Polynomial Divison and the Polynomial Factor Theorem

Objective: To be able to divide one polynomial by another, with remainder, and understand the Polynomial Factor Theorem

Warm-up:

1. Divide 138 by 11 with remainder.
2. Hence write 138 in the form $11q + r$ with $0 \leq r < 11$.
3. Multiply out $(x + 1)(x + 2)$.
4. Hence write $x^2 + 3x + 8$ in the form $(x + 1)q(x) + r$, where $q(x)$ is a polynomial and r is a constant.
5. Substitute $x = 10$ in your expression for $x^2 + 3x + 8$ above and compare with questions 1 and 2.
6. Evaluate $x^2 + 3x + 8$ when $x = 7$.
7. Evaluate your expression from question 4 when $x = 7$.
8. Using the previous two questions, write 78 in the form $8q + r$. Check this!

Theory - Long Division with Polynomials

Divide $x^2 + 7x + 19$ by $x + 10$ with remainder.

Divide $x^3 - 2x^2 + 6x - 4$ by $x - 1$ with remainder.

Practice:

1. Divide $x^2 - 5$ by $x + 3$ with remainder.
2. Divide $4x^2 + 6x - 3$ by $x - 2$ with remainder.
3. Divide $x^4 + 3x^2 - x + 1$ by $x - 2$ with remainder.
4. Divide $x^2 + 6x + 6$ by $x + 3$ with remainder. Hence factorise $x^2 + 6x + 9$.

Application - the Polynomial Factor Theorem:

Suppose $f(x)$ is a polynomial and can be factored as $f(x) = (x - a)g(x)$ for some constant a and polynomial $g(x)$. Show that $f(a) = 0$ - we say a is a root of f.

Now we go the other way around. Suppose that $f(x)$ is a polynomial and $f(a) = 0$. Prove that $f(x) = (x - a)g(x)$ for some polynomial $g(x)$.

Practice with the Polynomial Factor Theorem:

1. Factorise $x^2 - 6x + 8$.
2. Hence write down the two roots of $x^2 - 6x + 8$.
3. Let $f(x) = x^3 - 7x^2 + 14x - 8$. Evaluate $f(1)$. Hence find a such that $f(x) = (x - a)g(x)$ for some g.
4. With the notation as above, find g.
5. Hence factor f completely into the form $f(x) = (x - a)(x - b)(x - c)$.
6. Hence write down all three roots of f.

Key Points to Remember:

1. A **polynomial** is a function of a variable (x, say), which is built just from addition, subtraction, multiplication, and raising x to positive whole-number powers. For instance, $8x^3 - 7x^2 + 3$ is a polynomial, but $\sin(x) + x^3$ is not, because of the sin term.

2. The **degree** of a polynomial is the largest power of the variable appearing. For instance, the degree of $-x^3 + 4x^2 - 2x$ is 3.

3. A **root** of a polynomial $f(x)$ is a number a such that $f(a) = 0$ - *i.e.*, substituting a in place of x gives 0.

4. A polynomial $f(x)$ is a **factor** of another polynomial $g(x)$ if $f(x)$ can be multiplied by a polynomial to make $g(x)$ - just like a whole number f is a factor of another whole number g if f can be multiplied by some other number to make g.

5. Any degree-n polynomial can be divided by a degree-1 polynomial to give a degree-$(n-1)$ polynomial and a constant remainder.

6. To divide one polynomial by another, use long division, starting with the largest power and working down.

7. The **Polynomial Factor Theorem** says that a number a is a **root** of a polynomial if and only if $(x - a)$ is a **factor** of that polynomial.