Part I

From class slides, we know that for normal-normal model,

$$p(\theta|x) \propto e^{-\frac{1}{2}\left[\left(\sum_{i=1}^{n}\phi(x_i-\theta)^2\right)+\tau(\theta-\theta_0)^2\right]}$$

WTS $p(\theta|x) \propto e^{-\frac{1}{2}(\tau+n\phi)\left(\theta-\frac{1}{\tau+n\phi}\left(\tau\theta_0+\phi\sum_{i=1}^{n}x_i\right)\right)^2}$

$\Rightarrow$ WTS $e^{\underbrace{-\frac{1}{2}\left[\left(\sum_{i=1}^{n}\phi(x_i-\theta)^2\right)+\tau(\theta-\theta_0)^2\right]}_{\textcircled{1}}} \propto e^{\underbrace{-\frac{1}{2}(\tau+n\phi)\left[\theta-\frac{1}{\tau+n\phi}\left(\tau\theta_0+\phi\sum_{i=1}^{n}x_i\right)\right]^2}_{\textcircled{2}}}$

Note: WTS stands for Want to show

Note that $P(\theta|x)$ implies that we are also given θ_0, τ, ϕ,

$\Rightarrow p(\theta|x) = p(\theta|x, \theta_0, \phi, \tau)$

$$\textcircled{1} = -\frac{1}{2}\left[\left(\sum_{i=1}^{n}\phi(x_i-\theta)^2\right)+\tau(\theta-\theta_0)^2\right]$$

$$= -\frac{1}{2}\left[\left(\sum_{i=1}^{n}\phi(x_i^2-2\theta x_i+\theta^2)\right)+\tau(\theta^2-2\theta\theta_0+\theta_0^2)\right]$$

$$= -\frac{1}{2}\left[\left(\sum_{i=1}^{n}(\phi x_i^2-2\phi\theta x_i+\phi\theta^2)\right)+\tau\theta^2-2\tau\theta\theta_0+\tau\theta_0^2\right]$$

$$= -\frac{1}{2}\left[\left(\sum_{i=1}^{n}\phi x_i^2\right)-\left(\sum_{i=1}^{n}2\phi\theta x_i\right)+\left(\sum_{i=1}^{n}\phi\theta^2\right)+\tau\theta^2-2\tau\theta\theta_0+\tau\theta_0^2\right]$$

$$= -\frac{1}{2}\left[\phi\left(\sum_{i=1}^{n}x_i^2\right)-2\phi\theta\left(\sum_{i=1}^{n}x_i\right)+n\phi\theta^2+\tau\theta^2-2\tau\theta\theta_0+\tau\theta_0^2\right]$$

$$= -\frac{1}{2}\left[\phi\left(\sum_{i=1}^{n}x_i^2\right)-2\phi\theta\left(\sum_{i=1}^{n}x_i\right)+\theta^2(\tau+n\phi)-2\tau\theta\theta_0+\tau\theta_0^2\right]$$

$$= -\frac{1}{2}(\tau+n\phi)\left[\frac{\phi\left(\sum_{i=1}^{n}x_i^2\right)}{\tau+n\phi}-\frac{2\phi\theta\left(\sum_{i=1}^{n}x_i\right)}{\tau+n\phi}+\theta^2-\frac{2\tau\theta\theta_0}{\tau+n\phi}+\frac{\tau\theta_0^2}{\tau+n\phi}\right]$$

$$= -\frac{1}{2}(\tau+n\phi)\left[\theta^2-\frac{2\tau\theta\theta_0}{\tau+n\phi}-\frac{2\phi\theta\left(\sum_{i=1}^{n}x_i\right)}{\tau+n\phi}+\frac{\tau\theta_0^2}{\tau+n\phi}+\frac{\phi\left(\sum_{i=1}^{n}x_i^2\right)}{\tau+n\phi}\right]$$

$$= -\frac{1}{2}(\tau+n\phi)\left[\theta^2-\frac{2\theta}{\tau+n\phi}\left(\tau\theta_0+\phi\sum_{i=1}^{n}x_i\right)+\underbrace{\frac{1}{\tau+n\phi}\left(\tau\theta_0^2+\phi\left(\sum_{i=1}^{n}x_i^2\right)\right)}_{\textcircled{3}}\right]$$

Let's pause here and take a look at ②

$$\textcircled{2} = -\frac{1}{2}(\tau+n\phi)\left[\theta-\frac{1}{\tau+n\phi}\left(\tau\theta_0+\phi\sum_{i=1}^{n}x_i\right)\right]^2$$

$$= -\frac{1}{2}(\tau+n\phi)\left[\theta^2-\frac{2\theta}{\tau+n\phi}\left(\tau\theta_0+\phi\sum_{i=1}^{n}x_i\right)+\underbrace{\frac{1}{(\tau+n\phi)^2}\left(\tau\theta_0+\phi\sum_{i=1}^{n}x_i\right)^2}_{\textcircled{4}}\right]$$

Note that the only difference between ① and ② is the last terms, ③ and ④

Also note that both ③ and ④ do not contain θ, which means they are constant given θ_0, ϕ, τ, so we can use $\propto$ to simplify further, or add any terms not involving θ directly

$$① = -\frac{1}{2}(\tau+n\phi)\left[\theta^2 - \frac{2\theta}{\tau+n\phi}\left(\tau\theta_0 + \phi\sum_{i=1}^{n} x_i\right) + \frac{1}{\tau+n\phi}\left(\tau\theta_0^2 + \phi\left(\sum_{i=1}^{n} x_i^2\right)\right)\right]$$

$$\propto -\frac{1}{2}(\tau+n\phi)\left[\theta^2 - 2\theta\frac{1}{\tau+n\phi}\left(\tau\theta_0 + \phi\sum_{i=1}^{n} x_i\right)\right]$$ (last term removed)

$$\propto -\frac{1}{2}(\tau+n\phi)\left[\theta^2 - 2\theta\frac{1}{\tau+n\phi}\left(\tau\theta_0 + \phi\sum_{i=1}^{n} x_i\right) + \frac{1}{(\tau+n\phi)^2}\left(\tau\theta_0 + \phi\sum_{i=1}^{n} x_i\right)^2\right]$$ (Term added to complete the square)

$$= -\frac{1}{2}(\tau+n\phi)\left[\theta - \frac{1}{\tau+n\phi}\left(\tau\theta_0 + \phi\sum_{i=1}^{n} x_i\right)\right]^2$$

$$= ②$$

Therefore, I have

$$e^{-\frac{1}{2}\left[\left(\sum_{i=1}^{n}\phi(x_i-\theta)^2\right) + \tau(\theta-\theta_0)^2\right]} \propto e^{-\frac{1}{2}(\tau+n\phi)\left[\theta - \frac{1}{\tau+n\phi}\left(\tau\theta_0 + \phi\sum_{i=1}^{n} x_i\right)\right]^2}$$ by ① $\propto$ ②.