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ORIGINAL RESEARCH

The Limit Cycles of Discontinuous Piecewise Linear Differential Systems Formed by Centers and Separated by Irreducible Cubic Curves II

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Abstract

In this paper we provide a lower bound for the maximum number of crossing limit cycles of some class of planar discontinuous piecewise linear differential systems formed by centers and separated by an irreducible algebraic cubic curve. First we prove that the systems constituted by three zones can exhibit 0, 1, 2, 3 or 4 crossing limit cycles having four intersection points with the cubic of separation. Second we prove that the systems constituted by two zones can exhibit 0, 1, or 2 crossing limit cycles having four intersection points with the cubic of separation.

Keywords Limit cycles · Discontinuous piecewise linear differential systems · Linear differential centers · Irreducible cubic curves

Mathematics Subject Classification 34C29 · 34C25 · 47H11

Introduction

We can summarize the 16th Hilbert problem (see [11, 13, 17]) as follows: What are the possible configurations of limit cycles and an upper bound for their maximum number that the polynomial differential systems in the plane of a given degree can exhibit? In fact the possible configurations of limit cycles has been partially solved in [21], but for the moment there is no answer for such upper bound.

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The objective of the present paper is to solve the 16th Hilbert problem extended to the limit cycles of the discontinuous piecewise linear differential systems formed by centers and separated by irreducible cubic curves when these limit cycles intersect the cubic in four points, because if the limit cycles intersect the cubic in two points this problem has been solved in [3]. Of course we follow the rules of Filippov [7] for defining the discontinuous piecewise linear differential systems on the discontinuity curve.

Using the first integrals of the linear centers, which are quadratic functions in the cartesian coordinates (x, y) of the plane, and the expression of the irreducible cubics, we shall obtain a set of polynomial equations whose solutions provide the limit cycles of the the discontinuous piecewise linear differential systems that we study. Studying the solutions of these polynomial equations we obtain the exact maximum number of such limit cycles, which vary with the different irreducible cubic. Moreover, from such polynomial equations we also can describe the possible different configurations of limit cycles for each cubic.

Classification of the Irreducible Cubic Polynomials

A *cubic curve* is the set of points $(x, y) \in \mathbb{R}^2$ satisfying $P(x, y) = 0$ for some polynomial $P(x, y)$ of degree three. This cubic is *irreducible* (respectively *reducible*) if the polynomial $P(x, y)$ is irreducible (respectively reducible) in the ring of all real polynomials in the variables x and y.

A point (x_0, y_0) of a cubic $P(x, y) = 0$ is *singular* if $P_x(x_0, y_0) = 0$ and $P_y(x_0, y_0) = 0$. A cubic curve is *singular* if it has some singular point, as usual here P_x and P_y denote the partial derivaties of P with respect to the variables x and y respectively.

A *flex* of an algebraic curve Γ is a point p of Γ such that Γ is nonsingular at p and the tangent at p intersects Γ at least three times. The next theorem characterizes all the irreducible cubic algebraic curves.

Theorem 1 *The following statements classify all the irreducible cubic algebraic curves.*

(a) *A cubic curve is nonsingular and irreducible if and only if it can be transformed with affine transformations into one of the following two curves*;

$$c_1(x, y) = y^2 - x(x^2 + bx + 1) = 0 \qquad \text{with } b \in (-2, 2), \text{ or}$$
$$c_2(x, y) = y^2 - x(x-1)(x-r) = 0 \qquad \text{with } r > 1.$$

(b) *A cubic curve is singular and irreducible if and only if it can be transformed with affine transformations into one of the following three curves*:

$$\begin{aligned} c_3(x, y) =& y^2 - x^3 = 0, \quad \text{or} \\ c_4(x, y) =& y^2 - x^2(x-1) = 0, \quad \text{or} \\ c_5(x, y) =& y^2 - x^2(x+1) = 0. \end{aligned}$$

Statement (a) of Theorem 1 is proved in Theorem 8.3 of the book [4] under the additional assumption that the cubic has a flex, but in section 12 of that book it is shown that every nonsingular irreducible cubic curve has a flex. While statement (b) of Theorem 1 follows directly from Theorem 8.4 of [4].

Crossing Limit Cycles

For $k = 1, \dots, 5$ let C_k be the five classes of planar discontinuous piecewise linear differential systems formed by centers and separated by the irreducible cubic curve $c_k(x, y) = 0$, or simply the irreducible cubic curve c_k.

Piecewise linear differential systems appear in the characterization of many real processes such as switches in electronic circuits, see for instance [2, 16, 18].

In the qualitative theory of piecewise linear differential systems one of the important and difficult problems is the determination of the existance and the number of limit cycles. We recall that a crossing periodic orbit, is a periodic orbit of a piecewise linear differential system with a cubic separation curve c_k with at least two points in c_k. If this periodic orbit is isolated we call it a *crossig limit cycle*. In recent years, much progress has been made in studying the existance and lower and upper bounds of crossing limit cycles of piecewise linear differential systems, see for example [1, 5, 6, 8–10, 12, 20, 22, 24–26].

There are previous results on the number of crossing limit cycles of discontinuous piecewise linear differential centers separated by a curve Σ. More precisely, if Σ is one straight line then such systems have no crossing limit cycles see [19, 23]. If Σ is a conic the number of crossing limit cycles have been studied in [14], and if Σ is a reducible cubic curve the number of crossing limit cycles have been analyzed in [15].

In [3] we started the study of crossing limit cycles of the discontinuous piecewise linear differential centers in $\mathbb{R}^2$ separated by an irreducible algebraic cubic curve. We proved that these differential systems only can exhibit at most three crossing limit cycles having two intersection points with the cubic of separation.

The objective of this paper is to provide lower bounds for the maximum number $\mathcal{N}$ of crossing limit cycles for the planar discontinuous piecewise linear differential centers which intersect the irreducible cubic curves c_i, with $i = 1 \dots 5$, and have four points of intersection with the cubic of separation. First, we study the number $\mathcal{N}$ when the crossing limit cycles intersect the curves c_2 and c_5, in four points in three different zones. Second, we give $\mathcal{N}$ when the crossing limit cycles intersect c_i with $i = 1 \dots 5$, in four points in two different zones.

Figures 1 and 2 show the different regions separated by the cubic curves c_i, with $i = 1 \dots 5$.

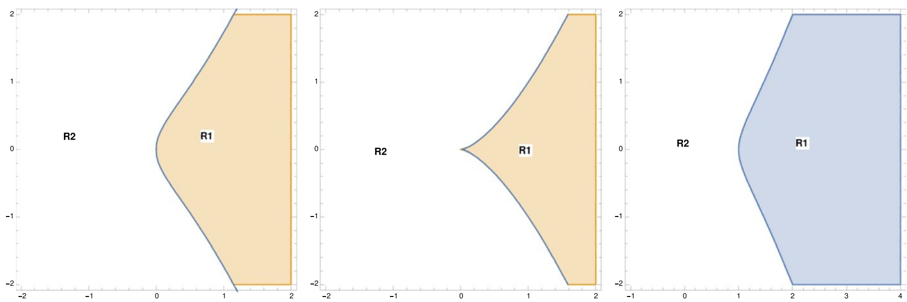


Fig. 1 The two regions R_1 and R_2 of the plane separated by the curves c_1 on the left, c_3 on the middle and c_4 on the right

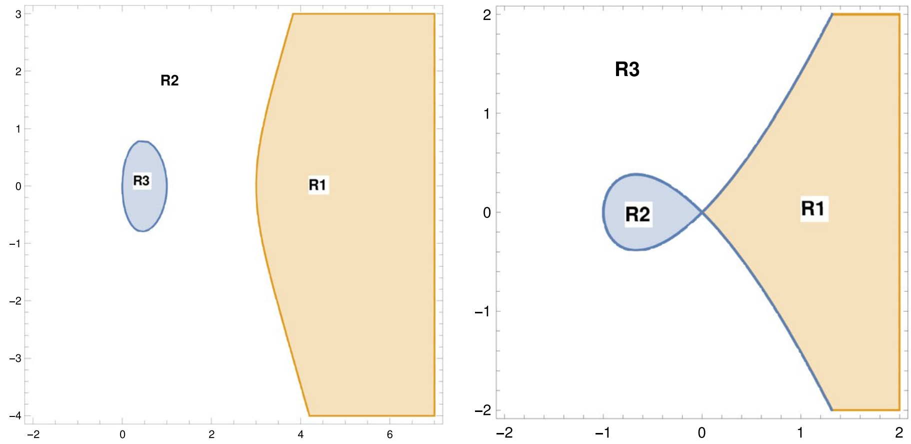


Fig. 2 The three regions R_1, R_2 and R_3 of the plane separated by the curves c_2 on the left and c_5 on the right

Crossing Limit Cycles Contained in Three Regions Intersecting the Two Cubic Curves c_2 and c_5 in Four Points

In this subsection we are interessted to provide lower bounds for the maximum number of *crossing limit cycles* of piecewise linear differential centers separated by the irreducible cubic curves c_2 and c_5, having four points of intersection with the cubic of separation. We note that such piecewise differential systems are formed by three pieces in each one there is a linear differential center.

We study the crossing limit cycles contained in three regions of the discontinuous piecewise linear centers in the classes C_2 or C_5 and having four points of intersection with the cubic of separation. Then our first main result is the following.

Theorem 2 *The following statements hold.*

(a) *There are systems in C_2 and in C_5 exhibiting exactly one crossing limit cycle which intersect c_2 or c_5 in four points. The class C_2 has one possible configuration see Fig. 3, while the class C_5 has two possible configurations see $(\mathcal{C}_5^1)$ and $(\mathcal{C}_5^2)$ of Fig. 4.*
(b) *There are systems in C_2 and in C_5 exhibiting exactly two crossing limit cycles which intersect c_2 or c_5 in four points. The class C_2 has one possible configuration see Fig. 5, while the class C_5 has three possible configurations see $(\mathcal{C}_5^1)$ of Fig. 5, and $(\mathcal{C}_5^2)$ and $(\mathcal{C}_5^3)$ of Fig. 6.*
(c) *There are systems in C_2 and in C_5 exhibiting exactly three crossing limit cycles which intersect c_2 or c_5 in four points. The class C_2 has one possible configuration see Fig. 7, while the class C_5 has four possible configurations see $(\mathcal{C}_5^1)$ and $(\mathcal{C}_5^2)$ of Fig. 8, and $(\mathcal{C}_5^3)$ and $(\mathcal{C}_5^4)$ of Fig. 9.*

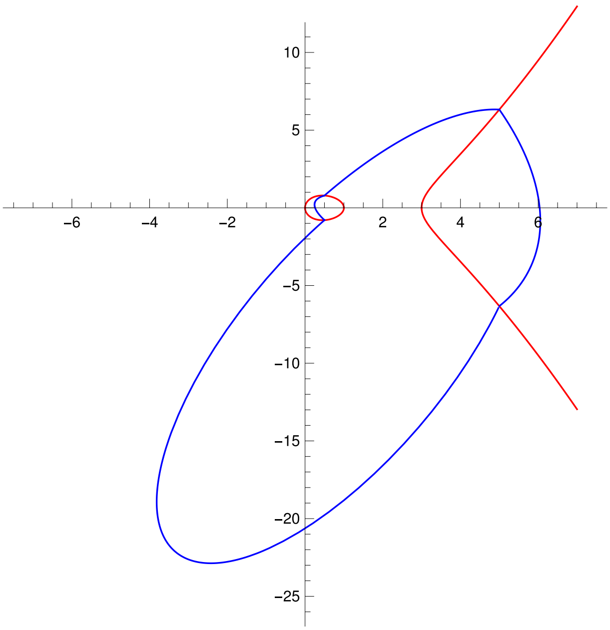
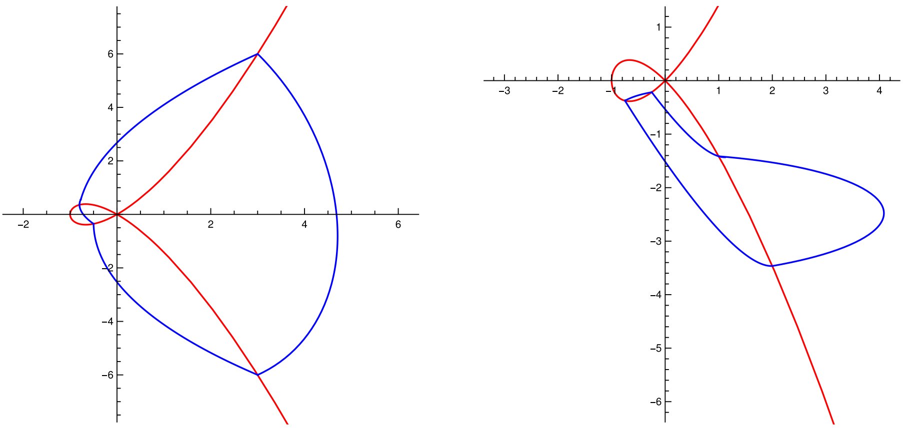


Fig. 3 The unique limit cycle of the discontinuous piecewise linear differential system (1)–(3) contained in three zones



$(\mathcal{C}_5^1)$ $(\mathcal{C}_5^2)$

Fig. 4 The unique limit cycle of the discontinuous piecewise linear differential system $(\mathcal{C}_5^1)$ for (5)–(7), and $(\mathcal{C}_5^2)$ for (9)–(11) contained in three zones

(d) *There are systems in C_2 and in C_5 exhibiting exactly four crossing limit cycles which intersect c_2 or c_5 in four points. The class C_2 has one possible configuration $(\mathcal{C}_2)$ of Fig.* 10, *and we give four configurations of the class C_5, see (C_5^1) of Fig.* 10, (C_5^2) *and (C_5^3) of Fig.* 11 , *and (C_5^4) of Fig.* 12.

Theorem 2 is proved in section 2.

