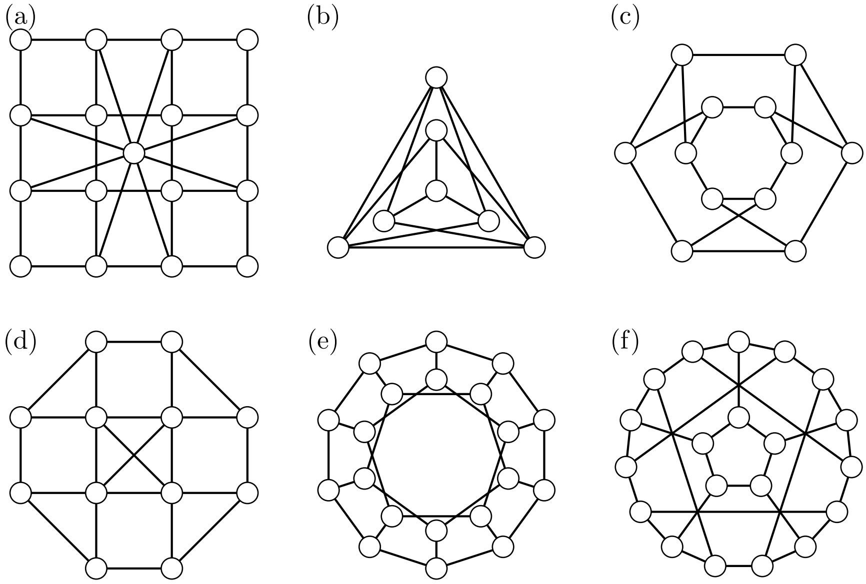
(4) Determine which graphs in the following classes are planar. Find a plane drawing or a forbidden substructure.
(a) $K_{r,s}$
(b) M_n
(c) Q_r
(d) series-parallel graphs

(5) Determine whether the following graphs are planar. Find a plane drawing or a forbidden substructure.
(a) $C_3 \square P_r$
(b) $K_3 \square K_3$
(c) $P_3 \square P_3 \square P_3$

(6) Determine whether the Mycielskian $M(C_n)$ is planar. Find a plane drawing or a forbidden substructure.

(7) Determine which cubic graphs of order 8 are planar. Find a plane drawing or a forbidden substructure.

(8) Determine which graphs drawn below are planar. Find a plane drawing or a forbidden substructure.



(9) Characterize when $G + H$ is planar in terms of properties of G and H.

(10) Characterize when $G \square H$ is planar in terms of properties of G and H.

(11) + Characterize when G^2 is planar in terms of properties of G.

(12) Characterize all graphs G for which G^3 is planar.

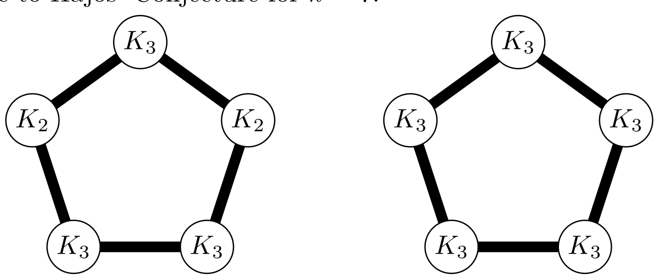
(13) + (Sedlacek [**1964**]) Show that a line graph $L(G)$ is planar if and only if G is planar, $\Delta(G) \leq 4$, and every vertex of degree 4 is a cut-vertex.

(14) Show that any graph formed by 3-sums of maximal planar graphs has no subdivision of K_5. (*Note*: Mader [**2005**] showed that these are the extremal graphs with no K_5-subdivision.)

(15) (Wagner [**1937**]) Show that if G is a 3-connected graph with $n \geq 6$ containing a subdivision of K_5, then G contains a subdivision of $K_{3,3}$.

(16) Find a minimal class of graphs $\mathbb{G}$ such that if a graph G has W_5 as a minor, then G contains a subdivision of some graph in $\mathbb{G}$.

(17) (Catlin [**1979**]) Construct a graph by substituting K_2, K_3, K_2, K_3, K_3 (in order) for the vertices of C_5 (see below left). Show that this graph is a counterexample to Hajos' Conjecture for $k = 7$.



(18) (Catlin [**1979**]) Construct a graph by substituting K_3's for all vertices of C_5 (see above right). Show that this graph is a counterexample to Hajos' Conjecture for $k = 8$. Modify this graph to disprove Hajos' Conjecture for all $k \geq 8$.

(19) (a) Let G be a maximal planar graph with an edge e contained in exactly two triangles. Show that G/e is also maximal planar.
(b) Show that any maximal planar graph with $n \geq 4$ has at least n such edges.

(20) + (Thomassen [**1980**]) Show that every 3-connected graph with $n \geq 5$ has an edge e such that G/e is 3-connected.

(21) State and prove a forbidden minor characterization for forests.

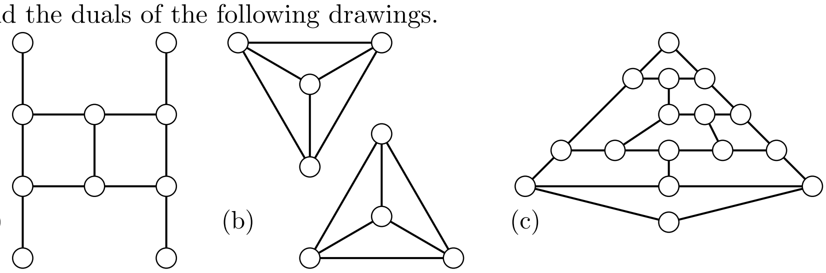
(22) A **linear forest** is a forest whose components are all paths. State and prove a forbidden minor characterization for linear forests.

(23) A cactus is a graph in which every block is a bridge or cycle. State and prove a forbidden minor characterization for cactus graphs.

(24) An **apex graph** G has $G - v$ planar for some vertex v.
(a) Find a sharp upper bound for the size of an apex graph of order n.
(b) Show that apex graphs are minor-closed.

Section 5.4:

(1) Find the duals of the following drawings.



(a) (b) (c)

(2) Find the dual graphs of the graphs in the following classes.
(a) double wheels
(b) C_{2k}^2
(c) $P_{n-2} + K_2$
(d) P_n^3

(3) Find all planar 3-connected cubic graphs of order 10 and their maximal planar duals.

(4) Show that $(G^*)^* = G$ if and only if G is connected.

(5) Let G be a connected planar graph. Show that G is bipartite if and only if G^* is Eulerian.

(6) + Let G be a connected planar graph. Show that G is bipartite if and only if every region of a plane drawing of G has even length.

(7) Show that edges of a planar graph G form a cycle in G if and only if the corresponding dual edges form a bond in G^*.

(8) Show that a set of edges of a planar graph G form a spanning tree of G if and only if the duals of the remaining edges form a spanning tree of G^*.

(9) Show that any subdivision of a 3-connected planar graph has a unique plane drawing up to rotation on a sphere (any plane drawing has the same regions).

(10) Let G be a graph with a single 2-vertex cutset $\{u, v\}$. Prove or disprove: G has a unique dual graph.

(11) Prove or disprove: Every theta graph has a unique plane drawing up to rotation on a sphere.

(12) Determine which graphs formed by identifying the ends of four nontrivial paths have a unique plane drawing up to rotation on a sphere.

(13) For each of the five Platonic solids, find the radius and diameter.

(14) For each of the five Platonic solids, find the independence number and chromatic number.

(15) Find a planar graph with $\delta(G) = 5$ and no adjacent vertices of degree 5.

(16) Find maximal planar graphs of orders 8 and 10 with four vertices of degree 3 and all other vertices with degree at least 6.

(17) Show that if G is self-dual and nontrivial, then $m = 2n - 2$, and G has at least four triangles.

(18) (Servatius/Christopher [**1992**]) Let G be a self-dual graph in which vertex v maps to face f and H be a self-dual graph in which vertex v' maps to face f'. Form a new graph identifying v and v' and f and f'. Show that this graph is self-dual.

(19) Show that a graph G is outerplanar if and only if $G + v$ is planar.

(20) Use the Four Color Theorem to show that any outerplanar graph is 3-colorable.

(21) (Chartrand/Harary [**1967**]) Show that a graph G is outerplanar if and only if G does not contain a subdivision of K_4 or $K_{2,3}$.

(22) Show that any 2-connected outerplanar graph is series-parallel.

(23) The **weak dual** of a graph G is formed by deleting the vertex representing the exterior region in G^*. Show that the weak dual of an outerplanar graph is a forest. Use this to show that a nontrivial outerplanar graph has at least two vertices of degree at most 2.

(24) State and prove an operation characterization of maximal outerplanar graphs (*Hint*: Modify the definition of 2-trees.)

(25) State and prove an upper bound for the size of triangle-free outerplanar graphs.

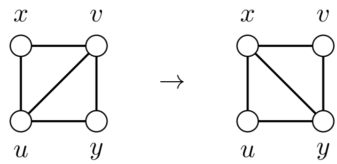
(26) Show that an outerplanar graph G has $m \leq 2n - 3$ using Exercise (19).

(27) Let G be a maximal outerplanar graph with $n \geq 4$ and with n_i vertices of degree i. Show that $6 = \sum (4 - i)\, n_i$.

(28) A planar graph is **nearly maximal planar** if every region is bounded by a cycle, at most one of which is not a triangle. Find sharp bounds on the size of a nearly maximal planar graph of order n.

(29) Let G be a maximal planar graph with edge $e = uv$, and let x and y be the other vertices of the triangular regions containing e. A **flip** of uv deletes uv and adds xy (assuming it is not already an edge). An edge is **flippable** if a

flip of it does not create parallel edges. (Bose/Hurtado [**2009**] surveys related concepts.)



(a) + (Wagner [**1936**]) Show that any maximal planar graph can be converted to (the 3-tree) $P_{n-2} + K_2$ via a sequence of flips.
(b) (Gao et al. [**2001**]) Show that a maximal planar graph with $n \geq 5$ may have as few as $n - 2$ flippable edges.

(30) Show that a 3-tree is (maximal) planar if and only if when constructed, each three vertices that are the neighborhood of a new vertex form a triangle and cannot be used again as the neighborhood of another new vertex.

(31) Show that the bound of the Art Gallery Theorem is sharp for each n.

(32) + (Kahn et al. [**1983**]) An **orthogonal art gallery** has every corner at a right angle. Show that an orthogonal art gallery with n vertices requires at most $\left\lfloor \frac{n}{4} \right\rfloor$ guards at its vertices to see all of its interior.

(33) Find a straight edge plane drawing of K_4 where edge has integer length.

(34) Find a straight edge plane drawing of $K_5 - e$ where edge has integer length.

(35) Determine which graphs in the following classes are penny graphs.
(a) $K_{1,s}$
(b) C_n
(c) W_n
(d) $G_{r,s}$

(36) (a) Show that every penny graph is 3-degenerate. (*Note*: This shows that they are 4-colorable without using the Four Color Theorem.)
(b) Find a 4-chromatic penny graph.

(37) Let T be a tree that is a penny graph with adjacent vertices u and v. Show that $\Delta(T) \leq 5$ and $d(u) + d(v) \leq 9$.

(38) Let G be a penny graph. Harborth [**1974**] showed that $m \leq \left\lfloor 3n - \sqrt{12n - 3} \right\rfloor$. Show that this bound is sharp.

(39) Find the smallest cubic matchstick graph.

(40) Prove or disprove: Every matchstick graph is a penny graph.

(41) Show that every tree is a matchstick graph. Further, show that such a drawing of a tree can be made inside an arbitrarily thin wedge with any vertex of the tree at the vertex of the wedge.

(42) Find a unit distance graph with order 10 and chromatic number 4 that does not contain the Moser spindle.

(43) Determine which graphs in the following classes are unit distance graphs.
(a) K_n
(b) W_n
(c) $G_{r,s}$

(44) Determine which graphs in the following classes are unit distance graphs.
(a) $K_{r,s}$
(b) Q_r
(c) the Petersen graph

(45) Determine which graphs in the following classes are rigid, flexible, or Laman graphs.
(a) K_n
(b) $K_{r,s}$
(c) C_n
(d) cubic graphs

(46) Determine which graphs in the following classes are rigid, flexible, or Laman graphs.
(a) W_n
(b) $G_{r,s}$
(c) Q_r
(d) theta graphs

(47) (Hass et al. [**2005**]) Show that the Henneberg construction of a Laman graph may start with any pair of vertices of the graph.

(48) (Haas et al. [**2005**]) Show that if a Laman graph contains three degree 3 vertices that induce a triangle, then the Henneberg construction may start with them.

(49) Show that Henneberg operation 1 preserves minimal rigid graphs.

(50) Show that the **double banana** $2K_3 + 2K_1$ satisfies $m = 3n - 6$ and that each subgraph with order $n' \geq 3$ has size $m' \leq 3n' - 6$, but it is not rigid in three dimensions.

Section 5.5:

(1) Find toroidal embeddings of K_4 whose regions have lengths
(a) 4 and 8.
(b) 3 and 9.

(2) Find toroidal embeddings of $K_{3,3}$ whose regions have lengths
(a) 6,6,6.
(b) 4,4,8.

(3) Find toroidal embeddings of K_5 whose regions have lengths
(a) 3,3,4,4,6.
(b) 3,3,3,4,7.
(c) 3,3,3,3,8, different from the example in the section.

(4) Show that there are three different embeddings of the Petersen graph on the torus.

(5) Find toroidal embeddings of the following graphs.
(a) $K_{4,4}$
(b) M_8
(c) Q_4

(6) Find toroidal embeddings of the following graphs.
 (a) K_6
 (b) $K_{3,6}$
 (c) $C_r \square C_s$

(7) Prove or disprove: Every disconnected graph has a 2-cell embedding on some surface.

(8) Generalize the generalized Euler Identity to graphs with k components.

(9) Show that if G is connected and triangle-free, then $\gamma(G) \geq \frac{m}{4} - \frac{n}{2} + 1$.

(10) Show that if G is a connected graph with girth g, then $\gamma(G) \geq \frac{m}{2}\left(1 - \frac{2}{k}\right) - \frac{n}{2} + 1$.

(11) Show that for $r, s \geq 2$, $\gamma(K_{r,s}) \geq \left\lceil \frac{(r-2)(s-2)}{4} \right\rceil$. (*Note*: Ringel [**1965**] showed that this is an equality.)

(12) + Show that for for $k \geq 2$, $\gamma(Q_k) = (k-4)\,2^{k-3} + 1$. (*Hint*: Use the definition.)

(13) Let $n \geq 9$ be not prime or twice a prime. Find a 6-regular graph of order n that embeds on the torus.

(14) (a) + (Duke/Haggard [**1972**]) Show that $K_8 - K_3$, $K_8 - (2K_2 \cup P_3)$, and $K_8 - K_{2,3}$ are the only minimal nontoroidal graphs of order 8.
 (b) + (Harary/Kainen et al. [**1974**]) Show that $K_8 - C_5$ is a maximal toroidal graph that does not triangulate the torus.

(15) Show that for any $k > 0$, there is a planar graph G so that $\gamma(G \square K_2) \geq k$.

(16) + (Battle/Harary/Kodama/Youngs [**1962**]) Show that for a graph G with blocks B_i, $\gamma(G) = \sum \gamma(B_i)$.

(17) The **skewness** $\mu(G)$ of a graph G is the minimum number of edges whose removal results in a planar graph. Show that for any graph G, $\gamma(G) \leq \mu(G) \leq \operatorname{cr}(G)$.

(18) (Nordhaus/Stewart/White [**1971**]) Let $\gamma_M(G)$ be the **maximum genus** of a surface on which G has a 2-cell embedding. Show that if G is a connected graph, then $\gamma_M(G) \leq \left\lfloor \frac{m-n+1}{2} \right\rfloor$, with equality if and only if G has a 2-cell embedding on a surface with maximum genus with one or two regions when $m - n + 1$ is odd or even, respectively.

Chapter 6

Hamiltonian Graphs

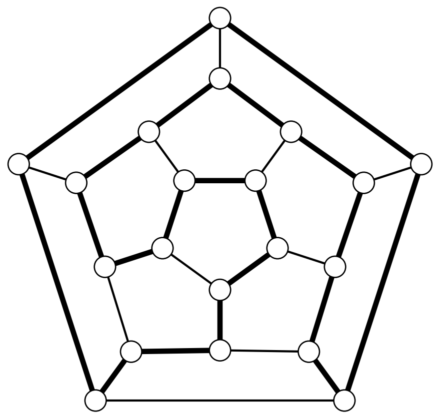
A traveling salesman must drive to a number of cities to sell his wares before returning home. He would like to visit each city exactly once, so as not to waste time. Is there a route satisfying these conditions? If so, how can it be found?

This problem may sound similar to the Chinese Postman Problem, but there is an important difference. When modeling it with a graph, we only need to visit every vertex, not every edge. The salesman has no desire to drive on every road!

Definition 6.1. A **Hamiltonian cycle** of a graph G is a spanning cycle of G. A graph with a Hamiltonian cycle is called a **Hamiltonian graph**. A **Hamiltonian path** of a graph G is a spanning path of G. A graph is **traceable** if it has a Hamiltonian path.

Any Hamiltonian graph necessarily contains a Hamiltonian path, but the converse does not hold.

Hamiltonian graphs are named after Irish mathematician William Rowan Hamilton, though they were actually introduced slightly earlier by English minister Thomas Kirkman. Hamilton popularized the problem in 1857 when he invented a game called the Icosian puzzle. He used a dodecahedron to represent the Earth, with vertices labeled with twenty different cities. The goal was to find a route that went through each city exactly once and returned to the starting point. The game did not sell well, probably because it was easy to solve and did not have much replayability.



Unlike the problem of Eulerian graphs, there is no simple characterization of Hamiltonian graphs. Instead, we must consider various necessary and sufficient conditions which solve the problem for some graphs.

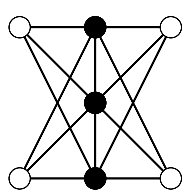
6.1. Finding Hamiltonian Cycles

6.1.1. Necessary Conditions. A graph G with a cut-vertex v cannot be Hamiltonian. Once a path left a component of $G - v$ through v, there would be no way to return without using v again. Thus any Hamiltonian graph must be 2-connected. When a graph is bipartite, a path must alternate between its partite sets. Thus any bipartite graph with partite sets with different cardinalities cannot be Hamiltonian. These observations generalize.

Proposition 6.2. *If G is a graph containing a set $S \subset V(G)$ such that $G - S$ has more than $|S|$ components, then G is not Hamiltonian.*

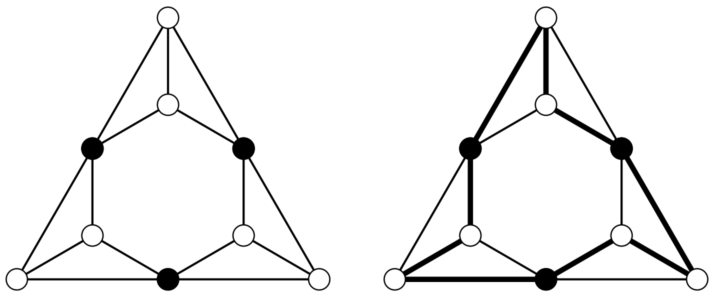
Proof. Any Hamiltonian cycle must use a distinct vertex of S each time it leaves S, so if G is Hamiltonian, $G - S$ has at most $|S|$ components. $\square$

Example. In the graph G below, let S be the three solid vertices. Now $G - S$ has four components, so G is not Hamiltonian.



For relatively small graphs, Hamiltonian cycles can be found through trial and error. However, using the structure of a graph can limit the number of cases that must be considered. When a graph contains a set S so that $G - S$ has exactly $|S|$ components, Proposition 6.2 implies that if G has a Hamiltonian cycle, it must alternate between single vertices of S and spanning paths of components of $G - S$.

Example. In the figure below, removing the three black vertices results in three components. Connecting these vertices through each component results in a Hamiltonian cycle.



Note that how a graph is drawn matters for finding a Hamiltonian cycle. If the graph is drawn with the cycle on the outside and the other edges inside, it can easily be seen to be Hamiltonian. A drawing may reveal (or conceal) symmetry in the graph that can reduce the number of cases to consider.

Minimal and maximal Hamiltonian graphs are C_n and K_n, respectively. Both are easily seen to be Hamiltonian; the graphs in between are harder to check for a Hamiltonian cycle. It is sometimes helpful to consider a Hamiltonian graph as just a cycle with some chords added.

The converse of Proposition 6.2 is false. There are other ways to show that a Hamiltonian cycle cannot exist or to find one when it does. We use the following observations.

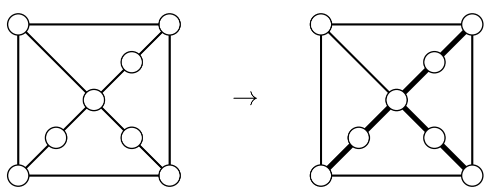
Remark 6.3. *Let e be an edge of a graph G such that $G-e$ has a cut-vertex. Then every Hamiltonian cycle of G contains e.*

In particular, the edges of any minimum 2–edge cut must be on any Hamiltonian cycle. That includes both edges incident with a vertex of degree 2.

Remark 6.4. *Let H be the graph induced by all edges that must be in any Hamiltonian cycle of G. If G is Hamiltonian, then $\Delta(H) \leq 2$, and H has no nonspanning cycle.*

This provides another way to show that a graph is not Hamiltonian.

Example. In the graph below, each thick edge is incident with a degree 2 vertex, and so must be in any Hamiltonian cycle. But three thick edges are incident with a single vertex, so no Hamiltonian cycle exists.



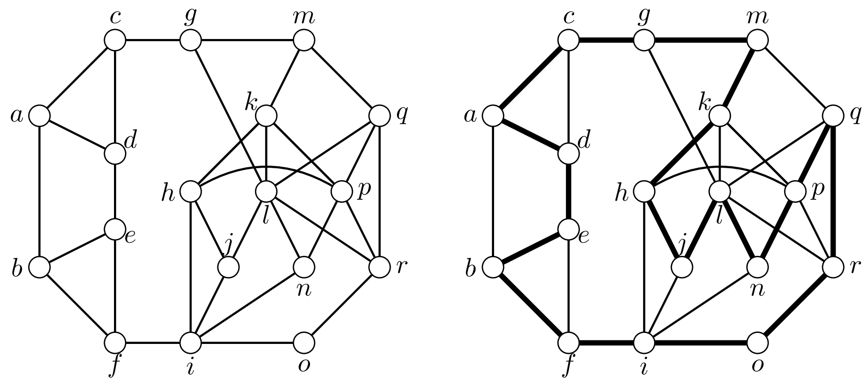
Remark 6.5. *If two edges incident with a vertex v of a graph G must be on any Hamiltonian cycle, then any other edges incident with v cannot be on any Hamiltonian cycle of G.*

When the degrees of a graph are not too large, these observations narrow the cases to determine whether a graph has a Hamiltonian cycle.

Example. Consider the graph below left. Note that cg and fi form a 2–edge cut, so they must be contained in any Hamiltonian cycle. Vertex o has degree 2, so edges io and or must be on any Hamiltonian cycle. Now we already have two edges incident with i, so edges ih, ij, and in cannot be on any Hamiltonian cycle. That leaves only two edges each incident with j and n, so hj, jl, ln, and np must be on any Hamiltonian cycle. The edge hp cannot be on a Hamiltonian cycle, since including it would create a nonspanning cycle, $hjlnph$. Since two edges incident with l have been used, edges lg, lk, lq, and lr cannot be used.

Three edges incident with k remain; there are three possible choices of two to use on a Hamiltonian cycle. Using hk and kp would create a nonspanning cycle, so only one of them can be used, and km must be used.

Similar reasoning shows that gm must be used, mq cannot be used, and pq and qr must be used. A $c-f$ path through the left side of the graph is easily found. A Hamiltonian cycle is shown at right.

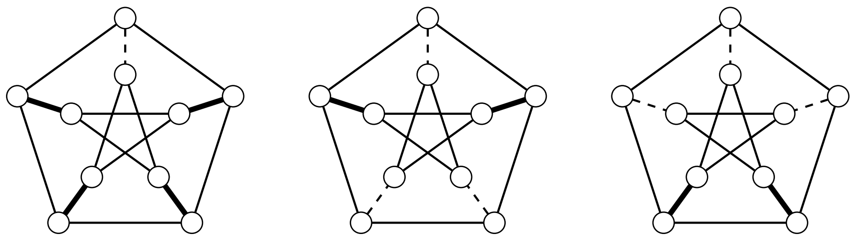


A similar argument can be used to show that a graph cannot contain a Hamiltonian cycle by finding either three edges incident with a single vertex that must be on any Hamiltonian cycle or a nonspanning cycle that must be on any Hamiltonian cycle.

When a graph is 3-connected, there is no easy starting point, and multiple cases must be considered. For example, there are three possible ways a Hamiltonian cycle can go through a vertex of degree 3. Some graphs have symmetry which allows the elimination of multiple cases.

Proposition 6.6. *The Petersen graph is non-Hamiltonian.*

Proof. Consider the drawing of the Petersen graph with two 5-cycles and five spokes. Any Hamiltonian cycle C must use either two or four spokes. The two spokes must be incident with vertices that are either adjacent or not on the outside cycle. In each case, excluding the other spokes would create a 5-cycle composed of edges that must be in C. $\square$



This completes the classification of the cycles of the Petersen graph begun in Chapter 1. It has cycles of length 5, 6, 8, and 9, and no cycles of length 3, 4, 7, and 10.

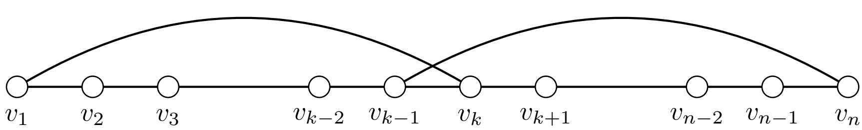
Definition 6.7. A graph G is **hypohamiltonian** if G is non-Hamiltonian, but for any vertex v, $G - v$ is Hamiltonian.

This definition is inspired by the Petersen graph, which is the hypohamiltonian graph of smallest order (the next smallest has order 13).

6.1.2. Sufficient Conditions. There are also sufficient conditions for a graph to be Hamiltonian.

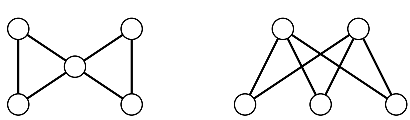
Theorem 6.8 (**Ore's Theorem**—Ore [**1960**]). *Let G be a graph with $n \geq 3$, and let $d(u) + d(v) \geq n$ for any pair of nonadjacent vertices u and v. Then G is Hamiltonian.*

Proof. Assume to the contrary that there is such a graph G that is not Hamiltonian. We may assume that G is maximal non-Hamiltonian; that is, if any edge is added, a Hamiltonian cycle would be created. Then there must be a Hamiltonian path P, $v_1 v_2 \cdots v_n$ between nonadjacent vertices v_1 and v_n of G. Now P has $n-3$ edges not adjacent to v_1 or v_n. Every other neighbor of v_1 and v_n is on P. For each edge $v_1 v_k$, mark the edge $v_{k-1} v_k$ of P. Similarly, mark $v_k v_{k+1}$ for each $v_k v_n$. Now at least $n-2$ edges are marked, so by the Pigeonhole Principle, some edge is marked twice. Then G has a Hamiltonian cycle of the form $v_1 v_k v_{k+1} \cdots v_n v_{k-1} v_{k-2} \cdots v_1$, a contradiction. □



Corollary 6.9 (**Dirac's Theorem**—Dirac [**1952**]). *Let G be a graph with $n \geq 3$ and $\delta(G) \geq \frac{n}{2}$. Then G is Hamiltonian.*

Both bounds are sharp. For example, $K_1 + 2K_k$ and $K_{k,k+1}$ are extremal graphs when $n = 2k+1$ is odd.



We can improve on Ore's Theorem somewhat.

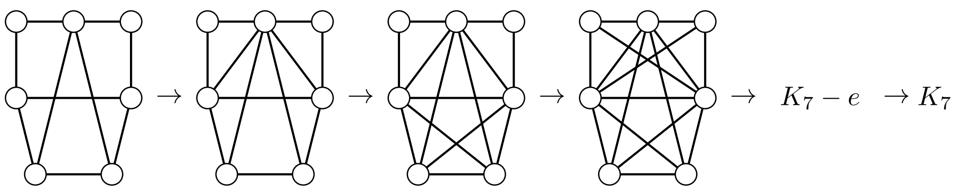
Lemma 6.10 (Ore [**1960**]). *Let u and v be nonadjacent vertices of a graph G with $d(u) + d(v) \geq n$. Then G is Hamiltonian if and only if $G + uv$ is Hamiltonian.*

Proof. ($\Rightarrow$) This is immediate since adding an edge cannot destroy a Hamiltonian cycle.

($\Leftarrow$) Assume $G + uv$ is Hamiltonian and, to the contrary, G is not. Then every Hamiltonian cycle in $G + uv$ contains uv, so G has a $u - v$ Hamiltonian path. Since $d(u) + d(v) \geq n$, the proof of Theorem 6.8 shows that G is Hamiltonian. $\square$

Definition 6.11. The **closure** of a graph G, $\mathrm{cl}(G)$, is the graph obtained from G by repeatedly adding edges between pairs of nonadjacent vertices whose degree sum is at least n until none remain.

Example. Starting with the graph below left, the closure is found in five iterations.



Lemma 6.12. *The closure of a graph is well-defined (it does not depend on the order the edges are added).*

Proof. Let $e_1, \ldots, e_s$ and $e'_1, \ldots, e'_t$ be sequences of edges added to form $\mathrm{cl}(G)$, obtaining graphs G_1 and G_2, respectively. If nonadjacent vertices u and v have $d(u) + d(v) \geq n$ at some point in the process, then uv will be added somewhere in the sequence. Thus e_1 belongs to G_2. Similarly, if $e_1, \ldots, e_i$ are in G_2, then so must be e_{i+1}. Thus each sequence contains all edges of the other, so $G_1 = G_2$. $\square$

Repeated application of Lemma 6.10 produces the following.

Theorem 6.13 (Bondy/Chvatal [**1976**]). *A graph is Hamiltonian if and only if its closure is Hamiltonian.*

This may be useful if the closure of a graph is complete or some other obviously Hamiltonian graph.

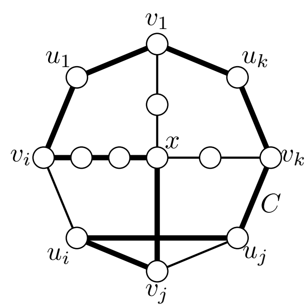
All the necessary conditions considered so far require about half of the possible edges to be present. Some additional assumptions can lower this requirement.

Theorem 6.14 (Jackson [**1980**], Hilbig [**1986**]). *If G is a 2-connected regular graph with $\delta(G) > \frac{n-1}{3}$, then G is Hamiltonian. The Petersen graph is the unique extremal graph.*

We have seen that Hamiltonian graphs must be 2-connected. There is a necessary condition comparing connectivity and independence number.

Theorem 6.15 (Chvatal/Erdos [**1972**]). *If G is a graph with $n \geq 3$ and $\kappa(G) \geq \alpha(G)$, then G is Hamiltonian.*

Proof. The hypothesis implies $\kappa(G) \geq 2$. Then G contains a cycle; let C be one of maximum length. By Theorem 2.47, there are at least $k = \kappa(G)$ vertices on C. Assume to the contrary that C is not a Hamiltonian cycle, and let x be a vertex not on C. By Lemma 2.46, there are k independent paths from x to $v_1, \ldots, v_k$ on C. If any of these vertices are consecutive, the cycle could be extended through x via two paths. If not, let u_i be the vertex following v_i for some orientation of C. No vertex u_i is adjacent to x, since otherwise the cycle could be extended through x. If there is an edge of the form $u_i u_j$, there is a longer cycle $v_1 v_2 \cdots v_i \cdots x \cdots v_j \cdots u_i u_j \cdots v_1$. If not, then $\{x, u_1, \ldots, u_k\}$ is an independent set of size $k+1$, a contradiction. $\square$



We have mentioned that there is no simple characterization of Hamiltonian graphs. In fact, there is no known efficient algorithm for determining whether a graph is Hamiltonian; there is an algorithm with complexity $\mathcal{O}(1.657^n)$ (Bjorklund [**2010**]). The problem of determining whether a graph is Hamiltonian is NP-complete.

6.1.3. Generalizations. There are several generalizations of Hamiltonian graphs. When a graph is not Hamiltonian, we may settle for something less, such as a long cycle or a Hamiltonian path.

Definition 6.16. The **circumference** $c(G)$ of a graph G is the length of its longest cycle.

A Hamiltonian graph has circumference n, while a hypohamiltonian graph has circumference $n-1$. Many results on Hamiltonian graphs generalize to results on circumference. Bondy generalized Ore's Theorem as follows.

Theorem 6.17 (Bondy [**1971**]). *If G is 2-connected and $d(u)+d(v) \geq s$ for every nonadjacent pair of vertices u and v, then $c(G) \geq \min\{n, s\}$.*

Fan used Bondy's theorem to prove a stronger theorem.

Theorem 6.18 (Fan [**1984**]). *If G is 2-connected and $d(u,v) = 2$ implies $\max\{d(u), d(v)\} \geq \frac{s}{2}$, then $c(G) \geq \min\{n, s\}$.*

Tian [**2004**] found a short proof of this theorem. When $s = n$, this yields a sufficient condition for Hamiltonian cycles stronger than previous theorems.

The problem of determining whether a graph contains a Hamiltonian path is closely related to that of finding a Hamiltonian cycle.

Proposition 6.19. *A graph G has a Hamiltonian path if and only if $G + K_1$ contains a Hamiltonian cycle.*

This follows since a Hamiltonian path of G can be extended to a cycle through the new vertex. Many necessary and sufficient conditions for the existence of a Hamiltonian cycle have corresponding versions for Hamiltonian paths. For example, $\delta(G) \geq \frac{n-1}{2}$ guarantees a Hamiltonian path. Corresponding versions of other theorems are explored in the Exercises.

When a graph is Hamiltonian, we can consider stronger conditions.

Definition 6.20. A graph is **Hamiltonian-connected** if for every pair of vertices u, v there is a Hamiltonian $u - v$ path.

Any Hamiltonian-connected graph is Hamiltonian, as can be seen by choosing an adjacent pair of vertices. There are similar (stricter) conditions that guarantee a graph is Hamiltonian-connected.

Theorem 6.21 (Ore [**1963**]). *Let G be a graph with $d(u) + d(v) \geq n + 1$ for any pair of nonadjacent vertices. Then G is Hamiltonian-connected.*

Proof (Bondy/Chvatal [**1976**]). Assume the hypotheses. The result is trivial for $n = 1$; assume $n \geq 2$. Now the $(n+1)$-closure of G, formed by recursively joining nonadjacent vertices with degree sum at least $n + 1$, is complete. Let u and v be any two vertices of G. Let H be formed by adding vertex x and edges ux and vx to G. Then H has order $n + 1$, so the vertices of G in $\mathrm{cl}(H)$ induce K_n. Now for all $y \in V(G)$, $d_{\mathrm{cl}(H)}y + d_{\mathrm{cl}(H)}x \geq n + 1$, so $\mathrm{cl}(H) = K_{n+1}$. By Theorem 6.13, H is Hamiltonian. Any Hamiltonian cycle of H uses ux and vx, so G has a Hamiltonian $u - v$ path. □

This theorem implies that $\delta(G) \geq \frac{n+1}{2}$ guarantees that a graph G is Hamiltonian-connected.

Related Terms: toughness, Hamiltonian walk, Hamiltonian number, upper Hamiltonian number, panconnected graph, pancyclic, vertex-pancyclic, dominating circuit, Goldner-Harary graph.

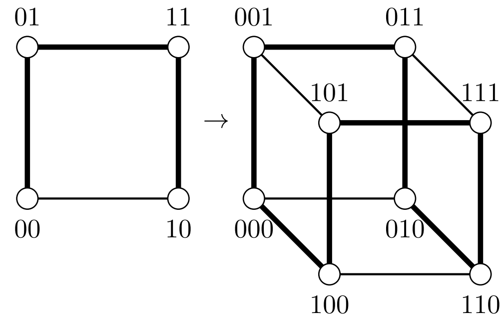
6.2. Hamiltonian Applications

6.2.1. Grey Codes. Suppose that we have k independent binary parameters and we need to check every possible combination of them. There are 2^k possible combinations. We could list them in lexicographic order. For example, when $k = 3$, we have 000, 001, 010, 011, 100, 101, 110, 111. But suppose there is a nontrivial cost to change one of the parameters. When we change from 011 to 100, all three parameters must change. It would be desirable to have an ordering for all bitstrings of length k so that consecutive strings (including the first and last) differ in only one digit.

Recall that the hypercube Q_k can be defined with vertices representing bitstrings of length k which are adjacent when they differ in a single digit. Thus our problem is to find a Hamiltonian cycle in Q_k.

Proposition 6.22. *For $k \geq 2$, Q_k is Hamiltonian.*

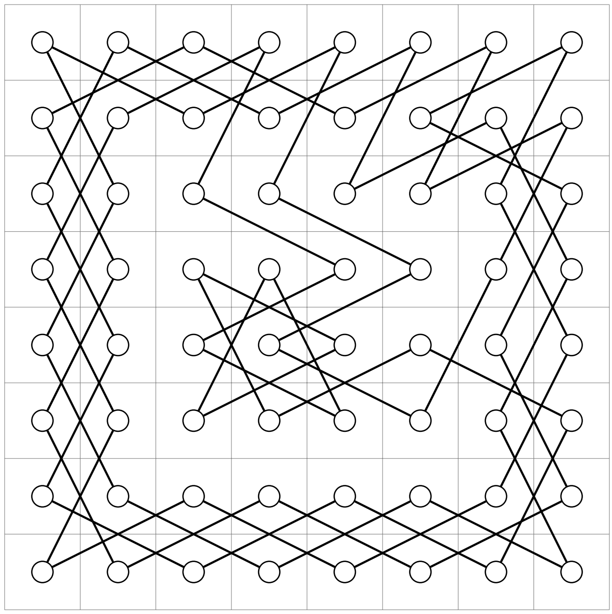
Proof. We use induction on k. For $k = 2$, $Q_k = C_4$ is a cycle. Assume for some $k \geq 2$ that Q_k is Hamiltonian, and hence traceable. Recall that Q_{k+1} can be formed from two copies of Q_k by adding edges between corresponding vertices. Follow a Hamiltonian path in one copy of Q_k, take an edge to the other copy, follow the corresponding path in reverse, and take an edge back to the start. Thus Q_{k+1} is Hamiltonian. $\square$



When $k = 3$, the ordering described is 000, 001, 011, 010, 110, 111, 101, 100. Such an ordering is known as a **Grey code** after Frank Grey, who studied this concept in 1947. The problem actually dates back to at least 1878. Grey dealt with the problem of a machine switching between bitstrings. Each digit is controlled by a separate switch, which in practice don't switch at exactly the same time. Thus between 01 and 10, we might briefly obtain the unwanted string 00 or 11. Grey codes eliminate this uncertainty.

6.2.2. Knight's Tours. An old problem in recreational mathematics is finding a knight's tour. The pieces on a chessboard have various rules for how they can be moved. The knight can only be moved two squares in one direction and one in the other. A **knight's tour** is a sequence of knight's moves on a chessboard so that each square is visited exactly once and the knight returns to the starting position.

This problem can be modeled with each vertex representing a square of the board and edges between squares that can be reached by knight's moves. The standard 8-by-8 chessboard has a knight's tour. Indeed, it has 26,534,728,821,064 different knight's tours. One of them is shown below.



This problem can be generalized to boards of different sizes.

Theorem 6.23 (**Schwenk's Theorem**—Schwenk [**1991**]). *An m-by-n board, $m \leq n$, has a knight's tour unless*

(1) *m and n are both odd.*

(2) *$m \in \{1, 2, 4\}$.*

(3) *$m = 3$ and $n \in \{4, 6, 8\}$.*

Parts of the proof are requested in the Exercises.

6.2.3. The Traveling Salesman Problem. Consider the traveling salesman who wants to visit a number of cities once and then return home. If we suppose that there are roads between each city, then there is no difficulty finding a Hamiltonian cycle. However, not all trips between two cities are equally difficult. We model this situation with a weighted complete graph, where the weights represent distance, travel time, or cost.

Definition 6.24. The **Traveling Salesman Problem** (TSP) is to find a Hamiltonian cycle with minimum total edge weight in a weighted complete graph.

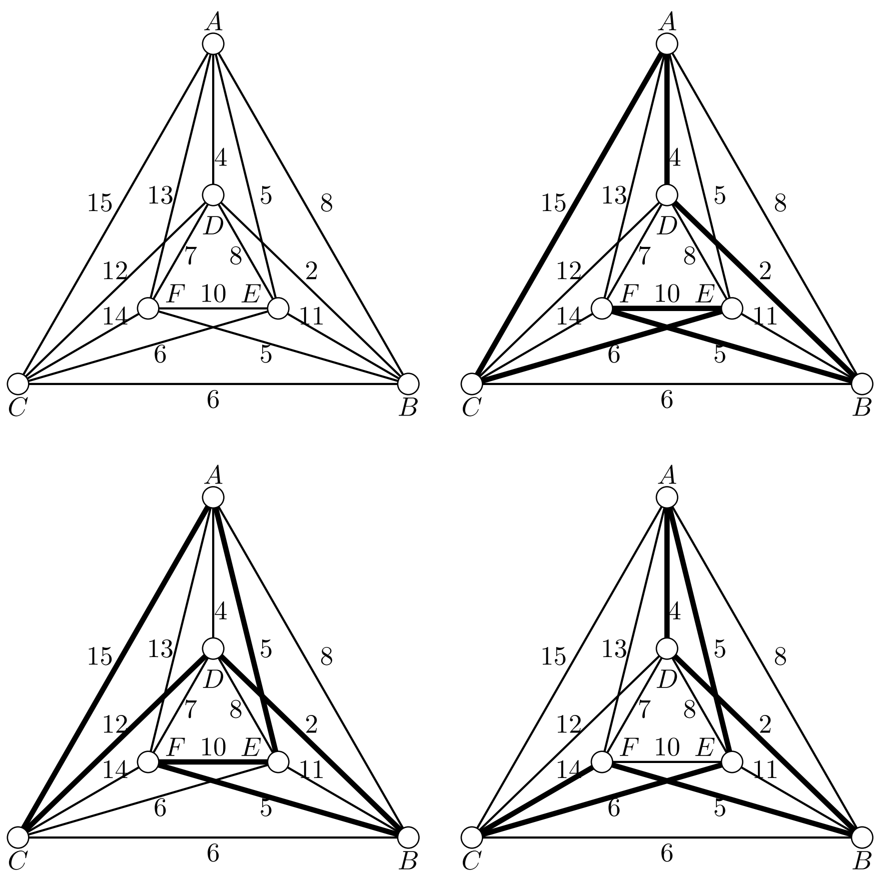
One way of addressing this problem would be to find all Hamiltonian cycles, add up their costs, and select the smallest. But how many Hamiltonian cycles are there in K_n? There are $n!$ distinct vertex orderings. The same cycle can be obtained by starting at n different vertices and traveling in two different directions. Thus there are $\frac{n!}{2n} = \frac{(n-1)!}{2}$ distinct cycles. Thus the number of cycles grows far too fast for it to be practical to check them all.

One of the best known algorithms for solving the TSP, the Held-Karp algorithm (Held/Karp [**1962**]), is only somewhat more efficient, namely $\mathcal{O}\left(n^2 2^n\right)$. In fact, the TSP is an example of an NP-complete problem. A solution to the TSP would imply an algorithm for determining whether a graph G has a Hamiltonian cycle. Specifically, define a complete graph with the edges of G labeled 1 and the other edges labeled infinity. Then solving the TSP on this complete graph would produce a Hamiltonian cycle in G, if one exists.

In the absence of a practical exact solution to the TSP, the best one can hope for is a cycle that is relatively close to the minimum total edge weight. We consider two possible ways of producing such a cycle.

Algorithm 6.25 (**Nearest Neighbor Algorithm**). *Start from a given vertex* (*home*) *and repeatedly visit the nearest neighbor not already visited* (*using the incident edge with smallest weight*). *When all vertices have been visited, return home.*

Example. Apply the Nearest Neighbor Algorithm starting at vertex A and D to the TSP for the graph at top left in the figure below. Vertex A is incident with edges labeled 15, 13, 4, 5, and 8. Use the edge with weight 4 to go to D. Vertex D is incident with edges with weights 2, 8, 7, and 12. Take the edge with weight 2 to B. Continuing, we find the Hamiltonian cycle A, D, B, F, E, C, A with total weight 41 (top right). Starting at D, we find the cycle D, B, F, E, A, C, D with total weight 49 (bottom left).



The result of the Nearest Neighbor Algorithm depends on the starting vertex. The second algorithm does not require a starting point.

Algorithm 6.26 (**Sorted Edges Algorithm**). *Repeatedly select the edge with smallest weight that does not create a vertex of degree* 3 *or non-Hamiltonian cycle in the graph induced by the selected edges. Stop when a Hamiltonian cycle is produced.*

Example. Apply the Sorted Edges Algorithm to the TSP for the graph in the previous example. Sorting the edges by weight, we find 2, 4, 5, 5, 6, 6, 7, 8, 8, 10, 11, 12, 13, 14, 15. We add the first four (BD, AD, AE, BF) with no diffuculty. We cannot add BC, since vertex B has already been used twice. We add CE. At this point, the only option is to add CF and complete the cycle. In order, the cycle is A, D, B, F, C, E, A (bottom right in the figure above) with total weight 36.

The Sorted Edges Algorithm is guaranteed to produce a Hamiltonian cycle. However, neither algorithm is guaranteed to produce a cycle with minimum total edge weight. Both can fail to produce good results when the last edge or two that must be selected have large weights. In fact, neither can even be guaranteed to produce a weight within a constant multiple of the correct answer!

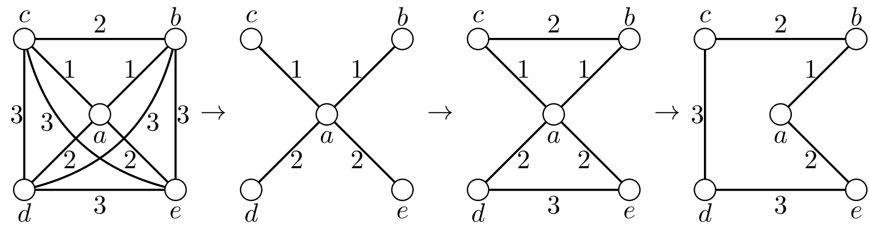
The Sorted Edges Algorithm is reminiscent of Kruskal's Algorithm for finding a minimum spanning tree. In fact, there is a connection between that problem and the TSP. Let G be a weighted complete graph, and let $T(G)$, $P(G)$, and $C(G)$ be the minimum weights of a spanning tree, spanning path, and spanning cycle, respectively. Now each spanning cycle contains a spanning path, which is a spanning tree. Thus $T(G) \leq P(G) \leq C(G)$, producing a lower bound for the TSP.

A minimum spanning tree is also useful for finding an upper bound in a special case. We assume that the weight function $l(e)$ satisfies the **triangle inequality**, $l(uw) \leq l(uv) + l(vw)$, for any vertices u, v, and w. This is certainly true for geometric distance.

We begin by finding a closed walk that contains all vertices (perhaps repeated) and then refine this walk to obtain a Hamiltonian cycle. To find the walk, begin with a minimum spanning tree T. Traversing each edge twice would produce the walk. However, a more efficient solution is to use k edges to travel between the $2k$ odd vertices of T. Thus we choose a minimum weight perfect matching M on the $2k$ odd vertices of T. The edges of T and M (duplicating those in both) make all degrees even, yielding an Eulerian trail.

To produce a Hamiltonian cycle, list the vertices of the trail in order and delete any that appear earlier in the trail until returning to the first vertex. This produces a spanning cycle of G. This process is known as **Christofides' Algorithm.**

Example. Consider the graph below left. The next graph is a minimum spanning tree. Now $\{bc, de\}$ is a minimum weight matching. Adding these edges yields the Eulerian trail *abcadea*. To find a Hamiltonian cycle, we replace *cad* with *cd*, producing the cycle *abcdea*.



Theorem 6.27 (Christofides [**1976**]). *Let G be a complete graph with a weight function $l(e)$ that satisfies the triangle inequality. Let $l(C)$ be the weight of the cycle C produced by Christofides' Algorithm. Then $C(G) \leq l(C) \leq \frac{3}{2}C(G)$.*

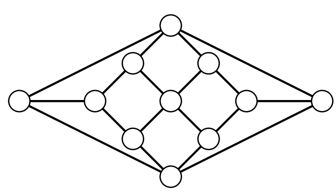
Proof. The first bound follows since Christofides' Algorithm produces a Hamiltonian cycle. Let C, W, T, and M be the cycle, trail, tree, and matching produced by the algorithm. Now the $2k$ odd vertices of T split a minimum spanning cycle into two sets of disjoint paths, one of which has at most half its weight. Thus $l(M) \leq \frac{1}{2}C(G)$. The triangle inequality guarantees that taking a more direct route cannot increase the weight. Thus

$$l(C) \leq l(W) = l(T) + l(M) \leq C(G) + \frac{1}{2}C(G) = \frac{3}{2}C(G). \quad \square$$

Perhaps surprisingly, this is essentially the best known bound on the TSP that is efficient to compute. Note that $\frac{3}{2}C(G)$ is the worst case for the upper bound; Christofides' Algorithm will often produce a closer bound.

6.3. Hamiltonian Planar Graphs

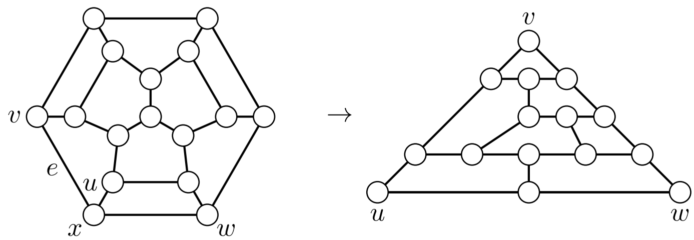
Any graph representing a convex polyhedron is planar and 3-connected. Since Hamiltonian graphs were introduced by showing that the dodecahedron is Hamiltonian, it is natural to ask whether other planar 3-connected graphs must be Hamiltonian. The **Herschel graph**, which is not regular, is not Hamiltonian.



