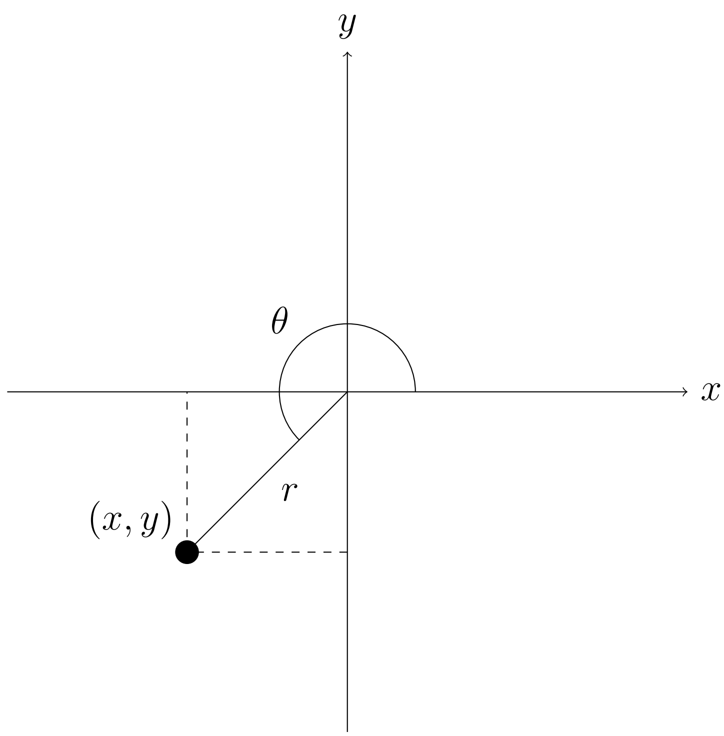
Plane Polar Coordinates and Inverse Trigonometric Functions

Objective: To understand inverse trig functions and polar coordinates, and be able to convert between polar and cartesian coordinates.

Recap of previous material:

Consider the diagram below, showing a general point with coordinates (x, y) (pictured here in the third quadrant, but it could be anywhere):



1. Express the length r from the origin to (x, y) in terms of x and y.
2. Express x in terms of r and θ, using a trigonometric function.
3. Express y in terms of r and θ, using a trigonometric function.
4. For the specific point $(-4, 2)$, find r.
5. For the specific point with $r = 2$, $\theta = \frac{4\pi}{3}$, find the (x, y)-coordinates. Plot this point on the above diagram.

Warm-up:

1. What angle does the point $(-1, 1)$ make with the positive x-axis?
2. What is the distance from the origin to $(-1, 1)$?
3. Hence find the point on the unit circle which makes the same angle with the positive x-axis as $(-1, 1)$.
4. Hence write down an angle θ such that $\cos(\theta) = \frac{-1}{\sqrt{2}}$ and $\sin(\theta) = \frac{1}{\sqrt{2}}$
5. Is there any other angle ϕ such that $\cos(\phi) = \frac{-1}{\sqrt{2}}$?
6. Is there any other angle ψ such that $\sin(\psi) = \frac{1}{\sqrt{2}}$?

Theory - Inverse Trigonometric Functions:

Given any angle θ, we can find $\cos(\theta)$ and $\sin(\theta)$ as the x- and y-coordinates respectively of the point on the unit circle at angle θ. We can then find $\tan(\theta)$ as the ratio of these two numbers:

