Second-Order Systems and Damping

Objective: To understand the derivation of RLC and mass-spring systems, and the three damping regimes.

Warm-Up: Mass-Spring-Damper Systems:

Let m be a mass on a spring. We require 3 equations from physics; firstly, Newton's 2nd Law of motion (for constant mass):

$$F = ma,$$

where F is the *total* force on an object, and a is its acceleration; secondly, Hooke's Law for Springs:

$$F_{\text{spring}} = -kx,$$

where F_{spring} is the force exerted by the spring on the mass, k is a constant measuring the stiffness of the spring, and x is the displacement of the mass from its rest position; and thirdly,

$$F_{\text{resistance}} = -\mu v,$$

where $F_{\text{resistance}}$ is the resistive (friction/damping) force, μ a constant (the drag coefficient), and v is the velocity. Hooke's Law says that the more you stretch or squash a spring, the harder it pulls/pushes you back to the rest position. The resistance equation says that the faster you move, the more resistance you encounter; this is not necessarily true, but provides a reasonable model for air resistance and the like.

Suppose we apply an external (time-varying) force $f(t)$ to the mass.

1. Write down an expression for the total force F acting on the mass.
2. Substituting this expression into Newton's 2nd Law, rearrange to show that
$$a + \frac{\mu}{m}v + \frac{k}{m}x = f(t).$$
3. Hence conclude that the position $x(t)$ of the mass is the solution to the 2nd order ODE
$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \frac{\mu}{m}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{k}{m}x = f(t).$$

Theory: RLC Circuits:

Consider a series RLC circuit. Let V_{in} be the voltage applied (as a function of time); let V_L, V_C, and V_R be the voltages across the three components. Let Q be the charge on the capacitor and I the current through the resistor; so I is the derivative of Q with respect to time. We require several equations: Ohm's Law

$$V_R = RI$$

(compare with the resistance equation from the mass-spring system); Faraday's Law of Induction

$$V_L = L\frac{\mathrm{d}I}{\mathrm{d}t}$$

(compare with Newton's 2nd Law); and the capacitor equation

$$V_C = \frac{1}{C}Q$$

(compare with Hooke's Law).

1. Show that
$$I = C\frac{\mathrm{d}V_C}{\mathrm{d}t}.$$

2. Hence show that
$$V_L = LC\frac{\mathrm{d}^2V_C}{\mathrm{d}t^2} \quad \text{and} \quad V_R = RC\frac{\mathrm{d}V_C}{\mathrm{d}t}.$$

3. Hence show that
$$\frac{\mathrm{d}^2V_C}{\mathrm{d}t^2} + \frac{R}{L}\frac{\mathrm{d}V_C}{\mathrm{d}t} + \frac{1}{LC}V_C = \frac{1}{LC}V_{\text{in}}$$
and compare with the mass-spring-damper equation.

4. Use the Laplace transform to show that the transfer function for an RLC circuit is
$$\frac{1/LC}{s^2 + \frac{R}{L}s + \frac{1}{LC}}.$$
Note: we could write the transfer function in a nicer form as
$$\frac{1}{LCs^2 + RCs + 1}.$$
However, it is convenient to have the leading coefficient of the denominator be 1 for finding poles and partial fraction decompositions.

Damping Regimes:

Recall the quadratic formula: the roots of $s^2 + bs + c$ are

$$\frac{-b}{2} \pm \frac{1}{2}\sqrt{b^2 - 4c}.$$

Consider an RLC circuit with transfer function

$$\frac{1/LC}{s^2 + \frac{R}{L}s + \frac{1}{LC}}.$$

1. Show that the denominator of the transfer function factors as

$$\left(s + \frac{R}{2L} + \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}\right)\left(s + \frac{R}{2L} - \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}\right).$$

2. Suppose

$$\frac{R^2}{4L^2} - \frac{1}{LC} < 0.$$

Considering the partial fraction decomposition of the transfer function, show that the impulse response has the form

$$H(t)e^{-Rt/2L}\left(A\sin(\omega t) + B\cos(\omega t)\right),$$

where A and B are constants and

$$\omega = \sqrt{\frac{1}{LC} - \frac{R^2}{4L}} \in \mathbb{R}.$$

3. Suppose

$$\frac{R^2}{4L^2} - \frac{1}{LC} > 0.$$

Considering the partial fraction decomposition of the transfer function, show that the impulse response has the form

$$H(t)e^{-Rt/2L}\left(Ae^{\gamma t} + Be^{-\gamma t}\right),$$

where

$$\gamma = \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}} \in \mathbb{R}.$$

4. Suppose $\frac{R^2}{4L^2} - \frac{1}{LC} = 0$. Show that the transfer function has partial fraction decomposition:

$$\frac{1/LC}{\left(s + \frac{R}{2L}\right)^2}.$$

This has inverse transform

$$\frac{H(t)}{LC}te^{-Rt/2L}.$$