A Heterogeneous Moran Process for the Analysis of Public Goods Games

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1 Introduction

2 Literature Review

Evolutionary game theory has long been at the forefront of the study of emergent cooperation, showing how a strategy that is seemingly unfavourable for the individual, but beneficial to the collective, can become dominant in systems of rational actors. This is often shown through the study of games such as the iterated prisoner's dilemma[**?**, **?**] and the snowdrift game[**?**]. These examples, however, only allow for pairwise interactions, which does not accurately represent many real-world scenarios. Therefore, we look at a common game used to model the sharing of resources and interactions between multiple people - the public goods game.

In order to further tailor the idea of a public goods game to real world scenarios, we introduce the idea of heterogeneity to the system. Often in the study of games we consider each individual to exist in the same conditions - having the same utility and probability of transitioning to a different action type. However, this does not accurately reflect many of the interactions that occur in the real world, where individuals may have different factors affecting which strategies they are able to take. For example, in a public goods game, one player may wish to contribute x units to the public good, however they do not have enough wealth available to spare such a contribution. Heterogeneity allows us to give varying attributes to the different actors in order to simulate such situations. These attributes can be encoded into the players themselves as separate to the game, for example the reputation in [**?**], exist within the strategies of the players [**?**, **?**], or they may exist within the actual payoffs which the players receive. We will look at multiple different ways in which heterogeneity can be implemented in order to model different scenarios.

In much of the literature, we see heterogeneous public goods game being played graphically. By this I mean that in much research, players are placed on graphs (often regular square lattices, as in [**?**, **?**]) and participate in multiple public goods games at the same time, with their payoff being the combined payoff from all their games. This gives even more weight to the decision to cooperate or defect, and allows players to interact in scenarios with many different levels of these heterogeneous attributes. We also see many ideas for what these attributes could be based upon. [**?**] takes the idea of that players could build a reputation through their decisions across multiple generations, with players refusing to contribute higher amounts to groups made up of those who have a reputation for defection. While this is a dynamically updating heterogeneous attribute, some have proposed static attributes based on inherent properties of the individuals, such as in [**?**] where the players participate in different amounts of games, and contribute different amounts based on this factor.

In both of the above cases, it is shown that an increased scrutiny of heterogeneous attributes in a spatial public goods game is beneficial to the emergence of cooperation, though for different reasons. In [**?**], we see that increasing such scrutiny obviously harms defectors, as they receive reduced payoffs based on their low reputation due to being unable to coerce high contributions from other players. [**?**], however, finds that if we encourage players to cooperate with those in a similar number of games to them, here done by tuning the β parameter, the high-degree defector nodes are unable to gather a large payoff, and so will be unable to spread their defection to their neighbours. If you encourage players to interact with those with different degrees, however, then middle-degree defectors will make heavy losses and be unable to survive.

[**?**] shows how the benefits of this heterogeneity are not always as pronounced as it may seem. When the contribution is not calculated by some attribute, but rather is an attribute in and of itself (essentially becoming a new strategy) then even in systems where contribution dominates, we will see low-value contributions act as a sort of defection against the most globally beneficial strategies. Therefore, we can see that the emergence of cooperation is not the end of the story in some cases, and we must look at what sort of cooperation it is that we have fostered the emergence of.

One common theme in the spatial public goods game is the manner in which cooperation spreads. An initial "invasion" of defectors leads to cooperation mainly existing within small clusters. These clusters, however, are much more profitable than their defecting neighbours. Therefore, the cooperating clusters will influence the nearby defectors to join in, which eventually spreads throughout the system. This relies on the ability of the clusters to resist the initial invasion of defectors - an occurrence which relies on a high enough r value (the positive multiplier of the contribution to the public good).

We also see that a potential climate club may soon be forming within the European Union. While it functions slightly differently to the classic example in [?], it follows the same principles of encouraging countries to join the group by punishing those outside of the club, in order to attempt to reduce overall carbon emissions. This is the "Carbon Border Adjustment Mechanism" [?]. Currently, all goods within the EU require taxation based on their carbon footprint, however imported goods from outside cannot be subject to such taxes. CBAM aims to prevent foreign goods from gaining benefits due to environmentally harmful practices by enforcing that EU companies report the carbon content of imported goods, and purchase certificates to cover the environmental cost of said goods. However, if a country already levies a tax on their own goods (similar to the EU's carbon tax), then this cost will be taken into consideration, and fewer certificates must be purchased. This discourages the importation of goods from countries which do not levy a similar tax on their own items, forming a structure which behaves very similarly to the "climate clubs" in [?].

3 The Model

In this section, we examine the underlying framework governing a heterogeneous game. We begin by defining a population of individuals and the actions which they may take, and then we see different methods by which we can model the evolution of a population over time.

