Solving Trig Equations

Objective: To be able to solve equations with trig functions.

Recap of previous material:

1. $\sin\left(\frac{\pi}{4}\right) =$
2. $\cos\left(\frac{\pi}{6}\right) =$
3. $\tan\left(\frac{\pi}{3}\right) =$
4. Sketch the graph of $y = \sec(x)$.
5. Sketch the graph of $y = \cot(x)$.

Warm-up:

1. For what values of x does $\sin(x) = 0$?
2. For what values of x does $\cos(x) = -1$?
3. Solve $\sin(x) = \frac{\sqrt{3}}{2}$ for $0 \leq x < 2\pi$.
4. Solve $\cos(x) = \frac{1}{2}$ for $0 \leq x < 2\pi$.
5. Find θ between 0 and 2π such that $\cos(\theta) = \frac{1}{2}$ and $\sin(\theta) = \frac{\sqrt{3}}{2}$.

Worked Examples - Solving Trig Equations:

Solve $\sec(t) - 9 = \cos(t)$ for $0 \leq t < 2\pi$:

Solve $\cot(\theta + 30^\circ) = \sqrt{3}$ for $0 \leq \theta < 360^\circ$:

Practice:

1. Solve $\sin(x) + 2 = 3$ for $0^\circ \leq x < 360^\circ$
2. Solve $2\cos^2(x) - \sqrt{3}\cos(x) = 0$ for $0 \leq x < 2\pi$.
3. Solve $\sin^2(\theta) - \sin(\theta) = 2$ for $0 \leq x < 2\pi$.
4. Solve $\cos^2(t) + \cos(t) = \sin^2(t)$ for $0 \leq t < 2\pi$.
5. Solve $\operatorname{cosec}(x) + a\sin(x) = b$ in terms of a and b, for $0^\circ \leq x < 360^\circ$.

Application—Orbital Motion:

The planet Zorg orbits its star in a circular orbit; its position at time t is given by

$$x(t) = r\cos\left(\frac{2\pi}{T}t\right), \qquad y(t) = r\sin\left(\frac{2\pi}{T}t\right),$$

where T is the orbital period (the length of one year on the planet), and r is the radius of the orbit. The coordinate axes are set up so that the sun is at the origin.

The planet Yarg orbits the same star with radius $\frac{r}{2}$ and period $\frac{T}{2}$, and out of phase. Its position at time t is given by

$$x(t) = \frac{r}{2}\cos\left(\frac{4\pi}{T}t + \frac{\pi}{2}\right), \qquad y(t) = \frac{r}{2}\sin\left(\frac{4\pi}{T}t + \frac{\pi}{2}\right).$$

An astronaut plans to fly from Zorg to Yarg, and wants to do so when the distance between the planets is minimum, to save fuel.

1. Write down an expression for Δx (the difference in x-coordinates of the two planets), and a similar expression for Δy, the difference in y-coordinates.

2. Hence write down an expression for the distance d between the two planets at time t. Simplify this expression.

3. Using the formulae

$$\cos(A)\cos(B) = \frac{1}{2}(\cos(A+B) + \cos(A-B)),$$
$$\sin(A)\sin(B) = \frac{1}{2}(\cos(A-B) - \cos(A+B)),$$

   simplify your expression for the distance further.

4. The distance is a minimum (respectively maximum) precisely when the square of the distance is minimum (respectively maximum), and the square of the distance is easier to work with. Show that, for $0 \le t < T$ the distance attains its minimum and maximum values when

$$\sin\left(\frac{2\pi}{T}t + \frac{\pi}{2}\right) = 0.$$

   Note: If you haven't yet done differentiating trig functions or maximum/minimum problems, skip this part.

