FHS Microeconometrics Notes

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Abstract

These are my Macroeconometrics notes made for my finals in 2022. They cover all of the topics. Feel free to use these notes and pass them on to others. Please note, however, that these have just been made by a student and not checked over. They likely contain errors, so it will be worth checking things for yourself. Thanks to Kevin Sheppard, Vanessa Berenguer Rico and Bent Nielsen - these notes are just my interpretation of their lectures and tutorials.

Contents

Need to know Theorems

Chebyshev iid LLN

Theorem *(Law of Large Numbers by Chebyshev)*

For $i = 1, ..., n$ let x_i be independent and identically distributed with finite mean, μ, and variance σ^2. Then, as $n \longrightarrow \infty$,

$$\bar{x}_n = \frac{1}{n} \sum_{i=1}^{n} x_i \xrightarrow{P} \mu.$$

Lindeberg-Levy iid CLT

Theorem *(Central Limit Theorem by Lindeberg-Levy)*

For $i = 1, ..., n$ let x_i be independent and identically distributed with finite mean, μ, and variance σ^2. Then, as $n \longrightarrow \infty$,

$$\frac{\sqrt{n}(\bar{x}_n - \mu)}{\sigma} \xrightarrow{D} N(0, 1)$$

Multivariate iid CLT

Theorem *(Multivariate Lindeberg-Levy CLT)*

Let Z_i for $i = 1, ..., n$ be independent and identically distributed m-dimensional random vectors with finite mean vector $\mu_Z = E[Z_i]$, and finite positive definite covariance matrix $\Sigma_Z = E[(Z_i - \mu_Z)(Z_i - \mu_Z)']$. Then

$$\sqrt{n}(\bar{Z}_i - \mu_Z) \xrightarrow{D} N(0_m, \Sigma_Z)$$

where $\bar{Z}_n = n^{-1} \sum_{i=1}^{n} Z_i$ and $N(0_m, \Sigma_Z)$ is multivariate normal.

Covariance Stationary LLN

Theorem *(Law of Large Numbers)*

Let y_t be a covariance-stationary process with $E(y_t) = \mu$ and $\gamma_h = \operatorname{Cov}(y_t, y_{t-h})$ and absolutely summable autocovariances so that $\sum_{h=0}^{\infty} |\gamma_h| < \infty$. Then as $T \longrightarrow \infty$,

$$\frac{1}{T} \sum_{t=1}^{T} y_t \xrightarrow{\mathrm{P}} \mu,$$

Why require absolutely summable autocovariences?

Because for a ***weakly stationary process*** a sufficient condition for ***mean square convergence*** (which implies convergence in probability) is,

$$\sum_{h=0}^{\infty} |\gamma_h| < \infty \text{ where } \gamma_h = \operatorname{Cov}(y_t, y_{t-h})$$

What this means is that a sufficient condition for a process to mean square converge is that the covariances, although can initially be non-zero, must at some point tend to zero - hence they are absolutely summable.

The intuition of this is that eventually covariances like $\gamma_{1000} = \operatorname{cov}(y_t, y_{t+1000})$ should be zero if the process is really stationary and $E[y_t]$ really converges to μ. This implies that the sum of all the covariances should be less than infinity.

Wold Decomposition CLT

Theorem *(Central Limit Theorem)*

If $y_t = \mu + \Psi(L)u_t$ where $u_t \sim iid\left(0,\ \sigma^2\right)$ and $\sum_{j=0}^{\infty} |\psi_j| < \infty$, then as $T \longrightarrow \infty$,

$$\sqrt{T}\left(\bar{y}_T - \mu\right) \xrightarrow{\text{D}} \text{N}\left(0, \sum_{h=-\infty}^{\infty} \gamma_h\right)$$

where $\sum_{h=-\infty}^{\infty} \gamma_j = \sigma^2 \Psi^2(1)$ is the long run variance.

Stationary AR CLT

Theorem *(Central Limit Theorem for AR processes)*

Let y_t be a stationary $AR(p)$ process with $E\left(y_t\right) = \mu$ and $\gamma_h = \text{Cov}\left(y_t, y_{t-h}\right)$. Then,

$$\sqrt{T}\left(\bar{y}_T - \mu\right) \xrightarrow{\text{D}} \text{N}\left(0, \sum_{h=-\infty}^{\infty} \gamma_h\right).$$

MDS LLN

Theorem *(Law of Large Numbers)*

Let $(m_t, \mathcal{I}_t)$ for $t \in \mathbb{N}$ be a martingale difference sequence. If one of the following conditions holds

(a) $\sum_{t=1}^{\infty} E\left|m_t\right|^{1+p} / t^{1+p} < \infty$ for some $p \in [0, 1]$,

(b) $\underset{T \longrightarrow \infty}{lim} \max_{t \leq T} E\left|m_t\right|^{1+p} < \infty$ for some $p \in [0, 1]$,

then, as $T \longrightarrow \infty$,

$$T^{-1} \sum_{t=1}^{T} m_t \xrightarrow{P} 0.$$

MDS CLT

Theorem *(Central Limit Theorem)*

Let $(m_t, \mathcal{I}_t)$ for $t \in \mathbb{N}$ be a martingale difference sequence satisfying $\text{E}m_t^2 < \infty$ for all t. Let $S_T^2 = \sum_{t=1}^{T} Em_t^2$. Suppose

$$(i) \quad \sum_{t=1}^{T} m_t^2 / S_T^2 \xrightarrow{\text{P}} 1,$$

$$(ii) \quad \sum_{t=1}^{T} \text{E}\left\{\left(m_t / S_T\right)^2 1_{(|m_t S_T| > \delta)}\right\} \longrightarrow 0 \text{ for all } \delta > 0,$$

then

$$\frac{1}{S_T} \sum_{t=1}^{T} m_t \xrightarrow{\text{D}} \text{N}(0, 1)$$

Remark: The Lindeberg condition *(ii)*

$$\sum_{t=1}^{T} \mathrm{E}\left\{ \left(m_t/S_T\right)^2 1_{(|m_t/S_T|>\delta)} \right\} \longrightarrow 0,$$

for all $\delta > 0$ follows from the Lyapounov condition

$$(ii') \quad \sum_{t=1}^{T} E\left|m_t/S_T\right|^{2+\delta} \longrightarrow 0$$

for some $\delta > 0$.

Slutsky's Theorem

Theorem *(Slutsky Theorem)*

Let $Y_n \xrightarrow{P} c$ and $X_n \xrightarrow{D} X$, then:

(a) $Y_n + X_n \xrightarrow{D} c + X$
(b) $Y_n X_n \xrightarrow{D} cX$
(c) $Y_n^{-1} X_n \xrightarrow{D} c^{-1} X$ if $c \neq 0$
(d) If $c = 0$ then $Y_n X_n \xrightarrow{P} 0$

Wold Decomposition

Theorem *(The Wold Decomposition)*

If x_t is stationary and non-deterministic, then

$$x_t = \sum_{j=0}^{\infty} \Psi_j u_{t-j} + z_t = \Psi(L) u_t + z_t$$

Where,

- $\Psi_0 = 1$ and $\sum_{j=0}^{\infty} \Psi_j^2 < \infty$
- u_t is $wn(0, \sigma^2)$
- z_t is deterministic
- $\mathrm{Cov}(u_t, z_t) = 0$ for all s and t
- Ψ_j and u_t are unique.

Some Laws, Tricks, and Terms

Series and Summations

The 'T' Rules

$$\sum_{t=1}^{T} 1 = T$$
$$\sum_{t=1}^{T} t = \sum_{t=1}^{T}(T+1-t) = \frac{T(T+1)}{2}$$
$$\sum_{t=1}^{T} t^2 = \frac{T(T+1)(2T+1)}{6}$$

Geometric Series

$$\text{Finite: } \sum_{k=0}^{n-1} ar^k = \frac{a\,(1-r^n)}{1-r} = \sum_{k=1}^{n} ar^{k-1} = \frac{a\left(1-r^{(n+1)}\right)}{1-r} \quad [r \neq 1]$$
$$\text{Infinite: } \sum_{k=0}^{\infty} ar^k = \frac{a}{1-r} = \sum_{k=1}^{\infty} ar^{k-1} = \frac{a}{1-r} \quad \Big[|r| < 1\Big]$$

Useful Result

$$\frac{1}{T}\sum_{t=1}^{T}\left(\frac{t}{T}\right)^{v} \longrightarrow \int_0^1 r^v dr = \frac{1}{v+1}$$
$$\frac{1}{T^{v+1}}\sum_{t=1}^{T}(t)^{v} \longrightarrow \int_0^1 r^v dr = \frac{1}{v+1}$$

Stationarity

Strict Stationarity

The time series $\{Y_t, t \in \mathbb{Z}\}$ is strictly stationary if the joint distributions $(Y_t, Y_{t+1}, \ldots, Y_{t+k}) \underset{D}{=} (Y_s, Y_{s+1}, \ldots, Y_{s+k})$ for all t, s and k.

Weak Stationarity

The time series $\{Y_t, t \in \mathbb{Z}\}$ is weakly stationary if:

(i) $E[Y_t] = m$ for all t

(ii) $Var(Y_t) = \sigma^2 < \infty$ for all t.

(iii) $Cov(Y_t, Y_s) = Cov(Y_{t+h}, Y_{s+h})$ for all $t, s, h \in \mathbb{Z}$.

(iii') $Cov(Y_t, Y_{t-h}) = \gamma_h$ for all h (this is equivalent to (iii)).

