Representations of Quivers

An introduction to hereditary algebras & Auslander-Reiten theory.

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I declare that this is my own work unless otherwise stated.

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Contents

0.1 Abstract

This report explores quiver representations, mostly as modules over the path algebra. We will begin by studying indecomposable decompositions, with a view to illuminating the structure of representations, and will then identify the importance of projective modules. This will lead us to the notion of a hereditary algebra and the standard resolution of a quiver representation, allowing us to observe a sense in which path algebras of quivers are almost semisimple. We will also consider the problem of classifying hereditary algebras. Finally, we will develop some of the ideas behind the Auslander-Reiten theory, culminating in the powerful 'knitting' algorithm for constructing information about the representation category of a quiver.

0.2 A Note on Terminology

Throughout this piece, we adopt the following conventions. All rings have a unity, but need not be commutative; by an 'algebra' we mean an associative, unital k-algebra for some field k. Initially, no assumptions are placed on fields, but before long we begin assuming that they are algebraically closed; it will be mentioned when this is to be assumed, or the reader may assume it from the start, if preferred.

0.3 Acknowledgments

I would like to thank my supervisor, Dr. Ed Segal, for guiding me on this project and helping me out whenever I got stuck.

The statements of most results in this report come from various sources, which are referenced in the statement, however some minor results are my own formulation (generally these are well-known or simple-to-observe lemmas). Where possible, I have tried to provide my own proofs; where I have needed to draw upon another's work, entirely or in part, this is indicated by a citation either after the word 'proof' if it is largely or entirely another's work or in the relevant part of the proof if only some ideas come from elsewhere. Regardless, when using another's ideas, I have always endeavoured to extract the important notions and then provide my own account, rather than simply regurgitating someone else's maths.

Chapter 1

Quivers & Path Algebras

1.1 Introduction

Broadly speaking, a quiver is a collection of vertices connected by arrows and a representation of a quiver assigns a vector space to each vertex and a linear map to each arrow. As such, the study of quiver representations is, at its most basic level, the study of general mapping problems in linear algebra and quiver representation theory can give rise to normal forms for maps or collections of maps. However, quiver representations also have applications in algebraic geometry, relating to moduli spaces, and physics, where they appear in string theory. The application of greatest interest to us, however, will be the application of quivers to representation theory generally. This takes two forms; firstly, quivers provide a special type of representation theory, in that Maschke's Theorem fails for quiver representations, but 'only just', in a sense which will be made precise. Secondly, modules over an algebra can be studied by means of their Auslander-Reiten quiver, to be introduced in chapter 4.

In this first chapter, we will study the basic definitions, as well as motivating the study with some examples of applications to linear mapping problems. We will finish by constructing an associative algebra related to a quiver, its *path algebra*, and identifying the representations of the quiver with the modules of this algebra. In chapter 2, we will delve deeper into the theory of modules over associative algebras, looking at indecomposable modules and projective modules, which form an important part of the study of quiver representations.

This will lead into chapter 3, where we will study the 'hereditary' nature of projective modules over certain algebras - the submodule of a projective module is itself projective. We will prove that path algebras of quivers are hereditary and also that path algebras are in some way representative of hereditary algebras, in a relationship known as *Morita equivalence*.

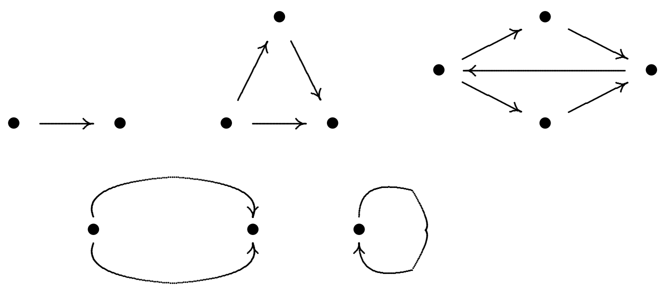
For the final chapter, we will take a different approach to studying quiver representations, introducing the Auslander-Reiten theory, which uses a special type of short exact sequence, called an *almost-split* sequence, to relate different indecomposable representations. These links between the indecomposable representations can be used to uncover an awful lot of information about the category of representations of a quiver, through a process called *knitting*, which will form the culmination of this report.

First, of course, we require some introductory definitions and preliminary results.

1.2 Quivers & their Representations

Definition 1.1 (Quivers)
A *quiver* is a directed multigraph; that is, a tuple (Q_0, Q_1, s, t), where Q_0 is a set, whose members are called *vertices*, Q_1 is a set, whose members are called *arrows*, and $s, t : Q_1 \to Q_0$ are functions assigning to each arrow its *source* and *target* respectively. We place no restrictions on our source and target functions - in particular, we allow *loops* (arrows whose source and target coincide) and multiple arrows between the same two vertices, in the same or different directions. We normally consider *finite quivers*; that is, quivers with finitely many vertices and arrows: $|Q_0| < \infty$, $|Q_1| < \infty$, so all quivers are to be assumed finite unless otherwise stated.

Example 1.2 (Some Quivers)
Each of the following is a quiver (as is the union of any number of them):



Definition 1.3 (Representations of Quivers)
Let $Q = (Q_0, Q_1, s, t)$ be a quiver and k a field. A *representation* of Q over k is a pair (V, f), where $V = (V_v)_{v \in Q_0}$ is a Q_0-indexed family of vector spaces over k and $f = (f_a)_{a \in Q_1}$ is a Q_1-indexed family of linear maps, where $f_a : V_{s(a)} \to V_{t(a)}$. That is, a representation assigns a vector space V_v to each vertex v and a linear map $f_a : V_\sigma \to V_\tau$ for each arrow a with source vertex σ and target τ, so that the quiver is instantiated as a diagram of linear maps between vector spaces. In particular, the *zero* or *trivial representation* assigns to each vertex the zero vector space and to each arrow the zero map.

A representation is said to be *finite-dimensional* if every vector space involved is finite-dimensional. Unless stated otherwise, all representations are to be assumed finite-dimensional. Given a quiver Q and a representation V of Q, we may order the vertices of Q, labelling them $v_1, \ldots, v_n$, and then form a length-n vector called the *dimension vector* of V and denoted $\dim(V)$, whose i^{th} entry is the dimension of the vector space attached to the vertex v_i in the representation V. Rather than specifying an ordering on Q_0 and then writing the dimension vector as a row or column vector, it is often convenient to write the vector in the shape of the quiver.

Example 1.4 (A Representation & its Dimension Vector)
The four-vertex quiver in the above example (top-right) has a representation

$$\begin{tikzcd}
 & k \arrow[dr, "\left(\begin{smallmatrix} 9 \\ 0 \end{smallmatrix}\right)"] & \\
k \arrow[ur, "7"] \arrow[dr, "\left(\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right)"'] & & k^2 \arrow[ll, "0"'] \\
 & k^2 \arrow[ur, "\left(\begin{smallmatrix} -2 & 4 \\ 7 & 1 \end{smallmatrix}\right)"'] &
\end{tikzcd}$$

with dimension vector $\begin{smallmatrix} & 1 & \\ 1 & & 2 \\ & 2 & \end{smallmatrix}$

Having defined a new type of object, we ought to now define the subobjects, products and morphisms.

Definition 1.5 (Subrepresentations & Morphisms)
Given a quiver Q and a representation $((V_v)_{v \in Q_0}, (f_a)_{a \in Q_1})$ of Q, a *subrepresentation* is a representation $((U_v)_{v \in Q_0}, (g_a)_{a \in Q_1})$ where for all $v \in Q_0$, U_v is a subspace of V_v and for all $a \in Q_1$, g_a is the restriction of f_a to $U_{s(a)}$; obviously, it is required that the image of f_a be contained in $U_{t(a)}$ to ensure that $g_a : U_{s(a)} \to U_{t(a)}$ is well-defined, so the subspaces U_v may not, in general, be chosen totally independently.

Given a quiver Q, a field k and two representations $U = ((U_v), (f_a))$ and $V = ((V_v), (g_a))$ of Q over k, a *morphism of representations* $\phi : U \to V$ is a family of linear maps $\phi = (\phi_v)_{v \in Q_0}$ where $\phi_v : U_v \to V_v$, such that the following diagram commutes for all arrows a:

$$\begin{tikzcd} U_{s(a)} \arrow[r, "f_a"] \arrow[d, "\phi_{s(a)}"'] & U_{t(a)} \arrow[d, "\phi_{t(a)}"] \\ V_{s(a)} \arrow[r, "g_a"] & V_{t(a)} \end{tikzcd}$$

This allows an alternative definition of a subrepresentation; instead of requiring that the linear maps be restrictions of those in the parent representation, we can require that the family of inclusion maps form a morphism of representations. It is easy to check that these are equivalent. The family (1_v), where 1_v is the identity map on U_v, is a morphism, called the identity morphism of the representation U and denoted 1_U. Given morphisms $\phi = (\phi_v) : U \to V$ and $\psi = (\psi_v) : V \to W$, we define the composition $\psi\phi = (\psi_v\phi_v) : U \to W$. This is easily shown to be a morphism; moreover, $\phi 1_U = 1_V \phi = \phi$. Thus, the representations of a quiver Q over a field k form a category, denoted $\mathrm{Rep}(Q, k)$.

Of course, a morphism which admits an inverse morphism is called an isomorphism. For a morphism ϕ to be an isomorphism, it is clearly necessary that all component linear maps ϕ_v be bijective; this is also sufficient, as then the family of inverse maps, $\phi^{-1} = (\phi_v^{-1})_{v \in Q_0}$ is also a morphism - the proof is trivial and omitted for brevity.

Definition 1.6 (Direct Sums; Irreducible & Indecomposable Representations)
We define the *direct sum* of two representations in the obvious way: if $U = ((U_v), (f_a))$ and $V = ((V_v), (g_a))$ are two representations of the same quiver over the same field, then the direct sum is the representation $U \oplus V = ((U_v \oplus V_v), (f_a \oplus g_a))$. A representation which is not isomorphic to a direct sum of proper, non-trivial subrepresentations is called *indecomposable*; a representation which *has no* proper, non-trivial subrepresentations is called *irreducible*.

Any irreducible representation is clearly indecomposable; however, the converse is, in general, false:

Example 1.7 (A Non-Irreducible Indecomposable)
Let Q be the quiver with two vertices, x and y, and one edge, f from x to y, which we depict:

$$x \xrightarrow{f} y$$

Then a representation of Q is a pair of vector spaces X and Y and a linear map $F : X \to Y$. Let $X = Y = k$, the one-dimensional space, and let F be the identity map. Then the representation $X' = 0$, $Y' = k$, $F' = 0$ is a subrepresentation, since:

$$\begin{tikzcd} k \arrow[r, "1"] & k \\ 0 \arrow[r] \arrow[u, hook] & k \arrow[u, hook] \end{tikzcd}$$

clearly commutes. However, there is no complementary subrepresentation; for if there were, it would need to have k as its x-vector space and 0 as its y-vector space, by dimension considerations, but the diagram:

$$\begin{tikzcd} k \arrow[r, "1"] & k \\ k \arrow[r] \arrow[u, hook] & 0 \arrow[u, hook] \end{tikzcd}$$

does not commute, so this is not a subrepresentation. There can be no other (proper, non-trivial) subrepresentations, as any subrepresentation of this representation is entirely determined by which vector spaces are non-zero. Therefore this representation is indecomposable, but it is not irreducible.

This shows a contrast between the representation theory of quivers and that of finite groups (over fields of characteristic prime to the order of the group). Any finite-dimensional representation of a group is semisimple, but this example shows that the corresponding result for quivers fails - there is no analogue of Maschke's Theorem for quivers. We will see in chapter 3 that quiver representation theory is, in some sense, the simplest representation theory where Maschke's Theorem fails.

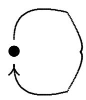
It is evident that any representation is either indecomposable or a direct sum of two (or more) subrepresentations. A general strategy for studying quiver representations is therefore to attempt to classify the indecomposable representations. As we shall see later, every finite-dimensional representations is uniquely expressible as a direct sum of indecomposables.

In general, however, classifying the indecomposable representations of a given quiver is very difficult. We shall see a tool for making headway on this problem in the final chapter, but it is not generally possible to solve. A quiver is called representation-finite if it has only finitely many indecomposables; a famous result of Pierre Gabriel classifies the representation-finite quivers as precisely those whose underlying, undirected graph is a simply laced Dynkin-diagram. See, for instance, Brion (2008) for details; we will not explore this result further.

Even when a quiver is not representation-finite, it can still be possible to classify its indecomposable representations in a reasonable way, putting them into finitely many one-parameter families in each dimension; such quivers are called 'tame'. There are, however, so-called 'wild' quivers, whose indecomposables cannot be parameterised by a single parameter. The indecomposables of a wild quiver cannot be classified. Again, we do not explore this aspect of the theory, but details can be found in Barot (2006).

In many simple cases, basic linear algebra is adequate to find all indecomposable representations and completely understand the representation structure of a quiver. In fact, studying quiver representations can be viewed as studying mapping problems in linear algebra. Some examples will illustrate this.

Example 1.8 (Jordan Normal Form)
Consider the quiver consisting of a single vertex and one loop:



A representation of this quiver is a vector space V of dimension n and a linear map $f : V \to V$. So the study of representations of this quiver is precisely the study of linear endomorphisms. If this representation splits as a direct sum of two subrepresentations, $V = U \oplus W$, then by taking the union of a basis for U with a basis for W, we obtain a basis for V in which f is represented as a block-diagonal matrix.

Over an algebraically closed field, there is a basis for V in which f is expressed in Jordan normal form. Each Jordan block corresponds to a subrepresentation, since it is closed under the action of f, and is indecomposable, since a Jordan block is a minimal f-invariant subspace. So the Jordan Normal Form Theorem can be regarded as a complete classification of the finite-dimensional representations of this quiver over an algebraically closed field: the representations (up to isomorphism) are in bijection with all possible Jordan normal matrices, with the Jordan blocks corresponding to indecomposables, and two representations are isomorphic if and only if the matrices of their respective maps f are conjugate.

Observe that in each dimension, the indecomposable representations are parameterised by the eigenvalue of the Jordan block matrix, so this is an example of the 'tame' case mentioned above.

Example 1.9 (The Rank-Nullity Theorem)
Consider the quiver consisting of two vertices and a single arrow:

$$\bullet \to \bullet$$

A representation of this quiver consists of a pair of vector spaces V and W and a linear map $f : V \to W$. We shall show that there are precisely three indecomposable representations, namely:

$$0 \to k \qquad k \to 0 \qquad k \xrightarrow{1} k$$

The first two of these are irreducible, since any proper subrepresentation must have dimension zero at each vertex, so in particular they are indecomposable. The last of these was shown to be indecomposable (but not irreducible) in an earlier example. Moreover, for any representation of this quiver

$$V \xrightarrow{f} W$$

we may take subspaces $U \subseteq V$ complementary to $\ker(f)$ and $X \subseteq W$ complementary to $f(V)$ and form subrepresentations:

$$\ker(f) \to 0 \qquad U \xrightarrow{f|_U} f(V) \qquad 0 \to X$$

Note that the possibility of forming such subrepresentations is precisely the proof of the Rank-Nullity Theorem. If $\ker(f)$ has dimension n, the first of these is isomorphic to n copies of $(k \to 0)$, while if X has dimension m, the last is isomorphic to m copies of $(0 \to k)$. Note that f is injective when restricted to U and surjects onto $f(V)$, so choosing a basis B for U, the set $f(B)$ is a basis for $f(V)$ and, in these bases, $f|_U$ is represented as the identity matrix; so the middle representation is isomorphic to $\dim(V) - n$ copies of $(k \xrightarrow{1} k)$. Moreover, these three subrepresentations intersect trivially, by construction, and their sum is the entire representation, again by construction, so the original representation splits:

$$V \xrightarrow{f} W = (k \to 0)^{\oplus n} \oplus (k \xrightarrow{1} k)^{\oplus \dim(V)-n} \oplus (0 \to k)^{\oplus m}$$

This firstly proves that the three indecomposable representations given are indeed every indecomposable (as any other putative indecomposable would split as a direct sum among these three) and also gives a new interpretation of the standard result of linear algebra that for a map between distinct vector spaces there is a choice of basis on each side which will give the map the block matrix

$$\left(\begin{array}{c|c} 1_{r\times r} & 0 \\ \hline 0 & 0 \end{array}\right)$$

where r is the rank of the map. Thus the representations of this quiver (up to isomorphism) are in bijection with the matrices of this form; in particular, they are entirely determined by three data: the dimensions of V and W and the rank of f.

These examples illustrate that, at its most elementary level, the study of quiver representations is nothing more than a reformulation of linear algebra and that problems in quiver representation theory can be solved by purely linear algebra considerations. As is often the case in mathematics, having two points of view on a problem is extremely useful, so we now develop a new perspective on quiver representations, in terms of modules over an associative algebra, so that we may henceforth switch between the representations/linear algebra perspective and the modules perspective at will.

1.3 The Path Algebra

Definition 1.10 (Paths)
Given a quiver Q and two vertices v and w, a *path of length* n from v to w is a consecutive sequence of arrows starting at v and ending at w; formally, it is a family $(a_i)_{i=1}^n$ of arrows, where $t(a_1) = w$, $s(a_n) = v$ and $t(a_i) = s(a_{i-1})$ for all $i \in \{2, 3, \ldots, n\}$. Note that this convention is 'backwards' so as to agree with function composition; if a and b are arrows, ab can be regarded as a path made by going first along b, then along a.

There is a natural partial operation on paths called *composition*: if $p = (a_i)_{i=1}^n$ and $q = (b_i)_{i=1}^m$ are two paths and $t(a_n) = s(b_1)$ we say *q and p are composable* and write qp for the composed path: $qp = (c_i)_{i=1}^{n+m}$, where $c_i = a_i$ for $1 \leq i \leq n$ and $c_i = b_{i-n}$ for $n < i \leq n+m$. We extend our source and target functions to apply to all paths, if $p = (a_i)_{i=1}^n$ is a path from v to w, then $s(p) = s(a_n) = v$ and $t(p) = t(a_1) = w$

It is clear that the arrows of a quiver are precisely the paths of length 1. It is convenient to regard individual vertices as paths of length 0, with the same start and endpoint; note, however, that in a general quiver the paths of length 0 need not be the only paths which start and end at the same vertex; any such path of strictly positive length is called an *(oriented) cycle*; the cycles of length 1 are called *loops*. A quiver with no cycles is called *acyclic*. Note that our definition of cycle is inherently oriented, so an acyclic quiver may still have unoriented cycles in its underlying non-directed graph. We define Q_i to be the set of all paths of length i, generalising our notation of Q_0 and Q_1; we define $Q_{\geq i}$ to be the set of all paths of length at least i.

Regarding a vertex v as a path allows us to compose it with another path p, if the two are composable together; we have the relations $vp = p$ for all paths p ending at v and $pv = p$ for all p starting at v. It is evident that composition of paths is associative. Moreover, if we take not just paths, but all formal linear combinations of paths over some field and define composition of linear combinations by extending linearly and also declaring the composition of two non-composable paths to be zero, then the formal sum of all the vertices is a two-sided identity. To see this, let p be any path; then:

$$\begin{aligned} p \sum_{v \in Q_0} v &= \sum_{v \in Q_0} pv \\ &= ps(p) + 0 \\ &= p \end{aligned}$$

and

$$\begin{aligned} \left(\sum_{v \in Q_0} v \right) p &= \sum_{v \in Q_0} vp \\ &= t(p)p + 0 \\ &= p \end{aligned}$$

and extending linearly proves the claim. This motivates the following:

Definition 1.11 (The Quiver Algebra)
Let Q be a quiver and k a field. The *quiver algebra* or *path algebra* kQ is the k-algebra consisting of formal k-linear combinations of paths in Q with multiplication given by path composition and extended linearly, where the product of two non-composable paths is taken to be the zero vector. As every path of length strictly greater than 1 is, by definition, a composition of length-1 paths, it is clear that the path algebra is generated by the paths of lengths 0 and 1 - *i.e.*, the vertices and arrows. Thus, equivalently, kQ is the k-algebra generated by $Q_0 \cup Q_1$ subject to the relations:

- $v^2 = v \ \ \forall v \in Q_0$
- $v_1 v_2 = 0 \ \ \forall v_1, v_2 \in Q_0 \ : \ v_1 \neq v_2$
- $as(a) = t(a)a = a \ \ \forall a \in Q_1$
- $ab = 0 \ \ \forall a, b \in Q_1 \ : \ t(b) \neq s(a)$

The identity of the path algebra, denoted 1_{kQ} or just 1 where no confusion is possible, is the sum of all the vertices.

