Complex Numbers–Summary

1 Key Points - Fill in the Blanks:

Fill in the blanks in the key points below with a word, phrase, or mathematical expression. The unblanked versions are on the next page. Note that the size of the blank does not indicate the size of the missing word or phrase!

1. The **imaginary unit** j (or i) is defined to be a solution to the equation .... The other solution to this equation is .....
2. A **complex number** is any expression of the form $a + bj$ where a and b are real numbers. This is called the .... form of the complex number; a is called the .... and b is called the ....
3. If $z = a + bj$, then the complex number $\bar{z}$ is equal to .... and is called the .... of z.
4. We can represent the complex number $a + bj$ by the point (a, b) in the ....
5. If z is a complex number, the non-negative real number $|z|$ is called the .... of z and the angle $\arg(z)$ is called the .... of z. These are the .... of the point (a, b) in the Argand diagram.
6. If $z = a + bj$, then $|z| =$........ and $\tan(\arg(z)) =$........
7. The abbreviation $\operatorname{cis}(\theta)$ is shorthand for ....
8. We can write any complex number z in the form $r \operatorname{cis}(\theta)$, where $r =$...... and $\theta =$....... This is called the .... of z.
9. To multiply two complex numbers in polar form, we ....
10. **De Moivre's Theorem** states that if $z = r \operatorname{cis}(\theta)$, then ....
11. When using de Moivre's Theorem with $b \neq 1$ (*e.g.*, to find roots of a complex number) we must always remember ....

2 Key Points to Remember

The statements from the previous page, with the blanks filled in.

1. The **imaginary unit** j (or i) is defined to be a solution to the equation $x^2 = -1$. The other solution to this equation is $-j$.
2. A **complex number** is any expression of the form $a + bj$ where a and b are real numbers. This is called the *cartesian* form of the complex number; a is called the *real part* and b is called the *imaginary part*.
3. If $z = a + bj$, then the complex number $\bar{z}$ is equal to $a - bj$ and is called the *complex conjugate* of z.
4. We can represent the complex number $a+bj$ by the point (a, b) in the *Argand diagram* (or *complex plane*).
5. If z is a complex number, the non-negative real number $|z|$ is called the *modulus* of z and the angle $\arg(z)$ is called the *argument* of z. These are the *polar coordinates* of the point (a, b) in the Argand diagram.
6. If $z = a + bj$, then $|z| = \sqrt{a^2 + b^2}$ and $\tan(\arg(z)) = \frac{b}{a}$.
7. The abbreviation $\operatorname{cis}(\theta)$ is shorthand for $\cos(\theta) + j\sin(\theta)$.
8. We can write any complex number z in the form $r\operatorname{cis}(\theta)$, where $r = |z|$ and $\theta = \arg(z)$. This is called the *polar form* of z.
9. To multiply two complex numbers in polar form, we *multiply the moduli and add the arguments*.
10. **De Moivre's Theorem** states that if $z = r\operatorname{cis}(\theta)$, then $z^{a/b} = r^{a/b}\operatorname{cis}\left(\frac{a\theta}{b}\right)$.
11. When using de Moivre's Theorem with $b \neq 1$ (*e.g.*, to find roots of a complex number) we must always remember *to add multiples of 2π to the argument to get all the roots.*

3 Revision Questions

1. Let $z = 1 - j$ and $w = 4 + 3j$. Compute $z + w$, zw, $\bar{z}$, $\bar{w}$, $\frac{z}{w}$ and z^2.
2. Express $5 - 12j$ in polar form.
3. Express $14 \operatorname{cis}\left(\frac{-5\pi}{6}\right)$ in cartesian form.
4. Let $z = \frac{1}{2}(\sqrt{3} - j)$. Compute z^{100}.
5. Find all cube roots of $-2 + j$.

4 Solutions

It is possible I've made a mistake or two in these, so if your answer is different from mine and after checking you can't find a mistake in your work, ask me about it!

1.

$$\begin{aligned}
z+w &= 5+2j \\
zw &= 7-j \\
\bar{z} &= 1+j \\
\bar{w} &= 4-3j \\
\frac{z}{w} &= \frac{(1-j)(4-3j)}{4^2+3^2} = \frac{1}{25} - \frac{7}{25}j \\
z^2 &= -2j
\end{aligned}$$

2. $|5-12j| = \sqrt{5^2+12^2} = \sqrt{169} = 13$; $\tan(\arg(5-12j)) = \frac{-12}{5}$ and we're in the bottom right quadrant, so $\arg(5-12j) = \tan^{-1}\left(\frac{-12}{5}\right) \approx -1.176$. So $5-12j = 13\operatorname{cis}(-1.176)$.

3. The real part is $14\cos\left(\frac{-5\pi}{6}\right)$; $\frac{-5\pi}{6}$ is in the bottom left quadrant, at an angle of $\frac{\pi}{6}$ below the negative real axis, so $\cos\left(\frac{-5\pi}{6}\right) = -\cos\left(\frac{\pi}{6}\right) = -\frac{\sqrt{3}}{2}$. So $\mathrm{Re}(14\operatorname{cis}\left(\frac{-5\pi}{6}\right) = -7\sqrt{3}$. The imaginary part is $14\sin\left(\frac{-5\pi}{6}\right) = 14\sin\left(\frac{\pi}{6}\right) = 7$ (again by considering the quadrants). So $14\operatorname{cis}\left(\frac{-5\pi}{6}\right) = -7\sqrt{3}+7j$.

4. We use de Moivre's Theorem. First we put z into polar form; $|z| = \sqrt{\frac{3}{4}+\frac{1}{4}} = 1$, $\tan(\arg(z)) = \frac{-1}{\sqrt{3}}$, and z is in the bottom right quadrant, so $\arg(z) = \tan^{-1}\left(\frac{-1}{\sqrt{3}}\right) = -\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{-\pi}{6}$. So $z = \operatorname{cis}\left(\frac{-\pi}{6}\right)$. Then $z^{100} = \operatorname{cis}\left(\frac{-100\pi}{6}\right) = \operatorname{cis}\left(\frac{-50\pi}{3}\right)$. We have $2\pi = \frac{6\pi}{3}$, so $\frac{48\pi}{3} = 8\times 2\pi$, so $\operatorname{cis}\left(\frac{-50\pi}{3}\right) = \operatorname{cis}\left(\frac{-2\pi}{3}\right)$. We can leave this as our final answer in polar form, or revert to cartesian form: $\frac{-1}{2}+\frac{\sqrt{3}}{2}j$.

5. Again, we use de Moivre's Theorem. $|-2+j| = \sqrt{4+1} = \sqrt{5}$, and $\tan(\arg(-2+j)) = \frac{1}{-2}$; we are in the top right quadrant, so $\arg(-2+j) = \pi + \tan^{-1}\left(\frac{-1}{2}\right) \approx 2.678$. So we have

$$-2+j = \sqrt{5}\operatorname{cis}(2.678) = \sqrt{5}\operatorname{cis}(8.961) = \sqrt{5}\operatorname{cis}(15.244)$$

by adding 2π and 4π to the argument. Now, $\sqrt[3]{\sqrt{5}} = \sqrt[6]{5} \approx 1.308$ and dividing the three arguments by 3 gives 0.893, 2.987, and 5.081 respectively. So the cube roots are:

$$1.308\,\mathrm{cis}(0.893), \qquad 1.308\,\mathrm{cis}(2.987), \qquad 1.308\,\mathrm{cis}(5.081).$$

We could convert these back to cartesian form if we wished.