Rules for Differentiation.

Objective: To understand rules for differentiating various functions.

Recap: Monotonicity and Extrema:

Let $f(t) = 4t - t^2$.

1. Find the derivative of $f(t)$ with respect to t.
2. Solve $f'(t) = 0$.
3. By examining whether $f'(t)$ is positive or negative, determine for which t values $f(t)$ is increasing, and for which it is decreasing.
4. Hence determine whether your solution to part 2 is a maximum or minimum.

Warm-up:

We will prove that

$$\frac{\mathrm{d}x^n}{\mathrm{d}x} = nx^{n-1}$$

for n a positive integer. To do this, we will need to evaluate

$$\lim_{h\to 0} \frac{(x+h)^n - x^n}{h}.$$

1. Consider

   $$(x+h)^n = (x+h)(x+h)\dots(x+h).$$

   Without knowing n, we cannot multiply this out—we do not know how many brackets there are. However, we can say (in terms of n) what the term with x^n in will be. What is it?

2. When multiplying out, we pick one item from each bracket, and do this in all possible ways. How many ways are there of picking out h from one bracket and x from all the othes? Therefore, what is the term involving hx^{n-1} in the expansion?

3. Every other term in the expansion must involve h at least twice—*i.e.*, must be a multiple of h^2. So we can write

   $$(x+h)^n = x^n + nhx^{n-1} + h^2F,$$

   where F is some unknown combination of x and h. Using this, show that

   $$\lim_{h\to 0} \frac{(x+h)^n - x^n}{h} = nx^{n-1}.$$

Theory: Linearity, and the Chain and Product Rules:

So far we have seen the definition of the derivative, and some uses - determining whether a function is increasing or decreasing, and finding maxima and minima. But we don't have any techniques for computing the derivative of a function, apart from directly applying the definition. Now we will study three extremely useful rules for differentiation.

Suppose $f(t)$ and $g(t)$ are functions such that $f'(t)$ and $g'(t)$ are known. Find the derivative of $af(t) + bg(t)$, where a and b are constants.

Suppose f and g are as above. Find the derivative of the product, $p(t) = f(t)g(t)$.

Suppose f and g are as above. Find the derivative of the composition $c(t) = f(g(t))$.

Examples of Computing Derivatives:

Let $y = x^{-n}$, where n is a positive integer. Compute $\frac{\mathrm{d}y}{\mathrm{d}x}$.

Let $y = x^{m/n}$, where m and n are positive integers. Compute $\frac{\mathrm{d}y}{\mathrm{d}x}$.

Suppose that $f(x)$ is an invertible function and $\frac{\mathrm{d}f}{\mathrm{d}x}$ is known. Find $\frac{\mathrm{d}f^{-1}}{\mathrm{d}x}$ by considering $g(x) = f(f^{-1}(x))$.

Theory: Derivatives of Trigonometric and Exponential Functions:

Differentiate $\sin(x)$ and $\cos(x)$, making use of the small-angle approximations:

$$\sin(h) \approx h \qquad \text{and} \qquad \cos(h) \approx 1 - \frac{h^2}{2}$$

for h close to 0.

Let $b > 0$, $b \neq 1$. Differentiate b^x with respect to x.

We will need the following Python code, which can be executed at

`https://www.onlinegdb.com/online_python_interpreter`

```
def f(base):
    print("base = %f" % base)
    for i in range(7):
        h=10**(-i)
        derivative_at_zero = (base**h - 1) / h
        print("h=%f, f'(0)=%f" % (h,derivative_at_zero))
    print("\n")

f(2)
f(3)
f(2.5)
f(2.7)
f(2.71828183)
```

Further Examples of Computing Derivatives:

From a previous page, we have that

$$\frac{\mathrm{d}x^n}{\mathrm{d}x} = nx^{n-1}$$

for any rational number n.

Differentiate $\log_e(x)$ with respect to x.

Differentiate $4x^2 + 3x - 2$ with respect to x.

Differentiate $(2x - 1)(x + 1)$ with respect to x.

Differentiate $(12t^2 + 2t - 3)^7$ with respect to x.

