THE INJECTIVE SPECTRUM OF A RIGHT NOETHERIAN RING

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The injective spectrum was introduced briefly by Gabriel, building on work of Matlis. It is a ringed space associated to a ring R that agrees with the usual Zariski spectrum when R is commutative noetherian, and allows the Zariski spectrum to be extended to the case of noncommutative rings (or even Grothendieck categories, as the injective spectrum depends only on the category of (right) modules, and not on the ring itself).

In this thesis, we study the injective spectra of right noetherian rings, establishing a number of basic topological properties, and relating the topological dimension of the spectrum to the Krull dimension of the ring (in the sense of deviation of the poset of right ideals). We also compute a number of examples, illustrating both the geometrically nice behaviour possible, and the more unpleasant behaviour that the injective spectrum can exhibit. We further establish partial results on functoriality of the injective spectrum, and explore links with the torsion spectrum developed by Golan, and consider sheaves of modules over the injective spectrum.

Declaration

No portion of the work referred to in the thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institute of learning. An eprint is being prepared by the author to submit to Arxiv, containing the main results of this thesis.

Copyright Statement

In loving memory of my sister,

Martha Taylor,

and my friend,

Phasu Chomsomboon.

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Chapter 1

Introduction and Background

1.1 Conventions and Notation

Throughout, all rings will be associative and unital, but not necessarily commutative, and all modules will be unital right modules, unless otherwise specified. The unit element of a ring will be denoted simply by 1. If R is a ring, Mod-R will denote the category of right R-modules and module homorphisms, and mod-R the full subcategory of finitely presented right modules. For left modules, we use R-Mod, respectively R-mod. We denote by **Ab** the category of abelian groups, Mod-$\mathbb{Z}$. We may declare that M is an R-module by writing $M \in$ Mod-R or simply referring to it as M_R. If R and S are rings, we denote an (R, S)-bimodule M by ${}_RM_S$. We shall tend to distinguish between R as a ring, and R_R as a right module over R.

All categories we consider will be at least preadditive, and all functors considered will be covariant additive functors. For a category containing objects A and B, we write (A, B) for the abelian group of morphisms from A to B. Similarly, for categories $\mathcal{C}$ and $\mathcal{D}$, with $\mathcal{C}$ (skeletally) small, we write $(\mathcal{C}, \mathcal{D})$ for the category of additive functors from $\mathcal{C}$ to $\mathcal{D}$. We write $\mathcal{C}^{\text{fp}}$ for the full subcategory of $\mathcal{C}$ consisting of finitely presented objects (so mod-$R = (\text{Mod-}R)^{\text{fp}}$). When there are no non-zero morphisms $A \to B$ for some objects A and B in an abelian category, we shall often say simply that A does not map to B.

When speaking of topological spaces, we will say that a set is **compact** if every open cover admits a finite subcover - we do not assume the Hausdorff condition. Indeed, the spaces we consider will almost never be Hausdorff. A topological space is **noetherian** if it has the ascending chain condition on open sets, equivalently the descending chain condition on closed sets. A space is **hereditarily compact** if every non-empty subset is compact. It is straightforward to show that a space is noetherian if and only if it is hereditarily compact.

By a **topological embedding**, we mean a continuous map between topological spaces which is homeomorphic onto its image. We say a topological space is **sober** if for any non-empty irreducible closed set C there is a **generic point**, *i.e.*, a point $p \in C$ such that $C = \mathrm{cl}(p)$, where cl denotes the closure.

Recall that a collection B of open sets in a topological space X is said to be a **basis (of open sets)** if $\bigcup B = X$ and for any $B_1, B_2 \in B$ and any $x \in B_1 \cap B_2$ there is $B_3 \in B$ such that $x \in B_3 \subseteq B_1 \cap B_2$. Given a basis, any open set is expressible (not necessarily uniquely) as a union of basic open sets. By a **basis of closed sets** for X, we mean a collection of closed sets whose complements form a basis of open sets. Equivalently, a collection C of closed sets is a basis of closed sets if $\bigcap C = \varnothing$ and for any $C_1, C_2 \in C$ and any $x \notin C_1 \cup C_2$, there is $C_3 \in C$ with $x \notin C_3$ and $C_1 \cup C_2 \subseteq C_3$. Given a basis of closed sets, any closed set is expressible (not necessarily uniquely) as an intersection of basic closed sets.

1.2 Injective and Uniform Modules

In this section, we fix an arbitrary ring R.

Recall that an R-module M is **injective** if for any module B, submodule $A \leq B$, and map $f : A \to M$, there exists a map $\hat{f} : B \to M$ extending f. The two following basic results are well-known.

Theorem 1.2.1 (Baer's Criterion, [13], 3.7)**.** *When checking injectivity, it suffices to consider the case $B = R_R$. In more detail, let M be an R-module. Then M is injective if and only if for every right ideal $I \leq R_R$ and map $f : I \to M$ there is a map $\hat{f} : R_R \to M$ extending f.*

Theorem 1.2.2 ([17], 2.4 & 2.5)**.** *Let M be an R-module. The following are equivalent:*

1. *M is injective.*

2. *The functor $(-, M) : (\text{Mod-}R)^{\mathrm{op}} \to \mathbf{Ab}$ is exact.*

3. Any embedding of modules $M \to N$ splits.

There is a further, more model-theoretic characterisation of injectivity, due to Eklof and Sabbagh [4, §3]. First, given an R-module M, a **system of equations** Σ in M consists of indexing sets I and J, and functions $m : I \to M : i \mapsto m_i$ and $r : I \times J \to R : (i, j) \mapsto r_{ij}$ such that for each $i \in I$, there are only finitely many $j \in J$ with $r_{ij} \neq 0$. We write such a system as

$$\Sigma = \left\{ \sum_{j \in J} x_j r_{ij} = m_i \right\}_{i \in I}$$

and call the symbols x_j for $j \in J$ the **variables** of Σ. We say that Σ is **consistent** if whenever $I_0 \subseteq I$ is finite and $s_i \in R$ for each $i \in I_0$ are such that

$$\sum_{i \in I_0} r_{ij} s_i = 0$$

for all $j \in J$, then

$$\sum_{i \in I_0} m_i s_i = 0.$$

That is, if whenever a linear combination of the left-hand sides of the equations is zero, so too is the corresponding linear combination of the right-hand sides. Note that any system of equations in M is automatically also a system of equations in any module N containing M, and that the system is consistent in M if and only if it is consistent in N.

Given a system Σ of equations in M and a module N containing M, a **solution** of Σ in N is a function $n : J \to N$ such that for each $i \in I$ the equation

$$\sum_{j \in J} n_j r_{ij} = m_i$$

is satisfied. Clearly, if Σ has a solution in some extension of M, then Σ must be consistent; the converse is also true [4, Lemma 3.2].

Theorem 1.2.3 ([4], §3)**.** *An R-module M is injective if and only if every consistent system of equations in M has a solution in M. By Baer's Criterion (Theorem 1.2.1), it is sufficient to consider systems of equations in just one variable.*

We give a rough outline of a proof. The idea is that a consistent system of equations Σ in M as above is equivalent to a map into M from the submodule of $R^{(J)}$ generated by the tuples $r_i = (r_{ij})_{j \in J}$; namely, the map $r_i \mapsto m_i$ for each $i \in I$. A solution $(m_j)_{j \in J}$ to Σ in M is then an extension of this map to a map $R^{(J)} \to M$, by sending the i^{th} standard basis vector of $R^{(J)}$ to m_j.

Note that, in particular, by solving one equation in one variable, any injective module is divisible; *i.e.*, if E is injective, $e \in E$, and $r \in R$ is such that $\operatorname{ann}_R(r) \subseteq \operatorname{ann}_R(e)$, then there is $x \in E$ such that $e = xr$. Indeed, it follows from Baer's criterion that over a principal right ideal ring, this is sufficient as well as necessary for a module to be injective. Injectivity may therefore be regarded as a sort of "generalised divisibility" condition.

Throughout this chapter, we shall use $\mathbb{Z}$ as a running example to illustrate the ideas expounded. In fact, we could use an arbitrary commutative principal ideal domain with essentially no change.

Example 1.2.4. *The abelian group ($\mathbb{Z}$-module) $\mathbb{Q}$ is injective.*

We can see this by Baer's criterion. Any ideal $n\mathbb{Z}$ in $\mathbb{Z}$ is a free $\mathbb{Z}$-module, so a map $f : n\mathbb{Z} \to \mathbb{Q}$ is determined by a choice of any single element q of $\mathbb{Q}$, with $f(n) = q$. Then extending f to $\hat{f} : \mathbb{Z} \to \mathbb{Q}$ means choosing $\hat{f}(1) = r$ such that $nr = q$. Taking $r = q/n$ suffices, so we can extend any map $n\mathbb{Z} \to \mathbb{Q}$ into a map $\mathbb{Z} \to \mathbb{Q}$, so $\mathbb{Q}_{\mathbb{Z}}$ is injective. This illustrates how, for a PID, injectivity is the same as divisibility.

Alternatively, we can think in terms of solving systems of equations. Any system of $\mathbb{Z}$-linear equations over $\mathbb{Q}$ may be regarded as a system of $\mathbb{Q}$-linear equations whose coefficients happen to lie in $\mathbb{Z}$. Since $\mathbb{Q}_{\mathbb{Q}}$ is a vector space, this system has a solution if and only if it is consistent. Moreover, it is consistent over $\mathbb{Q}$ if and only if it is consistent over $\mathbb{Z}$; for by clearing denominators, any $\mathbb{Q}$-linear combination can be made into a $\mathbb{Z}$-linear combination. So $\mathbb{Q}$ is injective as a $\mathbb{Z}$-module (and also as a $\mathbb{Q}$-module - indeed, this argument shows that for any commutative domain R, any vector space over the field of fractions of R is injective as an R-module). ■

A further result from [4, §3] is the following, where we take a language for R-modules consisting of the constant symbol 0, the binary function symbol $+$, and for each $r \in R$ the unary function symbol $\cdot r$.

Theorem 1.2.5 ([4], 3.19)**.** *Let R be a ring. Then the class of injective right modules is first-order axiomatisable if and only if R is right noetherian.*

For R right noetherian, we denote the first-order theory of injective right modules by T_{inj}.

Another useful characterisation of noetherianity in terms of injectives is due to Bass & Papp, and Matlis:

Theorem 1.2.6 ([13], 3.46 & 3.48)**.** *Let R be a ring. Then the following are equivalent:*

1. *R is right noetherian;*
2. *Any direct sum of injective right R-modules is injective;*
3. *Any countable direct sum of injective right R-modules is injective;*
4. *Any directed colimit of injective right R-modules is injective;*
5. *Any injective module can be expressed as a (necessarily unique) direct sum of indecomposable injective modules.*

We say that a submodule A of a module B is **essential** if every non-zero submodule A' of B has non-zero intersection with A: $A \cap A' \neq 0$. If every non-zero submodule of B is essential (*i.e.*, if any two non-zero submodules of B have non-zero intersection), we say B is **uniform**. Clearly, any uniform module is indecomposable, for any non-trivial summands would have trivial intersection. The converse is false, in general.

The following is a well-known and useful set of criteria for checking uniformity, which we shall henceforth use without comment.

Lemma 1.2.7. *Let R be any ring and M an R-module. Then the following are equivalent:*

1. *M is uniform;*
2. *For any non-zero submodule $N \leq M$ and any non-zero element $m \in M$, there is $r \in R$ with $0 \neq mr \in N$;*
3. *For any non-zero elements $n, m \in M$, there are $r, s \in R$ with $0 \neq mr = ns$.*

PROOF:

(1. $\Rightarrow$ 2.): Since N and mR are non-zero and M is uniform, $N \cap mR \neq 0$, so there is some $r \in R$ with $0 \neq mr \in N$.

(2. $\Rightarrow$ 3.): Take $N = nR$ in (2).

(3. $\Rightarrow$ 1.): Given non-zero submodules $L, N \leq M$, take $0 \neq n \in N$ and $0 \neq m \in L$ and apply (3) to obtain $0 \neq mr = ns$ in $L \cap N$. ■

Eckmann and Schopf [3] proved that any module M has a maximal essential extension and minimal injective extension, and that these coincide. To state this more precisely:

Theorem 1.2.8 ([3], §4). *Let M be an R-module. Then there is an injective module $E(M)$ which contains M as an essential submodule. Whenever B is an essential extension of M, there is a (necessarily essential) embedding of B into $E(M)$ which is the identity on M, and whenever E is an injective extension of M there is a (necessarily split) embedding of $E(M)$ into E which is the identity on M.*

A module with these properties is called an **injective hull** or **injective envelope** of M. Given an injective hull of M and another injective module E containing M as an essential submodule, it follows easily from the above that there is an isomorphism between $E(M)$ and E which is the identity on M. As such, we regard any two injective hulls of M as being the same and refer simply to "the" injective hull of M.

Theorem 1.2.3 gives some insight into how we might compute the injective hull of a module M. A module is injective if and only if every consistent system of equations has a solution, so we "throw in" to M solutions to consistent systems of equations to extend M to its injective hull. For instance, given $\mathbb{Z}_\mathbb{Z}$, to form the injective hull we would adjoin solutions to equations. The equation $bx = a$ should have solution $x = a/b$, so we expect to obtain $\mathbb{Q}_\mathbb{Z}$ as $E(\mathbb{Z}_\mathbb{Z})$; indeed, we shall prove properly that $\mathbb{Q}$ is the injective hull of $\mathbb{Z}$ in Example 1.2.10.

Recall that a module is **strongly indecomposable** if its endomorphism ring is local; as the name suggests, any strongly indecomposable module is indecomposable. We have the following important result:

Theorem 1.2.9 ([13], 3.52). *Let E be an injective R-module. Then the following are equivalent:*

1. *E is indecomposable.*

2. *E is strongly indecomposable.*

3. *E is uniform.*

4. *There is a uniform module M such that $E = E(M)$.*

5. *For any non-zero submodule $M \leq E$, $E = E(M)$.*

Indecomposable injective modules will play a vital rôle throughout this thesis; we shall make extensive use of the above characterisations without explicit reference.

Example 1.2.10. *The injective hull of $\mathbb{Z}_{\mathbb{Z}}$ is $\mathbb{Q}$, which is indecomposable.*

We showed in Example 1.2.4 that $\mathbb{Q}_{\mathbb{Z}}$ is injective, and certainly it contains $\mathbb{Z}$. Therefore $E(\mathbb{Z})$ embeds in $\mathbb{Q}$; since any embedding from an injective splits, $E(\mathbb{Z})$ is a summand of $\mathbb{Q}$. Therefore by proving indecomposability of $\mathbb{Q}$, we also obtain that $\mathbb{Q} = E(\mathbb{Z})$.

To prove indecomposability, we prove uniformity. Given any two non-zero elements a/b and c/d in $\mathbb{Q}$, we have $(a/b)bc = ac = (c/d)ad$ is a non-zero common multiple of a/b and c/d. So $\mathbb{Q}$ is a uniform abelian group, and hence is indecomposable and the injective hull of $\mathbb{Z}$. ■

1.3 Hereditary Torsion Theories

We shall also require the notion of torsion theory, which we recall here. All this introductory exposition can be found, with proofs, in [23], particularly chapters 6 and 7; see also [18, Chapter 11] for an exposition that is less detailed, but closer in spirit to the present discussion.

Let $\mathcal{A}$ be a Grothendieck abelian category throughout the following discussion.

Lemma 1.3.1 ([18], §11.1)**.** *Let $(\mathcal{T}, \mathcal{F})$ be a pair of classes of objects in $\mathcal{A}$ (which are each closed under isomorphism). The following are equivalent:*

1. *$\mathcal{T}$ is closed under subobjects, quotients, extensions, and arbitrary coproducts, and $\mathcal{F}$ consists of all objects which receive no non-zero morphism from any object of $\mathcal{T}$.*

2. *$\mathcal{F}$ is closed under subobjects, extensions, injective hulls, and arbitrary products, and $\mathcal{T}$ consists of all objects which do not map to any object of $\mathcal{F}$.*

3. *There are no non-zero morphisms from an object of $\mathcal{T}$ to an object of $\mathcal{F}$, the addition of any object to either class would break this property, and $\mathcal{T}$ is closed under subobjects.*

4. *There are no non-zero morphisms from an object of $\mathcal{T}$ to an object of $\mathcal{F}$, the addition of any object to either class would break this property, and $\mathcal{F}$ is closed under injective hulls.*

5. *There is a left exact subfunctor τ of the identity functor on $\mathcal{A}$ with the property that for any $A \in \mathcal{A}$, $\tau(A/\tau(A)) = 0$, and $\mathcal{T}$ consists of those objects A with $\tau(A) = A$ while $\mathcal{F}$ consists of those objects A with $\tau(A) = 0$.*

When the equivalent conditions of the above Lemma hold, we say that the pair $(\mathcal{T}, \mathcal{F})$ is a **hereditary torsion theory**; we call $\mathcal{T}$ the **(hereditary) torsion class** and $\mathcal{F}$ the **(hereditary) torsionfree class**. An object is said to be torsion (respectively torsionfree) if it belongs to $\mathcal{T}$ (respectively $\mathcal{F}$). We will consider only hereditary torsion theories in this thesis, so we henceforth drop the adjective "hereditary". The functor τ of the Lemma is called the **torsion functor** or **torsion radical** of the torsion theory. For any object A, $\tau(A)$ is the unique largest subobject of A contained in $\mathcal{T}$.

By the Lemma, to specify a torsion theory it suffices to give any one of the following data: the torsion class, the torsionfree class, and the torsion radical. When specifying a torsion theory, we shall give any one of these, and denote the others with subscripts. For instance, if we say "let $\mathcal{T}$ be a torsion class", then by $\mathcal{F}_{\mathcal{T}}$ we shall mean the corresponding torsionfree class and by $\tau_{\mathcal{T}}$ the corresponding torsion radical.

Given an object A, we say that a subobject B is **τ-dense** in A if A/B is torsion.

A **Serre subcategory** of an abelian category is a full subcategory closed under isomorphisms, extensions, subobjects, and quotients. The objects of a Serre subcategory are referred to as a **Serre class**. So a torsion class is simply a Serre class in a Grothendieck category that is also closed under coproducts.

Given abelian categories $\mathcal{A}$ and $\mathcal{B}$ and an exact functor $F : \mathcal{A} \to \mathcal{B}$, we denote by $\ker(F)$ the class of objects A in $\mathcal{A}$ such that $F(A) = 0$, and call this the **kernel** of F. It is straightforward to verify that the kernel of an exact functor is always a Serre class; we shall shortly see a converse to this in Proposition 1.3.2.

The principal significance of Serre subcategories and torsion theories comes from the following:

Proposition 1.3.2 (See Chapter 4, especially Sections 4.3 and 4.4, of [17]). *Let $\mathcal{A}$ be an abelian category and $\mathcal{S}$ a Serre subcategory. Then there exist an abelian category $\mathcal{A}/\mathcal{S}$ and a dense, exact functor $Q_\mathcal{S} : \mathcal{A} \to \mathcal{A}/\mathcal{S}$ with kernel $\mathcal{S}$ obeying the following universal property: whenever $\mathcal{B}$ is an abelian category and $F : \mathcal{A} \to \mathcal{B}$ is an exact functor such that $F(A) = 0$ for all $A \in \mathcal{S}$, then there exists a unique exact functor $\hat{F} : \mathcal{A}/\mathcal{S} \to \mathcal{B}$ such that $F = \hat{F} \circ Q_\mathcal{T}$.*

Now suppose $\mathcal{A}$ is Grothendieck. Then $\mathcal{S}$ is closed under coproducts (i.e., is a torsion class in $\mathcal{A}$) if and only if $Q_\mathcal{S}$ admits a right adjoint, which we denote $i_\mathcal{S}$. When $\mathcal{S}$ is a torsion class, $i_\mathcal{S}$ is fully faithful and $\mathcal{A}/\mathcal{S}$ is Grothendieck. Moreover, for any object $A \in \mathcal{A}$, the localisation $i_\mathcal{S} Q_\mathcal{S}(A)$ can be described as

$$i_\mathcal{S} Q_\mathcal{S}(A) = \pi^{-1}\left(\tau_\mathcal{S}\left(\frac{E(A/\tau_\mathcal{S}(A))}{A/\tau_\mathcal{S}(A)}\right)\right),$$

where π is the quotient map. That is, to localise an object of a Grothendieck category at a torsion class, we quotient out the torsion part to obtain a torsionfree object, then look at the part of the injective hull which becomes torsion modulo this torsionfree object.

We call $\mathcal{A}/\mathcal{S}$ the **quotient category** or **localisation** of $\mathcal{A}$ by $\mathcal{S}$, $Q_\mathcal{S}$ the **quotient functor** or **localisation functor**, and $i_\mathcal{S}$ the **adjoint inclusion functor**.

In the case $\mathcal{A} = \text{Mod-}R$ for some ring R, so we can talk about elements of an object, if $\mathcal{T}$ is a torsion class in Mod-R, we can rewrite the description of the localisation as

$$i_\mathcal{T} Q_\mathcal{T}(M) = \left\{ e \in E(M/\tau(M)) \,\middle|\, e + (M/\tau_\mathcal{T} M) \in \tau_\mathcal{T}\left(\frac{E(M/\tau_\mathcal{T} M)}{M/\tau_\mathcal{T} M}\right)\right\}$$

That is, to localise we quotient out the torsion, then take those elements of the injective hull which become torsion modulo $M/\tau_\mathcal{T} M$.

We will be particularly interested in torsion theories with an additional property.

Lemma 1.3.3 ([18], 11.1.12, 11.1.14, 11.1.26)**.** *Let $(\mathcal{T}, \mathcal{F})$ be a torsion theory in $\mathcal{A}$. Then the following are equivalent:*

1. *$\tau_{\mathcal{T}}$ commutes with directed colimits.*
2. *$i_{\mathcal{T}}$ commutes with directed colimits of monomorphisms.*
3. *$\mathcal{F}$ is closed under directed colimits.*

Moreover, if $\mathcal{A}$ is locally finitely presented (has a generating set of finitely presented objects), then these conditions are equivalent to $\mathcal{T}$ being generated as a torsion class by $\mathcal{T} \cap \mathcal{A}^{\mathrm{fp}}$, the finitely presented torsion objects.

When these equivalent conditions hold, if $A \in \mathcal{A}$ is finitely generated, then $Q_{\mathcal{T}}(A) \in \mathcal{A}/\mathcal{T}$ is finitely generated, and if $\mathcal{G}$ is a generating family for $\mathcal{A}$, then $Q_{\mathcal{T}}\mathcal{G}$ is a generating family for $\mathcal{A}/\mathcal{T}$. If $\mathcal{A}$ is locally finitely presented, then "finitely generated" can be replaced by "finitely presented" in this paragraph.

Such a torsion theory is said to be **of finite type**. Note that, for any ring R, Mod-R is locally finitely presented, with R_R as a finitely presented generating object.

A Grothendieck category is **locally noetherian** if it has a generating set of noetherian objects. For R a right noetherian ring, Mod-R is locally noetherian, as well as being locally finitely presented. In a locally noetherian category, all torsion theories are of finite type; this follows from [18, Proposition 11.1.14]. Therefore, since our focus here is on modules over right noetherian rings, all torsion theories we consider will be of finite type.

Given a torsion class $\mathcal{T}$ of finite type in a locally coherent Grothendieck category $\mathcal{A}$, the finitely presented objects of $\mathcal{T}$, form a Serre subcategory $\mathcal{T} \cap \mathcal{A}^{\mathrm{fp}}$ of $\mathcal{A}^{\mathrm{fp}}$; conversely, given a Serre subcategory of $\mathcal{A}^{\mathrm{fp}}$, closing under coproducts gives a torsion class in $\mathcal{A}$. In a locally noetherian category, this establishes an isomorphism of lattices between the lattice of torsion classes in $\mathcal{A}$ and the lattice of Serre subcategories of $\mathcal{A}^{\mathrm{fp}}$. We will mostly work with torsion classes directly, but when the torsion classes under consideration are of finite type, it is sometimes convenient to restrict to finitely presented objects and work with Serre subcategories of $\mathcal{A}^{\mathrm{fp}}$ instead.

It is easy to see from the closure conditions that any intersection of torsion (resp. torsionfree) classes is itself a torsion (resp. torsionfree) class. Therefore, given an indexing set I and a torsion theory $(\mathcal{T}_i, \mathcal{F}_i)$ for each $i \in I$, we can construct two new torsion theories. The first of these has torsion class $\bigcap_{i \in I} \mathcal{T}_i$; we denote the torsionfree class for this theory $\sum_{i \in I} \mathcal{F}_i$. The second has torsionfree class $\bigcap_{i \in I} \mathcal{F}_i$; we denote its torsion class $\sum_{i \in I} \mathcal{T}_i$.

It is not hard to check that these intersection and sum operations make the set of torsion classes (partially ordered under inclusion), into a complete lattice. Similarly, the set of torsionfree classes is a complete lattice, and these two lattices are dual to each other. See [8, §1] for details, though be aware that the notation there differs significantly from here.

Given a class C of objects in $\mathcal{A}$, we denote by $\mathcal{T}(C)$ the intersection of all torsion classes containing C and call it the **torsion class generated by** C. Similarly, we denote by $\mathcal{F}(C)$ the intersection of all torsionfree classes containing C and call it the **torsionfree class cogenerated by** C. When $C = \{A\}$ consists of a single object, we omit the braces, writing simply $\mathcal{T}(A)$ and $\mathcal{F}(A)$.

The following useful result is common knowledge.

Lemma 1.3.4. *Let R be a right noetherian ring and S a set of R-modules. Let $E(S)$ denote the set of injective hulls of modules in S, and P the product of all modules in S. Then $\mathcal{F}(S)$ consists of all submodules of direct products of objects of $E(S)$, $\mathcal{F}(S) = \mathcal{F}(E(S)) = \mathcal{F}(P)$, and $\mathcal{T}_{\mathcal{F}(S)}$ consists of those modules M such that $(M, E) = 0$ for all $E \in E(S)$.*

Proof:

Let C denote the class of submodules of products of elements of $E(S)$. Since torsionfree classes are closed under subobjects, products, and injective hulls, certainly $C \subseteq \mathcal{F}(S)$. For the reverse inclusion, we prove that C is a torsionfree class, and appeal to the fact that $\mathcal{F}(S)$ is, by definition, the smallest torsionfree class containing S. So we must show that C itself is closed under subobjects, products, and injective hulls; but this is easily verified.

Applying this characterisation to $\mathcal{F}(S)$ and $\mathcal{F}(E(S))$ shows that they are the same. To see that $\mathcal{F}(P)$ is also the same, observe that $\mathcal{F}(S)$ is closed under products, so $P \in \mathcal{F}(S)$, and $\mathcal{F}(P)$ is closed under submodules, so $S \subseteq \mathcal{F}(P)$; therefore $\mathcal{F}(S) = \mathcal{F}(P)$.

Finally, a module M is $\mathcal{F}(S)$-torsion if and only if $(M, F) = 0$ for all $F \in \mathcal{F}(S)$. So if $M \in \mathcal{T}_{\mathcal{F}(S)}$, then in particular $(M, E) = 0$ for all $E \in E(S)$. On the other hand, if $M \notin \mathcal{T}_{\mathcal{F}(S)}$, then $N := M/\tau_{\mathcal{F}(S)}(M)$ is a non-zero, $\mathcal{F}(S)$-torsionfree module, so N is a submodule of a product of elements of $E(S)$; in particular, $(N, E) \neq 0$ for some $E \in E(S)$, so $(M, E) \neq 0$ by restriction along the quotient map. So $M \in \mathcal{T}_{\mathcal{F}(S)}$ if and only if $(M, E) = 0$ for all $E \in E(S)$. ■

Example 1.3.5. *In* $\mathbf{Ab} = \mathbb{Z}\text{-Mod}$*, torsion-theoretic localisation at* $\mathcal{F}(\mathbb{Q})$ *is the same as classical localisation at the multiplicative set* $\mathbb{Z} \setminus 0$*. More precisely, there is an equivalence of categories* $\mathbb{Q}\text{-Mod} \cong \mathbf{Ab}/\mathcal{T}_{\mathcal{F}(\mathbb{Q})}$ *such that the following diagram commutes.*

$$\begin{tikzcd}
\mathbf{Ab} \arrow[d, "\mathbb{Q} \otimes_{\mathbb{Z}} -"'] \arrow[dr, "Q_{\mathcal{F}(\mathbb{Q})}"] & \\
\mathbb{Q}\text{-Mod} \arrow[r] & \frac{\mathbf{Ab}}{\mathcal{T}_{\mathcal{F}(\mathbb{Q})}}
\end{tikzcd}$$

Moreover, for any denominator set over any ring, there is a corresponding torsion theory such that the Ore and torsion-theoretic localisations coincide in the same sense.

We will not prove these assertions rigorously, but will outline the ideas. For the first assertion, we outline both a proof by concrete computation, to illustrate the localisation progress in action, and also a more abstract proof.

Both arguments rely on first computing $\mathcal{T}_{\mathcal{F}(\mathbb{Q})}$; we show that this consists precisely of the torsion abelian groups in the classical sense. By Lemma 1.3.4, $\mathcal{T}_{\mathcal{F}(\mathbb{Q})}$ consists of all abelian groups A such that $(A, \mathbb{Q}) = 0$. If A contains an element a of infinite order, then $a\mathbb{Z}$ is free, so $(a\mathbb{Z}, \mathbb{Q}) \neq 0$; since $\mathbb{Q}$ is injective, this implies that $(A, \mathbb{Q}) \neq 0$. On the other hand, if $f : A \to \mathbb{Q}$ is non-zero, then we can take $a \in A$ such that $f(a) \neq 0$, and then $f(a)$ generates a free subgroup of $\mathbb{Q}$, hence a generates a free subgroup of A. So we see that $A \in \mathcal{T}_{\mathcal{F}(\mathbb{Q})}$ if and only if A contains no elements of infinite order; *i.e.*, if and only if A is a torsion group in the classical sense.

Now we proceed with the explicit computation. For any $A \in \mathbf{Ab}$, $\tau_{\mathcal{F}(\mathbb{Q})}A$ is the largest subgroup of A which is torsion; *i.e.*, it is exactly the torsion subgroup. Therefore $A/\tau_{\mathcal{F}(\mathbb{Q})}A$ is a torsionfree group in the classical sense. For convenience, we shall write $\bar{A}$ for $A/\tau_{\mathcal{F}(\mathbb{Q})}(A)$. Now, Proposition 1.3.2 tells us that

$$i_{\mathcal{F}(\mathbb{Q})} Q_{\mathcal{F}(\mathbb{Q})}(A) = \pi^{-1}\left(\tau_{\mathcal{F}(\mathbb{Q})}\left(\frac{E(\bar{A})}{\bar{A}}\right)\right),$$

so we take the injective hull of $\bar{A}$. This is a direct sum of indecomposable injective modules, by Theorem 1.2.6, and $\bar{A}$ is essential in $E(\bar{A})$, hence has non-zero intersection with each summand. But $\bar{A}$ is torsionfree and we shall see in Example 1.4.2 that all indecomposable injective abelian groups except $\mathbb{Q}$ are torsion, so $E(\bar{A})$ must be a direct sum of copies of $\mathbb{Q}$.

Now we consider the quotient $E(\bar{A})/\bar{A}$. We will show that this is an $\mathcal{F}(\mathbb{Q})$-torsion group; *i.e.*, a classical torsion group. Write $E(\bar{A})$ as $\mathbb{Q}^{(I)}$ for some cardinal I, by the above, and take $a = (a_i)_{i\in I}$ a non-zero element of $\mathbb{Q}^{(I)}$. The i^{th} summand of $\mathbb{Q}^{(I)}$ has non-zero intersection with $\bar{A}$, since $\bar{A}$ is essential in $E(\bar{A})$; since $\mathbb{Q}$ is uniform, if $a_i \neq 0$, there is some $n_i \in \mathbb{Z}$ such that $0 \neq a_i n_i \in \bar{A}$. Since there are only finitely many i for which $a_i \neq 0$, we can take n to be the maximum of the n_i, and then $0 \neq an \in \bar{A}$. Therefore, in $E(\bar{A})/\bar{A}$, $a + \bar{A}$ is a torsion element. But a was arbitrary, so $E(\bar{A})/\bar{A}$ is torsion, as claimed.

Finally, taking the preimage of $\tau_{\mathcal{F}(\mathbb{Q})}(E(\bar{A})/\bar{A})$ under the quotient map, we conclude

$$i_{\mathcal{F}(\mathbb{Q})}Q_{\mathcal{F}(\mathbb{Q})}(A) = E(\bar{A}) = \mathbb{Q}^{(I)}.$$

We now show that this coincides with $A \otimes_{\mathbb{Z}} \mathbb{Q}$.

The canonical map $A \to A \otimes_{\mathbb{Z}} \mathbb{Q}$ has kernel consisting precisely of the (classical) torsion elements of A, so $\bar{A}$ embeds in $A \otimes_{\mathbb{Z}} \mathbb{Q}$. Moreover, this embedding is essential, since elements of $A \otimes_{\mathbb{Z}} \mathbb{Q}$ can be expressed as fractions with numerator in A, so any non-zero element can be multiplied by its denominator to obtain a non-zero element of the form $a/1$ for $a \in A$. So $A \otimes_{\mathbb{Z}} \mathbb{Q}$ is an essential extension of $\bar{A}$, and is injective, being a direct sum of copies of $\mathbb{Q}$, hence is the injective hull $E(\bar{A})$, by Theorem 1.2.8.

So $i_{\mathcal{F}(\mathbb{Q})}Q_{\mathcal{F}(\mathbb{Q})}(A) = E(\bar{A}) = A \otimes_{\mathbb{Z}} \mathbb{Q}$, showing that the torsion-theoretic and classical localisations coincide.

For the more abstract argument, we note that $\mathbb{Q} \otimes_{\mathbb{Z}} -$ is an exact functor to the Grothendieck category $\mathbb{Q}$-Mod, which sends to zero precisely the torsion abelian groups; *i.e.*, $\ker(\mathbb{Q} \otimes_{\mathbb{Z}} -) = \mathcal{T}_{\mathcal{F}(\mathbb{Q})}$. Therefore, by the universal property of Proposition 1.3.2, $\mathbb{Q} \otimes_{\mathbb{Z}} -$ factors through a unique exact functor $F : \mathbf{Ab}/\mathcal{T}_{\mathcal{F}(\mathbb{Q})} \to \mathbb{Q}$-Mod.

It is an easy exercise to check that the kernel of F is the class of zero objects in the quotient category, and that F is therefore faithful. Since $\mathbb{Q} \otimes_{\mathbb{Z}} -$ is full and dense, so too is F. Therefore F is an equivalence.

Now for the general statement about denominator sets. Let R be a ring and D a right denominator set in R; *i.e.*, a right-reversible, right Ore set. Recall that a module M is said to be D-torsion if every element of M is annihilated by some element of D; equivalently, if the Ore localisation MD^{-1} is zero. It is straightforward to check that the class of D-torsion modules is closed under subquotients, extensions, and coproducts; therefore, this class is a torsion class, $\mathcal{T}_D$.

To show that Ore localisation at D coincides with torsion-theoretic localisation at $\mathcal{T}_D$, we adapt the second argument used above to show that $\mathbb{Q} \otimes_{\mathbb{Z}} -$ is localisation at $\mathcal{F}(\mathbb{Q})$. That is, note that the Ore localisation $- \otimes_R RD^{-1}$ is a full, dense, exact functor whose kernel is precisely $\mathcal{T}_D$ and apply the universal property of torsion-theoretic localisation to factor $- \otimes_R RD^{-1}$ through a faithful functor $F : \text{Mod-}R/\mathcal{T}_D \to \text{Mod-}RD^{-1}$, which must be full and dense, and so is an equivalence. ∎

1.4 The Injective Spectrum

The Zariski spectrum of a commutative ring R is a locally ringed space, $\operatorname{Spec}(R)$, whose underlying set is the set of all prime ideals of R. For $r \in R$, we denote by $D(r)$ the set of primes p such that $r \notin p$. The Zariski topology on $\operatorname{Spec}(R)$ has $\{D(r) \mid r \in R\}$ as a basis of open sets. There are strong links between the geometric properties of $\operatorname{Spec}(R)$ and the algebraic properties of R.

