Fourier, Legendre, and Laplace Decompositions

Objective: To understand how orthonormal decompositions of vectors and functions

Warm-up: Orthonormal Vectors:

1. Let $e_1 = (1,0), e_2 = (0,1)$ in $\mathbb{R}^2$. Let $u = (-7,4)$ and $v = (a,b)$ for some unknown constants a and b.

   (a) Find $u \cdot e_1$ and $u \cdot e_2$ (where $\cdot$ denotes the usual dot product of vectors in $\mathbb{R}^n$).

   (b) Find scalars λ and μ such that $u = \lambda e_1 + \mu e_2$.

   (c) Find $v \cdot e_1$ and $v \cdot e_2$.

   (d) Find scalars λ and μ such that $v = \lambda e_1 + \mu e_2$.

2. Let $f_1 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, $f_2 = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ in $\mathbb{R}^2$. Let u and v be as above.

   (a) Show that $f_1 \cdot f_2 = 0$.

   (b) Show that $f_1 \cdot f_1 = 1 = f_2 \cdot f_2$.

   (c) Find $u \cdot f_1$ and $u \cdot f_2$.

   (d) Find scalars λ and μ such that $u = \lambda f_1 + \mu f_2$.

   (e) Find $v \cdot f_1$ and $v \cdot f_2$.

   (f) Find scalars λ and μ such that $v = \lambda f_1 + \mu f_2$.

Theory: Inner Products and Orthonormal Bases:

Let V be a real vector space. An **inner product** on V is a function $f \colon V^2 \to \mathbb{R}$, usually written $f(u, v) = \langle u \mid v \rangle$, satisfying three properties: for any $u, v, w \in V$ and $\lambda, \mu \in \mathbb{R}$:

$$\langle \lambda u + \mu v \mid w \rangle = \lambda \langle u \mid w \rangle + \mu \langle v \mid w \rangle \qquad \text{linearity in 1}^{\text{st}} \text{ argument}$$
$$\langle u \mid v \rangle = \langle v \mid u \rangle \qquad \text{symmetry}$$
$$\langle u \mid u \rangle > 0 \text{ if } u \neq 0 \qquad \text{positive-definiteness.}$$

Exercises: show that any inner product is linear in the second argument, and that $\langle v \mid v \rangle = 0$ for any v.

The standard example is the dot product (also called the Euclidean inner product) on $\mathbb{R}^n$. Given an inner product on V, we say that vectors $v_1, v_2 \ldots \in V$ are **orthogonal** if $\langle v_i \mid v_j \rangle = 0$ whenever $i \neq j$. If $v_1, v_2, \ldots \in V$ are orthogonal and also $\langle v_i \mid v_i \rangle = 1$ for all i, we say that they are **orthonormal**. We can express this concisely with the Kronecker delta symbol: $v_1, v_2, \ldots$ are orthonormal if

$$\langle v_i \mid v_j \rangle = \delta_{ij}.$$

Suppose $v_1, v_2, \ldots \in V$ are orthonormal and v is in the span of $v_1, v_2, \ldots$; then we have

$$v = \sum_i \lambda_i v_i$$

for some scalars λ_i. Because inner products are linear in each argument, we have for each j

$$\langle v \mid v_j \rangle = \left\langle \sum_i \lambda_i v_i \mid v_j \right\rangle$$
$$= \sum_i \lambda_i \langle v_i \mid v_j \rangle$$
$$= \sum_i \lambda_i \delta_{ij}$$
$$= \lambda_j.$$

Key point: if we write v as a linear combination of orthonormal vectors $v_1, v_2, \ldots$, we can find the coefficients by simplying taking the inner product of v with each v_i in turn!

Theory: Fourier Series:

Let $V = C[-\pi, \pi]$ be the set of continuous (real-valued) functions on the interval $[-\pi, \pi]$. This is an infinite-dimensional real vector space (exercise!). We can define an inner product on V:

$$\langle f(x) \mid g(x) \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) g(x) \, \mathrm{d}x.$$

You should think of this as just like taking the dot product of vectors, except that your "vectors" are actually functions. So just like if we have an orthonormal basis $v_1, \ldots, v_n$ on $\mathbb{R}^n$, we can write a vector v in terms of $v_1, \ldots, v_n$ by dotting it with each v_i in turn, if we can find an orthonormal basis $f_1(x), f_2(x), \ldots$ for $C[-\pi, \pi]$, then we can write any function $f(x) \in C[-\pi, \pi]$ in terms of $f_1(x), f_2(x), \ldots$ by simply integrating it against each $f_i(x)$ in turn!

