FHS Microeconomics Notes

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Abstract

These are my microeconomics notes made for my finals in 2022. They cover all of the topics. Probably the most useful part of these notes are the practise questions with solutions at the end of each topic. These questions cover the majority of Part A questions that you will find on the exam, and are taken from old problem sheets as well as past exams. Feel free to use these notes and pass them on to others. Please note, however, that these have just been made by a student and not checked over. They likely contain errors, so it will be worth checking things for yourself.

Contents

General Equilibrium

Exchange Economy

2 Agents (A, B), 2 goods $(1, 2)$

Endowments: $\boldsymbol{w}_a = \left(w_a^1, w_a^2\right),\ \boldsymbol{w}_b = \left(w_b^1, w_b^2\right)$

Consumption: $\boldsymbol{x}_a = \left(x_a^1, x_a^2\right),\ \boldsymbol{x}_b = \left(x_b^1, x_b^2\right)$

Utility: $u_a\left(x^1, x^2\right),\ u_b\left(x^1, x^2\right)$

Prices: $\boldsymbol{p} = (p_1, p_2)$

Demand: $x_a\left(p_1, p_2, \boldsymbol{p} \cdot \boldsymbol{w}_a\right), x_b\left(p_1, p_2, \boldsymbol{p} \cdot \boldsymbol{w}_b\right)$

Edgeworth Box

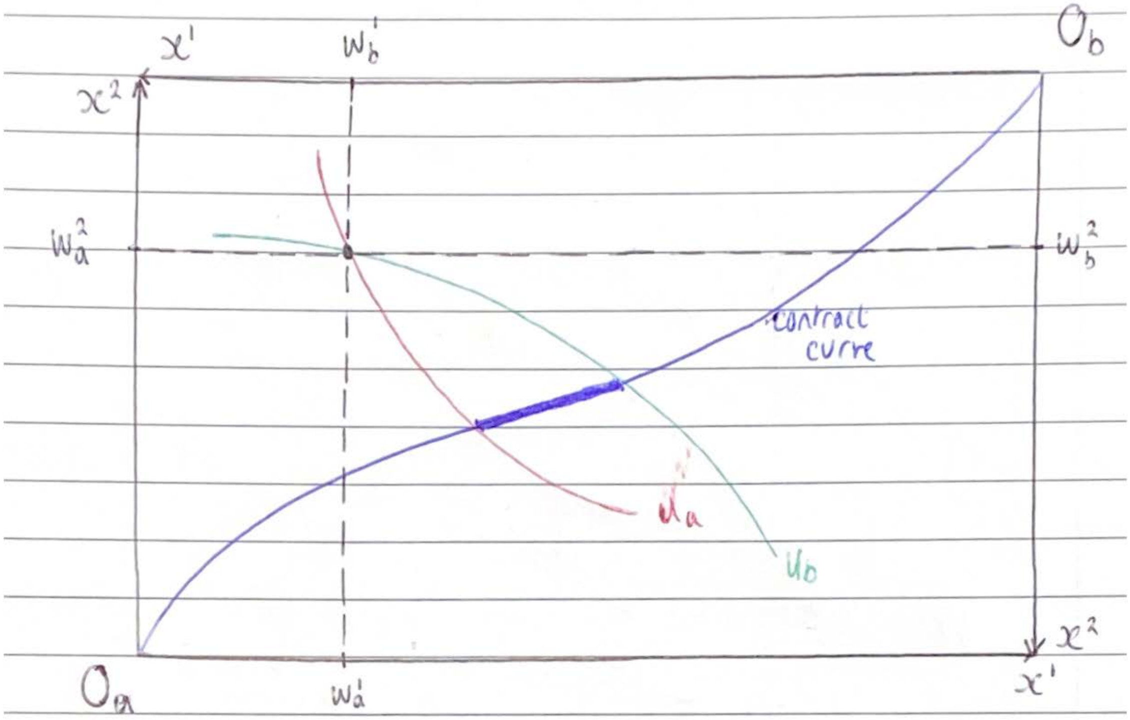

Figure 1: Edgeworth Box

Excess Demand

Good 1 aggregate excess demand: $z^1(\boldsymbol{p}) = \left(x_a^1(\boldsymbol{p}) - w_a^1\right) + \left(x_b^1(\boldsymbol{p}) - w_b^1\right)$

Good 2 aggregate excess demand: $z^2(\boldsymbol{p}) = \left(x_a^2(\boldsymbol{p}) - w_a^2\right) + \left(x_b^2(\boldsymbol{p}) - w_b^2\right)$

- At the equilibrium price vector $p^* > 0$ we want, $z^1\left(p^*\right) = z^2\left(p^*\right) = 0$
- That is: total demand = total amount of resources

Walras' Law

$$\boldsymbol{p} \cdot z(\boldsymbol{p}) = \boldsymbol{0}$$

Which in the n agent problem is written as

$$p_1 z^1(\boldsymbol{p}) + p_2 z^2(\boldsymbol{p}) + \ldots + p_K z^K(\boldsymbol{p}) = 0$$

Note this is for any p, not just p^*, and further note that if $(K-1)$ markets clear then the Kth market clears

Leisure/Labour Production Economy

Consumer who likes leisure and goods, firm who maximises profits and let the consumer owns the firm.
Agents: C, F (Consumer, Firm); Goods: X, R (good, leisure)

- Utility: $u(X, R)$ where $R = 1 - L$
- Production: $F(L)$
- Prices: p, w

Consumer Problem

$$max\ u(x, R) \text{ such that } px = w(1 - R) + \pi \Leftrightarrow px + wR = w + \pi$$

Firms Problem

$$max\ px - wL \text{ such that } x \leq F(L)$$

In terms of Leisure

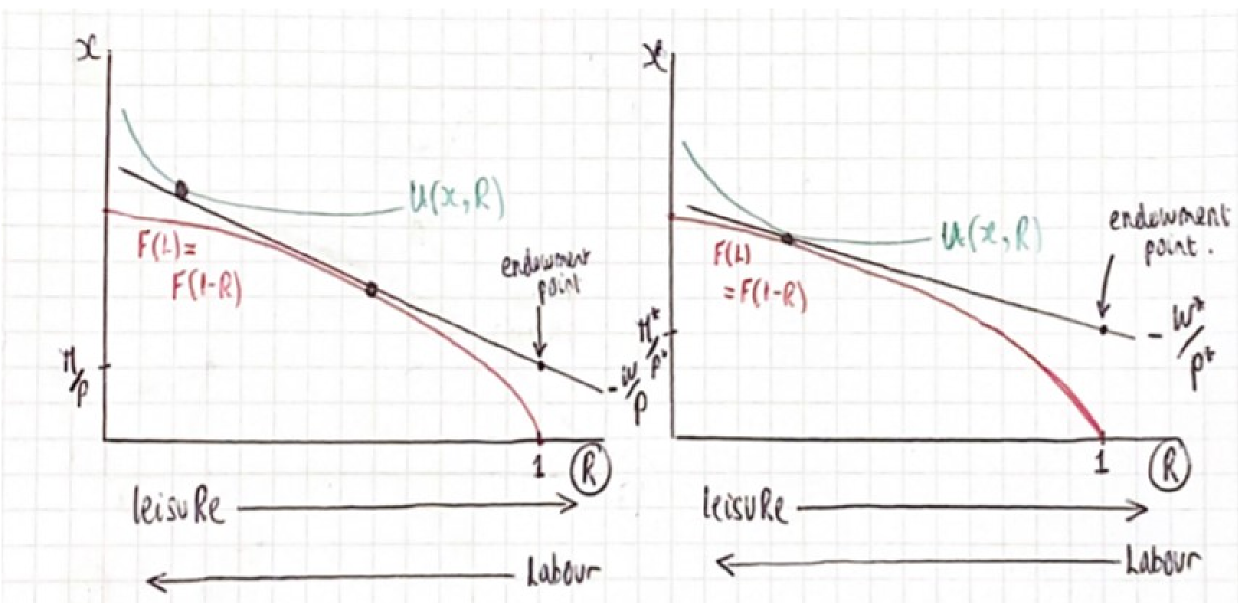

Figure 2: In terms of Leisure

In terms of Labour

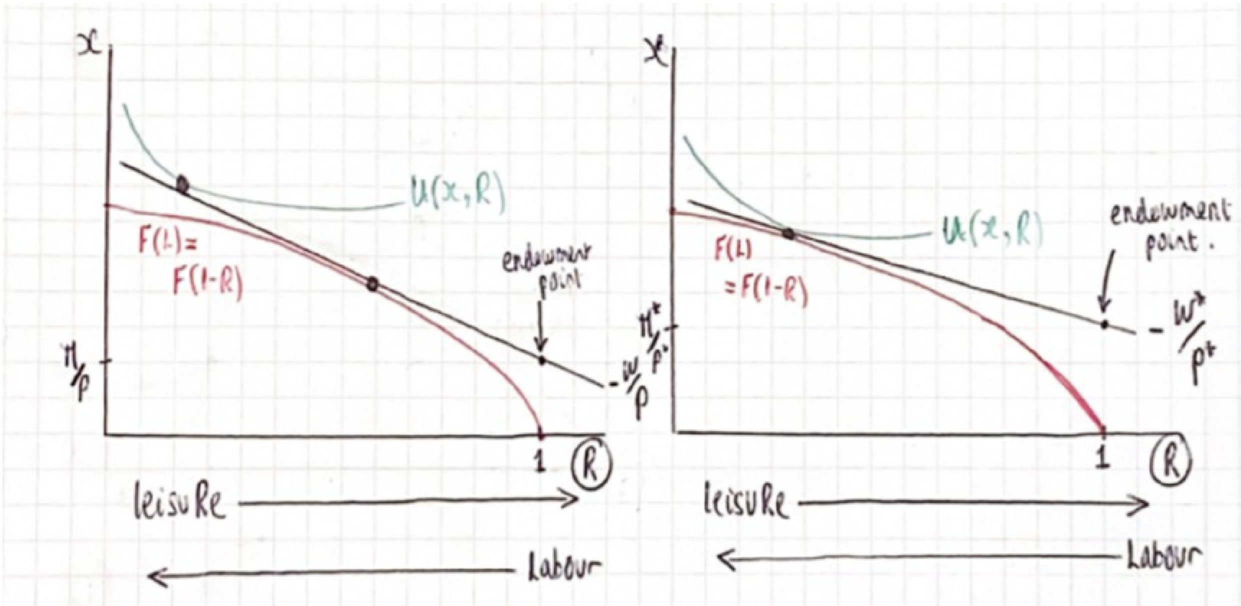

Figure 3: In terms of Labour

Simple Production Economy

2 factors (L, K) and 2 goods (X, Y)

Assumptions

(1) Constant returns to scale,
$$F_i(aK, aL) = aF_i(K, L)$$

- F is homogeneous of degree one.

(2) Decreasing marginal returns to each factor,
$$\frac{\partial^2 F_i}{\partial K^2} < 0 \ , \ \frac{\partial^2 F_i}{\partial L^2} < 0$$

- (Even though we have CRS, if you hold one input factor constant and increase the other, the marginal productivity of it will go down)

(3) Factors are perfectly mobile across sectors.

- This means the wage rate w and the rental rate of capital r will be the same across the sectors. (X and Y face the same w and r).

(4) Perfectly competitive markets for K and L,
$$w = p_i \frac{\partial F_i}{\partial L} \ , \ r = p_i \frac{\partial F_i}{\partial K}$$

Firm's Problem

$$\min_{K_i, L_i} \{wL_i + rK_i\} \quad \text{subject to } Y \leq F_i(K_i, L_i)$$

If we let,
$$ra_{K,i}(w, r) + wa_{L,i}(w, r) := \min_{K_i, L_i} \{wL_i + rK_i\} \ \text{ s.t. } 1 \leq F_i(K_i, L_i)$$

That is we find the minimum cost of outputting exactly one unit.

Under zero profits we know price = cost, so,

$$p_i = ra_{K,i}(w, r) + wa_{L,i}(w, r)$$

At optimum the Marginal Rate of Technical Substitution (MRTS) will be equal to the relative factor price:

$$\text{MRTS}_i = \frac{\frac{\partial F_i}{\partial L_i}}{\frac{\partial F_i}{\partial K_i}} = \frac{w}{r}$$

And due to CRS the optimal $\frac{K}{L}$ ratio does not depend on scale.

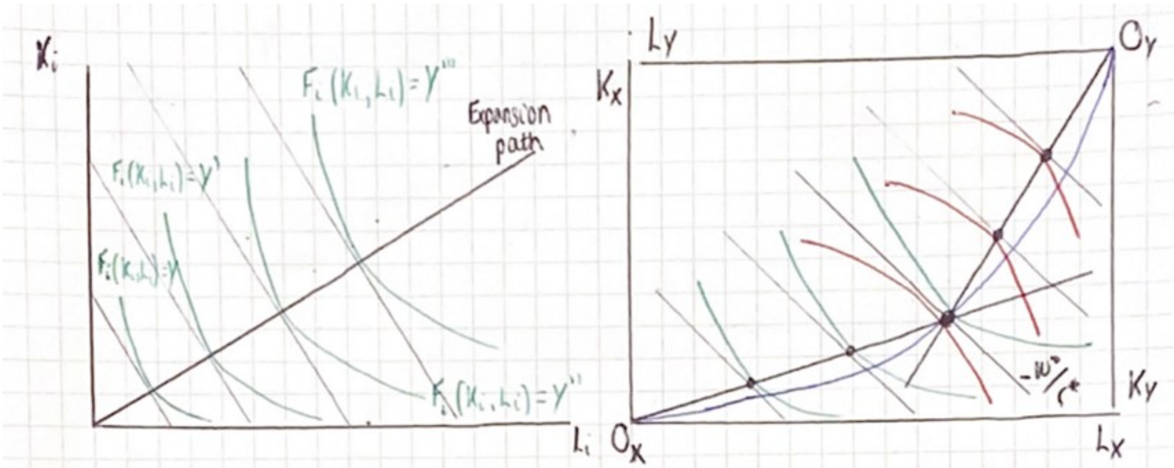

Figure 4: Left graph is in the case of a single inductry, the right graph considers two industries

Rybczynski Theorem

If,

(1) Relative prices are constant,

(2) Both goods continue to be produced,

Then,

- An increase in the supply of one factor will lead to an increase in output of the good that uses this factor more intensively and decrease the output of the other good.

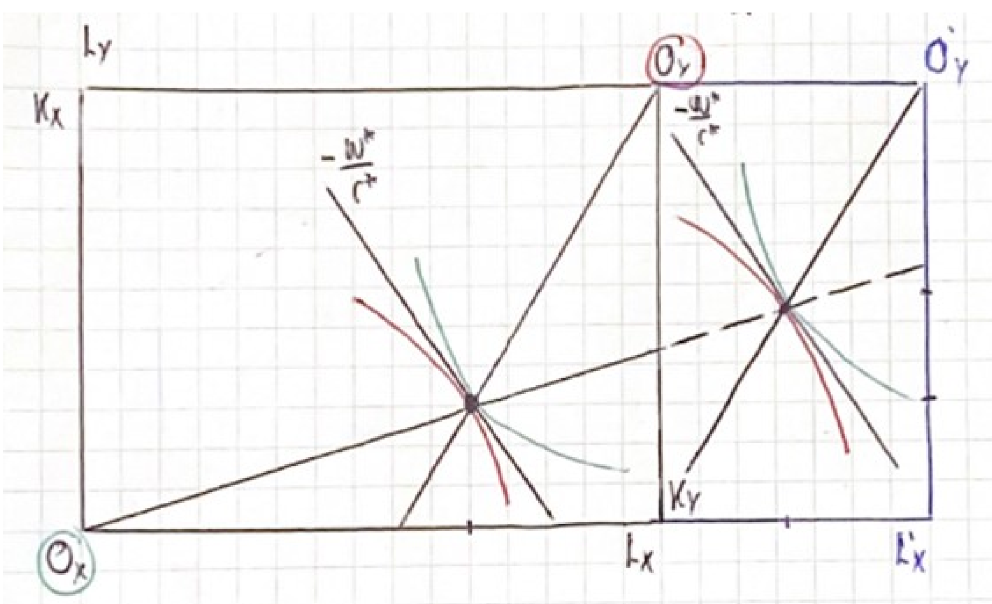

Figure 5: Rybczynski Theorem

Stolper-Samuelson Theorem

If,

(1) CRS,

(2) Both goods continue to be produced,

Then,

- A relative increase in the price of a good will increase the real return to the factor used intensively in that industry and reduce the return to the other factor.

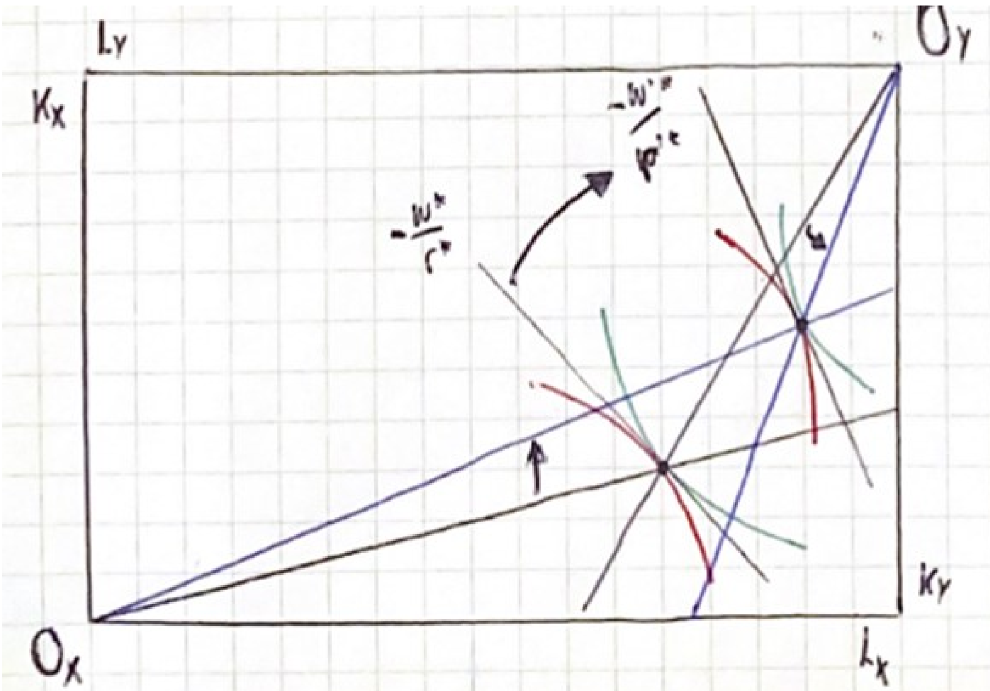

Figure 6: Stolper-Samuelson Theorem

'Specific Factors' Production Economy

Two producers $(1, 2)$ and one consumer.

Production function: $y_i = G_i(L_i)$

Labour: $L = L_1 + L_2$

Profit: $\pi_i = p_i G_i(L_i) - wL_i$

Utility: $u(x^1, x^2)$

Consumer Problem

$$max\ u(x^1, x^2) \text{ s.t. } p_1 x^1 + p_2 x^2 = wL + \pi_1 + \pi_2$$

Firms Problem

$$\max_{L_i} \pi_i = p_i G_i(L_i) - wL_i$$

Firm chooses labour allocation since the consumer controls the firm and hence decides how to best allocate her labour. Convex indifference curves ensures consumer always wants some of each good (need infinite amount of good 1 to take no good 2, and vice-versa).

Autarky

For the consumer: $MRS =$ Price ratio

$$\frac{\frac{\partial u}{\partial x^1}}{\frac{\partial u}{\partial x^2}} = \frac{p_1}{p_2}$$

For the firm: $MPL = MCL$ or $MRT =$ Price ratio

$$p_i G_i'(L_i) = w\ ;\ \frac{\frac{\partial y_1}{\partial L_2}}{\frac{\partial y_2}{\partial L_2}} = \frac{p_1}{p_2}$$

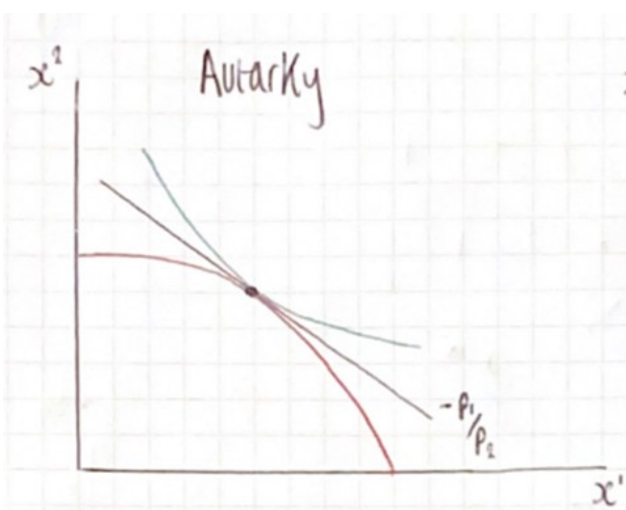

Figure 7: Autarky

Trade

Economy opens up for trade agents are exposed to new world prices p_1^w, p_2^w.

Suppose the economy has comparative advantage in good one: $p_1^w/p_2^w > p_1/p_2$

Hence exports good 1 and imports good 2.

Exporting the good that it has comparative advantage in allows the economy to move its consumer to a higher indifference curve.

Notice that in the graph consumption of both goods has increased, but the higher-than-autarky indifference curve could still be reached even if less of good 1 was consumed.

Notice that ANY PRICES OTHER THAN AUTARKY PRICES MAKES CONSUMER BETTER OFF! That is as long as the world price is not the same as the autarky price then the consumer is unambiguously better off.

- A: Production and Consumption in Autarky
- C: Consumption during trade
- D: Production during trade
- $A \to B$: Gains from liberalisation (trading but not specialising)
- $B \to C$: Gains from specialisation in good 1.

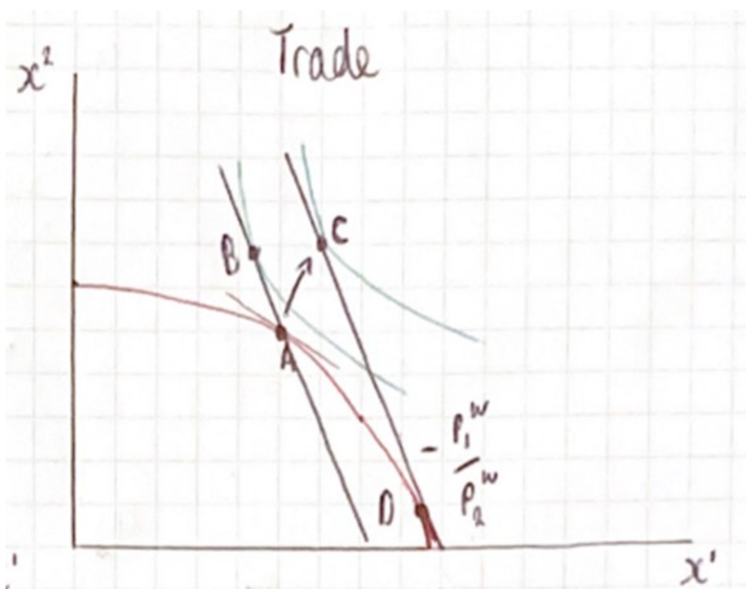

Figure 8: Trade

Comparative Advantage

This model shows two countries (red and blue), where red is better at producing good 2 and blue better at producing good 1.

Under Autarky the consumers (consumers in both countries have identical preferences) are at A.

Under trade the consumer is at a higher indifference curve at F.

It is true that consumers always gain from trade even if they have different preferences.

Every country has a comparative advantage in something (unless the countries are identical).

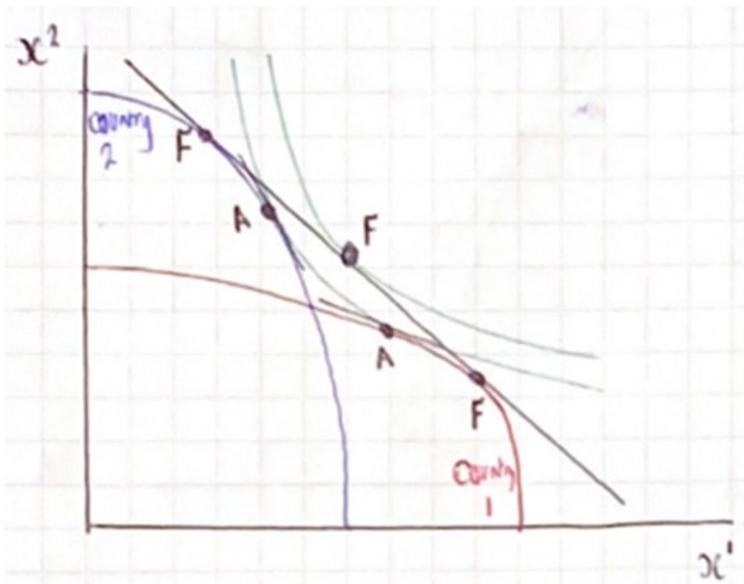

Figure 9: Comparative Advantage

So what are the losses from trade?

(1) Consumers *within a country*

- Notice how consumers in country 1 who liked good 1 lose out when it is exported, but consumers who like good 2 gain from the cheaper imports.

(2) Firms *within a country*

- $MB = MC$ (see firms FOCs) under autarky hence we are at A.
- The firm not producing the specialised good loses out – the other firm is selling more of their's at the new higher world price. Meanwhile this good is being imported more cheaply.