Theorem 1.12 (Module-Representation Equivalence (Brion 2008))
Let Q be a quiver and k a field. Then there is an equivalence of categories between $\operatorname{Rep}(Q,k)$ and kQ-Mod, the category of (left) kQ-modules.

PROOF:
Let $U = ((U_v),(f_a))$ and $V = ((V_v),(g_a))$ be arbitrary representations and $\phi : U \to V$ a morphism. Define a covariant functor $F : \operatorname{Rep}(Q,k) \to kQ$-Mod as follows. Let $F(U) = \bigoplus_{v\in Q_0} U_v$ as a vector space and define an action of kQ on $F(U)$ by letting each vertex $v \in Q_0$ act as projection onto the direct summand U_v and letting each arrow $a \in Q_1$ act as f_a between the summands $U_{s(a)}$ and $U_{t(a)}$ and as the zero map on all other summands. Now define $F(\phi) : F(U) \to F(V)$ by $F(\phi) = \bigoplus_{v\in Q_0} \phi_v$. That $F(\phi)$ is linear is apparent from the definition; kQ-linearity follows immediately from the fact that ϕ is a morphism of representations. Note that:

$$\begin{aligned} F(1_U) &= F((1_{U_v})_{v\in Q_0}) \\ &= \bigoplus_{v\in Q_0} 1_{U_v} \\ &= \sum_{v\in Q_0} v \\ &= 1_{F(U)} \end{aligned}$$

and, if $\psi : W \to X$ is another morphism:

$$\begin{aligned} F(\psi\phi) &= \bigoplus_{v\in Q_0} \psi_v\phi_v \\ &= \bigoplus_{v\in Q_0} \psi_v \bigoplus_{v\in Q_0} \phi_v \\ &= F(\psi)F(\phi) \end{aligned}$$

so that F is indeed a covariant functor.

Now define a functor $G : kQ\text{-Mod} \to \operatorname{Rep}(Q,k)$ as follows. Let M be a kQ-module and define $G(M)$ to be the representation $((U_v),(f_a))$ with $U_v = vM$ and f_a the linear map $s(a)M \to t(a)M$ induced by multiplication by a. Now let N be another kQ-module and let $\phi : M \to N$ be a kQ-linear map; define $G(\phi) = (\phi_v)_{v\in Q_0}$, where $\phi_v : vM \to vN$ is simply the restriction of ϕ to vM; that this is well-defined comes from kQ-linearity: $\phi(vm) = v\phi(m) \in vN$. It follows immediately from the definition that $G(1_M) = 1_{G(M)}$ and $G(\psi\phi) = G(\psi)G(\phi)$ for any two composable kQ-module homomorphisms ψ and ϕ, showing that G is indeed a covariant functor.

Now we sketch the proof that F and G give an equivalence of categories. Let U be a representation; then $F(U) = \bigoplus_{v\in Q_0} U_v$ and so the vertex family of $GF(U)$ is $(v \bigoplus_{w\in Q_0} U_w)_{v\in Q_0}$. Note that for any vertex v, $v \bigoplus_{w\in Q_0} U_w$ is simply the direct sum of U_v with $|Q_0| - 1$ copies of the zero space, as v is the projection operator onto U_v. Let $i^U_v : U_v \to v \bigoplus_{w\in Q_0} U_w$ be the obvious linear isomorphism and let $I_U = (i^U_v)_{v\in Q_0} : (U_v)_{v\in Q_0} \to (v \bigoplus_{w\in Q_0} U_w)_{v\in Q_0}$; so I_U is a family of maps from the vertex spaces of U to those of $GF(U)$.

It is then straightforward but tedious to verify that I_U is an isomorphism of representations and is natural for variation of U, making $(I_U)_{U\in\operatorname{Rep}(Q,k)}$ a natural isomorphism between GF and the identity functor on $\operatorname{Rep}(Q,k)$. A similar long but easy argument shows that the reverse composition is naturally isomorphic to the identity functor on kQ-Mod, proving the desired equivalence of categories. ∎

The significance of this result is that we can study quiver representations by studying the modules over the quiver algebra. For this reason, we now cover some algebraic results about modules.

Chapter 2

Algebras & their Modules

Now that we have an alternative viewpoint on quiver representations, that of modules over the path algebra, we study associative algebras and their modules in greater generality, to obtain results which we can then apply to quivers.

2.1 Indecomposables & the Krull-Schmidt Theorem

We now work towards a vitally important result for studying finite-dimensional modules, the Krull-Schmidt theorem, which is a unique decomposition result. First, we require some basic technical preliminaries from non-commutative algebra.

Definition 2.1 (The Jacobson Radical)
Let R be any ring (not necessarily commutative). The set of all elements x of R with the property that $1 - rxs$ is a unit for all $r, s \in R$ is called the *Jacobson radical* of R and denoted $J(R)$:

$$J(R) = \{x \in R | \forall r, s \in R \; 1 - rxs \in R^{\times}\}$$

Lemma 2.2
The Jacobson radical is a two-sided ideal.

Proof:
Let R be a ring with Jacobson radical J. First note that $0 \in J$, so J is non-empty. Moreover, if $x \in J$ and $t \in R$, then for any $r, s \in R$ $1-r(tx)s = 1-(rt)xs$ is invertible, since $x \in J$, so $tx \in J$. Similarly, $xt \in J$. Finally, let $x, y \in J$; take $r, s \in R$ and consider the element $1 - r(x + y)s = 1 - rxs - rys$. Since $x \in J$, there is an inverse χ of $1 - rxs$; then $\chi(1 - rxs - rys) = 1 - \chi rys$; since $\chi r \in R$ and $y \in J$, this too has an inverse, υ; so

$$\begin{aligned} \upsilon\chi(1 - rxs - rys) &= \upsilon(1 - \chi rys) \\ &= 1 \end{aligned}$$

so $1 - r(x + y)s$ is a unit and hence $x + y \in J$. Thus J is non-empty, closed under addition and absorbs multiplication, hence it is an ideal. ■

It is worth noting that there are many alternative characterisations of the Jacobson radical, but we will not have need of them here.

Lemma 2.3 (The Jacobson-Azumaya Lemma (Lam 2001))
Let R be a ring and J the Jacobson radical of R. Let M be any non-zero, finitely generated left R-module (the result holds equally for right modules); then the submodule JM of M is proper: $JM \neq M$.
This result is sometimes called the non-commutative Nakayama Lemma.

Proof:
Because M is finitely generated, it must have a minimal generating set (take a finite generating set, if there is an element whose removal does not reduce the span, discard it, and

continue until all such redundant elements are removed); so take a finite, minimal generating set $X = \{x_1, \ldots, x_n\}$. Assume for a contradiction that $JM = M$; then the sum of the x_i is an element of $M = JM$, so there exist an natural number m and elements $m_j \in M$, $r_j \in J$ such that

$$\sum_{j=1}^{m} r_j m_j = \sum_{i=1}^{n} x_i$$

expressing each m_j as an R-linear combination of the x_i and rearranging terms, we see that there are elements $s_i \in J$ such that

$$\sum_{i=1}^{n} (1 - s_i) x_i = 0$$

Since s_i is an element of the Jacobson radical, $1 - s_i$ is a unit, so we may multiply this equation by $(1 - s_1)^{-1}$ to obtain

$$x_1 = -(1 - s_1)^{-1} \sum_{i=2}^{n} (1 - s_i) x_i$$

which expresses x_1 as an R-linear combination of the other x_i, showing that $X \setminus \{x_1\}$ is still a generating set for M, contradicting minimality of X. ■

The reason we introduce the Jacobson radical is to obtain the following result, which we will require shortly:

Corollary 2.4 (Nilpotent Ideals)
Let A be a k-algebra (for k any field) and I a finite-dimensional, two-sided ideal in A. Then I is nilpotent (as an ideal) if and only if every element of I is nilpotent.

PROOF:
First note that if $I^n = 0$ for some natural number n, then for any $x \in I$, $x^n \in I^n$, so $x^n = 0$ and x is nilpotent.
Conversely, suppose that all elements of I are nilpotent. Observe that, for any nilpotent element x, $1 - x$ is a unit (its inverse is the sum of all non-zero powers of x). Therefore, $I \subseteq J(A)$; for if $x \in I$, $r, s \in A$, then $rxs \in I$, so rxs is nilpotent and hence $1 - rxs$ is a unit.
Consider the descending chain of ideals

$$I \supseteq I^2 \supseteq I^3 \supseteq \ldots$$

Because I is finite-dimensional, this must become stationary; so for some n, $I^n = I^{n+1}$. Now, $I \subseteq J(A)$, so if I^n is non-zero, the Jacobson-Azumaya Lemma (2.3) tells us that we have a proper containment: $I(I^n) \subsetneq I^n$. This is a contradiction, so $I^n = 0$, proving that I is a nilpotent ideal. ■

We have seen that not all indecomposable quiver representations are irreducible, therefore the quiver algebra is not semi-simple and admits indecomposable modules which are not simple (at least in general - for some specific quivers, the path algebra *is* semi-simple; *e.g.*, the quiver with one vertex and no arrows, whose path algebra is simply k). It is therefore important to know when a module is indecomposable.

Let A be any algebra and M a left A-module. Suppose M is not indecomposable, so that $M \cong X \oplus Y$ for some A-modules X and Y; then there is an A-linear projection f of M onto the submodule X, so f is a non-trivial idempotent element of $\mathrm{End}_A(M)$, the endomorphism algebra of M. Conversely, suppose $f \in \mathrm{End}_A(M)$ is an idempotent, $f \neq 1_M$. Then M decomposes as $M \cong \ker(f) \oplus \mathrm{Im}(f)$. So M is indecomposable if and only if $\mathrm{End}_A(M)$ has no non-trivial idempotent.

With some restrictions, further criteria for indecomposability are provided by the following:

Lemma 2.5 (Criteria for Indecomposability (Brion 2008))
Let k be an algebraically closed field, A an algebra over k and M a finite-dimensional left module over A. Then the following are equivalent:

1. M is indecomposable;

2. All A-endomorphisms of M are either invertible or nilpotent;

3. As a vector space, $\text{End}_A(M) = I \oplus k1_M$ for some nilpotent ideal I.

PROOF:
(1. $\Rightarrow$ 2.)(Jacobson 2009): This result is known as Fitting's Lemma. Let f be an A-endomorphism of M and suppose f is not invertible. Then f is not bijective (as the inverse function of a bijective A-module morphism is automatically a morphism itself); but f is a linear transformation of a finite-dimensional vector space, so f is neither injective nor surjective, by the Rank-Nullity Theorem. So $\ker(f)$ is non-trivial.

Consider the sequence of submodules $K_n := \ker(f^n)$. Clearly, $K_n \subseteq K_{n+1}$, as $f^n(x) = 0$ implies $f^{n+1}(x) = 0$, so (K_n) is an increasing sequence of submodules. As M is finite-dimensional, this sequence must become stationary, so there exists some N such that $K_n = K_N$ for all $n \geq N$. Similarly, let $I_n := \text{Im}(f^n)$ to obtain a decreasing sequence, which must also become stationary, so that for some L, $I_n = I_L$ for all $n \geq L$. Let $m = \max\{L, N\}$ and $g = f^m$. Note that $\ker(g^2) = \ker(g)$ and $\text{Im}(g^2) = \text{Im}(g)$.

The intersection of $\ker(g)$ and $\text{Im}(g)$ is trivial. For suppose $x \in \ker(g) \cap \text{Im}(g)$; then $x = g(y)$ for some y, so $0 = g(x) = g^2(y)$, hence $y \in \ker(g^2) = \ker(g)$, so $x = g(y) = 0$. Since M is finite-dimensional, the Rank-Nullity Theorem now implies that M decomposes as a vector space as

$$M \cong \ker(g) \oplus \text{Im}(g)$$

Since g is A-linear, its kernel and image are submodules, so we have decomposed M as a direct sum of A-modules. As M was assumed to be indecomposable, one of these modules must be trivial. By assumption, $\ker(g) = \ker(f^m)$ is non-trivial, so $\text{Im}(g) = (0)$, *i.e.*, $f^m(x) = 0$ for all $x \in M$, meaning that f is nilpotent, as required.

(2. $\Rightarrow$ 3.): Let I be the set of all nilpotent elements of $E := \text{End}_A(M)$. We first show that I is an ideal. Let $x, y \in I$ be nilpotent endomorphisms and $z \in E$ any endomorphism; then $\ker(x) \neq 0$, so $\ker(zx) \neq 0 \neq \ker(xz)$, where the second inequality follows from finite-dimensionality, since either z is surjective and so includes the kernel of x in its image or is not surjective and hence also not injective, by Rank-Nullity. So zx and xz are not invertible, hence they are nilpotent, so $zx, xz \in I$. Now suppose $x + y$ is not nilpotent; then it is invertible, so for some endomorphism w, $w(x + y) = 1_M$. But wy is nilpotent by the above, hence $wx = 1_M - wy$ is invertible; but wx is also nilpotent, giving a contradiction.

So I is an ideal whose every element is nilpotent. But then Corollary 2.4 tells us that any such ideal is nilpotent. So I is a nilpotent ideal, as was to be shown.

Now we show the direct sum decomposition. Let f be any A-endomorphism of M. By assumption that k is algebraically closed, f has an eigenvalue, λ, and $f - \lambda 1_M$ has non-trivial kernel, hence is not invertible and so must be nilpotent; *i.e.*, $f - \lambda 1_M = x$ for some $x \in I$. Moreover, the expression $f = x + \lambda 1_M$ for $x \in I$, $\lambda \in k$ is unique, since if $x + \lambda 1_M = y + \mu 1_M$ for $y \in I$, $\mu \in k$, then $x - y = (\mu - \lambda)1_M$; the left-hand expression is nilpotent, so we must have $\mu = \lambda$ and hence also $x = y$. We have shown that every endomorphism can be uniquely written as a nilpotent endomorphism plus a scalar multiple of the identity, which proves the desired decomposition.

(3. $\Rightarrow$ 1.): First note that if I is a nilpotent ideal then for any $x \in I$ and $n \geq 1$, $x^n \in I^n$, so we must have that x is nilpotent; that is, all elements of our nilpotent ideal are nilpotent. Suppose $f \in \text{End}_A(M)$ is idempotent. Write $f = x + \lambda 1_M$ for x nilpotent and $\lambda \in k$. If $\lambda = 0$, then f is nilpotent and, as it is also idempotent, $f = 0$. On the other hand, if $\lambda \neq 0$, then f is invertible, so from the idempotency equation $f^2 = f$, we get $f = 1_M$. Therefore the only

idempotent endomorphisms of M are the trivial ones (zero and identity) and, by the comment preceeding this lemma, that implies M is indecomposable. ■

Because of the extreme utility of this lemma, we henceforth assume at all times that our base field k is algebraically closed and that all modules/representations are finite-dimensional. At times we may emphasise these assumptions further in the statement of important results, but they are always to be assumed.

We are now in a position to prove the central result of this section:

Theorem 2.6 (The Krull-Schmidt Theorem (Brion 2008))
Let k be an algebraically closed field, A an algebra over k and M a finite-dimensional left A-module. Then M can be decomposed as:

$$M \cong \bigoplus_{i=1}^{r} M_i^{\oplus m_i}$$

where the M_i are pairwise non-isomorphic indecomposable left modules. Moreover, this decomposition is unique up to reordering of the factors.

Furthermore, we have a decomposition of vector spaces (not algebras):

$$\mathrm{End}_A(M) = I \oplus B$$

where I is a nilpotent ideal and B is a subalgebra of A with $B \cong \bigoplus_{i=1}^{r} \mathrm{Mat}_{m_i \times m_i}(k)$ as algebras.

Proof:
The existence of the decomposition is straightforward and totally analagous to the proof that integers factor as products of primes: if M is indecomposable, we're done; otherwise, it splits as a direct sum of two submodules; iterating, we write M as a direct sum of an increasing number of submodules. Since M is finite dimensional, the process must terminate after at most as many steps as the dimension of M.

For the uniqueness part of the proof, it is convenient to temporarily rewrite our decomposition as:

$$M \cong \bigoplus_{i=1}^{R} \bar{M}_i$$

where $R = \sum_{i=1}^{r} m_i$ and $\bar{M}_i \cong M_1$ for $1 \leq i \leq m_1$, $\bar{M}_i \cong M_2$ for $m_1 < i \leq m_1 + m_2$, etc. This amounts to splitting up the repeated factors to write *e.g.*, $M_1 \oplus M_1$ instead of $M_1^{\oplus 2}$. Now suppose we have an alternative decomposition:

$$M \cong \bigoplus_{i=1}^{s} N_i^{\oplus n_i} = \bigoplus_{i=1}^{S} \bar{N}_i$$

where again the N_i are pairwise non-isomorphic, but the $\bar{N}_i$ are not.

There is an isomorphism f from the first decomposition (in the $\bar{M}_i$) to the second (in the $\bar{N}_i$), with inverse g. If $R = 1$ (so $r = 1$ and $m_1 = 1$), then f gives a decomposition of the indecomposable module $M = \bar{M}_1$ as a direct sum of the $\bar{N}_i$, so we must have $S = 1$, so that $M = M_1 \cong N_1$ and we're done. Now we induct on R. Define f_{ij} as the composition:

$$\bar{M}_j \hookrightarrow \bigoplus_{\alpha=1}^{R} \bar{M}_\alpha \xrightarrow{f} \bigoplus_{\alpha=1}^{S} \bar{N}_\alpha \twoheadrightarrow \bar{N}_i$$

so that f is the matrix (f_{ij}). Similarly, write $g = (g_{ij})$, where g_{ij} is the composition:

$$\bar{N}_j \hookrightarrow \bigoplus_{\alpha=1}^{S} \bar{N}_\alpha \xrightarrow{g} \bigoplus_{\alpha=1}^{R} \bar{M}_\alpha \twoheadrightarrow \bar{M}_i$$

Defining these functions without the indexing getting messy is the sole reason for our use of $\bar{M}_i$ and $\bar{N}_i$. Then $1_M = gf = (h_{ij})$ where

$$h_{ij} = \sum_{\alpha=1}^{S} g_{i\alpha} f_{\alpha j}$$

Therefore, considering $\bar{M}_1$, the first copy of M_1 in the decomposition of M, we have that:

$$1_{M_1} = h_{11} = \sum_{\alpha=1}^{S} g_{1\alpha} f_{\alpha 1}$$

Each $g_{1\alpha} f_{\alpha 1}$ is an endomorphism of $\bar{M}_1$, so is either invertible or nilpotent, by the above lemma. The nilpotent endomorphisms form an ideal, so it cannot be that all the $g_{1\alpha} f_{\alpha 1}$ are nilpotent, since their sum is not; so at least one of these endomorphisms is invertible. By reordering the $\bar{N}_i$ if necessary, we may assume without loss of generality that $g_{11} f_{11}$ is invertible. Then $(g_{11} f_{11})^{-1} f_{11}$ is a left inverse of g_{11}, so g_{11} is a section (and, in particular, injective) and hence we have the following split, short exact sequence:

$$0 \to N_1 \xrightarrow{g_{11}} M_1 \to M_1/g_{11}(N_1) \to 0$$

But M_1 is indecomposable, so either $N_1 = 0$, which is a contradiction, or $M_1/g_{11}(N_1) = 0$, in which case g_{11} is surjective. But we already know that g_{11} is injective, so g_{11} is bijective and hence an isomorphism, giving $M_1 \cong N_1$.

Now, taking our isomorphism between the two decompositions of M and quotienting both sides by $M_1^{\oplus m_1}$ (assuming without loss of generality that $m_1 \leq n_1$), we have:

$$\bigoplus_{i=2}^{r} M_i^{\oplus m_i} \cong M_1^{\oplus(n_1 - m_1)} \oplus \bigoplus_{i=2}^{s} N_i^{\oplus n_i}$$

This has one term fewer on the left hand side, so by the inductive hypothesis, the two decompositions have the same number of terms and consist of the same indecomposable modules. Then we must have $n_1 = m_1$, as otherwise $M_1 \cong M_i$ for some $i > 1$, contradicting the assumption that the M_i are pairwise non-isomorphic. Hence we conclude that $r = s$ and (after reordering) $M_i \cong N_i$ and $n_i = m_i$ for all i.