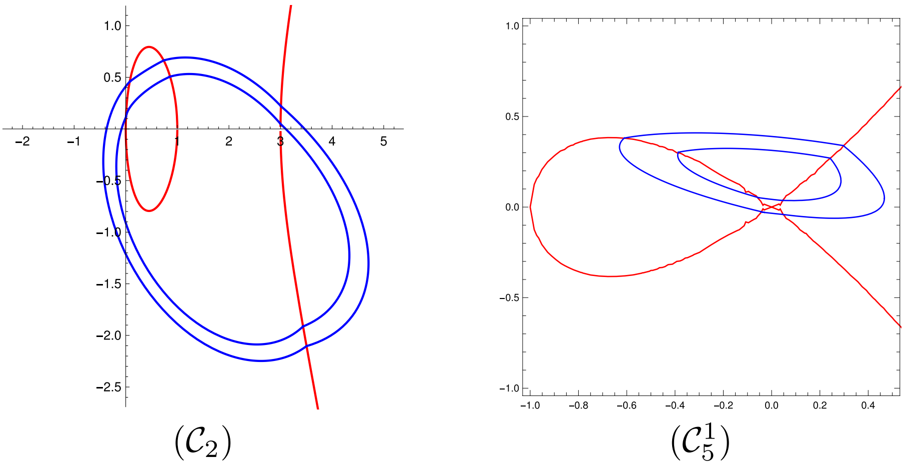


Fig. 5 The two limit cycles of the discontinuous piecewise linear differential system $(\mathcal{C}_2)$ for (12)–(14), and $(\mathcal{C}_5^1)$ for (15)–(17) contained in three zones

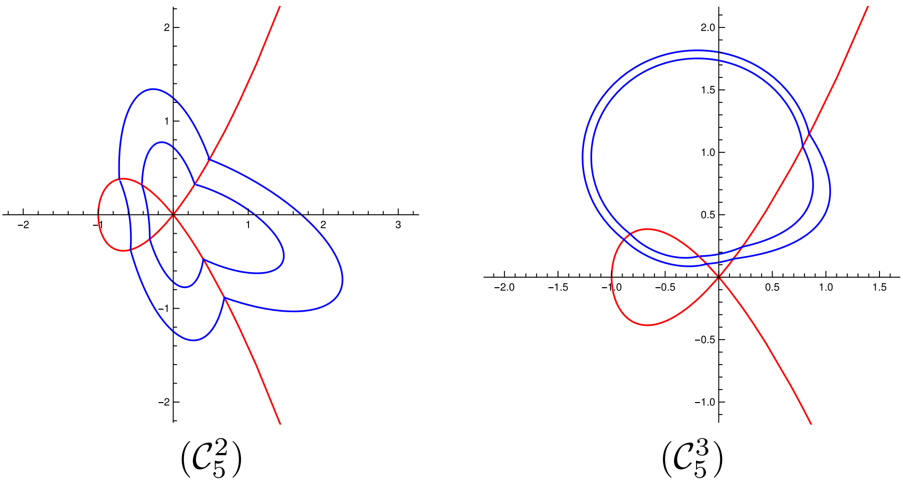


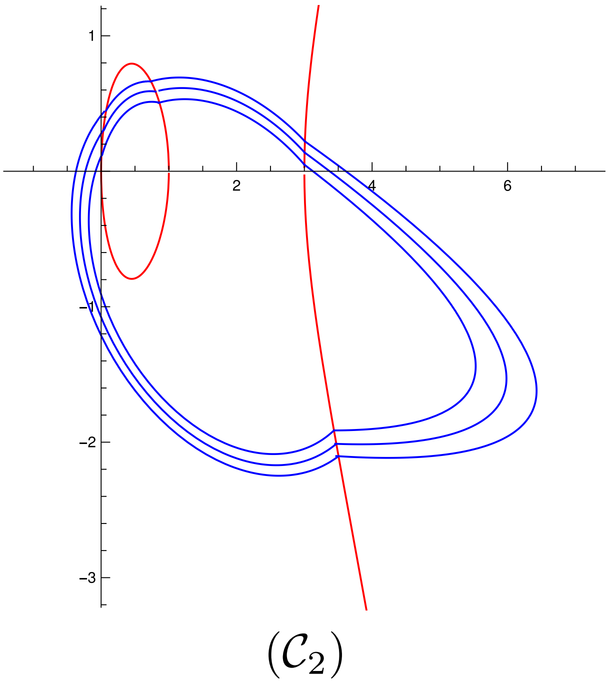
Fig. 6 The two limit cycles of the discontinuous piecewise linear differential system $(\mathcal{C}_5^2)$ for (18)–(20), and $(\mathcal{C}_5^3)$ for (21)–(23) contained in three zones

Crossing Limit Cycles Contained in two Regions Intersecting the Irreducible Cubic Curves c_i in Four Points

Now we give our second main result which provides information on the number of crossing limit cycles of the discontinuous piecewise linear differential systems formed by two centers and intersect the cubic curves c_i, with $i = 1 \ldots 5$ in four points.

We study the crosssing limit cycles contained only in two regions of the discontinuous piecewise linear centers in the class C_k for $k = 1 \ldots 5$, and having four intersection points with the cubic of separation. Then our second main result is the following.

Fig. 7 The three limit cycles of the discontinuous piecewise linear differential system (24)–(26) contained in three zones



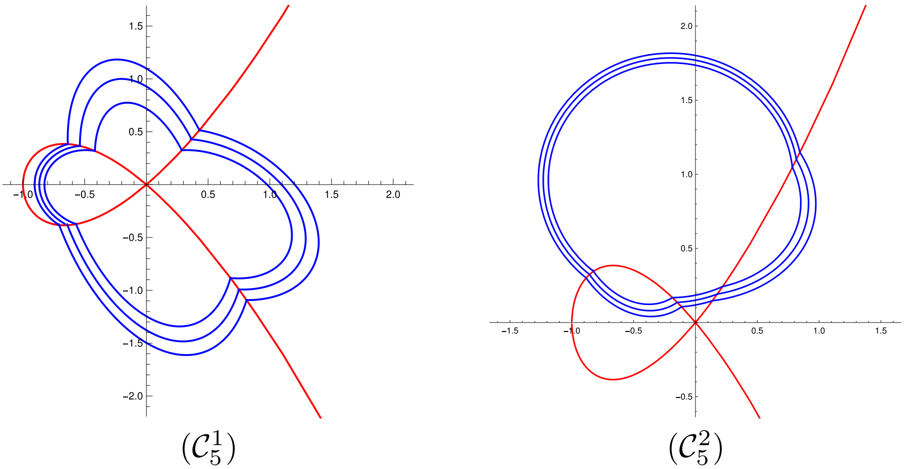


Fig. 8 The three limit cycles of the discontinuous piecewise linear differential system $(\mathcal{C}_5^1)$ for (27)–(29), and $(\mathcal{C}_5^2)$ for (30)–(32) contained in three zones

Theorem 3 *The following statements hold.*

(a) *There are systems in C_k exibiting exactly one crossing limit cycle intersecting the cubic curves c_i in four points. The classes C_1, C_3 and C_4 have one possible configuration, see $(\mathcal{C}_1)$, $(\mathcal{C}_3)$ of Fig.* 13 *and $(\mathcal{C}_4)$ of Fig.* 15, *respectively. The classes C_2 and C_5 have*

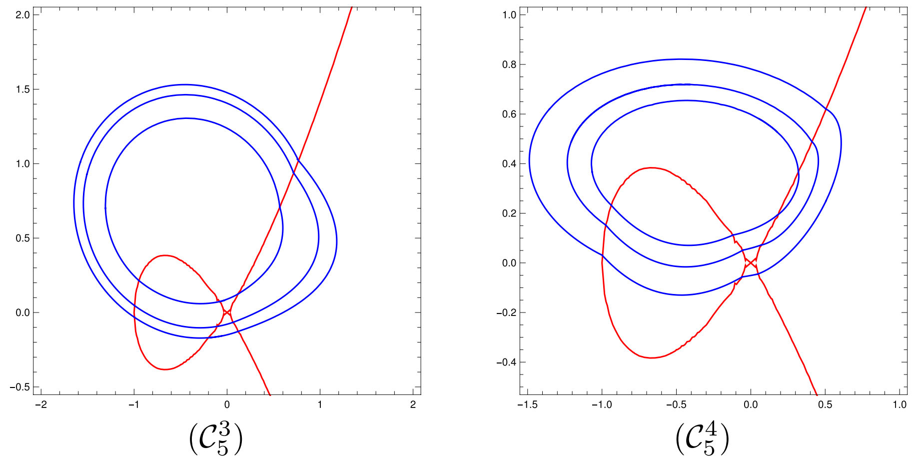


Fig. 9 The three limit cycles of the discontinuous piecewise linear differential system $(\mathcal{C}_5^3)$ for (33)–(35), and $(\mathcal{C}_5^4)$ for (36)–(38) contained in three zones

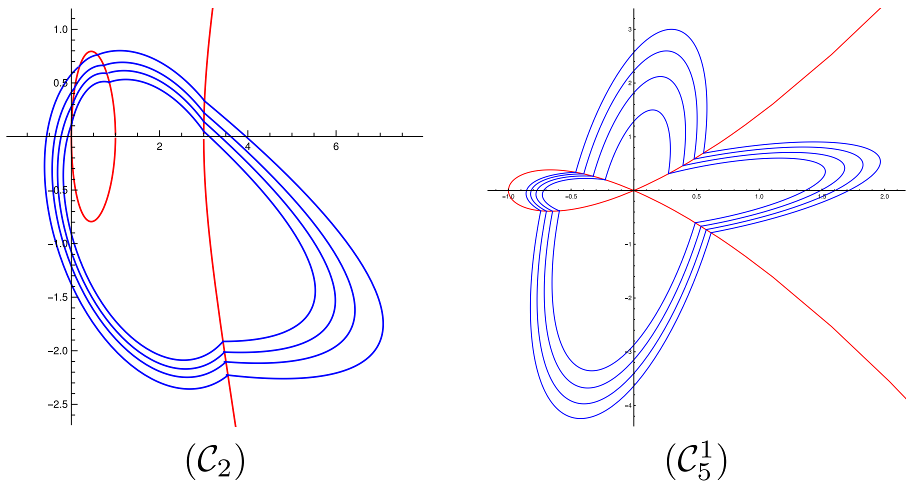


Fig. 10 The four limit cycles of the discontinuous piecewise linear differential system $(\mathcal{C}_2)$ for (39)–(41), and $(\mathcal{C}_5^1)$ for (42)–(44) contained in three zones

two possible different configurations see $(\mathcal{C}_2^1)$ *and* $(\mathcal{C}_2^2)$ *of Fig.* 14, *and* $(\mathcal{C}_5^1)$ *and* $(\mathcal{C}_5^2)$ *of Fig.* 16.

(b) *There are systems in* C_k *exibiting exactly two crossing limit cycles intersecting the cubic curves* c_i *in four points. The classes* C_1, C_3 *and* C_4 *have one possible configuration see* $(\mathcal{C}_1)$, $(\mathcal{C}_3)$ *and* $(\mathcal{C}_4)$ *of Fig.* 17, *respectively. The classes* C_2 *and* C_5 *have two possible different configurations see* $(\mathcal{C}_2^1)$ *and* $(\mathcal{C}_2^2)$ *of Fig.* 18 *for the class* C_2, *and* $(\mathcal{C}_5^1)$ *and* $(\mathcal{C}_5^2)$ *of Fig.* 19 *for the class* C_5.

Theorem 3 is proved in section 3.

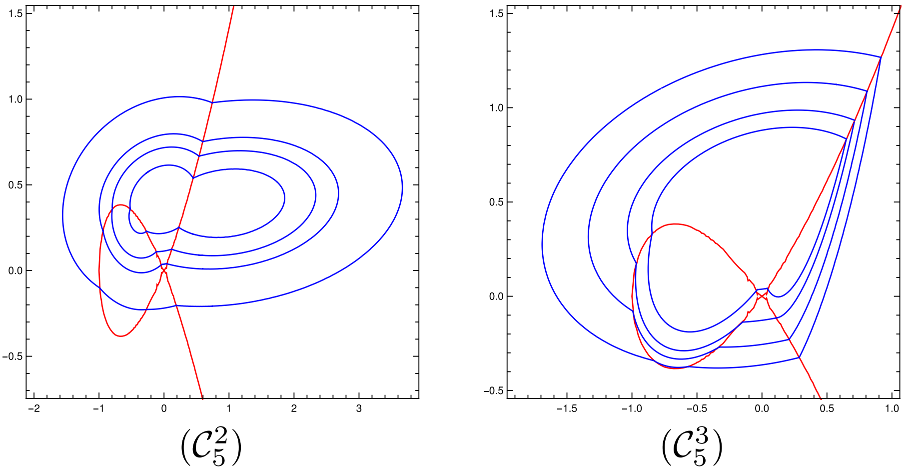
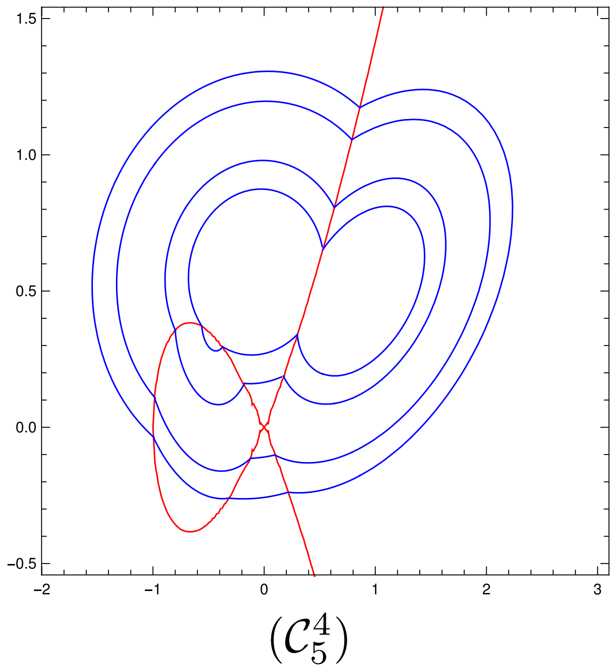


Fig. 11 The four limit cycles of the discontinuous piecewise linear differential system $(\mathcal{C}_5^2)$ for (45)–(47), and $(\mathcal{C}_5^3)$ for (48)–(50) contained in three zones

Fig. 12 The four limit cycles of the discontinuous piecewise linear differential system (51)–(53) contained in three zones



