A Heterogeneous Moran Process for the Analysis of Public Goods Games

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1 Introduction

2 Literature Review

Evolutionary game theory has long been at the forefront of the study of emergent cooperation, showing how a strategy that is seemingly unfavourable for the individual, but beneficial to the collective, can become dominant in systems of rational actors. This is often shown through the study of games such as the iterated prisoner's dilemma[1, 2] and the snowdrift game[3]. These examples, however, only allow for pairwise interactions, which does not accurately represent many real-world scenarios. Therefore, we look at a common game used to model the sharing of resources and interactions between multiple people - the public goods game.

In order to further tailor the idea of a public goods game to real world scenarios, we introduce the idea of heterogeneity to the system. Often in the study of games we consider each individual to exist in the same conditions - having the same utility and probability of transitioning to a different action type. However, this does not accurately reflect many of the interactions that occur in the real world, where individuals may have different factors affecting which strategies they are able to take. For example, in a public goods game, one player may wish to contribute x units to the public good, however they do not have enough wealth available to spare such a contribution. Heterogeneity allows us to give varying attributes to the different actors in order to simulate such situations. These attributes can be encoded into the players themselves as separate to the game, for example the reputation in [4], exist within the strategies of the players [5, 6], or they may exist within the actual payoffs which the players receive. We will look at multiple different ways in which heterogeneity can be implemented in order to model different scenarios.

In much of the literature, we see heterogeneous public goods game being played graphically. By this I mean that in much research, players are placed on graphs (often regular square lattices, as in [4, 6]) and participate in multiple public goods games at the same time, with their payoff being the combined payoff from all their games. This gives even more weight to the decision to cooperate or defect, and allows players to interact in scenarios with many different levels of these heterogeneous attributes. We also see many ideas for what these attributes could be based upon. [4] takes the idea of that players could build a reputation through their decisions across multiple generations, with players refusing to contribute higher amounts to groups made up of those who have a reputation for defection. While this is a dynamically updating heterogeneous attribute, some have proposed static attributes based on inherent properties of the individuals, such as in [5] where the players participate in different amounts of games, and contribute different amounts based on this factor.

In both of the above cases, it is shown that an increased scrutiny of heterogeneous attributes in a spatial public goods game is beneficial to the emergence of cooperation, though for different reasons. In [4], we see that increasing such scrutiny obviously harms defectors, as they receive reduced payoffs based on their low reputation due to being unable to coerce high contributions from other players. [5], however, finds that if we encourage players to cooperate with those in a similar number of games to them, here done by tuning the β parameter, the high-degree defector nodes are unable to gather a large payoff, and so will be unable to spread their defection to their neighbours. If you encourage players to interact with those with different degrees, however, then middle-degree defectors will make heavy losses and be unable to survive.

[6] shows how the benefits of this heterogeneity are not always as pronounced as it may seem. When the contribution is not calculated by some attribute, but rather is an attribute in and of itself (essentially becoming a new strategy) then even in systems where contribution dominates, we will see low-value contributions act as a sort of defection against the most globally beneficial strategies. Therefore, we can see that the emergence of cooperation is not the end of the story in

some cases, and we must look at what sort of cooperation it is that we have fostered the emergence of.

One common theme in the spatial public goods game is the manner in which cooperation spreads. An initial "invasion" of defectors leads to cooperation mainly existing within small clusters. These clusters, however, are much more profitable than their defecting neighbours. Therefore, the cooperating clusters will influence the nearby defectors to join in, which eventually spreads throughout the system. This relies on the ability of the clusters to resist the initial invasion of defectors - an occurrence which relies on a high enough r value (the positive multiplier of the contribution to the public good).

We also see that a potential climate club may soon be forming within the European Union. While it functions slightly differently to the classic example in [9], it follows the same principles of encouraging countries to join the group by punishing those outside of the club, in order to attempt to reduce overall carbon emissions. This is the "Carbon Border Adjustment Mechanism" [11]. Currently, all goods within the EU require taxation based on their carbon footprint, however imported goods from outside cannot be subject to such taxes. CBAM aims to prevent foreign goods from gaining benefits due to environmentally harmful practices by enforcing that EU companies report the carbon content of imported goods, and purchase certificates to cover the environmental cost of said goods. However, if a country already levies a tax on their own goods (similar to the EU's carbon tax), then this cost will be taken into consideration, and fewer certificates must be purchased. This discourages the importation of goods from countries which do not levy a similar tax on their own items, forming a structure which behaves very similarly to the "climate clubs" in [9].