It remains to prove the statement about the endomorphism space of M. We do this by the method laid out in Brion (2008). By an earlier lemma, for any i, there is a decomposition of vector spaces $\mathrm{End}_A(M_i) \cong I \oplus k1_{M_i}$, with I a nilpotent ideal; therefore, there is a surjection of rings $\mathrm{End}_A(M_i) \twoheadrightarrow k$ with kernel I. This induces a surjection

$$u_i : \mathrm{Mat}_{m_i \times m_i}(\mathrm{End}_A(M_i)) \twoheadrightarrow \mathrm{Mat}_{m_i \times m_i}(k)$$

This in turn allows us to define a surjection

$$u : \mathrm{End}_A(M) \twoheadrightarrow \prod_{i=1}^{r} \mathrm{Mat}_{m_i \times m_i}(k)$$

as follows. Any endomorphism of M can be written as a matrix of maps between the indecomposable summands M_i. This matrix can be split into blocks of size $m_i \times m_j$ of maps from M_j to M_i; the blocks on the leading diagonal are matrices of endomorphisms of the indecomposables summands. We can therefore apply each function u_i to its corresponding diagonal block to obtain an $m_i \times m_i$ matrix over k. This gives a map as stated, whose surjectivity follows from that of the u_i. This map is clearly linear, so it induces an isomorphism of vector spaces

$$\mathrm{End}_A(M) \cong I \oplus \prod_{i=1}^{r} \mathrm{Mat}_{m_i \times m_i}(k)$$

where I denotes the kernel of the map.

It remains to show that I is a nilpotent, two-sided ideal. To show that I is an ideal, it suffices to prove that u is a map of rings. To this end, let f and g be two A-endomorphisms of M and write them as matrices $f = (f_{ij})$, $g = (g_{ij})$ for f_{ij} and g_{ij}. Then $fg = (h_{ij})$, where

$$h_{ij} = \sum_{l=1}^{r} f_{il} g_{lj}$$

So $u(fg) = (u(h_{ij}))$, the matrix whose entries are $u(h_{ij})$, so it suffices to prove that for any i, j,

$$u\left(\sum_{l=1}^{r} f_{il} g_{lj}\right) = \sum_{l=1}^{r} u(f_{il}) u(g_{lj})$$

Since u is linear, we need only show that for all i, j, l, $u(f_{il}g_{lj}) = u(f_{il})u(g_{lj})$. It is convenient to drop the subscripts; so we now prove that if $f : M_i \to M_l$ and $g : M_l \to M_j$ are maps of indecomposables, then $u(fg) = u(f)u(g)$.

If $M_i \neq M_j$, this is immediate, since u vanishes on maps between distinct indecomposables by definition. If $M_i = M_j \neq M_l$, then $u(f) = u(g) = 0$ and $fg : M_i \to M_i$ must be nilpotent (as otherwise it would have an inverse h and so gh would be a right-inverse of f, making M_i a summand of M_l, which is a contradiction); but u vanishes on nilpotent endomorphisms, so $u(fg) = 0 = u(f)u(g)$. Finally, if $M_i = M_j = M_l$, then we can uniquely write $f = n_f + \lambda_f 1_{M_i}$ and $g = n_g + \lambda_g 1_{M_i}$ for nilpotent endomorphisms n_f, n_g and scalars λ_f, λ_g. But then $fg = n_f n_g + \lambda_f n_g + \lambda g n_f + \lambda_f \lambda_g 1_{M_1}$, so $u(fg) = \lambda_f \lambda_g = u(f)u(g)$.

Moreover, $u(1)$ is clearly 1. So u is indeed a map of rings and hence I is an ideal.

Finally, to prove that I is nilpotent, it suffices to prove that every element of I is nilpotent, by the Jacobson-Azumaya Theorem (as in our earlier proof). An element f of the kernel is a matrix of non-isomorphisms f_{ij} between indecomposables; raising to a power, f^n is a matrix whose entries are length-n compositions among the f_{ij}. Choosing $n = N_{ij} r$ for some N_{ij}, the pigeonhole principle guarantees that one of the indecomposables M_l appears at least N_{ij} times in the composition appearing as the ij-entry of f^n. But this composition then involves N_{ij} non-invertible endomorphisms of M_l; since the non-invertible endomorphisms of an indecomposable form a nilpotent ideal, by choosing N_{ij} suitably large, we can make the ij-entry of f^n zero. This holds for arbitrary i, j, so choosing N to be the maximum of the N_{ij}, we see that $f^{Nr} = 0$. ∎

So we see that if we wish to classify the left modules of an algebra of any given (finite) dimension, it suffices to classify the indecomposable left modules of that dimension and all smaller dimensions.

2.2 Projective Modules

We now apply the above result to A itself, viewed as an A-module, to find about more about the structure of an algebra. Since the above result assumes that the module being studied is finite-dimensional, we henceforth assume (in addition to algebraic closure of k and finite-dimensionality of all modules) that all algebras are also finite-dimensional. For the path algebra of a quiver, this amounts to assuming that the quiver is finite and acyclic.

Lemma 2.7 (Decomposition of an Algebra Itself (Brion 2008))
Let A be any algebra, regarded as an A-module by left multiplication. Then:

1. There is a bijective correspondence between decompositions of unity as a sum of orthogonal idempotents, $1_A = e_1 + \ldots + e_n$ and direct sum decompositions of A: $A = Ae_1 \oplus \ldots \oplus Ae_n$;

2. (Yoneda's Lemma) For any left A-module M and idempotent e of A, we have a (natural) isomorphism *of vector spaces*: $\mathrm{Hom}_A(Ae, M) \cong eM$ given by $f \mapsto f(e)$;

3. There is an algebra isomorphism $\mathrm{End}_A(Ae) \cong (eAe)^{\mathrm{op}}$, where eAe is viewed as an algebra with unity e;
4. Ae is an indecomposable A-module if and only if e cannot be decomposed as the sum of orthogonal idempotents; equivalently, e is the unique (non-zero) idempotent of eAe; if this is the case, we say that e is a *primitive* idempotent.

PROOF:

1. This follows immediately from the fact that $\mathrm{End}_A(A) \cong A^{\mathrm{op}}$ (the isomorphism assigns to $a \in A$ the right multiplication map $x \mapsto xa$, the inverse is the evaluation map at unity, $f \mapsto f(1)$), so that Ae is the result of applying an idempotent endomorphism (*i.e.*, a projection) to A.
2. $f(e) = f(e^2) = ef(e) \in eM$, so the given map is well-defined. Its linearity is clear. To show that it is injective, suppose that $f(e) = g(e)$; then for a general element xe of Ae, $f(xe) = xf(e) = xg(e) = g(xe)$ by A-linearity, so f and g agree on their domain and so are equal. Finally, to show that the map is surjective, let em be an arbitrary element of eM; note that if there exists an A-linear map $f : A \to M$ with $f(1) = m$, then $f|_{Ae} \in \mathrm{Hom}_A(Ae, M)$ and $f|_{Ae}\ (e) = f(e) = ef(1) = em$, so it suffices to prove that there is such an f; but this is straightforward to construct by A-linearly extending from the defining relation $f(1) = m$. We omit the proof that this isomorphism is natural for variation of M; it is not difficult.
3. By part 2., with $M = Ae$, we have a linear isomorphism: $\mathrm{End}_A(Ae) \cong eAe$, given by $f \mapsto f(e)$; to show that this is an algebra isomorphism when eAe is endowed with the opposite algebra structure, observe that $1_{Ae}(e) = e = 1_{eAe}$ and, for any two A-endomorphisms f and g of Ae, we have $g(e) \in Ae$, so $g(e) = \alpha e = \alpha e^2 = g(e)e$ for some $\alpha \in A$, whence $fg(e) = f(g(e)e) = g(e)f(e)$ by A-linearity, since $g(e) \in Ae \subseteq A$ and so $g(e)$ can be pulled out of an A-linear map such as f. So multiplying f with g in $\mathrm{End}_A(Ae)$ and then mapping to eAe corresponds to mapping them to eAe and then multiplying in the opposite order.
4. Ae is indecomposable if and only if there are no non-trivial idempotents in $\mathrm{End}_A(Ae)$; by part 3., $\mathrm{End}_A(Ae) \cong (eAe)^{\mathrm{op}}$, so Ae is indecomposable iff e is the unique idempotent of eAe (as taking the opposite algebra structure has no effect on idempotents). If $e = e_1 + \ldots + e_n$ for orthogonal idempotents $e_1, \ldots, e_n$, then $Ae = A(e_1 + \ldots + e_n) = Ae_1 \oplus \ldots \oplus Ae_n$, so Ae is not indecomposable. Finally, if $e_1 \neq e$ is another idempotent in eAe, then $(e - e_1)^2 = e^2 - e_1 e - ee_1 + e_1^2 = e - e_1$, as e is the identity of eAe, and $e_1(e - e_1) = e_1 e - e_1^2 = 0$, so $e = e_1 + (e - e_1)$ is a sum of orthogonal idempotents. ■

In the case of a quiver Q and its path algebra kQ, the identity 1_{kQ} is the sum of the vertices (the paths of length 0); if v is a vertex, then kQv is precisely the set of all linear combinations of paths starting at v. Then the endomorphism algebra of kQv is $vkQv^{\mathrm{op}}$, the set of all linear combinations of paths starting *and* finishing at v - *i.e.*, of all oriented cycles at v (with multiplication reversed). kQv corresponds to the representation of Q where the vector space on a vertex w is $wkQv$, the span of all paths from v to w, and an arrow a between vertices u and w is the function composing each path p from v to u with a on the left to obtain a path ap from v to w.

Returning to the general case, we now consider a special type of module called a *projective* module.

Lemma 2.8 (Projective Modules (Rotman 2009))
Let R be a ring and P a left R-module. Then the following are equivalent:

1. Given maps f and s of left R-modules as shown:

$$\begin{tikzcd} & X \arrow[d, "s"] \\ P \arrow[ur, dashed, "g"] \arrow[r, "f"'] & Y \end{tikzcd}$$

where s is a surjection, there exists a map $g : P \to X$ as shown making the diagram commute (we say that f *lifts through* s *via* g or that g *lifts* f *through* s).

2. The covariant Hom-functor $\mathrm{Hom}_R(P, \circ)$ is exact.

3. Every short exact sequence ending in P splits; that is, whenever $f : M \to P$ is a surjection, $M \cong P \oplus \ker(f)$.

4. P is a direct summand of a free R-module.

PROOF:

(1. $\Rightarrow$ 2.): Hom-functors are always left-exact, so the content of statement 2. is that given a surjection $s : X \to Y$, the induced map $s_* : \mathrm{Hom}_R(P, X) \to \mathrm{Hom}_R(P, Y)$ is also surjective; *i.e.*, given any map $f : P \to Y$, there exists $g : P \to X$ such that $f = s_*(g) = sg$. But this is precisely statement 1.

(2. $\Rightarrow$ 3.): Let $f : M \to P$ be surjective. Then $f_* : \mathrm{Hom}_R(P, M) \to \mathrm{Hom}_R(P, P)$ is surjective, so there exists $g : P \to M$ such that $f_*(g) = 1_P$, *i.e.*, $1_P = fg$ and f is a retraction, proving 3.

(3. $\Rightarrow$ 4.): P is certainly a quotient of a free module (as all modules are), but 3. tells us that whenever P is a quotient of a module, it is also a summand of that module, proving 4.

(4. $\Rightarrow$ 1.): We first prove that free modules have this lifting property. Let F be a free module, with R-basis $\{b_i\}_{i \in I}$ for some indexing set I, and consider a diagram:

$$\begin{tikzcd} & X \arrow[d, "s"] \\ F \arrow[r, "f"] & Y \end{tikzcd}$$

with s a surjection. Then for every $i \in I$ there exists $x_i \in X$ such that $s(x_i) = f(b_i)$. Define $g : F \to X$ by setting $g(b_i) = x_i$ and extending A-linearly. Then $sg = f$ on a basis of F and so by R-linearity, g lifts f through s everywhere. So free modules have the lifting property.

Now let F be a free module of which P is a direct summand. Given $f : P \to Y$ and $s : X \to Y$ as in the statement of the lemma, define a map $F \to Y$ as the composition $F \twoheadrightarrow P \to Y$ and, as F is free, lift this map through s to obtain a map $F \to X$. Then define $g : P \to X$ to be the restriction of this map to $P \subseteq F$; by construction, g lifts f through s, showing that P has property 1. ∎

Definition 2.9 (Projective Modules)
Let R be a ring. We say a left R-module is *projective* if it meets the equivalent conditions of the above lemma 2.8. In particular, we are interested in the case where R is in fact an algebra - *i.e.*, it is a ring and also a vector space.

A useful consequence of the fact that projective modules are precisely direct summands of free modules is that any summand of a projective module is itself projective, as is any direct sum of projectives.

Applying this to our earlier classification of the direct summands of A when regarded as an A-module, we see that for every idempotent e, Ae is a direct summand of the free A-module A, hence is projective. If A is finite-dimensional, this turns out to give us a complete classification of the projective A-modules:

Lemma 2.10 (Classification of Projective Modules (Brion 2008))
Let A be a finite dimensional algebra which decomposes as an A-module as:

$$A = \bigoplus_{i=1}^{r} M_i^{\oplus m_i}$$

for indecomposable and pairwise non-isomorphic modules M_i. Then:

1. There is an isomorphism *of vector spaces* $A \cong I \oplus B$ where I is a nilpotent ideal and B is a subalgebra isomorphic to $\bigoplus_{i=1}^{r} \mathrm{Mat}_{m_i \times m_i}(k)$ (and hence is semi-simple).

2. Let $S_i := M_i / IM_i$, for I as above. Then the modules S_i are simple and, moreover, every simple A-module is of this form.

3. Every projective and indecomposable A-module is isomorphic to one of the M_i.

4. Every projective A-module P can be uniquely decomposed:

$$P \cong \bigoplus_{i=1}^{r} M_i^{\oplus p_i}$$

where the p_i are non-negative integers.

Proof:

1. 1_A is an idempotent of A, so $\mathrm{End}_A(A) \cong A^{\mathrm{op}}$. Moreover, $\mathrm{End}_A(A) \cong B \oplus I$ as vector spaces, for B and I as in the statement. The result follows.

2. First note that $IM_i \subsetneq M_i$, as $I \subseteq A$ and M_i is an A-module, so $IM_i \subseteq M_i$, and if equality held, we would have $I^n M_i = M_i$ for all n, contradicting nilpotency of I; so S_i is well-defined and non-trivial. $A/I \cong B$ and I is in the kernel of the action of A on S_i, so each S_i may be regarded as a B-module. B is a subalgebra of A, so any A-submodule of S_i is also a B-submodule; so it suffices to prove that S_i is simple as a B-module.

   B is a semi-simple algebra, so S_i is a semi-simple B-module. S_i is a quotient module of M_i, and only the direct summand of B corresponding to M_i acts non-trivially on M_i, hence only this acts non-trivially on S_i - *i.e.*, S_i is in fact a $\mathrm{Mat}_{m_i \times m_i}(k)$-module or, which is the same thing, a direct sum of m_i-dimensional vector spaces, each acted on independently by $\mathrm{Mat}_{m_i \times m_i}(k)$ or, equivalently, by B. So to show that S_i is simple, it suffices to show that it has dimension m_i.

   By the definition of S_i, we see that $\dim(S_i) = \dim(M_i) - \dim(IM_i)$. Consider the map f defined by restricting to M_i the canonical projection $F : A \twoheadrightarrow A/I \cong B$. f has kernel $I \cap M_i$ and image isomorphic to k^{m_i} (to see this, note that the restriction of F to $M_i^{\oplus m_i}$ has image $\mathrm{Mat}_{m_i \times m_i}(k) \subseteq B$ and that, as F treats each copy of M_i the same, the restriction of F to $M_i^{\oplus m_i}$ has image $\mathrm{Im}(f)^{\oplus m_i}$). Hence, by the rank-nullity theorem, $\dim(M_i) - \dim(I \cap M_i) = m_i$, so it suffices to prove that $\dim(I \cap M_i) = \dim(IM_i)$.

   As I is a bi-ideal and M_i a left-ideal, we have that $IM_i \subseteq I \cap M_i$, so we need only show the reverse inclusion. Let x be an element of $I \cap M_i$ and n be the smallest integer such that $x^n = 0$ (as $x \in I$ is nilpotent). Let $p : A \twoheadrightarrow M_i$ be the canonical projection; then $p(x) = p(x1_A) = xp(1_A)$, but $x \in M_i$, so $x = p(x) = xp(1_A)$. Now, $p(1_A) \in M_i$ and $x^{n-1} \in M_i$, so $p(1_A) + x^{n-1} \in M_i$; we see that $x = x(p(1_A) + x^{n-1})$ is a factorisation of x as an element of I multiplied by an element of M_i, so $x \in IM_i$, proving the claim.

   To show that every simple A-module S has this form, simply note that S is also a B-module, so there is a decomposition of B-modules

$$S = \bigoplus S_i^{\oplus n_i}$$

   for some non-negative integers n_i not all zero. But each of these summands is also an A-module, so this is also an A-module decomposition. Since S is simple as an A-module, we must have that, for some j, $n_j = 1$ and $n_i = 0$ for all $i \neq j$, so $S = S_j$.

3. Let P be a projective, indecomposable module. Then P is a direct summand of a free A-module A^n for some n. We have a (unique) decomposition into indecomposables:

$$A^n \cong \bigoplus_{i=1}^{r} M_i^{\oplus nm_i}$$

so we see that P, being indecomposable, must be isomorphic to one of the M_i.

4. Now let P be any projective module. There is a unique decomposition into indecomposables:

$$P \cong \bigoplus_{i=1}^{s} N_i^{\oplus n_i}$$

P is a direct summand of a free A-module, so the N_i are too. Hence each N_i is projective as well as indecomposable, and so by part 3., $N_i \cong M_j$ is a summand of A, for some j. ■

We now look at the consequences of this result in the context of quivers. Let Q be a quiver with n vertices and no oriented cycles (so that the path algebra is finite dimensional); label the vertices $v_1, \ldots, v_n$ and let P_i be the kQ-module kQv_i. So $kQ \cong \bigoplus_{i=1}^n P_i$ and each P_i is spanned by the paths starting at v_i and is both indecomposable and projective (and all projective indecomposables have this form). Observe that each projective indecomposable has multiplicity 1 as a summand of kQ.

The endomorphism algebra $\mathrm{End}_{kQ}(P_i) \cong v_i kQ v_i^{\mathrm{op}}$ is the linear span of all oriented cycles at v_i; but, by assumption, there are none, except for the trivial cycle of length 0 corresponding to the vertex itself, so $\mathrm{End}_{kQ}(P_i) = k$.

Let us now find the ideal I and subalgebra B from the statement of the lemma. If we let $kQ_{\geq j}$ denote the span of all oriented paths in Q of length at least j, then $kQ_{\geq j}$ is an ideal of kQ, as it is closed under addition by definition and composing a path of length at least j with another path can only increase the length or give the zero element. Moreover, for $j \geq 1$, $(kQ_{\geq j})^n = kQ_{\geq jn}$; as we are assuming that there are no oriented cycles (so all paths have finite length), this means that $kQ_{\geq j}$ is a nilpotent ideal for all $j \geq 1$. Any element of kQ is a linear combination of vertices plus a linear combination of paths of length at least 1, so we have an isomorphism of vector spaces $kQ \cong kQ_0 \oplus kQ_{\geq 1}$, where Q_0 is the vertex set of Q. But kQ_0 is simply a vector space of dimension $|kQ_0|$, hence it is a direct sum of (1-dimensional) matrix algebras.

So we have decomposed kQ as a vector space into the direct sum of a nilpotent ideal $kQ_{\geq 1}$ with a direct sum of matrix algebras, as in the statement of the above lemma. Moreover, the matrix algebras which are the summands of kQ_0 are all 1-dimensional, corresponding to the fact that all indecomposable summands of kQ have multiplicity 1, as according to the theorem.

