Rotations and Compound Angle Formulae

Objective: To understand the effect of a rotation on the coordinates of a point, and be able to use the compound angle formulae for trigonometric functions

Recap of previous material:

1. Sketch the graph of $y = x^3$. Hence sketch the graph of $y = (2x - 3)^3 - 1$.
2. Identify the amplitude, angular frequency, ordinary frequency, and phase of the sinusoid $7\sin(\pi t - 1.42)$. Sketch its graph.
3. Identify the amplitude, angular frequency, ordinary frequency, and phase of the sinusoid $-3\cos(2(t-1))$. Sketch its graph. Be careful—this one is tricky!

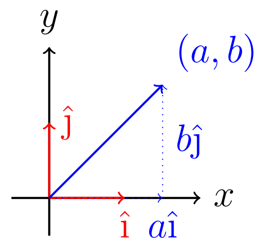
Warm-up:

Let's explore rotations of points about the origin.

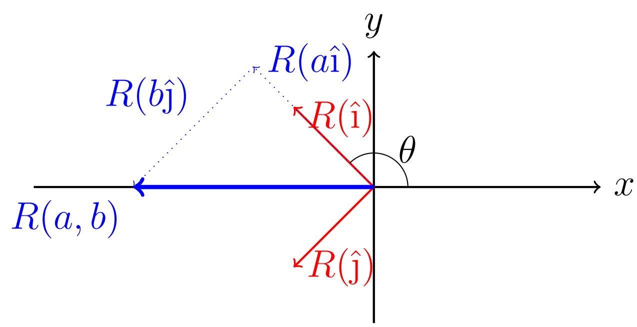
1. Suppose the point $(1, 0)$ is rotated anticlockwise by an angle θ. What are its new coordinates? Hint: think about the definition of sin and cos.
2. Suppose the point $(0, 1)$ is rotated anticlockwise by an angle θ. What are its new coordinates? Hint: compare with $(1, 0)$.
3. Suppose the points $(1, 0)$, $(0, 1)$, and (a, b) are rotated anticlockwise by $\frac{\pi}{2}$.
   (a) What are the coordinates of the rotations of $(1, 0)$ and $(0, 1)$?
   (b) The rotated coordinates of (a, b) are $(-b, a)$. We can relate the original coordinates to each other by the equation
   $$(a, b) = a(1, 0) + b(0, 1).$$
   Write a similar equation linking the coordinates of the rotated points.
4. Suppose the points $(1, 0)$, $(0, 1)$, and (a, b) are rotated anticlockwise by $\frac{3\pi}{4}$.
   (a) What are the coordinates of the rotations of $(1, 0)$ and $(0, 1)$?
   (b) The rotated coordinates of (a, b) are
   $$\left(\frac{-a}{\sqrt{2}} - \frac{b}{\sqrt{2}}, \frac{b}{\sqrt{2}} - \frac{a}{\sqrt{2}}\right).$$
   We can relate the original coordinates to each other by the equation
   $$(a, b) = a(1, 0) + b(0, 1).$$
   Write a similar equation linking the coordinates of the rotated points.

Theory—Vectors and Rotations:

Think of the point $(1, 0)$ as instructions "go one unit to the right." Call this instruction $\hat{\imath}$. Similarly, $(0, 1)$ is the instruction "go one unit upwards." Call this $\hat{\jmath}$. Then a general point (a, b) may be thought of as $a\hat{\imath} + b\hat{\jmath}$—"go a to the right and b upwards." We call $\hat{\imath}$ and $\hat{\jmath}$ the **standard unit vectors**.



Let R denote rotation by an angle θ. Then $R(\hat{\imath})$ is the instruction "move one unit in the direction θ above the positive x-axis" and $R(\hat{\jmath})$ is the instruction "move one unit in the direction θ anticlockwise of the positive y-axis." Rotate the whole setup above by θ:



Rotating everything by θ doesn't affect the *relative angles*. So if R is any rotation, we have

$$R(a\hat{\imath} + b\hat{\jmath}) = aR(\hat{\imath}) + bR(\hat{\jmath}).$$

So to understand a rotation, it is enough to understand what it does to the **unit vectors** $\hat{\imath}$ and $\hat{\jmath}$. We saw in the warm-up that, for rotation by angle θ:

$$R(\hat{\imath}) = (\cos(\theta), \sin(\theta)) = \cos(\theta)\hat{\imath} + \sin(\theta)\hat{\jmath}$$

$$R(\hat{\jmath}) = (-\sin(\theta), \cos(\theta)) = -\sin(\theta)\hat{\imath} + \cos(\theta)\hat{\jmath}.$$

Therefore we have:

$$\begin{aligned} R(a, b) &= aR(\hat{\imath}) + bR(\hat{\jmath}) \\ &= a(\cos(\theta)\hat{\imath} + \sin(\theta)\hat{\jmath}) + b(-\sin(\theta)\hat{\imath} + \cos(\theta)\hat{\jmath}) \\ &= (a\cos(\theta) - b\sin(\theta))\hat{\imath} + (a\sin(\theta) + b\cos(\theta))\hat{\jmath} \\ &= (a\cos(\theta) - b\sin(\theta),\ a\sin(\theta) + b\cos(\theta)) \end{aligned}$$

Practice:

Recall: The formula derived above is that if the point (a, b) is rotated by an angle θ, its new coordinates are:

$$(a\cos(\theta) - b\sin(\theta),\ a\sin(\theta) + b\cos(\theta))\,.$$

1. Find the new coordinates of the point $(7, 2)$ after a rotation by $\frac{\pi}{3}$.
2. Find the new coordinates of the point $(-4, 3)$ after a rotation by -1.97.
3. A point is rotated by an angle $\frac{\pi}{12}$, after which its coordinates are $(1.6, -2.9)$. What were the original coordinates of the point? Hint: think how geometrically to undo a rotation.

Application—Compound Angle Formulae:

We shall use rotations to derive the **compound angle formulae**. These are important trigonometric identities relating trig functions of two angles α and β with trig functions of their sum $\alpha+\beta$. The idea we will follow is to form an angle $\alpha+\beta$ by starting at an angle α and rotating by β; by doing this in both polar and cartesian coordinates, we shall obtain the compound angle formulae.

Recall: The formula derived above is that if the point (a, b) (in cartesian coordinates) is rotated by an angle θ, its new coordinates are:

$$(a\cos(\theta) - b\sin(\theta),\ a\sin(\theta) + b\cos(\theta))\,.$$

1. What are the cartesian coordinates of the point P whose polar coordinates are $(1, \alpha)$?

2. In polar coordinates, the formula for a rotation is much easier than in cartesian coordinates. Write down the polar coordinates of the point obtained by rotating P through an angle β. Call this rotated point $R(P)$.

3. Express $R(P)$ in cartesian coordinates.

4. By comparing your answer to question 3 with the cartesian rotation formula above, derive the following **compound angle formulae**:

$$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta),$$

$$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta).$$

5. Hence deduce the compound angle formula for tan:

$$\tan(\alpha+\beta) = \frac{\tan(\alpha)+\tan(\beta)}{1-\tan(\alpha)\tan(\beta)}.$$

Key Points to Remember:

1. The **standard unit vectors** are $\hat{\imath} = (1, 0)$ and $\hat{\jmath} = (0, 1)$, thought of as instructions to move one unit to the right or upwards, respectively.
2. Any point with cartesian coordinates (a, b) can be expressed in terms of the standard unit vectors as $a\hat{\imath} + b\hat{\jmath}$.
3. Applying a rotation R does not affect the *relative* positions of the vectors, so $R(a, b) = aR(\hat{\imath}) + bR(\hat{\jmath})$. So a rotation is *completely determined* by what it does to the standard unit vectors.
4. In **cartesian coordinates**, rotation by angle θ moves the point (a, b) to
$$(a\cos(\theta) - b\sin(\theta),\ a\sin(\theta) + b\cos(\theta))\,.$$
5. In **polar coordinates**, rotation by angle ϕ moves the point (r, θ) to $(r, \theta+\phi)$.
6. The **compound angle formulae** are:
$$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta),$$
$$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta).$$
$$\tan(\alpha+\beta) = \frac{\tan(\alpha) + \tan(\beta)}{1 - \tan(\alpha)\tan(\beta)}.$$