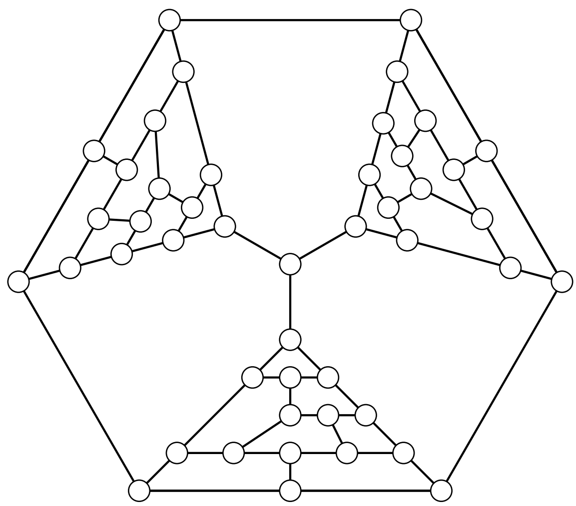
However, this problem is harder when restricted to cubic graphs. Peter Tait believed that all 3-connected cubic planar graphs are Hamiltonian. This came to be known as **Tait's Conjecture** (Tait [**1884**]). This conjecture is closely related to edge coloring and the Four Color Theorem, and it is explored in Section 7.4. Before long, other mathematicians noticed that this claim had not been proven. However, it was not until 1946 that William Tutte [**1946**] produced a counterexample.

To understand this example, first consider the graph T_{16} below left. It is Hamiltonian; indeed it contains an edge e that must be in any Hamiltonian cycle (Exercise 5). It is the smallest cubic 3-connected planar graph with this property (Holton/McKay [**1988**]). Deleting vertex x from it results in a graph called **Tutte's**

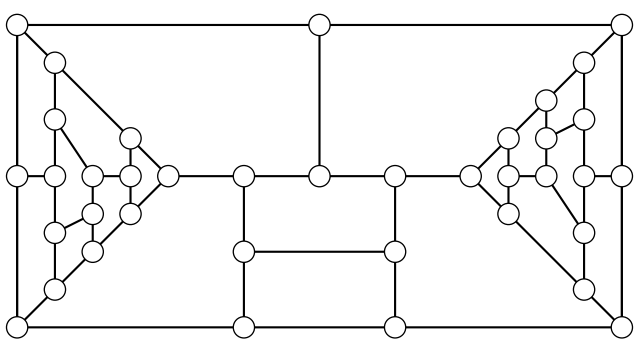
fragment (below right). It has no spanning $u-w$ path, since if it did, the original graph would have a Hamiltonian cycle not containing e.



Tutte's fragment does have spanning $u-v$ and $v-w$ paths, so when it is a subgraph of a cubic graph, any Hamiltonian cycle must use the other edge incident with v. Replacing three vertices of K_4 with Tutte's fragment as shown below produces the **Tutte graph**. Any Hamiltonian cycle would have to use the three edges incident with the center vertex, so no Hamiltonian cycle exists.



Holton and McKay [**1988**] showed that the smallest order of a 3-connected cubic planar non-Hamiltonian graph is 38. There are six such graphs that are formed by replacing two vertices of $C_5 \square K_2$ with copies of Tutte's fragment. One is shown below.

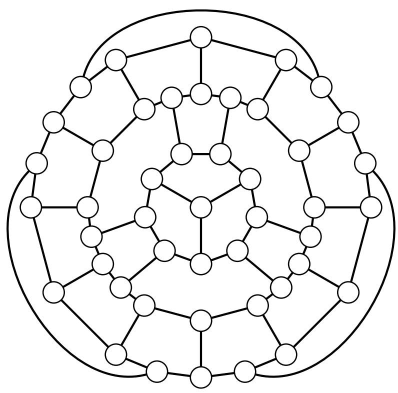


There is a necessary condition for planar Hamiltonian graphs.

Theorem 6.28 (**Grinberg's Theorem**—Grinberg [**1968**])**.** *Let G be a plane graph with Hamiltonian cycle C with r_i regions of length i inside C, and r_i' regions of length i outside C. Then $\sum_{i=3}^{n} (i-2)(r_i - r_i') = 0$.*

Proof. Consider the graph H induced by C and its interior edges. Euler's Formula says $n - m + r = 2$, so $2n - 4 = 2m - 2r$. Summing the lengths of the interior regions and exterior region shows $2m(H) = \sum i \cdot r_i + n$. Then $2n - 4 = n + \sum i \cdot r_i - 2(\sum r_i + 1)$, so $n - 2 = \sum (i-2) r_i$. Similarly, considering the outside regions shows $n - 2 = \sum (i-2) r_i'$. Equating these expressions shows $\sum (i-2)(r_i - r_i') = 0$. □

Example. Grinberg constructed the **Grinberg graph**, which has order 46, 21 regions of length 5, 3 of length 8, and one of length 9. If it is Hamiltonian, then taking the equation in Grinberg's Theorem mod 3 gives $7(r_9 - r_9') = 0$, so $-7 = 0$. Thus the Grinberg graph is non-Hamiltonian.



Grinberg's Theorem is most useful for non-Hamiltonian graphs with many (but not all) regions of length 5, 8, 11, ... It is more useful for constructing counterexamples than for testing a given graph.

While a planar graph being 3-connected is not enough to guarantee that it is Hamiltonian, being 4-connected does suffice. In fact, it guarantees even more.

Theorem 6.29 (Tutte [**1956**], Thomassen [**1983**])**.** *Every planar 4-connected graph is Hamiltonian-connected.*

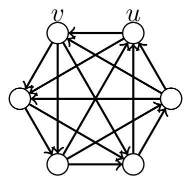
6.4. Tournaments

In a round-robin sports tournament, each team plays each other team exactly once. The results of the tournament can be expressed using a digraph with a directed edge from the winner to the loser of each game.

Definition 6.30. Let D be a digraph containing vertices u and v. The **outdegree** $d^+(v)$ is the number of edges from v. The **indegree** $d^-(v)$ is the number of edges to v. A **directed** $u - v$ **path** is a path in the underlying graph so that each vertex except v has outdegree 1. A **directed cycle** is a cycle in the underlying graph so that each vertex has outdegree 1. A (directed) **Hamiltonian path** or **Hamiltonian cycle** contains all vertices of D. The **distance** $d(u, v)$ is the length of the shortest directed $u - v$ path.

A **tournament** is an oriented complete graph. A **king** is a vertex v of a tournament such that $d(v, x) \leq 2$ for all $x \in V(G)$.

Example. In the tournament below, v is a king and u is not.



Proposition 6.31 (Landau [**1953**]). *Every vertex with maximum outdegree in a tournament is a king.*

Proof. Let v be a vertex of maximum outdegree. It is distance 1 from each of its neighbors. Let $u \neq v$ not be in the neighborhood of v. If u were adjacent to v and all neighbors of v, it would have a larger outdegree than v. Thus u is distance 2 from v, and the conclusion follows. $\square$

Landau [**1953**] investigated tournaments to study pecking orders of chickens. Chickens show dominance by pecking each other, with one of each pair pecking the other, but not both. This produces a tournament.

Proposition 6.32 (Redei [**1934**]). *Every tournament contains a Hamiltonian path.*

Proof. Let T be a tournament with order n containing path $v_1 v_2 \cdots v_r$. If this is not a Hamiltonian path, let v be a vertex not on the path. If $v \to v_1$ or $v_r \to v$, add v to the beginning or end of the path. If not, there is some i so that $v_i \to v$ and $v \to v_{i+1}$. Then $v_1 v_2 \cdots v_i v v_{i+1} v_r$ is a longer path. Thus the path can be extended until it is Hamiltonian. $\square$

This implies a simple algorithm for finding a Hamiltonian path. Some tournaments have only one Hamiltonian path.

Definition 6.33. A tournament is **transitive** if $u \to v$ and $v \to w$ implies $u \to w$ for all vertices u, v, w. The **score sequence** of a tournament is the nondecreasing sequence of outdegrees of its vertices.

Theorem 6.34. *For a tournament T, the following are equivalent:*

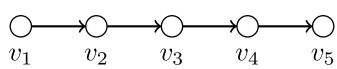
(1) *T is transitive.*

(2) *T has no cycles.*

(3) *The vertices of T can be ordered $v_1, v_2, \ldots, v_n$ so that $v_i \to v_j$ whenever $i > j$.*

(4) *T has score sequence $0, 1, \ldots, n - 1$.*

$$\Longrightarrow$$



Proof. $(2 \Rightarrow 1)$ (contrapositive) If T is not transitive, then $u \to v$, $v \to w$, and $w \to u$ for some vertices u, v, w. But this produces a cycle.

$(1 \Rightarrow 2)$ Suppose T is transitive and, to the contrary, contains a cycle $v_1 v_2 \cdots v_r v_1$. Then $v_1 \to v_2$ and $v_2 \to v_3$, so $v_1 \to v_3$. Similarly, $v_1 \to v_4$, $v_1 \to v_5$, and eventually $v_1 \to v_{r-1}$. But then v_{r-1}, v_r, and v_1 contradict transitivity.

$(3 \Rightarrow 1)$ Suppose the condition holds, and let $v_i \to v_j$ and $v_j \to v_k$. Then $k < j < i$, so $v_i \to v_k$. Thus T is transitive.

$(2 \Rightarrow 3)$ Let T be acyclic. We use induction on n. The result is obvious when $n = 1$. Assume the result holds for tournaments with order less than n. Now T must have a vertex v with outdegree 0. (If not, the end of a maximal directed path must be adjacent to a vertex on it, producing a directed cycle.) Now $T - v$ is also acyclic, so by induction its vertices can be ordered $v_2, v_3, \ldots, v_n$ so that $v_i \to v_j$ whenever $i < j$. Then $v = v_1$ can be added to the beginning of this list, proving the result.

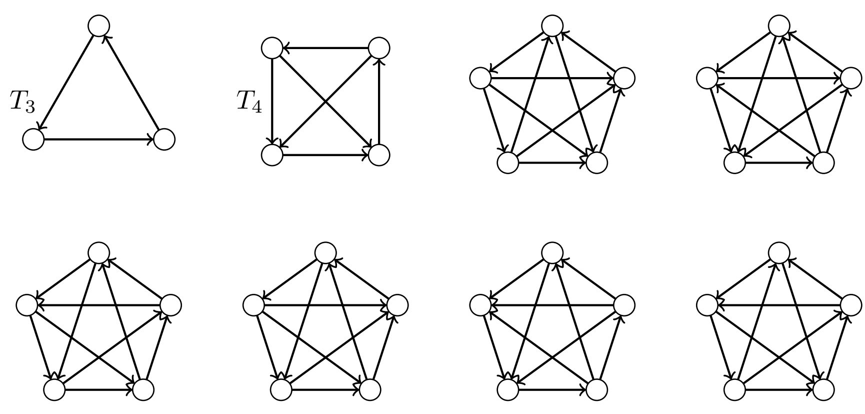
$(3 \Rightarrow 4)$ Suppose T has the stated ordering. Then v_i is adjacent to $i - 1$ vertices, so the score sequence is $0, 1, \ldots, n-1$.

$(4 \Rightarrow 2)$ Let T have score sequence $0, 1, \ldots, n-1$. The vertex v with outdegree 0 is not on a cycle. Deleting v reduces all the remaining outdegrees by 1, so $T - v$ has score sequence $0, 1, \ldots, n-2$. Iterating the same argument eventually eliminates all the vertices, so T is acyclic. $\square$

Technically, $2 \Rightarrow 1$ or $3 \Rightarrow 1$ are redundant in this proof, but these simple implications are worth proving without intermediate steps.

Definition 6.35. A digraph is **strong** (or **strongly connected**) if there are $u - v$ and $v - u$ paths for all vertices u and v. A **strong component** of a digraph is a maximal strong subgraph. The **condensation** of a digraph is formed by contracting every strong component to a vertex.

The condensation of a tournament is transitive, since if it contained a cycle, then all vertices of the corresponding subgraphs would be part of the same strong component. Thus the structure of a nonstrong tournament can be described with an ordered list of strong components. The only strong tournaments with $2 \le n \le 5$ are shown below.



The nonstrong tournaments of order 5 can be described as (T_4, K_1), (K_1, T_4), (T_3, K_1, K_1), (K_1, T_3, K_1), (K_1, K_1, T_3), $(K_1, K_1, K_1, K_1, K_1)$. A sequence for the number of tournaments of order n begins with 1, 1, 2, 4, 12, 56, 456, 6880, 191536, 9733056, ... (OEIS A000568). A sequence for the number of strong tournaments of order n begins with 1, 0, 1, 1, 6, 35, 353, 6008, 178133, 9355949, ... (OEIS A051337).

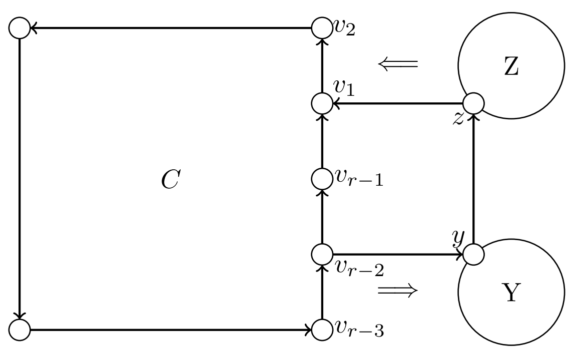
Theorem 6.34 implies that any vertex of a strong tournament is on a cycle. The following stronger result is valuable for understanding the structure of strong tournaments.

Theorem 6.36 (Moon [**1966**]). *Let T be a strong tournament. For each vertex v of T and each m with $3 \le m \le n$, v is contained in an m-cycle.*

Proof. Let T be a strong tournament of order $n \ge 3$ containing vertex v_1. We use induction on r, the length of the cycle. Since T is strong, there are edges both to and from v_1. There must be some u and w with $u \to v_1$, $v_1 \to w$ so that $w \to u$, or else the tournament is not strong. Thus v_1 is on a 3-cycle.

Assume v_1 is on an $r-1$-cycle $C = v_1 v_2 \cdots v_{r-1} v_1$, $r > 3$. If there is a vertex x with edges from and to vertices of C, there must be a vertex v_i on C adjacent to x whose successor v_{i+1} is adjacent from x. Then $v_1 \cdots v_i x v_{i+1} \cdots v_{r-1} v_1$ is an r-cycle containing v_1.

If there is no such vertex x, let Y contain all vertices that are adjacent from all vertices of C and Z contain all vertices adjacent to all vertices of C. Then Y and Z are nonempty and contain all vertices not on C. There must be $y \in Y$ and $z \in Z$ with $y \to z$, or else T is not strong. Then $v_1 \cdots v_{r-2} y z v_1$ is an r-cycle containing v_1. $\square$



If a tournament is Hamiltonian, it must be strong, since it has a directed path between any two vertices. The converse, first proved by Camion [**1959**], is implied by the previous theorem.

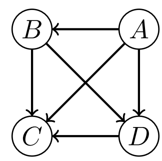
One application of tournaments is in voting theory. One way to conduct an election with several candidates is to have each voter submit a **preference list**—a ranking of all candidates from most to least preferred. This makes it possible to compare each pair of candidates to see which one is preferred. It is possible that there could be a tie between two candidates. But with a large number of voters, ties are unlikely, so we assume they do not happen. Thus we can construct a tournament with an edge from the winner to the loser of each pair.

Definition 6.37. Condorcet's Method of election is that a candidate wins if he would defeat every other candidate one-on-one. A **Condorcet winner** is such a candidate.

Example. The preference lists of five voters below produce the tournament at right, so A is the Condorcet winner.

1	A	B	C	D	D
2	B	A	A	A	B
3	C	D	B	C	C
4	D	C	D	B	A

$\rightarrow$

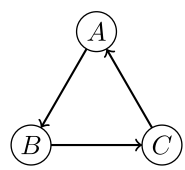


In a tournament, there is a Condorcet winner if there is a vertex with outdegree $n-1$. However, this may not occur. This is called **Condorcet's paradox**. Indeed there may be no Condorcet winner in a tournament with only three vertices.

Example. The preference lists of three voters below produce a strong tournament, so there is no Condorcet winner.

1	A	B	C
2	B	C	A
3	C	A	B

$\rightarrow$



When there is no Condorcet winner, another method is needed to determine a winner. There are several possibilities, including a **Borda count**, the **Hare system**, and **plurality runoff**. Each of these methods has advantages and disadvantages. One particularly flawed system deserves attention.

Definition 6.38. An **agenda** is an ordered list of all candidates. **Sequential pairwise voting** compares the first two candidates on the agenda, compares the winner with the next candidate on the agenda, and so on to determine a winner.

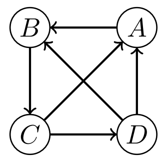
A Condorcet winner must win sequential pairwise voting no matter what agenda is used. But when there is no Condorcet winner, different agendas may produce different winners.

In fact, when a tournament is strong, each candidate wins using some agenda. Suppose we want v to win. Since a strong tournament has a Hamiltonian cycle, it has a Hamiltonian path beginning at v. Define an agenda by listing the vertices of this path in reverse order. Then each candidate will lose to the next on the agenda, ending with v winning the election.

Example. Consider the following election with three voters and four candidates. The tournament has cycle $ABCDA$. With agenda $DCBA$, A wins. With agenda $ADCB$, B wins. With agenda $BADC$, C wins. With agenda $CBAD$, D wins.

1	B	C	D
2	C	D	A
3	D	A	B
4	A	B	C

$\rightarrow$



Constructing such an agenda requires knowing the tournament of pairwise preferences. Even without this, however, candidates near the end of the agenda have an advantage over candidates near the beginning. Thus sequential pairwise voting should never be used in serious elections. It is sometimes used on game shows, where fairness is not required.

Exercises

Section 6.1:

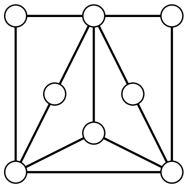
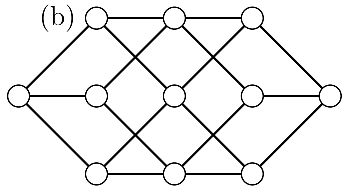
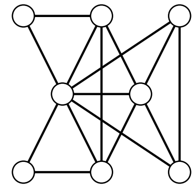
(1) Suppose a city worker needs to inspect every traffic light in an urban neighborhood after a power outage. How can the ideas of this section be used to model this situation?

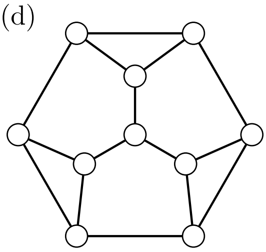
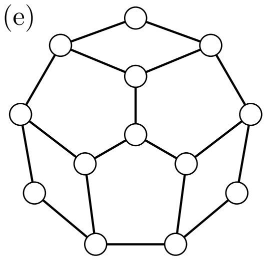
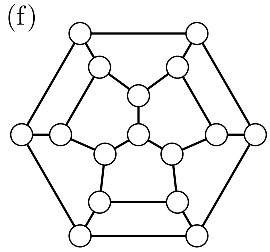
(2) The game of *Risk* has a board with regions, some of which are adjacent. Armies can attack and conquer regions to which they are adjacent. Explain how Hamiltonian cycles and paths are relevant in *Risk*.

(3) Determine which graphs in the following classes are Hamiltonian. For those that are not, determine which are traceable.
   (a) K_n
   (b) $K_{r,s}$
   (c) W_n
   (d) $G_{r,s}$

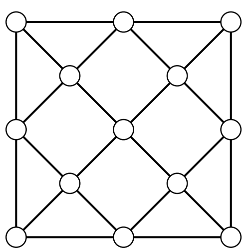
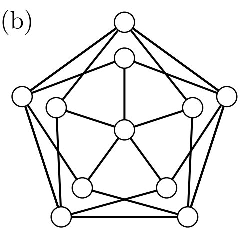
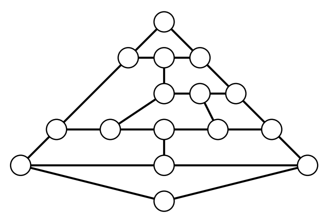
(4) Determine which graphs in the following classes are Hamiltonian. For those that are not, determine which are traceable.
   (a) trees
   (b) double wheels
   (c) the theta graph $\theta_{i,j,k}$
   (d) the triangular grid T_l

(5) Determine which of the following graphs are Hamiltonian. For those that are not, determine which are traceable.

(a)  (b)  (c) 

(d)  (e)  (f) 

(6) Determine which of the following graphs are Hamiltonian. For those that are not, determine which are traceable.

(a)  (b)  (c) 

(7) Show that every 5-vertex path in the dodecahedron lies on a Hamiltonian cycle.

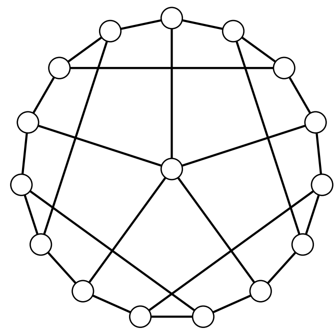
(8) Show that there is a 6-vertex path in the dodecahedron that is not contained in any Hamiltonian cycle.

(9) (Zamfirescu [**1976**]) Let G be a graph formed from the Petersen graph by deleting a vertex and adding leaves adjacent to each vertex in its neighborhood. Show no vertex of G is contained in all its maximum paths. (*Note*: Two maximum paths of a connected graph must share a vertex. Gallai asked in 1966 whether all maximum paths of a connected graph share a vertex.)

(10) Show that every hypohamiltonian graph is 3-connected.

(11) Show that no hypohamiltonian graph is bipartite.

(12) + (Lindgren [**1967**]) Show that the following graph is hypohamiltonian.



(13) Find a 2-connected bipartite graph with equal numbers of vertices in its partite sets that does not contain a Hamiltonian cycle.

(14) + Let G be a bipartite graph with partite sets both having size $k \geq 2$. Show that if $\delta(G) > \frac{k}{2}$, then G is Hamiltonian.

(15) Find an infinite class of graphs that are Hamiltonian but not Eulerian.

(16) Find an infinite class of graphs that are Eulerian but not Hamiltonian.

(17) Prove or disprove: If G is Hamiltonian, then $\overline{G}$ is not Hamiltonian.

(18) Prove or disprove: If G is not Hamiltonian, then $\overline{G}$ is Hamiltonian.

(19) Prove or disprove: If G is Hamiltonian, then any subdivision of G is Hamiltonian.

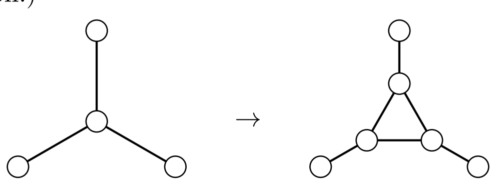
(20) Show that if a graph G is Eulerian, then the line graph $L(G)$ is Hamiltonian.

(21) Show that if a graph G is Hamiltonian, then the line graph $L(G)$ is Hamiltonian.

(22) (Harary/Nash-Williams [**1965**]) A **dominating circuit** is a circuit that contains at least one endpoint of each edge. Show that if a graph G contains a dominating circuit, then the line graph $L(G)$ is Hamiltonian. (*Note*: The converse also holds.)

(23) + For graphs G and H, find a condition that characterizes when $G + H$ is Hamiltonian.

(24) Let G be a graph with a degree 3 vertex v. We **replace** v **with a triangle** by deleting v and adding a triangle with each of its vertices adjacent to one neighbor of v. Show that G is Hamiltonian if and only if the graph formed by replacing a vertex of G with a triangle is Hamiltonian. (*Note*: Applying this operation to one vertex of the Petersen graph produces **Tietze's graph**. Hilbig [**1986**] showed that this is the unique 2-connected $\frac{n-3}{3}$-regular non-Hamiltonian graph.)



(25) (Fisher et al. [**1998**]) Show that if G is Hamiltonian, then the Mycielskian $M(G)$ is Hamiltonian.

(26) (Fisher et al. [**1998**]) Show that if G contains two leaves, then the Mycielskian $M(G)$ is not Hamiltonian.

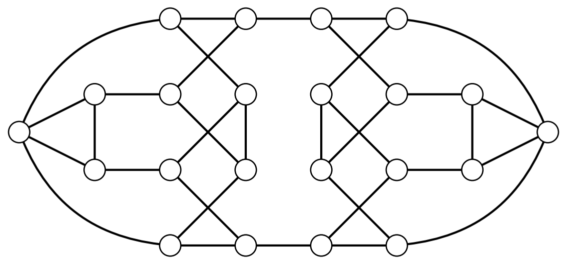
(27) Let G be a theta graph. Show that G^2 is Hamiltonian. (*Note*: Fleischner [**1976**] showed that if G is 2-connected, then G^2 is Hamiltonian.)

(28) (Harary/Schwenk [**1971**]) Let T be a tree. Show that T^2 is Hamiltonian if and only if T is a caterpillar.

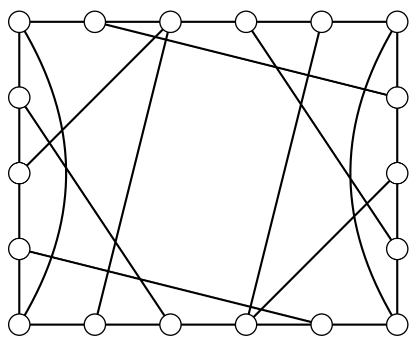
(29) Show that K_{2k} decomposes into k Hamiltonian paths.

(30) Show that K_{2k+1} decomposes into k Hamiltonian cycles.

(31) (Entringer/Swart [**1980**]) Show that the following graph has a unique Hamiltonian cycle. (*Note*: This can be generalized to a graph with a unique Hamiltonian cycle, $\delta(G) = 3$, and two vertices of degree 4 for any even $n \geq 22$.)



(32) + (Royle [**2016**]) Show that the following graph has a unique Hamiltonian cycle.



(33) (Sheehan [**1975**]) For all $n \geq 3$, find a graph with $m = \left\lfloor \frac{n^2}{4} \right\rfloor + 1$ that has a unique Hamiltonian cycle.

(34) Sheehan [**1975**] proved that the maximum size of a graph with a unique Hamiltonian cycle is $m = \left\lfloor \frac{n^2}{4} \right\rfloor + 1$. Prove the (slightly weaker) bound $m \leq \frac{n^2+n}{4}$ by considering crossing chords of a Hamiltonian cycle.

(35) (a) + (Tutte [**1946**]) Show that every edge of a graph with all odd degrees is on an even number of Hamiltonian cycles. (*Hint*: Define a graph whose edges are Hamiltonian paths starting with v_1v_2, and edges between paths with the forms $v_1v_2\cdots v_{n-1}v_n$ and $v_1v_2\cdots v_kv_nv_{n-1}\cdots v_{k+1}$. Consider the degrees in this graph.)

(b) Show that any Hamiltonian graph with all odd degrees contains at least three Hamiltonian cycles.

(36) Show that if a graph G has $n \geq 3$, $m \geq \binom{n-1}{2} + 2$, then G is Hamiltonian. What is the unique extremal graph?

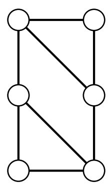
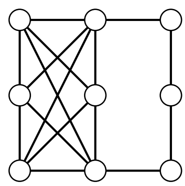
(37) Find all maximal non-Hamiltonian graphs
(a) of order 5.
(b) of order 6.

(38) Show that the Petersen graph is maximal non-Hamiltonian.

(39) Find an infinite class of graphs that satisfies the hypothesis of Ore's Theorem but doesn't satisfy the hypothesis of Dirac's Theorem.

(40) Show that for any k and n with $1 \leq k < \frac{n}{2}$, there is a non-Hamiltonian graph G with order n, minimum degree $\lfloor \frac{n-1}{2} \rfloor$ or $\lfloor \frac{n-2}{2} \rfloor$, and connectivity k.

(41) Find the closures of the following graphs.

(a)  (b) 

(42) + (Chvatal [**1972**]) Let G be a graph with $n \geq 3$ and degrees $d_1 \leq \cdots \leq d_n$. Show that if there is no integer $i < \frac{n}{2}$ for which $d_i \leq i$ and $d_{n-i} \leq n - i - 1$, then G is Hamiltonian. (*Hint*: Show that the closure is complete.)

(43) Find an infinite class of (Hamiltonian) graphs with $\kappa(G) = \alpha(G)$.

(44) Find an infinite class of non-Hamiltonian graphs with $\kappa(G) = \alpha(G) - 1$.

(45) The **toughness** $t(G)$ of a graph G is $t(G) = \min \frac{|S|}{k(G-S)}$, where $k(G)$ is the number of components of G and the minimum is taken over all vertex cuts S. Determine the toughness of
(a) C_n.
(b) $K_{r,s}$.
(c) the Petersen graph.
(d) trees.

(46) Show that the toughness of a graph satisfies $\frac{\kappa(G)}{\alpha(G)} \leq t(G) \leq \frac{\kappa(G)}{2}$.

(47) Find the circumference of the following graphs.
(a) $K_{r,s}$
(b) the theta graph $\theta_{i,j,k}$
(c) $G_{r,s}$

(48) + (Dirac [**1952**]) Without using Theorem 6.17 or Theorem 6.18, show that a 2-connected graph G has $c(G) \geq \min\{n, 2\delta(G)\}$.

(49) + (Bondy [**1971**]) A graph is **pancyclic** if it contains cycles of all lengths l, $3 \leq l \leq n$. Show that if $\delta(G) \geq \frac{n+1}{2}$, then G is pancyclic. (*Note*: Bondy showed that $K_{\frac{n}{2},\frac{n}{2}}$ is the only graph with $\delta(G) = \frac{n}{2}$ that is not pancyclic.)

(50) State and prove a result analogous to Proposition 6.2 for Hamiltonian paths.

(51) State and prove a result analogous to Theorem 6.8 for Hamiltonian paths.

(52) State and prove a result analogous to Theorem 6.15 for Hamiltonian paths.

(53) Determine which graphs in the following classes are Hamiltonian-connected.
(a) C_n
(b) W_n
(c) $C_r \square K_2$

(54) Show that any Hamiltonian-connected graph with $n \geq 4$ is 3-connected.

(55) Prove or disprove: No bipartite graph with $n \geq 3$ is Hamiltonian-connected.

(56) Show that if a graph G has $n \geq 3$, $m \geq \binom{n-1}{2} + 3$, then G is Hamiltonian-connected.

(57) Show that if a graph G is Hamiltonian, then $G + K_1$ is Hamiltonian-connected.

(58) + (Sekanina [**1960**], Karaganis [**1968**]) Let G be a connected graph. Show that G^3 is Hamiltonian-connected.

Section 6.2:

(1) Find a Grey code for bitstrings of length 4.

(2) Find the number of digits that flip when the bitstrings of length k are cycled through in lexicographic order (including returning to the start).

(3) Find the number of Hamiltonian cycles of Q_3.

(4) Show that if G is Hamiltonian, then so is $G \square K_2$.

(5) Show that if G and H are Hamiltonian, then so is $G \square H$.

(6) Show that if G and H are traceable, then so is $G \square H$.

(7) Let $m \leq n$. Show that there is no knight's tour of an m-by-n board when
(a) $m = 1$ or $m = 2$.
(b) m and n are both odd.
(c) $m = 4$.

(8) Show that there is a knight's tour of a board of dimensions
(a) 5 by 6.
(b) 3 by 10.
(c) 3 by n, $n \geq 10$ even.

(9) Find the number of Hamiltonian cycles in the following graphs.
(a) W_n
(b) $K_{r,r}$
(c) $C_n \square K_2$
(a) $G_{3,s}$

(10) Find the number of Hamiltonian paths in the following graphs.
(a) K_n
(b) C_n
(c) $K_{r,r}$
(d) $G_{2,s}$

(11) For the following graphs, find the cycle produced by the Nearest Neighbor Algorithm starting at each vertex. Find the total cost for each cycle.

$$\begin{bmatrix} 0 & 8 & 11 & 1 & 13 & 6 \\ 8 & 0 & 5 & 15 & 4 & 10 \\ 11 & 5 & 0 & 7 & 2 & 14 \\ 1 & 15 & 7 & 0 & 12 & 3 \\ 13 & 4 & 2 & 12 & 0 & 9 \\ 6 & 10 & 14 & 3 & 9 & 0 \end{bmatrix} \begin{bmatrix} 0 & 5 & 6 & 2 & 7 & 4 \\ 5 & 0 & 4 & 8 & 9 & 1 \\ 6 & 4 & 0 & 3 & 6 & 5 \\ 2 & 8 & 3 & 0 & 2 & 3 \\ 7 & 9 & 6 & 2 & 0 & 8 \\ 4 & 1 & 5 & 3 & 8 & 0 \end{bmatrix}$$

(12) For the preceding graphs, find the cycle produced by the Sorted Edges Algorithm. List the edges in the order added. Find the total cost for each cycle.

(13) Prove or disprove: The Nearest Neighbor Algorithm must use the edge of smallest weight.

(14) Prove or disprove: The Nearest Neighbor Algorithm cannot use the edge of largest weight.

(15) Prove or disprove: The Nearest Neighbor Algorithm can produce the cycle with the largest total weight.

(16) Find a weighted complete graph where the Nearest Neighbor Algorithm fails to find the minimum cost Hamiltonian cycle when started from any vertex.

(17) Prove or disprove: The Sorted Edges Algorithm cannot use the edge of largest weight.

(18) Prove or disprove: The Sorted Edges Algorithm can produce the cycle with the largest total weight.

(19) Prove or disprove: The minimum cost Hamiltonian cycle of a weighted complete graph must contain the edge with smallest weight.

(20) Prove or disprove: The minimum cost Hamiltonian cycle of a weighted complete graph cannot contain the edge with largest weight.

(21) For the graphs in Exercise 11, find the cycle produced by Christofides' Algorithm. Find the total cost for each cycle.

(22) + Show that Christofides' Algorithm may produce a cycle with weight arbitrarily close to $\frac{3}{2}C(G)$. (*Hint*: Consider a path with all edges with weight 1.)

Section 6.3:

(1) Find a cubic non-Hamiltonian planar graph with connectivity 2 and order less than 20.

(2) Let G be a maximal planar graph, and let H be the maximal planar graph formed by adding degree 3 vertices inside all regions of G. Determine all G so that H is Hamiltonian.

(3) Show that any maximal outerplanar graph with $n \geq 3$ has a unique Hamiltonian cycle.

(4) Show that a 2-tree is Hamiltonian if and only if it is outerplanar.

(5) Show that the graph T_{16} has an edge that is contained in any Hamiltonian cycle.

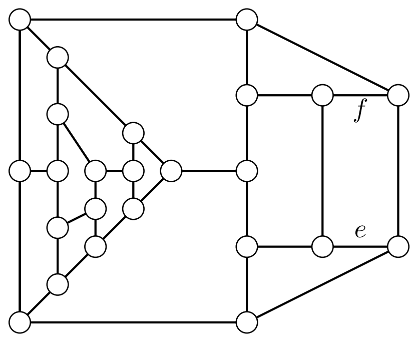
(6) Find the number of Hamiltonian cycles in the graph T_{16}.

(7) There are four 3-connected cubic planar Hamiltonian graphs with order $n \leq 8$. Show that all of them have the property that any pair of edges is contained in a Hamiltonian cycle.

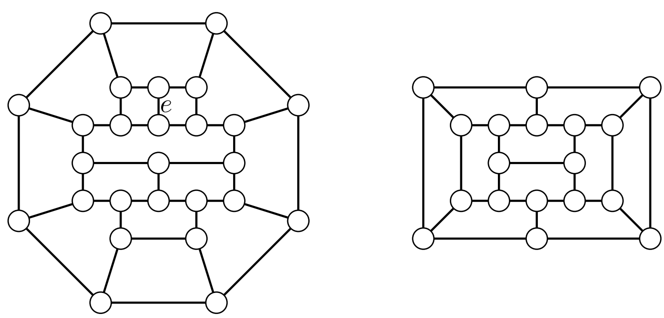
(8) Show that $C_5 \square K_2$ has a pair of edges that are not contained in a Hamiltonian cycle.

(9) (Lederberg [**1965**]) Show that there are six 3-connected cubic planar non-Hamiltonian graphs of order 38, formed by replacing two vertices of $C_5 \square K_2$ with copies of Tutte's fragment.

(10) (Holton/McKay [**1988**]) Show that the following graph has no Hamiltonian cycle containing edges e or f.



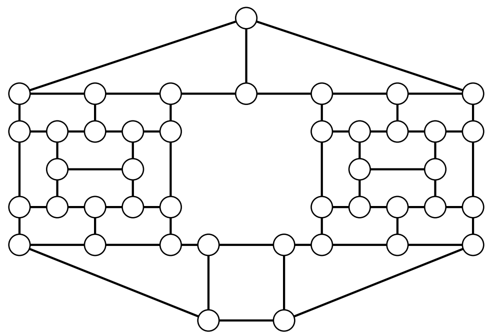
(11) (Holton/McKay [**1988**]) Show that the graph below left has no Hamiltonian cycle containing edge e.



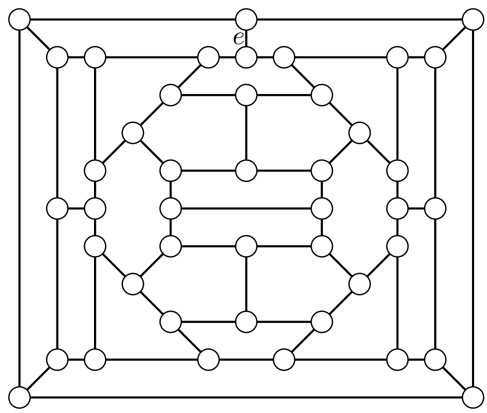
(12) Show that the graph above right contains an edge that must be in every Hamiltonian cycle.

(13) Use Grinberg's Theorem to show that the Herschel graph is non-Hamiltonian.

(14) (Faulkner/Younger [**1974**]) Show that the graph below, which has order 42, is non-Hamiltonian.



(15) (Grinberg [**1968**]) Show that **Grinberg's graph** (below), which has order 44, is non-Hamiltonian.



(16) (Grinberg [**1968**]) Form a graph of order 42 by deleting edge e from the graph above and contracting one edge incident to each of its ends. Show that this graph is non-Hamiltonian.

(17) + (Zaks [**1977**]) Show that there is a 3-connected cubic planar non-Hamiltonian graph of order 92 that has only regions of lengths 5 and 8. (*Hint*: Use a construction based on two copies of Grinberg's graph.)

(18) + Use Grinberg's Theorem to show that the Tutte graph is non-Hamiltonian.

Section 6.4:

(1) Find the number of tournaments on n labeled vertices.

(2) Prove Proposition 6.32 using
  (a) induction on n.
  (b) contradiction.

(3) Show that no tournament contains exactly two Hamiltonian paths. (*Note*: Redei [**1934**] proved that every tournament has an odd number of Hamiltonian paths.)

(4) Characterize tournaments with exactly three Hamiltonian paths.

(5) Take for granted the sequence 1, 0, 1, 1, 6, 35, ... of the number of strong tournaments of order n (OEIS A051337). Show that there are 56 tournaments of order 6.

(6) In a **regular tournament**, all vertices have the same outdegree. Determine all orders for which a regular tournament exists.

(7) Prove or disprove: Every edge of a strong tournament is contained in a Hamiltonian cycle.

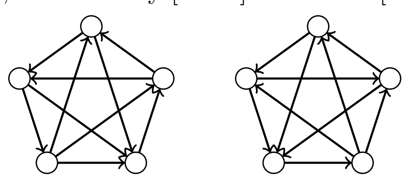
(8) Show that if T is a tournament containing vertex v on cycle C, then v is on a 3-cycle containing consecutive vertices of C.

(9) (Erdos/Moser [**1964**]) Show that every tournament of order n contains a transitive tournament of order $1 + \lfloor \log_2 n \rfloor$.

(10) (Erdos/Moser [**1964**]) Show that there is a tournament of order 7 containing no transitive tournament of order 4.

(11) (Erdos/Moser [**1964**]) Find the number of ways a tournament with k vertices can occur as a subtournament of a tournament with n labeled vertices. Use this to show that there is a tournament of order n without a transitive subtournament with $2 + 2\lfloor \log_2 n \rfloor$ vertices.

(12) (Stockmeyer [**1977**]) The **Reconstruction Conjecture** (for digraphs) is that every digraph D with vertex set $\{v_1, v_2, \ldots, v_n\}$, $n > 2$, has a distinct multiset $\{D - v_1, D - v_2, \ldots, D - v_n\}$ of vertex-deleted subgraphs. Show that this conjecture is false using the tournaments below. (*Note*: The corresponding conjecture for graphs, due to Kelly [**1957**] and Ulam [**1960**], remains open.)



(13) Show that a score sequence $s_1, \ldots, s_n$, $0 \le s_1 \le \cdots \le s_n$ satisfies $\sum_{i=1}^{k} s_i \ge \binom{k}{2}$ for $1 \le k \le n$, with equality for $k = n$. (*Note*: Landau [**1953**] showed the converse is true.)

(14) Show that the nondecreasing sequence $s_1, \ldots, s_n$ of nonnegative integers is a score sequence if and only if $s_1, \ldots, s_{s_n}, s_{s_n+1} - 1, \ldots, s_{n-1} - 1$ is a score sequence.

(15) (Maurer [**1980**], Reid [**1982**]) Let T be a tournament with no vertex with outdegree $n - 1$.
   (a) Show that if v is a king, there is another king in the in-neighborhood of v.
   (b) Show that no tournament has exactly two kings.

(16) (Maurer [**1980**], Reid [**1982**]) Kings of tournaments:
   (a) Show that there is a tournament with every vertex a king for any odd n.
   (b) + Show that there is a tournament with every vertex a king if and only if $n \notin \{2, 4\}$. (*Hint*: Use induction on n.)
   (c) + Show that there is a tournament with order n and k kings, $1 \le k \le n$, except when $k = 2$ or $n = k = 4$.

(17) For the preference list below left, the number on top indicates how many voters submitted this list. Draw the tournament corresponding to this election and determine whether there is a Condorcet winner.

	2	2	1	1	1
1	A	D	A	B	C
2	C	C	B	C	D
3	D	A	C	D	B
4	B	B	D	A	A

	3	2	2	1	1
1	C	A	B	A	B
2	D	D	C	D	A
3	A	B	D	C	D
4	B	C	A	B	C

(18) For the preference lists above right, the number on top indicates how many voters submitted this list. Draw the tournament corresponding to this election and determine the winner using agendas
(a) $ABCD$.
(b) $DCBA$.
(c) $CADB$.
(d) $BDAC$.

(19) For the preference lists below, find an agenda that will make the winner
(a) A.
(b) B.
(c) C.
(d) D.

	3	1	1	1	1	1	1
1	A	A	B	B	C	C	D
2	D	B	C	C	B	D	C
3	B	C	A	D	D	B	B
4	C	D	D	A	A	A	A

(20) Show that there is an agenda that makes a vertex v of tournament T the winner if and only if it is in the strong component whose corresponding vertex in the condensation of T is adjacent to all other vertices.

Chapter 7

Matchings

7.1. Bipartite Matchings

7.1.1. Hall's Theorem.

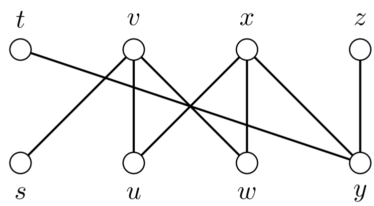
Example. An employer has a number of employees and an equal number of jobs that must be done. Each employee is qualified for some, but not all, of the jobs. The employer wants to know whether each employee can be assigned to a job so that every job can be completed simultaneously. If so, the employer would also like to know how to actually assign the jobs. This situation can be modeled with a bipartite graph. The partite sets are employees and jobs, so the graph must be bipartite. An edge joins an employee and a job when the employee is qualified for that job. To assign the jobs, we need a set of nonadjacent edges incident with every employee (and every job).

Example. An online advertising company has a number of client websites that run advertisements and a number of client advertisers that run ads. Not every advertisement is appropriate for every website, however. The company would like to assign ads to websites so that every website gets an ad. Again this problem can be modeled with a bipartite graph, with edges between sites and ads that are appropriate fits.

Both of these examples require solving the same graph theory problem. Many other examples lead to the same problem, including the problem of assignment of classes to teachers given at the beginning of Section 1.7.

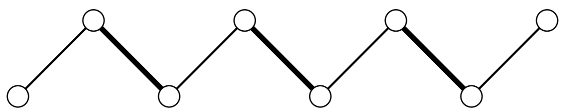
Definition 7.1. A set of edges are **independent** if none of them are adjacent, in which case they are called a **matching**. A **maximum matching** has largest possible size. A **perfect matching** includes an endpoint of every vertex. The vertices incident with the edges of a matching are **covered** by it.

Example. In the graph below, $\{uv, xy\}$ is a matching. It is maximal, but not maximum. A maximum matching is $\{uv, wx, yz\}$. This graph has no perfect matching.



For a graph to have a perfect matching, every component must have even order. This is not sufficient, as illustrated by $K_{1,3}$. To characterize bipartite graphs with a perfect matching, we first consider a preliminary result.

Definition 7.2. Given a matching M, an **M-alternating path** is a path whose edges alternate between those in M and not in M. An **M-augmenting path** is an M-alternating path for which M does not cover either end. The **symmetric difference** of two multigraphs is induced by the edges appearing in exactly one of them.



When a graph has an M-augmenting path, a matching can be expanded by trading the edges of M for the nonedges on the path.

Lemma 7.3. *Every component of the symmetric difference of two matchings is a path or even cycle.*

Proof. Let H be the symmetric difference of matchings M and N. Each vertex is incident with at most one edge of each, so $\Delta(H) \leq 2$. Thus the components of H are paths or cycles. Since the edges alternate between matchings, any cycle must be even. $\square$

Proposition 7.4 (Berge [**1957**])**.** *A matching M in a multigraph G is a maximum matching if and only if G has no M-augmenting path.*

Proof. ($\Rightarrow$) (contrapositive) If G has an M-augmenting path, then the matching can be enlarged by swapping edges on the path.

($\Leftarrow$) (contrapositive) Let N be a matching of G that is larger than M. Let H be the symmetric difference of M and N. They both have the same number of edges in any even cycle component of H. Thus some component of H must be a path with edges on both ends in N that is an M-augmenting path. $\square$

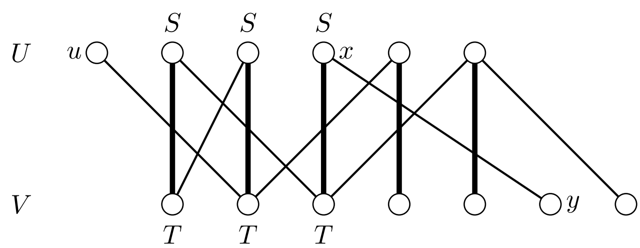
Definition 7.5. The **neighborhood of a set** S, $N_G(S)$ or just $N(S)$, is the set of vertices having a neighbor in S. For a graph with vertex set U, **Hall's Condition** is $|N(S)| \geq |S|$ for all $S \subseteq U$.

Any bipartite graph with a matching covering one of its partite sets must satisfy Hall's Condition. Certainly $|S|$ vertices must have at least this many neighbors for there to be a matching covering them. Hall's Theorem says that the converse is also true.

Theorem 7.6 (**Hall's Theorem**—Hall [**1935**]). *A bipartite multigraph with partite sets U and V has a matching that covers U if and only if $|N(S)| \geq |S|$ for all $S \subseteq U$.*

Proof. ($\Rightarrow$) The set S has at least $|S|$ neighbors in $N(S)$.

($\Leftarrow$) Assume the multigraph satisfies Hall's Condition. We show that a matching M that does not cover $u \in U$ has an M-augmenting path. Let $S \subseteq U$ be the vertices reachable from u by nontrivial M-alternating paths, and let $T \subseteq V$ be the vertices preceding them on these paths. Now S is matched to T by edges of M, so $|S| = |T|$, and $|S \cup \{u\}| > |T|$. By Hall's Condition, there must be some vertex $x \in S \cup \{u\}$ adjacent to $y \in V - T$, and y is not covered by M. Since there is an M-alternating $u-x$ path, it can be extended to an M-alternating $u-y$ path. Since M does not cover u and y, this is an M-augmenting path. Thus the matching can be expanded to include u. Iterating this process produces a matching that covers U. $\square$



Hall's Theorem is also known as the **Marriage Theorem**. Given two groups of men and women of equal number, some pairs find each other acceptable as spouses, and some do not. Then Hall's Theorem describes when it is possible for everyone in the groups to marry.

Corollary 7.7. *Every k-regular bipartite multigraph, $k > 0$, has a perfect matching.*

Proof. Let G be a k-regular bipartite multigraph with partite sets U and V. Summing degrees for each partite set shows $k|U| = k|V|$, so $|U| = |V|$. Let S be a subset of U or V. Summing degrees for S and $N(S)$, we find $k|S| \leq k|N(S)|$, so $|S| \leq |N(S)|$. Thus G satisfies Hall's Condition, so it has a perfect matching. $\square$

Deleting a perfect matching from a regular bipartite graph produces another regular bipartite graph. Thus applying this result repeatedly shows that any regular bipartite graph decomposes into perfect matchings.

7.1.2. Set Theory Applications. Matchings have several applications in set theory.

Definition 7.8. A **system of distinct representatives** (SDR) for a collection of sets $S_1, \ldots, S_k$ is a set $\{s_1, \ldots, s_k\}$ with $s_i \in S_i$ for all i.

Example. The collection $\{1,2\}$, $\{2,4\}$, $\{3,4,5\}$, $\{1,4\}$, $\{1,3,5\}$ has an SDR $\{1,2,3,4,5\}$. The collection $\{3,4,5\}$, $\{1,2\}$, $\{1,3\}$, $\{2,3\}$, $\{1,2,3\}$, $\{1,4,5,6\}$ has no SDR, because $\{1,2\}$, $\{1,3\}$, $\{2,3\}$, $\{1,2,3\}$ have union $\{1,2,3\}$, with only three elements.

