Polar Form of Complex Numbers

Objective: To understand the polar form of complex numbers and its uses. To be able to convert complex numbers between polar and cartesian forms.

Recap of previous material:

1. Let $z = -1.3 - 9.4j$. Write down $\mathrm{Re}(z)$, $\mathrm{Im}(z)$, and $\bar{z}$.
2. $(2 - 3j) + (4j - 7) =$
3. $j(1 + 5j) =$
4. $(1 - 2j)(1 + 2j) =$
5. $\frac{4-j}{1+4j} =$

Warm-up:

We will explore a link between complex numbers and their complex conjugates, and polar coordinates.

1. Let $z = 2 - 3j$.
   (a) Calculate $\sqrt{z\bar{z}}$.
   (b) We can represent z graphically by the point $(2, -3)$ in the (x, y)-plane. Convert this point into polar coordinates.
   (c) Compare the two answers above.
2. Let $z = a + bj$ for some real numbers a and b.
   (a) Calculate $z\bar{z}$ in terms of a and b.
   (b) Is $z\bar{z}$ real, imaginary, or complex? What about $\sqrt{z\bar{z}}$?
   (c) Represent z by the point (a, b) in the (x, y)-plane. Convert this point into polar coordinates.
   (d) Compare the answers above.

Theory—Cartesian and Polar Forms of Complex Numbers:

We have seen that a complex number z has the form $a+bj$ for two real numbers a and b—called the **real part** and **imaginary part** respectively. We can represent this as the point (a, b) in the (x, y)-plane. Because we are treating the real and imaginary parts of z as cartesian coordinates of a point, we call $a+bj$ the **cartesian form** of z. When we plot a point in the plane to represent a complex number, we call that diagram the **complex plane** or an **Argand diagram**.

Suppose a complex number z is represented in the Argand diagram by a point whose polar coordinates are (r, θ). Find the real and imaginary parts of z.

So we see that

$$z = r\cos(\theta) + rj\sin(\theta) = r\left[\cos(\theta) + j\sin(\theta)\right].$$

We sometimes write $\operatorname{cis}(\theta)$ as shorthand for $\cos(\theta) + j\sin(\theta)$ ("cis" stands for "$\cos + i\sin$", using i in place of j). So then $z = r\operatorname{cis}(\theta)$. This is called the **polar form** of z (whether written in terms of cis or longhand with sin and cos).

We saw in the second warm-up question that $z\bar{z}$ is always a non-negative real number, so has a non-negative real square root, which is precisely the r coordinate of z in polar coordinates. This number is called the **modulus** of z and written $|z|$. The θ-coordinate of z is called the **argument** of z and written $\arg(z)$. We have for $z = a + bj$:

$$|z| = \sqrt{z\bar{z}} = \sqrt{a^2 + b^2} \qquad \tan(\arg(z)) = \frac{b}{a}.$$

As always with polar coordinates, you have to think about the quadrant the point is in to figure out whether $\arg(z) = \tan^{-1}\left(\frac{b}{a}\right)$ or $\arg(z) = \pi + \tan^{-1}\left(\frac{b}{a}\right)$.

Practice:

We shall practise converting between cartesian and polar forms, then explore why polar form is useful.

1. Let $z = 1 - j$.
   (a) Calculate the modulus $|z|$.
   (b) Calculate the argument $\arg(z)$.
   (c) Hence express z in polar form.
2. Let $w = 2\,\mathrm{cis}\left(\frac{\pi}{3}\right)$.
   (a) Write down $|w|$.
   (b) Write down $\arg(w)$.
   (c) Calculate $\mathrm{Re}(w)$.
   (d) Calculate $\mathrm{Im}(w)$.
   (e) Hence express w in cartesian form.
3. Take z and w from the previous two questions.
   (a) Working in cartesian form, calculate the product zw.
   (b) Calculate $|zw|$. Compare with $|z|$ and $|w|$.
   (c) Calculate $\arg(zw)$. Compare with $\arg(z)$ and $\arg(w)$.
   (d) Now calculate $\frac{z}{w}$, again working in cartesian form.
   (e) Calculate $\left|\frac{z}{w}\right|$. Compare with $|z|$ and $|w|$.
   (f) Calculate $\arg\left(\frac{z}{w}\right)$. Compare with $\arg(z)$ and $\arg(w)$.

