Principle's of Real Analysis by Walter Rudin Solutions guide

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All exercises are from Principle's of Real Analysis by Walter Rudin. Any mistakes are my own.

1 Chapter 2 Basic of Topology

1: The empty set is a subset of every set

Proof. Assume there is a set X such that $\emptyset$ is not a subset of X. Then the emptyset contains an element not in X. By definition $\emptyset$ contains no element. Thus $\emptyset$ is a subset of every set. $\square$

2: The set of algebraic numbers is countable

Proof. Define $f : P_n \to \mathbb{Z}^{n+1}$ by $f(a_0z^n+a_1z^{n-1}+\cdots+a_n) = (a_0, a_1, \ldots, a_n)$. Where P_n denotes a polynomial of degree at most n with integer coefficients.

Suppose that $f(a_0z^n + a_1z^{n-1} + \cdots + a_n) = f(b_0z^n + b_1z^{n-1} + \cdots + b_n)$. Clearly $(a_0, a_1, \ldots, a_n) = (b_0, b_1, \ldots, b_n)$ hence the function is injective. Similarly, letting $y \in \mathbb{Z}^{n+1}$. The polynomial $y_0z^n + y_1z^{n-1} + \cdots + y_n$ applied to f, shows f is surjective. Hence f is bijective.

Since the integers are countable, we have that $\mathbb{Z}^{n+1}$ is countable and therefore P_n is countable. For each $p(z) \in P_n$, let $\theta_{p(z)}$ denote the set of all roots of $p(z)$. Clearly $\theta_{p(z)}$ is a finite set.

Defining $T = \bigcup_{p_z \in P_n} \theta_{p(z)}$ is at most countable. Observe that every positive integer is an algebraic number, hence the set of algebraic numbers is infinite and since it is a subset of T, it follows the set of algebraic numbers is countable. $\square$

3: There exist Real numbers which are not algebraic

Proof. Suppose all Real numbers were algebraic. Then the set of Real numbers would be countable. This is a contradiction. □

4: Is the set of all irrational numbers countable?

Proof. The set of irrational numbers is not countable. Suppose it was then this would mean $\mathbb{R} = (\mathbb{R} \setminus \mathbb{Q}) \cup \mathbb{Q}$ would be countable. Contradiction. □

5: Construct a bounded set of real numbers with exactly three limit points

Proof. Let $A = A_1 \cup A_2 \cup A_3$ where $A_1 = \{\frac{1}{n} : n \in \mathbb{N}\}$, $A_2 = \{10 + \frac{1}{n} : n \in \mathbb{N}\}$ and $A_3 = \{20 + \frac{1}{n} : n \in \mathbb{N}\}$ then clearly A_1, A_2, A_3 are all bounded sets and have limit points $0, 10, 20$ respectively. Hence $A = \{0, 10, 20\}$ □

6: Let E' be the set of all limit points of E. Prove that E' is closed. Prove that $\bar{E}$ and E' have the same limit points. Do E and E' always have the same limit points?

Proof. Let $p \in (E')'$ and $N_r(p)$ be a neighbourhood of p for some r. There exists $q \neq p \in N_r(p)$ such that $q \in E'$. Hence q is a limit point of E therefore there exists $t \neq q$ in $N_w(q)$ such that $t \in E$ for all $w > 0$. Let $w = \frac{1}{2}\min\{d(p,q), r - d(p,q)\}$, then $N_w(q)/subsetN_r(p)$.

We also have $t/neqp$. Suppose that $t = p$ then $p \in N_w(q)$. If $w = \frac{1}{2}d(p,q)$ this means $d(p,q) < w = \frac{1}{2}d(p,q)$. Contradiction. Also if $w = \frac{1}{2}(r - d(p,q))$ then $r - d(p,q) \leq d(p,q)$ but $d(p,q) < w = \frac{1}{2}(r - d(p,q))$. This implies that $2d(p,q) < r - d(p,q)$ contradiction.

Hence every neighbourhood of $N_r(p)$ contains a point s such that $s \neq p$ and $s \in E$. Thus p is a limit point of E and $p \in E'$ and E' is closed.