Applying the lemma with this decomposition in mind, we see that $S_i := P_i/kQ_{\geq 1}P_i$ (where $P_i = kQv_i$, as above) is simple for all i and that this is a complete classification of all simple kQ-modules. S_i is the span of all paths starting at v_i, modulo those of length at least 1; thus it is isomorphic to the span of all length 0 paths starting at v_i; but there is exactly one of these, v_i itself, so S_i is 1-dimensional. As a representation, S_i has a one-dimensional vector space at v_i and the zero vector space at all other vertices; all arrows are of course represented by the zero map.

We see from the above that the path algebras of acyclic quivers have no simple modules of dimension greater than 1. This is not true for quivers with oriented cycles. For consider the quiver with two vertices v and w and two arrows: $x : v \to w$ and $y : w \to v$:

$$\begin{tikzcd} v \arrow[r, bend left, "x"] & w \arrow[l, bend left, "y"] \end{tikzcd}$$

This has a simple representation corresponding to a 2-dimensional module where both vertices have a 1-dimensional vector space and both linear maps are the identity. To see that this is indeed simple, note that any (proper, non-trivial) subrepresentation must have one of the vector spaces (without loss of generality, v) be zero and the other 1-dimensional; then the following diagram for the arrow y does not commute, so this is not a subrepresentation:

$$\begin{tikzcd} k & k \arrow[l, "1_k"'] \\ 0 \arrow[u, hook] & k \arrow[l] \arrow[u, hook] \end{tikzcd}$$

Next, we shall look closer at projective modules and discover a sense in which path algebras of quivers are almost semi-simple.

Chapter 3

Hereditary Algebras

This chapter will explore the failure of Maschke's Theorem for quivers by introducing the notion of a hereditary algebra, which is in some sense 'almost semisimple'. We will see that path algebras of quivers are hereditary and that they are representative of all hereditary algebras.

3.1 Projective & Global Dimension

Definition 3.1 (Projective Resolutions)
Let A be a k-algebra and M and A-module. A *projective resolution* of M is an exact sequence:

$$\ldots \to P_n \to \ldots P_1 \to P_0 \to M \to 0$$

where every module P_i is projective. The largest m such that $P_m \neq 0$ is called the *length* of the resolution, when it exists; otherwise, the resolution is said to have infinite length. Note that a resolution $0 \to P_0 \to M \to 0$ is said to have length 0 - the length is *one less than* the number of projective modules P_i.

Any finitely generated module is a quotient of a free module; the kernel of this quotient map must itself be a quotient of a free module, and so *ad infinitam.* So we can construct an infinite exact sequence

$$\ldots F_1 \to F_0 \to M \to 0$$

for free modules F_n. Since all free modules are projective, this is a projective resolution for M, so all finitely generated modules have projective resolutions (possibly infinite). Since all modules we consider are even finite-dimensional, they have projective resolutions. They will often in fact turn out to have finite resolutions.

Definition 3.2 (Projective and Global Dimensions)
Let A and M be as above. The minimum length of any projective resolution of M is called the *projective dimension* of M. That is, the projective dimension $\mathrm{pd}(M)$ is the least integer such that all projective resolutions of M have length at least $\mathrm{pd}(M)$, or is infinite in the case that M admits no finite projective resolution. Clearly, a module M has projective dimension 0 if and only if it is itself projective, as then the resolution becomes simply an isomorphism between M and P_0.

The *global dimension* of A, $\mathrm{gd}(A)$, is the maximum projective dimension of any of its modules; that is, the least integer such that all A-modules have projective dimension at most $\mathrm{gd}(A)$, or infinity if A has a module of infinite projective dimension or modules of arbitrarily high projective dimension.

Lemma 3.3 (Global Dimension & Semi-Simplicity (Rotman 2009))
A finite-dimensional algebra has global dimension 0 if and only if it is semi-simple (which, by the Artin-Wedderburn Theorem, occurs if and only if it is a direct sum of matrix algebras).

Proof:

Let A be a finite-dimensional, semi-simple algebra. Then A is a direct sum of matrix algebras. A matrix algebra is a direct sum of its unique simple module (with multiplicity equal to its dimension) and the simple modules of a direct sum of algebras are simply those of the summands, so all simple modules of A are direct summands of A, the free module of rank 1, hence are projective. Then any other module is a direct sum of these projective, simple modules, hence is also projective. So any A-module M is projective and hence has projective dimension 0. All modules have projective dimension 0, so the global dimension of A is 0.

Conversely, let A be an algebra with $\mathrm{gd}(A) = 0$. By a previous lemma, A is isomorphic *as a vector space* to $I \oplus B$ for a nilpotent ideal I and a semi-simple subalgebra B. So it suffices to prove that I is 0, as then A has equal dimension to its subalgebra B and so is equal to it. Let M be any indecomposable A-module; by nilpotency, $IM \subsetneq M$, so $S := M/IM$ is non-trivial. We have a short exact sequence:

$$0 \to IM \to M \to S \to 0$$

But A has global dimension 0, so S has projective dimension 0, so S is projective; hence this sequence splits and we have $M \cong IM \oplus S$. M is indecomposable and $S \neq 0$, so we must have $IM = 0$. M was an arbitrary indecomposable and all modules are direct sums of indecomposables, so $IN = 0$ for any module N. In particular, $IA = 0$; but $I = I \times 1 \subseteq IA = 0$, so $I = 0$, completing the proof. ■

In light of this, global dimension may be regarded as a measure of how far an algebra is from being semisimple. We see that quiver algebras generally have positive global dimension, as we have seen an example of an indecomposable quiver representation which had a subrepresentation. Of course, some particular quiver algebras have global dimension 0. For instance, a quiver with n vertices and no arrows has as its path algebra the direct sum of n copies of k; this is semi-simple, so has global dimension 0. In fact, we shall shortly show that the global dimension of the path algebra of an acyclic quiver is never more than 1. This gives a sense in which quiver algebras are 'almost' semi-simple.

3.2 Hereditary Algebras

Before we introduce the main concept of this section, we require a technical result:

Lemma 3.4 (Schanuel's Lemma (Rotman 2009))
Let A be an algebra and M, L_i and P_i modules over A, for $i \in \{1, 2\}$, such that P_1 and P_2 are projective and the sequences

$$0 \to L_1 \xrightarrow{i} P_1 \xrightarrow{f} M \to 0 \quad \text{and} \quad 0 \to L_2 \xrightarrow{j} P_2 \xrightarrow{g} M \to 0$$

are exact. Then $L_1 \oplus P_2 \cong L_2 \oplus P_1$.

Proof:
We seek to construct a short exact sequence

$$0 \to L_1 \xrightarrow{\alpha} L_2 \oplus P_1 \xrightarrow{\beta} P_2 \to 0$$

As P_2 is projective, such a sequence must split, and the result follows immediately.

To define β, observe that f is surjective, so by the lifting property of projective modules applied to P_2, g lifts through f; that is, there exists a map $p : P_2 \to P_1$ such that $fp = g$. By a symmetrical argument, there exists $q : P_1 \to P_2$ lifting f through g (so $gq = f$). Now, given $z \in P_2$, we can write $z = (z - qp(z)) + qp(z)$. That $qp(z)$ is in the image of q is clear; moreover, $z - qp(z) = j(x_z)$ for some $x_z \in L_2$, since

$$\begin{aligned} g(z - qp(z)) &= g(z) - gqp(z) \\ &= g(z) - g(z) \\ &= 0 \end{aligned}$$

and $\ker(g) = \operatorname{Im}(j)$. Define $m : P_2 \to L_2$ by $m(z) = x_z$; this is well-defined, as j is injective, so x_z is unique. So every element of P_2 is the sum of an element in the image of j and an element in the image of q; thus we define $\beta : L_2 \oplus P_1 \to P_2$ by $\beta(x, y) = j(x) + q(y)$; the foregoing discussion shows that for $z \in P_2$, $z = \beta(m(z), p(z))$, so β is surjective.

To define the map α, we seek maps $L_1 \to L_2$ and $L_1 \to P_1$. For the second of these, the inclusion i is the obvious choice. For the first, take an element $x \in L_1$ and note that $gqi(x) = fi(x) = 0$, so $qi(x) \in \ker(g) = \operatorname{Im}(j)$, so there exists $y_x \in L_2$ with $j(y_x) = qi(x)$. This y_x is unique by the injectivity of j, so define $n : L_1 \to L_2$ by $n(x) = y_x$ and $\alpha : L_1 \to L_2 \oplus P_1$ by $\alpha(x) = (n(x), -i(x))$ (the reason for the minus sign will shortly become apparent). Since i is injective, $\alpha(x) = (0, 0)$ occurs only when $x = 0$, so α is injective.

We have our sequence; it remains to check that it is exact at $L_2 \oplus P_1$; *i.e.*, that $\ker(\beta) = \operatorname{Im}(\alpha)$. To show that $\operatorname{Im}(\alpha) \subseteq \ker(\beta)$, note that $\beta(\alpha(x)) = \beta(n(x), -i(x)) = jn(x) - qi(x)$. By definition of $n(x)$, $jn(x) = qi(x)$, so we see that $\beta\alpha = 0$ (hence the minus sign in the definition of α). For the reverse inclusion, suppose that $\beta(x, y) = 0$; then $j(x) = q(-y)$, so $0 = gj(x) = gq(-y) = f(-y)$, so $-y \in \ker(f) = \operatorname{Im}(i)$ and $-y = i(z)$ for some $z \in L_1$. Moreover, $n(z)$ is that unique element of L_2 such that $jn(z) = qi(z) = q(-y) = j(x)$, so $n(z) = x$. Thus $(x, y) = (n(z), -i(z)) = \alpha(z)$ for some $z \in L_1$ and our sequence is indeed exact, completing the proof. ■

Definition 3.5 (Hereditary Algebras)
An algebra A is said to be *left hereditary* if for every projective left module P, all submodules of P are also projective - that is, projectiveness is inherited by submodules.

Right hereditary algebras are defined similarly; in fact, these two conditions are equivalent, so we henceforth write simply 'hereditary'. We do not prove the symmetry here, as we make no use of it - any use of 'hereditary' in this text may be read as left- or right-hereditary as appropriate. For proof of the symmetry, see Assem, Simson, and Skowroński (2006).

Theorem 3.6 (Global Dimension & Heredity (Rotman 2009))
An algebra is hereditary if and only if its global dimension is at most 1.

Proof:

Let A be an algebra. If $\operatorname{gd}(A) = 0$, then all modules are projective, so the result is trivial. So suppose $\operatorname{gd}(A) = 1$ and let P be a projective module with a submodule M. There is a short exact sequence

$$0 \to M \to P \to P/M \to 0$$

If P/M has no projective resolution of length 1, then it must only have projective resolutions of length 0, so it is projective. Hence the above sequence splits and M is a direct summand of P and hence is projective. Otherwise, take a projective resolution of P/M of length 1:

$$0 \to Q_1 \to Q_0 \to P/M \to 0$$

This gives us a second short exact sequence terminating in P/M and with a projective middle term, so we may apply Schanuel's Lemma to see that $M \oplus Q_0 \cong Q_1 \oplus P$. $Q_1 \oplus P$ is a direct sum of projectives, so it too is projective, but then $M \oplus Q_0$ is projective and so M is again a direct summand of a projective module. This proves that an algebra of global dimension at most 1 is hereditary.

Conversely, suppose A is hereditary and let M be a module. Then M has a presentation; that is, there exists a short exact sequence

$$0 \to K \to F \to M \to 0$$

for F a free module. Free modules are projective, so F is projective; as A is hereditary, K is projective too; so this is a projective resolution for M of length 1, proving that M has projective dimension at most 1. M was arbitrary, so $\operatorname{gd}(A) \leq 1$. ■

We have seen that global dimension can be regarded as a measure of how far an algebra is from being semisimple. So, in light of this result, a hereditary algebra can be viewed as 'almost' semisimple.

3.3 The Standard Resolution

Let A be an algebra over a field k and suppose we have a decomposition of unity into primitive, pairwise orthogonal idemopotents $e_i \in A$:

$$1 = \sum_{i=1}^{n} e_i$$

Then for any left A-module M, we have a decomposition of vector spaces

$$M = 1M = \sum_{i=1}^{n} e_i M$$

Moreover, this sum is direct, since if $e_i m = e_j \mu$ for some $m, \mu \in M$, then

$$0 = e_i e_j \mu = e_i^2 m = e_i m = e_j \mu$$

so the intersection of any two of these spanning spaces is zero. So we can decompose any module M as a direct sum of subspaces $e_i M$; thus we can study M by studying the $e_i M$, though it is important to note that these will generally only be subspaces, not submodules.

In order to understand the action of A on M, we should consider the orbit of $e_i M$ under A. Since $e_i^2 = e_i$, A acts on $e_i M$ as Ae_i; that is, for $a \in A, m \in M$, $a(e_i m) = (ae_i)(e_i m)$, so if $ae_i = be_i$ for some other $b \in A$, then the actions of a and b on $e_i M$ are identical. So it suffices to study the restricted action of Ae_i on M.

Of course, the action of Ae_i on $e_i M$ has image in M, not just in $e_i M$; this gives a bilinear map $Ae_i \times e_i M \to M$. In order to keep all maps linear, it is more convenient to write this as a linear map from the tensor product space:

$$f_i : Ae_i \otimes_k e_i M \to M : ae_i \otimes e_i m \mapsto ae_i m$$

In fact, this map is readily seen to be A-linear with respect to the natural left A-module structure on $Ae_i \otimes_k e_i M$

We wish to consider this action for each i, so it is convenient to combine them all into one, rather than needing to consider each in turn. So we obtain a map

$$F = (f_1, \ldots, f_n) : \bigoplus_{i=1}^{n} Ae_i \otimes_k e_i M \to M$$

which is precisely the action of A on M - that is, for $a \in A$, $m \in M$, we can write $a = ae_1 + \ldots + ae_n$ and so

$$\begin{aligned} am &= \sum_{i=1}^{n} ae_i m \\ &= F(ae_1 \otimes e_1 m, \ldots, ae_n \otimes e_n m) \end{aligned}$$

In particular, taking $a = 1$ in the above, we see that F is surjective.

Applying this in the case where $A = kQ$ is the path algebra of a finite quiver Q, the vertices are pairwise orthogonal, primitive idempotents and their sum is unity, so for any kQ-module M we have a surjection

$$\bigoplus_{v \in Q_0} kQv \otimes_k vM \twoheadrightarrow M$$

The obvious question which now presents itself is: what is the kernel? For any arrow $\alpha : v \to w$ (for vertices v and w), we have $\alpha v = w\alpha$, so for any $m \in M$, $\alpha v m = \alpha m = w \alpha m$;

so for any $p \in kQ$, $p\alpha \otimes vm$ and $pw \otimes \alpha m$ both map to $p\alpha m$; so the kernel should contain all elements of the form $p\alpha \otimes vm - pw \otimes \alpha m$ for α an arrow from v to w. We can express all (linear combinations of) elements of this form as the image of a map

$$i : \bigoplus_{\alpha \in Q_1} kQt(\alpha) \otimes_k s(\alpha)M \to \bigoplus_{v \in Q_0} kQv \otimes_k vM$$

defined by:

$$pt(\alpha) \otimes s(\alpha)m \mapsto p\alpha \otimes s(\alpha)m - pt(\alpha) \otimes \alpha m$$

Saying that the image of this map lies in the kernel of F is just expressing the associativity $(p\alpha)m = p(\alpha m)$ in a complicated way. This complexity will pay dividends shortly! It turns out that, for a finite quiver, every element of the kernel arises in this way and that the map i is injective. We have the following:

Theorem 3.7 (The Standard Resolution (Brion 2008))
Let Q be a finite quiver and k a field. Then there is a short exact sequence

$$0 \to \bigoplus_{\alpha \in Q_1} kQt(\alpha) \otimes_k s(\alpha)M \xrightarrow{i} \bigoplus_{v \in Q_0} kQv \otimes_k vM \xrightarrow{F} M \to 0$$

where

$$i(p \otimes m) = p\alpha \otimes m - p \otimes \alpha(m)$$

for p a path starting at $t(\alpha)$ and $m \in s(\alpha)M$ and

$$F(p \otimes m) = pm$$

for p a path starting at v and $m \in vM$. Moreover, the first two terms in this sequence are projective, so this is a projective resolution for M.

PROOF:
(Crawley-Boevey 1992) The surjectivity of F and the fact that $F \circ i = 0$ were proved in the discussion above, so to prove exactness, it remains to show that i is injective and that $\ker(F) \subseteq \mathrm{Im}(i)$.

Firstly, take an arbitrary element x of the middle term in the sequence; we can write

$$x = \sum_{v \in Q_0} \sum_{p \in kQv} p \otimes m_p$$

for elements $m_p \in vM$ almost all zero. Define the *degree* of x to be the maximum length of a path p with $m_p \neq 0$. Given any path $p \in kQv$ of positive length l, we can write $p = q_p a_p$ for q_p a path of length $l-1$ and a_p an arrow starting at v; regarding $q_p \otimes m_p$ as an element of $kQt(a_p) \otimes_k vM$, we have

$$\begin{aligned} i(q_p \otimes m_p) &= q_p a_p \otimes m_p - q_p \otimes a_p m_p \\ &= p \otimes m_p - q_p \otimes a_p m_p \end{aligned}$$

We can use this to adjust the degree of x by adding an element of the image of i. If x has degree $d > 0$, then

$$x - \sum_{v \in Q_0} \sum_{p \in kQ_d v} i(q_p \otimes m_p)$$

has degree strictly less than d. That is, we take all paths of length d starting at a given vertex, write each as the composition of a shorter path q_p and an arrow, then subtract $i(q_p \otimes m_p)$ to cancel the $p \otimes m_p$ term in x; we proceed to do this over all vertices. By induction, this shows that every coset $x + \mathrm{Im}(i)$ contains an element of degree 0.

Hence, to prove that $\ker(F) \subseteq \mathrm{Im}(i)$, we can take $x \in \ker(F)$ and choose y with $y + \mathrm{Im}(i) = x + \mathrm{Im}(i)$ and $\deg(y) = 0$; moreover, $F(y) = F(x) = 0$, as $\mathrm{Im}(i) \subseteq \ker(F)$. So it suffices to prove that any element of $\ker(F)$ which has degree zero is in fact zero, as then $x+\mathrm{Im}(i) = y+\mathrm{Im}(i) = \mathrm{Im}(i)$.

So let $y \in \ker(F)$ have degree zero. Then y can be written

$$y = \sum_{v \in Q_0} v \otimes m_v$$

and we have

$$0 = F(y) = \sum_{v \in Q_0} vm_v$$

Multiplying this equation by any vertex v simply extracts vm_v from the right-hand side, forcing $vm_v = 0$ for all v. Since $vm_v = m_v$ (as $m_v \in vM$), y is in fact zero. This completes the proof of exactness at the middle term.

We now show that i is injective. An element x of the first term in the sequence can be written

$$x = \sum_{\alpha \in Q_1} \sum_{p \in kQt(\alpha)} p \otimes m_{\alpha p}$$

for $m_{\alpha p} \in s(\alpha)M$ almost all zero. Then

$$i(x) = \sum_{\alpha \in Q_1} \sum_{p \in kQt(\alpha)} (p\alpha \otimes m_{\alpha p} - p \otimes \alpha m_{\alpha p})$$

If $x \neq 0$, we can take a path p of maximal length with $m_{\alpha p} \neq 0$ for some α with $t(\alpha) = s(p)$. But then $i(x)$ contains the non-zero term $p\alpha \otimes m_{\alpha p}$ and no other term involving $p\alpha$ to cancel it, since there is no longer path to give a contribution involving $p\alpha$. So $i(x) \neq 0$; hence $\ker(i) = 0$.