For noncommutative rings, this spectrum does not work so well. For instance, over a field k of characteristic 0, the first Weyl algebra, $A_1(k)$, is a simple ring, so has only one prime ideal, hence has a one-point spectrum, but is too rich a ring to be described by such a simple space; for instance, it is not artinian, so based on the commutative theory, we would expect its spectrum to be at least 1-dimensional.

It is desirable then to find an alternative definition of the Zariski spectrum, with a better generalisation to noncommutative rings. The first step to doing this was completed by Matlis, who established the following

Theorem 1.4.1 ([14], 3.1)**.** *Let R be a commutative noetherian ring. Then there is a bijection between prime ideals of R and (isoclasses of) indecomposable injective* *modules, given by taking a prime p to $E(R/p)$ in one direction and taking E to the (provably unique) ideal which is maximal among annihilators of non-zero submodules of E, in the other.*

PROOF:

If p is prime, it is $\cap$-irreducible among right ideals, so R/p is uniform and hence $E(R/p)$ is indecomposable.

Conversely, given E indecomposable injective, the set of annihilators of non-zero submodules of E contains some maximal element $p = \mathrm{ann}_R(N)$ by noetherianity. We show that any other annihilator $I = \mathrm{ann}_R(M)$ is contained in p. For $N \cap M \neq 0$, by uniformity, and $I + p \subseteq \mathrm{ann}_R(N \cap M)$, but p is maximal among annihilators, so $I + p = p$. Moreover, if $rs \in p$, then $Nrs = 0$; so either $Nr = 0$, in which case $r \in p$, or $(Nr)s = 0$, so $s \in \mathrm{ann}_R(Nr)$, and Nr is a non-zero submodule of E, so $\mathrm{ann}_R(Nr) \subseteq p$. So p is prime.

Finally, we show that these two assignments are mutually inverse. On the one hand, given a prime ideal p, any non-zero submodule M of $E(R/p)$ has non-zero intersection with R/p, so $\mathrm{ann}_R(M) \subseteq \mathrm{ann}_R(M \cap R/p)$. Since p is prime, for any $r + p \in R/p$, if $(r + p)s = 0$, then $rs \in p$, so $r + p = p$ or $s \in p$; so the annihilator of any non-zero submodule of R/p is contained in p, so $\mathrm{ann}_R(M) \subseteq p$. So p is the maximal annihilator of non-zero submodules of $E(R/p)$.

On the other hand, given E indecomposable injective, let p be the maximal annihilator of non-zero submodules of E. Then there is some $M \subseteq E$ with $p = \mathrm{ann}_R(M)$. For $m \in M \setminus 0$, $p \subseteq \mathrm{ann}_R(m)$, but also $\mathrm{ann}_R(m) \subseteq p$, by unique maximality. So $p = \mathrm{ann}_R(m)$, and so $mR \cong R/p$, so R/p embeds in E and therefore $E = E(R/p)$. ■

Example 1.4.2. *The indecomposable injective abelian groups are $\mathbb{Q}$ and the Prüfer groups $\mathbb{Z}_{p^\infty}$ for p prime.*

By the above Theorem 1.4.1, the indecomposable injectives are precisely $E(\mathbb{Z}/(p))$ for (p) a prime ideal; *i.e.*, $p = 0$ or p prime. We saw in Example 1.2.10 that $\mathbb{Q}_{\mathbb{Z}} = E(\mathbb{Z})$ (and indeed that this is indecomposable), so it remains to show that $E(\mathbb{Z}/(p)) = \mathbb{Z}_{p^\infty}$ for p prime. We also illustrate how we could arrive at $\mathbb{Z}_{p^\infty}$ for the injective hull of $\mathbb{Z}/(p)$ without being given it.

By remarks after Theorem 1.2.8, to form the injective hull of a module, we must throw in solutions to consistent systems of equations. Consider the equation $ax = b$, where $a \in \mathbb{Z}$, $b \in \mathbb{Z}/(p)$. This is consistent whenever $a \neq 0$, since $\mathbb{Z}$ is a domain. For any $a \notin (p)$, this has a solution, since $\mathbb{Z}/(p)$ is a field. For $a = \alpha p^n$, where p does not divide α, it is sufficient to solve $p^n x = c$ where $\alpha c = b$. Of course, if we can solve $p^n x = 1 + (p)$, then cx solves $p^n(cx) = c$. So it suffices to solve $p^n x = 1 + (p)$.

There is an embedding $\mathbb{Z}/(p) \to \mathbb{Z}/(p^{n+1})$, taking $1 + (p)$ to $p^n + (p^{n+1})$. Then $1 + (p^{n+1})$ is a solution of $p^n x = p^n + (p^{n+1})$; so this is an extension of $\mathbb{Z}/(p)$ in which the equation $p^n x = 1 + (p)$ has a solution (after changing the name of $1 + (p)$ according to the embedding). Therefore, to solve all consistent equations in one variable, we need to solve $p^n x = 1 + (p)$ for all n, which suggests taking the directed colimit

$$\varinjlim \frac{\mathbb{Z}}{(p^n)} = \mathbb{Z}_{p^\infty}.$$

So this calculation suggests that $\mathbb{Z}_{p^\infty}$ is likely to be the injective hull of $\mathbb{Z}/(p)$. A simple application of Baer's Criterion (Theorem 1.2.1) proves that $\mathbb{Z}_{p^\infty}$ is indeed injective. Moreover, it is uniserial with socle $\mathbb{Z}/(p)$, so any non-zero submodule contains $\mathbb{Z}/(p)$. So $\mathbb{Z}_{p^\infty}$ is an essential extension of $\mathbb{Z}/(p)$, hence the injective hull. ■

The next step in generalising the Zariski spectrum was of course to describe the topology in terms of the indecomposable injective modules. This was accomplished by Gabriel [6, §VI.3], though [18, §14.1.1] contains a discussion closer in spirit to our own. We first require some notation; for a module M, we denote by (M) the set of indecomposable injective modules E such that $(M, E) \neq 0$, and by $[M]$ its complement in the set of indecomposable injectives: $[M] = \{E \text{ indecomposable injective} \mid (M, E) = 0\}$.

For a ring R, let $\mathrm{InjSpec}(R)$ denote the set of (isoclasses of) indecomposable injective modules, with the topology having $\{[M] \mid M \in \text{mod-}R\}$ as a basis of open sets. We call this topological space the (right) **injective spectrum** of R. Note that $\mathrm{InjSpec}(R)$ is indeed a set, not a proper class; for any indecomposable E is the injective hull of every non-zero submodule, and contains a cyclic submodule, and there is only a set of cyclic modules, hence only a set of injective hulls of cyclic modules.

Theorem 1.4.3 ([6], §VI.3)**.** *For R a commutative noetherian ring, the Matlis bijection is a homeomorphism between* $\mathrm{InjSpec}(R)$ *and* $\mathrm{Spec}(R)$*.*

This gives a topological space associated to any ring, which agrees with the Zariski spectrum in the commutative noetherian case. In fact, the definition of the injective spectrum now does not depend on the ring at all, only on the category Mod-R; therefore, for a Grothendieck category $\mathcal{A}$, we define $\mathrm{InjSpec}(\mathcal{A})$ to be the set of (isoclasses of) indecomposable injective objects of $\mathcal{A}$, endowed with the topology having $\{[A] \mid A \in \mathcal{A}^{\mathrm{fp}}\}$ as a basis. Of course, this is an abuse of notation, as now $\mathrm{InjSpec}(R)$ is actually equal to $\mathrm{InjSpec}(\text{Mod-}R)$. However, as R is not a Grothendieck category, this should cause no confusion.

Note also that $\mathrm{InjSpec}(R)$ depends only on the right module category of R; R also has a left injective spectrum, $\mathrm{InjSpec}(R\text{-Mod})$. We will not consider the left injective spectrum here; of course, it is included in the present results by switching to the opposite ring, so we lose nothing by restricting to the right spectrum.

Before computing an example, we require a simple result:

Lemma 1.4.4. *Let $\mathcal{A}$ be any Grothendieck category and*

$$0 \to A \to C \to B \to 0$$

a short exact sequence in $\mathcal{A}$. Then $[C] = [A] \cap [B]$.

Proof:

Let E be any injective object of $\mathcal{A}$. Then the functor $(-, E) : \mathcal{A}^{\mathrm{op}} \to \mathbf{Ab}$ is exact by Theorem 1.2.2, giving us a short exact sequence of abelian groups:

$$0 \to (B, E) \to (C, E) \to (A, E) \to 0.$$

The middle term in a short exact sequence is zero if and only if both outer terms are zero, so we see that $(C, E) = 0$ if and only if $(A, E) = 0$ and $(B, E) = 0$. Since this holds for all indecomposable injectives E, this means $[C] = [A] \cap [B]$. ■

Example 1.4.5. *We compute* $\mathrm{InjSpec}(\mathbb{Z})$ *and compare with* $\mathrm{Spec}(\mathbb{Z})$.

By Example 1.4.2, the points of $\mathrm{InjSpec}(\mathbb{Z})$ are $\mathbb{Q}$ and the Prüfer groups $\mathbb{Z}_{p^\infty}$ for each non-zero prime p.

The topology on $\mathrm{InjSpec}(\mathbb{Z})$ has basis of open sets consisting of the sets $[M] = \{E \in \mathrm{InjSpec}(\mathbb{Z}) \mid (M, E) = 0\}$ for M a finitely presented $\mathbb{Z}$-module. Any such M

is expressible as a finite direct sum of cyclic modules $\mathbb{Z}$ and $\mathbb{Z}/(p^n)$ for prime p and positive integer n. We first consider the basic open sets $[\mathbb{Z}]$ and $[\mathbb{Z}/(p^n)]$, then use the above Lemma 1.4.4.

Firstly, note that $\mathbb{Z}$ admits a non-zero map to any non-zero abelian group, in particular the points of $\mathrm{InjSpec}(\mathbb{Z})$, so $[\mathbb{Z}] = \varnothing$. For p prime and n positive, $\mathbb{Z}/(p^n)$ admits a non-zero map to an abelian group A if and only if A contains a non-zero element annihilated by p^n. In $\mathbb{Q}$, every non-zero element has annihilator 0, whereas in $\mathbb{Z}_{q^\infty}$, every element has annihilator some power of (q); so the only indecomposable injective receiving a non-zero map from $\mathbb{Z}/(p^n)$ is $\mathbb{Z}/p^\infty$. Therefore $[\mathbb{Z}/(p^n)] = \mathrm{InjSpec}(\mathbb{Z}) \smallsetminus \{\mathbb{Z}_{p^\infty}\}$.

Now, given a finitely presented abelian group A, we can easily compute $[A]$ by Lemma 1.4.4. If A has any elements of infinite order (*i.e.*, if A has a summand isomorphic to $\mathbb{Z}_\mathbb{Z}$), then $[A]$ is empty. Otherwise, $[A]$ consists of $\mathbb{Q}$ and the Prüfer groups $\mathbb{Z}_{p^\infty}$ for those primes p such that A has no p-torsion elements (*i.e.*, $[A]$ omits only those Prüfers at primes p such that $\mathbb{Z}/(p^n)$ is a summand of A for some n).

We see therefore that every non-empty basic open set includes $\mathbb{Q}$, and the non-empty basic open sets are all cofinite. Moreover, for any finite set of primes $p_1, \ldots, p_n$, the cofinite set $\mathrm{InjSpec}(\mathbb{Z}) \smallsetminus \{\mathbb{Z}_{p_1^\infty}, \ldots, \mathbb{Z}_{p_n^\infty}\}$ is precisely the basic open set $[A]$, where

$$A = \bigoplus_{i=1}^{n} \mathbb{Z}/(p_i),$$

so every cofinite set still including $\mathbb{Q}$ is basic open.

Since every open set is a union of basic open sets, and a union of cofinite sets is cofinite, we see that the open sets of $\mathrm{InjSpec}(\mathbb{Z})$ are precisely the cofinite sets which include $\mathbb{Q}$, and the empty set. Put another way, the closed sets are precisely the finite sets which exclude $\mathbb{Q}$, the empty set, and the whole space.

Compare now with $\mathrm{Spec}(\mathbb{Z})$. The points here are the primes 0 and (p) for p a prime number, and the closed sets are precisely the finite sets excluding 0, the empty set, and the whole space. Of course, 0 is the prime corresponding to $\mathbb{Q}_\mathbb{Z}$ under Matlis' bijection, and p corresponds to $\mathbb{Z}_{p^\infty}$, so we see that this bijection does indeed give a homeomorphism of topological spaces. ■

Of course, the Zariski spectrum is not just a topological space, it is a ringed space. Gabriel also developed a sheaf of rings on the injective spectrum of a ring, via localisation at hereditary torsion theories of finite type [6, §VI.3], see also [18, §14.1.4]. We describe this now.

Let R be any ring. Given a basic open set $[M]$ (for $M \in \text{mod-}R$), let $\mathcal{F}_M$ be the torsionfree class cogenerated by the indecomposable injectives in $[M]$. Note that an indecomposable injective E is in $\mathcal{F}_M$ if and only if $(M, E) = 0$, so $\mathcal{F}_M$ consists precisely of those indecomposable injectives which are torsionfree for the torsion class $\mathcal{T}(M)$; therefore $\mathcal{F}_M = \mathcal{F}_{\mathcal{T}(M)}$. Since M is finitely presented, and R_R is a finitely presented generator for Mod-R, so Mod-R is locally finitely presented, Lemma 1.3.3 tells us that $(\mathcal{T}(M), \mathcal{F}_M)$ is a torsion theory of finite type.

Let $(\text{Mod-}R)_M$ denote the localisation of Mod-R at this torsion theory, and denote by Q_M, i_M the associated quotient and adjoint inclusion functors. Denote by R_M the endomorphism ring of $(Q_M R, Q_M R)$ in $(\text{Mod-}R)_M$. Note that, since Q_M is left adjoint to i_M, this is isomorphic as an abelian group to $(R, i_M Q_M R)$, which in turn is isomorphic to the R-module $i_M Q_M R$, by Yoneda's Lemma. Moreover, Q_M gives a ring map $(R, R) \to (Q_M R, Q_M R)$, *i.e.*, $R \to R_M$, which we denote ρ_M.

So we have, for each basic open set $[M]$, a ring R_M and a ring map $R \to M$. If $[N] \subseteq [M]$, then $\mathcal{F}_N$ is cogenerated by a subset of (a cogenerating set of) $\mathcal{F}_M$, so $\mathcal{F}_N \subseteq \mathcal{F}_M$. Therefore $\mathcal{T}(M) \subseteq \mathcal{T}(N)$; so, by the universal property of localisation (Proposition 1.3.2), the quotient functor Q_N factors through Q_M by a unique exact functor $Q_{M,N}$:

$$\begin{tikzcd} \text{Mod-}R \arrow[d, "Q_M"'] \arrow[dr, "Q_N"] & \\ (\text{Mod-}R)_M \arrow[r, "Q_{M,N}"'] & (\text{Mod-}R)_N \end{tikzcd}$$

Therefore $Q_{M,N} Q_M R = Q_N R$, and so $R_N = (Q_{M,N} Q_M R, Q_{M,N} Q_M R)$. Moreover, $Q_{M,N}$ gives a ring map $R_M = (Q_M R, Q_M, R) \to (Q_{M,N} Q_M R, Q_{M,N} Q_M R) = R_N$, which we denote $\rho_{M,N}$. So for each inclusion of basic open sets $[N] \subseteq [M]$, we have a restriction map $R_M \to R_N$.

Similarly, if $[L] \subseteq [N] \subseteq [M]$, $Q_{M,L} : (\text{Mod-}R)_M \to (\text{Mod-}R)_L$ is the unique exact functor such that this diagram commutes:

$$\begin{tikzcd}
\text{Mod-}R \arrow[d, "Q_M"'] \arrow[dr, "Q_L"] & \\
(\text{Mod-}R)_M \arrow[r, "Q_{M,L}"'] & (\text{Mod-}R)_L
\end{tikzcd}$$

But the following diagram commutes, and all functors in it are exact:

$$\begin{tikzcd}
\text{Mod-}R \arrow[d, "Q_M"'] \arrow[dr, "Q_N"] \arrow[drr, "Q_L"] & & \\
(\text{Mod-}R)_M \arrow[r, "Q_{M,N}"'] & (\text{Mod-}R)_N \arrow[r, "Q_{N,L}"'] & (\text{Mod-}R)_L
\end{tikzcd}$$

Therefore $Q_{M,L} = Q_{N,L} \circ Q_{M,N}$. In particular, $\rho_{M,L} = \rho_{N,L} \circ \rho_{M,N}$. Therefore the assignment taking a basic open set $[M]$ to R_M and an inclusion of basic open sets $[N] \subseteq [M]$ to $\rho_{M,N}$ is a presheaf-on-a-basis on $\mathrm{InjSpec}(R)$. This is sufficient for the sheafification process to work, and so we obtain a sheaf of rings $\mathcal{O}_R$ on $\mathrm{InjSpec}(R)$, which we call the **sheaf of finite type localisations**, or simply the **structure sheaf**.

Of course, we must compare this to the usual structure sheaf in the commutative case. Indeed, we have the following:

Theorem 1.4.6 ([6], §VI.3)**.** *If R is commutative noetherian and* $\mathrm{InjSpec}(R)$ *is identified with* $\mathrm{Spec}(R)$ *via the Matlis bijection, then the sheaf of finite type localisations is isomorphic to the usual Zariski structure sheaf.*

Example 1.4.7. *Having illustrated the topological space* $\mathrm{InjSpec}(\mathbb{Z})$*, we now also illustrate the sheaf of finite type localisations.*

We showed in Example 1.4.5 that, given a finitely presented abelian group A, the basic open set $[A]$ was empty if A had a free summand and otherwise consisted of all indecomposable injectives except those $\mathbb{Z}_{p^\infty}$ where A had a p-torsion element. So suppose A has no free summand and let $p_1, \dots, p_n$ be the primes for which A has torsion. So the finite-type torsionfree class $\mathcal{F}_A$ is cogenerated by $\mathbb{Q}$ and those $\mathbb{Z}_{p^\infty}$ for $p \notin \{p_1, \dots, p_n\}$.

Now we localise $\mathbb{Z}_\mathbb{Z}$ at $\mathcal{F}_A$ to obtain $i_A Q_A \mathbb{Z}$. Recall that

$$i_A Q_A \mathbb{Z} = \pi^{-1}\left(\tau_A\left(\frac{E(\mathbb{Z}/\tau_A(\mathbb{Z}))}{\mathbb{Z}/\tau_A(\mathbb{Z})}\right)\right) \qquad (\star)$$

First we show that for any abelian group B, $\tau_A(B)$ consists of the elements of B whose order is finite and not divisible by any prime not in $\{p_1, \ldots, p_n\}$.

Suppose $b \in B$ has order m. Then $(b\mathbb{Z}, \mathbb{Q}) = 0$, and $b\mathbb{Z} \cong \mathbb{Z}/m\mathbb{Z}$ has a non-zero map precisely to those $\mathbb{Z}_{q^\infty}$ where q is a prime factor of m. But $b\mathbb{Z} \in \mathcal{T}_{\mathcal{F}_A}$ if and only if it maps to no torsionfree injective, if and only if it does not map to $\mathbb{Q}$ or to any $\mathbb{Z}_{q^\infty}$ with $q \notin \{p_1, \ldots, p_n\}$. So $b \in \tau_A(B)$ if and only if $b\mathbb{Z} \subseteq \tau_A(B)$ if and only if b has finite order whose prime factors are all in $\{p_1, \ldots, p_n\}$, as claimed.

In particular, $\tau_A(\mathbb{Z}) = 0$, so Equation $(\star)$ simplifies to $i_A Q_A \mathbb{Z} = \pi^{-1}(\tau_A(\mathbb{Q}/\mathbb{Z}))$. An element of $\mathbb{Q}$ can be uniquely expressed as an integer plus an element of $\mathbb{Q} \cap [0, 1)$, and this non-integer part can be uniquely decomposed by partial fractions into a sum of fractions with prime power denominator. It follows that elements of $\mathbb{Q}/\mathbb{Z}$ are uniquely expressible as finite sums of fractions of the form a/p^n for p prime, $0 < a < p^n$, and $p \nmid a$. Grouping terms with a common prime in the denominator, it follows that

$$\frac{\mathbb{Q}}{\mathbb{Z}} \cong \bigoplus_{p \text{ prime}} \mathbb{Z}_{p^\infty}.$$

Now, $\tau_A(\mathbb{Q}/\mathbb{Z})$ consists of those elements annihilated only by products of the p_i, $i \in \{1, \ldots, n\}$, so

$$\tau_A\left(\frac{\mathbb{Q}}{\mathbb{Z}}\right) \cong \bigoplus_{i=1}^{n} \mathbb{Z}_{p_i^\infty}.$$

Therefore $i_A Q_A \mathbb{Z}$ consists of those elements of $\mathbb{Q}$ whose fractional part has denominator having prime factors only among the p_i. That is, $i_A Q_A \mathbb{Z}$ is the localisation of $\mathbb{Z}$ at the multiplicative set generated by $p_1, \ldots, p_n$. That is, if we let $m = p_1 \times \ldots \times p_n$, then $i_A Q_A \mathbb{Z} = \mathbb{Z}[m^{-1}]$.

Comparing this with $\mathrm{Spec}(\mathbb{Z})$, $[A]$ corresponds to the complement of $\{p_1, \ldots, p_n\}$, which is the basic open set $D(m)$ given by m. The localisation of $\mathbb{Z}$ corresponding to $D(m)$ is $\mathbb{Z}[m^{-1}]$, the same as the ring found with the sheaf of finite-type localisations. It is routine but tedious to verify that the restriction maps are also the same between the two sheaves. ■

1.5 The Ziegler Spectrum

There is an alternative route to the injective spectrum, via the model theory of modules. For a ring R, let $\mathcal{L}_R$ be the first-order language (with equality) containing a constant symbol 0, a binary function symbol $+$, and, for each ring element $r \in R$, a unary function symbol $\cdot r$. The class of right R-modules is a definable class within this language, and of course we always work modulo the axioms of R-modules.

The class of **positive primitive (pp) formulae** is the smallest class of $\mathcal{L}_R$-formulae containing all atomic formulae and closed under conjunction and existential quantification. A general pp-formula in free variables $x_1, \ldots, x_n$ therefore has the form

$$\exists x_{n+1}, \ldots, x_m \bigwedge_{i=1}^{k} \left(\sum_{j=1}^{m} x_j r_{ij} = 0 \right)$$

modulo the theory of R-modules, where the r_{ij} are elements of R (hence unary function symbols of $\mathcal{L}_R$).

The following result is a well-known Corollary to the results of Eklof and Sabbagh that are given in this thesis as Theorems 1.2.3 and 1.2.5.

Proposition 1.5.1. *Let R be a right noetherian ring and $\phi(x_1, \ldots, x_n)$ a pp formula for right R-modules. Then there is a quantifier-free pp formula (i.e., a finite system of equations) $\theta(x_1, \ldots, x_n)$ such that $T_{\text{inj}} \vdash \phi \leftrightarrow \theta$. In other words, in any injective module E containing elements $e_1, \ldots, e_n$, $E \models \phi(e_1, \ldots, e_n)$ if and only if $E \models \theta(e_1, \ldots, e_n)$.*

This follows from, for instance, [18, Th^m 11.1.44 and following remarks], though that proves a much more general result. For a sketch proof, note that ϕ asserts the existence of a solution to a system of equations with parameters $x_1, \ldots, x_n$. By Theorem 1.2.3, this occurs if and only if the system is consistent, which occurs if and only if certain ring elements annihilate certain linear combinations of the $x_1, \ldots, x_n$; *i.e.*, if and only if the $x_1, \ldots, x_n$ satisfy certain equations. Since the ring is right noetherian, it suffices to take finitely many equations among the $x_1, \ldots, x_n$ to establish consistency, and so the desired formula θ is the conjunction of these equations.

A **pp-pair** ϕ/ψ is a pair of pp-formulae, ϕ and ψ, in the same free variables, such that in all modules M, $\psi(M) \subseteq \phi(M)$. We say that the pp-pair ϕ/ψ **closes** on the module M, or that M closes ϕ/ψ, if $\phi(M) = \psi(M)$; otherwise, we say that ϕ/ψ **opens** on M or that M opens ϕ/ψ.

An embedding of R-modules $i : A \hookrightarrow B$ is said to be **pure** if for every pp formula $\phi(x)$, we have $i\phi(A) = \phi(B) \cap i(A)$ Note that the forward inclusion is automatic, so the content of this definition is that $i\phi(A) \supseteq \phi(B) \cap i(A)$. That is, i is pure if whenever $a \in A$ and $B \models \phi(i(a))$, then $A \models \phi(a)$.

A module M is said to be **pure-injective** if every pure embedding starting from M splits. The set of all indecomposable pure-injective modules is denoted $\mathrm{pinj}(R)$. Note that every injective module is necessarily pure-injective, so $\mathrm{InjSpec}(R) \subseteq \mathrm{pinj}(R)$ as sets.

Ziegler [26] defined a topological space associated to a ring, now known as the (right) **Ziegler spectrum** and denoted Zg_R, whose point set is $\mathrm{pinj}(R)$ and whose topology has for a basis of open sets the sets $(\phi/\psi) := \{M \in \mathrm{pinj}(R) \mid \phi(M) \neq \psi(M)\}$ for ϕ/ψ pp-pairs. That is, the set of indecomposable pure-injective modules which open a given pp-pair form an open set. We introduce notation $(\phi/\psi) = \{M \in \mathrm{pinj}(R) \mid \phi(M) \neq \psi(M)$ and $[\phi/\psi] := \{M \in \mathrm{pinj}(R) \mid \phi(M) = \psi(M)\}$. The basic open sets (ϕ/ψ) are precisely the compact open sets of Zg_R [26, Thm. 4.9].

There is an alternative topology on $\mathrm{pinj}(R)$, based on Hochster's duality for spectral spaces (see Section 8.1) and the fact that the sets (ϕ/ψ) for pp-pairs ϕ/ψ are precisely the compact open sets of the Ziegler spectrum. Taking $\{[\phi/\psi] \mid \phi/\psi \text{ pp-pair}\}$ as a basis of open sets gives a topology called the **dual-Ziegler** topology; the space with this topology is called the dual-Ziegler spectrum and denoted $\mathrm{Zg}_R^{\mathrm{d}}$. Since all injectives are pure-injective, the injective spectrum is a subset of the Ziegler spectrum; it is natural therefore to ask how the topologies interact. We have the following

Theorem 1.5.2 ([19], 5.1)**.** *Let R be right coherent. Then the topology introduced above on* $\mathrm{InjSpec}(R)$ *is precisely the subspace topology inherited from the dual-Ziegler spectrum.*

In fact, for R right noetherian, the stronger result is true that the sets of the form $\mathrm{InjSpec}(R) \cap [\phi/\psi]$ for ϕ/ψ a pp-pair and the sets of the form $[M]$ for M finitely presented are precisely the same sets. We outline why this is true.

Proof:

On the one hand, given a finitely presented module M, we can take a generating set $m_1, \ldots, m_n$ and a finite conjunction of equations $\theta(x_1, \ldots, x_n)$ generating the relations on the m_i, and then a module E receives a non-zero map from M if and only if there are elements $e_1, \ldots, e_n \in E$, not all zero, such that $E \models \theta(e_1, \ldots, e_n)$. So $(M, E) = 0$ if and only if E closes the pp-pair

$$\frac{\theta(x_1, \ldots, x_n)}{\bigwedge_{i=1}^n x_i = 0}.$$

On the other hand, if ϕ/ψ is a pp-pair, by Proposition 1.5.1 each of ϕ and ψ is equivalent modulo T_{inj} to a quantifier-free pp formula; so without loss of generality we can assume that ϕ and ψ are quantifier-free, hence are simply finite systems of equations in variables $x_1, \ldots, x_n$. Define finitely presented modules A, respectively B, generated by $x_1, \ldots, x_n$ subject to the relations $\phi(x_1, \ldots, x_n)$, respectively $\psi(x_1, \ldots, x_n)$.

For any module M, the elements of $\phi(M)$ are precisely those tuples in M^n satisfying the relations ϕ, so are precisely the valid choices for where to send the generators of A to obtain a map $A \to M$. So there is an isomorphism of abelian groups (in fact, of functors) $\phi(M) \cong (A, M)$. Similarly, $\psi(M) \cong (B, M)$ for all M. So $[\phi/\psi] \cap \operatorname{InjSpec}(R)$ consists of those indecomposable injectives such that $\phi(M) = \psi(M)$; *i.e.*, such that $(A, M) = (B, M)$.

Since ϕ/ψ is a pp-pair, $\psi \to \phi$, so the generators of B satisfy the relations of the generators of A; so there is a surjection $B \to A$. Denote the (finitely presented) kernel by K. For any injective module E, the functor $(-, E)$ is exact (Theorem 1.2.2), so we have a short exact sequence

$$0 \to (A, E) \to (B, E) \to (K, E) \to 0.$$

So $(A, E) = (B, E)$ if and only if $(K, E) = 0$, and so $[\phi/\psi] \cap \operatorname{InjSpec}(R) = [K]$. ∎

There are therefore two meaningful topologies on the injective spectrum; the topology we first introduced, having $\{[M] \mid M \in \text{mod-}R\}$ for a basis of open sets, we call the **Zariski topology**; this is the subspace topology inherited from the dual-Ziegler spectrum. The subspace topology inherited from the Ziegler spectrum we call the **Ziegler topology**; this has $\{(M) \mid M \in \text{mod-}R\}$ as a basis of open sets.

When we use topological terms without specifying the topology, we will always mean the Zariski topology; however, it will sometimes be useful in proofs to switch to the Ziegler topology. The injective spectrum can be shown to be a closed subset of the Ziegler spectrum (for R right noetherian), essentially by Theorem 1.2.5 (but see also [18, §5.1.1]), so the Ziegler compact sets (ϕ/ψ) remain compact in the Ziegler topology on InjSpec(R). By Theorem 1.5.2, these are precisely Gabriel's sets (M) for M finitely presented.

Henceforth, when we write (ϕ/ψ) or $[\phi/\psi]$, we mean the intersection of this set with the injective spectrum, not the entire set in the Ziegler spectrum.

1.6 The Sheaf of Definable Scalars

Given a theory T of R-modules, a **definable scalar** for T is a pp formula $\rho(x, y)$ in 2 free variables satisfying the condition that

$$T \vdash \forall x \exists ! y\,(\rho(x, y))\,;$$

i.e., ρ defines a function on all models of T by taking a to the unique b such that $\rho(a, b)$ holds. Technically, a definable scalar is actually the equivalence class of such a formula under the equivalence relation of being logically equivalent modulo T, but for convenience we refer to a definable scalar as a specific formula.

Note that, for any ring element r, the formula $y = xr$ is a definable scalar for the theory of all R-modules, and hence also for any stronger theory. Therefore the ring of definable scalars includes all actual scalars, but may include more besides. However, modulo certain theories, distinct ring elements r and s could give the same definable scalar; clearly, any theory T containing an axiom $\forall x(xr = xs)$ will collapse r and s to the same definable scalar.

Now let R be right noetherian, so that injectivity is axiomatisable (Theorem 1.2.5). Given a pp-pair ϕ/ψ, there is an associated theory of R-modules, $T_{\mathrm{inj},[\phi/\psi]}$ obtained by adjoining to the theory of injectives T_{inj} an axiom saying that ϕ/ψ closes. Denote by $R_{[\phi/\psi]}$ the set of definable scalars for $T_{\mathrm{inj},[\phi/\psi]}$. There are addition and multiplication operations on $R_{[\phi/\psi]}$ defined as follows:

$$\rho(x, y) + \sigma(x, y) := \exists w, z(\rho(x, w) \wedge \sigma(x, z) \wedge (y = w + z))$$

$$\rho(x, y)\sigma(x, y) := \exists z(\rho(x, z) \wedge \sigma(z, y))$$

as well as a zero element $(x = x \wedge y = 0)$, and a unity $(y = x)$. This makes $R_{[\phi/\psi]}$ into a ring, called the **ring of definable scalars** for the basic dual-Ziegler open set $[\phi/\psi]$.

Moreover, if $[\phi/\psi] \subseteq [\xi/\eta]$, then every formula giving a definable scalar for $T_{\mathrm{inj},[\xi/\eta]}$ also gives a definable scalar for $T_{\mathrm{inj},[\phi/\psi]}$, so there is a function $R_{[\xi/\eta]} \to R_{[\phi/\psi]}$. Note that logical equivalence might be a finer equivalence relation modulo $T_{\mathrm{inj},[\phi/\psi]}$ than modulo $T_{\mathrm{inj},[\xi/\eta]}$, so this map need not be injective. It is easily checked that this function is a ring map, and gives a presheaf-on-a-basis of rings over the injective spectrum. Extending this to the entire topology and sheafifying gives the **sheaf of definable scalars** over the injective spectrum.

Theorem 1.6.1 ([21], Theorem 2.4.2)**.** *Let R be a right noetherian ring. Then the sheaf of definable scalars on* $\mathrm{InjSpec}(R)$ *is isomorphic to the sheaf of finite type localisations.*

In fact, the sheaf of definable scalars can be defined over the entire dual-Ziegler spectrum, and [21, Theorem 2.4.2] actually proves that if R is only right coherent, the restriction of the sheaf of definable scalars to $\mathrm{InjSpec}(R) \subseteq \mathrm{Zg}^{\mathrm{d}}_R$ agrees with the sheaf of finite type localisations. The restricted presentation given above makes the sheaf slightly easier to work with for our present purposes.

Example 1.6.2. *We illustrate definable scalars in the case $R = \mathbb{Z}$.*

Let $n \in \mathbb{Z}$ and consider the pp-pair $(nx = 0)/(x = 0)$. A module closes this pair if and only if it has no non-zero elements annihilated by n, if and only if it receives no map from $\mathbb{Z}/n\mathbb{Z}$; so $[(nx = 0)/(x = 0)] = [\mathbb{Z}/n\mathbb{Z}]$ in the injective spectrum, which consists of $\mathbb{Q}$ and all $\mathbb{Z}_{p^\infty}$ where p does not divide n. This corresponds to the basic open set $D(n)$ in $\mathrm{Spec}(\mathbb{Z})$.

Let $\rho(x, y)$ be the formula $ny = x$. On all abelian groups, this does not define a function; for instance, in $\mathbb{Z}/n\mathbb{Z}$, for $a \neq n\mathbb{Z}$, there is no b such that $\rho(a, b)$ holds, and every $b \in \mathbb{Z}/n\mathbb{Z}$ satsifies $\rho(n\mathbb{Z}, b)$; so the "function" defined by ρ fails to take values on most of the domain, and is multi-valued on the one point where it is defined.

However, the theory $T_{\mathrm{inj},[(nx=0)/(x=0)]}$ says that no non-zero element is annihilated by n and every non-zero element is divisible by n (since injectives are divisible, by Theorem 1.2.3 and the following remarks). Therefore, modulo $T_{\mathrm{inj},[(nx=0)/(x=0)]}$, ρ defines a function; so ρ is a definable scalar for this theory.

Therefore the ring of definable scalars $R_{[(nx=0)/(x=0)]}$ includes ρ, which can be thought of as the scalar $1/n$. Indeed, by checking the definitions for addition and multiplication of definable scalars, it is clear that ρ obeys the same arithmetic relationship to the integers in $R_{[(nx=0)/(x=0)]}$ as $1/n$ does in $\mathbb{Q}$. Indeed, one can check that $R_{[(nx=0)/(x=0)]} \cong \mathbb{Z}[n^{-1}]$, as expected under Theorem 1.6.1. ■

1.7 Pappacena's "Weak Zariski" Topology

In [16], Pappacena studies a topological space he calls the injective spectrum of a noncommutative space. This is closely related to the notion of injective spectrum we consider here, both in provenance and details, but is not the same. For the avoidance of confusion, we clarify here the distinction.

Pappacena defines a **noncommutative space** to be a Grothendieck abelian category, and focuses on those which are locally noetherian. This corresponds closely to the focus in this thesis on modules over noetherian rings. A **weakly closed subspace** is then defined to be a full subcategory which is itself Grothendieck, and is closed under subquotients, coproducts, and isomorphisms, and is such that the inclusion functor admits a right adjoint.

