Fourier Series and Filtering

Objective: To understand how Fourier series can be applied to understand the effect of a filter on a signal.

Recall that the Fourier series of a function $f(x)$ on the interval $[a, a+L]$ is

$$f_{\text{Fourier}}(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{2\pi n x}{L}\right) + b_n \sin\left(\frac{2\pi n x}{L}\right) \right],$$

where

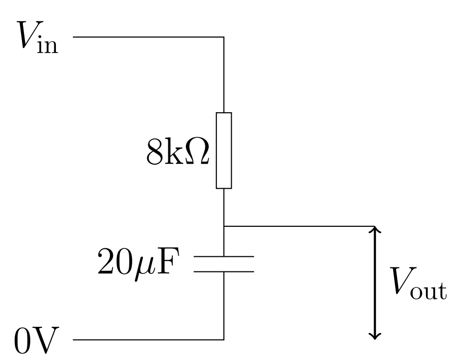
$$a_n = \frac{2}{L} \int_a^{a+L} f(x) \cos\left(\frac{2\pi n x}{L}\right) \, \mathrm{d}x$$
$$b_n = \frac{2}{L} \int_a^{a+L} f(x) \sin\left(\frac{2\pi n x}{L}\right) \, \mathrm{d}x.$$

Outside the interval $[a, a+L]$, f_{Fourier} approximates the periodic extension of f.

We will look at an application of Fourier series in electronics. Let $f(t)$ be a square wave function, defined by periodic extension of

$$f(t) = \begin{cases} 0: & 0 < t < 1 \\ 1: & 1 < t < 3. \end{cases}$$

An example would be the output of an astable. We will analyse the effect of passing this signal through the following low-pass filter:



1. For the square wave function $f(t)$ given above, show that the Fourier cosine coefficients are given by

$$a_n = \begin{cases} \frac{4}{3}: & n = 0 \\ \frac{-\sqrt{3}}{2\pi n}: & n = 3k+1 \\ \frac{\sqrt{3}}{2\pi n}: & n = 3k+2 \\ 0: & n = 3k,\ k \geq 1. \end{cases}$$

2. For the square wave function $f(t)$, show that the Fourier sine coefficients are given by

$$b_n = \begin{cases} 0: & n = 3k \\ \frac{-3}{2\pi n}: & n = 3k+1 \text{ or } n = 3k+2. \end{cases}$$

3. Hence write down the Fourier series of $f(t)$, up to the 12^{th} harmonic.

4. The impedance of a capacitor of capacitance C is $\frac{1}{j\omega C}$, where ω is the angular frequency of the voltage applied. In a voltage divider, as above, the output voltage for a sinusoidal input is found by dividing the input voltage in the ratio of the impedances:

$$V_{\text{out}} = V_{\text{in}} \frac{1/(j\omega C)}{R + 1/(j\omega C)}.$$

This gives a complex output voltage, whose modulus is the amplitude of the output and whose argument is the phase (relative to the input sinusoid). Using the estimate $\frac{1}{2\pi} \approx 0.16$, show that if a sinusoid $A \sin\left(\frac{2\pi n t}{3} + \phi\right)$ is input to the above voltage divider, then the output is

$$V_{\text{out}} = A \sin\left(\frac{2\pi n t}{3} + \phi\right) \frac{3}{3 + nj}.$$

The function "multiply by $\frac{3}{3+nj}$" is called the **transfer function** for this circuit.

5. Hence show that the output of the voltage divider for the above sinusoidal input has amplitude given by

$$\frac{3A}{\sqrt{9 + n^2}}$$

and phase

$$-\tan^{-1}\left(\frac{n}{3}\right),$$

relative to the input. Since we allowed any input phase ϕ, this result holds for both sine and cosine inputs.