3 The Model

3.1 An Initial System

In this section, we will look at the underlying model of the heterogeneous Moran process. We define the following:

- N **ordered** individuals
- k types $A_1, A_2, \dots A_k$
- A state space S given by the set of ordered N-tuples with entries of type $A_1, A_2, \dots A_k$. Note that $|S| = k^N$.
  A state $v = (v_1, v_2, \dots) \in S$ is called **absorbing** if $v_i = v_j \ \forall \ i, j \in \mathbb{N}^+ : i, j \leq N$, and the set of absorbing states is called S^Γ
- A strictly positive fitness function $f : S \to \mathbb{R}^N$

Let $h(v, u)$ denote the Hamming distance [7] between two states $v, u \in S$. We consider a Markov chain [8], where for the transition $v = (v_1, v_2, \dots, v_N) \to u = (u_1, u_2, \dots, u_N)$, the transition probability is defined as follows:

$$p_{v,u} = \begin{cases} \frac{\sum_{v_i = u_{i^*}} f(v_i)}{\sum_{v_i} f(v_i)} & \text{if } h(v,u) = 1, \text{ differing at position } i^* \\ 0 & \text{if } h(v,u) > 1 \\ 1 - \sum_{u \in S \setminus \{\mathrm{v}\}} P_{v,u} & \text{if } h(v,u) = 0 \end{cases} \tag{1}$$

Transitioning denotes something slightly different in the heterogeneous case of the Moran process as compared with the homogeneous. While we would traditionally say an *individual* was removed and a new individual was generated to replace them, we must now say an individual's *action type* was removed, and replaced with a new action type. This is because the individuals themselves are ordered in the heterogeneous Moran process, and possess certain attributes specific to themselves. Therefore, we cannot say that the individual themself is replaced, but rather that their action type is replaced.

The final case in (**??**) corresponds to a transition from v to itself. This is possible when a given individual has their action type removed and any other individual with the same type is chosen for duplication. A direct computation of $P_{v,v}$ is given by:

$$\frac{1}{N} \sum_{i=1}^{N} \frac{\sum_{v_j = v_i} f(v_i)}{\sum_{v_j} f(v_i)} \tag{2}$$

This would require at least $N^2 + N$ calculations. However, we can observe that a transition of some sort must occur, and so we simply take the probability of not transitioning to any **different** state as $P_{v,v}$.

An important case which proceeds from (**??**) is that of a transition $v \to u$ where u contains an individual of a *type* not found in v. For example, the transition (0,1) → (0,2). This would be forbidden by the intuition of a standard Moran process - i.e the new individual being the duplication of another individual in v. However, we can see that the standard formula yields $p_{v,u} = 0$ because $\nexists\ v_i \in V : v_i = u_{i^*}$. Following this, we see that (**??**) also correctly gives us $p_{v,u} = 0$ for any **absorbing** $v \neq u$.

This model allows for heterogeneity as we have a fitness function dependent on the individual, not dependent on the action taken. Therefore, we can give different individuals of the same action type a different fitness by passing different attributes to the fitness function itself. We will see examples of this in future sections.

As this forms a Markov chain, we can therefore define a transition matrix T for any given state space. This can be used to show the transition probabilities between any two given states, and therefore can be used to find the fixation probabilities for any given starting state by the following method:

Given a transition matrix T, we can calculate the state distribution after k iterations of the Moran process using the formula T^k [8]. Therefore, as $k \to \infty$, we will clearly acquire a distribution that tends towards the absorption probabilities for a given starting state, as the process will stabilise only if we enter an absorbing state (while subsets of S of states may have extremely high probabilities of transitioning to each other, the only way to have certainty of not transitioning away from a state or pair of states is to be in one of the previously defined absorbing states.)
Thus, our absorption probabilities starting at are given by the non-zero entries in the matrix $\lim_{k\to\infty} T^k$

4 The General Heterogenous Public Goods Game

4.1 The Standard Model and Transformations

In this section, we shall see how this model can be applied to the classical public goods game. In this game, individuals choose whether or not to contribute to a public resource pool. In the end, the total resource is multiplied by some factor $r > 1$ and distributed equally between each player. This model often encourages a behaviour known as "free-riding" [9], where players refuse to contribute to the pool and simply take the benefit provided by other individuals' contribution. The classic problem of a public goods game is to provide an environment where such free-riding is not a profitable strategy - or rather, is less profitable than contributing to the public good.