Given a noncommutative space X, Pappacena then defines the injective spectrum to be the set of isoclasses of indecomposable injective objects in X, with the **weak Zariski topology** defined as follows. For each weakly closed subspace Z, we define $V(Z)$ to be the set of indecomposable injectives containing a subobject from Z. Then the collection of all sets $V(Z)$ for Z weakly closed is a basis of closed sets for the weak Zariski topology. The space with this topology is what Pappacena refers to as the injective spectrum.

Clearly the points of Pappacena's injective spectrum coincide with ours. Moreover, for any module M, there is a weakly closed subspace $\sigma[M]$ (see Section 8.2 for this notation), which consists of all subquotients of coproducts of copies of M. Then the weak Zariski basic closed set $V(\sigma[M])$ is precisely the set $(M) = \{E \mid (M, E) \neq 0\}$. In particular, taking M to be finitely presented, every basic closed set in our topology is still basic closed for Pappacena; so, despite the name, the weak Zariski topology refines the Zariski topology.

This refinement is however strict, in general. To see this, work in the category of abelian groups and take M to be the direct sum of each simple group $\mathbb{Z}/(p)$ for p an odd prime. Then $V(\sigma[M])$ consists of all indecomposable injective abelian groups except $\mathbb{Q}$ and $\mathbb{Z}_{2^\infty}$. To prove this, show that $\mathbb{Q}$ and $\mathbb{Z}/(2)$ are torsionfree for the hereditary torsion class generated by M, so $(M, \mathbb{Q}) = 0 = (M, \mathbb{Z}_{2^\infty})$. But this set is not Zariski-closed, for it contains the generic point $\mathbb{Q}$ but not the whole space.

Pappacena also mentions a second topology on the injective spectrum, which he calls the **strong Zariski topology**. However, he points out that this topology is trivial for simple rings (remarks after Prop. 4.11 of [16]), in contrast with the Zariski topology studied here (see Theorem 4.2.2 and following remarks), so this topology is also different to ours.

1.8 Kanda's Atom Spectrum

A further topological space related to the injective spectrum is the atom spectrum. We give here a brief overview of this space and its relationship to the injective spectrum.

We begin with some definitions originally due to Storrer [24] and extended by Kanda [12]. Let $\mathcal{A}$ be an abelian category. An object A of $\mathcal{A}$ is **monoform** if for every non-zero subobject B of A, A and A/B have no common subobject. Two monoform objects are said to be **atom-equivalent** if they have a common non-zero subobject; an **atom** is an atom-equivalence class of monoform objects.

Kanda [12] then defines the **atom support** of an object $A \in \mathcal{A}$ to be the class of atoms represented by monoform subquotients of A, denoted $\mathrm{ASupp}(A)$, and the **atom spectrum** of $\mathcal{A}$, denoted $\mathrm{ASpec}(\mathcal{A})$ to be the set of all atoms together with a topology having as basis of open sets the atom supports of objects of $\mathcal{A}$. That is, for any $A \in \mathcal{A}$, $\mathrm{ASupp}(A) \subseteq \mathrm{ASpec}(\mathcal{A})$ is open. We then have the following result:

Theorem 1.8.1 ([11] Theorem 5.9). *Let $\mathcal{A}$ be a locally noetherian Grothendieck category (e.g., Mod-R for R right noetherian). Then there is a bijection* $\mathrm{ASpec}(\mathcal{A}) \to \mathrm{InjSpec}(\mathcal{A})$*, which is a homeomorphism when* $\mathrm{InjSpec}(\mathcal{A})$ *is equipped with the Ziegler topology. The bijection takes an atom $[A]$ represented by a monoform object A to the injective object $E(A)$.*

Since we are interested primarily in the dual-Ziegler (Zariski) topology, we do not use the notion of atoms, since the natural topology there is the Ziegler topology.

However, all the results of this thesis could be approached from an atom-theoretic standpoint.

There is an additional viewpoint on the injective spectrum, called the torsion spectrum, which will be useful to us. This forms the content of Chapter 5.

Chapter 2

Preliminary Results

2.1 Specialisation

First we consider the closure of a point in the injective spectrum. For $E, F \in \mathrm{InjSpec}(R)$, we write $E \rightsquigarrow F$ and say that E **specialises to** F if $F \in \mathrm{cl}(E)$ - *i.e.*, if every closed set containing E also contains F.

Lemma 2.1.1. *Let R be right noetherian. For $E, F \in \mathrm{InjSpec}(R)$, the following are equivalent:*

1. $E \rightsquigarrow F$
2. *For every finitely presented module M, if $(M, E) \neq 0$, then $(M, F) \neq 0$;*
3. *For every cyclic module M, if $(M, E) \neq 0$, then $(M, F) \neq 0$;*
4. $E \in \mathcal{F}(F)$;
5. *For every module M, if $(M, E) \neq 0$, then $(M, F) \neq 0$;*
6. *For every right ideal $I \leq R_R$, if $\mathrm{ann}_E(I) \neq 0$, then $\mathrm{ann}_F(I) \neq 0$;*

Proof:

(1. $\Leftrightarrow$ 2.): Since $\{[M] \mid M \in \text{mod-}R\}$ is a basis of open sets for the topology, their complements, $(M) := \{E \in \mathrm{InjSpec}(R) \mid (M, E) \neq 0\}$ form a basis of closed sets. Then $F \in \mathrm{cl}(E)$ if and only if F is in every closed set containing E, if and only if F is in every basic closed set containing E, and the result follows.

(2. $\Rightarrow$ 3.): Since R is right noetherian, the notions of finitely generated and finitely presented coincide; in particular, all cyclic modules are finitely presented.

(3. $\Rightarrow$ 4.): For any non-zero element $e \in E$, $(eR, E) \neq 0$; so, by (3), $(eR, F) \neq 0$. But for any element e of the $\mathcal{F}(F)$-torsion submodule $\tau_{\mathcal{F}(F)}(E)$, $eR \in \mathcal{T}_{\mathcal{F}(F)}$, so $(eR, F) = 0$, so $(eR, E) = 0$ and hence $e = 0$. Therefore $\tau_{\mathcal{F}(F)}(E) = 0$, and so $E \in \mathcal{F}(F)$.

(4. $\Rightarrow$ 5.): If $E \in \mathcal{F}(F)$, then $\mathcal{F}(E) \subseteq \mathcal{F}(F)$, since $\mathcal{F}(E)$ is the intersection of all torsionfree classes containing E. So $\mathcal{T}_{\mathcal{F}(F)} \subseteq \mathcal{T}_{\mathcal{F}(E)}$, since the lattice of torsion classes is dual to the lattice of torsionfree classes. Therefore for any module M, if $(M, F) = 0$, then $(M, E) = 0$. Taking the contrapositive gives the result.

(5. $\Rightarrow$ 2.): Obvious.

(3. $\Leftrightarrow$ 6.): Any cyclic module has the form R/I for some right ideal I. Consider the short exact sequence

$$0 \to I \to R \to R/I \to 0$$

and let $H \in \{E, F\}$; since H is injective, we have an exact sequence

$$0 \to (R/I, H) \to (R, H) \to (I, H) \to 0.$$

So $(R/I, H)$ is the submodule of (R, H) consisting of those maps which vanish on I. By Yoneda's Lemma, $(R, H) \cong H$, and, under this isomorphism, $(R/I, H) \cong \mathrm{ann}_H(I)$. The result follows. ■

In Section 6.1, statement (4) will be particularly useful to us, coupled with the fact that, for F injective, the modules in $\mathcal{F}(F)$ are precisely those which embed in a product of copies of F (Lemma 1.3.4).

Corollary 2.1.2. *Let $E, F \in \mathrm{InjSpec}(R)$ with $E \rightsquigarrow F$. Then, taking $M = E$ in statement (5) of the lemma, $(E, F) \neq 0$.*

The converse of this statement is true for commutative noetherian rings.

Proposition 2.1.3. *Let R be commutative noetherian and $E, F \in \mathrm{InjSpec}(R)$. Then $E \rightsquigarrow F$ if and only if $(E, F) \neq 0$.*

PROOF:

By the homeomorphism between $\text{InjSpec}(R)$ and $\text{Spec}(R)$ for commutative noetherian rings (Theorem 1.4.3), $E = E(R/p)$, $F = E(R/q)$ for some prime ideals p, q of R, and $E \rightsquigarrow F$ if and only if $p \rightsquigarrow q$ in $\text{Spec}(R)$. But the specialisation order on $\text{Spec}(R)$ is simply the inclusion order on prime ideals. So it suffices to prove that $p \subseteq q$ if and only if $(E, F) \neq 0$.

On the one hand, if $p \subseteq q$, then there is a non-zero map $R/p \to R/q$, and by injectivity this extends to give a non-zero map $E(R/p) \to E(R/q)$. On the other hand, suppose $\phi : E \to F$ is non-zero, and $e \in E$ is such that $\phi(e) \neq 0$. Matlis proved that every element of $E(R/p)$ has annihilator exactly p^n for some n [14, Theorem 3.4], so we have $ep^n = 0$ for some n, and so $\phi(e)p^n = 0$, so $p^n \subseteq \text{ann}_R(\phi(e))$. But since $\phi(e) \in F = E(R/q)$, Matlis' result again gives $p^n \subseteq q^m$ for some m. So $p^n \subseteq q$, and since q is prime it follows that $p \subseteq q$. ■

However, in the non-commutative case, it can occur that $(E, F) \neq 0$, but E does not specialise to F. We illustrate this below.

Example 2.1.4. *Let k be a field and R the path algebra over k of the quiver A_2; equivalently, the ring of 2×2 upper triangular matrices over k. Then $\text{InjSpec}(R)$ is discrete, but there is a non-zero map between distinct indecomposable injectives.*

By standard results from the theory of quiver representations, there are precisely 2 indecomposable injectives over R; as representations of the quiver, they are $k \to 0$ and $k \to k$ (where the map in the second representation is the identity). Both of these are closed points; for $\{k \to 0\} = (k \to 0)$, and $\{k \to k\} = (0 \to k)$. So the topology is discrete and neither point specialises to the other. However, there is a non-zero map from $k \to k$ onto $k \to 0$, namely the quotient map. ■

So we have two indecomposable injectives, with a non-zero map between them, but no specialisations between them. Contrasting with the commutative case above, we see that the problem is that $k \to k$ is an extension between two non-isomorphic simple modules, whereas the import of Matlis' result is that (after localising at p) $E(R/p)$ is built from copies of the single simple module $(R/p)_p$.

So in the commutative case, there is a homogeneity of indecomposable injectives, which are essentially each built entirely out of one part, whereas in the non-commutative case we can have an indecomposable injective built out of multiple different parts. In Section 6.2, we shall see an even more extreme case of this lack of homogeneity: we shall construct a uniform module which is built out of infinitely many simple modules, and whose injective hull fails to specialise to the hull of any of these simples.

2.2 Closed Points

In the commutative noetherian case, the closed points of $\mathrm{Spec}(R)$ are the maximal ideals, so the closed points of $\mathrm{InjSpec}(R)$ are the points of the form $E(R/m)$ for m a maximal ideal. Of course, m is maximal if and only if R/m is a simple module, so the closed points are those which contain a simple submodule. It would be desirable for this to hold more generally; indeed, we can obtain some results in this direction.

Proposition 2.2.1. *Let $E \in \mathrm{InjSpec}(R)$ and suppose that E has a simple submodule S. Then E is a closed point.*

Proof:

The embedding $S \hookrightarrow E$ gives a non-zero element of (S, E), so if $E \rightsquigarrow F$, then $(S, F) \neq 0$. But S is simple, so any non-zero map from S is an embedding, so $S \leq F$; but F is indecomposable injective, so F is the injective hull of S, which is E. So E is alone in its closure, hence is a closed point. ■

We can also obtain a partial converse to this.

Proposition 2.2.2. *Let E be a closed point in $\mathrm{InjSpec}(R)$ such that $\{E\} = (M)$ is basic closed (for some finitely presented module M). Then E has a simple submodule. Moreover, if $\mathrm{InjSpec}(R)$ is a noetherian space, then the additional hypothesis that $\{E\}$ be basic closed can be dropped.*

Proof:

Since M is a finitely generated module, it has at least one simple quotient. For any simple quotient S of M, we have $(M, E(S)) \neq 0$, so $E(S) \in (M) = \{E\}$, so $E = E(S)$.

Now suppose that InjSpec(R) is noetherian. Then, since E is a closed point, $\{E\}$ is an intersection of basic closed sets, and this intersection can be taken to be finite, by noetherianity. Now, each basic Zariski-closed set (M) is Ziegler-open, and so $\{E\}$ must be Ziegler-open. Hence $\{E\}$ can be written as a union of basic Ziegler-open sets (M). But, since $\{E\}$ is a singleton, it must in fact be equal to a basic Ziegler-open set, which is the same as a basic Zariski-closed set. So in this case the hypothesis that $\{E\}$ be basic closed is automatic. ■

This result is one of several places where the importance of noetherianity of the injective spectrum appears. We will see in Chapter 4 that the injective spectrum of a right noetherian domain of Krull dimension 0 or 1 is noetherian (see Chapter 3 for Krull dimension). However, Section 6.2 exhibits a right noetherian domain of Krull dimension 2, whose injective spectrum fails to be noetherian; moreover, there is a closed point which contains no simple submodule, in contrast with the above Proposition.

Question 1. *Under what conditions on the ring is it true that every closed point of the injective spectrum contains a simple submodule? Is there a characterisation of the closed points in the injective spectrum that holds for any ring and has a similar flavour to containing a simple submodule?*

2.3 Functoriality of the Injective Spectrum

Of course, the Zariski spectrum is not simply a topological space associated to a commutative ring. It is a contravariant functor from the category of commutative rings to the category of topological spaces (actually locally ringed spaces, but we will consider sheaves in Chapter 7).

Unfortunately, there are serious limits to what we can hope for when it comes to functoriality of InjSpec. In particular, we cannot hope for a duality of categories, as the following example, shown to me by Ryo Kanda, illustrates.

Example 2.3.1. *Let k be a field and $M_2(k)$ denote the ring of 2×2 matrices with coefficients in k. Consider the ring map $f : k \times k \to M_2(k)$, embedding the leading diagonal. We show that f cannot induce a natural map of ringed spaces* $\mathrm{InjSpec}(M_2(k)) \to$

InjSpec($k \times k$).

In fact, we will forget the ringed structure and simply work with topological spaces; this suffices, for if InjSpec were a functor to ringed spaces, composing with the forgetful functor would give a functor to topological spaces.

Since $M_2(k)$ is simple artinian, every module is injective. There is a unique indecomposable module, k^2, so InjSpec$(M_2(k)) = \{k^2\}$. On the other hand, indecomposable $k \times k$-modules are just indecomposable modules over each factor, and again all modules are injective, since the ring is semisimple artinian, so InjSpec$(k \times k)$ is a 2-point, discrete space.

So any map induced by f on injective spectra would need to take the single point of InjSpec$(M_2(k))$ to one of the two points of InjSpec$(k \times k)$. But there is nothing to distinguish these points; indeed, there is an automorphism of $k \times k$ swapping the two factors, so there is no natural way to pick one out to be the image of an induced map.

In particular, if there were a duality of categories between the category of rings and some subcategory of ringed spaces, as in the commutative case, then the monomorphism f would induce an epimorphism of ringed spaces, hence a topological epimorphism. But there is no epimorphism from a one point space to a discrete, two point space. ∎

Theorem 2.3.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be Grothendieck abelian categories and let $F : \mathcal{A} \to \mathcal{B}$ be a fully faithful functor with an exact left adjoint $L : \mathcal{B} \to \mathcal{A}$ which preserves finitely presented objects. Then F induces by restriction and corestriction a continuous injection* InjSpec$(\mathcal{A}) \to$ InjSpec$(\mathcal{B})$.

If, moreover, L is such that for any finitely presented object A of $\mathcal{A}$ there is $B \in \mathcal{B}$ finitely presented with $L(B) = A$, then the induced map is a topological embedding.

Note that if $\mathcal{B}$ is locally finitely presented, L is the localisation at a hereditary torsion theory of finite type, and F is the adjoint inclusion (see Section 1.3), then all of these assumptions are satisfied.

PROOF:

Since F is fully faithful, it preserves indecomposables. For if $A \in \mathcal{A}$ is such that $F(A)$ is decomposable, then there is a non-trivial idempotent endomorphism e of $F(A)$

(projection onto a summand). Then, since F is full, $e = F(\epsilon)$ for some $\epsilon : A \to A$, with $F(\epsilon^2 - \epsilon) = e^2 - e = 0$; but F is faithful, so $\epsilon^2 - \epsilon = 0$, so ϵ is an idempotent, and necessarily non-trivial. So A is decomposable.

Since F has an exact left adjoint, it preserves injectives. For if $E \in \mathcal{A}$ is an injective, then $(-, F(E)) = (L(-), E)$ is the composition of the exact functors $(-, E)$ and L, so is exact, implying that $F(E)$ is injective. So F gives a well-defined function $\mathrm{InjSpec}(\mathcal{A}) \to \mathrm{InjSpec}(\mathcal{B})$, which is injective since F is fully faithful, and hence injective on isoclasses of objects.

To show that this map is continuous, we show that the preimage of any basic open set $[B]$, for $B \in \mathcal{B}^{\mathrm{fp}}$, is open. To see this, note that $F^{-1}[B] = \{E \in \mathrm{InjSpec}(\mathcal{A}) \mid (B, F(E)) = 0\}$; but $(B, F(E)) = (L(B), E)$, so $F^{-1}[B] = [L(B)]$. Since L preserves finitely presented objects, this is open (even basic open).

Finally, note that for any $A \in \mathcal{A}^{\mathrm{fp}}$, $F[A] = \{F(E) \mid E \in \mathrm{InjSpec}(\mathcal{A}) \wedge (A, E) = 0\}$. If $A = L(B)$, for some $B \in \mathcal{B}^{\mathrm{fp}}$, then $(A, E) = (L(B), E) = (B, F(E))$, so then $F[A] = \{F(E) \mid E \in \mathrm{InjSpec}(\mathcal{A}) \wedge (B, F(E)) = 0\} = [B] \cap F(\mathrm{InjSpec}(\mathcal{A}))$. So if for any A we can find such a B, then the image of any basic open set is relatively open within the image and so F is an embedding of topological spaces. ■

We can obtain a useful Corollary to this; first, we require the following result due to Silver:

Lemma 2.3.3 ([22], Corollary 1.3). *Let $f : R \to S$ be a map of rings. Then f is a ring epimorphism if and only if the restriction of scalars functor* Mod-$S \to$ Mod-R *induced by f is full.*

Corollary 2.3.4. *Let R and S be rings and $f : R \to S$ a ring epimorphism such that S is flat when viewed as a left (sic) R-module. Then f induces a continuous injection $f^* : \mathrm{InjSpec}(S) \to \mathrm{InjSpec}(R)$ by restriction of scalars. If, moreover, every finitely presented S-module can be obtained by extending scalars on a finitely presented R-module, then f^* is a topological embedding.*

PROOF:

The restriction of scalars functor Mod-$S \to$ Mod-R is faithful and, since f is epi, is also full. It has a left adjoint $- \otimes_R S$, which is exact, since ${}_R S$ is flat. Therefore the above Theorem applies. ■

Question 2. *Is there a broader class of ring morphisms which induce maps between injective spectra? Or a class of rings properly including all commutative rings on which all ring maps induce maps on injective spectra?*

Chapter 3

Krull Dimension

3.1 Noncommutative Krull Dimension and Critical Dimension

Following on from the consideration of closed points in the last chapter, it is worth considering the height of a prime ideal in a commutative ring and attempting to relate this to indecomposable injectives. The closed points in the spectrum of a commutative ring are the primes of maximal height, and the lower the height of a prime, the bigger its closure. Height corresponds to the codimension of the associated variety (dimension in the sense of chains of irreducible closed sets). We therefore seek a dimension associated to an indecomposable injective module which will correspond to topological dimension.

As in the commutative case, we shall work with Krull dimension; however, we use the "noncommutative Krull dimension" developed by Gabriel and Rentschler [7], followed by Gordon and Robson [9]. We recall the definitions and basic results.

We first define the **deviation** of a poset (from being artinian), by a transfinite induction. Let P be a poset.

For $a, b \in P$, let $[a, b]$ denote the subposet of P defined by the formula $a \leq x \leq b$. If $a_0 > a_1 > \ldots$ is a chain in P of type ω^{op}, we call the subposets $[a_{i+1}, a_i]$ for $i < \omega$ the **factors** of the chain.

If P is trivial (has no comparable elements), we say P has deviation -1 (some authors prefer $-\infty$). If P is non-trivial and has the descending chain condition, then we say it has deviation 0. Having defined what it means to have deviation at least α, we say that P has deviation at least $\alpha + 1$ if there is an infinite descending chain in P of type ω^{op} all of whose factors have deviation at least α. If λ is a limit ordinal and P has deviation at least α for all $\alpha < \lambda$, we say that P has deviation at least λ. Finally, P has deviation α if it has deviation at least α and does not have deviation at least $\alpha + 1$.

Note that some posets do not have well-defined deviation. For instance, suppose $P = (\mathbb{Q} \cap [0, 1], <)$ has deviation α, where here $[0, 1]$ denotes the closed unit interval in $\mathbb{R}$. Then take any chain $a_0 > a_1 > \ldots$; each factor $[a_{i+1}, a_i]$ is isomorphic to P, so has deviation α; but then P has deviation at least $\alpha + 1$, a contradiction. Indeed, a poset fails to have deviation if and only if it contains a dense linear order [15, Proposition 6.1.12].

For M an R-module, we define the **Krull dimension** $K(M)$ to be the deviation of the poset of submodules of M. So $K(M)$ measures how far M is from being artinian. We define the (right) Krull dimension of a ring to be its dimension as a (right) module over itself: $K(R) := K(R_R)$.

For sufficiently large modules, the Krull dimension need not be defined. We show this below.

Lemma 3.1.1. *Let P be a poset whose deviation exists. Then for any convex subposet Q of P, the deviation of Q exists, and is no greater than the deviation of P.*

PROOF:

Any descending chain in Q is a descending chain in P with the same factors when regarded in Q or in P. The result follows immediately. ■

Proposition 3.1.2. *An infinite direct sum of non-zero modules does not have Krull dimension.*

PROOF:

This proof is an adaptation of [15, Lemma 6.2.6], which proves a similar result. Suppose for a contradiction that the result is false; then, by the well-order property of ordinals, we can choose a counterexample of minimal Krull dimension, α say. So there is a module M of Krull dimension α which is an infinite direct sum of non-zero modules, and α is the minimal Krull dimension among such modules. Without loss of generality, we can assume that M is a countable direct sum of non-zero modules; for certainly M contains a countable direct sum, and by Lemma 3.1.1 any submodule of M has Krull dimension at most α, so by minimality any infinite direct sum contained in M also has Krull dimension α. So we write

$$M = \bigoplus_{i<\omega} N_i$$

for some non-zero summands N_i.

Define a sequence of submodules of M

$$M_n := \bigoplus_{i<\omega} N_{2^n i}$$

for each $n < \omega$. Then $M_0 > M_1 > M_2 > \ldots$ is an infinite descending chain, with factors

$$\frac{M_n}{M_{n+1}} = \frac{\bigoplus_{i<\omega} N_{2^n i}}{\bigoplus_{i<\omega} N_{2^{n+1} i}} \cong \bigoplus_{i<\omega} N_{2^n(2i-1)}.$$

Each factor of this chain is itself an infinite direct sum of non-zero submodules. By Lemma 3.1.1, each factor has Krull dimension; so, by minimality of α, we have $K(M_n/M_{n+1}) = \alpha$ for all n. But then $K(M)$ must be at least $\alpha + 1$, a contradiction. ■

In the case where $K(M)$ fails to exist, we write $K(M) = \infty$. For the purposes of inequalities, we regard ∞ as being strictly greater than any ordinal.

Let α be an ordinal. We say that a non-zero module M is **α-critical** if $K(M) = \alpha$ and for any non-zero submodule N of M, $K(M/N) < \alpha$. We say that M is **critical** if there exists some α such that M is α-critical. Note that a 0-critical module is precisely a simple module.

Some basic facts about Krull dimension are summarised below. All can be found, with proofs, in [15, §§6.1-6.3]. These results will be used throughout without explicit reference.

Proposition 3.1.3 ([15], §§6.1-6.3)**.** *Let R be any ring and M an R-module.*

1. *If R is commutative and noetherian, then $K(R)$ is equal to the classical Krull dimension (the maximal length of a chain of prime ideals).*

2. *If M is noetherian, then $K(M)$ exists.*

3. *If L is a submodule of M, then $K(M) = \max\{K(L), K(M/L)\}$.*

4. *If M is finitely generated, then $K(M) \leq K(R)$.*

5. *If $M \neq 0$ and $K(M) \neq \infty$, then M has a critical submodule.*

6. *If M is α-critical, then any non-zero submodule of M is also α-critical.*

7. *If M is critical, then M is uniform.*

Critical modules will be particularly useful to us, so we sketch the proofs of parts 5, 6, and 7:

PROOF:

5. Suppose $M \neq 0$ and $K(M)$ exists. Then every submodule and quotient of M also has Krull dimension, by Lemma 3.1.1. Krull dimension is ordinal-valued, so the dimensions of submodules of M attain a minimum, α say; so let L be a submodule of Krull dimension α. If L is not critical, then there is some $L_1 < L$ such that $K(L/L_1) = K(L)$; by minimality of α, $K(L_1) = \alpha$. Similarly, if L_1 is not critical, there is $L_2 < L_1$ with $K(L_2) = \alpha = K(L_1/L_2)$. Proceeding in this way, if L contains no critical submodule we obtain a descending chain $L > L_1 > L_2 > \ldots$ such that each factor has dimension α, so $K(L) \geq \alpha + 1$, a contradiction.

6. Suppose M is α-critical and $0 < L \leq M$. By part (3), $\alpha = \max\{K(L), K(M/L)\}$, but $K(M/L) < \alpha$, so $K(L) = \alpha$. Now for any $0 < N \leq L$, L/N embeds in M/N, so $K(L/N) \leq K(M/N) < \alpha$, so L is α-critical.

7. Suppose M is not uniform. Then M contains a direct sum $L \oplus N$ with $L \neq 0 \neq N$. By part (3), $K(L \oplus N) = \max\{K(L), K(N)\}$; without loss of generality suppose $K(L) \leq K(N)$. Then $K(L \oplus N) = K(N)$, but N is a proper quotient of $L \oplus N$, so $L \oplus N$ is not critical. Therefore, by part (6), M is not critical. ■

In this section, we investigate the relationship between Krull dimension and the geometric structure of the injective spectrum. The aim is to develop a picture where the Krull dimension of the ring is equal to the dimension of the spectrum (in the sense of the length of a maximal chain of irreducible closed subsets) and the dimension of the closure of a point E in the spectrum is governed by a Krull dimension associated to E.

Unfortunately, this nice picture does not always work. We shall see it failing in Section 6.2. An open question is to establish conditions in which it does work.

Even before coming to such counterexamples to the nice picture we might hope for, a problem with the project proposed above is that injective modules, being 'big', do not generally have a well-defined Krull dimension. However, we can circumvent this issue, at least over a right noetherian ring, by defining a new, closely related notion of dimension as follows.

For an arbitrary non-zero module M over a right noetherian ring, let the **critical dimension** $\mathrm{cd}(M)$ denote the minimum Krull dimension of non-zero submodules of M. Note that, since every non-zero module has a finitely generated submodule and every finitely generated module is noetherian and so has Krull dimension, M must contain a non-zero submodule whose Krull dimension is defined. Then, since Krull dimension is ordinal-valued, and so obeys the well-order property, $\mathrm{cd}(M)$ is well-defined for any non-zero M. The name will be explained shortly.

We begin by noting two trivial but useful properties of critical dimension.

Lemma 3.1.4. *Let R be a right noetherian ring, M any non-zero R-module and N any R-module whose Krull dimension is defined. Then*

1. *M has a $\mathrm{cd}(M)$-critical submodule;*

2. *If $K(N) < \mathrm{cd}(M)$, then $(N, M) = 0$.*

PROOF:

1. By definition of $\mathrm{cd}(M)$ and since R is right noetherian, M has a non-zero submodule L with $K(L) = \mathrm{cd}(M)$. Any module which has Krull dimension contains a critical submodule, by Proposition 3.1.3, so there is $L' \leq L$ critical. Moreover, since $L' \leq M$, $K(L') \geq \mathrm{cd}(M)$, but since $L' \leq L$, $K(L') \leq K(L) = \mathrm{cd}(M)$; so L' is $\mathrm{cd}(M)$-critical.

2. Suppose $K(N) < \mathrm{cd}(M)$ and let $\phi : N \to M$ be a morphism. Then $\phi(N)$ has Krull dimension at most $K(N)$, but M has no non-zero submodule of Krull

dimension less than $\mathrm{cd}(M)$, so $\phi(N)$ must be zero. ■

Lemma 3.1.5. *Let U be a uniform module. Then any two critical submodules of U have the same Krull dimension.*

PROOF:

Suppose $A, B \leq U$ are critical. Then, by Proposition 3.1.3, any submodule of A is $K(A)$-critical and any submodule of B is $K(B)$-critical. But U is uniform, so A and B have a common submodule, and therefore $K(A) = K(B)$. ■

This critical dimension now allows us to apply Krull dimension to injective modules. By the above Lemma 3.1.5, for an indecomposable injective E, all critical submodules must have the same Krull dimension; in light of Lemma 3.1.4, this dimension will be precisely $\mathrm{cd}(E)$. This is the reason for terming it the critical dimension.

This chapter, and our subsequent uses of the results herein, should be compared with results of Pappacena in [16]. There, it is shown that for any dimension function on a Grothendieck category satisfying suitable axioms, the sum of all critical subobjects of a uniform object is itself critical. Pappacena then works with the resulting largest critical subobject and relates its dimension to topological notions of dimension on the injective spectrum in his "weak Zariski" topology (see Section 1.7). Of course, given a uniform object U, the dimension of its largest critical subobject is precisely the critical dimension as defined here; so Pappacena uses essentially the same tool as we do, but reached via a slightly different route.

3.2 Specialisation and Dimension

Lemma 3.2.1. *Let R be right noetherian and take $E, F \in \mathrm{InjSpec}(R)$ with $E \rightsquigarrow F$. Then $\mathrm{cd}(E) \geq \mathrm{cd}(F)$, with equality if and only if $E \cong F$.*

PROOF:

Take a $\mathrm{cd}(E)$-critical submodule C of E, by Lemma 3.1.4; then $E = E(C)$, since E is indecomposable. Now $E \rightsquigarrow F$ and $(C, E) \neq 0$, so $(C, F) \neq 0$, by Lemma 2.1.1. It follows by Lemma 3.1.4 that $\mathrm{cd}(E) = K(C) \geq \mathrm{cd}(F)$.

Take $\phi : C \to F$ non-zero. If ϕ is an embedding, then $F \cong E(C) = E$ (and certainly $\mathrm{cd}(E) = \mathrm{cd}(F)$). Otherwise, $\phi(C)$ is a proper quotient of C, hence $K(\phi(C)) < K(C) = \mathrm{cd}(E)$, by criticality of C. But $(\phi(C), F) \neq 0$, so, by Lemma 3.1.4, $\mathrm{cd}(F) \leq K(\phi(C)) < \mathrm{cd}(E)$. ■

Corollary 3.2.2. *For any right noetherian ring R,* $\mathrm{InjSpec}(R)$ *is T_0, i.e., Kolmogorov.*

PROOF:

If two points, E and F, are topologically indistinguishable, then $E \rightsquigarrow F$ and $F \rightsquigarrow E$, so $\mathrm{cd}(E) \geq \mathrm{cd}(F) \geq \mathrm{cd}(E)$; hence $\mathrm{cd}(E) = \mathrm{cd}(F)$, but this yields an isomorphism $E \cong F$ when combined with the fact that $E \rightsquigarrow F$. ■

Corollary 3.2.3. *Let R be right noetherian and denote $K(R)$ by d. Let α be an ordinal and let E be a specialisation chain in* $\mathrm{InjSpec}(R)$ *of order type α^{op}; i.e., $E : \alpha \to$* $\mathrm{InjSpec}(R)$ *is an injection with $E_\gamma \rightsquigarrow E_\delta$ for all $\delta < \gamma < \alpha$. Then $\alpha \leq d + 1$.*

PROOF:

By Lemma 3.2.1, for all $\delta < \gamma < \alpha$, $\mathrm{cd}(E_\gamma) > \mathrm{cd}(E_\delta)$. It follows by transfinite induction that $\mathrm{cd}(E_\gamma) \geq \gamma$. Moreover, since every indecomposable injective E_γ contains a finitely generated module, whose Krull dimension is therefore at most d, we see that $\mathrm{cd}(E_\gamma) \leq d$. So for each $\gamma < \alpha$, $d \geq \gamma$. If α is a limit ordinal, then it follows that $d \geq \alpha$.

If, on the other hand, α is a successor ordinal, say $\alpha = \gamma + 1$, then $d \geq \mathrm{cd}(E_\gamma) \geq \gamma$, so $d + 1 \geq \alpha$. ■

If $\mathrm{InjSpec}(R)$ is sober, then specialisation chains such as this correspond to chains of irreducible closed subsets. So for rings where $\mathrm{InjSpec}(R)$ is sober, the above corollary bounds the dimension of the space (in the sense of the length of a maximal chain of irreducible closed subsets) to be at most d. See Section 5.3 and Chapter 8 for more on sobriety of $\mathrm{InjSpec}(R)$, and Chapter 4 to see that noetherian rings of Krull dimension 0 and noetherian domains of Krull dimension 1 have sober injective spectra.

Note that, since α in the above Corollary counts the number of injectives in a specialisation chain, not the number of specialisations, the bound on topological dimension really is d, not $d + 1$. For instance, if $d = 1$, then take $\alpha = 2$, the maximum value allowed by the Corollary; then a chain as in the Corollary consists of indecomposable injectives $E_1 \rightsquigarrow E_0$, which would normally be called a chain of length 1. So although $\alpha = d + 1$, this does correspond to the maximum length of a specialisation chain being d.

Given this upper bound, it is natural to try to bound the dimension of the spectrum from below by d as well. We now turn to this.

Lemma 3.2.4. *Let R be any ring and fix an ordinal α. Then:*

1. *If M is a module with $K(M) = \alpha + 1$, there is a subquotient of M with Krull dimension exactly α;*

2. *If M is a non-zero noetherian module with $K(M) = \alpha$, there is an α-critical subquotient of M.*

PROOF:

1. Since $K(M) \geq \alpha + 1$, there is an ω^{op} descending chain $M_0 > M_1 > \ldots$ of submodules of M with $K(M_i/M_{i+1}) \geq \alpha$ for all sufficiently large i. If for all but finitely many i we had $K(M_i/M_{i+1}) \geq \alpha+1$, then we would have $K(M) \geq \alpha+2$, a contradiction, so there is certainly some i such that $K(M_i/M_{i+1}) = \alpha$. Then this M_i/M_{i+1} is a subquotient of M with Krull dimension α.

2. Let M be a non-zero noetherian module of Krull dimension α. Since M is noetherian, the set of submodules of M with Krull dimension strictly less than α (which contains 0, so is non-empty) has a maximal element, M_0, say. By Proposition 3.1.3, M/M_0 contains a critical submodule; we show that any non-zero submodule of M/M_0 has Krull dimension α, and therefore this critical submodule is α-critical, as required.

   So let L be a submodule of M strictly containing M_0, so that L/M_0 is a general non-zero submodule of M/M_0. Then, by maximality of M_0, $K(L) = \alpha$; but $K(L) = \max\{K(M_0), K(L/M_0)\}$. Since $K(M_0) < \alpha$, we must have $K(L/M_0) = \alpha$, completing the proof. ■

Lemma 3.2.5. *Let R be any ring and $L \leq M \leq N$ a chain of modules, so M/L is a generic subquotient of N. Then for any injective module E, if $(M/L, E) \neq 0$, then*

$(N, E) \neq 0$. *In particular, if N is finitely presented, there is an inclusion of basic closed sets of* InjSpec(R)*: $(M/L) \subseteq (N)$.*

PROOF:

Let $\phi : M/L \to E$ be a non-zero morphism. Then ϕ lifts to a non-zero map $M \to E$ and extends (by injectivity of E) to a non-zero map $N \to E$. ■

Lemma 3.2.6. *Let R be any ring and M a critical module. Then $E(M)$ is indecomposable and* $\mathrm{cd}(E(M)) = K(M)$.

