Even and Odd Functions

Objective: To understand the decomposition of a function into even and odd parts.

Theory: Even and Odd Functions:

Recall that a function $f(x)$ is **even** if $f(-x) = f(x)$ for any x, and is **odd** if $f(x) = -f(-x)$ for any x. The standard examples are that x^n is odd for n odd and even for n even (hence the use of the same adjectives!), and the normal and hyperbolic sine and tangent functions are odd, while the normal and hyperbolic cosine functions are even.

Graphically, f is even if and only if the graph of $y = f(x)$ is symmetrical under reflection in the y-axis; this is because reflection in the y-axis corresponds to switching x and $-x$, but $f(x) = f(-x)$ for an even function, so the function is unchanged by the reflection.

For odd functions, on the other hand, if you reflect the part of the graph for positive x in the y-axis and also the x-axis, you get the part of the graph for negative x. These two reflections have the same effect as a rotation by 180° about the origin, so a function is odd if and only if it has rotational symmetry of 180° about the origin.

Most functions are neither even nor odd. For instance, if $f(x) = x^2 + x$, then $f(-1) = 0$, whereas $f(1) = 2$, so f is neither even nor odd. However, we see that $f(x)$ can be written as an even function, x^2, plus an odd function, x. In fact, this is always true! For any function f, define the **even part** and **odd part** of f by:

$$f_{\text{even}}(x) = \frac{f(x) + f(-x)}{2}$$
$$f_{\text{odd}}(x) = \frac{f(x) - f(-x)}{2}.$$

Compare with the definitions of $\cosh(x)$ and $\sinh(x)$; these are precisely the even and odd parts respectively of the exponential function.

Practice:

1. Show that for any function f:
   (a) f_{even} is indeed even.
   (b) f_{odd} is indeed odd.
   (c) $f(x) = f_{\text{even}}(x) + f_{\text{odd}}(x)$.
2. Let $f(x) = 7x^4 - 9x^3 + 3x^2 + 12x + 8$. Find the even and odd parts of f.
3. Find the even and odd parts of $\cos\left(2\pi t + \frac{\pi}{3}\right)$.
4. Let f be an integrable function, and a a positive constant. Show that:
   (a) If f is even,
   $$\int_{-a}^{a} f(x)\,\mathrm{d}x = 2\int_{0}^{a} f(x)\,\mathrm{d}x.$$
   (b) If f is odd,
   $$\int_{-a}^{a} f(x)\,\mathrm{d}x = 0.$$
   (c) Hence, whatever f is,
   $$\int_{-a}^{a} f(x)\,\mathrm{d}x = 2\int_{0}^{a} f_{\text{even}}(x)\,\mathrm{d}x.$$
5. Let E_1 & E_2 be even functions and O_1 & O_2 odd functions, with $E_2(x)$ and $O_2(x)$ not constantly zero. Show that:
   (a) $O_1(x) = 0$.
   (b) $E_1(x)E_2(x)$ and $O_1(x)O_2(x)$ are even.
   (c) $E_1(x)O_1(x)$ is odd.
   (d) $E_1(x) + E_2(x)$ is even.
   (e) $O_1(x) + O_2(x)$ is odd.
   (f) $E_2(x) + O_2(x)$ is neither even nor odd.
6. Let $f(x)$ be a function which is both even and odd. Show that $f(x) = 0$ for all x.

Key Points to Remember:

1. An **even function** f is one such that $f(x) = f(-x)$.
2. An **odd function** f is one such that $f(x) = -f(-x)$.
3. Any function f can be decomposed into an **even part** and an **odd part**: $f(x) = f_{\text{even}}(x) + f_{\text{odd}}(x)$, where
$$\begin{aligned} f_{\text{even}}(x) &= \frac{f(x) + f(-x)}{2} \\ f_{\text{odd}}(x) &= \frac{f(x) - f(-x)}{2}. \end{aligned}$$
4. Functions are usually neither even nor odd. The only function which is both even and odd is the zero function.
5. For any integrable even function $E(x)$ and integrable odd function $O(x)$, and any positive constant a:
$$\begin{aligned} &\int_{-a}^{a} E(x)\,\mathrm{d}x = 2\int_{0}^{a} E(x)\,\mathrm{d}x \\ &\int_{-a}^{a} O(x)\,\mathrm{d}x = 0. \end{aligned}$$