3.1 An Initial System

- N **ordered** individuals who play actions from action sets $A_i = (a_{i,1}, a_{i,2}, ..., a_{i,k})$. In this paper, we shall assume that all action sets contain the same possible actions for each player.
- A state space S given by the set of ordered N-tuples with entries $\mathbf{a} = (a_1, a_2, a_3, \dots) \in A_1 \times A_2 \times A_3 \times ....$
- A strictly positive fitness function $f : S \to \mathbb{R}^N$. We define $f_i(\mathbf{a})$ as i^{th} entry of $f(\mathbf{a})$

Now let $h(\mathbf{a},\ \mathbf{b})$ denote the Hamming distance [?] between two states $\mathbf{a}, \mathbf{b} \in S$. The set of states $\mathbf{b}$ such that $h(\mathbf{a},\ \mathbf{b}) = 1$ is defined as Neb($\mathbf{a}$), the "Neighbourhood set of a state" [?]. Our system will be in the form of a Markov chain [?] which moves between states with a probability > 0 if and only if $h(\mathbf{a},\ \mathbf{b}) \leq 1$. We call the index at which $\mathbf{a}$ and $\mathbf{b}$ differ the *index of difference*, denoted $I(\mathbf{a},\mathbf{b})$ [?]. We shall look at different population dynamics by which we may model such a Markov chain.

3.2 Population Dynamics

3.2.1 The Moran Process

The Moran process [?] is a population corresponding to the following algorithm:

1. A player is chosen to be duplicated with a probability proportional to their fitness in the population
2. A player is chosen with probability $\frac{1}{N}$ to be removed from the population

Let (a_{-j}) denote the state from the perspective of a_j, so that $\mathbf{a} = (a_j, a_{-j})$. Then, the entry $T_{\mathbf{a},\mathbf{b}}$ of a transition matrix T in which each row and column correspond to a state in S is defined as the probability of a transition $\mathbf{a} = (a_1, a_2, \dots, a_N) \to \mathbf{b} = (b_1, b_2, \dots, b_N)$. For the Moran process, this is as follows:

$$T_{\mathbf{a},\mathbf{b}} = p_{a_{I(\mathbf{a},\mathbf{b})} \to b_{I(\mathbf{a},\mathbf{b})}}(a_{-j}) = \begin{cases} \frac{\sum_{a_i = b_{I(\mathbf{a},\mathbf{b})}} f(a_i)}{N \sum_{a_j} f(a_j)} & \text{if } \mathbf{b} \in \text{Neb}(\mathbf{a}), \text{ differing at position } I(\mathbf{a},\mathbf{b}) \\ 0 & \text{if } \mathbf{b} \notin \text{Neb}(\mathbf{a}) \text{ and } \mathbf{a} \neq \mathbf{b} \\ 1 - \sum_{\mathbf{b} \in S \setminus \{\mathbf{a}\}} p_{a_{I(\mathbf{a},\mathbf{b})} \to b_{I(\mathbf{a},\mathbf{b})}}(a_{-j}) & \text{if } \mathbf{a} = \mathbf{b} \end{cases} \tag{1}$$

An important case which proceeds from (**??**) is that of a transition $\mathbf{a} \to \mathbf{b}$ where $\mathbf{b}$ contains an individual of a *type* not found in $\mathbf{a}$. For example, the transition $(0,1) \to (0,2)$. This would be forbidden by the intuition of a standard Moran process, as the new individual is the duplication of another individual in $\mathbf{a}$. However, we can see that the standard formula yields $p_{v,u} = 0$ because $\nexists\ a_i \in \mathbf{a} : a_i = b_{i^*}$.

We can now discuss one of the key ideas in the Moran process. As no new action types can be introduced to the population once they are fully removed, we have a set of absorbing states $\mathbf{a}$ such that $a_1 = a_2 = ...$, for which we have $p_{\mathbf{a} \to \mathbf{a}} = 1$. We denote the set of absorbing states S^γ. We therefore have that the Moran process forms an absorbing Markov chain, and we can investigate the probability of the population being absorbed into each absorbing state.

To show how we use this, consider a game with two players, each playing one of two actions. We have a state space $\mathbf{a} = (0,0), \mathbf{b} = (0,1), \mathbf{c} = (1,0), \mathbf{d} = (1,1)$, and our transition matrix T would take the form

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{f_1(\mathbf{b})}{2f_1(\mathbf{b})+2f_2(\mathbf{b})} & 1 - \frac{f_1(\mathbf{b})}{2f_1(\mathbf{b})+2f_2(\mathbf{b})} - \frac{f_2(\mathbf{b})}{2f_1(\mathbf{b})+2f_2(\mathbf{b})} & 0 & \frac{f_2(\mathbf{b})}{2f_1(\mathbf{b})+2f_2(\mathbf{b})} \\ \frac{f_2(\mathbf{c})}{2f_1(\mathbf{c})+2f_2(\mathbf{c})} & 0 & 1 - \frac{f_1(\mathbf{c})}{2f_1(\mathbf{c})+2f_2(\mathbf{c})} - \frac{f_2(\mathbf{c})}{2f_1(\mathbf{c})+2f_2(\mathbf{c})} & \frac{f_1(\mathbf{c})}{2f_1(\mathbf{c})+2f_2(\mathbf{c})} \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2}$$

