Convergence in Probability Theory

Objective: To understand the different notions of convergence used in probability theory, and how they relate to one another.

Warm-up: Limits of Sequences and Functions:

1. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers. Define what it means for a_n to converge to a limit L as n tends to infinity.
2. Let (a_n) and (b_n) be sequences of real numbers, with $a_n \to A$ and $b_n \to B$ as $n \to \infty$. Prove that $a_n + b_n \to A + B$ and $a_n b_n \to AB$.
3. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function. Define what it means for $f(x)$ to tend to a limit L as x tends to some constant a.
4. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function. Prove that $f(x) \to L$ as $x \to a$ if and only if for every sequence (x_n) with $x_n \to a$, we have $f(x_n) \to L$.

Probabilistic Convergence:

Let (X_n) be a sequence of (real-valued) random variables, and X a random variable. Give the definition of each of the following types of convergence:

$$X_n \xrightarrow{d} X$$

$$X_n \xrightarrow{P} X$$

$$X_n \xrightarrow{L_n} X$$

Relationships between Different Types of Convergence:

1. First we prove that convergence in probability implies convergence in distribution:
$$\left(X_n \xrightarrow{P} X\right) \quad \Rightarrow \quad \left(X_n \xrightarrow{d} X\right).$$
   (a) First a Lemma. Prove that, for random variables Y and Z, and constants $a \in \mathbb{R}$ and $\epsilon > 0$:
$$P(Y \leq a) \leq P(Z \leq a + \epsilon) + P(|Y - Z| > \epsilon).$$
   Hint: show that $\{Y \leq a\} \subseteq \{Z \leq a + \epsilon\} \cup \{|Y - Z| > \epsilon\}$.

   (b) Let $a \in \mathbb{R}$ be such that F_X is continuous at a. We need to show that $F_{X_n}(a) \to F_X(a)$.

      i. Use the Lemma to show that
$$P(X \leq a-\epsilon) - P(|X_n - X| > \epsilon) \leq P(X_n \leq a) \leq P(X \leq a+\epsilon) + P(|X_n - X| > \epsilon).$$
      Hint: Use the Lemma twice—once with $Y = X$, once with $Y = X_n$.

      ii. Hence conclude that
$$F_X(a - \epsilon) \leq \lim_{n\to\infty} F_{X_n}(a) \leq F_X(a + \epsilon).$$
      iii. Hence conclude that $X_n \xrightarrow{d} X$.

2. Next we show that convergence in mean implies convergence in probability:
$$\left(X_n \xrightarrow{L_1} X\right) \quad \Rightarrow \quad \left(X_n \xrightarrow{P} X\right).$$
   (a) First we need **Markov's Inequality**: if Y is a non-negative random variable, then
$$P(Y \leq a) \geq \frac{E(Y)}{a}.$$
   Prove this by writing out the definition of $E(Y)$ and relating it to $P(Y \geq a)$.

   (b) Now suppose that $X_n \xrightarrow{L_1} X$ and fix $\epsilon > 0$. Show that
$$P(|X_n - X| > \epsilon) \to 0$$
   as $n \to \infty$; *i.e.*, that $X_n \xrightarrow{P} X$.

3. As a bonus, conclude that convergence in mean implies convergence in distribution:
$$\left(X_n \xrightarrow{L_1} X\right) \quad \Rightarrow \quad \left(X_n \xrightarrow{d} X\right).$$

The Weak Law of Large Numbers:

Let (X_n) be a sequence of i.i.d. random variables with finite mean μ, and let $\bar{X}_n$ be the sample mean:

$$\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i.$$

The **Weak Law of Large Numbers** (WLLN) states that

$$X_n \xrightarrow{P} \mu.$$

To prove this, we assume that the X_i have finite variance σ^2, though the result holds (with a harder proof) without this assumption. Assuming finite variance allows us to prove **Chebyshev's Inequality**: for a random variable X with mean μ and variance σ^2, and for any $\lambda > 0$, we have

$$P(|X-\mu| \geq \lambda\sigma) \leq \frac{1}{\lambda^2}.$$

Prove Chebyshev's Inequality by applying Markov's Inequality (see previous page) to the random variable $(X-\mu)^2$, with $a = (\lambda\sigma)^2$.

Apply Chebyshev's Inequality to $\bar{X}_n$ to prove the WLLN.