Corollary 7.9 (Hall [**1935**]). *A collection of sets has an SDR if and only if the union of any j sets has at least j elements, for each j, $1 \leq j \leq k$.*

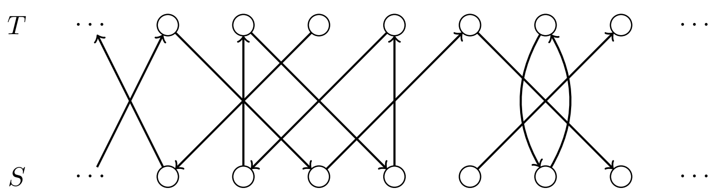
Proof. Define a bipartite graph G with partite sets $\{S_1, \ldots, S_k\}$ and $\bigcup S_i$. An edge joins S_i and s_j when $s_j \in S_i$. Now G has an SDR if and only if there is a matching of $\{S_1, \ldots, S_k\}$ into $\bigcup S_i$. By Hall's Theorem, this will occur exactly when the union of any j sets has at least j elements. $\square$

Georg Cantor showed that infinite sets may have different cardinalities. The cardinality of a finite set can be found by just counting the elements, but the cardinalities of infinite sets are compared in a different way.

Definition 7.10. Two sets S and T have the **same cardinality**, $|S| = |T|$, if there is a bijection between them. The cardinality of S is **less than or equal to the cardinality** of T, $|S| \leq |T|$, if there is a one-to-one function from S to T.

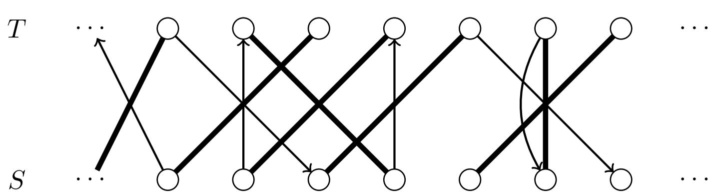
Another application of matchings is to show that two sets have the same cardinality. If $|S| \leq |T|$ and $|S| \geq |T|$, it seems reasonable to suspect that $|S| = |T|$. This is obvious for finite sets, but nontrivial for infinite sets.

Theorem 7.11 (**Schroder-Bernstein Theorem**). *If $|S| \leq |T|$ and $|S| \geq |T|$, then $|S| = |T|$.*



Proof (Konig [**1906**]). Assume $|S| \leq |T|$ and $|S| \geq |T|$, so there are one-to-one functions $f : S \to T$ and $g : T \to S$. Form an (infinite) bipartite digraph G with partite sets S and T and edges representing the mappings ($s \to t$ when $f(s) = t$, and $t \to s$ when $g(t) = s$). For $s \in S$ or $t \in T$, define a two-sided walk by applying f and g in one direction and f^{-1} and g^{-1} (if defined) in the other direction: $\ldots, f^{-1}\left(g^{-1}(s)\right), g^{-1}(s), s, f(s), g(f(s)), \ldots$.

On the left, this walk may terminate (if f^{-1} or g^{-1} are not defined) or not. Since f and g are one-to-one, each element of S and T occurs in exactly one distinct walk. Each walk must be a one-way infinite path, a two-way infinite path, or a (finite) even cycle. Each of these subgraphs contains a perfect matching (direction not important). Combining them produces a perfect matching in G, so $|S| = |T|$. $\square$



The Schroder-Bernstein Theorem can be used to show that two sets have the same cardinality by mapping each into the other, rather than constructing a bijection, which can be difficult.

Example. We show $|\mathbb{R}| = |\mathbb{R} \times \mathbb{R}|$. The mapping $f(x) = (x, 0)$ shows $|\mathbb{R}| \leq |\mathbb{R} \times \mathbb{R}|$. Let $x = \cdots a_2 a_1 . x_1 x_2 \cdots$, where we don't allow repeating 9's in the decimal expansion. The mapping that interlaces the digits of two coordinates, $g(\cdots a_2 a_1 . x_1 x_2 \cdots, \cdots b_2 b_1 . y_1 y_2 \cdots) = \cdots a_2 b_2 a_1 b_1 . x_1 y_1 x_2 y_2 \cdots$. shows $|\mathbb{R} \times \mathbb{R}| \leq |\mathbb{R}|$. The Schroder-Bernstein Theorem implies that $|\mathbb{R}| = |\mathbb{R} \times \mathbb{R}|$.

7.1.3. Independence and Covers. Independent sets of vertices and edges are closely related to vertex and edge covers.

Definition 7.12. A **vertex cover** of a multigraph is a set of vertices so that each edge has at least one end in it. A vertex cover **covers** the edges incident with it. An **edge cover** of a multigraph is a set of edges so that each vertex is one of their ends. The following parameters optimize independence and covering.

maximum size of independent set	$\alpha(G)$
maximum size of matching	$\alpha'(G)$
minimum size of vertex cover	$\beta(G)$
minimum size edge cover	$\beta'(G)$

The prime in the notation above indicates an "edge version" of a vertex parameter. Vertex and edge parameters are often related by line graphs. Thus $\alpha'(G) = \alpha(L(G))$. There are other natural relationships between these parameters.

Lemma 7.13. *In a multigraph G, S is an independent set if and only if $\overline{S}$ is a vertex cover. Thus $\alpha(G) + \beta(G) = n$.*

Proof. ($\Rightarrow$) If S is independent, then every edge is incident with some vertex of $\overline{S}$. Thus $\beta(G) \leq n - \alpha(G)$.

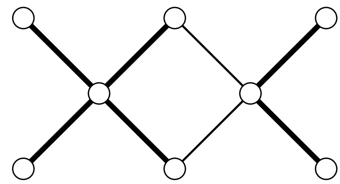
($\Leftarrow$) If $\overline{S}$ covers every edge, then no edge joins vertices in S. Thus $\alpha(G) \geq n - \beta(G)$.

Thus any maximum independent set is the complement of some minimum vertex cover. $\square$

Theorem 7.14 (Gallai [**1959**]). *If G is a multigraph with no isolated vertices, then $\alpha'(G) + \beta'(G) = n$.*

Proof. Given a matching of size $\alpha'(G)$, add one edge incident with each uncovered vertex. This produces an edge cover of size $n - \alpha'(G)$, so $\beta'(G) \leq n - \alpha'(G)$.

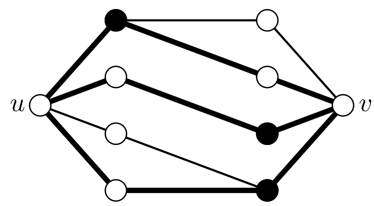
Let E be a minimum edge cover of G. Then each component of the graph induced by E is a star (else some edge could be deleted). If there are k components, then $|E| = n - k$. Choosing one edge from each star forms a matching of size k. Thus $\alpha'(G) \geq n - \beta'(G)$. $\square$



Restricted to bipartite graphs, there are additional relationships.

Theorem 7.15 (**Konig's Theorem**—Konig [**1931**], Egervary [**1931**]). *If G is a bipartite multigraph, then $\alpha'(G) = \beta(G)$.*

Proof. Let G be a bipartite multigraph with partite sets U and V. Form a multigraph H by adding vertices u and v adjacent to all vertices in U and V, respectively. Menger's Theorem says that the maximum number of independent $u-v$ paths in H equals the minimum size of a $u-v$ cut. Any $u-v$ cut must use the vertices of G, and a maximum cut must cover each edge. When restricted to G, independent $u-v$ paths form a matching. Thus Menger's Theorem implies that $\alpha'(G) = \beta(G)$. □



Konig's theorem can also be proved as a consequence of Hall's Theorem.

Corollary 7.16 (Konig [**1916**]). *If G is a bipartite multigraph with no isolated vertices, then $\alpha(G) = \beta'(G)$.*

Proof. Add the equations of Lemma 7.13 and Theorem 7.14 and subtract that of Theorem 7.15. □

The previous two results are all examples of dual optimization problems, where the maximum value of one parameter equals the minimum value of another parameter. Finding appropriate vertex or edge sets of equal size proves that both of them are optimal and, hence, gives the values of both parameters.

These results have applications to perfect graphs. The following definition is relevant to coloring complements of graphs.

Definition 7.17. The **clique cover number** $\mathrm{cc}(G)$ of a graph G in the minimum number of cliques that cover its vertices.

Theorem 7.18. *Bipartite graphs and their complements, and line graphs of bipartite graphs and their complements are perfect.*

Proof. Bipartite graphs are hereditary, and so are their line graphs, since deleting a vertex of a line graph corresponds to deleting an edge of the original graph. Nonempty bipartite graphs have $\chi(G) = \omega(G) = 2$, so they are perfect.

Let G be a bipartite graph. Any coloring of the complement of a bipartite graph G uses classes of size at most 2, and those of size 2 correspond to a matching in G. Thus $\chi\left(\overline{G}\right) = n - \alpha'(G) = \beta'(G) = \alpha(G) = \omega\left(\overline{G}\right)$.

Coloring $L(G)$ corresponds to partitioning $E(G)$ into matchings. Theorem 7.26 justifies the following equalities: $\chi(L(G)) = \Delta(G) = \omega(L(G))$.

Coloring a graph is equivalent to covering its complement with cliques, so $\chi(G) = \mathrm{cc}\left(\overline{G}\right)$. A clique in $L(G)$ consists of edges of G with a common endpoint. Thus a clique cover for $L(G)$ corresponds to a vertex cover of G. Finally, Konig's Theorem shows that $\chi\left(\overline{L(G)}\right) = \mathrm{cc}(L(G)) = \beta(G) = \alpha'(G) = \alpha(L(G)) = \omega\left(\overline{L(G)}\right)$. □

The results on complements also follow from the Perfect Graph Theorem. As mentioned earlier, these four classes are key to proving the Strong Perfect Graph Theorem.

Related Terms: matching graph, matching preclusion number, biregular graph.

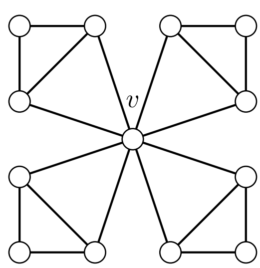
7.2. Tutte's 1-Factor Theorem

Suppose a college has a number of incoming students to assign to dorm rooms, two per room. Students are surveyed on their preferences for roommates, making some pairs compatible, and some incompatible. The college wishes to know whether it is possible to assign each student a roommate without conflict. If not, a student can be assigned to a single room by himself, but the college would like to minimize the number of times that this occurs. To model this situation using graph theory, we still need a maximum matching, but now there is no requirement that the graph is bipartite.

Definition 7.19. A **1-factor** is a spanning 1-regular subgraph; its edge set is a perfect matching. In a multigraph H, let $o(H)$ be the number of **odd components** (those having an odd number of vertices). **Tutte's condition** is that $o(G-S) \leq |S|$ for any vertex set S.

In a multigraph G, the **deficiency of a set** $\mathrm{def}_G(S)$ or $\mathrm{def}(S)$ is $o(G-S) - |S|$. A **Tutte set** is a vertex subset with positive deficiency. The **deficiency of a multigraph** is $\mathrm{def}(G) = \max_S \mathrm{def}(S)$.

Example. Let $S = \{v\}$ in the graph below. Then $\mathrm{def}(G) = \mathrm{def}(S) = 4 - 1 = 3$.



For a graph to have a 1-factor, Tutte's condition must hold, since each odd component must have some vertex matched to a vertex in S. More generally, any matching must miss $n - \operatorname{def}(G)$ vertices, so the size of a maximum matching is at most $\frac{1}{2}(n - \operatorname{def}(G))$. Tutte proved the converse of the first statement, characterizing graphs with 1-factors. Berge extended this result to prove that $\alpha'(G) = \frac{1}{2}(n - \operatorname{def}(G))$.

Theorem 7.20 (**Tutte-Berge Formula**—Berge [**1958**]). *If G is a multigraph, then $\alpha'(G) = \frac{1}{2}(n - \operatorname{def}(G))$.*

Proof (Schrijver [**2003**]). We have noted that $\frac{1}{2}(n - \operatorname{def}(G))$ is an upper bound.

We prove the reverse inequality by induction on n. The case $n = 1$ is trivial. We can assume that G is connected, as otherwise we can apply induction to the components of G.

First assume that there exists a vertex v covered by all maximum-size matchings. Then $\alpha'(G - v) = \alpha'(G) - 1$, and by induction there exists a subset U' of $V(G) - v$ with

$$\alpha'(G - v) = \frac{1}{2}\left((n-1) + |U'| - o(G - v - U')\right).$$

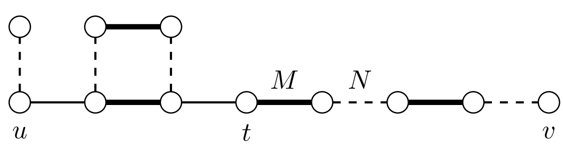
Then $U = U' \cup \{v\}$ gives equality.

So we can assume that there is no such v. In particular, $\alpha'(G) < \frac{n}{2}$. We show there is a matching of size $\frac{n-1}{2}$, which implies the theorem (with $U = \emptyset$).

Indeed suppose to the contrary that each maximum-size matching M misses at least two distinct vertices u and v. Among all such M, u, v, choose them such that the distance $d(u, v)$ of u and v in G is as small as possible.

If $d(u, v) = 1$, then u and v are adjacent, and hence we can augment M by uv, contradicting the maximality of $|M|$. So $d(u, v) \geq 2$, and hence we can choose an intermediate vertex t on a shortest $u - v$ path. By assumption, there exists a maximum-size matching N missing t.

Consider the component P of the graph induced by $M \cup N$ containing t. As N misses t, P is a path with end t. As M and N are maximum-size matchings, P contains an equal number of edges in M as in N. Since M misses u and v, P cannot cover both u and v. So by symmetry we can assume that P misses u. Exchanging M and N on P, M becomes a maximum-size matching missing both u and t. Since $d(u, t) < d(u, v)$, this contradicts the minimality of $d(u, v)$. □



When Tutte's condition holds, $\operatorname{def}(G) = 0$, so $\alpha'(G) = \frac{n}{2}$, and G has a 1-factor.

Corollary 7.21 (**Tutte's 1-Factor Theorem**—Tutte [**1947**]). *A multigraph G has a 1-factor if and only if $o(G - S) \leq |S|$ for any vertex set S.*

A naive algorithm to find a maximum matching would calculate the deficiency, which in turn would require checking all 2^n vertex subsets. This is impractical. A better alternative is Edmonds' Algorithm, which can be implemented in $\mathcal{O}(\sqrt{n}m)$ time for both bipartite and nonbipartite graphs (Vazirani [**1980**]). Theorem 7.14 implies that β' can be found in the same time.

Some graph classes can be shown to have a 1-factor using Tutte's Theorem.

Corollary 7.22 (Petersen [**1891**]). *Every bridgeless cubic multigraph contains a 1-factor.*

Proof. Let G be a bridgeless cubic multigraph, $S \subset V(G)$, and let H be an odd component of $G - S$. The number of edges from H to S cannot be 1 (since G is bridgeless) and cannot be 2 (since this would produce an odd number of odd vertices in H), so it is at least 3. Since G is cubic, each vertex in S is incident with at most three edges from H to S. If l is the number of edges joining S and odd components of $G - S$, then $3o(G - S) \leq l \leq 3|S|$, so $o(G - S) \leq |S|$. Tutte's condition is satisfied, so G has a 1-factor. □

Corollary 7.22 is due to Julius Petersen. Petersen has a strong claim to being the first graph theorist. While some graph theoretic results precede Petersen's 1891 paper, they were seen as isolated results in recreational mathematics. Petersen's paper translated a problem in abstract algebra into the language of graph theory. He proved that every cubic graph with at most two bridges has a 1-factor (Exercise (3) of Section 7.3) and every regular graph with positive even degree has a 2-factor (see below).

Definition 7.23. A **2-factor** of a multigraph is a spanning 2-regular subgraph (hence a disjoint union of cycles).

Deleting a 1-factor from a cubic graph leaves a 2-factor. A 2-regular graph has a 1-factor if and only if each cycle has even length. Peter Tait had claimed that every bridgeless cubic graph decomposes into three 1-factors. In an 1898 paper, Petersen [**1898**] cited the Petersen graph, which bears his name, as a counterexample. Any 2-factor of the Petersen graph is $2C_5$, which has no 1-factor.

Theorem 7.24 (Petersen [**1891**]). *Every regular multigraph with positive even degree has a 2-factor.*

Proof. Let G be a regular multigraph with positive even degree. Every component of G has an Eulerian circuit C. For each component, form a bipartite graph H with partite sets U and W, where u_i and w_i in H correspond to v_i in G. An edge joins u_i and w_j when C goes from v_i to v_j. Since G is regular, H is also regular. By Corollary 7.7, H has a perfect matching. This matching covers u_i and w_i exactly once, so the corresponding vertex v_i in G is covered exactly twice. This produces a 2-factor in G. □

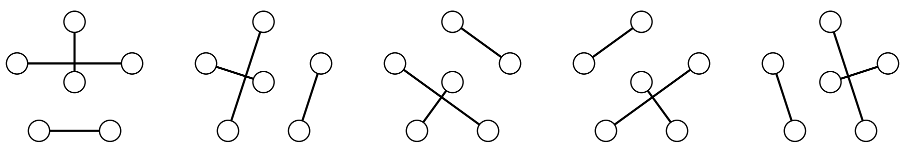
Deleting a 2-factor from a regular graph with positive even degree produces another such graph, or an empty graph. Thus any regular graph with positive even degree decomposes into 2-factors.

This text includes many theorems relating sets of paths and sets of vertices, such as Menger's Theorem (M), the Max-flow Min-cut Theorem (F), Hall's Theorem (H), Konig's Theorem (K), and the Tutte-Berge Formula (T). All of these theorems are equivalent in the sense that assuming one of them makes it easier to prove any of the others. In this book, we proved M, H, and T independently, as well as $M \Rightarrow F$ and $M \Rightarrow K$. Some other implications between these theorems are explored in the Exercises. Other combinatorial theorems, including Dilworth's Theorem on posets, are also equivalent to these theorems.

7.3. Edge Coloring

Example. Suppose we want to schedule a round-robin tournament in a sports league. If there are $2k$ teams, there can be at most k games simultaneously. We want each team to play each other team exactly once. In graph theory terms, we want a decomposition of K_{2k} into matchings.

Arrange the vertices of K_{2k} with one in the center and the rest on a circle. Match the center to one vertex, and match the rest with parallel edges. This perfect matching can be rotated to produce a decomposition of K_{2k} into perfect matchings.



With an odd number of teams in a tournament, we name one vertex "bye" and use the corresponding decomposition of K_{2k} to schedule games. Deleting the center vertex produces a decomposition of K_{2k-1} into k (nonperfect) matchings.

This example is reminiscent of coloring. However, it is the edges that are being colored rather than the vertices. Rather than split the vertex set into independent sets (of vertices), we split the edge set into independent sets of edges.

Definition 7.25. A **proper edge coloring** is a partition of the edge set into matchings. A graph is k**-edge-colorable** if it has an edge coloring with k matchings. The **edge chromatic number** (**chromatic index**) $\chi'(G)$ of a graph G is the minimum number of matchings in any proper edge coloring of G.

The chromatic number and edge chromatic number are related by line graphs. A proper edge coloring of a graph corresponds to a proper vertex coloring of its line graph. Thus $\chi'(G) = \chi(L(G))$, so determining edge chromatic numbers is essentially a subset of the problem of determining chromatic numbers. Unfortunately, this observation does not seem to be helpful for determining edge chromatic numbers.

The edges incident with a single vertex require distinct colors. This implies that $\chi'(G) \geq \Delta(G)$. Many graphs make this an equality.

Theorem 7.26 (Konig [**1916**])**.** *Every bipartite multigraph has* $\chi'(G) = \Delta(G)$.

Proof. Let G be a bipartite multigraph. If G is not regular, form a multigraph H by adding vertices to make the partite sets have equal size (if necessary), then add edges between vertices in different partite sets with less than maximum degree until H is regular. Now Corollary 7.7 says that H has a 1-factor. Deleting it produces another regular bipartite multigraph. Applying Corollary 7.7 repeatedly produces a decomposition of H into 1-factors. Restricting it to G shows $\chi'(G) = \Delta(G)$. $\square$

We will show that any graph requires at most $\Delta + 1$ colors, but a multigraph with repeated edges may require more colors.

Definition 7.27. Let G be a multigraph containing vertices u and v. The **multiplicity** of uv, $\mu(uv)$, is the number of edges joining u and v. The **multiplicity** of G, $\mu(G)$, is the maximum of $\mu(uv)$ over all pairs of vertices u and v.

Lemma 7.28. *Let G be a graph with vertex v, such that $d(v) \leq k$ and $d(u) + \mu(uv) \leq k + 1$ for all neighbors u of v, with equality for at most one neighbor. Then if $G - v$ is k-edge-colorable, also G is k-edge-colorable.*

Proof (Schrijver [**2003**]). We use induction on k. The result certainly holds for $k = 0$. We can assume that $d(u) + \mu(uv) = k$ for each neighbor u of v, except for one with $d(u) + \mu(uv) = k + 1$, since otherwise we can repeatedly add a new vertex w and an edge uw without violating the induction hypothesis.

Consider any k-edge-coloring of $G - v$. For $1 \leq i \leq k$, let X_i be the set of neighbors of v that are missed by color i. Choose a coloring such that $\sum_{i=1}^{k} |X_i|^2$ is minimized.

First assume that $|X_i| \neq 1$ for all i. As each $u \in N(v)$ is in precisely $2\mu(uv)$ of the X_i—except for one $u \in N(v)$ being in $2\mu(uv) - 1$ of the X_i—we know

$$\sum_{i=1}^{k} |X_i| = -1 + 2 \sum_{u \in N(v)} \mu(uv) = 2d(v) - 1 < 2k.$$

Hence there exist i, j with $|X_i| < 2$ and $|X_j|$ odd. So $|X_i| = 0$ and $|X_j| \geq 3$. Consider the subgraph H made by all edges of colors i and j, and consider a component of H containing a vertex in X_j. This component is a path P starting in X_j. Exchanging colors i and j on P reduces $|X_i|^2 + |X_j|^2$, contradicting our minimality assumption.

So we can assume $|X_k| = 1$, say $X_k = \{u\}$. Let G' be the graph obtained from G by deleting one of the edges vu and deleting all edges of color k. So $G' - v$ is $k - 1$-edge-colored. Moreover, in G', $d_{G'}(v) \leq k - 1$ and each neighbor w of v satisfies $d_{G'}(w) + \mu_{G'}(wv) \leq k$, with equality for at most one neighbor. So by the induction hypothesis, G' is $k - 1$-edge-colorable. Restoring color k and giving edge vu color k, gives a k-edge-coloring of G. $\square$

Setting $k = \Delta(G) + \mu(G)$ proves the following theorem.

Theorem 7.29 (**Vizing's Theorem**—Vizing [**1964**], Gupta [**1966**])**.** *For any multigraph G, $\chi'(G) \leq \Delta(G) + \mu(G)$. If G is a graph, $\chi'(G) \leq \Delta(G) + 1$.*

Vizing's Theorem can be used to prove a bound due to Shannon [**1949**] purely in terms of $\Delta(G)$, $\chi'(G) \leq \frac{3}{2}\Delta(G)$ (Exercise (26)). For graphs, Vizing's Theorem implies that there are only two possibilities for the edge chromatic number.

Definition 7.30. A graph is **class 1** if $\chi'(G) = \Delta(G)$ and **class 2** if $\chi'(G) = \Delta(G) + 1$. A graph is 1**-factorable** if it decomposes into 1-factors.

A graph is 1-factorable if and only if it is regular and class 1. We have seen that bipartite graphs are class 1. Lemma 7.28 implies that any graph G where the vertices of degree $\Delta(G)$ induce a forest is class 1 (Exercise 14). In fact, Vizing proved an even stronger result.

Definition 7.31. A graph is **χ'-critical** if any proper subgraph has smaller edge-chromatic number.

Theorem 7.32 (**Vizing's Adjacency Lemma**—Vizing [**1965**]). *If G is a χ'-critical graph and xy is an edge of G, then x is adjacent to at least* $\max\{2, \Delta(G) - d(y) + 1\}$ *vertices of maximum degree.*

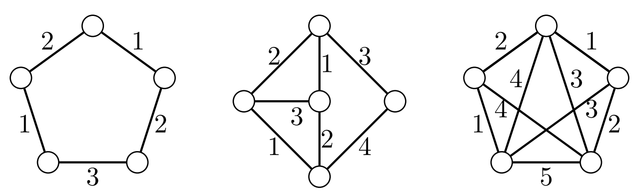
Kostochka [**2014**] found a reasonably short proof of this theorem. It implies that any class 2 graph has at least three vertices with maximum degree, which is attained by K_3.

The example at the beginning of the section shows that K_{2k} is class 1. A matching of K_{2k+1} uses at most k edges, so a Δ-edge-coloring of K_{2k+1} would use at most $\Delta k = 2k^2$ edges, but $m(K_{2k+1}) = 2k^2 + k$. Combined with the decomposition at the beginning of the section, this shows K_{2k+1} is class 2.

Generalizing, we see that if a graph with odd order n and maximum degree Δ has size larger than $\Delta \frac{n-1}{2}$, it cannot be Δ-edge-colorable.

Definition 7.33. A graph G contains an **overfull subgraph** H if $2m(H) > \Delta(G)(n(H) - 1)$.

Any graph with an overfull subgraph must be class 2. The following graphs with order 5 are overfull.



Conjecture 7.34 (**Overfull Conjecture**—Chetwynd/Hilton [**1986**]). *If $\Delta(G) > \frac{n}{3}$, then G is class 2 if and only if G has an overfull subgraph.*

The Overfull Conjecture would provide a short proof of the following conjecture about regular graphs, first published in Chetwynd/Hilton [**1985**]. This conjecture has been proved for all sufficiently large n (see Csaba/Kuhn et al. [**2016**]).

Conjecture 7.35 (**1-factorization Conjecture**). *Let $r \geq 2\left\lceil \frac{n}{4} \right\rceil - 1$. Then every r-regular graph of even order n is class 1.*

Proposition 7.36 (Petersen [**1898**]). *The Petersen graph is class 2.*

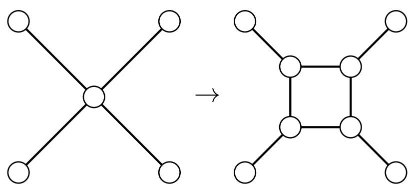
Proof. The Petersen graph has a 1-factor. Removing it leaves a 2-factor, which is a union of cycles. Since the Petersen graph has cycles of length 5, 6, 8, and 9 only, the 2-factor is $2C_5$, which is not 2-edge-colorable. Thus it is class 2. $\square$

The Petersen graph has even order and no overfull subgraph, but it is class 2. Thus the Overfull Conjecture cannot be extended to sparce graphs. Dense graphs can be checked quickly for overfull subgraphs (Niessen [**2001**]), so if true, the Overfull Conjecture would lead to a quick test of Δ-edge-colorability for these graphs. There is a polynomial algorithm to produce a $\Delta + 1$-edge-coloring based on Vizing's Theorem. However, the general problem of Δ-edge-colorability is NP-complete.

7.4. Tait Coloring

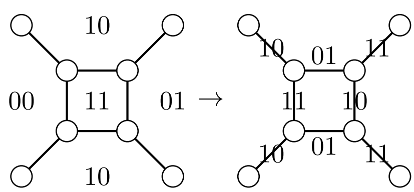
The Four Color Problem was originally stated in terms of coloring regions of maps. It was transformed into a problem of coloring vertices using the dual map operation. There is another way to model the Four Color Problem with a graph. Let edges be the boundaries between regions, and let vertices be the intersections of at least three regions. Peter Tait considered an edge coloring of this graph and proved a surprising relationship between it and map colorings.

First we note that the problem can be reduced to edge coloring of cubic graphs. If more than three edges meet at a vertex, another region can be inserted to create a cubic graph. The graph must be planar and bridgeless, since there cannot be an edge between a region and itself.



Definition 7.37. A **cubic map** is a bridgeless cubic planar graph. A **Tait coloring** is a 3-edge-coloring of a cubic map.

Theorem 7.38 (Tait's Theorem—Tait [**1880**]**).** *A cubic map has a 4-region-coloring if and only if it has a 3-edge-coloring.*

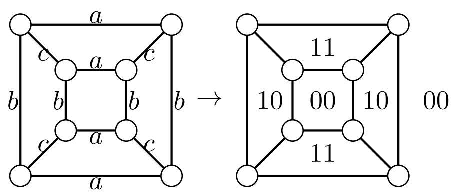


Proof. ($\Rightarrow$) Let G be a cubic map with a 4-region-coloring. Let the four colors be 00, 01, 10, and 11 (these can be viewed as elements of the Klein 4-group, $\mathbb{Z}_2 \times \mathbb{Z}_2$). Given the colors of two adjacent regions, add each digit mod 2 and assign this color to the edge between them (e.g., 01+11=10). Since the colors on two adjacent regions are distinct, 00 is never assigned to an edge.

The three regions adjacent to a vertex have distinct colors. If two adjacent edges receive the same color, then they are both adjacent to the same region. If the common region has color a and other regions have colors b and c, then $a + b = a + c$, so $b = c$. Thus two adjacent regions have the same color, a contradiction.

($\Leftarrow$) Let G have a 3-edge-coloring using colors a, b, c, with edge sets E_a, E_b, and E_c. The sets are all perfect matchings, so the union of any two is 2-regular, hence a disjoint union of cycles. Let $G_1 = E_a \cup E_b$ and $G_2 = E_a \cup E_c$. Assign to each region of G a color whose first digit is the parity of the number of cycles of G_1 that contain it and whose second digit is the parity of the number of cycles of G_2 that contain it.

Regions sharing an edge e are distinct and are separated by a cycle in at least one of G_1 and G_2. All other cycles are unaffected on opposite sides of e. Thus the parity of the number of cycles containing these regions is different in one or both coordinates, so they receive distinct colors. □



Thus Tait successfully proved that Tait colorings were equivalent to 4-colorings of maps. To prove the Four Color Theorem, he needed to show that every bridgeless cubic planar graph is 3-edge-colorable. Tait's original paper seemed to claim that all bridgeless cubic graphs are 3-edge-colorable. Julius Petersen [**1898**] refuted this claim by showing that the Petersen graph is not 1-factorable. However, the Petersen graph is nonplanar, so this did not really address Tait's core claim.

Tait believed that all 3-connected cubic planar graphs are Hamiltonian. This came to be known as Tait's Conjecture (Tait [**1884**]). A Hamiltonian cubic graph is 3-edge-colorable, since two colors can alternate on the edges of the cycle (which must have even order) and the remaining edges use the third color. Before long, other mathematicians noticed this this claim had not been proven. However, it was not until 1946 that William Tutte produced a counterexample, the Tutte graph (Section 6.3).

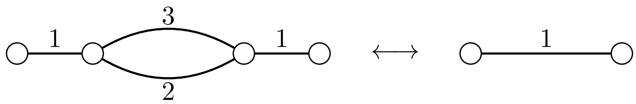
The Tutte graph does not disprove the Four Color Theorem (it has a 3-edge-coloring), only Tait's attempted proof of it. To prove the Four Color Theorem, it would suffice to prove that every class 2 bridgeless cubic planar graph is nonplanar. Any class 2 bridgeless cubic planar graph can be constructed from a smaller class of graphs using operations described in the following result.

Definition 7.39. A **snark** is a class 2 cubic 3-connected graph with no trivial 3–edge cut and girth at least 5.

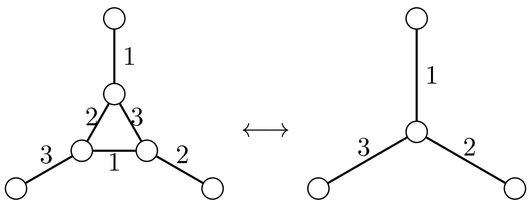
Proposition 7.40. *If there is a bridgeless cubic planar class* 2 *multigraph, then there is a planar snark.*

Proof. Let G be a cubic planar class 2 multigraph. We describe several operations that when applied to G produce a smaller multigraph H. In each case, if H has a 3-edge-coloring, then so does G. Thus the assumption that G is class 2 implies that H is also.

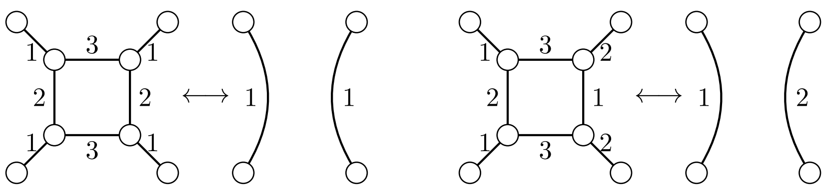
If G contains parallel edges, form H by contracting them as below.



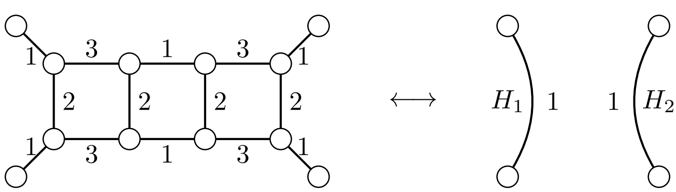
If G contains a triangle, form H by contracting the triangle to a single vertex as below.



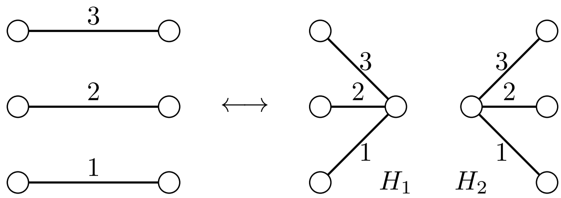
If G contains a 4-cycle and no triangle, replace it and the neighboring edges with two edges. There are two possibilities for how these edges may be colored, both of which would produce 3-edge-colorings of G.



If G contains a 2-edge-cut, then it contains a ladder ($P_k \Box K_2$ with the neighboring vertices of degree 2 on each end joined to nonadjacent vertices). This follows since the ends of the two edges are either adjacent (in which case the ladder continues) or not. Eventually, the ladder must end at two nonadjacent vertices. Form H_1 and H_2 by deleting the ladder and adding edges between the nonadjacent pairs of vertices at its ends. If H_1 and H_2 are both 3-edge-colorable, then so is G.

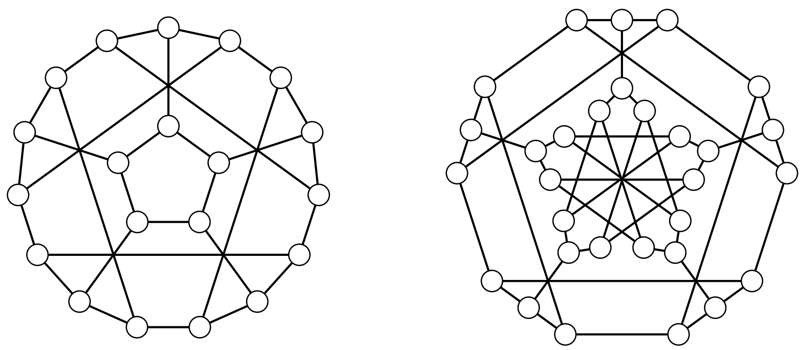


If G is 3-connected and contains a nontrivial 3–edge cut, the edges of the cut must be independent. Contracting the subgraph on either side of the cut produces two graphs H_1 and H_2. If both are 3-edge-colorable, then so is G.



Repeatedly applying the operations above must produce a cubic graph that is 3-connected with no trivial 3–edge cut and girth at least 5. Each operation preserves planarity in both directions, so it is a planar snark. □

The Four Color Theorem (via Tait's Theorem) implies that any bridgeless cubic planar graph is class 1. Thus any bridgeless cubic class 2 graph (hence any snark) is nonplanar. The operations in this proof show how to construct any class 2 bridgeless cubic graph from a snark. Thus snarks are the interesting cases.



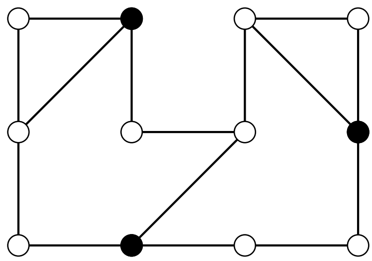
Until 1946, the Petersen graph was the only known snark. By 1975, only three more were discovered. After that, several infinite classes were discovered, along with operations to construct snarks from smaller snarks. A **flower snark** and the **double star snark** are shown above. They and many other snarks resemble the Petersen graph.

Conjecture 7.41 (**Tutte's Conjecture**—Tutte [**1966**]). *Any snark contains a subdivision of the Petersen graph.*

The Four Color Theorem implies that any snark is nonplanar. Tutte's Conjecture is a stronger claim, since the Petersen graph is a specific nonplanar graph. In 1999, Robertson, Sanders, Seymour, and Thomas announced a proof of this conjecture. However, as of 2020 components of the proof are still being published.

7.5. Domination

Example. A museum wants to position guards so that each room is observable. (Alternately, the "guards" could be security cameras.) Not every room needs a guard, but each room without a guard must be adjacent to one with a guard. What is the smallest number of guards that the museum can hire to observe all rooms? Let vertices represent rooms, with edges between pairs of rooms that observe each other. We need a set of vertices so that each vertex not in the set is adjacent to one that is. Given the following graph, we see that the black vertices are such a set. It appears that no smaller set is possible.

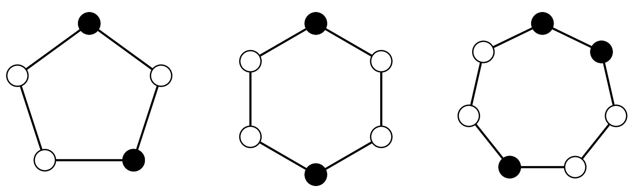


Example. The government builds transmitters in some towns in a rural area. Some towns are within transmission range of other towns (and vice versa) and others are not. How can transmitters be located so that every town is within range of some transmitter and the smallest number of transmitters is used? Let vertices represent towns, with edges between towns in transmission range. Again we need a set of vertices so that each vertex not in the set is adjacent to one that is.

Definition 7.42. A **dominating set** of a graph G is a set S of vertices so that every vertex not in S is adjacent to a vertex in S. A vertex in S is said to **dominate** itself and its neighbors. The **domination number** $\gamma(G)$ is the minimum size of a dominating set of G.

Note that $\gamma(G)$ is also the notation for the genus of a graph. Context should make it clear which meaning is intended.

Example. We show $\gamma(C_n) = \left\lceil \frac{n}{3} \right\rceil$. Number the vertices 0 through $n-1$, and let $S = \{v_0, v_3, v_6, \dots\}$. Then S is a dominating set. Each vertex can dominate at most three vertices, so a smaller set would fail to dominate some vertex.



The domination number is not easy to calculate in general, so we consider several bounds. Certainly $1 \le \gamma(G) \le n$. The extremal graphs for the lower bound are those with a vertex of degree $n-1$. Empty graphs are the extremal graphs for the upper bound.

Considering vertex degrees lead to better bounds. Since any isolated vertex must be in any dominating set of a graph, consider excluding graphs with isolated vertices.

Proposition 7.43 (Ore [**1962**])**.** *If G is a graph with no isolated vertices, then $\gamma(G) \le \frac{n}{2}$.*

Proof. Consider a spanning forest of G, which is bipartite. Each partite set dominates the other. Choosing the smallest set shows $\gamma(G) \le \frac{n}{2}$. $\square$

This bound is sharp. The extremal graphs are explored in the Exercises. The hypothesis of Proposition 7.43 could be stated as $\delta(G) \geq 1$. A larger minimum degree will reduce the upper bound.

Theorem 7.44. *Let G be a graph.*

If $\delta(G) \geq 1$, then $\gamma(G) \leq \frac{1}{2}n$ (Ore [**1962**]).

If $\delta(G) \geq 2$, then $\gamma(G) \leq \frac{2}{5}n$ (if $n > 7$) (McQuaig/Shepherd [**1989**]).

If $\delta(G) \geq 3$, then $\gamma(G) \leq \frac{3}{8}n$ (Reed [**1996**]).

If $\delta(G) \geq 4$, then $\gamma(G) \leq \frac{4}{11}n$ (Sohn/Xudong [**2009**]).

If $\delta(G) \geq 5$, then $\gamma(G) \leq \frac{5}{14}n$ (Xing, Sun, and Chen [**2006**]).

If $\delta(G) \geq 6$, then $\gamma(G) \leq \frac{6}{17}n$ (Jianxiang et al. [**2008**]).

If $\delta(G) \geq k$, then $\gamma(G) \leq \left[1 - k\left(\frac{1}{k+1}\right)^{1+\frac{1}{k}}\right] n$ (Caro/Roditty [**1985**, **1990**]).

The bounds for $1 \leq k \leq 6$ all satisfy $\gamma(G) \leq \frac{k}{3k-1}n$. For $\delta(G) \geq 7$, the final bound is superior to this formula. The bounds for $1 \leq k \leq 3$ are known to be sharp.

Vertex degrees also produce lower bounds.

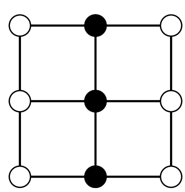
Definition 7.45. Let G be a graph with degrees $d_1 \geq d_2 \geq \cdots \geq d_n$. The **Slater number** of G, $\mathrm{sl}(G)$, is the smallest integer k so that $\sum_{i=1}^{k} d_i \geq n - k$.

Proposition 7.46. *A graph G has $\gamma(G) \geq \mathrm{sl}(G)$. In particular, $\gamma(G) \geq \frac{n}{1+\Delta(G)}$.*

Proof. If S is a minimum dominating set of G, then $v \in S$ dominates $1 + d(v)$ vertices. Thus $n \leq \sum_{v \in S}(1 + d(v)) \leq \sum_{i=1}^{|S|}(1 + d_i) \leq \gamma(G) \cdot (1 + \Delta(G))$. The third expression gives the first result. The last expression gives the latter result. ☐

This result does not take the structure of a graph into account. It may not be exact, since the vertices with large degree may have many common neighbors.

Example. Consider $G_{3,3}$, with degree sequence 433332222. We find $\gamma(G) \geq \mathrm{sl}(G) = 2$, since $4+3 = 9-2$. (The other bound gives $\gamma(G) \geq \frac{9}{1+4} = 1.8$.) However, this is not exact since the degree 4 vertex is adjacent to all degree 3 vertices. Picking the degree 4 vertex and a neighbor leaves two undominated vertices. In fact, $\gamma(G_{3,3}) = 3$, as seen below.



To show that $\gamma(G) = k$, we must show

(1) $\gamma(G) \leq k$. Find a minimum dominating set, or use an upper bound.

(2) $\gamma(G) \geq k$. Use a lower bound, or find a contradiction to show that $\gamma(G) < k$ is impossible.

The latter step is often very difficult, even for well-known classes of graphs. Indeed, determining the domination number of a graph is NP-complete.

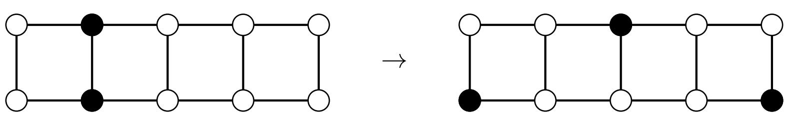
A greedy approach to finding a dominating set is to add a vertex of maximum degree and iterate this step on the graph induced by the undominated vertices. This usually does not produce a minimum dominating set, but the result is not too far from optimal.

Determining the domination number of grids has received considerable attention. Note that any grid has $\Delta(G) \leq 4$, so $\gamma(G_{r,s}) \geq \frac{n}{5}$. For large grids, this is close but not exact due to the boundary vertices. A complete classification of the values of $\gamma(G_{r,s})$, which has 23 separate cases, is given in (Goncalves et al. [**2011**]), with references. The following result illustrates the techniques used to find the domination number.

Proposition 7.47 (Jacobsen/Kinch [**1983**]). *We have* $\gamma(G_{2,s}) = \left\lceil \frac{s+1}{2} \right\rceil$.

Proof. Denote the vertices of $G_{2,s}$ as (i,j), $1 \leq i \leq s$, $1 \leq j \leq 2$. Let $S = \{(1,1), (3,2), (5,1), \ldots\}$ except when s is even; we add one more vertex to cover the final corner. Then S is a dominating set with $|S| = \left\lceil \frac{s+1}{2} \right\rceil$. Each vertex dominates at most four vertices, but the two on the ends dominate fewer.

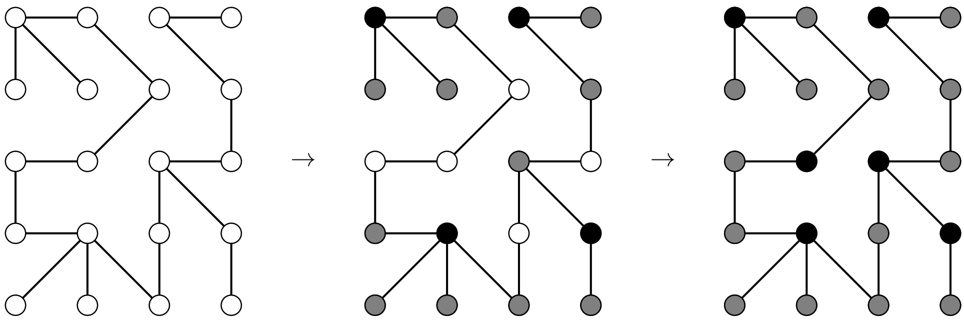
We claim that some minimum dominating set contains $(1,1)$ (or $(1,2)$). If not, then it contains $(2,1)$ and $(2,2)$, which together dominate six vertices. Replacing them with $(1,1)$ and $(3,2)$ dominates seven vertices, including the same six as before. Thus any dominating set of k vertices dominates at most $4k-2$ vertices. Thus $n = 2s \leq 4k-2$, so $k = \left\lceil \frac{2s+2}{4} \right\rceil$. $\square$



There is an algorithm to determine the domination number of a tree exactly, but it does not lead to a convenient formula. We note that when $n \geq 3$, there is no need to use a leaf in the dominating set, since any such leaf could be replaced by its neighbor.

Algorithm 7.48. *Begin with a copy of a tree T, and a set S that will be the dominating set, and a set D of dominated vertices. If T is a star with all leaves in D, add the center to S and stop. Else add all neighbors of leaves to S and add all neighbors of vertices in S to D. Then delete all leaves and vertices in S, and iteratively delete all leaves in D. While any vertices remain, repeat these steps again on each component.*

Example. Consider the tree at left. The vertices added to S are black and the vertices added to D are gray. The first iteration is shown in the middle, and the second (final) iteration is shown at right.



When a graph has a 1-shell (trees rooted on the 2-core), the same approach to finding a minimum dominating set can be found. A 2-monocore graph can be constructed by adding ears and cycles (Theorem 3.34), which can also be used to construct a dominating set.

There are many, many variations of domination. Perhaps a museum does not fully trust its security guards, so it wants each guard to be observed by another guard. Perhaps the transmitters in a rural area need to be able to relay a message between any two transmitters. These variations impose additional restrictions on the dominating set, so $\gamma(G)$ is a lower bound for each of them.

Definition 7.49. An **independent dominating set** of a graph G is a dominating set that is independent. The **independent domination number** $i(G)$ is the minimum size of an independent dominating set of G.

A **connected dominating set** of a connected graph is a dominating set that is connected. The **connected domination number** $\gamma_c(G)$ is the minimum size of a connected dominating set of G.

A **total dominating set** of a graph with no isolated vertices is a dominating set that requires each vertex in the set to be dominated by another vertex. The **total domination number** $\gamma_t(G)$ is the minimum size of a total dominating set of G.

We briefly note a few results on these parameters. The following result on the independent domination number implies that $\gamma(G) \leq i(G) \leq \alpha(G)$.

Proposition 7.50. *A vertex set is an independent dominating set if and only if it is a maximal independent set.*

Proof. An independent set S is maximal if and only if every vertex not in S has a neighbor in S, in which case S is a dominating set. $\square$

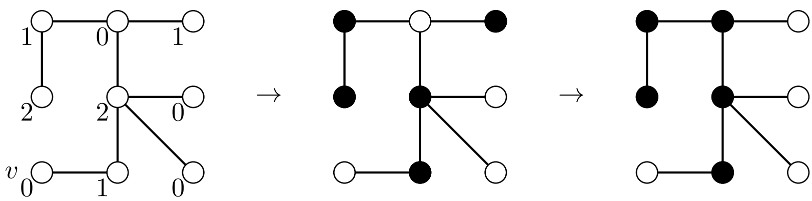
Theorem 7.51 (Allan/Laskar [**1978**])**.** *If G is a claw-free graph, then $\gamma(G) = i(G)$.*

Proof (Goddard/Henning [**2013**]). Let G be a claw-free graph. Let S be a minimum dominating set so that $G[S]$ has minimum size. Suppose S is not independent. Then there exist adjacent vertices u and v in S. Let $P_v = \{w \in V(G) \mid N(w) \cap S = \{v\}\}$ be the **private neighbors** of v. By the minimality of S, the set P_v is nonempty. Since G is claw-free, the set P_v is a clique. Thus for any $v' \in P_v$, the set $S' = S - \{v\} \cup \{v'\}$ is a minimum dominating set such that $G[S']$ has fewer edges that $G[S]$, a contradiction. □

For total domination, we again have upper bounds based on minimum degree.

Theorem 7.52 (Cockayne/Dawes/Hedetniemi [**1980**]). *Let G be a connected graph with $n \geq 3$. Then $\gamma_t(G) \leq \frac{2}{3}n$.*

Proof (Bickle [**2013**]). Let T be a spanning tree of G, and let v be a leaf of T. Label each vertex of T with its distance from v mod 3. This produces three sets that partition the vertices of G. Then some set contains at least one third of the vertices of G, and the union S of the other two contains at most two thirds of the vertices. Each internal vertex of T is adjacent to a vertex in each of the other sets. If S contains an isolated leaf, replace it with its neighbor. Then S is a total dominating set. □



The extremal graphs for this bound are characterized in the Exercises.

