Exponential Form of Fourier Series

Objective: To be able to express a periodic function as a series of exponential functions.

Recap/Warm-up: Fourier Series:

Recall that the Fourier series of a function $f(t)$ on the interval $[a, a+L]$ is

$$f_{\text{Fourier}}(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{2\pi n t}{L}\right) + b_n \sin\left(\frac{2\pi n t}{L}\right) \right],$$

where

$$a_n = \frac{2}{L} \int_a^{a+L} f(t) \cos\left(\frac{2\pi n t}{L}\right) \, \mathrm{d}t$$
$$b_n = \frac{2}{L} \int_a^{a+L} f(t) \sin\left(\frac{2\pi n t}{L}\right) \, \mathrm{d}t.$$

Outside the interval $[a, a+L]$, f_{Fourier} approximates the periodic extension of f.

1. Let $f(t)$ be the periodic extension of the function $\cos(t)$ on the interval $[0, \pi]$. Show that the Fourier series of f is

$$f_{\text{Fourier}}(t) = \sum_{n=1}^{\infty} \frac{8n}{(4n^2 - 1)\pi} \sin(2nt).$$

2. Use Euler's Equation to show that

$$\sin(2nt) = \frac{e^{2njt} - e^{-2njt}}{2j}.$$

3. Hence show that the Fourier series can be rewritten as

$$f_{\text{Fourier}}(t) = \sum_{n=-\infty}^{\infty} \frac{4n}{(4n^2 - 1)\pi j} e^{2njt}.$$

Theory: Exponential Fourier Series:

We have seen how rearranging Euler's equation gives us the identities

$$\cos(t) = \frac{e^{jt} + e^{-jt}}{2}, \qquad \sin(t) = \frac{e^{jt} - e^{-jt}}{2j}.$$

Using these, we can rewrite a Fourier series as a sum of exponentials; however, to do this, we must allow negative values of n, so our sum goes from $-\infty$ to ∞. Given a Fourier series

$$F(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{2\pi nt}{L}\right) + b_n \sin\left(\frac{2\pi nt}{L}\right)\right],$$

we substitute our above identities, and set $b_0 = 0$:

$$\begin{aligned}
F(t) &= \frac{a_0}{2} - \frac{b_0 j}{2} + \sum_{n=1}^{\infty} \left[a_n \frac{e^{2\pi njt/L} + e^{-2\pi njt/L}}{2} + b_n \frac{e^{2\pi njt/L} - e^{-2\pi njt/L}}{2j} \right] \\
&= \frac{a_0}{2} - \frac{b_0 j}{2} + \sum_{n=1}^{\infty} \left[\frac{a_n}{2}\left(e^{2\pi njt/L} + e^{-2\pi njt/L}\right) - \frac{b_n j}{2}\left(e^{2\pi njt/L} - e^{-2\pi njt/L}\right)\right] \\
&= \frac{a_0 - b_0 j}{2} + \sum_{n=1}^{\infty} \frac{a_n - b_n j}{2} e^{2\pi njt/L} + \sum_{n=1}^{\infty} \frac{a_n + b_n j}{2} e^{-2\pi njt/L}
\end{aligned}$$

Since we have negative powers of e occuring, it makes sense to allow n to take negative values; how do our Fourier coefficients behave with negative values of n? We see that, simply substituting $-n$ into the definition:

$$\begin{aligned}
a_{-n} &= \frac{2}{L}\int_a^{a+L} f(t) \cos\left(\frac{-2\pi nt}{L}\right)\, \mathrm{d}t \\
&= \frac{2}{L}\int_a^{a+L} f(t) \cos\left(\frac{2\pi nt}{L}\right)\, \mathrm{d}t \\
&= a_n,
\end{aligned}$$

since cosine is an even function. For the sine coefficients, on the other hand:

$$\begin{aligned}
b_{-n} &= \frac{2}{L}\int_a^{a+L} f(t) \sin\left(\frac{-2\pi nt}{L}\right)\, \mathrm{d}t \\
&= -\frac{2}{L}\int_a^{a+L} f(t) \sin\left(\frac{2\pi nt}{L}\right)\, \mathrm{d}t \\
&= -b_n,
\end{aligned}$$

since sine is an odd function. So we can extend our definition of the Fourier coefficients to negative values of n, with the simple relations $a_{-n} = a_n$, and $b_{-n} = -b_n$.

Therefore, in the second infinite series of our rearranged Fourier series above, we can write

$$\sum_{n=1}^{\infty} \frac{a_n + b_n j}{2} e^{-2\pi njt/L} = \sum_{n=1}^{\infty} \frac{a_{-n} - b_{-n} j}{2} e^{-2\pi njt/L}$$
$$= \sum_{n=-\infty}^{1} \frac{a_n - b_n j}{2} e^{2\pi njt/L}.$$