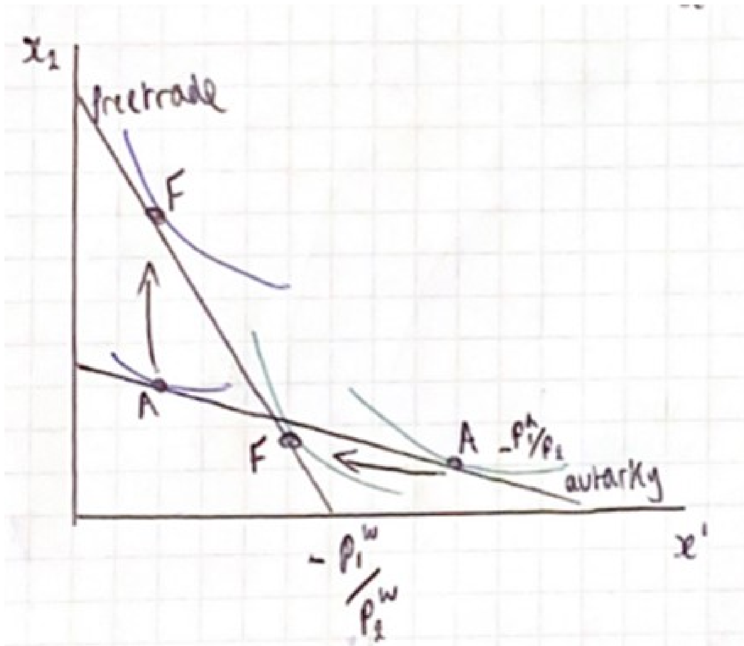

Figure 10: Consumers within a country

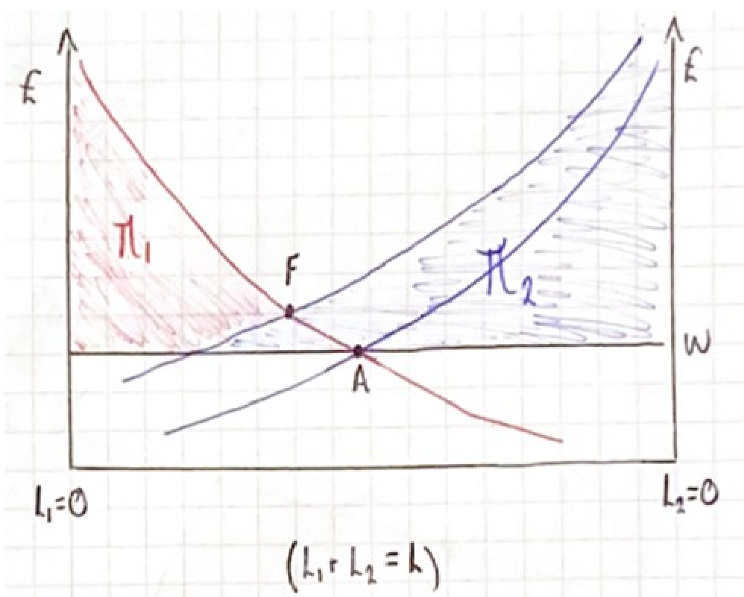

Figure 11: Consumers within a country

Evaluating CE

1. Uniqueness?

- Not Globally unique – there may be many Walrasian equilibria for a given specification of preferences and endowments.
- Locally unique – there is a finite number of equilibria.

2. Getting to equilibrium?

- Tatonnement (Walrasian Auctioneer)
  i. Gets everyone to report demand and endowment.
  ii. Calculates excess demand for each good.
  iii. Sets prices in $t+1$ such that,
  $$\frac{\partial p^k(t)}{\partial t} = a^k z^k(\boldsymbol{p(t)})$$
  iv. This doesn't even always work – sometimes a competitive equilibrium still isn't reached as dynamics can get stuck.
- Need to assume that goods are gross substitutes (increase in price of one increases demand for the other) in order for this to work.

3. Vernon Smith's Experiment

- Using:
  i. Double oral auction (buyers and sellers shout out prices),
  ii. Repeat experiment over several periods.
- His students eventually converged to competitive equilibrium.

Worked Examples

Example: Simple Exchange Economy

Consider an economy with two goods, 1 and 2, and two consumers, a and b. The preferences of consumers are represented by the utility functions:

$$u_a(x_a, y_a) = 2lnx_a + 3lny_a \text{ and } u_b(x_b, y_b) = 2lnx_b + lny_b$$

Let consumer a be endowed with 20 units of x and consumer b be endowed with 12 units of y.

(a) Find demand functions.

(b) Find excess demand functions.

(c) Confirm Walras' Law holds.

(d) Find optimal price vector and Walrasian equilibrium.

Demand Functions,

$$L = 2lnx_a + 3lny_a + \lambda(20p_x - p_x x_a - p_y y_a)$$

FOC,

$$L_{x_a} = \frac{2}{x_a} - \lambda p_x = 0$$
$$L_{y_a} = \frac{3}{y_a} - \lambda p_y = 0$$
$$p_x x_a + p_y y_a = 20p_x$$

Solve,

$$20p_x = \frac{2}{\lambda} + \frac{3}{\lambda}$$
$$\lambda = \frac{1}{4p_x}$$

Hence we get the demand functions,

$$x_a = 8 \ , \ y_a = 12\frac{p_x}{p_y}$$

$$L = 2lnx_b + lny_b + \lambda(12p_y - p_x x_b - p_y y_b)$$

FOC,

$$L_{x_b} = \frac{2}{x_b} - \lambda p_x = 0$$
$$L_{y_b} = \frac{1}{y_b} - \lambda p_y = 0$$
$$p_x x_b + p_y y_b = 12p_y$$

Solve,

$$12p_y = \frac{2}{\lambda} + \frac{1}{\lambda}$$
$$\lambda = \frac{1}{4p_y}$$

Hence we get the demand functions,

$$x_b = 8\frac{p_y}{p_x} \ , \ y_a = 4$$

Excess Demand,

$$z_y(p_x, p_y) = 12\frac{p_x}{p_y} + 4 - 12 = 12\frac{p_x}{p_y} - 8$$
$$z_x(p_x, p_y) = 8 + 8\frac{p_y}{p_x} - 20 = 8\frac{p_y}{p_x} - 12$$

Walras' Law

$$p_y z_y(p_x, p_y) + p_x z_x(p_x, p_y) = 0$$
$$p_y[12\frac{p_x}{p_y} - 8] + p_x[8\frac{p_y}{p_x} - 12] = 0$$
$$12p_x - 8p_y + 8p_y - 12p_x = 0$$

Optimal Price Vector

$$z_y(p_x^*, p_y^*) = 0\ ,\ z_x(p_x^*, p_y^*) = 0$$
$$12\frac{p_x^*}{p_y^*} - 8 = 0\ ,\ 8\frac{p_y^*}{p_x^*} - 12 = 0$$
$$\frac{p_x^*}{p_y^*} = \frac{2}{3}\ ,\ \frac{p_y^*}{p_x^*} = \frac{3}{2}$$

Walrasian Equilibrium

$$\frac{p_x^*}{p_y^*} = \frac{2}{3}\ ,\ X^a(x_a, y_a) = (8, 8)\ ,\ X^b(x_b, y_b) = (12, 4)$$

Example: Quasi-Linear Utility Simple Exchange Economy

Consider a pure exchange economy with two goods, x and y, and two consumers, a and b, that trade the goods. The preferences of consumer a are represented by the utility function:

$$u_a(x_a, y_a) = x_a + lny_a$$

and the preferences of consumer b are represented by the utility function:

$$u_b(x_b, y_b) = x_b + lny_b$$

(a) Calculate the Walrasian equilibrium for endowments $(w_x^a, w_y^a) = (4, 0)$ and $(w_x^b, w_y^b) = (0, 4)$.

(b) Illustrate your findings in an Edgeworth box, and clearly indicate all the Pareto efficient allocations.

Walrasian Equilibrium

$$L = x_a + lny_a + \lambda(m_a - p_x x_a - p_y y_a) \text{ where } m_a = p_x w_x^a + p_y w_y^a$$

FOC,

$$\begin{aligned} L_{x_a} &= 1 - \lambda p_x = 0 \\ L_{y_a} &= \frac{1}{y_a} - \lambda p_y = 0 \\ m_a &= p_x x_a + p_y y_a \end{aligned}$$

Hence we get the demand functions for a, and by symmetry for b,

$$\begin{aligned} x_a &= w_x^a + \frac{p_y}{p_x} w_y^a - 1 \ , \ y_a = \frac{p_x}{p_y} \\ x_b &= w_x^b + \frac{p_y}{p_x} w_y^b - 1 \ , \ y_a = \frac{p_x}{p_y} \end{aligned}$$

From which we are able to start solving,

$$\begin{aligned} x_a^* &= 4 + \frac{p_y}{p_x} 0 - 1 \\ x_a^* &= 3 \end{aligned}$$

Since $x_a + x_b = 4$ we hence know that $x_b^* = 1$. This will now help us find the optimum price vector,

$$\begin{aligned} x_b^* &= 1 \\ x_b^* &= 0 + \frac{p_y^*}{p_x^*} 4 - 1 \\ 2 &= 4 \frac{p_y^*}{p_x^*} \\ \frac{p_y^*}{p_x^*} &= \frac{1}{2} \end{aligned}$$

From this we can then calculate that $y_a^* = y_b^* = 2$.

Finally,

$$X_a(x_a^*, y_a^*) = (3, 2) \ , \ X_b(x_b^*, y_b^*) = (1, 2)$$

Example: Production Economy (One firm & one good)

Consider a simple production economy with one good and one firm, which is owed by the consumer - hence the consumer recieves its profits. Let the firm have the DRS production function,

$$x = L^{\frac{1}{2}}$$

And the consumer have the utility function which depends on consumption and leisure,

$$u(x, R) = \alpha lnx + (1-\alpha)lnR$$

In all the parts below you may asssume that p, the price of good x, is 1.

(a) Find the Labour Demand.

(b) Find the Consumer Demands.

(c) Find the Labour Supply.

(d) Find the market clearing outcome

Labour Demand,

$$\max \pi(L) = L^{\frac{1}{2}} - wL$$

FOC,

$$0 = \frac{1}{2}L^{-\frac{1}{2}} - w$$

Hence,

$$\begin{aligned} L^* &= (\frac{1}{2w})^2 \\ x(L^*) &= \frac{1}{2w} \\ \pi(L^*) &= \frac{1}{2w} - \frac{w}{4w^2} = \frac{1}{4w} \end{aligned}$$

Consumer demands,

$$\max u(x, R) = \alpha lnx + (1-\alpha)lnR \text{ such that } px + wR \leq w + \pi(w)$$

Lagrangian,

$$L = \alpha lnx + (1-\alpha)lnR - \lambda(px + wR - w - \pi(w))$$

FOC,

$$\begin{aligned} L_x &= \frac{\alpha}{x} - \lambda p = 0 \\ L_R &= \frac{1-\alpha}{R} - \lambda w = 0 \\ px + wR &= w + \pi(w) \end{aligned}$$

Which imply,

$$\begin{aligned} px &= \frac{\alpha}{\lambda} \\ wR &= \frac{1-\alpha}{\lambda} \\ \frac{\alpha}{\lambda} + \frac{1-\alpha}{\lambda} &= w + \pi(w) \\ \lambda &= \frac{1}{w + \pi(w)} \end{aligned}$$

Hence the demands for the good and for leisure,

$$x(p, w) = \alpha \frac{w + \pi(w)}{p}$$
$$R(p, w) = (1 - \alpha) \frac{w + \pi(w)}{w}$$

Labour supply,

Is given by $1 -$ Demand for Leisure, since we assume that each consumer is endowed with one unit of labour which they can commit to work or to leisure.

Market Clearing,

Assuming $p = 1$ for simplicity,

$$1 - (1{-}\alpha) \frac{w + \pi(w)}{w} = \frac{1}{4w^2}$$
$$w^* = (\frac{2 - \alpha}{4\alpha})^{\frac{1}{2}}$$
$$\pi(w^*) = \frac{1}{4}(\frac{4\alpha}{2 - \alpha})^{\frac{1}{2}}$$

Example: Production Economy (Two firms & two goods)

Consider a production economy two goods, c_1 & c_2, and two firms. Let the firm have the CRS production functions,

$$y_i = \frac{L_i}{\alpha_i}$$

And recall that CRS implies the firms do not make a profit.

Let the consumer have the utility function which depends on consumption of the two goods,

$$u(c_1, c_2) = \alpha ln c_1 + (1-\alpha) ln c_2$$

(a) Find the Consumer Demand.

(b) Find the optimal wage using zero profits.

(c) Find the market clearing outcome.

Consumer Demands,

$$\max\, u(c_1, c_2) = \alpha ln c_1 + (1-\alpha) ln c_2 \;\text{ such that }\; p_1 c_1 + p_2 c_2 \leq wL$$

Lagrangian,

$$L = \alpha ln c_1 + (1-\alpha) ln c_2 - \lambda(p_1 c_1 + p_2 c_2 - wL)$$

FOC,

$$\begin{aligned} L_{c_1} &= \frac{\alpha}{c_1} - \lambda p_1 = 0 \\ L_{c_2} &= \frac{1-\alpha}{c_2} - \lambda p_2 = 0 \\ p_1 c_1 + p_2 c_2 &= wL \end{aligned}$$

Which imply,

$$\begin{aligned} p_1 c_1 &= \frac{\alpha}{\lambda} \\ p_2 c_2 &= \frac{1-\alpha}{\lambda} \\ \frac{\alpha}{\lambda} + \frac{1-\alpha}{\lambda} &= wL \\ \lambda &= \frac{1}{wL} \end{aligned}$$

Hence the demand for each good,

$$\begin{aligned} c_1 &= \frac{\alpha w L}{p_1} \\ c_2 &= \frac{(1-\alpha) w L}{p_2} \end{aligned}$$

Zero profits and the assumption that the firm makes a non-zero amount of the good gives us the optimal wage,

$$\begin{aligned} \pi = p_i y_i - w L_i &= 0 \\ p_i y_i &= w L_i \\ p_i \frac{L_i}{\alpha_i} &= w L_i \\ w &= \frac{p_i}{\alpha_i} \end{aligned}$$

In market clearing $y_i = c_i$,

$$
\begin{aligned}
c_1 &= \frac{\alpha \frac{p_1}{\alpha_1} L}{p_1} = \alpha \frac{L}{\alpha_1} \\
y_1 &= \alpha \frac{L}{\alpha_1} = \frac{L_1}{\alpha_1} \\
c_2 &= \frac{(1-\alpha) \frac{p_2}{\alpha_2} L}{p_2} = (1-\alpha) \frac{L}{\alpha_2} \\
y_2 &= (1-\alpha) \frac{L}{\alpha_2} = \frac{L_2}{\alpha_2}
\end{aligned}
$$

Welfare Economics & Social Choice

Pareto Criterion

Pareto Domination: Allocation $\boldsymbol{x}$ *Pareto-dominates* allocation $\boldsymbol{x'}$ if everyone weakly prefers $\boldsymbol{x}$ to $\boldsymbol{x'}$ and at least one agent *strictly* prefers $\boldsymbol{x}$ to $\boldsymbol{x'}$,

$$\boldsymbol{x} P \boldsymbol{x'} \text{ if } \boldsymbol{x} \succsim \boldsymbol{x'} \ \forall i \text{ and } \exists i \text{ such that } \boldsymbol{x} \succ \boldsymbol{x'}$$

Pareto Efficiency: An allocation is *Pareto-efficient* if it is not *Pareto-dominated.*

Utility Possibility Frontier

- Fix a level of resources at $\boldsymbol{x}$, where $\boldsymbol{x} = (\boldsymbol{x}_1, ..., \boldsymbol{x}_n)$, and where $\boldsymbol{x}_i = (x_i^1, ..., x_i^n)$.
- We have utility profile of individuals at $\boldsymbol{x}$, $\boldsymbol{u}(\boldsymbol{x}) = (u_1(\boldsymbol{x}), ..., u_n(\boldsymbol{x}))$
- The *utility possibility frontier* describes the utility of agents when we reallocate these resources, that is it describes the utility of agents when we reallocate resources from one to another. The frontier shows the utility gain of one individual at the expense of utility loss of another individual.
- The frontier is hence the utility maximisation given an economies' endowment and technology.

$$\boldsymbol{U} = \{\boldsymbol{u}(\boldsymbol{y}) \text{ such that } \sum_i \boldsymbol{y}_i = \sum_i \boldsymbol{x}_i\}$$

Derivation in the two agent case:

$$u_a(x_a, y_a)\ ,\ u_b(x_b, y_b)\ ,\ x_a + x_b = w_x\ ,\ y_a + y_b = w_y$$

The UPF is the set of all utility pairs (μ_a, μ_b) such that there exists a Pareto optimal allocation with the property that $\mu_a = u_a(x_a, y_a)\ ,\ \mu_b = u_b(x_b, y_b)$.

Necessary conditions are,

(1) Optimality:

$$\text{MRS}_a = \text{MRS}_b \ \Leftrightarrow\ \frac{\frac{\partial u_a}{\partial x_a}}{\frac{\partial u_a}{\partial y_a}} = \frac{\frac{\partial u_b}{\partial x_b}}{\frac{\partial u_b}{\partial y_b}}$$

(2) Feasibility

$$x_a + x_b = w_x\ ,\ y_a + y_b = w_y$$

Social Welfare Function & Welfare Maximisation

What is the 'best' point on the UPF? Each SWF says a different Pareto efficient point is in fact also optimal.

Example functions,

(I) Concern for the worst off: (Rawlsian)

$$W(\boldsymbol{u}) = \min_i\{u_i(\boldsymbol{x})\}$$

(II) Indifferent to inequality: (Benthamite/Utilitarian)

$$W(\boldsymbol{u}) = \sum_i u_i(\boldsymbol{x})$$

(III) Inequality-averse:

$$W(\boldsymbol{u}) = \prod_i u_i(\boldsymbol{x})^{a_i} \text{ where } \sum_i a_i = 1$$

Social Welfare Maximisation,

We maximise social welfare by choosing some SWF (from above) and optimising it with respect to the UPF - the set of maximised utilities - by first,

(1) Fixing a resource constraint,

(2) Derive the UPF,

(3) Maximise the SWF with respect to the UPF.

First Fundamental Theorem of Welfare Economics

If $(\boldsymbol{x}^*, \boldsymbol{p}^*)$ is a competitive equilibrium in the exchange economy, Then $(\boldsymbol{x}^*, \boldsymbol{p}^*)$ is Pareto-efficient.

Second Fundamental Theorem of Welfare Economics

Suppose $\boldsymbol{x}^* > 0$ is a Pareto-efficient allocation in the exchange economy and that preferences are convex, continuous, and monotonic. Then $(\boldsymbol{x}^*, \boldsymbol{p}^*)$ is a competitive equilibrium for the initial endowments $\boldsymbol{p}^*\boldsymbol{w} = \boldsymbol{p}^*\boldsymbol{x}^*$.

Social Choice

Individuals' preference orderings must be:

(1) **Complete:** All social states can be compared against one another – for any two states x and y, either x is at least as good as y, or y is at least as good as x, or both.

$$x \succsim y \ , \ y \succsim x \text{ or both, that is } x \sim y$$

(2) **Transitive:** If social state x is weakly preferred to y, and y to z, then x needs to be weakly preferred to z.

$$\text{If } x \succsim y \text{ and } y \succsim z, \text{ then } x \succsim z$$

(3) **Continuous:** The set of all social states that are at least as good as x and the set of states that are no better than x are both closed sets.

Preference Aggregation Rule:

Maps all preference profiles into a preference ordering for society,

$$F(\succsim_1, ..., \succsim_n) = \succsim_*$$

Arrows Impossibility Theorem & Counterarguments

Arrows Impossibility Theorem states that there is ***no complete, transitive purely ordinal social choice rule*** for at least three social states whereby all the following conditions are satisfied:

- Unrestricted domain: F should produce a social ordering $\succsim_*$ for any profile of individual preferences $(\succsim_1, ..., \succsim_n)$.
- Pareto-principle: If $x \succsim_i y \;\; \forall i,$ then $x \succsim_* y$.
- Independence of irrelevant alternatives: If the individual preference orderings over x and y do not change, F should produce the same ordering over x and y even if preferences over other alternatives change.
- No-dictatorship: There exists no agent i, such that irrespective of the preferences of others $\forall x, y$ if $x \succsim_i y$, then $x \succsim_* y$.

Examples:

(1) Dictatorial Rule,

$$x \succsim_i y \;\Leftrightarrow\; x \succsim_* y$$

This ordering is complete and transitive but it does not honour 'no-dictatorship'.

(2) Majority Rule

$$x \succsim_* y \;\Leftrightarrow\; \{i \in N \text{ such that } x \succsim_i y\} \geq \{i \in N \text{ such that } y \succsim_i x\}$$

This holds except if there are more than three social states as then it is not necessarily transitive. It may be the case that a majority of people prefer x to y and a majority of people prefer y to z, but also a majority of people prefer z to x. See the table below for proof,

Person A	Person B	Person C
x	y	z
y	z	x
z	x	y

(3) Scoring Rule

$$x \succsim_* y \;\Leftrightarrow\; \sum_{i=1}^{n} p(K(\succsim_i, x)) \geq \sum_{i=1}^{n} p(K(\succsim_i, y))$$

Where $K()$ denotes social states place in i's ordering, and the function $p()$ assign's points given a place in the ordering. Therefore if x gets more points than y then x is socially preferred to y.

This fails, however, as it does not honour the independence of irrelevant alternatives,

Person A	Person B
x	y
y	z
z	x

Consider the table for now without z,

A gives x 2 point and y 1 point, B does the alternate, Hence both outcomes x and y get 3 points - *they are equal*.

Now introducing z, x gets 3 and 1 (4 points) and y gets 2 and 3 (6 points).

Hence y is now *preferred* by x due to the introduction of an irrelevant alternative.

Rank voting can be manipulated by introducing new alternatives that change the final ranks.

Counterarguments to Arrow

- Relax the unrestricted domain.
- Replace transitivity with quasi-transitivity (i.e. transitivity only for strict relations)
- Relax IIA - what is so bad about having lots of options? The more options might change the rankings but that's only because we now know much more about the people's preferences.

Median Voter Theorem

“If alternatives are one-dimensional and all voters have single peaked preferences, the outcome of majority voting is the preferred option of the median voter”

Arrows Impossibility Theorem required an unrestricted domain, that F should produce a social ordering $\succsim_*$ for any profile of individual preferences, $\succsim_1, ..., \succsim_n$

What if we relaxed this assumption?

Person A	Person B	Person C
A	B	C
B	C	B
C	A	A

Restrict domain such that only single peaked preferences are allowed.

If all preferences are such as above and the majority rule is applied then there are no violations of transitivity and B, the median choice wins.

We can further add a truth telling requirement to our preference aggregation rule, which no longer need produce a complete and transitive ordering. Make it such that this social choice function,

(1) Respects citizens’ sovereignty: every possible ranking of social states is possible for some preference profile.

(2) Strategy-proofness: no agent can report different preferences and be better off under the social choice function,

$$\forall i \;\; F(\succsim_i, \succsim_{-i}) \succsim_i F(\trianglerighteq_i, \succsim_{-i})$$

Given at least 3 social states there is no social function that meets these two criteria. ***Except, if agents have single peaked preferences, then majority voting is strategy-proof!***

Worked Examples

Example: General Expression for a Contract Curve

Find an expression for the contract curve given utility functions,

$$u_a(x_a, y_a) = x_a y_a^3 \ , \ u_b(x_b, y_b) = 16x_b y_b$$

And the constraints,

$$x_a + x_b = 8 \ , \ y_a + y_b = 8$$

(1) Optimality: Find all Pareto Efficient allocations

Using $MRS_a = MRS_b \Leftrightarrow \frac{\frac{\partial u_a}{\partial x_a}}{\frac{\partial u_a}{\partial y_a}} = \frac{\frac{\partial u_b}{\partial x_b}}{\frac{\partial u_b}{\partial y_b}}$

$$\frac{y_a^3}{3x_a y_a^2} = \frac{16y_b}{16x_b}$$
$$\frac{y_a}{3x_a} = \frac{y_b}{x_b}$$

(2) Feasibility: Pareto Efficient allocations within the constraints

Using $8 = x_a + x_b$, $8 = y_a + y_b$ and $\frac{y_a}{3x_a} = \frac{y_b}{x_b}$

$$y_a = 3x_a \frac{y_b}{x_b}$$
$$y_a = 3x_a \frac{8 - y_a}{8 - x_a}$$
$$8y_a - y_a x_a = 24x_a - 3x_a y_a$$
$$y_a(8 - x_a + 3x_a) = 24x_a$$
$$y_a = \frac{24x_a}{8 + 2x_a}$$

Giving the Contract Curve,

$$y_a = \frac{24x_a}{8 + 2x_a}$$

Having derived the contract curve it should then be possible to derive the UPF. We can input the contract curve into the utility functions for the agents to get utility only in terms of x_a. Then by varying x_a between 0 and 8 - the range of possible values for x_a - we will get the UPF.

Applied Welfare, Externalities, and Public Goods

Core Concepts

Income & Substitution Effects

- Income: How price changes affect disposable income, which affects consumption.
- Substitution: Changes in purchasing based on price changes. (Substitutes expensive good for more of the cheaper good)

Marshallian Demand

$$\boldsymbol{x}^*(\boldsymbol{p}, m)$$

Which is found by solving,

$$\max u(\boldsymbol{x}) \text{ such that } \boldsymbol{px} \geq m$$

- Describes how consumption varies with prices and income.
- The Marshallian demand allows us to obtain the indirect utility function,

$$v(\boldsymbol{p}, m) \text{ where } v(\boldsymbol{p}, m) = u(\boldsymbol{x}^*(\boldsymbol{p}, m)) = \bar{u}$$

Hicksian Demand

$$\boldsymbol{x}^* = h(\boldsymbol{p}, \bar{u})$$

Which is found by solving,

$$\min \boldsymbol{px} \text{ such that } u(\boldsymbol{x}) \geq \bar{u}$$

- Describes how consumption varies with price and utility, hence Hicksian demand looks at consumption patterns when relative income is held constant.
- There is no income effect in Hicksian demand when prices change.
- From the Hicksian demand we can obtain the expenditure function,

$$e(\boldsymbol{p}, \bar{u})$$

- It gives the minimum amount of money needed to be spent in order to achieve some level of utility.
- In the two good case,

$$e(p_1, p_2, \bar{u}) = p_1 h_1(p_1, p_2, \bar{u}) + p_2 h_2(p_1, p_2, \bar{u})$$

Evaluating a Change in Welfare

Let $\boldsymbol{x} = (x_1, x_2)$ and $\boldsymbol{p}_0 = (p_1, p_2)$. Suppose then there is a change in price such that $\boldsymbol{p_1} = (p_1', p_2')$. How might we evaluate this change in welfare?

