Laplace Transforms

Objective: To understand the Laplace transform as a modification of the Fourier transform.

Warm-up: Fixing Divergence of the Fourier Transform:

We have seen that when finding the Fourier transform of $f(t) = e^{2\pi\alpha j t}$, (for α a real constant), the integral

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(t) e^{-2\pi j \xi t} \, \mathrm{d}t$$

diverges. We have looked at one way to fix this—broadening our notion of functions to include functionals, and introducing the Dirac delta—now we will see another.

Intuitively, the problem with evaluating the above integral is that the integrand doesn't decay to 0 as t goes to $\pm\infty$, so we end up with an infinite amount of area. So why not try multiplying by a function that decays rapidly to 0, such as $e^{-2\pi t}$? Of course, $e^{-2\pi t} \to 0$ as $t \to \infty$, but not as $t \to -\infty$, so let's also restrict our integration to positive t; since we're often only interested in the future behaviour of a function, not its past, this shouldn't be a problem. So we consider replacing the Fourier transform of f by the integral

$$\int_{0}^{\infty} f(t) e^{-2\pi t} e^{-2\pi j \xi t} \, \mathrm{d}t. \qquad (\star)$$

1. For $f(t) = e^{2\pi j \alpha t}$, evaluate the integral $(\star)$. Call the result $F(\xi)$.
2. Of course, there's nothing special about $e^{-2\pi t}$; any $e^{-2\pi\alpha t}$ will do for $\alpha > 0$ (and maybe, for some functions $f(t)$, even for $\alpha \leq 0$). Then the $e^{-2\pi\alpha t}$ and $e^{-2\pi j \xi t}$ terms can be combined as e^{-st} where $s = 2\pi(\alpha + j\xi)$. Evaluate

   $$\int_{0}^{\infty} f(t) e^{-st} \, \mathrm{d}t$$

   as a function of the complex number s.

Theory: The Laplace Transform:

We saw above how multiplying a function by a decaying exponential and restricting the integral to positive t can allow us to deal with a function that would otherwise fail to have a Fourier transform. So instead of integrating $f(t)e^{-2\pi j\xi t}$, we integrate $f(t)e^{-\alpha t}e^{-2\pi j\xi t} = f(t)e^{-(\alpha+2\pi\xi j)t}$. Therefore we can consider this as multiplying $f(t)$ by an arbitrary complex exponential and then integrating. So we define the **Laplace transform** of $f(t)$ to be the function $F(s)$ defined by

$$F(s) = \int_0^\infty f(t)e^{-st}\,\mathrm{d}t,$$

where s is a complex number.

This is analagous to the Fourier transform, but the Fourier transform $\hat{f}(\xi)$ is a function that takes a real input ξ and gives complex outputs, whereas the Laplace transform takes a complex input s and gives a complex output. The Fourier variable ξ was thought of as frequency; the Fourier transform splits a function $f(t)$ up as a combination of sinusoids $e^{2\pi j\xi t}$, with the amplitude of the sinusoid at frequency ξ given by $\hat{f}(\xi)$. We can think of the Laplace transform in a similar way, but now we include amplitude information directly with the sinusoid. If $s = a+bj$, then e^{-at} represents the amplitude of the sinusoid, and b the (angular) frequency.

For a Fourier transform, we can imagine infinitely many violins playing each frequency ξ at a fixed amplitude $\hat{f}(\xi)$ for all time, and the sinusoidal sound waves combining to give $f(t)$. For a Laplace transform, on the other hand, we imagine infinitely many tuning forks, one for each frequency, with damping, so that after the tuning fork of frequency ξ is struck with amplitude $F(a + 2\pi\xi j)$ at time 0, its oscillations die down due to friction at a rate of e^{-at}; then by combining all the sounds from the tuning forks, we recover the original function.

If the real part of s is far enough to the left, we do not generally expect the Laplace transform $F(s)$ to converge. This is the same problem as for Fourier transforms with divergent integrals. But when $\mathrm{re}(s)$ is large enough, the exponential decay should be big enough to make the integral converge.

To recover $f(t)$ from its Laplace transform $F(s)$, we have **Mellin's inverse formula**:

$$f(t) = \frac{1}{2\pi j}\int_{-\infty}^{\infty} F(a+\omega j)e^{(a+\omega j)t}\,\mathrm{d}\omega,$$

for any a sufficiently large that $F(a+\omega j)$ converges for all ω.

So we can pick any sufficiently large a (friction coefficient for our tuning forks) and recover the original function by combining the decaying tuning fork notes at that rate of decay and all different frequencies.

Practice:

The Laplace transform of $f(t)$ is the function $\mathcal{L}(f) = F(s)$ (of a complex variable s) defined by

$$F(s) = \int_0^\infty f(t)e^{-st}\,\mathrm{d}t.$$

1. Let $\alpha = a + bj \in \mathbb{C}$ be a constant. Show that

$$\mathcal{L}\left(e^{\alpha t}\right) = \frac{1}{s-\alpha},$$

for values of s such that $\mathrm{re}(\alpha - s) < 0$.

2. Let $H(t)$ be the Heaviside step function and τ a positive real constant. Show that

$$\mathcal{L}(H(t-\tau)) = \frac{e^{-\tau s}}{s}$$

for $\mathrm{re}(s) > 0$.

3. Let τ be a positive real constant and δ the Dirac delta. Show that

$$\mathcal{L}(\delta(t-\tau)) = e^{-\tau s}.$$

4. Let $f(t)$ be a differentiable function. Show that

$$\mathcal{L}\left(\frac{\mathrm{d}f}{\mathrm{d}t}\right) = s\mathcal{L}(f) - f(0^+),$$

where $f(0^+)$ denotes the limit of $f(t)$ as t tends to 0 from above only. What assumptions about f must you make for this to hold? Compare this with the analagous result for Fourier transforms.

5. Let f be a function, τ a real constant, and H the Heaviside step function. Show that

$$\mathcal{L}(f(t-\tau)H(t-\tau)) = e^{-\tau s}\mathcal{L}(f).$$

Key Points to Remember:

1. The **Laplace transform** of a function $f(t)$ is the function of the *complex* variable s:
$$\mathcal{L}(f)(s) = \int_0^\infty f(t)e^{-st}\,\mathrm{d}t,$$
for values of s where this converges.

2. We can think of the imaginary part of s as a frequency, like in the Fourier transform, and the real part as a decay rate for the sinusoids which make up f.

3. The Laplace transform tends to give a well-defined function (at least for some values of s) on a broad range of input functions f, unlike the Fourier transform, whose integral only converges for a fairly narrow class of functions.