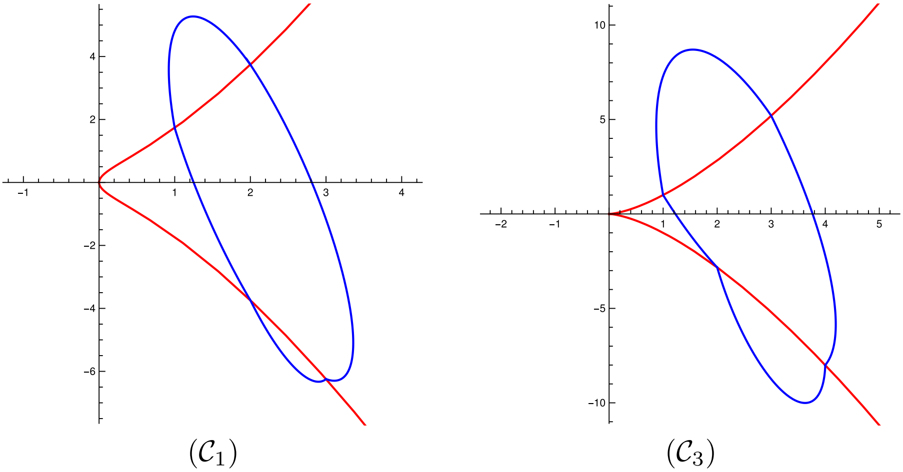


Fig. 13 The unique crossing limit cycle of the discontinuous piecewise linear differential system $(\mathcal{C}_1)$ for (54)–(55), and $(\mathcal{C}_3)$ for (61)–(62) contained in two zones

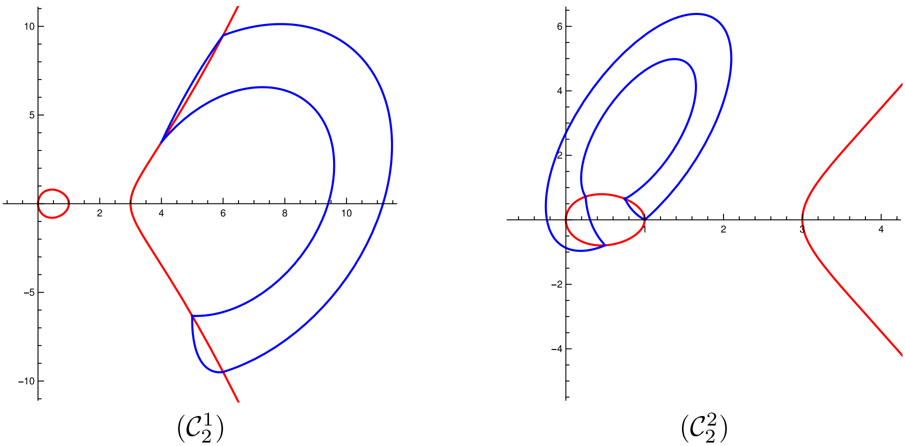


Fig. 14 The unique crossing limit cycle of the discontinuous piecewise linear differential system $(\mathcal{C}_2^1)$ for (57)–(58), and $(\mathcal{C}_2^2)$ for (59)–(60) contained in two zones

Proof of Theorem 2

The following lemma provides a normal form for an arbitrary linear differential system having a center. For a proof see Lemma 1 of [23].

Lemma 1 *A linear differential system having a center can be written as*

$$\dot{x} = -bx - \frac{4b^2 + \omega^2}{4a}y + d, \qquad \dot{y} = ax + by + c,$$

with $a > 0$ *and* $\omega > 0$.

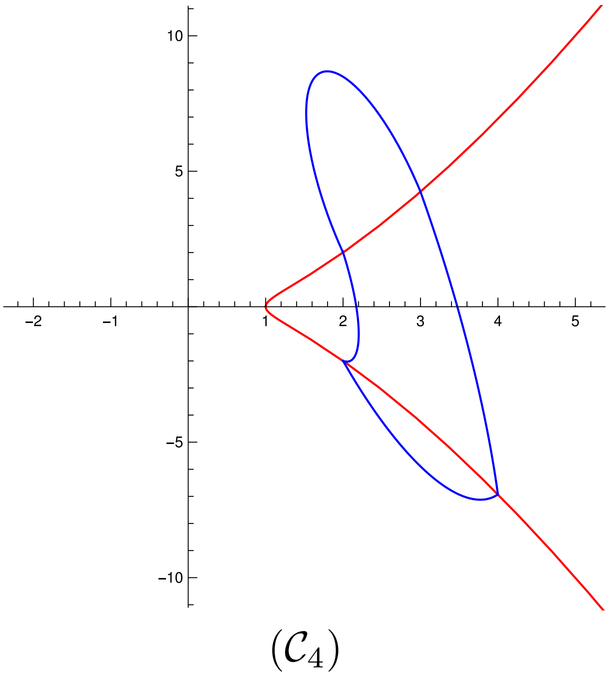


Fig. 15 The unique crossing limit cycle of the discontinuous piecewise linear differential system (63)–(64) contained in two zones

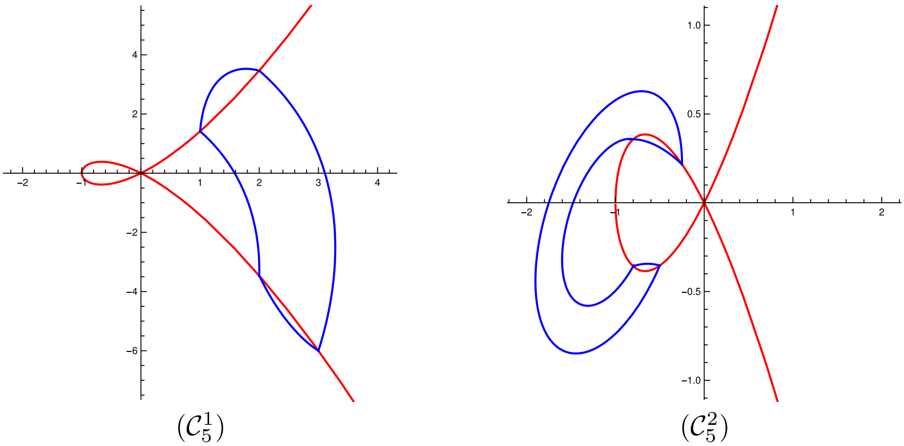


Fig. 16 The unique crossing limit cycle of the discontinuous piecewise linear differential system $(\mathcal{C}_5^1)$ for (65)–(66), and $(\mathcal{C}_5^2)$ for (67)–(68) contained in two zones

Proof of statement (a) of Theorem 2 First we prove the statement for the class C_2. We consider the first linear differential center in the region R_1

$$\dot{x} = -\frac{x}{4} - \frac{5y}{16} + \frac{5}{4}, \quad \dot{y} = x + \frac{y}{4} + \frac{1}{2}, \tag{1}$$

this system has the first integral

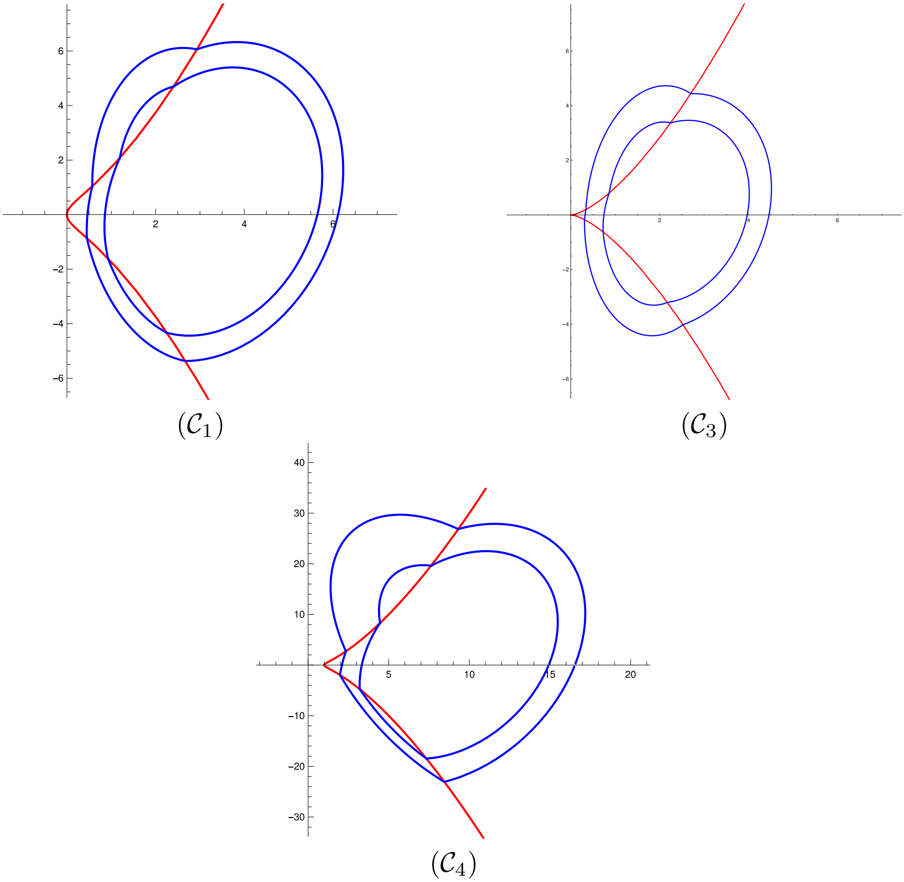


Fig. 17 The two crossing limit cycles of the discontinuous piecewise linear differential system $(\mathcal{C}_1)$ for (69)–(70), $(\mathcal{C}_3)$ for (76)–(77), and $(\mathcal{C}_4)$ for (78)–(79) contained in two zones

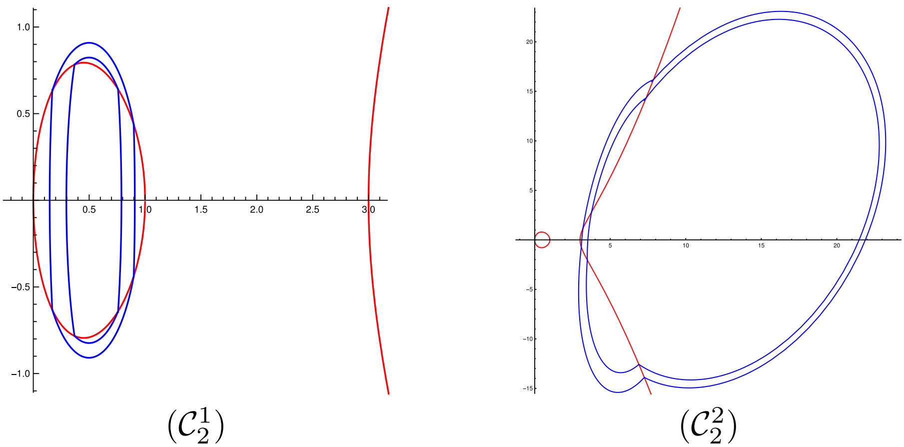


Fig. 18 The two crossing limit cycles of the discontinuous piecewise linear differential system $(\mathcal{C}_2^1)$ for (72)–(73), and $(\mathcal{C}_2^2)$ for (74)–(75) contained in two zones

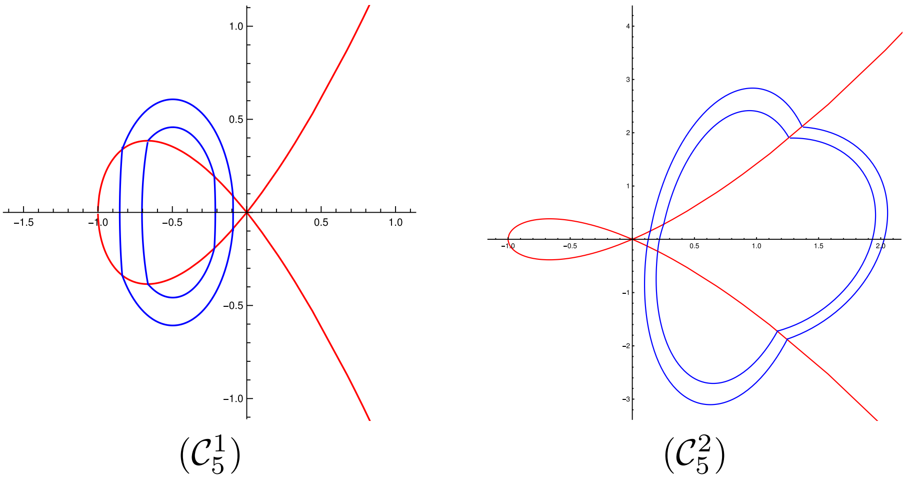


Fig. 19 The two crossing limit cycles of the discontinuous piecewise linear differential system $(\mathcal{C}_5^1)$ for (80)–(81), and $(\mathcal{C}_5^2)$ for (82)–(83) contained in two zones

$$H_1(x, y) = 4\left(x + \frac{y}{4}\right)^2 + 8\left(\frac{x}{2} - \frac{5y}{4}\right) + y^2.$$

The second linear differential center in the region R_2 is

$$\dot{x} = \frac{1}{56}(14x - 7y - 79), \quad \dot{y} = x - \frac{y}{4} - \frac{211}{64}, \tag{2}$$

this differential system has the first integral

$$H_2(x, y) = 4x^2 + x\left(-2y - \frac{211}{8}\right) + \frac{1}{14}y(7y + 158).$$

Now we consider the third linear differential center in the region R_3

$$\dot{x} = -x - \frac{5y}{4} + \frac{1}{2}, \quad \dot{y} = x + y - 2, \tag{3}$$

this differential system has the first integral

$$H_3(x, y) = 4(x + y)^2 + 8\left(-2x - \frac{y}{2}\right) + y^2.$$

The discontinuous piecewise linear differential system formed by the linear differential centers (1), (2) and (3) has exactly one crossing limit cycle, because the system of equations

$$\begin{aligned}
&H_1(\alpha,\beta)-H_1(\gamma,\delta)=0,\\
&H_2(\alpha,\beta)-H_2(f,g)=0,\\
&H_2(\gamma,\delta)-H_2(h,k)=0,\\
&H_3(f,g)-H_3(h,k)=0,\\
&\beta^2-\alpha(\alpha-1)(\alpha-3)=0,\\
&\delta^2-\gamma(\gamma-1)(\gamma-3)=0,\\
&g^2-f(f-1)(f-3)=0,\\
&k^2-h(h-1)(h-3)=0,
\end{aligned}\tag{4}$$

has a unique real solution $(\alpha,\beta,\gamma,\delta,f,g,h,k)=(5,-2\sqrt{10},5,2\sqrt{10},1/2,-\sqrt{5}/2\sqrt{2}, 1/2,\sqrt{5}/2\sqrt{2})$. This completes the proof of statement (a) for the class C_2.