Our transition matrix represents the probability of transitioning from the state represented by the row to the state represented by the column. Here our columns represent $\mathbf{a}$,$\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ from left to right, and the rows represent these states in the same order from top to bottom.
Now, by taking the fundamental matrix of T and performing a right multiplication by R, the matrix with entries which represent transitions from transitive states to absorbing states[**?**], we can obtain the absorption matrix for this system. For the above game, we find the following equation:

$$\left(I - \begin{bmatrix} 1 - \frac{f_1(\mathbf{b})}{2f_1(\mathbf{b})+2f_2(\mathbf{b})} - \frac{f_2(\mathbf{b})}{2f_1(\mathbf{b})+2f_2(\mathbf{b})} & 0 \\ 0 & 1 - \frac{f_1(\mathbf{c})}{2f_1(\mathbf{c})+2f_2(\mathbf{c})} - \frac{f_2(\mathbf{c})}{2f_1(\mathbf{c})+2f_2(\mathbf{c})} \end{bmatrix} \right)^{-1} \begin{bmatrix} \frac{f_1(\mathbf{b})}{2f_1(\mathbf{b})+2f_2(\mathbf{b})} & \frac{f_2(\mathbf{b})}{2f_1(\mathbf{b})+2f_2(\mathbf{b})} \\ \frac{f_2(\mathbf{c})}{2f_1(\mathbf{c})+2f_2(\mathbf{c})} & \frac{f_1(\mathbf{c})}{2f_1(\mathbf{c})+2f_2(\mathbf{c})} \end{bmatrix} \tag{3}$$

And solve to find our absorption matrix:

$$\begin{bmatrix} \frac{f_1(\mathbf{b})}{\left(\frac{f_1(\mathbf{b})}{2f_1(\mathbf{b})+2f_2(\mathbf{b})} + \frac{f_2(\mathbf{b})}{2f_1(\mathbf{b})+2f_2(\mathbf{b})}\right)(2f_1(\mathbf{b})+2f_2(\mathbf{b}))} & \frac{f_2(\mathbf{b})}{\left(\frac{f_1(\mathbf{b})}{2f_1(\mathbf{b})+2f_2(\mathbf{b})} + \frac{f_2(\mathbf{b})}{2f_1(\mathbf{b})+2f_2(\mathbf{b})}\right)(2f_1(\mathbf{b})+2f_2(\mathbf{b}))} \\ \frac{f_2(\mathbf{c})}{\left(\frac{f_1(\mathbf{c})}{2f_1(\mathbf{c})+2f_2(\mathbf{c})} + \frac{f_2(\mathbf{c})}{2f_1(\mathbf{c})+2f_2(\mathbf{c})}\right)(2f_1(\mathbf{c})+2f_2(\mathbf{c}))} & \frac{f_1(\mathbf{c})}{\left(\frac{f_1(\mathbf{c})}{2f_1(\mathbf{c})+2f_2(\mathbf{c})} + \frac{f_2(\mathbf{c})}{2f_1(\mathbf{c})+2f_2(\mathbf{c})}\right)(2f_1(\mathbf{c})+2f_2(\mathbf{c}))} \end{bmatrix} \tag{4}$$

In this matrix, the rows represent our initial transitive state (in our system, row 1 represents $\mathbf{b}$ and row 2 represents $\mathbf{c}$), and our columns represent which state the system is absorbed into (with $\mathbf{a}$ on the left, and $\mathbf{d}$ on the right). It is immediately clear from this result that the probability of the system ending up in a particular absorbing state does not rely on the fitness of individuals in the absorbing state, but rather their fitness in the transitive states. Thus, players can remain in a state which could be bettered by a single change, however they will not make such a change if no individual of another type exists in the population.