Finally, observe that, as a kQ-module, $kQv \otimes_k N$ is the direct sum of $\dim(N)$ copies of kQv for any vertex v and module N; since kQv is projective, so is $kQv \otimes_k N$. Moreover, for any vertex v, we have

$$kQv = kv \oplus \bigoplus_{\alpha \in kQ_1 v} kQ\alpha$$

since any path starting at v is either of length zero (and hence in kv, the simple module concentrated at the vertex v) or of positive length (and hence in $kQ\alpha$ for some unique arrow α with source v). So for any arrow α, $kQ\alpha$ is a direct summand of the projective module $kQs(\alpha)$, hence is projective; but then for any module N $kQ\alpha \otimes_k N$ is a direct sum of $\dim(N)$ copies of $kQ\alpha$, hence is also projective. So this short exact sequence is indeed a projective resolution, as claimed. ∎

Corollary 3.8 (Path Algebras are Hereditary)
Let Q be any quiver (finite, but not necessarily acyclic) and k any algebraically closed field. Then the path algebra kQ is hereditary, since every module has a length 1 projective resolution, as above.

∎

3.4 The Classification of Hereditary Algebras

Definition 3.9 (Complete Sets of Idempotents & Basic Algebras)
Let A be a k-algebra. A set $\{e_1, \ldots, e_n\} \subseteq A$ is called a *complete set of pairwise orthogonal, primitive idempotents*, if each e_i is a primitive idempotent (an idempotent that cannot be written as a sum of non-zero idempotents), for any $i \neq j$ $e_ie_j = 0$ and

$$1_A = \sum_{i=1}^{n} e_i$$

Corresponding to any idempotent $e \in A$ is a projective (left) ideal in A, namely Ae; this is indecomposable if and only if e is primitve, so a complete set of pairwise orthogonal, primitive idempotents corresponds to a decomposition of A into a direct sum of indecomposable A-modules; since such a decomposition always exists and is unique for any finite-dimensional A, by the Krull-Schmidt Theorem, complete sets of pairwise orthogonal, primitive idempotents always exist and the ideals their elements generate are unique - in particular, the cardinality of the complete set is fixed.

An algebra A is called *basic* if each indecomposable summand of A occurs with multiplicity one; this is equivalent to requiring that in any complete set of pairwise orthogonal, primitive idempotents $\{e_1, \dots, e_n\}$, $Ae_i \ncong Ae_j$ for any $i \neq j$. We have observed that this is true for path algebras of quivers, so path algebras are always basic.

Definition 3.10 (Morita Equivalence)
Let R and S be any two rings-with-unity. If the categories of left R-modules and of left S-modules are equivalent as categories, then R and S are said to be left Morita equivalent. There is a similar notion of right Morita equivalence. It can in fact be shown (see *e.g.*, Meyer (1997)) that left and right Morita equivalence are equivalent conditions; we do not prove this here, but nor will we make use of this fact, so we henceforth simply refer to Morita equivalence without specifying a chirality, but one may be imposed if the reader desires.

We now work towards classifying hereditary algebras up to Morita equivalence.

Definition 3.11 (Representative Subsets)
Let A be an algebra and $E = \{e_1, \dots, e_n\}$ a complete set of pairwise orthogonal, primitive idempotents. Call a subset $S \subseteq E$ a *representative subset* if for any $e_i, e_j \in S$, $e_i \neq e_j$, we have $Ae_i \ncong Ae_j$ and for every $e_i \in E$ there exists $e_j \in S$ with $Ae_i \cong Ae_j$. That is, the ideals generated by the elements of S should include all projective indecomposables, each with multiplicity one. Thus, A is basic if and only if E is itself a representative subset.

Theorem 3.12 (The Basic Algebra of A (Assem, Simson, and Skowroński 2006))
Let A be a finite-dimensional k-algebra and $E = \{e_1, \dots, e_n\}$ a complete set of pairwise orthogonal, primitive idempotents in A. Suppose $\{e_1, \dots, e_r\} \subseteq E$ is a representative subset and define

$$e = \sum_{i=1}^{r} e_i$$

Then the A-module eAe is a basic k-algebra with unity e and is Morita equivalent to A; moreover, any different choice of representative subset will give rise to an isomorphic basic algebra. We call eAe the *basic algebra of A*.

PROOF:
First we prove that eAe is an algebra and $\{e_1, \dots, e_r\}$ is a complete set of pairwise orthogonal, primitive idempotents in eAe. Since eAe is a subset of A, it inherits a multiplicative structure; it is clearly closed under this structure and e acts as unity, so eAe is indeed an associative algebra with unity e. Since the e_i are pairwise orthogonal, $e_i \in eAe$ if and only if $i \leq r$ - that is, if and only if e_i is in the chosen representative subset; this is because $ee_ie = 0$ for all $i > r$, while $ee_ie = e_i$ for all $i \leq r$. So $\{e_1, \dots e_r\}$ is a set of pairwise orthogonal idempotents in eAe; moreover, they are primitive, since any idempotent of eAe is an idempotent of A, so any decomposition into a sum of idempotents which holds in eAe also holds in A. This set of pairwise orthogonal idempotents is a complete set in eAe, since the unity e is precisely the sum of the e_i for $i \leq r$.

Now we prove the Morita equivalence. Let M be any left A-module; there is a natural eAe-module structure on eM given by $(eae)(em) = eaem$. Moreover, any A-linear map ϕ :

$M \to N$ restricts to a map $eM \to eN$, since $\phi(em) = e\phi(m) \in eN$, and this is an eAe-module homomorphism. This defines a covariant functor from left A-modules to left eAe-modules.

For the quasi-inverse, let M be a left eAe-module; since Ae is naturally an (A, eAe)-bimodule, we can form the tensor product of modules $Ae \otimes_{eAe} M$ and endow this with a left A-module structure. To describe the action of this functor on morphisms, let $\phi : M \to N$ be a map of eAe-modules; then $1_A \otimes \phi$ is a map of A-modules $Ae \otimes_{eAe} M \to Ae \otimes_{eAe} N$.

Given any eAe-module M, the result of applying these two functors successively is $eAe \otimes_{eAe} M$. By standard results about tensor products, this is naturally isomorphic to M. Indeed, there is a function $eAe \otimes_{eAe} M \to M$ given by $eae \otimes m \mapsto eaem$; this is easily checked to be an isomorphism (its inverse is $m \mapsto e \otimes m$) and natural for variation of M.

Now we consider the opposite composition, which takes an A-module M to $Ae \otimes_{eAe} eM$. Define a map $Ae \otimes_{eAe} eM \to M$ by $ae \otimes em \mapsto aem$. It is far from clear that this is even surjective, let alone a natural isomorphism, but we shall argue that it is. By tensor-hom adjunction, we have an isomorphism

$$\mathrm{Hom}_A(Ae \otimes_{eAe} eM, M) = \mathrm{Hom}_{eAe}(eM, \mathrm{Hom}_A(Ae, M))$$

We apply Yoneda's Lemma to deduce that $\mathrm{Hom}_A(Ae, M) = eM$, so the right-hand side becomes $\mathrm{End}_{eAe}(eM)$. Consider the identity map $1_{eM} \in \mathrm{End}_{eAe}(eM)$; this corresponds under the Yoneda isomorphism to the map in $\mathrm{Hom}_{eAe}(eM, \mathrm{Hom}_A(Ae, M))$ which takes an element em to the "multiply by m" map $Ae \to M : ae \mapsto aem$. Now, under the tensor-hom adjunct isomorphism, this corresponds to the map $Ae \otimes_{eAe} eM \to M : ae \otimes em \mapsto aem$, which is precisely the map claimed to be an isomorphism. As it corresponds to the identity map and both the Yoneda and tensor-hom isomorphisms are natural, this map must therefore be a natural isomorphism.

Finally we show that eAe is indeed a basic algebra. We have shown that a complete set of pairwise orthogonal, primitive idempotents is $\{e_1, \ldots e_r\}$ and so the indecomposable summands of eAe are $eAee_i = eAe_i$ for $i \in \{1, \ldots, r\}$. Suppose for some i, j that $eAe_i \cong eAe_j$ as eAe-modules. Take the A-modules Ae_i and Ae_j; applying our functor A-Mod $\to eAe$-Mod gives us eAe_i and eAe_j, which are assumed to be isomorphic, but then applying the quasi-inverse $Ae \otimes_{eAe} \circ$ must take us back to Ae_i and Ae_j (up to a natural isomorphism), so $Ae_i \cong Ae_j$. Since $\{e_1, \ldots, e_r\}$ is a representative subset, this implies $i = j$. So the indecomposable summands of eAe have no repetitions, so eAe is indeed basic. ■

Theorem 3.13 (Basic Hereditary Algebras (Ringel 2012))
Let A be a finite-dimensional, hereditary, basic algebra over an algebraically closed field k and write $A = B \oplus I$ for B a semisimple subalgebra and I a nilpotent ideal (by an earlier lemma). Let $\{e_1, \ldots, e_r\}$ be a complete set of pairwise-orthogonal, primitive idempotents. Then there exists a quiver Q such that A is isomorphic to the path algebra kQ.

Proof:

First we determine the structure of the subalgebra B. It is the direct sum of matrix algebras of dimensions given by the multiplicities of the indecomposable summands of A; since A is basic, these multiplicities are all 1, so in fact $B = k^r$. Of course, B is spanned by the e_i and these project to the standard basis of k^r.

For a path algebra, a complete set of pairwise orthogonal, primitive idempotents is precisely the set of vertices, so it is clear that we should take the vertices of Q to be $\{e_1, \ldots, e_r\}$. For clarity, we denote the vertices in majuscule: $Q_0 = \{E_1, \ldots, E_r\}$ and reserve miniscule for the elements of A.

Now we need to construct the arrow set Q_1; the arrows between vertices v and w in a quiver are precisely the paths from v to w which have length 1; they project down to a basis of wJv/wJ^2v, where $J = kQ_{\geq 1}$ is the nilpotent ideal in the decomposition of kQ. So for idempotents e_i, e_j in A, choose a set $\{a_1^{ij}, \ldots, a_{n(i,j)}^{ij}\}$ in e_jIe_i which projects to a basis modulo

$e_jI^2e_i$ and take these to be the arrows from E_i to E_j (again, when regarding them as arrows, denote them in majuscule: $\{A_1^{ij}, \ldots, A_{n(i,j)}^{ij}\}$).

We now have our quiver Q and by construction a map $\phi : kQ \to A$ given by $E_i \mapsto e_i$ and $A_l^{ij} \mapsto a_l^{ij}$; since the vertices and arrows generate kQ as an algebra, this map is defined on all kQ. Moreover, it is surjective. For $A/I = k^r$, so given any $x \in A$ we may write $x + I = (x_1, \ldots, x_r) \in k^r$, whence it follows that

$$x - \sum_{i=1}^{r} x_i e_i \in I$$

Since the e_i are in the image of ϕ, it suffices to prove that I is in the image, for which it is enough to show that the a_l^{ij} generate I. We do this by induction: we show that if the a_l^{ij} generate I^{n+1}, then they generate I^n; since I is nilpotent, I^n is (vacuously) generated for some large n and so the claim follows.

So suppose that I^{n+1} is generated by the a_l^{ij} and $x \in I^n$. I^n is linearly spanned by n-fold products of elements of I, so we may assume without loss of generality that x has the form

$$x = \prod_{i=1}^{n} y_i$$

for some $y_i \in I$. Since the a_l^{ij} generate I/I^2, we can write each y_i as $\alpha_i + z_i$, where $\alpha_i \in I$ is generated by the a_l^{ij} and $z_i \in I^2$ is an error term. But then, multiplying out, we see that

$$x = \prod_{i=1}^{n} \alpha_i + z$$

for $z \in I^{n+1}$, since every term in z is a product of n terms, at least one of which is a $z_i \in I^2$ and the rest of which are $\alpha_j \in I$. By the inductive hypothesis, z is therefore generated by the a_l^{ij}, and so x is generated by them as well. This completes the proof of surjectivity.

Now, the dimension of kQ is the number of paths in Q, which is the sum over all n of the number of paths of length n; a path of length n is an n-fold product of arrows. Similarly, we can define a *path in A of length n* to be any non-zero product of n elements a_l^{ij}, for positive n, or simply a "vertex" e_i for $n = 0$. We have shown above that A is spanned by its paths; the linear independence of the a_l^{ij} modulo I^2 can be easily used to show that the paths actually form a basis for A. Moreover, there is a correspondence between paths in kQ and paths in A, so the dimensions of these two algebras are equal (and, in particular, kQ must be finite-dimensional, so Q is acyclic).

We have a surjection between vector spaces of the same finite dimension, so it is an isomorphism. Hence $A \cong kQ$. ■

Corollary 3.14 (Classification of Hereditary Algebras)
Let A be a finite-dimensional, hereditary algebra over an algebraically closed field. Since A is Morita equivalent to its basic algebra B, this is also hereditary, so B is isomorphic to the path algebra of an acyclic quiver Q. So A is Morita equivalent to the path algebra kQ. Thus, the finite-dimensional hereditary algebras are, up to Morita equivalence, precisely the path algebras of acyclic quivers.

■

Chapter 4

Auslander-Reiten Theory

We now take a different tack. This chapter is an introduction to some of the ideas of the Auslander-Reiten theory, which is a hugely important area of representation theory. Its application is far broader than presented here, but we focus mainly on hereditary algebras, such as path algebras of quivers, both to simplify the theory and so that the theory is immediately applicable to our present concern of understanding quiver representations.

We begin by studying a special type of morphism, called a *radical map*, and introduce a tool for summarising information about a module category, the *Auslander-Reiten quiver*. Then we look closer at the internal structure of the Auslander-Reiten quiver, which contains a special type of short exact sequence, called an *almost-split sequence*. This leads to a procedure known as *knitting with dimension vectors*, which provides a powerful way to construct Auslander-Reiten quivers, at least partially.

For this section, a basic knowledge of Ext functors is assumed, though as much as possible their use has been supressed. All results needed about Ext, both for the present discussion and for more detailed accounts, can be found in, for instance, Rotman (2009).

4.1 Radical & Irreducible Morphisms

Definition 4.1 (Radical Morphisms)
Let A be a finite-dimensional algebra over a field k and let M and N be finite-dimensional left A-modules. A morphism $f : M \to N$ is called *radical* if, when expressed as a matrix of morphisms between indecomposables, no entry is an isomorphism. In particular, if M and N are indecomposable, f is radical if and only if it is not an isomorphism. The subset of $\mathrm{Hom}_A(M, N)$ consisting of all radical morphisms from M to N is denoted $\mathrm{Rad}(M, N)$.

Lemma 4.2 (The Radical Subspace (Barot 2006))
For any two modules M and N, $\mathrm{Rad}(M, N)$ is a vector subspace of $\mathrm{Hom}_A(M, N)$.

PROOF:
Let $f, g : M \to N$ be two radical morphisms. Then, writing them as matrices, it is clear that λf has no component an isomorphism for any scalar $\lambda \in k$, so λf is radical. Moreover, the ij-entry of the matrix of $f + g$ is the sum of the ij-entries of f and g: $(f + g)_{ij} = f_{ij} + g_{ij}$; f_{ij} and g_{ij} are non-isomorphisms between indecomposables, so it suffices to prove that the sum of two non-isomorphisms between indecomposables M and N is again a non-isomorphism.

In the case $M \cong N$, this follows from the decomposition of vector spaces $\mathrm{End}_A(M) = k1_M \oplus I$, where I is a nilpotent ideal consisting of all non-isomorphisms; ideals are closed under addition, so indeed $f_{ij} + g_{ij}$ is not invertible.

If $M \neq N$, f and g can be viewed as endomorphisms of $M \oplus N$; $\mathrm{End}_A(M \oplus N) = k^2 \oplus I$ for a nilpotent ideal I and the maps f and g map strictly between the summands, so are contained in I. Again, since this is an ideal, $f + g \in I$ is also a non-isomorphism. ■

The fact that the radical endomorphisms of an indecomposable are precisely the elements of the nilpotent ideal I is the reason for the term "radical morphism" - I is the Jacobson radical of

$\mathrm{End}_A(M)$ and $I = \mathrm{Rad}(M,M)$. For distinct modules M and N, $\mathrm{Rad}(M,N)$ can be embedded in the Jacobson radical of $\mathrm{End}(M \oplus N)$ as the subspace of strictly upper triangular matrices with respect to the decomposition $M \oplus N$, but this subspace is not the entire Jacobson radical. In light of this, though, it comes as no surprise that $\mathrm{Rad}(M,N)$ behaves somewhat like an ideal in that it absorbs multiplication:

Lemma 4.3 (Compositions of Radical Maps (Barot 2006))
Let $f : M \to N$ be a radical morphism and $g : L \to M$, $h : N \to L$ be any morphisms, for any module L. Then the compositions fg and hf are radical: $fg \in \mathrm{Rad}(L,N)$ and $hf \in \mathrm{Rad}(M,L)$.

PROOF:

We prove the result for fg; the corresponding result for hf is proved similarly. We first deal with the special case where all modules are indecomposable, so let $f : M \to N$, $g : L \to M$ be non-isomorphisms between indecomposables and suppose for a contradiction that $fg : L \to N$ is an isomorphism, with inverse $\phi : N \to L$. Then $g\phi : N \to M$ is a right-inverse of f, so f is a retraction and we have a decomposition

$$M \cong N \oplus \ker(f)$$

As M is irreducible, either $N = 0$, which cannot occur (as N is indecomposable) or $\ker(f) = 0$, which must therefore occur. So f is injective, but it is also surjective (it is a retraction), so f is a bijection and hence an isomorphism, contradicting the fact that f is radical.

Now let L, M and N be any modules (not necessarily indecomposable) and write f and g as matrices $f = (f_{ij})$, $g = (g_{ij})$. Then $fg = (\phi_{ij})$, where

$$\phi_{ij} = \sum_r f_{ir} g_{rj}$$

with the index of summation ranging over appropriate values. By definition, fg is radical if and only if no ϕ_{ij} is an isomorphism; each f_{ir} and g_{rj} is a non-isomorphism between indecomposables, so by the above each $f_{ir}g_{rj} : L \to N$ is radical. Since $\mathrm{Rad}(L,N)$ is a vector space, the sum ϕ_{ij} must be radical too, so is not an isomorphism. ■

In particular, in the case where $L = M = N$, the above lemma is precisely the multiplicative property of the nilpotent ideal I in the decomposition $\mathrm{End}_A(M) = k1_M \oplus I$. So even when considering maps between non-isomorphic modules, there is a subspace of the Hom-space which is "like an ideal" in terms of its multiplicative behaviour. This allows us to make the following definition:

Definition 4.4 (The Square-Radical & Higher Radical Powers (ibid.))
Let M and N be modules over a k-algebra A. The *square-radical* is the subspace of $\mathrm{Rad}(M,N)$ defined by:

$$\mathrm{Rad}^2(M,N) = \sum_L \mathrm{Rad}(L,N)\mathrm{Rad}(M,L)$$

the span of all radical morphisms which can be factored through radical morphisms $M \to L$ and $L \to N$. The above lemma shows that this is indeed contained in $\mathrm{Rad}(M,N)$. It follows immediately from the above lemma that $\mathrm{Rad}^2(M,N)$ also "absorbs multiplication": for $f \in \mathrm{Rad}^2(M,N)$, $g \in \mathrm{Hom}_A(L,M)$, $fg \in \mathrm{Rad}^2(L,N)$. Inductively, we define the n^{th} *radical power* as

$$\mathrm{Rad}^n(M,N) = \sum_L \mathrm{Rad}(L,N)\mathrm{Rad}^{n-1}(M,L)$$

the span of morphisms which can be written as a composition of n radical morphisms. Moreover, we define the *infinite radical* as

$$\mathrm{Rad}^\infty(M,N) = \bigcap_{n=1}^{\infty} \mathrm{Rad}^n(M,N)$$

the span of morphisms which can be written as a composition of arbitrarily many radical morphisms. Of particular interest to us will be those quivers such that the infinite radical between any two representations is zero, meaning that every morphism can be factored as a composition of at most finitely many radical morphisms. These are particularly amenable to the methods we will develop in this section.

Definition 4.5 (Irreducible Morphisms)
A radical morphism $f \in \mathrm{Rad}(M, N)$ is called *irreducible* if it cannot be factored through other radicals: $f \notin \mathrm{Rad}^2(M, N)$. In particular, if the infinite radical between two modules is zero, then every radical morphism between those modules can be written as the composition of finitely many irreducible morphisms.

Lemma 4.6 (Irreducible Maps Between Indecomposables (Barot 2006))
Let $f : M \to N$ be an irreducible morphism between indecomposables. Then f is either injective or surjective, but not both.