Now we prove the existence of two different configurations of one crossing limit cycle for the class C_5. For the first possible configuration we consider the linear differential center

$$\dot{x}=-\frac{x}{8}-\frac{17y}{64}+\frac{3}{8},\quad \dot{y}=x+\frac{y}{8}-1,\tag{5}$$

in the region R_1, with its first integral

$$H_1(x,y)=4\left(x+\frac{y}{8}\right)^2+8\left(-x-\frac{3y}{8}\right)+y^2.$$

In the region R_3 we consider the linear center

$$\begin{aligned}
\dot{x}&=\frac{x}{4}-\frac{101y}{16}+\frac{3800\sqrt{2}+3584\sqrt{5}+2203113}{320\left(95\sqrt{2}+56\sqrt{5}-4380\right)},\\
\dot{y}&=x-\frac{y}{4}+\frac{-448604\sqrt{2}-315662\sqrt{5}+48\sqrt{10}+22560621}{160\left(95\sqrt{2}+56\sqrt{5}-4380\right)},
\end{aligned}\tag{6}$$

with its first integral

$$\begin{aligned}
H_2(x,y)=\;&\frac{1}{40(95\sqrt{2}+56\sqrt{5}-4380)}\\
&(160(95\sqrt{2}+56\sqrt{5}-4380)x^2+x(-80(95\sqrt{2}+56\sqrt{5}-4380)y\\
&-897208\sqrt{2}-631324\sqrt{5}+96\sqrt{10}+45121242)+y(1010(95\sqrt{2}\\
&+56\sqrt{5}-4380)y-3800\sqrt{2}-3584\sqrt{5}-2203113).
\end{aligned}$$

In the region R_2 we consider the following linear differential center

$$\dot{x}=-x-10y-\frac{1}{2},\quad \dot{y}=x+y-2,\tag{7}$$

this differential system has the first integral

$$H_3(x,y) = 4(x+y)^2 + 8\left(\frac{5}{12}\left(-3\sqrt{2} + \frac{81}{50} - \frac{288}{25\sqrt{5}}\right)x - y\right) + 16y^2.$$

The discontinuous piecewise linear differential system formed by the linear differential centers (5), (6) and (7) has exactly one crossing limit cycle, because the system of equations

$$\begin{aligned}
&H_1(\alpha,\beta) - H_1(\gamma,\delta) = 0,\\
&H_2(\alpha,\beta) - H_2(f,g) = 0,\\
&H_2(\gamma,\delta) - H_2(h,k) = 0,\\
&H_3(f,g) - H_3(h,k) = 0,\\
&\beta^2 - \alpha^2(\alpha+1) = 0,\\
&\delta^2 - \gamma^2(\gamma+1) = 0,\\
&g^2 - f^2(f+1) = 0,\\
&k^2 - h^2(h+1) = 0,
\end{aligned} \tag{8}$$

has a unique real solution $(\alpha,\beta,\gamma,\delta,f,g,h,k) = (3,-6,3,6,-1/2,-1/(2\sqrt{2}),-4/5,4/(5\sqrt{5}))$.

For the second configuration. In the region R_1 we consider the linear differential center

$$\dot{x} = \frac{x}{7} - \frac{785y}{49} + \frac{70\sqrt{50-8\sqrt{6}} + 40083}{490\left(\sqrt{2}-2\sqrt{3}\right)}, \quad \dot{y} = x - \frac{y}{7} + \frac{1}{5}, \tag{9}$$

which has the first integral

$$H_1(x,y) = 4\left(x - \frac{y}{7}\right)^2 + 8\left(\frac{x}{5} + \frac{\left(70\sqrt{2} - 280\sqrt{3} - 40083\right)y}{490\left(\sqrt{2}-2\sqrt{3}\right)}\right) + 64y^2.$$

In the region R_3 we consider the linear differential center

$$\begin{aligned}
\dot{x} &= -\frac{3x}{7} - \frac{373y}{1764} + \frac{-23556784\sqrt{2} - 26814823\sqrt{3} + 14402472\sqrt{6} + 83393133}{80960544},\\
\dot{y} &= x + \frac{3y}{7} + \frac{-4861464\sqrt{2} + 3092483\sqrt{3} + 1548288\sqrt{6} - 1763763}{56448\left(88\sqrt{2} - 91\sqrt{3} + 15\right)},
\end{aligned} \tag{10}$$

with its first integral

$$H_2(x,y) = 4\left(x + \frac{3y}{7}\right)^2 + \frac{y^2}{9} + \frac{A}{7056\left(88\sqrt{2} - 91\sqrt{3} + 15\right)\left(16\sqrt{3} - 3\right)},$$

where

$$A = 88902216\sqrt{2} - 37497657\sqrt{3} - 82428288\sqrt{6} + 153730473)x - 8(7597532\sqrt{3} \\ + 266112\sqrt{6(259 - 32\sqrt{3})} - 23982207)y.$$

In the region R_2 we consider the linear differential system

$$\dot{x} = \frac{x}{8} - \frac{577y}{64} + \frac{-10025\sqrt{3} - 29787}{9216}, \quad \dot{y} = x - \frac{y}{8} - \frac{1}{3}, \tag{11}$$

which has the first integral

$$H_3(x, y) = 4\left(x - \frac{y}{8}\right)^2 + 8\left(\frac{\left(10025\sqrt{3} + 29787\right)y}{9216} - \frac{x}{3}\right) + 36y^2.$$

The real solutions of the system of equations (8) with the values of $H_i(x, y)$ with $i = 1, 2, 3$ given for this second configuration is $(\alpha, \beta, \gamma, \delta, f, g, h, k) = (1, -\sqrt{2}, 2, -2\sqrt{3}, -1/4, -\sqrt{3}/8, -3/4, -3/8)$, then the discontinuous piecewise linear differential system (9), (10) and (11) has one crossing limit cycle ($\mathcal{C}_5^2$) of Fig. 4. This completes the proof of statement (a) for the class C_5.

Proof of statement (b) of Theorem 2 First we prove the statement for the class C_2. We consider the first linear differential center in the region R_1

$$\dot{x} = -x - 5y - 1.8035, \quad \dot{y} = x + y - 0.664282, \tag{12}$$

this system has the first integral

$$H_1(x, y) = 4(x + y)^2 + 8(1.8035y - 0.664282x) + 16y^2.$$

The second linear differential center in the region R_2 is

$$\dot{x} = -\frac{x}{2} - \frac{10y}{4} - 1, \quad \dot{y} = x + \frac{y}{2} - \frac{3}{2}, \tag{13}$$

this differential system has the first integral

$$H_2(x, y) = 4\left(x + \frac{y}{2}\right)^2 + 8\left(y - \frac{3x}{2}\right) + 9y^2.$$

The third linear differential center in the region R_3 is

$$\dot{x} = -2x - 8y + 0.837903, \quad \dot{y} = x - 2y - 0.169396, \tag{14}$$

this differential system has the first integral

$$H_3(x, y) = 4(x - 2y)^2 + 8(-0.169396x - 0.837903y) + 16y^2.$$

The discontinuous piecewise linear differential system formed by the linear differential centers (12), (13) and (14) have exactly two crossing limit cycles, because the system of equations (4) has two real solutions $(\alpha_1, \beta_1, \gamma_1, \delta_1, f_1, g_1, h_1, k_1) = (3.00039\ldots, 0.0485802\ldots,$

$3.43695\ldots, -1.91305\ldots, 0.860569\ldots, 0.506666\ldots, 0.00442503\ldots, 0.114878\ldots)$ and $(\alpha_2, \beta_2, \gamma_2, \delta_2, f_2, g_2, h_2, k_2) = (3.00805\ldots, 0.220494\ldots, 3.50419\ldots, -2.1034\ldots, 0.735716\ldots, 0.663523\ldots, 0.0779996\ldots, 0.458408\ldots)$. This completes the proof of statement (b) for the class C_5.

Now we prove the existence of three different configurations of two crossing limit cycles for the class C_5. To obtain the first configuration we consider in the region R_1 the linear differential center

$$\dot{x} = -1.77296\ldots x - 3.39338\ldots y + 1, \quad \dot{y} = x + 1.77296\ldots y - 0.15026\ldots, \tag{15}$$

with its first integral

$$H_1(x, y) = 4x^2 + 14.1837\ldots xy - 1.20208\ldots x + 13.5735\ldots y^2 - 8y.$$

In the region R_3 we consider the linear center

$$\dot{x} = -\frac{3x}{2} - \frac{45y}{4} + 2, \quad \dot{y} = x + \frac{3y}{2} - 0.3, \tag{16}$$

with its first integral

$$H_2(x, y) = 4x^2 + 12xy - \frac{12x}{5} + 45y^2 - 16y.$$

In the region R_2 we consider the linear differential center

$$\dot{x} = -1.77845\ldots x - 9.41287\ldots y + 2, \quad \dot{y} = x + 1.77845\ldots y - 0.640432\ldots, \tag{17}$$

its first integral is

$$H_3(x, y) = 4x^2 + 14.2276\ldots xy - 5.12346\ldots x + 37.6515\ldots y^2 - 16y.$$

The system of equations (8) has two real solutions $(\alpha_1, \beta_1, \gamma_1, \delta_1, f_1, g_1, h_1, k_1) = (0.297668\ldots, 0.33909\ldots, 0.0389395\ldots, -0.0396904\ldots, -0.610209\ldots, 0.380973\ldots, -0.0300808\ldots, -0.0296249\ldots)$ and $(\alpha_2, \beta_2, \gamma_2, \delta_2, f_2, g_2, h_2, k_2) = (0.242325\ldots, 0.270095\ldots, 0.0366105\ldots, 0.0372747\ldots, -0.386234\ldots, 0.302588\ldots, -0.0530128\ldots, 0.0515885\ldots)$. Then the discontinuous piecewise linear differential system formed by the linear differential centers (15), (16) and (17) have exactly two crossing limit cycles.

For the second configuration we consider the linear differential center

$$\dot{x} = -\frac{6x}{5} - \frac{369y}{100} + 0.152456, \quad \dot{y} = x + \frac{6y}{5} - 0.365572, \tag{18}$$

in the region R_1, with its first integral

$$H_1(x, y) = 4\left(x + \frac{6y}{5}\right)^2 + 8(-0.365572x - 0.152456y) + 9y^2.$$

In the region R_3 we consider the linear center

$$\dot{x} = -\frac{x}{5} - \frac{29y}{100}, \quad \dot{y} = x + \frac{y}{5}, \tag{19}$$

with its first integral

$$H_2(x,y) = 4\left(x + \frac{y}{5}\right)^2 + y^2.$$

In the region R_2 we consider the following linear differential center

$$\dot{x} = -y - 0.251587, \quad \dot{y} = x + 1.93017, \tag{20}$$

this differential system has the first integral

$$H_3(x,y) = 4x^2 + 8(1.93017x + 0.251587y) + 4y^2.$$

The discontinuous piecewise linear differential system formed by the linear differential centers (12), (13) and (14) have exactly two crossing limit cycles, because the system of equations (8) has two real solutions $(\alpha_1, \beta_1, \gamma_1, \delta_1, f_1, g_1, h_1, k_1) = (0.286549\ldots, 0.325022\ldots, 0.400999\ldots, -0.474638\ldots, -0.416749\ldots, 0.318275\ldots, -0.313035\ldots, -0.259454\ldots)$ and $(\alpha_2, \beta_2, \gamma_2, \delta_2, f_2, g_2, h_2, k_2) = (0.484398\ldots, 0.590171\ldots, 0.681486\ldots, -0.883698\ldots, -0.719407\ldots, 0.381077\ldots, -0.569542\ldots, -0.373673\ldots)$.

Finally to obtain the third configuration we consider in the region R_1 the linear differential center

$$\dot{x} = -2x - \frac{25y}{4} + 6.3854, \quad \dot{y} = x + 2y + 0.134162, \tag{21}$$

which has the first integral

$$H_1(x,y) = 4(x+2y)^2 + 8(0.134162x - 6.3854y) + 9y^2.$$

In the region R_3 we consider the linear differential center

$$\dot{x} = \frac{3}{2} - \frac{25y}{16}, \quad \dot{y} = x + \frac{1}{5}, \tag{22}$$

with its first integral $H_2(x,y) = 4x^2 + 8\left(\frac{x}{5} + \frac{3}{2}\right) + \frac{25y^2}{4}$.

In the region R_2 we consider the linear differential system

$$\dot{x} = -\frac{x}{2} - \frac{5y}{2} + 0.902952, \quad \dot{y} = x + \frac{y}{2} + 0.234228, \tag{23}$$

which has the first integral

$$H_3(x,y) = 4\left(x + \frac{y}{2}\right)^2 + 8(0.234228x - 0.902952y) + 9y^2.$$

The discontinuous piecewise linear differential system formed by the linear differential centers (21), (22) and (23) have exactly two crossing limit cycles, because the system of equations (8) has two real solutions $(\alpha_1, \beta_1, \gamma_1, \delta_1, f_1, g_1, h_1, k_1) = (0.844854\ldots, 1.14753\ldots, 0.137908\ldots, 0.14711\ldots, -0.884123\ldots, 0.300962\ldots, -0.112734\ldots, 0.106189\ldots)$ and $(\alpha_2, \beta_2, \gamma_2, \delta_2, f_2, g_2, h_2, k_2) = (0.783948\ldots, 1.04708\ldots, 0.219822\ldots, 0.242783\ldots, -0.824415\ldots, 0.345453\ldots, -0.186379.., 0.168115\ldots)$. This completes the proof of statement (b) for the class C_5.