That is the mean, variance, and covariance do no depend on t. They are *time invariant.*

If a process is Gaussian Normal then strict and weak stationarity coincide

Martingale Difference Sequence

Let m_t be a sequence of random scalars with $E(m_t) = 0$ and let $\mathcal{I}_t$ be the information available at date t, so $\mathcal{I}_t$ will include current and past values of $\{m_t\}$, as well as current and past values of any other random sequences, such as perhaps $\{x_t\}$.

$$\mathcal{I}_t = \{m_t, m_{t-1}, m_{t-2}, \ldots, x_t, x_{t-1}, x_{t-2}, \ldots\}$$

If $E(m_t \mid \mathcal{I}_{t-1}) = 0$ then $\{m_t\}$ is said to be a martingale difference sequence with respect to $\{\mathcal{I}_t\}$.

$E(m_t \mid \mathcal{I}_{t-1}) = 0$ implies that $\{m_t\}$ is serially uncorrelated (stronger assumption than uncorrelatedness but weaker than independence.)

Lag Operator

Properties

$$
\begin{aligned}
&Ly_t = y_{t-1} \\
&L\left(Ly_t\right) = L\left(y_{t-1}\right) = y_{t-2} \text{ hence } L^j\left(y_t\right) = y_{t-j} \\
&L\mu = \mu \\
&L^j \mu y_t = \mu y_{t-j} \\
&L\left(y_t + x_t\right) = y_{t-1} + x_{t-1}
\end{aligned}
$$

Using Lag Polynomials

Let's consider the AR(1) case,

$$y_t = \phi y_{t-1} + u_t$$

We can rewrite the model using the lag operator after doing some basic algebra,

$$
\begin{aligned}
y_t - \phi y_{t-1} &= u_t \\
(1 - \phi L) y_t &= u_t \\
\Phi(L) y_t &= u_t \text{ where } \Phi(L) = 1 - \phi L
\end{aligned}
$$

This doesn't immediately seem useful, but it allows us to simplify more complicated models, for example the AR(p) model,

$$
\begin{aligned}
y_t &= \phi_1 y_{t-1} + \phi_2 y_{t-2} + \ldots + \phi_p y_{t-p} + u_t \\
y_t - \phi_1 y_{t-1} - \phi_2 y_{t-2} - ... - \phi_p y_{t-p} &= u_t \\
(1 - \phi_1 L - \phi_2 L^2 - ... - \phi_p L^p) y_t &= u_t \\
\Phi_p(L) y_t &= u_t \text{ where } \Phi_p(L) = 1 - \phi_1 L - \phi_2 L^2 - ... - \phi_p L^p
\end{aligned}
$$

And the ARMA(p,q) model,

$$
\begin{aligned}
y_t &= \underbrace{\phi_1 y_{t-1} \ldots + \phi_p y_{t-p}}_{AR(p)} + \underbrace{u_t + \theta_1 u_{t-1} + \ldots + \theta_q u_{t-q}}_{MA(q)} \\
y_t - \phi_1 y_{t-1} - \phi_2 y_{t-2} - \ldots - \phi_p y_{t-p} &= u_t + \theta_1 u_{t-1} + \theta_2 u_{t-2} + \ldots + \theta_q u_{t-q} \\
\left(1 - \phi_1 L - \phi_2 L^2 - \ldots - \phi_p L^p\right) y_t &= \left(1 + \theta_1 L + \theta_2 L^2 + \ldots + \theta_q L^q\right) u_t \\
\Phi_p(L) y_t &= \Theta_q(L) u_t \\
\text{Where } \Phi_p(L) &= \left(1 - \phi_1 L - \phi_2 L^2 - \ldots - \phi_p L^p\right), \\
\text{and } \Theta_q &= \left(1 + \theta_1 L + \theta_2 L^2 + \ldots + \theta_q L^q\right).
\end{aligned}
$$

Inverse Lag Polynomial

Using lag polynomials also allows us to 'switch' between representations, such as AR (auto-regressive) and MA (moving-average). We can do this by *inverting* the lag polynomial. Recall in the AR(1) case

$$\Phi(L)y_t = u_t \text{ where } \Phi(L) = 1 - \phi L$$

Well if we want to get the MA representation (y_t in terms of the lags of u_t), we simply invert $\Phi(L)$,

$$y_t = \{\Phi(L)\}^{-1} u_t \text{ where } \Phi(L) = 1 - \phi L$$

Conditions for Invertibility

In order to be able to invert a lag polynomial we *must have roots of the polynomial* ***outside of the unit circle***.

If the roots of a polynomial are inside the unit circle then the coefficients do not decay sufficiently fast for the error term to be a well-defined random variable with finite variance. More technically when the roots are inside the unit circle the infinite sum does not converge in mean square.

Note that this only really matters for AR model, since as long as an MA model is finite then you can **always find an equivalent and invertible MA polynomial**. Of course you still need to check roots for both models though; it is just that if an MA polynomial is not invertible you can always find an equivalent model that is invertible.

Finding Roots of Lag Polynomial

The roots of the lag polynomial are very simply just the values of L when $\Phi(L) = 0$. We will consider two very simple cases for finding these.

(1) First Order Difference Equation

$$\begin{aligned} y_t &= \phi y_{t-1} + u_t \\ (1 - \phi L)y_t &= u_t \\ \Phi(L)y_t &= u_t \end{aligned}$$

So we have the lag polynomial,

$$\Phi(L) = 1 - \phi L$$

Which has roots when,

$$\Phi(L) = 0 \Rightarrow 1 - \phi L = 0 \Rightarrow L = \frac{1}{\phi}$$

Hence the roots of this polynomial are at $L = \frac{1}{\phi}$.

(2) Second Order Difference Equation

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \omega_t$$

Using lags,

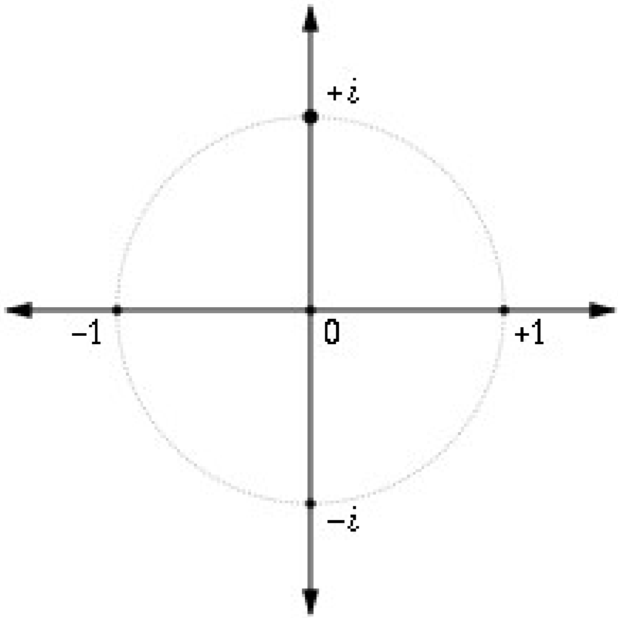
$$\left(1 - \phi_1 L - \phi_2 L^2\right) y_t = \omega_t$$

Find roots of lag polynomial, $1 - \phi_1 L - \phi_2 L^2 = 0$, using,

$$L = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The Unit Circle

To be able to invert a polynomial it must be the case that **all** the polynomials roots are outside of the unit circle. That is we need all the values of L at $\Phi(L) = 0$ to be outside the unit circle.



An imaginary number bi is outside the unit-circle if $|b| > 1$.

A real number a is outside the unit-circle if $|a| > 1$.

A number with a real and imaginary part $a + bi$ is outside the unit circle if $\sqrt{b^2 + a^2} > 1$

Inverting the Lag Polynomial in the AR(1) case

$$\begin{aligned} y_t &= \phi y_{t-1} + u_t \\ (1 - \phi L) y_t &= u_t \\ \Phi(L) y_t &= u_t \end{aligned}$$

We can only invert the polynomial $\Phi(L)$ if its roots are outside of the unit circle. The roots of the lag polynomial, that is the value of L when $\Phi(L) = 0$, are given by,

$$0 = \Phi(L) \Rightarrow 0 = 1 - \phi L \Rightarrow L = \frac{1}{\phi}$$

The roots being outside unit circle requires that $|L| > 1$, hence the lag polynomial $\Phi(L)$ is only invertible in the case in which $|\phi| < 1$.

Supposing that we are in the case in which $|\phi| < 1$, therefore,

$$y_t = [\Phi(L)]^{-1} u_t$$

Where,

$$[\Phi(L)]^{-1} = \frac{1}{1 - \phi L}$$

Inverse Lag Polynomial: Infinite Sum

We can always turn an inverted lag polynomial into an infinite sum lag polynomial. This is useful when switching between forms, such as from AR to MA, since leaving an inverted lag polynomial has little meaning to us.

