# Fourier Series

**Objective: To understand how an even periodic function can be approximated by a series of cosines, and any periodic function can be approximated by a series of sines and cosines.**

### Recap/Warm-up: Fourier Sine Series and Orthonormality of Cosines:

Recall that the Fourier sine series of a function $f(x)$ on the interval $[a, a+L]$ is

$$f_{\sin}(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi n x}{L}\right),$$

where

$$b_n = \frac{2}{L}\int_a^{a+L} f(x)\sin\left(\frac{2\pi n x}{L}\right)\,\mathrm{d}x.$$

Outside the interval $[a, a+L]$, $f_{\sin}$ approximates the periodic extension of $f$.

1. Let $E(x)$ be an even function defined on $\left[-\frac{L}{2}, \frac{L}{2}\right]$. Show that for all $n$, $b_n = 0$.
2. Show that the functions
$$\cos\left(\frac{2\pi n x}{L}\right)$$
for $n \geq 1$, and also the constant function $\frac{1}{2}$, are orthonormal with respect to the inner product on $[a, a+L]$ defined by
$$\langle f \mid g\rangle = \frac{2}{L}\int_a^{a+L} f(x)g(x)\,\mathrm{d}x.$$
3. Show that for any $n$ and $m$ (whether equal or not):
$$\left\langle \sin\left(\frac{2\pi n x}{L}\right) \,\middle|\, \cos\left(\frac{2\pi m x}{L}\right)\right\rangle = 0$$
with respect to the integral inner product on $[a, a+L]$. That is, that the cosine and sine functions of any harmonics of $L$ are orthogonal. Hint: use question 1!
4. Show also that the constant function $\frac{1}{2}$ is orthogonal to the sines of the harmonics of $L$.

**Theory: Fourier Cosine Series:**

We have seen in the warm-up that the Fourier sine series of an even function (or rather a function whose periodic extension is even) is always 0; essentially, since the sine functions of various harmonics are all odd, they cannot ever give a good approximation to an even function. We also saw that the cosine functions

$$\cos\left(\frac{2\pi n x}{L}\right)$$

and the constant function $\frac{1}{2}$ are orthonormal; so we can approximate a function by a linear combination of cosines:

$$f(x) \approx \left\langle f(x) \,\middle|\, \frac{1}{2} \right\rangle \frac{1}{2} + \sum_{n=1}^{N} \left\langle f(x) \,\middle|\, \cos\left(\frac{2\pi n x}{L}\right) \right\rangle \cos\left(\frac{2\pi n x}{L}\right).$$

As with the Fourier sine series, we take the limit as $N$ tends to infinity. So we write the **Fourier cosine series** of $f(x)$:

$$f_{\cos} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n x}{L}\right),$$

where

$$a_n = \frac{2}{L} \int_a^{a+L} f(x) \cos\left(\frac{2\pi n x}{L}\right) \, \mathrm{d}x.$$

Just as a Fourier sine series is zero for an even function, a Fourier cosine series is zero for an odd function:

Show that if $O(x)$ is an odd function on the interval $\left[-\frac{L}{2}, \frac{L}{2}\right]$, then $a_n = 0$ for all $n$.

**Theory: Fourier Series for Arbitrary Functions:**

We have seen how an even function can be approximated by a series of cosines, and an odd function by a series of sines. It is now clear how to extend to a general (square-integrable) function: we know that any function $f(x)$ can be decomposed as $f(x) = f_{\text{even}}(x) + f_{\text{odd}}(x)$, where

$$f_{\text{even}}(x) = \frac{f(x) + f(-x)}{2}$$
$$f_{\text{odd}}(x) = \frac{f(x) - f(-x)}{2}.$$

Then, when finding our Fourier coefficients,

$$a_n = \frac{2}{L} \int_a^{a+L} f(x) \cos\left(\frac{2\pi n x}{L}\right)\, \mathrm{d}x = \frac{2}{L} \int_a^{a+L} f_{\text{even}}(x) \cos\left(\frac{2\pi n x}{L}\right)\, \mathrm{d}x$$
$$b_n = \frac{2}{L} \int_a^{a+L} f(x) \sin\left(\frac{2\pi n x}{L}\right)\, \mathrm{d}x = \frac{2}{L} \int_a^{a+L} f_{\text{odd}}(x) \sin\left(\frac{2\pi n x}{L}\right)\, \mathrm{d}x.$$

So the Fourier sine series of $f$ is the same as the Fourier sine series of $f_{\text{odd}}$, and the Fourier cosine series of $f$ is the same as the Fourier cosine series of $f_{\text{even}}$. Then we simply add these two series together to obtain the **Fourier series** of $f$ on the interval $[a, a+L]$:

$$f_{\text{Fourier}}(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[ a_n \cos\left(\frac{2\pi n x}{L}\right) + b_n \sin\left(\frac{2\pi n x}{L}\right) \right].$$

As with the sine and cosine series, the general Fourier series is periodic of period dividing $L$; so outside the interval $[a, a+L]$, it approximates the periodic extension of $f$.