PROOF:

That f is not bijective is immediate, since any irreducible morphism is radical and so not an isomorphism. We can factor f as

$$f : M \xrightarrow{\phi} f(M) \xrightarrow{\psi} N$$

where ϕ is a surjection defined by $\phi(m) = f(m)$ and ψ is the inclusion map, so injective. If ϕ is injective, then $f = \psi\phi$ is the composition of two injections, so is injective and we're done; similarly if ψ is surjective, so too is f. So suppose ϕ is non-injective and ψ is non-surjective.

If ϕ is not radical, then one of its entries as a matrix is an isomorphism; therefore $f(M)$ has an indecomposable summand isomorphic to M; but then the dimension of $f(M)$ is at least $\dim(M)$, so ϕ is a surjection onto a higher-dimensional space and so we must have equality of dimensions and injectivity of ϕ, contrary to assumption. On the other hand, if ψ is not radical, then one of its entries is an isomorphism and so N appears as an indecomposable factor of $f(M)$ and then $f(M)$ has dimension no smaller than $\dim(N)$, so ψ is an injection into a lower-dimensional space, forcing equality of dimensions and making ψ a surjection, which again is a contradiction.

So now both ϕ and ψ must be radical; but then f has been factored through two radical maps, contradicting irreducibility. ■

4.2 The Auslander-Reiten Quiver

Definition 4.7 (The Auslander-Reiten Quiver)
Let Q be a quiver and k a field. Then the associated *Auslander-Reiten (A-R) quiver* is the quiver whose vertices are the isomorphism classes of indecomposable representations and where the number of arrows from a vertex $[M]$ (the class of the indecomposable M) to the vertex $[N]$ is the dimension of the quotient space $\mathrm{Rad}(M, N)/\mathrm{Rad}^2(M, N)$. In practice, it is often convenient to choose representatives for each isomorphism class and a basis for each radical quotient space, so that the A-R quiver has particular, pairwise non-isomorphic indecomposables as its vertices and as its arrows particular irreducible morphisms which form a basis modulo the square-radical.

Example 4.8 (The Linear Quiver (ibid.))
We compute the Auslander-Reiten quiver for the linear quiver:

$$1 \xrightarrow{a_1} 2 \xrightarrow{a_2} \to \ldots \to (n-1) \xrightarrow{a_{n-1}} n$$

First we compute the indecomposable representations; this takes several steps. To this end, let $V = (V_i, f_j)$ be an indecomposable representation, where i and j range over the sets $\{1, \ldots, n\}$

and $\{1, \ldots, n-1\}$ respectively, the V_i are k-vector spaces and the f_j are linear maps. First we show that if one of the f_j is not injective, then its range V_{j+1} and all subsequent V_i are zero. Then we show that if one of the f_j is not surjective, then its domain V_j and all previous V_i are zero. Thence we deduce that V has the form:

$$0 \to \ldots \to 0 \to k \xrightarrow{1} k \xrightarrow{1} \ldots \xrightarrow{1} k \to 0 \to \ldots \to 0$$

That is, for some integers $a < b$, $V_i = 0$ for all $i < a$ & $i > j$ and $V_i = k$ for $a \leq i \leq b$, while each f_j is the identity if mapping between two 1-dimensional spaces ($a \leq j \leq b-1$) and must be the zero map otherwise. We denote this representation $[a, b]$.

To begin then, suppose that f_α is non-injective and, without loss of generality, that α is the least such index, so f_i is injective for all $i < \alpha$. Let $W_\alpha = \ker(f_\alpha)$ and, for $i < \alpha$, define inductively $W_i = f_i^{-1}(W_{i+1})$, so that W_i is the set of all vectors in V_i which are mapped to zero by the time they reach V_α. Further, for $i < \alpha$, define $W_i = 0$. Then, by construction, for all i, $f_i(W_i) \subseteq W_{i+1}$, so $W = (W_i, f_j|_{W_j})$ is a subrepresentation of V.

Now we construct a subrepesentation complementary to W. First we show that if U_i is a subspace of V_i complementary to W_i (so that $V_i = U_i \oplus W_i$), then there is a subspace U_{i+1} of V_{i+1} which is complementary to W_{i+1}. If $v \in f_i(U_i) \cap W_{i+1}$, v is in the image of f_i and so has a preimage $\hat{v} \in U_i$, which must also be contained in $W_i = f_i^{-1}(W_{i+1})$, which implies $\hat{v} = 0$, so $v = f_i(\hat{v}) = 0$. So $f_i(U_i) \cap W_{i+1} = 0$ and so, if B is a basis for $f_i(U_i)$ and C a basis for W_{i+1}, then $B \cup C$ is linearly independent and may be extended to a basis D for V_{i+1}; then $D \setminus C$ is a basis for a subspace complementary to W_{i+1} and containing $f_i(U_i)$. Therefore we can take a complementary subspace $U_1 \subseteq V_1$ of W_i and extend to a sequence of subspaces $U_i \subseteq V_i$ complementary to W_i for all i and such that $f_i(U_i) \subseteq U_{i+1}$, so $U = (U_i, f_j|_{U_j})$ is a subrepresentation of V. By construction, $V = U \oplus W$, but V is indecomposable; since f_α is non-injective and $W_\alpha = \ker(f_\alpha)$, $W \neq 0$, so U is the zero representation. But for $i > \alpha$, $W_i = 0$ and so $V_i = W_i \oplus U_i = 0 \oplus 0 = 0$. This completes the first step.

Now suppose that f_α is non-surjective and, without loss of generality, that α is the greatest such index. We prove that $V_i = 0$ for all $i \leq \alpha$, by a method similar to the above. Let $W_{\alpha+1} = f_\alpha(V_\alpha)$ and inductively define $W_i = f_{i-1}(W_{i-1})$ for $i \geq \alpha+2$; for $i \leq \alpha$, define $W_i = V_i$. Given a subspace $U_{i+1} \subseteq V_{i+1}$ with $V_{i+1} = U_{i+1} \oplus W_{i+1}$, we can find $U_i \subseteq V_i$ complementary to W_i and such that $f_i(U_i) \subseteq U_{i+1}$. For if $v \in f_i^{-1}(U_{i+1}) \cap W_i$, then $f_i(v) \in U_{i+1} \cap W_{i+1}$, so $f_i(v) = 0$, whence $v \in \ker(f_i)$; but, by the above, either f_i is injective or $V_{i+1} = 0$; if f_i is injective, then $v = 0$ and $f_i^{-1}(U_{i+1}) \cap W_i = 0$, so there exists U_i as claimed; if instead $V_{i+1} = 0$, then $V_i = \ker(f_i)$, so any subspace U_i complementary to W_i will suffice. So $(U = (U_i, f_j|_{U_j})$ is a subrepresentation of V and $V = U \oplus W$. $U_{\alpha+1}$ is non-zero, since it is complementary to $W_{\alpha+1}$ which is strictly contained in $V_{\alpha+1}$, by non-surjectivity of f_α; hence $W = 0$, but for $i \leq \alpha$, $V_i = W_i = 0$, completing the second step.

Now we conclude that V has the stated structure. Let α be the greatest index such that f_α is not surjective and β the smallest index such that f_β is not injective. Then $V_i = 0$ for all $i \leq \alpha$ and $i > \beta$; moreover, for $\alpha < i \leq \beta$, f_i is both injective and surjective, hence is a linear isomorphism. So V has the following form:

$$0 \to \ldots \to 0 \to V_{\alpha+1} \xrightarrow{\sim} \ldots \xrightarrow{\sim} V_\beta \to 0 \to \ldots \to 0$$

Choose a basis $\{v_1, \ldots, v_n\}$ for $V_{\alpha+1}$; then applying $f_{\alpha+1}$ gives a basis for $V_{\alpha+2}$ *&c.*, so we obtain bases for each of the isomorphic V_i in the middle such that the isomorphisms may all be regarded as the identity map. Then it is apparent that, if $\dim(V_{\alpha+1}) = d$, V is the direct sum of d copies of the representation

$$[\alpha+1, \beta] = 0 \to \ldots \to 0 \to k \xrightarrow{1} \ldots \xrightarrow{1} k \to 0 \ldots \to 0$$

Since V is indecomposable, we thus conclude that $d = 1$ and so in fact $V = [\alpha+1, \beta]$, as was to be proved.

We have proved that any indecomposable representation of Q has the form $[a, b]$; we now show that all representations $[a, b]$ are indeed indecomposable. Suppose for a contradiction that $[a, b] = V \oplus W$ for some non-trivial representations $V = (V_i, f_j)$ and $W = (W_i, g_j)$. Then without loss of generality, we may assume that $V_a = k$ and $W_a = 0$ (otherwise just swap V and W); if $a = b$, this forces $V = [a, b]$ and $W = 0$, which is a contradiction. Therefore $f_a + g_a = 1$, since the $(a+1)^{\text{th}}$ vertex of $[a, b]$ is also non-zero; but $g_a = 0$, so $f_a = 1$, hence $V_{a+1} = k$ and so $W_{a+1} = 0$. Proceeding by induction, $W_i = 0$ for all $a \le i \le b$; but W_i is also zero for all i outside this range, since here $[a, b]_i = 0$. So $W = 0$, which is a contradiction.

Moreover, it is clear that $[a, b] \cong [\alpha, \beta]$ if and only if $a = \alpha$ and $b = \beta$, since otherwise there is a vertex at which these representations have different dimensions. Therefore we conclude that a complete list of pairwise non-isomorphic indecomposable representations of Q is given by $[a, b]$ for $1 \le a \le b \le n$. In particular, Q is representation finite - it has a finite set of finite-dimensional indecomposables.

Now we proceed with computing the A-R quiver of Q; to do this, we need to find the radical morphisms between indecomposables. Consider a morphism between indecomposables $\phi : [a, b] \to [\alpha, \beta]$; this is the same as a rectangular commuting diagram with top row $[a, b]$, bottom row $[\alpha, \beta]$ and descending arrows ϕ_i. Excluding the 'boring' squares where two or more objects involved are zero, we can have the following squares which must commute:

$$\begin{tikzcd} k \arrow[r, "1"] \arrow[d] & k \arrow[d, "\phi_\alpha"] \\ 0 \arrow[r] & k \end{tikzcd} \qquad \begin{tikzcd} k \arrow[r, "1"] \arrow[d, "\phi_{i-1}"'] & k \arrow[d, "\phi_i"] \\ k \arrow[r, "1"'] & k \end{tikzcd} \qquad \begin{tikzcd} k \arrow[r, "1"] \arrow[d, "\phi_\beta"'] & k \arrow[d] \\ k \arrow[r] & 0 \end{tikzcd}$$

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & k \arrow[d, "\phi_a"] \\ k \arrow[r, "1"'] & k \end{tikzcd} \qquad \begin{tikzcd} k \arrow[r] \arrow[d, "\phi_b"'] & 0 \arrow[d] \\ k \arrow[r, "1"'] & k \end{tikzcd}$$

Studying these diagrams, we see that if $\alpha > a$, $\phi_\alpha = 0$ (from the top-left square), if $a, \alpha \le i-1$ and $i < b, \beta$, then $\phi_{i-1} = \phi_i$ (top-middle square), if $\beta < b$, ϕ_β can be any map $k \to k$ (top-right square), if $\alpha < a$, ϕ_a can be any map $k \to k$ (bottom-left square) and if $b < \beta$, $\phi_b = 0$. From this we conclude that if $\alpha > a$, then $\phi_\alpha = 0$ and consequently $\phi_i = 0$ for all i; similarly if $\beta > b$, $\phi_b = 0$ and consequently $\phi_i = 0$ for all i; so there is no non-zero morphism $[a, b] \to [\alpha, \beta]$ if $\alpha > a$ or $\beta > b$. Moreover, if $a \ge \alpha$ and $b \ge \beta$, non-zero morphisms are possible, and all maps ϕ_i for $a \le i \le b$ must be equal to each other but otherwise can be freely chosen.

Putting this together, the space of morphisms $[a, b] \to [\alpha, \beta]$ is:

$$\operatorname{Hom}_{kQ}([a, b], [\alpha, \beta]) = \begin{cases} k\gamma_{ab}^{\alpha\beta} & \text{if } a \ge \alpha \text{ and } b \ge \beta \\ 0 & \text{otherwise} \end{cases}$$

where $\gamma_{ab}^{\alpha\beta} : [a, b] \to [\alpha, \beta]$ is the morphism defined at the i^{th} vertex by:

$$\gamma_i = \begin{cases} 1 & \text{if } a \le i \le b \\ 0 & \text{otherwise} \end{cases}$$

That is, γ is the identity wherever it can be and zero wherever it must be; to illustrate this, we show $\gamma : [2, 3] \to [1, 2]$ in the case $n = 3$:

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & k \arrow[r, "1"] \arrow[d, "1"] & k \arrow[d] \\ k \arrow[r, "1"'] & k \arrow[r] & 0 \end{tikzcd}$$

Since these are all the morphisms between indecomposables and $\gamma_{ab}^{\alpha\beta}$ is an isomorphism if and only if $a = \alpha, b = \beta$, we see that there is a radical morphism $[a, b] \to [\alpha, \beta]$ precisely when $a \geq \alpha$ and $b \geq \beta$ and equality holds in at most one of these two expressions, and that it is a multiple of $\gamma_{ab}^{\alpha\beta}$. Moreover, it is clear that $\gamma_{ab}^{\alpha\beta}\gamma_{xy}^{\chi\upsilon} = \gamma_{xy}^{\alpha\beta}$, whence it follows that the irreducible morphisms between indecomposables are precisely the multiples of $\gamma_{ab}^{a-1,b}$ or $\gamma_{ab}^{a,b-1}$; that is, the irreducible morphisms either kill the last 1-dimensional space in the representation or add on one more at the beginning, as illustrated by the two irreducible morphisms below, in the case of three vertices:

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & k \arrow[r, "1"] \arrow[d, "1"] & k \arrow[d, "1"] \\ k \arrow[r, "1"'] & k \arrow[r, "1"'] & k \end{tikzcd} \qquad \begin{tikzcd} 0 \arrow[r] \arrow[d] & k \arrow[r, "1"] \arrow[d, "1"] & k \arrow[d] \\ 0 \arrow[r] & k \arrow[r] & 0 \end{tikzcd}$$

We therefore see the structure of the Auslander-Reiten quiver of the linear quiver; it has vertices the indecomposable representations $[a, b]$ and arrows the irreducible morphisms $\gamma_{ab}^{a-1,b}$ (which are all injective) and $\gamma_{ab}^{a,b-1}$ (which are all surjective). The A-R quiver for the linear quiver on four vertices is illustrated below. For clarity, it is convenient to represent each indecomposable by its dimension vector, in the shape of the quiver; for example, the representation $[2, 3]$ is denoted 0110; the irreducible morphisms are not labelled, as there is no risk of ambiguity.

$$\begin{tikzcd}[column sep=small] & & & 1111 \arrow[dl] & & & \\ & & 1110 \arrow[dl] & & 0111 \arrow[ul] \arrow[dl] & & \\ & 1100 \arrow[dl] & & 0110 \arrow[ul] \arrow[dl] & & 0011 \arrow[ul] \arrow[dl] & \\ 1000 & & 0100 \arrow[ul] & & 0010 \arrow[ul] & & 0001 \arrow[ul] \end{tikzcd}$$

Notice that every map going upwards in this diagram is an injection and every map going downwards is a surjection; moreover, the diagram commutes. In addition, we know from earlier considerations that the simple representations are precisely those which are concentrated at a single vertex - *i.e.*, those which form the bottom row of this A-R quiver.

The projective indecomposables are also prominently displayed in this diagram; they are those representations consisting of the span of paths from a given vertex and thus are precisely those shown along the right-hand edge of the diagram. Interestingly, these are precisely those indecomposables which are not the domains of surjective irreducible morphisms (*i.e.*, they are the only vertices in the A-R quiver which are not on the receiving end of an arrow coming down). It is not surprising that the projectives have this property - any surjection ending in a projective splits, so cannot have come from a non-isomorphic indecomposable.

Just as the right-hand edge consists of indecomposable modules which receive no surjective irreducible map, the left-hand edge consists of indecomposables which source no injective irreducible. We shall see (in example 4.20) that these are also modules of a special type, called the *injective* indecomposables, which are dual to projectives. For now, we present only the definition:

Definition 4.9 (Injective Modules)
Let R be a ring. A (left) R-module M is called *injective* if it satisfies the following, equivalent conditions:

1. Given any injection of R-modules $i : X \hookrightarrow Y$ and any map $f : X \to M$, there is a map

$g : Y \to M$ making the diagram commute:

$$\begin{tikzcd} M & \\ X \arrow[u, "f"] \arrow[r, hook, "i"'] & Y \arrow[ul, dashed, "g"'] \end{tikzcd}$$

2. The contravariant Hom-functor $\mathrm{Hom}_R(\circ, M)$ is exact
3. Any short exact sequence starting with M splits.

Compare with Definition 2.9 and Lemma 2.8, which define projective modules; this definition is dual, but there is no analogue of the fact that projective modules are precisely the summands of free modules. The proof of the equivalence of these conditions is very similar to the corresponding proof for projective modules, so is omitted; see (Rotman 2009) for details.

Just as a projective cannot receive a surjection from an indecomposable, so too can no injective be the source of an injection into an indecomposable, as such an injection would have to split. So the only possible candidates for injective indecomposables in this example are those along the left-hand edge of the diagram. We shall see in example 4.20 that these are in fact all injective. However, this does not entirely generalise - although any projective indecomposable cannot receive any surjective irreducible maps in the A-R quiver, not all such representations are generally projective; similarly, any injective cannot source an injective irreducible, but an indecomposable which sources no injective irreducible need not generally be injective.

This example can be extended to the linear quiver on any number of vertices; the diagram simply grows downward in much the same way and the indecomposable projectives are still precisely those along the right-hand edge, while the left-hand edge consists of the injective indecomposables.

Moreover, the A-R quiver contains all the information about the finite-dimensional representations of the linear quiver; every representation decomposes as a direct sum of indecomposables (all of which are represented in the A-R quiver) and each map splits as a matrix of maps between indecomposables, each of which is either an isomorphism (and so may be taken to be the identity) or is radical and so is a composition of finitely many irreducible morphisms (which are all represented in the A-R quiver). Thus every representation and every morphism of representations is built in finitely many steps from information readily available in the Auslander-Reiten quiver.

This remains true of any quiver of finite representation type such that the infinite radical between any two indecomposables is always zero. Thus, in these cases, the A-R quiver is a powerful tool for summarising the entire category of finite-dimensional representations. Even for other quivers, the A-R quiver carries a lot of information about the category of finite-dimensional representations, though no longer all of it.

4.3 Almost-Split Sequences

We now begin working toward the main result of Auslander-Reiten theory, which will enable us to develop a technique known as "knitting", with which we will be able to compute the A-R quiver of a given quiver very efficiently, without the fiddly calculations deployed in the above example. A key ingredient in the A-R theory is the idea of an *almost-split exact sequence*, which we here develop. First, a fresh look at split exact sequences:

Lemma 4.10 (The Splitting Lemma Revisited)
Given a short exact sequence of left modules over a ring R

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$$

the following are equivalent:

1. The sequence splits

2. For any module M and any map $f : M \to C$, there is a map $g : M \to B$ such that the following diagram commutes:
$$\begin{tikzcd} B \arrow[r, "p"] & C \\ M \arrow[u, dashed, "g"] \arrow[ur, "f"'] & \end{tikzcd}$$

3. For any M and any map $f : I \to M$, there is a map $g : B \to M$ such that the following diagram commutes:
$$\begin{tikzcd} & M \\ A \arrow[ur, "f"] \arrow[r, "i"'] & B \arrow[u, dashed, "g"'] \end{tikzcd}$$

Proof:

(1. $\Leftrightarrow$ 2.): If the sequence splits, there is a section s along p (so $ps = 1_C$), so define $g = sf : M \to B$. Then $pg = psf = f$, so the diagram commutes. Conversely, take $M = C$ and $f = 1_C$, then g is a section along p and so the sequence splits.