What this means is, if we consider a pure AR(p) model,

$$\Phi_p(L)x_t = \varepsilon_t$$

We can also write this as,

$$x_t = \{\Phi_p(L)\}^{-1}\varepsilon_t$$

And we can write this inverted lag polynomial as an infinite lag polynomial, such that we get an MA∞ model,

$$x_t = \Psi_\infty(L)\varepsilon_t$$

In other words it is always the case that

$$\{\Phi_p(L)\}^{-1} = \frac{1}{1+\phi_1 L+\phi_2 L^2 ... +\phi_p L^p} = 1+\psi_1 L+\psi_2 L^2+... = \Psi_\infty(L)$$

Proof

The proof of this really just comes from the law for infinite geometric sums stated above,

$$\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r} \quad \text{for } |r|<1$$

This law shows that an inverse polynomial of degree one can be written as an infinite sum. This implies that in our AR(1) case,

$$\frac{1}{1-\phi} = \sum_{k=0}^{\infty}\phi^k = 1+\phi+\phi^2+\phi^3+... \quad \text{for } |\phi|<1$$

In the case of a finite polynomial of degree p we just need to write it in its factorised form,

$$\Phi_p(L) = 1+\phi_1 L+\phi_2 L^2+...\phi_p L^p = \prod_{i=1}^{p}\left(1-\frac{L}{r_i}\right)$$

Where $r_1, ..., r_p$ are the roots of the polynomial.

Having written the p^{th} degree polynomial as a product of first degree polynomials, invert and apply the law from above,

$$\Big\{\Phi_p(L)\Big\}^{-1} = \prod_{i=1}^{p}\left(1-\frac{L}{r_i}\right)^{-1} = \prod_{i=1}^{p}\left(\sum_{j=0}^{\infty}\left(\frac{L}{r_i}\right)^j\right) = \sum_{j=0}^{\infty}\psi_j L^j = 1+\psi_1 L+\psi_2 L^2+... = \Psi_\infty(L)$$

Which holds because $\left|\frac{L}{r_i}\right| < 1$ for all $i = 1, ..., p$.

Hence,

$$\{\Phi_p(L)\}^{-1} = \frac{1}{1+\phi_1 L+\phi_2 L^2 ... +\phi_p L^p} = 1+\psi_1 L+\psi_2 L^2+... = \Psi_\infty(L)$$

Example: General Case

We have already shown the AR(1) case, so lets consider the $AR(2)$ case,

$$\begin{aligned}
y_t &= \phi_1 y_{t-1} + \phi_2 y_{t-2} + u_t \\
y_t - \phi_1 y_{t-1} - \phi_2 y_{t-2} &= u_t \\
(1 - \phi_1 L - \phi_2 L^2) y_t &= u_t \\
y_t &= \frac{1}{1 - \phi_1 L - \phi_2 L^2} u_t \\
&= \Big\{ \Phi_2(L) \Big\}^{-1} u_t
\end{aligned}$$

We now know that is possible to write this model as an MA∞ model by writing the inverted polynomial of degree two as an infinite geometric sum.

$$\Big\{ \Phi_2(L) \Big\}^{-1} = \frac{1}{1 - \phi_1 L - \phi_2 L^2} = \psi_0 + \psi_1 L + \psi_2 L + \psi_3 L^3 + ... = \Psi_\infty(L)$$

To calculate the values of ϕ_1, ϕ_2, etc we know it will the case that,

$$\left(1 - \phi_1 L - \phi_2 L^2\right) \left(\psi_0 + \psi_1 L + \psi_2 L^2 + \psi_3 L^3 + \ldots\right) = 1$$

Or perhaps more obviously that,

$$\left(1 - \phi_1 L - \phi_2 L^2\right) \left(\psi_0 + \psi_1 L + \psi_2 L^2 + \psi_3 L^3 + \ldots\right) = 1 + 0L + 0L^2 + 0L^3 + \ldots$$

By multiplying out the brackets and matching up coefficients of L we find that,

$$\begin{aligned}
&\Rightarrow \psi_0 = 1 \\
&\Rightarrow \psi_1 - \phi_1 \psi_0 = 0 \\
&\Rightarrow \psi_2 - \psi_1 \phi_1 - \phi_2 \psi_0 = 0 \\
&\Rightarrow \psi_3 - \psi_2 \phi_1 - \psi_1 \phi_2 = 0 \\
&\qquad \ldots \\
&\Rightarrow \psi_j - \psi_{j-1} \phi_1 - \psi_{j-2} \phi_2 = 0
\end{aligned}$$

Example: Specific Case

Now we can consider a specific example. Take this ARMA(2,2) model and express it into MA form,

$$x_t = 1.1 x_{t-1} - 0.8 x_{t-2} + u_t - 1.7 u_{t-1} + 0.72 u_{t-2}$$

We can begin by writing it using lag polynomials,

$$\left(1 - 1.1L + 0.8L^2\right) x_t = \left(1 - 1.7L + 0.72L^2\right) u_t$$

Of course before we try and invert the polynomial on x_t we need to check that it can be inverted: that the roots are outside of the unit circle. Given that, in this case, they are, we can write it as,

$$x_t = \frac{\left(1 - 1.7L + 0.72L^2\right)}{\left(1 - 1.1L + 0.8L^2\right)} u_t$$

Where we know it will be the case that,

$$\begin{aligned}
\frac{\left(1 - 1.7L + 0.72L^2\right)}{\left(1 - 1.1L + 0.8L^2\right)} &= \left(\delta_0 + \delta_1 L + \delta_2 L^2 + \delta_3 L^3 + \ldots\right) \\
\left(1 - 1.7L + 0.72L^2\right) &= \left(1 - 1.1L + 0.8L^2\right) \left(\delta_0 + \delta_1 L + \delta_2 L^2 + \delta_3 L^3 + \ldots\right)
\end{aligned}$$

Multiplying out the brackets on the RHS,

$$
\begin{aligned}
&= \delta_0 \quad + \delta_1 L + \quad \delta_2 L^2 \quad + \delta_3 L^3 + \dots \\
&\qquad - 1.1\delta_0 L - 1.1\delta_1 L^2 - 1.1\delta_2 L^3 + \dots \\
&\qquad\qquad\qquad + 0.8\delta_0 L^2 + 0.8\delta_1 L^3 + \dots \\
&= \delta_0 + (\delta_1 - 1.1\delta_0)\, L + (\delta_2 - 1.1\delta_1 + 0.8\delta_0)\, L^2 + (\delta_3 - 1.1\delta_2 + 0.8\delta_1)\, L^3 + ...
\end{aligned}
$$

Finally we can set these coefficients equal to the coefficients that we know from the LHS,

$$
\begin{aligned}
\delta_0 &= 1 \\
\delta_1 - 1.1\delta_0 &= -1.7 \\
\delta_2 - 1.1\delta_1 + 0.8\delta_0 &= 0.72 \\
\delta_3 - 1.1\delta_2 + 0.8\delta_1 &= 0
\end{aligned}
$$

And so this implies,

$$
\begin{aligned}
&\delta_0 = 1 \\
&\Rightarrow \delta_1 = -0.6 \\
&\Rightarrow \delta_2 = -0.74 \\
&\Rightarrow \delta_3 = -1.614 \\
&\Rightarrow \delta_j = 1.1\delta_{j-1} - 0.8\delta_{j-2} \;\; (j > 2)
\end{aligned}
$$

AR(p)

AR(p) is an autoregressive model of order p. That is we model y_t as a function of its last p lags $y_{t-1}, ..., y_{t-p}$, a constant α, and an error u_t.

$$y_t = \alpha + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \ldots + \phi_p y_{t-p} + u_t$$

Usefulness of the Model

- Forecasting,
- Stochastic properties of the data,
- Modelling time series.

Stationary AR(1)

We will studies the properties of the stationary AR(1) process,

$$y_t = \alpha + \phi y_{t-1} + u_t \ , \ u_t \sim iid\left(0, \sigma^2\right)$$

For simplicity we omit the constant α, but it could be included. When $\alpha = 0$ the model above is,

$$y_t = \phi y_{t-1} + u_t$$

Using the fact that $y_{t-i} = \phi y_{t-i-1} + u_{t-i}$ and backward substitution, the AR(1) model can be represented as,

$$\begin{aligned}
y_t &= \phi y_{t-1} + u_t \\
&\qquad y_{t-1} = \phi y_{t-2} + u_{t-1} \\
y_t &= \phi\big(\phi y_{t-2} + u_{t-1}\big) + u_t \\
&= \phi^2 y_{t-2} + \phi u_{t-1} + u_t \\
&= \phi^2\big(\phi y_{t-3} + u_{t-2}\big) + \phi u_{t-1} + u_t \\
&= \phi^3 y_{t-3} + \phi^2 u_{t-2} + \phi u_{t-1} + u_t \\
&\ldots \\
y_t &= \phi^h y_{t-h} + \sum_{i=0}^{h-1} \phi^i u_{t-i} \\
y_t &= \phi^t y_0 + \sum_{i=0}^{t-1} \phi^i u_{t-i}
\end{aligned}$$

Requirements for Stationarity

For an AR(1) to be stationary we require,

(1) $|\phi| < 1$,

(2) $E\left[y_0\right] = 0, \ \operatorname{Var}\left(y_0\right) = \frac{\sigma^2}{1-\phi^2}$.