Proof of statement (c) of Theorem 2 First we prove the statement for the class C_2. We consider the first linear differential center in the region R_1

$$\dot{x} = -3.95435x - 19.6368y - 6.39482, \quad \dot{y} = x + 3.95435y + 4.13635, \tag{24}$$

this system has the first integral

$$H_1(x, y) = 4(x + 3.95435y)^2 + 8(4.13635x + 6.39482y) + 16y^2.$$

The second linear differential center in the region R_2

$$\dot{x} = -\frac{x}{2} - \frac{5y}{2} - 1, \quad \dot{y} = x + \frac{y}{2} - \frac{3}{2}, \tag{25}$$

this differential system has the first integral

$$H_2(x, y) = 4\left(x + \frac{y}{2}\right)^2 + 8\left(y - \frac{3x}{2}\right) + 9y^2.$$

Now we consider the third linear differential center in the region R_3

$$\dot{x} = -0.241343x - 1.00852y - 0.358353, \quad \dot{y} = x + 0.241343y - 0.869754, \tag{26}$$

this differential system has the first integral

$$H_3(x, y) = 4(x + 0.241343y)^2 + 8(0.358353y - 0.869754x) + 4y^2.$$

The discontinuous piecewise linear differential system formed by the linear differential centers (24), (25) and (26) have exactly three crossing limit cycles, because the system of equations (4) has three real solutions $(\alpha_1, \beta_1, \gamma_1, \delta_1, f_1, g_1, h_1, k_1) = (3.00039\dots, 0.0485802\dots, 3.43695\dots, -1.91305\dots, 0.860569\dots, 0.506666\dots, 0.00442503\dots, 0.114878\dots)$, $(\alpha_2, \beta_2, \gamma_2, \delta_2, f_2, g_2, h_2, k_2) = (3.00318\dots, 0.138419\dots, 3.47181\dots, -2.01219\dots, 0.803882\dots, 0.588414\dots, 0.0319264\dots, 0.302877\dots)$ and $(\alpha_3, \beta_3, \gamma_3, \delta_3, f_3, g_3, h_3, k_3) = (3.00805\dots, 0.220494\dots, 3.50419\dots, -2.1034\dots, 0.735716\dots, 0.663523\dots, 0.0779996\dots, 0.458408\dots)$.

Now we prove the existence of four different configurations of three crossing limit cycles for the class C_5. For the first configuration we consider the linear differential center

$$\dot{x} = -0.345578x - 1.11942y - 0.128163, \quad \dot{y} = x + 0.345578y - 0.440337, \tag{27}$$

in the region R_1, with its first integral

$$H_1(x, y) = 4(x + 0.345578y)^2 + 8(0.128163y - 0.440337x) + 4y^2.$$

In the region R_3 we consider the linear center

$$\dot{x} = -\frac{x}{5} - \frac{29y}{100}, \quad \dot{y} = x + \frac{y}{5}, \tag{28}$$

with its first integral

$$H_2(x, y) = 4\left(x + \frac{y}{5}\right)^2 + y^2.$$

In the region R_2 we consider the following linear differential center

$$\dot{x} = -0.0923038x - 1.00852y - 0.0805185, \quad \dot{y} = x + 0.0923038y + 0.46371, \tag{29}$$

this differential system has the first integral

$$H_3(x, y) = 4(x + 0.0923038y)^2 + 8(0.46371x + 0.0805185y) + 4y^2.$$

The discontinuous piecewise linear differential system formed by the linear differential centers (27), (28) and (29) has exactly three crossing limit cycles, because the system of equations (8) has three real solutions $(\alpha_1, \beta_1, \gamma_1, \delta_1, f_1, g_1, h_1, k_1) = (0.286549\ldots, 0.325022\ldots, 0.681486\ldots, -0.883698\ldots, -0.416749\ldots, 0.318275\ldots, -0.569542\ldots, -0.373673\ldots)$, $(\alpha_2, \beta_2, \gamma_2, \delta_2, f_2, g_2, h_2, k_2) = (0.366248\ldots, 0.428095\ldots, 0.749792\ldots, -0.991822\ldots, -0.538507\ldots, 0.365825\ldots, -0.639545\ldots, -0.383969\ldots)$ and $(\alpha_3, \beta_3, \gamma_3, \delta_3, f_3, g_3, h_3, k_3) = (0.429988\ldots, 0.514188\ldots, 0.81153\ldots, -1.09226\ldots, -0.636378\ldots, 0.383743\ldots, -0.706638\ldots, -0.382736\ldots)$.

For the second configuration. In the region R_1 we consider the linear differential center

$$\dot{x} = -0.121241x - 2.2647y + 1.93257, \quad \dot{y} = x + 0.121241y + 0.00611219, \tag{30}$$

which has the first integral

$$H_1(x, y) = 4(x + 2y)^2 + 8(0.134162x - 6.3854y) + 9y^2.$$

In the region R_3 we consider the linear differential center

$$\dot{x} = \frac{3}{2} - \frac{25y}{16}, \quad \dot{y} = x + \frac{1}{5}, \tag{31}$$

with its first integral

$$H_2(x, y) = 4x^2 + 8\left(\frac{x}{5} - \frac{3y}{2}\right) + \frac{25y^2}{4}.$$

In the region R_2 we consider the linear differential system

$$\dot{x} = -0.217737x - 1.04741y + 0.515416, \quad \dot{y} = x + 0.217737y + 0.350398, \tag{32}$$

which has the first integral

$$H_3(x, y) = 4(x + 0.217737y)^2 + 8(0.350398x - 0.515416y) + 4y^2.$$

The discontinuous piecewise linear differential system formed by the linear differential centers (30), (31) and (32) have exactly three crossing limit cycles, because the system of equations (8) has three real solutions $(\alpha_1, \beta_1, \gamma_1, \delta_1, f_1, g_1, h_1, k_1) = (0.844854\ldots, 1.14753\ldots, 0.137908\ldots, 0.14711\ldots, -0.884123\ldots, 0.300962\ldots, -0.112734\ldots, 0.106189\ldots)$, $(\alpha_2, \beta_2, \gamma_2, \delta_2, f_2, g_2, h_2, k_2) = (0.816222\ldots, 1.1\ldots, 0.177162\ldots, 0.192216\ldots, -0.856934\ldots, 0.324128\ldots, -0.147429\ldots, 0.136128\ldots)$ and $(\alpha_3, \beta_3, \gamma_3, \delta_3, f_3, g_3, h_3, k_3) = (0.783948\ldots, 1.04708\ldots, 0.219822\ldots, 0.242783\ldots, -0.824415\ldots, 0.345453\ldots, -0.186379\ldots, 0.168115\ldots)$.

For the third configuration. In the region R_1 we consider the linear differential center

$$\dot{x} = -6.11391\ldots x - 39.6298\ldots y + 26.1607\ldots, \quad \dot{y} = x + 6.11391\ldots y + 6.97061\ldots, \tag{33}$$

its first integral is

$$H_1(x,y) = 4x^2 + 14.1837\ldots xy - 1.20208\ldots x + 13.5735\ldots y^2 - 8y.$$

In the region R_3 we consider the linear differential center

$$\dot{x} = -\frac{x}{10} - \frac{113y}{50} + \frac{3}{2}, \quad \dot{y} = x + \frac{y}{10} + \frac{3}{10}, \tag{34}$$

with its first integral

$$H_2(x,y) = 4x^2 + \frac{4xy}{5} + \frac{12x}{5} + \frac{226y^2}{25} - 12y.$$

In the region R_2 we consider the linear differential system

$$\dot{x} = 0.0191156\ldots x - 1.56287\ldots y + 1.64182\ldots, \quad \dot{y} = x - 0.0191156\ldots y + 0.287842\ldots, \tag{35}$$

which has the first integral

$$H_3(x,y) = 4x^2 + 14.2276\ldots xy - 5.12346\ldots x + 37.6515\ldots y^2 - 16y.$$

The discontinuous piecewise linear differential system formed by the linear differential centers (33), (34) and (35) have exactly three crossing limit cycles, because the system of equations (8) has three real solutions $(\alpha_1, \beta_1, \gamma_1, \delta_1, f_1, g_1, h_1, k_1) = (0.566406\ldots, 0.708892\ldots, 0.117204\ldots, 0.123882\ldots, -0.948744\ldots, 0.214793\ldots, -0.0766687\ldots, 0.0736711\ldots)$, $(\alpha_2, \beta_2, \gamma_2, \delta_2, f_2, g_2, h_2, k_2) = (0.715789\ldots, 0.937598\ldots, 0.0641072\ldots, -0.0661301\ldots, -0.997735\ldots, 0.0474848\ldots, -0.0974716\ldots, -0.0925994\ldots)$ and $(\alpha_3, \beta_3, \gamma_3, \delta_3, f_3, g_3, h_3, k_3) = (0.767868\ldots, 1.02097\ldots, 0.119835\ldots, -0.126812\ldots, -0.999038\ldots, -0.0309942\ldots, -0.187296\ldots, -0.168847)$.

Finally we give an example for the fourth configuration. In the region R_1 we consider the linear differential center

$$\dot{x} = 0.69208\ldots x - 1.47897\ldots y + 0.293478\ldots, \quad \dot{y} = x - 0.69208\ldots y + 0.00888246\ldots, \tag{36}$$

which has the first integral

$$H_1(x,y) = 4x^2 - 5.53664\ldots xy + 0.0710597\ldots x + 5.9159\ldots y^2 - 2.34783\ldots y.$$

In the region R_3 we consider the linear differential center

$$\dot{x} = -\frac{x}{5} - \frac{629y}{100} + \frac{23}{10}, \quad \dot{y} = x + \frac{y}{5} + \frac{3}{10}, \tag{37}$$

its first integral is

$$H_2(x,y) = 4x^2 + \frac{8xy}{5} + \frac{12x}{5} + \frac{629y^2}{25} - 18.4y.$$

In the region R_2 we consider the linear differential system

$$\dot{x} = 0.242453\ldots x - 0.308783\ldots y + 1.11385\ldots, \quad \dot{y} = x - 0.242453\ldots y + 0.433519\ldots, \tag{38}$$

it has the first integral

$$H_3(x,y) = 4x^2 - 1.93962\ldots xy + 3.46816\ldots x + 1.23513\ldots y^2 - 8.91079\ldots y.$$

The system of equations (8) has three real solutions $(\alpha_1,\beta_1,\gamma_1,\delta_1,f_1,g_1,h_1,k_1) = (0.319705\ldots, 0.367272\ldots, 0.17682\ldots, 0.191816\ldots, -0.938407\ldots, 0.232893\ldots, -0.119654\ldots, 0.112267\ldots)$, $(\alpha_2,\beta_2,\gamma_2,\delta_2,f_2,g_2,h_2,k_2) = (0.411196\ldots, 0.488475\ldots, 0.0696268\ldots, 0.07201\ldots, -0.975404\ldots, 0.152973\ldots, -0.0528588\ldots, 0.0514428\ldots)$ and $(\alpha_3,\beta_3,\gamma_3,\delta_3,f_3,g_3,h_3,k_3) = (0.505367\ldots, 0.620052\ldots, 0.0444858\ldots, -0.0454646\ldots, -0.999041\ldots, 0.0309382\ldots, -0.0582745\ldots, -0.0565511\ldots)$. Hence the discontinuous piecewise linear differential system formed by the linear differential centers (36), (37) and (38) have exactly three crossing limit cycles. This completes the proof of statement (c) for the class C_5.

Proof of statement (d) of Theorem 2 First we prove the statement for class C_2. We consider the first linear differential center in the region R_1

$$\dot{x} = -3.95435x - 19.6368y - 6.39482, \quad \dot{y} = x + 3.95435y + 4.13635, \tag{39}$$

this system has the first integral

$$H_1(x,y) = 4(x + 3.95435y)^2 + 8(4.13635x - 6.39482y) + 16y^2.$$

The second linear differential center in the region R_2

$$\dot{x} = -\frac{x}{2} - \frac{5y}{2} - 1, \quad \dot{y} = x + \frac{y}{2} - \frac{3}{2}, \tag{40}$$

this differential system has the first integral

$$H_2(x,y) = 4\left(x + \frac{y}{2}\right)^2 + 8\left(y - \frac{3x}{2}\right) + 9y^2.$$

Now we consider the third linear differential center in the region R_3

$$\dot{x} = -0.241343x - 1.00852y - 0.358353, \quad \dot{y} = x + 0.241343y - 0.869754, \tag{41}$$

this differential system has the first integral

$$H_3(x,y) = 4(x + 0.241343y)^2 + 8(0.358353y - 0.869754x) + 4y^2.$$

The discontinuous piecewise linear differential system formed by the linear differential centers (39), (40) and (41) have exactly four crossing limit cycles, because the system of equations (4) has four real solutions $(\alpha_1,\beta_1,\gamma_1,\delta_1,f_1,g_1,h_1,k_1) = (3.00039\ldots, 0.0485802\ldots, 3.43695\ldots, -1.91305\ldots, 0.860569\ldots, 0.506666\ldots, 0.00442503\ldots, 0.114878\ldots)$, $(\alpha_2,\beta_2,\gamma_2,\delta_2,f_2,g_2,h_2,k_2) = (3.00318\ldots, 0.138419\ldots, 3.47181\ldots, -2.01219\ldots, 0.803882\ldots, 0.588414\ldots, 0.0319264\ldots, 0.302877\ldots)$, $(\alpha_3,\beta_3,\gamma_3,\delta_3,f_3,g_3,h_3,k_3) = (3.00805\ldots, 0.220494\ldots, 3.50419\ldots, -2.1034\ldots, 0.735716\ldots, 0.663523\ldots, 0.0779996\ldots, 0.458408\ldots)$ and $(\alpha_4,\beta_4,\gamma_4,\delta_4,f_4,g_4,h_4,k_4) = (3.01815\ldots, 0.332542\ldots, 3.5491\ldots, -2.22883\ldots, 0.593338\ldots, 0.762036\ldots, 0.192725\ldots, 0.66088\ldots)$.

