Roots of Polynomials

Warm-up: The Polynomial Factor Theorem

1. Let $f(x) = x^3 - 6x^2 + 11x - 6$. Divide $f(x)$ by $(x - 2)$.
2. Hence show that $f(2) = 0$.
3. Let $g(x) = x^3 - 6x^2 + 11x + \alpha$ for some constant α. Divide $g(x)$ by $(x - 3)$ with remainder.
4. Given that $g(3) = 0$, what must the remainder be? Hence find the value of α.

Theory: The Polynomial Factor Theorem:

Suppose $f(x)$ is a polynomial and can be factored as $f(x) = (x - a)g(x)$ for some constant a and polynomial $g(x)$. Show that $f(a) = 0$ - we say a is a root of f.

Now we go the other way around. Suppose that $f(x)$ is a polynomial and $f(a) = 0$. Prove that $f(x) = (x - a)g(x)$ for some polynomial $g(x)$.

THEOREM:

Let $f(x)$ be a polynomial. Then $f(\alpha) = 0$ if and only if $f(x) = (x - \alpha)g(x)$ for some polynomial g.

Worked Examples:

Let $f(x) = x^2 + x - 6$, and let α and β be the roots. Show that $\alpha + \beta = -1$ and $\alpha\beta = -6$. By solving these equations simultaneously, find α and β.

Now let $f(x) = ax^2 + bx + c$, and let α and β be the roots. Show that

$$\alpha + \beta = \frac{b}{a} \quad \text{and} \quad \alpha\beta = \frac{c}{a}.$$

Hence show that the distance between α and β is given by

$$|\alpha - \beta| = \frac{\sqrt{b^2 - 4ac}}{a}.$$

Geometrically, α and β can be found by starting halfway between them and then moving up or down by half the distance between them:

$$\alpha, \beta = \frac{\alpha + \beta}{2} \pm \frac{1}{2}|\alpha - \beta|.$$

Hence show that

$$\alpha, \beta = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Exercises:

1. Let $f(x) = x^3 + px^2 + qx - 15$, where p and q are real constants. The roots of f are
$$\alpha, \frac{5}{\alpha}, \text{ and } \left(\alpha + \frac{5}{\alpha} - 1\right)$$
for some complex constant α.
   (a) Solve the equation $f(x) = 0$.
   (b) Hence find the value of p.

2. Let $g(t) = t^4 + at^3 - t^2 + bt + c$. The roots of g are
$$\alpha, -\alpha, \frac{1}{\alpha}, \frac{-1}{\alpha}$$
for some complex number α.
   (a) Show that $a = b = 0$.
   (b) Show that $c = 1$.
   (c) Hence show that α^2 and $\frac{1}{\alpha^2}$ are the roots of the polynomial
$$s^2 - s + 1.$$
   Why can we conclude from this that $\alpha^* = \frac{1}{\alpha}$?
   (d) Hence find
$$\alpha^2 + \frac{1}{\alpha^2}.$$
   (e) Hence show that
$$\left(\alpha + \frac{1}{\alpha}\right)^2 = 3$$
   and
$$\left(\alpha - \frac{1}{\alpha}\right)^2 = -1.$$
   (f) Hence show that all of the roots of g have real part $\frac{\pm\sqrt{3}}{2}$ and imaginary part $\frac{\pm 1}{2}$.
   (g) Hence write down the four roots of g.

3. Let $h(y) = y^3 - 11y^2 + ry - 169$, where r is a real constant. The roots of h are
$$\alpha, \frac{169}{\alpha}, \beta,$$
where α and β are constants.

(a) Show that $\beta = 1$.

(b) Hence find a quadratic equation whose roots are α and $\frac{169}{\alpha}$.

(c) Hence solve $h(y) = 0$ and find the value of r.

4. Consider a quadratic $ax^2 + bx + c$, where $c \neq 0$. Show that the roots are given by

$$x = \frac{2c}{-b \pm \sqrt{b^2 - 4ac}}.$$

Hint: First consider the special case $x^2 + x + 1$; evaluate the above expression, and express in standard form of complex numbers. Can you generalise the manipulations you did to the solutions of any quadratic?

