Ordered Pairs, Relations, and Functions

Objective: To understand how the ZF axioms allow us to define relations and functions, and work with these.

Ordered Pairs:

Recall that $\langle a, b\rangle$ is defined to be $\{\{a\}, \{a, b\}\}$.

1. Prove that $\langle a, b\rangle = \langle c, d\rangle$ if and only if $a = c$ and $b = d$.
2. Let's define $[a, b]$ to be $\{a, \{a, b\}\}$.
   (a) Prove that $[a, b] = [c, d]$ if and only if $a = c$ and $b = d$.
   (b) Recall that we define the natural numbers by $0 = \varnothing$ and $n + 1 = n \cup \{n\}$. What is the difference between 2 and $[0, 0]$?
3. Suggest a definition for the ordered triple $\langle a, b, c\rangle$. Prove that this satisfies the property you would expect of an ordered triple.
4. Prove that if A and B are sets, then $A \times B = \{\langle a, b\rangle \mid a \in A \wedge b \in B\}$ is a set.

Relations:

Recall that a **binary relation** on a set A is a subset of $A \times A$.

Draw a picture of the relation $\leq$ on $\{0, 1, 2, 3, 4\}$.

Assume $\mathbb{R}$ is a set with all the familiar properties. Draw a picture of the relation $\sim$ on $\mathbb{R}$ defined by $x \sim y$ if and only if $x^2 \leq y \wedge y < 2x + 4$.

Exercises:

1. Let R be a relation on A. Define what it means for R to be:
   (a) Reflexive
   (b) Symmetric
   (c) Transitive
   (d) Antisymmetric
   (e) A partial order
   (f) A total order (linear order)
   (g) A well-order
2. Let $(P, \leq)$ be a poset and define a new relation $\sim$ on P by $a \sim b$ if and only if $b \leq a$. Prove that $\sim$ is a partial order, and is linear if and only if $\leq$ is linear.
3. Let A be a set and $\equiv$ an equivalence relation (a symmetric, transitive, reflexive relation) on A. For each $a \in A$, let $[a]$ be the equivalence class of A—the class of b in A such that $a \equiv b$. Prove that:
   (a) For each $a \in A$, $[a]$ is indeed a set.
   (b) For any a and b in A, $[a] = [b]$ or $[a] \cap [b] = \varnothing$, where $[a] \cap [b] = \{x \in A \mid x \in [a] \wedge x \in [b]\}$.
   (c) Prove that $A/\equiv$, the collection of all equivalence classes of $\equiv$ in A, is a set.
4. Let A be a set and $\sim$ a relation on A which is transitive and reflexive (we call such a relation a pre-order). Prove that:
   (a) The relation $\equiv$ on A defined by $a \equiv b$ if and only if $a \sim b \wedge b \sim a$ is an equivalence relation.
   (b) The relation $\leq$ on $A/\equiv$ defined by $[a] \leq [b]$ if and only if $a \sim b$ is well-defined and a partial order.

Functions:

Let A and B be sets. Give the definition of a **function** from A to B.

Let $(A, \leq)$ and $(B, \sim)$ be posets. Define what it means for $f : A \to B$ to be **order-preserving**.

Define what it means for a function $f : A \to B$ to be **injective**, **surjective**, and **bijective**.

Exercises:

1. Let $(P, \leq)$ be a poset and Q a subset of P. Prove that the inclusion function $i : Q \to P : q \mapsto q$ is order-preserving with respect to the restricted order on Q.
2. Let $f : A \to B$ and $g : B \to C$ be functions. Prove that $g \circ f : A \to C$ is a function.
3. Let A and B be non-empty sets and $f : A \to B$ a function. Say that f is a **section** if there is a function $g : B \to A$ such that $g \circ f$ is the identity function on A, and say that f is a **retraction** if there is a function $h : B \to A$ such that $f \circ h$ is the identity function on B.
   (a) Prove that if f is a section, then f is injective.
   (b) Prove the converse: if f is injective, then f is a section.
   (c) Prove that if f is a retraction, then f is surjective.
   (d) Attempt to prove that if f is surjective, then f is a retraction. What goes wrong?