# General Heterogeneous Moran Processes

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## 1 Introduction

## 2 Literature Review

## 3 Model

In this section, we will look at the underlying model of the heterogeneous Moran process. We define the following:

- $N$ **ordered** individuals
- $k$ types $A_1, A_2, \dots A_k$
- A state space S given by the set of ordered $N$-tuples with entries of type $A_1, A_2, \dots A_k$. Note that $|S| = k^N$. A state $v = (v_1, v_2, \dots) \in S$ is called **absorbing** if $v_i = v_j \ \forall \ i, j \in \mathbb{N}^+ : i, j \leq N$
- A fitness function $f : S \to \mathbb{R}^N$

Let $h(v, u)$ denote the Hamming distance [1] between two states $v, u \in S$. We consider a Markov chain [2], where for the transition $V = (v_1, v_2, \dots, v_N) \to U = (u_1, u_2, \dots, u_N)$, the transition probability is defined as follows:

$$p_{v,u} = \begin{cases} \frac{\sum_{v_i = u_{i^*}} f(v_i)}{\sum_{v_i} f(v_i)} & \text{if } h(v, u) = 1, \text{ differing at position } i^* \\ 0 & \text{if } h(v, u) > 1 \\ 1 - \sum_{u \in S \setminus \{\text{v}\}} P_{v,u} & \text{if } h(v, u) = 0 \end{cases} \tag{1}$$

The final case corresponds to a transition from $v$ to itself. This is possible when a given individual has their action type removed and any other individual with the same type is chosen for duplication. A direct computation of $P_{v,v}$ is given by:

$$\frac{1}{N} \sum_{i=1}^{N} \frac{\sum_{v_j = v_i} f(v_i)}{\sum_{v_j} f(v_i)} \tag{2}$$

This would require at least $N^2 + N$ calculations. However, we can observe that a transition of some sort must occur, and so we simply take the probability of not transitioning to any **different** state as $P_{v,v}$.

An important case which proceeds from (1) is that of a transition $v \to u$ where $u$ contains an individual of a *type* not found in $v$. For example, the transition (0,1) $\to$ (0,2). This would be forbidden by the intuition of a standard Moran process - i.e the new individual being the duplication of another individual in $v$. However, we can see that the standard formula yields $p_{v,u} = 0$ because $\nexists \ v_i \in V : v_i = u_{i^*}$. Following this, we see that (1) also correctly gives us $p_{v,u} = 0$ for any **absorbing** $v \neq u$.

## References


[1] Dave K. Kythe, Prem K. Kythe (2012) - Algebraic and Stochastic Coding Theory

[2] William J. Stewart (2009) - Probability, Markov Chains, Queues, and Simulation: The Mathematical Basis of Performance Modeling