Theory—Multiplication in Polar Form:

Let $z = r\,\text{cis}(\theta)$ and $w = s\ \text{cis}(\phi)$ be two complex numbers written in polar form. Multiply them together and collect real and imaginary terms to find an expression for zw. Hence find $|zw|$.

Use the compound angle formulae to expand $rs\ \text{cis}(\theta + \phi)$. Compare with the above.

So we see that if we multiply two complex numbers in polar form, we can use the compound angle formulae to simplify the answer to a single complex number in polar form, whose modulus is $|z| \times |w|$ and whose argument is $\arg(z) + \arg(w)$. This leads to the following simple rule for multiplying complex numbers:

<u>M</u>ultiply the <u>M</u>oduli and <u>A</u>dd the <u>A</u>rguments.

As an equation,

$$(r\,\text{cis}(\theta))(s\ \text{cis}(\phi)) = rs\ \text{cis}(\theta + \phi).$$

Theory—Dividing in Polar Form:

We shall now apply the above method for multiplication to division of complex numbers in polar form.

Suppose we have two complex numbers, z and w, and want to compute $\frac{z}{w}$. Let $y = \frac{z}{w}$; then y is defined by the property that $wy = z$. Since to multiply two complex numbers we <u>M</u>ultiply the <u>M</u>oduli and <u>A</u>dd the <u>A</u>rguments, we must have

$$|w| \times |y| = |z| \qquad \text{and} \qquad \arg(w) + \arg(y) = \arg(z).$$

Rearranging these, we see that

$$|y| = \frac{|z|}{|w|} \qquad \text{and} \qquad \arg(y) = \arg(z) - \arg(w).$$

So to divide complex numbers, we **divide the moduli and subtract the arguments.** As an equation,

$$\frac{r\,\mathrm{cis}(\theta)}{s\,\mathrm{cis}(\phi)} = \frac{r}{s}\,\mathrm{cis}(\theta - \phi).$$

In particular, if we want to find $\frac{1}{z}$ for some complex number z, we use that $|1| = 1$ and $\arg(1) = 0$, so

$$\left|\frac{1}{z}\right| = \frac{1}{|z|} \qquad \text{and} \qquad \arg\left(\frac{1}{z}\right) = -\arg(z).$$

Next time we shall see *de Moivre's Theorem*—a simple but powerful use of this multiplication/division rule in polar form, to find powers and roots of complex numbers quickly and easily.

Key Points to Remember

1. The **cartesian form** of a complex number z is its normal expression as $a+bj$ for some real numbers a and b.
2. The point $z = a + bj$ can be represented on the **complex plane** (or **Argand diagram**) by plotting it as the point (a, b).
3. If we express (a, b) in polar coordinates (r, θ), then r is called the **modulus** of z, written $|z|$, and θ the **argument** of z, written $\arg(z)$.
4. The **polar form** of z is the expression $|z|(\cos[\arg(z)] + j\sin[\arg(z)])$, often abbreviated to $|z| \operatorname{cis}(\arg(z))$.
5. To multiply complex numbers in polar form, we use the rule: **<u>M</u>ultiply the <u>M</u>oduli and <u>A</u>dd the <u>A</u>rguments**:
$$|zw| = |z| \times |w| \qquad \text{and} \qquad \arg(zw) = \arg(z) + \arg(w).$$
6. To divide complex numbers in polar form, we use this in reverse, we **divide the moduli and subtract the arguments**:
$$\left|\frac{z}{w}\right| = \frac{|z|}{|w|} \qquad \text{and} \qquad \arg\left(\frac{z}{w}\right) = \arg(z) - \arg(w).$$
7. Cartesian form is (much) better to use when adding or subtracting complex numbers. Though multiplication and division aren't too hard in cartesian form, they're very easy in polar form.