Theorem 7.53 (Henning/Yeo [**2007**]). *Let G be a graph without isolated vertices.*

If $\delta(G) \geq 1$, then $\gamma_t(G) \leq \frac{2}{3}n$, provided G is connected with $n \geq 3$.

If $\delta(G) \geq 2$, then $\gamma_t(G) \leq \frac{4}{7}n$, ($G$ connected and not C_3, C_5, C_6, or C_{10}).

If $\delta(G) \geq 3$, then $\gamma_t(G) \leq \frac{1}{2}n$.

If $\delta(G) \geq 4$, then $\gamma_t(G) \leq \frac{3}{7}n$.

If $\delta(G) \geq k$, then $\gamma_t(G) \leq \frac{1+\ln k}{k}n$.

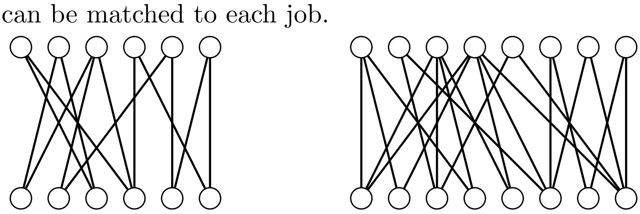
Domination is the second most popular research topic in graph theory, after coloring. It is explored in a book (Haynes/Hedetniemi/Slater [**1998**]).

Related Terms: corona, brush, roman domination, domatic number, fractional domination number, Vizing's Conjecture, irredundance number, upper irredundance number, upper domination number, weak product.

Exercises

Section 7.1:

(1) The graph below left represents applicants for various jobs. Determine whether applicants can be matched to each job.



(2) The graph above right represents advertisements and websites. Determine whether ads can be matched to each site.

(3) Determine whether the sets $\{1,3,6\}$, $\{1,4,5\}$, $\{2,6\}$, $\{2,3,5\}$, $\{3,4\}$, $\{4,6\}$ have a system of distinct representatives.

(4) Determine whether the sets $\{1,5,7\}$, $\{2,3,5\}$, $\{3,6\}$, $\{1,4,7\}$, $\{2,3,6\}$, $\{5,6\}$, $\{2,3,5,6\}$ have a system of distinct representatives.

(5) Show that $|\mathbb{Z}| = |\mathbb{Z} \times \mathbb{Z}|$.

(6) Show that $|\mathbb{R}| = |\mathbb{R}^n|$.

(7) Let G be a connected bipartite graph with partite sets U and V. Show that if all vertices of U have distinct degrees, then G has a perfect matching.

(8) Define a bipartite graph to be **biregular** if vertices in the same partite set have the same degree. Show that every biregular graph has a matching that covers its smaller partite set.

(9) Show that every tree has at most one perfect matching.

(10) Show that there is a bipartite graph of order $2k$ and size $\binom{k+1}{2}$ with a unique perfect matching.

(11) + Prove Hall's Theorem using induction on order. (Consider two cases, one where $|N(S)| > |S|$ for all nonempty proper subsets S of U.)

(12) + (Ford/Fulkerson [**1958**]) A **common system of distinct representatives** of families of sets $A = \{A_1, \ldots, A_m\}$ and $B = \{B_1, \ldots, B_m\}$ is a set of m elements that is an SDR for A and for B (the indices need not be the same). Use Menger's Theorem to show that A and B have a common system of distinct representatives if and only if for each pair I, $J \subseteq [m]$,

$$\left| \left(\bigcup_{i \in I} A_i \right) \cap \left(\bigcup_{j \in J} B_j \right) \right| \geq |I| + |J| - m.$$

(13) Find the number of perfect matchings in the labeled graph $K_{r,r}$.

(14) Find the number of perfect matchings in the labeled graph K_{2r}.

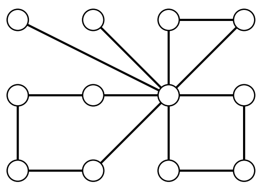
(15) Show that every edge of the Petersen graph is contained in four 5-cycles. Use this to count the number of perfect matchings it contains.

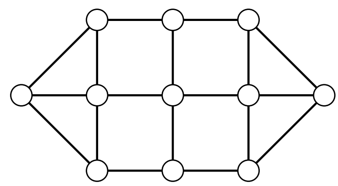
(16) Show that Q_k, $k \geq 2$, has at least $2^{2^{k-2}}$ perfect matchings.

(17) Determine α, α', β, and β' for the graphs in the following classes.
(a) P_n
(b) W_n
(c) Q_k
(d) T_l

(18) Determine α, α', β, and β' for the graphs in the following classes.
(a) C_n
(b) $K_{r,s}$
(c) I_n
(d) $G_{r,s}$

(19) Determine α, α', β, and β' for the graph below left. Find vertex and edge sets to illustrate each value.





(20) Determine α, α', β, and β' for the graph above right. Find vertex and edge sets to illustrate each value.

(21) A bipartite graph G with no isolated vertices has order 23 and $\alpha(G) = 8$. Determine α', β, and β'.

(22) A bipartite graph G with no isolated vertices has order 78 and $\beta'(G) = 35$. Determine α, α', and β.

(23) (Bickle [**2014**]) Show that if G is a graph with $\delta(G) \geq k$, then $\alpha'(G) \geq \left\lceil \frac{k}{2} \right\rceil$, and the graphs for which this is an equality are exactly empty graphs, stars, complete graphs, and $K_{2i+1} - tK_2$, $1 \leq t \leq i$.

(24) (Weinstein [**1963**]) Let G be a graph of order n without isolated vertices. Show that $\alpha'(G) \geq \frac{n}{1+\Delta(G)}$.

(25) Show that for any graph G, $\beta(G) \leq 2\alpha'(G)$. Show that this bound is sharp.

(26) Prove or disprove: For any graph G, $\beta'(G) = \beta(L(G))$.

(27) Characterize the graphs for which α, α', β, and β' equal 1.

(28) Find sharp upper bounds for α, α', β, and β' on graphs of order n.

Section 7.2:

(1) Show that a tree T has a perfect matching if and only if $o(T - v) = 1$ for every vertex.

(2) Find a cubic graph of smallest order that does not contain a 1-factor.

(3) (Petersen [**1891**]) Show that every cubic graph with at most two bridges contains a 1-factor.

(4) Show that every cubic graph with all bridges on a path contains a 1-factor.

(5) Show that every edge of a bridgeless cubic graph is contained in some 1-factor.

(6) Show that any bridgeless cubic graph decomposes into copies of P_4.

(7) Show that in any cubic graph with a 1-factor, every bridge is contained in the 1-factor.

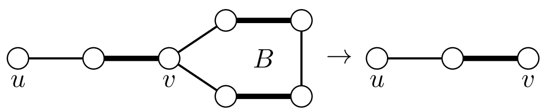
(8) Show that a k-regular graph G with $\kappa'(G) \geq k-1$ has a 1-factor.

(9) Show that the largest size of a graph of order $2k$ with a unique perfect matching is k^2.

(10) + (Sumner [**1974**]) Show that every connected claw-free graph of even order contains a perfect matching.

(11) Show that if G has order n and $S \subseteq V(G)$, then $o(G-S) - |S| \equiv n \bmod 2$. (*Note*: This implies that if S is a Tutte set and n is even, then $o(G-S) \geq |S| + 2$.)

(12) + (Edmonds [**1965**]) Let G be a graph with a matching M. A **blossom** B is an odd cycle in G consisting of $2k+1$ edges, of which exactly k belong to M, and where one of the vertices v of the cycle is such that there exists an M-alternating path of even length from v to an uncovered vertex u. Form H by contracting B to a single vertex, and let N be the matching corresponding to M. Show that M is a maximum matching of G if and only if N is a maximum matching of H.



(13) + Prove Menger's Theorem using Konig's Theorem. (*Hint*: Use induction on n and consider whether G has a minimum $u-v$ cut other than $N(u)$ and $N(v)$.)

(14) Prove Hall's Theorem using Menger's Theorem.

(15) Prove Hall's Theorem using Tutte's 1-Factor Theorem.

(16) + Prove Konig's Theorem using Hall's Theorem. (*Hint*: Split the vertex cover between the partite sets, and construct disjoint matchings covering each of them.)

(17) + In graph theory terms, **Dilworth's Theorem** (Dilworth [**1950**]) states that for a transitive digraph D, the minimum number of paths that partition the vertices equals the independence number $\alpha(D)$. Prove Dilworth's Theorem by applying Konig's Theorem to the split of D.

(18) Prove Konig's Theorem using Dilworth's Theorem. (*Hint*: Consider a transitive orientation of a bipartite graph.)

Section 7.3:

(1) Five people want to play doubles tennis with all possible teams of two playing each other (the fifth person will chase down stray tennis balls). How many different matches will there be, and how many days must they play if no team can play more than once per day?

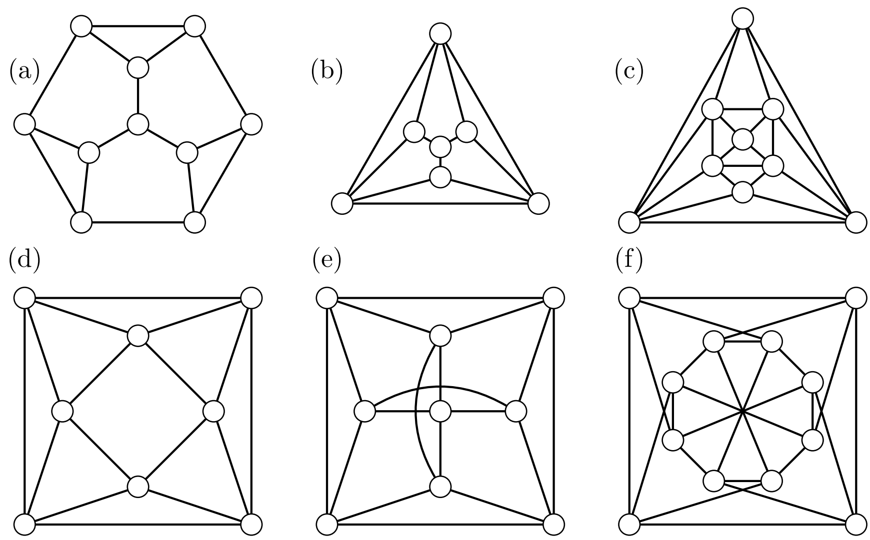
(2) A business has a number of objects to manufacture, each requiring some tasks to be performed on different machines (in any order). If each task takes the

same amount of time and each machine can perform one task at a time, how many time periods are required for all the objects to be manufactured?

(3) Determine which graphs in the following classes are class 1 or class 2.
   (a) C_n
   (b) Q_k
   (c) I_n
   (d) T_l

(4) Determine which graphs in the following classes are class 1 or class 2.
   (a) trees
   (b) W_n
   (c) theta graphs
   (d) $G_{r,s}$

(5) Determine whether the graphs below are class 1 or class 2.



(6) Determine whether each of the Platonic solids are class 1 or class 2.

(7) Characterize all graphs with $\chi'(G) = 2$.

(8) Without using Vizing's Theorem, find a short proof that $\chi'(G) \leq 2\Delta(G) - 1$.

(9) Let G be a nonempty regular graph with odd order. Determine whether G is class 1 or class 2.

(10) Let G be formed by subdividing an edge of a regular graph with even order. Determine whether G is class 1 or class 2.

(11) Show that a regular graph with a cut-vertex is class 2.

(12) Show that if G is class 1, then $G \square H$ is class 1.

(13) Show that a graph is χ'-critical if and only if G is a star (which is class 1) or G is class 2 and $G - e$ is class 1 for all $e \in E(G)$.

(14) Use Lemma 7.28 (not Vizing's Adjacency Lemma) to show the following.

(a) If the vertices of a graph G can be sucessively deleted so that when deleted, each vertex is adjacent to at most one vertex with degree $\Delta(G)$, then G is class 1.

(b) (Vizing [**1965**]) A class 2 χ'-critical graph has every vertex adjacent to at least two vertices of degree $\Delta(G)$.

(15) Find an infinite class of class 2 graphs with three vertices of maximum degree.

(16) Let G be a graph with D the maximum degree in G of the vertices in the 1-shell of G. Show that $\chi'(G) = \max\{D, \chi'(C_2(G))\}$.

(17) (a) (a) + (Goufei [**2003**]) Use Vizing's Adjacency Lemma to show that every k-degenerate graph with $\Delta \geq 2k$ is class 1.

(b) Show that if G is planar with $\Delta(G) \geq 10$, then G is class 1. (*Note*: Vizing [**1965**] conjectured this holds for $\Delta \geq 6$. He proved it for $\Delta \geq 8$, and Sanders/Zhao [**2001**] proved it for $\Delta = 7$. The case $\Delta = 6$ remains open.)

(18) Find class 2 planar graphs for each maximum degree with $\Delta \in \{2, 3, 4, 5\}$.

(19) Use the graph formed by deleting a vertex of the Petersen graph to show that the degree condition in the Overfull Conjecture cannot be improved.

(20) Show that the Overfull Conjecture implies the 1-Factorization Conjecture. (*Hint*: Show that a subgraph of a regular graph of even order is overfull if and only if the subgraph induced by the other vertices is overfull.)

(21) Show that the 1-Factorization Conjecture is sharp.

(22) Show that for any multigraph G, $\chi'(G) \geq \frac{m}{\alpha'(G)}$.

(23) Show that any graph G with $m > \Delta(G)\,\alpha'(G)$ is class 2.

(24) Show that for any graph G, $\alpha'(G) \geq \frac{m}{1+\Delta(G)}$.

(25) (a) Find an infinite class of multigraphs with order 3 and $\chi'(G) = \frac{3}{2}\Delta(G)$.

(b) Find an infinite class of multigraphs with order 3 and $\chi'(G) = \Delta(G) + \mu(G)$.

(26) + (Shannon [**1949**]) Use Vizing's Theorem to prove $\chi'(G) \leq \frac{3}{2}\Delta(G)$. (*Hint*: Consider a minimal counterexample G containing $e = uv$, and count how many colors are used at the ends of $G - e$.)

Section 7.4:

(1) Show that every Hamiltonian plane graph is 4-region-colorable without using the Four Color Theorem.

(2) Show that any uniquely 3-edge-colorable graph has exactly three Hamiltonian cycles.

(3) Show that any graph formed from K_4 by iterating the operation of replacing a vertex with a triangle is uniquely 3-edge-colorable. (*Note*: Fowler [**1998**] showed that any planar uniquely 3-edge-colorable graph can be formed this way.)

(4) Determine whether Mobius ladders are class 1 or class 2.

(5) Show that the Tutte graph is class 1.

(6) + (Isaacs [**1975**]) Show that the flower snark of order 20 is class 2.

(7) The **Cycle Double Cover Conjecture** (Szekeres [**1973**], Seymour [**1980**]) asserts that any bridgeless graph has a collection of cycles that together contain every edge exactly twice.
  (a) Show that the Petersen graph has a cycle double cover.
  (b) Show that any Eulerian graph has a cycle double cover.
  (c) Show that any planar bridgeless cubic graph has a cycle double cover.

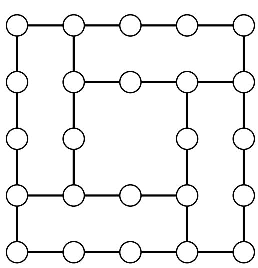
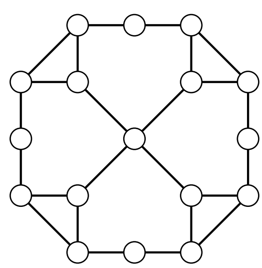
(8) (a) Show that if the Cycle Double Cover Conjecture holds for bridgeless cubic graphs, then it holds for all bridgeless graphs.
  (b) Show that if the Cycle Double Cover Conjecture holds for snarks, then it holds for all bridgeless cubic graphs.

(9) + (Robertson [**1968**], Thomasen [**1982**], Schwenk [**1989**]) For the generalized Petersen graph $P(r,k)$,
  (a) Show that when $r \geq 3$ and $r \not\equiv 5 \bmod 6$, $P(r,2)$ is Hamiltonian.
  (b) Show that when $r \equiv 5 \bmod 6$, $P(r,2)$ is not Hamiltonian.
  (c) Show that when $r \equiv 3 \bmod 6$, $P(r,2)$ has exactly three Hamiltonian cycles.
  (d) Show that when $r \geq 15$ and $r \equiv 3 \bmod 6$, $P(r,2)$ has more than one 3-edge-coloring.

(10) (a) (Tutte [**1976**]) Show that $P(9,2)$ has a unique 3-edge-coloring.
  (b) (Belcastro/Haas [**2014**]) Let G and H be cubic graphs with a and b 3-edge-colorings, respectively. Let $G \curlyvee H$ be formed by deleting one vertex from each graph and adding a matching between their neighborhoods (this operation is not unique). Show that $G \curlyvee H$ has ab 3-edge-colorings.
  (c) (Belcastro/Haas [**2015**]) Show that there are infinitely many nonplanar triangle-free uniquely 3-edge-colorable graphs.

Section 7.5:

(1) Each graph below represents the rooms of a museum. Find the smallest number of security cameras that can observe the whole museum, and where they should be placed.

(a)  (b) 

(2) Find the domination numbers of all Platonic solids.

(3) Find the domination number of the graphs in the following classes.
  (a) K_n
  (b) C_n^2
  (c) $K_{r,s}$
  (d) $\theta_{i,j,k}$

(4) Find the domination number of the graphs in the following classes.
(a) W_n
(b) P_n
(c) $K_{r,s,t}$
(d) I_n

(5) Domination number of grids.
(a) Find $\gamma(G_{3,s})$.
(b) Show $\gamma(G_{4,s}) \leq s$, $s \geq 4$, unless $s = 5, 6, 9$.
(c) Show $\gamma(G_{5,s}) \leq \left\lceil \frac{6s-4}{5} \right\rceil$, $s \geq 5$, unless $s = 7$.

(6) Find the domination number of the following products.
(a) $C_r \Box K_2$
(b) $C_r \Box P_3$
(c) $C_r \Box C_3$
(d) $C_{5r} \Box C_{5s}$
(e) $K_r \Box K_r$

(7) + (Georges/Lin/Mauro [**2014**]) Show that $\gamma(K_r \Box K_r \Box K_r) = \left\lceil \frac{r^2}{2} \right\rceil$.

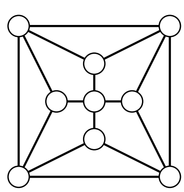
(8) Find the domination number of the following graphs.
(a) the Petersen graph
(b) the Grotzch graph
(d) Q_4

(9) Find the domination number of the trees in the following classes.
(a) stars
(b) double stars
(c) spiders
(d) brooms

(10) Prove that Algorithm 7.48 produces a minimum dominating set.

(11) Let G be a connected graph with order n and diameter d. Show that $\frac{1+d}{3} \leq \gamma(G) \leq n - \frac{2d}{3}$.

(12) Find the domination number and diameter of the graph below. (*Note*: Goddard/Henning [**2002**] showed this is the only planar graph with these parameters.)



(13) Show that $\gamma(G) \leq n - \Delta(G)$. Characterize the extremal graphs for this bound.

(14) Bounds on $\gamma(G)$. Let G be a graph without isolated vertices.
(a) Show that $\gamma(G) \leq \beta(G)$.
(b) Show that $\gamma(G) \leq \beta'(G)$.
(c) Show that $\gamma(G) \leq \alpha'(G)$.

(15) Determine sharp bounds on $\gamma(G - v)$ in terms of $\gamma(G)$ and n.

(16) Determine sharp bounds on $\gamma(G-e)$ in terms of $\gamma(G)$.

(17) Determine $\gamma(G+H)$ in terms of properties of G and H.

(18) Determine $\gamma(G\cup H)$ in terms of properties of G and H.

(19) **Vizing's Conjecture** (Vizing [**1968**]) is that $\gamma(G\square H)\geq\gamma(G)\cdot\gamma(H)$. Show that Vizing's Conjecture is true when $\gamma(G)=1$, and when $G,H\in\{P_4,C_4\}$.

(20) (Vizing [**1968**]) Show that $\gamma(G\square H)\leq\min\{\gamma(G)\,n(H),\gamma(H)\,n(G)\}$.

(21) (Payan/Xuong [**1982**]) The **corona** of a graph is a graph that adds a leaf adjacent to each existing vertex. Show that a graph with $\delta(G)\geq 1$ has $\gamma(G)=\frac{n}{2}$ if and only if $G=C_4$ or G is the corona of some graph.

(22) + Characterize graphs with $\delta(G)\geq 1$ and $\gamma(G)=\frac{n-1}{2}$.

(23) (McQuaig/Shepherd [**1989**]) Find an infinite class of connected graphs with $\delta(G)=2$ and $\gamma(G)=\frac{2}{5}n$.

(24) Find seven graphs of order at most 7 with $\delta(G)=2$ and $\gamma(G)>\frac{2}{5}n$.

(25) Find all graphs of order 10 with $\delta(G)=2$ and $\gamma(G)=4$.

(26) Find the domination numbers of all cubic graphs of order 8. (Those with $\gamma(G)=3$ are extremal graphs for $\gamma(G)\leq\frac{3}{8}$ when $\delta(G)\geq 3$.)

(27) Characterize trees with $\gamma(G)=\frac{n}{1+\Delta(G)}$.

(28) For all $r\geq 2$ and $n=k(1+r)$, show that there is a connected r-regular graph with $\gamma(G)=\frac{n}{1+r}$.

(29) (Ore [**1962**]) Show that the complement of a minimal dominating set of G is a dominating set G. Use this to prove Proposition 7.43.

(30) Use Proposition 7.50 to prove Proposition 7.43.

(31) Find the independent domination number of the graphs in the following classes.
- (a) C_n
- (b) $K_{r,s}$
- (c) double stars

(32) Find the connected domination number of the graphs in the following classes.
- (a) C_n
- (b) C_n^2
- (c) $G_{3,s}$
- (d) $G_{4,s}$

(33) Find the total domination number of the graphs in the following classes.
- (a) C_n
- (b) $\theta_{i,j,k}$
- (c) $G_{2,s}$
- (d) $K_n\square K_n$

(34) Find the total domination number of the graphs in the following classes.
- (a) W_n
- (b) C_n^2
- (c) $G_{3,s}$
- (d) $C_r\square K_2$

(35) Show that for a connected graph G, $\gamma_c(G)$ is n minus the maximum number of leaves of a spanning tree of G.

(36) Show that for a connected graph G, $1 \leq \gamma_c(G) \leq n-2$. Characterize the extremal graphs for both bounds.

(37) Show that $\gamma_t(G) \leq 2\gamma(G)$ and that this bound is sharp.

(38) + (Brigham et al. [**2000**], Bickle [**2013**]) A **brush** is a graph formed by starting with some graph G and identifying a leaf of a copy of P_3 with each vertex of G. Let G be a connected graph with $n \geq 3$. Show that $\gamma_t(G) = \frac{2}{3}n$ exactly when G is C_3, C_6, or a brush.

(39) The **domatic number** $\mathrm{dom}(G)$ of a graph G is the maximum size of partition of the vertices into dominating sets. Show that for a graph with no isolated vertices, $2 \leq \mathrm{dom}(G) \leq \min\left\{\delta(G)+1, \frac{n}{\gamma(G)}\right\}$.

(40) Find the domatic number of the graphs in the following classes.
  (a) K_n
  (b) C_n
  (c) trees
  (d) $K_{r,s}$

(41) Find the domatic numbers of the cubic graphs of order 8.

(42) In chess, a queen can move vertically, horizontally, or diagonally. Find a placement of five queens on an 8-by-8 chessboard that cannot attack each other but can attack all other squares.

Chapter 8

Generalized Graph Colorings

Recall the definition of chromatic number.

Definition 8.1. A k**-coloring** of a graph is a proper vertex coloring using colors $1, \ldots, k$ (not necessarily all of them). The **chromatic number** $\chi(G)$ is the minimum number of colors used in any k-coloring of a graph G.

Consider the following equivalent definitions.

Definition 8.2. The following are definitions of chromatic number.

(1) The **chromatic number** $\chi(G)$ is the minimum number k such that the vertices of G can be properly colored from identical lists of k elements.

(2) The **chromatic number** $\chi(G)$ is the minimum number of sets in a partition of the vertex set so that each set induces an empty graph.

(3) The **chromatic number** $\chi(G)$ is the minimum number of colors required to greedily color its vertices for all possible vertex orders.

(4) The **chromatic number** $\chi(G)$ is the minimum number of colors required to color the vertices of G so that vertices distance one apart receive distinct colors.

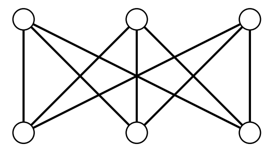
(5) The **chromatic number** $\chi(G)$ is the minimum number of colors required to assign sets of colors of size one to the vertices of G so that adjacent vertices are assigned disjoint sets.

Each of these definitions is equivalent to the original definition. The advantage in stating the definition in different ways is that each statement suggests different alterations or generalizations of the chromatic number. This chapter examines a few of the many variations on vertex coloring.

8.1. List Coloring

The first definition of chromatic number, Definition 8.2(1), suggests the idea of coloring a graph by choosing lists of possible colors for each vertex, then choosing one color from each list for the corresponding vertices. When all the lists are identical, this is just the problem of finding a k-coloring. When the lists are all completely different, the vertices can be colored by making an arbitrary choice at each vertex. However, when the lists only partially overlap, lists of length $\chi(G)$ may not suffice.

Example. Consider the following assignment of lists of length 2 to $K_{3,3}$. No matter what choices are made, two distinct colors must appear on vertices of each partite set, forcing two vertices with the same color to be adjacent. Thus it cannot be guaranteed that $K_{3,3}$ can be colored using elements of lists of length 2.



Definition 8.3. A **list coloring** of a graph begins with lists $L(v_i)$ of length k assigned to each vertex and chooses a color from each list to obtain a proper vertex coloring. A graph G is k**-choosable** if any assignment of lists of length k to the vertices permits a proper coloring. The **list chromatic number** $\chi_l(G)$, is the smallest k such that G is k-choosable.

There are natural bounds for list chromatic number involving chromatic number and degeneracy.

Theorem 8.4. *For any graph G, $\chi(G) \leq \chi_l(G) \leq 1 + D(G)$.*

Proof. The lower bound holds since the lists could be identical.

For the upper bound, establish a construction sequence for G. If a vertex v has degree $d(v)$, a list of $1 + d(v)$ colors guarantees v can be colored. Thus $\chi_l(G) \leq 1 + D(G)$. □

For graphs with $\chi(G) = 1 + D(G)$, this theorem immediately gives the list chromatic number. It implies that every tree is 2-choosable. It also implies that if G has a 2-core, then $\chi_l(G) = \chi_l(C_2(G))$.

Recall that the θ-graph $\theta_{i,j,k}$ is the graph formed by identifying the endpoints of three paths of lengths i, j, and k.

Theorem 8.5 (Erdos/Rubin/Taylor [**1979**]). *A connected graph G is 2-choosable if and only if it is a tree or its 2-core is an even cycle or $\theta_{2,2,2k}$, $k \geq 1$.*

Thus every 2-monocore graph G that is not an even cycle or $\theta_{2,2,2k}$, $k \geq 1$, has $\chi_l(G) = 3$. The proof of Theorem 8.5 is outlined in the Exercises.

The list chromatic may be arbitrarily far away from the chromatic number.

Proposition 8.6 (Erdos/Rubin/Taylor [**1979**]). *Let $r = \binom{2k-1}{k}$. Then $\chi_l(K_{r,r}) > k$.*

Proof. For each partite set, use each list of k elements of $[2k-1]$ on a vertex. Coloring each partite set requires k colors, since if fewer colors are used, a vertex with a list of k other colors cannot be colored. The Pigeonhole Principle implies that some color must be used on both partite sets, so $K_{r,r}$ is not k-choosable. □

This implies that $K_{10,10}$ is not 3-choosable, however this result is not sharp. In fact, determining the list chromatic number for complete bipartite graphs turns out to be a difficult problem.

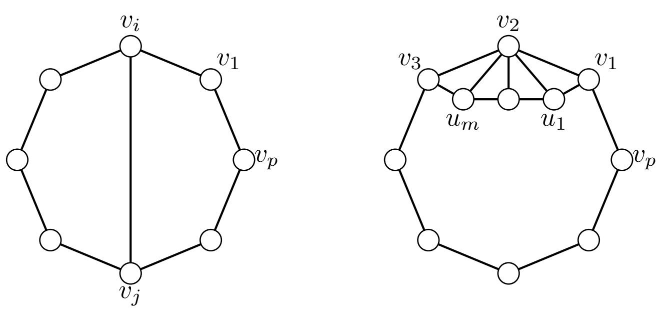
Planar graphs are 5-degenerate; it follows immediately that they are 6-choosable. Vizing [**1976**] and Erdos, Rubin, and Taylor [**1979**] independently conjectured that planar graphs are 5-choosable and that planar graphs with list chromatic number 5 exist. Both conjectures have been proven.

Theorem 8.7 (Thomassen [**1994**])**.** *Planar graphs are 5-choosable.*

Proof. Given a plane drawing of a graph G, add edges until G is **nearly maximal planar**, that is, all regions except possibly the outside region are triangles. We use induction on the stronger assumption that G is 5-choosable when G has two adjacent vertices v_1 and v_p on the outside cycle $C = v_1 v_2 \cdots v_p v_1$ with distinct assigned colors and all other vertices on C with lists of length 3. The base case is a triangle with two fixed colors and a third vertex with at least one available color.

Case 1. C has a chord $v_i v_j$. In this case, split G into two graphs G_1 (containing v_1 and v_p) and G_2 that overlap on $v_i v_j$. By induction, G_1 is 5-choosable. Given a coloring of G_1, fix the colors of v_i and v_j. By induction again, G_2 is 5-choosable, so G is 5-choosable.

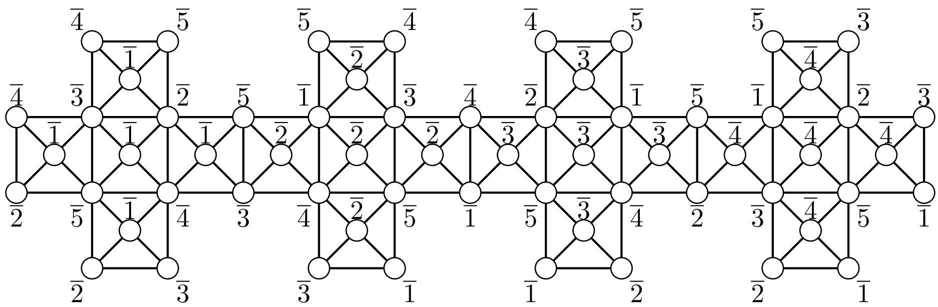
Case 2. C has no chord. Now v_2 has neighbors $v_1, u_1, \ldots, u_m, v_3$, which form a path $v_1 u_1 \cdots u_m v_3$. Now $G - v_2$ is bounded by a cycle. Let x and y be two colors in $L(v_2) - L(v_1)$. Delete x and y from the lists of each u_i, and any others arbitrarily so that $|L(u_i)| = 3$. By induction, $G - v_2$ is 5-choosable. Then x or y can be assigned to v_2, so G is 5-choosable. □



This proof is a great example of finding exactly the right induction hypothesis. Sometimes, proving a stronger result actually leads to a simpler proof. This theorem also yields another proof of the Five Color Theorem. This proof does not use degeneracy and does not require recoloring, unlike the proof based on Kempe chains.

Margit Voigt [**1993**] found an example of a planar graph with list chromatic number 5 and order 238. Maryam Mirzakhani [**1996**] found an example with only 63 vertices. (Mirzakhani (1977–2017) was the first woman to win the Fields Medal, the highest honor in mathematics, for her work in geometry.)

The **Mirzakhani graph** is formed from the graph shown below by adding a vertex v adjacent to every vertex on the outside region. The vertices are labeled with lists showing that it is not 4-choosable. ($\overline{n}$ means $\{1,2,3,4,5\} - \{n\}$, and v is labeled $\overline{5}$.)

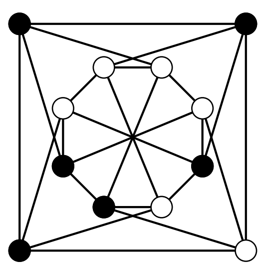


8.2. Vertex Arboricity

The second definition of chromatic number, Definition 8.2(2), is the smallest number of sets in a partition of the vertex set so that each set induces an empty graph. This suggests considering partitions that induce other types of graphs. For such a coloring to be defined for all graphs, the class of induced graphs should be hereditary. Forests are one such class.

Definition 8.8. The **vertex-arboricity** $a(G)$ of a graph G is the minimum number of sets in a partition of the vertex set so that each set induces a forest.

Example. The sets of black and white vertices in the Chvatal graph below each induce forests. Since it is not a forest, its vertex-arboricity is 2.



Determining the vertex arboricity is NP-hard, so finding bounds is desirable. The lower bound $a(G) \geq \left\lceil \frac{\omega(G)}{2} \right\rceil$ follows immediately from the fact that $a(K_n) = \left\lceil \frac{n}{2} \right\rceil$.

The upper bound $a(G) \leq \chi(G)$ holds since any empty graph is a forest. There is a bound using degeneracy that is usually superior.

Theorem 8.9 (Chartrand/Kronk/Wall [**1968**]). *For any graph G, $a(G) \leq 1 + \left\lfloor \frac{1}{2} D(G) \right\rfloor$.*

Proof. Let $k = D(G)$, and greedily color a construction sequence of G using $1 + \left\lfloor \frac{1}{2} k \right\rfloor$ vertex sets inducing forests (if nonempty). A vertex has at most k neighbors when colored, and it can be added to one of the vertex sets unless it is adjacent to two vertices in each set. But then it is adjacent to at least $2\left(1+\left\lfloor\frac{1}{2}k\right\rfloor\right)\geq k+1$ colored vertices, a contradiction. □

This bound is easy to compute. It is exact whenever $0\leq D(G)\leq 3$ since any 2-core contains a cycle. Since $a(K_n)=\left\lceil\frac{n}{2}\right\rceil$ and $D(K_n)=n-1$, this bound is sharp for arbitrarily large degeneracies.

Theorem 8.9 implies that $a(G)\leq 1+\frac{1}{2}\Delta(G)$. There is a version of Brooks' Theorem for vertex arboricity that characterizes the extremal graphs for this bound. A proof is outlined in the Exercises.

Theorem 8.10 (**Brooks' Theorem for vertex arboricity**—Kronk/Mitchem [**1975**]). *If G is connected, then $a(G)=1+\frac{1}{2}\Delta(G)$ if and only if G is a cycle or a complete graph of odd order.*

Theorem 8.9 can be used to bound the vertex arboricity of specific graph classes. Any planar graph G has $D(G)\leq 5$, so $a(G)\leq 1+\left\lfloor\frac{5}{2}\right\rfloor=3$. To show that this bound is sharp, we need the following result.

Proposition 8.11 (Stein [**1971**]). *Let G be a maximal plane graph with $n\geq 4$. Then $a(G)=2$ if and only if G^* is Hamiltonian.*

Proof. A plane drawing of G^* has a Hamiltonian cycle C exactly when no vertex is inside or outside C. This occurs exactly when the edges of the dual graph G that do not cross C induce no cycle, i.e., $a(G)=2$. □

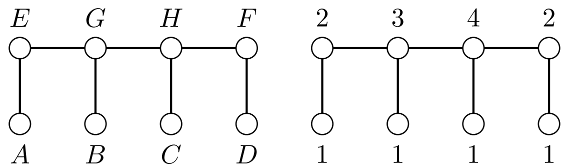
The dual of a maximal planar graph is cubic, planar, and 3-connected. The Tutte graph is an example of such a graph that is also non-Hamiltonian. Thus the dual of the Tutte graph has vertex arboricity 3 (Chartrand/Kronk [**1969**]).

8.3. Grundy Numbers

Definition 8.2(3) of the chromatic number is the minimum number of colors required by greedy coloring over all possible vertex orders of a graph. This works since some vertex order must use the minimum number of colors. Proving this requires starting with a minimum coloring, however. Finding an ideal vertex order is not easy in general. This raises the question of how bad a result greedy coloring could produce.

Definition 8.12. A **Grundy k-coloring** of a graph G is a proper k-coloring of vertices in G such that each vertex is colored by the smallest integer which has not appeared as a color of any of its neighbors. The **Grundy number** $\Gamma(G)$ is the largest integer k, for which there exists a Grundy k-coloring for G.

Greedy coloring need not produce a minimum coloring. A greedy coloring of the tree below left with vertices ordered alphabetically produces the coloring at right.



The bounds $\chi(G) \leq \Gamma(G) \leq 1+\Delta(G)$ are immediate since in any vertex order, each vertex has at most $\Delta(G)$ neighbors. In fact, there is a characterization of the Grundy number in terms of forbidden induced subgraphs.

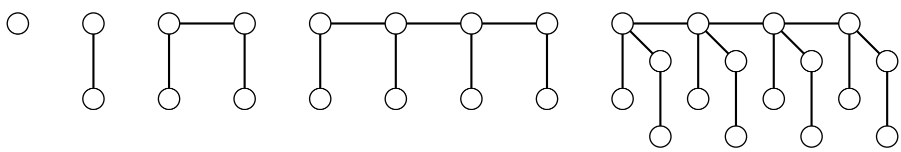
Definition 8.13. The class $\mathbb{G}_t$ of graphs called ***t*-atoms** are defined recursively as follows:

• $\mathbb{G}_1 = \{K_1\}$ and $\mathbb{G}_2 = \{K_2\}$.

• $G \in \mathbb{G}_t$ can be constructed from $H \in \mathbb{G}_{t-1}$ by adding a perfect matching between $\overline{K}_r$, $1 \leq r \leq t$, and $U \subseteq V(H)$ and joining each vertex in $V(H) - U$ to exactly one vertex in $\overline{K}_r$.

The **canonical partition** of G is the (unique) t-coloring where vertices in $\overline{K}_r$ receive color 1, and all other vertices receive color 1 more than in their canonical partition in H.

It follows immediately that $\mathbb{G}_3 = \{K_3, P_4\}$. There is a sequence of trees that are t-atoms; each is the corona of the previous.



Theorem 8.14 (Zaker [**2006**]). *For a given graph G, $\Gamma(G) \geq t$ if and only if G contains a t-atom so that the color classes of the canonical partition are independent in G.*

Proof. ($\Leftarrow$) It is easily seen by the construction of t-atoms that these atoms have a Grundy coloring with t colors. So the Grundy number of a graph which contains such an atom is at least t.

($\Rightarrow$) Consider a graph G with $\Gamma(G) \geq t$. We prove by induction on t that G contains a t-atom. This is obvious when $t = 1$. Let $C_1, \ldots, C_t$ be the color classes of a Grundy coloring of G with t colors, so that C_i consists of the vertices colored by i. Let $H = G - C_1$. Obviously, $\Gamma(H) \geq t - 1$. By induction we conclude that H includes a $t-1$-atom, say F. Now since C_1 is a maximal independent set in G, every vertex of F has a neighbor in C_1. Hence it is clear that we can obtain a t-atom inside G, by excluding, if necessary, some vertices in C_1 or edges between F and C_1. $\square$

The fact that K_3 and P_4 are the only 3-atoms immediately implies the following.

Corollary 8.15. *A connected graph G has $\Gamma(G) = 2$ if and only if $G = K_{r,s}$.*

Any t-atom has order at most 2^{t-1}. For fixed t, it is possible to search for all of them in polynomial time. Thus the problem of determining the Grundy number is said to be **fixed-parameter tractable**. However, for unbounded t it is NP-complete.

8.4. Distance and Sets

8.4.1. Distance k Coloring. Definition 8.2(4) of the chromatic number involving distance has a natural generalization.

Definition 8.16. The **distance k chromatic number** $\chi_k(G)$ is the minimum number of colors required to color the vertices of G so that vertices distance at most k apart receive distinct colors.

This parameter has a natural interpretation in terms of the original chromatic number.

Proposition 8.17. *For any graph G, $\chi_k(G) = \chi\left(G^k\right)$.*

Proof. In a distance k coloring, no pair of vertices with distance at most k receives the same color. In G^k, there is an edge between vertices with distance at most k in G. Thus the restrictions on coloring pairs of vertices are the same in both cases. □

Thus we see that this generalization of the chromatic number is actually a special case of the original problem. This does not necessarily make it uninteresting, since determining the chromatic number is still a difficult problem in general. However, it is not a truly different problem.

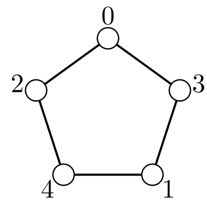
8.4.2. L(2,1) Coloring. When assigning channels for radio or television stations, stations that are closer together must be assigned channels that are further apart. This suggests another type of coloring.

Definition 8.18. For nonnegative integers h and k, an $L(h,k)$ **coloring** c of a graph G is an assignment of colors (nonnegative integers) to the vertices of G such that if u and w are adjacent vertices of G, then $|c(u) - c(w)| \geq h$, while if $d(u,w) = 2$, then $|c(u) - c(w)| \geq k$.

Hence an $L(1,0)$ coloring is just a proper vertex coloring, and an $L(1,1)$ coloring is a distance 2 coloring. Aside from these cases, most effort has gone into the study of $L(2,1)$ colorings. Determining the minimum number of colors required in an $L(h,k)$ coloring is not an interesting problem, since for h and k both positive, this is just $\chi\left(G^2\right)$. Since $L(h,k)$ coloring is concerned with the distance between colors, the minimum length of an interval containing the colors is of interest.

Definition 8.19. Given an $L(h,k)$ coloring c of a graph G, the **c-span** of G is $\lambda_{h,k}(c) = \max_{u,w \in G} |c(u) - c(w)|$. The **L-span** of G is $\lambda_{h,k}(G) = \min\{\lambda_{h,k}(c)\}$. Denote $\lambda_{2,1}(G)$ by $\lambda(G)$.

Example. The cycle C_5 has diameter 2, so all colors must be distinct. This combined with the coloring below shows that $\lambda(C_5) = 4$.



Griggs and Yeh [**1992**] showed that if G has maximum degree Δ, $\lambda(G) \leq \Delta^2 + 2\Delta$. They further conjectured that $\lambda(G) \leq \Delta^2$ for all graphs, and proved this for graphs of diameter 2. There are examples of graphs achieving this conjectured bound. Goncalves [**2005**] improved the bound to $\lambda(G) \leq \Delta^2 + \Delta - 2$. Degeneracy provides an upper bound that is usually better.

Theorem 8.20 (Bickle [**2010**]). *For any graph G, $\lambda(G) \leq D\left(G^2\right) + 2D(G)$.*

Proof. Color G using a construction sequence. Assign the first vertex color 0. Each vertex v added has at most $D(G)$ neighbors and at most $D\left(G^2\right) - D(G)$ **second-neighbors** (vertices distance 2 away). Thus when assigning a color to v, we must avoid three colors for its neighbors and one color for each vertex distance 2 away. Thus we must avoid at most $D\left(G^2\right) + 2D(G)$ colors for v, so at least one of the $D\left(G^2\right) + 2D(G) + 1$ colors between 0 and $D\left(G^2\right) + 2D(G)$ is available, so $\lambda(G) \leq D\left(G^2\right) + 2D(G)$. □

The previous theorem is applicable to trees.

Corollary 8.21. *Let T be a tree with maximum degree Δ. Then $\Delta + 1 \leq \lambda(T) \leq \Delta + 2$.*

Proof. We have $\lambda(K_{1,\Delta}) = \Delta + 1$ since every leaf must have a distinct label and the center vertex must be two away from all of them. Now $D\left(T^2\right) = \Delta$, so T can be colored with a span of at most $\Delta + 2$ colors. □

This also implies the general lower bound of $\lambda(G) \geq \Delta + 1$.

8.4.3. Set Coloring. Rather than assign only one color to a vertex, as in Definition 8.2(5), it may be desirable to assign multiple colors.

Definition 8.22. A **k-set coloring** assigns to each vertex of a graph G a set of k colors so that adjacent vertices receive disjoint sets of colors. Let $\chi^k(G)$ be the minimum number of colors in an k-set coloring of G.

There is a graph whose vertices represent all possible k-element sets which is relevant to this type of coloring.

Definition 8.23. The **Kneser graph** $KG_{r,k}$ has vertices representing the k-element subsets of $[r]$ and edges between disjoint subsets.

The Petersen graph is $KG_{5,2}$. The Kneser graph has a $r - 2k + 2$-coloring. For $1 \leq i \leq r - 2k + 1$, let V_i be the vertices with smallest color i, and let V_{i+1} be the remaining vertices. Then these sets partition the vertices into color classes. Kneser [**1955**] conjectured that $\chi(KG_{r,k}) = r - 2k + 2$. This was proved by Lovasz

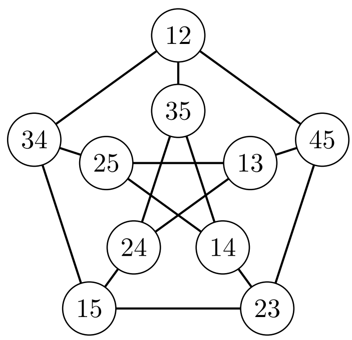
[**1978**] using algebraic topology. Improved topological proofs were produced by Barany [**1978**] and Greene [**2002**]. A purely combinatorial proof was discovered by Matousek [**2004**].

Theorem 8.24 (Lovasz [**1978**]). *We have* $\chi(KG_{r,k}) = r - 2k + 2$.

The Kneser graph $KG_{r,k}$ clearly has a k-set coloring using r colors. These graphs provide a key bound for set coloring.

Theorem 8.25. (Bollobas/Thomason [**1979**]) *The minimum of* $\chi^k(G)$ *over all graphs with chromatic number* $k + 2r - 2$ *is* r.

8.4.4. Pair Colorings. Recall that the Petersen graph is defined by a vertex labeling where every vertex has two distinct numbers and where adjacent vertices have no common numbers. This suggests considering vertex coloring with both sets of colors at each vertex and restrictions on distances greater than 1.



Definition 8.26. A **pair k-coloring** of a graph G assigns two distinct colors (a **label**) to each vertex so that adjacent vertices have no common colors. A graph is **pair k-colorable** if it has a pair k-coloring. The **pair chromatic number** of a graph G, $\operatorname{pc}(G)$, is the smallest k for which it has a pair k-coloring.

Proposition 8.27 (Bickle [**2013**]). *The following are results on pair coloring.*

(a) *A graph G is pair k-colorable if and only if it is contained in $KG_{r,2}$.*

(b) *If* $n > \binom{k}{2}$, $\operatorname{pc}(G) > k$. *Thus* $\operatorname{pc}(\overline{K}_n) = \left\lceil \frac{1+\sqrt{1+8n}}{2} \right\rceil$.

(c) *We have* $\operatorname{pc}(K_{r,s}) = \left\lceil \frac{1+\sqrt{1+8r}}{2} \right\rceil + \left\lceil \frac{1+\sqrt{1+8s}}{2} \right\rceil$, *so* $\operatorname{pc}(K_{1,s}) = \left\lceil \frac{5+\sqrt{1+8s}}{2} \right\rceil$.

Part (a) implies that there are finitely many critical graphs with $\operatorname{pc}(G) = k$. For $k = 5$, these are $\overline{K}_7$ and P_3. For $k = 6$, there are eleven such graphs, the minimal forbidden subgraphs of the Petersen graph.

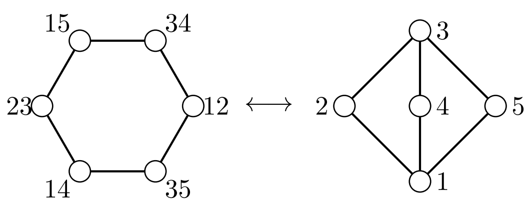
Theorem 8.28 (Bickle [**2013**]). *Let G have degeneracy $k = D(G)$. Then*

$$\left\lceil \frac{1+\sqrt{1+8n}}{2} \right\rceil \le \operatorname{pc}(G) \le 2k + \left\lceil \frac{1+\sqrt{1+8(n-k)}}{2} \right\rceil.$$

Proof. The lower bound follows from the previous proposition. Color G with a construction sequence. Each vertex v has $j \le k$ neighbors which exclude at most

$2j$ colors. There are at most $n-j-1$ labels that have already been used on nonneighbors of v. To guarantee there is an available label for v, we need r extra colors, where $\binom{r}{2} \geq n-j$. Solving, we find $r \geq \frac{1+\sqrt{1+8(n-j)}}{2}$. Thus we need at most $2j + \left\lceil \frac{1+\sqrt{1+8(n-j)}}{2} \right\rceil$ colors to label v, which is maximized when $j=k$. □

One reason why pair colorings are significant is their connection to decompositions. If the vertices of K_n are labeled 1 to n, each edge can be labeled with the pair of labels of its vertices. The $\binom{n}{2}$ possible labels occur exactly once. Thus a pair k-coloring of a graph corresponds to a (usually different) edge-induced subgraph of K_k. For example, the subgraph $K_{2,3}$ corresponds to C_6 (see below). Thus a pair k-coloring of a disconnected graph $G_1 \cup \cdots \cup G_r$ exists if and only if the corresponding factors $H_1, \ldots, H_r$ are contained in some decomposition of K_k.



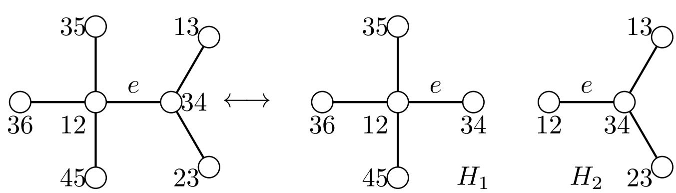
8.4.5. 2-tone Colorings. One variation on pair coloring allows labels to be repeated on vertices that are more than distance 2 apart.

Definition 8.29. A **proper 2-tone coloring** of a graph G assigns two distinct colors to each vertex so that adjacent vertices have no common colors, and vertices at distance 2 have at most one common color. The **2-tone chromatic number** of G, $\tau_2(G)$, is the smallest k for which G has a proper 2-tone k-coloring.