PROOF:

Critical modules are uniform by Proposition 3.1.3, so certainly $E(M)$ is indecomposable. Since $M \leq E(M)$, $\mathrm{cd}(E(M)) \leq K(M)$. Moreover, if $N \leq E(M)$ has Krull dimension strictly less than $K(M)$, then $0 \neq N \cap M \leq M$ and $K(N \cap M) \leq K(N) < K(M)$. But every non-zero submodule of M has Krull dimension $K(M)$, by criticality, giving a contradiction. So $E(M)$ has no non-zero submodule with Krull dimension strictly less than $K(M)$, so $\mathrm{cd}(E(M)) \geq K(M)$. The result follows. ■

Corollary 3.2.7. *Let R be any ring and let M and N be finitely presented R-modules such that $(M) \subseteq (N)$. Then $K(M) \leq K(N)$.*

PROOF:

Since M is noetherian, it has a $K(M)$-critical subquotient M_0, by Lemma 3.2.4. Then $E(M_0) \in (M_0) \subseteq (M) \subseteq (N)$, so $(N, E(M_0)) \neq 0$, and so by Lemma 3.1.4, $K(N) \geq \mathrm{cd}(E(M_0)) = K(M_0) = K(M)$. ■

Corollary 3.2.8. *Let R be right noetherian and suppose that $K(R) = d < \omega$. Then there are finitely presented critical modules M_i ($0 \leq i \leq d$) such that there is a proper chain of basic closed sets $(M_0) \subset (M_1) \subset \ldots \subset (M_d)$.*

PROOF:

Since $K(R_R) = d$ and R_R is noetherian, R_R has a d-critical (and noetherian) subquotient M_d, by Lemma 3.2.4. By repeated applications of the same Lemma, we obtain M_i an i-critical subquotient of M_{i+1} for each i. By Lemma 3.2.5, $(M_i) \subseteq (M_{i+1})$

for each i. Moreover, by Lemma 3.2.6, $E(M_{i+1})$ is indecomposable with $\mathrm{cd}(E(M_{i+1})) = i+1$ and so, by Lemma 3.1.4, $E(M_{i+1}) \notin (M_i)$, but of course $E(M_{i+1}) \in (M_{i+1})$; so the inclusions are strict. ■

Collecting the above results for a right noetherian ring R, if $\mathrm{InjSpec}(R)$ is sober, Corollary 3.2.3 bounds the length of chains of irreducible closed subsets from above by $d = K(R)$; if (M) is irreducible for every critical module M and $K(R) < \omega$, then Corollary 3.2.8 bounds the length of such chains from below by d. So if all of these conditions hold (*viz.* if R has finite Krull dimension, $\mathrm{InjSpec}(R)$ is sober, and for any finitely presented critical module M, (M) is irreducible), then $\mathrm{InjSpec}(R)$ has dimension exactly d.

However, we shall see in Section 6.2 an example of a finitely presented critical module M such that (M) is not irreducible. This problem notwithstanding, we do have the following results.

Lemma 3.2.9. *Let M be a finitely presented module. Then (M) can be expressed as a finite union*

$$(M) = \bigcup_{i=1}^{n} (N_i)$$

for some critical modules N_i (which can be taken to be subquotients of M).

PROOF:

Let C denote the set of all critical subquotients of M. We show first that

$$(M) = \bigcup_{N \in C} (N).$$

By Lemma 3.2.5, $(N) \subseteq (M)$ for any $N \in C$, so we have the right-to-left inclusion. For the converse, let $E \in (M)$; then $E = E(M/L)$ for some submodule L of M, and M/L has a critical submodule $N \in C$, so $E = E(N) \in (N)$.

Now we work in the Ziegler topology on $\mathrm{InjSpec}(R)$. In this topology, (M) is compact open and the above union shows that $\{(N) \mid N \in C\}$ is an open cover for (M), so it has a finite subcover, as required. ■

Corollary 3.2.10. *If (M) is an irreducible basic closed set in* $\mathrm{InjSpec}(R)$*, then there is a critical subquotient N of M such that $(M) = (N)$.*

PROOF:

By Lemma 3.2.9, each (M) can be written as a finite union of (N_i) for critical subquotients N_i of M; but (M) is irreducible, so must be equal to one of the (N_i). ■

An important direction for further work is therefore to examine when the conditions considered above hold:

Question 3. *Under what conditions on a ring R is* $\mathrm{InjSpec}(R)$ *sober?*

Question 4. *Are there conditions on a ring which guarantee that for any critical module M, (M) is irreducible?*

In Chapter 4, we shall see that these conditions are met for 0-dimensional (*i.e.*, artinian) rings and 1-dimensional noetherian domains.

3.3 Points of Maximal Dimension

A noetherian module is artinian - *i.e.* has Krull dimension 0 - if and only if it has a composition series, the factors of which are simple - *i.e.*, 0-critical - modules. This generalises to higher dimensions, though with a weakening of the uniqueness.

For a module M, a **critical composition series** is a finite sequence $M = M_n > M_{n-1} > \ldots > M_1 > M_0 = 0$ of submodules such that for each i, M_{i+1}/M_i is a critical module, and $K(M_{i+2}/M_{i+1}) \geq K(M_{i+1}/M_i)$.

Proposition 3.3.1 ([15], 6.2.19-6.2.22)**.** *Let R be any ring and M a noetherian module. Then M has a critical composition series. Moreover, any two critical composition series for M have the same length, and their composition factors can be paired so that corresponding factors have a non-zero isomorphic submodule.*

Note that, since the composition factors are critical, and hence uniform, the uniqueness condition of the Proposition guarantees that the injective hulls of the composition factors are uniquely determined, even though the composition factors themselves need not be.

In a commutative noetherian ring, there are finitely many minimal prime ideals - *i.e.*, finitely many irreducible closed sets in the Zariski spectrum of maximal dimension. We now show an analagous result for the injective spectrum.

Theorem 3.3.2. *Let R be right noetherian, with $K(R_R) = d$. Then there is at least one, and only finitely many, indecomposable injective modules of critical dimension d.*

PROOF:

First note that R_R is a noetherian module; hence, by Lemma 3.2.4, there is a d-critical subquotient of R_R. The injective hull of this therefore has critical dimension d.

Now take a critical composition series for R_R:

$$R_R = M_n > \ldots > M_0 = 0$$

and let E be an indecomposable injective of critical dimension d. First we show that for some i, $(M_{i+1}/M_i, E) \neq 0$. Certainly $(M_n, E) \neq 0$ and $(M_0, E) = 0$, so there is some index i such that $(M_{i+1}, E) \neq 0$, but $(M_i, E) = 0$. Then all maps $M_{i+1} \to E$ must vanish on M_i, hence induce maps on the quotient, so $(M_{i+1}/M_i, E) \neq 0$, as claimed.

Now let $\phi : M_{i+1}/M_i \to E$ be a non-zero map. Since $\mathrm{cd}(E) = d$, which is the maximum possible value for the Krull dimension of a finitely presented module, we must have $K(\phi(M_{i+1}/M_i)) = d$, so $K(M_{i+1}/M_i) = d$. But M_{i+1}/M_i is critical, so ϕ must be an embedding. Therefore $E = E(M_{i+1}/M_i)$. Since the critical composition series has finite length and uniquely determines the injective hulls of its composition factors, there can only be finitely many such E. Indeed, the number of points of maximal critical dimension will be precisely the number of inequivalent factors of maximal dimension in any critical composition series for R_R. ■

There is also of course the question of when the injective spectrum is irreducible and, in this case, if it has a generic point. We have the following result.

Theorem 3.3.3. *If R is a right noetherian domain, then* $\mathrm{InjSpec}(R)$ *is irreducible and has $E(R_R)$ as a generic point.*

PROOF:

By classical results of Goldie, any right noetherian domain is right Ore, and so has uniform dimension 1; *i.e.*, R_R is uniform. So certainly $E(R_R)$ is indecomposable. We show that $E(R_R)$ is contained in every non-empty open set, which proves that it is a generic point and therefore the space is irreducible. It suffices to show that $E(R_R)$ is contained in every non-empty basic open set $[M]$ for M finitely presented.

First we show that we can reduce to the case where M is cyclic. Let M be a finitely presented module such that $[M] \neq \varnothing$ and suppose for a contradiction that there is some $\phi : M \to E(R_R)$ non-zero. Then there is some $m \in M$ such that $\phi(m) \neq 0$, so $(mR, E(R_R)) \neq 0$. Moreover, $[mR] \neq \varnothing$, since if $(mR, E) \neq 0$ for all $E \in \mathrm{InjSpec}(R)$, then, by injectivity, $(M, E) \neq 0$ for all E, so $[M] = \varnothing$, a contradiction. So if there is M finitely presented with $E(R_R) \notin [M] \neq \varnothing$, then there is mR cyclic such that $E(R_R) \notin [mR] \neq \varnothing$.

So let I be a proper right ideal of R, so that $[R/I]$ is non-empty, and suppose for a contradiction that $(R/I, E(R_R)) \neq 0$. By enlarging I, we may assume without loss of generality that R/I embeds in $E(R_R)$; write f for such an embedding. Then $f(R/I)$ has non-zero intersection with R_R in $E(R_R)$; so there are some $r \in R \setminus I$ and $s \in R \setminus 0$ such that $f(r + I)$ and s coincide in $E(R_R)$.

Then $\mathrm{ann}_R(r + I) = \mathrm{ann}_R(f(r + I)) = \mathrm{ann}_R(s) = 0$, since R is a domain. So $(r + I)R \cong R_R$ is a free module of rank 1 inside R/I. We show that this cannot occur; *i.e.*, that no proper quotient of R_R contains an isomorphic copy of R_R.

Let $\phi_1 : R_R \to R/I$ be an embedding and set $J_1 := \phi_1(I)$ and I_1 to be the lift of J_1 along the quotient $R_R \to R/I$. So I_1 is a right ideal properly containing I, and $\phi_1(R)/J_1 \cong R/I$. Then R/J_1 contains $\phi_1(R)/J_1 \cong R/I$, which contains a free module of rank 1, so let $\phi_2 : R_R \to R/J_1$ be an embedding, and set $J_2 := \phi_2(I)$ and I_2 to be the lift of J_2 back to R. Since J_2 properly contains J_1, I_2 properly contains I_1. Moreover, $\phi_2(R)/J_2 \cong R/I$.

Proceeding in this way, we construct an infinite, strictly ascending chain of right ideals $I < I_1 < I_2 < \ldots$, a contradiction. So we conclude that for any proper right ideal I, $(R/I, E(R_R)) = 0$; *i.e.*, $E(R_R) \in [R/I]$. ■

Corollary 3.3.4. *Let R be a right noetherian domain. Then R_R is a critical module.*

PROOF:

Suppose for a contradiction that $0 < I < R_R$ is such that $K(R/I) = K(R)$. Then, by Lemma 3.2.4, R/I has a $K(R)$-critical subquotient, say J/K for some right ideals J, K, with $J > K \geq I > 0$. Then $E(J/K)$ is indecomposable, so $E(R_R) \rightsquigarrow E(J/K)$, but $\mathrm{cd}(E(R_R)) = K(R) = \mathrm{cd}(E(J/K))$, so $E(R_R) = E(J/K)$, by Lemma 3.2.1.

So $(J/K, E(R_R)) \neq 0$, and hence $(R/K, E(R_R)) \neq 0$. Therefore $[R/K]$ does not contain $E(R_R)$; by the Theorem, this forces $K = R_R$, but this is a contradiction, since $J > K$. ∎

Chapter 4

First Examples

In this chapter, we develop some examples of the theory at its nicest, where the topology is well-behaved, Krull dimension corresponds nicely to topological dimension, and a more-or-less complete picture of the injective spectrum can be obtained.

This chapter has been greatly simplified by suggestions from Mike Prest, for which I am very grateful.

4.1 Spectra of Right Artinian Rings

This short section addresses the simplest possible case; a ring R is right artinian if and only if $K(R) = 0$; so we consider 0-dimensional rings, which should, of course, be expected to have 0-dimensional spectra. This is indeed the case.

Proposition 4.1.1. *Let R be right artinian. Then $\mathrm{InjSpec}(R)$ is a finite discrete space.*

PROOF:

Let E be an indecomposable injective module. Then there is a finitely presented module M such that $E = E(M)$. Then M is artinian, so has a simple submodule S, so $E = E(S)$. Any simple module is annihilated by the Jacobson radical $J(R)$, and $R/J(R)$ is a finite direct sum of matrix rings over division rings, by Artin-Wedderburn, hence has finitely many simple modules. So there are only finitely many points in $\mathrm{InjSpec}(R)$.

Moreover, the injective hull of a simple module is a closed point, by Lemma 2.2.1, and a finite space where every point is closed must be discrete. ■

4.2 Spectra of 1-Critical Rings

Having dealt with 0-dimensional rings, in this section we consider rings R such that R_R is a 1-critical module. Recall Corollary 3.3.4, which says that if R is a right noetherian domain, then R_R is critical. So any 1-dimensional, right noetherian domain is covered by the results of this section. We also have the following result, giving further examples.

Lemma 4.2.1 ([15], 6.2.8). *Let R be a hereditary noetherian prime ring, then R is either artinian or is 1-critical.*

Theorem 4.2.2. *Let R be a right noetherian ring such that R_R is 1-critical. Then the indecomposable injective R-modules are $E(R_R)$ and the injective hulls of the simple modules. The open sets in* $\mathrm{InjSpec}(R)$ *are precisely the cofinite sets including $E(R_R)$, and of course the empty set. Therefore, $E(R_R)$ is a generic point in* $\mathrm{InjSpec}(R)$.

Proof:

Since R_R is 1-critical, $E(R_R)$ is indecomposable. All the other indecomposable injectives are $E(R/I)$ for some non-zero right ideal I; but each such R/I is artinian, since R_R is 1-critical, so each indecomposable injective except $E(R_R)$ is the hull of a simple module.

For the topology, we first show that the basic open sets $[M]$ for M finitely presented are all either cofinite and include $E(R_R)$, or are empty. We proceed by induction on the number of generators in a generating set for M. Recall Lemma 1.4.4, which says that if

$$0 \to A \to B \to C \to 0$$

is a short exact sequence, then $[B] = [A] \cup [C]$.

The base case is when $M \cong R/I$ is cyclic. If $I = 0$, $R/I = R$, which maps to everything, so $[R/I]$ is the empty set. If $I \neq 0$, then R/I is artinian, since R_R is 1-critical, so R/I has a composition series, and the indecomposable injectives receiving a map from R/I, are precisely the injective hulls of the composition factors of R/I, by repeated applications of Lemma 1.4.4. So $[R/I]$ is cofinite. Moreover, since $K(R/I) < 1 = K(R) = \mathrm{cd}(E(R_R))$, we must have $(R/I, E(R_R)) = 0$, so $E(R_R) \in [R/I]$. This completes the base case.

Now suppose that the result is proved for n-generated modules and let M be generated by $m_1, \ldots, m_{n+1}$. Define

$$N := \sum_{i=1}^{n} m_i R,$$

so that we have a short exact sequence

$$0 \to N \to M \to M/N \to 0$$

and M/N is cyclic. Then the inductive hypothesis covers both N and M/N, and Lemma 1.4.4 again gives us that $[M] = [N] \cup [M/N]$ is cofinite and includes $E(R_R)$ or is empty, completing the induction.

Each open set is a union of basic open sets, hence also is cofinite and includes $E(R_R)$ (or is empty). So all open sets fit the description given.

Finally, let $U \subseteq \mathrm{InjSpec}(R)$ be a cofinite set including $E(R_R)$; we show that U is open, by showing that $C := \mathrm{InjSpec}(R) \setminus U$ is closed. Since C is a finite set excluding $E(R_R)$, it contains only the injective hulls of some finitely many simple modules. By Lemma 2.2.1, the hull of a simple is a closed point; since a finite union of closed sets is closed, we see that C is indeed a closed set. ∎

In particular, let k be a field of characteristic 0, and let $A_1(k)$ be the first Weyl algebra over k; that is:

$$A_1(k) := k\langle x, \partial \mid \partial x - x\partial = 1 \rangle.$$

Then $A_1(k)$ is a noetherian domain of Krull dimension 1 (see *e.g.*, [15, 6.6.8]), so its injective spectrum fits the above description.

Note also that this description of the topology, having a collection of closed points and one generic point, is that of the affine line over any field. So the injective spectrum of a 1-critical ring is homeomorphic to an affine line as topological spaces, though there may not be any canonical way to identify it, even when there is a "natural" field to consider, and they will not generally by homeomorphic as ringed spaces. For instance, $\mathrm{InjSpec}(A_1(k)) \cong \mathrm{Spec}(k[x])$, but simply for the reason that they have the same cardinality and the same, very simple topology - we also have $\mathrm{InjSpec}(A_1(k)) \cong \mathrm{Spec}(k(t)[x])$, for the same reason. It will sometimes (see Section 6.1) be convenient to picture the injective spectrum of a 1-critical ring as a line for the purposes of visualising the injective spectra of more complicated rings, but it should always be borne in mind that this is a fiction for ease of visualisation.

Observe that in both the 0-dimensional (artinian) and 1-critical cases, the injective spectrum is a noetherian topological space. We will see in Section 6.2 that this can fail by dimension 2.

4.3 Piecing Together Spectra with Localisations

When I initially computed the injective spectrum of the first Weyl algebra, it was by a very different approach to the above. I first used results from the model theory of modules to prove that a principal left and right ideal domain has the topology described in Theorem 4.2.2, then used Ore localisations to transfer this information to the Weyl algebra. Although the proof above is both more straightforward and more general, the use of localisations in this fashion might be of use for further examples, and so is worth illustrating.

The key to the method is Corollary 2.3.4. Recall that this states that if $f : R \to S$ is a ring epimorphism such that ${}_RS$ is flat, then f induces a continuous injection $\mathrm{InjSpec}(S) \to \mathrm{InjSpec}(R)$, which is moreover a topological embedding if every finitely presented S-module can be written as $M \otimes_R S$ for some finitely presented R-module. Note than any Ore localisation satisfies the hypotheses of the Corollary, and so induces an embedding on injective spectra:

Lemma 4.3.1. *Let R be a right noetherian ring and D a right Ore set in R. Then there is an embedding of topological spaces* $\mathrm{InjSpec}(RD^{-1}) \to \mathrm{InjSpec}(R)$.

In the Weyl algebra $A_1(k)$, both $k[x] \setminus 0$ and $k[\partial] \setminus 0$ are right Ore sets. We denote the localisations at these sets by A_x and A_∂ respectively. Then A_x and A_∂ are readily shown to be principal left and right ideal domains (in fact, they are Euclidean domains, and even isomorphic to each other). Moreover, it follows from Block's classification of the simple $A_1(k)$-modules [1] that every indecomposable injective module is the injective hull of either an A_x-module or an A_∂-module.

Therefore $\mathrm{InjSpec}(A_1(k))$ is the union of $\mathrm{InjSpec}(A_x)$ with $\mathrm{InjSpec}(A_\partial)$. Having described the spectrum of a principal ideal domain, the description of $\mathrm{InjSpec}(A_1(k))$ then follows straightforwardly.

This suggests a general strategy for obtaining at least partial information about the injective spectrum of a ring: find Ore sets and localise to produce a simpler ring, whose spectrum can be better understood. Then glue together the parts of the spectrum coming from different localisations. We shall see a version of this strategy in play in Section 6.1, where we will obtain one part of the spectrum of the Heisenberg algebra from an Ore localisation, and the rest from a different approach.

Chapter 5

The Torsion Spectrum

Golan [8] discusses a number of topologies on the lattice of hereditary torsion theories in the module category over a noncommutative ring R and a particular subset thereof, consisting of the prime torsion theories. This allows the definition of the 'torsion spectrum' of a ring, which turns out, for R noetherian, to be homeomorphic to the injective spectrum.

Note that, since we consider only noetherian rings, all torsion theories are of finite type, hence determined by the intersection of the torsion class with the (skeletally small) category mod-R; as such, the class of all torsion theories is indeed a set.

5.1 Golan's Torsion Spectrum

We begin by explaining the ideas of Golan [8]; the notation and terminology is significantly changed from that paper to fit in better with the other concepts in this thesis.

A module M is called **torsion-critical** if every proper quotient of M is $\mathcal{F}(M)$-torsion.

Lemma 5.1.1. *Let M be a torsion-critical module. Then*

1. *M is uniform;*
2. *Any non-zero submodule of M is torsion-critical;*
3. *For any non-zero submodule N of M, $\mathcal{F}(N) = \mathcal{F}(M)$.*

PROOF:

1. Suppose for a contradiction that L and N are non-zero submodules of M with $L \cap N = 0$. Then M embeds in $M/L \oplus M/N$; but M/L and M/N are both $\mathcal{F}(M)$-torsion, hence so is M, a contradiction.

2. Let $N \leq M$ be non-zero. Note that, since $N \in \mathcal{F}(M)$, $\mathcal{F}(N) \subseteq \mathcal{F}(M)$. Suppose for a contradiction that N has a proper quotient L which is not $\mathcal{F}(N)$-torsion. Then
$$K := \frac{L}{\tau_{\mathcal{F}(N)}(L)}$$
is a proper, non-zero quotient of N which is $\mathcal{F}(N)$-torsionfree and hence $\mathcal{F}(M)$-torsionfree. But then K has the form I/J for some $J < I \leq M$, so M/J has a non-zero, $\mathcal{F}(M)$-torsionfree submodule, so is not $\mathcal{F}(M)$-torsion, a contradiction.

3. We prove the more general result that if L is an essential submodule of an arbitrary module N, then $\mathcal{F}(L) = \mathcal{F}(N)$; by part (1.), this suffices. Since L is essential in N, $E(L) = E(N)$, and the result then follows by Lemma 1.3.4, as this gives $\mathcal{F}(L) = \mathcal{F}(E(L)) = \mathcal{F}(E(N)) = \mathcal{F}(N)$. ■

The torsion theories of the form $\mathcal{F}(M)$ for M torsion-critical are called **prime torsion theories**. The set of all such is called the (right) **torsion spectrum** of R and denoted $\mathrm{TorSpec}(R)$.

Golan's definition actually only considers those torsion theories of the form $\mathcal{F}(M)$ for M torsion-critical and cyclic. However, by Lemma 5.1.1, this is equivalent to our definition. For any non-zero, cyclic submodule of a torsion-critical module M is also torsion-critical and cogenerates the same torsionfree class.

The torsion spectrum is endowed with a topology as follows. For $\mathcal{T}$ any torsion class, let $[\mathcal{T}]$ denote the set of prime torsion theories for which $\mathcal{T}$ is contained in the torsion class - *i.e.*, the intersection of $\mathrm{TorSpec}(R)$ with the principal filter generated by $\mathcal{T}$ in the lattice of torsion classes. The set of all $[\mathcal{T}(M)]$ where M ranges over finitely presented R-modules is a basis of open sets for a topology on $\mathrm{TorSpec}(R)$, called the finitary order topology. Henceforth, by $\mathrm{TorSpec}(R)$ we shall mean the set endowed with this particular topology. We denote by $(\mathcal{T})$ the complement of $[\mathcal{T}]$ in $\mathrm{TorSpec}(R)$.

5.2 Torsion Theories and the Injective Spectrum

We now relate Golan's torsion spectrum to the injective spectrum.

Theorem 5.2.1. *Let R be a right noetherian ring. Then there is a homeomorphism* $\mathrm{TorSpec}(R) \cong \mathrm{InjSpec}(R)$.

Proof:

First we establish a bijection. By Lemma 5.1.1, for M torsion-critical, M is uniform, hence $E(M)$ is indecomposable, so we define a map $h : \mathrm{TorSpec}(R) \to \mathrm{InjSpec}(R)$ by $h : \mathcal{F}(M) \mapsto E(M)$. To show that this map is well-defined and injective, we show that for two torsion-critical modules M and N, $E(M) \cong E(N)$ if and only if $\mathcal{F}(M) = \mathcal{F}(N)$.

For $E, F \in \mathrm{InjSpec}(R)$, $E \rightsquigarrow F$ if and only if $\mathcal{F}(E) \subseteq \mathcal{F}(F)$, by Lemma 2.1.1. Since $\mathcal{F}(M) = \mathcal{F}(E(M))$ (and similarly for N), $\mathcal{F}(M) = \mathcal{F}(N)$ if and only if each of $E(M)$ and $E(N)$ specialises to the other in $\mathrm{InjSpec}(R)$. Since $\mathrm{InjSpec}(R)$ is T_0 by Corollary 3.2.2, this occurs if and only if $E(M) \cong E(N)$, as required.

Next we show that h is surjective. For this it suffices to observe the following: if M is critical (in the sense of Krull dimension), then M is torsion-critical. For if $L \leq M$ is non-zero, then $K(M/L) < K(M)$, but $\mathrm{cd}(E(M)) = K(M)$, so $(M/L, E(M)) = 0$, and hence $M/L \in \mathcal{T}_{\mathcal{F}(M)}$. So every point of $\mathrm{InjSpec}(R)$ is indeed the injective hull of a torsion-critical module, and so h is onto.

Now we show that h is a homeomorphism. The sets of the form $[M]$ for M finitely presented form a basis of open sets for the topology on $\mathrm{InjSpec}(R)$, while those of the form $[\mathcal{T}(M)]$ form a basis for the topology on $\mathrm{TorSpec}(R)$. We show that $h([\mathcal{T}(M)]) = [M]$; since h is a bijection, this suffices.

First note that if $E(N) \in [M]$, where N is torsion-critical, then $(M, E(N)) = 0$, so $M \in \mathcal{T}_{\mathcal{F}(N)}$. Therefore $\mathcal{T}(M) \subseteq \mathcal{T}_{\mathcal{F}(N)}$, so $\mathcal{T}_{\mathcal{F}(N)}$ is contained in the filter of torsion theories generated by $\mathcal{T}(M)$, *viz.* $[\mathcal{T}(M)]$. So we see that, for $\mathcal{T} \in \mathrm{TorSpec}(R)$, if $h(\mathcal{T}) \in [M]$, then $\mathcal{T} \in [\mathcal{T}(M)]$; so $[M] \subseteq h([\mathcal{T}(M)])$.

For the reverse inclusion, suppose that $\mathcal{F}(N) \in [\mathcal{T}(M)]$ for some N torsion-critical. Then $\mathcal{T}(M) \subseteq \mathcal{T}_{\mathcal{F}(N)}$; in particular, $M \in \mathcal{T}_{\mathcal{F}(N)}$, so for any L in $\mathcal{F}(N)$, $(M, L) = 0$. But $E(N) \in \mathcal{F}(N)$, so taking $L = E(N)$ shows that $E(N) \in [M]$. This completes the proof. ■

Golan's definition of what he calls the finitary order topology used the sets $[\mathcal{T}(M)]$ for M cyclic as a basis, rather than for M finitely presented. However, since the sets $[M]$ for M cyclic form a basis for the topology on the injective spectrum, this shows that the $[\mathcal{T}(M)]$ for M cyclic form a basis for the topology on the torsion spectrum (essentially by repeated applications of Lemma 1.4.4), so our definition is equivalent to Golan's. The approach adopted here, however, has the advantage that it depends only on the module category, not on the ring itself; as such, the results apply to any suitable category. Indeed, the above Theorem, with the same proof, applies to any locally noetherian Grothendieck category.

We showed in Lemma 2.1.1 that, for $E, F \in \mathrm{InjSpec}(R)$, $E \rightsquigarrow F$ if and only if $\mathcal{F}(E) \subseteq \mathcal{F}(F)$. This is particularly natural in light of the above Theorem, for it amounts to the easily verified fact that the specialisation order on the torsion spectrum is precisely the inclusion order.

5.3 Irreducibility and Sobriety

In this Section, we explore questions of irreducible closed sets and sobriety of the injective/torsion spectrum. Now that we are working with torsion theories, it makes sense to move to the greater generality of locally noetherian Grothendieck categories. We begin with a number of well-known technical Lemmas.

Lemma 5.3.1. *Let $\{\mathcal{S}_i \mid i \in I\}$ be a collection of Serre subcategories of an abelian category. Then their join, the Serre subcategory*

$$\sum_{i \in I} \mathcal{S}_i,$$

consists precisely of those objects admitting a finite filtration whose factors each lie in some $\mathcal{S}_i$.

PROOF:

Denote by $\mathcal{S}$ the class of all objects admitting a finite filtration each of whose factors lies in some $\mathcal{S}_i$. Clearly each $\mathcal{S}_i$ is contained in $\mathcal{S}$, and $\mathcal{S}$ is contained in any Serre class containing all $\mathcal{S}_i$. So it suffices to prove that $\mathcal{S}$ is itself a Serre class.

Let $0 \to A \to B \to C \to 0$ be a short exact sequence. Suppose that we have filtrations $A = A_n > \ldots > A_1 > A_0 = 0$ and $C = C_m > \ldots > C_1 > C_0 = 0$ with each successive factor in some $\mathcal{S}_i$. Let B_i denote the full preimage of C_i under the projection $B \to C$ for each $i \in \{0, \ldots, m\}$. Then $B = B_m > \ldots > B_0 = A = A_n > \ldots > A_0 = 0$ is a finite filtration of B. Each factor is either A_{i+1}/A_i, and hence in some $\mathcal{S}_j$, or is

$$\frac{B_{i+1}}{B_i} \cong \frac{B_{i+1}/A}{B_i/A} \cong \frac{C_{i+1}}{C_i},$$

and so again lies in one of the $\mathcal{S}_j$. So $B \in \mathcal{S}$.

On the other hand, suppose that $B \in \mathcal{S}$, witnessed by the filtration $B = B_n > \ldots > B_0 = 0$. Then A has a filtration $A = A \cap B_n \geq \ldots \geq A \cap B_0 = 0$ (where some terms may be repeated, if $A \cap B_i = A \cap B_{i+1}$). Then we have

$$\begin{aligned} \frac{A \cap B_{i+1}}{A \cap B_i} &= \frac{A \cap B_{i+1}}{(A \cap B_{i+1}) \cap B_i} \\ &\cong \frac{(A \cap B_{i+1}) + B_i}{B_i} \\ &\leq \frac{B_{i+1}}{B_i}. \end{aligned}$$

Since this last term is in some $\mathcal{S}_j$, and $\mathcal{S}_j$ is closed under subobjects, we see that $A \in \mathcal{S}$.

Similarly, setting C_i to be the image of $B_i + A$ under the projection $B \to C$, we have $C = C_n \geq \ldots \geq C_0 = 0$ and

$$\begin{aligned} \frac{C_{i+1}}{C_i} &\cong \frac{(B_{i+1} + A)/A}{(B_i + A)/A} \\ &\cong \frac{B_{i+1} + A}{B_i + A} \\ &= \frac{B_{i+1} + (B_i + A)}{B_i + A} \\ &\cong \frac{B_{i+1}}{B_{i+1} \cap (B_i + A)} \\ &= \frac{B_{i+1}}{B_i + (A \cap B_{i+1})} \end{aligned}$$

where we have used the modular law to obtain the last line, which is a quotient of B_{i+1}/B_i, which lies in some $\mathcal{S}_j$. So we see that $C \in \mathcal{S}$.

So $\mathcal{S}$ is closed under extensions, subobjects, and quotients, and so is a Serre subcategory. ■

Lemma 5.3.2 ([18], 11.1.14, 11.1.15)**.** *Let $\mathcal{A}$ be a locally noetherian Grothendieck category. Then every torsion theory in $\mathcal{A}$ is of finite type and every torsion-theoretic quotient of $\mathcal{A}$ is also locally noetherian.*

Lemma 5.3.3. *Let $\mathcal{A}$ be a locally noetherian Grothendieck category. Then there is an inclusion-preserving bijection between Ziegler-closed subsets of the injective spectrum of $\mathcal{A}$ and hereditary torsionfree classes in $\mathcal{A}$.*

PROOF:

Given $C \subseteq \mathrm{InjSpec}(\mathcal{A})$ Ziegler-closed, we associate the torsionfree class $\mathcal{F}(C)$. Given $\mathcal{F}$ a torsionfree class, we define $\mathcal{C}(\mathcal{F}) = \mathrm{InjSpec}(\mathcal{A}) \cap \mathcal{F}$, the set of indecomposable injectives in $\mathcal{F}$.

First we take $C \subseteq \mathrm{InjSpec}(\mathcal{A})$ Ziegler-closed and show that $\mathcal{C}(\mathcal{F}(C)) = C$. It is clear that $C \subseteq \mathcal{C}(\mathcal{F}(C))$, so we prove the reverse inclusion. We have $C = \bigcap_{i \in I}[A_i]$ for some collection of finitely presented objects A_i, with I some indexing set. If $E \in \mathcal{C}(\mathcal{F}(C))$, then $E \in \mathcal{F}(C)$, so $(T, E) = 0$ for all $T \in \mathcal{T}_{\mathcal{F}(C)}$. Now, each A_i has $(A_i, F) = 0$ for all $F \in C$, so each $A_i \in \mathcal{T}_{\mathcal{F}(C)}$, so if $E \in \mathcal{C}(\mathcal{F}(C))$, then $(A_i, E) = 0$ for all i and so $E \in C$. Therefore $\mathcal{C}(\mathcal{F}(C)) = C$, as required.

Now let $\mathcal{F}$ be a torsionfree class. We show first that $\mathcal{C}(\mathcal{F})$ is a Ziegler-closed set. Let $E \in \mathrm{InjSpec}(\mathcal{A}) \setminus \mathcal{C}(\mathcal{F})$. Then E is not torsionfree for $\mathcal{F}$, so there is an $\mathcal{F}$-torsion submodule M of E (which can be taken to be finitely presented, without loss of generality, as any non-zero submodule of $\tau_{\mathcal{F}}(E)$ will suffice for our argument). Then $(M, \mathcal{F}) = 0$, so for all $F \in \mathcal{C}(\mathcal{F})$, $(M, F) = 0$, so $\mathcal{C}(\mathcal{F}) \subseteq [M]$; but $(M, E) \neq 0$, so $E \in (M) \subseteq \mathrm{InjSpec}(\mathcal{A}) \setminus \mathcal{C}(\mathcal{F})$, showing that $\mathrm{InjSpec}(\mathcal{A}) \setminus \mathcal{C}(\mathcal{F})$ is Ziegler-open.

Now we show that $\mathcal{F} = \mathcal{F}(\mathcal{C}(\mathcal{F}))$. Since $\mathcal{C}(\mathcal{F}) \subseteq \mathcal{F}$, certainly $\mathcal{F}(\mathcal{C}(\mathcal{F})) \subseteq \mathcal{F}$. Conversely, suppose $M \in \mathcal{F}$. Then $E(M) \in \mathcal{F}$, but all injectives are direct sums of indecomposable injectives, and $\mathcal{F}$ is closed under submodules, so $E(M)$ is a direct sum of elements of $\mathcal{C}(\mathcal{F})$. Hence $E(M) \in \mathcal{F}(\mathcal{C}(\mathcal{F}))$, and hence so too is M.

Finally, it is clear that this preserves the inclusion ordering. ■

Recall that a torsionfree class is prime if and only if it is cogenerated by a single indecomposable injective object (see proof of Theorem 5.2.1).

Corollary 5.3.4. *Every torsionfree class in a locally noetherian Grothendieck category $\mathcal{A}$ is a sum of prime torsionfree classes:*

$$\mathcal{F} = \sum \{\mathcal{F}(E) \mid E \in \mathcal{F} \cap \mathrm{InjSpec}(\mathcal{A})\}.$$

Lemma 5.3.5 ([18], 11.1.10)**.** *Let $\mathcal{A}$ be a locally noetherian Grothendieck category and $\mathcal{T}$ a torsion class in $\mathcal{A}$. Then there is an inclusion preserving bijection between torsion classes in $\mathcal{A}$ which contain $\mathcal{T}$ and torsion classes in $\mathcal{A}/\mathcal{T}$.*

The proof of this is not difficult, but requires a few extra technicalities, so we omit it.

Now we are almost ready to give an alternative characterisation of prime torsion theories. We just require one more notion; let us say that a torsion class is **simple** if it properly contains no other torsion class except 0.