(1. $\Leftrightarrow$ 3.): Just as above: if the sequence splits, there is a retraction r along i and $g = fr$ will give the desired commuting triangle; conversely, factor $1_A : A \to A$ through i to show that i is a section. ∎

Though not very deep, this lemma motivates the following definition:

Definition 4.11 (Almost-Split Exact Sequences (Auslander 1987))
A short exact sequence of left R-modules

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$$

is called *almost-split* if it meets the following four conditions:

1. A and C are indecomposable

2. For any module M and any *non-isomorphism* $f : M \to C$ there is a map g making the following diagram commute:
$$\begin{tikzcd} B \arrow[r, "p"] & C \\ M \arrow[u, dashed, "g"] \arrow[ur, "f"'] & \end{tikzcd}$$

3. For any module M and any *non-isomorphism* $f : A \to M$ there is a map g making the following diagram commute:
$$\begin{tikzcd} & M \\ A \arrow[ur, "f"] \arrow[r, "i"'] & B \arrow[u, dashed, "g"'] \end{tikzcd}$$

4. The sequence is *not* split - we want almost-split to mean *strictly* almost split.

Note that both split and almost-split sequences satisfy conditions 2. and 3.; the difference is that split sequences also satisfy those conditions for isomorphisms and almost-split sequences do not, and that split sequences are also defined when the outer terms are not indecomposable. In fact, almost-split sequences can be defined with the outer terms not indecomposable, but with extra constraints; see, *e.g.*, (Angeleri Hügel 2006). As we have no need of this greater generality for our purposes, we assume that the outer terms are indecomposable.

Split exact sequences are obviously uniquely determined by their outer two terms. What is a less obvious is that for almost-split sequences we have the even stronger result the sequence is entirely determined by either one of its outer two terms:

Lemma 4.12 (Uniqueness of Almost-Split Sequences (Auslander 1987))
Given two almost-split sequences:

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$$

$$0 \to A' \xrightarrow{j} B' \xrightarrow{q} C' \to 0$$

the following are equivalent:

1. There is an isomorphism of short exact sequences between them
2. A and A' are isomorphic
3. C and C' are isomorphic.

Proof:
(1. $\Rightarrow$ 2. & 3.): obvious.

(2 $\Rightarrow$ 1.): For convenience, assume without loss of generality that the isomorphism is actual equality: $A = A'$. First observe that neither of the outer two terms of an almost-split sequence can be zero, as then the sequence would split. Therefore i and j are not surjective and, in particular, are not isomorphisms. So we have a non-isomorphism $j : A \to B'$ which must therefore factor through i and a non-isomorphism $i : A \to B$ which must factor through j. So we have maps $f : B \to B'$ and $g : B' \to B$ such that $fi = j$ and $gj = i$. It follows that $gfi = i$ and $fgj = j$.

Identify C and C' with $B/i(A)$ and $B'/j(A)$ respectively; then f and g induce well-defined maps $\hat{f} : C \to C'$ and $\hat{g} : C' \to C$ defined by $\hat{f}(b + i(A)) = f(b) + j(A)$ and $\hat{g}(b' + j(A)) = g(b') + i(A)$. The composition $\hat{g}\hat{f}$ is an endomorphism of C; since C is indecomposable, its endomorphisms are either invertible or nilpotent. Suppose for a contradiction that $\hat{g}\hat{f}$ is nilpotent so that for some integer n $(\hat{g}\hat{f})^n(b + i(A)) = i(A)$ for all $b \in B$; *i.e.*, $(gf)^n(b) \in i(A)$ for all b. Then, since i is injective, for every $b \in B$ there is a unique $a_b \in A$ such that $(gf)^n(b) = i(a_b)$, so there is a well-defined function $h : B \to A : b \mapsto a_b$. It is easy to check that h is R-linear, and it is a retraction along i, since $(gf)^n i(a) = i(a)$, by induction on the fact that $gfi = i$. This implies that the first short exact sequence splits, which is a contradiction.

So $\hat{g}\hat{f}$ is invertible, from which it follows that $\hat{f}$ is injective, so that $i(A) = \ker(\hat{f}) = \{b + i(A) | f(b) \in j(A)\}$. So if $f(b) \in j(A)$, then $b \in i(A)$. In particular, since 0 is in $j(A)$, if $f(b) = 0$, we have $b \in i(A)$, so $b = i(a)$ for some a; but $f(b) = fi(a) = j(a)$ and j is injective, so $a = 0$ and hence $b = 0$. So f is injective.

By symmetry, $\hat{f}\hat{g}$ is an invertible endomorphism of C', so $\hat{f}$ is surjective. Since $\hat{f}$ is injective, this makes $\hat{f}$ an isomorphism, so $C \cong C'$; furthermore, we can show from this that f is also surjective and hence an isomorphism. To do this, note that surjectivity of $\hat{f}$ means that $\{b' + j(A) | b' \in B'\} = \hat{f}(C) = \{f(b) + j(A) | b \in B\}$. So for any $b' \in B'$, there is some $b \in B$ with $f(b) - b' \in j(A)$, so $f(b) - b' = j(a) = fi(a)$ for some $a \in A$. Hence $b' = f(b - i(a))$, so we have constructed an element of B which maps to b' under f, proving surjectivity.

So we have isomorphisms $f : B \xrightarrow{\sim} B'$ and $\hat{f} : C \xrightarrow{\sim} C'$; moreover, it is apparent from the definitions of these maps that they make the diagram below commute and so constitute an isomorphism of short exact sequences, proving 1.:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "i"] \arrow[d, "1"] & B \arrow[r, "p"] \arrow[d, "f"] & C \arrow[r] \arrow[d, "\hat{f}"] & 0 \\
0 \arrow[r] & A' \arrow[r, "j"] & B' \arrow[r, "q"] & C' \arrow[r] & 0
\end{tikzcd}$$

(3. $\Rightarrow$ 1.): this is proved similarly to the above, so details are omitted. Use the almost-split property to induce maps $f : B \to B'$ and $g : B' \to B$ such that $qf = p$ and $pg = q$, then pass to maps $\hat{f}$ and $\hat{g}$ between A and A' by restriction and use the fact that A and A' are indecomposable and the sequence does not split to deduce that $\hat{f}\hat{g}$ and $\hat{g}\hat{f}$ are invertible, then deduce bijectivity of $\hat{f}$ and apply the above to conclude that the two almost-split sequences are isomorphic. ∎

We see then that almost-split sequences starting or ending in a given module are unique, when they exist; we have not, however, shown that any do exist. In fact, we can immediately see that sometimes they do not; any short exact sequence ending in a projective module or starting in an injective module splits, so cannot be almost-split. So for any projective, indecomposable module P, there is certainly no almost-split sequence ending in P and dually there is no almost-split sequence starting in an injective indecomposable I. However, we shall see that these are the only cases in which an indecomposable module does not lie in an almost-split sequence.

4.4 Duality & the Nakayama Functor

The following discussion holds (with some modifications) in far greater generality than we treat here (and with a far stronger presence of Ext functors), but we assume henceforth that all rings are in fact hereditary algebras, such as quiver algebras.

Definition 4.13 (The Opposite Quiver)
Given a quiver Q, define the *opposite quiver* Q^{op} to be the quiver with the same vertices as Q but all arrows reversed. That is, $Q^{\mathrm{op}} = (Q_0, Q_1, s^{\mathrm{op}}, t^{\mathrm{op}})$, where $s^{\mathrm{op}}(\alpha) = t(\alpha)$ and $t^{\mathrm{op}}(\alpha) = s(\alpha)$ for all $\alpha \in Q_1$. Clearly, $(Q^{\mathrm{op}})^{\mathrm{op}} = Q$. A representation of Q^{op} is the same as a left module over $(kQ)^{\mathrm{op}}$, the opposite algebra of kQ, which is the same as a right module over kQ.

Given a quiver Q and a representation $V = (V_i, f_j)$, for $i \in Q_0, j \in Q_1$, there is a representation of Q^{op} given by $\mathrm{D}V = (V_i^*, f_j^*)$, where the stars denote dualisation over the ground field k. Given also a morphism of representations $\phi = (\phi_i) : V \to W$, there is an induced morphism given by $\mathrm{D}\phi = (\phi_i^*) : \mathrm{D}W \to \mathrm{D}V$. Thus, we have a contravariant functor from the category of representations of Q to the representations of Q^{op}. Since any finite-dimensional vector space is naturally isomorphic to its double dual, D^2 is naturally isomorphic to the identity functor on finite-dimensional representations. Viewed in terms of modules over the path algebra, D dualises the underlying vector space of the module M and, as each element a of the path algebra acts as a linear endomorphism of M, a acts as the dual endomorphism on $\mathrm{D}M$; this makes left modules into right modules *& vice versa.*

Lemma 4.14 (Projective-Injective Duality of D)
Let $\mathrm{proj}(Q)$ denotes the full subcategory of Q-Mod consisting of projective modules and $\mathrm{inj}(Q)$ the full subcategory of injective modules. Then D restricts to dualities $\mathrm{proj}(Q) \to \mathrm{inj}(Q^{\mathrm{op}})$ and $\mathrm{inj}(Q) \to \mathrm{proj}(Q^{\mathrm{op}})$.

PROOF:
Forgetting the module structure, D is a Hom-functor on vector spaces, hence is exact; adding the extra structure of a kQ-module does not change this, so D is an exact functor.

Any exact duality between abelian categories takes projectives to injectives. For suppose P is a projective object in an abelian catgeory C and D is another abelian category. Suppose also we have functors $F : C \to D$ and $G : D \to C$, which are exact and form a duality (so both are contravariant and their two compositions are naturally isomorphic to their respective identity functors). Then, given a diagram in D:

$$\begin{tikzcd} & N \\ F(P) & M \arrow[l] \arrow[u] \\ & 0 \arrow[u] \end{tikzcd}$$

with an exact column, applying G gives a diagram

$$\begin{tikzcd} & G(N) \arrow[d] \\ GF(P) \arrow[r] \arrow[ur, dashed] & G(M) \arrow[d] \\ & 0 \end{tikzcd}$$

Since G is exact, the column is still exact. Since $GF(P) = P$ is projective, there is a lifting morphism as shown making the diagram commute. Now applying F gives us the diagram

$$\begin{tikzcd} & FG(N) \arrow[dl, dashed] \\ FGF(P) & FG(M) \arrow[l] \arrow[u] \\ & 0 \arrow[u] \end{tikzcd}$$

Since F is exact, the column is still exact and, since FG is the identity functor, this shows that, in the original diagram, there was a morphism $N \to F(P)$ extending the given morphism $M \to F(P)$. But this is precisely the definition of an injective object, so $F(P)$ is injective. The argument that F also takes injectives to projectives is virtually identical.

Since D is an exact duality, the lemma is proved. ■

Similarly to dualising over the ground field k, we can dualise over kQ; it turns out that this also gives a contravariant functor kQ-Mod $\to$ Mod-kQ (*i.e.*, between representations of Q and representations of Q^{op}. To this end, let A be an algebra. Since we are now considering both left and right modules together, it is convenient to denote by ${}_A\text{Hom}(M, N)$ the space of left A-linear maps between left modules M and N, while $\text{Hom}_A(M, N)$ denotes the space of right A-linear maps between right modules M and N.

In general, ${}_A\text{Hom}(M, N)$ is only a vector space, but in the case $N = A$, we can endow ${}_A\text{Hom}(M, A)$ with the structure of a right A-module. We define a right action of A on ${}_A\text{Hom}(M, A)$ by $f^a(m) = f(m)a$ for $m \in M, a \in A, f : M \to A$. This is easily seen to preserve left A-linearity and to be a valid right action (*i.e.*, $f^{ab} = (f^a)^b$). Similarly, we define a left A-module structure on $\text{Hom}_A(M, A)$ by ${}^a f(m) = af(m)$.

In fact, this assignment is functorial; for given a map of left A-modules $f : M \to N$, we can define $f^\dagger : {}_A\text{Hom}(N, A) \to {}_A\text{Hom}(M, A)$ by $f^\dagger(g) = g \circ f : M \to A$ for any $g : N \to A$. Similarly for right modules M and N. This gives us a contravariant functor A-Mod $\to$ Mod-A and another Mod-$A \to A$-Mod.

Definition 4.15 (The Nakayama Functor (Schiffler 2014))
Composing these two functors we have just constructed gives a covariant functor $\text{D}_A\text{Hom}(\circ, A)$ from left A-modules to left A-modules and another, $\text{DHom}_A(\circ, A)$ from right A-modules to right A-modules. Abusing notation, both of these functors are called the *Nakayama functor* and denoted ν.

Now we study the behaviour of the Nakayama functor on hereditary algebras, such as path algebras of quivers. Let A be a hereditary algebra and let M be an indecomposable left A-module which is not projective. Then for any projective indecomposable P, there is no non-zero map $f : M \to P$. For any such map would factor as a surjection s followed by an injection i, so that $f = is$; but then $s(M)$ injects into the projective P and so is itself projective. Then $s : M \to s(M)$ is a surjection onto a projective, so it splits; since M was assumed to be indecomposable, this implies that s is an isomorphism, contradicting the assumption that M is not projective.

It follows from this that there is no non-zero map $M \to A$ for M a non-projective indecomposable; since all indecomposable summands of A are projective, any map $M \to A$ is a vector of maps from M to the indecomposale projectives; by the above, all of these maps must be zero. So if M is now any module, ${}_A\mathrm{Hom}(M, A)$ records only information about the projective indecomposable summands of M - all non-projective summands are killed. The same therefore applies to the Nakayama functor. It suffices then to study the effect on projective indecomposables (noting that both of the functors involved in the definition of ν respect direct sums, so $\nu(M)$ is the direct sum of $\nu(P_i)$ for P_i the projective indecomposable summands of M).

Therefore let Q be a quiver and let $v \in Q_0$ be any vertex, so that kQv is a projective indecomposable left kQ-module. Then ${}_{kQ}\mathrm{Hom}(kQv, kQ) = vkQ$ as a vector space, by Yoneda's Lemma; vkQ is the space of paths ending in v, but after applying ${}_{kQ}\mathrm{Hom}(\circ, kQ)$, it is a right-module, which is the same as a left module over the opposite quiver, so vkQ should be viewed as the space of paths in Q^{op} starting in v; this is precisely the projective indecomposable representation of Q^{op} based at the vertex v. Indeed, the defined right action of kQ on vkQ is consistent with this. So ${}_{kQ}\mathrm{Hom}(\circ, kQ)$ takes projective indecomposable representations of Q to projective indecomposable representations of Q^{op}. Since Hom functors respect direct sums, we can restrict to a functor $\mathrm{proj}(Q) \to \mathrm{proj}(Q^{\mathrm{op}})$.

We have already seen that D restricts to a functor $\mathrm{proj}(Q^{\mathrm{op}}) \to \mathrm{inj}(Q)$, so we see that the Nakayama functor restricts to a functor $\mathrm{proj}(Q) \to \mathrm{inj}(Q)$. The projective indecomposable kQv maps to $\mathrm{D}(vkQ)$, which we therefore see is an injective indecomposable; given any representation M, $\nu(M)$ is the direct sum of the injective indecomposables corresponding to the projective indecomposable summands of M.

Similar considerations, *mutatis mutandis*, show that $\mathrm{Hom}_{kQ}(\circ, kQ)$ gives a contravariant functor $\mathrm{proj}(Q) \to \mathrm{inj}(Q^{\mathrm{op}})$, so we also have $\nu : \mathrm{proj}(Q) \to \mathrm{inj}(Q)$. Since D and ${}_{kQ}\mathrm{Hom}(\circ, kQ)$ are both dualities on the proj and inj categories, their composition ν is an equivalence of categories with quasi-inverse given by $\nu^{-1} = \mathrm{Hom}(\circ, kQ)\mathrm{D}$, the composition in the opposite order. Indeed, we have

$$\begin{aligned}
\nu\nu^{-1} &= \mathrm{D}\mathrm{Hom}_{kQ}(\circ, kQ)_{kQ}\mathrm{Hom}(\circ, kQ)\mathrm{D} \\
&= \mathrm{D}1_{\mathrm{proj}(Q)}\mathrm{D} \\
&= \mathrm{D}^2 \\
&= 1_{\mathrm{proj}(Q)}
\end{aligned}$$

and similarly for the opposite composition.

4.5 The Auslander-Reiten Theorem

Definition 4.16 (The Auslander-Reiten Translate (Schiffler 2014))
Let Q be an acyclic quiver and M an indecomposable representation, with a projective resolution

$$0 \to P_1 \to P_0 \to M \to 0$$

The dual functor D is exact and Hom-functors are left-exact, so we see that the Nakayama functor ν is right-exact. Applying the Nakayama functor gives us an exact sequence

$$\nu(P_1) \to \nu(P_0) \to \nu(M) \to 0$$

Let $\tau(M)$ be the kernel of the leftmost map, so we have an exact sequence

$$0 \to \tau(M) \to \nu(P_1) \to \nu(P_0) \to \nu(M) \to 0$$

Of course, *a priori*, $\tau(M)$ depends on our choice of projective resolution; a routine homological algebra argument shows that it is actually independent of this choice. In brief, given two projective resolutions, the identity $M \to M$ lifts to a chain map between the resolutions, unique up to chain homotopy (see Rotman (2009) for details), which then passes to a map between the two kernels; this map is then shown to be an isomorphism. The details are omitted for brevity.

If M is projective, we may take $P_1 = 0$, so that $\nu(P_1) = 0$ and then $\tau(M) = 0$. So we restrict our attention to the case where M is a non-projective indecomposable. So then $\nu(M) = 0$, as ν kills all non-projective indecomposables, so we have a short exact sequence

$$0 \to \tau(M) \to \nu(P_1) \to \nu(P_0) \to 0$$

If $\tau(M)$ is injective, then this sequence splits, so $\nu(P_1) \cong \tau(M) \oplus \nu(P_0)$, but then $P_1 \cong \nu^{-1}\tau(M) \oplus P_0$. Since P_1 is also a submodule of P_0, this implies that $\nu^{-1}\tau(M) = 0$ and $P_1 \cong P_0$; but then in our original projective resolution for M, there is an injection of a finite-dimensional module P_1 into itself, which must therefore be an isomorphism, so M is the zero module, which is a contradiction. So we see that $\tau(M)$ is not injective.

Now suppose that $\tau(M) = X \oplus Y$ splits into two non-trivial modules X and Y. Then, since P_i is projective for $i \in \{0, 1\}$, $\nu(P_i)$ is injective, so the sequence

$$0 \to \tau(M) \to \nu(P_1) \to \nu(P_0) \to 0$$

is an injective resolution of $\tau(M)$. It therefore splits as the direct sum of an injective resolution for X and an injective resolution for Y. In particular, $\nu(P_1)$ and $\nu(P_0)$ split as direct sums in a way that is respected by the map between them; as ν is an equivalence of categories, this splitting still holds among P_1 and P_0, so the original projective resolution for M splits, showing that M is decomposable; but this is a contradiction. So $\tau(M)$ is indecomposable.

Finally then, we see that we have a function τ from the set of non-projective indecomposable representations to the set of non-injective indecomposables. This function is called the *Auslaner-Reiten translate.*

Similarly, given an indecomposable M and an injective resolution

$$0 \to M \to I_0 \to I_1 \to 0$$

we can apply ν^{-1} and take the cokernel, which we call $\tau^{-1}(M)$, obtaining an exact sequence

$$0 \to \nu^{-1}(M) \to \nu^{-1}(I_0) \to \nu^{-1}(I_1) \to \tau^{-1}(M) \to 0$$

Dual arguments to the above show that τ^{-1} is a function from the non-injective indecomposables to the non-projective indecomposables, called the *inverse Auslander-Reiten translate.* Of course, this assumes that there exists an injective resolution of length 1; but this is guaranteed. For the global dimension is at most 1, so all Ext^2 spaces are zero, so all modules admit an injective resolution of length at most 1. Indeed, similar to the standard projective resolution, there is a standard injective resolution for path algebras of quivers; see Hubery (s.a.) for details.

Given the suggestive notation, the next result will come as no surprise:

Lemma 4.17
The two maps τ and τ^{-1} are mutually inverse bijections.