(1) Ordinary Model of Welfare Change,

- We can very simply quantify the change in welfare using indirect utility:

$$\Delta\text{welfare} = v(\boldsymbol{p}_0, m) - v(\boldsymbol{p}_1, m) = u(\boldsymbol{x}(\boldsymbol{p}_0, m)) - u(\boldsymbol{x}(\boldsymbol{p}_1, m))$$

- Problem: Utility is an ordinal concept, but it is important that we have a cardinal measure of welfare since we need to be able to say by 'how much' a change in circumstance has improved or reduced welfare.

(2) Cardinal Model of Welfare Change

- Instead we can consider Hicksian demand and the expenditure function,

$$e(\boldsymbol{p}, \bar{u}) = \min \boldsymbol{px} \text{ such that } u(\boldsymbol{x}) \geq \bar{u}$$

$$e(\boldsymbol{p}, \bar{u}) = \boldsymbol{px^*} = m$$

- Suppose the price change $\boldsymbol{p}_0$ to $\boldsymbol{p}_1$ moves the consumers indirect utility from $v(\boldsymbol{p}_0, m) = u_0$ to $v(\boldsymbol{p}_1, m) = u_1$.
- For an arbitrary price vector $\boldsymbol{p'}$ we can measure:
  - $e(\boldsymbol{p'}, u_1)$: wealth level needed to reach u_1 when the price vector is p'.
  - $e(\boldsymbol{p'}, u_0)$: wealth level needed to reach u_0 when the price vector is p'.

$$\Delta\text{welfare} = e(\boldsymbol{p'}, u_1) - e(\boldsymbol{p'}, u_0)$$

This cardinal model of welfare change uses an arbitrary price vector $\boldsymbol{p'}$, but recall our example was of a price change from $\boldsymbol{p}_0$ to $\boldsymbol{p}_1$. This provides us with two key measures of welfare to use - dependent on which price vector we use.

(I) Compensating Variation (CV)

- Old Utility @ New Prices

$$CV(\boldsymbol{p}_1, m) = e(\boldsymbol{p}_1, u_1) - e(\boldsymbol{p}_1, u_0) = m - (\boldsymbol{p}_1, u_0)$$

(II) Equivalent Variation (EV)

- New Utility @ Old Prices

$$EV(\boldsymbol{p}_0, m) = e(\boldsymbol{p}_0, u_1) - e(\boldsymbol{p}_0, u_0) = (\boldsymbol{p}_0, u_1) - m$$

For example, if prices fell such that utility increased, the EV would be the amount one would need to be paid to get this new higher utility at the old prices, while the CV would be the amount you would pay to have this economic change happen.

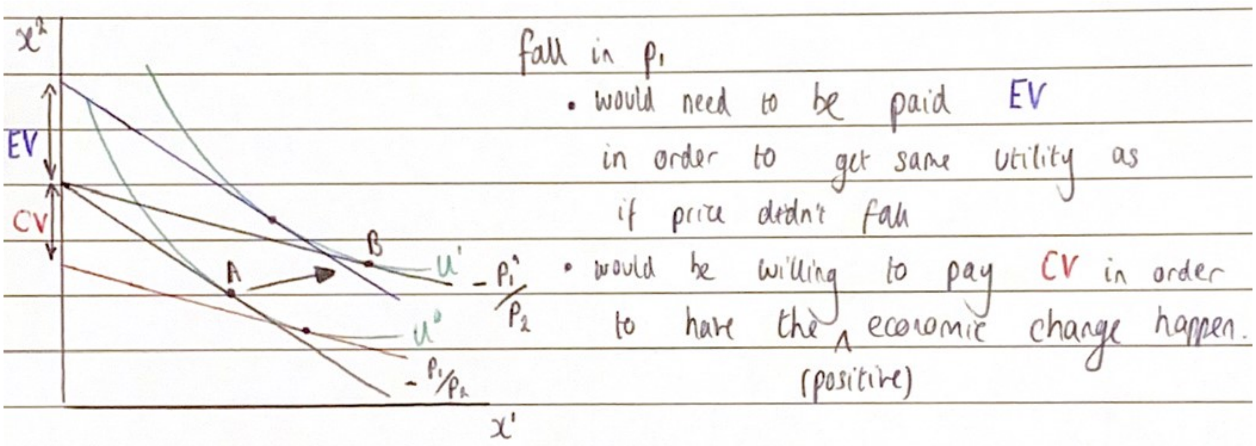

Figure 12: Compensating vs Equivalent Variation

Quasi-Linear Framework

Quasi-linear utility functions are utility functions that are linear in one argument. In our case in which utility functions tend to take two arguments, x and y, one of these is hence linear. We will also generally consider the linear good to be a numeraire.

$$U(x, y) = u(x) + y \text{ and } px + y = m$$
$$U(x, y) = u(x) + m - px$$

FOCs,

$$U'(x, y) = u'(x) - p = 0$$
$$u'(x) = p$$

The marginal utility of good 1 must be equal to its price. That is the demand for good 1 depends only on price and not income (providing income is high enough). So there is no income effect!

For example, say good x is chewing gum, good y is money spent on all other goods. I only ever buy the same quantity of chewing gum so when my income goes up it all goes onto other goods (not chewing gum).

Benefits of this framework:

- There is ***no income effect*** which means that $CV = \Delta CS = EV$
  - Unless M is very low.
- We can do ***partial equilibrium analysis:*** just focus on one good and treat the other as a composite good.
- We can aggregate easily: aggregate indirect utility is equal to the sum individual indirect utilities.
  - Hence in cases when we are solving maximisation problems of multiple agents **we can find pareto optimality by summing the utility functions**.
- Utility = money (Strong assumption)

Welfare & Commodity Tax

Consumer Side:

$$max\ U(x_1, x_2) = u(x_1) + x_2 \text{ such that } px_1 + x_2 = m$$
$$max\ U(x_1, m) = u(x_1) + m - px_1$$

FOC:

$$U'(x_1) = u'(x_1) - p = 0$$
$$u'(x_1) = p$$

Producer Side:

$$max\ \pi(y_1) = py_1 - c(y_1)$$

FOC

$$\pi'(y_1) = p - c'(y_1) = 0$$
$$p = c'(y_1)$$

At equilibrium $x_1 = y_1$ and $\pi = m$, hence,

$$u'(x_1^*) = p = c'(y_1^*)$$
$$U(x_1) = u(x_1) - c(x_1)$$

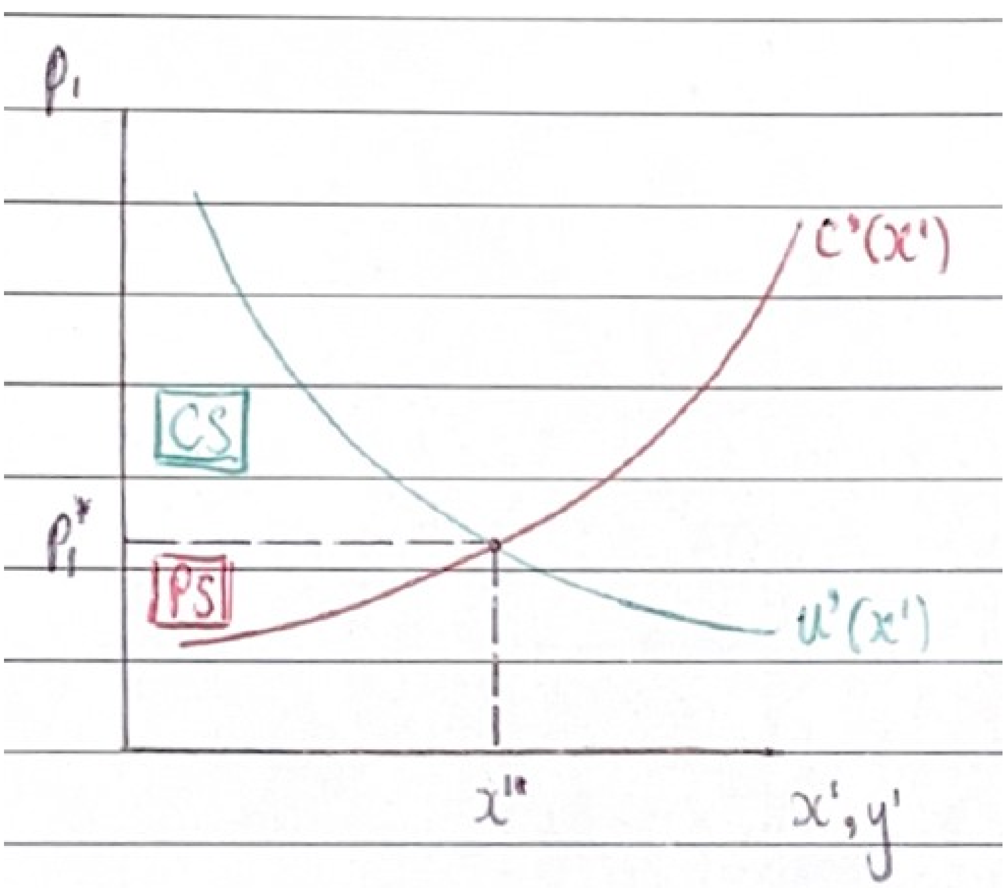

Figure 13: Equilibrium outcome

With a Commodity Tax

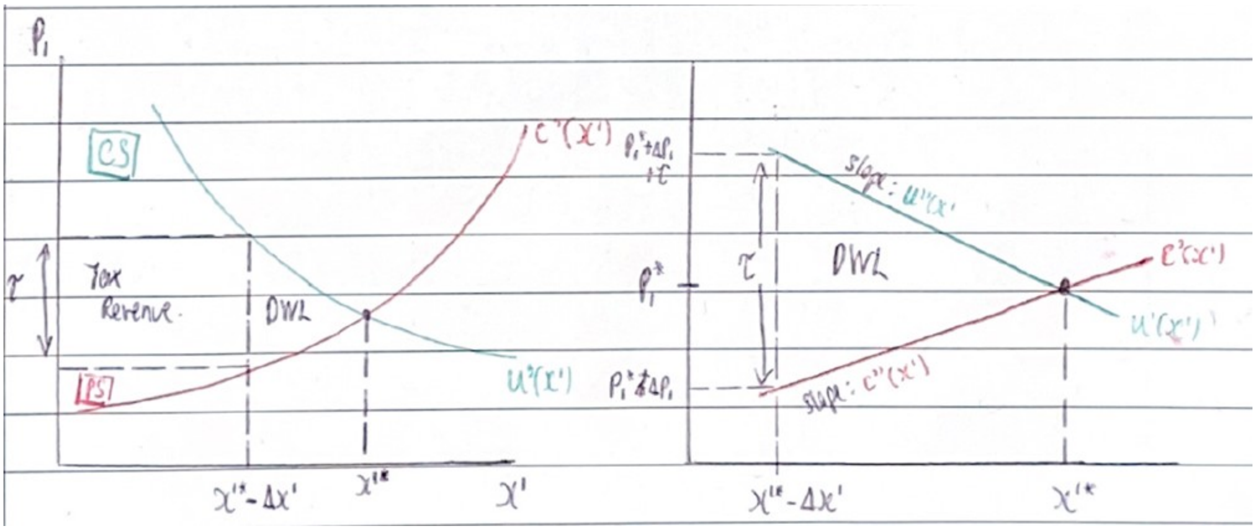

Figure 14: Commodity Tax: left shows the actual tax; right shows a linear approximation of the left

Zooming in locally we can think of things as linear and approximate the tax revenue

$$u'\left(x^1\right) = p$$
$$u''\left(x'\right) = \frac{\Delta p_1 + \tau}{\Delta x^1}\ ,\ \varepsilon_D = \frac{p_1}{x^1}\frac{dx_1}{dp_1}\ ,\ \frac{dp_1}{dx^1} = \frac{p_1}{x^1 \varepsilon_D}$$
$$u''\left(x'\right) = \frac{p_1}{x^1 \varepsilon_D}$$

$$c'\left(x^1\right) = p$$
$$c''\left(x'\right) = \frac{\Delta p_1}{\Delta x^1}\ ,\ \varepsilon_s = \frac{p_1}{x^1}\frac{dx_1}{dp_1}\ ,\ \frac{dp_1}{dx^1} = \frac{p_1}{x^1 \varepsilon_S}$$
$$c''\left(x^1\right) = \frac{p_1}{x^1 \varepsilon_s}$$

DWL is given by,

$$u''\left(x^1\right)\Delta x^1 = \Delta p_1 + \tau \text{ and } c''\left(x^1\right)\Delta x' = \Delta p_1$$
$$\left[u''\left(x^1\right) - c''\left(x^1\right)\right]\Delta x^1 = \tau$$
$$\Delta x^1 = \frac{\tau}{u''\left(x^1\right) - c''\left(x^1\right)}$$
$$\Delta x^1 = \frac{\tau}{\frac{p_1}{x^1 \varepsilon_D} - \frac{p_1}{x^1 \varepsilon_S}} = \frac{\tau x^{\varepsilon_s} \varepsilon_D}{p_1\left(\varepsilon_s - \varepsilon_D\right)}$$
$$DWL = \frac{\Delta x^1 \tau}{2} \times \left(\frac{\tau}{p_1}\right)^2 \times p_1 x^1 \times \frac{\varepsilon_S \varepsilon_D}{\left(\varepsilon_S - \varepsilon_D\right)}$$

Externalities

An externality is present when an action of one agent directly affects the utility of a third party with no say in the original transaction.

Negative Externalities Model

(1) **Actual Outcome**

Suppose a consumer who solves her maximisation problem,

$$max\ U(x,y) = u(x) - L(y) + z \text{ such that } px + z = m$$
$$max\ U(x,y) = u(x) - L(y) + m - px$$

Where x is a demanded good; $L(y)$ a negative externality caused by the production of x; and z is all other demanded goods, which we think of as a numeraire.

The firm which produces x by outputting y solves,

$$max\ \pi = py - C(y)$$

Where y is the output of the good x, and $C(y)$ is the cost associated with outputting y.

Notice that $L(y)$ does not appear in the firms profit equation, since the externality doesn't affect the firm and further that the consumer cannot choose y, hence she has no influence over $L(y)$.

FOCs,

$$\frac{\partial U(x,y)}{\partial x} = u'(x) - p = 0$$
$$u'(x) = p$$
$$\frac{\partial \pi(y)}{\partial y} = -C'(y) + p = 0$$
$$C'(y) = p$$

In equilibrium $x = y$, hence x^* solves,

$$u'(x^*) = C'(x^*)$$

That is that ***marginal benefit = marginal private cost (MB=MC)***

(2) **Socially Optimum Outcome**

In the socially optimal case we can think of the consumer as owning the profits of the firm and managing how the firm is run. Hence the consumer cares both about profit but also about the externality. Assume then that the consumer/firm still gets utility from consuming x, but disutility from its production, y. Now, however, rather than having an exogenous income m the consumers income is exactly the profit π, where $\pi = py - C(y)$.

Utility in x and y is hence,

$$U(x,y) = u(x) - L(y) - px + py - C(y)$$

Recalling that at equilibrium $x = y$,

$$U(x) = u(x) - L(x) - C(x)$$

FOC gives that x^{**} solves,

$$u'(x^{**}) = C'(x^{**}) + L'(x^{**})$$

That is, ***marginal benefit = marginal private cost + marginal external cost (MB=MSC)***

Solutions to Negative Externalities

(1) **Taxation: Pigouvian tax**

Introduce a tax on the firms output, in which the consumer receives the tax revenue such that her income is now,

$$m + ty$$

Where t is the tax and y is the firms output. The consumer's FOC is unchanged since she cannot choose y, therefore,

$$\frac{\partial U(x,y)}{\partial x} = u'(x) - p = 0$$
$$u'(x) = p$$

The firms new profit equation is,

$$\pi = py - C(y) - ty$$

Giving the FOC,

$$\frac{\partial \pi(y)}{\partial y} = -C'(y) + p - t = 0$$
$$C'(y) + t = p$$

At equilibrium $x = y$, and so,

$$u'(x) = C'(x) + t$$

In order for ***MB=MSC***, that is for $u'(x^{**}) = C'(x^{**}) + L'(x^{**})$ we therefore just need to set a tax t^* such that,

$$t^* = L'(y)$$

(2) **Property rights: Pollution trading**

Total efficient production level: $y*$

Two firms A and B are endowed with permits $\bar{y}_i$ which they can trade at price z, hence,

$$\pi_i(y) = py - C_i(y) - z(y - \bar{y}_i)$$

FOC,

$$\frac{\partial \pi_i(y)}{y} = p - C_i'(y) - z = 0$$

Hence,

$$p = C_i'(y) - z$$

By the same reasoning as in the tax section then we simply need to set the permit price z equal to the ***MSC*** to ensure that ***MB=MSC***. So,

$$z^* = L'(y^*)$$

Coase Theorem: Irrespective of allocation of property rights, frictionless bargaining produces an efficient outcome in the presence of externalities.

Notice that the Pigouvian tax and Property rights have the same outcome, hence which one we use theoretically doesn't matter! The both get us to the social optima.

(3) **Problem of Uncertainty**

One might ask then which should we use, property rights or pigouvian taxation? In theory it make no difference: they both generate the same outcome.

In a perfect world this is the case, however there is a problem of uncertainty in the marginal benefit - it could be higher or lower than expected.

INSERT DIAGRAM

Solution: Set tax or quantity fixing depends on elasticity of MSC,

- Flat MSC: price fix $\succ$ quantity fix, hence ***Flat MSC*** $\Rightarrow$ ***Tax***.
- Steep MSC: quantity fix $\succ$ price fix, hence ***Steep MSC*** $\Rightarrow$ ***Quota***.

[There is no equivalent problem with uncertainty in the marginal cost - but I can't remember why that is the case? I think it is because the firm always produces on the MB, and wherever the Gov sets the tax/quota the firm will produce at the intersection of that and the MB. Given that the tax & quota both generate the same point, if the real MSC is different from the Gov's expectation then the distance between the tax/quota intersection with MB and the MB-MSC intersection is the same, regardless of whether or not we used a tax or quota.]

DRAW DIAGRAM OF TAX INTERSECTING MB AND ANOTHER ONE OF QUOTA INTERSECTING MB SUCH THAT OUTCOME IS THE SAME POINT. NOTICE THAT REGARDLESS OF WHERE ACTUAL MSC IS, THE MARKET WILL NOT MOVE TO THE MB-MSC INTERSECTION, IT WILL STAY AT THE TAX-MB OR QUOTA-MB INTERSECTION. HENCE THE UNCERTAINTY DOESN'T CHANGE WHETHER A TAX OR QUOTA IS BETTER.

Public, Common, and Club Goods

We can distinguish between types of goods into four categories based on two characteristics: excludability and rivalrousness. Excludability refers to whether or not people can be prevented from consuming them, while rivalrousness is whether individuals can consume them without affecting their availability to others.

Table 1: Taxonomy from Excludability and Rivalrousness

	Rival	Non-rival
Excludable	(Private goods) Toothpaste Underwear	(Club goods) Online news subscription Computer software
Non-excludable	(Common goods) Fish Water Atmosphere	(Public goods) National defence Street lighting Wikipedia

Common goods

Common goods are goods (or resources) that are non-excludable but rival. That is anyone can use them, but if one individual consumes them, their availability to other individuals is reduced. This combination of characteristics can result in an overuse of common resources.

Fishing Lake Model:
Consider a common lake, with a finite (rival), but non-excludable resource of fish.

Let, $q =$ output , $L_i =$ time spent by i fishing , $w =$ opportunity cost of time and $F(L)$ be the function of time spent fishing that defines output q.

Total catch is given by,

$$q = F(L) = F(\sum_{i=1}^{n} L_i) \text{ and } F'' < 0 \text{ (DRS)}$$

Fisherman i's catch,

$$q_i = \frac{L_i}{L}F(L)$$

and i's profit,

$$\pi_i = pq_i - wL_i$$

Possible Market Outcomes:

INSERT DIAGRAM

(1) Actual Equilibrium

In equilibrium all profit opportunities are exploited as there are no barriers to entry. That is if agents are making profit then more enter the market (lake) until profit is 0. Alternatively if agents are making losses they leave until profit if 0.

This means that the industry profits are zero.

$$\pi = pF(L) - wL = 0$$
$$L = \frac{p}{w}F(L)$$

(2) Total Catch Maximisation

Total catch is maximised at L^{**}, where,

$$F'(L) = 0$$

(3) Socially Efficient Outcome

The socially efficient outcome $L*$ is also in fact the monopolistic outcome.

It solves the profit maximisation, $MR = MC$,

$$\frac{\partial \pi}{\partial L} = pF'(L) - w = 0$$
$$pF'(L) = w$$

Conclusion

The free market exploitation and overfishing, whereas fishing rights or even having a monopolist fisherman would be more sustainable.

Public goods

Public goods are non-excludable and non-rival. That means no one can be prevented from consuming them, and that when individuals do use them it does not reduce their availability to others.

Street Light Model:
Consider two consumers A and B, who are deciding on the amount of street lighting to purchase for their private road.

Let x be the public good (street lighting) and y be other private goods. Suppose that, for street lighting, $MB_B > MB_A$.

Each consumer $i \in A, B$ solves,

$$max\ u_i(x, y_i) = v_i(x) + y_i \text{ such that } x + y_i = m_i$$

The FOCs are,

$$MRS_{x,y_i} = \frac{v_i'(x)}{1} = MB_i(x) = 1$$

Hence,

$$v_i(x) = 1$$

Ideally they would jointly demand x^*, but A knows if she does not provide any of the public good, given that it is non-excludable, she can enjoy x_B^* at no cost. Hence ***in equilibrium there is an under-provision***.

Funding Public Goods
In order to supply x^* the government may tax A and B, but if A and B pay an equal amount in tax then A is paying relatively more than B, that is A is paying more for the public good than the amount that she wants the public good. Hence we might instead use,

(1) Lindahl pricing, The government taxes each consumer in order to alter their budget constraints such that,

$$t_i x + y_i = m \text{ and where } \sum_i t_i = 1$$

Consumers then solve their maximisation problem, giving

$$v_i'(x) = t_i$$

Call $i's$ demand for the public good given the tax $x_i(t_i)$. The government then goes ahead and asks each consumer for their demand for the public good given the tax and then aggregates these demands and sets taxes at a level such that, for all consumers i and j.

$$x_i(t_i) = x_j(t_j) = x^* \text{ and } \sum_i MB_i(x^*) = \sum_i t_i = 1$$

The problem with this pricing mechanism is: what if A lies and claims that $u_A'(x) = 0$? In this case A is not taxed, and hence we return to the original inefficient equilibrium where B supplies what she demands and A uses it for free.

(2) Vickrey-Clarke-Groves Mechanisms General form:

$$t_i = f(x_{-i}) - \sum_{j \neq i} v_j(x^*)$$

Where $-i$ means that i is not included in the calculation, and $f(x_{-i})$ is any function that doesn't depend on $i's$ report. For example,

$$t_i = \sum_{j \neq i} v_j(x_{-i}^*) - \sum_{j \neq i} v_j(x^*)$$

Hence consumers only pay if they affect the outcome.

Club goods

Club goods are non-rival but are excludable. Individuals can be prevented from consuming them, but their consumption does not reduce their availability to others. Club goods can sometimes be thought of as 'artificially scarce' - their scarcity is created rather than natural.

Streaming Service Model

Consider a streaming service that charges a single-entry fee but then is free to use. The efficient allocation is at x^* but the club wants to maximise profit. If they charge a higher price, however, they may exclude some consumers who do not demand the good at this price. This generates a DWL.

Solution: If the club has market power then they can price discriminate = no DWL = efficient.

Worked Examples THESE NEED FINISHING

Example: Marshallian, Hicksian, Expenditure, Indirect Utility Functions

Consider the utility function,

$$u(x,y) = x^{\frac{3}{5}} y^{\frac{2}{5}}$$

And the budget constraint,

$$p_x x + p_y y = m$$

(a) Find the Marshallian Demand functions

(b) Find the Hicksian Demand functions.

(c) Find the Expenditure function.

(d) Find the Indirect Utility function.

(e) What is the relationship between all of these functions?