Now we give four different configurations of four limit cycles for the class C_5. To obtain the first configuration we consider the linear differential center

$$\dot{x} = 0.897851x - 2.12174y - 0.62272, \quad \dot{y} = x - 0.897851y - 0.620434, \tag{42}$$

in the region R_1, with its first integral

$$H_3(x,y) = 4(x - 0.897851y)^2 + 8(0.62272y - 0.620434x) + 5.2624y^2.$$

In the region R_3 we consider the linear center

$$\dot{x} = \frac{x}{10} - \frac{17y}{450}, \quad \dot{y} = x - \frac{y}{10}, \tag{43}$$

with its first integral

$$H_2(x,y) = 4\left(x - \frac{y}{10}\right)^2 + \frac{y^2}{9}.$$

In the region R_2 we consider the following linear differential center

$$\dot{x} = -0.225709x - 1.26066y - 0.265909, \quad \dot{y} = x + 0.225709y + 0.334228, \tag{44}$$

this differential system has the first integral

$$H_3(x,y) = 4(x + 0.225709y)^2 + 8(0.334228x + 0.265909y) + 4.83887y^2.$$

The discontinuous piecewise linear differential system formed by the linear differential centers (42), (43) and (44) have exactly four crossing limit cycles, because the system of equations (8) has four real solutions $(\alpha_1, \beta_1, \gamma_1, \delta_1, f_1, g_1, h_1, k_1) = (0.275692\ldots, 0.311385\ldots, 0.490206\ldots, -0.598415\ldots, -0.22775\ldots, 0.200142\ldots, -0.593285\ldots, -0.378363\ldots)$, $(\alpha_2, \beta_2, \gamma_2, \delta_2, f_2, g_2, h_2, k_2) = (0.391233\ldots, 0.461461\ldots, 0.535875\ldots, -0.664112\ldots, -0.324107\ldots, 0.266457\ldots, -0.647454\ldots, -0.38443\ldots)$, $(\alpha_3, \beta_3, \gamma_3, \delta_3, f_3, g_3, h_3, k_3) = (0.480359\ldots, 0.584453\ldots, 0.577721\ldots, -0.72566\ldots, -0.399001\ldots, 0.309322\ldots, -0.696708\ldots, -0.383691\ldots)$ and $(\alpha_4, \beta_4, \gamma_4, \delta_4, f_4, g_4, h_4, k_4) = (0.555792\ldots, 0.693247\ldots, 0.616543\ldots, -0.783893\ldots, -0.46287\ldots, 0.339233\ldots, -0.742\ldots, -0.376889\ldots)$.

Now we give an example for the second configuration. In the region R_1 we consider the linear differential center

$$\dot{x} = 0.679261\ldots x - 20.2696\ldots y + 7.31483\ldots, \quad \dot{y} = x - 0.679261\ldots y - 0.688849\ldots, \tag{45}$$

which has the first integral

$$H_1(x,y) = 4x^2 - 5.43409\ldots xy - 5.5108\ldots x + 81.0784\ldots y^2 - 58.5187\ldots y.$$

In the region R_3 we consider the linear differential center

$$\dot{x} = \frac{2x}{5} - \frac{269569y}{40000} + \frac{14}{5}, \quad \dot{y} = x - \frac{2y}{5} + \frac{17}{100}, \tag{46}$$

with its first integral

$$H_2(x,y) = 4x^2 - \frac{16xy}{5} + \frac{34x}{25} + \frac{269569y^2}{10000} - \frac{112y}{5}.$$

In the region R_2 we consider the linear differential system

$$\dot{x} = -0.0259413\ldots x - 3.37607\ldots y + 1.05365\ldots, \quad \dot{y} = x + 0.0259413\ldots y + 0.350162\ldots, \tag{47}$$

which has the first integral

$$H_3(x,y) = 4x^2 + 0.20753\ldots xy + 2.8013\ldots x + 13.5043\ldots y^2 - 8.4292\ldots y.$$

The discontinuous piecewise linear differential system formed by the linear differential centers (45), (46) and (47) have exactly four crossing limit cycles, because the system of equations (8) has three real solutions $(\alpha_1, \beta_1, \gamma_1, \delta_1, f_1, g_1, h_1, k_1) = (0.448209\ldots, 0.539381\ldots, 0.227449\ldots, 0.251992\ldots, -0.519234\ldots, 0.360023\ldots, -0.268115\ldots, 0.229373\ldots)$, $(\alpha_2, \beta_2, \gamma_2, \delta_2, f_2, g_2, h_2, k_2) = (0.53821\ldots, 0.667513\ldots, 0.117618\ldots, 0.124343\ldots, -0.7962\ldots, 0.359438\ldots, -0.116994\ldots, 0.109938\ldots)$, $(\alpha_3, \beta_3, \gamma_3, \delta_3, f_3, g_3, h_3, k_3) = (0.595573\ldots, 0.752304\ldots, 0.0393207\ldots, 0.0400863\ldots, -0.938909\ldots, 0.232066\ldots, -0.0361079\ldots, 0.0354501\ldots)$ and $(\alpha_4, \beta_4, \gamma_4, \delta_4, f_4, g_4, h_4, k_4) = (0.741939\ldots, 0.979229\ldots, 0.187029\ldots, -0.20377\ldots, -0.989482\ldots, -0.101478\ldots, -0.264651\ldots, -0.226945\ldots)$.

For the third configuration we consider in the region R_1 the linear differential center

$$\dot{x} = 0.216672\ldots x - 0.113954\ldots y + 0.0710259\ldots, \quad \dot{y} = x - 0.216672\ldots y - 0.125477\ldots, \tag{48}$$

which has the first integral

$$H_1(x,y) = 4x^2 - 1.73337\ldots xy - 1.00381\ldots x + 0.455816\ldots y^2 - 0.568207\ldots y.$$

In the region R_3 we consider the linear differential center

$$\dot{x} = \frac{9x}{20} - \frac{1681y}{400} + \frac{193}{100}, \quad \dot{y} = x - \frac{9y}{20} + \frac{17}{100}, \tag{49}$$

with its first integral

$$H_2(x,y) = 4x^2 - \frac{18xy}{5} + \frac{34x}{25} + \frac{1681y^2}{100} - \frac{386y}{25}.$$

We consider in the region R_2 the linear differential system

$$\dot{x} = 0.469674\ldots x - 0.937777\ldots y + 0.544803\ldots, \quad \dot{y} = x - 0.469674\ldots y + 0.465659\ldots, \tag{50}$$

which has the first integral

$$H_3(x,y) = 4x^2 - 3.75739\ldots xy + 3.72527\ldots x + 3.75111\ldots y^2 - 4.35843\ldots y.$$

The discontinuous piecewise linear differential system formed by the linear differential centers (48), (49) and (50) have exactly four crossing limit cycles, because the system of equations (8) has three real solutions $(\alpha_1, \beta_1, \gamma_1, \delta_1, f_1, g_1, h_1, k_1) = (0.649616\ldots, 0.83435\ldots, 0.0409878\ldots, 0.0418194\ldots, -0.844622\ldots, 0.332933\ldots, -0.0357296\ldots, 0.0350855\ldots)$, $(\alpha_2, \beta_2, \gamma_2, \delta_2, f_2, g_2, h_2, k_2) = (0.713072\ldots, 0.933299v, 0.109221\ldots, -0.115031\ldots, -0.966853\ldots, 0.176027\ldots, -0.148221\ldots, -0.136796\ldots)$, $(\alpha_3, \beta_3, \gamma_3, \delta_3, f_3, g_3, h_3, k_3) = (0.808668\ldots, 1.08755\ldots, 0.208142\ldots, -0.22878\ldots, -0.993682\ldots, -0.0789808\ldots, -0.328436\ldots, -0.26915\ldots)$ and $(\alpha_4, \beta_4, \gamma_4, \delta_4, f_4, g_4, h_4, k_4) = (0.915369\ldots, 1.26684\ldots, 0.286357\ldots, -0.324779\ldots, -0.831632\ldots, -0.341241\ldots, -0.5652\ldots, -0.372689\ldots)$.

Finally for the fourth configuration we consider the linear differential center

$$\dot{x} = 0.754941 \ldots x - 3.40524 \ldots y + 1.04524 \ldots, \quad \dot{y} = x - 0.754941 \ldots y - 0.492103 \ldots, \tag{51}$$

in the region R_1. Its the first integral is

$$H_1(x, y) = 4x^2 - 6.03952 \ldots xy - 3.93682 \ldots x + 13.621 \ldots y^2 - 8.36195 \ldots y.$$

In the region R_3 we consider the linear differential center

$$\dot{x} = \frac{7x}{5} - \frac{149y}{25} + \frac{23}{10}, \quad \dot{y} = x - \frac{7y}{5} + \frac{3}{20}, \tag{52}$$

with its first integral

$$H_2(x, y) = 4x^2 - \frac{28xy}{25} + \frac{6x}{5} + \frac{10049y^2}{625} - \frac{92y}{5}.$$

We consider the linear differential system

$$\dot{x} = -0.110936 \ldots x - 1.51937 \ldots y + 0.51995 \ldots \quad \dot{y} = x + 0.110936 \ldots y + 0.40335 \ldots, \tag{53}$$

in the region R_2, which has the first integral

$$H_3(x, y) = 4x^2 + 0.88749 \ldots xy + 3.2268 \ldots x + 6.07749 \ldots y^2 - 4.1596 \ldots y.$$

The discontinuous piecewise linear differential system formed by the linear differential centers (51), (52) and (53) have exactly four crossing limit cycles, due to the fact that the system of equations (8) has three real solutions $(\alpha_1, \beta_1, \gamma_1, \delta_1, f_1, g_1, h_1, k_1) = (0.527644 \ldots, 0.652157 \ldots, 0.298869 \ldots, 0.340614 \ldots, -0.560417 \ldots, 0.371562 \ldots, -0.373144 \ldots, 0.295434 \ldots)$, $(\alpha_2, \beta_2, \gamma_2, \delta_2, f_2, g_2, h_2, k_2) = (0.631385 \ldots, 0.806441 \ldots, 0.173646 \ldots, 0.18812 \ldots, -0.800529 \ldots, 0.357534 \ldots, -0.177469 \ldots, 0.160953 \ldots)$, $(\alpha_3, \beta_3, \gamma_3, \delta_3, f_3, g_3, h_3, k_3) = (0.789235 \ldots, 1.0557 \ldots, 0.096985 \ldots, -0.101579 \ldots, -0.987466 \ldots, 0.110553 \ldots, -0.121412 \ldots, -0.113803 \ldots)$ and $(\alpha_4, \beta_4, \gamma_4, \delta_4, f_4, g_4, h_4, k_4) = (0.860054 \ldots, 1.17297 \ldots, 0.216049 \ldots, -0.238247 \ldots, -0.998835 \ldots, -0.0340962 \ldots, -0.31406 \ldots, -0.260109 \ldots)$. This completes the proof of statement (d).

Proof of Theorem 3

Proof of statement (a) of Theorem 3 First we prove the statement for the class C_1. We consider the first linear differential center in the region R_2

$$\begin{aligned} \dot{x} &= -\frac{x}{7} - \frac{113y}{3136} + \frac{\left(\sqrt{13}+1\right)\left(-187\sqrt{3} + 192\sqrt{13} - 64\right)}{5376}, \\ \dot{y} &= x + \frac{y}{7} + \frac{B}{75264}, \end{aligned} \tag{54}$$

with $B = \left(\sqrt{13}+1\right)\left(-10651\sqrt{13} + 896\sqrt{14} - 1792\sqrt{39} - 2618\sqrt{42} + 896\sqrt{182} + 18505\right)$. This system has the first integral

$$H_1(x,y)=\frac{1}{784(\sqrt{13}-1)}(3136(\sqrt{13}-1)x^2+y(113(\sqrt{13}-1)y+14(187\sqrt{3}\\ -192\sqrt{13}+64))+x(896(\sqrt{13}-1)y-10651\sqrt{13}+896\sqrt{14}\\ -1792\sqrt{39}-2618\sqrt{42}+896\sqrt{182}+18505)).$$

The second linear differential center in the region R_1 is

$$\begin{aligned}\dot{x}&=-\frac{x}{6}-\frac{25y}{576}-\frac{-288\sqrt{13}+\frac{751}{\sqrt{3}}+96}{576(\sqrt{13}-1)},\\ \dot{y}&=x+\frac{y}{6}-\frac{6009\sqrt{13}+576\sqrt{14}+1152\sqrt{39}-1502\sqrt{42}+576\sqrt{182}-10515}{3456(\sqrt{13}-1)},\end{aligned} \tag{55}$$

this differential system has the first integral

$$H_2(x,y)=\frac{1}{432(\sqrt{13}-1)}(1728(\sqrt{13}-1)x^2-x(-576(\sqrt{13}-1)y+6009\sqrt{13}\\ +576\sqrt{14}+1152\sqrt{39}-1502\sqrt{42}+576\sqrt{182}-10515)+y(75(\sqrt{13}\\ -1)y+2(751\sqrt{3}-864\sqrt{13}+288))).$$

For the piecewise linear differential system (54)–(55) the unique real solution of the system of equations

$$\begin{aligned}&H_1(\alpha_1,\beta_1)-H_1(\gamma_1,\delta_1)=0,\\ &H_1(\alpha_2,\beta_2)-H_1(\gamma_2,\delta_2)=0,\\ &H_2(\alpha_1,\beta_1)-H_2(\alpha_2,\beta_2)=0,\\ &H_2((\gamma_1,\delta_1)-H_2(\gamma_2,\delta_2)=0,\\ &c_i(\alpha_1,\beta_1)=0, c_i(\alpha_2,\beta_2)=0,\\ &c_i(\gamma_1,\delta_1)=0, c_i(\gamma_2,\delta_2)=0,\end{aligned} \tag{56}$$

when $i=1$, is $(\alpha_1,\beta_1,\alpha_2,\beta_2,\gamma_1,\delta_1,\gamma_2,\delta_2)=(1,\sqrt{3},2,-\sqrt{14},2,\sqrt{14},3,-\sqrt{39})$.

Now we prove the statement for the class C_2. We consider the first linear differential center in the region R_1

$$\dot{x}=\frac{x}{6}-\frac{5y}{18}-1,\quad \dot{y}=x-\frac{y}{6}+\frac{1}{18}\left(12\sqrt{3}+6\sqrt{10}-151\right), \tag{57}$$

this differential system has the first integral $H_1(x,y)=\frac{2}{9}(18x^2+2x(-3y+12\sqrt{3}+6\sqrt{10}-151)+y(5y+36))$.

The second linear differential center in the region R_2 is

$$\begin{aligned}\dot{x}&=\frac{x}{6}-\frac{13y}{144}+\frac{-82\sqrt{3}-205\sqrt{10}-96\sqrt{30}-5664}{4896},\\ \dot{y}&=x-\frac{y}{6}+\frac{-432\sqrt{10}+\sqrt{30\left(3936\sqrt{10}+24721\right)}-17964}{2448},\end{aligned} \tag{58}$$

this system has the first integral

$$
\begin{aligned}
H_2(x,y) = {} & \frac{1}{36(2\sqrt{3}-5\sqrt{10})}(144(2\sqrt{3}-5\sqrt{10})x^2 - 4x(24\sqrt{3}y - 60\sqrt{10}y \\
& + 1117\sqrt{3} - 574 - 2649\sqrt{10} + 96\sqrt{30} - 720) + y(26\sqrt{3}y - 65\sqrt{10}y \\
& + 384\sqrt{3} - 1632\sqrt{10})).
\end{aligned}
$$

The unique real solution of the system of equations (56) for $i = 2$, for the piecewise linear differential system (57)–(58) is $(\alpha_1, \beta_1, \alpha_2, \beta_2, \gamma_1, \delta_1, \gamma_2, \delta_2) = (4, 2\sqrt{3}, 5, -2\sqrt{10}, 6, 3\sqrt{10}, 6, -3\sqrt{10})$
.