6. Hence show that if we apply $f(t)$ as V_{in}, then the Fourier coefficients of V_{out} are given by

$$a_n = \begin{cases} \frac{4}{3} : & n = 0 \\ \frac{-3\sqrt{3}}{2\pi n\sqrt{9+n^2}} : & n = 3k+1 \\ \frac{3\sqrt{3}}{2\pi n\sqrt{9+n^2}} : & n = 3k+2 \\ 0 : & n = 3k,\ k \geq 1 \end{cases}$$

$$b_n = \begin{cases} 0 : & n = 3k \\ \frac{-9}{2\pi n\sqrt{9+n^2}} : & n = 3k+1 \text{ or } n = 3k+2, \end{cases}$$

with each sinusoid phase shifted by $-\tan^{-1}\left(\frac{n}{3}\right)$.

7. Hence write down a trigonometric series of V_{out}, up to the 12^{th} harmonic.

8. Suppose we switch the resistor and capacitor, to give a high-pass filter, instead of the low-pass filter we considered thus far. Show that now the output voltage for an input of $A\sin\left(\frac{2\pi nt}{3} + \phi\right)$ is given by

$$V_{\text{out}} = A\sin\left(\frac{2\pi nt}{3} + \phi\right)\frac{nj}{3+nj}.$$

In other words, show that the transfer function is multiplication by $\frac{nj}{3+nj}$.

9. Hence show that each sinusoidal input has output with amplitude multiplied by

$$\frac{n}{\sqrt{9+n^2}}$$

and phase shifted by

$$\frac{\pi}{2} - \tan^{-1}\left(\frac{n}{3}\right)$$

relative to the input.

10. Hence show that the Fourier coefficients of the output from the high-pass filter with input $f(t)$ are given by

$$a_n = \begin{cases} 0 : & n = 3k \\ \frac{-\sqrt{3}}{2\pi\sqrt{9+n^2}} : & n = 3k+1 \\ \frac{\sqrt{3}}{2\pi\sqrt{9+n^2}} : & n = 3k+2 \end{cases}$$

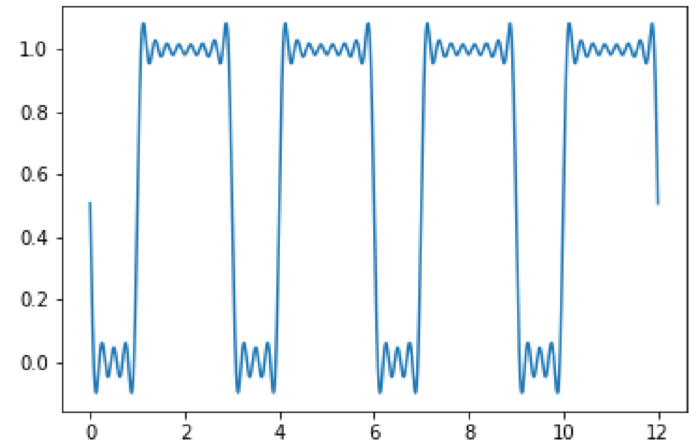
$$b_n = \begin{cases} 0 : & n = 3k \\ \frac{-3}{2\pi\sqrt{9+n^2}} : & n = 3k+1 \text{ or } n = 3k+2, \end{cases}$$

with each sinusoid phase shifted by $\frac{\pi}{2} - \tan^{-1}\left(\frac{n}{3}\right)$.

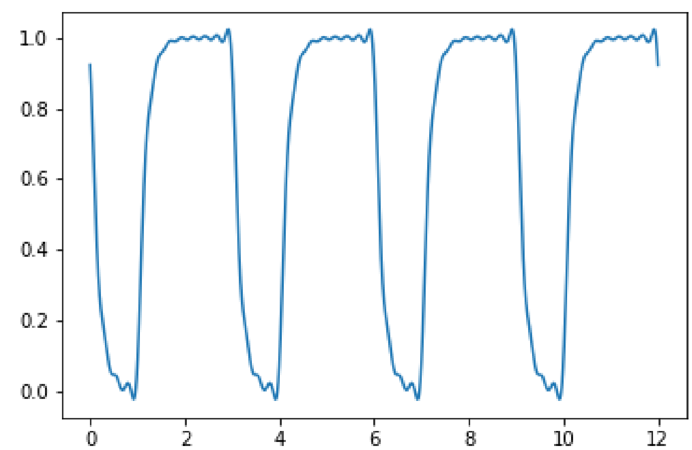
11. Hence write down a trigonometric series of V_{out} from a high-pass filter, up to the 12^{th} harmonic.