Marshallian Demand functions:

$$max\ u(x,y) = x^{\frac{3}{5}} y^{\frac{2}{5}} \text{ such that } p_x x + p_y y = m$$

FOC,

$$\text{MRS}_{x,y} = \frac{(3/5)x^{-2/5}y^{2/5}}{(2/5)y^{-3/5}x^{3/5}} = \frac{3y}{2x} = \frac{p_x}{p_y}$$

We now have two equations and two unknowns, $\frac{3y}{2x} = \frac{p_x}{p_y}$ and $p_x x + p_y y = m$. Rewriting the former as $y = x\frac{2p_x}{3p_y}$ latter as $y = -\frac{p_x}{p_y}x + \frac{m}{p_y}$ we can solve for the Marshallian demand,

$$\begin{aligned} y &= -\frac{p_x}{p_y}x + \frac{m}{p_y} \\ x\frac{2p_x}{3p_y} &= -\frac{p_x}{p_y}x + \frac{m}{p_y} \\ 2xp_x &= -3p_x x + 3m \\ 5xp_x &= 3m \\ x &= \frac{3m}{5p_x} \end{aligned}$$

And then subbing back into $y = x\frac{2p_x}{3p_y}$,

$$\begin{aligned} y &= x\frac{2p_x}{3p_y} \\ y &= \frac{3m}{5p_x}\frac{2p_x}{3p_y} \\ y &= \frac{2m}{5p_y} \end{aligned}$$

Hence the solution,

$$x = \frac{3m}{5p_x}\ ,\ y = \frac{2m}{5p_y}$$

Hicksian Demand functions:

$$min\ p_x x + p_y y \text{ such that } u(x,y) = x^{\frac{3}{5}} y^{\frac{2}{5}} = \bar{u}$$

FOC,

$$\text{MRS}_{x,y} = \frac{(3/5)x^{-2/5}y^{2/5}}{(2/5)y^{-3/5}x^{3/5}} = \frac{3y}{2x} = \frac{p_x}{p_y}$$

We now have two equations and two unknowns, $\frac{3y}{2x} = \frac{p_x}{p_y}$ and $x^{\frac{3}{5}} y^{\frac{2}{5}} = \bar{u}$. Rewriting the former as $y = x\frac{2p_x}{3p_y}$ we can solve for the Hicksian demand,

$$\begin{aligned}
\bar{u} &= x^{\frac{3}{5}} y^{\frac{2}{5}} \\
\bar{u} &= x^{\frac{3}{5}} (x\frac{2p_x}{3p_y})^{\frac{2}{5}} \\
\bar{u} &= x(\frac{2p_x}{3p_y})^{\frac{2}{5}} \\
x &= \bar{u}(\frac{3p_y}{2p_x})^{\frac{2}{5}}
\end{aligned}$$

And then subbing back into $y = x\frac{2p_x}{3p_y}$,

$$\begin{aligned}
y &= x\frac{2p_x}{3p_y} \\
y &= \bar{u}(\frac{3p_y}{2p_x})^{\frac{2}{5}} \frac{2p_x}{3p_y} \\
y &= \bar{u}\frac{(3p_y)^{\frac{2}{5}} 2p_x}{(2p_x)^{\frac{2}{5}} 3p_y} \\
y &= \bar{u}\frac{(2p_x)^{\frac{3}{5}}}{(3p_y)^{\frac{3}{5}}} \\
y &= \bar{u}(\frac{2p_x}{3p_y})^{\frac{3}{5}}
\end{aligned}$$

Hence the solution,

$$x = \bar{u}(\frac{3p_y}{2p_x})^{\frac{2}{5}}\ ,\ y = \bar{u}(\frac{2p_x}{3p_y})^{\frac{3}{5}}$$

Expenditure function:

The expenditure function is given by,

$$e(p_x, p_y, \bar{u}) = p_x h_x(p_x, p_y, \bar{u}) + p_y h_y(p_x, p_y, \bar{u})$$

Where $h_x(p_x, p_y, \bar{u}) = x$ and $h_y(p_x, p_y, \bar{u}) = y$.

Therefore,

$$\begin{aligned}
e(p_1, p_2, \bar{u}) &= p_x \bar{u}(\frac{3p_y}{2p_x})^{\frac{2}{5}} + p_y \bar{u}(\frac{2p_x}{3p_y})^{\frac{3}{5}} \\
&= \bar{u} p_x^{3/5} p_y^{2/5} [(\frac{3}{2})^{2/5} + (\frac{2}{3})^{3/5}]
\end{aligned}$$

Indirect Utility functions:

$$v(p_x, p_y, m) = u(x^*, y^*) = (\frac{3m}{5p_x})^{3/5} (\frac{2m}{5p_y})^{2/5}$$

Relationships:

$$e(p_x, p_y, \bar{u}) = m = e(p_x, p_y, v(p_x, p_y, m))$$

(Expenditure is the inverse of the indirect utility function)

$$h(p_x, p_y, \bar{u}) = x(p_x, p_y, e(p_x, p_y, \bar{u}))$$

(Expenditure gives income, hence can be subbed into Marshallian demand)

$$h(p_x, p_y, v(p_x, p_y, m)) = x(p_x, p_y, m)$$

(Indirect utility gives u hence can be subbed in)

Example: Private Provision of a Public Good

Two students, a and b, share a house. Heat for this house is a pure public good. Student a feels the cold. Her (derived) benefit from x units of heat is:

$$\phi^a(x) = 20x - \frac{x^2}{2}$$

Student b is a hardier soul. Her (derived) benefit from x units of heat is:

$$\phi^b(x) = 15x - x^2$$

The heating system runs on prepayment cards. Each student can provide units of heat by purchasing cards from a single profit-maximising heat supply firm. This firm's cost of supplying q units of heat is:

$$c(q) = 2q^2$$

Let x_a denote the units of heat purchased by student a, and x_b the units of heat purchased by student b. Total heat provision is then $x = x_a + x_b$. Assume that both students, as well as the firm, take the market price of heat as given. Assume, also, that each student takes her housemate's purchase of heat as given.

(a) At a competitive (Nash-Cournot) equilibrium involving price p^*, student a's purchase of the public good, x_a^*, must maximise her utility given x_b^*. Write down this utility maximisation problem and obtain the first order condition. Repeat this exercise for student b. Will both students purchase heat?

(b) At a competitive equilibrium involving price p^*, the firm's public good supply, q^*, must maximise its profits. Write down the firm's profit maximisation problem and obtain the first order condition.

(c) Use your answers above, together with a market clearing condition, to solve for the competitive equilibrium price, p^*, and provision of heat, q^*.

(d) Is the competitive equilibrium effcient? What would be a Pareto optimal provision of heat, q^o?

Example: Fishing Lake Problem

On an island there are 2 lakes and 16 fishermen. Each fisherman can fish on either lake. When L_1 men fish on lake 1, the total number of fish caught there is

$$F_1(L_1) = 8L_1 - \frac{1}{2}(L_1)^2$$

and when L_2 men fish on lake 2, the total number of fish caught there is

$$F_2(L_2) = 4L_2$$

Assume that all men on a given lake catch the same number of fish.

(a) In this unregulated situation, how may men will fish on lake 1, how many on lake 2, and how many fish will be caught altogether?

(b) The island chief believes that she can increase the total number of fish caught by restricting the number of men fishing on lake 1. What is the number of men fishing on lake 1 that maximises the number of fish caught altogether, and what is the number of fish caught in this situation? Explain why the chief is correct.

(c) Since she is opposed to coercion, the chief decides to issue permits in order to fish on lake 1. If she is to bring about the optimal allocation of labour, how much should a permit cost (in terms of fish)?

Example: Distribution of Externality

Consider an economy in which wealth is transferable, and where consumer a imposes an externality, h, on consumer b, with $0 \leq h \leq 50$. At this level of the externality, and when consumers a and b have wealth w_a and w_b, respectively, consumer a has utility $10h^{1/2} + w_a$, and consumer b has utility $10(50 - h)^{1/2} + w_b$.

(a) What is the level of h that maximises consumer a's utility?

(b) What is the level of h in any Pareto optimal allocation?

(c) Suppose b has rights to no externality, hence a must pay b in order to h, what level of h is emitted then?

Consumer a Utility Max,

$$\begin{aligned} max\ u_a(h, w_a) &= 10h^{1/2} + w_a \\ \frac{\partial u_a(h, w_a)}{\partial h} &= 5h^{-1/2} > 0 \ \text{ for } 0 < h \leq 50 \end{aligned}$$

Hence u_a is strictly increasing in h for $0 < h \leq 50$. Therefore the optimal h for a is $h = 0$.

Pareto Optimal Allocation,

Since we are considering quasi-linear utility functions we can sum them in order to solve for the pareto optimal solution,

$$\begin{aligned} max\ u_a(h, w_a) + u_b(h, w_b) &= 10h^{1/2} + w_a + 10(50 - h)^{1/2} + w_b \\ \frac{\partial}{\partial h}[u_a(h, w_a) + u_b(h, w_b)] &= 5h^{-1/2} - 5(50 - h)^{-1/2} = 0 \\ h^{-1/2} &= (50 - h)^{-1/2} \\ h &= 25 \end{aligned}$$

Consumer b Holds Rights

Suppose b has rights to no externality, hence a must pay b at some price p_h in order to h.

For consumer a,

$$\begin{aligned} max\ u_a(h_a, w_a, p_h) &= 10h_a^{1/2} + w_a - p_h h_a \\ \frac{\partial u_a(h_a, w_a, p_h)}{\partial h} &= 5h_a^{-1/2} - p_h = 0 \\ p_h &= 5h_a^{-1/2} \end{aligned}$$

For consumer b,

$$\begin{aligned} max\ u_b(h_b, w_b, p_h) &= 10(50 - h_b)^{1/2} + w_b + p_h h_b \\ \frac{\partial u_b(h_b, w_b, p_h)}{\partial h} &= -5(50 - h_b)^{-1/2} + p_h = 0 \\ p_h &= 5(50 - h_b)^{-1/2} \end{aligned}$$

At equilibrium $h_a = h_b$, hence,

$$\begin{aligned} 5h^{-1/2} = p_h &= 5(50 - h)^{-1/2} \\ h^{-1/2} &= (50 - h)^{-1/2} \\ 2h &= 50 \\ h &= 25 \end{aligned}$$

Giving the equilibrium allocation $h = 25$ and price $p_h = 1$.

Example: Externalities (Taxes & Quotas)

A profit-maximising firm is deciding how much pollution to emit. Its profit depends on the level of emission h, and the amount of money w_f it has: its profit function is $\pi(h) = w_f + 24h - 2h^2$. The pollution affects a consumer living nearby. The consumer's payoff depends on the level of emission h, a parameter θ capturing how badly the pollution affects her, and the amount of money w_c she has: her preferences are represented by the utility function $\phi(h;\theta) = w_c + 6\theta h - h^2$.

(a) What level of pollution will the firm choose to emit?

(b) Show that the Pareto optimal level of pollution is $4 - \theta$.

(c) Obtain an expression for the optimal Pigouvian tax as a function of θ.

Firm's Optimal Pollution Emissions,

$$\begin{aligned}
max\ \pi(h) &= w_f + 24h - 2h^2 \\
\frac{\partial \pi(h)}{\partial h} &= 24 - 4h = 0 \\
h &= 0
\end{aligned}$$

Pareto Optimal Pollution,

Since we are considering quasi-linear utility functions we can sum them in order to solve for the pareto optimal solution,

$$\begin{aligned}
max\ \pi(h) + \phi(h;\theta) &= w_f + 24h - 2h^2 + w_c - 6\theta h - h^2 \\
\frac{\partial}{\partial h}[\pi(h) + \phi(h;\theta)] &= 24 - 4h - 6\theta - 2h = 0 \\
6h &= 6(4 - \theta) \\
h &= 4 - \theta
\end{aligned}$$

Pigouvian Tax,

$$\begin{aligned}
max\ \pi_2(h) &= w_f + 24h - 2h^2 - th \\
\frac{\partial \pi_2(h)}{\partial h} &= = 24 - 4h - t = 0 \\
t &= 24 - 4h \ \ \text{(recall } h^* = 4 - \theta) \\
t &= 24 - 4(4 - \theta) \\
t &= 8 + 4\theta
\end{aligned}$$

Game Theory

Core Concepts

Defining a Game

- n players $i = 1, 2, ..., n$
- Payoff function $u_i(s_i, s_{-i})$ that indicates a players payoff (utility/profit/etc) that depend on her own strategy as well as the strategy of other players.

Strategies

- A **(pure) strategy** for player i is a plan of action, denoted s_i
- A **mixed strategy** for player i is a probability distribution over i's set of possible actions.
- A **strategy profile** $(s_1, s_2, ..., s_n)$ specifies a strategy for each player $i = 1, 2, ..., n$.
- It will be useful to define the strategy profile of all players except i as,

$$s_{-i} = (s_1, ..., s_{i-1}, s_{i+1}, ..., s_n)$$

- A **best response** strategy to s_{-i} for player i is a strategy s_i that maximises her payoff function $u_i(s_i, s_{-i})$, such that,

$$\forall s_i^{'} \;\; u_i(s_i, s_{-i}) \geq u_i(s_i^{'}, s_{-i})$$

- A **(strictly) dominant strategy** for player i is a (strict) best response to every strategy profile s_{-i} of the other players, such that,

$$\forall s_{-i} \text{ and } \forall s_i^{'} \neq s_i \;\; u_i(s_i, s_{-i}) \geq u_i(s_i^{'}, s_{-i})$$

Dominant Strategy Equilibria

- The strategy profile $s*$ is a dominant strategy equilibrium if, for every player i,

$$u_i(s_i^*, s_{-i}) \geq u_i(s_i, s_{-i}) \;\; \forall s = (s_i, s_{-i})$$

- Not all games have a dominant strategy equilibrium.

Nash Equilibrium

- A **Nash equilibrium** is a strategy profile $s = (s_1, s_2, ..., s_n)$ such that each player's strategy is a best response to the strategies of the other players. That is, $s = (s_1, s_2, ..., s_n)$ is a Nash equilibrium if *for each player* $i = 1, 2, ..., n$,

$$u_i(s_i, s_{-i}) \geq u_i(s_i^{'}, s_{-i}) \;\; \forall s_i^{'}$$

- At a Nash equilibrium no player can do better by changing her strategy.
- A game can have multiple pure strategy Nash equilibria.
- A **Nash equilibrium in mixed strategies** is when player i chooses actions with some probability and the other players best respond with some probability.

Is Nash equilibrium a sensible concept?

- Rational to play your best response, once at a Nash equilibrium there is no incentive to deviate from it.

Existence of Nash equilibrium?

- As long as finite players with continuous payoff functions and either each player has finite actions or actions are in a closed interval, then the game has a Nash equilibrium.

Why Nash equilibrium?

- Players learn to play them (can predict how others will play and then best respond as such (or randomise)).
- Nash equilibria are stable solutions.

Useful/Commonplace Games

Prisoner's Dilemma

	No Ads	Ads
No Ads	50, 50	20, 60
Ads	60, 20	40, 40

- Advertising only increases market share if the other company doesn't advertise (assuming that adverts are equally good).
- Nash equilibrium is (Ad, Ad), even though pareto optimal strategy is (No Ad, No Ad).

Battle of the Sexes

	B	F
B	2, 1	0, 0
F	0, 0	1, 2

- Here player 1 and 2 choose between B, F, where player one prefers B, player 2 prefers F, but they both prefer going somewhere together than going alone.
- Pure strategy Nash equilibria are (B, B) and (F, F) - if any player plays B or F, the other player cannot do better than play the same, and in the case in which both play the same action, neither can improve her position.
- Mixed strategy Nash equilibrium to play (B with probability 1/3, F with probability 2/3) for player 2 and (B with probability 2/3, F with probability 1/3) for player 1.
  - Player 1 plays B with p_1 to make Player 2 indifferent
$$u_2(B,\ p_1) = u_2(F,\ p_1)$$
$$\begin{aligned} 1(p_1) + 0(1-p_1) &= 0(p_1) + 2(1-p_1) \\ 3p_1 &= 2 \\ p_1 &= \frac{2}{3} \end{aligned}$$
  - Player 2 plays B with p_2 to make Player 1 indifferent
$$u_1(B,\ p_2) = u_1(F,\ p_2)$$
$$\begin{aligned} 2(p_2) + 0(1-p_2) &= 0(p_2) + 1(1-p_2) \\ 3p_2 &= 1 \\ p_2 &= \frac{1}{3} \end{aligned}$$
- Why randomise like this?
  - Player 2 needs to make player 1 indifferent between playing B and F in order for player 1 to actually randomise. If player 1 is not indifferent she can increase utility by playing which ever strategy she prefers with probability 1.

Matching Pennies

	H	T
H	-1, 1	1, -1
T	1, -1	-1, 1

- No pure strategy Nash equilibrium - if Column plays H, Row will play T, but if Row plays T, Column should play T, if Column plays T, however, then Row should play H, and if Row plays H then so should Column. . .
- Mixed strategy Nash equilibrium where player 1 plays (H with $\mathbb{P} = 1/2$, T with $\mathbb{P} = 1/2$) and player 2 plays (H with $\mathbb{P} = 1/2$, T with $\mathbb{P} = 1/2$).
  - Player 1 plays B with p_1 to make Player 2 indifferent

$$u_2(H,\ p_1) = u_2(T,\ p_1)$$

$$\begin{aligned} 1(p_1) - 1(1-p_1) &= -1(p_1) + 1(1-p_1) \\ 2p_1 - 1 &= 1 - 2p_1 \\ p_1 &= \frac{1}{2} \end{aligned}$$

  - Player 2 plays B with p_2 to make Player 1 indifferent

$$u_1(H,\ p_2) = u_1(T,\ p_2)$$

$$\begin{aligned} -1(p_2) + 1(1-p_2) &= 1(p_2) - 1(1-p_2) \\ 1 - 2p_2 &= 2p_2 - 1 \\ p_2 &= \frac{1}{2} \end{aligned}$$

Location Game

$$A_i = [0, 1]$$

$$u_i(a_1, a_2) = \begin{cases} \frac{1}{2}(a_1 + a_1) & \text{if } a_i < a_j \\ \frac{1}{2} & \text{if } a_i = a_j \\ 1 - \frac{1}{2}(a_1 + a_2) & \text{if } a_i > a_j \end{cases}$$

- Inhabitants are distributed uniformly on a beach, modeled as the interval [0,1]. On said beach there are two ice-cream sellers who can select a location on the beach.
- Inhabitants will only buy from the nearest seller.
- The games Nash equilibrium is at both vendors placing their carts exactly half way along the beach ($a_1 = \frac{1}{2}$ and $a_2 = \frac{1}{2}$).
- Why?
  - Imagine that both sellers set up either side of the point 3/4 on the line [0,1] then player 1 who has the lowers value (all less than 3/4) gets (3/4+3/4)1/2=3/4 utility and player 2 gets 1-3/4 which is ¼ utility.

  - Hence players 2 can improve by moving towards the centre.
  - Also once both players are at the centre, say player 2 moves right she now gets all the customers to the right of her but only half of the customers in between her and player 1 (less than before).
  - In a similar way if player 2 moves to the left of the centre line she gets all the customers below her but only half of the customers between her and player 1 (less than before) hence in both situations given that player 1 is at ½ it is not optimal for player 2 to move.

- Hence 1/2 is the highest payoff that any player can guarantee herself.
- When the sellers choose different locations deviations towards the other seller always improves the payoff.
- Not sure if I have explained this game well, but the video linked here does a very good job: TED-Ed: Why do competitors open their stories next to one another?

Dynamic Games

- In previous games players choose their actions simultaneously/without observing others' choices. Now we will consider sets of games in which players observe other players' actions before they must make their own choices.
- We represent such games in **extensive form**.
- A **subgame perfect equilibrium** is a Nash equilibrium which induces a Nash equilibrium in each subgame (it is in the set of Nash equilibrium, hence SPE is a refinement of Nash equilibrium).

Credibility and Commitment

- Players may be able to take pre-emptive actions to deter actions even if they do not 'play first' in the game.
- For example suppose in a Cournot market the incumbent pre-emptively increases capital, which is an irreversible sunk cost (hence credible). This lowers its mc and hence its BR curve shifts right, meaning the profit of a new entrant would be lower.

For example,

- Consider the case in which firm E can choose to enter or not enter a market and firm I can retaliate either by fighting against firm E or accommodating them into the market.
- If we first consider the game in matrix notation we find two (pure) Nash equilibrium at (Enter, accommodate if enter) and (Don't enter, Fight if enter).
- When we consider the game sequentially, however, (Don't enter, fight if enter) is not actually a reasonable prediction. If E chooses to enter then it is always in I's interest to accommodate, hence E should always enter and get the payoff of one rather than zero.
- Givens E's first mover advantage the threat by I to fight is not credible. It is never in I's interest to stick with that threat once E chooses to enter the market.

Finitely vs Infinitely repeated games

Finite games

- Imagine a Bertrand style competition game in which there is a total of 10,000 consumers, 1000 of which are loyal and 8000 of which are switchers (will choose the best price).

	Low	High
Low	20K, 20K	36K, 5K
High	5K, 36K	25K, 25K

- Nash equilibrium (Low, Low), but both firms do better if they can collude and maintain a high price – although there is incentive to deviate.
- If the game is played for a finite (but potentially large) number of periods we can solve using backward induction.
- In the last period the final subgame is exactly a one-shot prisoner's dilemma, hence both firms choose Low.
- In the penultimate period both players now they will defect next period, hence there is no credible threat of future punishment to stop them from defecting now, hence again they choose Low.
- This can be repeated until the first subgame.
- SPE = (Low, Low) in each game.

Infinitely Repeated Games

- Imagine the game is repeated an infinite number of times with the discount factor , which represents how the players weight the future against the present (they care less about the future than the present.
- A **trigger strategy** is one of the form: 'Play X as long as no player chooses Y, in which case play Z (which could be the same as Y)'.
- A **grim trigger strategy** is one of the form: 'once play of Z is triggered, play Z forever'

	Low	High
Low	20K, 20K	36K, 5K
High	5K, 36K	25K, 25K

- Hence we can calculate the present values of different outcomes:

$$PV_{coop} = 25000 + 25000\delta + 25000\delta^2 + ...$$

$$PV_{coop} = \frac{25000}{1-\delta}$$

$$PV_{cheat} = 36000 + 20000\delta + 20000\delta^2 + ...$$

$$PV_{cheat} = 36000 + \frac{20000}{1-\delta}$$

- Collusion can be sustained when,

$$PV_{coop} > PV_{cheat}$$

$$\frac{25000}{1-\delta} > 36000 + \frac{20000}{1-\delta}$$

$$\delta > \frac{11}{16}$$

- Grim trigger strategies can sustain collusive outcomes when,
  - Firms are sufficiently patient (δ is close to 1)
  - Temptation is not too high (36,000 vs 25,000)
  - The gains from long term collusion are sufficiently large (25,000 vs 20,000)

General Analysis

- A payoff pair is **feasible** if there is a pair of strategies that generate it
- A **minmax punishment** is the worst that one player can do to the other, given the other is responding optimally. When j optimally responds to i's minmax punishment she receives her minmax payoff.
- **Folk theorem** says that any feasible pair which gives each player at least her minmax payoff can be supported as a Nash equilibrium of an infinitely repeated game if players are sufficiently patient, etc.
- **SPE Folk theorem** says that any feasible payoff pair which gives each player at least her stage game equilibrium payoff can be supported as a subgame perfect equilibrium of an infinitely repeated if players are sufficiently patient, etc.

Worked Examples

Example: Pure & Mixed Strategies

Consider the following game:

	Up	Down
Left	5, 3	1, 1
Right	0, 0	3, 5

(a) Identify the Nash Equilibria in pure strategies.

(b) Is there a Nash Equilibrum in mixed strategies, where Column plays *Right* with probability p and Row plays *Up* with probability q? If so, what is each player's expected payoff? Is this better or worse than a player's average payoff across the two pure-strategy Nash Equilibria? Why?

Pure Strategy Nash Equilibrium:

(Up, Left) and (Down, Right)

Mixed Strategy Nash Equilibrium:

- There are two methods that we can use to solve for the mixed strategy Nash equilibrium:
  - The Indifference method, and
  - The Maximising Expected Payoff method.

(1) Indifference Method

- Row plays *Up* with q to make Column indifferent,

$$u_{row}(\text{Left},\ q) = u_{row}(\text{Right},\ q)$$

$$\begin{aligned} 3(q) + 0(1-q) &= 1(q) + 5(1-q) \\ 3q &= 5 - 4q \\ q &= \frac{5}{7} \end{aligned}$$

- Column plays *Right* with p to make Row indifferent,

$$u_{col}(\text{Up},\ p) = u_{col}(\text{Down},\ p)$$

$$\begin{aligned} 5(1-p) + 1(p) &= 0(1-p) + 3(p) \\ 5 - 4p &= 3p \\ p &= \frac{5}{7} \end{aligned}$$

- Hence the equilibrium:

$$(\text{Row plays Up with probability } \frac{5}{7}, \text{ Column plays Right with probability } \frac{5}{7})$$

(2) Maximisise Expected Payoffs Method

$$\begin{aligned} EP_{col} &= 3q(1-p) + 1qp + 0(1-q)(1-p) + 5(1-q)p \\ &= 3q - 3qp + 1qp + 5p - 5qp \\ &= 3q + 5p - 7qp \end{aligned}$$

- And,

$$\begin{aligned} EP_{row} &= 5q(1-p) + 1qp + 0(1-q)(1-p) + 3(1-q)p \\ &= 5q - 5qp + 1qp + 3p - 3qp \\ &= 5q + 3p - 7qp \end{aligned}$$

- Then notice that *Row* chooses q and *Column* chooses p, hence they will maximise their payoffs with respect to only the parameters they can choose. This will then return the optimal strategy for the other player.

$$\begin{aligned} \frac{\partial EP_{col}}{\partial p} &= -7q + 5 = 0 \\ q &= \frac{5}{7} \\ \frac{\partial EP_{row}}{\partial q} &= 5 - 7p = 0 \\ p &= \frac{5}{7} \end{aligned}$$

- Hence the equilibrium:

$$(\text{Row plays Up with probability } \frac{5}{7}, \text{Column plays Right with probability } \frac{5}{7})$$

Expected Payoffs:

$$EP_{col} = 3q + 5p - 7qp$$

$$\begin{aligned} &= 3\frac{5}{7} + 5\frac{5}{7} - 7\frac{5}{7}\frac{5}{7} \\ &= \frac{15}{7} \end{aligned}$$

$$EP_{row} = 5q + 3p - 7qp$$

$$\begin{aligned} &= 5\frac{5}{7} + 3\frac{5}{7} - 7\frac{5}{7}\frac{5}{7} \\ &= \frac{15}{7} \end{aligned}$$

Example: Repeated Game & Folk Theorem

Consider the following Prisoner's Dilemma one-shot game,

	C	D
C	5, 5	1, 8
D	8, 1	4, 4

Where C stands for cooperate, and D for defect.