PROOF:

Let M be a non-projective indecomposable and take a projective resolution for M

$$0 \to P_1 \xrightarrow{f} P_0 \to M \to 0$$

Applying ν gives an injective resolution of $\tau(M)$

$$0 \to \tau(M) \to \nu(P_1) \xrightarrow{\nu f} \nu(P_0) \to 0$$

and applying ν^{-1} gives

$$0 \to P_1 \xrightarrow{f} P_0 \to \tau^{-1}\tau(M) \to 0$$

showing that $\tau^{-1}\tau(M) = \text{coker}(f) = M$. The argument that $\tau\tau^{-1}(M) = M$ for M a non-injective indecomposable is similar. ■

Theorem 4.18 (The Auslander-Reiten Theorem on Existence of Almost-Split Sequences (Angeleri Hügel 2006))
Let M be a non-projective, indecomposable left A-module. Then there exists a module B and an almost-split short exact sequence:

$$0 \to \tau M \xrightarrow{i} B \xrightarrow{p} M \to 0$$

Similarly, if M is instead a non-injective indecomposable, then there exist a module B and an almost-split short exact sequence:

$$0 \to M \to B \to \tau^{-1}M \to 0$$

Proof:
We present only a very brief outline of two proofs.

The 'proper' way to prove this result is as follows. First construct a category related to A-Mod called the stable category modulo projectives, whose objects are A-modules and whose morphisms are classes of A-module homomorphisms modulo those which factor through projectives; denote the Hom-sets as Hom_A^P. Similarly, define the stable category modulo injectives, to have the same objects, but morphisms $\text{Hom}_A^I(M,N)$ given by $\text{Hom}_A(M,N)$ modulo morphisms which factor through injectives. Then derive the Auslander-Reiten Formulae, which are the isomorphisms

$$\text{Hom}_A^P(N, \tau M) \cong \text{DExt}_A^1(M,N)$$
$$\text{DHom}_A^I(M,N) \cong \text{Ext}_A^1(N, \tau M)$$

Finally, the result can be deduced by taking $1_M \in \text{End}_A(M) = \text{Hom}_A(M,M)$, relating it to an element of $\text{Ext}_A^1(M, \tau M)$ and viewing this element as an extension of M by τM in the usual fashion; this sequence turns out to be almost-split.

Developing the theory to flesh out these very bare bones would easily double the length of this project, so we shall say nothing more about this proof. The interested reader may consult Assem, Simson, and Skowroński (2006) for a more detailed exposition.

There is an alternative, more elementary proof, developed by Zimmermann (1983); however, though it is simpler and more direct than the proof by the Auslander-Reiten Formulae, it is also less illuminating, and so not worth giving in full detail. It centres around taking a projective resolution for M,

$$0 \to P_1 \xrightarrow{i} P_0 \xrightarrow{p} M \to 0$$

and constructing a map $\phi : P_1 \to M$ and a short exact sequence based on the pushout diagram of ϕ and i as shown below:

$$\begin{tikzcd}
0 \arrow[r] & P_1 \arrow[r, "i"] \arrow[d, "\phi"] & P_0 \arrow[r, "p"] \arrow[d, dashed] & M \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & \tau M \arrow[r, dashed] & B \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

The proof then proceeds by showing that ϕ does not factor through i, but that the composition $n\phi$ does for any nilpotent endomorphism n of τM, and then showing that these two properties imply that the bottom row of this diagram is an almost-split sequence. The details are quite fiddly and not very enlightening, so are omitted; they can be found in Angeleri Hügel (2006).

■

4.6 Knitting with Dimension Vectors

In this section we present a powerful application of the Auslander-Reiten theorem which provides a method known as knitting for very quickly and efficiently constructing the A-R quiver of a given quiver. First, a result linking the A-R quiver and the almost-split sequences of the A-R theorem.

Lemma 4.19 (The Middle Term of an Almost-Split Sequence (Barot 2006))
Let M be an indecomposable, non-projective representation and let

$$0 \to \tau M \xrightarrow{a=\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}} \bigoplus_{i=1}^{n} E_i \xrightarrow{b=(b_1,\ldots,b_n)} M \to 0$$

be an almost-split sequence, as given in the Auslander-Reiten theorem, where the E_i are indecomposables (possibly with repetitions). Then the maps a_i and b_i meet the conditions to be chosen as arrows in the A-R quiver (they are irreducible and linearly independent modulo the square-radical) and, in fact, the a_i are then all the arrows coming from τM, occurring once each, and the b_i are all the arrows going to M, occurring once each.

PROOF:
(ibid.) First observe that each a_i is radical, as otherwise it would be an isomorphism and hence a would be a section; similarly, each b_i is radical. Now let $f : \tau M \to X$ be an irreducible morphism for any indecomposable module X; then f is not an isomorphism, so f factors through a by the almost-split property:

$$\begin{tikzcd} & X \\ \tau M \arrow[ur, "f"] \arrow[r, "a"] & \bigoplus_{i=1}^{n} E_i \arrow[u, dashed, "g = (g_1, \ldots, g_n)"'] \end{tikzcd}$$

So we have

$$f = \sum_{i=1}^{n} g_i a_i$$

If all g_i are radical, then f is a sum of products of two radicals (g_i and a_i), hence is square-radical: $f \in \mathrm{Rad}^2(\tau M, X)$; but f was assumed to be irreducible, so at least one of the g_i is an isomorphism; by reordering, assume that $g_1, \ldots, g_t$ are isomorphisms and $g_{t+1}, \ldots, g_n$ are radical; by composing the first t of the a_i with an isomorphism, we may assume without loss of generality that $E_1 = E_2 = \ldots = E_t = X$ and $g_i = \lambda_i + \nu_i$ for some λ_i in the base field k and some nilpotent endomorphism ν_i of X, since g_i is an automorphism of an indecomposable.

So we can write

$$f = \sum_{i=1}^{t} \lambda_i a_i + h$$

where $h \in \mathrm{Rad}^2(\tau M, X)$ consists of the $g_i a_i$ for $i > t$ and the $\nu_i a_i$ (note that ν_i and a_i are radical, so $\nu_i a_i \in \mathrm{Rad}^2(\tau M, X)$. The form of this expression does not depend on f, only the particular values of the scalars λ and the square-radical map h do; so as we allow f to vary over the set of all irreducible morphisms, we see that $\{a_i, \ldots, a_t\}$ projects to a spanning set of $\mathrm{Rad}(\tau M, X)/\mathrm{Rad}^2(\tau M, X)$ under the canonical projection. In particular, the a_i are irreducible (for $i \leq t$).

We now show that this spanning set is in fact a basis. Suppose that, for some λ_i

$$\phi = \sum_{i=1}^{t} \lambda_i a_i \in \mathrm{Rad}^2(\tau M, X)$$

and that $\lambda_i \neq 0$ for some i. Then the map $l = (\lambda_1, \ldots, \lambda_n) : E_1^{\oplus t} \to E_1$ is a retraction, so up to isomorphism it is projection onto the first component, and $\phi = la$, which becomes a_1 when l is treated as projection. But ϕ is square-radical and a_1 is irreducible; a contradiction.

So $a_1, \ldots, a_t$ do indeed project to a basis of $\mathrm{Rad}(\tau M, X)/\mathrm{Rad}^2(\tau M, X)$, and can therefore be taken as arrows in the A-R quiver. They must include all arrows from τM to X, since they span, and must each occur with multiplicity one, by linear independence. Varying X, we see that each arrow coming from τM is represented exactly once.

The statements regarding the maps b_i are proved similarly. ■

Example 4.20 (Knitting with Dimension Vectors)
We now illustrate the technique of *knitting* an A-R quiver; let us take the example of the linear quiver on 4 vertices, for which we calculated the A-R quiver earlier, in Example 4.8 - so we can check our answer.

$$v_1 \to v_2 \to v_3 \to v_4$$

As before, we denote the indecomposable representations by their dimension vectors, in the shape of the quiver. The first step is to calculate the indecomposable projectives; we know that these are of the form kQv, where v is a vertex and kQ the path algebra, so they are:

$$P_1 = 1111 \quad P_2 = 0111 \quad P_3 = 0011 \quad P_4 = 0001$$

We also compute the spaces of morphisms between these modules; this is easy to do using the Yoneda Lemma: $\mathrm{Hom}_A(Ae, M) = eM$ for any algebra A, idempotent e and module M. So we have $\mathrm{Hom}_{kQ}(P_i, P_j) = v_i kQ v_j$; this is the space of paths from v_j to v_i. So the endomorphism space of each projective indecomposable is 1-dimensional and so must be spanned by the identity; so there are no radical endomorphisms of the projective indecomposables. Moreover, there is precisely one path from P_i to P_j for $i > j$ and no paths for $i < j$; so there is a 1-dimensional space of radical maps P_i to P_j for $i > j$ and it is clear that these are irreducible precisely when $i = j + 1$. So we deduce that part of the A-R quiver is:

$$0001 \to 0011 \to 0111 \to 1111$$

which was the right-hand edge of the A-R quiver we previously constructed for this linear quiver.

Now, each projective except 1111 is the source of a radical injection, so each except 1111 must be non-injective (we will shortly be able to deduce that 1111 is injective, though we do not know this yet). Then we know by the Auslander-Reiten theorem that there is an almost-split exact sequence starting in each of these non-injective indecomposables and, by the above, that the maps it involves can all be taken to be arrows in the A-R quiver. So we have, for P_4:

$$0 \to 0001 \to E \to \tau^{-1}(0001) \to 0$$

for some E which has P_3 as a summand and has no other projective summand; moreover, if F is any other indecomposable summand of E, then as F is non-projective there is an almost-split sequence

$$0 \to \tau F \to G \to F \to 0$$

for some G which has P_4 as a summand (since F and P_4 must be connected by an irreducible map), so there is a non-zero radical map $\tau F \to P_4$; this map cannot be surjective, as if it were the indecomposable τF would split, so it is injective and τF is a proper submodule of P_4. But P_4 is simple, so this is a contradiction. Therefore $E = P_3$ and we have an almost-split sequence:

$$0 \to 0001 \to 0011 \to \tau^{-1} P_4 \to 0$$

and hence $\tau^{-1} P_4$ has dimension vector $0011 - 0001 = 0010$.

Now considering P_3, it is the first term in an almost-split sequence with middle term H, say. H consists of all indecomposables which receive a map from P_3; there is a radical morphism $P_3 \to P_2$ and also $P_3 = E$ is the middle term of the last sequence considered, so it has a radical morphism to $\tau^{-1}P_4 = 0010$. So H has P_2 and 0010 as summands. Suppose that I is another indecomposable summand of H; then there is an almost-split sequence

$$0 \to \tau I \to J \to I \to 0$$

for some J with P_3 as a summand, so by reasoning similar to the above, τI is a proper submodule of P_3. But path algebras are hereditary, so this implies that τI is projective; the only projective proper submodule of P_3 is P_4, so $\tau I = P_4$. But then $I = \tau^{-1}(P_4) = 0010$ and this is already accounted for - I was assumed distinct from this. Contradiction. So the two summands of H found are the only ones and so H has dimension vector $0111 + 0010 = 0121$ and so $\tau^{-1}P_3$ has dimension vector $0121 - 0011 = 0110$.

Now the almost-split sequence starting with P_2 has middle term $\tau^{-1}P_3 \oplus P_1$, since the previous working shows that both of these are included (they receive irreducible maps from P_2) and the same reasoning as before rules out any other summands. So $\tau^{-1}P_2$ has dimension vector $0110 + 1111 - 0111 = 1110$.

We have repeatedly claimed that P_1 is injective, which means that it has no A-R translate, but we have not proved this, so let us blindly attempt to proceed as before. 1111 is the first term in a sequence whose middle term involves 1110 and no other summands, by similar reasoning to the above, so $\tau^{-1}(1111)$ has dimension vector $1110 - 1111 = (0, 0, 0, -1)$ (the brackets and commas are included because otherwise the minus sign makes it unclear what is meant). This cannot be true, as the entries in a dimension vector must be positive integers. If 1111 were non-injective, this procedure would have worked, so in fact the procedure tells us that 1111 is injective, as claimed. So we see that knitting is, in some sense, self-correcting.

So far, we have constructed the next layer of the A-R quiver; we have, with the Auslander-Reiten translate shown as dashed arrows:

$$\begin{tikzcd}
 & 1111 \arrow[dl] & & & \\
1110 \arrow[rr, dashed, "\tau"] & & 0111 \arrow[ul] \arrow[dl] & & \\
 & 0110 \arrow[ul] \arrow[rr, dashed, "\tau"] & & 0011 \arrow[ul] \arrow[dl] & \\
 & & 0010 \arrow[ul] \arrow[rr, dashed, "\tau"] & & 0001 \arrow[ul]
\end{tikzcd}$$

It should now be clear why this process is called knitting - it starts with the projectives, which form one edge of the diagram, then works its way across, layer-by-layer. The next step, then, is to construct the inverse A-R translates of the three newly constructed indecomposables.

Start with 0010; this is the first term in an almost-split sequence whose middle term must have a summand of 0110; suppose it has another summand K; then K is not projective (as we already know everything about the projectives and their irreducibles, and they receive no maps from 0010), so we have:

$$0 \to \tau K \to L \to K \to 0$$

for some L with a summand of 0010; so there is a radical map $\tau K \to 0010$; this cannot be a surjection, as any surjection to 0010 is involved in the almost-split sequence

$$0 \to 0001 \to 0011 \to 0010 \to 0$$

which we have already constructed and so τK would have to be 0011, but then $K = 0110$, yet we assumed K was distinct from this. So the radical $\tau K \to 0010$ is an injection; but the dimension vector of 0010 clearly allows no submodules - this is a simple representation. So no such K exists. Therefore $\tau^{-1}(0010)$ has dimension vector $0110 - 0010 = 0100$.

For $\tau^{-1}(0110)$, we have a sequence with middle term involving $1110 \oplus 0100$; any other summand is non-projective and has A-R translate which injects radically into 0110 by the same reasoning as above, but by dimension considerations, only 0100 & 0010 can possibly inject radically into 0110; as both of these are accounted for, there is no other summand. So $\tau^{-1}(0110)$ has dimension vector $1110 + 0100 - 0110 = 1100$.

As before, we have claimed that 1110 is injective; if we try to knit its inverse translate anyway, we find it has dimension vector $1100 - 1110 = (0, 0, -1, 0)$; again, the procedure tells us that 1110 is injective by giving an impossible, negative dimension.

Finally, we knit the last layer of the quiver and find that $\tau^{-1}(0100) = 1000$ and that 1100 and 1000 are injective, so there is no indecomposable left which is non-injective and whose inverse translate is not yet computed. So the procedure is complete and we have our finished diagram, again with τ shown as dashed arrows:

$$\begin{tikzcd}
 & & & 1111 \arrow[dl] & & & \\
 & & 1110 \arrow[dl] \arrow[rr, dashed, "\tau"] & & 0111 \arrow[ul] \arrow[dl] & & \\
 & 1100 \arrow[dl] \arrow[rr, dashed, "\tau"] & & 0110 \arrow[ul] \arrow[dl] \arrow[rr, dashed, "\tau"] & & 0011 \arrow[ul] \arrow[dl] & \\
1000 \arrow[rr, dashed, "\tau"] & & 0100 \arrow[ul] \arrow[rr, dashed, "\tau"] & & 0010 \arrow[ul] \arrow[rr, dashed, "\tau"] & & 0001 \arrow[ul]
\end{tikzcd}$$

Of course, we have only constructed the dimension vectors of the indecomposable representations, but we know the actual structure of the projective indecomposables and the maps between them, and all the rest were built from these with short exact sequences, so we can easily follow the sequences through the diagram to deduce the actual structure of each indecomposable representation. Thus, in general, it is enough to work with dimension vectors, though more care may be needed if it happens that two distinct indecomposables share a dimension vector. Nonetheless, their positions in the diagram should make it clear which is which.

It is worth noting that the knitting procedure does not always terminate in finite time; indeed, if the quiver in question is representation-infinite, it cannot. It is, however, often possible to argue by induction and deduce much of the structure of the quiver.

It also often occurs that the A-R quiver is not connected; it is easily seen from the definitions that two indecomposables are in the same connected component if and only if there is a radical map between them which is not in the infinite-radical. If there are multiple connected components, even inductively knitting from an initial vertex will not produce the whole quiver. It is then necessary to perform the procedure multiple times, knitting each connected component separately, which may prove difficult, as there can be uncountably many connected components! (in, *e.g.*, 'wild' quivers). Since the procedure starts with a projective indecomposable (or it works just as well working backwards from an injective), connected components which contain no projective nor injective can be particularly troublesome.

Nonetheless, the knitting procedure is, as illustrated, a very powerful tool for swiftly deducing some (and often all) of the structure of the category of representations of a given quiver.

Example 4.21 (The Kronecker Quiver K_2)
We now briefly study a more complicated Auslander-Reiten quiver, with multiple connected components, each of which is infinite. Consider the quiver

$$\begin{tikzcd} \bullet \arrow[r, bend left] \arrow[r, bend right] & \bullet \end{tikzcd}$$

The projective indecomposables have dimension vectors 01 and 12 and it is easily seen that there should be two arrows between them in the A-R quiver, so we start with:

$$\begin{tikzcd} 01 \arrow[r, bend left] \arrow[r, bend right] & 12 \end{tikzcd}$$

It is interesting to note that this is in fact a copy of the quiver itself (similarly, the projective indecomposables of the linear quiver formed a copy of the quiver along one edge of the A-R diagram). This always occurs for finite, connected, acyclic quivers (Assem, Simson, and Skowroński 2006).

The simple representation 01 cannot surject onto anything except itself and can only inject into another projective (by heredity), so these are all arrows coming from 01, so we have an almost-split sequence

$$0 \to 01 \to 12 \oplus 12 \to \tau^{-1}(01) \to 0$$

whence $\tau^{-1}(01) = 23$. An easy induction argument now shows that we have an infinite connected component of the A-R quiver as shown:

$$\begin{tikzcd} 01 \arrow[r, bend left] \arrow[r, bend right] & 12 \arrow[r, bend left] \arrow[r, bend right] & 23 \arrow[r, bend left] \arrow[r, bend right] & 34 \arrow[r, bend left] \arrow[r, bend right] & 45 \arrow[r, bend left] \arrow[r, bend right] & 56 \ldots \end{tikzcd}$$

where the Auslander-Reiten translate has two orbits, consisting of alternating representations.

This is, however, not the entire A-R quiver. For instance, it contains only one of the two simple representations; the other, 10, is injective, so we can knit its component as well. It is easy to check that another (in fact, the only other) injective indecomposable has dimension vector 21, namely:

$$\begin{tikzcd} k^2 \arrow[r, bend left, "{(1,0)}"] \arrow[r, bend right, "{(0,1)}"'] & k \end{tikzcd}$$

and that $\mathrm{Rad}(21, 10)/\mathrm{Rad}^2(21, 10)$ is two-dimensional, so we have in the A-R quiver:

$$\begin{tikzcd} 21 \arrow[r, bend left] \arrow[r, bend right] & 10 \end{tikzcd}$$

We then have an almost-split sequence

$$0 \to \tau(10) \to 21 \oplus 21 \to 10 \to 0$$

whence $\tau(10) = 32$. Inductively, we knit:

$$\begin{tikzcd} \ldots 65 \arrow[r, bend left] \arrow[r, bend right] & 54 \arrow[r, bend left] \arrow[r, bend right] & 43 \arrow[r, bend left] \arrow[r, bend right] & 32 \arrow[r, bend left] \arrow[r, bend right] & 21 \arrow[r, bend left] \arrow[r, bend right] & 10 \end{tikzcd}$$

We now have two connected components of the A-R quiver, but still not the whole diagram. It is not hard to show that, for any $m \times m$ Jordan block $J = J_m(\lambda)$ with eigenvalue λ, the representation

$$\begin{tikzcd} k^m \arrow[r, bend left, "1"] \arrow[r, bend right, "J"'] & k^m \end{tikzcd}$$

is indecomposable and that these indecomposables are pairwise non-isomorphic, for distinct m, λ; see Assem, Simson, and Skowroński (ibid.) for details. Since its two vector spaces have equal dimension, a property which none of the indecomposables we have knitted enjoys, no such indecomposable can be found in our knitted components.

This example shows that there are limitations to the knitting procedure. Nonetheless, it does quickly and easily provide us with a lot of information about the indecomposable representations and radical maps of a quiver, making it a useful tool for representation theory.

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