Graphs are shown overleaf.

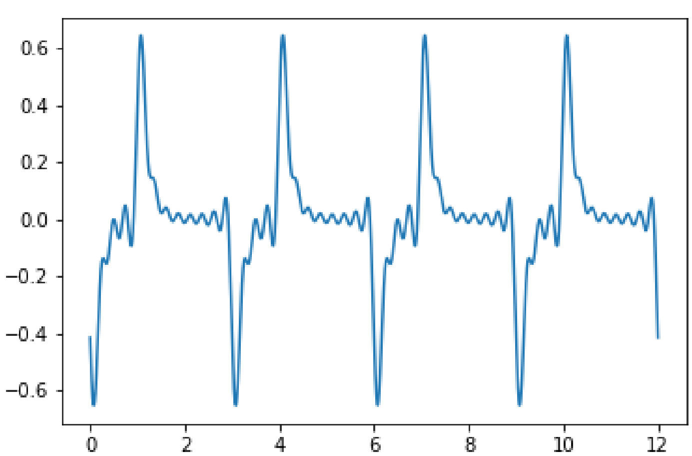
Taking twelve terms of the Fourier series:
Unfiltered:



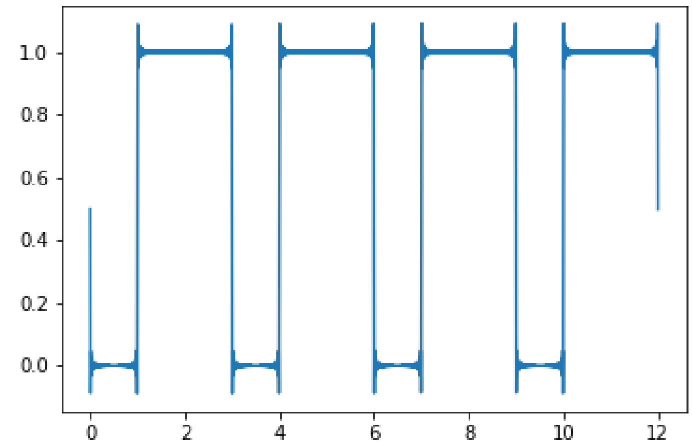
Low-pass:



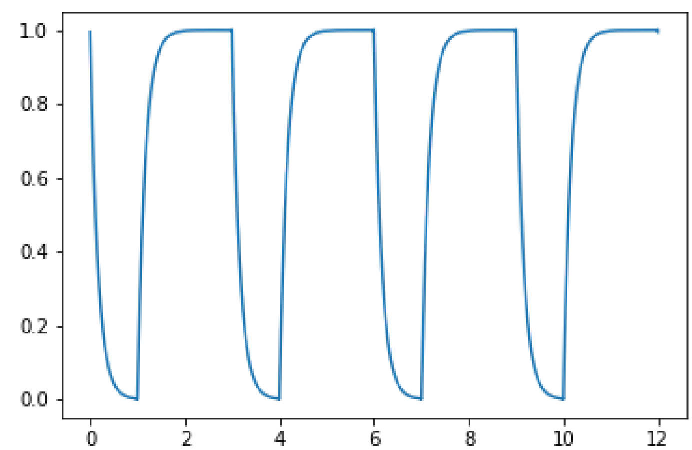
High-pass:



Taking two hundred terms of the Fourier series:
Unfiltered:



Low-pass:



High-pass: