ch8. 5.

$w_{n+1}=\prod_{i=0}^{n}(x-x_i)$, let $N=\frac{n}{2}$, the nodes can be written as $x_i=-1+ih$, $i=0,1,2,\dots,n$, $h=\frac{2}{n}$

$\therefore w_{n+1}=\prod_{i=0}^{n}(x-x_i)=\prod_{i=0}^{n}(x-(-1+ih))=\prod_{i=0}^{n}(x+1-ih)$, $\because$ the set of nodes $\{-Nh,-(N-1)h,\dots,Nh\}$

is symmetric at 0, $\therefore w_{n+1}=\prod_{i=0}^{n}(x+1-ih)=(x+Nh)(x+(N-1)h)\cdots(x+h)\cdot x\cdot(x-h)\cdots(x-Nh)$

$=(x+1)(x+(N-1)h)\cdots(x+h)\cdot x\cdot(x-h)\cdots(x-1)$

then let $x=rh$, $x_{n-1}=1-h=(N-1)h$; $x_n=1=Nh$, with $r\in(N-1,N)$

Hence, $w_{n+1}(rh)=\prod_{j=-N}^{N}(rh-jh)=h^{2N+1}\prod_{j=-N}^{N}(r-j)=h^{n+1}\prod_{j=-N}^{N}(r-j)$, $\therefore |w_{n+1}(rh)|=h^{n+1}|\prod_{j=-N}^{N}(r-j)|$

$\because r\in(N-1,N)$, the smallest two value in the product are $(r-(N-1))$ and $(N-r)$

$\therefore |w_{n+1}(rh)|=h^{n+1}[(r-(N-1))(N-r)]\prod_{j=-N}^{N-2}|r-j|$

And $\because |r-j|$ with $j\le N-2$ & $j\ge -N$, each j between for integer (k) and $(k+1)$, $k\in\mathbb{Z}$

$\therefore$ We have $(n-1)!\le\prod_{j=-N}^{N-2}|r-j|\le n!$ $\left(\text{or } (n-1)!\le\prod_{j=-\frac{n}{2}}^{\frac{n}{2}-2}|r-j|\le n!\right)$

Thus, $(n-1)!\,h^{n+1}(r-(N-1))(N-r)\le h^{n+1}[(r-(N-1))(N-r)]\prod_{j=-N}^{N}|r-j|\le n!\,h^{n+1}(r-(N-1))(N-r)$

$\therefore (n-1)!\,h^{n+1}(r-(N-1))(N-r)\le|w_{n+1}(rh)|\le n!\,h^{n+1}(r-(N-1))(N-r)\rightarrow(*)$

Rewrite for $(r-(N-1))(N-r)\cdot h\cdot h=(r-(N-1))h\cdot(N-r)h=(x-x_{n-1})(x-x_n)$

$\therefore (*)$ will be $(n-1)\,h^{n-1}|(x-x_{n-1})(x-x_n)|\le|w_{n+1}(x)|\le n!\,h^{n+1}|(x-x_{n-1})(x-x_n)|$, $x\in(x_{n-1},x_n)$

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ch8.6 consider $\dfrac{w_{n+1}(x+h)}{w_{n+1}(x)}=\dfrac{(x+h+Nh)(x+h+(N-1)h)\cdots(x+2h)(x+h)\cdots(x+h-Nh)}{(x+Nh)(x+(N-1)h)\cdots(x+h)\cdot x\cdot(x-h)\cdots(x-Nh)}$

$\therefore\left|\dfrac{w_{n+1}(x+h)}{w_{n+1}(x)}\right|=\left|\dfrac{(x+(N+1)h)(x+Nh)\cdots(x+2h)(x+h)x\cdots(x-(N-2)h)(x-(N-1)h)}{(x+Nh)(x+(N-1)h)\cdots(x+h)x(x-h)\cdots(x-(N-1)h)(x-Nh)}\right|$

$\left(N=\frac{n}{2},h=\frac{2}{n}\right)=\left|\dfrac{(x+(N+1)h)}{(x-Nh)}\right|=\left|\dfrac{(x+1+h)}{(x-1)}\right|$, $x\in(0,x_{n-1})\Rightarrow$ $x+h>0$ i.e. $x+\frac{2}{n}>0$, $x-1<0$

suppose $\left|\dfrac{(x+1+h)}{(x-1)}\right|\le1\Leftrightarrow\dfrac{x+1+\frac{2}{n}}{1-x}\le1\Leftrightarrow x+1+\frac{2}{n}\le1-x\Leftrightarrow2x+\frac{2}{n}\le0\Leftrightarrow x+\frac{1}{n}\le0$ $(\rightarrow\leftarrow)$

$\therefore$ by contradiction, $\left|\dfrac{w_{n+1}(x+h)}{w_{n+1}(x)}\right|>1$. for $x\in(0,x_{n-1})$, i.e. w_{n+1} is increasing for $x\in(0,x_{n-1})$

$\because w_{n+1}$ is even fun, and also symmetric at 0, $\therefore|w_{n+1}(x)|$ has maximum when $x\in(x_{n-1},x_n)$

(but in nodes point is 0, so it has max when it approaches to x_n)

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Ch8.8 Let $Hf \in \mathbb{P}_n$ satisfying $(Hf)^{(k)}(x_0) = f^{(k)}(x_0)$, $k=0,1,2,\dots,n$

suppose that $Hf(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \dots + a_n(x-x_0)^n$ which is about the node x_0, $a_i \in \mathbb{R}$ $\forall 0 \le i \le n$