Expectation

$$E\left[y_t\right] = \phi^t E\left[y_0\right] + \sum_{j=0}^{t-1} \phi^j E\left[u_{t-j}\right] = 0$$

Variance

$$\begin{aligned}
\operatorname{Var}\left(y_t\right) &= \phi^{2t} \operatorname{Var}\left(y_0\right) + \operatorname{Var}\left(\sum_{j=0}^{t-1} \phi^j u_{t-j}\right) \\
&= \phi^{2t} \operatorname{Var}\left(y_0\right) + \sum_{j=0}^{t-1} \phi^2 \operatorname{Var}\left(u_{t-j}\right) + 2 \sum_{j=0}^{t-2} \sum_{i=j+1}^{t-1} \operatorname{Cov}\left(u_{t-j}, u_{t-i}\right) \\
&= \phi^{2t}\left(\frac{\sigma^2}{1-\phi^2}\right) + \sigma^2 \sum_{j=0}^{t-1} \phi^{2j} \\
&= \phi^{2t}\left(\frac{\sigma^2}{1-\phi^2}\right) + \sigma^2 \left(\frac{1-\phi^{2t}}{1-\phi^2}\right) \\
&= \frac{\sigma^2}{1-\phi^2}
\end{aligned}$$

Covariance

$$\begin{aligned}
\operatorname{Cov}\left(y_t, y_{t-h}\right) &= \operatorname{Cov}\left(\phi^h y_{t-h} + \sum_{j=0}^{h-1} \phi^j u_{t-j},\ y_{t-h}\right) \\
&= \phi^h \operatorname{Cov}\left(y_{t-h}, y_{t-h}\right) + \underbrace{\operatorname{Cov}\left(\sum_{j=0}^{h-1} \phi^j u_{t-j},\ y_{t-h}\right)}_{=\operatorname{Cov}(u_t+\phi u_{t-1}+\ldots+\phi^{h-1}u_{t-h+1})=0} \\
&= \phi^h \operatorname{Var}\left(y_{t-h},\ y_{t-h}\right) \\
&= \phi^h \left(\frac{\sigma^2}{1-\phi^2}\right)
\end{aligned}$$

In case it isn't obvious why $\operatorname{Cov}(u_t+\phi u_{t-1}+\ldots+\phi^{h-1}u_{t-h+1}) = 0$, this is because y_{t-h} is only correlated with error terms that come temporally *before* it, that is before period $(t-h)$. In our case here we are considering error terms from period $t-h+1$ to period t, and notice that both of these periods are temporally *after* $t-h$, hence there is no covariances between y_{t-h} and these error terms.

Autocorrelation Function

$$\begin{aligned}
\operatorname{Corr}\left(y_t, y_{t-h}\right) &= \frac{\operatorname{Cov}\left(y_t, y_{t-h}\right)}{\operatorname{Var}\left(y_t\right)} \\
&= \frac{\phi^h \left(\frac{\sigma^2}{1-\phi^2}\right)}{\left(\frac{\sigma^2}{1-\phi^2}\right)} \\
&= \phi^h
\end{aligned}$$

Converting AR(1) to MA

$$
\begin{aligned}
y_t &= \phi y_{t-1} + u_t \\
y_t - \phi y_{t-1} &= u_t \\
(1-\phi L) y_t &= u_t \\
y_t &= \frac{1}{1-\phi L} u_t \\
&\quad \text{Where } \frac{1}{1-\phi L} = \left(1 + \phi L + \phi^2 L^2 + \phi^3 L^3 + \ldots\right) \\
y_t &= \sum_{j=0}^{\infty} \phi^j L^j u_t \\
y_t &= \sum_{j=0}^{\infty} \phi^j u_{t-j}
\end{aligned}
$$

Wold Decomposition

Recall the theorem,

Theorem *(The Wold Decomposition)*

If x_t is stationary and non-deterministic, then

$$
x_t = \sum_{j=0}^{\infty} \Psi_j u_{t-j} + z_t = \Psi(L) u_t + z_t
$$

Where,

- $\Psi_0 = 1$ and $\sum_{j=0}^{\infty} \Psi_j^2 < \infty$
- u_t is $wn(0, \sigma^2)$
- z_t is deterministic
- $\text{Cov}(u_t, z_t) = 0$ for all s and t
- Ψ_j and u_t are unique

In the case of the stationary AR(1) process (with no constant) the Wold Decomposition is

$$
y_t = \sum_{j=0}^{\infty} \phi^j u_{t-j}
$$

Which is also the same as the MA representation.

ARMA(p,q)

$$y_t = \underbrace{\phi_1 y_{t-1} + \phi_2 y_{t-2} + \ldots + \phi_p y_{t-p}}_{AR(p)} + \underbrace{\alpha}_{constant} + \underbrace{u_t + \theta_1 u_{t-1} + \theta_2 u_{t-2} + \ldots + \theta_q u_{t-q}}_{MA(q)}$$

Lag Polynomials

We can write an ARMA(p,q) using lag polynomials as,

$$\begin{aligned} y_t - \phi_1 y_{t-1} - \phi_2 y_{t-2} - \ldots - \phi_p y_{t-p} &= \alpha + u_t + \theta_1 u_{t-1} + \theta_2 u_{t-2} + \ldots + \theta_q u_{t-q} \\ \left(1 - \phi_1 L - \phi_2 L^2 - \ldots - \phi_p L^p\right) y_t &= \alpha + \left(1 + \theta_1 L + \theta_2 L^2 + \ldots + \theta_q L^q\right) u_t \\ \Phi_p(L) y_t &= \alpha + \Theta_q(L) u_t \end{aligned}$$

Where we define $\Phi_p(L) = \left(1 - \phi_1 L - \phi_2 L^2 - \ldots - \phi_p L^p\right)$ and $\Theta_q(L) = \left(1 + \theta_1 L + \theta_2 L^2 + \ldots + \theta_q L^q\right)$.

MA Representation

Must check for **stationarity** of the polynomial we are going to invert, in this case $\Phi_p(L)$, before trying to invert it. This is done by checking that the roots of the polynomial $\Phi_p(L)$ are outside of the unit circle.

$$\begin{aligned} y_t &= \{\Phi_p(L)\}^{-1} \alpha + \{\Phi_p(L)\}^{-1} \Theta_q(L) u_t \\ &= \alpha_{MA} + \{\Phi_p(L)\}^{-1} \Theta_q(L) u_t \end{aligned}$$

Where we know that,

$$\{\Phi_p(L)\}^{-1} \Theta_q(L) = \frac{1 + \theta_1 L + \theta_2 L^2 + \ldots + \theta_q L^q}{1 - \phi_1 L - \phi_2 L^2 - \ldots - \phi_p L^p} = 1 + \psi_1 L + \psi_2 L^2 + \psi_3 L^3 + \ldots$$

The α_{MA} term is just a constant, since the Lag operator passes through constants,

$$\{\Phi_p(L)\}^{-1} \alpha = \frac{1}{1 - \phi_1 L - \phi_2 L^2 - \ldots - \phi_p L^p} \alpha = \{\Phi_p(1)\}^{-1} \alpha = \frac{\alpha}{1 - \phi_1 - \phi_2 - \ldots - \phi_p} = \alpha_{MA}$$

Wold Decomposition

Recalling the theorem from above, for a stationary and non-deterministic y_t, then

$$y_t = \sum_{j=0}^{\infty} \Psi_j u_{t-j} + z_t = \Psi(L) u_t + z_t$$

In the ARMA(p,q) case, the Wold Decomposition is,

$$y_t = \{\Phi_p(L)\}^{-1} \Theta_q(L) u_t + \{\Phi_p(L)\}^{-1} \alpha$$

Which is exactly the MA representation.

AR Representation

Must check for **stationarity** of the polynomial we are going to invert, in this case $\Theta_p qL)$, before trying to invert it. This is done by checking that the roots of the polynomial $\Theta_q(L)$ are outside of the unit circle. Note, however, that even if this MA polynomial is non-invertible we can find an equivalent and invertible MA polynomial since it is finite.

$$\begin{aligned}\{\Theta_q(L)\}^{-1}\Phi_p(L)y_t - \{\Theta_q(L)\}^{-1}\alpha &= u_t \\ \{\Theta_q(L)\}^{-1}\Phi_p(L)y_t - a_{AR} &= u_t\end{aligned}$$

Where we know that,

$$\{\theta_q(L)\}^{-1}\Phi_p(L) = \frac{1-\phi_1 L - \phi_2 L^2 - \ldots - \phi_p L^p}{1-\theta_1 L - \theta_2 L^2 - \ldots - \theta_q L^q} = 1 + \Gamma_1 L + \Gamma_2 L^2 + \Gamma_3 L^3 + \ldots$$

The α_{AR} term is just a constant, since the Lag operator passes through constants

$$\{\Theta_q(L)\}^{-1}\alpha = \frac{1}{1-\theta_1 L - \theta_2 L^2 - \ldots - \theta_q L^q}\alpha = \{\Theta_q(1)\}^{-1}\alpha = \frac{\alpha}{1-\theta_1-\theta_2-\ldots-\theta_q} = \alpha_{AR}$$

Autocorrelation Function

Consider a stationary ARMA(1,1), that is where $|\phi| < 1,\ |\theta| < 1,\ \phi \neq 0$,

$$x_t = \phi x_{t-1} + u_t - \theta u_{t-1} \text{ with } u_t \sim iid(0,\sigma^2)$$

Using lag polynomials this can be rewritten as,

$$\begin{aligned}x_t - \phi x_{t-1} &= u_t - \theta u_{t-1} \\ (1-\phi L)x_t &= (1-\theta L)u_t \\ x_t &= \frac{1-\theta L}{1-\phi L}u_t \\ &= (1-\theta L)\frac{1}{1-\phi L}u_t\end{aligned}$$

Recall that,

$$\frac{1}{1-\phi L} = 1 + \phi L + \phi^2 L^2 + \phi^3 L^3 + ...$$

Hence we can write the ARMA(1,1) as,

$$\begin{aligned}x_t &= (1-\theta L)\frac{1}{1-\phi L}u_t \\ &= (1-\theta L)(1+\phi L+\phi^2 L^2+\ldots)u_t \\ &= (1+\phi L+\phi^2 L^2+\ldots \\ &\quad -\theta L - \theta\phi L^2 - \theta\phi^2 L^3 + \ldots)u_t \\ &= (1+\phi L - \theta L + \phi^2 L^2 - \theta\phi L^2 + \ldots)u_t \\ &= u_t + (\phi-\theta)u_{t-1} + (\phi-\theta)\phi u_{t-2} + \ldots \\ x_t &= u_t + (\phi-\theta)\sum_{j=1}^{\infty}\phi^{j-1}u_{t-j}\end{aligned}$$

This form above will be useful in showing the autocovariance function.

The autocovariance function is found starting from,

$$x_t = \phi x_{t-1} + u_t - \theta u_{t-1}$$

Then multiplying through by x_{t-h} and taking expectations,

$$\begin{aligned} x_t x_{t-h} &= \phi x_{t-1} x_{t-h} + u_t x_{t-h} - \theta u_{t-1} x_{t-h} \\ E\left[x_t x_{t-h}\right] &= \phi E\left[x_{t-1} x_{t-h}\right] + E\left[u_t x_{t-h}\right] - \theta E\left[u_{t-1} x_{t-h}\right] \end{aligned}$$

Given that $u_t \sim iid(0, \sigma^2)$, we know from the MA form of the ARMA(1,1) that $E(x_t) = 0$,

$$E[x_t] = E[u_t] + (\phi - \theta) \sum_{j=1}^{\infty} \phi^{j-1} E[u_{t-j}] = 0$$

This implies that the autocovariance which usually is given by $\gamma_x(h) = E[x_t x_{t-h}] - E[x_t]E[x_{t-h}]$ is given by $\gamma_x(h) = E[x_t x_{t-h}]$ instead. Therefore,

$$\gamma_x(h) = \phi \gamma_x(h-1) + E\left[u_t x_{t-h}\right] - \theta E\left[u_{t-1} x_{t-h}\right]$$

And so,

$$\begin{aligned} h = 0 : \gamma_x(0) &= \phi \gamma_x(1) + E\left[u_t x_t\right] - \theta E\left[u_{t-1} x_t\right] \\ h = 1 : \gamma_x(1) &= \phi \gamma_x(0) + E\left[u_t x_{t-1}\right] - \theta E\left[u_{t-1} x_{t-1}\right] \\ h \geq 2 : \gamma_x(h) &= \phi \gamma_x(h-1) + E\left[u_t x_{t-h}\right] - \theta E\left[u_{t-1} x_{t-h}\right] \end{aligned}$$

Which gives, using the fact that $x_t = u_t + (\phi - \theta) \sum_{j=1}^{\infty} \phi^{j-1} u_{t-j}$ and $x_t - 1 = u_t - 1 + (\phi - \theta) \sum_{j=1}^{\infty} \phi^{j-1} u_{t-1-j}$ from earlier,

$$\begin{aligned} h = 0 : \gamma_x(0) &= \phi \gamma_x(1) + E\left[u_t x_t\right] - \theta E\left[u_{t-1} x_t\right] \\ &= \phi \gamma_x(1) + E\left[u_t^2 + (\phi - \theta) \sum_{j=1}^{\infty} \phi^{j-1} u_{t-j} u_t\right] - \theta E\left[u_{t-1} u_t + (\phi - \theta) \sum_{j=1}^{\infty} \phi^{j-1} u_{t-j} u_{t-1}\right] \\ \gamma_x(0) &= \phi \gamma_x(1) + \sigma^2 - \theta(\phi - \theta)\sigma^2 \end{aligned}$$

$$\begin{aligned} h = 1 : \gamma_x(1) &= \phi \gamma_x(0) + E\left[u_t x_{t-1}\right] - \theta E\left[u_{t-1} x_{t-1}\right] \\ &= \phi \gamma_x(0) + E\left[u_t u_{t-1} + (\phi - \theta) \sum_{j=1}^{\infty} \phi^{j-1} u_{t-1-j} u_t\right] - \theta E\left[u_{t-1} u_{t-1} + (\phi - \theta) \sum_{j=1}^{\infty} \phi^{j-1} u_{t-1-j} u_{t-1}\right] \\ \gamma_x(1) &= \phi \gamma_x(0) + 0 - \theta \sigma^2 \end{aligned}$$

$$\begin{aligned} h \geq 2 : \gamma_x(h) &= \phi \gamma_x(h-1) + E\left[u_t x_{t-h}\right] - \theta E\left[u_{t-1t-h}\right] \\ \gamma_x(h) &= \phi \gamma_x(h-1) \end{aligned}$$

Giving the autocorrelation function,

$$\rho_x(h) = \begin{cases} 1 & h = 0 \\ \frac{(\phi - \theta)(1 - \phi\theta)}{1 + \theta^2 - 2\phi\theta} & h = 1 \\ \phi \rho_x(h-1) & h = 2 \end{cases}$$

ARDL(p,r)

$$y_t = \underbrace{\phi_1 y_{t-1} + \phi_2 y_{t-2} + \ldots + \phi_p y_{t-p}}_{AR(p)} + \underbrace{\mu}_{constant} + \underbrace{\gamma_0 x_t + \gamma_1 x_{t-1} + \ldots + \gamma_r x_{t-r}}_{DL(r)} + \underbrace{\varepsilon_t}_{error}$$

Static Model

$$y_t = \mu + \gamma_0 x_t + \varepsilon_t$$

We call this model static since all parameters are contemporaneous.

If ε_t iid, $E\left[\varepsilon_t \mid x_t, x_{t-1}, \ldots\right] = 0$, and $\operatorname{Var}\left(\varepsilon_t \mid x_t, x_{t-1}, \cdots\right) = \sigma_u^2$ then we can use standard inference.

But if ε_t is temporally dependent (depends on their past values - hence are not iid) then while estimator is still consistent, variance gets messy,

We can either:

(1) Correct for Temporal Dependence,

(2) Model it (Dynamic Model).

Dynamic Model

$$y_t = \mu + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \ldots + \phi_p y_{t-p} + \gamma_0 x_t + \gamma_1 x_{t-1} + \ldots + \gamma_r x_{t-r} + \varepsilon_t$$

Often adding just one lag is enough for the errors to now be iid (ε_t iid). If this is the case and if it is also the case that $E\left[\varepsilon_t \mid x_t, x_{t-1}, \ldots\right] = 0$ and $\operatorname{Var}\left(\varepsilon_t \mid x_t, x_{t-1}, \ldots\right) = \sigma_u^2$ then we get standard inference.

Note that although the same term ε_t is used in these two models it is capturing different errors.

Lag Form

$$\begin{aligned} (1 - \gamma_1 L - ... - \gamma_p L^p) y_t &= \mu + (\beta_0 + \beta_1 L + \cdots + \beta_r L^r) x_t + \varepsilon_t \\ C_p(L) y_t &= \mu + B_r(L) x_t + \varepsilon_t \end{aligned}$$

Stablility: The ARDL model is stable iff the roots of $C_p(L)$ are outside of the unit circle, and hence $C_p(L)$ is invertible.

$$\frac{B_r(L)}{C_p(L)} = D_\infty(L)$$

is convergent if the model is stable.

DL Representation

$$\begin{aligned} y_t &= \frac{\mu}{C_p(L)} + \frac{B_r(L)}{C_p(L)} x_t + \frac{1}{C_p(L)} \varepsilon_t \\ y_t &= \alpha + D_\infty(L) x_t + u_t \\ y_t &= \alpha + \sum_{j=0}^{\infty} \delta_j x_{t-j} + u_t \end{aligned}$$

Note that the new error term $u_t = \frac{\varepsilon_1}{C_p(L)}$ is autocorrelated.

Also the constant $\alpha = \frac{\mu}{C_p(L)} = \frac{\mu}{C_p(1)}$ is finite.

ECM Representation

$$\Delta y_t = (\gamma - 1)(y_{t-1} - \theta x_{t-1} - \tau) + \beta \Delta x_t + \varepsilon_t$$

Example: ARDL(1,1)

$$\begin{aligned}
y_t &= \mu + \gamma_1 y_{t-1} + \beta_0 x_t + \beta_1 x_{t-1} + \varepsilon_t \\
y_t - y_{t-1} &= \mu + (\gamma_1 - 1) y_{t-1} + \beta_0 x_t + \beta_1 x_{t-1} + \varepsilon_t \\
\Delta y_t &= \mu + (\gamma_1 - 1) y_{t-1} + \beta_0 x_t - \beta_0 x_{t-1} + \beta_0 x_{t-1} + \beta_1 x_{t-1} + \varepsilon_t \\
\Delta y_t &= \mu + (\gamma_1 - 1) y_{t-1} + \beta_0 \Delta x_t + (\beta_0 + \beta_1) x_{t-1} + \varepsilon_t \\
\Delta y_t &= (\gamma_1 - 1)\left[y_{t-1} + \frac{(\beta_0 + \beta_1)}{(\gamma_1 - 1)} x_{t-1} + \frac{\mu}{(\gamma_1 - 1)} \right] + \beta_0 \Delta x_t + \varepsilon
\end{aligned}$$

The LR expectation of this given that y_t and x_t are stationary must be that:

$$\begin{aligned}
&E[y_{t-1} - \theta x_{t-1} - \tau] = 0 \\
&\text{or} \\
&E[y_{t-1} - \theta x_{t-1}] = \tau
\end{aligned}$$

Example: ARDL(2,2)

ECM Form

$$\begin{aligned}
y_t &= \gamma_1 y_{t-1} + \gamma_2 y_{t-2} + \beta_0 x_t + \beta_1 x_{t-1} + \beta_2 x_{t-2} + u_t \\
y_t - y_{t-1} &= (\gamma_1 - 1) y_{t-1} + \gamma_2 y_{t-2} + \beta_0 x_t - \beta_0 x_{t-1} + \beta_1 x_{t-1} + \beta_0 x_{t-1} + \beta_2 x_{t-2} + u_t \\
\Delta y_t &= (\gamma_1 - 1) y_{t-1} + (\beta_1 + \beta_0) x_{t-1} + \gamma_2 y_{t-2} + \beta_0 \Delta x_t + \beta_2 x_{t-2} + u_t \\
\Delta y_t &= (\gamma_1 - 1)\left[y_{t-1} + \frac{(\beta_0 + \beta_1)}{(\gamma_1 - 1)} x_{t-1} \right] + \gamma_2 y_{t-2} + \beta_0 \Delta x_t + \beta_2 x_{t-2} + u_t
\end{aligned}$$

Assume $y(t)$ and $x(t)$ are stationary processes and let $E[x(t)] = \mu$. Given that $E[x_t] = \mu$ and $E[u_t] = 0$, and the fact that x_t and y_t are stationary, then it must be the case that,

$$E[\Delta y_t] = E[\Delta x_t] = 0 \text{ and } E[y_{t-i}] = E[y_t] \text{ and } E[x_{t-i}] = E[x_t] \text{ for all } i$$

$$\begin{aligned}
E[\Delta y_t] &= (\gamma_1 - 1)\left[E[y_{t-1}] + \frac{(\beta_0 + \beta_1)}{(r_1 - 1)} E[x_{t-1}] \right] + \gamma_2 E[y_{t-2}] + \beta_0 E[\Delta x_t] + \beta_2 E[x_{t-2}] + E[u_t] \\
0 &= (\gamma_1 - 1)\left[E[y_t] + \frac{(\beta_0 + \beta_1)}{(\gamma_1 - 1)} \mu \right] + \gamma_2 E[y_t] + \beta_0 (0) + \beta_2 \mu + (0) \\
0 &= (\gamma_1 - 1) E[y_t] + (\beta_0 + \beta_1) \mu + \gamma_2 E[y_t] + \beta_2{}^{\mu} \\
E[y_y] &= \frac{-(\beta_0 + \beta_1 + \beta_2)}{\gamma_2 + \gamma_1 - 1} \mu = \frac{(\beta_0 + \beta_1 + \beta_2)}{(1 - \gamma_2 - \gamma_1)} \mu
\end{aligned}$$

Multipliers

Recall for this that we are considering an ARDL(p,r),

$$\begin{aligned} y_t &= \frac{\mu}{C_p(L)} + \frac{B_r(L)}{C_p(L)} x_t + \frac{1}{C_p(L)} \varepsilon_t \\ &= \frac{\mu}{C_p(L)} + D_\infty(L) x_t + \frac{1}{C_p(L)} \varepsilon_t \end{aligned}$$

Where,

$$\begin{aligned} C_p(L) &= 1 - \gamma_1 L - ... - \gamma_p L^p \\ B_r(L) &= \beta_0 + \beta_1 L + \cdots + \beta_r L^r \\ D_\infty(L) &= \sum_{j=0}^{\infty} \delta_j L^j = \delta_0 + \delta_1 L + \delta_2 L^2 + ... \end{aligned}$$

And further recall that,

$$\begin{aligned} A_q(L) &= \left(1 + \alpha_1 L + \alpha_2 L^2 + \ldots + \alpha_q L^q\right) \\ A_q(0) &= \left(1 + \alpha_1 0 + \alpha_2 0^2 + \ldots + \alpha_q 0^q\right) = 1 \\ A_q(1) &= (1 + \alpha_1 + \alpha_2 + \ldots + \alpha_q) \end{aligned}$$

Impact (Contemporaneous) Multiplier

'How does today's x_t influence today's y_t'

$$m_0 = \frac{\delta y_t}{\delta x_t} = D_\infty(0) = \delta_0 = \frac{B_r(0)}{C_p(0)} = \beta_0$$

J-th lag Multiplier

'How does x_t j days ago influence today's y_t'

$$m_j = \frac{\delta y_t}{\delta x_{t-j}} = \delta_j \neq \beta_j$$

Total/Long-run Multiplier

'Total effect'

$$m_{total} = \sum_{j=0}^{\infty} m_j = D(1) = \sum_{j=0}^{\infty} \delta_j = \frac{B_r(1)}{C_p(1)}$$

Transmission Effects

(Assume that $\delta_j \geq 0$)

Mean Lag

'How concentrated (or diluted) the effect of x_t on y_t is'.

Earlier lags get a higher weight (remember $t-4$ is earlier than t).

$$Meanlag = \frac{\sum_{j=0}^{\infty} j\delta_j}{\sum_{j=0}^{\infty} \delta_j} = \frac{D'(1)}{D(1)} = \frac{B'(1)}{B(1)} - \frac{C'(1)}{C(1)}$$

Where

$$D'(1) = \left. \frac{dD(L)}{dL} \right|_{L=1}$$

Median Lag

'The time when y_t has accumulated 50% of the total effect'

$$Medianlag = \min_q \left\{ \frac{\sum_{j=0}^{q} \delta_j}{\sum_{j=0}^{\infty} \delta_j} \geq 0.5 \right\}$$

If you add $\delta_0, \delta_1, \delta_2$ and get an answer > 0.5 then the median is 2

Asymptotics & Estimation: AR(p)

Here we will consider a stationary AR(1), that is a model,

$$y_t = \phi y_{t-1} + u_t$$

Where $u_t \sim iid(0, \sigma_u^2)$, $E[u_t^4] = \mu_4 < \infty$, $E[y_0] = 0$, $\text{Var}(y_0) = \frac{\sigma^2}{1-\phi^2}$ and where y_0 is independent of all u_t.

All this reasoning should extend to the AR(p).

Asymptotics: Sample Mean

$$\bar{y}_t = \frac{1}{T}\sum_{t=1}^{T} y_t = \frac{1}{T}\sum_{t=1}^{T}\sum_{i=1}^{\infty}\phi^i u_{t-i}$$

Consistency

We will use the Covariance-Stationary LLN in this case, which requires that we check that the autocovariances are absolutely summable.

$$\begin{aligned}
\gamma_h = \phi^{|h|}\frac{\sigma_u^2}{1-\phi^2} \\
\sum_{h=0}^{\infty}|\gamma_h| &= \sum_{h=0}^{\infty}\left|\phi^{|h|}\frac{\sigma_u^2}{1-\phi^2}\right| = \sum_{h=0}^{\infty}|\phi|^{|h|}\frac{\sigma_u^2}{|1-\phi^2|} \\
&= \frac{\sigma_u^2}{|1-\phi^2|}\sum_{h=0}^{\infty}|\phi|^h = \frac{\sigma_u^2}{|1-\phi^2|}\left(1+|\phi|^1+|\phi|^2+\dots\right) \\
&= \frac{\sigma_u^2}{|1-\phi^2|}\frac{1}{1-|\phi|} < \infty
\end{aligned}$$

Having shown that they are, hence we can use the Covariance-Stationary LLN,

$$\bar{y}_t = \frac{1}{T}\sum_{t=1}^{T} y_t \xrightarrow{P} \mu = E\left[y_t\right] = 0$$

Asymptotic Normality

Here we will use the Wold Decomposition CLT on the Wold Decomposition of the AR(1),

$$y_t = \sum_{i=0}^{\infty}\phi^i u_{t-i}$$

Given we have already assumed $u_t \sim iid(0, \sigma_u^2)$, we just need to check the condition that the coefficients on the error term are absolutely summable, that is,

$$\sum_{i=0}^{\infty}|\psi_i| < \infty$$

In our case,

$$\begin{aligned}
\sum_{i=0}^{\infty}|\psi_j| &= \sum_{i=0}^{\infty}|\phi^i| = 1+|\phi|+|\phi^2|+|\phi^3|+ ... \\
&= \frac{1}{1-|\phi|} < \infty
\end{aligned}$$

Hence the Wold Decomposition CLT applies and so,

$$\sqrt{T}\big(\bar{y}_t-\mu\big)\xrightarrow{D}N\bigg(0,\ \sum_{h=-\infty}^{\infty}\gamma_h\bigg)$$

Where $\mu=0$ and where $\sum_{h=-\infty}^{\infty}\gamma_h=\sigma_u^2\Psi^2(1)$. Recall that in this case,

$$\Psi(L)=\frac{1}{1-\phi L}$$
$$\Psi(1)=\frac{1}{1-\phi}$$

Therefore,

$$\sum_{h=-\infty}^{\infty}\gamma_h=\sigma_u^2\Psi^2(1)=\sigma_u^2\bigg(\frac{1}{1-\phi}\bigg)^2$$

Using all of this and the Wold Decomposition CLT,

$$\sqrt{T}\big(\bar{y}_t\big)\xrightarrow{D}N\bigg(0,\ \sigma_u^2\Big(\frac{1}{1-\phi}\Big)^2\bigg)$$

Estimation: OLS

We estimate ϕ in the AR(1) model by OLS,

$$\widehat{\phi}_T=argmin\ \sum_{t=1}^{T}u_t^2=argmin\ \sum_{t=1}^{T}\big(y_t-\phi y_{t-1}\big)$$

Giving the FOC,

$$0=-2\sum_{t=1}^{T}y_{t-1}\big(y_t-\widehat{\phi}_T y_{t-1}\big)$$
$$0=\sum_{t=1}^{T}y_{t-1}y_t-\widehat{\phi}_T\sum_{t=1}^{T}y_{t-1}^2$$

Hence,

$$\widehat{\phi}_T=\frac{\sum_{t=1}^{T}y_{t-1}y_t}{\sum_{t=1}^{T}y_{t-1}^2}=\frac{\sum_{t=1}^{T}y_{t-1}\left(\phi y_{t-1}+u_t\right)}{\sum_{t=1}^{T}y_{t-1}^2}=\phi+\frac{\sum_{t=1}^{T}y_{t-1}u_t}{\sum_{t=1}^{T}y_{t-1}^2}$$

Asymptotics: OLS Estimator

Consistency

$$\widehat{\phi}_T-\phi=\frac{T^{-1}\sum_{t=1}^{T}y_{t-1}u_t}{T^{-1}\sum_{t=1}^{T}y_{t-1}^2}$$

Consider the Numerator to start, and notice that, where $\mathcal{I}_{t-1}$ is the information set we have at $t-1$, $E[y_{t-1}u_t\mid\mathcal{I}_{t-1}]=y_{t-1}E[u_t\mid\mathcal{I}_{t-1}]=0$. We have a Martingale Difference Series (MDS), so we will apply the MDS LLN.

Check condition (b) of MDS LLN for $p=1$ and $m_t=y_{t-1}u_t$,

$$\begin{aligned}\lim_{T\to\infty}\max_{t\le T}E\left|y_{t-1}u_t\right|^2&=\lim_{T\to\infty}\max_{t\le T}E\left[{y_{t-1}}^2{u_t}^2\right]\\&=\lim_{T\to\infty}\max_{t\le T}=E\left[y_{t-1}^2E\left(u_t^2\mid\mathcal{I}_{t-1}\right)\right]\\&=\frac{\sigma^2}{1-\phi^2}\sigma^2=\frac{\sigma^4}{1-\phi^2}<\infty\end{aligned}$$

The condition holds, so,

$$T^{-1} \sum_{t=1}^{T} y_{t-1} u_t \xrightarrow{P} 0$$

Now considering the Denominator,

$$T^{-1} \sum_{t=1^T}^{T} y_{t-1}^2 = T^{-1} \left(y_0^2 + y_1^2 + \ldots + y_{T-1}^2 \right) = T^{-1} \left(y_0^2 + y_1^2 + \ldots + y_{T-1}^2 + y_T^2 - y_T^2 \right)$$
$$= T^{-1} \sum_{t=1}^{T} y_t^2 + T^{-1} \left(y_0^2 - y_T^2 \right)$$

And we can work out, simply by squaring the AR(1) model,

$$\{y_t\}^2 = \{\phi y_{t-1} + u_t\}^2$$
$$y_t^2 = \phi^2 y_{t-1}^2 + u_t^2 + 2\phi y_{t-1} u_t$$

Now substituting this in,

$$T^{-1} \sum_{t=1}^{T} y_{t-1}^2 = T^{-1} \sum_{t=1}^{T} \left(\phi^2 y_{t-1}^2 + u_t^2 + 2\phi y_{t-1} u_t \right) + T^{-1} \left(y_0^2 - y_T^2 \right)$$
$$= \phi^2 T^{-1} \sum_{t=1}^{T} y_t^2 + T^{-1} \sum_{t=1}^{T} \left(u_t^2 + 2\phi y_{t-1} u_t \right) + T^{-1} \left(y_0^2 - y_T^2 \right)$$
$$T^{-1} \sum_{t=1}^{T} y_{t-1}^2 - \phi^2 T^{-1} \sum_{t=1}^{T} y_{t-1}^2 = T^{-1} \sum_{t=1}^{T} u_t^2 + 2\phi T^{-1} \sum_{t=1}^{T} y_{t-1} u_t + T^{-1} \left(y_0^2 - y_T^2 \right)$$
$$(1-\phi^2) T^{-1} \sum_{t=1}^{T} y_{t-1}^2 = T^{-1} \sum_{t=1}^{T} u_t^2 + 2\phi T^{-1} \sum_{t=1}^{T} y_{t-1} u_t + T^{-1} \left(y_0^2 - y_T^2 \right)$$
$$T^{-1} \sum_{t=1}^{T} y_{t-1}^2 = \frac{1}{1-\phi^2} \left[T^{-1} \sum_{t=1}^{T} u_t^2 + 2\phi T^{-1} \sum_{t=1}^{T} y_{t-1} u_t + T^{-1} \left(y_0^2 - y_T^2 \right) \right]$$

Now analysising each term,

$$T^{-1} \sum_{t=1}^{T} u_t^2 \xrightarrow{P} \sigma^2 \ (iid\ LLN)$$
$$T^{-1} \sum_{t=1}^{T} y_{t-1} u_t \xrightarrow{P} 0 \ (mds\ LLN)$$
$$T^{-1} \left(y_0^2 - y_T^2 \right) \xrightarrow{P} 0$$

Hence the denominator,

$$T^{-1} \sum_{t=1}^{T} y_t^2 \xrightarrow{P} \frac{\sigma^2}{(1-\phi^2)}$$

So overall,

$$\widehat{\phi}_T = \phi + \frac{T^{-1} \sum_{t=1}^{T} y_{t-1} u_t}{T^{-1} \sum_{t=1}^{T} y_{t-1}^2} \xrightarrow{P} \phi + \frac{0}{\sigma^2 / (1-\phi^2)} = \phi$$

Asymptotics Normality

$$\sqrt{T}\left(\widehat{\phi}_T - \phi\right) = \frac{T^{-1/2}\sum_{t=1}^{T} y_{t-1}u_t}{T^{-1}\sum_{t=1}^{T} y_{t-1}^2}$$

Denominator,

$$T^{-1}\sum_{t=1}^{T} y_t^2 \xrightarrow{P} \frac{\sigma^2}{1-\phi^2}$$

Numerator,

MDS CLT applies (argumentation is tedious)

$$\frac{1}{T^{1/2}}\sum_{t=1}^{T} y_{t-1}u_t \xrightarrow{D} N\left(0, E\left[y_{t-1}^2 u_t^2\right]\right) = N\left(0,\ \frac{\sigma^4}{1-\phi^2}\right)$$

And so overall,

$$\sqrt{T}\big(\widehat{\phi}_T - \phi\big) = \frac{T^{-1/2}\sum_{t=1}^{T} y_{t-1}u_t}{T^{-1}\sum_{t=1}^{T} y_{t-1}^2} \xrightarrow{D} \frac{N\left(0,\ \frac{\sigma^4}{1-\phi^2}\right)}{\frac{\sigma^2}{1-\phi^2}} = N\Bigg(0,\ \frac{\frac{\sigma^4}{1-\phi^2}}{\left(\frac{\sigma^2}{1-\phi^2}\right)^2}\Bigg) = N\Big(0,\ (1-\phi^2)\Big)$$

Asymptotics, Estimation & Selection: ARMA(p,q)

Asymptotics: Sample Mean

Consistency

Use Covariance-Stationary LLN for a stationary ARMA model.

Asymptotic Normality

Use the Wold Decomposition (the MA representation).

Estimation

We get our estimates $\widehat{\phi}$, $\widehat{\theta}$, and $\widehat{\sigma}$, in the ARMA(1,1) model from the Yule-Walker equations,

$$\begin{aligned} h = 0 &: \gamma_x(0) = \phi\gamma_x(1) + \sigma^2 - \theta(\phi - \theta)\sigma^2 \\ h = 1 &: \gamma_x(1) = \phi\gamma_x(0) + 0 - \theta\sigma^2 \\ h \geq 2 &: \gamma_x(h) = \phi\gamma_x(h-1) \end{aligned}$$

The ARMA(p,q) is estimated by ML.

Selection

How many lags should we use in the model? Having many lags implies a more fleixble model with less bias, but fewer lags means a lower variance...

(1) Stepwise Testing Down Procedure

- Start with some p lags
  (1) Perform a t-test $H_0 \ : \ \phi_p = 0$
  (2) If we accept $H_0 \mid \phi_p = 0$ then repeat $p-1$ lags until we reject H_0 (as long as stationary and weakly dependent $\widehat{\phi}_i$ is asymptotically normal).
- Problem: $\widehat{\phi}_i$ could be significant by chance.

(2) Information Criteria

$$\min \ IC(k) = \log(\widehat{\sigma}_k) + k\frac{P(T)}{T}$$

- Where:
  - There are $\bar{k}$ alternative models, $M_1, \ldots, M_{\bar{k}}$ where $k = 1, \ldots, \bar{k}$ represent the number of parameters in the model,
  - $\widehat{\sigma}_k$ is the variance of the residuals of model M_k,
  - T is the sample size,
  - P(T) is a penalty for including too many lags.