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}.$$

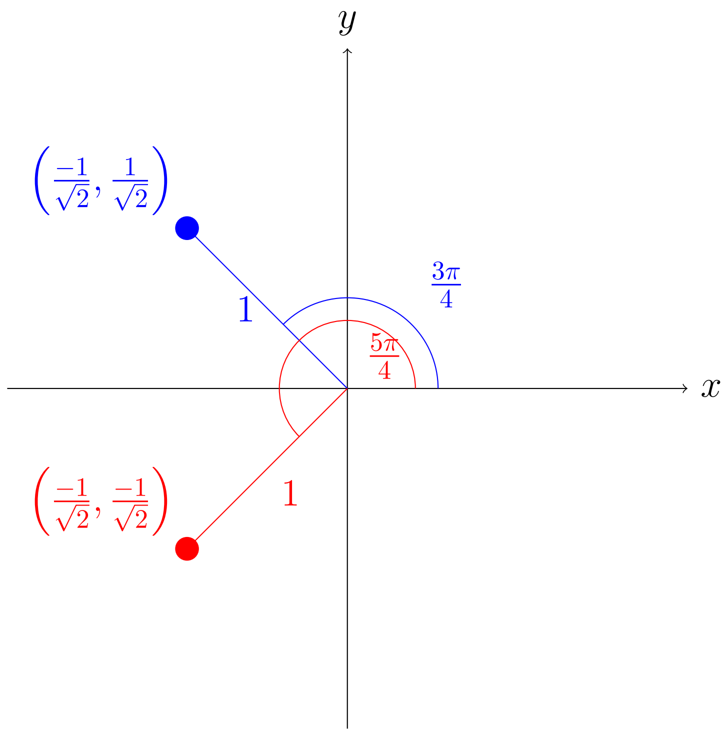
Can we go the other way? If we know $\sin(\theta)$ and $\cos(\theta)$, we can plot the point with coordinates $(\cos(\theta), \sin(\theta))$, which will lie on the unit circle, and measure the angle it forms with the positive x-axis. There are computational methods, such as Newton-Raphson, for figuring out the precise angle θ without the inaccurate approach of plotting the point and measuring the angle with a protractor.

What if we just know $\sin(\theta)$, without knowing $\cos(\theta)$? Then there are two possible angles θ, because there are two points on the unit circle with the same y-coordinate. Similarly, if we know $\cos(\theta)$, there are two points with the same x-coordinate, so two possible values of θ. This means that the trigonometric functions **cannot properly be inverted**.

However, we can partly invert them. The **inverse trigonometric function** arccos (often denoted $\cos^{-1}$) returns the **smallest** non-negative angle with a given cosine, always in the range $0 \leq \theta \leq \pi$. The inverse trigonometric functions arcsin and arctan (often denoted $\sin^{-1}$ and $\tan^{-1}$ respectively) return the unique angle between $\frac{-\pi}{2}$ and $\frac{\pi}{2}$ having a given sine or tangent respectively.

So $\sin(\sin^{-1}(x)) = x$, but $\sin^{-1}(\sin(\theta))$ might not equal θ, but is instead the angle closest to zero with the same sine as θ. Similarly, $\cos(\cos^{-1}(x)) = x$, but $\cos^{-1}(\cos(\theta))$ might not equal θ, and $\tan(\tan^{-1}(x)) = x$, but $\tan^{-1}(\tan(\theta))$ might not equal θ.

Example - arccos:



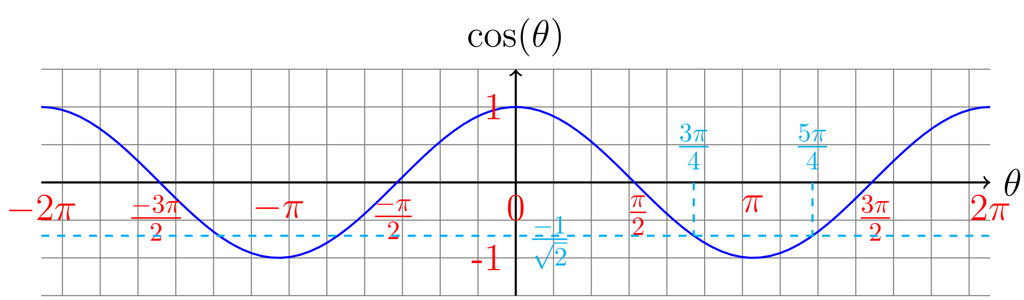
The two points shown both lie on the unit circle and have the same x-coordinate, $x = \frac{-1}{\sqrt{2}}$. Therefore

$$\cos\left(\frac{3\pi}{4}\right) = \cos\left(\frac{5\pi}{4}\right) = \frac{-1}{\sqrt{2}}$$