5. Solve the equation in part 4 for $0 \le t < T$.

6. Hence say when the astronaut should make their flight.

Solution:

1.
$$\Delta x = r\left(\cos\left(\frac{2\pi}{T}t\right) - \frac{1}{2}\cos\left(\frac{4\pi}{T}t + \frac{\pi}{2}\right)\right)$$
$$\Delta y = r\left(\sin\left(\frac{2\pi}{T}t\right) - \frac{1}{2}\sin\left(\frac{4\pi}{t}t + \frac{\pi}{2}\right)\right)$$

2. For clarity, write $\alpha = \frac{2\pi}{T}t$ and $\beta = \frac{4\pi}{T}t + \frac{\pi}{2}$.

$$\begin{aligned} d &= \sqrt{(\Delta x)^2 + (\Delta y)^2} \\ &= r\sqrt{\cos^2(\alpha) - \cos(\alpha)\cos(\beta) + \frac{1}{4}\cos^2(\beta) + \sin^2(\alpha) - \sin(\alpha)\sin(\beta) + \frac{1}{4}\sin^2(\beta)} \\ &= r\sqrt{\cos^2(\alpha) + \sin^2(\alpha) + \frac{1}{4}(\cos^2(\beta) + \sin^2(\beta)) - \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)} \\ &= r\sqrt{\frac{5}{4} - \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)} \end{aligned}$$

3.
$$\begin{aligned} d &= r\sqrt{\frac{5}{4} - \frac{1}{2}(\cos(\alpha+\beta) + \cos(\alpha-\beta)) - \frac{1}{2}(\cos(\alpha-\beta) - \cos(\alpha+\beta))} \\ &= r\sqrt{\frac{5}{4} - \frac{1}{2}(2\cos(\alpha-\beta))} \\ &= r\sqrt{\frac{5}{4} - \cos\left(\frac{-2\pi}{T}t - \frac{\pi}{2}\right)} \\ &= r\sqrt{\frac{5}{4} - \cos\left(\frac{2\pi}{T}t + \frac{\pi}{2}\right)} \end{aligned}$$

4.
$$r^2 = \frac{5}{4} - \cos\left(\frac{2\pi}{T}t\frac{\pi}{2}\right)$$
$$\frac{d\left(r^2\right)}{dt} = \frac{2\pi}{T}\sin\left(\frac{2\pi}{T}t + \frac{\pi}{2}\right)$$

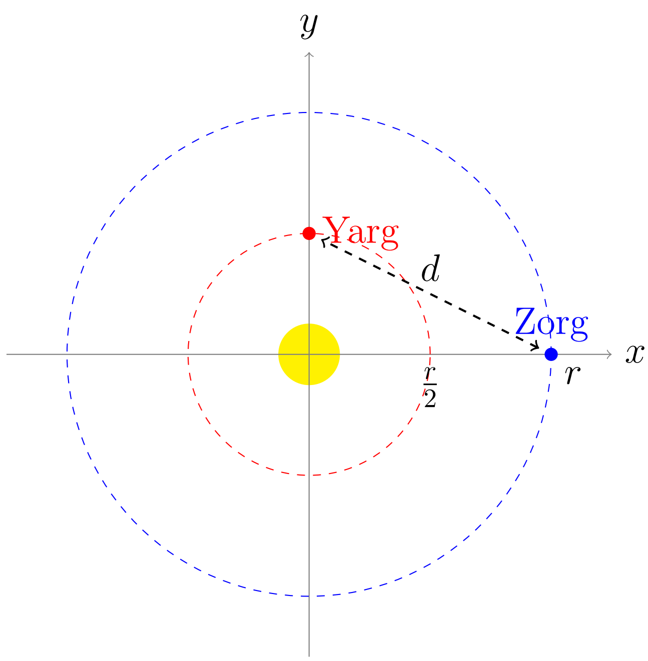
Maxima and minima occur when the derivative is 0, so when
$$\frac{2\pi}{T}\sin\left(\frac{2\pi}{T}t + \frac{\pi}{2}\right) = 0.$$

Cancelling the factor of $\frac{2\pi}{T}$ gives the desired result.