We have $\tau_2(G) \leq \mathrm{pc}(G)$, and if $\mathrm{diam}(G) \leq 2$, then this is an equality. Thus $\tau_2(G) = \mathrm{pc}(G)$ for complete multipartite graphs and, in particular, stars.

Lemma 8.30 (Bickle/Phillips [**2018**]). *Let G be a graph with a bridge $e = uv$. Let F_1 and F_2 be the components of $G-e$ containing u and v, respectively, and let $H_1 = G[V(F_1) \cup v]$ and $H_2 = G[V(F_2) \cup u]$. Then $\tau_2(G) = \max\{\tau_2(H_1), \tau_2(H_2)\}$.*

Proof. Color each subgraph separately. Permute the colors on one of the subgraphs to agree on u and v. Color G consistently with the colorings of these two subgraphs. Now the result is a 2-tone coloring since it works for all pairs of vertices at distances 1 and 2, and larger distances have no impact. Now $\tau_2(G) \geq \max\{\tau_2(H_1), \tau_2(H_2)\}$, and this is a coloring with $\max\{\tau_2(H_1), \tau_2(H_2)\}$ colors. □



Theorem 8.31 (Fonger et al. [**2009**]). *Let T be a nontrivial tree with maximum degree Δ. Then*

$$\tau_2(T) = \left\lceil \frac{5+\sqrt{1+8\Delta}}{2} \right\rceil .$$

Proof (Bickle/Phillips [**2018**]). The formula holds for K_2. By the previous lemma,

$$\tau_2(T) = \max_{v\in T} \tau_2\left(K_{1,d(v)}\right) = \max_{v\in T} \left\lceil \frac{5+\sqrt{1+8d(v)}}{2} \right\rceil = \left\lceil \frac{5+\sqrt{1+8\Delta}}{2} \right\rceil ,$$

where the maximum is taken over all internal vertices of T. □

Theorem 8.32 (Fonger et al. [**2009**], Bickle/Phillips [**2018**]). *We have*

$$\tau_2(C_n) = \begin{cases} 6 & n = 3,4,7 \\ 5 & \textit{else}. \end{cases}$$

Proof. Certainly $\tau_2(C_n) \geq \tau_2(P_3) = 5$. Now $C_3 = K_3$, and $C_4 = K_{2,2}$, so $\tau_2(C_3) = \tau_2(C_4) = 6$. Now C_7 is not a subgraph of the Petersen graph and can be labeled as below. The cycles of length 5, 6, 8, and 9 are subgraphs of the Petersen graph, with labelings below represented as broken cycles.

$$-12-34-51-23-45-$$

$$-12-34-15-32-14-35-$$

$$-12-34-56-13-24-35-46-$$

$$-12-34-15-23-14-25-13-45-$$

$$-12-34-15-32-14-25-13-24-35-$$

Finally, for $n \geq 10$, the cycle can be constructed by breaking and attaching together cycles of length 5, 6, 8, and 9, which can be done because these labelings above agree on the first three vertices. □

Degeneracy also provides a good upper bound for 2-tone coloring.

Theorem 8.33 (Bickle [**2013**]). *Let G be a graph with degeneracy k and maximum degree $\Delta = \Delta(G)$. Then $\tau_2(G) \leq 2k + \left\lceil \frac{1+\sqrt{9+8(2\Delta k-\Delta-k^2)}}{2} \right\rceil$.*

Proof. Color G with a construction sequence. When colored, a vertex v has $j \leq k$ neighbors already colored, which excludes $2j$ colors. Each of the j neighbors has at most $\Delta - 1$ second-neighbors. Then v has at most $\Delta - j$ uncolored neighbors, each of which has at most $k-1$ colored second-neighbors. Thus there are at most $j(\Delta-1) + (\Delta-j)(k-1)$ second-neighbors already colored, which is maximized when $j = k$. Thus we need r extra colors, where $\binom{r}{2} \geq 2\Delta k - \Delta - k^2 + 1 \geq j(\Delta-1) + (\Delta-j)(k-1) + 1$. Solving, we find $r \geq \frac{1+\sqrt{9+8(2\Delta k-\Delta-k^2)}}{2}$. □

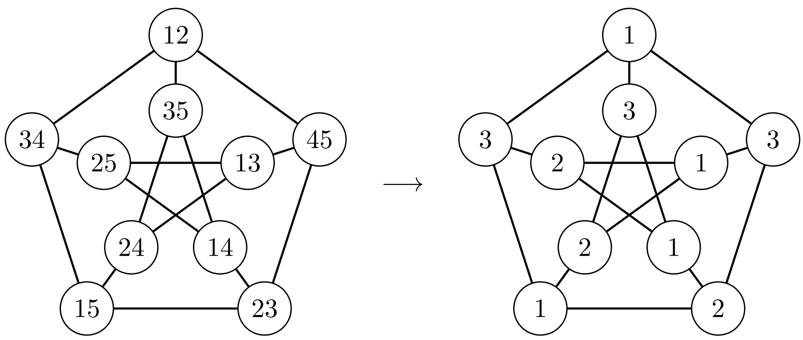
For cubic graphs, this implies an upper bound of 9 for the 2-tone chromatic number. There is a better bound, which is attained by K_4.

Theorem 8.34 (Cranston et al. [**2013**]). *If G is cubic, $\tau_2(G) \leq 8$.*

There are two trivial lower bounds for 2-tone coloring analogous to the basic lower bounds for chromatic number. There is also a lower bound relating it to the chromatic number.

Theorem 8.35 (Bickle/Phillips [**2018**]). *For any nonempty graph G, $\tau_2(G) \geq \chi(G) + 2$, and for each $k \geq 4$, this is an equality for $KG_{r,2}$.*

Proof. Let G be a graph with $\tau_2(G) = k \geq 4$, and let c be a 2-tone k-coloring of G. Form proper vertex coloring c' by deleting the largest color from the label of each vertex of G and by recoloring any vertex v with label $\{k-1, k\}$ in c using $k-2$. Any neighbor u of v has largest color at most $k-2$ in c and color less than $k-2$ in c'. Thus u and v have distinct colors in c'. □



Related Terms: list chromatic index, List Coloring Conjecture, total coloring, Total Coloring Conjecture, vertex linear arboricity, point partition number, complete coloring, achromatic number, parsimonious ϕ-coloring, ochromatic number, Hoffman-Singleton graph, Erdos-Ko-Rado Theorem, covering graph, precoloring extension, harmonious coloring, harmonic coloring, strong edge coloring, irregular coloring, Channel Assignment Problem, T-coloring, radio coloring, Hamiltonian coloring.

Exercises

Section 8.1:

(1) Determine the list chromatic number of the graphs in the following classes.
 (a) K_n
 (b) C_{2k+1}
 (c) $G_{r,s}$
 (d) maximal k-degenerate graphs

(2) Show that any chordal graph has $\chi(G) = \chi_l(G)$.

(3) Show that if G has a 3-core, then $\chi_l(G) = \chi_l(C_3(G))$.

(4) Let $k = \omega(G)$. Show that $\chi_l(G) = \chi_l(C_{k-1}(G))$.

(5) Let $s \geq r^r$. Show that $K_{r,s}$ is not r-choosable.

(6) + (Erdos/Rubin/Taylor [**1979**]) Show that $K_{7,7}$ is not 3-choosable.

(7) (Erdos/Rubin/Taylor [**1979**]) Show that the following graphs are 2-choosable.
 (a) C_{2k}

(b) $\theta_{2,2,2k}$

(8) (Erdos/Rubin/Taylor [**1979**]) Show that the following graphs are not 2-choosable.
(a) $\theta_{1,3,3}$
(b) $K_{2,4}$
(c) two 4-cycles with an edge between vertices of distinct cycles
(d) two 4-cycles with two vertices of distinct cycles identified

(9) + (Erdos/Rubin/Taylor [**1979**]) Proof of Theorem 8.5.
(a) Let G be a graph with a vertex v with $d(v) = 2$. Form a graph G' by deleting v and identifying its neighbors (G' must be a graph for the operation to be defined). Show that if G is 2-choosable, then G' is 2-choosable.
(b) Use part (a) and the previous two problems to prove Theorem 8.5.

(10) Show that any planar graph G with list chromatic number 5 and smallest order has $\delta(G) \geq 4$.

(11) Let M be the Mirzakhani graph with vertex v having degree 42.
(a) Determine $\chi(M)$.
(b) Determine whether M and $M - v$ are Hamiltonian.
(c) Determine whether $M - v$ has a perfect matching.

(12) + (Mirzakhani [**1996**]) Show that the Mirzakhani graph is not 4-choosable. (*Hint*: Start with a 17-vertex subgraph with lists of length 3 or 4.)

Section 8.2:

(1) Determine the vertex-arboricity of the graphs in the following classes.
(a) W_n
(b) $G_{r,s}$
(c) $K_{r,s}$
(d) irregular graphs I_n

(2) Determine the vertex-arboricity of the graphs in the following classes.
(a) C_n^2
(b) outerplanar graphs
(c) triangular grids
(d) $C_r \square C_s$

(3) Show that $a(G + H) \leq a(G) + a(H)$, and determine when this is an equality.

(4) Characterize graphs that are maximal with respect to vertex arboricity.

(5) Write an algorithm to partition the vertices of a graph into at most $1 + \left\lfloor \frac{1}{2} D(G) \right\rfloor$ sets that induce forests.

(6) A graph is **critical with respect to vertex-arboricity** if $a(G - v) < a(G)$ for all vertices of G. Show that if G has $a(G) = k \geq 2$ and is critical with respect to vertex-arboricity, then $\delta(G) \geq 2k - 2$.

(7) Let $k = D(G)$. Show that if $0 \leq k \leq 4$, then $a(G) = a(C_k(G))$.

(8) Show that $a(G) \leq \max_k \min \left\{ \left\lfloor \frac{k+2}{2} \right\rfloor, a(C_k(G)) \right\}$.

(9) Let $f(G)$ be the maximum order of an induced forest of G. Find a lower bound for $a(G)$ involving $f(G)$.

(10) + (Bickle [**2019**]) A short proof of Brooks' Theorem for Vertex Arboricity
   (a) Show that if G is an r-regular 2-connected noncomplete graph, $r \geq 4$, then G has a vertex v with three neighbors x, y, and z, not all adjacent, such that $G - x - y - z$ is connected. (*Hint*: Consider a minimum vertex cut S of G so that $G - S$ has a component H of smallest order.)
   (b) Prove Brooks' Theorem for Vertex Arboricity using part (a). (*Hint*: Generalize the proof of Brooks' Theorem.)

(11) (Raspaud/Wang [**2008**]) The minimum order of a 3-connected cubic planar non-Hamiltonian graph is 38. Find the minimum order of a maximal planar graph G with $a(G) = 3$.

(12) Show that any triangle-free planar graph has $a(G) \leq 2$.

(13) (Lick/White [**1970**]) The **point partition number** $\rho_k(G)$ is the minimum number of sets into which the vertices of G can be partitioned so that each set induces a k-degenerate graph. Thus $\chi(G) = \rho_0(G)$ and $a(G) = \rho_1(G)$. Show that $\rho_k(G) \leq 1 + \left\lfloor \frac{1}{k+1} D(G) \right\rfloor$.

(14) (Bickle [**2012**]) Let G be a graph, and let $k_1, \ldots, k_t$ be nonnegative integers with $\sum k_i \geq D(G) - t + 1$. Show that the vertices of G can be partitioned into sets $V_1, \ldots, V_t$ so that $D(G[V_i]) \leq k_i$.

Section 8.3:

(1) Find the Grundy number of the graphs in the following classes.
   (a) C_n
   (b) W_n
   (c) $G_{r,s}$

(2) Show that the Grundy number and the degeneracy bound $1 + D(G)$ are incomparable; that is, there are graphs for which each is larger than the other.

(3) Find all 4-atoms.

(4) Show that $\Gamma(G + K_1) = \Gamma(G) + 1$.

(5) A **complete coloring** is a vertex coloring in which every pair of colors appears on at least one pair of adjacent vertices. The **achromatic number** $\psi(G)$ of a graph G is the maximum number of colors possible in any complete coloring of G. Show that $\Gamma(G) \leq \psi(G)$.

(6) + (Erdos/Hare et al. [**1987**]) A **parsimonious ϕ-coloring** colors a graph G starting with coloring the first vertex 1. Each succeeding vertex must use an existing color, if possible, and the color assigned must result in the smallest number of colors needed to color G. Else the next smallest unused color must be assigned. The **ordered chromatic number** (**ochromatic number**) $\chi^o(G)$ of G is the largest number of colors required in any parsimonious ϕ-coloring of G. Show that $\Gamma(G) = \chi^o(G)$.

Section 8.4:

(1) Determine the distance k chromatic number for the graphs in the following classes.
  (a) P_n
  (b) C_n

(2) Let G be a graph with diameter k. Determine $\chi_k(G)$.

(3) Determine the L-span $\lambda(G)$ for the graphs in the following classes.
  (a) K_n
  (b) C_n
  (c) W_n

(4) Determine the L-span $\lambda(G)$ for the trees in the following classes.
  (a) paths
  (b) double stars

(5) Show that $D\left(G^2\right) + 2D(G) \leq \Delta^2 + 2\Delta$, and if G is connected with $\Delta \geq 2$, then equality holds exactly for regular graphs with girth at least 5.

(6) Show that $\lambda(G) \leq \max_{H \subseteq G}\left(\delta\left(H^2\right) + 2\delta(H)\right)$. Show that this bound is superior to $\lambda(G) \leq D\left(G^2\right) + 2D(G)$ for $K_1 + (K_2 \cup K_1)$.

(7) Show that if $D(G) \leq \Delta - 2$, then $\lambda(G) \leq \Delta^2$.

(8) Find a minimal tree with maximum degree Δ and $\lambda(G) = \Delta + 2$.

(9) (Griggs/Yeh [**1992**]) Show that for any graph G, $\lambda(G) \leq n + \chi(G) - 2$. Show that this is an equality for complete $\chi(G)$-partite graphs.

(10) (Griggs/Yeh [**1992**]) Proof that $\lambda(G) \leq \Delta^2$ for diameter 2 graphs.
  (a) Show that the result holds for $\Delta = 2$.
  (b) Show that the result holds for $\Delta \geq \frac{n-1}{2}$. (*Hint*: Use the previous problem.)
  (c) Show that the result holds for $\Delta \leq \frac{n-2}{2}$. (*Hint*: Show that the complement contains a Hamiltonian path.)

(11) There are graphs with maximum degree Δ, diameter 2, and order $\Delta^2 + 1$ when $\Delta \in \{2, 3, 7\}$ and possibly when $\Delta = 57$. For $\Delta = 3$ the only such graph is the Petersen graph, and for $\Delta = 7$ the only such graph is the **Hoffman-Singleton graph**. Determine the L-span of these graphs.

(12) Show that the Heawood graph, which is the unique cubic graph with order 14 and girth 6, has $\lambda(G) = \Delta^2 - \Delta$. (*Note*: The same bound holds for any incidence graph of a projective plane.)

(13) The **Erdos-Ko-Rado Theorem** (Erdos/Ko/Rado [**1961**]) states that in a set of cardinality r, a family of distinct subsets of cardinality k, no two of which are disjoint, can have at most $\binom{r-1}{k-1}$ members.
  (a) + Prove this theorem using double counting.
  (b) What does this theorem imply about $\alpha(KG_{r,k})$?

(14) Without using Theorem 8.24, show that $\chi(KG_{r,2}) = r - 2$.

(15) Determine $\chi^2(G)$ of the graphs in the following classes.
  (a) K_n
  (b) bipartite graphs
  (c) C_n

(16) Show that $\chi^2(G) \le 2\chi(G)$, and this need not be an equality.

(17) Let G be a connected graph with maximum degree $\Delta \ge 1$. Show that

$$\mathrm{pc}(G) \le 2\Delta - 1 + \left\lceil \frac{1+\sqrt{1+8(n-\Delta+1)}}{2} \right\rceil.$$

(18) Let F be a forest. Show that $\mathrm{pc}(F) \le 2 + \left\lceil \frac{1+\sqrt{1+8(n-1)}}{2} \right\rceil$, and this bound is sharp.

(19) Show that for $n \ge 3$, $\mathrm{pc}(P_n) = \begin{cases} 5 & 3 \le n \le 10 \\ \left\lceil \frac{1+\sqrt{1+8n}}{2} \right\rceil & n \ge 11. \end{cases}$

(20) + Show that $\mathrm{pc}(C_n) = \begin{cases} 5 & n = 5, 6, 8, 9 \\ 6 & n = 3, 4, 7, 10-15 \\ \left\lceil \frac{1+\sqrt{1+8n}}{2} \right\rceil & n \ge 11. \end{cases}$

(21) (a) + Let $a_1, \ldots, a_k$ be integers with $1 \le a_1 \le \cdots \le a_k \le \left\lfloor \frac{n}{2} \right\rfloor$ and $\sum a_i \le \binom{n}{2}$. Show that K_n has a decomposition using matchings of sizes a_i.
(b) Let $a_1, \ldots, a_k$ be integers with $1 \le a_1 \le \cdots \le a_k$ and $n = \sum a_i$. Show that $\mathrm{pc}\left(\bigcup_i K_{a_i}\right) = \max\left\{2a_k, \left\lceil \frac{1+\sqrt{1+8n}}{2} \right\rceil\right\}$.

(22) Show that any graph G has finitely many critical forbidden subgraphs.

(23) Show that $\tau_2(G) \ge 2\omega(G)$ and $\tau_2(G) \ge \frac{2n}{\alpha(G)}$.

(24) Let G be a connected graph with a 2-core and D be the maximum degree in G of the vertices of the 1-shell of G. Show that

$$\tau_2(G) = \max\left\{\left\lceil \frac{5+\sqrt{1+8D}}{2} \right\rceil, \tau_2(C_2(G))\right\}.$$

(25) Show that $\tau_2(G+H) \ge \tau_2(G) + \tau_2(H)$, and this is an equality if G and H have diameter at most 2.

(26) Show that if a graph G has diameter at most 4, then G is 2-tone 5-colorable if and only if G is a subgraph of the Petersen graph.

(27) Let G be a nonempty graph with $u, v \in V(G)$ not adjacent and set $e = uv$. Show that $\tau_2(G+e) - \tau_2(G) \le 1$.

(28) Let G be a graph so that $\overline{G}$ is triangle-free, and let m be the largest size of a subgraph H of $\overline{G}$ with $\Delta(H) \le 2$. Show that $\tau_2(G) = 2n - m$.

(29) (Cranston et al. [**2013**], Bickle [**2013**]) Let G be a graph with maximum degree $\Delta \ge 2$. Show that $\tau_2(G) \le 2\Delta - 1 + \left\lceil \frac{1+\sqrt{1+8\Delta(\Delta-1)}}{2} \right\rceil$.

(30) (Cranston et al. [**2013**], Bickle [**2013**]) Let a nonempty bipartite graph G have maximum degree $\Delta = \Delta(G)$. Show that $\tau_2(G) \le 2\left\lceil \frac{1+\sqrt{1+8\Delta(\Delta-1)}}{2} \right\rceil + 1$.

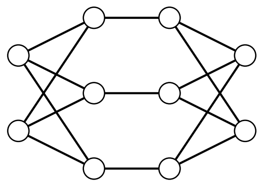
(31) A **covering graph** (or **graph cover**) is a graph G for which there is an onto homomorphism f from G to a graph H with the property that for each vertex v of G the neighborhood of v maps bijectively onto the neighborhood of $f(v)$.

Show that for $r \geq 5$, an $\binom{r-2}{2}$-regular graph G has $\tau_2(G) = r$ if and only if G is a $KG_{r,2}$-cover.

(32) Show that G is a $KG_{r,2}$-cover, $r \geq 5$, if and only if it can be obtained by starting with k disjoint copies of a $KG_{r,2}$ and performing some number of 2-switches on pairs of edges that join vertices with the same labels.

(33) + (Cranston et al. [**2013**]) Show that the Heawood graph is not 2-tone 6-colorable.

(34) Let G be the cubic graph with a matching between two $K_{2,3}$'s. Show that G is not 2-tone 6-colorable.



(35) Show that for $n \geq k+2$, $\tau_2\left(P_n^k\right) = 2k+3$.

(36) Let T be a tree with maximum degree $\Delta \geq 3$. Show that $\tau_2\left(T^2\right) = 2(\Delta+1)$.

(37) Let G be a planar graph with maximum degree $\Delta = \Delta(G) \geq 3$. Show that $\tau_2(G) \leq 10 + \left\lceil \frac{1+\sqrt{72\Delta-191}}{2} \right\rceil$.

(38) Let G be a nonempty outerplanar graph with maximum degree $\Delta = \Delta(G)$. Show that $\tau_2(G) \leq 4 + \left\lceil \frac{1+\sqrt{24\Delta-23}}{2} \right\rceil$.

(39) (Fonger et al. [**2009**]) Show that $\tau_2(C_n \Box K_2) = 6$.

(40) Let $m \leq n$. Show that $\tau_2(K_m \Box K_n) = \begin{cases} 6 & m = n = 2 \\ 2n & \text{else.} \end{cases}$

(*Hint*: Consider Latin squares.)

(41) Determine $\tau_2(P_m \Box P_n)$ and $\tau_2(C_{3i} \Box C_{3j})$.

(42) Let $i, j \geq 3$. Show that $6 \leq \tau_2(C_i \Box C_j) \leq 7$.

(43) + Show that $\tau_2(C_3 \Box C_i) = \begin{cases} 6 & i = 3, 6,\ i \geq 8 \\ 7 & i = 4, 5, 7. \end{cases}$

(44) (a) If a Cartesian product of cycles has a 2-tone 6-coloring, show that no vertex has both its two neighbors in its row sharing a common color and its two neighbors in its column sharing a common color.

(b) + Let $i \geq 3$. Show that $\tau_2(C_4 \Box C_i) = 7$.

Chapter 9

Decompositions

We have already encountered many results on decompositions.

- A graph and its complement decompose a complete graph.
- The Cartesian product $G \square H$ decomposes into copies of G and H.
- The blocks of a graph decompose it.
- G is a line graph of some graph H if and only if G decomposes into cliques so that each vertex of G appears in at most two cliques.
- A multigraph decomposes into cycles if and only if its degrees are even.
- The minimum number of open trials that decompose a connected non-Eulerian multigraph with exactly $2k$ odd vertices is k.
- The vertex covering number is the minimum number of stars that decompose a graph.
- Any regular graph with positive even degree decomposes into 2-factors.
- The edge chromatic number is the minimum number of matchings that decompose a graph.

Any decomposition can be thought of as a (nonproper) edge coloring in which the color classes induce disjoint subgraphs.

Definition 9.1. A **decomposition** of G is a set of nonempty subgraphs, called **factors**, whose edge sets partition $E(G)$. The subgraphs are said to **decompose** G. A **k-decomposition** of a graph G is a decomposition of G into k subgraphs. If a graph G decomposes into k_i copies of G_i, $1 \leq i \leq r$, we write $G \to \{k_1 [G_1], \ldots, k_r [G_r]\}$. If $G \to \{k [H]\}$, we say H **decomposes** G.

The general question addressed in this chapter is given a graph G and class $\mathbb{G}$, when does G decompose into factors in $\mathbb{G}$? Answering this question requires both finding constructions to show that decompositions exist and arguments to prove when decompositions do not exist.

9.1. Decomposing Complete Graphs

Decompositions of complete graphs are especially interesting, since for a given set of vertices, each possible edge will occur in exactly one factor. There are two basic necessary conditions that a decomposition of K_n must satisfy.

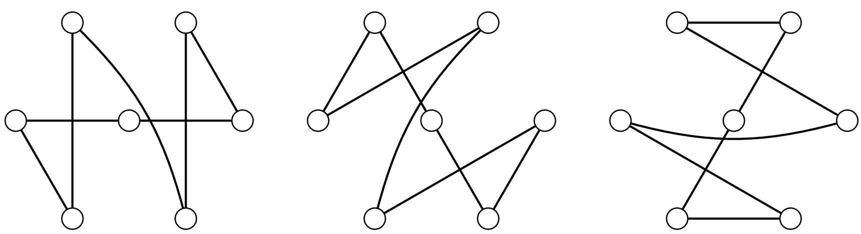
(1) The sizes of the factors must sum to the size of K_n. If H decomposes K_n, then $m(H)$ divides $\frac{n(n-1)}{2}$.

(2) The degrees of the factors must partition into subsets that sum to $n-1$. If an r-regular graph H decomposes K_n, then r divides $n-1$.

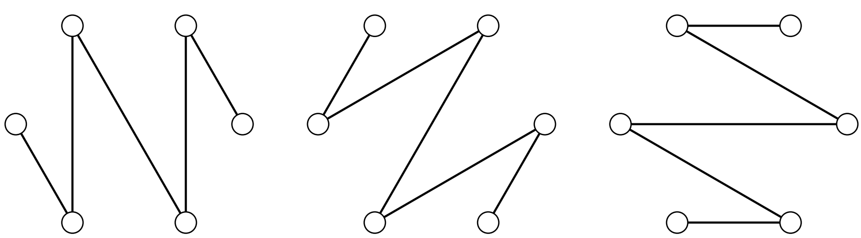
9.1.1. Cyclic Decompositions. Consider decomposing K_n into Hamiltonian cycles. The first necessary condition requires that $n|\frac{n(n-1)}{2}$, so $2|(n-1)$, and n is odd. The second necessary condition says that $2|(n-1)$, which imposes no additonal restriction. We begin with a common technique for constructing decompositions.

Definition 9.2. A decomposition of K_n into copies of G is a **cyclic decomposition** if there is a permutation of the vertices that induces a cyclic permutation of the factors.

Given a cyclic decomposition, the vertices can be arranged so that one factor can be rotated onto each of the other factors. One example of this is given in Section 7.3, showing that K_{2k} has a cyclic decomposition into perfect matchings. Below, a cyclic decomposition of K_{2k+1} into Hamiltonian cycles is illustrated. Note that the vertex permutation need not be cyclic; the center vertex is a fixed point.

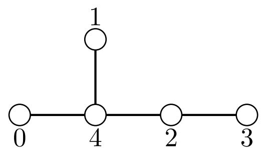


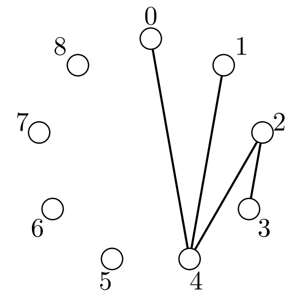
Deleting the center vertex in the decomposition above produces a decomposition of K_{2k} into Hamiltonian paths.



Consider numbering the vertices of K_{2k+1} from 0 to $2k$. The edges can be grouped into classes based on the difference between their ends; each class contains $2k+1$ edges. If a graph can use exactly one edge from each class, it can cyclically decompose K_{2k+1}.

Definition 9.3. A labeling $f(v) : V(G) \to \{0, \ldots, m\}$ of the vertices of a graph G is a **graceful labeling** if each value in $\{1, \ldots, m\}$ occurs exactly once as a difference $|f(u) - f(v)|$ for $uv \in E(G)$. A graph is **graceful** if it has a graceful labeling.





Proposition 9.4 (Rosa [**1967**]). *If a graph G with size m is graceful, then there is a cyclic decomposition of K_{2m+1} into copies of G.*

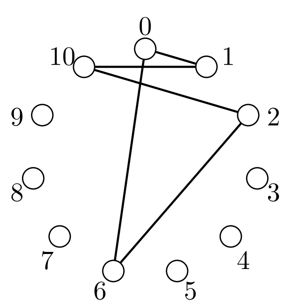
Proof. Number the vertices of K_{2m+1} from 0 to $2m$. Use the graceful labeling of G to define a factor on the vertices of K_{2m+1}. Define the other factors by adding k, $0 \le k \le 2m$, to each vertex label (mod $2m+1$). Since each difference in vertex labels is used exactly once in G and each occurs $2m+1$ times in K_{2m+1}, each edge of K_{2m+1} occurs exactly once in a factor. This produces a cyclic decomposition. $\square$

Conjecture 9.5 (**Graceful Tree Conjecture**—Rosa [**1967**]). *Every tree is graceful.*

(**Ringel's Conjecture**—Ringel [**1964**]) *Any tree T with size m decomposes K_{2m+1}.*

The Graceful Tree Conjecture implies Ringel's Conjecture, since any graceful graph with size m cyclically decomposes K_{2m+1}. The Graceful Tree Conjecture has been verified by computer search for all trees with $n \le 35$ (Fang [**2010**]) and for some special classes of trees. A simple induction argument does not work, since if v is a leaf of T, there is no obvious connection between graceful labelings of T and $T - v$.

Some other graph classes have graceful labelings. Even when no graceful labeling exists (as for C_5), a graph may still cyclically decompose some complete graph.

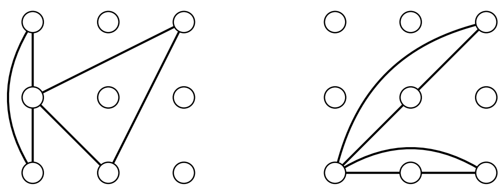


While cyclic decompositions are a useful tool, there are also noncyclic decompositions. In the Exercises, you are asked to show that $K_6 \to \{3\,[P_6]\}$ in two ways, one cyclic and one noncyclic. Also, $K_{10} \to \{9\,[K_4 - e]\}$, and the decomposition must be noncyclic (Alavi et al. [**1988**]).

9.1.2. Block Designs. When does a complete graph decompose into smaller complete graphs? Clearly, K_2 decomposes any graph. For K_3 to decompose K_n, the necessary conditions are $3|\frac{n(n-1)}{2}$ and $2|\,(n-1)$. This implies that $n \equiv 1$ or 3 mod 6.

Definition 9.6. A **Steiner triple system** of order n is a K_3 decomposition of K_n.

There is a Steiner triple system of order 9. Take the triples to be all rows, columns, and forward and backward diagonals of a 3×3 grid of vertices.

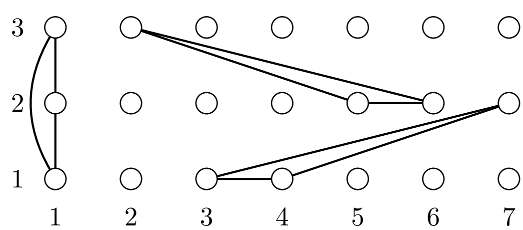


Theorem 9.7 (Kirkman [**1847**]). *A Steiner triple system exists when $n \equiv 1$ or 3 mod 6.*

Next we sketch the proof of this theorem.

(Bose [**1939**]) If $n = 6k+3$, denote the vertices as $\{1, 2, \ldots, 2k+1\} \times \{1, 2, 3\}$. Let $(\{1, 2, \ldots, 2k+1\}, \circ)$ be an idempotent, commutative quasigroup of odd order. This can easily be shown to exist for all odd orders by relabeling the group table for the group $\mathbb{Z}_{2k+1}$. Let the triples be $\{(i,1), (i,2), (i,3)\}$, $1 \le i \le 2k+1$, and for $1 \le i < j \le 2k+1$, $\{(i,1), (j,1), (i \circ j, 2)\}$, $\{(i,2), (j,2), (i \circ j, 3)\}$, and $\{(i,3), (j,3), (i \circ j, 1)\}$. This results in $2k+1+\binom{2k+1}{2} = \frac{n(n-1)}{6}$ triples. Each edge is contained in some triple, so they form a decomposition of K_n.

	1	2	3	4	5	6	7
1	1	5	2	6	3	7	4
2	5	2	6	3	7	4	1
3	2	6	3	7	4	1	5
4	6	3	7	4	1	5	2
5	3	7	4	1	5	2	6
6	7	4	1	5	2	6	3
7	4	1	5	2	6	3	7



(Skolem [**1958**]) If $n = 6k+1$, denote the vertices as $\{v\} \cup \{1, 2, \ldots, 2k\} \times \{1, 2, 3\}$. Let $(\{1, 2, \ldots, 2k\}, \circ)$ be a half-idempotent, commutative quasigroup of even order. This can easily be shown to exist for all even orders by relabeling the group table for the group $\mathbb{Z}_{2k}$. Let the triples be

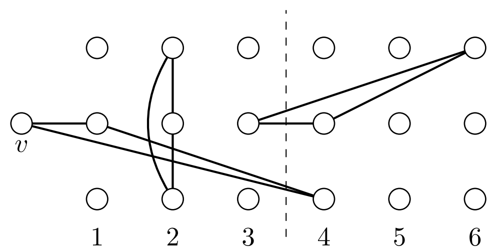
(1) For $1 \le i \le k$, $\{(i,1), (i,2), (i,3)\}$.

(2) For $1 \le i \le k$, $\{v, (k+i, 1), (i, 2)\}$, $\{v, (k+i, 2), (i, 3)\}$, and $\{v, (k+i, 3), (i, 1)\}$.

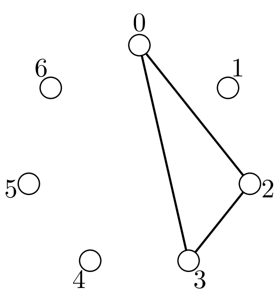
(3) For $1 \le i < j \le 2k$, $\{(i, 1), (j, 1), (i \circ j, 2)\}$, $\{(i, 2), (j, 2), (i \circ j, 3)\}$, and $\{(i, 3), (j, 3), (i \circ j, 1)\}$.

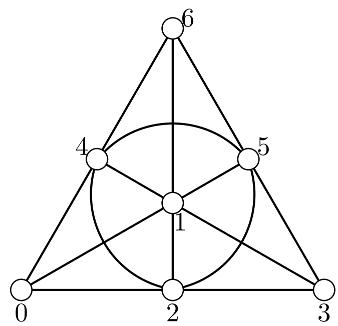
It can be checked that this results in $\frac{n(n-1)}{6}$ triples, and each edge is contained in some triple, so they form a decomposition of K_n.

	1	2	3	4	5	6
1	1	4	2	5	3	6
2	4	2	5	3	6	1
3	2	5	3	6	1	4
4	5	3	6	1	4	2
5	3	6	1	4	2	5
6	6	1	4	2	5	3

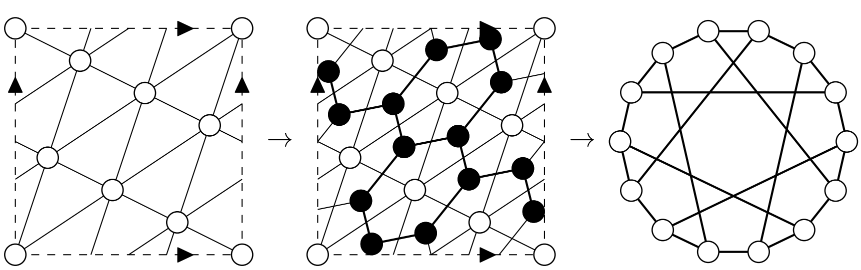


The Steiner triple system of order 7 is cyclic.





This is essentially the same object as the **Fano Plane**, a projective plane of order 7 with seven lines, each of which contain three points. It is illustrated above, with the lines drawn as straight line segments, except for one drawn as a circle. A different way to illustrate this object is to embed K_7 on a torus, which results in a triangulation. Choosing seven triangles with no edge in common produces the Steiner triple system of order 7; there are two ways to make this choice. Note that the dual graph of K_7 on the torus is the **Heawood graph**, the unique cubic graph with girth 6 and order 14 (it is bipartite).



Steiner triple systems are a special case of a more general combinatorial object.

Definition 9.8. A **balanced incomplete block design (BIBD)** B is a family b blocks, that are k-element subsets ($k < v$) of a set X, $|X| = v$, such that any $x \in X$ is contained in r blocks, and any pair of distinct points x and y in X is contained in λ blocks. The design is called a (v, k, λ)**-design**.

It is easily checked that $bk = vr$ and $\lambda(v-1) = r(k-1)$. When $\lambda = 1$, a BIBD is a decomposition of K_v into K_k; when also $k = 3$, it is a Steiner triple system. The necessary conditions are sufficient when $2 \le k \le 5$.

Theorem 9.9 (Hanani [**1975**]). *Existence of block designs:*

If $n \equiv 1$ *or* $4 \bmod 12$, K_4 *decomposes* K_n.

If $n \equiv 1$ *or* $5 \bmod 20$, K_5 *decomposes* K_n.

Richard Wilson [**1975**] has shown that the necessary conditions for existence of (v, k, λ)-designs are almost always sufficient. However, there are some exceptional cases.

Definition 9.10. A **projective plane** is a design with $r = k$, $\lambda = 1$, and $r > \lambda+1$. The elements of X are called **points** and the blocks are called **lines**.

Any projective plane has $v - 1 = k(k-1)$, so it has $v = k(k-1) + 1$ points. Projective planes are known to exist when $k-1$ is prime or a power of a prime. It is known that no projective plane exists when $k-1$ is 6 or 10. The nonexistence of a projective plane with $k-1 = 6$ implies that no $(43, 7, 1)$-design exists. The existence of a projective plane for any value of $k-1$ that is not a prime power is unknown. Thus the problem of decomposing a complete graph into smaller complete graphs has yet to be completely solved.

9.1.3. Class Decompositions.

We can also consider decompositions of K_n into factors that need not be isomorphic but must be in the same class.

Theorem 9.11 (de Bruijn/Erdos [**1948**]). *Any decomposition of* K_n *into proper complete graphs has at least* n *factors. The extremal decompositions are* $\{K_{n-1}, (n-1)[K_2]\}$ *and when* $n = k(k-1)+1$, $\{n[K_k]\}$, *where each vertex occurs in exactly* k *of the* K_k*'s.*

Proof. Denote the vertices by v_i and the r factors by G_j. Let k_i be the number of factors containing v_i, and let $n_j = n(G_j)$. By counting the number of incidences in two ways, we see $\sum_{j=1}^{r} n_j = \sum_{j=1}^{n} k_i$. Further, if G_j does not contain v_i, then $n_j \le k_i$, since v_i has edges to all n_j vertices of G_j, and each of these edges are in different factors (else two factors would have a common edge).

Assume now that k_n is the smallest k_i and that $G_1, \ldots, G_{k_n}$ are factors containing v_n. Assume each factor contains at least two vertices, since else it could be omitted. Also $k_n > 1$, for otherwise there is only one factor. We may assume $v_i \in G_i$, $1 \le i \le k_n < n$. Also if $i < j \le k_n$, then v_i is not in G_j, for otherwise G_i and G_j would have two vertices in common. Thus $n_2 \le k_1$, $n_3 \le k_2, \ldots, n_{k_n} \le k_{k_n - 1}$, $n_1 \le k_{k_n}$, and $n_j \le k_n$ for $j > k_n$. Summing, we find $\sum_{j=1}^{n} = \sum_{j=1}^{r} n_j \le \sum_{j=1}^{r} k_i$, so $r \ge n$.

We now determine the cases where $r = n$. If $r = n$, then the previous inequalities have to be equalities. Consequently, we can renumber the vertices so that

$n_i = k_i$, $1 \le i \le n$. We may suppose that $k_1 \ge k_2 \ge \cdots \ge k_n > 1$. There are two cases:

(1) $k_1 > k_2$. Hence by $n_1 = k_1 > k_i$, $2 \le i \le n$, all vertices except v_1 are in G_1. Thus the decomposition is $\{K_{n-1}, (n-1)[K_2]\}$.

(2) $k_1 = k_2$. If no k_i is less than k_1, then clearly $k_i = n_j$, $1 \le i, j \le n$. We shall show that this is the only possibility. If $k_j < k_1$, then v_j is in both G_1 and G_2. Hence k_n is the only k which can be less than k_1. Now $n_n = k_n$ different factors contain v_n. Any factor containing v_n contains another vertex, and all but one contain two more vertices, since $k_1 = k_2 = \cdots = k_{n-1} > k_n \ge 2$. Thus there are at least two factors which do not contain v_n; for both of these factors we have $n_j \le k_n$. This contradicts $n_1 = n_2 = \cdots = n_{n-1} > k_n$.

Apart from case (1), we only have the case where $n_i = k_i = k$, $1 \le i, j \le n$. Then $n = k(k-1) + 1$, and any pair of factors intersect at one vertex. For if G_i does not intersect G_j, and if v_l lies on G_i, then we infer that $k_l \ge n_j + 1$ which is not possible since $k_l = n_j = k$. $\square$

Decompositions that satisfy the latter condition include (finite) projective planes. This theorem has a geometric interpretation: n points in a real projective plane, not all collinear, determine at least n lines.

There is a similar lower bound for decompositions into complete bipartite graphs.

Theorem 9.12 (**Graham-Pollak Theorem**—Graham/Pollak [**1972**]). *Any decomposition of K_n into complete bipartite graphs has at least $n-1$ factors.*

Proof (Vishwanathan [**2013**]). Suppose to the contrary there is a decomposition K_n, with vertex set $[n]$, into $t \le n-2$ complete bipartite graphs, with parts specified as $(L_1, R_1), \ldots, (L_t, R_t)$. Let $\sigma : [n] \to [k]$ be a labeling of the vertices, where $k > n^n$. Associate with σ a pattern $(\rho_1, \ldots, \rho_{t+1})$ defined by $\rho_i = \sum_{j \in L_i} \sigma(j)$ for $i \le t$ and $\rho_{t+1} = \sum_{j=1}^{n} \sigma(j)$.

The number of possible patterns is at most $(kn)^{t+1}$, and the number of possible labelings is k^n. Since $t \le n-2$ and $k > n^n$, $(kn)^{t+1} \le (kn)^{n-1} < \left(k\left(k^{\frac{1}{n}}\right)\right)^{n-1} < k^n$. The Pigeonhole Principle implies that some two labelings σ_1 and σ_2 yield the same pattern. Let $\tau = \sigma_1 - \sigma_2$. Since the patterns are the same, for $1 \le i \le t$ we have $\sum_{j \in L_i} \tau(j) = 0$. Also $\sum_{j=1}^{n} \tau(j) = 0$, because the patterns are also equal in the last coordinate. Since the labelings are different, τ is nonzero on at least one vertex. In the equality

$$\left(\sum_{j=1}^{n} \tau(j)\right)^2 = \sum_{j=1}^{n} \tau(j)^2 + 2 \sum_{1 \le i < j \le n} \tau(i)\,\tau(j),$$

the left side is zero and the first term on the right is nonzero. We obtain a contradiction to the assumption that $t \le n-2$ by showing that the second term on the right equals zero. Since each edge ij, $i < j$, occurs once in the decomposition, for any function f we have

$$\sum_{1\le i<j\le n} f(i)\, f(j) = \sum_{k=1}^{t} \left(\sum_{l\in L_i} f(l) \right) \left(\sum_{r\in R_i} f(r) \right).$$

With $f = \tau$, the first factor in each term on the right is zero. □

Graham and Pollak's original proof of this theorem used linear algebra, as did several subsequent proofs.

9.1.4. Thickness. When designing a circuit board, it is desirable to avoid crossings. Since many circuits are nonplanar, one way of addressing this is to split the circuit into separate planar levels.

Definition 9.13. The **thickness** $\theta(G)$ of a graph G is the minimum number of planar graphs that decompose G.

Certainly, $\theta(G) = 1$ if and only if G is planar. Since any planar graph with $n \ge 3$ has at most $3n-6$ edges, a natural lower bound is $\theta(G) \ge \frac{m}{3n-6}$. The thickness of many nonplanar graphs is not known exactly. Here we focus on complete graphs.

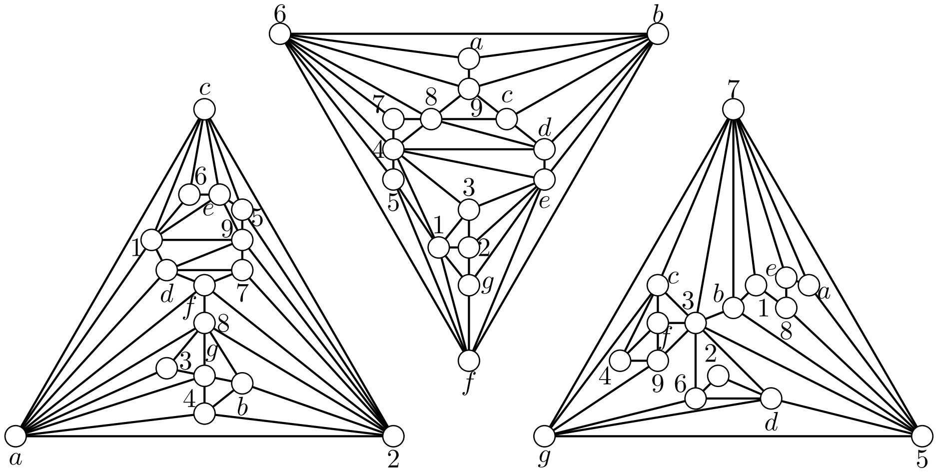
Theorem 9.14. *We have* $\theta(K_n) = \left\lfloor \frac{n+7}{6} \right\rfloor$ *unless* $n \in \{9, 10\}$*, and* $\theta(K_9) = \theta(K_{10}) = 3$.

The proof of this theorem is spread over a number of papers. The result holds for $1 \le n \le 4$, when K_n is planar. For a lower bound, we have

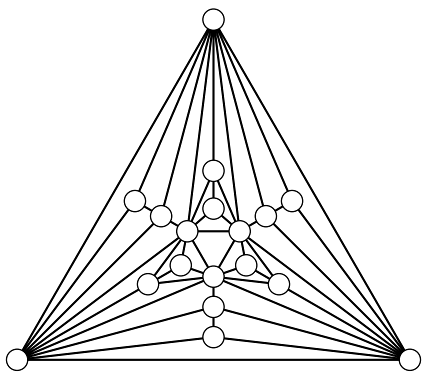
$$\begin{aligned} \theta(K_n) &\ge \left\lceil \frac{\frac{n(n-1)}{2}}{3n-6} \right\rceil = \left\lfloor \frac{\frac{n(n-1)}{2} - 1}{3(n-2)} + 1 \right\rfloor \\ &= \left\lfloor \frac{n^2-n-2}{6(n-2)} + 1 \right\rfloor = \left\lfloor \frac{n+1}{6} + 1 \right\rfloor = \left\lfloor \frac{n+7}{6} \right\rfloor. \end{aligned}$$

The case of K_9 was settled by Battle, Harary, and Kodoma [**1962**] and Tutte [**1963**]. Tutte checked all 50 maximal planar graphs of order 9, showing that none of their complements are planar. Thus $\theta(K_9) \ge 3$.

For the upper bound, note that $K_3 \square K_3 - v$ is planar and self-complementary, so $\theta(K_8) = 2$. Jean Mayer [**1972**], a professor of French literature, found the following decomposition to show $\theta(K_{16}) = 3$.



A general construction showing $\theta(K_n) = \left\lfloor \frac{n+7}{6} \right\rfloor$ was found by Beineke and Harary [**1965**] for $n \not\equiv 4 \bmod 6$. They found a decomposition of K_{6r} into r planar graphs formed by replacing six regions of $K_{2,2,2}$ with $P_r + K_2$, as illustrated below.



They used a special Latin square to make the decomposition work. They also showed how to add three vertices and one more planar graph to decompose K_{6r+3} into $r+1$ planar graphs. Finally, Vasok [**1976**] and Alekseev and Gonchakov [**1976**] modified this construction to decompose K_{6r+4} into $r+1$ planar graphs, completing the proof of the theorem.

9.2. General Decompositions

To decompose a general (not necessarily complete) graph G, the two basic necessary conditions are the following.

(1) The sizes of the factors must sum to the size of G. If all the factors are copies of H, then $m(H)$ divides $m(G)$.

(2) The degrees of the factors must sum to the degrees of G. If G is r-regular and the factors are all s-regular, then s divides r.

Sometimes, these are the only necessary restrictions.

9.2.1. Arboricity. Since forests are a hereditary class of graphs, any graph decomposes into forests.

Definition 9.15. A **p-forest decomposition** of a multigraph is a decomposition of the edge set into p forests. The minimum p is called the **(edge) arboricity** of G and is denoted $a'(G)$. A graph is **critical with respect to the arboricity** (a'-critical) if $a'(G-e) < a'(G)$ for each edge e.

Certainly the arboricity of a graph is at least $\frac{m}{n-1}$. This must be true for any nontrivial subgraph H of G, so $a'(G) \geq \max_{H \subseteq G} \left\lceil \frac{m(H)}{n(H)-1} \right\rceil$. In fact, this is an equality. To prove this, we need a lemma involving a'-critical graphs.

Lemma 9.16 (Chen/Matsumoto et al. [**1994**]). *Let G be connected and a'-critical with $a'(G) > 1$. Then for any $e \in E(G)$, any $a'(G) - 1$-forest decomposition of $G - e$ is a decomposition into $a'(G) - 1$ spanning trees of G.*

Proof. Suppose this is false, and let $E_1, \ldots, E_{a'-1}$ be a forest decomposition of $G - e$ where E_1 is not a spanning tree. Since $E_1 + e$ must contain a cycle, both ends of e are in a connected component T of E_1. Let $K = G[V(T)]$. From the assumption on E_1, we deduce $V(T) \neq V(G)$. Since G is connected, $E(G) - E(K)$ is not empty and, by the criticality of G, K has a $a' - 1$-forest decomposition $E(K) = A_1 \cup \cdots \cup A_{a'-1}$.