(a) What is the unique Nash equilibrium?

(b) The players have the same discount factor δ, and the game is repeated n times. Find a Nash equilibrium of this repeated game. Are there any others? If not, why not. If there are, what are they?

(c) Suppose instead that the game is repeated an infinite number of times. Show that when the players are sufficiently patient, the following trigger strategies constitute a Nash equilibrium: player 1 plays C in the first period and continues to do so until player 2 plays D, in which case player 1 switches to D forever; player 2 plays C in the first period and continues to do so until player 1 plays D, in which case player 2 switches to D forever. What is the critical level of δ? Do the above strategies constitute a subgame perfect equilibrium? Why, or why not?

(d) What is a 'minmax punishment'? What is meant by saying that a strategy is 'individually rational'? What is meant by saying that a payoff pair is 'feasible'? What does the Folk Theorem have to say about infinitely repeated games, such as the one above?

Nash Equilibrium,

$$(D, D)$$

Finitely Repeated Game,

- SPE of each subgame is the Nash equilibrium.
- Players will play (D, D) in the final stage, hence will play the unique Nash equilibrium in the penultimate stage, and the one before that...
- Players play (D, D) in all stages.

Infinitely Repeated Game with Grim Trigger,

- If they cooperate and play (C, C) then the payoff for each player is 5. If one player cheats in the round she cheats she gets 8, but after that the grim trigger induces the outcome (D, D) and hence a payoff of 4.

$$PV_{coop} = 5 + 5\delta + 5\delta^2 + ...$$
$$PV_{coop} = \frac{5}{1-\delta}$$

$$PV_{cheat} = 8 + 4\delta + 4\delta^2 + ...$$
$$PV_{cheat} = 8 + \frac{4\delta}{1-\delta}$$

- Cooperation is sustained when the present value of cooperating (PV_{coop}) is greater than or equal to the present value of deviating (PV_{cheat}), hence,

$$
\begin{aligned}
PV_{coop} &\geq PV_{cheat} \\
\frac{5}{1-\delta} &\geq 8 + \frac{4\delta}{1-\delta} \\
5 &\geq 8(1-\delta) + 4\delta \\
5 - 8 &\geq -4\delta \\
\delta &\geq \frac{3}{4}
\end{aligned}
$$

Minmax punishment

- Player 1 minmax punishes player 2 by giving player 2 the lowest possible payoff given that player 2 is best responding.

Folk Theorem

(1) Individually rational

- A strategy is 'individually rational' if it guarantees the player a payoff at least as good as her minmax payoff.

(2) Feasible

- Payoff pair is feasible if there is a pair of strategies, one for each player, that generates it

(3) Folk Theorem:

- Any feasible pair which gives each player at least her minmax payoff can be supported as a Nash equilibrium of an infinitely repeated game if players are sufficiently patient, etc.

Industrial Organisation

Oligopolistic markets are made up of a ***small number of firms*** each with ***significant market power***. In such a market each firms profit depends on its own behaviour as well as the behaviour of other firms. Hence interactions are strategic. In the models below we consider the strategic interactions of two firms: $i = 1, 2$.

Basic Structure

Cost functions:

$$C_i = c_i q_i$$

- In the case in which the firms are identical then we assume that $c_1 = c_2 = c$

Inverse Demand functions:

$$p_i = \alpha_i - \beta_i q_i - \gamma q_j \text{ where } i \neq j$$

- Further $\gamma > 0$ since the goods are assumed to be substitutes - that is as firm j increases its output it drives down the price for firm i.
- In the case in which the goods are perfect substitutes then $\alpha_1 = \alpha_2 = \alpha$ and $\beta_1 = \beta_2 = \gamma$ hence total demand is given by,

$$p = \alpha - \gamma(q_i + q_j)$$

Profit (Quantity):

$$\begin{aligned}
\pi_i(q_i, q_j) &= p_i(q_i, q_j) \cdot q_i - c_i q_i \\
\pi_i(q_i, q_j) &= (\alpha_i - \beta_i q_i - \gamma q_j) q_i - c_i q_i \\
\pi_i(q_i, q_j) &= (\alpha_i - c_i - \gamma q_j) q_i - \beta_i q_i^2
\end{aligned}$$

- Notice that, for a given q_j this function is strictly concave in q_i. We are therefore able to find a unique global maximum in q_i for a given level of q_j,

$$\begin{aligned}
\frac{\partial \pi_i}{\partial q_i} &= (\alpha_i - c_i - \gamma q_j) - 2\beta_i q_i \\
0 &= (\alpha_i - c_i - \gamma q_j) - 2\beta_i q_i \\
q_i &= \frac{\alpha_i - c_i - \gamma q_j}{2\beta_i} \\
BR_i(q_j) &= \frac{\alpha_i - c_i - \gamma q_j}{2\beta_i}
\end{aligned}$$

Demand functions:

$$q_i = a_i - b_i p_i + \phi p_j \text{ where } i \neq j$$

- It will also be useful for Bertrand competition to write quantity as a function of price. That is $q_i(p_i, p_j)$ rather than $p_i(q_i, q_j)$.
- Notice that had we used the same parameters as we did for inverse demand, we could have made a substitution for q_j and solved for q_i to write the demand function as,

$$\begin{aligned}
q_i &= \frac{\gamma(p_j - \alpha_j) + \beta_j(\alpha_i - p_i)}{\beta_i \beta_j - \gamma^2} \\
q_i &= \frac{\alpha_i \beta_j - \alpha_j \gamma}{\beta_i \beta_j - \gamma^2} - \frac{\beta_j}{\beta_i \beta_j - \gamma^2} p_i + \frac{\gamma}{\beta_i \beta_j - \gamma^2} p_j
\end{aligned}$$

Profit (Prices):

$$\begin{aligned}
\pi_i(p_i,p_j) &= p_i \cdot q_i(p_i,p_j) - c_i \cdot q_i(p_i,p_j) \\
\pi_i(p_i,p_j) &= p_i(a_i - b_ip_i + \phi p_j) - c_i(a_i - b_ip_i + \phi p_j) \\
\pi_i(p_i,p_j) &= p_i(a_i + \phi p_j) - c_i(a_i + \phi p_j) + b_ic_ip_i - b_ip_i^2
\end{aligned}$$

- Again solving for the FOC,

$$\begin{aligned}
\frac{\partial \pi_i}{\partial p_i} &= (a_i + \phi p_j) + b_ic_i - 2b_ip_i \\
0 &= a_i + \phi p_j + b_ic_i - 2b_ip_i \\
p_i &= \frac{a_i + b_ic_i + \phi p_j}{2b_i} \\
BR_i(p_j) &= \frac{a_i + b_ic_i + \phi p_j}{2b_i}
\end{aligned}$$

Differentiation/Homogeneity

- We either consider the goods to be homogeneous, or we consider them to be ***horizontally differentiated***. That is the goods differ in some dimension other than quality. In this case the goods differ marginally, and importantly subjectively such that some consumers purchase goods from one firm, while others purchase goods from the other. Were the goods ***vertically differentiated***, that is differentiated objectively on quality, then when sold at the same price everyone would buy from the higher quality producer, hence the market would not be oligopolistic.
- For examples of this think of Coca Cola vs Pepsi as horizontally differentiated products - some people prefer Coke to Pepsi and vice-versa. For £1 a can people would buy some would be one and others the other. For vertically differentiated products think of a 2009 model Audi vs a 2022 model Audi. At the same price everyone would buy the newer - better quality - car.

Cournot Competition

Cournot-Nash Equilibrium

Firms simultaneously decide what **output** to produce.

$$BR_i(q_j) = \frac{\alpha_i - c_i - \gamma q_j}{2\beta_i} \quad \text{for } i = 1, 2$$

The intersection of these best response functions gives,

$$\begin{aligned}
q_i^c &= \frac{\alpha_i - c_i - \gamma q_j^c}{2\beta_i} \\
&= \frac{\alpha_i - c_i - \gamma \frac{\alpha_j - c_j - \gamma q_i^c}{2\beta_j}}{2\beta_i} \\
&= \frac{\frac{2\beta_j(\alpha_i - c_i) - \gamma(\alpha_j - c_j - \gamma q_i^c)}{2\beta_j}}{2\beta_i} \\
&= \frac{2\beta_j(\alpha_i - c_i) - \gamma(\alpha_j - c_j - \gamma q_i^c)}{4\beta_j\beta_i} \\
&= \frac{2\beta_j(\alpha_i - c_i) - \gamma(\alpha_j - c_j) + \gamma^2 q_i^c}{4\beta_j\beta_i} \\
&= \frac{2\beta_j(\alpha_i - c_i) - \gamma(\alpha_j - c_j)}{4\beta_j\beta_i} + \frac{\gamma^2 q_i^c}{4\beta_j\beta_i} \\
q_i^c(1 - \frac{\gamma^2}{4\beta_j\beta_i}) &= \frac{2\beta_j(\alpha_i - c_i) - \gamma(\alpha_j - c_j)}{4\beta_j\beta_i} \\
q_i^c &= \frac{\frac{2\beta_j(\alpha_i - c_i) - \gamma(\alpha_j - c_j)}{4\beta_j\beta_i}}{1 - \frac{\gamma^2}{4\beta_j\beta_i}}
\end{aligned}$$

Hence finally,

$$q_i^c = \frac{2\beta_j(\alpha_i - c_i) - \gamma(\alpha_j - c_j)}{4\beta_j\beta_i - \gamma^2}$$

In the case in which outputs are homogeneous and the firms are identical, that is $\alpha_1 = \alpha_2 = \alpha$; $\beta_1 = \beta_2 = \gamma$; and $c_1 = c_2 = c$, then,

$$\begin{aligned}
q_i^c &= \frac{2\beta_j(\alpha_i - c_i) - \gamma(\alpha_j - c_j)}{4\beta_j\beta_i - \gamma^2} \\
&= \frac{2\gamma(\alpha - c) - \gamma(\alpha - c)}{4\gamma^2 - \gamma^2} \\
&= \frac{\gamma(\alpha - c)}{3\gamma^2} \\
q_i^c &= \frac{(\alpha - c)}{3\gamma}
\end{aligned}$$

Hence it is the case (due to symmetry of firms) that,

$$q_1^c = q_2^c = \frac{(\alpha - c)}{3\gamma}$$

How does this outcome compare with the perfectly competitive and monopolistic outcomes?

(1) For perfect competition we know that $p = mc$,

In the differentiated case,

$$\begin{aligned} p_i = \alpha_i - \beta_i q_i^{pc} - \gamma q_j^{pc} &= c_i \\ q_i^{pc} &= \frac{\alpha_i - c_i - \gamma q_j^{pc}}{\beta_i} \\ &= \frac{\alpha_i - c_i - \gamma \frac{\alpha_j - c_j - \gamma q_i^{pc}}{\beta_j}}{\beta_i} \\ &= \frac{\beta_j(\alpha_i - c_i) - \gamma(\alpha_j - c_j) + \gamma^2 q_i^{pc}}{\beta_j \beta_i} \\ q_i^{pc}(1 - \frac{\gamma^2}{\beta_i \beta_j}) &= \frac{\beta_j(\alpha_i - c_i) - \gamma(\alpha_j - c_j)}{\beta_j \beta_i} \\ q_i^{pc} &= \frac{\beta_j(\alpha_i - c_i) - \gamma(\alpha_j - c_j)}{\beta_j \beta_i - \gamma^2} \end{aligned}$$

And in the homogeneous case,

$$\begin{aligned} p = \alpha - \gamma(q_i^{pc} + q_j^{pc}) &= c \\ q_i^{pc} + q_j^{pc} &= \frac{\alpha - c}{\gamma} \end{aligned}$$

(2) For the monopolistic case we imagine the firms combine into one firm and maximise $\pi_1 + \pi_2$, with the FOC,

In the differentiated case,

$$\begin{aligned} 0 &= \alpha_i - c_i - 2\gamma q_j^m - 2\beta_i q_i^m \\ q_i^m &= \frac{\alpha_i - c_i - 2\gamma q_j^m}{2\beta_i} \\ q_i^m &= \frac{\alpha_i - c_i - 2\gamma(\frac{\alpha_j - c_j - 2\gamma q_i^m}{2\beta_j})}{2\beta_i} \\ q_i^m &= \frac{\beta_j(\alpha_i - c_i) - \gamma(\alpha_j - c_j)}{2\beta_j \beta_i - 2\gamma^2} \\ q_i^m &= \frac{\beta_j(\alpha_i - c_i) - \gamma(\alpha_j - c_j)}{2(\beta_j \beta_i - \gamma^2)} \end{aligned}$$

And in the homogeneous case,

$$\begin{aligned} 0 &= \alpha - c - 2\gamma(q_j^m + q_i^m) \\ q_j^m + q_i^m &= \frac{\alpha - c}{2\gamma} \end{aligned}$$

	Differentiated	Homogeneous
Cournot-Nash	$q_i^c = \frac{2\beta_j(\alpha_i - c_i) - \gamma(\alpha_j - c_j)}{4\beta_j \beta_i - \gamma^2}$	$q_i^c = \frac{\alpha - c}{3\gamma}$
Perfect Competition	$q_i^{pc} = \frac{\beta_j(\alpha_i - c_i) - \gamma(\alpha_j - c_j)}{\beta_j \beta_i - \gamma^2}$	$q_i^{pc} + q_j^{pc} = \frac{\alpha - c}{\gamma}$
Monopoly	$q_i^m = \frac{\beta_j(\alpha_i - c_i) - \gamma(\alpha_j - c_j)}{2(\beta_j \beta_i - \gamma^2)}$	$q_j^m + q_i^m = \frac{\alpha - c}{2\gamma}$

In the differentiated case,

$$q_i^m < q_i^c < q_i^{pc}$$

Hence Cournot returns a greater output (and lower price) than the monopolist, but less output (and a higher price) than the perfectly competitive outcome.

If we sum $q_i^c + q_j^c$ to get $\frac{2(\alpha-c)}{3\gamma}$ we can also see in the homogeneous case that,

$$\frac{\alpha - c}{2\gamma} < \frac{2(\alpha - c)}{3\gamma} < \frac{\alpha - c}{\gamma}$$

Our conclusion is as we would expect. The Oligopolist's do better in the Cournot-Nash equilibrium than they would do in the Perfectly Competitive one, but if they could collude to the monopolistic quantity they would do even better. Of course collusion is not always easy to sustain...

Stackleberg Competition

Firms still make output decisions, but now firm 1 **announces her output first**, firm 2 then follows. We call firm 1 the market leader, and once her output has been announced it cannot be changed.

Firm 1 knows that for any given q_1 firm 2 will best respond, hence firm 1 needs to optimise her own profit given that she knows that firm 2 will best respond. That is,

$$\max \pi_1(q_1, q_2) \text{ such that } q_2 = \frac{\alpha_2 - c_2 - \gamma q_1}{2\beta_2}$$

$$\pi_1(q_1, q_2) = (\alpha_1 - c_1 - \gamma q_2)q_1 - \beta_1 q_1^2$$

We can use the Lagrangian,

$$L = (\alpha_1 - c_1 - \gamma q_2)q_1 - \beta_1 q_1^2 - \lambda(\alpha_2 - c_2 - \gamma q_1 - 2\beta_2 q_2)$$

Which has the FOCs,

$$\begin{aligned} L_{q_1} &= (\alpha_1 - c_1 - \gamma q_2) - 2\beta_1 q_1 + \lambda\gamma = 0 \\ L_{q_2} &= -\gamma q_1 + 2\lambda\beta_2 = 0 \\ L_\lambda &= \alpha_2 - c_2 - \gamma q_1 - 2\beta_2 q_2 = 0 \end{aligned}$$

(The reason firm 1 is optimising with respect to *both* q_1 and q_2 is that firm 1 chooses q_1 directly, but further given that firm 1 knows that firm 2 will best respond she can indirectly choose q_2, since she knows the value of q_2 that will be chosen for each q_1 she chooses. In this sense then firm 1 is in fact choosing q_2 as well as q_1).

We can solve these FOCs and find,

$$\begin{aligned} q_1^s &= \frac{4\beta_2(\alpha_1 - c_1) - \gamma(\alpha_2 - c_2)}{4\beta_1\beta_2 - 2\gamma^2} \\ q_1^s &= \frac{\alpha - c}{2\gamma}\ ,\ q_2^s = \frac{\alpha - c}{4\gamma}, \end{aligned}$$

In the differentiated and homogeneous cases respectively.

The outcome is that, while firm 1's profit is greater than in the Cournot-Nash equilibrium, total profit is actually lower. I show this only for the homogeneous case,

$$\begin{aligned} \pi_1(q_1^s, q_2^s) &= (\alpha - c - \gamma\frac{\alpha - c}{4\gamma})\frac{\alpha - c}{2\gamma} - \gamma(\frac{\alpha - c}{2\gamma})^2 \\ &= (\alpha - c - \frac{(\alpha - c)}{4})\frac{\alpha - c}{2\gamma} - \gamma\frac{(\alpha - c)^2}{4\gamma^2} \\ &= \frac{(\alpha - c)^2}{2\gamma} - \frac{(\alpha - c)^2}{8\gamma} - \frac{(\alpha - c)^2}{4\gamma} \\ &= \frac{(\alpha - c)^2}{8\gamma} \\ \pi_2(q_1^s, q_2^s) &= (\alpha - c - \gamma\frac{\alpha - c}{2\gamma})\frac{\alpha - c}{4\gamma} - \gamma\frac{(\alpha - c)^2}{16\gamma^2} \\ &= \frac{(\alpha - c)^2}{4\gamma} - \frac{(\alpha - c)^2}{8\gamma} - \frac{(\alpha - c)^2}{16\gamma} \\ &= \frac{(\alpha - c)^2}{16\gamma} \end{aligned}$$

$$
\begin{aligned}
\pi_1(q_1^c, q_2^c) &= (\alpha - c - \gamma\frac{\alpha - c}{3\gamma})\frac{\alpha - c}{3\gamma} - \gamma\frac{(\alpha - c)^2}{9\gamma^2} \\
&= \frac{(\alpha - c)^2}{3\gamma} - \frac{2(\alpha - c)^2}{9\gamma} \\
&= \frac{(\alpha - c)^2}{9\gamma} \\
\pi_2(q_1^c, q_2^c) &= \frac{(\alpha - c)^2}{9\gamma}
\end{aligned}
$$

And I hope from all of these equations it is obvious enough that,

$$
\pi_1(q_1^s, q_2^s) = \frac{(\alpha - c)^2}{8\gamma} > \frac{(\alpha - c)^2}{9\gamma} = \pi_1(q_1^c, q_2^c)
$$
$$
\pi_1(q_1^c, q_2^c) + \pi_2(q_1^c, q_2^c) = \frac{2(\alpha - c)^2}{9\gamma} > \frac{3(\alpha - c)^2}{16\gamma} = \pi_1(q_1^s, q_2^s) + pi_2(q_1^s, q_2^s)
$$

Bertrand Competition

Firms simultaneously decide what **price** to sell at.

In the homogeneous case. That is in the case in which the goods are perfect substitutes; the equilibrium prices are,

$$p_1^b = p_2^b = c$$

This is the case since if $p_1 > p_2$ then, given the homogeneity of the products, all the consumers will buy the cheaper product and so $q_2 = Q$ and $q_1 = 0$, where $Q =$ total supply. Similarly if $p_2 > p_1$ then all the consumers will buy the cheaper product and so $q_1 = Q$ and $q_2 = 0$. Firms hence do best by slightly undercutting their competitors, that is setting the price $p_i = p_j - \epsilon$. This reasoning iterates down until $p_1 = p_2 = c$.

In the differentiated case,

$$BR_i(p_j) = \frac{a_i + b_i c_i + \phi p_j}{2b_i}$$

The intersection of these best response functions gives,

$$\begin{aligned}
p_i^b &= \frac{a_i + b_i c_i + \phi p_j^b}{2b_i} \\
&= \frac{a_i + b_i c_i + \phi \frac{a_j + b_j c_j + \phi p_i^b}{2b_j}}{2b_i} \\
&= \frac{2b_j(a_i + b_i c_i) + \phi(a_j + b_j c_j + \phi p_i^b)}{4b_j b_i} \\
&= \frac{2b_j(a_i + b_i c_i) + \phi(a_j + b_j c_j)}{4b_j b_i)} + \frac{\phi^2 p_i^b}{4b_j b_i} \\
p_i^b(1 - \frac{\phi^2}{4b_j b_i}) &= \frac{2b_j(a_i + b_i c_i) + \phi(a_j + b_j c_j)}{4b_j b_i} \\
p_i^b &= \frac{\frac{2b_j(a_i + b_i c_i) + \phi(a_j + b_j c_j)}{4b_j b_i}}{1 - \frac{\phi^2}{4b_j b_i}}
\end{aligned}$$

Hence finally,

$$p_i^b = \frac{2b_j(a_i + b_i c_i) + \phi(a_j + b_j c_j)}{4b_j b_i - \phi^2}$$

We can simplify this further while maintaining that the goods are not perfect substitutes by setting $a_1 = a_2 = a$; $b_1 = b_2 = b$; and $c_1 = c_2 = c$, but ensuring that $b \neq \phi$,

$$\begin{aligned}
p_i^b &= \frac{2b_j(a_i + b_i c_i) + \phi(a_j + b_j c_j)}{4b_j b_i - \phi^2} \\
p_i^b &= \frac{2b(a + bc) + \phi(a + bc)}{4b^2 - \phi^2} \\
p_i^b &= \frac{(a + bc)(2b + \phi)}{(2b - \phi)(2b + \phi)} \\
p_i^b &= \frac{a + bc}{2b - \phi}
\end{aligned}$$

Worked Examples

Example: Collusion

There are n firms in an industry and they have identical cost functions $c(q_k) = 2q_p$, $k = 1, 2, ..., n$; the inverse demand function in the industry is $p(q) = 10 - q$, where $q = \sum_{k=1}^{n} q_k$.

(a) If $n = 1$, i.e. there is a monopolist, what is the profit maximising quantity, q^M, what will be the resulting price, p^M, and how much profit, π^M, will the firm make in each period?

(b) *Price setting:* Each of them considers playing the following trigger strategy: charge p^M in each period (the resulting market quantity being determined by demand) as long as all the other firms have done so in the past; if any firm deviates, then charge c for evermore. How large does δ have to be for these trigger strategies to constitute a subgame perfect equilibrium in which collusion is sustained?

(c) *Quantity setting:* As an alternative, each of them considers playing the following trigger strategy: supply $\frac{q^M}{n}$ in each period (the resulting market price being determined by demand) as long as all the other firms have done so in the past; if any firm deviates, then supply the 'Cournot quantity' for evermore. When n = 2, how large does δ have to be for these trigger strategies to constitute a subgame perfect equilibrium in which collusion is sustained? What if n = 3?

(d) Hence show that when $n = 2$ there are values of δ for which collusion is sustainable in the price-setting game of part (b) but not in the quantity-setting game of part (c), and that when $n = 3$ this conclusion is reversed.

Monopolist Case: $n = 1$

$$max \;\; \pi(p, q) = (10 - q)p - 2q = 8q - q^2$$

- FOC,

$$\frac{\partial \pi(p, q)}{\partial q} = 8 - 2q^M$$
$$0 = 8 - 2q^M$$
$$q = 4^M$$

- Giving the solutions,

$$q^M = 4$$
$$p^M = 6$$
$$\pi(q^M, p^M) = 16$$

- SOC,

$$\frac{\partial^2 \pi(p, q)}{\partial q^2} = -2 < 0$$

- Hence we have a maximum.

Price Setting Case:

- Collusive Profit:
  $\pi_k = \frac{\pi(p^M)}{n}$ for all firms k.
- Deviating Profit
  $\pi_k = \pi(p^M - \epsilon) \approx \pi(p^M)$ for some very small ϵ, and just for the deviating firm k, other firms get 0 profit.

- Punishment Profit:
  $\pi_k(p = c) = 0$

- Sustaining Collusion:

$$\begin{aligned} PV_{coop} &> PV_{cheat} \\ \frac{\pi(p^M)}{n} + \delta\frac{\pi(p^M)}{n} + \delta^2\frac{\pi(p^M)}{n} + ... &> \pi(p^M) \\ \frac{\pi(p^M)}{n(1-\delta)} &> \pi(p^M) \\ \delta &> 1 - \frac{1}{n} \end{aligned}$$

- So as the number of firms increases, sustaining collusion becomes harder and harder. This seems like a very intuitive conclusion.

