Vector Spaces

Vector Spaces:

A vector space is, roughly speaking, somewhere you can add things and multiply them by scalars. More precisely, a vector space is a set V with operations $+ : V \times V \to V$ and $\times : \mathbb{R} \times V \to V$ such that:

Examples of vector spaces include:

Subspaces:

Given a vector space V, a subspace is just a subset which is itself a vector space, with the same operations. To check that $U \subseteq V$ is a subspace of V, we need to check that:

For each of the examples of a vector space above, try to find an example of a subspace:

Span:

Let V be a vector space and S a subset of V. Define $\langle S\rangle$ to be the intersection of all subspaces of V which contain S:

$$\langle S\rangle = \bigcap\{U \subseteq V \mid U \text{ is a subspace of } V \text{ and } S \subseteq U\}.$$

Prove that $\langle S\rangle$ is a subspace of V and that for any subspace U, $S \subseteq U$ if and only if $\langle S\rangle \subseteq U$.

Let S be a finite set, $S = \{v_1, \ldots .v_n\}$. Prove that

$$\langle S\rangle = \left\{\sum_{i=1}^{n} \lambda_i v_i \mid \lambda_1, \ldots, \lambda_n \in \mathbb{R}\right\};$$

that is, prove that $\langle S\rangle$ consists of all linear combinations of $v_1, \ldots, v_n$. Hint: show that the set above is a subspace and contains S, then use the property of $\langle S\rangle$ we proved above.

For each of the examples of a subspace U given above, find a set which spans U.

Span and Matrices:

Consider $\mathbb{R}^3$ and the vectors $v_1 = (3, -4, 7)$ and $v_2 = (-1, 2, 6)$. What is the span of these vectors?

$$\begin{aligned}
\langle v_1, v_2 \rangle &= \{\alpha v_1 + \beta v_2 \mid \alpha, \beta \in \mathbb{R}\} \\
&= \{(3\alpha - \beta, -4\alpha + 2\beta, 7\alpha + 6\beta) \mid \alpha, \beta \in \mathbb{R}\} \\
&= \left\{ (\alpha, \beta) \begin{pmatrix} 3 & -4 & 7 \\ -1 & 2 & 6 \end{pmatrix} \mid \alpha, \beta \in \mathbb{R} \right\} \\
&= \left\{ v \begin{pmatrix} 3 & -4 & 7 \\ -1 & 2 & 6 \end{pmatrix} \mid v \in \mathbb{R}^2 \right\}.
\end{aligned}$$

In general, if $v_1, \ldots, v_n$ are vectors in $\mathbb{R}^m$, then we can form an $n \times m$ matrix A with rows $v_1, \ldots, v_n$; then

$$\langle v_1, \ldots, v_n \rangle = \{uA \mid u \in \mathbb{R}^n\}.$$

Looking at this the other way around, if A is any $n \times m$ matrix, then $\{uA \mid u \in \mathbb{R}^n\}$ is a subspace of $\mathbb{R}^m$ spanned by the rows of A. Can you see why?

Given an $n \times m$ matrix A, there is a function $f_A : \mathbb{R}^n \to \mathbb{R}^m$ defined by $f_A(u) = uA$. Then the subspace $\{uA \mid u \in \mathbb{R}^n\}$ of $\mathbb{R}^m$ is exactly the image of this function f_A. When you see linear maps, you will see that this is actually true for any vector space V! Any subspace of V is the image of a linear map from some other vector space, and the image of any linear map is a subspace. So the idea here is that a subspace is the image of another vector space within V; in the case of $\mathbb{R}^m$, a subspace is the image of $\mathbb{R}^n$ in $\mathbb{R}^m$ under the linear map defined by a matrix.

Let A be an $n \times m$ matrix and let $f_A : \mathbb{R}^n \to \mathbb{R}^m$ be the function $f_A(u) = uA$. Let e_i be the i^{th} standard basis vector of $\mathbb{R}^n$ (so e_i has a 1 in the i^{th} coordinate and 0 everywhere else). Show that $f_A(e_i)$ is the i^{th} row of A.

The upshot of all this is that the map f_A is surjective if and only if the rows of A span $\mathbb{R}^m$.

Linear Independence:

What does it mean to say that vectors $v_1, \ldots, v_n$ are linearly independent?

Prove that $v_1, \ldots, v_n$ are linearly independent if and only if no proper subset of $v_1, \ldots, v_n$ spans $\langle v_1, \ldots . v_n \rangle$. In other words, if we remove any v_i, the span becomes strictly smaller. Hint: prove that if they are linearly independent and we remove v_i, then the span of the remaining vectors does not include v_i; then prove that if they are linearly dependent, then there is some i such that v_i is the in the span of the remaining vectors.

Is $\{1 + x, 1 - x, 2 + 2x + x^2\}$ linearly independent in the vector space of polynomials?

Linear Independence and Matrices:

We have seen that the span of some vectors in $\mathbb{R}^m$ is the image of the matrix whose rows are those vectors. Now we show a similar result for linear independence. Let $v_1, \ldots, v_n$ be vectors in $\mathbb{R}^m$ and let A be the $n \times m$ matrix whose i^{th} row is v_i. Show that $v_1, \ldots, v_n$ are linearly independent if and only if the matrix equation $uA = 0$ (for $u \in \mathbb{R}^n$) has the unique solution $u = 0$.

Let $f_A : \mathbb{R}^n \to \mathbb{R}^m$ be the function $f_A(u) = uA$. Show that f_A is injective if and only if the rows of A are linearly independent. Hint: first show that $f_A(u) = f_A(v)$ if and only if $f_A(u - v) = 0$.

So we have seen that the rows of A span $\mathbb{R}^m$ if and only if f_A is surjective, and the rows are linearly independent if and only if f_A is injective.

Bases:

Define a basis of a vector space V.

Show that $v_1, \ldots, v_n$ in $\mathbb{R}^m$ are a basis if and only if the map $f_A : \mathbb{R}^n \to \mathbb{R}^m$ is bijective, where A is the $n \times m$ matrix whose i^{th} row is v_i and f_A is defined by $f_A(u) = uA$.