- consider $Hf(x_0) = a_0 = f(x_0)$

And $Hf'(x) = a_1 + 2a_2(x-x_0) + \dots + na_n(x-x_0)^{n-1} \Rightarrow Hf'(x_0) = a_1 = f^{(1)}(x_0)$ (by 題目)

$Hf''(x) = 2a_2 + 6a_3(x-x_0) + \dots + n(n-1)a_n(x-x_0)^{n-2} \Rightarrow Hf''(x_0) = 2a_2 = f^{(2)}(x_0)$

$\vdots$

$\therefore$ for $Hf^{(n)}(x) = n(n-1)(n-2)\cdots 2\cdot 1\cdot a_n = n!\,a_n \Rightarrow Hf^{(n)}(x_0) = n!\,a_n = f^{(n)}(x_0)$

Hence $Hf(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \dots + a_n(x-x_0)^n$

$$= f(x_0) + f^{(1)}(x_0)(x-x_0) + \frac{1}{2}f^{(2)}(x_0)(x-x_0)^2 + \dots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n = \sum_{k=0}^{n}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$$

$$\therefore Hf(x) = \sum_{k=0}^{n} a_k(x-x_0)^k = \sum_{k=0}^{n}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k = \sum_{j=0}^{n}\frac{f^{(j)}(x_0)}{j!}(x-x_0)^j, \quad j=0,1,2,\dots,n \quad \#$$

題目: Show that for $n+1$ Chebyshev points of the second kind, the barycentric weights are (after rescaling) $w_j = (-1)^j$, $j=1,2,\dots,n-1$ and $w_0 = 1/2$, $w_n = (-1)^n/2$

sol: The barycentric weight: $w_j = \dfrac{1}{\prod_{k\ne j}(x_j-x_k)} \Rightarrow w_j = \dfrac{1}{\prod_{k=0, k\ne j}^{n}\left(\cos\left(\frac{j\pi}{n}\right) - \cos\left(\frac{k\pi}{n}\right)\right)}$, by $\cos A - \cos B = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$

$$\therefore w_j = \frac{1}{\prod_{k=0, k\ne j}^{n}\left[-2\sin\frac{(j+k)\pi}{2n}\sin\frac{(j-k)\pi}{2n}\right]} = \frac{1}{(-2)^n\prod_{k=0, k\ne j}^{n}\sin\frac{(j+k)\pi}{2n}\sin\frac{(j-k)\pi}{2n}} \Rightarrow \text{let } A = \prod_{k=0, k\ne j}^{n}\sin\frac{(j-k)\pi}{2n},\ B = \prod_{k=0, k\ne j}^{n}\sin\frac{(j+k)\pi}{2n}$$

For A, if $k>j$, $\frac{(j-k)\pi}{2n} > 0$; $k<j$, $\frac{(j-k)\pi}{2n} < 0$ $\Rightarrow A = (-1)^{n-j}\prod_{k=0, j\ne k}^{n}\left|\sin\frac{(j-k)\pi}{2n}\right|$, so for $k>j$, let $m=k-j$, $m=1,2,\dots,j$; $k<j$, let $m=j-k$, $m=1,2,\dots,n-j$

$$\therefore A = (-1)^{n-j}\prod_{m=1}^{j}\sin\frac{m\pi}{2n}\prod_{m=1}^{n-j}\sin\frac{m\pi}{2n}.$$

For B, its terms are $\sin\frac{(j+0)\pi}{2n}$, $\sin\frac{(j+1)\pi}{2n}$, ..., $\sin\frac{(j+n)\pi}{2n}$, and for $k=j \Rightarrow \sin\frac{2j\pi}{2n} = \sin\frac{j\pi}{n}$

$$\therefore B = \frac{\prod_{m=j}^{j+n}\sin\frac{m\pi}{2n}}{\sin\frac{j\pi}{n}}.$$

$$\therefore \prod_{k\ne j}^{n}(x_j-x_k) = \prod_{k=0, k\ne j}^{n}\left[-2\sin\frac{(j+k)\pi}{2n}\sin\frac{(j-k)\pi}{2n}\right] = (-2)^n(-1)^{n-j}\left(\prod_{m=1}^{j}\sin\frac{m\pi}{2n}\right)\left(\prod_{m=1}^{n-j}\sin\frac{m\pi}{2n}\right)\left(\frac{\prod_{m=j}^{j+n}\sin\frac{m\pi}{2n}}{\sin\frac{j\pi}{n}}\right) \rightarrow (*)$$

Using the property $= \prod_{m=1}^{n-1}\sin\frac{m\pi}{n} = \frac{n}{2^{n-1}}$ and $\prod_{m=1}^{n-1}\sin\frac{m\pi}{2n} = \frac{\sqrt{n}}{2^{n-1}}$, so by the properties, we will have

$$(*) \Rightarrow \begin{cases} \dfrac{1}{w_j} = (-1)^j\dfrac{n}{2^{n-1}}, & \text{for } 1\le j\le n-1 \\ \dfrac{1}{w_0} = \dfrac{n}{2^{n-2}}, & \text{for } j=0 \\ \dfrac{1}{w_n} = (-1)^n\dfrac{n}{2^{n-2}}, & \text{for } j=n \end{cases} \Rightarrow \text{conclude this we will get} \Rightarrow w_j = \begin{cases} (-1)^j\cdot\frac{1}{2}, & j=0 \text{ or } j=n \\ (-1)^j, & 1\le j\le n-1 \end{cases}$$

$$\therefore w_0 = \frac{1}{2},\ w_j = (-1)^j,\ w_n = \frac{(-1)^n}{2} \quad \divideontimes$$