Suppose that $p \in (\bar{E})'$ Then p is a limit point of $\bar{E} = E \cup E'$. Hence for all $r > 0$ there exists $q \in N_r(p), q \neq p$ such that $q \ \in \bar{E} = E \cup E'$. If $q \in E$ then p is a limit point of E already and therefore $p \in E'$. If $q \in E'$ by the same argument as above $p \in E'$. Thus $(barE)' \subset E'$.

Conversely suppose that $p \in E'$ and $N_r(p)$. Since p is a limit point of E there exists a point $q \neq p$ in $N_r(p)$ such that $q \in E$. Hence $q \in \bar{E}$ therefore p is a limit point of $\bar{E}$. Hence $E' \subset (\bar{E})'$. Thus $\bar{E}' = E'$.

The sets E and E' do not need to contain the same limit points. For example, $E = \{\frac{1}{n} : n \in \mathbb{N}\}$ has the limit point $\{0\} = E'$. Then $E'' = \emptyset$ (finite sets have no limit points). □

7A: Let $A_1, A_2, \ldots$ be subsets of a metric space. If $B_n = \bigcup_{i=1}^{n} A_i$ prove that $\bar{B_n} = \bigcup_{i=1}^{n} \bar{A}_i, n = 1, 2, 3, \ldots$

Proof. Let $x \in \bar{B_n}$, then $x \in B_n$ or $x \in B_n'$. If $x \in B_n$ then $x \in A_i$ for some i so $x \in \bar{A}_i$. Hence $x \in \bigcup_{i=1}^n \bar{A}_i$. If $x \in B_n'$ then x is a limit point of B_n. Therefore $x \in A_i'$ for some i. To see this observe that if $x \notin A_i'$ for all i, then for every i there exists a neighbourhood $N_{r_i}(x)$ for some $r_i > 0$, such that $N_{r_i}(p) \cap A_i = \emptyset$. Denote the minimum of the r_i by r i.e. $r = \min_{i \in [1,n]}\{r_i\} > 0$. Hence $N_r(x) \cap A_i = \emptyset$ for all $i \in \{1, 2, \ldots, n\}$, thus $N_r(x) \cap B_n = \emptyset$. Therefore, $x \in A_i'$ for some i and $\bar{B_n} \subseteq \bigcup_{i=1}^n \bar{A}_i$.

On the other hand, if $x \in \bigcup_{i=1}^n \bar{A}_i$ then $x \in \bar{A}_i$ and $x \in A_i$ or A_i' for some $i \in \{1, 2, \ldots, n\}$. If $x \in A_i$ then $x \in B_n \subseteq \bar{B_n}$. Similarly, if $x \in A_i'$ there exists $y \in N_r(x)$ for all $r > 0$ and $y \neq x$ such that $y \in A_i$. By definition, $A_i \subseteq B_i$ so x is also a limit point of B_n. This implies that $\bigcup_{i=1}^n \bar{A}_i \subseteq \bar{B_n}$. Therefore we have that $\bar{B_n} = \bigcup_{i=1}^n \bar{A}_i$

□

7B: If $B = \bigcup_{i=1}^\infty A_i$ prove that $\bar{B} \supset \bigcup_{i=1}^\infty \bar{A}_i$. Show the inclusion may be proper.

Proof. The main result follows in a similar fashion to 7A so is omitted. For the proper inclusion consider $A_i = \{\frac{1}{i}\}$ $i \in \mathbb{N}$. Then $B = \{1, \frac{1}{2}, \ldots\}$. It follows that $A_i' = \emptyset$ and therefore $\bar{A}_i = \{\frac{1}{i}\}$. Hence, $\bigcup_{i=1}^\infty \bar{A}_i = \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots\right\}$. But, $\bar{B} = \{0, 1, \frac{1}{2}, \ldots\}$ therefore $\bar{B} \supset \bigcup_{i=1}^\infty \bar{A}_i$. □

8: Is every point of every open set $E \subset \mathbb{R}^2$ a limit point of E? Answer the same question for closed sets in $\mathbb{R}^2$.

Proof. For the first part of the question, yes it is true. To see this, consider the following. Let $p = (p_1, p_2)$ be in the open set E. Then, since E is open in $\mathbb{R}^2$, $N_s(p) \subseteq E$ for some $s > 0$. Let $r > 0$ then without loss of generality suppose $s \leq r$. Clearly $(q_1, q_2) \in N_s(p) \subseteq N_r(p)$. If instead, $r < s$ let $p' = (p_1 + \frac{r}{2}, p_2)$. Then

$$0 < \sqrt{(p_1 + \frac{r}{2} - p_1)^2 + (p_2 - p_2)^2} = \frac{r}{2} < r$$

we have that $p' \in N_r(p), p' \neq p$. Hence p is a limit point of E.