3.2.2 Fermi Imitation Dynamics

Another method by which we can calculate the probability of one player copying the strategy of another is by using the Fermi function. Consider a Markov chain where at each time step, one player is chosen with probability $\frac{1}{N}$ to be the "focal" player a_i, and they compare their strategy with another player a_j chosen with probability $\frac{1}{N-1}$. Then, a_i copies the strategy of a_j with a probability according to the Fermi function:

$$\phi(f_i(\mathbf{a}) - f_j(\mathbf{a})) = \frac{1}{1 + e^{\left(\frac{f_i(\mathbf{a}) - f_j(\mathbf{a})}{\beta}\right)}} \tag{5}$$

Where β is the selection intensity of the system. This method has been commonly used in studies of evolutionary games, for example [**?**, **?**, **?**].
We now define our transition matrix according to the Fermi function:

$$T_{\mathbf{a},\mathbf{b}} = p_{a_{I(\mathbf{a},\mathbf{b})} \to b_{I(\mathbf{a},\mathbf{b})}}(a_{-j}) = \begin{cases} \frac{1}{N(N-1)} \sum_{a_j = b_{I(\mathbf{a},\mathbf{b})}} \phi(f_{\mathrm{I}(\mathbf{a},\mathbf{b})}(a) - f_j(\mathbf{a}))) & \text{if } \mathbf{b} \in \mathrm{Neb}(\mathbf{a}) \\ 0 & \text{if } \mathbf{b} \notin \mathrm{Neb}(\mathbf{a}) \text{ and } \mathbf{a} \neq \mathbf{b} \\ 1 - \sum_{\mathbf{b} \in S \setminus \{\mathbf{a}\}} p_{a_{I(\mathbf{a},\mathbf{b})} \to b_{I(\mathbf{a},\mathbf{b})}}(a_{-j}) & \text{if } \mathbf{a} = \mathbf{b} \end{cases} \tag{6}$$

Here we see how the Fermi function can be used to define the transition probability between states in a Markov chain. Importantly, we see that the first case of equation **??** is always in the interval $(0, 1)$, as the Fermi function always takes a value on $(0, 1)$, and the summation term is at most $N - 1$ values of said function, and thus takes a value $< N - 1$.

Now, let us look once more at our state space $\mathbf{a} = (0, 0), \mathbf{b} = (0, 1), \mathbf{c} = (1, 0), \mathbf{d} = (1, 1)$. We can define a transition matrix:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2}\phi(f_2(\mathbf{b}) - f_1(\mathbf{b})) & \frac{1}{2} & 0 & \frac{1}{2}\phi(f_1(\mathbf{b}) - f_2(\mathbf{b})) \\ \frac{1}{2}\phi(f_1(\mathbf{c}) - f_2(\mathbf{c})) & 0 & \frac{1}{2} & \frac{1}{2}\phi(f_2(\mathbf{c}) - f_1(\mathbf{c})) \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{7}$$

and using our previous method, we can acquire our absorption matrix:

$$\begin{bmatrix} \phi(f_2(\mathbf{b}) - f_1(\mathbf{b})) & \phi(f_1(\mathbf{b}) - f_2(\mathbf{b})) \\ \phi(f_1(\mathbf{c}) - f_2(\mathbf{c})) & \phi(f_2(\mathbf{c}) - f_1(\mathbf{c})) \end{bmatrix} \tag{8}$$

This function shows that once again, our absorption probabilities rely on the fitness of players in transitive states and not those of players in the absorbing states.

3.2.3 Introspection Dynamics

We will now look at a model of population dynamics in which players do not consider the payoffs of other individuals in the population, but instead at their own possible payoffs. In this dynamic, the following process take place [**?**]

1. A random player is chosen with probability $\frac{1}{N}$ to reconsider their strategy $a_{i,k}$
2. Said player chooses another possible strategy, $a_{i,l}$, from the set of all possible strategies.
3. The chosen player replaces their strategy with probability $p_{a_{I(\mathbf{a},\mathbf{b})} \to b_{I(\mathbf{a},\mathbf{b})}}(a_{-I(\mathbf{a},\mathbf{b})}) = \phi(\Delta f_{I(\mathbf{a},\mathbf{b})})$, where $\Delta f_{I(\mathbf{a},\mathbf{b})} = f_{I(a,b)}(\mathbf{a}) - f_{I(a,b)}(\mathbf{b})$ is the difference between the possible fitness of the new strategy and the fitness of the player's current strategy.

This process is especially strong in a heterogeneous population. This is because in some cases using imitation dynamics, a player may copy a strategy due it's fitness when played by another individual in the population, despite the fact that such a strategy actually has a lower payoff for the focal player than their current strategy. In introspection dynamics, players do not consider the fitness of other individuals, and so cannot be fooled by high fitness strategies with attributes different to that of our focal player.

Couto [?] defines the transition matrix for a process according to introspection dynamics as follows:

$$T_{\mathbf{a},\mathbf{b}} = \begin{cases} \dfrac{1}{N(m_j - 1)} p_{a_j \to b_j}(a_{-j}) & \text{if } \mathbf{b} \in \mathrm{Neb}(\mathbf{a}) \text{ and } j = I(\mathbf{a},\mathbf{b}), \\ 0 & \text{if } \mathbf{b} \notin \mathrm{Neb}(\mathbf{a}) \text{and } \mathbf{a} \neq \mathbf{b}, \\ 1 - \displaystyle\sum_{c \neq b} T_{a,c} & \text{if } \mathbf{a} = \mathbf{b}. \end{cases}$$

where m_j is the number of actions available to individual j. Since we assume that all individuals have the same action set with size k, we can replace $m_j - 1$ with $k - 1$.