Consider the set S of all a'-forest decompositions of G that are of the form $\left\{E_1', \ldots, E_{a'-1}', e'\right\}$ so that a component of E_1' is a spanning tree of K and $e' \in K$. The above $\{E_1, \ldots, E_{a'-1}, e\}$ shows that $S \neq \emptyset$. Let $\left\{\overline{E}, \ldots, \overline{E}_{a'-1}, \overline{e}\right\}$ be an element of S that maximizes

$$J\left(\overline{E}\right) = \sum_{i=1}^{a'-1} m\left(A_i \cap \overline{E}_i\right).$$

Since $\overline{e} \in E(K)$, $\overline{e} \in A_t$ for some t. Now $\overline{E}_t + \overline{e}$ must contain a cycle $C \ni \overline{e} \in K$. If $t = 1$, then C is contained in K by the assumption on E_1. If $t \neq 1$ and C is not contained in K, then we can take an edge $f \in E(C)$ with one end in $V(K)$ and the other end in $V(G) - V(K)$ (note that K is an induced subgraph). Since a connected component of $\overline{E}_1$ spans K, $\overline{E}_1 + f$ is acyclic, and $\left\{\overline{E}_1 + f, \ldots, \overline{E}_t + \overline{e} - f, \ldots, E_{a'-1}\right\}$ leads to a contradiction. Thus, we may assume $C \subset K$. Since A_t is acyclic, there exists an edge $f \in E(C) - A_t \subset E(K)$. Now $\left\{\overline{E}_1, \ldots, \overline{E}_t + \overline{e} - f, \ldots, E_{a'-1}, f\right\} \in S$ increases J by one, which contradicts the assumption on maximality. $\square$

Theorem 9.17 (**Nash-Williams Theorem**—Nash-Williams [**1961**]). *For any multigraph G, $a'(G) = \max_{H \subseteq G} \left\lceil \frac{m(H)}{n(H)-1} \right\rceil$, where the maximum is taken over all nontrivial subgraphs.*

Proof (Chen/Matsumoto et al. [**1994**]). We have seen that

$$a'(G) \geq \max_{H \subseteq G} \left\lceil \frac{m(H)}{n(H)-1} \right\rceil.$$

Let G be a counterexample that minimizes $n+m$, so $a'(G) > \max_{H\subseteq G}\left\lceil \frac{m(H)}{n(H)-1}\right\rceil$. Obviously, G is connected with $a'(G) > 1$ and a'-critical. Lemma 9.16 implies

$$m(G) - 1 = m(G-e) = (n(G)-1)(a'(G)-1),$$

which leads to the contradiction,

$$a'(G) > \left\lceil \frac{m(G)}{n(G)-1}\right\rceil = \left\lceil a'(G) - 1 + \frac{1}{n(G)-1}\right\rceil = a'(G). \qquad \square$$

The proof of Theorem 9.17 does not translate directly into an algorithm for calculating the arboricity of a graph. Nonetheless, the arboricity can be calculated in polynomial time using ideas from matroid theory.

The Nash-Williams Theorem can be used to easily calculate the arboricity of some special classes of graphs. If G is maximal k-degenerate, it is not hard to show that $a'(G) \le k$. The arboricity may be smaller if n is small relative to k.

Corollary 9.18 (Patil [**1984**]). *Let G be maximal k-degenerate. Then*

$$a'(G) = \left\lceil k - \binom{k}{2}\frac{1}{n-1}\right\rceil.$$

Proof (Bickle [**2010**]). A maximal k-degenerate graph of order n has size $m = k\cdot n - \binom{k+1}{2}$. Then $\frac{m}{n-1} = \left[k\cdot n - \binom{k+1}{2}\right]\frac{1}{n-1} = k + \left[k - \binom{k+1}{2}\right]\frac{1}{n-1} = k - \binom{k}{2}\frac{1}{n-1}$. Note that this function is monotone with respect to n. Now k-degenerate graphs are hereditary, so if $H \subset G$, then $\frac{m(H)}{n(H)-1} \le \frac{m(G)}{n(G)-1}$. Then by the Nash-Williams Theorem, $a'(G) = \left\lceil k - \binom{k}{2}\frac{1}{n-1}\right\rceil$. $\square$

9.2.2. Decomposition Conjectures. When a set of graphs satisfy the two basic conditions to decompose a graph G, it may not be clear whether such a decomposition actually exists. That is the theme of many unresolved conjectures on decompositions.

As a consequence of the characterization of Eulerian graphs, we know that any graph with all even degrees decomposes into cycles. However, the theorem does not give any information about the lengths or number of cycles. Similarly, any graph can be decomposed into some number of paths.

Conjecture 9.19 (**Gallai's Conjecture**). *Every connected graph with order n can be decomposed into $\left\lceil \frac{n}{2}\right\rceil$ paths.*

(Hajos 1968) *Every graph with order n and all even degrees decomposes into $\left\lfloor \frac{n}{2}\right\rfloor$ cycles.*

A weaker result has been proven.

Theorem 9.20 (Lovasz [**1968**]). *Every graph with order n decomposes into $\left\lfloor \frac{n}{2}\right\rfloor$ paths and cycles.*

For a graph to decompose into Hamiltonian cycles, it must be regular with even degree. This is not sufficient when that degree is small (the graph may not even be connected). Dirac's Theorem says that $\delta(G) \ge \frac{n}{2}$ implies that G contains a

Hamiltonian cycle. Surprisingly, the same degree in regular graphs may guarantee a Hamiltonian decomposition.

Conjecture 9.21 (**Hamiltonian Decomposition Conjecture**—Nash-Williams [**1971**]). *If G is an r-regular graph, $r \geq \frac{n}{2}$, then G has a decomposition into Hamiltonian cycles and at most one perfect matching.*

This conjecture has been proved for all sufficiently large n (Csaba/Kuhn et al. [**2016**]), in a text that relates it to the 1-factorization Conjecture.

Brian Alspach [**1981**] conjectured that the obvious necessary conditions on cycles of arbitrary lengths decomposing a complete graph are sufficient. This has been proven by Bryant, Horsley, and Pettersson.

Theorem 9.22 (**Alspach's Conjecture**—Bryant et al. [**2014**]). *Let $3 \leq k_1 \leq \cdots \leq k_r \leq n$ be integers with $\sum k_i = \frac{n(n-1)}{2}$ for n odd or with $\sum k_i = \frac{n(n-2)}{2}$ for n even. Then $C_{k_1}, \ldots, C_{k_r}$ decompose K_n (n odd) or $K_n - \frac{n}{2}K_2$ (n even).*

Suppose that a conference has several dinners in a dining room. It would be desirable for each person to sit next to each other person once during the conference. In graph theory terms, we would like a decomposition of K_n into some 2-factor.

Problem 9.23 (**The Oberwolfach Problem**—Ringel 1967). *Let G be a 2-factor. When does G decompose K_n (n odd) or $K_n - \frac{n}{2}K_2$ (n even)?*

No solutions exist when the 2-factor is $2K_3$, $4K_3$, $C_4 \cup C_5$, and $2K_3 \cup C_5$. Otherwise, it is suspected that all possibilities work. This has been proven when $n \leq 40$ (Deza et al. [**2010**]), and for some special classes of 2-factors.

We have already seen several conjectures on decompositions into trees. The fact that $1 + 2 + 3 + \cdots + (n-1) = \binom{n}{2}$ suggests another.

Conjecture 9.24 (**Tree Packing Conjecture**—Gyarfas [**1976**]). *Any set of $n-1$ trees T_i, $1 \leq i \leq n-1$, with $m(T_i) = i$ decomposes K_n.*

These conjectures and many others continue to inspire graph theory research.

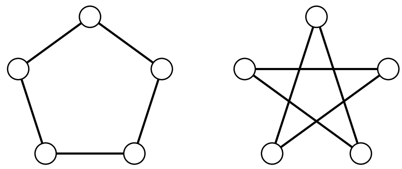
Related Terms: Sylvester-Gallai Theorem, edge-partition number, star arboricity, linear arboricity, linear arboricity conjecture, Kirkman's schoolgirl problem.

9.3. Ramsey Numbers

9.3.1. Classical Ramsey Numbers. Consider a group of six people. Some of them know each other and some do not. Either some three of them all know each other, or some three of them all don't know each other. Why?

Consider one person. Among the five other people, he either knows at least three of them or does not know at least three of them. This follows by the Pigeonhole Principle. Suppose the former. If any two of the three people he knows know each other, we are done. If not, all three of them don't know each other. The other case follows by symmetry.

In graph theory terms, we have shown that in any 2-decomposition of K_6, one of the factors contains a triangle. The order 6 is best possible, since C_5 is self-complementary and triangle-free. Note that this decomposition is unique.



The basic idea of Ramsey theory is to take some mathematical structure and split it into a fixed number of substructures. If the structure is sufficiently large, at least one of the substructures must contain a smaller substructure of a given size. There must be a minimum size that forces this to happen. A familiar example of this concept is the Pigeonhole Principle.

Theorem 9.25 (**The Pigeonhole Principle**)**.** *If more than nk objects are distributed into n boxes, some box has at least $k+1$ objects.*

Proof (contrapositive). If every box has at most k objects, there are at most nk objects in n boxes. □

The Pigeonhole Principle can be formulated in a different way.

Theorem 9.26. *If a set with at least $nk+1$ elements is partitioned into n subsets, some subset has at least $k+1$ elements* (*and $nk+1$ is the smallest number of elements that guarantees this conclusion*)*.*

In this case, elements of sets are the objects being partitioned. Frank Ramsey adapted this idea to partitions of subsets of a set.

Definition 9.27. The **Ramsey number** $R(s,t)$ is the smallest order n such that in any 2-decomposition of K_n, the first factor contains K_s or the second factor contains K_t.

Ramsey theory has been described by Theodore Motzkin as saying that complete disorder is impossible. Every sufficiently large structure can be broken into pieces, one of which must contain some orderly structure of a given size.

There are several ways to interpret the Ramsey number $R(s,t)$. Since a graph of order n and its complement form a 2-decomposition, $R(s,t)$ is the minimum order such that for all graphs G of order n, either G contains K_s or $\overline{G}$ contains K_t. However, this interpretation might leave the incorrect impression that one factor is more important than the other.

Another interpretation involves (nonproper) edge colorings. We assume that red is the first color and blue is the second. The Ramsey number $R(s,t)$ is the minimum order n such that any 2-coloring of K_n produces a red K_s or a blue K_t.

The definition implies that Ramsey numbers are symmetric: $R(s,t) = R(t,s)$. Note that $R(1,t) = 1$ since a single vertex satisfies the first condition. Also, $R(2,t) = t$ since for t vertices, either the first factor has an edge or every edge is in the second factor. We showed earlier that $R(3,3) = 6$. However, we have yet to show that all Ramsey numbers exist.

Theorem 9.28 (**Ramsey's Theorem**—Ramsey [**1930**]). *The Ramsey number $R(s,t)$ exists, and for $s,t \geq 2$, $R(s,t) \leq R(s-1,t) + R(s,t-1)$. If both terms on the right are even, the inequality is strict.*

Proof. We use induction on $s+t$. We have seen that $R(1,t) = R(t,1)$ exists. Assume the result holds when $s+t < k = R(s-1,t)+R(s,t-1)$, and consider a 2-coloring of K_k. A vertex v has degree $k-1$, so the Pigeonhole Principle implies that it is either incident with $R(s-1,t)$ red edges or $R(s,t-1)$ blue edges. Suppose the former. Let H be the clique induced by vertices joined to v by red edges. Then there is a red K_{s-1} or a blue K_t in H, and thus a red K_s or a blue K_t in $H+v$. The other case is similar.

If the bound is an equality, there is a 2-decomposition with every vertex incident with $R(s-1,t)-1$ red edges and $R(s,t-1)-1$ blue edges. Then we have two regular factors with order $R(s-1,t)+R(s,t-1)-1$. If both terms are even, we have a regular graph with odd order and odd degrees, a contradiction. □

Ramsey actually proved a more general result, allowing not just more than two factors in the decomposition, but also colorings of subsets of a set with size larger than two.

To show that $R(s,t) = N$, we must show

(1) Any 2-decomposition of K_N has a red K_s or a blue K_t.

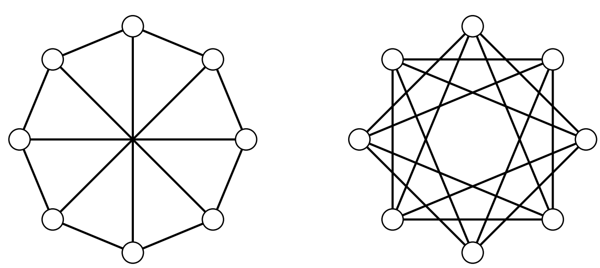
(2) Some 2-decomposition of K_{N-1} has no red K_s or blue K_t.

Now we consider some small examples whose values can be proved using the recursive upper bound.

Proposition 9.29 (Greenwood/Gleason [**1955**]). *We have $R(3,4) = 9$.*

Proof. For an upper bound, we have $R(3,4) \leq R(2,4)+R(3,3)-1 = 4+6-1 = 9$.

For a lower bound, the Mobius ladder M_8 is triangle-free and its complement C_8^2 is K_4-free. □



To construct graphs that provide good lower bounds for Ramsey numbers, we employ some ideas from number theory.

Definition 9.30. Let F_n be a finite field of order n, where n is a prime power, $n \equiv 1 \bmod 4$. A **power residue graph** F_n^k has vertices $\{0,1,\ldots,n-1\}$ and edges between vertices that differ by a power residue, an integer q such that $x^k \equiv q \bmod n$ for some x. For $k=2$, this is called a **quadratic residue graph** or **Paley graph**.

Power residue graphs are vertex-transitive and hence regular. Properties of Paley graphs are summarized in Elsawy [**2009**]. Note that $C_5 = F_5^2$ is a Paley graph.

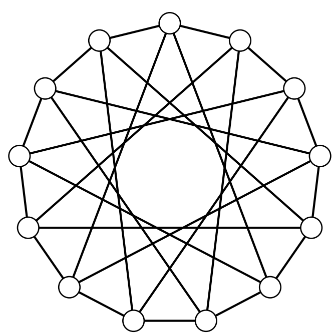
Proposition 9.31. (Sachs [**1962**]) *Any Paley graph is self-complementary.*

Proof. Let G be a Paley graph, and consider the map $f(x) = rx$ from $V(G)$ to itself, where r is a quadratic nonresidue mod q. If xy is an edge of G, then $x - y$ is a quadratic residue, so $r(x - y)$ is not, and $\{rx, ry\}$ is an edge of $\overline{G}$. Similarly, each edge of $\overline{G}$ maps to an edge of G. If $f(x) = f(y)$, then $rx = ry$, so $x = y$. Thus f is one-to-one. Now $\gcd(r, q) = 1$, so $qa + rb = 1$ for some integers a, b, implying $rb \equiv 1 \bmod q$. Thus $f(bx) = rbx = x$, so f is onto and a bijection. $\square$

Proposition 9.32 (Greenwood/Gleason [**1955**]). *We have* $R(3,5) = 14$.

Proof. For an upper bound, we have $R(3,5) \leq R(2,5) + R(3,4) = 5 + 9 = 14$.

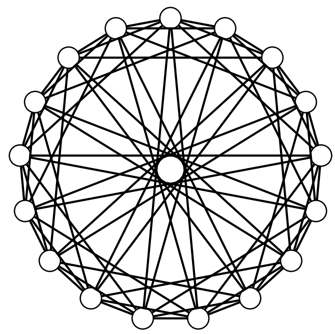
For a lower bound, consider the cubic residue graph F_{13}^3. The cubic residues are $\{\pm 1, \pm 5\}$. It is certainly triangle-free. To verify that $\overline{F_{13}^3}$ has no K_5, assume that there is a K_5 containing 0. It must be adjacent to four vertices in the three sets $\{2,3,4\}$, $\{6,7\}$, and $\{9,10,11\}$. By the Pigeonhole Principle, 0 must be adjacent to two vertices in one of the groups. They must be nonadjacent, so by symmetry, assume 0 is adjacent to 2 and 4. This eliminates all but 6 and 11, which are adjacent. $\square$



Proposition 9.33 (Greenwood/Gleason [**1955**]). *We have* $R(4,4) = 18$.

Proof. For an upper bound, we have $R(4,4) \leq R(3,4) + R(4,3) = 9 + 9 = 18$.

For a lower bound, consider the quadratic residue graph F_{17}^2. The quadratic residues are $\{\pm 1, \pm 2, \pm 4, \pm 8\}$. Does it contain K_4? If so, there are positive residues a, b, c, such that $a + b$, $b + c$, and $a + b + c$ are also residues. Checking cases, we see that this is impossible. $\square$



It can be shown that the decompositions in Propositions 9.32 and 9.33 are unique.

Only nine nontrivial Ramsey numbers are known exactly. When the upper bound fails to be exact, researchers resort to tedious computer searches, which become impractical by 40 or more vertices. Finding constructions for the lower bounds is also difficult. Known values and bounds for Ramsey numbers are contained in the following table from the Dynamic Survey of Small Ramsey Numbers, which contains relevant citations (Radziszoksk [**2017**]).

r\s	3	4	5	6	7	8	9	10	11
3	6	9	14	18	23	28	36	40-42	47-50
4		18	25	36-41	49-61	59-84	73-115	92-149	102-191
5			43-48	58-87	80-143	101-216	133-316	149-442	183-633
6				102-165	115-298	134-495	183-780	204-1171	256-1804
7					205-540	217-1031	252-1713	292-2826	405-4553

Among the unknown Ramsey numbers, $R(5,5)$ has attracted the most interest. Significant evidence points to 43 as its value. Commenting on the difficulty of calculating Ramsey numbers, Paul Erdos remarked (Graham/Spencer [**1990**]),

> Suppose aliens invade the earth and threaten to obliterate it in a year's time unless human beings can find the Ramsey number for red five and blue five. We could marshal the world's best minds and fastest computers, and within a year we could probably calculate the value. If the aliens demanded the Ramsey number for red six and blue six, however, we would have no choice but to launch a preemptive attack.

The known bounds on Ramsey numbers are quite far apart. The recursive upper bound in Ramsey's Theorem can be used to prove an explicit upper bound of $R(s,t) \leq \binom{s+t-2}{s-1}$. When $s = t$, this yields an asymptotic upper bound of $C\frac{4^s}{\sqrt{s}}$. Erdos proved a lower bound of $\frac{s}{\sqrt{2}e}2^{s/2}$. The bounds

$$\frac{s}{\sqrt{2}e}\left(\sqrt{2}\right)^s \leq R(s,s) \leq C\frac{4^s}{\sqrt{s}}$$

are still essentially the best known. We prove a slightly weaker version of the lower bound.

Theorem 9.34 (Erdos [**1947**])**.** *For $s \geq 3$, $R(s,s) > \left\lfloor \sqrt{2}^s \right\rfloor$.*

Proof. Let $n = \left\lfloor \sqrt{2}^s \right\rfloor$. Then $n^s \leq 2^{s^2/2} < \frac{1}{2} s! 2^{\binom{s}{2}}$. There are $2^{\binom{n}{2}}$ labeled graphs with order n. Each clique with order s occurs in $2^{\binom{n}{2}-\binom{s}{2}}$ of them. The number N of graphs with vertex set $[n]$ that contain K_s satisfies

$$N \leq \binom{n}{s} 2^{\binom{n}{2}-\binom{s}{2}} < \frac{n^s}{s!} 2^{\binom{n}{2}-\binom{s}{2}} < \frac{1}{2} 2^{\binom{n}{2}}.$$

The set of all graphs on $[n]$ containing K_s and their complements contains $2N < 2^{\binom{n}{2}}$ graphs. Thus some graph on $[n]$ has no K_s in it or its complement. □

The preceding proof is an existence proof, and there is no explicit construction that achieves, or even comes close to, this bound.

Ramsey numbers generalize naturally to decompositions with more than two factors.

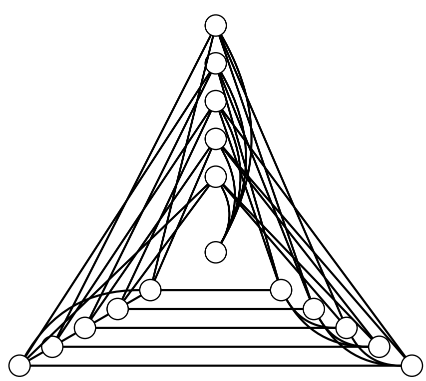
Definition 9.35. The **Ramsey number** $R(s_1, s_2, \ldots, s_k)$ is the smallest order n such that in any k-decomposition of K_n, there is some i such that factor G_i contains K_{s_i}.

Ramsey's Theorem and the upper bound also generalize naturally, and it is easy to show that $R(s_1, s_2, \ldots, s_k, 2) = R(s_1, s_2, \ldots, s_k)$. Yet determining exact values is very difficult.

Proposition 9.36. (Greenwood/Gleason [**1955**]) *We have* $R(3,3,3) = 17$.

Proof. For an upper bound, we have $R(3,3,3) \leq R(3,3,2) + R(3,2,3) + R(2,3,3) - 1 = 6 + 6 + 6 - 1 = 17$.

For a lower bound, consider a (cyclic) 3-decomposition of K_{16} into three copies of the **Clebsch graph**, which is triangle-free. □



The Clebsch graph may be constructed by adding edges between opposite vertices of Q_4. It is the cubic residue graph F_{16}^3. Deleting a vertex and its neighbors from it produces the Petersen graph.

For 60 years, $R(3,3,3)$ was the only nontrivial 3-factor Ramsey number whose value was known exactly. Finally, Codish et al. [**2016**] proved that $R(3,3,4) = 30$.

9.3.2. Graph Ramsey Theory. Ramsey's original theorem was more general than graph theory, but when restricted to graph theory it implies the existence of cliques in factors of decompositions. Cliques are certainly not the only interesting graphs, so we may ask what other graphs the factors of a decomposition are forced to contain.

Definition 9.37. Given graphs G and H, the **graph Ramsey number** $R(G,H)$ is the smallest order n such that in any 2-decomposition of K_n, the first factor contains G or the second factor contains H. Given graph classes $\mathbb{G}$ and $\mathbb{H}$, $R(\mathbb{G},\mathbb{H})$ is the smallest order n such that in any 2-decomposition of K_n, the first factor contains a graph in $\mathbb{G}$ or the second factor contains a graph in $\mathbb{H}$.

If G has order s and H has order t, then $R(G,H) \leq R(s,t)$, so $R(G,H)$ is defined. Note that $R(K_s,K_t) = R(s,t)$. If $G \in \mathbb{G}$ and $H \in \mathbb{H}$, then $R(\mathbb{G},\mathbb{H}) \leq R(G,H)$, so $R(\mathbb{G},\mathbb{H})$ is also defined.

Example. We show $R(K_{1,3},C_4) = 6$. The decomposition $\{C_5,C_5\}$ of K_5 shows $R(K_{1,3},C_4) > 5$. Consider a 2-decomposition of K_6 with no red $K_{1,3}$. Thus the red graph G has $\Delta(G) \leq 2$. The only maximal graphs of order 6 with this property are C_6, $C_5 \cup K_1$, $C_4 \cup K_2$, and $2K_3$. Each of their complements contains C_4.

Given the difficulty of calculating classical Ramsey numbers, it may be surprising that many graph Ramsey numbers are not hard to determine.

Theorem 9.38 (Chvatal [**1977**]). *If T is any tree with order s, then*

$$R(T,K_t) = (s-1)(t-1)+1.$$

Proof. Since the result holds for $s = t = 1$, assume $s,t \geq 2$. The lower bound follows from the decomposition $\{(t-1)K_{s-1}, K_{s-1,\ldots,s-1}\}$, which has no red component of order s and no blue t-clique.

Let $n = (s-1)(t-1)+1$, and let G be the red subgraph of a 2-decomposition of K_n. If $K_t \nsubseteq \overline{G}$, then $\alpha(G) \leq t-1$. Then $\chi(G) \geq \left\lceil \frac{n}{\alpha(G)} \right\rceil \geq \left\lceil \frac{(s-1)(t-1)+1}{t-1} \right\rceil \geq s$. Thus G contains an s-critical subgraph H, so $\delta(H) \geq s-1$. By Theorem 2.7, $T \subseteq H \subseteq G$. □

The same formula holds for another class of graph Ramsey numbers.

Theorem 9.39 (Chartrand/Polimeni [**1974**]). *Let $\mathbb{G}$ be the set of graphs with $\chi(G) \geq s$, and let $\mathbb{H}$ be the set of graphs with $\chi(H) \geq t$. Then*

$$R(\mathbb{G},\mathbb{H}) = (s-1)(t-1)+1.$$

Proof. The decomposition $\{(t-1)K_{s-1}, K_{s-1,\ldots,s-1}\}$ proves the lower bound, since the chromatic numbers of the factors are $s-1$ and $t-1$, respectively.

Let $n = (s-1)(t-1)+1$, and consider a 2-decomposition of K_n with factors G and H. Assume $\chi(H) < t$. Shift edges to H, making it maximal $t-1$-chromatic, so it is a complete multipartite graph. The Pigeonhole Principle implies that some partite set of H has at least s vertices, so $\chi(G) \geq s$. □

What if the graph classes are graphs with k-cores? We need the following definitions.

Definition 9.40. Given nonnegative integers $t_1, t_2, \ldots, t_k$, the **Ramsey core number** $rc\,(t_1, t_2, \ldots, t_k)$ is the smallest n such that for all edge colorings of K_n with k colors, there exists an index i such that the subgraph induced by the ith color, H_i, has a t_i-core. Given $T = \sum t_i$, define a function as follows:

$$B\,(t_1, \ldots, t_k) = \left\lceil \frac{1}{2} - k + T + \sqrt{T^2 - \sum t_i^2 + (2-2k)\,T + k^2 - k + \frac{9}{4}} \right\rceil .$$

A graph with no k-core is $k-1$-degenerate. The following theorem deals with decompositions into degenerate graphs.

Theorem 9.41 (Klein/Schonheim [**1992**])**.** *Any complete graph with order* $n < B\,(t_1, \ldots, t_k)$ *has a decomposition into* k *subgraphs with degeneracies at most* $t_1 - 1, \ldots, t_k - 1$.

This is the key to determining Ramsey core numbers.

Theorem 9.42 (Bickle [**2012**])**.** *We have* $\mathrm{rc}\,(t_1, t_2, \ldots, t_k) = B\,(t_1, \ldots, t_k)$.

Proof. The size of a maximal k-core-free graph of order n is $(k-1)\,n - \binom{k}{2}$. Now by the Pigeonhole Principle, some H_i has a t_i-core when

$$\binom{n}{2} \geq \sum_{i=1}^{k} \left((t_i - 1)\,n - \binom{t_i}{2} \right) + 1.$$

This can be solved for n using the quadratic formula (we omit the details). We obtain

$$n \geq \left\lceil \frac{1}{2} - k + T + \sqrt{T^2 - \sum t_i^2 + (2-2k)\,T + k^2 - k + \frac{9}{4}} \right\rceil = B\,(t_1, \ldots, t_k)\,.$$

Thus we have an upper bound $\mathrm{rc}\,(t_1, \ldots, t_k) \leq \min\{n \,|\, n \geq B\,(t_1, \ldots, t_k)\} = B\,(t_1, \ldots, t_k)$.

By Theorem 9.41, there exists a decomposition of the complete graph of order $B\,(t_1, \ldots, t_k) - 1$ such that subgraph H_i has degeneracy $t_i - 1$, and hence has no t_i-core. Thus $\mathrm{rc}\,(t_1, t_2, \ldots, t_k) > B\,(t_1, \ldots, t_k) - 1$, so $\mathrm{rc}\,(t_1, t_2, \ldots, t_k) = B\,(t_1, \ldots, t_k)$. $\square$

A different variation of graph Ramsey theory is to make the graph that is decomposed something other than a complete graph. Since any 2-decomposition of K_6 forces a K_3, we ask if there is any K_6-free graph so that any 2-decomposition of it forces a monochromatic K_3. Ron Graham [**1968**] showed that $C_5 + K_3$ has this property.

Definition 9.43. The **Folkman number** $f\,(r, k, l)$, $k < l$, is the minimum order of a K_l-free graph with the property that every r-coloring of the edges of G must yield at least one monochromatic copy of K_k.

Graham showed that $f(2,3,6)=8$. Folkman [**1970**] showed the existence of Folkman numbers for $r=2$. It is known that $f(2,3,5)=15$. The bound on $f(2,3,4)$ was initially extremely large (it contained an iterated tower of exponentials), but has been gradually reduced. Dudek and Rodl [**2008**] showed that the power residue graph F_{941}^{5} (with order 941) is K_4-free and any 2-decomposition of it forces a monochromatic K_3.

Related Terms: Ramsey multiplicity, diagonal Ramsey number.

9.4. Nordhaus-Gaddum Theorems

One common way to study a graph parameter $p(G)$ is to examine the sum $p(G)+p(\overline{G})$ and product $p(G)\cdot p(\overline{G})$. A theorem providing sharp upper and lower bounds for this sum and product is known as a theorem of the Nordhaus-Gaddum class.

Such theorems can be contrasted with Ramsey numbers. To calculate a Ramsey number, you fix the size of the clique(s) you want, then find the smallest order such that any decomposition has a factor containing one of them. In a Nordhaus-Gaddum theorem, you fix the order and optimize a parameter in some way. In fact, the lower bound for $\omega(G)+\omega(\overline{G})$ depends on Ramsey numbers (Exercise (19)).

We will examine Nordhaus-Gaddum theorems for degeneracy and chromatic number in this section. An extensive survey of Nordhaus-Gaddum theorems for other parameters appears in Aouchiche/Hansen [**2013**]; many of these results are considered in the Exercises.

Lemma 9.44. *For any graph G, $D(G)+D(\overline{G})\le n-1$.*

Proof. Let $p=D(G)$ and suppose $\overline{G}$ has an $n-p$-core. These cores use at least $(p+1)+(n-p+1)=n+2$ vertices and, hence, they share a common vertex v. But then $d_G(v)+d_{\overline{G}}(v)\ge p+(n-p)=n$, a contradiction. $\square$

Theorem 9.45 (**Nordhaus-Gaddum Theorem**—Nordhaus/Gaddum [**1956**])**.** *Let G be a graph. Then*

$$2\sqrt{n}\le\chi(G)+\chi(\overline{G})\le n+1,$$
$$n\le\chi(G)\cdot\chi(\overline{G})\le\left(\frac{n+1}{2}\right)^2.$$

Proof. We have $\chi(G)+\chi(\overline{G})\le 1+D(G)+1+D(\overline{G})\le n-1+2=n+1$.

Let $k=\chi(G)$, and let H be a maximal k-chromatic graph containing G. Then H is complete multipartite and $\overline{H}$ is a disjoint union of cliques. Thus $\chi(G)\cdot\chi(\overline{G})\ge\chi(H)\cdot\chi(\overline{H})\ge k\cdot\frac{n}{k}=n$.

Note that $\sqrt{xy}\le\frac{x+y}{2}$ with equality exactly when $x=y$. Thus $\chi(G)\cdot\chi(\overline{G})\le\left(\frac{\chi(G)+\chi(\overline{G})}{2}\right)^2\le\left(\frac{n+1}{2}\right)^2$ and $2\sqrt{n}\le 2\sqrt{\chi(G)\cdot\chi(\overline{G})}\le\chi(G)+\chi(\overline{G})$. $\square$

The extremal decompositions for the Nordhaus-Gaddum Theorem have been characterized. The sum upper bound is dealt with here; the others are considered in the Exercises. Note that if a 2-decomposition of K_n achieves $\chi(G)+\chi(\overline{G})=n+1$, then we can easily construct a 2-decomposition of K_{n+1} with $\chi(G')+\chi\left(\overline{G'}\right)=n+2$, by joining a vertex to all the vertices of a color-critical subgraph of G or $\overline{G}$, and allocating the extra edges arbitrarily. Conversely, we may be able to delete some vertex v of K_n so that $\chi(G-v)+\chi\left(\overline{G-v}\right)=n$. If this is impossible, we say that an extremal decomposition is fundamental.

Definition 9.46. A 2-decomposition of K_n with $\chi(G)+\chi\left(\overline{G}\right)=n+1$, such that no vertex v of K_n can be deleted so that $\chi(G-v)+\chi\left(\overline{G-v}\right)=n$, is called a **fundamental decomposition**.

Theorem 9.47. *The fundamental* 2*-decompositions for* $\chi(G)+\chi\left(\overline{G}\right)\le n+1$ *are* $\{K_1,K_1\}$ *and* $\{C_5,C_5\}$.

Proof. It is easily seen that $\chi(K_1)+\chi(K_1)=2$ and $\chi(C_5)+\chi(C_5)=6$, so these decompositions are extremal. They are fundamental since no vertex can be deleted from the first, and deleting a vertex from the second produces $\{P_4,P_4\}$, and $\chi(P_4)+\chi(P_4)=4$.

Consider a fundamental 2-decomposition $\left\{G,\overline{G}\right\}$. Then both graphs are connected. Let $\chi(G)=r$, so that $\chi\left(\overline{G}\right)=n+1-r$. Then $\delta(G)\ge r-1$ and $\delta\left(\overline{G}\right)\ge n-r$. But then G and $\overline{G}$ must be regular, since $n-1=d_G(v)+d_{\overline{G}}(v)\le D(G)+D\left(\overline{G}\right)\le n-1$. Brooks' Theorem says the only connected regular graphs achieving $\chi(G)=1+D(G)$ are cliques and odd cycles. The only such graphs whose complements are cliques and odd cycles are K_1 and C_5. Thus the fundamental 2-decompositions are as stated. $\square$

The extremal 2-decompositions for the upper bound of the Nordhaus-Gaddum Theorem were first characterized by H. J. Finck [**1968**]. Below we offer a similar characterization with a shorter proof. For the rest of the section, we list the critical subgraphs of each factor in a decomposition; any extra edges can be arbitrarily allocated to either factor.

Corollary 9.48. *The extremal* 2*-decompositions for* $\chi(G)+\chi\left(\overline{G}\right)\le n+1$ *are exactly* $\{K_p,K_{n-p+1}\}$ *and* $\{C_5+K_p,C_5+K_{n-p-5}\}$.

Proof. It is immediate that these are extremal 2-decompositions. Assume that we have an extremal 2-decomposition $\left\{G,\overline{G}\right\}$ with order n, and let G be r-critical. If the critical subgraphs overlap on a single vertex and $G=K_r$, then $\overline{G}=K_{n-r}+\overline{K}_r$, which is uniquely $n-r+1$-colorable. Deleting any edge of the copy of K_{n-r} would reduce the chromatic number, so K_{n-r+1} is the only possible $n-r+1$-critical subgraph. If $G\ne K_r$ has order $p\ge r+2$, then the critical subgraph of $\overline{G}$ is contained in $K_{n-p}+\overline{K}_p$, which is impossible. If the critical subgraphs overlap on C_5, the argument is similar. $\square$

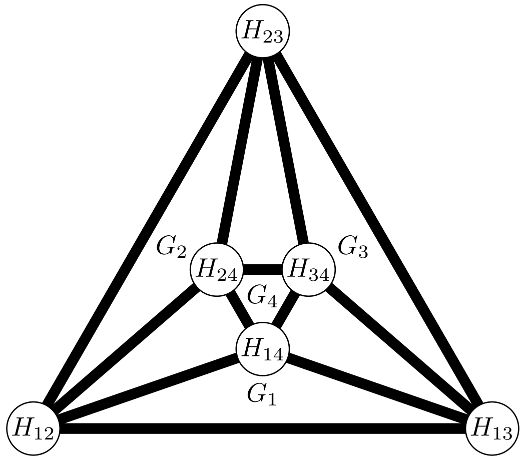
An international round-robin sports tournament is held between n teams. The games are split between k locations in different countries, which can host multiple games simultanously. The teams can travel to different locations to play, but it is impractical for the fans to visit more than one location. In this situation, it is reasonable to want teams that play at a given location to play as many games there as possible so that local fans can see them as much as possible. More precisely, we can compute the minimum number of games played by the teams at that location. We then wish to maximize the sum of these minimum numbers over all the locations in the tournament. In graph theory terms, we want to maximize $\sum D(G_i)$ over all factors of a decomposition. Thus we consider Nordhaus-Gaddum Theorems for decompositions into more than two factors.

Definition 9.49. For a graph parameter p, let $p(k;G)$ denote the maximum of $\sum_{i=1}^{k} p(G_i)$ over all k-decompositions of G. A k-decomposition of K_n, with $K = \sum_{i=1}^{k} \chi(G_i)$ achieving the maximum possible such that no vertex v of K_n can be deleted so that $\sum_{i=1}^{k} \chi(G_i - v) = K - 1$, is called a **fundamental decomposition**.

For degeneracy, consider k-decompositions with the restriction that each vertex is contained in exactly two factors. Consider the following construction.

Algorithm 9.50. *Let $r_1, \ldots, r_k$ be nonnegative integers at most one of which is odd. Let G_{ij}, $1 \le i < j \le k$, be an r_i-regular graph of order $r_i + r_j + 1$, and let $G_{ji} = \overline{G}_{ij}$. Let $G_i = \bigoplus_{j,j\neq i} G_{ji}$, the join of all G_{ji}'s. Let S_k be the set of all k-decompositions of the form $\{G_1, \ldots, G_k\}$ in this fashion.*

A 4-decomposition is illustrated below, where thick edges indicate that all possible edges between two graphs are present.



Theorem 9.51 (Bickle [**2012**]). *A k-decomposition with order $n > 1$ and every vertex in exactly two factors has $\sum D(G_i) \le \left(\frac{2k-3}{k-1}\right) n - \frac{k}{2}$, and equality holds exactly for those decompositons in the set S_k.*

Proof. Since each vertex is contained in exactly two of the k factors, we can partition them into $\binom{k}{2}$ distinct classes. Let $H_{ij} = V(G_i) \cap V(G_j)$, and let $n_{ij} = |H_{ij}|$ for $i \neq j$, $n_{ii} = 0$. Hence $n = \sum_{i,j} n_{ij}$. For $v \in H_{ij}$, we have $D(G_i) \le d_{G_i}(v) \le d_{G_i[H_{ij}]}(v) + \sum_{t=1}^{k} n_{it}$. Sum for each of the two factors and each of the $\binom{k}{2}$ classes. Then $(k-1)\sum_{i=1}^{k} D(G_i) \le 2(k-2)\sum_{i,j} n_{ij} + \sum_{i,j,i\neq j}(n_{ij} - 1) = (2k-3)n - \binom{k}{2}$, so $\sum_{i=1}^{k} D(G_i) \le \left(\frac{2k-3}{k-1}\right) n - \frac{k}{2}$.

($\Rightarrow$) If this bound is an equality, then all k of the factors must be regular. Let $r_{ij} = d_{G_i[H_{ij}]}(v)$ for $v \in G_i[H_{ij}]$. Also, all edges between two classes sharing a common factor must be in that factor, so it is a join of $k-1$ graphs. A join of graphs is regular only when they are all regular. Now since G_i is regular, its complement must also be regular. But this implies that all the constants r_{ji}, $j \neq i$ are equal. Let r_i be this common value. Then $n_{ij} = r_i + r_j + 1$, so $n = (k-1)\sum r_i + \binom{k}{2}$. This implies that at most one of r_i and r_j is odd, so at most one of all the r_i's is odd.

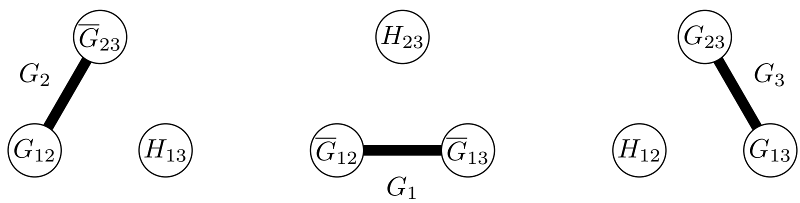
($\Leftarrow$) Let G_i be a factor of a decomposition constructed using the algorithm. It is easily seen that G_i is regular of degree $(k-3)r_i + (k-2) + \sum_j r_j$. Summing over all the factors, we find that $\sum D(G_i) = \left(\frac{2k-3}{k-1}\right) n - \frac{k}{2}$. $\square$

Now consider 3-decompositions.

Theorem 9.52 (Furedi et al. [**2005**], Bickle [**2012**]). *We have*

$$D(3; K_n) = \left\lfloor \frac{3}{2}(n-1) \right\rfloor,$$

and the extremal decompositions that achieve $\sum_{i=1}^{3} D(G_i) = \frac{3}{2}(n-1)$ *all consist of three* $\frac{n-1}{2}$*-regular graphs. For* $n = 1$, $\{K_1, K_1, K_1\}$ *is the only extremal* 3*-decomposition, and for odd order* $n > 1$ *they are exactly those in the set* S_3*.*



Proof. Let G_1, G_2, and G_3 be the three factors of an extremal decomposition for $D(3; K_n)$. Clearly, $\{K_1, K_1, K_1\}$ is the only possibility for $n = 1$, so let $n > 1$. The previous theorem shows that $\sum_{i=1}^{3} D(G_i) \leq \frac{3}{2}(n-1)$.

Now any vertex can be contained in at most two of the three factors, since its degrees in the three graphs sum to at most $n-1$. Now adding a vertex with adjacencies so that it is contained in exactly one of the three factors increases n by one and $\sum_{i=1}^{3} D(G_i)$ by at most one, so this cannot violate the bound. Thus deleting a vertex of an extremal decomposition contained in only one of the three factors would decrease n by one and $\sum_{i=1}^{3} D(G_i)$ by at most one. For n odd, this is a contradiction and for n even it can occur only when it is the only such vertex.

If there are only two distinct classes, then add a vertex joined to all the vertices of the two disjoint factors. This increases n by one and $\sum_{i=1}^{3} D(G_i)$ by two. Hence if the new decomposition satisfies the bound, so does the original, and if the orginal decomposition attains the bound, then n must be even.

Thus by the previous theorem, those decompositions with $\sum_{i=1}^{3} D(G_i) = \frac{3}{2}(n-1)$ are exactly those in S_3. Further, by the previous proof the factors of such a decomposition are all $1 + \sum r_j$-regular. Now $2(r_1 + r_2 + r_3) = \sum (n_{ij} - 1) = n - 3$, so $\sum_j r_j = \frac{n-3}{2}$. Thus the factors are all $\frac{n-1}{2}$-regular.

Finally, note that joining a vertex to all vertices of one factor of an extremal decomposition of odd order attains the bound for even order, so $D(3; K_n) = \left\lfloor \frac{3}{2}(n-1) \right\rfloor$ for even orders as well. $\square$

Jan Plesnik made the following conjecture concerning $\chi(k; K_n)$.

Conjecture 9.53 (**Plesnik's Conjecture**—Plesnik [**1978**]). *For $n \geq \binom{k}{2}$, $\chi(k; K_n) = n + \binom{k}{2}$.*

For $k = 2$, this is just the Nordhaus-Gaddum Theorem. Plesnik proved the conjecture for $k = 3$ and determined an upper bound for $\chi(k; K_n)$. There is a simple construction that shows $\chi(k; K_n) \geq n + \binom{k}{2}$. Take the line graph $L(K_k)$ with order $\binom{k}{2}$, and decompose it into k copies of K_{k-1}. For any additional vertex, make it adjacent to all the vertices of one of the cliques in the decomposition and allocate any extra edges arbitrarily.

Plesnik proved a recursive upper bound of $\chi(k; K_n) \leq n + t(k)$, where $t(2) = 1$ and $t(k) = \sum_{i=2}^{k-1} \binom{k}{i} t(i)$ (thus $t(3) = 3$ and $t(4) = 18$). This implies a worse explicit bound of $\chi(k; K_n) \leq n + 2^{\binom{k+1}{2}}$. The upper bound was improved to $\chi(k; K_n) \leq n + \frac{k!}{2}$ by Watkinson [**1985**] and to $\chi(k; K_n) \leq n + 7^k$ by Furedi et al. [**2005**]. All of these bounds remain far from Plesnik's conjecture, however.

We can describe many fundamental decompositions for $k \geq 3$ using the following construction.

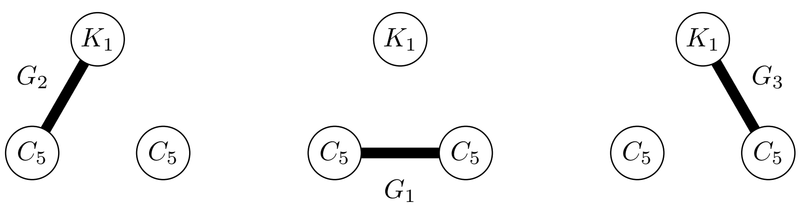
Algorithm 9.54 (**Construction of fundamental k-decompositions**). *For $k \geq 3$ and $n \geq \binom{k}{2}$, construct a decomposition of K_n as follows.*

(1) *Start with the line graph $L(K_k)$ decomposed into k copies of K_{k-1}.*

(2) *Replace each vertex by either K_1 decomposed into $\{K_1, K_1\}$ or K_5 decomposed into $\{C_5, C_5\}$.*

(3) *Join each factor to the other factors corresponding to the same copy of K_{k-1} in the decomposition of $L(K_k)$.*

(4) *Allocate any remaining edges arbitrarily.*



An example of a fundamental 3-decomposition is given above. We will see below that the graphs produced by this algorithm attain the bound of Plesnik's conjecture. This algorithm produces all such graphs for $k = 2$ but not all for $k = 3$.

Lemma 9.55.

(1) *For $k \geq 3$, let D be a k-decomposition with every vertex contained in exactly two color-critical subgraphs of the decomposition that maximizes $\sum_{i=1}^{k} \chi(G_i)$. Then $\sum_{i=1}^{k} \chi(G_i) = n + \binom{k}{2}$.*

(2) *The k-decompositions produced by the preceding algorithm satisfy* $\sum_{i=1}^{k} \chi(G_i) = n + \binom{k}{2}$.

Proof. Assume the hypothesis, and let H_i be the critical subgraphs of the k graphs. Thus we can partition the n vertices into $\binom{k}{2}$ classes: $V_{ij} = V(H_i) \cap V(H_j)$. Now the edges between V_{ij} and V_{il} may as well be in H_i since this is the only critical subgraph with vertices in both classes. Similarly, if V_{ij} and V_{lm} have no common indices, then no edges between them are contained in a critical subgraph. Then $\chi(H_i) \leq \sum_j \chi(H_i[V_{ij}])$, where $1 \leq j \leq k$, $i \neq j$. Then $n + \binom{k}{2} \leq \sum \chi(H_i) \leq \sum_{i,j} \chi(H_i[V_{ij}]) \leq \sum (n(V_{ij}) + 1) = n + \binom{k}{2}$, with the last inequality following from the Nordhaus-Gaddum Theorem. But then we have equalities, which implies that $\sum_{i=1}^{k} \chi(G_i) = n + \binom{k}{2}$, and the two graphs that decompose $K_n[V_{ij}]$ form an extremal 2-decomposition. Since $\{K_1, K_1\}$ and $\{C_5, C_5\}$ are fundamental 2-decompositions, Algorithm 9.54 produces fundamental k-decompositions. □

Now we can prove Plesnik's Conjecture for $k = 3$.

Theorem 9.56 (Plesnik [**1978**]). *For $n \geq 3$, $\chi(3; K_n) = n + 3$.*

Proof. Assume that some fundamental decomposition of K_n into three factors yields $\chi(G_1) = a + 1$, $\chi(G_2) = b + 1$, and $\chi(G_3) = c + 1$, with $a + b + c = n$. We may consider the critical subgraphs H_i of the three graphs, which are a, b, and c-cores, respectively. Now no vertex of K_n can be contained in all three of the H_i's, since this would imply that K_n has at least $a + b + c + 1 = n + 1$ vertices.

Since deleting a vertex from a k-critical graph produces a $k-1$-chromatic graph and the decomposition is fundamental, every vertex is contained in exactly two of the three critical subgraphs. Then by Lemma 9.55, $\chi(3; K_n) = n + 3$. □

Exercises

Section 9.1:

(1) Determine all graphs that decompose K_4.

(2) Determine all graphs that decompose K_5.

(3) (a) Find all spanning trees that decompose K_6.
(b) + Determine all graphs that decompose K_6.

(4) + Find all triples of (not necessarily isomorphic) trees of order 6 that decompose K_6.

(5) Show that K_{2r} decomposes into spanning double stars.

(6) Find all decompositions $K_5 \to \{5[P_3]\}$, both cyclic and noncyclic.

(7) Show that $K_6 \to \{3[P_6]\}$ in two ways, one cyclic and one noncyclic.

(8) (Alavi et al. [**1988**]) Show that $K_{10} \to \{9[K_4 - e]\}$, and the decomposition must be noncyclic.

(9) Let PG be the Petersen graph. Find a graph H so that $K_{10} \to \{2[PG], H\}$. (*Note*: Schwenk [**1983**] showed that $K_{10} \nrightarrow \{3[PG]\}$ using linear algebra.)

(10) Show that any decomposition of a complete graph into two copies of a self-complementary graph must be cyclic.

(11) (Rosa [**1967**]) Show that an Eulerian graph with size $m \equiv 1$ or $2 \bmod 4$ is not graceful. (*Hint*: Sum the edge labels.)

(12) (Rosa [**1967**]) Show that C_n is graceful if and only if $n \equiv 0$ or $3 \bmod 4$.

(13) Show that K_n is not graceful if $n \geq 5$.

(14) Show that $K_{r,s}$ is graceful.

(15) Show that caterpillars are graceful. (*Hint*: Place the vertices on two parallel lines so all edges join vertices on different lines.)

(16) Show that wheels are graceful.

(17) Show that if the Graceful Tree Conjecture holds, then any tree T with size m decomposes K_{2m}.

(18) (Wilson [**1975**]) Show that for any graph G, there is a regular graph H so that G decomposes H. (*Hint*: Generalize the concept of a graceful graph so that it applies to any graph.)

(19) Find a Steiner triple system of order 13.

(20) Find a Steiner triple system of order 15.

(21) Show that $K_{11} \to \{K_5, 15\,[K_3]\}$.

(22) Verify that for a BIBD, $bk = vr$ and $\lambda(v-1) = r(k-1)$.

(23) (a) (Melchior [**1941**]) Prove the **Sylvester-Gallai Theorem**: Given a finite number of points in the Euclidean plane, not all collinear, there is a line that contains exactly two points. (*Hint*: Among all pairs of a point and a line not containing it, consider one for which the distance between the point and line is smallest.)

(b) Use the Sylvester-Gallai Theorem (not Theorem 9.11) to prove that n points in a real projective plane, not all collinear, determine at least n lines.

(24) Show that for any complete bipartite graph G with order n, there is a decomposition of K_n into $n-1$ complete bipartite factors, one of which is G.

(25) Show that for $n \geq 4$, there is a decomposition of K_n into $n-1$ nonspanning complete bipartite factors.

(26) (Schwenk/Zhang [**1998**]) Show that there is a decomposition of K_n into $n-1$ complete bipartite graphs that are not stars when $n = 9$ but not when $n < 9$.