For the second configuration of the class C_2, we consider the linear differential center in the region R_2

$$
\begin{aligned}
\dot{x} &= \frac{x}{5} - \frac{41y}{400} + \frac{1440\sqrt{3} - 640\sqrt{10} - 160\sqrt{33} + 2503}{3200\left(-3\sqrt{3} + 2\sqrt{10} + \sqrt{33}\right)}, \\
\dot{y} &= x - \frac{y}{5} + \frac{160\sqrt{10} + \dfrac{2503\sqrt{10} + 960\sqrt{30} - 3200}{-3\sqrt{3} + 2\sqrt{10} + \sqrt{33}} - 4390}{6400},
\end{aligned}
\tag{59}
$$

this differential system has the first integral

$$
\begin{aligned}
H_1(x,y) = {} & \frac{1}{800(-3\sqrt{3} + 2\sqrt{10} + \sqrt{33})}(3200(-3\sqrt{3} + 2\sqrt{10} + \sqrt{33})x^2 + x(13170\sqrt{3} \\
& - 6277\sqrt{10} + 480\sqrt{30} - 4390\sqrt{33} + 160\sqrt{330} - 1280(-3\sqrt{3} + 2\sqrt{10} + \\
& \sqrt{33})y) + 2y(164(-3\sqrt{3} + 2\sqrt{10} + \sqrt{33})y - 1440\sqrt{3} + 640\sqrt{10} + 160\sqrt{33} \\
& - 2503)).
\end{aligned}
$$

The second linear differential center in the region R_3 is

$$
\begin{aligned}
\dot{x} &= \frac{x}{5} - \frac{13y}{100} + \frac{360\sqrt{3} - 160\sqrt{10} - 40\sqrt{33} + 579}{800\left(-3\sqrt{3} + 2\sqrt{10} + \sqrt{33}\right)}, \\
\dot{y} &= x - \frac{y}{5} - \frac{-3873\sqrt{3} + 4898\sqrt{10} + 1440\sqrt{11} + 480\sqrt{30} + 2449\sqrt{33}}{3200\left(-3\sqrt{3} + 2\sqrt{10} + \sqrt{33}\right)},
\end{aligned}
\tag{60}
$$

this system has the first integral

$$
\begin{aligned}
H_2(x,y) = {} & \frac{1}{400(-3\sqrt{3} + 2\sqrt{10} + \sqrt{33})}1600(-3\sqrt{3} + 2\sqrt{10} + \sqrt{33})x^2 - x(640(-3\sqrt{3} \\
& + 2\sqrt{10} + \sqrt{33})y - 3873\sqrt{3} + 4898\sqrt{10} + 1440\sqrt{11} + 480\sqrt{30} + 2449\sqrt{33}) \\
& + 4y\left(52\left(-3\sqrt{3} + 2\sqrt{10} + \sqrt{33}\right)y - 360\sqrt{3} + 160\sqrt{10} + 40\sqrt{33} - 579\right).
\end{aligned}
$$

The unique real solution of the system of equations (56) for $i = 2$, for the piecewise linear differential system (59)–(60) is $(\alpha_1, \beta_1, \alpha_2, \beta_2, \gamma_1, \delta_1, \gamma_2, \delta_2) = (1/4, \sqrt{33}/8, 1/2, -\sqrt{5}/(2\sqrt{2}), 3/4, 3\sqrt{3}/8, 1, 0)$.

We prove the statement for the class C_3. We consider the first linear differential center in the region R_1

$$
\begin{aligned}
\dot{x} &= -\frac{x}{6} - \frac{25y}{576} + \frac{-128\sqrt{2}+288\sqrt{3}+483}{-384\sqrt{2}+576\sqrt{3}+1344},\\
\dot{y} &= x + \frac{y}{6} + \frac{-96618\sqrt{2}+9\sqrt{3(58745128\sqrt{2}+89093337)}-500192}{192384},
\end{aligned} \tag{61}
$$

this differential system has the first integral

$$
\begin{aligned}
H_1(x,y) = {} & \frac{1}{144(2\sqrt{2}-3\sqrt{3}-7)}(576(2\sqrt{2}-3\sqrt{3}-7)x^2 + x(192(2\sqrt{2}-3\sqrt{3}\\
& -7)y - 3770\sqrt{2}+6861\sqrt{3}+1152\sqrt{6}+14875) + y(25(2\sqrt{2}-3\sqrt{3}\\
& -7)y - 768\sqrt{2}+1728\sqrt{3}+2898).
\end{aligned}
$$

The second linear differential center in the region R_2 is

$$
\begin{aligned}
\dot{x} &= -\frac{x}{9} - \frac{145y}{5184} + \frac{134932\sqrt{2}-69489\sqrt{3}-77682\sqrt{6}+250949}{577152},\\
\dot{y} &= x + \frac{y}{9} + \frac{-23197\sqrt{2}+45834\sqrt{3}-1728\sqrt{6}+79814}{5184(2\sqrt{2}-3\sqrt{3}-7)},
\end{aligned} \tag{62}
$$

this system has the first integral

$$
\begin{aligned}
H_2(x,y) = {} & \frac{1}{648(-2\sqrt{2}-18\sqrt{3}+6\sqrt{6}-20)}(5184(-\sqrt{2}-9\sqrt{3}+3\sqrt{6}-10)x^2\\
& + x(1152(-\sqrt{2}-9\sqrt{3}+3\sqrt{6}-10)y + 7645\sqrt{2}+193608\sqrt{3}+332692\\
& - 67863\sqrt{6}) + y(145(-\sqrt{2}-9\sqrt{3}+3\sqrt{6}-10)y + 2304\sqrt{2}+10755\sqrt{3}\\
& - 6912\sqrt{6}+41343)).
\end{aligned}
$$

This piecewise linear differential centers has a unique real solution of the system of equations (56) for $i = 3$, which is $(\alpha_1, \beta_1, \alpha_2, \beta_2, \gamma_1, \delta_1, \gamma_2, \delta_2) = (1, 1, 2, -2\sqrt{2}, 3, 3\sqrt{3}, 4, -8)$.

Now we prove the statement for the class C_4. We consider the linear differential center in the region R_1

$$
\dot{x} = -\frac{x}{9} - \frac{181y}{8100} + \frac{2}{9}, \quad \dot{y} = x + \frac{y}{9} + \frac{\sqrt{2}}{3} - \frac{2071}{540} + \frac{8}{3\sqrt{3}}, \tag{63}
$$

this differential system has the first integral

$$
H_1(x,y) = 4x^2 + \frac{2}{135}x\left(60y + 180\sqrt{2} + 480\sqrt{3} - 2071\right) + \frac{y(181y-3600)}{2025}.
$$

The second linear differential center in the region R_2 is

$$
\begin{aligned}
\dot{x} &= -\frac{x}{8} - \frac{y}{32} + \frac{66\sqrt{2}-62\sqrt{3}-18\sqrt{6}+525}{1128},\\
\dot{y} &= x + \frac{y}{8} + \frac{-600\sqrt{6}+8\sqrt{6\left(232\sqrt{6}+2555\right)}-12627}{4512},
\end{aligned} \tag{64}
$$

this system has the first integral

$$H_2(x,y) = \frac{1}{8(-15\sqrt{2}-4\sqrt{3}+6\sqrt{6}+24)}32(-15\sqrt{2}-4\sqrt{3}+6\sqrt{6}+24)x^2 \\ +2x(4(-15\sqrt{2}-4\sqrt{3}+6\sqrt{6}+24)y+1521\sqrt{2}+556\sqrt{3}-666\sqrt{6} \\ -2456)+y((-15\sqrt{2}-4\sqrt{3}+6\sqrt{6}+24)y-8(-51\sqrt{2}-16\sqrt{3} \\ +24\sqrt{6}+76)).$$

In this case the unique real solution of the system of equations (56) for $i=4$, is $(\alpha_1,\beta_1,\alpha_2,\beta_2,\gamma_1,\delta_1,\gamma_2,\delta_2)=(2,2,2,-2,3,3\sqrt{2},4,-4\sqrt{3})$.

We prove the statement for the first configuration of the class C_5. We consider the linear differential center in the region R_3

$$\begin{aligned} \dot{x} &= -\frac{x}{6}-\frac{13y}{144}+\frac{53\sqrt{2}+1134}{4896}, \\ \dot{y} &= x+\frac{y}{6}+\frac{-114\sqrt{2}+91\sqrt{3}-24\sqrt{6}+843}{72(\sqrt{2}-6)}, \end{aligned} \tag{65}$$

this system has the first integral

$$H_1(x,y) = \frac{1}{36(\sqrt{2}-6)}(144(\sqrt{2}-6)x^2-4x(-12(\sqrt{2}-6)y+114\sqrt{2}-91\sqrt{3} \\ +24\sqrt{6}-843)+y(13(\sqrt{2}-6)y-48\sqrt{2}+394)).$$

The second linear differential center in the region R_1 is

$$\begin{aligned} \dot{x} &= \frac{x}{6}-\frac{5y}{18}+\frac{1}{612}\left(-89\sqrt{2}-636\right), \\ \dot{y} &= x-\frac{y}{6}+\frac{1}{306}(432\sqrt{3}+178\sqrt{3(\sqrt{3}+2)}-795), \end{aligned} \tag{66}$$

this differential system has the first integral

$$H_2(x,y) = \frac{1}{9(\sqrt{2}-6)}(36(\sqrt{2}-6)x^2-4x(3(\sqrt{2}-6)y+141\sqrt{2}+142\sqrt{3} \\ +6\sqrt{6}-312)+2y(5(\sqrt{2}-6)y+6\sqrt{2}-214)).$$

For this piecewise linear differential centers the unique real solution of system (56) when $i=5$ is $(\alpha_1,\beta_1,\alpha_2,\beta_2,\gamma_1,\delta_1,\gamma_2,\delta_2)=(1,\sqrt{2},2,-2\sqrt{3},2,2\sqrt{3},3,-6)$. Hence, the discontinuous piecewise linear differential system (65)–(66) has a unique crossing limit cycle, see $(\mathcal{C}_5^1)$ of Fig. 16.

Finally we prove the statement for the second configuration of the class C_5 .In the region R_3 we consider the linear differential center

$$\dot{x} = \frac{x}{2}-\frac{5y}{4}+\frac{2}{5}, \quad \dot{y} = x-\frac{y}{2}+\frac{1}{640}(96\sqrt{2}+88\sqrt{3}+365), \tag{67}$$

this system has the first integral

$$H_1(x,y) = 4x^2 + \frac{1}{80}x\left(-320y + 96\sqrt{2} + 88\sqrt{3} + 365\right) + \frac{1}{5}y(25y - 16).$$

In the region R_2 we consider the linear center

$$\begin{aligned}
\dot{x} &= -\frac{x}{4} - \frac{y}{8} + \frac{-22000\sqrt{2} - 3000\sqrt{3} + 34816\sqrt{5} + 24585}{640(275\sqrt{2} + 75\sqrt{3} - 272\sqrt{5}}, \\
\dot{y} &= x + \frac{y}{4} + \frac{2400\sqrt{2} - 2200\sqrt{3} - \dfrac{4125(149\sqrt{11 - 4\sqrt{6}} + 200)}{275\sqrt{2} + 75\sqrt{3} - 272\sqrt{5}} + 62593}{108800},
\end{aligned} \tag{68}$$

this system has the first integral

$$\begin{aligned}
H_2(x,y) = {} & \frac{1}{800(275\sqrt{2} + 75\sqrt{3} - 272\sqrt{5})}(3200(275\sqrt{2} + 75\sqrt{3} - 272\sqrt{5})x^2 \\
& + x(1600(275\sqrt{2} + 75\sqrt{3} - 272\sqrt{5})y + 940225\sqrt{2} + 312300\sqrt{3} \\
& - 1001488\sqrt{5} - 25000\sqrt{6} - 38400\sqrt{10} + 35200\sqrt{15}) + 10y(40(275\sqrt{2} \\
& + 75\sqrt{3} - 272\sqrt{5})y + 22000\sqrt{2} + 3000\sqrt{3} - 34816\sqrt{5} - 24585)).
\end{aligned}$$

For this piecewise linear differential centers the unique real solution of system (56) when $i = 5$ is $(\alpha_1, \beta_1, \alpha_2, \beta_2, \gamma_1, \delta_1, \gamma_2, \delta_2) = (-1/4, \sqrt{3}/8, -4/5, 4/(5\sqrt{5}), -1/2, -1/(2\sqrt{2}), -4/5, -4/(5\sqrt{5})$. Then the discontinuous piecewise linear differential system (67)–(68) has exactly one crossing limit cycle, see $(\mathcal{C}_5^2)$ of Fig. 16. This completes the proof of statement (a) of Theorem 3.