Now, Couto also calculated in [?] the explicit strategy distribution for an asymmetric interaction between two players. We can adapt these results to define our transition matrix for our example state space **a,b,c,d**:

Unlike in the case of imitation dynamics, there are no absorbing states for a Markov chain transitioning according to imitation dynamics. Therefore, we define a steady state vector **u** according to the method Couto uses in [?].

3.2.4 Introspective Imitation Dynamics

Introspection dynamics allows players to learn based on their own fitness, resulting in a reduced probability of state changes that worsen the focal individual's fitness in a heterogeneous population. Now we consider a population dynamic that utilises this property of introspection dynamics but within the context of an imitative process. We define the introspective imitation process as follows:

1. A focal player is chosen at random to reconsider their strategy
2. Another player a_j in the population is chosen with a probability proportional to their fitness within the population to have their strategy considered
3. The focal player changes their strategy with probability $\phi(\Delta(f_{I(\mathbf{a},\mathbf{b})})$

We can define a transition matrix T of an introspective imitation process as follows:

$$T_{\mathbf{a},\mathbf{b}} = \begin{cases} \frac{1}{N} \frac{\sum_{a_j = b_{I(\mathbf{a},\mathbf{b})}} f_j(\mathbf{a})}{\sum_k f_k(\mathbf{a})} \phi(\Delta(f_{I(\mathbf{a},\mathbf{b})})) & \text{if } \mathbf{b} \in \mathrm{Neb}(\mathbf{a}) \\ 0 & \text{if } \mathbf{b} \notin \mathrm{Neb}(\mathbf{a}) \text{ and } \mathbf{a} \neq \mathbf{b} \\ 1 - \sum_{\mathbf{b} \in \mathrm{Neb}(\mathbf{a})} p_{a_{I(\mathbf{a},\mathbf{b})} \to b_{I(\mathbf{a},\mathbf{b})}}(a_{-j}) & \text{if } \mathbf{a} = \mathbf{b} \end{cases} \tag{9}$$

We see that players can only copy the actions present in the state, however they determine whether or not to copy a player based on their own fitness rather than by considering the fitness of another player. This models a behaviour where individuals learn from each other, but do not blindly trust that the strategy will work for themself just because it worked for another player. Instead, players look to others to see what action they *could* play, but judge based on their own fitness to decide what action they *should* play.

Let us now see this model applied to our ongoing example in the state space $S = \{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}\}$. Let $\eta_{i,j}(\mathbf{x},\mathbf{y}) = 1 + e^{\frac{(f_i(\mathbf{x}) - f_j(\mathbf{y}))}{\beta}}$ be the denominator of the Fermi function. We first see our transition matrix:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{f_1(b)}{2\eta_{2,2}(\mathbf{b},\mathbf{a})(f_1(b)+f_2(b))} & 1 - \frac{f_2(b)}{\eta_{1,1}(\mathbf{b},\mathbf{d})(2f_1(b)+2f_2(b))} - \frac{f_1(b)}{\eta_{2,2}(\mathbf{b},\mathbf{a})(2f_1(b)+2f_2(b))} & 0 & \frac{f_2(b)}{2\eta_{1,1}(\mathbf{b},\mathbf{d})(f_1(b)+f_2(b))} \\ \frac{f_2(c)}{2\eta_{1,1}(\mathbf{c},\mathbf{a})(f_1(c)+f_2(c))} & 0 & 1 - \frac{f_1(c)}{\eta_{2,2}(\mathbf{c},\mathbf{d})(2f_1(c)+2f_2(c))} - \frac{f_2(c)}{\eta_{1,1}(\mathbf{c},\mathbf{a})(2f_1(c)+2f_2(c))} & \frac{f_1(c)}{2\eta_{2,2}(\mathbf{c},\mathbf{d})(f_1(c)+f_2(c))} \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{10}$$

And, using the same method as in the other methods of imitation dynamics, we obtain the absorption matrix:

$$\begin{bmatrix} \frac{\eta_{1,1}(\mathbf{b},\mathbf{d}) f_1(b)}{\eta_{1,1}(\mathbf{b},\mathbf{d}) f_1(b) + \eta_{2,2}(\mathbf{b},\mathbf{a}) f_2(b)} & \frac{\eta_{2,2}(\mathbf{b},\mathbf{a}) f_2(b)}{\eta_{1,1}(\mathbf{b},\mathbf{d}) f_1(b) + \eta_{2,2}(\mathbf{b},\mathbf{a}) f_2(b)} \\ \frac{\eta_{2,2}(\mathbf{c},\mathbf{d}) f_2(c)}{\eta_{1,1}(\mathbf{c},\mathbf{a}) f_1(c) + \eta_{2,2}(\mathbf{c},\mathbf{d}) f_2(c)} & \frac{\eta_{1,1}(\mathbf{c},\mathbf{a}) f_1(c)}{\eta_{1,1}(\mathbf{c},\mathbf{a}) f_1(c) + \eta_{2,2}(\mathbf{c},\mathbf{d}) f_2(c)} \end{bmatrix} \tag{11}$$