(27) Find a cyclic decomposition of K_9 into three copies of a planar graph G.

(28) + Show that all ten self-complementary graphs of order 8 are planar.

(29) + (Tutte [**1963**]) Generate all 50 maximal planar graphs of order 9. Verify that each of their complements are nonplanar.

(30) Show that $\theta\left(K_{r,s}\right) \geq \left\lceil \frac{rs}{2(r+s-2)} \right\rceil$. (*Note*: Beineke/Harary [**1964**] proved this is an equality for most values of r and s, with a few cases undecided.)

Section 9.2:

(1) Show that if G decomposes H and H decomposes K, then G decomposes K.

(2) Show that $K_{1,r}$ decomposes an r-regular graph G if and only if G is bipartite.

(3) (Harary/Hsu/Miller [**1977**]) Show that any graph G decomposes into $\lceil \log_2 \chi(G) \rceil$ bipartite graphs.

(4) + Show that P_3 decomposes any connected graph with even size.

(5) Show that P_4 decomposes a bridgeless cubic graph if and only if it has a 1-factor.

(6) Determine whether P_5 decomposes any cubic graph.

(7) Determine all graphs that decompose the Petersen graph.

(8) Determine all graphs that decompose $K_{3,3}$.

(9) + Determine all graphs that decompose Q_3.

(10) + Determine all graphs that decompose $K_{2,2,2}$.

(11) Show that $K_{2r,2r}$ decomposes into Hamiltonian cycles.

(12) Show that $K_{2r,2r-1}$ decomposes into Hamiltonian paths.

(13) Determine the arboricity of the graphs in the following classes.
(a) W_n
(b) $G_{r,s}$
(c) $K_{r,s}$
(d) maximal planar graphs

(14) Determine the arboricity of the graphs in the following classes.
(a) K_n
(b) outerplanar graphs
(c) triangular grids
(d) $C_r \square C_s$

(15) (Burr [**1986**]) Show that $a'(G) \geq \left\lceil \frac{1+D(G)}{2} \right\rceil$. Use this to show that $a(G) \leq a'(G)$.

(16) Find the graph of smallest order for which $a(G) \neq a'(G)$.

(17) Let H be a subgraph of a graph G that maximizes $\frac{m}{n-1}$ over all nontrivial subgraphs. Show that $\delta(H) \geq \left\lceil \frac{1}{2} D(G) \right\rceil$.

(18) For any graph G, show that there exists a subgraph $H \subseteq G$ that maximizes $\frac{m}{n-1}$ over all nontrivial subgraphs so that $\delta(H) \geq \left\lceil \frac{1}{2}(D(G)+1) \right\rceil$.

(19) The **edge-partition number**, $\rho'_k(G)$ is the minimum number of k-degenerate graphs into which G can be decomposed. Show that $\left\lceil \frac{1+D(G)}{2k} \right\rceil \leq \rho'_k(G) \leq \left\lceil \frac{D(G)}{k} \right\rceil$.

(20) Let G be a nonempty graph. Prove or disprove:

$$\rho'_k(G) = \max_{H \subseteq G} \left\lceil \frac{m(H)}{k \cdot n(H) - \binom{k+1}{2}} \right\rceil,$$

where the maximum is taken over all induced subgraphs of G.

(21) The **star arboricity** $\mathrm{st}(G)$ of a graph G is the minimum number of forests, all of whose components are stars, into which a graph can be decomposed. Show that $a(G) \leq \mathrm{st}(G) \leq 2a(G)$.

(22) The **linear arboricity** $\mathrm{la}(G)$ of a graph G is the minimum number of linear forests (whose components are all paths) into which a graph can be decomposed. Show that $\mathrm{la}(G) \geq \left\lceil \frac{\Delta(G)}{2} \right\rceil$, and $\mathrm{la}(G) \geq \left\lceil \frac{d+1}{2} \right\rceil$ for d-regular graphs. (*Note*: The **linear arboricity conjecture** asserts that $\mathrm{la}(G) = \left\lceil \frac{d+1}{2} \right\rceil$ for d-regular graphs.)

(23) Show that Gallai's Conjecture holds for graphs with at most one even vertex.

(24) Verify Alspach's Conjecture for $n = 5$ and $n = 6$.

(25) Show that the Oberwolfach Problem has no solution for $2K_3$.

(26) Show that the Oberwolfach Problem has a solution for $K_3 \cup C_4$.

(27) Show that the Oberwolfach Problem has a solution for $2C_4$.

(28) Show that the Oberwolfach Problem has a solution for $K_3 \cup C_5$.

(29) **Kirkman's schoolgirl problem** asks for a grouping of fifteen schoolgirls into rows of three in seven different ways so that each pair of girls appears once in each triple. This is equivalent to the Oberwolfach Problem for $5K_3$. Solve this problem.

(30) Verify that the Tree Packing Conjecture holds for $2 \leq n \leq 5$.

Section 9.3:

(1) (a) Find all maximal triangle-free graphs of order 6.
 (b) Find the minimum number of triangles in any 2-decomposition of K_6.
 (c) Does there exist a 2-decomposition of K_6 with exactly one triangle in each factor?

(2) Find all 2-decompositions that jusitify the lower bound for $R(3,4)$.

(3) Use Ramsey's Theorem to prove upper bounds for the following Ramsey numbers.
 (a) $R(3,10)$
 (b) $R(3,11)$
 (c) $R(3,12)$

(4) Use Ramsey's Theorem to prove upper bounds for the following Ramsey numbers.
 (a) $R(5,5)$
 (b) $R(6,6)$
 (c) $R(7,7)$

(5) Draw the Paley graphs of orders 9 and 13.

(6) + Use the Paley graph of order 101 to show $R(6,6) \geq 102$.

(7) Show that any Paley graph of prime order decomposes into Hamiltonian cycles.

(8) + Show that the Paley graph of order n is strongly regular. Determine its parameters.

(9) Show that $R(3,t) \leq \frac{t^2+3}{2}$.

(10) Show that $R(s,t) \leq \binom{s+t-2}{s-1}$.

(11) + (Ramsey [**1930**]) Show that $R(r_1, r_2, \ldots, r_k)$ is defined, and find an upper bound for it.

(12) + (Ramsey [**1930**]) Let $t, n_1, \ldots, n_k$ be integers. Show that there exists an integer N such that if each t-set of $[N]$ is colored with colors from $[k]$, then for some i there is a subset $S \subseteq [N]$ containing n_i elements so that every t-set of S is colored i.

(13) Show that $R(s_1, s_2, \ldots, s_k, 2) = R(s_1, s_2, \ldots, s_k)$.

(14) + Form a graph with the vertices of Q_5 and edges between vertices whose distance in Q_5 is 2. Show that this graph has two components that are both the complement of the Clebsch graph.

(15) Evaluate the following graph Ramsey numbers.
 (a) $R(P_3, P_3)$
 (b) $R(2K_2, P_3)$
 (c) $R(2K_2, 2K_2)$
 (d) $R(P_4, P_4)$

(16) Evaluate the following graph Ramsey numbers.
 (a) $R(2K_2, P_3)$
 (b) $R(K_{1,3}, P_3)$
 (c) $R(C_4, C_4)$
 (d) $R(2K_3, 2K_3)$

(17) Determine $R(K_{1,s}, K_{1,t})$.

(18) Show that $R(sK_2, sK_2) = 3s - 1$.

(19) Determine $R(P_3, G)$ as a function of properties of G.

(20) Show that $R(K_{s_1}, \ldots, K_{s_k}, T) = (R(s_1, \ldots, s_k) - 1)(t-1) + 1$ for any tree T with order t.

(21) Let G and H be graphs with r the order of the largest component of G and $k = \chi(H)$. Show that $R(G,H) \geq (r-1)(k-1)+1$.

(22) Let $\mathbb{G}$ be the class of graphs containing G as a subgraph, and let $\mathbb{H}$ be the class of graphs containing H as a subgraph. Show that $R(\mathbb{G}, \mathbb{H}) = R(G,H)$.

(23) (Chartrand/Polimeni [**1974**]) Let $\mathbb{G}_{s_i}$ be the class of graphs with $\chi(G) \geq s_i$. Show that $R(\mathbb{G}_{s_1}, \ldots, \mathbb{G}_{s_k}) = 1 + \prod (s_i - 1)$.

(24) (Chartrand/Polimeni [**1974**]) Let $\mathbb{G}$ be the class of graphs with vertex-arboricity $a(G) \geq s$, and let $\mathbb{H}$ be the class of graphs with $a(H) \geq t$. Determine $R(\mathbb{G}, \mathbb{H})$.

(25) Let $t_1 = t_2 = \cdots = t_k = 2$. Find an explicit construction to show that $\mathrm{rc}(t_1, t_2, \ldots, t_k) = 2k+1$.

(26) + (Bickle [**2012**]) Explicit construction for $\mathrm{rc}(2,t)$.
 (a) Show that $\mathrm{rc}(t_1 + 1, t_2, \ldots, t_k) \geq \mathrm{rc}(t_1, \ldots, t_k) + 1$.
 (b) Let $t = \binom{r}{2} + q$, $1 \leq q \leq r$. Show that $B(2,t) = t + r + 1$.
 (c) Let $t = \binom{r}{2} + q$, $1 \leq q \leq r$. Show that $\mathrm{rc}(2,t) = B(2,t)$ using the caterpillar whose nonleaves (in order) have degrees r, r, $r-1$, $r-2, \ldots,$ 4, 3, 2.

(27) Show that every K_6-free graph with $n < 8$ has a 2-edge-coloring with no monochromatic triangle.

(28) (Graham [**1968**]) Show that $C_5 + K_3$ is K_6-free and every 2-edge-coloring produces a monochromatic triangle.

(29) (Erdos/Szekeres [**1935**]) Show that any list of more than n^2 distinct numbers has a monotone sublist of length more than n. Show that this result is sharp. (*Hint*: Label each number with a pair indicating the length of the longest increasing and decreasing sublists.)

(30) + (Schur [**1916**]) Show that for any k-coloring of the positive integers, there are three integers x, y, z (possibly $x = y$) with $x + y = z$. (*Hint*: Use Ramsey numbers.)

Section 9.4:

(1) Prove that $\chi(G) \cdot \chi(\overline{G}) \geq n$ using an appropriate two-element labeling of the vertices.

(2) Use the bound $\chi(G) \leq \frac{\omega(G)+n+1-\alpha(G)}{2}$ (Brigham/Dutton [**1985**]) to prove the sum upper bound of the Nordhaus-Gaddum Theorem.

(3) (Stewart [**1969**]) Show that for every pair of positive integers a and b satisfying $a + b \leq n + 1$ and $a \cdot b \geq n$, there exists a graph G such that $\chi(G) = a$ and $\chi(\overline{G}) = b$.

(4) Show that the extremal decompositions for $\chi(G) \cdot \chi(\overline{G}) \leq \left(\frac{n+1}{2}\right)^2$ are exactly $\left\{K_{\frac{n+1}{2}}, K_{\frac{n+1}{2}}\right\}$ and $\left\{C_5 + K_{\frac{n-5}{2}}, C_5 + K_{\frac{n-5}{2}}\right\}$.

(5) Show that if G is regular and $\chi(G) + \chi(\overline{G}) = n$, then the 2-decompositions that satisfy this equation are $\{C_7, \overline{C}_7\}$ and $\{C_4, 2K_2\}$.

(6) Starr and Turner [**2008**] proved that $\chi(G)+\chi(\overline{G}) = n+1$ if and only if $V(G)$ can be partitioned into three sets S, T, and $\{x\}$ such that $G[S] = K_{\chi(G)-1}$ and $G[T] = K_{\chi(\overline{G})-1}$. Show that this is equivalent to Corollary 9.48.

(7) (Watkinson [**1985**]) Find a 4-decomposition of K_7 with $\sum_{i=1}^{4} \chi(G_i) = 13$ so that three critical graphs share a vertex.

(8) Find a sharp upper bound for $\chi_l(G) + \chi_l(\overline{G})$.

(9) Show that the graphs for which $D(G)+D(\overline{G}) = n-1$ are exactly the graphs constructed by starting with a regular graph and iterating the following operation: Given $k = D(G)$, H a k-monocore subgraph of G, add a vertex adjacent to at least $k+1$ vertices of H, and all vertices of degree k in H (or similarly for $\overline{G}$).

(10) Show that $0 \leq D(G) \cdot D(\overline{G}) \leq \left(\frac{n-1}{2}\right)^2$. Characterize the extremal decompositions for the lower bound. Show that the upper bound is an equality exactly when $D(G) + D(\overline{G}) = n - 1$ and $D(G) = D(\overline{G})$.

(11) + (Borodin [**1993**]) Show that $D(G) + D(\overline{G}) \geq 2n - 1 - \left\lfloor\sqrt{2n^2 - 2n + 1}\right\rfloor$.

(12) + (Furedi et al. [**2005**], Bickle [**2012**]) Let n, r, a, b, c, and s be nonnegative integers with $n = 3r+1$, $a+b+c = s-1$ and a, b, c, even if s is odd. Let G_1,

G_2, G_3 be a, b, c-regular graphs, respectively, of order s. Let G_4, G_5, G_6 be $r-s$-regular graphs of orders $r-a$, $r-b$, $r-c$, respectively. Let S be the set of all decompositions of the form $\left\{G_1+G_4, G_2+G_5, G_3+G_6, \overline{G}_3+\overline{G}_4+\overline{G}_6\right\}$. Show that $D(4;K_n) = \left\lfloor \frac{5}{3}(n-1)\right\rfloor$, and for $n=1$, $\{K_1,K_1,K_1,K_1\}$ is the only extremal 4-decomposition and the extremal decompositions of order $n = 3r+1>1$ that achieve $\sum_{i=1}^{4} D(G_i) = \frac{5}{3}(n-1)$ are exactly those in S.

(13) (Furedi et al. [**2005**]) Show that $D(k;K_n) \geq \left\lfloor \frac{2k-3}{k-1}(n-1)\right\rfloor$.

(14) Show that $D(5;K_n) \geq \left\lfloor \frac{11}{6}n-2\right\rfloor$.

(15) Show that $D(6;K_n) \geq 2n-2$.

(16) (Bickle [**2012**]) Suppose there is a k-decomposition of K_n into regular subgraphs G_i and $\sum_{i=1}^{k} D(G_i) = c(n-1)$. Show that there are infinitely many other k-decompositions with order n' and $\sum_{i=1}^{k} D(G_i) = c(n'-1)$.

(17) + (Furedi et al. [**2005**]) Show that for all $n,k \in \mathbb{Z}^+$, $D(k;K_n) \leq -\frac{k}{2} + \sqrt{\frac{k^2}{4}+kn(n-1)}$. Show that this is an equality exactly when there is a decomposition of K_n into k cliques of equal size.

(18) (Bickle [**2012**]) Block designs and Nordhaus-Gaddum Theorems for degeneracy.
   (a) Show that $D\left(\binom{n}{2};K_n\right) = \binom{n}{2}$.
   (b) Show that if $n \equiv 1$ or $3 \bmod 6$, $D\left(\frac{1}{3}\binom{n}{2};K_n\right) = \frac{2}{3}\binom{n}{2}$.
   (c) If $n \equiv 1$ or $4 \bmod 12$, K_n decomposes into K_4's (Hanani [**1975**]). Show that if $n \equiv 1$ or $4 \bmod 12$, $D\left(\frac{1}{6}\binom{n}{2};K_n\right) = \frac{1}{2}\binom{n}{2}$.
   (d) If $n \equiv 1$ or $5 \bmod 20$, K_n decomposes into K_5's (Hanani [**1975**]). Show that if $n \equiv 1$ or $5 \bmod 20$, $D\left(\frac{1}{10}\binom{n}{2};K_n\right) = \frac{2}{5}\binom{n}{2}$.

(19) Find a sharp lower bound for $\omega(G)+\omega\left(\overline{G}\right)$ that involves Ramsey numbers.

(20) + (Furedi et al. [**2005**]) Show that for all positive integers n and k with $n \geq \binom{k}{2}$, $\omega(k;K_n) = n+\binom{k}{2}$.

(21) + (Bickle/White [**2013**]) Let γ represent genus. The Graham-Pollak Theorem guarantees that $K_{10} \nrightarrow 5\left[K_{3,3}\right]$. Use this to show that K_n decomposes into at most $\gamma(K_n)$ nonplanar graphs.

(22) (Bickle/White [**2013**]) Let γ represent genus.
   (a) Let $\left\{G,\overline{G}\right\}$ be a 2-decomposition of order n. Let the factors have h K_2-components together and j components with orders $n_i \geq 3$ together. Show that $\gamma(G)+\gamma\left(\overline{G}\right) \geq j+\frac{1}{6}\left(\binom{n}{2}-h\right)-\frac{1}{2}\sum_{i=1}^{j} n_i$.
   (b) Show that for $n \geq 4$, there is a 2-decomposition that minimizes $\gamma(G)+\gamma\left(\overline{G}\right)$ for which both factors are connected.
   (c) Show that for $n \geq 3$, any 2-decomposition $\left\{G,\overline{G}\right\}$ has $\gamma(G)+\gamma\left(\overline{G}\right) \geq \left\lceil \frac{1}{12}\left(n^2-13n+24\right)\right\rceil$.

(23) (Jaeger/Payan [**1972**], Borowiecki [**1976**]) Let γ represent domination number. Show that $\gamma(G)+\gamma\left(\overline{G}\right) \geq 2$, and characterize the 2-decompositions with $\gamma(G)+\gamma\left(\overline{G}\right) = 2$ and $\gamma(G)+\gamma\left(\overline{G}\right) = 3$.

(24) (Borowiecki [**1976**], Cockayne/Hedetniemi [**1977**]) Let γ represent domination number. Show that $\gamma(G) + \gamma\left(\overline{G}\right) \leq n + 1$, and characterize the 2-decompositions with $\gamma(G) + \gamma\left(\overline{G}\right) = n + 1$.

(25) + (Jaeger/Payan [**1972**], Borowiecki [**1976**], Payan/Xuong [**1982**]) Let γ represent domination number. Show that $\gamma(G) \cdot \gamma\left(\overline{G}\right) \leq n$, and show that equality holds exactly when $\left\{\gamma(G), \gamma\left(\overline{G}\right)\right\} \in \left\{\{1, n\}, \left\{2, \frac{n}{2}\right\}\right\}$.

(26) (Cockayne/Daves/Hedeetniemi [**1980**]) Let G and $\overline{G}$ have no isolated vertices. Show that $\gamma_t(G) + \gamma_t\left(\overline{G}\right) \leq n + 2$, with equality exactly for $\left\{\frac{n}{2}K_2, \overline{\frac{n}{2}K_2}\right\}$.

(27) (Mitchem [**1971**]) Show that $a(G) + a\left(\overline{G}\right) \leq \frac{n+3}{2}$, and determine the extremal decompositions.

(28) (Mitchem [**1971**]) Show that $a(G) \cdot a\left(\overline{G}\right) \geq \frac{n}{4}$.

(29) (Bondy [**1968**], Bosak/Rosa/Znam [**1966**]) Let G and $\overline{G}$ be connected, and let $n \geq 6$. Show that $4 \leq \operatorname{diam}(G) + \operatorname{diam}\left(\overline{G}\right) \leq n + 1$, and both bounds are sharp.

(30) (Bosak [**1990**]) Let G and $\overline{G}$ be connected, and let $n \geq 4$. Show that $4 \leq \operatorname{rad}(G) + \operatorname{rad}\left(\overline{G}\right) \leq \frac{n+4}{2}$, and both bounds are sharp.

(31) (Xu [**1987**]) Let G and $\overline{G}$ both contain cycles, and let $n \geq 6$. Show that $6 \leq \operatorname{girth}(G) + \operatorname{girth}\left(\overline{G}\right) \leq n + 3$. Characterize the extremal decompositions for the upper bound.

(32) (Xu [**1987**]) Let G and $\overline{G}$ both contain cycles, and let $n \geq 6$. Show that $n + 2 \leq c(G) + c\left(\overline{G}\right) \leq 2n$. Show that both bounds are sharp.

(33) Show that $\alpha(k; K_n) = (k-1)n + 1$.

(34) Show that $\beta(k; K_n) = \left\lfloor \frac{n}{2} \right\rfloor \min\{k, \chi'(K_n)\}$.

(35) Show that $\Delta(k; K_n) = \binom{n}{2} - \binom{n-k}{2}$.

(36) + Show that $\chi'(k; K_n) = \Delta(k; K_n) + 1$ when $n - k \geq 2$ is even and that $\chi'(k; K_n) = \Delta(k; K_n)$ otherwise. (*Hint*: Use induction on $n + k$.)

Chapter 10

Appendices

10.1. Proofs

Why do mathematicians care so much about proofs? One reason is that a proof is the only way to be certain that a mathematical result is true. No matter how many special cases you check, without a proof you cannot be sure that the next one isn't an exception. For example, Pell's equation $x^2 - 109y^2 = 1$ has smallest positive integer solution $(x, y) = (158070671986249, 15140424455100)$. There are many examples of results that were claimed to be true but were false, or unjustified, based on flaws in their purported proofs. This occurred several times in the history of the Four Color Theorem outlined in this book.

Even if we could be certain that a result is true, mathematicians would still want to know WHY it is true. Beyond simply understanding a particular result, the proof of one result may help us to conjecture and prove new results. This is because many proofs contain techniques that can be applied to other problems. This also explains why mathematicians are interested in different proofs of the same result. One proof may generalize in a way that another does not. Some famous results like the Pythagorean Theorem have hundreds of different proofs.

Before going any further, we should clarify the names of various mathematical statements.

Result—a true mathematical statement

Theorem—a significant mathematical result with a known proof

Proposition—a less significant result with a known proof

Lemma—a result that is mainly useful for proving one or more theorems

Corollary—a result that follows easily from a theorem

The distinction between the four types of results is subjective. They may be classified differently in different texts. Some books do not use propositions at all. A lemma typically arises when some portion of a proof occurs repeatedly, at which point it makes sense to split it off as a separate result. A corollary may be a special case of a theorem, or it may follow from a slight modification of the proof of a theorem.

A **conjecture** is a mathematical statement that has been suggested to be true, but is not known to be true or false. If it is proved, it becomes a theorem. If it is disproved, it also ceases to be a conjecture.

A **proof** is a convincing argument demonstrating the truth of a theorem. It may use axioms, definitions, and previously proved theorems. What exactly constitutes a convincing argument varies somewhat depending on the audience. Proofs in mathematical journals are written for professional mathematicians, and they tend to be written more tersely and assume more knowledge from the reader. Students using this textbook should try to write proofs that can be understood by fellow students. Justifications of truly obvious statements may be omitted. However, it is better to err on the side of providing too much justification rather than too little, since omitting essential justification is worse than including unnecessary information.

There are several basic proof techniques. The rest of this appendix provides brief explanations and examples of the most common techniques.

10.1.1. Direct Proof. A **direct proof** is something of a default option that does not use techniques such as contradiction or contrapositive. Simpler examples have each statement follow clearly from the previous, but direct proofs can still be quite difficult.

Many theorems are **conditionals** (or **implications**) that have the form "if P, then Q" (symbolically, $P \Rightarrow Q$). A direct proof would begin by assuming P and conclude with Q, using as many intermediate steps as necessary. If it is not immediately clear what to do, a good idea is to start by rereading (or writing) all definitions and theorems that relate to the statement that you are asked to prove. This may help to piece together the steps of the proof.

Example. Let a, b, and $m > 0$ be integers. Show that if $a \equiv b \bmod m$, then $ka \equiv kb \bmod m$.

Strategy. Recall (or look up) the definition of congruence: a **is congruent to** $b \bmod m$ ($a \equiv b \bmod m$) if $m \mid (a-b)$. Then we need the definition of divisibility: r **divides** s ($r \mid s$) if $s = kr$ for some integer k. Working forward, we can start by assuming the hypothesis and translate it into an algebraic equation ($a \equiv b \bmod m \Rightarrow m \mid (a-b) \Rightarrow a - b = km$).

We can also work backwards from the conclusion, surmising what must be true for it to hold. We find $ka \equiv kb \bmod m \Leftarrow m \mid (ka - kb) \Leftarrow ka - kb = lm$. Finally, we put the pieces together, using a little algebra in the middle. We must be careful not to use the same variable for more than one purpose.

Proposition 10.1. *Let a, b, and $m > 0$ be integers. If $a \equiv b \bmod m$, then $ka \equiv kb \bmod m$.*

Proof. Assume $a \equiv b \bmod m$. Then $m \mid (a-b)$, so $a - b = lm$, where l is an integer. Then $ka - kb = klm$, so $m \mid (ka - kb)$, so $ka \equiv kb \bmod m$. $\square$

In the past, it was common to end a proof with QED, short for "*quod erat demonstrandum*", a Latin term meaning roughly "which (is what) was to be shown". Now it is more common to end proofs with a special symbol such as $\square$ or something similar.

Mathematicians conducting research often conjecture statements that they wish to prove or disprove. A student who is new to proofs may be tempted to provide an example as a proof.

Example. Prove or disprove: If n an integer, then $5n-1$ is odd.

Bad Proof: If $n=2$, then $5n-1=9$, so the statement is true.

Disproof: If $n=3$, then $5n-1=14$, so the statement is false.

Note that a single example does not prove the statement, since it is claimed for all integers, not just one. However, a single example suffices to disprove the claim that it is true for all integers.

Working one or more examples may be a good strategy, since it may reveal a pattern that will lead to a proof. A proof of a statement about a set of many objects requires a **general example**, that is, an example that does not assume anything more than the hypothesis.

Example. Prove that if n is even, then $3n+5$ is odd.

Strategy. A number n is even if $n=2k$, where k is some integer. A number n is odd if $n=2k+1$, where k is some integer. We begin by assuming that n is even and see what this implies about $3n+5$.

Proposition 10.2. *If n is even, then $3n+5$ is odd.*

Proof. Assume n is even, so $n=2k$, $k\in\mathbb{Z}$. Then $3n+5=3(2k)+5=6k+5=2(3k+2)+1$. Since $3k+2$ is an integer, $3n+5$ is odd. $\square$

Sometimes a proof uses multiple cases.

Example. Show that for every integer n, n^2-5n is even.

Strategy. It is not immediately clear how to prove this. Perhaps an additional assumption would help. We could assume that n is even or odd. These two cases cover all integers, so if the conclusion holds in both cases, it holds in general.

Proposition 10.3. *For every integer n, n^2-5n is even.*

Proof. *Case* 1. Let n be even, so $n=2k$. Then $n^2-5n=(2k)^2-5(2k)=4k^2-10k=2\left(2k^2-5k\right)$, so n^2-5n is even.

Case 2. Let n be odd, so $n=2k+1$. Then $n^2-5n=(2k+1)^2-5(2k+1)=4k^2+4k+1-10k-5=2\left(2k^2-3k-2\right)$, so n^2-5n is even.

Thus in either case, n^2-5n is even. $\square$

Alternatively, we could note that $n^2-5n=n(n-5)$. Thus one factor must be even and the other odd, so their product is even. But this still implicitly uses cases since we don't know which factor is which.

Next we consider a direct proof in graph theory.

Example. Show that for every connected graph, $\operatorname{diam}(G)\le 2\operatorname{rad}(G)$.

Strategy. We start by recalling relevant definitions and facts.

- The **distance** between vertices u and v, $d(u,v)$, is the length of the shortest $u-v$ path.
- A $u-v$ **geodesic** is a $u-v$ path of length $d(u,v)$.
- The **diameter** of a graph G, $\operatorname{diam}(G)$, is the maximum length of a geodesic in G.
- The **eccentricity** of a vertex $e(v)$ is the maximum length of a geodesic starting at v.
- The **radius** of a graph G, $\operatorname{rad}(G)$, is the minimum eccentricity of its vertices.
- Distance satisfies the triangle inequality $d(u,w) \leq d(u,v) + d(v,w)$ since concatenating the $u-v$ and $v-w$ paths produces a $u-w$ walk, which contains a $u-w$ path by Lemma 1.40.

We then consider a general example of two vertices u and v at the ends of a geodesic of maximum length and apply the definitions, obtaining a chain of inequalities.

Proposition 10.4. *For every connected graph,* $\operatorname{diam}(G) \leq 2\operatorname{rad}(G)$.

Proof. Let u and v be vertices with $d(u,v) = \operatorname{diam}(G)$, and let w be a vertex with $e(w) = \operatorname{rad}(G)$. By the triangle inequality,

$$\operatorname{diam}(G) = d(u,v) \leq d(u,w) + d(w,v) \leq 2e(v) = 2\operatorname{rad}(G). \qquad \square$$

10.1.2. Proof by Contrapositive. Given an implication "if P, then Q", its **contrapositive** is "if not Q, then not P". An implication and its contrapositive are logically equivalent, which can be verified with a truth table. When proving an implication, the contrapositive should be checked whenever the proof is not immediately obvious. Some statements that are difficult to prove directly have easy proofs by contrapositive. In this text, we indicate a proof by contrapositive with "(contrapositive)" beginning the proof, leaving the statement of the contrapositive to the reader.

Example. Show that if a^2 is even, then a is even.

Strategy. We could let $a^2 = 2k$. But then it is not clear what to do, unless we have results from number theory about unique factorization of integers. Instead, try the contrapositive, which allows us to start with the simpler expression. The contrapositive is "If a is odd, then a^2 is odd".

Lemma 10.5. *If a^2 is even, then a is even.*

Proof (contrapositive). Let a be odd, so $a = 2k+1$. Then $a^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2\left(2k^2 + 2k\right) + 1$. Thus a^2 is odd. $\square$

For the implication "if P, then Q", its **converse** is "if Q, then P". The converse of a true statement need not be true. For example, the statement "if you live in Kalamazoo, then you live in Michigan" is true, but the converse statement "if you live in Michigan, then you live in Kalamazoo" is false.

Sometimes the converse of a true statement is also true. Then the **biconditional** "if P, then Q and if Q, then P" is true. In this case, we write "P if and only if Q", sometimes abbreviated "P iff Q" (symbolically, $P \Leftrightarrow Q$). To prove a biconditional, we must prove both directions. We need two proofs in one, which may use difference techniques. The forward direction $P \Rightarrow Q$ is called **necessity**; we write ($\Rightarrow$) at its beginning. The backward direction $Q \Rightarrow P$ is called **sufficiency**; we write ($\Leftarrow$) at its beginning.

Example. Show that n is even if and only if $3n - 7$ is odd.

Strategy. The forward direction is straightforward. We assume n is even, and plug $2k$ into $3n - 7$. For the converse, we could assume that $3n - 7 = 2k + 1$, but it is unclear where to go from there. Instead, we prove the contrapositive: if n is odd, then $3n - 7$ is even. We then begin by assuming n is odd and plug $2k + 1$ into $3n - 7$.

Proposition 10.6. *An integer n is even if and only if $3n - 7$ is odd.*

Proof. ($\Rightarrow$) Assume n is even, so $n = 2k$. Then $3n - 7 = 3(2k) - 7 = 6k - 7 = 2(3k - 4) + 1$. Thus $3n - 7$ is odd.

($\Leftarrow$) (contrapositive) Assume n is odd, so $n = 2k+1$. Then $3n-7 = 3(2k + 1) - 7 = 6k - 4 = 2(3k - 2)$. Thus $3n - 7$ is even. □

Care must be taken when negating statements containing quantifiers. The following table shows the correct negations of such statements.

Statement	Negation
for all x, $P(x)$	there exists x, such that not $P(x)$
there exists x such that $P(x)$	for all x, not $P(x)$

Example. Prove the Pigeonhole Principle: If more than nk objects are distributed into n boxes, then some box has at least $k + 1$ objects.

Strategy. The word "some" indicates an existential quantifier. The contrapositive of the statement is "If every box has at most k objects, then at most nk objects are distributed into n boxes." The proof of the statement is a simple consequence of a counting rule.

Theorem 10.7 (**The Pigeonhole Principle**). *If more than nk objects are distributed into n boxes, then some box has at least $k + 1$ objects.*

Proof (contrapositive). If every box has at most k objects, there are at most nk objects in n boxes. □

10.1.3. Proof by Contradiction. The old saying that "you can't prove a negative" is certainly not true in general. In fact, any true statement can be expressed as a negation, since A and "not not A" are logically equivalent. However, when it is claimed that something does not exist, this is difficult to prove directly. A **proof by contradiction** of not A uses the following steps.

(1) Begin by assuming A.

(2) Together with other known mathematical truths, deduce a contradiction.

(3) This implies that the assumption A must be false, so not A is true.

We indicate a proof by contradiction by writing "Assume to the contrary..." prior to the false assumption. Contradiction is also called "*reductio ad absurdum*", that is, reduction to absurdity.

Example. Prove that there are infinitely many prime numbers.

Strategy. Since infinite means "not finite", proof by contradiction is a reasonable strategy. We assume that there are finitely many primes and try to show that there must be another prime.

Proposition 10.8. *There are infinitely many prime numbers.*

Proof. Assume to the contrary that there are finitely many prime numbers $p_1, \ldots, p_r$. Let $P = p_1 \cdot \ldots \cdot p_r + 1$. Now none of $p_1, \ldots, p_r$ can be factors of P, so it must be prime. This is a contradiction, so there are infinitely many prime numbers. $\square$

Note that given primes $p_1, \ldots, p_r$, it is not the case that $p_1 \cdot \ldots \cdot p_r + 1$ must be prime, only that all of its prime factors are not amongst $p_1, \ldots, p_r$. Note that this is an example of an **existence proof**—it shows that infinitely many primes exist, but does not provide a list of these primes. In fact, only finitely many primes are known at present.

Example. Prove that $\sqrt{2}$ is irrational.

Strategy. Since irrational means "not rational", try proof by contradiction. We assume that $\sqrt{2}$ is rational. Since any rational number has multiple fractional representations, we are free to also assume that the fraction is reduced (has no common factors). We then try some algebra and look for a contradiction.

Theorem 10.9. *The number $\sqrt{2}$ is irrational.*

Proof. Assume to the contrary that $\sqrt{2}$ is rational, so $\sqrt{2} = \frac{a}{b}$, $a, b \in \mathbb{Z}$, and the fraction is reduced. Then $2 = \frac{a^2}{b^2}$, so $2b^2 = a^2$. Thus a^2 is even, so by Lemma 10.5, a is even. Thus $a = 2k$, $k \in \mathbb{Z}$, so $2b^2 = (2k)^2$, and $b^2 = 2k^2$. Thus b^2 is even, so by Lemma 10.5 again, b is even. But then a and b are both even, so the fraction is not reduced. This is a contradiction, so $\sqrt{2}$ is irrational. $\square$

While proof by contradiction is logically sound, such proofs are often somewhat unsatisfying. This is because some or all of the statements in the proof are untrue, so it may not give as clear an answer as to WHY a statement is true.

Some proofs can be rewritten to avoid the use of contradiction. An implication that can be proved using contradiction can also be proved using contrapositive.

Example. Prove that if $3a + 1$ is odd, then a is even.

Here is a proof by contradiction:

Proposition 10.10. *If $3a + 1$ is odd, then a is even.*

Proof. Assume that $3a + 1$ is odd, and to the contrary, that a is odd. Then $a = 2k + 1$, so $3a + 1 = 3(2k + 1) + 1 = 6k + 4 = 2(3k + 2)$. Thus $3a + 1$ is even, a contradiction. $\square$

Here is a proof by contrapositive:

Proof (contrapositive). Assume that a is odd. Then $a = 2k + 1$, so $3a + 1 = 3(2k + 1) + 1 = 6k + 4 = 2(3k + 2)$. Thus $3a + 1$ is even. □

Note that the proof by contrapositive is shorter because it eliminates the unnecessary contradiction. When an unnecessary contradiction can be eliminated from a proof, it should be.

If we assume that there is a counterexample to some statement, we may be able to make an additional assumption. If the examples are positive integers, or have some parameter that has positive integer values, then there must be some counterexample that has a minimum value. (This follows from the **Well-ordering Principle**, which says that any subset of the positive integers has a minimum value.) One can then find a contradiction by showing that there must be a smaller counterexample. This method is known as **proof by minimum counterexample**. In graph theory, order, size, or some other parameter may be minimized.

10.1.4. Mathematical Induction. If there is a long line of dominoes set up, what will guarantee that they all fall over? There are two requirements—that the first domino falls, and that the dominoes are close enough together that each one knocks over the next one. This is the intuition behind mathematical induction.

Theorem 10.11 (**Principle of Mathematical Induction**). *If $P(n)$ is a statement defined on the positive integers, it can be proved for all n by showing*

(1) $P(1)$ *is true.*

(2) *for any $k \in \mathbb{N}$, if $P(k)$ is true, then $P(k+1)$ is true.*

This can be proved using the Well-ordering Principle. In fact, it is logically equivalent to it. The following steps are a guide to using the Principle of Mathematical Induction.

(1) Prove the **base case** $P(1)$ (we could start with a different integer).

(2) Assume the **induction hypothesis** $P(k)$ (for some fixed k).

(3) Consider $P(k+1)$. Reduce it to $P(k)$, and use this to prove $P(k+1)$.

(4) Conclude that $P(n)$ holds in general.

Example. Show that the sum of the first n positive integers is $\frac{n(n+1)}{2}$.

Strategy. The sum of the first $k + 1$ integers can be broken into the sum of the first k and one more term. This reduces $P(k+1)$ to $P(k)$, so the induction hypothesis can be used (in the second equality in the chain below).

Proposition 10.12. *The sum of the first n positive integers is $\frac{n(n+1)}{2}$.*

Proof. We use induction on n.

Base case: For $n = 1$, $1 = \frac{1 \cdot 2}{2}$.

Induction: Assume that for some fixed $k \geq 1$, $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$. Now $\sum_{i=1}^{k+1} i = k + 1 + \sum_{i=1}^{k} i = k + 1 + \frac{k(k+1)}{2} = \frac{(k+1)(k+2)}{2}$. Thus the sum of the first n positive integers is $\frac{n(n+1)}{2}$. □

Induction can also be used when the base case is not $P(1)$.

Example. Show that for $n \geq 4$, $n! > 2^n$.

Strategy. Recall that $(k+1)! = (k+1) \cdot k!$ and $2^{k+1} = 2 \cdot 2^k$. These observations will be used in the induction step to reduce $P(k+1)$ to $P(k)$.

Proposition 10.13. *For $n \geq 4$, $n! > 2^n$.*

Proof. We use induction on n.

Base case: When $n = 4$, $4! = 24 > 16 = 2^4$.

Induction: Assume that for some fixed $k \geq 4$, $k! > 2^k$. Now $(k+1)! = (k+1) \cdot k! > (k+1)\, 2^k > 2 \cdot 2^k = 2^{k+1}$. Thus for $n \geq 4$, $n! > 2^n$. □

Example. Show that for $n \geq 0$, 3 divides $4^n - 1$.

Strategy. The induction step can begin by writing out the definition of "divides" and noting that $4^{k+1} = 4 \cdot 4^k$, which provides a place to use the induction hypothesis.

Proposition 10.14. *For $n \geq 0$, 3 divides $4^n - 1$.*

Proof. We use induction on n.

Base case: Certainly for $n = 0$, $3 \mid \left(4^0 - 1\right)$.

Induction: Assume that for some fixed $k \geq 0$, $3 \mid \left(4^k - 1\right)$, so $4^k - 1 = 3x$, $x \in \mathbb{N}$. Now $4^{k+1} - 1 = 4 \cdot 4^k - 1 = 4(3x+1) - 1 = 3(4x+1)$. Thus $3 \mid \left(4^{k+1} - 1\right)$, so $3 \mid (4^n - 1)$ for $n \geq 0$. □

Each of the preceding examples uses only the previous value of k in the induction step. But for some proofs, we need more. Strong induction allows us to assume all previous values of n, not just one. This can also allow us to avoid using the dummy variable k in proofs, which often makes them simpler.

Theorem 10.15 (**Strong Induction**). *If $P(n)$ is a statement defined on the positive integers, it can be proved by showing*

(1) *$P(1)$ is true.*

(2) *for all $n > 1$, if $P(k)$ is true for $1 \leq k < n$, then $P(n)$ is true.*

The earlier form of induction is sometimes called **weak induction** in distinction with strong induction. Despite the name, the two ideas are logically equivalent. For many mathematicians, it is traditional to use weak induction when it is sufficient and to reserve strong induction for when it is necessary. However, this is not required. The following steps are a guide to using Strong Induction.

(1) Prove the base case $P(1)$ (we could start with a different integer).

(2) Assume the induction hypothesis $P(k)$ (for all k with $1 \leq k < n$).

(3) Consider $P(n)$. Reduce it to one or more previous cases, and use this to prove $P(n)$.

Example. Show that any positive integer can be expressed as a product of prime numbers.

Strategy. This is half of the **Fundamental Theorem of Arithmetic** (the other half is that the factorization is unique). Since a number is either prime or composite, we consider two cases. Since the sizes of the factors of the number are unknown, strong induction is necessary.

Theorem 10.16 (**Fundamental Theorem of Arithmetic**). *Any integer $n > 1$ can be expressed as a product of prime numbers.*

Proof. We use induction on n.

Base case: Since 2 is prime, it is a product of primes.

Induction: Assume that all integers k with $2 \leq k < n$ can be expressed as a product of prime numbers. If n is prime, the result is immediate. If n is composite, $n = a \cdot b$, $2 \leq a \leq b < n$. By the induction hypothesis, both a and b can be expressed as products of prime numbers. Thus n can be also. □

Example. A jigsaw puzzle is assembled by successively linking groups of pieces. One piece may be added at a time, or larger groups of connected pieces may be linked together. Show that no matter how a puzzle with n pieces is assembled, $n-1$ links must be made.

Strategy. Since groups of pieces of any size may be linked, Strong Induction is necessary. We assume the formula holds for each smaller group and add these quantities and one more link.

Proposition 10.17. *No matter how a puzzle with n pieces is assembled, $n-1$ links must be made.*

Proof. We use Induction on n, the number of puzzle pieces.

Base case: If there is one piece, zero links are required.

Induction: Assume that any puzzle with k pieces, $1 \leq k < n$, requires $k-1$ links no matter how it is assembled. Consider a puzzle with n pieces, split into groups of size r and $n-r$. Then $(r-1)+(n-r-1)+1 = n-1$ links are required to assemble the puzzle. □

In graph theory, there are many theorems in which the induction step involves removing a portion of the graph of unspecified size or splitting it into pieces of unspecified size. In these theorems, Strong Induction is essential.

Example. Show that every $u-v$ walk contains a $u-v$ path.

Strategy. A walk allows vertices and edges to be repeated; a path does not. Thus we may be able to start with a walk and delete some portion of it to obtain a path. Since there is no way to know in general how much must be deleted, Strong Induction is essential.

Lemma 10.18. *Every $u-v$ walk contains a $u-v$ path.*

Proof. We use induction on the length l of the walk.

Base case: A walk of length 0 is a single vertex, which is trivially a path.

Induction: We assume that any $u-v$ walk of length k, $0 \leq k < l$, contains a $u-v$ path. Let W be a $u-v$ walk of length l. If W is a path, we a done. If not, W

must repeat some vertex w. Deleting all the vertices and edges on the walk between the occurrences of w produces a shorter $u-v$ walk. By the induction hypothesis, this walk contains a $u-v$ path, so W does also. □

Any proof by induction can be converted into a proof by minimum counterexample. The negation of the (weak) induction hypothesis "for any $k \in \mathbb{N}$, if $P(k)$ is true, then $P(k+1)$ is true" is "for some k, $P(k)$ is true and $P(k+1)$ is false". Thus we can construct a proof by contradiction by assuming we have a minimum counterexample and reworking the induction step to show that there must be a smaller counterexample (or otherwise finding a contradiction). Next we reprove the statement from the previous example using proof by minimum counterexample.

Lemma 10.19. *Every $u-v$ walk contains a $u-v$ path.*

Proof. Assume to the contrary that there is a $u-v$ walk that does not contain a $u-v$ path, and let W be such a walk with minimum length. Then W must repeat some vertex w. Deleting all the vertices and edges on the walk between the occurrences of w produces a shorter $u-v$ walk. Either it is a $u-v$ path or it is a smaller counterexample. Either way, there is a contradiction. □

Some graph classes have operation characterizations. Such characterizations can be proved using induction. They also facilitate proofs by induction, as a statement can be proved for all graphs in the class if it satisfies the base case(s) and the operation.

10.1.5. Tips for Writing Proofs. There is no general method for constructing a proof. However, there are some techniques that are often helpful. The following list contains some of these tips.

- Start be rereading (or writing) all definitions and theorems that relate to the statement that you are asked to prove.
- Try to reason forward, piecing together available definitions and theorems.
- Try to reason backward from the conclusion, inferring what must be true for the conclusion to hold.
- Consider writing a proof for a special case. Try to generalize this to a proof that works for all values or objects.
- If an argument only works in some cases, try using multiple cases to complete the proof.
- If the statement is an implication, write the contrapositive and consider whether proving it is easier.
- Consider a proof by contradiction, especially when asked to prove that something does not exist.
- Try a proof by minimum counterexample to more easily find a contradiction.
- When the statement is integer-valued, try a proof by induction.
- To show that two sets are equal, $S = T$, consider showing that $S \subseteq T$ and $T \subseteq S$.

- To show that two quantities are equal, $a = b$, consider showing that $a \leq b$ and $b \leq a$.
- To show that a quantity is unique, assume that there are two and prove that they must be identical.

Various mathematical disciplines have more specialized proof techniques. The combinatorial proof techniques of **Counting Two Ways** and **Counting by Bijection** are covered in the next appendix. For graph theory, some tips include the following.

- Draw a picture to illustrate that statement that you are asked to prove.
- Prove the statement for a familiar graph or class of graphs. Try to generalize this to a proof for all graphs in the statement.
- Assume that a graph or subgraph is extreme (minimal or maximal) in some way, and use this to derive a contradiction.

Writing good proofs takes practice. One final tip is to read the proofs written by professional mathematicians and try to imitate their writing style.

Exercises. All variables in the Exercises for this section are integers, unless otherwise stated.

(1) Prove that if n is even, then $4n - 3$ is odd.

(2) Prove that if n is odd, then $3n + 11$ is even.

(3) Prove that if n is an integer, $n^2 + n$ is even.

(4) Prove that if n is an integer, $n^2 + n + 1$ is odd.

(5) Prove that $6 \mid \big(n^3 - n\big)$.

(6) Prove that $12 \mid \big(n^4 - n^2\big)$.

(7) Prove that if $a \mid b$ and $b \mid c$, then $b \mid c$.

(8) Prove that if $a \mid b$ and $a \mid c$, then $a \mid (kb + lc)$.

(9) Prove that congruences are
   (a) reflexive ($a \equiv a \bmod m$).
   (b) symmetric (if $a \equiv b \bmod m$, then $b \equiv a \bmod m$).
   (c) transitive (if $a \equiv b \bmod m$ and $b \equiv c \bmod m$, then $a \equiv c \bmod m$).

(10) Let $a \equiv b \bmod m$ and $c \equiv d \bmod m$. Prove that
   (a) $a + c \equiv b + d \bmod m$.
   (b) $a - c \equiv b - d \bmod m$.
   (c) $ac \equiv bd \bmod m$.

(11) Prove or disprove: If $ac \equiv bc \bmod m$, then $a \equiv b \bmod m$.

(12) Prove that if $a \equiv b \bmod m$, then $a^k \equiv b^k \bmod m$.

(13) Prove that if $7n - 5$ is even, then n is odd.

(14) Prove that if $5n + 9$ is odd, then n is even.

(15) Prove that if n is even if and only if $5n - 3$ is odd.

(16) Prove that if n is odd if and only if $7n + 5$ is even.

(17) Prove that a^2 is even if and only if a is even.

(18) Prove that a^3 is even if and only if a is even.

(19) Use the Pigeonhole Principle to show that any set of $n+1$ integers from $[2n]$ contains two relatively prime integers.

(20) + Use the Pigeonhole Principle to show that any set of $n+1$ integers from $[2n]$ contains two integers so that one divides another.

(21) Prove that if $2^m - 1$ is prime, then m is prime.

(22) Prove that every integer $n > 1$ has a prime divisor.

(23) Prove that any composite integer $n > 1$ has a prime factor at most $\sqrt{n}$.

(24) (a) Prove that the product of two numbers congruent to 1 mod 4 is also congruent to 1 mod 4.
(b) Prove that there are infinitely many primes of the form $4k+3$.

(25) The **division algorithm** states that, given integers a and $b \neq 0$, there exist unique integers q and r with $a = bq + r$ and $0 \leq r < b$.
(a) Prove the existence portion of this theorem.
(b) Prove the uniqueness portion of this theorem.

(26) Prove that $\sqrt{3}$ is irrational.

(27) Prove that if p is prime, $\sqrt{p}$ is irrational.

(28) Suppose that we mimic the proof that $\sqrt{2}$ is irrational but replace 2 with 4. Where does the proof fail?

(29) Show that the product of an irrational number and a nonzero rational number is irrational.

(30) Show that the quotient of an irrational number and a nonzero rational number is irrational.

(31) Prove the Principle of Mathematical Induction from the Well-ordering Principle.

(32) Prove the Well-ordering Principle from the Principle of Mathematical Induction.

(33) Prove the Principle of Strong Induction from the Well-ordering Principle.

(34) Prove that the Principle of Mathematical Induction and Strong Induction are logically equivalent.

(35) Prove that the sum of the n smallest positive odd integers is n^2.

(36) State and prove a formula for the sum of the smallest n positive even integers.

(37) Use induction to prove the formula for the sum of a geometric series: $a + ar + ar^2 + \cdots + ar^n = \frac{ar^{n+1}-a}{r-1}$, where a and r are real numbers.

(38) Prove that $\sum_{i=1}^{n} i \cdot i! = (n+1)! - 1$.

(39) Prove that $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.

(40) Prove that $\sum_{i=1}^{n} i^3 = \left(\frac{k(k+1)}{2}\right)^2$.

(41) Prove that $2^n \geq n^2$ for $n \geq 0$, $n \neq 3$.

(42) Prove that $n! > 3^n$ for all sufficiently large values of n.

(43) The **Fibonacci numbers** are defined as $f_1 = f_2 = 1$ and $f_n = f_{n-1} + f_{n-2}$. Show that $f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$.