This makes the summands equal to those of the first infinite series, but the limits now go from $-\infty$ to 1, instead of 1 to ∞. The constant term also has the same form, so we can treat it as

$$\frac{a_n - b_n j}{2} e^{2\pi njt/L}$$

when $n = 0$. Therefore every term we are adding has the same form, so we can combine them all into a single infinite sum:

$$F(t) = \sum_{n=-\infty}^{\infty} \frac{a_n - b_n j}{2} e^{2\pi njt/L}.$$

Now consider the coefficients in the above series. From the definitions, we have

$$\begin{aligned}
\frac{a_n - b_n j}{2} &= \frac{1}{2}\left[\frac{2}{L}\int_a^{a+L} f(t)\cos\left(\frac{2\pi nt}{L}\right)\,\mathrm{d}t - \frac{2j}{L}\int_a^{a+L} f(t)\sin\left(\frac{2\pi nt}{L}\right)\,\mathrm{d}t\right] \\
&= \frac{1}{L}\int_a^{a+L} f(t)\left[\cos\left(\frac{2\pi nt}{L}\right) - j\sin\left(\frac{2\pi nt}{L}\right)\right]\,\mathrm{d}t \\
&= \frac{1}{L}\int_a^{a+L} f(t)e^{-2\pi njt/L}\,\mathrm{d}t.
\end{aligned}$$

So we define

$$c_n = \frac{1}{L}\int_a^{a+L} f(t)e^{-2\pi njt/L}\,\mathrm{d}t,$$

and write the **exponential Fourier series** of f over $[a, a+L]$ as

$$f_{\exp}(t) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi njt/L}.$$

Theory: The Coefficient Function:

Consier the exponential Fourier coefficients, c_n. We can view this as a function which takes an integer n as input and returns a complex number c_n as output. Since this depends on the original function f, let us denote this coefficient function $\hat{f}$: $\hat{f}(n) = c_n$.

This process of "putting a hat" on a function f is therefore a type of **operator**. We have seen operators before, such as differentiation; just as a function takes a number and gives you a number, an operator takes a function and gives you a new function. In this case, we start with a function f and obtain a function $\hat{f}$. There is a crucial difference between this "hat operator" and the differentiation operator, however. Differentiation takes a function that has real inputs and real outputs, and it gives a function out that has real inputs and real outputs. The hat operator, on the other hand, takes a function with real inputs and outputs, and gives out a function with integer inputs and complex outputs!

The original function f often has time as an input, in applications. The integer inputs to $\hat{f}$ are the harmonics over L, so represent frequencies. We say that f is a function on the **time domain**, whereas $\hat{f}$ is a function on the **frequency domain**. In general, any real number is possible as the frequency of a sinusoid (or a complex exponential!), but $\hat{f}$ takes only integer multiples of the fundamental frequency on L. We say $\hat{f}$ is a function on the **discrete frequency domain**.

We shall see a generalisation of Fourier series later, the Fourier transform, which allows any real frequency as input, so is a function on the entire frequency domain, not just the integer multiples of a fixed fundamental frequency. The Fourier transform is a complex-valued function on the *continuous* frequency domain.

It is worth looking more closely at the definition of our hat operator. It takes a function $f(t)$ and returns the function $\hat{f}(n)$, where

$$\hat{f}(n) = \frac{1}{L}\int_a^{a+L} f(t)e^{-2\pi njt/L}\,\mathrm{d}t.$$

So we introduce a new variable n, then integrate with respect to the old variable to leave an expression using only the new variable. We multiply $f(t)$ by a function $e^{-2\pi njt/L}$ which involves *both t and n*, then integrate in the time domain to leave a function in the frequency domain. So the function $e^{-2\pi njt/L}$ acts as a bridge between the time and frequency domains.

To recover the original function, via its exponential Fourier series, we multiply by a similar "bridge" function, $e^{2\pi njt/L}$, and sum over all value of n—we sum over the frequency domain. This leaves us with $f_{\exp}(t)$, a function in the time domain, which (for sufficiently nice f), will be equal to the original function f.

Practice:

$$f_{\text{exp}}(t) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi n j t/L},$$

where

$$c_n = \frac{1}{L}\int_a^{a+L} f(t) e^{-2\pi n j t/L}\, \mathrm{d}t.$$

1. Let $f(t)$ be the square wave function obtained by periodic extension of the Heaviside function on $[-1, 1]$. Find the exponential Fourier series of f.
2. Let $T(t)$ be the triangle wave function obtained by periodic extension of $|t|$ on $[-1, 1]$. Find the exponential Fourier series of T.
3. Let $S(t)$ be the sawtooth function obtained by periodic extension of t on $[-1, 1]$. Find the exponential Fourier series of S.

Key Points to Remember:

1. The **exponential Fourier series** of a function $f(x)$ on the interval $[a, a+L]$ is the complex exponential series
$$f_{\exp}(t) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi njt/L},$$
where
$$c_n = \frac{1}{L}\int_a^{a+L} f(t)e^{-2\pi njt/L}\,\mathrm{d}t.$$

2. If we substitute for the complex exponentials with Euler's equation, and split the coefficients c_n into real and imaginary parts $c_n = a_n - b_n j$, we recover the trigonometric Fourier series.

3. There is an **operator** taking the real-valued **time-domain** function f to the complex-valued **discrete frequency-domain** function $\hat{f}$, defined by
$$\hat{f}(n) = c_n = \frac{1}{L}\int_a^{a+L} f(t)e^{-2\pi njt/L}\,\mathrm{d}t.$$