Unlike other imitation dynamics, we can see that the absorption probability in an introspective imitation process relies on the fitness that at least one individual will possess in the absorbing state. This means that players will enter absorbing states with a higher probability if it improves their own fitness, despite any heterogeneity between payoffs for the same action played by different players. This solves the problem of players making worsening moves based on the fitness of another individual in the population, while still retaining the property of imitation dynamics that new strategies cannot be introduced to the system.

3.3 Classification of Population Dynamics

We will now classify our population dynamics into one of two categories. The first are population dynamics which accept strategies from the population based on the fitness of said strategies when played by other individuals. These dynamics are poorly suited to a heterogeneous population, as individuals do not consider how the change will affect them. However, these do offer a good model for biological processes such as the evolution of traits in nature, as in these cases the fitness of an individual causes them to pass on their traits. The second are population dynamics which consider their own change in payoff in the case that they accept a new strategy. These dynamics are more well suited to a heterogeneous population, modelling state changes where individuals make an active decision to change state in order to improve their own fitness.

Below, we see a diagram showing the classification of the population dynamics shown in this section:

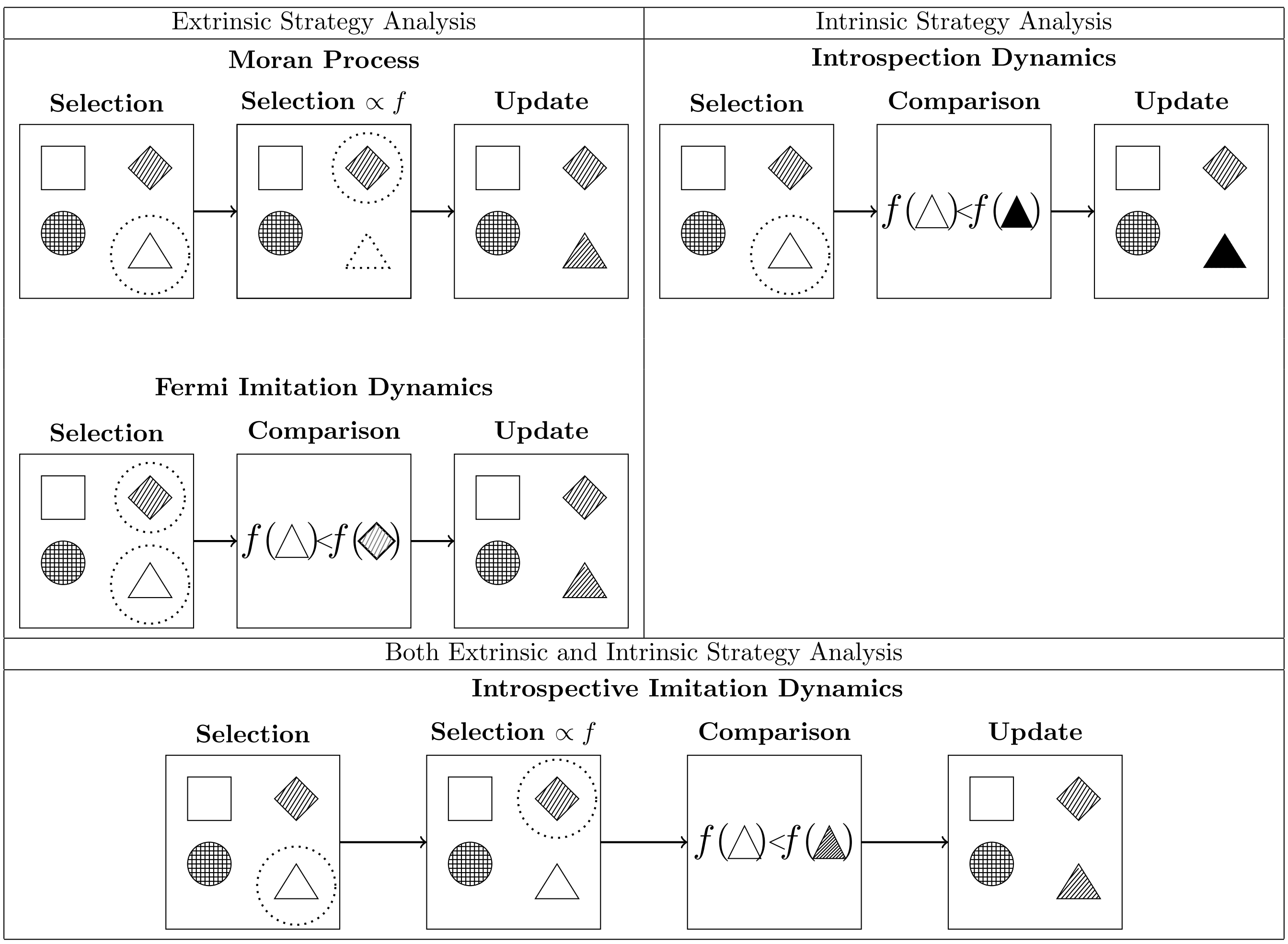

Table 1: A comparison of different population dynamics. We assume in all cases that the triangle is chosen for replacement with a probability $\frac{1}{N}$ and the diamond is chosen with a varying probability depending on the process. In the case of Introspection dynamics, we assume that the strategy represented by the shape being filled in is chosen with probability $\frac{1}{k-1}$

4 The Public Goods Game

4.1 The Standard Model and Transformations

In this section, we shall see how different population dynamics can be applied to the classical public goods game. In this game, individuals choose whether or not to contribute to a public resource pool. In the end, the total resource is multiplied by some factor $r > 1$ and distributed equally between each player. This model often encourages a behaviour known as "free-riding" [?], where players refuse to contribute to the pool and simply take the benefit provided by other individuals' contribution. The classic problem of a public goods game is to provide an environment where such free-riding is less profitable than contributing to the public good.

We can model a public goods game by providing the following payoff function:

$$\pi_i(\mathbf{a}) = \frac{r\sum_{j=1}^{N}\alpha_j}{N} - \alpha_i \tag{12}$$