For closed sets the question is false. As an example let $E = \{(1, 0), (3, 0)\}$ then E is closed but $E' = \emptyset$. □

9A: Prove the Interior of E, E° is always open

Proof. Let $x \in E^\circ$. We must show that x is an interior point of E°. By definition x is an interior point of E so there exists a neighbourhood N_x of x such that $N_x \subseteq E$. For each $y \in N_x$, there exists a neighbourhood N_y of y such that $N_y \subseteq N_x \subseteq E$. Therefore every point of N_x is an interior point of E. Thus x is an interior point of E°. $\square$

9B: Prove that E is open if and only if $E^\circ = E$

Proof. We must have $E^\circ \subseteq E$ by definition. Furthermore E is open if and only if every point of E is an interior point of E i.e. $E \subseteq E^\circ$. The result follows. $\square$

9C: If $G \subset E$ and G is open, prove that $G \subset E^\circ$

Proof. Let $x \in G$. Then there exists a neighbourhood N of x such that $N \subseteq G \subseteq E$. Hence x is an inerior point of E. Thus $x \in E^\circ$ thus $G \subseteq E^\circ$ $\square$

9D: Prove that the complement of E° is the closure of the complement of E

Proof. Suppose $x \in (E^\circ)^c$. Then x is not an interior point of E. If $x \notin E$ then $x \in E^c \subseteq \bar{E}^c$. On the other hand, if $x \in E$ then every neighbourhood N of x satisfies $N \cap E^c \neq \emptyset$. Therefore, x is a limit point of E^c and $x \in (E^c)' \subseteq \bar{E}^c$. Therefore $(E^\circ)^c \subseteq \bar{E}^c$.

On the other hand, if $x \in \bar{E}^c$ then $x \in E^c$ or $x \in (E^c)'$. If we have $x \in E^c, x \notin E$. Since $E^\circ \subseteq, x \notin E^\circ$ therefore $x \in (E^\circ)^c$. If $x \in (E^c)'$ then x is a limit point of E^c. If $x \notin (E^\circ)^c$ then we have $x \in E^\circ$. Therefore x is an interior point of E so there is a neighbourhood N of x such that $N \subseteq E$. Therefore, $N \cap E^c = \emptyset$. This contradicts the fact x is a limit point of E^c. Hence we have $x \in (E^\circ)^c, \bar{E}^c \subseteq (E^\circ)^c$.

Combining the two results together we have that $(E^\circ)^c = \bar{E}^c$. $\square$

9E: Do E and $\bar{E}$ always have the same interiors?

Proof. This is false. As a counterexample let $E = (-1, 0) \cup (0, 1) \in \mathbb{R}$. Then $E^\circ = E$ but $\bar{E} = E \cup E' = [-1, 1]$. Hence $\bar{E}^\circ = (-1, 1)$. Thus $E^\circ \neq \bar{E}^\circ$ $\square$

9F: Do E and E° always have the same closures?

Proof. This is false. Let $E = \{2\} \in \mathbb{R}$. Then E is closed thus $\bar{E} = E$. But $E^\circ = \emptyset$ thus $\bar{E}^\circ = \emptyset$. $\square$

10: Let X be an infinite set. For $p \in X$ and $q \in X$ define

$$d(p,q) = \begin{cases} 1 & p \neq q \\ 0 & p = q \end{cases}$$

Prove this is a metric. Which subsets of the resulting metric space are open? Which are closed? Which are compact?