Lemma 5.3.6. *Let $\mathcal{A}$ be a locally noetherian Grothendieck category. Then for any simple object $S \in \mathcal{A}$, $\mathcal{T}(S)$ is a simple torsion class. Given two simple objects S_1, S_2, $\mathcal{T}(S_1) = \mathcal{T}(S_2)$ if and only if $S_1 \cong S_2$.*

Proof:

The class $\mathcal{F}_{\mathcal{T}(S)}$ consists of those objects F such that $(S, E(F)) = 0$, by Lemma 1.3.4; but S is simple, so if $(S, E(F)) \neq 0$, then S embeds in $E(F)$. Since F is essential in $E(F)$, the image of S in $E(F)$ has non-zero intersection with F, so is contained in F, by simplicity again. So $(S, E(F)) \neq 0$ if and only if $(S, F) \neq 0$. So $\mathcal{F}_{\mathcal{T}(S)}$ consists of those F such that $(S, F) = 0$.

Therefore $\mathcal{T}(S)$ consists of those objects T such that $(T, F) = 0$, whenever $(S, F) = 0$. Since S is simple, $(S, F) = 0$ precisely when F does not contain S as a subobject. So if any quotient of T fails to contain S as a submodule, that quotient is torsionfree but receives a map from T, a contradiction. So $\mathcal{T}(S)$ consists of objects whose every non-zero quotient has S as a submodule.

But then any torsion class containing any non-zero object of $\mathcal{T}(S)$, being closed under subquotients, must contain S, and so contains all of $\mathcal{T}(S)$. So $\mathcal{T}(S)$ is a simple torsion class.

Now suppose that S_1 and S_2 are simple objects with $\mathcal{T}(S_1) = \mathcal{T}(S_2)$. Then $S_1 \in \mathcal{T}(S_2)$, so S_1 has S_2 as a subobject, by the above. But S_1 is simple, so $S_1 \cong S_2$, as claimed. ■

Theorem 5.3.7. *Let $\mathcal{A}$ be a locally noetherian Grothendieck category and $(\mathcal{T}, \mathcal{F})$ a torsion theory. Then the following are equivalent:*

1. *$(\mathcal{T}, \mathcal{F})$ is prime;*

2. *$\mathcal{F}$ is $+$-irreducible in the lattice of torsionfree classes;*

3. *$\mathcal{T}$ is $\cap$-irreducible in the lattice of torsion classes.*

PROOF:

(2. $\Leftrightarrow$ 3.): Since the lattice of torsion classes is dual to the lattice of torsionfree classes, this is obvious.

(1. $\Rightarrow$ 2.): Let $\mathcal{F}$ be a prime torsionfree class, cogenerated by the indecomposable injective E. Suppose that $\mathcal{F} = \mathcal{F}_1 + \mathcal{F}_2$ is the join of some torsionfree classes $\mathcal{F}_1$, $\mathcal{F}_2$ (necessarily contained in $\mathcal{F}$). Let $\mathcal{E}_i$ denote the set of injective objects (up to isomorphism) of $\mathcal{F}_i$. Then $\mathcal{E}_1 \cup \mathcal{E}_2$ cogenerates $\mathcal{F}$.

Since $E \in \mathcal{F}$, E embeds in a product of elements of $\mathcal{E}_1 \cup \mathcal{E}_2$. So we can take $E_i \in \mathcal{E}_i$ such that E embeds in $E_1 \times E_2$. Let K_i be the kernel of the composition $E \to E_1 \times E_2 \to E_i$. So the kernel of $E \to E_1 \times E_2$ is $K_1 \cap K_2$; but this map is an embedding, so $K_1 \cap K_2 = 0$. Since E is uniform, we therefore have $K_1 = 0$ or $K_2 = 0$. So E embeds in either E_1 or in E_2, and so $E \in \mathcal{F}_1$ or $E \in \mathcal{F}_2$. So $\mathcal{F} = \mathcal{F}_1$ or $\mathcal{F} = \mathcal{F}_2$, proving that $\mathcal{F}$ is $+$-irreducible.

(3. $\Rightarrow$ 1.): Let $\mathcal{T}$ be a $\cap$-irreducible torsion class, with associated torsionfree class $\mathcal{F}$. We show that $\mathcal{A}/\mathcal{T}$ contains a unique simple object S and that $i_{\mathcal{T}}E(S)$ is an indecomposable injective cogenerator for $\mathcal{F}$, showing that $\mathcal{F}$ is prime.

First note that, by Lemma 5.3.5, $\mathcal{A}/\mathcal{T}$ has at most one simple torsion class, since the intersection of two simple classes is necessarily 0, but 0 is $\cap$-irreducible in $\mathcal{A}/\mathcal{T}$.

Now note that, since $\mathcal{A}$ is locally noetherian, it certainly contains a noetherian object, N say. Then N has a maximal proper subobject, and hence a simple quotient. So $\mathcal{A}$ contains a simple object, S. Then by Lemma 5.3.6, $\mathcal{T}(S)$ is simple and, since two non-isomorphic simple objects must generate different torsion classes, we see that $\mathcal{A}/\mathcal{T}$ has exactly one simple object, S.

Since the coproduct of the injective hulls of the simple objects form a cogenerating set for $\mathcal{A}/\mathcal{T}$ (by the fact that every object has a simple subquotient - see [13, Theorem 19.8], though the proof there deals specifically with module categories), we see that $E(S)$ is an injective cogenerator for all of $\mathcal{A}/\mathcal{T}$.

Consider the quotient functor $Q_{\mathcal{T}} : \mathcal{A} \to \mathcal{A}/\mathcal{T}$ and its right adjoint inclusion $i_{\mathcal{T}}$. Since $i_{\mathcal{T}}$ is fully faithful, it preserves indecomposables, and since it has an exact left adjoint, it preserves injectives (see Theorem 2.3.2). So $i_{\mathcal{T}}(E(S))$ is an indecomposable injective object of $\mathcal{A}$. We show that this object cogenerates $\mathcal{F}$, and therefore that $\mathcal{F}$ is prime.

Every object F of $\mathcal{F}$ embeds in its localisation $i_{\mathcal{T}}Q_{\mathcal{T}}(F)$, by Proposition 1.3.2. Since $E(S)$ cogenerates $\mathcal{A}/\mathcal{T}$, there is some cardinal λ such that $Q_{\mathcal{T}}(F)$ embeds in $E(S)^{\lambda}$. Since $i_{\mathcal{T}}$ is a right adjoint, it is left exact and preserves products, so $i_{\mathcal{T}}Q_{\mathcal{T}}(F) \hookrightarrow i_{\mathcal{T}}(E(S))^{\lambda}$. So every object of $\mathcal{F}$ is cogenerated by the indecomposable injective $i_{\mathcal{T}}(E(S))$, and hence $\mathcal{F}$ is prime. ■

This allows us to identify points of $\mathrm{InjSpec}(\mathcal{A}) = \mathrm{TorSpec}(\mathcal{A})$ purely in the lattice of torsion classes, without any reference to the actual objects of $\mathcal{A}$. However, it does not yet let us give a description of the topology, since this is given in terms of torsion classes generated by finitely presented objects. We will shortly address this, but first, we extract a Corollary from the above Proposition.

Corollary 5.3.8. *Let $\mathcal{A}$ be a locally noetherian Grothendieck category. Then the injective spectrum* $\mathrm{InjSpec}(\mathcal{A})$ *is sober in its Ziegler topology.*

PROOF:

Let $C \subseteq \mathrm{InjSpec}(\mathcal{A})$ be an irreducible Ziegler-closed set and let $\mathcal{F} = \mathcal{F}(C)$ be the torsionfree class it cogenerates. Since the lattice of Ziegler-closed subsets of $\mathrm{InjSpec}(\mathcal{A})$ is isomorphic to the lattice of torsionfree classes, by Lemma 5.3.3, we see that $\mathcal{F}$ is +-irreducible, hence prime. So there is an indecomposable injective E which cogenerates $\mathcal{F}$.

Now if M is a finitely presented object of $\mathcal{A}$ and $(M, E) = 0$, so $E \in [M]$, then $M \in \mathcal{T}_{\mathcal{F}}$, so $(M, \mathcal{F}) = 0$, so for all $F \in C$, $(M, F) = 0$; so $C \subseteq [M]$. Therefore C is contained in the Ziegler-closure of E. For the reverse inclusion, note that $E \in \mathcal{F}(C) \cap \mathrm{InjSpec}(R)$, which is C, by Lemma 5.3.3; since C is Ziegler-closed and contains E, it contains the Ziegler-closure of E. ■

Unfortunately, we are more interested in sobriety of the injective spectrum in its Zariski topology for the purposes of this thesis. This will be addressed in Theorem

5.3.13, though not completely resolved. Sobriety in the Ziegler topology will however be useful in Section 8.1.

Theorem 5.3.7 lets us describe the points of the torsion spectrum purely in terms of the lattice of torison classes, without reference to the objects of $\mathcal{A}$. We now extend this to describe the topologies in these terms too.

Lemma 5.3.9. *Let $\mathcal{A}$ be a locally noetherian Grothendieck category. Then the Ziegler-closed sets of* $\mathrm{TorSpec}(\mathcal{A})$ *are precisely those of the form $[\mathcal{T}]$ for any torsion class $\mathcal{T}$.*

PROOF:

A basis of closed sets for the Ziegler topology is given by the $[\mathcal{T}(A)]$ for A finitely presented; so each Ziegler-closed set has the form

$$\bigcap_{i\in I}[\mathcal{T}(A_i)]$$

for some finitely presented objects A_i. For any torsion class $\mathcal{T}$, $\mathcal{T}$ contains all $\mathcal{T}(A_i)$ if and only if $\mathcal{T}$ contains $\sum_{i\in I}\mathcal{T}(A_i)$; *i.e.*, we have

$$\bigcap_{i\in I}[\mathcal{T}(A_i)] = \left[\sum_{i\in I}\mathcal{T}(A_i)\right].$$

So all Ziegler-closed sets have the desired form. On the other hand, in a locally noetherian category all torsion theories are of finite type, by Lemma 5.3.2, and hence determined by their finitely presented objects, by Lemma 1.3.3, so any torsion class $\mathcal{T}$ can be written as the sum of the torsion classes generated by the finitely presented objects of $\mathcal{T}$, and so

$$[\mathcal{T}] = \left[\sum_{A\in\mathcal{T}\cap\mathcal{A}^{\mathrm{fp}}}\mathcal{T}(A)\right] = \bigcap_{A\in\mathcal{T}\cap\mathcal{A}^{\mathrm{fp}}}[\mathcal{T}(A)],$$

showing that every set of this form is Ziegler-closed. ■

Proposition 5.3.10. *Let $\mathcal{A}$ be a locally noetherian Grothendieck category. Then the torsion classes $\mathcal{T}(M)$ for $M \in \mathcal{A}^{\mathrm{fp}}$ are precisely those which are compact elements of the lattice of torsion classes; i.e., those $\mathcal{T}$ such that if $\mathcal{T}$ is contained in a sum of a set of torsion classes, it is already contained in the sum of some finite subset.*

PROOF:

First suppose that $\mathcal{T}$ is compact in the lattice. Since $\mathcal{A}$ is assumed to be locally noetherian, each torsion class $\mathcal{T}$ is determined by the finitely presented objects it contains, by Lemmas 1.3.3 and 5.3.2. So

$$\mathcal{T} = \sum_{A \in \mathcal{T}^{\mathrm{fp}}} \mathcal{T}(A).$$

Since $\mathcal{T}$ is compact, there are some finitely presented objects $A_1, \ldots, A_n \in \mathcal{T}$ such that

$$\mathcal{T} = \sum_{i=1}^{n} \mathcal{T}(A_i) = \mathcal{T}\left(\bigoplus_{i=1}^{n} A_i\right),$$

so $\mathcal{T}$ is generated by a single finitely presented object.

Conversely, suppose M is finitely presented and the torsion classes $\mathcal{T}_i$ for i in some indexing set I are such that

$$\mathcal{T}(M) \subseteq \sum_{i \in I} \mathcal{T}_i.$$

Intersecting with finitely presented objects, we have an inclusion of Serre subcategories of $\mathcal{A}^{\mathrm{fp}}$

$$\mathcal{T}(M) \cap \mathcal{A}^{\mathrm{fp}} \subseteq \sum_{i \in I} (\mathcal{T}_i \cap \mathcal{A}^{\mathrm{fp}}).$$

In particular, M is contained in the right-hand side. By Lemma 5.3.1, therefore, M admits a finite filtration, each of whose factors lies in some $\mathcal{T}_i \cap \mathcal{A}^{\mathrm{fp}}$; so there are finitely many $\mathcal{T}_i$ whose sum already contains M and hence $\mathcal{T}(M)$. So $\mathcal{T}(M)$ is compact. ∎

So we now have a description of the torsion spectrum of a locally noetherian category purely in terms of the lattice of torsion classes. The points are the $\cap$-irreducible elements of the lattice, while a basis of open sets for the Zariski topology is the set of $[\mathcal{T}]$ where $\mathcal{T}$ ranges over compact elements of the lattice. The Ziegler topology has closed sets precisely the $[\mathcal{T}]$ for any $\mathcal{T}$.

We now turn to the sobriety of $\mathrm{TorSpec}(\mathcal{A}) \cong \mathrm{InjSpec}(\mathcal{A})$ in the Zariski topology. We first require a topological preliminary. Let X be a topological space with a basis of closed sets B. Say that a closed set $C \subseteq X$ is **basic-irreducible** (with respect to B) if whenever $A_1, A_2 \in B$ and $C \subseteq A_1 \cup A_2$, we have $C \subseteq A_1$ or $C \subseteq A_2$.

Lemma 5.3.11. *Let X be a topological space with a basis of closed sets B. Then a closed set $C \subseteq X$ is basic-irreducible with respect to B if and only if it is irreducible.*

PROOF:

Certainly if C is irreducible, it is *a fortiori* basic-irreducible. Conversely, suppose C is basic-irreducible and that $C \subseteq A \cup D$ for some closed sets A, D. Writing A and D in terms of B, we have

$$C \subseteq \left(\bigcap_i A_i \right) \cup \left(\bigcap_j D_j \right) = \bigcap_{i,j} (A_i \cup D_j)$$

for some $A_i, D_j \in B$. So for each i, j, $C \subseteq A_i \cup D_j$; but C is basic-irreducible, so for each i, j, $C \subseteq A_i$ or $C \subseteq D_j$.

Now suppose that $C \nsubseteq A$; then for some i_0, $C \nsubseteq A_{i_0}$. Then for all j, $C \subseteq A_{i_0} \cup D_j$, so $C \subseteq D_j$; so $C \subseteq D$. ■

Applying this to the torsion spectrum gives the following, where we write $\mathbb{C}$ for the set of compact torsion classes of $\mathcal{A}$, $\mathbb{P}$ for the set of all prime torsion classes (the set underlying $\mathrm{TorSpec}(\mathcal{A})$), and $\mathbb{T}$ for the set of all torsion classes. For the purposes of the statement and proof of this result, we adopt the following convention: a subscript i is taken to range over the set $\{1, 2\}$, and any statement about an object subscripted with i is taken to hold for all i. For instance, the statement "$\mathcal{S}_i \in \mathbb{C}$" is to be read as "$\mathcal{S}_1, \mathcal{S}_2 \in \mathbb{C}$", and "$\mathcal{S}_i \subseteq \mathcal{P}_i$" is to be read as "$\mathcal{S}_1 \subseteq \mathcal{P}_1 \wedge \mathcal{S}_2 \subseteq \mathcal{P}_2$".

Lemma 5.3.12. *Let $\mathcal{T} \in \mathbb{C}$ be a compact torsion class in the locally noetherian Grothendieck category $\mathcal{A}$. Then the following are equivalent:*

1. *The basic closed set $(\mathcal{T})$ is irreducible.*

2. *Whenever $\mathcal{S}_i \in \mathbb{C}$ and $\mathcal{P}_i \in \mathbb{P}$ are such that $\mathcal{S}_i \subseteq \mathcal{P}_i$ and $\mathcal{T} \nsubseteq \mathcal{P}_i$, then there is a single $\mathcal{P} \in \mathbb{P}$ with $\mathcal{S}_i \subseteq \mathcal{P}$ and $\mathcal{T} \nsubseteq \mathcal{P}$.*

3. *Whenever $\mathcal{S}_i \in \mathbb{C}$ and $\mathcal{R}_i \in \mathbb{T}$ are such that $\mathcal{S}_i \subseteq \mathcal{R}_i$ and $\mathcal{T} \nsubseteq \mathcal{R}_i$, then there is a single prime torsion class $\mathcal{P} \in \mathbb{P}$ with $\mathcal{S}_i \subseteq \mathcal{P}$ and $\mathcal{T} \nsubseteq \mathcal{P}$.*

Moreover, these imply that whenever $\mathcal{S}_i \in \mathbb{T}$ are such that $\mathcal{T} \subseteq \mathcal{S}_1 + \mathcal{S}_2$, then $\mathcal{T} \subseteq \mathcal{S}_1$ or $\mathcal{T} \subseteq \mathcal{S}_2$; i.e., $\mathcal{T}$ is $+$-irreducible among all torsion classes.

PROOF:

(1. $\Leftrightarrow$ 2.): We have

$$
\begin{aligned}
(\mathcal{T}) \text{ irreducible} \Leftrightarrow (\mathcal{T}) \text{ basic-irreducible} & \\
&\Leftrightarrow \forall \mathcal{S}_i \in \mathbb{C}\big((\mathcal{T}) \subseteq (\mathcal{S}_1) \cup (\mathcal{S}_2) \to (\mathcal{T}) \subseteq (\mathcal{S}_1) \vee (\mathcal{T}) \subseteq (\mathcal{S}_2)\big) \\
&\Leftrightarrow \forall \mathcal{S}_i \in \mathbb{C}\big([\mathcal{S}_1] \cap [\mathcal{S}_2] \subseteq [\mathcal{T}] \to [\mathcal{S}_1] \subseteq [\mathcal{T}] \vee [\mathcal{S}_2] \subseteq [\mathcal{T}]\big) \\
&\Leftrightarrow \forall \mathcal{S}_i \in \mathbb{C}\big([\mathcal{S}_i] \nsubseteq [\mathcal{T}] \to [\mathcal{S}_1] \cap [\mathcal{S}_2] \nsubseteq [\mathcal{T}]\big) \\
&\Leftrightarrow \forall \mathcal{S}_i \in \mathbb{C}\big(\exists \mathcal{P}_i \in \mathbb{P}(\mathcal{T} \nsubseteq \mathcal{P}_i \wedge \mathcal{S}_i \subseteq \mathcal{P}_i) \to \exists \mathcal{P} \in \mathbb{P}(\mathcal{T} \nsubseteq \mathcal{P} \wedge \mathcal{S}_i \subseteq \mathcal{P})\big)
\end{aligned}
$$

which is the second statement.

(2. $\Rightarrow$ 3.): Suppose $\mathcal{S}_i \in \mathbb{C}$, $\mathcal{R}_i \in \mathbb{T}$ are given as in statement 3. By Corollary 5.3.4, there are prime torsion classes $\mathcal{P}_j$, $\mathcal{Q}_k$ for some indices j, k, such that $\mathcal{R}_1 = \bigcap_j \mathcal{P}_j$ and $\mathcal{R}_2 = \bigcap_k \mathcal{Q}_k$.

Since $\mathcal{T} \nsubseteq \mathcal{R}_i$, there are j_0, k_0 such that $\mathcal{T} \nsubseteq \mathcal{P}_{j_0}$ and $\mathcal{T} \nsubseteq \mathcal{Q}_{k_0}$. On the other hand, since $\mathcal{S}_i \subseteq \mathcal{R}_i$, we have $\mathcal{S}_1 \subseteq \mathcal{P}_{j_0}$ and $\mathcal{S}_2 \subseteq \mathcal{Q}_{k_0}$. Then, applying statement 2, we conclude the existence of a prime $\mathcal{P} \in \mathbb{P}$ such that $\mathcal{S}_i \subseteq \mathcal{P}$ and $\mathcal{T} \nsubseteq \mathcal{P}$, as required.

(3. $\Rightarrow$ 2.): Obvious.

Finally, suppose that the above equivalent conditions on $\mathcal{T}$ hold and take $\mathcal{T}_i \in \mathbb{T}$ such that $\mathcal{T} \subseteq \mathcal{T}_1 + \mathcal{T}_2$. Since all torsion classes are of finite type by Lemma 5.3.2,

$$\mathcal{T}_i = \sum_{A \in \mathcal{T}_i \cap \mathcal{A}^{\mathrm{fp}}} \mathcal{T}(A);$$

i.e., $\mathcal{T}_i$ is generated by its finitely presented objects. So

$$\mathcal{T} \subseteq \sum_{A \in \mathcal{T}_1 \cap \mathcal{A}^{\mathrm{fp}}} \mathcal{T}(A) + \sum_{B \in \mathcal{T}_2 \cap \mathcal{A}^{\mathrm{fp}}} \mathcal{T}(B);$$

but $\mathcal{T}$ is compact, so it is already contained in the sum of finitely many of these classes. So there are some finitely presented objects $A_1, \ldots, A_n \in \mathcal{T}_1$, $B_1, \ldots, B_m \in \mathcal{T}_2$ such that

$$\mathcal{T} \subseteq \sum_{j=1}^{n} \mathcal{T}(A_j) + \sum_{k=1}^{m} \mathcal{T}(B_k).$$

Write

$$\mathcal{S}_1 = \sum_{j=1}^{n} \mathcal{T}(A_j) = \mathcal{T}\left(\bigoplus_{j=1}^{n} A_j\right), \mathcal{S}_2 = \sum_{k=1}^{m} \mathcal{T}(B_k) = \mathcal{T}\left(\bigoplus_{k=1}^{m} B_k\right).$$

Since each $\mathcal{S}_i$ is generated by a single finitely presented object, it is compact (Proposition 5.3.10). Now suppose for a contradiction that $\mathcal{T} \nsubseteq \mathcal{S}_i$, then applying statement 3 with $\mathcal{R}_i = \mathcal{S}_i$, we conclude that there is a prime $\mathcal{P} \in \mathbb{P}$ such that $\mathcal{S}_i \subseteq \mathcal{P}$ and $\mathcal{T} \nsubseteq \mathcal{P}$. But then $\mathcal{S}_1 + \mathcal{S}_2 \subseteq \mathcal{P}$, so $\mathcal{T} \nsubseteq \mathcal{S}_1 + \mathcal{S}_2$, a contradiction. So in fact we must have either $\mathcal{T} \subseteq \mathcal{S}_1 \subseteq \mathcal{T}_1$, or $\mathcal{T} \subseteq \mathcal{S}_2 \subseteq \mathcal{T}_2$, as required. ■

Note that for a compact torsion class, being $+$-irreducible is the same as being Σ-irreducible (meaning that whenever it is contained in an arbitrary sum of torsion classes, it is already contained in one of the summands). So the final conclusion of the Lemma could be rephrased as saying that $\mathcal{T}$ is Σ-irreducible.

After all these technical Lemmas, we are finally in a position to prove the following:

Theorem 5.3.13. *Let $\mathcal{A}$ be a locally noetherian Grothendieck category and let $\mathcal{T}$ be a compact torsion class in $\mathcal{A}$ such that the basic closed set $(\mathcal{T})$ is irreducible. Then $(\mathcal{T})$ has a generic point.*

PROOF:

Let $\mathcal{S}$ be the sum of all torsion classes not containing $\mathcal{T}$. By Lemma 5.3.12, $\mathcal{T}$ is Σ-irreducible, so $\mathcal{T} \nsubseteq \mathcal{S}$. Suppose that $\mathcal{S}$ is properly contained in some torsion classes $\mathcal{T}_1$ and $\mathcal{T}_2$. Then, since the inclusion is proper, both $\mathcal{T}_i$'s contain $\mathcal{T}$, so $\mathcal{T} \subseteq \mathcal{T}_1 \cap \mathcal{T}_2$; so $\mathcal{S} \neq \mathcal{T}_1 \cap \mathcal{T}_2$. So $\mathcal{S}$ is $\cap$-irreducible, hence prime by Theorem 5.3.7. So $\mathcal{S}$ is a point in $\mathrm{TorSpec}(\mathcal{A})$.

Finally, points in $(\mathcal{T})$ are prime torsion classes not containing $\mathcal{T}$; any such class $\mathcal{P}$ is contained in $\mathcal{S}$, by construction, and so $\mathcal{S} \rightsquigarrow \mathcal{P}$ (see Lemma 2.1.1, and remarks after Theorem 5.2.1). So $\mathcal{S}$ is generic in $(\mathcal{T})$. ■

This provides a partial answer to Question 3 (when is $\mathrm{InjSpec}(R)$ sober?), but does not tell us about sobriety for non-basic irreducible closed sets.

Chapter 6

Further Examples

In this chapter we develop a further example of "nice" behaviour of the injective spectrum, in the Heisenberg algebra, which can be thought of as the Weyl algebra with some centre added. The description of the underlying set and parts of the topology uses essentially the techniques we have seen before, but to describe specialisation between different parts of the spectrum we make use of some of the torsion theoretic ideas developed in Section 5.2.

We then give an example of "bad" behaviour, the quantum plane, where critical dimension fails to describe topological dimension in the way we would like.

6.1 The Heisenberg Algebra

Let k be an algebraically closed field of characteristic 0 and let $\mathfrak{h}$ denote the (first) Heisenberg algebra; *viz.* the 3-dimensional Lie algebra with basis $\{p, q, z\}$, where $[p, q] = z$, and $[z, -] = 0$. Let H be the universal enveloping algebra of $\mathfrak{h}$; so

$$H = k[z]\langle p, q \mid [p, q] = z\rangle.$$

We will obtain a complete description of the points of $\mathrm{InjSpec}(H)$, and an extensive description of the topology.

First we require some Lemmas to extend our functoriality results from Section 2.3.

Lemma 6.1.1. *Let R be a right noetherian ring, I a two-sided ideal of R and F an indecomposable injective R/I-module. Suppose that $z \in I \cap Z(R)$ is a central element of I. Then for any $x \in E(F_R)$, there is a natural number n such that $xz^n = 0$.*

PROOF:

This proof is adapted from an argument originally due to E. Noether, as presented in Theorem 3.78 of [13].

If $x = 0$, there's nothing to prove; so suppose $x \neq 0$ and let $Q := \mathrm{ann}_R(x)$. Then Q is a proper right ideal of R and $xR \cong R/Q$. Since F is uniform, so is $E(F_R)$, and therefore so is xR, so Q is $\cap$-irreducible among right ideals of R.

Also by uniformity, $xR \cap F_R \neq 0$; so let D be the right ideal of R containing Q such that D/Q corresponds to $xR \cap F_R$ under the isomorphism $R/Q \cong xR$. Since F is an R/I-module, $F_R I = 0$, $(D/Q)I = 0$, so $DI \subseteq Q$; in particular, $Dz \subseteq Q$. Moreover, D strictly contains Q, since $xR \cap F_R \neq 0$.

Now consider the sets $(Q : z^i) := \{r \in R \mid rz^i \in Q\}$. These are right ideals, since z is central, and $(Q : z^i) \subseteq (Q : z^{i+1})$ for all i. Since R is right noetherian, this chain stabilises, and so there is some n such that $(Q : z^n) = (Q : z^{n+1})$. We show that, for this particular n,

$$Q = (Q + z^n R) \cap D.$$

The left-to-right inclusion is clear. Let y be an element of the right-hand side; so for some $q \in Q$ and $r \in R$, $y = q + rz^n$ and also $y \in D$. Then $yz = qz + rz^{n+1} \in Q$, since $Dz \subseteq Q$. Hence $rz^{n+1} = yz - qz \in Q$. So $r \in (Q : z^{n+1}) = (Q : z^n)$, so $rz^n \in Q$. But then $y = q + rz^n \in Q$, proving the claim.

Now, since D strictly contains Q and Q is $\cap$-irreducible, we must have $Q = Q + z^n R$, so $z^n \in Q = \mathrm{ann}_R(x)$, so $xz^n = 0$. ■

Lemma 6.1.2. *Let R and S be rings and let $f : R \to S$ be a surjection. Then f induces an injective function $f^* : \mathrm{InjSpec}(S) \to \mathrm{InjSpec}(R)$ given by $F \mapsto E(F_R)$, where E denotes the injective hull (taken, in this case, in* Mod-R*).*

This function takes closed sets in $\mathrm{InjSpec}(S)$ to relatively closed sets in $\mathrm{InjSpec}(R)$ (i.e., the intersection of a closed set with the image of f^*), and also preserves the specialisation ordering: if $F \rightsquigarrow G$ in $\mathrm{InjSpec}(S)$, then $E(F_R) \rightsquigarrow E(G_R)$ in $\mathrm{InjSpec}(R)$.

If, moreover, $\ker(f)$ is generated by a single central element, then f^* is continuous in the Zariski topologies, and therefore is a topological embedding of $\mathrm{InjSpec}(S)$ into $\mathrm{InjSpec}(R)$.

Note that, although I can only prove continuity of f^* under the above assumption on $\ker(f)$ (or the assumption that ${}_RS$ be flat, in which case Corollary 2.3.4 pertains), I do not know any examples where continuity fails.

PROOF:

Since f is surjective, the submodule structure of any S-module is the same when scalars are restricted along f to R. In particular, for any uniform S-module U, U_R is uniform. Therefore, for $F \in \mathrm{InjSpec}(R)$, $E(F_R)$ is indecomposable. So f^* is well-defined.

Moreover, if $F, G \in \mathrm{InjSpec}(S)$ and $E(F_R) = E(G_R)$, then F_R and G_R have a common submodule (up to isomorphism), by uniformity. So there are $M_R \leq F_R$, $N_R \leq G_R$ with $M_R \cong N_R$. But F and G have the same submodules over R and over S, so M_R and N_R are indeed restrictions of some S-submodules M and N, and the isomorphism between them is also an isomorphism of S-modules, since f is onto. So $F_S = E(M_S)$ and $G_S = E(N_S)$ are isomorphic and so f^* is injective.

Next we show that f^* maps closed sets to relatively closed sets. So let $M \in$ mod-S, so that (M) is a basic closed set in $\mathrm{InjSpec}(S)$. We show that $f^*(M) = (M_R) \cap \mathrm{Im}(f^*)$. Since f is a surjection, a finite generating set for M as an S-module will still generate it over R and a finite set of relations over S lifts to a finite set over R, so M_R is finitely presented and hence (M_R) is (basic) closed.

On the one hand, if $E(F_R) \in f^*(M)$, then $(M, F) \neq 0$, so $(M_R, F_R) \neq 0$ (since restriction of scalars is faithful) and so $(M_R, E(F_R)) \neq 0$ and so $E(F_R) \in (M_R) \cap \mathrm{Im}(f^*)$.

On the other hand, if $E(F_R) \in (M_R) \cap \mathrm{Im}(f^*)$, then $(M_R, E(F_R)) \neq 0$, so M_R has a submodule N_R such that $(N_R, F_R) \neq 0$. Since M has the same submodule structure over R and over S, M has an S-submodule N restricting to N_R and any non-zero map $N_R \to F_R$ is also S-linear, so $(N, F) \neq 0$. Then injectivity of F allows us to extend any non-zero, S-linear map $N \to F$ to a map $M \to F$, showing that $F \in (M)$ and so $E(F_R) \in f^*(M)$.

Finally, since f^* is injective, for any sets A_i, $f^*\left(\bigcap A_i\right) = \bigcap f^*(A_i)$, so this extends from basic closed sets to arbitrary closed sets.

Now we consider the specialisation order. Recall from Lemma 2.1.1 that $F \rightsquigarrow G$ if and only if F is torsionfree for the torsion theory cogenerated by G, which by Lemma 1.3.4 occurs if and only if F embeds in a direct sum of copies of G.

Suppose $F \rightsquigarrow G$ in InjSpec(S). Then F embeds in the product G^λ for some cardinal λ, and hence F_R embeds in $E(G_R)^\lambda$. Since $F_R \leq E(F_R)$ and $E(G_R)^\lambda$ is injective, this embedding extends to a map $E(F_R) \to E(G_R)^\lambda$, which must also be injective, since F is essential in $E(F_R)$. So $E(F_R) \rightsquigarrow E(G_R)$ in InjSpec(R).

Finally, we assume that $\ker(f) = zR$ for some $z \in Z(R)$ and prove that f^* is continuous. It suffices to prove that for any finitely presented R-module M, the preimage of the basic closed set (M) under f^* is basic closed. The preimage is $(f^*)^{-1}((M)) = \{F \in \text{InjSpec}(S) \mid (M, E(F_R)) \neq 0\}$. We show that $(f^*)^{-1}((M)) = (M \otimes_R S)$, which is basic closed in InjSpec(S), since tensoring preserves finitely presented modules.

Certainly if $F \in (M \otimes_R S)$, then $(M, F_R) = (M \otimes_R S, F) \neq 0$, so after embedding in $E(F_R)$, we see that $(M, E(F_R)) \neq 0$, so $F \in (f^*)^{-1}((M))$.

Conversely, if $(M, E(F_R)) \neq 0$, take $\phi : M \to E(F_R)$ a non-zero map and let $m_1, \ldots, m_n$ be a generating set for M. By Lemma 6.1.1, for each i there is some smallest integer $\nu_i \geq 0$ such that $\phi(m_i)z^{\nu_i} = 0$. Let $\nu = \max_{i=1}^n\{\nu_i\}$. We cannot have all ν_i zero, since then ϕ would be zero, a contradiction; so $\nu \geq 1$.

So $\phi(M)z^\nu = 0$ and $\phi(M)z^{\nu-1} \neq 0$; so $\phi(M)z^{\nu-1}$ is a non-zero submodule of $A := \text{ann}_{E(F_R)}(z)$. We show that $A = F_R$. Certainly $F_R \subseteq A$, since F is a module over $S \cong R/zR$, so $F_R z = 0$. Moreover, $Az = 0$, so A naturally has an S-module structure. But $F_R \leq A \leq E(F_R)$, so A is an essential extension of F_R; since $f : R \to S$ is surjective, the submodule structure of A does not depend on whether we regard it as an R-module or S-module, so A_S is an essential extension of F. But F is injective, so admits no proper essential extension as an S-module; therefore $A = F$, as claimed.

So $\phi(M)z^{\nu-1}$ is a non-zero submodule of F_R. Let $\psi : E(F_R) \to E(F_R)$ be the "multiplication by $z^{\nu-1}$" map, which is R-linear, since z is central. Then $\psi \circ \phi : M \to E(F_R)$ is non-zero and has image in F_R, so $(M, F_R) \neq 0$, and therefore $F \in (M \otimes_R S)$, completing the proof. ■

Now we apply these results to the study of InjSpec(H). For convenience, we let $H_\alpha := H/(z-\alpha)H$ denote the quotient ring, for $\alpha \in k$, and let $f_\alpha : H \to H_\alpha$ denote the quotient map.

First we describe the points of $\mathrm{InjSpec}(H)$. Let E be an indecomposable, injective H-module. There are two cases to consider; either there is some non-zero element of $k[z]$ which acts non-invertibly on E, or there isn't.

In the first case, suppose that $f(z)$ is such an element of minimal degree, and monic, without loss of generality. Because E is injective (hence divisible, by Theorem 1.2.3 and the following remarks) and H is a domain, $f(z)$ must act surjectively on E, so to be non-invertible, its action must be non-injective. So there is some $e \in E \setminus 0$ such that $ef(z) = 0$.

Since k is algebraically closed, we can factor $f(z)$ as a product of linear factors $(z - \alpha_i)$, where the α_i, $i \in \{1, \dots, n\}$ are the roots of f. Then we have

$$e(z - \alpha_1)(z - \alpha_2) \dots (z - \alpha_n) = 0.$$

Let j be the minimal index such that $e(z-\alpha_1)\dots(z-\alpha_j) = 0$. Then $e(z-\alpha_1)\dots(z-\alpha_{j-1}) \neq 0$ and is annihilated by the linear polynomial $(z-\alpha_j)$. Since f was assumed to have minimal degree among polynomials acting non-injectively, we must have $\deg(f) = 1$.

