Partial Fractions

Objective: To be able to compute partial fraction decompositions.

Warm-up:

1. Simplify
$$\frac{6}{7} - \frac{1}{2}.$$

2. Hence write
$$\frac{1}{14}$$
in the form
$$\frac{A}{7} + \frac{B}{2}$$
where A and B are real numbers (not necessarily integers!).

3. Write
$$\frac{6}{n+3} - \frac{1}{n-2}$$
as a single fraction in simplest terms.

4. Hence write
$$\frac{n-3}{n^2+n-6}$$
in the form
$$\frac{A}{n+3} + \frac{B}{n-2},$$
where A and B are real numbers.

5. Evaluate your expressions from questions 3 & 4 at $n = 4$ and compare with your answers to questions 1 & 2.

Theory: Partial Fractions:

The idea of partial fractions is essentially the reverse of combining fractions over a common denominator. When given two (or more) fractions that are being added together or subtracted, it is often convenient to combine them as a single fraction. It can also be useful to do the reverse: to split a single fraction into a sum of two (or more) simpler fractions; this is particularly the case for fractions of algebraic expressions.

An expression of the form

$$\frac{f(x)}{g(x)},$$

where f and g are polynomials, is called a **rational function** in x. Given a rational function $\frac{f(x)}{g(x)}$, an equation of the form

$$\frac{f(x)}{g(x)} = p_0(x) + \frac{A_1}{p_1(x)} + \frac{A_2}{p_2(x)} + \ldots + \frac{A_n}{p_n(x)}$$

where $A_1, \ldots, A_n$ are constants and $p_0, \ldots, p_n$ are distinct polynomials, is called a **partial fraction decomposition** of $\frac{f(x)}{g(x)}$.

Find a partial fraction decomposition of $\frac{x}{x^2-1}$.

Write $\frac{2t^3+3t^2+7}{t^2+t-2}$ in partial fractions.

Practice:

Compute partial fraction decompositions for the following rational functions.

1. $\frac{17x-53}{x^2-2x-15}$.
2. $\frac{34-12y}{3y^2-10y-8}$.
3. $\frac{s^5-16s^3+2s^2+70s-33}{s^2-9}$.
4. $\frac{2t+1}{t^2+2t+1}$.

We saw on the last page that if we attempt to write

$$\frac{2t+1}{t^2+2t+1}$$

in partial fractions, we struggle. This is a problem that occurs when we have powers of a polynomial in the denominator; here it arises because $t^2+2t+1 = (t+1)^2$. However, we can write

$$\frac{2t+1}{t^2+2t+1} = \frac{2(t+1)-1}{(t+1)^2} = \frac{2}{t+1} - \frac{1}{(t+1)^2}.$$

So when finding a partial fraction decomposition of a rational function whose denominator factors with a power of a simpler polynomial, we have to consider all smaller powers of that polynomial. For instance, if the original denominator is $(x-3)(2x+1)^3$, then our partial fraction decomposition will involve fractions with denominators $(x-3)$, $(2x+1)$, $(2x+1)^2$, and $(2x+1)^3$.

Decompose

$$\frac{6x^4-2x^3+x^2-7}{x^3}$$

into partial fractions.

Decompose

$$\frac{s^3}{(3s-1)(s+2)^2}$$

into partial fractions.

Practice and Applications:

1. Find a partial fractions decomposition for

$$\frac{2t^4 - 9t^3 + t^2 - 2t + 1}{t^3 + 2t^2}.$$

   Hence find

$$\int \frac{2t^4 - 9t^3 + t^2 - 2t + 1}{t^3 + 2t^2}\, \mathrm{d}t.$$

2. Find a partial fractions decomposition for

$$\frac{s}{s^2 + \omega^2},$$

   where ω is a real constant. (Hint: you will need to use complex numbers to factorise the denominator). Hence find the inverse Laplace transform of

$$\frac{s}{s^2 + \omega^2}.$$