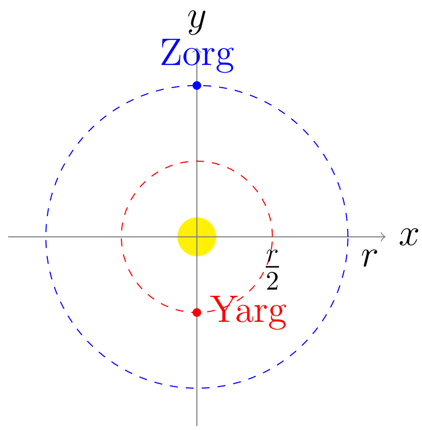
5. We need to solve
$$\sin\left(\frac{2\pi}{T}t + \frac{\pi}{2}\right) = 0.$$
for $0 \leq t < T$. Let $z = \frac{2\pi}{T}t + \frac{\pi}{2}$, so we need to solve $\sin(z) = 0$ for $\frac{\pi}{2} \leq z < \frac{5\pi}{2}$. The solutions to this are $z = \frac{n\pi}{2}$ for $n = 1, 2, 3, 4$. Hence the t-solutions are:
$$t = \frac{(n-1)T}{4} \qquad \text{for} \qquad n = 1, 2, 3, 4.$$

6. We need to identify which of the above times gives a global minimum. We can do this by evaluating the distance (or the squared distance) at each of those times, and picking the smallest value. (Note: if you know the 2nd derivative test, we could also use that first to reduce the possibilities to check). We have:

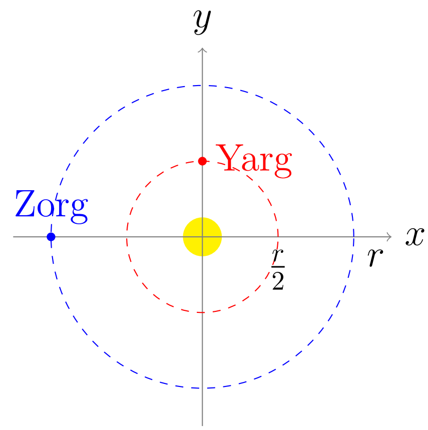
n	t	z	d
1	0	$\frac{\pi}{2}$	$\frac{\sqrt{5}}{2}r$
2	$\frac{T}{4}$	π	$\frac{3}{2}r$
3	$\frac{T}{2}$	$\frac{3\pi}{2}$	$\frac{\sqrt{5}}{2}r$
4	$\frac{3T}{4}$	2π	$\frac{1}{2}r$



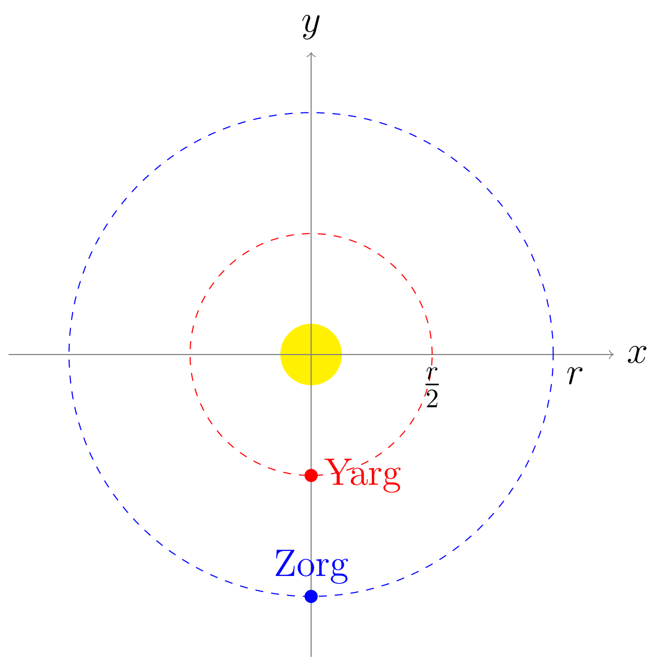
$t = 0$



$t = \frac{T}{4}$



$t = \frac{T}{2}$



$t = \frac{3T}{4}$