(44) Show that for the Fibonacci numbers, $\sum_{k=1}^{n} f_k = f_{n+2} - 1$.

(45) Define a sequence by $a_1 = 1$, and $a_2 = 3$, and $a_n = 2a_{n-1} + a_{n-2}$. Show that all terms of this sequence are odd.

(46) Define a sequence by $a_1 = 1$, and $a_2 = 8$, and $a_n = a_{n-1} + 2a_{n-2}$. Show that $a_n = 3 \cdot 2^{n-1} + 2\,(-1)^n$.

(47) The **Fermat number** F_n is $F_n = 2^{2^n} + 1$.
  (a) Show that $F_0 F_1 \cdot \ldots \cdot F_{n-1} = F_n - 2$.
  (b) Show that for $n > m \geq 0$, F_n and F_m are relatively prime.
  (c) Use the previous parts to show that there are infinitely many primes.

(48) Show that every positive integer n has a binary expansion $d_r d_{r-1} \cdots d_1 d_0$, where $d_i \in \{0, 1\}$.

(49) A chocolate bar consists of unit squares arranged in a $p \times q$ rectangular grid. You may split the bar into individual unit squares, by breaking along the lines. Show that $pq - 1$ breaks are required.

(50) Show that a $2^n \times 2^n$ checkerboard with one square removed can be tiled with triominoes (three squares in an L shape).

(51) A country has n cities. Any two cities are connected by a one-way road. Show that there is a route that passes through all n cities.

(52) Suppose that n people all know some information that none of the others know. The **gossip problem** asks for the minimum number of two-person telephone calls that will lead to everyone learning all of the information. Prove that for $n \geq 4$, this number is at most $2n - 4$. (*Note*: In fact, it is equal to $2n - 4$.)

(53) Suppose there are n lines in the plane such that every two have a common point and no three have a common point. Prove that they divide the plane into $1 + \frac{n(n+1)}{2}$ regions.

(54) (a) + Show that any polygon with at least four sides has an interior diagonal.
  (b) Use part (a) to show that a simple polygon with $n \geq 3$ sides can be triangulated into $n - 2$ triangles.

(55) Show that if a round-robin tournament has a cycle of $r \geq 3$ teams so that each defeats the next in the tournament, then it has a cycle of three teams so that each defeats the next.

(56) + An odd number of people stand at distinct distances from each other. Each person throws a water balloon at the person nearest to him. Show that someone does not have a balloon thrown at him.

(57) In the game of **Nim**, there are two players and two piles of objects, initially with equal size. The players alternate turns, and on each turn a player must remove some positive number of objects from one of the piles. The goal is to avoid taking the last object. Prove that the second player has a winning strategy.

(58) Determine what amounts of postage can be made from stamps with the following values. Use Strong Induction to prove your answer.

(a) 2 cents and 3 cents
(b) 2 cents and 5 cents
(c) 3 cents and 5 cents
(d) 4 cents and 7 cents

(59) Find the flaw in the following "proof" that all horses are the same color.

Base case: For a set of one horse, there is one color.

Induction: Assume that within any set of $n-1$ horses, there is only one color. Now look at any set of n horses. The sets $\{1, 2, \ldots, n-1\}$ and $\{2, 3, \ldots, n\}$ both contain $n-1$ horses; therefore within each there is only one color. Since the two sets overlap, so there is only one color among all n horses.

(60) Determine whether the following "proof" of the Fundamental Theorem of Arithmetic is valid.

Base case: Since 2 is prime, it has a unique expression as a product of primes.

Induction: Assume that all integers k with $2 \leq k < n$ can be expressed uniquely as a product of prime numbers. If n is prime, the result is immediate. If n is composite, $n = a \cdot b$, $2 \leq a \leq b < n$. By the induction hypothesis, both a and b can be expressed uniquely as products of prime numbers. Thus n can be also.

10.2. Counting Techniques and Identities

10.2.1. Ordered Counting. In combinatorics, we often wish to count the number of objects in a set. In graph theory, we may wish to find the number of some type of graph or the number of some type of subgraph of a graph. We begin with a basic fact.

Theorem 10.20 (The Counting Product Rule). *If there are k independent choices that can be made in n_i ways, $1 \leq i \leq k$, then there are $\prod n_i$ total ways to make the choices.*

Example. A Michigan license plate has three letters followed by four numbers. There are $26^3 10^4 = 175,760,000$ possible license plates.

Example. How many positive integer factors does 360 have?

Solution. The unique prime factorization of 360 is $2^3 3^2 5$. There are four possibilities (0, 1, 2, or 3) for how many 2's can be in a factor. Similarly, there are three possibilities for the 3's, and two for the five. Thus there are $4 \cdot 3 \cdot 2 = 24$ total factors. Note that even if we listed all 24, we could not be sure we had them all without a counting argument.

Since any positive integer has a unique prime factorization $n = p_1^{n_1} \cdots p_k^{n_k}$, the number of positive integer factors $\tau(n)$ of n is $\tau(n) = (n_1 + 1) \cdots (n_k + 1)$.

Example. How many subsets does a set with n elements have?

Solution. For each element, there are two possibilities. It is either in or out of a subset. This is true for each element, so there are 2^n subsets.

Example. There are n^{n-2} sequences of length $n-2$ with elements from $[n] = \{1, \ldots, n\}$. This is also the number of labeled trees, as there is a bijection between such sequences and labeled trees.

Suppose that we want to know how many ways there are to select k objects out of n possible objects. There are two questions we need to ask.

- Are repetitions allowed?
- Is order important?

If repetitions are allowed and order is important, then there are n possibilities for the first choice, n for the second choice, etc. Thus there are $n^k = n \cdot \ldots \cdot n$ ways to select the k objects.

Suppose repetitions are not allowed and order is important, and consider the special case $n = k$. Such an arrangement of n objects is called a **permutation**. There are n possibilities for the first choice. There are $n-1$ for the second choice since one object has already been picked and is no longer available. Continue similarly down to one possibility for the last choice. Thus there are a total of $n\,(n-1)\,(n-2)\cdots 3\cdot 2\cdot 1 = n!$ ways to order the objects.

Corollary 10.21. *There are $\frac{n!}{(n-k)!}$ ways to list k of n distinct objects. In particular, there are $n!$ permutations of n objects.*

Proof. There are $n\,(n-1)\cdots(n-k+1) = n\,(n-1)\cdots(n-k+1)\,\frac{(n-k)\cdots 3\cdot 2\cdot 1}{(n-k)\cdots 3\cdot 2\cdot 1} = \frac{n!}{(n-k)!}$ possible lists. □

Example. In a race with ten horses, there are $10\cdot 9\cdot 8 = 720$ possibilities for win, place, and show.

Factorials can be defined recursively as $0! = 1$ and $n! = n\cdot(n-1)!$ for any positive integer n. Computing the first few values, we have the following.

n	1	2	3	4	5	6	7	8
$n!$	1	2	6	24	120	720	5040	40320

It is apparent that factorials grow very quickly. **Stirling's approximation** for $n!$ is $n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$. This can be used to show that $n!$ grows faster than any exponential function, but slower than n^n.

10.2.2. Unordered Counting. What if there are repeated elements in a permutation? Suppose a of n elements are the same. If we treat each as distinct, there are $n!$ permutations. But the a elements are permuted in $a!$ ways, so each ordering occurs $a!$ times. Thus we must divide by $a!$, so there are $\frac{n!}{a!}$ distinct permutations. We can generalize this.

Theorem 10.22. *Suppose n objects with k distinct types repeated a_i times, $1 \le i \le k$, with $a_1 + \cdots + a_k = n$ are ordered. There are $\frac{n!}{a_1!\cdots a_k!}$ ways to order them.*

Example. A multiset is a collection of objects where repetition is allowed. Consider the multiset $\{A, A, A, B, B, C\}$. There are $\frac{6!}{3!2!1!} = 60$ orders for this multiset.

How many ways can k objects be chosen from a set of n objects if repetition is not allowed and order is unimportant? We call this the number of **combinations** "n choose k", with notation $\binom{n}{k}$. We have k chosen and $n-k$ unchosen elements, which justifies the following.

Corollary 10.23. *The number of ways k objects can be chosen from a set of n objects if repetition is not allowed and order is unimportant is*

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!} = \frac{n\,(n-1)\cdot\ldots\cdot(n-k+1)}{k!}.$$

Example. In a class of seven students, there are $\binom{7}{3} = \frac{7\cdot 6\cdot 5}{3\cdot 2\cdot 1} = 35$ possible groups of three students.

Example. How many labeled graphs with order n are there?

Solution. There are $\binom{n}{2}$ possible edges. Each edge can be included or not in a graph. Thus there are $2^{\binom{n}{2}}$ labeled graphs of order n.

Example. How many 3-cubes are contained in the n-cube?

Solution. Think of the vertices as bit strings. Any 3-cube must vary three of the bits. There are $\binom{n}{3}$ ways to do this. There are 2^{n-3} choices for the other bits. Thus there are $\binom{n}{3}2^{n-3}$ 3-cubes in the n-cube.

How many ways can k objects be chosen from a set of n objects if repetition is allowed and order is unimportant? Every choice of k objects from a multiset can be thought of as putting x's in bins corresponding to the elements. Equivalently, we could find the number of orderings of k x's and $n-1$ dividers. This justifies the following result.

Corollary 10.24. *The total number of unordered choices of k elements from a set of n elements, allowing repetition, is $\binom{k+n-1}{k}$.*

Suppose we want to find how many ways are there to write a positive integer N as a sum of p whole numbers. Let the p whole numbers be bins and distribute N 1's. Then there are $\binom{N+p-1}{N}$ possible sums.

Example. If $N = 4$ and $p = 3$, we have $\binom{4+3-1}{4} = \binom{6}{4} = 15$ sums.

The formulas for the number of ways to select k of n objects are summarized in the following table.

	Repetitions Allowed	No Repetitions
Order Important	n^k	$\frac{n!}{(n-k)!}$
Order Unimportant	$\binom{n+k-1}{k}$	$\binom{n}{k} = \frac{n!}{k!(n-k)!}$

10.2.3. Combinatorial Identities. Consider the expression $(x+y)^n$ (n an integer). We can use the distributive property to completely expand it out. Each term of the resulting sum has one factor being either x or y from each of the n binomials. For example,

$$(x+y)^3 = xxx+xxy+xyx+xyy+yxx+yxy+yyx+yyy = x^3+3x^2y+3xy^2+y^3.$$

Thus there will be 2^n terms, some of which will be equivalent. The term x^ky^{n-k} will occur every time exactly k of the x's are chosen from the n binomials. Thus it will occur $\binom{n}{k}$ times. This justifies the following theorem.

Theorem 10.25 (Binomial Theorem). *Let n be a nonnegative integer. Then*

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

The Binomial Theorem can also be proved by induction. It has several immediate consequences.

Corollary 10.26. *Let n be a nonnegative integer. Then* $\sum_{k=0}^{n} \binom{n}{k} = 2^n$.

Proof. Plug $x = y = 1$ into the Binomial Theorem. $\square$

Corollary 10.27. *Let n be a nonnegative integer. Then* $\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots = 2^{n-1}$.

Proof. Plug $x = -1$ and $y = 1$ into the Binomial Theorem. Then

$$0 = (-1+1)^n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \binom{n}{4} - \cdots,$$

so $\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots$. Now both sides together sum to 2^n, so each individually sums to 2^{n-1}. $\square$

The numbers $\binom{n}{k}$ are known as the **binomial coefficients**. They can be arranged as follows, where the rows give increasing values of n and the diagonals give increasing values of k.

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}$$

This is called **Pascal's triangle**. Corollary 10.26 can be interpreted as finding the sums of the rows of Pascal's triangle. There are many other identities related to Pascal's triangle and binomial coefficients. We introduce two techniques for generating and proving combinatorial identities.

10.2.4. Counting Two Ways.

Definition 10.28. A **partition** of a set S is a collection of nonempty, mutually disjoint subsets S_i of S whose union is S. The **Counting Sum Rule** says that for any partition of S, the cardinality of S is the sum of the cardinalities of the subsets in the partition, $|S| = \sum |S_i|$.

The first new technique is partitioning a set in two different ways, and using each partition to find an expression for the cardinality of the set. Since both expressions count the same cardinality, they must be equal, proving a combinatorial identity. This technique is known as **Counting Two Ways**.

Pascal's triangle can be simply generated by noting that $\binom{n}{0} = \binom{n}{n} = 1$, and each number in its interior is the sum of the two numbers diagonally above it. For example, $10 = 4 + 6$. This works because of the following identity.

Proposition 10.29 (Pascal's Formula). *For any integers $n \geq 1$ and k,* $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$.

Proof. Consider the set of all k-element subsets of $[n]$. There are $\binom{n}{k}$ of them. Partition the set into the subsets containing 1 and not containing 1. There are $\binom{n-1}{k-1}$ subsets containing 1 and $\binom{n-1}{k}$ subsets not containing 1. Summing these cardinalities proves the identity. □

This identity, and all the others in this section, can also be proved algebraically. However, there is good reason to prefer the combinatorial approach. A combinatorial argument gives a natural explanation of the result that algebra does not. Counting arguments are also a natural technique for generating new combinatorial identities. Algebra can prove existing identities, but it does not naturally suggest new identities.

The row sum identity (Corollary 10.26) proved above also has a natural combinatorial interpretation and proof.

Proposition 10.30. *Let n be a nonnegative integer. Then $\sum_{k=0}^{n} \binom{n}{k} = 2^n$.*

Proof. Consider the set of all subsets (the **power set**) of $[n]$, which has cardinality 2^n. Partition this set based on the cardinalities of the subsets. There are $\binom{n}{k}$ subsets of cardinality k, so summing from 0 to n counts all subsets. □

Another identity comes from summing a diagonal of Pascal's triangle up to a point.

Proposition 10.31. *We have $\binom{k}{k} + \binom{k+1}{k} + \binom{k+2}{k} + \cdots + \binom{n}{k} = \binom{n+1}{k+1}$.*

Proof. Consider the set of all $k+1$-element subsets of $[n+1]$, which has cardinality $\binom{n+1}{k+1}$. The largest number chosen must be between $k+1$ and $n+1$. Partition the set based on the largest number. If it is $k+i+1$, there are $\binom{k+i}{k}$ ways to choose the other k elements. Summing between k and n counts all the subsets. □

If $k = 1$, this identity reduces to $1 + 2 + 3 + \cdots + n = \binom{n+1}{2}$. This identity is a common example used to teach proof by mathematical induction. While this is a valid method of proof, it does not explain where the identity comes from or what it means, as does the counting argument.

Counting Two Ways is used to prove the First Theorem of Graph Theory.

Theorem 10.32. *For a graph G, we have $\sum d(v_i) = 2m$.*

Proof. Consider the set of all vertex-edge incidences (the "ends" of edges) in a graph. Partitioning the set by vertices shows its cardinality is the sum of the degrees. Partitioning the set by edges shows each edge appears twice, so its cardinality is $2m$. Thus $\sum d(v_i) = 2m$. □

10.2.5. Counting by Bijection. In this counting technique, we find a bijection between the elements of two distinct finite sets. This shows that they have the same cardinality. If we can count both sets, we establish a combinatorial identity. If we can count only one of the sets, we find a formula for the other set's cardinality.

This technique is different from counting two ways. In that technique, we partition one set in two different ways; in counting by bijection, we find a bijection between two (usually) different sets. In the first technique, the problem is finding the partitions; in the second, the problem is finding the bijection and then counting one or both sets.

Proposition 10.33. *Pascal's triangle is symmetric:* $\binom{n}{k} = \binom{n}{n-k}$.

Proof. Establish a bijection between the set of k-element subsets of $[n]$ and the set of $n-k$-element subsets of $[n]$ where each set is mapped to its complement in $[n]$. The first set has $\binom{n}{k}$ elements, and second has $\binom{n}{n-k}$ elements. The bijection shows these quantities are equal. □

The alternating sum identity for Pascal's triangle also has a combinatorial proof.

Proposition 10.34. *We have* $\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots$.

Proof. Consider the set of subsets of $[n]$ with even cardinality and the set of subsets of $[n]$ with odd cardinality. They have cardinalities $\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots$ and $\binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots$, respectively. Establish a bijection between them by deleting n from a subset if it appears and adding it if it does not. This function always changes the cardinality by one and it is its own inverse, so it is a bijection. □

Counting by bijection is a common tool in graph theory. A bijection between the set of labeled trees and the set of Prufer codes is used to count labeled trees. When some class of graphs has large sizes, it may be easier to consider the bijection to their complements and count them instead. Bijections can also be used to count subgraphs within a graph.

Proposition 10.35. *The Petersen graph contains fifteen 8-cycles.*

Proof. Any 8-cycle omits two vertices. They must be adjacent or else they would have a common neighbor that could not be on the cycle. Deleting two adjacent vertices from the Petersen graph results in a graph that clearly contains a single 8-cycle. The Petersen graph has fifteen edges, and hence fifteen pairs of adjacent vertices, so it has fifteen 8-cycles. □

Many graph theory counting problems appear throughout the Exercises of this text.

Related Terms: Inclusion-Exclusion Formula, derangement, partition number, Bell number, Catalan number, Stirling number, generating function, Euler's phi function.

Exercises.

(1) Simplify $\frac{(n+1)!}{n!}$.
(2) Which is bigger, $(2n)!$ or $2 \cdot n!$?
(3) Is there a natural choice for the value of $(-1)!$? If so, what is it?
(4) Express the product $(2n-1)(2n-3)\cdots 5 \cdot 3 \cdot 1$ using factorials.
(5) How many different positive integer factors do the following numbers have?
    (a) 105

   (b) $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11$
   (c) 2^{10}
   (d) 120

(6) How many different positive integer factors do the following numbers have?
   (a) 48
   (b) 1729
   (c) 2^k
   (d) 10^k

(7) How many seven-digit numbers have
   (a) no 0?
   (b) no repeated digits?
   (c) no consecutive repeated digits?

(8) A math class has seven students.
   (a) How many different orders are there for three students to make class presentations?
   (b) How many different groups of three students could make a presentation?

(9) How many n-digit positive integers are there?

(10) A combination lock requires three distinct numbers out of 20 be entered. If you forgot the combination, how many possible combinations are there to try?

(11) Out of a group of ten students, how many different groups are there that could go on a field trip?

(12) Out of eight runners, how many possibilities are there for the winners of the gold, silver, and bronze medals?

(13) Find the number of five-card hands that can be selected from a standard 52-card deck.

(14) A standard 52-card deck has four suits and thirteen kinds. Find the number of five-card hands that are
   (a) a straight (five cards with consecutive kinds).
   (b) three of a kind.
   (c) a flush (all of one suit).
   (d) a full house (three of one kind, two of another).

(15) Find the number of squares (of any size) in an 8×8 chessboard.

(16) Find the number of rectangles (of any size) in an 8×8 chessboard.

(17) How many paths of length $2n$ are there from the point $(0,0)$ to (n,n) in a grid?

(18) The **Catalan numbers** are a sequence that occurs in many counting problems with formula $C(n) = \frac{1}{n+1}\binom{2n}{n}$.
   (a) Calculate $C(n)$ for $0 \leq n \leq 5$.
   (b) Use Stirling's approximation to find an approximation for $C(n)$.

(19) How many ways are there to write 5 as the sum of three whole numbers?

(20) How many ways are there to write 8 as the sum of four whole numbers?

(21) Expand the following binomials.
   (a) $(x-y)^2$

(b) $(x + 2y)^3$
(c) $\left(5 + x^2\right)^4$

(22) Expand the following binomials.
(a) $(x + 3)^3$
(b) $\left(x^2 - y^3\right)^4$
(c) $(2x + 3y)^5$

(23) Generate the next two rows of Pascal's triangle.

(24) Color the odd and even numbers in Pascal's triangle differently. What pattern do you find?

(25) Verify $\binom{n}{k} = \binom{n}{n-k}$ algebraically.

(26) Verify $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$ algebraically.

(27) Find the sums of the first ten "$\frac{1}{3}$ slope" diagonals of Pascal's triangle: 1, 1, 1+1, 1+2, 1+3+1, ... What familiar pattern do you find? Explain why this works.

(28) Show that the rows of Pascal's triangle increase from $k = 0$ to $k = \left\lfloor \frac{n+1}{2} \right\rfloor$ and decrease from $k = \left\lceil \frac{n+1}{2} \right\rceil$ to $k = n$.

(29) Partition the power set of $[n]$ according to the largest element in each subset. What combinatorial identity follows from this?

(30) Consider mapping each subset of $[n]$ to its complement. Can this be used to prove Proposition 10.34?

(31) Show that $k\binom{n}{k} = n\binom{n-1}{k-1}$ using
(a) counting two ways (*Hint*: Consider choosing committees with a chairman.)
(b) algebra.

(32) Show that $\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}$ using
(a) counting two ways. (*Hint*: Consider choosing committees with a chairman.)
(b) calculus and the Binomial Theorem.

(33) (**VanderMonde's Identity**) Show that $\binom{n+m}{k} = \sum_{i=0}^{k} \binom{n}{i}\binom{m}{k-i}$ using Counting Two Ways.

(34) Show that $\binom{n}{m}\binom{n-m}{k} = \binom{n}{k}\binom{n-k}{m}$, $k + m \leq n$, using Counting Two Ways.

(35) Show that $\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2$.

(36) + Use induction on n to prove the Binomial Theorem.

(37) In a single-elimination tournament, each team that loses a game is eliminated (there are no ties). How many games are played in a tournament with n teams?

(38) The number subsets of $[n]$ containing 1 equals the number that doesn't contain 1. Find two different bijections to prove this.

(39) + A theater sells tickets (with assigned seating) for all n seats. The first person to enter the theater loses his ticket, so he picks a seat at random. After that, each person sits in his assigned seat when open, and picks a random seat

otherwise. Determine the probability that the last person gets to sit in his assigned seat by
(a) finding a bijection with some collection of sets.
(b) considering the last person to be seated.

(40) + How many paths of length $2n$ are there from the point $(0, 0)$ to (n, n) in a grid that cross the line $y = x$? (*Hint*: Find a bijection.)

10.3. Computational Complexity

10.3.1. Introduction. Suppose a traveling salesman wants to visit several cities and return home. A natural question is what route he should take to minimize cost. In theory, there is a perfectly good answer to this question. There are $\frac{(n-1)!}{2}$ possible routes through n cities. For each route, the costs can easily be added up, and the lowest can be selected. This is an example of a **brute force algorithm**—examining every possible alternative to select the best one. But is this practical? If there were another algorithm requiring $n \log n$ or n^n computations, would that be better or worse?

In recent decades, much more attention has been paid to the issue of how efficiently problems can be solved. Theoretical mathematics often implicitly assumes that there is an infinite amount of time, space, and precision available to solve a problem. However, this is not a realistic assumption. With the growth of computer power and its use to solve mathematical problems have come increasing research into the issue of how efficiently problems can be solved, and how these issues can even be described precisely.

One issue is whether a problem can be solved at all. That is, is there an algorithm that will always give the right answer? Most problems can be expressed as decision problems.

Definition 10.36. A **decision problem** asks whether a given statement is true. If there is an algorithm to solve it, we say it is **decidable**; else it is **undecidable**.

Some problems are undecidable. One example is the **Halting Problem**, which asks whether a computer program eventually halts (stops) or continues to run forever. It would certainly be nice to know this before trying to run a program. However, no algorithm can decide this for any possible program.

When a problem is decidable, there may be many algorithms that solve it. How can we describe the efficiency of an algorithm, and compare different algorithms? Perhaps we would like a function that gives the time the algorithm takes in terms of the size of the algorithm's input. But this time will depend on the processing speed of the computer being used. Even the same computer may run more slowly or quickly depending on its age and what other programs it is running. A better goal would be a function that describes the number of calculations that an algorithm must perform, which will roughly determine how fast it will run on a given computer.

Definition 10.37. The **time complexity** of an algorithm is the number of operations used by that algorithm. The **complexity** of a problem is the minimum worst-case running time over all solution algorithms.

One could also ask about the amount of memory required by an algorithm (the **space complexity**) and the average running time, as the worst case may be unusual for some algorithms. However, this section will not address these issues. Instead, we consider how to describe and compare complexity mathematically.

10.3.2. Growth Rates of Functions. As the input to a problem grows larger, the time required to solve it will grow to infinity. There are many functions that go to infinity as their input goes to infinity, but some grow faster than others.

Definition 10.38. A function $f(x)$ **grows faster than** $g(x)$ (or g grows slower than f) if

$$\lim_{x\to\infty}\frac{f(x)}{g(x)}=\infty \quad \text{or} \quad \lim_{x\to\infty}\frac{g(x)}{f(x)}=0.$$

In this case we write $f \gg g$ or $g \ll f$.

Example. The calculations below repeatedly apply L'Hopital's Rule to show that $e^x \gg x^n$:

$$\lim_{x\to\infty}\frac{e^x}{x^n}=\lim_{x\to\infty}\frac{e^x}{nx^{n-1}}=\lim_{x\to\infty}\frac{e^x}{n(n-1)x^{n-2}}=\cdots=\infty.$$

Many common functions can be ordered by their growth rates, as follows.

Theorem 10.39.

$$\ln\ln n \ll \ln n \ll \underset{p\in(0,1)}{n^p} \ll n \ll n\ln n \ll \underset{p>1}{n^p} \ll e^n \ll n! \ll n^n \ll 2^{2^n}.$$

Proving this is a nice exercise in calculating limits. One complication is the factorial function, which is common in counting problems. Since it is nondifferentiable, it cannot be used directly. However, Stirling's approximation, $n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$, can be used for these calculations.

Additional functions can be inserted into this order. For example, additional exponential functions $(1.1)^n \ll 2^n \ll e^n \ll 3^n$ could be inserted in place of e^n. Not any function fits into this order; one could construct a function that crosses more than one of these functions infinitely many times. However, these are the most common functions describing the complexity of computer algorithms.

A function describing the complexity of an algorithm could potentially be very complicated. However, we really only need an approximation, since there is no hope of an exact answer to the time question. The following notation is called **big-O notation**.

Definition 10.40. Let f and g be functions. We say

$g \in \mathcal{O}(f)$ if $|g(x)| \le c\,|f(x)|$ for some c and sufficiently large x.

$g \in \Omega(f)$ if $|g(x)| \ge c\,|f(x)|$ for some c and sufficiently large x.

$g \in \Theta(f)$ if $g \in \mathcal{O}(f)$ and $g \in \Omega(f)$.

If $g \in \mathcal{O}(f)$, we say g is $\mathcal{O}(f)$. This means that g grows at most as fast as f. Similarly, $g \in \Omega(f)$ means g grows at least as fast as f, and $g \in \Theta(f)$ means f and g grow at the same rate.

For example, n^3, n^2, and $\ln n$ are all in $\mathcal{O}\left(n^4\right)$, but e^n is not. It is easy to show that if $f \ll g$, then $g+f \in \mathcal{O}(g)$. Thus n^2-n and n^2-3n+2 are in $\mathcal{O}\left(n^2\right)$. These functions all have the same leading term. This is what really matters, since when n gets large, all other terms become much smaller than the leading term. When describing the efficiency of algorithms, it is typical to chop off the trailing terms and just report the leading term, which is a basic function like those in Theorem 10.39.

The brute force algorithm for the Traveling Salesman Problem stated above has complexity $\Theta(n!)$. There is another algorithm for the TSP that has complexity $\Theta\left(n^2 2^n\right)$. The latter algorithm is better, but still not very good.

10.3.3. P versus NP. When the complexity of a problem does not grow too quickly, we can hope to solve it by using more resources (more time or a faster computer). However, for a problem with no efficient algorithm, using more resources will yield little progress.

Definition 10.41. A problem is **tractable** if it has polynomial complexity, $\mathcal{O}(n^p)$. Else it is **intractable**.

The class P contains all problems with polynomial time solutions. The class NP (for nondeterministic polynomial) contains all problems for which a solution can be checked in polynomial time.

For example, finding an Eulerian circuit is in P, since a polynomial algorithm exists for this problem. Finding a Hamiltonian cycle is in NP, since it is easy to verify that Hamiltonian cycle is as stated. However, this problem is not known to be in P, since finding a Hamiltonian cycle may be difficult.

Thus we have P $\subseteq$ NP. A natural question is whether P = NP. That is, do nondeterministic polynomial problems have polynomial solutions? This is perhaps the most significant unanswered question in computer science. In 2000, the Clay Mathematics Institute named it one of seven Millennium Prize Problems, with a \$1,000,000 prize for a solution. (As of 2020, only one of the seven problems has been solved.)

One major focus of research into computer algorithms is the idea that a solution to one problem could lead to a solution to other problems.

Definition 10.42. A problem is NP**-hard** if a polynomial-time algorithm for it could be used to construct a polynomial-time algorithm for each problem in NP. A problem is **NP-complete** if it belongs to NP and is NP-hard.

SATISFIABILITY is a problem that asks whether a logical expression has an assignment of truth values for its variables that makes the expression true.

SATISFIABILITY is in NP, since verifying that given truth values produce a true outcome is easy. However, a brute force algorithm requires $\Omega(2^n)$ operations for n variables. SATISFIABILITY has been shown to be an NP-hard problem. This was done by Steven Cook in [**1971**]. The next year, Richard Karp [**1972**] showed that 20 different problems are NP-complete. Since then, thousands of other problems have also been shown to be NP-complete.

Proving that a problem is NP-complete requires showing that a solution to it can be converted to a solution to a known NP-complete problem in polynomial time, and vice versa. Since the product of polynomials is a polynomial, if any of them have a polynomial-time solution, they all do. Given the number of problems involved, some of which have been studied very extensively, it seems very unlikely that such a polynomial-time algorithm exists. Thus most computer scientists believe that $\mathrm{P} \neq \mathrm{NP}$.

The difficultly in proving this is that while it is straightforward to find the efficiency of a given algorithm, it is unclear how to show that no efficient algorithm exists for a problem.

10.3.4. Complexity in Graph Theory. Graphs have two fundamental parameters, order n and size m, so the complexity of graph algorithms may be expressed in terms of either of them. The size of a complete graph is $\mathcal{O}\left(n^2\right)$, so n^2 could be substituted for m to obtain an upper bound just in terms of n. However, some graph classes, like planar graphs, have $\mathcal{O}(n)$ edges.

The number of labeled graphs of order n is $2^{\binom{n}{2}} \in \mathcal{O}\big(2^{n^2}\big)$. The number of unlabeled graphs is only slightly less than this. Even many smaller classes like trees grow exponentially or worse. Thus any brute force algorithm that depends on checking each graph will be impractical.

The same is true for any algorithm that must check every possible vertex or edge order. One could check whether a graph is Eulerian by examining all $m!$ edge orders. However, Heirholzer's algorithm, implied in the proof of Theorem 3.2, can be implemented in $\mathcal{O}(m)$ time. We have seen that a brute force approach to the TSP has complexity $\mathcal{O}(n!)$. Unlike the Eulerian circuit problem, it seems unlikely that a significantly better algorithm exists.

Decision problems are traditionally named in all caps. For example, EULERIAN CYCLE asks whether a graph has an Eulerian cycle as yes or no question. An optimization problem like TSP can be formulated as "Does G have a Hamiltonian cycle with weight at most k?", and this question is asked repeatedly until k is minimized.

A **good characterization** is one that provides a condition checkable in polynomial time. Some characterizations are theoretically interesting but not conveniently checkable. For example, the characterization of planar graphs as those not containing a subdivision of K_5 or $K_{3,3}$ is not conveniently checkable since such a subdivision could be arbitrarily large. Other algorithms exist to check planarity in $\mathcal{O}(m)$ time. A characterization with subgraphs of a fixed size, such as that for line graphs, is checkable in polynomial time.

Many common problems in graph theory are NP-complete. These include the following.

CHINESE POSTMAN PROBLEM

TRAVELING SALESMAN PROBLEM

HAMILTONIAN CYCLE

CIRCUMFERENCE

LONGEST PATH
SUBGRAPH ISOMORPHISM
TREEWIDTH
VERTEX COVER
3-COLORABILITY
CLIQUE COVER
ART GALLERY PROBLEM
Δ-EDGE COLORABILITY
MAXIMUM INDEPENDENT SET
MAXIMUM CLIQUE
MAXIMUM BIPARTITE SUBGRAPH
DOMINATION NUMBER
GENUS

In some cases, it is easy to show that two problems are equivalent. For example, MAXIMUM INDEPENDENT SET and MAXIMUM CLIQUE can be reduced to each other by considering the complement of a graph. Other reductions are more complicated. See West [**2001**] for NP-completeness proofs for some common graph theory problems.

Other problems can be solved in polynomial time. These include MINIMUM SPANNING TREE, CONNECTIVITY, DEGENERACY, and PLANARITY.

Some problems are not known to be either P or NP-complete. One example is the graph isomorphism problem. In 2017, Laszlo Babai announced a quasipolynomial time algorithm (that has complexity $2^{\mathcal{O}((\log n)^c)}$ for some $c > 0$). GRAPH ISOMORPHISM (**GI**) has its own class of problems.

When a problem is NP-complete, and thus likely has no efficient solution, we may try to solve a weaker problem.

Definition 10.43. A **heuristic algorithm** guarantees an answer that will be within some fixed percentage of the exact answer.

There is a heuristic algorithm for the Traveling Salesman Problem when the distances satisfy the triangle inequality, namely Christofides' Algorithm (Theorem 6.27).

In this text, we will not calculate the efficiency of algorithms, but we will comment on the complexity of various algorithms and common graph theory problems. A basic understanding of computational complexity is valuable in many areas of mathematics.

10.4. Bounds and Extremal Graphs

Much of graph theory involves the study of graph parameters.

Definition 10.44. A **graph parameter** $f(G)$ is a function that is from some or all graphs to the real numbers.

Some parameters are easy to calculate, while others are NP-hard. In the latter case, it may not be practical to calculate a parameter exactly, so it is desirable to bound it.

Definition 10.45. A function $g(G)$ is an **upper bound** for a function $f(G)$ if $f(G) \leq g(G)$ for all graphs in the domain of $f(G)$.

There are many upper bounds for a function. A bound need not be a good approximation; we would like to express whether it is close to the true value of a parameter.

Definition 10.46. The bound $f(G) \leq g(G)$ is **attained** (by G) if there is a graph G for which it is an equality. The bound $f(G) \leq g(G)$ is **sharp** if there are infinitely many graphs that make it an equality. The bound $f(G) \leq h(G)$ is **sharper** than the bound $f(G) \leq g(G)$ if $h(G) \leq g(G)$ and $h(G) < g(G)$ for some graph G. The bound $f(G) \leq g(G)$ is **exact** if it is an equality for all graphs.

Example. The bound $\chi(G) \leq n^2$ is attained by K_1 only, so it is not sharp. The bound $\chi(G) \leq n$ is sharp, as it is attained by all K_n. The sharper bound $\chi(G) \leq 1 + \Delta(G)$ is attained by complete graphs and odd cycles.

Definition 10.47. The **extremal graphs** for the bound $f(G) \leq g(G)$ are the graphs that make it an equality.

Determining the extremal graphs for a bound helps to understand the parameter. It also leads to an improved bound. If $\mathbb{G}$ is the class of extremal graphs for $f(G) \leq g(G)$, then the bound could be stated as

$$\text{if } G \notin \mathbb{G}, \text{ then } f(G) < g(G) \quad \text{or}$$

$$\text{if } G \notin \mathbb{G}, \text{ then } f(G) \leq g(G) - 1 \text{ (if } f(G) \text{ is integer-valued).}$$

Extremal graph theory is concerned with the question of finding the extreme value (maximum or minimum) of a parameter over some class of graphs. Perhaps the most common is example is finding the maximum size over a given class.

Definition 10.48. The **extremal number** $\mathrm{ex}(n, \mathbb{G})$ is the maximum size among all graphs of order n that do not contain any graph $G \in \mathbb{G}$ as a subgraph. We write $\mathrm{ex}(n, G)$ when $\mathbb{G} = \{G\}$.

Thus a graph with size $\mathrm{ex}(n, \mathbb{G}) + 1$ must contain some $G \in \mathbb{G}$ as a subgraph. The following table catalogues many extremal numbers, both throughout this book and elsewhere, along with the extremal graphs. (The extremal number is always the floor of the listed quantity, but the extremal graphs are only for the quantity itself.)

Class/Property	$\mathrm{ex}(n, \mathbb{G})$ (large n)	Extremal Graphs
contains P_3	$\left\lfloor \frac{n}{2} \right\rfloor$	$\left\lfloor \frac{n}{2} \right\rfloor K_2$
contains cycle	$n-1$	trees
contains more than one cycle	n	unicyclic graphs
two edge-disjoint cycles	$n+4$	2-core is $K_{3,3}$ subdivision
not a cactus	$\frac{3(n-1)}{2}$	every block is K_3
nonplanar or contains triangle	$2n-4$	planar, all regions are C_4
not outerplanar	$2n-3$	maximal outerplanar
contains K_4 subdivision	$2n-3$	2-trees
not a penny graph	$3n-\sqrt{12n-3}$	hexagonal grids
nonplanar	$3n-6$	maximal planar
contains K_5 subdivision	$3n-6$	3-sums of max. planar
two disjoint cycles	$3n-6$	$K_3+\overline{K}_{n-3}$
does not embed on S_k	$3(n-2+2k)$	triangulation of S_k
not an apex graph	$4n-10$	join v to maximal planar
contains $K_{1,k+1}$	$\frac{kn}{2}$	k-regular graphs
contains $k+1$-core	$kn-\binom{k+1}{2}$	maximal k-degenerate
has $\alpha(G) < n-k+1$	$kn-\binom{k+1}{2}$	$K_k+\overline{K}_{n-k}$
contains $(k+1)K_2$	$kn-\binom{k+1}{2}$	$K_k+\overline{K}_{n-k}$
contains odd cycle	$\left\lfloor \frac{n^2}{4} \right\rfloor$	$K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}$
contains K_{k+1}	$\left(1-\frac{1}{k}\right)\frac{n^2}{2}-\frac{b(k-b)}{2k}$ $b=n-k\left\lfloor \frac{n}{k} \right\rfloor$	Turan graph $T_{n,k}$
connected/Hamiltonian path	$\binom{n-1}{2}$	$K_{n-1}\cup K_1$
Hamiltonian	$\binom{n-1}{2}+1$	$K_1+(K_{n-2}\cup K_1)$
Hamiltonian connected	$\binom{n-1}{2}+2$	$K_n-K_{1,n-3}$
k-connected	$\binom{n-1}{2}+k-1$	$K_n-K_{1,n-k}$

For many graphs, including C_k and $K_{r,s}$, the extremal number is not known exactly. Only approximations are known.

In some of these examples, the size required to guarantee a property is quite large, since many edges could be bunched together in one part of a graph. Fewer edges may be required if they were forced to spread out by a restriction on the minimum degree. Note that $\delta(G) \geq \frac{2m}{n}$ implies G has size at least m. When a property requires something of every vertex (e.g., Hamiltonian), a smaller minimum degree may suffice.

Class/Property	required minimum degree
domination number $\gamma(G) \leq \frac{1}{2}n$	1
domination number $\gamma(G) \leq \frac{3}{8}n$	3
K_4 subdivision	3
$\alpha'(G) \geq \lceil \frac{k}{2} \rceil$	k
contains all trees of size k	k
connected/Hamiltonian path	$\frac{n-1}{2}$
Hamiltonian	$\frac{n}{2}$
Hamiltonian connected	$\frac{n+1}{2}$
contains triangle/contains odd cycle	$\frac{n+1}{2}$
k-connected	$\lceil \frac{n+k-2}{2} \rceil$

There are many other bounds relating other parameters. Once a sharp bound has been found, the next problem is to characterize the extremal graphs. These problems give insight into how the parameters work by showing what graphs drive them to extreme values. Many extremal classes are familiar graphs (complete, cycles, complete bipartite) but sometimes less common graphs also arise.

10.5. Graph Characterizations

When we encounter some class of graphs, such as the extremal graphs for a bound, we would like a characterization that makes it easy to understand what graphs are in the class. Certain types of characterizations occur repeatedly. Examples are given for each of them, along with some discussion.

(1) **Forbidden Subgraphs**

A forbidden subgraph characterization characterizes a graph class by what subgraphs the graphs do not contain. Thus any such class is hereditary.

- A graph is a forest if and only if is does not contain a cycle.
- A graph is bipartite if and only if it does not contain an odd cycle.
- A graph is planar if and only if it does not contain a subdivision of K_5 or $K_{3,3}$.
- A maximal k-degenerate graph is a k-tree if and only if it contains no subdivision of K_{k+2}.

If such a subgraph can be identified, it is immediate that the graph is not in the given class. When the number of forbidden subgraphs is finite, then their size is necessarily bounded, so an efficient algorithm can check all subgraphs up to that size. When the number of subgraphs is infinite, this is not practical. However, an efficient algorithm may exist that uses some other method.

(2) **Forbidden Induced Subgraphs**

Adding the requirement that a forbidden subgraph must be induced means that adding an edge to a graph not in the class may result in a graph that is in the class. Thus these classes are not hereditary.

- A bipartite graph is complete bipartite if and only if it does not contain P_4 as an induced subgraph.

- A graph is chordal if and only if it does not contain a cycle of length at least 4 as an induced subgraph.
- A graph is perfect if and only if it contains no induced C_{2k+1} or $\overline{C_{2k+1}}$, $k > 1$.
- A graph is a line graph if and only if it does not contain one of nine forbidden induced subgraphs.

(3) **Forbidden Minors**

- A graph is a forest if and only if it has no K_3 minor.
- A 2-connected graph is series-parallel if and only if it has no K_4 minor.
- A graph is outerplanar if and only if it has no K_4 or $K_{2,3}$ minor.
- A graph is planar if and only if it has no K_5 or $K_{3,3}$ minor.

The Graph Minor Theorem says that any minor-closed graph class has a finite set of forbidden minors. This is an existence proof that does not provide the set, which could be very large. If the set is known, there is a polynomial-time algorithm for checking whether a graph is in a minor-closed graph class.

(4) **Local Structure**

A local structure characterization is one in which any vertex, or small subset of vertices, must be contained in some subgraph (which may be only a small part of the entire graph).

- A graph is a tree if and only if there is a unique path between any two vertices.
- A graph has $\delta(G) \geq 2$ if and only if every vertex is on a cycle or a path between cycles.
- A graph is 2-connected if and only if every two vertices are on a common cycle.
- A nontrivial connected graph is Eulerian if and only if every edge is contained in an odd number of cycles.

(5) **Operation Characterization**

An operation characterization (or recursive characterization) specifies a graph class using specific graphs and one or more operations that can be used to construct graphs in the class from graphs in the class. Such a characterization is typically proved via induction. Operation characterizations are useful for induction proofs of other theorems, since verifying a property holds under the operation (and on the base cases) proves it for all graphs in the class.

- A graph is a tree if and only if it can be constructed from the trivial graph by iteratively adding a new vertex joined to an existing vertex.
- The hypercube Q_k is defined recursively by $Q_1 = K_2$ and $Q_k = Q_{k-1} \Box K_2$.
- Irregular graphs can be constructed with $I_2 = K_2$, $I_3 = P_3$, and $I_n = (I_{n-2} \cup K_1) + K_1$.

There are many more complicated operation characterizations for classes such as 2-monocore graphs.

(6) **Extremal Characterization**

Some classes can be characterized as the extremal graphs for certain bounds or properties.

- A graph is a tree if and only if it is maximal with respect to the property of being acyclic.

- A graph is a tree if and only if it is minimal with respect to the property of being connected.
- A bipartite graph is complete bipartite if and only if it is maximal with respect to the property of being bipartite.

(7) **Decompositions**

Some classes can be characterized by decompositions into certain subgraphs.

- A graph is 2-regular if and only if it is a disjoint union of cycles.
- A multigraph can be decomposed into cycles if and only if all vertices have even degree.
- For a graph G, there is a graph H with $L(H) = G$ if and only if G decomposes into cliques so that each vertex of G appears in at most two cliques.

(8) **Degree Sequences**

Some graph classes can be characterized solely by their degree sequences.

- A connected graph is Eulerian if and only if every vertex has even degree.
- A connected graph has an Eulerian trail if and only if exactly two vertices have odd degree.
- Let G be a connected graph. Then $D(G) = \Delta(G)$ if and only if G is regular.

(9) **Other Characterizations**

Still other characterizations involve vertex orders, colorings, or drawings.

- A graph is a cograph if and only if every vertex order is a perfect order.
- A graph G is maximal planar if and only if a plane drawing of G is a triangulation.
- A cubic map has a 4-region-coloring if and only if it has a 3-edge-coloring.
- A multigraph G has a 1-factor if and only if $o(G - S) \leq |S|$ for any vertex set S.

Any nontrivial characterization provides insight about a graph class, and so is a worthy contribution to graph theory.

Nomenclature

$\tau_2(G)$	2-tone chromatic number
$\psi(G)$	achromatic number
$A(G)$	adjacency matrix
$u \leftrightarrow v$	adjacent vertices
$G \square H$	Cartesian product of graphs
$\chi(G)$	chromatic number
$c(G)$	circumference
$\mathrm{cc}(G)$	clique cover number
$\omega(G)$	clique number
$\mathrm{cl}(G)$	closure
$\overline{G}$	complement
$K_{r,s}$	complete bipartite graph
K_n	complete graph
$K_{n_1,\ldots,n_k}$	complete multipartite graph
$a \equiv b \bmod m$	congruence relation
$\gamma_c(G)$	connected domination number
G/H	contraction
$C(v)$	core number of a vertex
$C_k(G)$	k-core
$\mathrm{cr}(G)$	crossing number
C_n	cycle with n vertices
$\mathrm{def}(G)$	deficiency of a graph
$\mathrm{def}(S)$	deficiency of a set
$D(G)$	degeneracy

$d(v)$	degree of a vertex
$\mathrm{diam}\,(G)$	diameter of G
D	digraph
$d\,(u,v)$	distance between u and v
$\chi_k\,(G)$	distance k chromatic number
$\gamma\,(G)$	domination number
$E\,(G)$	edge set
$e\,(v)$	eccentricity of v
$G+e$	edge addition
$a'\,(G)$	edge arboricity
$\chi'\,(G)$	edge chromatic number
$\beta'\,(G)$	edge cover number
$G-e$	edge deletion
$\alpha'\,(G)$	edge independence number
$\rho'_k\,(G)$	edge partition number
$\overline{K}_n$	empty graph
$\mathrm{ex}\,(n,\mathbb{G})$	extremal number
$f\,(r,k,l)$	Folkman number
$f \gg g$	function grows faster
$G=H$	isomorphic graphs
$P\,(r,k)$	generalized Petersen graph
$\gamma\,(G)$	genus of a graph
$\mathbb{G}$	graph class
$R\,(G,H)$	graph Ramsey number
G	graph
$G_{r,s}$	grid
$g \in \mathcal{O}\,(f)$	growth rate of function
$\Gamma\,(G)$	Grundy number
$G \to \{k\,[H]\}$	H decomposes G
Q_k	hypercube
$d^-\,(v)$	indegree
$\alpha\,(G)$	independence number
$i\,(G)$	independent domination number
$G\,[S]$	induced subgraph
$G=H$	isomorphic graphs (unlabeled)
$G \cong H$	isomorphic graphs (labeled)
$G+H$	join of graphs
kG	k copies of G

$\chi^k(G)$	k-set chromatic number
$KG_{r,k}$	Kneser graph
$\lambda(G)$	L(2,1)-span
$\lambda_{h,k}(G)$	L-span
$\mathrm{la}(G)$	linear arboricity
$\chi_l(G)$	list chromatic number
$\Delta(G)$	maximum degree
$\gamma_M(G)$	maximum genus
$p(k;G)$	maximum sum of p over k-decompositions
$\delta(G)$	minimum degree
M_n	Mobius ladder
$\mu(G)$	multiplicity of a graph
$\mu(uv)$	multiplicity of an edge
$M(G)$	Mycielskian
$N(S)$	neighborhood of a set
$N(v)$	neighborhood
$u \not\leftrightarrow v$	nonadjacent vertices
$o(H)$	number of odd components
$\chi^o(G)$	ochromatic number
$n(G)$	order (number of vertices)
$d^+(v)$	outdegree
$\mathrm{pc}(G)$	pair chromatic number
P_n	path with n vertices
$\rho_k(G)$	point partition number
$r \mid s$	r divides s
$\mathrm{rad}(G)$	radius
$\mathrm{rc}(t_1, t_2, \ldots, t_k)$	Ramsey core number
$R(s,t)$	Ramsey number
r	region
$[n]$	set $\{1, \ldots, n\}$
$m(G)$	size (number of edges)
$\mu(G)$	skewness
$\mathrm{sl}(G)$	Slater number
$\mathrm{st}(G)$	star arboricity
$K_{1,s}$	star
$H \subseteq G$	subgraph relation
S_k	surface with genus k
$G \times H$	tensor product of graphs

$\theta_{i,j,k}$	theta graph
$\theta(G)$	thickness
$\gamma_t(G)$	total domination number
$\mathrm{tr}(A)$	trace of a matrix
T	tree
$\mathrm{tw}(G)$	treewidth
T_l	triangular grid
$T_{n,k}$	Turan graph
$u \to v$	u adjacent to v
$G \cup H$	union of graphs
$V(G)$	vertex set
$a(G)$	vertex arboricity
$\beta(G)$	vertex cover number
$G - v$	vertex deletion
$w(e)$	weight of an edge
W_n	wheel with n vertices

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Graph theory is a fascinating and inviting branch of mathematics. Many problems are easy to state and have natural visual representations, inviting exploration by new students and professional mathematicians. The goal of this textbook is to present the fundamentals of graph theory to a wide range of readers. The book contains many significant recent results in graph theory, presented using up-to-date notation. The author included the shortest, most elegant, most intuitive proofs for modern and classic results while frequently presenting them in new ways. Major topics are introduced with practical applications that motivate their development, and which are illustrated with examples that show how to apply major theorems in practice. This includes the process of finding a brute force solution (case-checking) when an elegant solution is not apparent. With over 1200 exercises, internet resources (e.g., the OEIS for counting problems), helpful appendices, and a detailed guide to different course outlines, this book provides a versatile and convenient tool for the needs of instructors at a large variety of institutions.

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