So if any non-zero element of $k[z]$ acts non-invertibly on E, then there exists some $\alpha \in k$ such that $(z - \alpha)$ acts non-invertibly. Since $(z - \alpha)$ is central, the annihilator in E of $(z - \alpha)$ is a submodule; so E has a submodule S which is a H_α-module. So E has the form $E(M_H)$ for some $M \in \text{Mod-}H_\alpha$. Since any essential extension of M in $\text{Mod-}H_\alpha$ remains essential over H, we can replace M by its injective hull over H_α, which must be indecomposable, since M is uniform. So E has the form $E(F_H)$ for some $F \in \mathrm{InjSpec}(H_\alpha)$.

In other words, those modules on which a non-zero element of $k[z]$ acts non-invertibly are of the form $f_\alpha^*(F)$ for some $F \in \mathrm{InjSpec}(H_\alpha)$. By Lemma 6.1.2, f_α^* is an embedding of topological spaces, so the internal topology of the set of points of $\mathrm{InjSpec}(H)$ containing an element annihilated by $(z - \alpha)$ is precisely the topology on $\mathrm{InjSpec}(H_\alpha)$.

Moreover, if E contains an element annihilated by $z - \alpha$ and also an element annihilated by $z - \beta$, for some $\alpha, \beta \in k$, then the submodules $\mathrm{ann}_E(z - \alpha)$ and $\mathrm{ann}_E(z - \beta)$ intersect, by uniformity. So E contains a non-zero element annihilated by $z - \alpha$ and $z - \beta$, and hence also by $\alpha - \beta$. Therefore we must have $\alpha = \beta$. So the set of indecomposable injectives of the form $f^*(F)$ for $F \in \mathrm{InjSpec}(H_\alpha)$ splits as a disjoint union over the different values of $\alpha \in k$.

In the second case, every non-zero element of $k[z]$ acts invertibly on E. Let $S := k[z] \smallsetminus 0$; this is a central multiplicative (hence Ore) set in H; let H_S denote the corresponding localisation. Then E is an indecomposable, injective H_S-module, so lies in the image of the continuous injection $\mathrm{InjSpec}(H_S) \to \mathrm{InjSpec}(H)$ induced by the (flat, epimorphic) localisation map (Corollary 2.3.4).

Since every point in $\mathrm{InjSpec}(H)$ must fall into (exactly) one of the above two cases, we see that, as a set, $\mathrm{InjSpec}(H)$ is the disjoint union of $\mathrm{InjSpec}(H_\alpha)$ as α ranges over k, and also $\mathrm{InjSpec}(H_S)$.

We now describe each of these parts of the spectrum. Note that H_α has presentation $k\langle p, q \mid [p, q] = \alpha\rangle$. In the case where $\alpha \neq 0$, replacing p by p/α, we obtain the first Weyl algebra over k, whose injective spectrum was described in Theorem 4.2.2 and the remarks immediately thereafter. If instead $\alpha = 0$, p and q commute in the quotient, so we get $H_\alpha = k[p, q]$, whose spectrum is, of course, the affine plane.

We will henceforth refer to $\mathrm{InjSpec}(H_\alpha)$ (with $\alpha \neq 0$) and the corresponding subset of $\mathrm{InjSpec}(H)$ as being a line, for convenience; do not take this too literally however, as it is simply a topological space of cardinality $|k|$ whose closed sets are the finite sets omitting the generic. There is no canonical homeomorphism with the affine line on k, nor any canonical bijection between the closed points and any 1-dimensional vector space. It is simply convenient to visualise as linear.

Each of these subsets of $\mathrm{InjSpec}(H)$ for different $\alpha \in k$ is closed; for the image of $\mathrm{InjSpec}(H_\alpha)$ is precisely the basic closed set $(H/(z - \alpha)H)$ (including for $\alpha = 0$). By Lemma 6.1.2, the subspace topology on this closed set is precisely the (known) topology on $\mathrm{InjSpec}(H_\alpha)$.

Now consider H_S. This has presentation $k(z)\langle p, q \mid [p, q] = z\rangle$. Replacing p by p/z, we get the first Weyl algebra over $k(z)$. By Corollary 2.3.4, the map $\mathrm{InjSpec}(H_S) \to \mathrm{InjSpec}(H)$ given by restricting scalars along the localisation map is an embedding of topological spaces; see also Section 4.3. The image of this embedding, however, is not a closed set, as we shall see.

We now consider specialisation within InjSpec(H). Our tool here is Lemma 2.1.1; to show that $E \rightsquigarrow F$, we show that E is torsionfree for $\mathcal{F}(F)$; for this, it suffices to prove that M embeds in a direct product of copies of F, where M is any module such that $E = E(M)$.

First we observe by Theorem 3.3.3 that, since H is a noetherian domain, InjSpec(H) is irreducible and $E(H)$ is generic in InjSpec(H). Observe that, as H embeds in H_S, $E(H) = E(H_S)$, which is the generic point in the line InjSpec(H_S) embedded in InjSpec(H). In fact we shall see that the points in InjSpec(H_S) are generics over certain irreducible sets in the rest of the spectrum, and then $E(H_S)$ sits above all as the "generic over the generics".

Let $\alpha \in k$ be non-zero. We show that for each closed point E in InjSpec(H_α), there is a point in InjSpec(H_S) (other than the generic) specialising to E. Let I_α be a maximal right ideal in $H_\alpha \cong A_1(k)$, so that $E(H_\alpha/I_\alpha)$ is a closed point over H_α. Since $H_S \cong A_1(k(z)) \cong A_1(k) \otimes_k k(z)$, $I_\alpha \otimes_k k(z)$ is a maximal proper right ideal of H_S (after applying an isomorphism), so $E(H_S/(I_\alpha \otimes_k k(z)))$ is a closed point over H_S.

We show that $E(H_S/(I_\alpha \otimes_k k(z))) \rightsquigarrow E(H_\alpha/I_\alpha)$. Since H embeds in H_S, we can consider the submodule

$$\frac{H + I_\alpha \otimes_k k(z)}{I_\alpha \otimes_k k(z)} \cong H/I,$$

where $I := H \cap (I_\alpha \otimes_k k(z))$. Then

$$E\left(\frac{H_S}{I_\alpha \otimes_k k(z)}\right) = E(H/I).$$

So it suffices to prove that H/I embeds in a direct product of copies of $E(H_\alpha/I_\alpha)$.

Let $\phi_n : H \to H/(I + (z-\alpha)^n H)$ be the quotient map. The intersection of the kernels of these ϕ_n over all $n \in \mathbb{N}$ is I, so if it can be shown that each $H/(I+(z-\alpha)^n H)$ embeds in $E(H_\alpha/I_\alpha)$ under some map ψ_n, then the product of the $\psi_n \circ \phi_n$'s will embed H/I in $E(H_\alpha/I_\alpha)^{\aleph_0}$.

So we show that $H/(I+(z-\alpha)^n H)$ embeds in $E(H_\alpha/I_\alpha)$. This amounts to finding an element of $E(H_\alpha/I_\alpha)$ whose annihilator is exactly $I + (z-\alpha)^n H$. By solubility of equations in injective modules (Theorem 1.2.3), there exists some $e \in E(H_\alpha/I_\alpha)$ such that $e(I + (z-\alpha)^n H) = 0$ and $e(z-\alpha)^{n-1} = 1 + I_\alpha$ (which forces $e \neq 0$). Write $\xi_n(x)$ for the formula

$$x(I + (z-\alpha)^n H) = 0 \wedge x(z-\alpha)^{n-1} = 1 + I_\alpha.$$

We prove by induction on n that if $E(H_\alpha/I_\alpha) \models \xi_n(e)$, then $\mathrm{ann}_H(e) = I + (z-\alpha)^n H$.

When $n = 1$, $\xi_1(e)$ says that $e(I+(z-\alpha)H) = 0$ and $e = 1+I_\alpha$. This has a unique solution, $1 + I_\alpha$, and $\mathrm{ann}_{H_\alpha}(1 + I_\alpha) = I_\alpha$, so $\mathrm{ann}_H(1 + I_\alpha)$ is the preimage of I_α under f_α, which is $I + (z-\alpha)H$ (since a generating set for I_α lifts to a generating set for I), and so the base case is done.

For general $n \geq 2$, suppose any element satisfying ξ_{n-1} has annihilator exactly $I + (z-\alpha)^{n-1}H$, and that e satisfies ξ_n. Certainly $I + (z-\alpha)^n H \subseteq \mathrm{ann}_H(e)$, so we need only show the reverse inclusion. Suppose that J is a right ideal containing $I + (z-\alpha)^n H$ and that $eJ = 0$. We show that $J = I + (z-\alpha)^n H$.

In H_α, $f_\alpha(I + (z-\alpha)H) = I_\alpha$, which is maximal, so $I + (z-\alpha)H$ is maximal in H, and hence $J + (z-\alpha)H = I + (z-\alpha)H$ or $J + (z-\alpha)H = H$.

If $J + (z-\alpha)H = H$, then there is some $h \in H$ such that $1 - (z-\alpha)h \in J$. Then $(z-\alpha)^{n-1} - (z-\alpha)^n h \in J$, and

$$e[(z-\alpha)^{n-1} - (z-\alpha)^n h] = e(z-\alpha)^{n-1} - 0 = 1 + I_\alpha \neq 0,$$

so J contains an element not annihilating e, a contradiction.

So $J + (z-\alpha)H = I + (z-\alpha)H$. Now, $e' := e(z-\alpha)$ satisfies

$$e'(I + (z-\alpha)^{n-1}H) = e((z-\alpha)I + (z-\alpha)^n H) = 0$$

and

$$e'(z-\alpha)^{n-2} = e(z-\alpha)^{n-1} = 1 + I_\alpha,$$

so e' satisifes ξ_{n-1} and hence by the inductive hypothesis, $\mathrm{ann}_H(e') = I + (z-\alpha)^{n-1}H$.

For any $h \in J$, $h + (z-\alpha)g \in I$ for some $g \in H$, since $J + (z-\alpha)H = I + (z-\alpha)H$. Now, $eh = 0$, but $e(z-\alpha)^{n-1} = 1 + I_\alpha$, so $e(z-\alpha) \neq 0$, as $n \geq 2$. But $eI = 0$, so we must have $e(z-\alpha)g = 0$; *i.e.*, $e'g = 0$. Therefore $g \in \mathrm{ann}_H(e') = I + (z-\alpha)^{n-1}H$. So

$$(z-\alpha)g \in I(z-\alpha) + (z-\alpha)^n H \subseteq I + (z-\alpha)^n H,$$

so $h = (h + (z-\alpha)g) - (z-\alpha)g \in I + (z-\alpha)^n H$, so $J = I + (z-\alpha)^n H$, completing the proof.

Note that, since each H_α for $\alpha \neq 0$ is isomorphic to $A_1(k)$, and we started with a maximal right ideal I_α in H_α, we can take the corresponding right ideal in each H_β for $\beta \neq 0$. So not only does $E(H/I)$ specialise to $E(H_\alpha/I_\alpha)$, but also to each corresponding point in each other fibre over $\beta \neq 0$. So, viewing each fibre as a vertical line, we can think of the closure of $E(H/I)$ as a horizontal line cutting across the fibres.

So the picture we have developed of the injective spectrum of the Heisenberg algebra is that we have, for each $\alpha \in k^{\times}$, a copy of the line $\mathrm{InjSpec}(A_1(k))$, together with an affine plane $\mathrm{InjSpec}(k[p,q])$ at $\alpha = 0$, and a line $\mathrm{InjSpec}(A_1(k(z)))$ of generic points specialising across these $A_1(k)$-lines.

We can visualise this by imagining a copy of k as a z-axis, with a fibre over each $\alpha \in k$, which is a line for non-zero α, and a plane for $\alpha = 0$. Each fibre is an irreducible closed set with its own generic point, and there is a "line of generics" which specialise across the fibres. The line of generics has its own "big" generic, $E(H)$, which specialises to every point. See Fig. 1.

There are further points in the line of generics, with different closures cutting across the fibres. For instance, given any rational function $f(z)/g(z) \in k(z)$ (in lowest terms), the right ideal $(p - f(z)/g(z))H_S = (g(z)p - f(z))H_S$ is maximal in H_S, hence there is a point $E_{f/g} := E(H/(g(z)p - f(z))H)$ in the line of generics. For any α such that $g(\alpha) \neq 0$, we can take the $\cap$-irreducible right ideal $(g(\alpha)p - f(\alpha))H_\alpha$ of H_α, and obtain the point $E(H_\alpha/(g(\alpha)p - f(\alpha))H_\alpha) = E(H/[(z-\alpha)H + (g(z)p - f(z))H])$, which we write as $E_{f/g,\alpha}$.

Then $E_{f/g} \rightsquigarrow E_{f/g,\alpha}$ for each $\alpha \in k$ such that $g(\alpha) \neq 0$. To prove this, we show that $H/(g(z)p - f(z))H$ embeds in a direct product of copies of $E_{f/g,\alpha}$, so that $E_{f/g} = E(H/(g(z)p - f(z))H)$ is torsionfree for $\mathcal{F}(E_{f/g,\alpha})$. It suffices to show that for each $n \geq 1$, $H/((z-\alpha)^n H + (g(z)p - f(z))H)$ embeds in $E_{f/g,\alpha}$; *i.e.*, that $E_{f/g,\alpha}$ contains an element e_n whose annihilator is precisely $(z-\alpha)^n H + (g(z)p - f(z))H$.

Let $\xi_n(x)$ be the system of equations

$$x(z-\alpha)^{n-1} = 1 + (z-\alpha)H + (g(\alpha)p - f(\alpha))H \wedge x(g(z)p - f(z)) = 0.$$

By solubility of systems of equations in injective modules (Theorem 1.2.3), ξ_n has a solution in $E_{f/g,\alpha}$, which must be non-zero, since $g(\alpha) \neq 0$ implies that $1 + (z-\alpha)H + (g(\alpha)p - f(\alpha))H \neq 0$. We take e_n to be any such solution and prove by induction that all solutions of ξ_n have annihilator exactly $(z-\alpha)^n H + (g(z)p - f(z))H$.

The base case $n = 1$ is clear, as the only solution to ξ_1 is $1 + (z - \alpha)H + (g(\alpha)p - f(\alpha))H$, whose annihilator is exactly

$$(z - \alpha)H + (g(\alpha)p - f(\alpha))H = (z - \alpha)H + (g(z)p - f(z))H.$$

For $n \geq 2$, note that certainly $(z - \alpha)^n H + (g(z)p - f(z))H$ is contained in the annihilator of any solution e_n of ξ_n, so we need only prove the reverse inclusion.

Suppose $h \in H$ is such that $e_n h = 0$; we show that $h \in (z - \alpha)^n H + (g(z)p - f(z))H$. Certainly $e_n(z-\alpha)h = 0$, and $e_n(z-\alpha)$ satisfies ξ_{n-1}; hence, by the inductive hypothesis, $h \in (z - \alpha)^{n-1}H + (g(z)p - f(z))H$. So we can write

$$h = (z - \alpha)^{n-1}h_1 + (g(z)p - f(z))h_2$$

for some $h_1, h_2 \in H$. Then

$$0 = e_n h = e_n(z - \alpha)^{n-1}h_1 + e_n(g(z)p - f(z))h_2 = e_n(z - \alpha)^{n-1}h_1,$$

by ξ_n. But

$$e_n(z - \alpha)^{n-1} = 1 + (z - \alpha)H + (g(\alpha)p - f(\alpha))H,$$

and

$$(z - \alpha)H + (g(\alpha)p - f(\alpha))H = (z - \alpha)H + (g(z)p - f(z))H,$$

so this implies that $h_1 = (z - \alpha)h_3 + (g(z)p - f(z))h_4$, for some $h_3, h_4 \in H$. So

$$h = (z - \alpha)^n h_3 + (g(z)p - f(z))(h_2 + (z - \alpha)^{n-1}h_4) \in (z - \alpha)^n H + (g(z)p - f(z))H,$$

as required.

So the point $E_{f/g}$ specialises to $E_{f/g,\alpha}$ whenever $g(\alpha) \neq 0$, as claimed. Note that when $g(\alpha) = 0$, $(g(\alpha)p - f(\alpha))H = f(\alpha)H = H$ (since f/g is assumed to be in lowest terms, so $f(\alpha) \neq 0$). So when $g(\alpha) = 0$, $E_{f/g,\alpha} = 0$, so is not a point in $\mathrm{InjSpec}(H)$.

If $\alpha \neq 0$, then $H_\alpha \cong A_1(k)$ and, if $g(\alpha) \neq 0$, $(g(\alpha)p - f(\alpha))H_\alpha$ is a maximal ideal, so $E_{f/g,\alpha}$ is a closed point in the "line" $\mathrm{InjSpec}(H_\alpha)$. At $\alpha = 0$, $H_0 \cong k[p, q]$ and, if $g(0) \neq 0$, $(g(0)p - f(0))H_0$ is the prime ideal corresponding to the line $p = f(0)/g(0)$ in the affine plane $\mathrm{InjSpec}(H_0) \cong \mathrm{Spec}(k[p, q])$. So $E_{f/g}$ specialises to the single closed point $E_{f/g,\alpha}$ for all $\alpha \neq 0$ such that $g(\alpha) \neq 0$, and to the generic point $E_{f/g,0}$ of the line $p = f(0)/g(0)$ (and hence to all the closed points of this line too) if $g(0) \neq 0$. Write

$E_{f/g,0,\beta}$ for the closed point $E(H_0/((q-\beta)H_0+(g(0)p-f(0))H_0))$ in the closure of $E_{f/g,0}$, in the event that $g(0)\neq 0$.

Moreover, we now prove that these are the only points to which $E_{f/g}$ specialises, and in fact comprise the basic closed set $(H/(g(z)p-f(z))H)$. That is

$$\left(\frac{H}{(g(z)p-f(z))H}\right)=\mathrm{cl}(E_{f/g})=\{E_{f/g}\}\cup\{E_{f/g,\alpha}\mid g(\alpha)\neq 0\}\cup\{E_{f/g,0,\beta}\mid \beta\in k\},$$

where the final disjunct only occurs if $g(0)\neq 0$.

To see this, note that if E is an indecomposable injective receiving a map from $H/(g(z)p-f(z))H$, then E contains a non-zero element e annihilated by $g(z)p-f(z)$. If E lies in $\mathrm{InjSpec}(H_S)$, then $(g(z)p-f(z))H_S$ is a maximal right ideal of H_S, so $\mathrm{ann}_{H_S}(e)=(g(z)p-f(z))H_S$, and so $H_S/(g(z)p-f(z))H_S$ embeds in E, and hence $E=E_{f/g}$.

If instead E lies in $\mathrm{InjSpec}(H_\alpha)$, then by the proof of Lemma 6.1.2 $E=E(F_H)$ for some indecomposable injective H_α-module F which receives a non-zero map from $H_\alpha/(g(z)p-f(z))H_\alpha=H_\alpha/(g(\alpha)p-f(\alpha))H_\alpha$. If $g(\alpha)=0$ then $f(\alpha)\neq 0$, since we assume that f/g is in lowest terms, and so $H_\alpha/(g(\alpha)p-f(\alpha))H_\alpha=0$; so for those α such that $g(\alpha)=0$, there are no points of $(H/(g(z)p-f(z))H)$ lying in the fibre over α.

So suppose $g(\alpha)\neq 0$. If $\alpha\neq 0$, $H_\alpha/(g(\alpha)p-f(\alpha))H_\alpha$ is simple, so the only possibility for E is $E_{f/g,\alpha}$. If $\alpha=0$, $H_0\cong k[p,q]$ and $H_0/(g(0)p-f(0))H_0\cong k[p,q]/(p-f(0)/g(0))$, so $(H_0/(g(0)p-f(0))H_0)$ consists of the injective hulls of $k[p,q]/(p-f(0)/g(0))$ and $k[p,q]/(p-f(0)/g(0),q-\beta)$ for $\beta\in k$; but these points are precisely $E_{f/g,0}$ and $E_{f/g,0,\beta}$.

So every point in $(H/(g(z)p-f(z))H)$ is among the points listed, hence this is the entire set. Moreover, since $E_{f/g}\in(H/(g(z)p-f(z))H)$ and all the listed points are in $\mathrm{cl}(E_{f/g})$, we must also have that this set is equal to $\mathrm{cl}(E_{f/g})$.

Similarly, we can consider $E(H_S/(g(z)q-f(z))H_S)$ and show that it specialises to each $E(H_\alpha/(g(\alpha)q-f(\alpha))H_\alpha)$ in each fibre for $\alpha\neq 0$ and $g(\alpha)\neq 0$ (and hence also to each closed point below this at $\alpha=0$), and that this is precisely the basic closed set $(H/(g(z)q-f(z))H)$.

Of course, not every maximal right ideal of $H_S\cong A_1(k(z))$ can be written as $(p-f(z)/g(z))A_1(k(z))$ or $(q-f(z)/g(z))A_1(k(z))$ for some rational function f/g; so there are additional points in the line of generics, with perhaps more exotic closures. However, since each fibre is a closed set and the fibres at $\alpha \neq 0$ are topologically affine lines, the closure of a point can only pick out either finitely many points from each fibre over $\alpha \neq 0$, or the whole fibre. Over $\alpha = 0$, the closure of a point can pick out a subvariety of the plane.

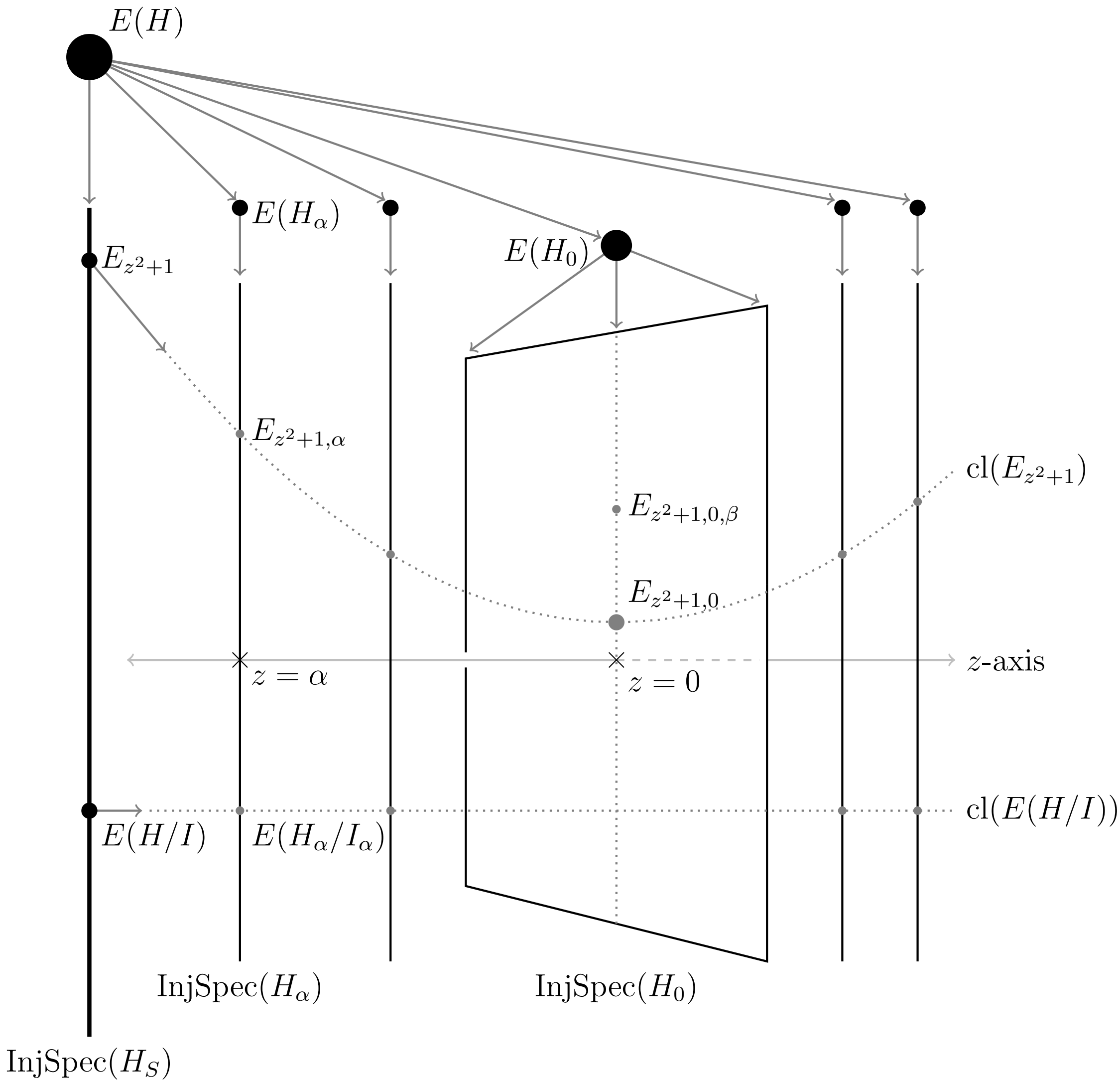

Figure 6.1: The injective spectrum of the Heisenberg algebra, shown as a collection of closed "fibres" at each value of z in k, with a generic point over each fibre whose size indicates "how generic it is", and a "line of generics" to the left, with the "biggest" generic at the top. Specialisation is shown with grey arrows. The horizontal dotted grey line shows a point $E(H_\alpha/I_\alpha)$ at $z = \alpha \neq 0$ being lifted to a point $E(H/I)$ in the line of generics, which specialises to the original point and to the "copies" of it in the isomorphic fibres at other non-zero values of z. The dotted grey parabola and vertical line show the closure of a point E_{z^2+1} in the line of generics to a single point from each fibre away from zero, and the line $p = 0^2 + 1$ in $\mathrm{InjSpec}(H_0) \cong \mathrm{Spec}(k[p, q])$; both the generic point of this line ($E_{z^2+1,0}$) and the closed point at $q = \beta$ ($E_{z^2+1,0,\beta}$) are shown.

6.2 The Quantum Plane

Having seen examples in Chapter 4 and Section 6.1 of good behaviour from the injective spectrum, we now illustrate that the theory is not always so well-behaved. The following example is of a ring with Krull dimension 2 whose injective spectrum is not noetherian and where there is a closed point of critical dimension 1, and a critical module M whose injective hull is not generic in (M).

Let k be an algebraically closed field and $q \in k$ a non-zero element that is not a root of unity. The quantum plane A_q is the k-algebra

$$A_q := k\langle x, y \mid xy = qyx\rangle,$$

which is the skew polynomial ring over $k[x]$ with new variable y, endomorphism defined by $x \mapsto q^{-1}x$, and derivation 0. Note that A_q is therefore noetherian, and has Krull dimension 2 (see [15], §6.9).

For any $\lambda \in k$, there is a simple A_q-module which we denote k_λ. This is a 1 dimensional k-vector space, where x acts as scaling via λ and y via 0. Each of these simple modules for different values of λ is distinct; for two distinct eigenvalues of x cannot occur in a one-dimensional space.

For any $\lambda \neq 0$, we will give an example of a finitely presented, 1-critical module M_λ such that $E(k_\lambda) \in (M_\lambda)$, but $E(M_\lambda)$ does not specialise to $E(k_\lambda)$. This is a counterexample to the thought that for any finitely presented and critical module M, $E(M)$ should be generic in (M) (cf. remarks after Corollary 3.2.8). Moreover, $E(M_\lambda)$ will be a closed point, despite not containing a simple submodule (cf. Propositions 2.2.1 and 2.2.2).

To construct M_λ, take a k-vector space with basis $\{v_i \mid i < \omega\}$ and define the action of A_q by $v_i x = q^{-i}\lambda v_i$ and $v_i y = v_{i+1}$. Then $v_i yx = v_{i+1}x = q^{-i-1}\lambda v_{i+1}$, and $v_i xy = q^{-i}\lambda v_i y = q^{-i}\lambda v_{i+1} = q(v_i yx)$, so the commutation relation of A_q is satisfied and this is indeed an A_q-module.

We see that v_0 is a generator for M_λ as an A_q-module, so it is indeed finitely presented. Moreover, we will show that it is uniserial, with submodules $M_\lambda^{(n)} := v_n A_q \cong M_{\lambda q^{-n}}$ for each $n < \omega$. To see this, suppose that N is a submodule, containing some non-zero element

$$v = \sum_{i<\omega} \alpha_i v_i$$

for some $\alpha_i \in k$ almost all zero. For convenience, write $v = [\alpha_i]$.

Now $vx = [\alpha_i \lambda q^{-i}]$, so for any n we have $v(x - q^{-n}\lambda) = [\alpha_i \lambda(q^{-i} - q^{-n})]$. But, since $\lambda \neq 0$ and q is not a root of unity, $\alpha_i \lambda(q^{-i} - q^{-n})$ is zero if and only if $\alpha_i = 0$ or $i = n$. So, taking n such that $\alpha_n \neq 0$, N contains an element $v(x - q^{-n}\lambda)$ whose representation in terms of the basis involves precisely one vector fewer than v did. Repeating this process, we can eliminate all but one term from v to obtain a non-zero element in N which is a multiple of a single basis vector v_n for some n. Moreover, we may do this for any n such that v_n is involved in a non-zero element of N; in particular, for the least such n.

So if n is the least index such that v_n is involved in an element of N, then $v_n \in N$. But then, by repeatedly applying y, we see that $v_i \in N$ for all $i \geq n$. So N is precisely the k-linear span of $\{v_i \mid i \geq n\}$, or equivalently the cyclic submodule generated by v_n. Denote this cyclic submodule by $M_\lambda^{(n)}$. So the non-zero submodules of M_λ are precisely the $M_\lambda^{(n)}$ for $n < \omega$.

Now, the $M_\lambda^{(n)}$ form an infinite descending chain of submodules of M_λ, so $K(M_\lambda) \geq 1$. Moreover, any quotient of M_λ is $M_\lambda / M_\lambda^{(n)}$ for some n, which has k-dimension n, so is artinian. Therefore M_λ is 1-critical.

The quotient $M_\lambda / M_\lambda^{(1)}$ is spanned by $\bar{v_0} = v_0 + M_\lambda^{(1)}$, with $\bar{v_0} x = \lambda x$ and $\bar{v_0} y = 0$, so this quotient is the simple module k_λ. So certainly $E(k_\lambda) \in (M_\lambda)$.

We will now show that $E(M_\lambda)$ does not specialise to $E(k_\lambda)$. We know that for $E, F \in \operatorname{InjSpec}(A_q)$, $E \rightsquigarrow F$ if and only if E is contained in the torsionfree class cogenerated by F, by Lemma 2.1.1. So it suffices to prove that $M_\lambda^{(1)} < E(M_\lambda)$ is torsion for the torsion theory cogenerated by $E(k_\lambda)$; *i.e.*, that $(M_\lambda^{(1)}, E(k_\lambda)) = 0$.

Any proper, non-zero quotient of $M_\lambda^{(1)}$ has the form $M_\lambda^{(1)} / M_\lambda^{(n+1)}$ for some $n \geq 1$. But this contains as a submodule $M_\lambda^{(n)} / M_\lambda^{(n+1)} \cong k_{\lambda q^{-n}}$; so if $f : M_\lambda^{(1)} \to E(k_\lambda)$ is non-zero, then either it is injective or its image contains the simple module $k_{\lambda q^{-n}}$ for some $n \geq 1$. But $E(k_\lambda)$ has k_λ as an essential, simple submodule, so if $k_{\lambda q^{-n}} \in E(k_\lambda)$, then $k_{\lambda q^{-n}} \cong k_\lambda$, but this can only occur for $\lambda q^{-n} = \lambda$, contrary to the assumptions that λ be non-zero and q be not a root of unity.

So the only possibility for a non-zero map $M_\lambda^{(1)} \to E(k_\lambda)$ is an embedding. But then $M_\lambda^{(1)}$ must contain k_λ as a submodule, whereas all submodules of $M_\lambda^{(1)}$ have the form $M_\lambda^{(n)}$ and so are infinite-dimensional.

Therefore we see that $(M_\lambda^{(1)}, E(k_\lambda)) = 0$ and so $M_\lambda^{(1)}$ is $\mathcal{F}(k_\lambda)$-torsion, and therefore $E(M_\lambda)$ is not $\mathcal{F}(k_\lambda)$-torsionfree, so does not specialise to $E(k_\lambda)$.

Furthermore, the point $E(M_\lambda)$, although of critical dimension 1, is a closed point, giving an example of a closed point which does not contain a simple submodule and thereby establishing that the converse to Lemma 2.2.1 is false, in general. For if $E(M_\lambda) \rightsquigarrow E \neq E(M_\lambda)$, then $\mathrm{cd}(E) < \mathrm{cd}(E(M_\lambda))$, so $\mathrm{cd}(E) = 0$, and hence $E = E(S)$ for some simple module S. But then $E(M_\lambda)$ is torsionfree for $\mathcal{F}(E(S))$, so $(M_\lambda, E(S)) \neq 0$, and so S is a subquotient of M_λ. But the only simple subquotients of M_λ are $k_{\lambda q^{-n}}$ for $n < \omega$, and a straightforward adaptation of the above argument shows that $M_\lambda^{(n+1)}$ is torsion for $\mathcal{F}(E(k_{\lambda q^{-n}}))$, so $E(M_\lambda) = E\left(M_\lambda^{(n+1)}\right)$ cannot specialise to $E(k_{\lambda q^{-n}})$. So $E(M_\lambda)$ specialises to nothing (except of course itself).

Proposition 2.2.2 tells us that if $\mathrm{InjSpec}(R)$ is noetherian, then every closed point is the hull of a simple module; so this shows that $\mathrm{InjSpec}(A_q)$ cannot be noetherian. Indeed, the basic closed sets $\left(M_\lambda^{(n)}\right)$ form an infinite, strictly descending chain of closed sets, whose intersection is $\{E(M_\lambda)\}$.

If (M_λ) were irreducible, the only possibility for a generic point, by dimension considerations, would be $E(M_\lambda)$, which we have shown is not generic in (M_λ). Therefore, by Theorem 5.3.13, (M_λ) cannot be irreducible. Indeed, for any n, we have the decomposition into proper closed subsets

$$(M_\lambda) = \left(M_\lambda^{(n)}\right) \cup \{E(k_{\lambda q^{-i}}) \mid 0 \leq i < n\}.$$

Chapter 7

Sheaves

7.1 The Structure Sheaf

Having considered the injective spectrum purely as a topological space thus far, we now turn to the sheaf of definable scalars/finite type localisations. A key property of the Zariski spectrum of a commutative ring is that the ring of global sections of the structure sheaf is simply the original ring. We begin with an example to show that this can fail for the injective spectrum.

Example 7.1.1. *Let k be a field and $R = kA_2$ be the path algebra over k of the quiver $A_2 : 1 \to 2$; then the ring of global sections of the sheaf of definable scalars is $k \oplus M_2(k)$, not R.*

By standard results on quiver representations, R has exactly two indecomposable injectives, namely the representations $(k \to 0)$ and $(k \to k)$; the topology on the injective spectrum is discrete, since both these points are artinian. The ring of global sections is therefore simply the direct sum of the two stalks.

For the stalk at $(k \to 0)$, we take the torsionfree class cogenerated by $(k \to 0)$ and localise $R_R = (k \to k) \oplus (0 \to k)$; obtaining $(k \to 0)$. We then take the endomorphism ring of this, which is simply k. For the stalk at $(k \to k)$, we localise R_R at the torsionfree class cogenerated by $(k \to k)$, obtaining $(k \to k)^2$, which has endomorphism ring $M_2(k)$, the 2×2 matrix ring.

So the ring of global sections over $\mathrm{InjSpec}(kA_2)$ is $k \oplus M_2(k)$, which is not isomorphic - or even Morita equivalent - to kA_2. ■

Having shown that, even for a very mild-mannered ring, the structure sheaf on the injective spectrum can fail to fulfill our expectations from the commutative case, we now show that it nonetheless often does.

Theorem 7.1.2. *Let R be a right uniform, right noetherian ring whose injective spectrum X is irreducible as a topological space. Then the ring of global sections of the structure sheaf of X is precisely R.*

PROOF:

Since X is irreducible and basic closed (since $X = (R_R)$), it has a generic point, by Theorem 5.3.13, which must have strictly greater critical dimension than any other indecomposable injective, and so must be $E(R_R)$. So $E(R_R)$ and hence R_R are torsionfree for every non-trivial torsion theory. Therefore the localisation of R_R at any torsion theory is the largest submodule of $E(R_R)$ which becomes torsion modulo R_R, by Proposition 1.3.2. So the presheaf of localisations of R_R over X associates to each open set a submodule of $E(R_R)$ (with the structure of a ring via its endomorphisms), and the restriction maps are simply inclusions into ever larger submodules of $E(R_R)$.