In our model, we can simulate a public goods game by providing the following payoff function:

$$\sigma(v_i) = \frac{r\sum_{j=1}^{N} C_{v_j}}{N} - C_{v_i} \tag{3}$$

(where C_{v_j} is the contribution by individual j.)
In the homogeneous public goods game, we can see that C_{v_j} can only take one of two values: some constant α if individual j cooperates, and 0 if not. However, the heterogeneous game will require a more tailored C_{v_j}.

Note that $\sigma(v_j)$ can be negative which would then not lead to sensible probability values as needed in **??**. Therefore, we must look at some transformation of σ to use as our fitness function.

A common method [10] for this is to apply the exponential function to our payoff function. By using $e^{\kappa\sigma}$ as our fitness function we will have positive values, however this particular approach gives:

$$\frac{\sum_{v_i=u_{i^*}} e^{\frac{\kappa r\sum_{j=1}^{N} C_{v_j}}{N} - C_{v_i}}}{\sum_{v_i} e^{\frac{\kappa r\sum_{j=1}^{N} C_{v_j}}{N} - C_{v_i}}} \tag{4}$$

Now, as e^r does not rely on v, we can take this out of the sum and acquire the following:

$$\frac{e^{\frac{\kappa r \sum_{j=1}^{N} C_{v_j}}{N}} \sum_{v_i=u_{i^*}} e^{-C_{v_i}}}{e^{\frac{\kappa r \sum_{j=1}^{N} C_{v_j}}{N}} \sum_{v_i} e^{-C_{v_i}}} = \frac{\sum_{v_i=u_{i^*}} e^{-C_{v_i}}}{\sum_{v_i} e^{-C_{v_i}}} \tag{5}$$

and we see that in this case, $p_{v,u}$ would not rely on r. Therefore, let us consider some other methods of guaranteeing a positive fitness function.

The first of these is known as the "shifted linear" transformation. We take, for some small tunable ϵ:

$$f(v_i) = \sigma(v_i) - \min_{v_j \in v} \sigma(v_j) + \epsilon \tag{6}$$

This guarantees a positive value, with the lowest fitness taking the value ϵ, a parameter that can be chosen based on the system in question.

Another type of mapping that we can use is the "Affine-linear mapping". In this, for some tunable ϵ, we take the following:

$$f(v_i) = 1 + \epsilon\sigma(v_i) \tag{7}$$

4.2 The Homogeneous Case

The most common type of public goods game is the homogeneous public goods game. In this game, we consider that we have a constant $C_{v_i} = \alpha$ for all players v_i. In this case, we take the payoff function:

$$\sigma(v_i) = \frac{r \sum_{j=1}^{N} k_j \alpha}{N} - k_i \alpha = (Kr - k_i)\alpha \tag{8}$$

Where $k_j = 1$ if j contributes and 0 if not, and K is the fraction of the population who contribute.

However, we cannot use 4.1 transformation or else we once again run into a problem with too many factors cancelling down and becoming unreliant on r:

$$(K * r - k_i)\alpha - \min_j (K * r - k_j)\alpha + \epsilon = (1 - k_i)\alpha + \epsilon = \begin{cases} \epsilon & \text{if } v_i \text{ contributes} \\ \alpha + \epsilon & \text{if not} \end{cases} \tag{9}$$

Not only does our choice of ϵ become too powerful, but also we cancel out all reliance on r

Therefore, we take the affine-linear mapping shown in 4.1 in order to see our fixation probabilities.

We will begin by looking at the state space with $N = 4$ and see how cooperation emerges based on different values of r and α.

Here we can see how the various parameters can be tuned to affect our fixation probabilities. As r increases, we see an increase in cooperation as individuals benefit more from their own contributions. However, as α increases, individuals must pay more to contribute to the public good, which also benefits defectors who receive higher payoffs without risking their own contribution - leading to a wider gap between the payoffs of cooperators and defectors in the game.

5 Public Goods Games with Heterogeneous Returns

When we look at a public goods game, we often generalise the players to all contribute the same amount. However, in the real world this is often not the case, as different individuals may have different attributes which affect their ability to contribute as much as others. In this section, we look at how a heterogeneous population with respect to contribution can affect the fixation probability.

We begin by defining a maximum value M, which is the total contribution to the public good in the case that all players contribute. The responsibility for the contribution of such M is then split between all players according to some rule. This rule will take the form of some λ, such that $\sum_{j=1}^{N} j\lambda = M$.
We look at 6 such rules for sharing the contribution between all players.