Quantity Setting Case:

- Collusive Profit:
  $\pi_k = \frac{\pi(q^M)}{n}$ for all firms k.
- Deviating Profit:
  Recall from an earlier part that $q^m = 4$. We start with the simple assumption that the deviating firm will set q_k and all other firms continue with the monopolistic quantity q^M,

$$\begin{aligned} Q &= q_k + \sum_{j=1,\ (j\neq k)}^{n-1} \frac{q^M}{n} \\ &= q_k + (n-1)\frac{4}{n} \\ &= q_k + 4(1 - \frac{1}{n}) \end{aligned}$$

  Now find the price as a function of q_k,

$$\begin{aligned} P(Q) &= 10 - Q \\ &= 10 - q_k - 4(1 - \frac{1}{}n) \\ &= 6 - q_k + \frac{4}{n} \end{aligned}$$

  Now we can maximise profit for the deviating firm k,

$$\begin{aligned} max\ \ \pi_k(q_k) &= (6 - q_k + \frac{4}{n})q_k - 2q_k \\ &= 4q_k - q_k^2 + \frac{4}{n}q_k \\ &= 4(1 + \frac{1}{n})q_k - q_k^2 \end{aligned}$$

  Giving the FOC,

$$\begin{aligned} \frac{\partial \pi_k(q_k)}{\partial q_k} &= 4(1 + \frac{1}{n}) - 2q_k^* \\ 0 &= 4(1 + \frac{1}{n}) - 2q_k^* \\ q_k^* &= 2(1 + \frac{1}{n}) \end{aligned}$$

Hence the deviating profit is,

$$\pi_k(q_k^*)) = 4(1+\frac{1}{n})2(1+\frac{1}{n}) - 4(1+\frac{1}{n})^2$$
$$\pi_k(q_k^*)) = 8(1+\frac{1}{n})^2 - 4(1+\frac{1}{n})^2$$
$$\pi_k(q_k^*)) = 4(1+\frac{1}{n})^2$$

- Punishment Profit:

  All firms are identical, and at Cournot-Nash equilibrium they are all best responding to one another. It will be useful to define for this solution Q_{-k}, which is the quantity supplied by all firms bar firm k.

$$Q = \sum_{j=1}^{n} q_j = q_k + Q_{-k} \text{ where } Q_{-k} = \sum_{j=1\ (j \neq k)}^{n-1} q_j$$

  Hence each firm solves,

$$\begin{aligned} max\ \ \pi_k(q_k) &= (10 - q_k - Q_{-k})q_k - 2q_k \\ &= (8 - Q_{-k})q_k - q_k^2 \end{aligned}$$

  FOC,

$$\begin{aligned} \frac{\partial \pi_k(q_k)}{\partial q_k} &= (8 - Q_{-k}) - 2q_k^c \\ 0 &= (8 - Q_{-k}) - 2q_k^c \\ q_k^c &= 4 - \frac{1}{2}Q_{-k} \end{aligned}$$

  Given that each firm is best responding to one another, and that $Q_{-k} = \sum_{j=1\ (j \neq k)}^{n-1} q_j$

$$\begin{aligned} Q_{-k}^c &= \sum_{j=1\ (j \neq k)}^{n-1} q_j^c = (n-1)q_k^c \\ q_k^c &= 4 - \frac{1}{2}(n-1)q_k^c \\ q_k^c &= \frac{8}{n+1} \\ Q^c &= \frac{8n}{n+1} \end{aligned}$$

  Finally finding the profit level,

$$\begin{aligned} \pi(q_k^c) &= (10 - \frac{8n}{n+1})\frac{8}{n+1} - 2\frac{n}{n+1} \\ &= \frac{64}{n+1} - \frac{64n}{(n+1)^2} \\ &= \frac{64}{(n+1)^2} \end{aligned}$$

- Sustaining Collusion:

$$\begin{aligned} PV_{coop} &> PV_{cheat} \\ \frac{16}{n} + \delta\frac{16}{n} + \delta^2\frac{16}{n} + ... &> 4(1+\frac{1}{n})^2 + \delta\frac{64}{(n+1)^2} + \delta^2\frac{64}{(n+1)^2} + ... \\ \frac{16}{n(1-\delta)} &> 4(1+\frac{1}{n})^2 + \frac{64\delta}{(n+1)^2(1-\delta)} \end{aligned}$$

Example: Horizontal/Vertical Differentiation and Merging

Suppose that two firms produce horizontally differentiated products. The firms compete for a single period and choose prices simultaneously. Each firm has the same constant marginal cost of production $c \geq 0$, and fixed costs are zero. The linear demand functions are given by,

$$q_1(p_1, p_2) = 100 - \alpha p_1 + \beta p_2$$
$$q_2(p_1, p_2) = 100 - \alpha p_2 + \beta p_1$$

Assume that $\alpha > \beta > 0$ and that $100 > \alpha c$.

Differentiation:

- Horizontal: Differentiated on preference
  - At the same price some people will buy one, other people will buy the other.
  - E.g. Coca Cola vs Pepsi.
- Vertical: Differentiated on objective quality
  - At the same price everyone will buy one over the other.
  - E.g. 2007 Volvo vs 2022 Volvo.

Bertrand-Nash Equilibrium:

- Best-Respond Functions:

$$\begin{aligned} max\ \ \pi_1(p_1, p_2) &= (p_1 - c)q_1 = (p_1 - c)(100 - \alpha p_1 + \beta p_2) \\ &= (100p_1 - \alpha p_1^2 + \beta p_2 p_1) - (100c - \alpha p_1 c + \beta p_2 c) \\ &= (100 + \beta p_2 + \alpha c)p_1 - (100 - \beta p_2)c - \alpha p_1^2 \end{aligned}$$

Hence FOCs,

$$\begin{aligned} \frac{\partial \pi_1(p_1, p_2)}{\partial p_1} &= 100 + \beta p_2 + \alpha c - 2\alpha p_1^b \\ 0 &= (100 + \beta p_2 + \alpha c) - 2\alpha p_1^b \\ p_1^b &= \frac{100 + \alpha c + \beta p_2}{2\alpha} \end{aligned}$$

And so by symmetry we know,

$$BR_1(p_2) = \frac{100 + \alpha c + \beta p_2}{2\alpha}$$
$$BR_2(p_1) = \frac{100 + \alpha c + \beta p_1}{2\alpha}$$

At equilibrium,

$$\begin{aligned} p_1^b &= \frac{100 + \alpha c + \beta(\frac{100 + \alpha c + \beta p_1^b}{2\alpha})}{2\alpha} \\ &= \frac{2\alpha(100 + \alpha c) + \beta(100 + \alpha c + \beta p_1^b)}{4\alpha^2} \\ &= \frac{(100 + \alpha c)(2\alpha + \beta)}{4\alpha^2} + \frac{\beta^2}{4\alpha^2} p_1^b \\ p_1^b(1 - \frac{\beta^2}{4\alpha^2}) &= \frac{(100 + \alpha c)(2\alpha + \beta)}{4\alpha^2} \\ p_1^b &= \frac{(100 + \alpha c)(2\alpha + \beta)}{4\alpha^2 - \beta^2} \end{aligned}$$

And again by symmetry we find,

$$p_1^b = \frac{(100+\alpha c)(2\alpha+\beta)}{(2\alpha+\beta)(2\alpha-\beta)}$$
$$p_1^b = p_2^b = \frac{100+\alpha c}{2\alpha-\beta}$$

- Quantities:

$$\begin{aligned} q_1^b = q_2^b &= 100 - \alpha\frac{100+\alpha c}{2\alpha-\beta} + \beta\frac{100+\alpha c}{2\alpha-\beta} \\ &= \frac{100(2\alpha-\beta) - \alpha(100+\alpha c) + \beta(100+\alpha c)}{2\alpha-\beta} \\ &= \frac{\alpha(100+\beta c-\alpha c)}{2\alpha-\beta} \end{aligned}$$

  Notice that this number must be positive. For the denominator, $\alpha > \beta$, and for the numerator $100 > \alpha c$, $\beta > 0$, and $c > 0$.

Monopolist:

$$\begin{aligned} max\ \ \pi_m(p) &= (p-c)2(100+p(\beta-\alpha)) \\ \frac{\partial \pi_m(p)}{\partial p} &= 200 + 4p^m(\beta-\alpha) - 2c(\beta-\alpha) \\ 0 &= 200 + 4p^m(\beta-\alpha) - 2c(\beta-\alpha) \\ p^m &= \frac{100+c(\alpha-\beta)}{2(\alpha-\beta)} \end{aligned}$$

Example: Envelope Theorem

Consider a duopoly, with two firms producing goods that are perfect substitutes; they play a quantity-setting game for only one period. The firms differ in their cost functions: the total cost to firm k of producing a quantity q_k is $c_k q_k$, for $k = 1, 2$. The inverse demand function in the market is,

$$p(q) = a - q \text{ where } q = q_1 + q_2$$

. Assume that the firms move simultaneously.

(a) Derive the best response function of each firm. Hence find the Nash equilibrium of the game.

(b) Show that a marginal change in its own cost parameter, c_1, affects its profit through three channels: (i) the direct effect of the change in its own costs, (ii) an indirect effect via the change in its own quantity, and (iii) a second indirect effect via the change in the other firm's quantity.

Best Responses and Nash Equilibrium:

Optimisation Problem

$$\begin{aligned} \pi_1(q_1, q_2) &= (a - q_1 - q_2)q_1 - c_1 q_1 \\ \frac{\partial \pi_1(q_1, q_2)}{q_1} &= a - 2q_1 - q_2 - c_1 \\ 0 &= a - 2q_1 - q_2 - c_1 \\ q_1 &= \frac{a - c_1 - q_2}{2} \end{aligned}$$

Best Response Functions,

$$\begin{aligned} BR_1(q_2) &= \frac{a - c_1 - q_2}{2} \\ BR_2(q_1) &= \frac{a - c_2 - q_1}{2} \end{aligned}$$

Solutions,

$$\begin{aligned} q_1^c &= \frac{a - c_1 - \frac{a - c_2 - q_1}{2}}{2} \\ q_1^c &= \frac{2(a - c_1) - (a - c_2) + q_1^c}{4} \\ q_1^c &= \frac{a - 2c_1 + c_2}{3} \\ q_2^c &= \frac{a - 2c_2 + c_1}{3} \end{aligned}$$

$$\begin{aligned} q^c = q_1^c + q_2^c &= \frac{2a - c_1 - c_2}{3} \\ p^c &= \frac{a + c_1 + c_2}{3} \end{aligned}$$

Firm 1's equilibrium profit

$$\pi_1(q_1^c, q_2^c, c_1) = (a - q_1^c - q_2^c - c_1)q_1^c \;\; , \; q_1^c = q_1^c(c_1) \;\; , \; q_2^c = q_2^c(c_1)$$

Marginal Change in cost parameters:

Now we consider how profit changes when c_1 changes. That is the change in profit wrt to c_1. We must consider **all** the effects c_1 has on profit. This includes both its direct effect on profit, $\frac{\partial \pi_1}{\partial c_1}$, but also the indirect effects it has on the optimal quantities, $\frac{\partial q_1^c}{\partial c_1}$ and $\frac{\partial q_2^c}{\partial c_1}$.

Accounting for all these changes using the Envelope Theorem,

$$\frac{d\pi_1}{dc_1} = \frac{\partial \pi_1}{\partial c_1} + \frac{\partial \pi_1}{\partial q_1^c}\frac{\partial q_1^c}{\partial c_1} + \frac{\partial \pi_1}{\partial q_2^c}\frac{\partial q_2^c}{\partial c_1}$$

Calculating individually, we know that π_1^c is optimised with respect to q_1, hence $\frac{\partial \pi_1}{\partial q_1^c} = 0$, further,

$$\begin{aligned}\frac{\partial \pi_1}{\partial c_1} &= -q_1^c \\ \frac{\partial \pi_1}{\partial q_2^c} &= -q_1^c \\ \frac{\partial q_2^c}{\partial c_1} &= \frac{1}{3}\end{aligned}$$

Hence overall,

$$\begin{aligned}\frac{d\pi_1}{dc_1} &= \frac{\partial \pi_1}{\partial c_1} + \frac{\partial \pi_1}{\partial q_1^c}\frac{\partial q_1^c}{\partial c_1} + \frac{\partial \pi_1}{\partial q_2^c}\frac{\partial q_2^c}{\partial c_1} \\ &= -q_1^c + 0 - \frac{1}{3}q_1^c \\ &= -\frac{4}{3}q_1^C\end{aligned}$$

For more on problems like this, specifically on the Implicit Function Theorem, Envelope Theorem, etc, it is worth checking my Microeconomic Analysis Notes. The basic premise though is that if we have optimised a function, say a profit function, with respect to q but for *a given value* of some parameter c, then in essence the function has a different optimum value of q, call it q^*, for each possible value of c. In this sense then we can think of q^* as a function that *depends on* c. We say that q^* is implicitly a function of c. Hence when we are considering the effect of c on the optimised profit function we need to take into account not only c's direct effect, but further the direct effects it has on q^*, which generate indirect effects on profit. As I said my Microeconomic Analysis Notes do a much better job of explaining this.

Competition Policy

Competition policy is all about the FFTW, since competition (under the appropriate conditions) maximises the total surplus.

Threats to competition: Collusion & Cartels

If effective leads to monopoly pricing. Although obviously illegal, tacit collusion is hard to prove.

An example of tacit collusion might be 'never knowingly undersold' (John Lewis), as this is an immediate form of retaliation and hence incentives collusion.

Factors affecting the likelihood of collusion:

(i) Number of Firms:

- Assume an infinitely repeating Bertrand game. If you are unsure about the notation below it will be worth reading the prior sections (Game Theory & Industrial Organisation), which explain both the outcomes of Bertrand competition, as well as collusion vs cheating in infinitely repeated games.
- Industry profit if all the firms collude, that is set the monopoly price p_M, is $\pi(p_M)$. This implies that, for n firms, each firm makes $\frac{\pi(p_M)}{n}$.
- he payoffs *each period* are,
  - Collusion
  $$\frac{\pi(p_M)}{n} : \text{By setting monopoly price } p_M$$
  - Deviation
  $$\pi(p_M) : \text{By firm } i \text{ undercutting all other firms with price } \ p_M - \epsilon$$
  - Punishment
  $$0 : \text{Bertrand Equilibrium}$$
- Given grim trigger collusion is sustained when,

$$PV_{coop} > PV_{cheat}$$
$$\frac{\pi(p_M)}{n} + \delta\frac{\pi(p_M)}{n} + \delta^2\frac{\pi(p_M)}{n} + ... > \pi(p_M) + 0 + 0 + ...$$
$$\frac{\pi(p_M)}{n(1-\delta)} > \pi(p_M)$$
$$\delta > 1 - \frac{1}{n}$$

- So it is harder to collude when there are more firms - this seems reasonable.

(ii) Frequency of Sales

- If goods are only sold every other period then the PV of industry profits when firms cooperate is given by,

$$\pi(p_M)[1 + \delta^2 + \delta^4 + ...] = \frac{\pi(p_M)}{1-\delta^2}$$

- Hence for an individual firm the PV_{coop} is given by,

$$PV_{coop} = \frac{\pi(p_M)}{n(1-\delta^2)}$$

- And again collusion is sustained when,

$$PV_{coop} > PV_{cheat}$$

$$\frac{\pi(p_M)}{n(1-\delta^2)} > \pi(p_M)$$

$$\delta^2 > 1 - \frac{1}{n}$$

- Since δ is the discount factor and $\delta < 1$ then $\delta^2 < \delta$. Therefore it is harder to sustain collusion when transactions are less regular.
- This is intuitive: imagine the case in which a transaction was once a decade - it would be almost impossible to sustain collusion.

(iii) Ease of Detection

- Suppose in a collusive agreement a cheating firm will not be detected by the other firms for two periods (i.e. you can make the 'cheating' payoff for two periods before you get punished for breaking collusive agreement).
- PV of payoffs for each firm is,
  - Colluding:
$$\frac{\pi(p_M)}{n(1-\delta)}$$
  - Cheating:
$$\pi(p_M)(1+\delta)$$
- Again collusion is sustained when,
$$\delta^2 > 1 - \frac{1}{n}$$
- So collusion is harder to sustain when cheating is harder to detect.

(iv) Price Transparency

- It can make cheating easier to detect hence actually increase ability to punish and hence ability to collude.

(v) Multi-market Contract

- If firms deal in multiple markets then they can punish cheaters in multiple markets, hence loss from cheating is higher, hence collusion is easier to sustain.

(vi) Cost Asymmetry

- If firms are identical with no capacity constraints, but firm 1 has lower marginal cost than firm 2 then full collusion would have firm 2 shutting down and firm 1 producing the monopoly quantity - perhaps paying firm 2 side payments for this?

(vii) Other Asymmetries

- Smaller firms/new entrants are more impatient hence more likely to cheat.
- Small firms with capacity constraints may not be able to punish deviating/cheating firms hence harder to sustain collusion.
- Large firms selling different product varieties have more incentive to keep price higher as dropping one price means they should drop others.

Detecting Collusion

- Look at price and cost changes over time.
- Look at evidence of sharing price information (e.g. school fee fixing scandal)
- Look for patterns that are hard to explain if there is no collusion

Threats to competition: Anti-competitive Mergers

Types of mergers:

- Horizontal - merger between firms in the same part of the supply chain, for example Heinz and Kraft.
- Vertical - can be forwards or backwards, and refers to the merger between firms that operate in different parts of the supply chain
- Conglomerate - merger/takeover of a firm in a completely different/unrelated sector.

Market power vs Efficiency gains

- Merger's increase market power but can also reduce costs.
- In the diagram the merger allows the new firm to big pricing as a monopolist at , but given that costs have fallen to the producer surplus has increase while the consumer surplus has fallen.

INSERT DIAGRAM

Measuring market power

(1) Lerner index:

$$L = \frac{P - MC}{P}$$

- Firm has market power if can raise price above marginal cost.
  - Monopoly: $L = \frac{1}{\epsilon}$ (That is the inverse of elasticity of demand)
  - Perfect competition: $L = 0$
  - Cournot: $L = \frac{1}{n\epsilon_I}$ (That is $\frac{1}{n\cdot(\text{the industry elasticity})}$)

(2) Herfindahl Index:

$$H = \sum_i s_i^2 = \sum_i (\frac{q_i}{Q})$$

- Firm has market power if its market share is high.
  - Monopoly: $H = 1$
  - Perfect Competition: $H = 0$
  - Cournot/Bertrand: $H = \frac{1}{n}$

Do mergers always mean price increases?

(1) The Diversion Effect

- Firms A,B,C,. . . ,Z set prices to maximise profit.
- If A and B merge and A raises it price then A loses some demand and it goes to the other firms.
- But the demand that is diverted to B contributes to the joint profits of A and B.

- Hence it is in the interest to merge and raises prices as this will likely increase profit.
- As a rule of thumb A will increase price in merger with B if,
- Where the diversion ratio is the fraction of sales lost by A that go to B.
- A simple estimate of this is the market share of B divide by the total market share of all firms (except A who is increasing her price).
- Hence the larger the existing market share of the new partner, the greater the price-increasing effect of the merger.
- Another thought: the relevant market
  - A merger between Coca Cola and Pepsi monopolises the Cola market, but likely doesn't have a huge impact on the wider beverage market, especially for other products like beer, juice, etc.
  - We hence need to,
    * Estimate the own price elasticity for A and then own-price elasticity for A+B.
    * If just A is inelastic (less than 1) then A is the relevant market and price increase would be profitable.
    * Even if a price increase in A wouldn't be profitable, if a own price elasticity of A+B is less than 1 (inelastic) then monopolising after the merger would be profitable.
    * E.g. CMA only looked into Nestle-Perrier in terms of bottled water, since that is the only market in which the merger would create a monopoly. They decided to allow the merger but Nestle had to divest some water brands.

(2) Increase Collusion

- Reducing actors in the market and potentially making competition asymmetric could increase prices via collusion.

Are mergers always anti-competitive?

- Double marginalisation: If two monopolists from different levels in the supply chain merge then the supplier will no longer charge the seller monopoly prices, since that just eats into the overall firms profits, hence we might lose one set of monopoly prices.

Threats to Competition: Abuse of Market Power

Monopolies

- Monopolists can increase prices and reduces quantities, leading to DWL.

Are they always bad?

- Natural monopolies.
- Large sunk costs and the need for economies of scale means that we need a monopoly in order to be able to get the efficient level of output (or perhaps any output).
- R&D funding: Monopolies research and introduce new ideas.

Threats to Competition: Government

- Government intervention is a form of market failure, although sometimes Government intervenes in order to correct already occurring market failure.

Decisions Under Risk

Core Concepts

- A **(simple) lottery** is a function over a set of payoffs that assigns a probability to a finite number of the members of the set of payoffs and zero to all other members.

$$\mathbb{L} = [p_1, ..., p_n \ ; \ x_1, ..., x_n], \quad x_i \in X, \quad p_i \geq 0, \quad \sum_{i=1}^{n} p_i = 1$$

- A **compound lottery** is a function over a set of simple lotteries that assigns a probability to a finite number of the members of the set of simple lotteries and zero to all other members.

$$\mathbb{C} = [p_1, ..., p_n \ ; \ \mathbb{L}_1, ..., \mathbb{L}_n]$$

  - Every compound lottery can actually be thought of as simple lottery $\mathbb{L}_{\mathbb{C}}$ simply be enumerating all possible prizes and their corresponding probabilities.

- The **expected value** of a lottery is given as,

$$\mathbb{EV}_{\mathbb{L}} = \sum_{i=1}^{n} p_i x_i$$

- The **expected utility** of a lottery is given as,

$$\mathbb{EU}_{\mathbb{L}} = \sum_{i=1}^{n} p_i u(x_i)$$

- The **certainty equivalent** is the amount of money which would be as good to an individual as playing the lottery,

$$[1 \ ; \ \mathbb{CE}(\mathbb{L})] \sim \mathbb{L}$$

$$u(\mathbb{CE}(\mathbb{L})) = \sum_{i=1}^{n} p_i u(x_i)$$

$$\mathbb{CE}(\mathbb{L}) = u^{-1}(\sum_{i=1}^{n} p_i u(x_i))$$

- The **risk premium** is the amount a risk averse individual will pay to get rid of the risk,

$$\mathbb{RP}(\mathbb{L}) = \mathbb{EV}(\mathbb{L}) - \mathbb{CE}(\mathbb{L})$$

  - If the risk premium is zero the individual is risk neutral.

- We say an individual is **risk averse** if,

$$\forall \mathbb{L} \ \mathbb{EU}(\mathbb{L}) \leq u(\mathbb{EV}(\mathbb{L}))$$

$$\forall \mathbb{L} \ \mathbb{CE}(\mathbb{L}) \leq \mathbb{EV}(\mathbb{L})$$

$$\forall \mathbb{L} \ \mathbb{RP}(\mathbb{L}) \geq 0$$

$$u''(.) < 0 \ \ (u(.) \textit{ is concave})$$

- We can measure risk aversion (concavity) via **Absolute** and **Relative Arrow-Pratt:**

$$A(x) = -\frac{u''(x)}{u'(x)}$$

$$R(x) = -\frac{u''(x)}{u'(x)}x$$

  - A higher $A(x)$ means you are more risk averse, $A(x)$ may be increasing, constant, or decreasing in wealth, x. Say it is decreasing in wealth then the wealthier you are the less risk averse you get.
  - A higher $R(x)$ means you are more risk averse. If $R(x)$ does depend on wealth then we can say that as wealth increases you would hold a smaller (larger) percentage of your wealth in risky assets.

- We can compare two peoples risk aversion by checking that $u_1(.)$ is more concave than $u_2(.)$. Agent 1 is more risk averse than 2 if $u_1(.)$ is a **concave transformation** of $u_2(.)$

$$u_1(.) = f(u_2(.)) \;\; with \;\; f''(.) < 0$$

- **Risk pooling** is where two individuals are facing independent lotteries and hence share the risk by transferring half of the winnings from the winner to the loser in the case that just one individual faces the loss. This technique squares the probability of loss (because both must now lose) which reduces risk.

  - "Dividing up the return (+risk) from many investments(lotteries)"

- **Risk sharing** is where individuals buy a portion of a risky investment to reduce the spread of deviations away from the mean.

  - "Dividing up the return (+risk) from one investment (lottery)"

- Lottery $\mathbb{L}_1$ **first order stochastically dominates** lottery $\mathbb{L}_2$ if, for every y,

$$F_{\mathbb{L}_1}(y) \leq F_{\mathbb{L}_2}(y)$$

  - Notice if the probability of receiving y or less is smaller for lottery $\mathbb{L}_1$ than $\mathbb{L}_2$, then $\mathbb{L}_1$ must return *more* than y with a higher probability.

- Given two lotteries $\mathbb{L}_1$ and $\mathbb{L}_2$ with the *same* expected value, we say that $\mathbb{L}_1$ **second order stochastically dominates** $\mathbb{L}_2$ if there exists a monetary value y* such that

$$(1)\; F_{\mathbb{L}_1}(y) \leq F_{\mathbb{L}_2}(y) \quad \forall y \leq y*$$

$$(2)\; F_{\mathbb{L}_1}(y) \geq F_{\mathbb{L}_2}(y) \quad \forall y \geq y*$$

  - That is the safer lottery crosses once from below.

Graphical Representation

We will consider for this analysis the lottery,

$$\mathbb{L} = [p, (1-p) \; ; \; w_L, w_{NL}]$$

where the outcome L = loss, and the outcome NL = not loss. Hence we can think of p as the probability of a loss, that is $p = \mathbb{P}(loss)$.

Importantly for all of this section we will define that outcome ***L is on the y-axis and NL on the x-axis***. Of course we could define this the other way round. In a sense it makes no difference bar for slopes of lines, etc. The reason I choose to define it this way around is that this becomes useful/commonplace when we start looking at insurance problems in information economics.

Core Concepts

1. The **Indifference Curve** is given by the iso-expected utility line.

$$\mathbb{EU} = pu(w_L) + (1-p)u(w_{NL})$$

   - With gradient:

$$MRS = -\frac{\frac{\partial \mathbb{EU}}{\partial u(w_{NL})}}{\frac{\partial \mathbb{EU}}{\partial u(w_L)}} = -\frac{(1-p)u'(w_{NL})}{pu'(w_L)}$$

   - Hence when we are on the 45° line (the certainty line) we know that $w_{NL} = w_L$ (there is no uncertainty), and hence we know that at this specific point,

$$MRS|_{w_{NL}=w_L} = -\frac{(1-p)}{p}$$

2. The **Fair odds line** is given by the iso-expected value line.

$$\mathbb{EV} = pw_L + (1-p)w_{NL}$$

   - With gradient:

$$-\frac{(1-p)}{p}$$

   - Hence the indifference curves will be tangental to the fair odds line at the certainty line.

3. The **Certainty line** is the 45° line which contains all certain consumption bundles.
   - On this line there is no risk.