Proof. Clearly $d(p,q) > 0, p \neq q, d(p,p) = 0$. Furthermore clearly $d(p,q) = d(q,p)$. There exists $r \in X$ (since X is infinite) such that $r \neq p$ and $r \neq q$. Therefore, $r\ inX$ has several cases. Case 1: $p = q = r$. Then $d(p,q) = d(p,r) = d(r,q) = 0$ thus $d(p,q) = d(p,r) + d(r,q)$. Case 2: $p = q, r \neq p$. Then we have $d(p,q) = 0$ and $d(p,r) = d(r,q) = 1$ thus $d(p,q) < d(p,r) + d(r,q)$. Case 3: $p \neq q$ and $r = p$. Then we have $d(p,q) = d(r,q) = 1$ and $d(p,r) = 0$ so we have $d(p,q) = d(p,r) + d(r,q)$ Case 4: $p \neq q$ and $r = q$. Then we have $d(p,q) = d(p,r) = 1$ and $d(r,q) = 0$ so we have $d(p,q) = d(p,r) + d(r,q)$ Case 5: $p \neq q$ and $r \neq q$. Then we have $d(p,q) = d(p,r) = d(r,q) = 1$ so we have $d(p,q) < d(p,r) + d(r,q)$.

Thus d is a metric.

Let $x \in X$ and consider $N_r(x) = \{y \in X : d(x,y) < r\}$. Then $N_r(x) \subset \{x\}$, for $0 < r \leq$, the set $\{x\}$ is open in X for all $x \in X$. Thus every subset of X is open in X, since unions of open sets are open. Hence every subset of X is also closed. Clearly every finite subset of X is compact. Now assume $K \subseteq X$ is compact and infinite. Then for $x \in K$ let $G_x = \{x\}$ is open in X and $K \subseteq \bigcup_{x \in K} G_x$, then the collection $\{G_x\}$ is an open cover of K. Now since K is compact there exists an $n > 0$ such that $K \subseteq \bigcup_{i=1}^{n} G_i = \{x_1, \ldots, x_n\}$.

This contradicts K being infinite. Hence K is compact in X if and only if K is a finite subset of X. □

11A: Let $x, y\ in\mathbb{R}$ is $d_1(x,y) = (x-y)^2$ a metric?

Proof. This is not a metric. Consider $d_1(4,1) = 9, d_1(4,2) = 4, d_1(2,1) = 1$. Thus $d_1(4,1) > d_1(4,2) + d_1(2,1)$. Hence not a metric. □

11B: $d_2(x,y) = \sqrt{|x-y|}$

Proof. This is a metric. We omit the proof of the 2 simplier properties as they are clear. For the triangle inequality property, observe for any non-negative p, q we have $p + q \leq p + 2\sqrt{pq} + q$. Thus $\sqrt{p+q} \leq \sqrt{p} + \sqrt{q}$, thus for any $x, y, z \in \mathbb{R}$ $d_2(x,y) = \sqrt{|x-y|} \leq \sqrt{|x-z| + |z-y|} \leq \sqrt{|x-z|} + \sqrt{|z-y|} = d_2(x,z) + d_2(z,y)$. Hence we have a metric. □

11C: $d_3(x,y) = |x^2 - y^2|$

Proof. This is not a metric. As a counterexample $d_3(1,-1) = 0$ $\square$

11D: $d_4(x,y) = |x - 2y|$

Proof. This is not a metric. Consider $d_4(1,1) = 1$ $\square$

11E: $d_5 = (x,y) = \frac{|x-y|}{1+|x-y|}$

Proof. This is a metric. Clearly $d_5(x,x) = 0, d_5(x,y) > 0, x \neq y$. Also $d(x,y) = d(y,x)$.

We now recall that for $p,q,r \geq 0, \leq q+r$ then $\frac{p}{1+p} \leq \frac{q}{1+q} + \frac{r}{1+r}$ Letting $p = |x-y|, q = |x-z|, r = |y-z|$ into the above we have that $d_5(x,y) \leq d_z(x,z) + d_5(z,y)$.

$\square$

12: Let $K \subset \mathbb{R}$ consist of 0 and the numbers $\frac{1}{n}$ prove that K is compact directly from the definition.

Proof. Let $K = \{0, 1, \frac{1}{2}, \dots\}$ and let $\{G_\alpha\}$ be open subsets of $\mathbb{R}$ such that $K \subseteq \bigcup_\alpha G_\alpha$. Then $0 \in G_{\alpha_1}$ for some α_1. Since G_{α_1} is open in the reals, 0 is an interior point of G_{α_1} . Therefore there exists an interval (a,b) such that $(a,b), a < 0 < b, \subseteq G_{\alpha_1}$. By the Archemedian property, there exists N such that $Nb > 1$. Thus $\frac{1}{n} \in (a,b) \subseteq G_{\alpha_1}$ for all positive integers $n \geq N$.