5.1 The Linear Case

In this case, players simply contribute an amount equal to their position within the state. For example, a state (1,1,0,1) for $M = 10$ will contribute (1, 2, 0, 4) respectively. For a given N, M, we have that here, $\lambda = \frac{2M}{N(N+1)}$

6 The Dirichlet Distribution

6.1 The Basics

We begin with a beta distribution. Imagine you have tossed a coin N times, observing H heads and T tails. Our beta distribution takes two parameters: α and β with values $H + 1$ and $T + 1$ respectively. The "peak" (mode) of the beta distribution occurs at $\frac{H}{H+T}$.

Now the Dirichlet distribution extends this problem into K components. Imagine you have a bag containing different coloured balls - red, blue, and green. After a certain number of samples, you observe R, B, and G balls of each colour respectively. Then, similarly to our beta distribution, our dirichlet distribution takes the values $\alpha_1 = R + 1, \alpha_2 = B + 1, \alpha_3 = G + 1$.
Now, unlike the beta distribution, the Dirichlet distribution returns a vector of probabilities $X = (x_1, x_2, ..., x_K)$ such that $\sum_{i=1}^{K} x_i = 1$, rather than a single probability. While theoretically the beta distribution can be used to return a probability vector which sums to 1 by taking the compliment of it's value for a given parameter (in our above example, it could return $[\frac{H}{H+T}, \frac{T}{H+T}]$), the distribution is not often used this way. For the Dirichlet distribution, we always obtain such a vector. The PDF is therefore defined by this vector as follows:

$$f(x_1, ..., x_k, \alpha_1, ..., \alpha_k) = \frac{1}{B(\alpha_1, \alpha_2, ...)} \Pi_{i=1}^{K} x_i^{\alpha_i} \tag{10}$$

where $B(\alpha_1, \alpha_2, ...)$ is the normalisation function

$$\frac{\Pi_{i=1}^{K} \Gamma(\alpha_i)}{\Gamma(\sum_{i=1}^{K} \alpha_i)} \tag{11}$$

and Γ is the Gamma distribution. This exists to ensure that $\int_{-\infty}^{\infty} f(x_1, ..., x_k, \alpha_1, ..., \alpha_k) = 1$

Now let us look at the expected value of the Dirichlet distribution. We have the following:

$$E(X_i) = \frac{\alpha_i}{\sum_j \alpha_j} \tag{12}$$

So as $\sum_{j=1}^{K} \alpha_j = n$, we have that if we take a Dirichlet distribution D_1 with parameters $(\alpha_1, \alpha_2, ...)$ and another D_2 with parameters $(z\alpha_1, z\alpha_2, ...)$ for a constant z, they will retain the same mean for each component due to the following:

$$E(D_{2,i}) = \frac{n_i}{n} = \frac{z\alpha_i}{\sum_{j=1}^{K} z\alpha_j} = \frac{\alpha_i}{\sum_{j=1}^{K} \alpha_j} = E(D_{1,i}) \tag{13}$$

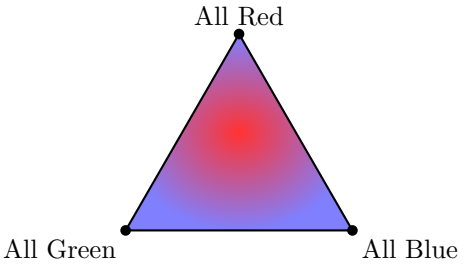
Now, as our number of observations n increases, we get that our expected value converges to $\frac{n_i}{n}$, where X_i is observed n_i times. The variance of the data also behaves in a similar way. The explicit formula is given by:

$$Var(X_i) = \frac{\frac{\alpha_i}{\sum_{j=1}^{K}\alpha_j}(1 - \frac{\alpha_i}{\sum_{j=1}^{K}\alpha_j})}{\sum_{j=1}^{K}\alpha_j + 1} \tag{14}$$

which tends to $\frac{x_i(1-x_1)}{n}$ as $n \to \infty$. We see that this is the result that we would intuitively expect by normal statistical methods. We also notice that as this is strictly decreasing as $n \to \infty$, and so we would acquire the result that $Var(D_{2_i}) < Var(D_{1,i})$. This fits with our intuition once again, as if we obtain more data for each case in the same ratio (so for example, having data (2 red, 3 blue, 1 green) and then obtaining the additional data (4 red, 6 blue, 2 green)), we will retain the same mean value in our posterior distribution, however our variance will be reeduced.