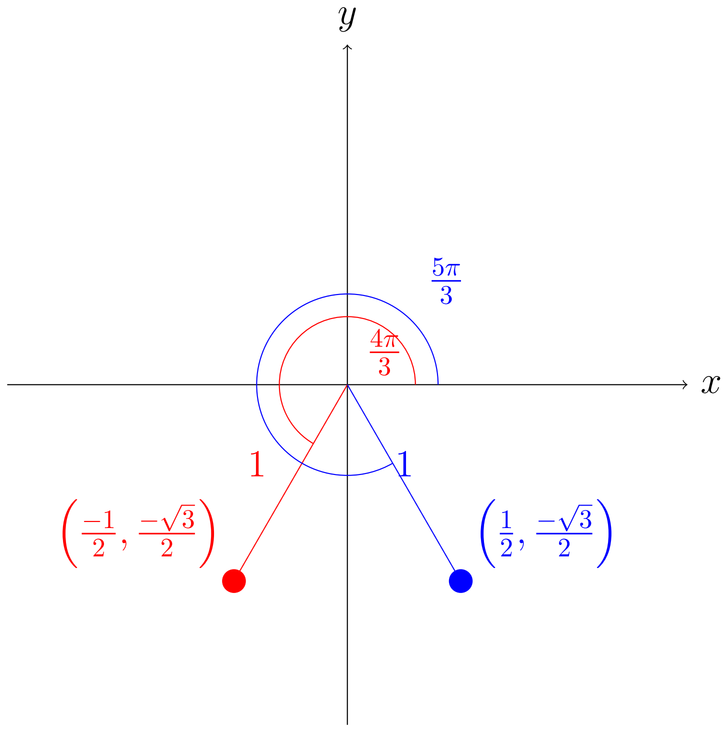
The arccos function returns the smallest non-negative angle having a given cosine, so

$$\cos^{-1}\left(\frac{-1}{\sqrt{2}}\right) = \frac{3\pi}{4}$$

Looking on the graph of the cosine function, we can see the two angles between 0 and 2π which have cosine $\frac{-1}{\sqrt{2}}$. We also see that there are other angles having the same cosine, because the cos function is 2π-**periodic**: it repeats exactly every 2π, so $\cos(\theta + 2\pi) = \cos(\theta)$.



Example - arcsin:



The two points shown both lie on the unit circle and have the same y-coordinate, $y = \frac{-\sqrt{3}}{2}$. Therefore

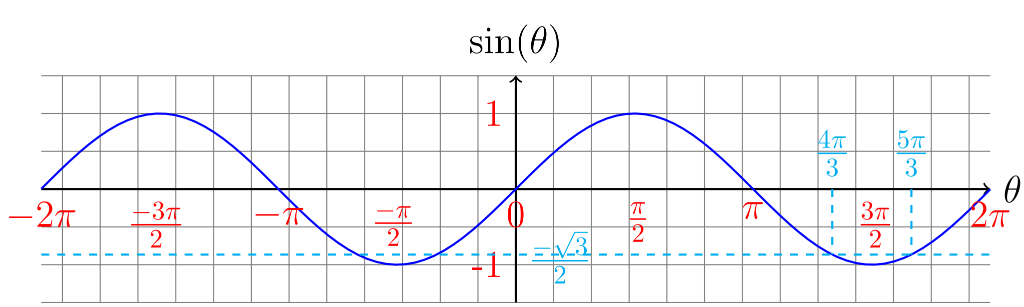
$$\sin\left(\frac{4\pi}{3}\right) = \sin\left(\frac{5\pi}{3}\right) = \frac{-\sqrt{3}}{2}$$