So for any global section σ of the structure sheaf, and any point E, there is an open set $U_E \ni E$ and an element $e_E \in E(R_R)$ such that $\sigma(F) = e_E$ for all $F \in U_E$. Given any two points E, F, $U_E \cap U_F \neq \varnothing$, since X is irreducible, so there is $G \in U_E \cap U_F$, and $\sigma(E) = \sigma(G) = \sigma(F)$. So in fact there is a single element $e \in E(R_R)$ such that $\sigma(E) = e$ for all $E \in X$.

It remains to show that $e \in R_R$. For any $E \in X$, the submodule of $E(R_R)/R_R$ generated by $e + R_R$ must be $\mathcal{F}(E)$-torsion, since the localisation of R_R is the submodules of $E(R_R)$ becoming torsion modulo R_R, by Proposition 1.3.2. So $((e + R_R)R, E) = 0$. Let $I = \mathrm{ann}_R(e + R_R)$, so $(e + R_R)R \cong R/I$. If $e \notin R$, then $I \neq R$, so R/I is non-zero; but then R/I has a simple quotient S, so $(R/I, E(S)) \neq 0$, a contradiction. So indeed $e \in R$. ■

Note that, by Theorem 3.3.3, any right noetherian domain satisfies the hypotheses of this Theorem. In particular, if R is any Weyl algebra or Heisenberg algebra, over any field, or indeed the quantum plane, then the ring of global sections over $\mathrm{InjSpec}(R)$ is R itself.

We now extend our partial functoriality result, Corollary 2.3.4, to consider the structure sheaf.

Theorem 7.1.3. *Let R and S be right noetherian rings and $f : R \to S$ an epimorphism such that ${}_RS$ is flat. Then the continuous map $f^* : \mathrm{InjSpec}(S) \to \mathrm{InjSpec}(R)$ induced by f is a morphism of ringed spaces.*

PROOF:

We work with the model-theoretic viewpoint of describing the topology via pp-pairs and the structure sheaves via definable scalars. We must prove that for any basic open set $[\phi/\psi]$ there is an induced ring map $R_{[\phi/\psi]} \to S_{(f^*)^{-1}[\phi/\psi]}$ which coheres with the restriction maps.

First we describe $(f^*)^{-1}[\phi/\psi]$. For any pp formula ξ over R, write ξ_f for the pp formula over S obtained by applying f to each element of R appearing in ξ. That is, if ξ is the formula

$$\exists x_{n+1}, \ldots, x_m \bigwedge_{i=1}^{k} \left(\sum_{j=1}^{m} x_j r_{ij} = 0 \right),$$

then ξ_f is the formula

$$\exists x_{n+1}, \ldots, x_m \bigwedge_{i=1}^{k} \left(\sum_{j=1}^{m} x_j f(r_{ij}) = 0 \right).$$

Then, for any S-module M and pp-pair for R-modules ϕ/ψ, it is clear that $\phi(M_R) = \psi(M_R)$ if and only if $\phi_f(M) = \psi_f(M)$. Therefore $(f^*)^{-1}[\phi/\psi] = [\phi_f/\psi_f]$.

Write T_{inj_R} for the theory of injective right R-modules and T_{inj_S} for the theory of injective right S-modules. Suppose ρ is a definable scalar for the theory $T_{\mathrm{inj}_R,[\phi/\psi]}$. That means on any injective R-module E closing ϕ/ψ, ρ defines a function. Given an injective S-module F which closes ϕ_f/ψ_f, $f^*(F) = F_R$ closes ϕ/ψ and is injective, so ρ defines a function on F_R. Therefore ρ_f defines a function on F. So ρ_f is a definable scalar for $T_{\mathrm{inj}_S,[\phi_f/\psi_f]}$.

Suppose that ρ and σ are pp formulae which are equivalent modulo $T_{\mathrm{inj}_R,[\phi/\psi]}$. Then for every injective R-module E closing ϕ/ψ, $\rho(E) = \sigma(E)$. In particular, for $E = F_R$, with F an injective S-module closing ϕ_f/ψ_f; so we see that $\rho \mapsto \rho_f$ gives a well-defined function $R_{[\phi/\psi]} \to S_{[\phi_f/\psi_f]}$. By the definition of addition and multiplication of definable scalars, it is clear that this function is a map of rings.

So we have the desired ring maps, and need only check coherence with the restriction maps. If $[\xi/\eta] \subseteq [\phi/\psi]$, and ρ is a definable scalar modulo $[\phi/\psi]$, then we can restrict ρ to $[\xi/\eta]$ to obtain a class of formulae represented by ρ, and then apply f to obtain a class in $S_{[\xi_f/\eta_f]}$ represented by ρ_f, or we can apply f first to obtain a class in $S_{[\phi_f/\psi_f]}$ and then restrict modulo $[\xi_f/\eta_f]$. But in either case we obtain an equivalence class of formulae represented by ρ_f, so these two processes must give the same element of $S_{[\xi_f/\eta_f]}$. So the maps of rings $R_{[\phi/\psi]} \to S_{[\phi_f/\psi_f]}$ for each pp-pair ϕ/ψ cohere with the restriction maps, and hence give a morphism of presheaves, which, upon sheafifying, gives a morphism between the associated sheaves. ■

7.2 Sheaves of Modules

Let $\mathcal{O}_R$ denote the sheaf of finite-type localisations/definable scalars on $\mathrm{InjSpec}(R)$. We now consider sheaves of $\mathcal{O}_R$-modules; let $\mathrm{Sh}(R)$ denote the category of all such sheaves, with morphisms of sheaves as arrows.

We begin by describing two functors $\text{Mod-}R \to \mathrm{Sh}(R)$. The first we call the **tensor sheaf functor**; for $M \in \text{Mod-}R$, we write $\mathcal{M}_\otimes$ for the tensor sheaf of M, and for $f : M \to N$ a map of R-modules, we write $f_\otimes : \mathcal{M}_\otimes \to \mathcal{N}_\otimes$ for the induced map.

The tensor sheaf functor is defined as follows: given an R-module M and an open set U in $\mathrm{InjSpec}(R)$, we form $M \otimes_R \mathcal{O}_R(U)$. Given $U \subseteq V$ an inclusion of open sets in $\mathrm{InjSpec}(R)$, the restriction map $\mathcal{O}_R(V) \to \mathcal{O}_R(U)$ induces a restriction map $M \otimes_R \mathcal{O}_R(V) \to M \otimes_R \mathcal{O}_R(U)$. This gives a presheaf of $\mathcal{O}_R$-modules, whose sheafification we define to be $\mathcal{M}_\otimes$, the tensor sheaf of M.

Given $f : M \to N$ in $\text{Mod-}R$, we have for each open set U an induced map $f \otimes_R \mathcal{O}_R(U) : M \otimes_R \mathcal{O}_R(U) \to N \otimes_R \mathcal{O}_R(U)$. Since this acts on the first factor of the tensor product and the restriction maps act on the second factor, these two maps commute, and so we have a morphism of presheaves. Sheafification then gives a morphism of sheaves $f_\otimes : \mathcal{M}_\otimes \to \mathcal{N}_\otimes$.

The fact that this tensor sheaf construction is functorial is trivial to verify.

It will be useful at times to reach this functor by a slightly different route. Since $\mathcal{O}_R$

is the sheafification of the presheaf-on-a-basis $[A] \mapsto R_{\mathcal{T}(A)}$, we can form a presheaf-on-a-basis of modules $[A] \mapsto M \otimes_R R_{\mathcal{T}(A)}$ for any $M \in \text{Mod-}R$, and then sheafify this and extend to the whole topology to obtain a sheaf of $\mathcal{O}_R$-modules.

To see that these two constructions of the sheaf $\mathcal{M}_\otimes$ are the same, we consider the presheaf-on-a-basis $[A] \mapsto M \otimes_R R_{\mathcal{T}(A)}$ and the sheaf-on-a-basis $[A] \mapsto M \otimes_R \mathcal{O}_R([A])$. Since $\mathcal{O}_R$ is the sheafification of $[A] \mapsto R_{\mathcal{T}(A)}$, there is a natural map from $[A] \mapsto R_{\mathcal{T}(A)}$ to $\mathcal{O}_R$, which becomes an isomorphism when sheafified. Tensoring with M gives a natural map from the presheaf $[A] \mapsto M \otimes_R R_{\mathcal{T}(A)}$ to the sheaf $[A] \mapsto M \otimes_R \mathcal{O}_R$, which becomes an isomorphism when sheafified. Therefore the sheaf-on-a-basis version of these two sheaves associated to M are canonically isomorphic, and hence so too are the full sheaves.

Our second functor $\text{Mod-}R \to \text{Sh}(R)$ we call the **torsion sheaf functor**; for $M \in \text{Mod-}R$, we write $\mathcal{M}_{\text{tors}}$ for the torsion sheaf of M, and for $f : M \to N$ a map of R-modules, we write $f_{\text{tors}} : \mathcal{M}_{\text{tors}} \to \mathcal{N}_{\text{tors}}$ for the induced map.

To define the torsion sheaf functor, we first consider torsion-theoretic localisation of modules. Recall from the remarks preceding Theorem 1.4.6 the construction of the sheaf of finite type localisations $\mathcal{O}_R$. Given a torsion theory $\mathcal{T}$ in $\text{Mod-}R$ (particularly one of the form $\mathcal{T}(M)$ for $M \in \text{mod-}R$), we can take a ring $R_\mathcal{T} = (Q_\mathcal{T} R, Q_\mathcal{T} R)$, the endomorphism ring of the image of R_R in the quotient category. There is a natural isomorphism of abelian groups $R_\mathcal{T} \cong i_\mathcal{T} Q_\mathcal{T} R$, and a canonical ring map $R \to R_\mathcal{T}$. The sheaf of definable scalars was obtained by sheafifying the presheaf-on-a-basis $[A] \mapsto R_{\mathcal{T}(A)}$.

We will now mirror this construction with modules to obtain an $R_\mathcal{T}$-module structure on $M_\mathcal{T} := (Q_\mathcal{T} R, Q_\mathcal{T} M)$ for any $M \in \text{Mod-}R$. This will gives us a presheaf-on-a-basis of modules over the presheaf-on-a-basis of rings of finite type localisations, which is enough information for sheafification to give us a sheaf of modules over $\mathcal{O}_R$.

Lemma 7.2.1. *Let $\mathcal{T}$ be a torsion class in* $\text{Mod-}R$*. Then there is a functor $(-)_\mathcal{T}$:* $\text{Mod-}R \to \text{Mod-}R_\mathcal{T}$*. Moreover, if $\mathcal{S} \subseteq \mathcal{T}$ is an inclusion of torsion classes, then there are restriction maps of abelian groups* $\text{res}^M_{\mathcal{S},\mathcal{T}} : M_\mathcal{S} \to M_\mathcal{T}$ *such that* $\text{res}^R_{\mathcal{S},\mathcal{T}}$ *is a ring map and each* $\text{res}^M_{\mathcal{S},\mathcal{T}}$ *is $R_\mathcal{S}$-linear when $M_\mathcal{T}$ has its $R_\mathcal{S}$ structure given by the ring map* $\text{res}^R_{\mathcal{S},\mathcal{T}}$.

Proof:

We set $M_{\mathcal{T}} = (Q_{\mathcal{T}}R, Q_{\mathcal{T}}M)$. Since $R_{\mathcal{T}} = (Q_{\mathcal{T}}R, Q_{\mathcal{T}}R)$, we have a pairing $M_{\mathcal{T}} \times R_{\mathcal{T}} \to M_{\mathcal{T}} : (\mu, \rho) \to \mu \circ \rho$. This satisfies the axioms to make $M_{\mathcal{T}}$ into an $R_{\mathcal{T}}$-module by preadditivity of $(\text{Mod-}R)/\mathcal{T}$.

Given $f : M \to N$ in Mod-R, define $f_{\mathcal{T}} : M_{\mathcal{T}} \to N_{\mathcal{T}} : \mu \mapsto (Q_{\mathcal{T}}f) \circ \mu$. It is trivial from functoriality of $Q_{\mathcal{T}}$ and preadditivity to verify that this defines an additive functor Mod-$R \to$ Mod-$R_{\mathcal{T}}$.

Now let $\mathcal{S} \subseteq \mathcal{T}$ be an inclusion of torsion classes. By the universal property of torsion-theoretic localisation (Proposition 1.3.2), there is a unique exact functor $Q_{\mathcal{S},\mathcal{T}} : (\text{Mod-}R)/\mathcal{S} \to (\text{Mod-}R)/\mathcal{T}$ such that $Q_{\mathcal{T}} = Q_{\mathcal{S},\mathcal{T}} \circ Q_{\mathcal{S}}$. Therefore $Q_{\mathcal{S},\mathcal{T}}$ induces maps $(Q_{\mathcal{S}}R, Q_{\mathcal{S}}R) \to (Q_{\mathcal{T}}R, Q_{\mathcal{T}}R)$ and $(Q_{\mathcal{S}}R, Q_{\mathcal{S}}M) \to (Q_{\mathcal{T}}R, Q_{\mathcal{T}}M)$; *i.e.*, $R_{\mathcal{S}} \to R_{\mathcal{T}}$ and $M_{\mathcal{S}} \to M_{\mathcal{T}}$. These are our restriction maps. Again, the linearity properties follow easily from properties of the functors. ■

Now, given an R-module M, we define a presheaf-on-a-basis by assigning to the basic open set $[A]$ the $R_{\mathcal{T}(A)}$-module $M_{\mathcal{T}(A)}$. By the above Lemma, this is indeed a presheaf. Its sheafification is the tensor sheaf $\mathcal{M}_{\text{tors}}$ associated to M.

Given a map $f : M \to N$ in Mod-R, we construct a morphism of presheaves between the presheaves-on-a-basis associated to M and N. Sheafification then turns this into a morphism of sheaves. For each basic open set $[A]$, we have $f_{\mathcal{T}(A)} : M_{\mathcal{T}(A)} \to N_{\mathcal{T}(A)}$ as in the Lemma. We need to check that these cohere with the restriction maps; *i.e.*, that if $[A] \subseteq [B]$ (so $\mathcal{T}(B) \subseteq \mathcal{T}(A)$), we have a commuting diagram

$$\begin{tikzcd}
M_{\mathcal{T}(B)} \arrow[r, "f_{\mathcal{T}(B)}"] \arrow[d, "\mathrm{res}^M_{\mathcal{T}(B),\mathcal{T}(A)}"'] & N_{\mathcal{T}(B)} \arrow[d, "\mathrm{res}^N_{\mathcal{T}(B),\mathcal{T}(A)}"] \\
M_{\mathcal{T}(A)} \arrow[r, "f_{\mathcal{T}(A)}"'] & N_{\mathcal{T}(A)}
\end{tikzcd}$$

To show that this diagram does indeed commute, take $\mu \in M_{\mathcal{T}(B)}$. Following the diagram anticlockwise, μ maps first to $Q_{\mathcal{T}(B),\mathcal{T}(A)}(\mu) \in M_{\mathcal{T}(A)}$, then to $(Q_{\mathcal{T}(A)}f) \circ (Q_{\mathcal{T}(B),\mathcal{T}(A)}\mu) \in N_{\mathcal{T}(A)}$. Following clockwise, μ maps first to $(Q_{\mathcal{T}(B)}f) \circ \mu \in N_{\mathcal{T}(B)}$ and then to $Q_{\mathcal{T}(B),\mathcal{T}(A)}((Q_{\mathcal{T}(B)}f) \circ \mu) = (Q_{\mathcal{T}(A)}f) \circ (Q_{\mathcal{T}(B),\mathcal{T}(A)}\mu)$, the same as when going anticlockwise.

So we have a functor from Mod-R to presheaves-on-a-basis of modules. Sheafifying then gives the desired torsion sheaf functor Mod-$R \to \mathrm{Sh}(R) : M \mapsto \mathcal{M}_{\mathrm{tors}}$.

So we have two functors Mod-$R \to \mathrm{Sh}(R)$, the torsion sheaf functor and the tensor sheaf functor. We now consider the relationship between them.

Lemma 7.2.2. *For any M in* Mod-R *and $\mathcal{T}$ a torsion class in* Mod-R*, there is a morphism of $R_{\mathcal{T}}$-modules $\theta_{M,\mathcal{T}} : M \otimes_R R_{\mathcal{T}} \to M_{\mathcal{T}}$. If $\mathcal{S} \subseteq \mathcal{T}$, then there is a commuting diagram*

$$\begin{tikzcd}
M \otimes_R R_{\mathcal{S}} \arrow[r, "\theta_{M,\mathcal{S}}"] \arrow[d, "M \otimes_R \mathrm{res}^R_{\mathcal{S},\mathcal{T}}"'] & M_{\mathcal{S}} \arrow[d, "\mathrm{res}^M_{\mathcal{S},\mathcal{T}}"] \\
M \otimes_R R_{\mathcal{T}} \arrow[r, "\theta_{M,\mathcal{T}}"'] & M_{\mathcal{T}}
\end{tikzcd}$$

Therefore θ_M gives a morphism of presheaves-on-a-basis from the presheaf underlying $\mathcal{M}_{\otimes}$ to the presheaf underlying $\mathcal{M}_{\mathrm{tors}}$.

Proof:

We have a Yoneda isomorphism of R-modules $y_M : M \to (R_R, M)$ given by $y_M(m) : r \mapsto mr$. Also, $Q_{\mathcal{T}}$ gives a morphism of abelian groups $(R_R, M) \to (Q_{\mathcal{T}}R, Q_{\mathcal{T}}M) = M_{\mathcal{T}}$. So define $\theta_{M,\mathcal{T}}(m \otimes \rho) = (Q_{\mathcal{T}}(y_M(m))) \circ \rho : Q_{\mathcal{T}}R \to Q_{\mathcal{T}}M$.

If $\mathcal{S} \subseteq \mathcal{T}$, take $m \otimes \rho$ in $M \otimes_R R_{\mathcal{S}}$. Following the diagram anticlockwise, $m \otimes \rho$ maps first to $m \otimes Q_{\mathcal{S},\mathcal{T}}(\rho) \in M \otimes_R R_{\mathcal{T}}$, and then to $(Q_{\mathcal{T}}(y_M(m))) \circ Q_{\mathcal{S},\mathcal{T}}(\rho) \in M_{\mathcal{T}}$. Going clockwise, $m \otimes \rho$ maps first to $(Q_{\mathcal{S}}(y_M(m))) \circ \rho \in M_{\mathcal{S}}$ and then to $Q_{\mathcal{S},\mathcal{T}}(Q_{\mathcal{S}}(y_M(m)) \circ \rho) = (Q_{\mathcal{T}}(y_M(m))) \circ Q_{\mathcal{S},\mathcal{T}}(\rho)$, as before. ■

Sheafifying, we therefore obtain a morphism of sheaves $\Theta_M : \mathcal{M}_{\otimes} \to \mathcal{M}_{\mathrm{tors}}$. This of course raises the question of what happens when we change modules along a map $f : M \to N$.

Proposition 7.2.3. *There is a natural transformation Θ from the tensor sheaf functor to the torsion sheaf functor, whose component at a module M is Θ_M.*

Proof:

We must show that for any morphism $f : M \to N$, the following diagram commutes

$$\begin{tikzcd}
\mathcal{M}_{\otimes} \arrow[r, "f_{\otimes}"] \arrow[d, "\Theta_M"'] & \mathcal{N}_{\otimes} \arrow[d, "\Theta_N"] \\
\mathcal{M}_{\mathrm{tors}} \arrow[r, "f_{\mathrm{tors}}"'] & \mathcal{N}_{\mathrm{tors}}
\end{tikzcd}$$

To do this, we show that for any basic open set $[A]$, the following diagram commutes

$$\begin{tikzcd}
M \otimes_R R_{\mathcal{T}(A)} \arrow[r, "f \otimes_R R_{\mathcal{T}(A)}"] \arrow[d, "\theta_{M,\mathcal{T}(A)}"'] & N \otimes_R R_{\mathcal{T}(A)} \arrow[d, "\theta_{N,\mathcal{T}(A)}"] \\
M_{\mathcal{T}(A)} \arrow[r, "f_{\mathcal{T}(A)}"'] & N_{\mathcal{T}(A)}
\end{tikzcd}$$

Commutativity of this diagram establishes commutativity of the relevant diagram of presheaves-on-a-basis; as sheafification is functorial, it preserves commutativity of diagrams, and hence we obtain commutativity of the desired diagram of sheaves.

So to establish commutativity in our second diagram, we take $m \otimes \rho \in M \otimes_R R_{\mathcal{T}(A)}$. Following the diagram anticlockwise, we obtain first $(Q_{\mathcal{T}(A)} y_M(m)) \circ \rho \in M_{\mathcal{T}(A)}$, and then $(Q_{\mathcal{T}(A)} f) \circ (Q_{\mathcal{T}(A)} y_M(m)) \circ \rho \in N_{\mathcal{T}(A)}$. Going clockwise, $m \otimes \rho$ maps first to $f(m) \otimes \rho \in N \otimes_R R_{\mathcal{T}(A)}$, and then $(Q_{\mathcal{T}(A)} y_N(f(m))) \circ \rho \in N_{\mathcal{T}(A)}$. So we must show that $(Q_{\mathcal{T}(A)} f) \circ (Q_{\mathcal{T}(A)} y_M(m)) = (Q_{\mathcal{T}(A)} y_N(f(m)))$.

Since $Q_{\mathcal{T}(A)}$ is functorial, $(Q_{\mathcal{T}(A)} f) \circ (Q_{\mathcal{T}(A)} y_M(m)) = Q_{\mathcal{T}(A)}(f \circ y_M(m))$; so it suffices to prove that $f \circ y_M(m) = y_N(f(m))$. But this is precisely naturality of the Yoneda maps, completing the proof. ■

Corollary 7.2.4. *For any torsion class $\mathcal{T} \in$ Mod-R, there is a natural transformation $\theta_{\mathcal{T}} : - \otimes_R R_{\mathcal{T}} \to (-)_{\mathcal{T}}$.*

Proof:

The component of $\theta_{\mathcal{T}}$ at the module M is of course $\theta_{M,\mathcal{T}}$. The diagram whose commutativity needs checking is precisely the second diagram in the above proof. ■

So we have two functors turning R-modules into sheaves of $\mathcal{O}_R$-modules, and a natural transformation between them. In the commutative noetherian case, we expect that torsion-theoretic localisation should be the same as localisation at a multiplicative set and hence that these two sheaf functors should coincide, so Θ should be an isomorphism. Indeed, we shall see a proof of this in Corollary 7.2.11.

To address the question of when Θ is an isomorphism, we require the notion of a Gabriel filter. This is an alternative viewpoint on torsion-theoretic localisation, of which we give a brief overview based on Chapter VI of [23].

Let R be a ring and $\mathcal{T}$ a torsion class in Mod-R. Then a module M is $\mathcal{T}$-torsion if and only if every cyclic submodule of M is $\mathcal{T}$-torsion. For, on the one hand, $\mathcal{T}$ is closed under subobjects, so any cyclic submodule of a $\mathcal{T}$-torsion module is $\mathcal{T}$-torsion; on the other hand, if M is an R-module whose every cyclic submodule is $\mathcal{T}$-torsion, then M can be expressed as a quotient of the direct sum of all its cyclic submodules, and so M is $\mathcal{T}$-torsion.

Any cyclic module has the form R/I for some right ideal I, so $\mathcal{T}$ is entirely determined by the set of right ideals I such that R/I is $\mathcal{T}$-torsion. We shall denote this set by $\mathcal{G}_{\mathcal{T}}$ and call it the **Gabriel filter** associated to $\mathcal{T}$ (this terminology will be explained shortly). Recall that, for I a right ideal and $r \in R$, $(I : r)$ denotes the right ideal $\{x \in R \mid rx \in I\}$; *i.e.*, the annihilator of $r + I$ in the quotient module R/I.

Lemma 7.2.5 ([23], SSVI.4, VI.5)**.** *Let R be any ring and $\mathcal{T}$ a torsion class in* Mod-R*. Let $\mathcal{G}_{\mathcal{T}}$ be the associated Gabriel filter:*

$$\mathcal{G}_{\mathcal{T}} = \{I \leq R_R \mid R/I \in \mathcal{T}\}.$$

Then $\mathcal{G}_{\mathcal{T}}$ has the following three properties:

1. *$\mathcal{G}_{\mathcal{T}}$ is a filter of right ideals of R - i.e., it is closed under finite intersection and upwards inclusion;*
2. *If $I \in \mathcal{G}_{\mathcal{T}}$ and $r \in R$, then $(I : r) \in \mathcal{G}_{\mathcal{T}}$;*
3. *If I is a right ideal and there is $J \in \mathcal{G}_{\mathcal{T}}$ such that for all $j \in J$, $(I : j) \in \mathcal{G}_{\mathcal{T}}$, then $I \in \mathcal{G}_{\mathcal{T}}$.*

PROOF:

1. Suppose $I, J \in \mathcal{G}_{\mathcal{T}}$ (so $R/I, R/J \in \mathcal{T}$), and K is a right ideal with $I \leq K$. Then R/K is a quotient of R/I, and $R/(I \cap J)$ embeds in $R/I \oplus R/J$, so R/K and

$R/(I \cap J)$ are both $\mathcal{T}$-torsion. Therefore K and $I \cap J$ are both elements of $\mathcal{G}_{\mathcal{T}}$, as required.

2. Suppose $I \in \mathcal{G}_{\mathcal{T}}$ and $r \in R$. Then $R/(I : r)$ is isomorphic to the cyclic submodule of R/I generated by $r + I$, hence is $\mathcal{T}$-torsion, so $(I : r) \in \mathcal{G}_{\mathcal{T}}$.

3. Suppose $I \leq R_R$ and $J \in \mathcal{G}_{\mathcal{T}}$ are such that for all $j \in J$, $(I : j) \in \mathcal{G}_{\mathcal{T}}$. Then for all $j \in J$, $R/(I : j) \in \mathcal{T}$. But $R/(I : j)$ is isomorphic to the submodule of R/I generated by $j + I$; so each cyclic submodule of J/I is $\mathcal{T}$-torsion, and therefore $J/I \in \mathcal{T}$. Now there is a short exact sequence

$$0 \to J/I \to R/I \to R/J \to 0$$

where both outer terms are $\mathcal{T}$-torsion. Since $\mathcal{T}$ is closed under extensions, this implies that $R/I \in \mathcal{T}$, and so $I \in \mathcal{G}_{\mathcal{T}}$. ■

Any collection of right ideals of R satisfying the above 3 properties is called a **Gabriel filter** on R, hence why $\mathcal{G}_{\mathcal{T}}$ is called the Gabriel filter associated to $\mathcal{T}$. Not only can we associate a Gabriel filter to any torsion theory on Mod-R, but we can also associate a torsion theory to any Gabriel filter $\mathcal{G}$ by declaring the cyclic torsion modules to be those of the form R/I where $I \in \mathcal{G}$. We thus have the following:

Theorem 7.2.6 ([23], Theorem VI.5.1)**.** *For any ring R there is a bijective correspondence between torsion theories on* Mod-R *and Gabriel filters on R.*

Theorem 7.2.7 ([23], Proposition XI.3.4)**.** *Let R be any ring and $\mathcal{T}$ a torsion class in* Mod-R. *Then the following are equivalent:*

1. *The functor $(-)_{\mathcal{T}}$ is exact and the functor* $(\text{Mod-}R)/\mathcal{T} \to \text{Mod-}R_{\mathcal{T}}$ *induced by passing $(-)_{\mathcal{T}} =$ to the quotient is an equivalence of categories;*

2. *The right adjoint inclusion* $i_{\mathcal{T}} : (\text{Mod-}R)/\mathcal{T} \to \text{Mod-}R$ *itself has a right adjoint;*

3. *The functor* $(-)_{\mathcal{T}} : \text{Mod-}R \to \text{Mod-}R_{\mathcal{T}} : M \mapsto M_{\mathcal{T}}$ *is exact and preserves coproducts;*

4. *The Gabriel filter $\mathcal{G}_{\mathcal{T}}$ has a filter base of finitely generated right ideals, and $(-)_{\mathcal{T}}$ is exact;*

5. *The natural transformation* $\theta_{\mathcal{T}} : - \otimes_R R_{\mathcal{T}} \to (-)_{\mathcal{T}}$ *is an isomorphism of functors;*

6. *For each* $M \in \text{Mod-}R$*, the kernel of the canonical map* $M \to M \otimes_R R_{\mathcal{T}}$ *is precisely* $\tau_{\mathcal{T}}(M)$*;*

7. *The restriction map* $\text{res}^R_{\mathcal{T}} : R \to R_{\mathcal{T}}$ *is a ring epimorphism making* $R_{\mathcal{T}}$ *into a flat left* R*-module, and* $\mathcal{G}_{\mathcal{T}} = \{I \leq R_R \mid \text{res}^R_{\mathcal{T}}(I)R_{\mathcal{T}} = R_{\mathcal{T}}\}$*.*

We call a torsion class $\mathcal{T}$ satisfying the above equivalent conditions a **perfect** torsion class.

Theorem 7.2.8. *Let* R *be a right noetherian ring. Then the natural transformation* Θ *from the tensor sheaf functor to the torsion sheaf functor is a natural isomorphism if and only if every prime torsion class is perfect, if and only if for every prime torsion class* $\mathcal{T}$ *the functor* $(-)_{\mathcal{T}} : \text{Mod-}R \to \text{Mod-}R_{\mathcal{T}}$ *is exact.*

PROOF:

Since Θ is an isomorphism if and only if its every component Θ_M is an isomorphism, it suffices to consider when $\Theta_M : \mathcal{M}_{\otimes} \to \mathcal{M}_{\text{tors}}$ is an isomorphism of sheaves. A map of sheaves is an isomorphism if and only if the induced maps on stalks are all isomorphisms. The stalks are the localisations at torsionfree classes cogenerated by single indecomposable injectives; *i.e.*, at prime torsion theories. So we see that Θ is an isomorphism if and only if $\theta_{M,\mathcal{T}}$ is an isomorphism for each module M and prime torsion theory $\mathcal{T}$.

But $\theta_{M,\mathcal{T}}$ is precisely the component at M of the natural transformation $\theta_{\mathcal{T}} : - \otimes_R R_{\mathcal{T}} \to (-)_{\mathcal{T}}$; so Θ is an isomorphism if and only if for each prime torsion theory $\mathcal{T}$, $\theta_{\mathcal{T}}$ is an isomorphism. By condition (5) of Theorem 7.2.7, this occurs if and only if each prime torsion class is perfect.

Finally, we apply condition (4) of Theorem 7.2.7. Since R is right noetherian, every Gabriel filter has a filter base of finitely generated right ideals, so we see that Θ is an isomorphism if and only if $(-)_{\mathcal{T}}$ is exact for all prime torsion theories $\mathcal{T}$. ∎

We claimed above that over a commutative noetherian ring, the two sheaf functors are isomorphic along Θ. We are now almost in a position to prove this, by proving that over such a ring all prime torsion classes are perfect; in fact, the stronger result holds that all torsion classes are perfect. First, though, we require some well-known preliminaries about torsion theories over commutative noetherian rings.

Lemma 7.2.9 ([6], Proposition V.5.10). *Let R be a commutative noetherian ring. Then for any torsion theory $(\mathcal{T}, \mathcal{F})$ in* Mod-R *and any prime ideal p, either $R/p \in \mathcal{T}$ or $R/p \in \mathcal{F}$. Moreover, $R/p \in \mathcal{T}$ if and only if $E(R/p) \in \mathcal{T}$.*

Proof:

Suppose $R/p \notin \mathcal{T}$. Then R/p is not torsion, so there is some proper ideal I with $p \subseteq I$ and $R/I \in \mathcal{F}$. Then $(R/p, E(R/I)) \neq 0$, so there is some prime q with $E(R/q)$ a summand of $E(R/I)$ (and hence $E(R/q) \in \mathcal{F}$) and $(R/p, E(R/q)) \neq 0$. By Proposition 2.1.3, therefore, $E(R/p) \rightsquigarrow E(R/q)$ in InjSpec(R). But this is equivalent to $E(R/p) \in \mathcal{F}(E(R/q))$, by Lemma 2.1.1, so $R/p \in \mathcal{F}(E(R/q)) \subseteq \mathcal{F}$.

For the final statement, first note that if $E(R/p) \in \mathcal{T}$, then certainly $R/p \in \mathcal{T}$. Conversely, suppose $R/p \in \mathcal{T}$. Then for any $e \in E(R/p)$, $\mathrm{ann}_R(e) = p^n$ for some n, by [14, Theorem 3.4], or an adaptation of Lemma 6.1.1. Therefore the cyclic submodule eR is isomorphic to R/p^n, which is built from R/p by extensions. Since $\mathcal{T}$ is closed under extensions, $eR \in \mathcal{T}$. Since every cyclic submodule of $E(R/p)$ is in $\mathcal{T}$, so too is $E(R/p)$ itself. ■

If a torsion theory $(\mathcal{T}, \mathcal{F})$ has the property that every indecomposable injective is either in $\mathcal{T}$ or in $\mathcal{F}$, we say that it is a **stable** torsion theory. The above Lemma shows that over a commutative noetherian ring, all torsion theories are stable.

The following result is well-known.

Lemma 7.2.10. *Let R be a commutative noetherian ring and $\mathcal{T}$ a torsion class in* Mod-R*. Then there is a multiplicative set $D \subseteq R$ such that $\mathcal{T} = \mathcal{T}_D$, the torsion class defined by*

$$\mathcal{T}_D = \{M \in \text{Mod-}R \mid \forall m \in M \exists d \in D (md = 0)\}.$$

Proof:

Let $\mathcal{T}$ be a torsion class. Then $\mathcal{T}$ is determined by the Ziegler-closed set $\mathcal{F}_{\mathcal{T}} \cap$ InjSpec(R), by Lemma 5.3.3. By Matlis' bijection (Theorem 1.4.1),

$$\mathcal{F}_{\mathcal{T}} \cap \mathrm{InjSpec}(R) = \{E(R/p) \mid p \in \mathrm{Spec}(R) \wedge R/p \in \mathcal{F}_{\mathcal{T}}\}.$$

Let D be the complement in R of $\bigcup\{p \in \mathrm{Spec}(R) \mid R/p \in \mathcal{F}_{\mathcal{T}}\}$. As the complement of a union of primes, D is a multiplicative set, so we have a torsion theory $\mathcal{T}_D$ associated to D, where $M \in \mathcal{T}_D$ if and only if every element of M is annihilated by some element of D. We shall show that for a prime q, $R/q \in \mathcal{F}_{\mathcal{T}}$ if and only if $R/q \in \mathcal{F}_{\mathcal{T}_D}$.

For any prime q, R/q is $\mathcal{T}_D$-torsionfree if and only if no element of D annihilates any element of R/q, if and only if $D \cap q = 0$, if and only if $q \subseteq \bigcup\{p \in \mathrm{Spec}(R) \mid R/p \in \mathcal{F}_{\mathcal{T}}\}$. Since q is finitely generated, if $q \subseteq \bigcup\{p \in \mathrm{Spec}(R) \mid R/p \in \mathcal{F}_{\mathcal{T}}\}$, then there are finitely many $p_1, \ldots, p_n$ such that $R/p_i \in \mathcal{F}_{\mathcal{T}}$ and $q \subseteq \bigcup_{i=1}^{n} p_i$. By the Prime Avoidance Lemma, q is therefore contained in a single p such that $R/p \in \mathcal{F}_{\mathcal{T}}$.

So if R/q is $\mathcal{T}_D$-torsionfree, then q is contained in some p such that $R/p \in \mathcal{F}_{\mathcal{T}}$. Therefore R/q surjects onto R/p, which is $\mathcal{T}$-torsionfree, so R/q cannot be $\mathcal{T}$-torsion. Therefore, by Lemma 7.2.9, $R/q \in \mathcal{F}_{\mathcal{T}}$, so we have shown that if R/q is $\mathcal{T}_D$-torsionfree, then R/q is $\mathcal{T}$-torsionfree.