Proof of statement (b) of Theorem 3 First we prove the statement for class C_1. In the region R_1 we consider the linear differential center

$$\dot{x} = -0.0327708x - 0.167012y + 0.202324, \quad \dot{y} = x + 0.0327708y - 2.82026, \tag{69}$$

its first integral is

$$H_1(x,y) = 4(x + 0.0327708y)^2 + 8(-2.82026x - 0.202324y) + 0.663752y^2.$$

The second linear differential center in the region R_2 is

$$\dot{x} = \frac{x}{10} - \frac{13y}{50} - \frac{1}{5}, \quad \dot{y} = x - \frac{y}{10} - \frac{16}{5}, \tag{70}$$

its first integral is

$$H_2(x,y) = 4\left(x - \frac{y}{10}\right)^2 + 8\left(\frac{y}{5} - \frac{16x}{5}\right) + y^2.$$

For the piecewise linear differential system (69)–(70) the real solutions of the system of equations

$$\begin{aligned}
&H_1(\alpha_i,\beta_i)-H_1(\gamma_i,\delta_i)=0,\\
&H_1(f_i,g_i)-H_1(h_i,k_i)=0,\\
&H_2(\alpha_i,\beta_i)-H_1(f_i,g_i)=0,\\
&H_2(\gamma_i,\delta_i)-H_1(h_i,k_i)=0,\\
&c_s(\alpha_i,\beta_i)=c_s(\gamma_i,\delta_i)=0, i=1,2,\\
&c_s(f_i,g_i)=c_s(h_i,k_i)=0.
\end{aligned}\tag{71}$$

when $s=1$, is $(\alpha_1,\beta_1,\gamma_1,\delta_1,f_1,g_1,h_1,k_1)=(0.571172\ldots,1.04103\ldots,2.93153\ldots,6.0596\ldots,0.449711\ldots,-0.861917\ldots,2.66907\ldots,-5.36724\ldots)$ and $(\alpha_2,\beta_2,\gamma_2,\delta_2,f_2,g_2,h_2,k_2)=(1.19066\ldots,2.07275\ldots,2.40283\ldots,4.69567\ldots,0.928007\ldots,-1.60885\ldots,2.25189\ldots,-4.32923\ldots)$. Hence these linear differential centers have exactly two crossing limit cycles, see $(\mathcal{C}_1)$ of Fig. 17.

Now we prove the statement for the class C_2. In the region R_2 we consider the linear differential center

$$\dot{x}=-0.252669y,\quad \dot{y}=x-0.498272,\tag{72}$$

its first integral is

$$H_1(x,y)=4x^2-3.98617x+1.01067y^2.$$

The second linear differential center in the region R_3 is

$$\dot{x}=-\frac{y}{25},\quad \dot{y}=x-\frac{1}{2},\tag{73}$$

its first integral is

$$H_2(x,y)=4x^2-4x+\frac{4y^2}{25}.$$

For these centers the real solutions of system (71) when $s=2$ are $(0.898773\ldots,0.43723\ldots,0.169784\ldots,0.631617\ldots,0.898773\ldots,-0.43723\ldots,0.169784\ldots,-0.631617\ldots)$ and $(0.758688\ldots,0.640578\ldots,0.368993\ldots,0.782685\ldots,0.758688\ldots,-0.640578\ldots,0.368993\ldots,-0.782685\ldots)$. Hence the discontinuous piecewise linear differential system (72)–(73) has two crossing limit cycles, see $(\mathcal{C}_2^1)$ of Fig. 18.

For the second configuration of the class C_2 we consider the linear center

$$\dot{x}=0.057143x-0.0550264y-0.0521176,\quad \dot{y}=x-0.057143y-0.805757,\tag{74}$$

in the region R_2, its first integral is

$$H_1(x,y)=4(x-0.057143y)^2+8(0.0521176y-0.805757x)+0.207044y^2.$$

The second linear differential center in the region R_1 is

$$\dot{x}=\frac{x}{10}-\frac{17y}{80}-\frac{1}{10},\quad \dot{y}=x-\frac{y}{10}-\frac{11}{10},\tag{75}$$

its first integral is

$$H_2(x,y)=4\left(x-\frac{y}{10}\right)^2+8\left(\frac{y}{10}-\frac{11x}{10}\right)+\frac{81y^2}{100}.$$

The real solutions of system (71) for these centers are (0.253068 … , 0.283285 … , 1.26272 … , 1.89942 … , 0.204637 … , −0.224601 … , 1.16927 … , −1.72215 …) and (0.145203 … , 0.155388 … , 1.36881 … , 2.10673 … , 0.119104 … , −0.125997 … , 1.24855 … , −1.87223 …) . Hence the discontinuous piecewise linear differential system (74)–(75) has two crossing limit cycles, see ($\mathcal{C}_2^2$) of Fig. 18.

For the class C_3 and in the region R_1 we consider the center

$$\dot{x}=0.0333015x-0.132045y-0.0410184,\quad \dot{y}=x-0.0333015y-1.97694, \tag{76}$$

its first integral is

$$H_1(x,y)=4(x-0.0333015y)^2+8(0.0410184y-1.97694x)+0.523744y^2.$$

The second linear differential center in the region R_2 is

$$\dot{x}=0.1x-0.26y-0.2,\quad \dot{y}=x-0.1y-2.3, \tag{77}$$

its first integral is

$$H_2(x,y)=4(x-0.1y)^2+8(0.2y-2.3x)+y^2.$$

The real solutions of system (71) when $s=3$ are (0.340737 … , 0.198898 … , 2.70162 … , 4.44056 … , 0.308678 … , −0.171498 … , 2.52616 … , −4.01504 …) and (0.862714 … , 0.801309 … , 2.24826 … , 3.37109 … , 0.727481 … , −0.620487 … , 2.17185 … , −3.2007 …) . Hence the discontinuous piecewise linear differential system (76)–(77) has two crossing limit cycles, see ($\mathcal{C}_3$) of Fig. 17.

We prove the statement for the class C_4. In the region R_1 we consider the center

$$\dot{x}=-0.142331x-0.0908868y+1.59716,\quad \dot{y}=x+0.142331y-9.92891, \tag{78}$$

its first integral is

$$H_1(x,y)=4(x+0.142331y)^2+8(-9.92891x-1.59716y)+0.282515y^2.$$

We consider the second linear differential center in the region R_2

$$\dot{x}=\frac{x}{10}-\frac{y}{10}-\frac{7}{10},\quad \dot{y}=x-\frac{y}{10}-\frac{44}{5}, \tag{79}$$

its first integral is

$$H_2(x,y)=4\left(x-\frac{y}{10}\right)^2+8\left(\frac{7y}{10}-\frac{44x}{5}\right)+\frac{9y^2}{25}.$$

The two real solutions of system (71) when $s=4$ are (2.34486 … , 2.71929 … , 9.31289 … , 26.851 … , 1.97922 … , −1.95854 … , 8.44735 … , −23.0527 …) and (4.45798 … , 8.2899 … , 7.62439 … , 19.6236 … , 3.20672 … , −4.7636 … , 7.33329 … , −18.455 …). Hence the discontinuous piecewise linear differential system (78)–(79) has two crossing limit cycles, see ($\mathcal{C}_4$) of Fig. 17.

For the first configuration of the class C_5. We consider the linear differential center

$$\dot{x} = -0.453205y, \quad \dot{y} = x + 0.497252, \tag{80}$$

in the region R_3 with its first integral

$$H_1(x, y) = 4x^2 + 3.97802x + 1.81282y^2.$$

We consider the second linear differential center in the region R_2

$$\dot{x} = -\frac{y}{10}, \quad \dot{y} = x + \frac{1}{2}, \tag{81}$$

its first integral is

$$H_2(x, y) = 4x^2 + 4x + \frac{2y^2}{5}.$$

The two real solutions of system (71) when $s = 5$ are $(-0.0927079\ldots, 0.088306\ldots, -0.837019\ldots, 0.337912\ldots, -0.0927079\ldots, -0.088306\ldots, -0.837019\ldots, -0.337912\ldots)$ and $(-0.217826\ldots, 0.192647\ldots, -0.663868\ldots, 0.38489\ldots, -0.217826\ldots, -0.192647\ldots, -0.663868\ldots, -0.38489\ldots)$. Hence, the discontinuous piecewise linear differential system (80)–(81) has two crossing limit cycles, see $(\mathcal{C}_5)$ of Fig. 19.

Finally we prove the statement for the second configuration of the class C_5. In the region R_3 we consider the linear differential center

$$\dot{x} = 0.057143x - 0.0550264y - 0.0521176, \quad \dot{y} = x - 0.057143y - 0.805757, \tag{82}$$

its first integral is

$$H_1(x, y) = 4(x - 0.057143y)^2 + 8(0.0521176y - 0.805757x) + 0.207044y^2.$$

We consider the second linear differential center in the region R_1

$$\dot{x} = \frac{x}{10} - \frac{17y}{80} - \frac{1}{10}, \quad \dot{y} = x - \frac{y}{10} - \frac{11}{10}, \tag{83}$$

its first integral is

$$H_2(x, y) = 4\left(x - \frac{y}{10}\right)^2 + 8\left(\frac{y}{10} - \frac{11x}{10}\right) + \frac{81y^2}{100}.$$

The two real solutions of system (71) when $s = 5$ are $(0.253068\ldots, 0.283285\ldots, 1.26272\ldots, 1.89942\ldots, 0.204637\ldots, -0.224601\ldots, 1.16927\ldots, -1.72215\ldots)$ and $(0.145203\ldots, 0.155388\ldots, 1.36881\ldots, 2.10673\ldots, 0.119104\ldots, -0.125997\ldots, 1.24855\ldots, -1.87223\ldots)$. Hence the discontinuous piecewise linear differential system (82)–(83) has two crossing limit cycles, see $(\mathcal{C}_5^2)$ of Fig. 19.

Conclusions

In Theorems 1 and 2 we have solved the extension of the 16th Hilbert problem to the discontinuous piecewise linear differential systems formed by centers and separated by an irreducible cubic curve restricted to the crossing limit cycles which intersect the cubic curve in four points. We recall that this problem restricted to the crossing limit cycles which intersect the cubic curve in two points already has been solved in [3], and that this problem for reducible cubic curves has been solved in [15].

In other words we have provided the maximum number of crossing limit cycles that intersect any irreducible cubic curve C in four points for all discontinuous piecewise linear differential systems formed by centers and separated by the irreducible cubic curve C. Moreover we also have provided the different configurations that such crossing limit cycles can exhibit.

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References

1. Artes, J.C., Llibre, J., Medrado, J.C., Teixeira, M.A.: Piecewise linear differential systems with two real saddles. Math. Comput. Simul. **95**, 13–22 (2014)
2. Banerjee, S., Verghese, G.: Nonlinear phenomena in power electronics. Attractors, bifurcations chaos and nonlinear control. Wiley-IEEE Press, New York (2001)
3. Benterki, R., Llibre, J.: The limit cycles of discontinuous piecewise linear differential systems formed by centers and separated by irreducible cubic curves I. Dyn. Contin. Discrete Impuls. Syst. Ser. A (2021)
4. Bix, R.: Conics and cubics. Undergraduat Texts in Mathematics, 2nd edn. Springer, New York (2006)
5. Buica, A., Llibre, J., Makarenkov, O.: A note on forced oscillations in differential equations with jumping nonlinearities. Differ. Equations Dyn. Syst. **23**(4), 415–421 (2015)
6. Euzébio, R.D., Llibre, J.: On the number of limit cycles in discontinuous piecewise linear differential systems with two pieces separated by a straight line. J. Math. Anal. Appl. **424**, 475–486 (2015)
7. Filippov, A.F.: Differential equations with discontinuous righthand sides. Kluwer Academic Publishers Group, Dordrecht (1998)
8. Freire, E., Ponce, E., Rodrigo, F., Torres, F.: Bifurcation sets of continuous piecewise linear systems with two zones. Int. J. Bifurc. Chaos. **8**, 2073–2097 (1998)
9. Freire, E., Ponce, E., Torres, F.: Canonical discontinuous planar piecewise linear systems. SIAM J. Appl. Dyn. Syst. **11**, 181–211 (2012)
10. Han, M., Zhang, W.: On Hopf bifurcation in non-smooth planar systems. J. Differ. Equations. **248**, 2399–2416 (2010)
11. Hilbert, D.: Mathematische Probleme, Lecture, Second Internat. Congr. Math. (Paris, 1900). Nachr. Ges. Wiss. Göttingen Math. Phys. KL., 253–297 (1900) [English transl., Bull. Amer. Math. Soc. 8 (1902), 437–479; Bull. (New Series) Amer. Math. Soc. 37 (2000), 407–436]
12. Huan, S.M., Yang, X.S.: On the number of limit cycles in general planar piecewise linear systems. Discrete Cont. Dyn. Syst. **32**, 2147–2164 (2012)
13. Ilyashenko, Y.: Centennial history of Hilbert's 16th problem. Bull. Am. Math. Soc. **39**, 301–354 (2002)
14. Jimenez, J., Llibre, J. and Medrado, J.C.: Crossing limit cycles for a class of piecewise linear differential centers separated by a conic. Electron. J. Diffe Equ **41**, 36 (2020)
15. Jimenez, J., Llibre, J., Medrado, J.C.: Crossing limit cycles for piecewise linear differential centers separated by a reducible cubic curve. Electron. J. Qual. Theory Differ. Equations **19**, 48 (2020)
16. Leine, R.I., Nijmeijer, H.: Dynamics and bifurcations of non-smooth mechanical systems. Lecture Notes in Applied and Computational Mechanics, vol. 18. Springer, Berlin (2004)
17. Li, J.: Hilbert's 16th problem and bifurcations of planar polynomial vector fields. Int. J. Bifur. Chaos Appl. Sci. Eng. **13**, 47–106 (2003)
18. Liberzon, D.: Switching in systems and control: foundations and applications. Birkhuser, Boston (2003)
19. Llibre, J., Novaes, D.D., Teixeira, M.A.: Maximum number of limit cycles for certain piecewise linear dynamical systems. Nonlinear Dyn. **82**, 1159–1175 (2015)

20. Llibre, J., Ponce, E.: Three nested limit cycles in discontinuous piecewise linear differential systems with two zones. Dyn. Contin. Discrete Impuls. Syst. Ser. B **19**, 325–335 (2012)
21. Llibre, J., Rodríguez, G.: Configurations of limit cycles and planar polynomial vector fields. J. Differ. Equations **198**, 374–380 (2004)
22. Llibre, J., da Silva, C.E.L., da Silva, P.R.: Piecewise bounded quadratic systems in the plane. Differ. Equations Dyn. Syst. **24**, 51–62 (2016)
23. Llibre, J., Teixeira, M.A.: Piecewise linear differential systems with only centers can create limit cycles? Nonlinear Dyn. **91**, 249–255 (2018)
24. Lum, R., Chua, L.O.: Global properties of continuous piecewise-linear vector fields. Part I: simplest case in $\mathbb{R}^2$. Int. J. Circuit Theory Appl. **19**, 251–307 (1991)
25. Lum, R., Chua, L.O.: Global properties of continuous piecewise linear vector fields. Part II: simplest case in $\mathbb{R}^2$. Int. J. Circuit Theory Appl. **19**, 9–46 (1992)
26. Shui, S., Zhang, X., Li, J.: The qualitative analysis of a class of plana Filippov systems. Nonlinear Anal. **73**, 1277–1288 (2010)