Key Derivatives to Know:

$$\frac{\mathrm{d}x^n}{\mathrm{d}x} = nx^{n-1}$$

$$\frac{\mathrm{d}\sin(x)}{\mathrm{d}x} = \cos(x)$$

$$\frac{\mathrm{d}\cos(x)}{\mathrm{d}x} = -\sin(x)$$

$$\frac{\mathrm{d}e^x}{\mathrm{d}x} = e^x$$

$$\frac{\mathrm{d}\log_e(x)}{\mathrm{d}x} = \frac{1}{x}$$

Practice:

The chain rule:

$$\text{if } h(t) = f(g(t)), \text{ then } h'(t) = g'(t)f'(g(t)).$$

The product rule:

$$\text{if } h(t) = f(t)g(t), \text{ then } h'(t) = f(t)g'(t) + f'(t)g(t).$$

Linearity, for constants a and b:

$$\frac{\mathrm{d}\left(af(t) + bg(t)\right)}{\mathrm{d}t} = a\frac{\mathrm{d}f(t)}{\mathrm{d}t} + b\frac{\mathrm{d}g(t)}{\mathrm{d}t}.$$

1. Let $f(t) = 2t + 1$ and $g(t) = 4t - 6$.
   (a) Compute $f'(t)$.
   (b) Compute $g'(t)$.
   (c) Use the product rule to differentiate $h(t) = f(t)g(t)$.
   (d) Use the chain rule to differentiate $i(t) = f(g(t))$.
2. Differentiate $(x^2 + 2x + 3)^{10}$ with respect to x (hint: apply the chain rule with $f(x) = x^{10}$, $g(x) = x^2 + 2x + 3$).
3. Differentiate $(7x - 3)^{500}$ with respect to x (hint: chain rule).
4. Differentiate $(4x - 1)(2x + 1)$ with respect to x (hint: apply the product rule with $f(x) = 4x - 1$, $g(x) = 2x + 1$).
5. Differentiate $(13x^2 - 2)(4x + 6)$ with respect to x (hint: product rule).
6. Differentiate $(4x^2 + 6x - 2)^5(x^3 + 1)$ with respect to x (hint: use both the product and chain rules).
7. Let $y = \sin(4x + 1)$. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$.
8. Let $y = 2\sin(x)\cos(x)$.
   (a) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ by the product rule.
   (b) By the compound angle formula for sine, $y = \sin(2x)$. Hence find $\frac{\mathrm{d}y}{\mathrm{d}x}$ by the chain rule.
   (c) Show that your two answers are the same.
9. Differentiate $\tan(x)$ with respect to x (hint: write $\tan(x) = \sin(x)(\cos(x))^{-1}$ and apply the product and chain rules).

Application:

Suppose a 10V DC supply is used to drive current through a variable resistor. The resistance is varied according to $R = \frac{1000}{(t^2-10t+26)(t+1)}$.

1. Find the current I through the resistor at time t.
2. By differentiating, find the rate of change of current with respect to time, $\frac{\mathrm{d}I}{\mathrm{d}t}$.
3. Solve $\frac{\mathrm{d}I}{\mathrm{d}t} = 0$.
4. By considering whether $\frac{\mathrm{d}I}{\mathrm{d}t}$ is positive or negative to either side of the solutions from the previous part, determine whether these points are maxima, minima, or points of inflexion.
5. Find the maximum and minimum current through the resistor at any time between $t = 0$ and $t = 10$, and the times these currents occur.

Key Points to Remember:

1. Linearity:

$$\frac{\mathrm{d}ax + by}{\mathrm{d}t} = a\frac{\mathrm{d}x}{\mathrm{d}t} + b\frac{\mathrm{d}y}{\mathrm{d}t}.$$

for any constants a and b.

2. The product rule:

$$\frac{\mathrm{d}xy}{\mathrm{d}t} = x\frac{\mathrm{d}y}{\mathrm{d}t} + y\frac{\mathrm{d}x}{\mathrm{d}t}.$$

3. The chain rule:

$$\text{if } h(t) = f(g(t)), \text{ then } h'(t) = f'(g(t))g'(t).$$

4.

$$\frac{\mathrm{d}x^n}{\mathrm{d}x} = nx^{n-1}.$$

5.

$$\frac{\mathrm{d}\sin(x)}{\mathrm{d}x} = \cos(x).$$

6.

$$\frac{\mathrm{d}\cos(x)}{\mathrm{d}x} = -\sin(x).$$