So $K = \{1, \frac{1}{2}, \dots, \frac{1}{N-1}\} \cup \{0, \frac{1}{N}, \frac{1}{N+1}, \dots\}$. By the above we have $\{0. \frac{1}{N}, \frac{1}{N+1}, \dots\} \subseteq G_{\alpha_1}$.

Additionally, since we have $\{1, \frac{1}{2}, \dots, \frac{1}{N-1}\}$ is finite there are finitely many G_{α_m} such that $\{1, \frac{1}{2}, \dots, \frac{1}{N-1}\} \subseteq G_{\alpha_2} \cup G_{\alpha_3} \cup \dots \cup G_{\alpha_m}$.

Thus we have $K \subseteq \bigcup_{i=1}^m G_{\alpha_i}$. Hence K is compact. $\square$

13: Construct a compact set of real numbers whose limit points form a countable set.

Proof. Let $A_0 = \{0, 1, \frac{1}{2} \dots, \}$ and $A_n = \{\frac{1}{x} + \frac{1}{y} : x = y, y+1 \dots, \}$, for $x \in \mathbb{N}$. We define $G = \bigcup_{n=0}^\infty A_n$. It is clear that A_0, A_n have the limit points 0 and $\frac{1}{n}$ respectively. Therefore $\{0, 1, \frac{1}{2}, \dots\} \subseteq G'$. Let $p \in \mathbb{R}$ be a limit point of G. Then if $p < 0$ let $\delta = \frac{|p|}{2}$ hence $(p-\delta, p+\delta) \cap G = \emptyset$. If instead $p > 2$ we let $\delta = \frac{p-2}{2}$ hence $(p-\delta, p+\delta) \cap G = \emptyset$. The remaining case, $1 < p \leq 2, \delta = \frac{1}{2}\min(p-1, 2-p)$ therefore $(p-\delta, p+\delta) \cap G$ contains finitely many points of G. Hence $G' \subseteq [0,1]$.

Now suppose $p \in [0,1] \setminus A_0$, then there exists a positive integer k such that $\frac{1}{k+1} < p < \frac{1}{k}$. Since $\frac{1}{x} + \frac{1}{x} \geq \frac{1}{x} + \frac{1}{y}$ and also $\frac{1}{x} + \frac{1}{x} \geq \frac{1}{y} + \frac{1}{y}$, for all $y \geq x$ then the maximum of the sets $G_{k+y} \cup G_{k+y+1} \cup G_{k+y+2} \cup \dots$ is , $\frac{2}{k+y}$. Letting $\delta = \frac{1}{2}\min(p - \frac{1}{k+1}, \frac{1}{k} - p)$ then $(p-\delta, p+\delta) \subset (\frac{1}{k+1}, \frac{1}{k})$.

If we have $\delta = \frac{1}{2}(\frac{1}{k} - p)$ then $p - \delta = \frac{3p}{2} - \frac{1}{2k} > \frac{1}{2}(\frac{3}{k+1} - \frac{1}{k}) = \frac{2k-1}{2(k+1)^2} > \frac{2}{k+y}$ for all $y > \frac{4(k+1)^2}{2k-1} - k$. In this situation $(p-\delta, p+\delta)$ will contain only a finite number of points from $G_1 \cup G_2 \cup \dots \cup G_{y-1}$.

Letting $\delta = \frac{1}{2}\min(p - \frac{1}{k+1}, \frac{1}{k} - p)$ so that $(p, \delta, p+\delta) \subset (\frac{1}{k+1}, \frac{1}{k})$. On the other hand letting $\delta = \frac{1}{2}(\frac{1}{k} - p)$ it follows that $p - \delta = \frac{3p}{2} - \frac{1}{2k} > \frac{1}{2}(\frac{3}{k+1} - \frac{1}{k}) = \frac{2k-1}{2(k+1)^2} > \frac{2}{+y}$ for all $y > \frac{4(k+1)^2}{2k-1} - k$. Then $(p-\delta, p+\delta)$ contains a finite number of points of $G_1 \cup G_2 \cup \dots \cup G_{y-1}$. In a similar fashion if $\delta = \frac{1}{2}(p - \frac{1}{k+1}), p - \delta = \frac{p}{2} + \frac{1}{2(k+1)} > \frac{1}{k+1} > \frac{2}{k+y}$ for all $y > k+2$. Again $(p-\delta, p+\delta)$ contains only a finite number of points of $G_1 \cup G_2 \cup \dots \cup G_{y-1}$. Hence in either case $(p-\delta, p+\delta)$ can only contain a finite number of points of G and therefore is not a limit point of G. Thus $G' = G_0 \subset G$ and it follows G is closed.