[INSERT DIAGRAM]

4. The **Actuarially fair insurance** case is the situation in which insurer profit is zero.
   - We suppose that the insurer will offer insurance at price π per unit.
   - The agent facing the lottery insures a total amount Q, up to their total loss.

$$\mathbb{L} = [p, (1-p) \; ; \; w - L, w]$$

$$\begin{aligned} \mathbb{L}_{insur} &= [p, (1-p) \; ; \; w - L - \pi Q + Q, w - \pi Q] \\ &= [p, (1-p) \; ; \; w - L + (1-\pi)Q, w - \pi Q] \end{aligned}$$

- We assume the insurer's profits are given by total earnings πQ less total loss pQ. What this really means is that the insurer receives π per unit of insurance, but has to pay out on those units insured with probability p. In the case in which profit is zero we can solve for π,

$$0 = \pi Q - pQ$$

$$\pi = p$$

  [INSERT DIAGRAM]

5. The **General insurance case** is when the insurer makes some positive profit and hence it is the case that $\pi Q - pQ > 0$ and so $0 < p < \pi < 1$. In this case of course the insured person does not earn her $\mathbb{EV}$.
   [INSERT DIAGRAM]

Expected Utility Theory

Properties of Expected Utility

I) Independence

$$\mathbb{L}_1 \succeq \mathbb{L}_2 \;\Leftrightarrow\; [p, (1-p)\;;\; x, \mathbb{L}_1] \succeq [p, (1-p)\;;\; x, \mathbb{L}_2]$$

II) Continuity

- Given $x_H > x_M > x_L$ there exists a $p* \in (0,1)$ such that,

$$[1\;;\; x_M] \sim [p*, (1-p*)\;;\; x_H, x_L]$$

- Obviously for very low values of p you would prefer the first lottery, and for very high values you would prefer the latter, but there should be some intermediate value in which you are indifferent between the two.

Expected Utility Theorem

The following two statements are equivalent,

(1) The preferences $\succeq$ satisfy reduction (all compound lotteries can be reduced to simple lotteries), continuity, and independence.
(2) The preferences $\succeq$ corresponds to some expected utility $\mathbb{EU}(\mathbb{L})$ that uses some function $u(.)$.

Risk Aversion

Absolute Arrow-Pratt (AAP):

$$A(x) = -\frac{u''(x)}{u'(x)}$$

Relative Arrow-Pratt (RAP):

$$R(x) = -\frac{u''(x)}{u'(x)}x$$

The reason these concavity measures are normalised with the first derivative is because we allow linear transformations of expectated utility such that we can have $u(.)$ and $v(.) = \alpha + \beta u(.)$. The problem with measuring concavity is that $v''(.) = \beta u''(.)$, hence $u''(.) \neq v''(.)$. Normalising with the first deriviate gets us around this problem.

Arrow-Pratt Theorem

The following statements are equivalent,

(1) For every lottery, the risk premium created by u_1 is larger than that of u_2;
(2) u_1 is more concave than u_2;
(3) The Absolute Arrow-Pratt measure of u_1 is, as a function, greater than that of u_2.

Utility functions

Constant Absolute Risk Aversion (CARA)

$$A(x) = a$$

For example,

$$u(x) = -e^{ax} \ , \ a \neq 0$$

Decreasing Absolute Risk Aversion (DARA)

$$A'(x) < 0$$

For example,

$$u(x) = \sqrt{x} \ , \ A(x) = \frac{1}{2x}$$

Constant Relative Risk Aversion (CRRA)

$$R(x) = r$$

For example,

$$u(x) = \frac{1}{1-r}x^{1-r} \ , \ r < 1$$

Taylor Approximations

Let $\tilde{z}$ be a mean-zero random variable and π be the risk premium (the amount you would pay to get rid of $\tilde{z}$). Given $\tilde{z}$ is mean zero we can call the expected value of this lottery x, as $\tilde{z}$ does not affect it.

Recall,

$$\mathbb{RP}(\mathbb{L}) = \mathbb{CE}(\mathbb{L}) - \mathbb{EV}(\mathbb{L})$$
$$u(\mathbb{CE}(\mathbb{L})) = \mathbb{EU}(\mathbb{L})$$

$$u(\mathbb{CE}(\mathbb{L})) = u(\mathbb{EV}(\mathbb{L}) - \mathbb{RP}(\mathbb{L})) = \mathbb{EU}(\mathbb{L})$$
$$u(x - \pi) = \mathbb{E}[u(x + \tilde{z})] \quad (1)$$

Formula for a Taylor approximation of $f(x)$ about a:

$$\begin{aligned} f(x) &\approx f(a) + f'(a)(x-a) \\ &\approx f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 \\ &\approx etc... \end{aligned}$$

Taking a first-order Taylor approximation of the LHS of (1) about x:

$$\begin{aligned} u(x-\pi) &\approx u(x) + u'(x)(x-\pi-x) \\ &\approx u(x) + -\pi u'(x) \end{aligned}$$

Taking a second-order Taylor approximation of the RHS of (1) about x:

$$\begin{aligned} \mathbb{E}[u(x+\tilde{z})] &\approx \mathbb{E}[u(x) + u'(x)(x+\tilde{z}-x) + \frac{1}{2}u''(x)(x+\tilde{z}-x)^2] \\ &\approx u(x) + u'(x)\mathbb{E}[\tilde{z}] + \frac{1}{2}u''(x)\mathbb{E}[\tilde{z}^2] \\ &\approx u(x) + u'(x)(0) + \frac{1}{2}u''(x)(\sigma^2) \\ &\approx u(x) + \frac{1}{2}u''(x)\sigma^2 \end{aligned}$$

Hence overall from (1) we get,

$$\begin{aligned} u(x) + -\pi u'(x) &\approx u(x) + \frac{1}{2}u''(x)\sigma^2 \\ -\pi u'(x) &\approx \frac{1}{2}u''(x)\sigma^2 \\ \pi &\approx -\frac{1}{2}\frac{u''(x)}{u'(x)}\sigma^2 \\ \pi &\approx \frac{1}{2}A(x)\sigma^2 \end{aligned}$$

So we can approximate the $\mathbb{RP}(\mathbb{L})$ when wealth x is subject to mean zero fluctuations with variance σ^2 using the equation above.

Stochastic Dominance

Cumulative Distribution Function (CDF)

An increasing real-valued function which outputs the probability of experiencing a result below or equal to any given value y, denoted $F(y)$.

Example:

- Lotteries:
$$\mathbb{L}_1 = [\frac{1}{2}, \frac{1}{2}\ ;\ 30, 50]$$
$$\mathbb{L}_2 = [\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\ ;\ 10, 20, 40]$$
- CDF's

	10	20	30	40	50
CDF 1	0	0	1/2	1/2	1
CDF 2	1/3	2/3	2/3	1	1

- Plot
[INSERT DIAGRAM]

First Order Stochastic Dominance (FOSD)

Lottery $\mathbb{L}_1$ FOSDs lottery $\mathbb{L}_2$ if, for every y,

$$F_{\mathbb{L}_1}(y) \leq F_{\mathbb{L}_2}(y)$$

Notice if the probability of receiving y or less is smaller for lottery $\mathbb{L}_1$ than $\mathbb{L}_2$, then $\mathbb{L}_1$ must return more than y with a higher probability.

Example:

	10	20	30	40	50
CDF 1	0	0	1/2	1/2	1
CDF 2	1/3	2/3	2/3	1	1

Notice that the CDF of $\mathbb{L}_1$ is less than (or equal to) the CDF of $\mathbb{L}_2$. That is if we plotted these CDFs on the same graph the CDF of $\mathbb{L}_1$ would be below (or equal to) the CDF of $\mathbb{L}_2$ for all values of y. Hence we can say that $\mathbb{L}_1$ FOSDs $\mathbb{L}_2$.

Mean Preserving Spread

If we have a compound lottery $\mathbb{L}_2$ obtained by,

(1) Playing first the simple lottery $\mathbb{L}_1$
(2) Adding some *mean-zero* noise to the prize obtained.

Then in such case, we say that $\mathbb{L}_2$ is a mean-preserving spread of $\mathbb{L}_1$. Hence $\mathbb{L}_1$ is preferred to $\mathbb{L}_2$ by risk averse expected utility maximisers.

Interestingly the converse is also true, if $\mathbb{L}_1$ and $\mathbb{L}_2$ have the same expected value and all expected utility maximisers prefer $\mathbb{L}_1$ to $\mathbb{L}_2$ then $\mathbb{L}_2$ is a mean preserving spread of $\mathbb{L}_1$.

Second Order Stochastic Dominance (SOSD)

Given two lotteries $\mathbb{L}_1$ and $\mathbb{L}_2$ with the same expected value, we say that $\mathbb{L}_1$ SOSDs $\mathbb{L}_2$ if there exists a monetary value $y*$ such that

$$(1)\ F_{\mathbb{L}_1}(y) \leq F_{\mathbb{L}_2}(y) \quad \forall y \leq y*$$
$$(2)\ F_{\mathbb{L}_1}(y) \geq F_{\mathbb{L}_2}(y) \quad \forall y \geq y*$$

This is more usefully interpreted as “the safer lottery crosses once from below”. Plot out the CDF and check by eye.

Worked Examples

Example: Investment Problem

Charlie is an expected utility maximiser, with utility function $u(y) = lny$ over final wealth y. Assume that his initial wealth is 10 (we will just call his initial wealth y).

He has the opportunity to invest an amount $c > 0$ in a risky project: with probability $\frac{1}{2}$ it succeeds and he gets back his investment *plus* a profit of c; with probability $\frac{1}{2}$ it fails and he gets back half of what he invested (thus making a loss).

(a) Show that he is risk-averse, with constant relative risk aversion.

(b) What is his final wealth if he invests and the project is a success? What is his final wealth if he invests and the project is a failure? Write down the expression for his expected utility, as a function of the investment level c. What investment level c will he choose?

Is Charlie risk averse?

$u''(y) = -\frac{1}{y^2} < 0$ hence risk averse.

$A(y) = -\frac{-\frac{1}{y^2}}{\frac{1}{y}} = \frac{1}{y}$

Charlie has DARA preferences, hence his attitudes towards lotteries is more favourable when y is large.

$R(y) = -\frac{-\frac{1}{y^2}}{\frac{1}{y}}y = \frac{1}{y}y = 1$

Charlies preferences are also CRRA, so relative risk aversion is independent of wealth.

Investing in the project:

$$\begin{aligned}\mathbb{L} =&[\frac{1}{2}, \frac{1}{2}\ ;\ y - c + 2c, y - c + \frac{1}{2}c] \\ =&[\frac{1}{2}, \frac{1}{2}\ ;\ y + c, y - \frac{1}{2}c]\end{aligned}$$

$$\mathbb{EV}(\mathbb{L}) = \frac{1}{2}(y + c) + \frac{1}{2}(y - \frac{1}{2}c)$$

$$\mathbb{EU}(\mathbb{L}) = \frac{1}{2}ln(y + c) + \frac{1}{2}ln(y - \frac{1}{2}c)$$

Optimise $\mathbb{EU}(\mathbb{L})$:

$$max_c\ \ \mathbb{EU}(\mathbb{L}) = max_c\ \ [\frac{1}{2}ln(y + c) + \frac{1}{2}ln(y - \frac{1}{2}c)]$$

$$\frac{\partial \mathbb{EU}(\mathbb{L})}{\partial c} = \frac{1}{2}\frac{1}{y + c} + \frac{1}{2}\frac{-\frac{1}{2}}{y - \frac{1}{2}c} = 0$$

$$(y - \frac{1}{2}c) = \frac{1}{2}(y + c)$$

$$c = \frac{y}{2}$$

Example: Insurance Problem

Perdita is a risk averse expected utility maximiser. Her initial wealth is w and her preferences over monetary outcomes, y, are summarised by the utility function $u(y)$. She is living in uncertain times { she might incur a loss of $L > 0$, or she might lose nothing at all. The loss occurs with probability π, but insurance is available at a cost of pq for an amount q of cover, i.e. she can pay a premium of pq and, in the event of a loss, she will receive a reimbursement of q.

(a) What can you say about the signs of u' and u''? What does this imply for the comparison of $u'(w-L)$ with $u'(w)$?

(b) Briefly y explain why the case in which $p = \pi$ is called *actuarially fair*.

(c) When the amount of insurance cover she buys is q, what is her expected utility? Show that the first-order condition for her optimal choice, $q*$, is

$$\pi(1-p)u'(w-L+(1-p)q*) - (1-\pi)pu'(w-pq*) = 0$$

.

(d) By using the FOC, or otherwise, show that when the insurance premium is actuarially fair she will fully insure, i.e. $p = \pi \; => \; q* = L$.

(e) What if $p > \pi$?

Utility function:

$u'(y) > 0$ since utility is increasing in y.

$u''(y) < 0$ since utility is concave - recall we are told the agent is risk averse.

Actuarially fair case:

In this situation the insurers profit is zero ($0 = \pi Q - pQ$, hence $p = \pi$) and the insured person receives the expected value with certainty. Expected gain or loss for the insurer is zero.

Insurance case:

$$\mathbb{L} = [\pi, (1-\pi) \; ; \; y - L + q(1-p), y - pq]$$

$$\mathbb{EU}(\mathbb{L}) = \pi u(y - L + q(1-p)) + (1-\pi)u(y-pq)$$

$$max_q \; \mathbb{EU}(\mathbb{L})$$

$$\frac{\partial \mathbb{EU}(\mathbb{L})}{\partial q} = \pi(1-p)u'(y-L+(1-p)q*) - p(1-\pi)u'(y-pq*) = 0$$

$$\frac{\pi(1-p)}{(1-\pi)p} = \frac{u'(y-pq*)}{u'(y-L+(1-p)q*)}$$

What if $p = \pi$?

$$\frac{\pi(1-p)}{(1-\pi)p} = \frac{p(1-p)}{(1-p)p} = 1$$

$$1 = \frac{u'(y - pq*)}{u'(y - L + (1-p)q*)}$$

$$y - pq* = y - L + (1-p)q*$$
$$-pq* = -L + (1-p)q*$$
$$-pq* = -L + q * -pq*$$
$$q* = L$$

What if $p > \pi$?

$$(1-\pi)p = p - p\pi \text{ and } (1-p)\pi = \pi - p\pi$$
$$p > \pi \quad => \quad p - p\pi > \pi - p\pi \quad => \quad (1-\pi)p > (1-p)\pi$$
$$1 > \frac{\pi(1-p)}{(1-\pi)p}$$

$$1 > \frac{u'(y - pq*)}{u'(y - L + (1-p)q*)}$$

$$u(y - L + (1-p)q*) > u'(y - pq*)$$

Recall that $u'(.)$ is strictly decreasing, hence for the inequality above to hold it must be the case that,

$$y - pq* > y - L + (1-p)q*$$
$$-pq* > -L + q * -pq*$$
$$q* < L$$

She underinsures in this case.

Example: Bilateral Insurance Contracts

Arthur is risk averse, and his income tomorrow depends on which of two possible states occurs; each state is equally likely. His income will be 8 if state 1 occurs, but only 2 if state 2 occurs.

Norma is risk neutral. Her income will be 3 if state 1 occurs, and 7 if state 2 occurs. Suppose that they can write contracts of the form "Arthur will give Norman an amount x iff state 1 occurs, and Norman will give Arthur an amount y iff state 2 occurs."

(a) Draw Arthur's indifference curves in state-contingent income space, and explain how he could be better off if he were able to buy insurance.

(b) What shape are Norma's indifference curves? From Arthur's point of view, what is the best contract that Norma would and acceptable, and why?

(c) From Norma's point of view, what is the best contract that Arthur would and acceptable?

(d) Illustrate your results from part (b) and part (c) in an Edgeworth box, and highlight the set of effcient risk-sharing contracts.

Example: Stochastic Dominance

Tom the thief gets caught and fined an amount f with probability p. His initial wealth is w and his preferences over monetary outcomes, y, are summarised by the utility function $u(y)$. Thus his expected utility from committing a crime is

$$pu(w-f)+(1-p)u(w)$$

The authorities plan to increase the amount Tom expects to pay from pf to $1.01pf$, by either,

(a) increasing p by 1% to $1.01p$, or

(b) increasing f by 1% to $1.01f$.

DRAW CDF

Show that (a) crosses once from below, hence (a) SOSDs (b), and therefore as a utility maximiser Tom prefers (a) - so (b) is the better deterent.

Example: Risk Pooling & Risk Sharing 1

Bill's initial wealth is 22. He can invest it all in a risky project that has a 50:50 chance of succeeding. If he invests and the project succeeds, his final wealth will be 40, but if it fails, his final wealth will be 10. Bill's preferences over monetary outcomes, y, are summarised by the utility function $u(y) = lny$.

(a) For Bill, what is the certainty equivalent of this project? What is the risk premium? Why should he reject the opportunity to invest in the project?

(b) Ben has the same preferences and initial wealth as Bill, and suggests to him that they each invest 11 in the project and divide the proceeds equally, each ending with his remaining wealth plus either 40/2 after a success or 10/2 after a failure. Should Bill accept this offer to share the risk?

(c) Ben now finds an opportunity of his own to invest in a project with the same characteristics as Bill's. (The success or failure of one project is independent of that of the other.) Obviously he wouldn't do it on his own, but they could agree that each of them take on their project and then divide the proceeds equally between them. Should they pool their risks and invest in the projects?

Bill's Lottery, $\mathbb{CE}$ and $\mathbb{RP}$,

$$\mathbb{L} = [\frac{1}{2}, \frac{1}{2} \ ; \ 40, 10]$$

$$\mathbb{EU}(\mathbb{L}) = \frac{1}{2}ln(40) + \frac{1}{2}ln(10) = \frac{1}{2}ln(400)$$

$$\mathbb{EV}(\mathbb{L}) = \frac{1}{2} \cdot 40 + \frac{1}{2} \cdot 10 = 20 + 5 = 25$$

We know that $u(\mathbb{CE}(\mathbb{L})) = \mathbb{EU}$, hence,

$$ln(\mathbb{CE}(\mathbb{L})) = \frac{1}{2}ln400$$

$$\mathbb{CE}(\mathbb{L}) = e^{\frac{1}{2}ln400} = 20$$

Further we know that $\mathbb{RP}(\mathbb{L}) = \mathbb{EV}(\mathbb{L}) - \mathbb{CE}(\mathbb{L})$, hence,

$$\mathbb{RP}(\mathbb{L}) = 25 - 20 = 5$$

Bill will not participate in this lottery, since his $\mathbb{CE}$ is the amount of money he would accept not to play the lottery, and in this case the $\mathbb{CE} <$ Bill's Wealth. Therefore Bill would rather keep his wealth of 22 then play the lottery.

Risk Sharing Case:

$$\mathbb{L} = [\frac{1}{2}, \frac{1}{2} \ ; \ 20 + 11, 5 + 11] = [\frac{1}{2}, \frac{1}{2} \ ; \ 31, 16]$$

This is the case because Bill is now only investing half of his wealth and Ben is investing half - they are risk sharing. After the lottery they will have the wealth the didn't invest (11) plus whatever the lottery returns.

$$\mathbb{EU}(\mathbb{L}) = \frac{1}{2}ln(31) + \frac{1}{2}ln(16) = \frac{1}{2}ln(496)$$

$$ln(\mathbb{CE}(\mathbb{L})) = \frac{1}{2}ln496$$

$$\mathbb{CE}(\mathbb{L}) = e^{\frac{1}{2}ln496} = 22.27$$

$22.27 > 22$ hence the $\mathbb{CE}$ is greater than initial wealth, so it is now optimal to play the lottery.

Risk Pooling Case:

$$\mathbb{L} = [\frac{1}{4}, \frac{1}{2}, \frac{1}{4} \ ; \ 10, 25, 40]$$

In the risk pooling case the lotteries are independent, hence whether one wins/loses does not depend on the other. There is now just a $\frac{1}{4}$ chance ($\frac{1}{2} \cdot \frac{1}{2}$) that *both* lose; a $\frac{1}{2}$ one wins and one loses ($\frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2}$ - since Bill can win and Ben lose *and* Ben can win and Bill lose); and a $\frac{1}{4}$ chance ($\frac{1}{2} \cdot \frac{1}{2}$) that *both* win. As stated in the question the two divide the proceeds equally, which has no impact when they both win or both lose, but returns $10 + 40 = 50$ combined in the case in which one wins and the other loses. The winner then transfers over 15 so that they both go home with 25.

$$\mathbb{EU}(\mathbb{L}) = \frac{1}{4}ln(10) + \frac{1}{2}ln(25) + \frac{1}{4}ln(40) = \frac{1}{4}ln(10 \cdot 25^2 \cdot 40)$$

$$ln(\mathbb{CE}(\mathbb{L})) = \frac{1}{4}ln(25000)$$

$$\mathbb{CE}(\mathbb{L}) = e^{\frac{1}{4}ln(25000)} = 22.36$$

Again $22.36 > 22$ hence the $\mathbb{CE}$ is greater than initial wealth, so it is now optimal to play the lottery.

Example: Risk Pooling & Risk Sharing 2

Janet's broad attitude to risk (risk averse, risk neutral, or risk loving) is independent of her wealth. She has initial wealth w and is offered the opportunity to buy a lottery ticket. If she buys it, her final wealth will be either $w + 4$ or $w - 2$, each equally likely. She is indifferent between buying the ticket and not buying it.

(a) Explain whether she is risk averse, risk neutral, or risk loving. Draw this situation in state-contingent income space, with the 'good' outcome on the horizontal axis and the 'bad' outcome on the vertical axis.

(b) She offers her friend Sam (who has the same initial wealth and attitude to risk as she does) the following proposition: they buy the ticket together, and share the cost and proceeds equally. Should Sam accept? Illustrate your answer in a diagram.

(c) Sam has another idea: they buy two tickets (whose outcomes are independent), again sharing the costs and proceeds equally. Is this better than buying no tickets?

Is she risk averse?

$$\mathbb{L} = [\frac{1}{2}, \frac{1}{2} \ ; \ w - 2, w + 4]$$

Given we know that Janet indifferent between buying the ticket and not then we know $w = \mathbb{CE}(\mathbb{L})$. This is because the circumstance in which she is indifferent between playing and not is the one in which the certainty equivalent of the lottery $\mathbb{CE}(\mathbb{L})$ is exactly equal to her wealth w. Now we just get what is larger, the $\mathbb{CE}$ or the $\mathbb{EV}$ to deduce her risk attitude.

$$\mathbb{EV}(\mathbb{L}) = \frac{1}{2}(w - 2) + \frac{1}{2}(w + 4) = w + 1$$

Since $w < w + 1$ then her risk premium is positive ($\mathbb{RP}(\mathbb{L}) = \mathbb{EV}(\mathbb{L}) - \mathbb{CE}(\mathbb{L})$), so Janet is risk averse, and $u(.)$ is of course concave.

Risk Sharing Case:

In the risk sharing case Janet and Sam split the ticket and hence half the losses and gains from purchasing a ticket.

$$\mathbb{L} = [\frac{1}{2}, \frac{1}{2} \ ; \ w - 1, w + 2]$$

Concavity of any function implies that $f((1 - \lambda)x + \lambda y) \geq (1 - \lambda)f(x) + \lambda f(y)$. Notice we can use this definition for $u(.)$ and choosing appropriate values for λ, namely that $\lambda = \frac{1}{2}$, to show,

$$u(w - 1) > \frac{1}{2}u(w) + \frac{1}{2}u(w - 2)$$

$$u(w + 2) > \frac{1}{2}u(w) + \frac{1}{2}u(w + 4)$$

If this doesn't make immediate sense to you perhaps it is easier to show it as a more clear application of the concavity formula:

$$u(w - 1) = u(\frac{1}{2}(w) + \frac{1}{2}(w - 2)) > \frac{1}{2}u(w) + \frac{1}{2}u(w - 2)$$

$$u(w + 2) = u(\frac{1}{2}(w) + \frac{1}{2}(w + 4)) > \frac{1}{2}u(w) + \frac{1}{2}u(w + 4)$$

Using the fact that $u(\mathbb{CE}(\mathbb{L})) = \mathbb{EU}(\mathbb{L})$ and further from part (a) that $\mathbb{CE}(\mathbb{L}) = w$, we know that $u(w) = \mathbb{EU}(\mathbb{L}) = \frac{1}{2}u(w-2) + \frac{1}{2}u(w+4)$. Using this fact and also the inequalities above, it must be the case that,

$$\begin{aligned}
\frac{1}{2}u(w-1) + \frac{1}{2}u(w+2) &> \frac{1}{4}u(w) + \frac{1}{4}u(w-2) + \frac{1}{4}u(w) + \frac{1}{4}u(w+4) \\
&> \frac{1}{2}u(w) + \frac{1}{2}[\frac{1}{2}u(w-2) + \frac{1}{2}u(w+4)] \\
&> \frac{1}{2}u(w) + \frac{1}{2}[u(w)] \\
\frac{1}{2}u(w-1) + \frac{1}{2}u(w+2) &> u(w)
\end{aligned}$$

Given that $u(w)$ was just the $\mathbb{EU}$ of the original lottery, and Janet was indifferent between the orginal and her wealth w, it is now optimal to play this lottery.

Risk Pooling Case:

$$\mathbb{L} = [\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\ ;\ w-2, w+1, w+4]$$

And recognise the $\mathbb{EU}$ is,

$$\mathbb{EU}(\mathbb{L}) = \frac{1}{4}u(w-2) + \frac{1}{2}u(w+1) + \frac{1}{4}u(w+4)$$

Then notice that the inequality below holds by the fact that $u(.)$ is strictly increasing and that in the first line the LHS and RHS differ by just $+1$. From this we will be able to show that the $\mathbb{EU}$ is greater than $u(w)$.

$$\begin{aligned}
\frac{1}{4}u(w-2) + \frac{1}{2}u(w+1) + \frac{1}{4}u(w+4) &> \frac{1}{4}u(w-2) + \frac{1}{2}u(w) + \frac{1}{4}u(w+4) \\
&> \frac{1}{2}u(w) + \frac{1}{2}[\frac{1}{2}u(w-2) + +\frac{1}{2}u(w+4)] \\
&> \frac{1}{2}u(w) + \frac{1}{2}[u(w)] \\
\frac{1}{4}u(w-2) + \frac{1}{2}u(w+1) + \frac{1}{4}u(w+4) &> u(w)
\end{aligned}$$

Given that $u(w)$ was just the $\mathbb{EU}$ of the original lottery, and Janet was indifferent between the orginal and her wealth w, it is now optimal to play this lottery.