When computing the Fourier coefficients $a_n$ and $b_n$, we can either use $f$ in our inner product integral, or $f_{\text{even}}$ and $f_{\text{odd}}$ respectively, whichever is more convenient. For instance, if $f(x) = x^2 + x$, the decomposition of $f$ into even and odd parts is very straightforward, so we would use the even part $(x^2)$ to find the $a_n$ and the odd part $(x)$ to find the $b_n$. However, for some functions, it may be easier simply to leave the original function in than find its even and odd parts and integrate with those; it's really a matter of what makes the integrals easiest to evaluate.

**Practice:**

$$f_{\text{Fourier}}(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[ a_n \cos\left(\frac{2\pi n x}{L}\right) + b_n \sin\left(\frac{2\pi n x}{L}\right) \right],$$

where

$$a_n = \frac{2}{L}\int_a^{a+L} f(x)\cos\left(\frac{2\pi n x}{L}\right)\,\mathrm{d}x$$

$$b_n = \frac{2}{L}\int_a^{a+L} f(x)\sin\left(\frac{2\pi n x}{L}\right)\,\mathrm{d}x.$$

1. Let $f(x)$ be the "staircase" function defined by

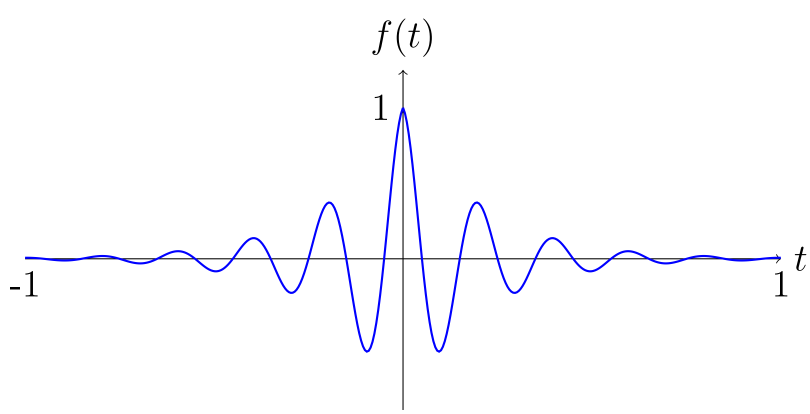
$$f(x) = \begin{cases} 0: & 0 \le x \le 1 \\ 1: & 1 \le x \le 2 \\ 2: & 2 \le x \le 3 \\ 3: & 3 \le x \le 4. \end{cases}$$

   Sketch the graph of the periodic extension of $f$ and find its Fourier series.

2. Let $f(t)$ be the decaying sinusoid function defined by

$$f(x) = e^{-|t|}\cos(10\pi t)$$

   on the interval $[-1, 1]$; the graph of $f$ is shown below. Find the Fourier series of $f$.



**Key Points to Remember:**

1. The **Fourier series** of a function $f(x)$ on the interval $[a, a+L]$ is the trigonometric series

$$f_{\text{Fourier}}(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[ a_n \cos\left(\frac{2\pi n x}{L}\right) + b_n \sin\left(\frac{2\pi n x}{L}\right) \right],$$

where

$$a_n = \frac{2}{L} \int_a^{a+L} f(x) \cos\left(\frac{2\pi n x}{L}\right) \, \mathrm{d}x$$
$$b_n = \frac{2}{L} \int_a^{a+L} f(x) \sin\left(\frac{2\pi n x}{L}\right) \, \mathrm{d}x.$$

2. The $N^{\text{th}}$ partial sum of the Fourier series is the best approximation to $f$ by a linear combination of the orthonormal functions

$$\frac{1}{2}, \cos\left(\frac{2\pi x}{L}\right), \sin\left(\frac{2\pi x}{L}\right), \ldots, \cos\left(\frac{2\pi N x}{L}\right), \sin\left(\frac{2\pi N x}{L}\right)$$

(with respect to the integral inner product on $[a, a+L]$).

3. The cosine coefficients $a_n$ depend only on the even part of the original function, while the sine coefficients $b_n$ depend only on the odd part.