Finally, since for all $t \in G$ we have $|t| \leq 2$ (say), G is a bounded set, thus be Heine-Borel G is a compact set. □

14: Give an example of an open cover of the segment $(0,1)$ which has no finite sub-cover.

Proof. Consider the open sets $G_n = (\frac{1}{n}, 1), n = 2, 3, \dots$. If $x \in (0,1)$ then by the Archemedian property there exists an n such that $nx > 1$ i.e. $x \in G_n$. Also $(0,1) \subseteq \bigcup_{n=2}^{\infty} G_n$. Hence $\{G_2, G_3, \dots\}$ is an open cover of the segment $(0,1)$.

Now assume that $\{G_{n_1}\}, G_{n_2}, \dots, G_{n_k}\}$ is a finite subcover of $(0,1), n_1, n_2, \dots, n_k \in \mathbb{Z}^+$ and $2 \leq n_1 < n_2 < \dots < n_k$. Hence $(0,1) \subseteq \bigcup_{i=1}^{k} G_{n_i} \subseteq G_{n_k}$, this contradicts that $\frac{1}{2n_k} \in (0,1)$ but $\frac{1}{2n_k} \notin (\frac{1}{n_k}, 1)$. Thus we do not have a finite subcover. □

15:Show that Theorem 2.36 and its Corollary become false in $\mathbb{R}$ if the word compact is replaced by closed or by bounded

Proof. Let $K_n = [n, \infty), n \in \mathbb{N}$. Then K_n is closed. Consider $D = \bigcap_{i=1}^{k} K_{n_i}$ such that $1 \leq n_1 < n_2 < \dots < n_k$, then $D[n_k, \infty) \neq \emptyset$. However we have $\bigcap_{n=1}^{\infty} K_n = \emptyset$, since for $x \in \mathbb{R}$, $x > n$ for all n. This is a contradiction.

Consider $K_n = (0, \frac{1}{n})$, $n \in \mathbb{N}$. Then for each n we have K_n is bounded. For $1 \leq n_1 < n_2 < \cdots < n_k$ then $\bigcap_{i=1}^{k} K_{n_i} = (0, \frac{1}{n_k}) \neq \emptyset$ however $\bigcap_{n=1}^{\infty} K_n = \emptyset$ since for $x \in \mathbb{R}$ and $x \in \bigcap_{n=1}^{\infty} K_n$ then $x < \frac{1}{n}$ for all $n \in \mathbb{N}$.

Observe that in both examples, $K_{n+1} \subseteq K_n$ thus Theorem 2.36 and the corollary are false if we have closed or bounded. $\square$

16: Regard $\mathbb{Q}$ as a metric space with $d(p, q) = |p - q|$. Let E be the set of all $p \in Q$ such that $2 < p^2 < 3$. Show that E is closed and bounded in $\mathbb{Q}$ but E is not compact. Is E open in $\mathbb{Q}$?

Proof. $E = \{p \in \mathbb{Q} : 2 < p^2 < 3\} = (-\sqrt{3}, -\sqrt{2}) \cup (\sqrt{2}, \sqrt{3})$. Observe that $0 \in \mathbb{Q}$ and also $|p| < 10$ for every $p \in E$. Hence it is clear E is bounded. Now let $p \in \mathbb{Q}$ be a limit point of E. Hence for every $r > 0$ there exists $q \in E, q \neq p$ such that $|p - q| < r$. Setting $r = \frac{1}{n}$ for n sufficiently large we obtain $0 < q - \frac{1}{n} < p < q + \frac{1}{n}$. Since $2 < q^2 < 3$ we have that $2 < q^2 - \delta$ and $q^2 + \delta < 3$ for some $\delta > 0$.