Information Economics

In the standard GCE (General competitive equilibrium) model we make several assumptions, including that of full, symmetric, information and no uncertainty.

The phenomenon of information asymmetries is, however, common place. These lead to inefficiencies that require corrective measures.

Adverse Selection: Used Car Market

Suppose a market of sellers and buyers of used cars, with an equal split of 'lemons' and 'plums' (poor quality and high quality used cars).

The buyers value the cars as least as much as the sellers hence there are gains to be had from trade.

	Plums	Lemons
Buyers	£1200	£200
Sellers	£700	£200

Case (1): 500 plums, 1500 lemons, >2000 buyers, symmetric information

- In this case buyers & sellers can both tell the lemons and plums apart.
- As the diagram shows the supply is limited in comparison to demand hence buyers purchase at the price they exactly value the car at.

Case (2): 500 plums, 1500 lemons, >2000 buyers, asymmetric information

- In this case only sellers know the quality of the cars.
- Since all goods look the same, all will sell at the same price.
- Maximum WTP for buyers: $\frac{1}{4}1200 + \frac{3}{4}200 = £450$
- At this price sellers with plums will not sell, hence the only cars in the market will be lemons.
- High quality goods aren't sold - the market unravels from the top.

Applications of Asymmetric Information

- ***Financial Crisis 2008*** – CDOs were disproportionately becoming lemons (full of subprime loans) and only really the lawyers and bankers who constructed them knew the real value of the underlying loans. The problem was that these bankers and lawyers tended not to assume any risk for these products that they were making.
- ***Health Insurance*** - The insured person knows how likely they are to claim (i.e. are they ill, likely to be in an accident, etc). Given that health insurers can't tell between low-risk and high-risk individuals they set premiums high. These high premiums unravel the market as the low-risk individuals will not pay for insurance at such a high price.
- ***Credit Market*** - Suppose A & B both wish to borrow from a bank, where A's project pays of 1.15x with certainty, but B's pays of 1.3x or 0.9x with 50% chance. Given that the bank can't tell the projects A and B apart, should the bank charge interest rates higher than 15% A will not borrow and the bank is left lending to just the riskier project. Higher interest rates hence only attract high-risk, high-return projects.

General points on Solutions to Asymmetric Information

We will consider two solutions to problems of asymmetric information: Signalling and Screening.

Signalling is where the *high-quality seller* takes an action that is too costly for low quality seller to imitate. For example high-quality workers undertake education, which has a lower cost - think effort - for them than low-quality workers, in order to differentiate themselves from the low-quality workers and hence recieve a higher wage.

Screening is where *uninformed party* (or third party) takes action to screen out 'bad' types. For example when you take out a low the bank checks credit history to assess the risk of default. Similarly when an individual takes out health insurance they may have to fill in a questionnaire to assess their likelihood of making a claim.

In the next two pages we will consider these two solutions using the model of workers and education for signalling, and insurance for screening.

Worker/Education Problems

- Workers of high- or low-productivity can either choose to get an education or not.
- Firms would ideally like to hire high-productivity workers, but can't tell them apart.
- In order to distinguish themselves from the low-productivity workers, the high-productivity workers must take an education to get the higher wage, *but* the low-productivity workers must *not* want an education themselves.
- That is the education must be costly enough that the low productivity worker doesn't want it.

Insurance Problems

-

Principal-Agent Problems

- A principal wishes to high an agent to work for them, where the agent can exert effort e_H which increases the likelihood of high profit, or e_L, which reduces the likelihood of high profit.
- If the principal wants the agent to exert e_L she should just pay some fixed which makes it such that the agent wants to work. We call this satisfying the agent's ***Individual Rationality*** constraint.
- If the principal wants the agent to exert e_H she needs to pay a variable wage package: a higher wage if profit is high and a lower one if profit is low. This variable wage incentivises the agent to exert more effort to generate a higher profit, and therefore get the higher wage. More specifically the wage packet needs to be such that the agent wants to work (satisfies ***IR***) and further that the agent does best by exerting e_H. That is the agent shouldn't do better by secretly exerting e_L rather than e_H - we call this the ***Incentive Compatibility*** constraint.
- Then the principal just needs to see in which of these circumstances she expects to make the most profit.

Signalling: Labour Force

Signalling is something the party with superior information can do to reduce adverse selection problems. As a seller of a high-quality good you would benefit from eliminating information asymmetry since then you would receive a higher price. The problem here is that sellers of the low-quality good would also like to persuade buyers that theirs are of high-quality and hence receive the high price. You hence must take an action that is too costly for low-quality sellers to imitate.

Model: Firms choosing over two types of worker

Suppose CRS and two types of worker, θ_L and θ_H, where $0 < \theta_L < \theta_H$. The proportion of type H workers is λ, that is $\lambda = \mathbb{P}(\theta = \theta_H)$.

We assume in this model that the firm is unable to distinguish the type L worker from the type H worker without a signal. We say that workers type is *unobservable*.

We also assume in this model that education does not affect productivity. That is that education is only a signal, and does not benefit the worker in anyway.

Worker's utility is given by,

$$w - c(e, \theta)$$

Cost depends on level of education and type of worker with the properties:

- $c(0, \theta) = 0$ - no education means there is no cost
- $c_e(e, \theta) > 0$ - Cost is increasing in education
- $c_\theta(e, \theta) < 0$ - Cost is decreasing in productivity - i.e. education is cheaper for more productive workers, hence at the same wage more productive workers can afford more education.
- $c_{ee}(e, \theta) > 0$ - 2nd derivative is positive.
- $c_{e\theta}(e, \theta) < 0$ - the indifference curves can cross only once.

If workers choose the same education level or it is possible to imitate the signal then.

$$w = \mathbb{E}[\theta]$$

If workers choose different education levels and signal is informative then firms pay.

$$w(e_L) = \theta_L \quad , \quad w(e_H) = \theta_H$$

Case 1: θ is Publicly Observable

- The optimal education level for all workers is zero ($e(\theta_L) = e(\theta_H) = 0$) and firms earn zero profits since workers are paid exactly their productivity ($w(e_L) = \theta_L$, $w(e_H) = \theta_H$).
- Obviously the problem with this case is that if type was not observable then all workers would choose $e = 0$ and claim to be type H. Firms would therefore lose money as the wage per worker would exceed the expected productivity per worker.
- This could not be an equilibrium in our case in which type is unobservable, since firms would rather not hire any workers in this circumstance and make zero profit than make a loss.
- We will now study the actual equilibria of this model, considering cases when θ is not observable.

Case 2: Pooling Equilibria

- Pooling = both types get the same level of education.
- The workers will both receive

$$w* = \mathbb{E}[\theta] = \lambda\theta_H + (1-\lambda)\theta_L$$

  That is the optimal wage to pay a worker is her expected productivity, since the firm cannot tell which type of worker she is. What education level, however, is it optimal for workers to choose?

(A) $e = 0$

- In the first case both might choose $e = 0$.
- If the wage curve offered by firms is too high then workers sit on the highest indifference curve without purchasing this high education level.

(B) $e = e'$

- In this scenario workers choose the same positive level of education.
- This is because the firm assumes the worker is a low productivity type if they choose a level of education below this. Again assume that the level of education at which the firm assumes they are high-productivity is too high.

So at equilibrium: $e^* \in [0, e']$

- Pareto Rankings?
  - $\pi = 0$ for all equilibria
  - $w = \mathbb{E}[\theta]$ for all equilibria
  - Assuming a disutility of education workers would prefer an equilibrium such that $e = 0$.
- If $e = 0$ is pareto-preferred why would $[0, e']$ be the set of all equilibria?
  - Because if firms assume that below some which is not zero a worker is a low productivity worker, then anyone who doesn't get this non-zero education level will only receive. It is therefore in the interest to get some education at this equilibrium

Case 3: Separating Equilibria

- Types of workers receive wages equal to their productivity,

$$w^*(e^*(\theta_t)) = \theta_t \text{ for } t = H, L$$
$$e^*(\theta_L) = 0 \text{ , hence } w^*(0) = \theta_L$$

(A) $e = \hat{e}$

- Here the low productivity workers are indifferent between getting education and not, assuming disutility of education they will accept $e = 0$ and the lower wage, while higher-productivity workers receive the higher wage.

(B) $e = \tilde{e}$

- At this extreme the high productivity workers are indifferent between high wages and education and low wages and no education, hence this is the highest level of education they would opt to receive.

So at equilibrium: $e \in [\hat{e}, \tilde{e}]$

- We can Pareto-rank these equilibria since the high productivity workers would prefer lower levels of education.
  - $\pi = 0$ for all equilibria
  - $e(\theta_L) = 0$ and hence $w(\theta_L) = \theta_L$
  - So here only the high-quality worker matters. Again assuming a disutility of education high-productivity workers would prefer an equilibrium such that $e = \hat{e}$.

Is everyone better off from signalling?

- Firms never make profit hence are in different between outcomes.
- Low productivity workers are (weakly) worse off from signalling.
  - They are indifferent between no signalling ($e = 0$) and pooling equilibria, but strictly worse off in separating since now they only receive $w = \theta_L$.
- High productivity workers depend on λ
  - When λ is low then expected productivity is low, hence ID curve is higher at higher-productivity wage and some education.
  - When λ is high then expected productivity is high, hence ID at high-productivity wage may be lower than at 0 education and expected productivity wage.
  - The problem is, is that if higher productivity workers set education to 0 she may be assumed to be low productivity, hence will not receive the expected productivity wage.

Screening: Insurance

Where the **uninformed party** takes advance action to distinguish the types of individuals with hidden information.

Model: Accident Prone vs Safe Individuals Health Insurance

Suppose consumers are risk-averse expected utility maximisers with some wealth W and some potentially loss M which different consumers experience with different probabilities.

Consumers H have a high risk of incurring loss and consumers L have a low risk. Let λ be the fraction of workers who are type L. Hence

$$\lambda = \mathbb{P}(\text{worker} = L)$$

The average probability of incurring a loss is given by,

$$\tilde{p} = \lambda p_L + (1 - \lambda) p_H$$

Suppose that firms are risk neutral and competitive – hence they make zero profit. (Insurance in this case is actuarially fair – The line on which insurance contracts are offered is the same line as the fair odds line from the endowment point)

Insurance contracts can be thought of as (Q, R) or w_0, w_1, where: (1) Q is the premium and R is the reimbursement, or; (2) w_0 is wealth if no loss occurs, and w_1 is wealth if loss occurs.

This contracts are in essence the same thing,

$$(w_0, w_1) = (W - Q,\ W - Q - M + R)$$

Case (1): Risk is publicly observable

- From the endowment point there are three different zero profit lines.
  - $0\pi_{p_L}$: zero profit when only low-risk individuals get insured
  - $0\pi_{p_H}$: zero profit when only high-risk individuals get insured
  - $0\pi_{\tilde{p}}$: zero profit when both individuals get insured
- All types here receive full insurance so they are on the certainty line (because of perfect competition).
  - Hence they are on a higher indifference curve than they would have been on without insurance.
  - They are now receiving EV.

Case (2): Pooling Equilibria

- Here the point where H and L indifference curves cross on the $0\pi_{\tilde{p}}$ is a pooling equilibrium.
- This equilibrium is not tenable, however, since (due to single crossing) a firm could offer a contract in the shaded region which would be at a,
  - Higher ID curve for type L
  - Lower ID curve for type H
  - Below $0\pi_{\tilde{p}}$ and hence would make positive profits.
- Given that such a point exists no rational firm would offer a pooling contract as they would be left with only type H individuals since another firm could take all the type L's as described above.

Case (3): Separating Equilibria

- Type H:
  - The H type ID curve MUST be tangential to the $0\pi_{p_H}$. Were this not the case there would be space above the H ID curve but below the zero profit line which consumers would obviously prefer and in which firms would make profit.
  - Slope of $0\pi_{p_H}$ is,

$$-\frac{1-p_H}{p_H}$$

  - Recall that we have actuarially fair insurance which is identical to being on the fair odds line.
- Type L:
  - Now for the L-types workers we have two conditions:
    (1) Firms make zero profit hence contract is on $0\pi_{p_H}$ line (otherwise incentive for firms to deviate).
    (2) Type H's must be indifference between this and their own contract.
- Overall:
  - Points H^* and L^* on the diagram give the type H contract and the type L contract. Here type L workers' pay their actuarially fair premium but are not fully insured (if they were fully insured then type H workers would rather pay this lower premium for full insurance).
- But a separating policy may not exist: The existence of a separating equilibria depends on λ - the proportion of individuals who are type L.
- When λ is high:
  - The $0\pi_{\tilde{p}}$ curve is steeper.
  - If it is steep enough then it will intersect the L type ID curve.
  - In this case the grey shaded area will be created, and in this area:
    * L type workers prefer contracts
    * H type workers prefer contracts
    * Firms make positive profits.
- Hence in this situation no separating (or pooling) equilibrium exists.

Moral Hazard: Principal Agent

Now we consider that while there might be symmetric information at the time of contracting there might be asymmetries that arise after contracting. The principal is the firm's owner and the agent the manager. The principal cannot observe how much effort the agent exerts.

Model: The Principal-Agent Problem

Principle is risk neutral and agent is risk averse.

Agent chooses either high effort: $e = e_H$, or low effort: $e = e_L < e_H$.

Gross profit,

$$\pi(e) \ : \ \pi = \pi_H \text{ or } \pi = \pi_L < \pi_H$$

High effort makes high profit more likely,

$$\mathbb{E}[\pi \mid e_H] > \mathbb{E}[\pi \mid e_L]$$

Agent's utility,

$$u(w, e) = v(w) - g(e), \text{ with } v' > 0, \ v'' \leq 0, \ g' > 0$$

Agent's reservation utility $\bar{u}$

The problem requires the principal to choose,

$$e \in (e_L, e_H) \text{ and } w(\pi) \text{ to maximise } \mathbb{E}[\pi - w(\pi) \mid e] \text{ such that,}$$

(1) The agent wishes to participate: ***Individual Rationality (IR)***

$$\mathbb{E}[v(w(\pi)) \mid e] - g(e) \geq \bar{u}$$

- The agents expected utility from their wage (dependent on their profit) less the disutility of effort should be as good as the utility they get from doing nothing.
- Rather minimally the agent should at least want to work for the principal.

(2) The agent wishes to choose same e as chosen by the principal: ***Incentive Compatibility (IC)***

$$e \text{ maximises } \mathbb{E}[v(w(\pi)) \mid e] - g(e)$$

- In order to induce e_H the expected utility from their wage less their disutility of working should be higher for the higher effort than it is for the lower effort level.
- That is, it must be optimal for the agent to exert the effort level that the principal wants her to exert.
- If the principal wants her to exert e_H then it must be the case that the worker does best in terms of her expected wages by exerting high effort.

Case (1): Effort is observable and risk averse agent ($v'' < 0$)

- Observable effort implies that you can ignore IC.

$$\min \mathbb{E}[w(\pi) \mid e] \text{ such that } \mathbb{E}[v(w(\pi))] - g(e) \geq \bar{u}$$

  - Minimise the expected wage so that the worker still gets a higher utility than they would doing nothing/doing their alternate option.

- IR binds and as agent is risk averse it is optimal to fix $w(\pi)$ at $w^*_{e_H}$ or $w^*_{e_L}$ such that,

$$v(w^*_{e_H}) - g(e_H) = \bar{u} \text{ and } v(w^*_{e_L}) - g(e_L) = \bar{u}$$

- Then find optimal e

$$\max_{e \in \{e_L, e_H\}} \{\mathbb{E}[\pi \mid e_L] - w^*_{e_L} \ , \ \mathbb{E}[\pi \mid e_H] - w^*_{e_H}\}$$

  - Essentially what we are doing here is finding the two wages for the two effort levels and then the principal should choose the wage that will maximise her profit.

Case (2): Effort not observable and risk neutral agent

- Risk neutral implies that $v'' = 0$
- The wage in this case is conditioned only on profit.
- We can implement effort levels by 'selling' the project to the agent at a price α.
- That is $w(\pi) = -\alpha + \pi$ and the agent choose e to maximise,

$$\mathbb{E}[\pi - \alpha \mid e] - g(e)$$

Case (3): Effort not observable and risk averse agent

- To implement e_L offer fixed wage $w^*_{e_L}$ such that the IR constraint binds,

$$v(w^*_{e_L}) - g(e_L) = \bar{u}$$

- To implement e_H we pay a variable wage that depends on profit, $w(\pi)$. The higher wage is paid if the outcome of the agents work is π_H and a lower wage is paid if the outcome is π_L.
  - In this case we need the IR & IC both to bind:
    (1) The worker must want to work (IR),
    (2) Further the worker must do better (in terms of expected utility) by exerting e_H than exerting e_L (IC). Recall the principal can't tell what effort level the agent is exerting, but here the principal wants to induce e_H.
  - In the case in which there are two effort levels the IC gives,

$$\mathbb{E}[v(w(\pi)) \mid e_H] - g(e_H) \geq \mathbb{E}[v(w(\pi)) \mid e_L] - g(e_L)$$

Worked Examples

Example: Adverse Selection

There are three qualities of second-hand bicycles available in equal numbers. Type X consumers and type Y consumers have different values for each quality of bicycle,

Quality	Type X	Type Y
High	90	110
Medium	80	85
Low	70	60

(a) What is the social planner allocation?

Now we will assume that type X consumers are sellers (they own the bicycles), and type Y consumers do not (they are buyers).

(b) What is the outcome with perfect information?

(c) What is the outcome when *neither* seller's nor buyers know the quality?

(d) Finally, suppose that buyers do not know the quality of any particular bicycle for sale, but sellers do know the quality of the one that they own. What will be the market equilibrium outcome?

Social Planner will allocate High to Y, Medium to Y, and Low to X.

Under Perfect Information High sells for 90-110; Medium sells for 80-85; and Low does not sell. Whether these sell at the upper range, lower range, or somewhere in the middle of the range depends on two things. First is whether we have excess demand or supply, second is how the auction/bargain is being conducted.

Under Symmetric Imperfect Information,

$$\mathbb{E}[Y] = \frac{1}{3}110 + \frac{1}{3}85 + \frac{1}{3}60 = 85$$
$$\mathbb{E}[X] = \frac{1}{3}90 + \frac{1}{3}80 + \frac{1}{3}70 = 80$$

So the bikes will sell for a price between 80-85, again dependent on supply/demand, and further the auction scheme.

Under Asymmetric Information,

The $\mathbb{E}[Y] = 85$, hence type X sellers will not sell the high-quality bikes.

Then the $\mathbb{E}[Y \mid \text{no high bikes}] = 72.5$, hence medium quality bikes won't sell either.

The $\mathbb{E}[Y \mid \text{low}] = 60$, so low bikes won't sell either since sellers value then at 70.

Therefore no bikes will sell and the market unravels from the top.

Example: Signalling & Education

There are two types of worker: L-types with low productivity θ_L, and H-types with high productivity θ_H; each worker knows their own type. Many risk-neutral firms compete for the services of the workers, but they cannot observe a worker's type.

(a) Explain why some of the workers would like to be able to signal their type to the firms. What would make the signal credible?

Suppose that $\theta_L = 80$, $\theta_H = 100$, and that there are three times as many L-types as H-types; the workers have no outside opportunities.

(b) What is the market equilibrium outcome if no signal is available?

(c) Suppose the workers can acquire an education, observable by the firms. This would cost 22 for an L-type, but only 12 for an H-type. Characterise the unique market equilibrium.

(d) How would the analysis in part (c) change if education costs were 18 for an L-type, and 16 for an H-type?

How and why signal?

H would like to signal in order to get a wage equal to her productivity, rather than just receive a wage equal to average productivity.

In order to be feasible for H the cost of the signal must be less than the difference between wage for H's productivity and the average wage, and in order to be credible the signal must be too expensive for L to copy.

If no signal is available then,

$$w^* = \mathbb{E}[\theta] = \frac{1}{4}100 + \frac{3}{4}80 = 85$$

The separating equilibrium is,

$$\begin{aligned} u_H &= w(\theta_H) - c(1, \theta_H) = 100 - 12 = 88 \\ u_L &= w(\theta_L) - c(0, \theta_L) = 80 - 0 = 80 \end{aligned}$$

This is optimal for type H's, since utility of 88 is better than a utility of 80 - recall that if a type H doesn't get an education then she is assumed to be a type L and hence receives the θ_L wage.

Notice that L wouldn't deviate either. Should she decide to copy the signal and get an education her utility is,

$$\begin{aligned} u_L &= w(\theta_L) - c(1, \theta_L) \\ &= 100 - 22 = 78 \end{aligned}$$

Which is lower than had she not taken an education at all.

What if education costs 18 for type L's, and 16 for type H's?

$$\begin{aligned} u_H &= w(\theta_H) - c(1, \theta_H) = 100 - 16 = 84 \\ u_L &= w(\theta_L) - c(0, \theta_L) = 80 - 0 = 80 \end{aligned}$$

But notice that now it would be optimal for type L's to copy the signal, since,

$$\begin{aligned} u_L &= w(\theta_L) - c(1, \theta_L) \\ &= 100 - 18 = 82 \end{aligned}$$

With no separating equilibrium there is a pooling equilibrium of 85, which no one would want to deviate from.

Example: Principal-Agent Problem

A risk-neutral principal can hire a risk-averse agent to undertake a project. There are two possible profit outcomes, π_H and π_L such that $\pi_H > \pi_L > 0$; assume throughout that $\pi_L = 60$. The agent can exert effort $e = 0$ or $e = 1$. Importantly,

$$\mathbb{P}[\pi_H \mid e = 0] = \frac{2}{5}$$
$$\mathbb{P}[\pi_H \mid e = 1] = \frac{3}{5}$$

The agent's utility function based on wages and effort is, $u = \sqrt{w} - e$ and the agent has reservation utility $\bar{u} = 8$.

(a) Find the optimal contract given Observable Effort and $[\pi \mid e_H] = 195$.

(b) Find the optimal contract given Unobservable Effort and $[\pi \mid e_H] = 195$.

Observable Effort,

By 'observable' we mean contractible - that is the principal can contract the agent to exert a certain effort level. Hence we can ignore the IC constraint since the agent will always comply with the contract. Only IR matters, and whats more in order to maximise profit it must bind.

Wages, where w_1 is the wage given $e = 1$ and w_0 the wage given $e = 0$,

$$\begin{aligned}\sqrt{w_1} - 1 &= 8\\ w_1 &= 81\\ \sqrt{w_0} - 0 &= 8\\ w_0 &= 64\end{aligned}$$

Supposing $[\pi \mid e_H] = 195$ and $[\pi \mid e_L] = 60$,

$$\begin{aligned}\mathbb{E}[\pi \mid e = 1] &= \frac{3}{5}195 + \frac{2}{5}60 = 141\\ \mathbb{E}[\pi \mid e = 0] &= \frac{2}{5}195 + \frac{3}{5}60 = 114\\ \mathbb{E}[\pi \mid e = 1] - w_1 &= 141 - 81 = 60\\ \mathbb{E}[\pi \mid e = 0] - w_0 &= 114 - 64 = 50\end{aligned}$$

Since $\mathbb{E}[\pi \mid e = 1] - w_1 > \mathbb{E}[\pi \mid e = 0] - w_0$ then high effort is optimal to induce.

Unobservable Effort,

We can induce $e = 0$ simply by offering the same wage as in the oberservable effort contract. This is because to work with minimal effort we only need the agent to weakly prefer working to doing otherwise. Hence the utility of her work must be as good as her reservation utility. That is only IR need hold.

To induce $e = 1$,

Let $v_j = \sqrt{w(\pi_j)}$, and recall our constraints: IR and IC,

$$\text{IR} : \mathbb{E}[v(w(\pi)) \mid e] - g(e) \geq \bar{u}$$

$$\text{IC} : \max_e \mathbb{E}[v(w(\pi)) \mid e] - g(e)$$

In order to optimally induce $e = 1$ we need it to be the case that the agent would rather work at the higher wage and exert effort $e = 1$ than get her reservation utility - hence IR must hold. Similarly it must be the case that the agent would rather exert $e = 1$ than take the variable wage packet offered to induce $e = 1$, but secretly only exert $e = 0$ - hence IC must bind.

IR implies,

$$\frac{3}{5}v_H + \frac{2}{5}v_L - 1 \geq 8$$
$$3v_H + 2v_L \geq 45$$

IC implies,

$$\frac{3}{5}v_H + \frac{2}{5}v_L - 1 \geq \frac{2}{5}v_H + \frac{3}{5}v_L - 0$$
$$v_H - v_L \geq 5$$

Recognise that, since the principal sets wages and hence v_i, further the principal wishes to maximise π, both of these constraints will bind.

We can then solve simultaneously,

$$v_H - v_L = 5$$
$$3v_H + 2v_L = 45$$
$$v_H = 11 \ , \ v_L = 6$$

Hence,

$$w(\pi_H) = 121 \ , \ w(\pi_L) = 36$$

When $\pi_H = 195$ will the principal want to induce $e = 0$ or $e = 1$?

$$\mathbb{E}[\pi - w(\pi) \mid e = 1] = \frac{3}{5}(195 - 121) + \frac{2}{5}(60 - 36) = 54$$
$$\mathbb{E}[\pi - w(\pi) \mid e = 0] = \frac{2}{5}(195 - 36) + \frac{2}{5}(60 - 36) = 50$$

Since $\mathbb{E}[\pi - w(\pi) \mid e = 1] > \mathbb{E}[\pi - w(\pi) \mid e = 0]$ the principal will pay the variable wage packet $(121, 36)$ to induce $e = 1$.