The Fourier basis functions: The functions $a_0 = \frac{1}{2}$, $a_n(x) = \cos(nx)$, and $b_n(x) = \sin(nx)$ for $n \geq 1$ form an orthonormal basis of $C[-\pi, \pi]$. So *any* continuous function $f(x)$ on $[-\pi, \pi]$ can be written as a series of sines and cosines, with coefficients given by the integral inner product:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{2\pi n x}{L}\right) + b_n \sin\left(\frac{2\pi n x}{L}\right) \right],$$

where

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) \, \mathrm{d}x$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) \, \mathrm{d}x.$$

There is nothing special about the interval $[-\pi, \pi]$; we can do the same thing over any interval $[a, a + L]$ by using the functions $a_n(x) = \cos\left(\frac{2\pi n x}{L}\right)$ and $b_n(x) = \sin\left(\frac{2\pi n x}{L}\right)$, and taking the inner product to be

$$\langle f(x) \mid g(x) \rangle = \frac{2}{L} \int_{a}^{a+L} f(x) g(x) \, \mathrm{d}x.$$

We can also work with piecewise continuous functions, instead of just continuous, although at points of discontinuity the Fourier series will not generally converge to the correct value.

Theory: Legendre Polynomials:

So Fourier series are about decomposing functions into sums of sines and cosines. If you think about sound, a pure note is a sine wave of pressure in the air; several pure notes can combine to create more complicated pressure waves; *e.g.*, three pure notes can combine to make a chord. You can think of a Fourier series as if you have an orchestra of musicians, each playing a different note; the coefficients a_n and b_n tell each musician how loud to play so that the notes will combine to make a pressure wave that follows the function $f(x)$—and for any continuous function $f(x)$ on a bounded interval, it is possible to play it with an orchestra in this way!

What about other orthnormal bases for the space of continuous functions, or similar vector spaces? Legendre polynomials are an alternative orthogonal basis (not usually ortho*normal*, but that's less important). They can be defined by the properties: each $P_n(x)$ is a polynomial of degree n with leading coefficient 1, the P_n's are orthogonal, and $P_n(1) = 1$ for each n.

The Legendre polynomials are not orthonormal, but instead have $\langle P_n(x) \mid P_n(x) \rangle = \frac{2}{2n+1}$. This is fine, it just means that when finding coefficients to write a general function in terms of Legendre polynomials, we have to divide by this. Sofor any continuous function $f(x)$ on $[-1, 1]$, we have a decomposition, the **Fourier-Legendre series**:

$$f(x) = \sum_{n=0}^{\infty} a_n P_n(x),$$

where

$$a_n = \frac{2n+1}{2} \int_{-1}^{1} f(x) P_n(x) \, \mathrm{d}x.$$

As with the Fourier series, think of this integral as a dot product of $f(x)$ with a basis "vector" to extract the coefficient of that basis vector in $f(x)$.

The **Legendre generating function** is

$$G(x, r) = \frac{1}{\sqrt{1 - 2rx + r^2}} = \left(1 - 2rx + r^2\right)^{-1/2}.$$

This is just a convenient way of carrying information about all the Legendre polynomials at once, because the Taylor series of $G(x, r)$ in the variable r is

$$G(x, r) = \sum_{n=0}^{\infty} P_n(x) r^n.$$

This can be used to derive recurrence relations involving the Legendre polynomials, and compute values of them at specific values of x.

Practice:

1. Let $f(x)$ be the triangle function on $[0, 2]$ defined by

$$f(x) = \begin{cases} x : & 0 \leq x \leq 1 \\ 1 - x : & 1 < x \leq 2. \end{cases}$$

   Find the Fourier series of $f(x)$.

2. Let $f(x)$ be as above and recall that the first four Legendre polynomials are given by $P_0(x) = 1$, $P_1(x)x$, $P_2(x) = \frac{1}{2}\left(3x^2 - 1\right)$, and $P_3(x) = \frac{1}{2}\left(5x^3 - 3x\right)$. Find the Fourier-Legendre series of $f(x)$ up to and including the $n = 3$ term.

3. Differentiate the Legendre generating function with respect to r to show that

$$(n + 1)P_{n+1}(x) = (2n + 1)xP_n(x) - nP_{n-1}(x).$$