(where α_j is the contribution by individual j.) In the homogeneous public goods game, we have that α_j can only take one of two values: some constant α if individual j cooperates, and 0 if not. However, the heterogeneous game will require a more tailored α_j.

Note that $\pi(v_j)$ can be negative, and such cannot be used as our fitness function for the Moran process. Therefore, we must look at some transformation of π to use as our fitness function.

A common method [?] for this is to apply the exponential function to our payoff function. By using $e^{\beta\sigma}$ as our fitness function we will have positive values, however this particular approach gives, in the case of the Moran process:

$$\frac{\sum_{a_i = b_{I(\mathbf{a},\mathbf{b})}} e^{\frac{\beta r\sum_{j=1}^{N}\alpha_j}{N} - \alpha_i}}{\sum_{a_i} e^{\frac{\beta r\sum_{j=1}^{N}\alpha_j}{N} - \alpha_i}} \tag{13}$$

Now, as e^r does not rely on v, we can take this out of the sum and acquire the following:

$$\frac{e^{\frac{\beta r\sum_{j=1}^{N}\alpha_j}{N}} \sum_{a_i = b_{I(\mathbf{a},\mathbf{b})}} e^{-\alpha_i}}{e^{\frac{\beta r\sum_{j=1}^{N}\alpha_j}{N}} \sum_{a_i} e^{-C_{v_i}}} = \frac{\sum_{a_i = b_{I(\mathbf{a},\mathbf{b})}} e^{-\alpha_i}}{\sum_{a_i} e^{-\alpha_i}} \tag{14}$$

and we see that in this case, $p_{a_j \to b_j}(a_{-j})$ would not rely on r. Therefore, let us consider some other methods of guaranteeing a positive fitness function.

The first of these is known as the "shifted linear" transformation. We take, for some small tunable ϵ:

$$f_i(\mathbf{a}) = \pi_i(\mathbf{a}) - \min_j \pi_j(\mathbf{a}) + \beta \tag{15}$$

This guarantees a positive value, with the lowest fitness taking the value ϵ, a parameter that can be chosen based on the system in question. However, in the homogeneous case, as we can see in appendix[?], this mapping creates a fitness function which does not rely on r, and so we cannot use this method.

Another type of mapping that we can use is the "Affine-linear mapping". In this, for some tunable ϵ, we take the following:

$$f_i(\mathbf{a}) = 1 + \epsilon\pi_i(\mathbf{a}) \tag{16}$$

As ϵ is a selection intensity parameter, this method both provides us a strictly positive fitness function for a well defined ϵ, and also allows us to control the selection intensity of the Moran process. Unlike in other population dynamics, this is not already defined within the Moran process, and so this is another benefit of using the affine-linear mapping. Thus, we take this mapping of π as our fitness function.
In order to avoid having two selection intensity parameters, we can define $\beta = 1$ in this case, and allow $\epsilon = \frac{1}{\beta}$, in order that we have $\pi_i = 1 + \frac{f_i}{\beta}$. Thus in the Fermi function we preserve the power of $e^{\frac{\Delta(f)}{\beta}}$ for some selection intensity β. However, we will now have a bound on β depending on the values of α in our public goods game, as we must define a strictly positive fitness function.