Conversely, if R/q is $\mathcal{T}$-torsionfree, then q itself *is* a p such that $R/p \in \mathcal{F}_{\mathcal{T}}$, so R/q is $\mathcal{T}_D$-torsionfree. This proves the claim.

Now we note that, since torsionfree classes are closed under injective hulls and submodules, $\mathcal{F}_{\mathcal{T}}$ and $\mathcal{F}_{\mathcal{T}_D}$ must contain the same indecomposable injectives. But then, by Lemma 5.3.3, they are equal; so, taking torsion classes, $\mathcal{T} = \mathcal{T}_D$. ■

Finally we are able to prove that for R commutative noetherian, the natural transformation Θ is always an isomorphism.

Corollary 7.2.11. *Let R be a commutative noetherian ring. Then every torsion class in* Mod-R *is perfect.*

PROOF:

Let $\mathcal{T}$ be a torsion class in Mod-R. Then $\mathcal{T} = \mathcal{T}_D$ for some multiplicative set D, by Lemma 7.2.10, and the classical localisation at D is an exact, full, and dense functor Mod-$R \to$ Mod-$D^{-1}R$ with kernel exactly $\mathcal{T}$, so is equivalent to the torsion-theoretic localisation functor $Q_{\mathcal{T}}$, by the universal property of localisation (Proposition 1.3.2). More precisely, there is an equivalence of categories $F : (\text{Mod-}R)/\mathcal{T} \to \text{Mod-}D^{-1}R$ such that $F \circ Q_{\mathcal{T}}$ is the classical localisation functor.

This equivalence makes the adjoint inclusion $i_{\mathcal{T}}$ into the restriction of scalars functor Mod-$D^{-1}R \to$ Mod-R, which has a right adjoint, namely the coinduced module functor $(D^{-1}R_R, -)$. So we meet condition (2) of Theorem 7.2.7, and so $\mathcal{T}$ is a perfect torsion class. ∎

So for a commutative noetherian ring, the two sheaves associated to a module coincide, and hence the two functors Mod-$R \to \mathrm{Sh}(R)$ coincide too. Of course, these are simply the usual way of turning a module over a commutative ring into a sheaf over $\mathrm{Spec}(R)$. We now turn to the consideration of noncommutative rings where these two sheaf functors coincide.

Lemma 7.2.12 ([23], Proposition XI.3.3)**.** *Let R be any ring and $\mathcal{G}$ a Gabriel filter on R having a filter base of projective right ideals. Let $\mathcal{T}$ be the torsion class associated to $\mathcal{G}$. Then $(-)_{\mathcal{T}}$: Mod-$R \to$ Mod-$R_{\mathcal{T}}$ is exact.*

Proof:

This result requires more machinery of Gabriel filters than we have developed here, so we give only a sketch proof. A complete proof can be found in [23, Proposition XI.3.3].

First we show that $(-)_{\mathcal{T}}$ is left exact without any conditions on $\mathcal{G}$. If A in Mod-R, then $A_{\mathcal{T}} = (Q_{\mathcal{T}}R, Q_{\mathcal{T}}A)$; since $Q_{\mathcal{T}}$ is exact and $(Q_{\mathcal{T}}R, -)$ is left exact, $(-)_{\mathcal{T}}$ is left exact.

Now we prove right exactness, under the hypothesis that $\mathcal{G}$ have a filter base of projectives. Let $f : A \to B$ be an epimorphism of R-modules. Let $\bar{A}, \bar{B}$ denote $A/\tau_{\mathcal{T}}(A)$ and $B/\tau_{\mathcal{T}}(B)$ respectively. Since quotients of torsion objects are torsion, $f(\tau_{\mathcal{T}}(A)) \subseteq \tau_{\mathcal{T}}(B)$, so f induces a map $\bar{f} : \bar{A} \to \bar{B}$. Since f is onto, so too is $\bar{f}$.

Now we use that

$$(Q_{\mathcal{T}}R, Q_{\mathcal{T}}A) = \varinjlim_{I \in \mathcal{G}} (I, \bar{A})$$

and similarly for $(Q_{\mathcal{T}}R, Q_{\mathcal{T}}B)$. This holds by [23, Lemma IX.1.6] and the remarks thereafter.

So take $\beta \in B_{\mathcal{T}} = (Q_{\mathcal{T}}R, Q_{\mathcal{T}}B)$; we shall show that β factors through $f_{\mathcal{T}}$. There exists some $I \in \mathcal{G}$ such that β can be represented by some map $b : I \to \bar{B}$. Since $\mathcal{G}$ has a filter base of projectives, we can assume without loss of generality that I is projective. Then b lifts along the surjection $\bar{f} : \bar{A} \to \bar{B}$, giving a map $a : I \to \bar{A}$ such that $b = f \circ a$.

Now let $\alpha : Q_{\mathcal{T}}R \to Q_{\mathcal{T}}A$ be the map represented by a. Passing to the directed colimit, we obtain $\beta = (Q_{\mathcal{T}}f \circ \alpha$. So any element β of $B_{\mathcal{T}}$ factors through $Q_{\mathcal{T}}f$, proving $f_{\mathcal{T}} = (Q_{\mathcal{T}}f) \circ -$ is onto. ■

Corollary 7.2.13 ([23], Corollary XI.3.6)**.** *Let R be a right noetherian, right hereditary ring. Then every torsion class in* Mod-R *is perfect.*

PROOF:

Immediate from the above Lemma with part (4) of Theorem 7.2.7. ■

Therefore, for any right noetherian, right hereditary ring, such as a principal right ideal ring or the first Weyl algebra over a field of characteristic 0, the torsion sheaf functor and the tensor sheaf functor are naturally isomorphic. There is therefore a single sensible notion of the sheaf associated to a module, opening the way to exploration of further analogues with commutative algebraic geometry.

For a general ring, however, the localisations involved in the sheaf of finite-type localisations might fail to be perfect, in which case it is not clear which is the "correct" notion of the sheaf associated to a module. It may of course be that different contexts require considering either tensor sheaves or torsion sheaves.

Recall that, given a ringed space $(X, \mathcal{O}_X)$, a sheaf of $\mathcal{O}_X$-modules $\mathcal{M}$ is **quasicoherent** if it has everywhere a local presentation. That is, if for any point $x \in X$, there is a neighbourhood U, sets I, J, and an exact sequence of sheaves:

$$\mathcal{O}_X^{(I)}|_U \to \mathcal{O}_X^{(J)}|_U \to \mathcal{M} \to 0,$$

where $-|_U$ denotes the restriction of a sheaf on X to a sheaf on U. Write $\mathrm{QCoh}(R)$ for the full subcategory of $\mathrm{Sh}(R)$ consisting of the quasicoherent sheaves on $\mathrm{InjSpec}(R)$.

Lemma 7.2.14. *Let R be a ring such that each prime torsion class is perfect. Then the torsion sheaf functor (equivalently the tensor sheaf functor)* Mod-$R \to \mathrm{Sh}(R)$ *has image in* $\mathrm{QCoh}(R)$.

PROOF:

We will show that for any module M there is in fact a global presentation for $\mathcal{M}_{\mathcal{T}}$. Take a presentation for M as an R-module:

$$R^{(I)} \to R^{(J)} \to M \to 0.$$

Applying the torsion sheaf functor, we obtain a sequence of sheaves

$$\mathcal{O}_R^{(I)} \to \mathcal{O}_R^{(J)} \to \mathcal{M}_{\mathcal{T}} \to 0;$$

we need only show that this sequence is exact. For this it suffices to show exactness on stalks. A stalk is given by localisation at a torsionfree class cogenerated by a single indecomposable injective; *i.e.*, at a prime torsion theory, by Theorem 5.2.1. But, by hypothesis, these torsion theories are perfect, and so by part (3) of Theorem 7.2.7, the localisation is exact. ■

Therefore, for rings over which all prime torsion classes are perfect, we have a functor Mod-$R \to$ QCoh(R). In the commutative case, this is an equivalence of categories. This result can certainly fail in the noncommutative case, as we now show.

Example 7.2.15. *Let $R = kA_2$, the path algebra over a field k of the quiver A_2. Then R is right noetherian and right hereditary, so all torsion classes in* Mod-R *are perfect, by Corollary 7.2.13. However, the tensor sheaf functor is not an equivalence of categories between* Mod-R *and* QCoh(R).

Recall Example 7.1.1, where we showed that the ring of global sections of $\mathcal{O}_R$ was $k \oplus M_2(k)$. We show that QCoh$(R) \cong$ Mod-$(k \oplus M_2(k))$; *i.e.*, that quasicoherent sheaves are equivalent to modules over the ring of global sections; since $k \oplus M_2(k)$ is not Morita equivalent to R, this proves that the tensor sheaf functor cannot be an equivalence.

First observe that, as InjSpec(R) is a 2-point discrete space, all sheaves are quasicoherent. Indeed, take any sheaf $\mathcal{M} \in$ Sh(R) and any point $E \in$ InjSpec(R); then $\{E\}$ is open, and $\mathcal{M}(\{E\})$ is simply an $\mathcal{O}_R(\{E\})$-module, hence has a presentation. So QCoh$(R) =$ Sh(R).

Write $E_1 = (k \to 0)$ and $E_2 = (k \to k)$ for the two indecomposable injective R-modules. Given a $(k \oplus M_2(k))$-module M, which can be naturally written as $M_1 \oplus M_2$, for $M_1 \in$ Mod-k, $M_2 \in$ Mod-$M_2(k)$, define a sheaf $\mathcal{M}$ by $\mathcal{M}(\{E_i\}) = M_i$. A map

$M \to N$ of $(k \oplus M_2(k))$-modules can be expressed as a pair of maps (f_1, f_2), with $f_1 : M_1 \to N_1$, $f_2 : M_2 \to N_2$; this gives a morphism of sheaves $\mathcal{M} \to \mathcal{N}$. This defines a functor $\text{Mod-}(k \oplus M_2(k)) \to \text{Sh}(R)$.

Conversely, given $\mathcal{M} \in \text{Sh}(R)$, taking global sections gives us a $(k \oplus M_2(k))$-module, and morphisms of sheaves induce morphisms of modules. It is trivial to verify that these functors give the desired equivalence of categories $\text{QCoh}(R) \cong \text{Mod-}(k \oplus M_2(k))$. Essentially this is all because the topology is discrete, and so sheaves are just direct sums of modules over the stalks of the structure sheaf. ■

Of course, there is a different tensor sheaf functor taking modules over the ring of global sections $\mathcal{O}_R(\text{InjSpec}(R))$ to sheaves on $\text{InjSpec}(R)$; in the above Example, we see that this functor will actually be an equivalence of categories. So we are led to ask, in general:

Question 5. *Is there an equivalence of categories between* $\text{Mod-}\mathcal{O}_R(\text{InjSpec}(R))$ *and* $\text{QCoh}(R)$ *by taking a module over the ring of global sections to its associated tensor sheaf?*

If the answer to this question is yes, then the tensor sheaf functor will give an equivalence $\text{Mod-}R \to \text{QCoh}(R)$ in cases where the ring of global sections is R itself, such as noetherian domains, by Theorem 7.1.2, and of course commutative rings, where we already know we have this equivalence of categories.

Although this example shows that the global sections functor is not generally quasi-inverse to the tensor sheaf functor, we do at least have an adjunction between them, as we shall now show. For $M \in \text{Mod-}R$, let $\underline{M}$ denote the constant presheaf associated to M. Thus, $\underline{M}(U) = M$ for any open set U, and all restriction maps are the identity on M.

Proposition 7.2.16. *Let R be a ring and let $\Gamma : \text{Sh}(R) \to \text{Mod-}R$ denote the global sections functor. Then the tensor sheaf functor is left adjoint to Γ.*

Proof:

Since sheafification is left adjoint to the forgetful functor from sheaves to presheaves, it suffices to work with the presheaf $M \otimes_R \mathcal{O}_R$ assigning to an open set U the $\mathcal{O}_R(U)$-module $M \otimes_R \mathcal{O}_R(U)$. For if $\mathcal{N}$ is any sheaf on $\text{InjSpec}(R)$, then there is a natural isomorphism $(\mathcal{M}_\otimes, \mathcal{N}) \cong (M \otimes_R \mathcal{O}_R, \mathcal{N})$; so we need only show the existence of a natural isomorphism $(M \otimes_R \mathcal{O}_R, \mathcal{N}) \cong (M, \Gamma(\mathcal{N}))$.

A map $f : M \otimes_R \mathcal{O}_R \to \mathcal{N}$ consists of a map $f_U : M \otimes_R \mathcal{O}_R(U) \to \mathcal{N}(U)$ for each open set U, such that whenever $U \subseteq V$ the diagram below commutes.

$$\begin{tikzcd}
M \otimes_R \mathcal{O}_R(V) \arrow[r, "f_V"] \arrow[d, "M \otimes_R \mathrm{res}^{\mathcal{O}_R}_{V,U}"'] & \mathcal{N}(V) \arrow[d, "\mathrm{res}^{\mathcal{N}}_{V,U}"] \\
M \otimes_R \mathcal{O}_R(U) \arrow[r, "f_U"'] & \mathcal{N}(U)
\end{tikzcd}$$

By tensor-hom adjunction, $f_V : M \otimes_R \mathcal{O}_R(V) \to \mathcal{N}(V)$ corresponds to the map $M \to (\mathcal{O}_R(V), \mathcal{N}(V))_R : m \mapsto f_V(m \otimes -)$, and similarly for f_U. Of course, we have that $(\mathcal{O}_R(V), \mathcal{N}(V))$ is naturally isomorphic to $\mathcal{N}(V)$, and under this isomorphism, $f_V(m \otimes -)$ is identified with $f_V(m \otimes 1)$, which we shall denote $\hat{f}_V(m)$.

By commutativity of the above diagram, we see that $\mathrm{res}^{\mathcal{N}}_{V,U} \circ \hat{f}_V = \hat{f}_U \circ \mathrm{res}^{\mathcal{O}_R}_{V,U}$, which is the map sending $m \in M$ to $f_U(m \otimes \mathrm{res}^{\mathcal{O}_R}_{V,U}(1)) = f_U(m \otimes 1) = \hat{f}_U(m)$.

So $f : M \otimes_R \mathcal{O}_R \to \mathcal{N}$ corresponds to a map $\hat{f}_U : M \to \mathcal{N}(U)$ for each open set U such that whenever $U \subseteq V$ we have $\mathrm{res}^{\mathcal{N}}_{V,U} \hat{f}_V = \hat{f}_U$. But this is precisely the same as a map from the constant presheaf $\underline{M}$ to $\mathcal{N}$.

So sheaf maps $M \otimes_R \mathcal{O}_R \to \mathcal{N}$ correspond to presheaf maps $\underline{M} \to \mathcal{N}$; but these correspond naturally to maps $M \to \mathcal{N}(\mathrm{InjSpec}(R))$, since every component $\hat{f}_U$ of $\hat{f} : \underline{M} \to \mathcal{N}$ is just obtained from $\hat{f}_{\mathrm{InjSpec}(R)}$ by the formula

$$\hat{f}_U(m) = \mathrm{res}^{\mathcal{N}}_{\mathrm{InjSpec}(R),U} \circ \hat{f}_{\mathrm{InjSpec}(R)}.$$

This establishes the isomorphism $(\mathcal{M}_\otimes, \mathcal{N}) = (M, \Gamma(\mathcal{N}))$. Naturality follows from naturality of all the intermediate steps. ∎

The following questions are some of the key points to address in further study of sheaves on injective spectra.

Question 6. *Under what conditions on R is every prime torsion class perfect, so that the tensor sheaf functor and torsion sheaf functor are naturally isomorphic?*

Question 7. *Under what conditions on R is either of these sheaf functors an equivalence of categories between* Mod-R *and* QCoh(R)*?*

Chapter 8

Spectral Spaces and Noetherianity

We have already remarked on cases where it would be desirable for InjSpec(R) to be noetherian. In this chapter, we consider in more detail what this would mean, and also the consequences if InjSpec(R) is spectral.

8.1 Spectral Spaces

Recall that a topological space X is **spectral** if it is compact, T_0, sober, and the family of compact open sets of X is closed under finite intersection and forms a basis of open sets for X. By a Theorem of Hochster [10, Theorem 6], a space is spectral if and only if it is homeomorphic to the Zariski spectrum of some commutative ring. Moreover, given a spectral space X, there is an alternative "dual" topology on the same underlying set as X, where the complements of the compact open sets of X are taken to be a basis of open sets for the dual topology. This dual space is also spectral, and its dual is the original topology on X.

The Ziegler spectrum of a ring is not generally spectral. However, it is "close enough" to spectral to allow the dual topology to be defined, and this is the dual-Ziegler topology, which restricts to the Zariski topology on the injective spectrum (see Section 1.5). In general, the dual-Ziegler topology can be even further from being spectral than the Ziegler topology. However, it is possible that restricting to the injective points of the Ziegler spectrum resolves this. Certainly, some of the ways in which the whole Ziegler spectrum can fail to be spectral cannot occur on the injective spectrum. To show this, we first require the following

Lemma 8.1.1. *The specialisation order for the Ziegler topology on* InjSpec(R) *is simply the reverse ordering of the specialisation order in the Zariski topology.*

PROOF:

Let $E, F \in$ InjSpec(R) be indecomposable injectives. Then E Zariski-specialises to F if and only if every basic Zariski-closed set containing E contains F. This means that for all finitely presented modules M we have $(M, E) \neq 0$ implies $(M, F) \neq 0$. This occurs if and only if for all finitely presented M we have $(M, F) = 0$ implies $(M, E) = 0$, which is precisely the statement that every basic Ziegler-closed set containing F contains E. That is, that F Ziegler-specialises to E. ■

For any ring R, the Ziegler spectrum is compact [18, 5.1.23]; if R is right coherent, then InjSpec(R) is a closed subset of the Ziegler spectrum [18, 5.1.11], hence is also compact in its Ziegler topology. By Corollary 3.2.2, InjSpec(R) is T_0 in the Zariski topology for R right noetherian; by Lemma 8.1.1, this implies that it is T_0 in the Ziegler topology too (note that the whole Ziegler spectrum is not generally T_0, [18, §8.2.12]). In the Ziegler spectrum, the sets (ϕ/ψ), for ϕ/ψ a pp-pair, are precisely the compact open sets [26, 4.9], and by definition they form a basis [18, 5.1.22]; this therefore remains true for the Ziegler topology on InjSpec(R). By Corollary 5.3.8, InjSpec(R) is sober in its Ziegler topology for any right noetherian ring R.

Therefore the only condition of a spectral space that can fail for the Ziegler topology on the injective spectrum of a right noetherian ring is that the intersection of compact open sets be compact open. If this condition holds, *i.e.*, if InjSpec(R) is spectral in its Ziegler topology, then the Zariski topology, being the Hochster dual, is also spectral. In particular, this would prove sobriety of the injective spectrum in its Zariski topology. At present, no examples are known where the intersection of compact Ziegler-open sets of InjSpec(R) fails to be compact (in the whole Ziegler spectrum, examples are known [18, 5.2.5]). So it is possible that the injective spectrum of a right noetherian ring is always a spectral space.

Given Hochster's result that all spectral spaces occur as Zariski spectra of commutative rings, if the injective spectrum of a right noetherian ring is always spectral, it would mean that a failure of commutativity cannot give anything new topologically, and that spectra of noncommutative rings differ only from those of commutative rings in the structure sheaf.

Question 8. *For R right noetherian, is* InjSpec(R) *a spectral space? If not, are there necessary and/or sufficient conditions on R for* InjSpec(R) *to be spectral?*

8.2 Isolating Closed Sets

In order to prove statements about closed sets in injective spectra, it may be useful to isolate them; *i.e.*, given a Zariski-closed set C in the injective spectrum of some ring, to construct a Grothendieck category whose injective spectrum is homeomorphic to C. We will show that this can be done for basic closed sets if InjSpec(R) is spectral in its Ziegler topology, and for arbitrary closed sets if InjSpec(R) is also noetherian in its Zariski topology.

Consider first the case of a commutative noetherian ring R, so that the injective spectrum is the usual Zariski spectrum. A general closed set in Spec(R) is Spec(R/I) for some $I \trianglelefteq R$. In the injective spectrum, this corresponds to those indecomposable injectives E which are the hulls of modules of the form R/p for $p \in$ Spec(R) with $I \subseteq p$. This corresponds precisely to the basic closed set (R/I) in InjSpec(R). Moreover, Mod-R/I is a full subcategory of Mod-R, consisting of those modules which are quotients of direct sums of copies of R/I - *i.e.*, it is the full subcategory generated by R/I.

This suggests, then, for a basic closed set (M) in the injective spectrum of an arbitrary noetherian ring R, to take the full subcategory of Mod-R generated by M, in the hopes that the injective spectrum of this category will be homeomorphic to (M). There is a problem, however; in general, this subcategory need not have a well-defined injective spectrum; indeed, it might not even be abelian. Following Wisbauer [25, §15], we consider the full subcategory $\sigma[M]$ of Mod-R *subgenerated* by M; *viz.*, that consisting of all *sub*quotients of direct sums of copies of M. This is the smallest Grothendieck subcategory of Mod-R containing M.

We record in the next Theorem some results from [25] that will be useful. Recall first that a module E is said to be M-injective if for any submodule $N \leq M$ and map $f : N \to E$, f extends to a map $M \to E$. For R-modules A and B, define the trace of

A in B by

$$\mathrm{Tr}(A,B) := \sum_{f\in(A,B)} f(A).$$

If A is fixed, then $\mathrm{Tr}(A,-)$ is functorial (acting by restriction and corestriction on morphisms).

Theorem 8.2.1 ([25], 15.1, 16.3, 16.8, 17.9)**.** *For R any ring and M any R-module, we have:*

1. *The module*

$$G_M := \bigoplus\{U \leq M^{(\aleph_0)} \mid U \textit{ is finitely generated}\}$$

is a generator for $\sigma[M]$ and the trace functor $\mathrm{Tr}(G_M,-)$ is right adjoint to the inclusion $i_M : \sigma[M] \to \mathrm{Mod}\text{-}R$;

2. *The injective objects of $\sigma[M]$ are precisely those M-injective R-modules which lie in $\sigma[M]$;*

3. *For $N \in \sigma[M]$, if $E_R(i_M N)$ is its injective hull in $\mathrm{Mod}\text{-}R$ and $E_M(N)$ is its injective hull in $\sigma[M]$, then $E_M(N) = \mathrm{Tr}(M, E_R(i_M N))$.*

In particular, for R right noetherian and M finitely presented, the summands of G_M are noetherian, so $\sigma[M]$ has a generating set of noetherian objects; *i.e.*, $\sigma[M]$ is locally noetherian.

We now consider how to use this to isolate a closed set. We keep the notation introduced above.

Theorem 8.2.2. *Let R be any ring and M a finitely presented R-module. Then there is a bijection $j : \mathrm{InjSpec}(\sigma[M]) \to (M) \subseteq \mathrm{InjSpec}(R) : F \mapsto E_R(i_M F)$, with inverse $j^{-1} = \mathrm{Tr}(M,-)$.*

Proof:

Let $F \in (M)$; then there is a non-zero map $M \to F$, so $\mathrm{Tr}(M,F) \neq 0$. Since $i_M\mathrm{Tr}(M,F)$ is a non-zero submodule of F, $F = E_R(i_M\mathrm{Tr}(M,F))$; note that this already proves that $j \circ j^{-1}$ is the identity on (M), under the assumption that j and j^{-1} are well-defined. If $\mathrm{Tr}(M,F)$ were decomposable, then, since i_M is fully faithful, so too would be $i_M\mathrm{Tr}(M,F)$ (see proof of Theorem 2.3.2), and so $E_R(i_M\mathrm{Tr}(M,F))$ would be decomposable, which is a contradiction. So $\mathrm{Tr}(M, F)$ is non-zero and indecomposable, and is injective by part (3) of Theorem 8.2.1. So $\mathrm{Tr}(M, -)$ does indeed give a well-defined function $(M) \to \mathrm{InjSpec}(\sigma[M])$.

For any $F \in \mathrm{InjSpec}(\sigma[M])$, F is uniform in $\sigma[M]$. Since $\sigma[M]$ is a full subcategory of Mod-R and is closed under submodules, this implies that $i_M F$ is uniform in Mod-R, so $E_R(i_M F)$ is indecomposable. So j is well-defined.

Finally, by part (3) of Theorem 8.2.1, $\mathrm{Tr}(M, -) \circ j$ is the identity function on $\mathrm{InjSpec}(\sigma[M])$. ■

Henceforth, we identify $\sigma[M]$ as a category in its own right with its image under i_M, except where this might cause confusion. Note that some categorical constructions do depend on whether we are working in $\sigma[M]$ or in Mod-R; for instance, products in $\sigma[M]$ are not the same as in Mod-R [25, 15.1]. For $N \in \sigma[M]$, the notation $[N]_R$ will refer to the basic open set of $\mathrm{InjSpec}(R)$ determined by N, whereas $[N]_M$ will denote the basic open set in $\mathrm{InjSpec}(\sigma[M])$. A similar convention will be adopted for basic closed sets.

We now wish to prove that j is a homeomorphism. This is where we require $\mathrm{InjSpec}(R)$ to be spectral in the Ziegler topology. We first require the following

Lemma 8.2.3. *Let R be any ring, E an indecomposable injective R-module and M any R-module. Then* $\mathrm{Tr}(M, E) = \mathrm{Tr}(G_M, E)$.

PROOF:

Since M is a summand of G_M, we certainly have $\mathrm{Tr}(M, E) \subseteq \mathrm{Tr}(G_M, E)$. For the converse, let $i : \mathrm{Tr}(G_M, E) \to E$ be the inclusion; then, given $A \leq M$ and $f : A \to \mathrm{Tr}(G_M, E)$, $i \circ f$ is a map $A \to E$ in Mod-R. This therefore extends to a map $g : M \to E$, whose image must lie in $\mathrm{Tr}(M, E) \subseteq \mathrm{Tr}(G_M, E)$; so the corestriction of g is a morphism $M \to \mathrm{Tr}(G_M, E)$ extending f. Therefore $\mathrm{Tr}(G_M, E)$ is M-injective and hence, by part (2) of Theorem 8.2.1, $\mathrm{Tr}(G_M, E)$ is an injective object of $\sigma[M]$.

Now, since $\mathrm{Tr}(M, E)$ is injective in $\sigma[M]$, by part (3) of the same Theorem (or, indeed, by a minor adaptation of the proof just given for $\mathrm{Tr}(G_M, E)$), it is a summand of $\mathrm{Tr}(G_M, E)$. So to prove equality, it suffices to show that $\mathrm{Tr}(G_M, E)$ is indecomposable. But, in Mod-R, it is a subobject of the uniform module E, so it certainly is indecomposable. ■

Theorem 8.2.4. *Suppose that* InjSpec(R) *is spectral in its Ziegler topology. Then the above bijection j is a homeomorphism when (M) has the subspace topology inherited from the Zariski topology on* InjSpec(R).

PROOF:

First we prove that j^{-1} is continuous. By Theorem 25.1 of [25], finitely presented R-modules in $\sigma[M]$ are finitely presented as objects of $\sigma[M]$, and for any $N \in \sigma[M]^{\mathrm{fp}}$, N is a subquotient of a finite direct sum of copies of M. But for M finitely presented over a right noetherian ring R, this implies that N is finitely presented as an R-module. So $\sigma[M]^{\mathrm{fp}} = \sigma[M] \cap \text{mod-}R$.

Now, a basic open set of InjSpec$(\sigma[M])$ has the form $[N]_M$ for $N \in \sigma[M]^{\mathrm{fp}}$. We show that $j[N]_M = [N]_R \cap (M)$, which suffices for continuity of j^{-1}. This means showing that

$$\{E_R(F) \mid F \in \mathrm{InjSpec}(\sigma[M]) \wedge (N,F) = 0\} = \{E \in (M) \mid (N,E) = 0\},$$

which amounts to showing that, for $F \in \mathrm{InjSpec}(\sigma[M])$, $(N,F) = 0$ if and only if $(N, E_R(F)) = 0$.

Since $(N,-)$ is left-exact and $F \subseteq E_R(F)$, we certainly have that if $(N, E_R(F)) = 0$, then $(N,F) = 0$. Conversely, if $f : N \to E_R(F)$ is non-zero, then $f(N) \cap F \neq 0$, since $E_R(F)$ is uniform, so there is $n \in N$ such that $0 \neq f(n) \in F$. Then $f|_{nR} : nR \to F$ is a non-zero morphism in $\sigma[M]$, but F is injective in $\sigma[M]$, so this extends to a non-zero morphism $N \to F$. Note that we have shown that j^{-1} is continuous not only in the Zariski topology, but also in the Ziegler topology; this will be essential for proving that j is Zariski-continuous.

Now we prove continuity of j. Let $N \in \text{mod-}R$; we show that $j^{-1}((N)_R \cap (M)_R)$ is closed. Note that $j^{-1}((N)_R \cap (M)_R) = \{F \in \mathrm{InjSpec}(\sigma[M]) \mid (N, E_R(F)) \neq 0\}$.

First we deal with the case where $N \in \sigma[M]$, so $(N)_R \subseteq (M)_R$. Then we have $(N, E_R(F)) = (N, Tr(G_M, E_R(F))$, by part (1) of Theorem 8.2.1, and this is (N,F), by Theorem 8.2.2 and Lemma 8.2.3, so $j^{-1}(N)_R = (N)_M$ in this case.

Now we deal with general N. Let S be the set of subquotients of N which lie in $\sigma[M]$. We claim that

$$j^{-1}((N)_R \cap (M)_R) = j^{-1}\left(\bigcup_{L\in S}(L)_R\right) = \bigcup_{L\in S} j^{-1}(L)_R. \qquad (\star)$$

Given this, each $L \in S$ lies in $\sigma[M]$, so $j^{-1}(L)_R = (L)_M$, by what we showed above. Moreover, $(N)_R$ and $(M)_R$ are compact open in the Ziegler topology, so $(N)_R \cap (M)_R$ is compact, since the Ziegler topology is assumed to be spectral. Since j^{-1} is Ziegler-continuous and the continuous image of a compact set is compact, $j^{-1}((N)_R \cap (M_R))$ is Ziegler-compact. But each $(L)_M$ is Ziegler-open in $\mathrm{InjSpec}(\sigma[M])$, so the union on the right-hand side of the equations $(\star)$ can be replaced by a finite union.

So $j^{-1}(N)_R$ is equal to a finite union of sets of the form $j^{-1}(L)_R = (L)_M$ for $L \in \sigma[M]^{\mathrm{fp}}$, hence is Zariski-closed, as required. So j is Zariski-continuous, and hence a homeomorphism between (M) and $\mathrm{InjSpec}(\sigma[M])$.

It remains only to prove the claim $(*)$. The second equality is a standard fact about images of unions (since j is invertible, $j^{-1}(L)_R$ is the image of $(L)_R$ under j^{-1}, not just the preimage of $(L)_R$ under j). For the first equality, take $F \in \mathrm{InjSpec}(\sigma[M])$. Then $(N, E_R(F)) \neq 0$ if and only if there is a subquotient L of N which embeds in F, so $F \in (L)_R$. Since $\sigma[M]$ is closed under subobjects and $F \in \sigma[M]$, we see that $L \in \sigma[M]$, so $L \in S$.. Conversely, if any subquotient L of N has a non-zero map to F, then $(L, E_R(F)) \neq 0$, and by injectivity we conclude $(N, E_R(F)) \neq 0$. This proves the claim. ■

So if $\mathrm{InjSpec}(R)$ is spectral in its Ziegler topology, then for any basic closed set (M) there is a locally noetherian category $\sigma[M]$ whose injective spectrum is homeomorphic to $(M) \subseteq \mathrm{InjSpec}(R)$. What about arbitrary closed sets? These are intersections of basic closed sets; if $\mathrm{InjSpec}(R)$ is noetherian (in its Zariski topology), then every closed set is a *finite* intersection of basic closed sets, and is therefore a single basic closed set, by the spectrality assumption. So for $\mathrm{InjSpec}(R)$ Zariski-noetherian and Ziegler-spectral, the above result covers all closed sets.

Chapter 9

Conclusions and Future Directions

The results of this thesis may broadly be classified into four areas: topological considerations (including the study of critical dimension and the torsion spectrum), functoriality, computation of examples, and sheaves. Of course, each of these areas relates to the others. Within each area, a number of open questions remain, some particularly highlighted by the results obtained thus far.

As far as the topology goes, perhaps our most important results are that the injective spectrum of a right noetherian ring is T_0 (Corollary 3.2.2), that every irreducible basic closed set has a generic point (Theorem 5.3.13), and that the injective spectrum is sober in the Ziegler topology (Corollary 5.3.8). There are also key topological results relating to critical dimension, most notably that specialisation reduces critical dimension (Lemma 3.2.1) and that there are only finitely many points of maximal dimension (Theorem 3.3.2).

There are a number of key questions raised by these results. In particular, it would be desirable to know when the injective spectrum is sober and when the basic closed set (M) given by a finitely presented module M is irreducible; for when these conditions hold and R has finite Krull dimension d, $\text{InjSpec}(R)$ has topological dimension exactly d (remarks after Corollary 3.2.8). In particular, if the injective spectrum is spectral in its Ziegler topology, then sobriety in the Zariski topology follows automatically, and in fact the injective spectrum would be homeomorphic to the Zariski spectrum of some commutative ring (Section 8.1).

A further key topological question is noetherianity; if $\text{InjSpec}(R)$ is noetherian, then every closed point is the injective hull of a simple module (Proposition 2.2.2) and, coupled with spectrality, we would be able to isolate any closed set as the injective spectrum of a Grothendieck category (Theorem 8.2.4 and following remarks). Given that the quantum plane (Section 6.2) gives an example of a seemingly nice ring where noetherianity of the spectrum fails, this might be a difficult question to resolve.

As far as functoriality goes, we have quite limited results, concerning only flat epimorphisms (Corollary 2.3.4) and quotients by a single central element (Lemma 6.1.2). This is unfortunate, since functoriality could be extremely useful for computing new examples by relating them to old ones. It is natural to ask if any other ring maps induce maps on spectra. In light of Example 2.3.1, which exhibits a ring map $k \times k \to M_2(k)$ that cannot feasibly induce a map on spectra, it is also worth considering the possibility of generalising the notion of continuous map or map of ringed spaces; for the restriction of scalars in that example does naturally take the unique indecomposable injective over $M_2(k)$ to *both* the points over $k \times k$, so it is perhaps worth considering induced continuous correspondences, rather than continuous functions. Another option in this vein would be to disregard the points altogether and consider a morphism of locales, or "pointless topological spaces".

As far as computing examples go, our main example is the first Heisenberg algebra, for which we have a fairly complete picture (Section 6.1). Useful directions for future computation would be the higher Weyl and Heisenberg algebras, as well as the universal enveloping algebras of other Lie algebras and other skew polynomial rings. Since the quantum plane provides an example of "bad" behaviour of the injective spectrum (Section 6.2), a more detailed examination of its spectrum would doubtless be very illuminating; perhaps also other quantum deformation algebras would be interesting and useful to examine.

With regard to sheaves, there is a great deal of room for further exploration. The development of tools for the computation of the structure sheaf would be extremely useful. Investigation of rings over which all prime torsion theories are perfect, and quasicoherent sheaves are likely to be fruitful avenues of enquiry to further the results of Section 7.

It would also be nice to be able to define a structure sheaf sensibly and canonically for the injective spectrum of any (suitably nice) Grothendieck category; without an underlying ring to follow through the finite type localisations, this seems impossible, so it might be worth considering simply the localisations of the category and obtaining a stack of finite type localisations of the whole category, rather than a sheaf of localisations of the ring.

As such, it seems to me that the crucial directions for future research in this area are:

1. To compute further examples, particularly skew polynomial rings, universal enveloping algebras of Lie algebras, and quantum deformations of such rings;

2. To develop further functoriality results, possibly by relaxing the notion of topological map (and if further functoriality results are obtained, this is likely to be helpful in computation of examples);

3. To explore spectrality and noetherianity and establish neccessary/sufficient conditions for them;

4. To understand better how the structure sheaf relates to the original ring, and how sheaves of modules over it relate to modules over the original ring, and perhaps to shift to working with stacks of categories instead of sheaves of rings.

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