The arcsin function returns the angle closest to 0 having a given sine, so

$$\sin^{-1}\left(\frac{-\sqrt{3}}{2}\right) = \frac{-\pi}{3}$$

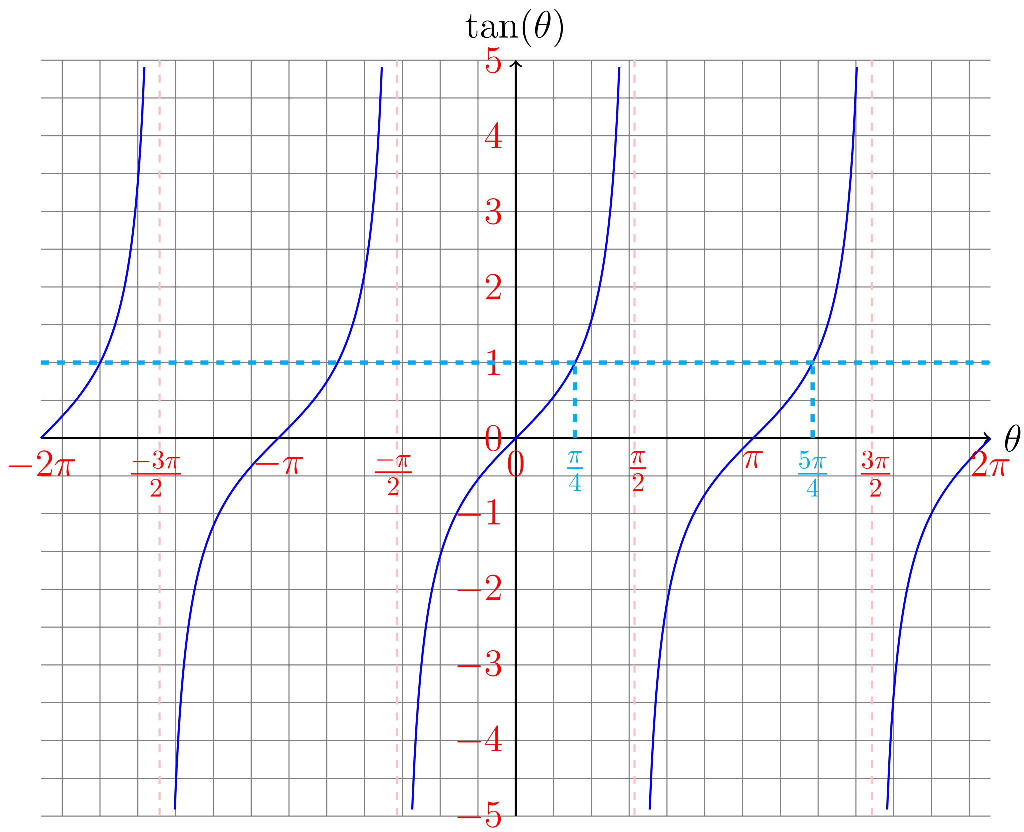
Shifting by 2π to get a positive angle gives us $\frac{5\pi}{3}$.

Looking on the graph of the sine function, we can see the two angles between 0 and 2π which have sine $\frac{-\sqrt{3}}{2}$. We also see that there are other angles having the same sine, because the sin function is 2π-**periodic**: it repeats exactly every 2π, so $\sin(\theta + 2\pi) = \sin(\theta)$.



