# The Zermelo-Fraenkel Axioms

**Objective: To understand the axioms of ZF set theory.**

For each axiom, we shall give an intuitive description of what it means.

**The Empty Set Axiom:**

$$\exists x \forall z (z \notin x)$$

**The Axiom of Extensionality:**

$$\forall x \forall y (x = y \leftrightarrow \forall z (z \in x \leftrightarrow z \in y))$$

**The Axiom of Foundation:**

$$\forall x(x \neq \varnothing \to \exists y \in x(\forall z \in y(z \notin x)))$$

**The Axiom Scheme of Separation:**

For each formula $\phi$ with free variables among $x$, $z$, $w_1, \ldots, w_n$, we have

$$\forall z \forall w_1 \ldots \forall w_n \exists y \forall x(x \in y \leftrightarrow x \in z \wedge \phi(x, z, w_1, \ldots, w_n))$$

**The Pairing Axiom:**

$$\forall x \forall y \exists z \forall w(w \in z \leftrightarrow (w = x \vee w = y))$$

**The Union Axiom:**

$$\forall F \exists A \forall x (x \in A \leftrightarrow \exists Y \in F (x \in Y))$$

**The Axiom Scheme of Replacement:**

For each formula $\phi$ with free variables among $x$, $y$, $A$, $w_1, \ldots, w_n$:

$$\begin{aligned}\forall A \forall w_1 \ldots \forall w_n ((\forall x \in A \exists ! y \, \phi(x, y, A, w_1, \ldots, w_n)) \\ \rightarrow \exists Y \forall x \in A \exists y \in Y \, \phi(x, y, A, w_1, \ldots, w_n))\end{aligned}$$

**The Axiom of Infinity:**

$$\exists x(\varnothing \in x \wedge \forall y \in x(y \cup \{y\} \in x))$$

**The Power Set Axiom:**

$$\forall x \exists y \forall z(z \in y \leftrightarrow z \subseteq x)$$

**Exercises:**

1. Prove that there is no set of all sets.
2. Prove that no set contains itself as an element.
3. Prove that for any set $x$, $x \cup \{x\}$ is a set.
4. Prove that for any set $x$, there is a set containing $\{y\}$ for every $y \in x$ and nothing else (*i.e.*, the set of all singleton subsets of $x$).