Integration by Change of Variables

Objective: To practise identifying which method to use to integrate a function, and applying the different techniques we have learnt.

A Checklist for Integration:

1. Is the integrand continuous? If not, will need to split the domain of integration and miss out a strip or width ϵ around the discontinuity, then take the limit as $\epsilon \to 0$.
2. Is it a standard integral, such as $\sin(x)$, $\cos(x)$, x^n, e^x, $\sinh(x)$, or $\cosh(x)$? Possibly with constants multiplying/added to the variable.
3. Can I rewrite the integrand in some more convenient form? For instance, factorising or expanding out brackets, rewriting roots as fractional powers, using trig identities, etc.
4. Is it a chain rule integral? That is, can I split the integrand into a function $g(f(x))$, times the derivative of the inside function, $f'(x)$? Then this came from the chain rule and so an antiderivative is $f(g(x))$. Might have to multiply by a constant at the end to make it work out exactly.
5. Is there a subsitution I could make that might simplify the integral? Trig or hyperbolic substitutions are often useful when there are expressions involving squares, because of all the trig identities with squared functions in.
6. Can I split the integrand as a product of two functions, u and $\frac{\mathrm{d}v}{\mathrm{d}x}$, and use integration by parts?

Practice:

1.
$$\int (7x^3 + 2e^{3x} - 9\cos(x))\ \mathrm{d}x =$$

2.
$$\int_{-2}^{2} \frac{1}{\sqrt{x}}\ \mathrm{d}x =$$

3.
$$\int t^3(t^2 + t^{1/2})(t^7 - 3t^{-3/2})\ \mathrm{d}t =$$

4.
$$\int (2\sin(t)\cos(t)\sin(2t) + cos^2(2t))\ \mathrm{d}t =$$

5.
$$\int t^2(t^3 - 4)^9\ \mathrm{d}t =$$

6.
$$\int y^{-1/2}\cos\left(\sqrt{y}\right)\ \mathrm{d}y =$$

7.
$$\int \left(y^2 - \frac{2}{3}y\right) e^{y^3 - y^2}\ \mathrm{d}y =$$

8.
$$\int \tan(\theta)\ \mathrm{d}\theta =$$

9.
$$\int \frac{1}{\sqrt{1 + z^2}}\ \mathrm{d}z =$$

10.
$$\int \frac{\sqrt{z^3 - 1}}{z}\ \mathrm{d}z =$$

11.
$$\int ze^z\ \mathrm{d}z =$$

12.
$$\int u\cos(2u)\ \mathrm{d}u =$$

Some standard useful tricks for particular integrals:

1. To integrate $\ln(x)$, write as $1.\ln(x)$ and use parts, with $u = \ln(x)$, $\frac{\mathrm{d}v}{\mathrm{d}x} = 1$.
2. To integrate $\sin^2(x)$ or $\cos^2(x)$, use the double angle formulae for cosine to rewrite in terms of $\cos(2x)$.
3. To integrate products of exponentials and trig functions, use Euler's Equation to rewrite in terms of real or imaginary parts of complex exponentials.