4.2 The Homogeneous Case

The most simple type of public goods game is the homogeneous public goods game. In this game, we consider that we have a constant $\alpha_i = \alpha$ for all players a_i. In this case, we take the payoff function:

$$\sigma(v_i) = \frac{r\sum_{j=1}^{N} \delta_j \alpha}{N} - \delta_j \alpha = (\kappa_c(\mathbf{a})r - \delta_j)\alpha \tag{17}$$

Where $\delta_j = 1$ if j contributes and 0 if not, and $\kappa_c(\mathbf{a})$ is the fraction of the population who contribute.

We will begin by looking at the state space with $N = 4$ and see how cooperation emerges based on different values of r and α.
We can assign the value 0 to non-contribution and 1 contribution in the state space $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$, and in doing so obtain the following absorption matrices for each population dynamic:

Moran Process	Fermi Imitation Dynamics
$\begin{bmatrix} \frac{\frac{\alpha r}{2}+\beta}{\alpha r-\alpha+2\beta} & \frac{\frac{\alpha r}{2}-\alpha+\beta}{\alpha r-\alpha+2\beta} \\ \frac{\frac{\alpha r}{2}+\beta}{\alpha r-\alpha+2\beta} & \frac{\frac{\alpha r}{2}-\alpha+\beta}{\alpha r-\alpha+2\beta} \end{bmatrix}$	$\begin{bmatrix} \frac{e^{\frac{\alpha}{\beta}}}{e^{\frac{\alpha}{\beta}}+1} & \frac{1}{e^{\frac{\alpha}{\beta}}+1} \\ \frac{e^{\frac{\alpha}{\beta}}}{e^{\frac{\alpha}{\beta}}+1} & \frac{1}{e^{\frac{\alpha}{\beta}}+1} \end{bmatrix}$
Introspection Dynamics	**Introspective Imitation Dynamics**
TODO	$\begin{bmatrix} \frac{(\alpha\epsilon r+2)\left(e^{\frac{\alpha\epsilon(2-r)}{2\beta}}+1\right)}{(\alpha\epsilon r+2)\left(e^{\frac{\alpha\epsilon(2-r)}{2\beta}}+1\right)+(\alpha\epsilon(r-2)+2)\left(e^{\frac{\alpha\epsilon(r-2)}{2\beta}}+1\right)} & \frac{(\alpha\epsilon(r-2)+2)\left(e^{\frac{\alpha\epsilon(r-2)}{2\beta}}+1\right)}{(\alpha\epsilon r+2)\left(e^{\frac{\alpha\epsilon(2-r)}{2\beta}}+1\right)+(\alpha\epsilon(r-2)+2)\left(e^{\frac{\alpha\epsilon(r-2)}{2\beta}}+1\right)} \\ \frac{(\alpha\epsilon r+2)\left(e^{\frac{\alpha\epsilon(2-r)}{2\beta}}+1\right)}{(\alpha\epsilon r+2)\left(e^{\frac{\alpha\epsilon(2-r)}{2\beta}}+1\right)+(\alpha\epsilon(r-2)+2)\left(e^{\frac{\alpha\epsilon(r-2)}{2\beta}}+1\right)} & \frac{(\alpha\epsilon(r-2)+2)\left(e^{\frac{\alpha\epsilon(r-2)}{2\beta}}+1\right)}{(\alpha\epsilon r+2)\left(e^{\frac{\alpha\epsilon(2-r)}{2\beta}}+1\right)+(\alpha\epsilon(r-2)+2)\left(e^{\frac{\alpha\epsilon(r-2)}{2\beta}}+1\right)} \end{bmatrix}$

Table 2: A table of absorption probabilities for different population dynamics in the homogeneous public goods game. The row correspond to which state $\mathbf{b}$ or $\mathbf{c}$ we begin in, and the column corresponds to which state $\mathbf{a}$ or $\mathbf{d}$ we are absorbed into. Thus the right hand row is the absorption probability for full contribution, and the left hand row is the absorption probability for no contribution.

We see that in the homogeneous case of the 2 player public goods game, the absorption probabilities do not depend on which transitive state you begin in. This is clear, as the states $\mathbf{b} = (0, 1)$ and $\mathbf{c} = (1, 0)$ are equivalent due to there being no difference between players 1 and 2.

4.3 Numerical Results

We can now see how the different population dynamics affect the absorption probabilities in the homogeneous public goods game. We first consider the effect of r on our population in various α values:

./src/img/absorption_probabilities_N_eq_2_homogeneous_fermi_r_alpha/main.pdf

Figure 1: Four images in a 2×2 arrangement.

5 Public Goods Games with Heterogeneous Returns

When we look at a public goods game, we often generalise the players to all receive the same return. However, in the real world this is often not the case.

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