Can you suggest a reason why this alternative to the usual quadratic formula might be useful?

A Trigonometric Method for Solving Cubics

Let $f(x) = x^3 + ax^2 + bx + c$ be a general cubic equation. We will derive a formula for the roots which uses trig functions.

Note: really, a general cubic should be $dx^3 + ax^2 + bx + c$, for some $d \neq 0$; but when solving, we can divide through by d. So we lose nothing by starting with the assumption that $d = 1$; we just have to remember if ever using this method to solve a cubic that we first have to divide by the leading coefficient.

1. Before we even look at the cubic, we do some trigonometry.
   (a) Writing $3\theta = 2\theta + \theta$, apply the compound angle formula for cosine to $\cos(3\theta)$.
   (b) Apply the double angle formula for cosine to the expression you got in the previous part, to express $\cos(3\theta)$ in terms of $\cos(\theta)$ and $\sin(\theta)$.
   (c) Use the Pythagorean identity ($\sin^2(\theta) + \cos^2(\theta) = 1$) to eliminate $\sin(\theta)$ from your formula above and get a formula for $\cos(3\theta)$ in terms of $\cos(\theta)$
2. Now we tackle our cubic. The first step is to "complete the cube"—like completing the square, only in degree 3.
   (a) Expand $\left(x + \frac{a}{3}\right)^3$.
   (b) Hence write $f(x)$ in the form $\left(x + \frac{a}{3}\right)^3 + \beta x + \gamma$ for some β and γ.
   (c) Hence write $f(x)$ in the form $\left(x + \frac{a}{3}\right)^3 + \delta\left(x + \frac{a}{3}\right) + \epsilon$ for some δ and ϵ.
   (d) Let $y = \left(x + \frac{a}{3}\right)$ to express $f(x)$ as $y^3 + \delta y + \epsilon$.
   (e) If $f(x) = x^3 - 5x^2 - 57x + 189$, what will δ and ϵ be?
3. Now we have removed the squared term from $f(x)$ by rewriting in terms of y, our next step is to modify the coefficients to make them look like the coefficients in our triple angle formula for cosine from the start of the question. To do this, we let
$$z = yj\sqrt{\frac{3}{4\delta}}.$$
   (a) Show that $y^3 = 4z^3 j\sqrt{\frac{4\delta^3}{27}}$.
   (b) Show that $\delta y = -3zj\sqrt{\frac{4\delta^3}{27}}$.
   (c) Hence write $f(x)$ in the form $\left(4z^3 - 3z + \eta\right) j\sqrt{\frac{4\delta^3}{27}}$ for some η.
   (d) If $f(x) = x^3 - 5x^2 - 57x + 189$, as before, what will η be?

4. Now we make yet another substitution; this time, we substitute $z = \cos(\theta)$.

   (a) Show that $f(x) = 0$ if and only if $4\cos^3(\theta) - 3\cos(\theta) = -\eta$, with η as above.

   (b) Hence, using the formula derived in part 1, show that $f(x) = 0$ if and only if $\cos(3\theta) = -\eta$.

   (c) Hence solve $x^3 - 5x^2 - 57x + 189 = 0$.

Complex Conjugation—Looking Deeper

Let $p(x)$ be a polynomial with real coefficients; *i.e.*, an expression of the form

$$p(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_{n-1} x^{n-1} + a_n x^n,$$

where all the a_i are real numbers.

1. Show that for any complex numbers z and w, $\overline{z+w} = \bar{z} + \bar{w}$.
2. Show that for any complex numbers z and w, $\overline{zw} = \bar{z}\bar{w}$.
3. Show that, if z is any complex number, $p(\bar{z}) = \overline{p(z)}$.
4. Hence show that if z is a complex root of p (*i.e.*, if $p(z) = 0$), then $\bar{z}$ is also a root.
5. Show that $2 - j$ is a root of the polynomial $x^2 - 4x + 5$. Hence write down the other root.
6. Let $p(x) = x^2 + jx + 2$. Show that j is a root of p, but that $-j$ is not a root. Why does this not contradict the result we proved above?