Pick $n > \max\{\frac{3q}{\delta}, \frac{1}{q}\}$ then it follows $p^2 < (q + \frac{1}{n})^2 < q^2 + \frac{3q}{n} < q^2 + \delta < 3$. We also have, $p^2 > (q - \frac{1}{n})^2 > q^2 - \frac{2q}{n} > q^2 - \delta > 2$.

The above inequalities imply that $p \in E$ hence E is closed. Now assume that E is compact in $\mathbb{Q}$. Then for each n consider the followng sets: $U_n = \{p \in \mathbb{Q} : 2 - \frac{1}{n} < p^2 < 3.\}, U_n^+ = (\sqrt{2 - \frac{1}{n}}, \sqrt{3})$ and $U_n^- = (-\sqrt{3}, -\sqrt{2 - \frac{1}{n}})$. Clearly $U_n = U_n^+ \cup U_n^-$.

Now, if $p \in E$ we have $-\sqrt{3} < p < -\sqrt{2}$ or $\sqrt{2} < p < \sqrt{3}$. Thus either $-\sqrt{3} < p < -\sqrt{2 - \frac{1}{n}}$ or $\sqrt{2 - \frac{1}{n}} < p < \sqrt{3}$ for some $n \in \mathbb{N}$.

Therefore, $p \in U_n^+ \cup U_n^- = U_n$ for some $n \in \mathbb{N}$ thus $E \subseteq \bigcup_{n=1}^{\infty} U_n$.

Define the midpoints of the intervals $(-\sqrt{3}, -\sqrt{2 - \frac{1}{n}})$ and $(\sqrt{2 - \frac{1}{n}}, \sqrt{3})$ by x^- and x^+ respectively. Then if $q \in U_n$ means $q \in U_n^+$ or $q \in U_n^-$. If $q \in U_n^-$ there exists a positive real number $t > 0$ such that $d(x^-, q) = r - t$, given $r = |\sqrt{3} - \sqrt{2 - \frac{1}{n}}|$. Now for all s such that $d(q, s) < t$ we have by definition $d(x^-, s) \leq d(x^-, q) + d(q, s) < r - t + t = r$. Hence $s \in U_n^-$.

In a similar fashion, we also have $N_t(q) \subseteq U_n^+$ if $q \in U_n^+$. Thus it follows that q is an interior point of U_n and $\{U_n\}$ is an open cover of E.

By the compactness of E we must have $E \subseteq U_{n_1} \cup U_{n_2} \cup \cdots \cup U_{n_k}$ for positive integers $n_1, n_2, \ldots, n_k$ and $n_1 < n_2 < \cdots < n_k$. We observe this gives a contradiction since $U_{n+1} \subseteq U_n$ for all positive integers n thus $E \subseteq U_{n_1}$. A similar argument shows E is open. $\square$

17:Let E be the set of all $x \in [0, 1]$ whose decimal expansion contains only the digits 4 and 7. Is E countable? Is E dense in $[0,1]$? is E compact? Is E perfect?

Proof. $x \in E$ is of the form $x = 0.x_1x_2\dots, x_n \in \{4,7\}$ for all n. E is uncountable. Let $x = 0.1$ and $y = 0.2$, then by definition we have $x < y < z, \forall z \in E$. Then E is not dense in $[0,1]$. Observe that $0.4 \leq x < 0.8$ and $[0,1]$ is bounded it follows that E is bounded. Let x be a limit point of E. Clearly $0.4 < x < 0.8$. Suppose $x \notin E$ and let x_n be the first digit of x which isn't 4 or 7. Let $h = 0.00\dots01$ then the nth and $n+1$st digit of any number contained in $(x-h, x+h)$ is not 4 or 7. Hence $(x-h, x+h) \cap E = \emptyset$. and x is not a limit point of E. Contradiction. Hence, $x \in E$ and E is cloesd and compact by Heine Borel. E is closed. Let $x = 0.x_1x_2\dots$ and $N_h(x) \in E$ and a neighbourhood of E for $h > 0$. Letting $0 < h < 1$ and h_n the first non-zero digit of h. Letting $y = 0.y_1y_2\dots$ be the number such that if the $n+1$st digit of x is a 4 or 7 then the $n+1$st digit of y is a 7 or 4 and all other digits are maintained as in x. Clearly $y \neq x, |x-y| = 0.0\dots03 < h$ thus $y \in N_h(x)$. Clearly $y \in E$ and x is a limit point of E. Thus E is a perfect set.

□

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