Example - arctan:

Consider the graph of the tangent function, and suppose we want to find angles θ such that $\tan(\theta) = 1$:



We have that $\tan\left(\frac{\pi}{4}\right) = 1$, so $\tan^{-1}(1) = \frac{\pi}{4}$. However, the tan function is π-periodic (compared with 2π-periodic for sin and cos), so $\tan\left(\frac{5\pi}{4}\right) = 1$ also, and, in fact, for any whole number n:

$$\tan\left(\frac{\pi}{4} + n\pi\right) = 1.$$

So we have seen that for all three trig functions, there are **infinitely many solutions** θ to the equation $\sin(\theta) = x$, or $\cos(\theta) = x$, or $\tan(\theta) = x$, and **two** of these solutions will satisfy $0 \leq \theta < 2\pi$. Either of these can be the "correct" solution to a given problem, so care must be taken to choose the right one. The infinitely many other solutions are just copies of these two as the functions repeat themselves.

Practice:

1. $\sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}$. What is $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$?

2. $\cos\left(-\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$. What is $\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$?

3. $\tan(\frac{4\pi}{3}) = \sqrt{3}$. What is $\tan^{-1}(\sqrt{3})$?

Application: Plane Polar Coordinates:

We have seen that if we know the distance r of a point from the origin and the angle θ that point makes with the positive x-axis, we can find the (x, y)-coordinates of the point by the formulae

$$x = r\cos(\theta) \qquad y = r\sin(\theta).$$

We have also seen that the distance r can be found by Pythagoras' Theorem:

$$r = \sqrt{x^2 + y^2}.$$

Now that we have the inverse trig functions, we can also find θ from x and y. We have:

$$\begin{aligned} \frac{y}{x} &= \frac{r\sin(\theta)}{r\cos(\theta)} \\ &= \frac{\sin(\theta)}{\cos(\theta)} \\ &= \tan(\theta) \end{aligned}$$

Now, it **does not follow** that $\theta = \tan^{-1}(y/x)$, because the arctan function returns the smallest angle having the given tangent. So either $\theta = \tan^{-1}(y/x)$ or $\theta = \pi + \tan^{-1}(y/x)$. We can find which it is by considering whether x and y are positive or negative, and therefore which quadrant the point must lie in.

So given (x, y)-coordinates for a point (we call these the **cartesian coordinates** of the point), we can find (r, θ)-coordinates (we call these **plane polar coordinates** or just **polar coordinates**) using

$$r = \sqrt{x^2 + y^2} \qquad \tan(\theta) = \frac{y}{x},$$

and given polar coordinates (r, θ) for a point, we can find the cartesian coordinates by

$$x = r\cos(\theta) \qquad y = r\sin(\theta).$$

So cartesian and polar coordinates are equivalent ways to describe a point. In some problems, one or other coordinate system makes it easier to find the solution. We will particularly see this when we look at complex numbers and later at calculus.

Note: some people call cartesian coordinates **rectangular coordinates** instead.

Practice:

1. Convert the following points from cartesian coordinates to plane polars:
   (a) $(0, 4)$
   (b) $(-3, 7)$
   (c) $(12, -2)$
   (d) $(-1, -8)$
2. Convert the following points from polar coordinates to cartesians:
   (a) $\left(5, \frac{\pi}{2}\right)$
   (b) $(7, 3.729)$

Key Points to Remember:

1. The **inverse trigonometric functions** arcsin, arccos, and arctan return the **angle closest to 0** (and positive for arccos) θ having a given sine, cosine, or tangent, respectively, but there will be infinitely many other angles possible, and two will lie between 0 and 2π.

2. **Cartesian coordinates** (or **rectangular coordinates**) are the usual (x, y)-coordinates for describing a point's position.

3. **Plane polar coordinates** describe a point's position by its distance from the origin r and the angle θ that it makes with the positive x-axis.

4. To convert from polar coordinates to cartesian coordinates, use the formulae:

$$x = r\cos(\theta) \qquad y = r\sin(\theta).$$

5. To convert from cartesian coordinates to polars, use the formulae:

$$r = \sqrt{x^2 + y^2} \qquad \tan(\theta) = \frac{y}{x},$$

and consider the quadrant to decide whether

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) \text{ or } \pi + \tan^{-1}\left(\frac{y}{x}\right).$$