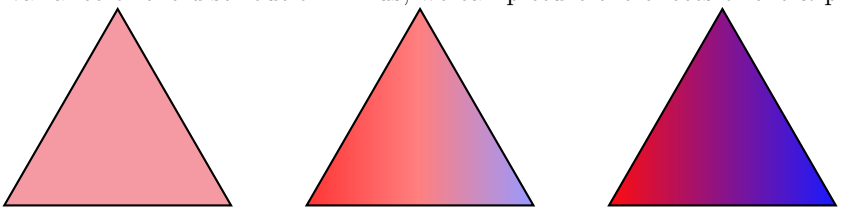
6.2 A Visual Example of Sampling from the Dirichlet Distribution

Now for a deeper look into sampling from the Dirichlet distribution. Consider each case as a vertex on the probability simplex as follows (we take K=3 as an example, however we would consider a K-sided simplex for a Dirichlet distribution with K components):



The points on this simplex correspond to different probability vectors, with the vertices representing the extreme cases in which our vectors are simply 1 for some entry, and 0 elsewhere. The distribution gives us which regions of the simplex have a higher probability of being the true values (for example, 5 red, 4 blue, 4 green being in the bag). This is a visual description of the PDF of the K-dimensional Dirichlet distribution.

Now how do the α parameters control the spread of probabity across the simplex? Well, as proven in the last section, we know that the ratio between the αs determines the expected value of the distribution, and the total of the αs determines the variance of the distribution. Thus, we can picture the effects of the α parameters visually as follows:



Where the red regions represent areas of high probability, and blue regions are those of low probability. We see that each α corresponds to a different vertex on the probability simplex, with the distribution being concentrated towards a vertex depending on it's corresponding α value. To illustrate the reason behind this, consider a set of αs where $\alpha_i = 0 \forall i \neq 1, \alpha_1 = 1$. In this case, our probability would be entirely concentrated on the vertex corresponding to α_1, with no variance. This is obvious from the expected value formula above. In a similar case of $\alpha_i = 1 \forall i \neq 1, \alpha_1 = 100$, we would see a very heavy concentration of probability towards the vertex corresponding to α_1, however we would not see it entirely concentrated into one point as before, instead there would be some variance, who's source is the other non-zero α values, that causes the probability to become more distributed across the simplex.

This illustration shows how the αs affect the probability distribution in practice, and serves as a visual demonstration of how the formulas in the previous section behave. Now, we shall move on to look at how the Dirichlet distribution applies in practice to our public goods game.

6.3 The Dirichlet distribution and a Markov Process

Consider now a Markov process. For a transition matrix T, $T_{A,B} = p_{A,B}$, the probability of transitioning from state A to state B (in this example, we have ordered the states in our state space, and A, B are integers). When we look at the the steady state of a given transition matrix, we are left with the vector $\mu = (\mu_1, \mu_2, ...)$, where once the steady state is reached, μ_i gives the proportion of time steps that, on average, will be spent in state i. This is an eigenvector for T. This is because for any state distribution vector $\tilde{\mu}$, we have that $\tilde{\mu}T$ gives us the probability of being in each state at the next timestep. For the vector μ, we get $\mu T = \mu$.

Now, upon obtaining this vector for a given Markov chain, we can see the folllowing properties:

- The entries of μ are positive
- The sum of the μ_is is equal to 1

Now, these probabilities can be meaningfully interpreted as the realisations of components of a Dirichlet distribution. Therefore, we can say that $\mu \sim Dirichlet(\alpha_1, \alpha_2, ..., \alpha_{k^N})$. Now, if we have fixed probabilities for the entries of a given transition matrix T, then the μ that we obtain is the average proportion of time steps spent in each state. Thus, we can say that the obtained vector $\mu = E(X) = (E(X_1), E(X_2), ...)$, where X is a random variable with a Dirichlet distribution. We can therefore fit a Dirichlet distribution according to this.

In the case of a heterogeneous moran process, our steady state is trivial - we always remain in one of the absorbing states $v \in S^\Gamma$. Therefore, for any given starting state, we have that our μ only has non-zero entries for μ_is corresponding to absorbing states. This means that we would model this Dirichlet distribution with many $\alpha_i = 0$ (specifically all but k entries of μ are equal to 0). However, as the Dirichlet distribution is parameterised by strictly positive α values, we omit these and simply write μ with only the entries according to their absorbing states. In this case, our μ corresponds with the probability of entering each absorbing state in a given markov process.

For the case of introspection dynamics, we have a more standard interpretation of μ as described before, where μ corresponds to the average number of timesteps spent in each state once we have reached the steady state of the process. Now, given this case, let us look further into how this works with regards to a certain type of heterogeneity - that being where the attributes of each player are assigned according to a random variable H. In this case, the transition probabilities $T_{A,B}$ are some function $t(H)$. Then, we must look at μ in this context as a random vector.

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