Asymptotics & Estimation: ARDL(p,r)

Estimation

With iid Errors

$$y_t = \mu + \beta_0 x_t + \beta_1 x_{t-1} + \alpha y_{t-1} + \varepsilon_t$$

Where $\varepsilon_t \sim iid\ N\left(0, \sigma^2\right)$, x_t is stationary AR(1) and ε_t of all past $x_t,\ x_{t-1},\ x_{t-2}, \ldots$ and past $y_{t-1},\ y_{t-2}, \ldots$

Use OLS to estimate,

$$\hat{\beta} - \beta = (X'X)^{-1}(X'\varepsilon)$$

With Autocorrelated Errors

Consider ARDL $(1,0)$

$$y_t = \beta_0 x_t + \gamma_1 y_{t-1} + \varepsilon_t$$

With $|\gamma_1| < 1$ and where ε_t is autocorrelated.

Stability allows us to write,

$$y_t = \frac{\beta_0}{1-\gamma_1 L} x_t + \frac{\varepsilon_t}{1-\gamma_1 L}$$

Mean independence of ε_t and regressors fails

$$\begin{aligned}
E\left(\varepsilon_p y_{t-1}\right) &= E\left[\varepsilon_t \frac{\beta_0}{1-\gamma_1 L} x_{t-1} + \varepsilon_t \frac{\varepsilon_{t-1}}{1-\gamma_1 L}\right] \\
&= E\left[\varepsilon_t \frac{\beta_0}{1-\gamma_1 L} x_{t-1}\right] + E\left[\varepsilon_t \frac{\varepsilon_{t-1}}{1-\gamma_1 L}\right] \\
&= 0 + E\left[\varepsilon_t \frac{\varepsilon_{t-1}}{1-\gamma_1 L}\right] = 0 + E\left[\varepsilon_t\left(\varepsilon_{t-1} + \gamma_1 \varepsilon_{t-2} + \gamma_1^2 \varepsilon_{t-3} + \ldots\right)\right] \neq 0
\end{aligned}$$

If the errors are AR(1), that is if ε_t is autocorrelated such that $\varepsilon_t = \phi\varepsilon_{t-1} + u_t$ where $u_t \sim idd(0, \sigma_u^2)$, then we can solve this by adding more lags.

Here is our ARDL(1,0) model,

$$y_t = \beta x_t + \gamma_1 y_{t-1} + \varepsilon_t$$

Using the AR(1) autocorrelated error and by adding the lags y_{t-2} and x_{t-1},

$$\begin{aligned}
&\varepsilon_t = \phi\varepsilon_{t-1} + u_t \quad , \quad u_t \sim idd(0, \sigma_u^2) \\
&u_t = \varepsilon_t - \phi\varepsilon_{t-1} \quad , \quad \varepsilon_t = y_t - \beta x_t - \gamma_1 y_{t-1} \quad , \quad \varepsilon_{t-1} = y_{t-1} - \beta x_{t-1} - \gamma_1 y_{t-2} \\
&u_t = y_t - \beta x_t - \gamma_1 y_{t-1} - \phi\big[y_{t-1} - \beta x_{t-1} - \gamma_1 y_{t-2}\big]
\end{aligned}$$

Which means finally,

$$y_t = (\gamma_1 + \phi) y_{t-1} - \gamma_1 y_{t-2} + \beta x_t - \beta x_{t-1} + u_t$$

with $u_t \sim iid(0, \sigma_u^2)$.

If, however, the errors are MA(1) autocorrelated, that is if $\varepsilon_t = u_t + \phi u_{t-1}$, $u_t \sim iid(0, \sigma_u^2)$ then no such trick can be used. Instead we can use IVs.

Asymptotics

Study asymptotics with mds LLN and CLT.

Time Trends

$$y_t = \alpha + \delta t + u_t \quad , \quad u_t \sim iid\left(0, \sigma_u^2\right)$$

Stochastic Properties

Expectation

$$E\left[y_t\right] = E\left[\alpha + \delta t + u_t\right] = \alpha + \delta t$$

Variance

$$\text{Var}\left(y_t\right) = E\left[\left\{y_t - E\left(y_t\right)\right\}^2\right] = E\left[u_t^2\right] = \sigma_u^2$$

Covariance

$$\text{Cov}\left(y_t, y_{t-h}\right) = E\left[\left\{y_t - E\left(y_t\right)\right\}\left\{y_{t-h} - E\left(y_{t-h}\right)\right\}\right] = E\left[u_t u_{t-h}\right] = 0$$

Correlation

$$\text{Corr}\left(y_t, y_{t-h}\right) = \frac{\text{cov}\left(y_t, y_{t-h}\right)}{\sqrt{\text{var}\left(y_t\right)}\sqrt{\text{var}\left(y_{t-h}\right)}} = 0$$

OLS Estimation

$$Y = XB + U \text{ where, } \underset{(T\times 1)}{Y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_t \end{pmatrix} \; , \; \underset{(T\times 2)}{X} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ \vdots & \vdots \\ 1 & T \end{pmatrix} \; , \; \underset{(2\times 1)}{B} = \begin{pmatrix} \alpha \\ \delta \end{pmatrix} \; , \; \underset{(T\times 1)}{U} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_t \end{pmatrix}$$

We can just use the usual OLS matrix estimator,

$$\underset{(2\times 1)}{\widehat{B}_T} = \left(X'X\right)^{-1} X'Y$$

And in our case,

$$\underset{(2\times 1)}{\widehat{B}_T} = \begin{pmatrix} \widehat{a}_T \\ \widehat{\delta}_T \end{pmatrix}$$

$$\left(\widehat{B}_T - B\right) = \begin{pmatrix} \widehat{a}_T - a \\ \widehat{\delta}_T - \delta \end{pmatrix} = \begin{pmatrix} \sum_{t=1}^T 1 & \sum_{t=1}^T t \\ \sum_{t=1}^T & \sum_{t=1}^T t^2 \end{pmatrix}^{-1} \begin{pmatrix} \sum_{t=1}^T u_t \\ \sum_{t=1}^T t u_t \end{pmatrix}$$

Where,

$$X'X = \begin{bmatrix} \sum_{t=1}^T & \sum_{t=1}^T t \\ \sum_{t=1}^T t & \sum_{t=1}^T t^2 \end{bmatrix} = \begin{bmatrix} T & \frac{T(T+1)}{2} \\ \frac{T(T+1)}{2} & \frac{T(T+1)(2T+1)}{6} \end{bmatrix}$$

Asymptotics

Need to first stabilise the denominator: $(X'X)^{-1}$

$$\gamma_T = \begin{pmatrix} T^{\frac{1}{2}} & 0 \\ 0 & T^{\frac{3}{2}} \end{pmatrix}$$

$$\gamma_T^{-1}(X'X)^{-1}\gamma_T^{-1} = \begin{pmatrix} T^{-1}T & T^{-2}\frac{T(T+1)}{2} \\ T^{-2}\frac{T(T+1)}{2} & T^{-3}\frac{T(T+1)(2T+1)}{6} \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} \end{pmatrix} := Q$$

Where Q is invertible.

Now we need γ_T 's to cancel out, this is how we do it,

$$\gamma_T\left(\widehat{B}_T - B\right) = \left\{\gamma_T^{-1}\left(X'X\right)\gamma_T^{-1}\right\}^{-1}\gamma_T^{-1}\left(X'U\right)$$

Hence we just need to consider,

$$\gamma_T(X'U) = \begin{bmatrix} T^{\frac{1}{2}} & 0 \\ 0 & T^{\frac{3}{2}} \end{bmatrix}^{-1}\begin{bmatrix} \sum_{t=1}^T u_t \\ \sum_{t=1}^T tu_t \end{bmatrix} = \begin{bmatrix} T^{-\frac{1}{2}}\sum_{t=1}^T u_t + 0\sum_{t=1}^T tu_t \\ 0\sum_{t=1}^T u_t + T^{-\frac{3}{2}}\sum_{t=1}^T tu_t \end{bmatrix} = \begin{bmatrix} T^{-\frac{1}{2}}\sum_{t=1}^T u_t \\ T^{-\frac{1}{2}}T^{-1}\sum_{t=1}^T tu_t \end{bmatrix}$$
$$= \begin{bmatrix} \frac{1}{\sqrt{T}}\sum_{t=1}^T u_t \\ \frac{1}{\sqrt{T}}\sum_{t=1}^T \frac{t}{T}u_t \end{bmatrix}$$

Row 1:

Given that $u_t \sim iid\left(0, \sigma^2\right)$ we can use the iid CLT so that,

$$\left(\frac{1}{\sqrt{T}}\right)\sum_{t=1}^T u_t \longrightarrow N\left(0, \sigma^2\right)$$

Row 2:

$$\text{Var}\left(\frac{t}{T}u_t\right) = \frac{t^2}{T^2}\sigma^2$$

hence we are dealing with heteroskedastic errors, which means we can't use the iid CLT but we can show this to be an MDS instead,

$$E\left[\frac{t}{T}u_t \mid \mathcal{I}_{t-1}\right] = \frac{t}{T}E\left[u_t \mid \mathcal{I}_{t-1}\right] = 0$$

Hence we just need to check conditions (i) and $(ii)'$ for the MDS CLT, which hold (I'm not going to show it). For this row we get out as our answer,

$$\frac{1}{\sqrt{T}}\sum_{t=1}^T \frac{t}{T}u_t \xrightarrow{D} N\left(0,\ \frac{\sigma^2}{3}\right)$$

Overall then,

$$\gamma_T(X'U) \xrightarrow{D} N(0,\ \sigma^2 Q)$$

Unit Roots

Spurious Regression & Cointegration