Applications of Integration: Arc Length

Objective: To understand how integration may be used to find the arc length along a curve.

Warm-up: Arc Length:

Consider the curve $y = \sqrt{x^2 - 1}$ for $-1 \leq x \leq 1$; this is a semicircle of unit radius—it is the top half of the unit circle, the bottom half being given by the equation $y = -\sqrt{x^2 - 1}$. We will find the length of this curve; of course, the circumference of a circle is $2\pi r$, where r is the radius, so the unit semicircle must have arc length π; we shall find this by an alternative route. The key idea is that distance travelled is the integral of velocity; so if we can "take a walk" along the semicircle with a known velocity, then integrating this velocity will give us the length of the curve.

1. Show that a walk from time $t = 0$ to $t = \pi$, such that our x- and y-positions at time t are given by $x = \cos(t)$, $y = \sin(t)$, traverses the semicircle.
2. Over a short time interval from t to $t + h$, our speed can be approximated as constant, so the x-distance we cover is roughly $x'(t)h$, while the y-distance we cover is roughly $y'(t)h$. Use this to find an estimate of the total distance travelled between t and $t+h$, and divide by h to show that our speed over the interval t to $t + h$ is approximately
$$\sqrt{(x')^2 + (y')^2}.$$
Since the approximation used to derive this becomes more accurate as $h \to 0$, we see that our instantaneous speed at time t is given by this expression. So if $s(t)$ is the arc length travelled along the semicircle by time t, then
$$\frac{\mathrm{d}s}{\mathrm{d}t} = \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}.$$
3. Integrate the above expression from $t = 0$ to $t = \pi$, using the x and y functions defined in part 1, to show that the total arc length of the semicircle is π, as expected.

Theory: Arc Length:

Suppose we have a curve C and wish to find the **arc length** s of this curve—s is traditionally used, and is short for the Latin *spatium*, meaning "distance." Suppose we can find functions $x(t)$ and $y(t)$, such that as t varies from a to b, the point $(x(t), y(t))$ moves along the curve C (in either direction). Then the velocity of this point as it moves is the rate of change of arc length travelled, so is $s'(t)$. Therefore, by the Fundamental Theorem of Calculus, the total arc length of C is given by the integral

$$s = \int_a^b \frac{\mathrm{d}s}{\mathrm{d}t}\,\mathrm{d}t.$$

So to find the arc length, we want to find the velocity $\frac{\mathrm{d}s}{\mathrm{d}t}$ of the point with coordinates $(x(t), y(t))$. For any time t, we can take Taylor expansions:

$$\begin{aligned} x(t+h) &= x(t) + x'(t)h + \dots, \\ y(t+h) &= y(t) + y'(t)h + \dots, \end{aligned}$$

and therefore the change in x-value and in y-value from time t to time $t+h$ is given by

$$\begin{aligned} x(t+h) - x(t) &= x'(t)h + \dots, \\ y(t+h) - y(t) &= y'(t)h + \dots \end{aligned}$$

By Pythagoras' Theorem, the distance between the position of the point at time t and its position at time $t+h$ is therefore given by

$$\sqrt{[x(t+h) - x(t)]^2 + [y(t+h) - y(t)]^2} = \sqrt{(x'(t)h)^2 + (y'(t)h)^2 + \dots},$$

where the terms indicated by the ellipsis are of degree at least 4 in h.

For very small values of h, the actual path travelled along C will be very close to the straight line distance between them given by the above expression (at least if C is smooth enough—if C has a discontinuity or a sharp corner, this will not be true). Therefore we have for the change in arc length travelled:

$$s(t+h) - s(t) \approx \sqrt{(x'(t)h)^2 + (y'(t)h)^2 + \dots},$$

where the approximation grows more accurate as h tends to 0. Dividing through by h, we have

$$\frac{s(t+h) - s(t)}{h} \approx \sqrt{(x'(t))^2 + (y'(t))^2 + \dots},$$

where the terms indicated by the ellipsis have degree at least 2 in h. Therefore, as h tends to 0:

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \lim_{h\to 0} \frac{s(t+h)-s(t)}{h} = \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}.$$

Integrating:

$$s = \int_a^b \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t.$$

This method allows us to find the arc length of any (smooth) curve so long as we can find functions $x(t)$ and $y(t)$ such that for t from a to b, the point $(x(t), y(t))$ traces out the curve. Such a pair of functions is called a **parametrisation** of the curve, and the variable t is called the **parameter**. It is often helpful to think of a parameter as time (even if it is given a different letter) and a parametrisation as instructions for how to "take a walk" along the curve.

What if we aren't given a parametrisation and can't easily find one, but are instead given the curve in the form $y = f(x)$, say? Suppose there exists a parametrisation in which $x'(t)$ is never negative (even if we don't know what such a parametrisation would be!). We can use the chain rule, which tells us that

$$\frac{\mathrm{d}y}{\mathrm{d}t} \div \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}t} \times \frac{\mathrm{d}t}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}x},$$

in our arc length formula:

$$\begin{aligned} s = \int_a^b \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t &= \int_a^b \left[\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\right] \frac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t \\ &= \int_{x(a)}^{x(b)} \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x, \end{aligned}$$

where the last line comes from the subtitution rule for integration. This gives us a formula for the arc length which does not depend on having a parametrisation. Alternatively, if we are given x as a function of y (instead of y as a function of x), we can show by a similar argument that

$$s = \int_{y(a)}^{y(b)} \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2 + 1}\,\mathrm{d}y.$$

Practice:

Recall the arc length formulae:

$$\begin{aligned} s &= \int_a^b \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t \\ &= \int_{x(a)}^{x(b)} \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x \\ &= \int_{y(a)}^{y(b)} \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2 + 1}\,\mathrm{d}y. \end{aligned}$$

1. Find the length of the parabola $y = x^2$ from $x = 0$ to $x = 5$. Hint: to evaluate the integral, use the substitution $x = \frac{1}{2}\sinh(u)$, then use Osborn's Rule on the double angle formula for cosine to rewrite the integral after the substitution in a more amenable form.

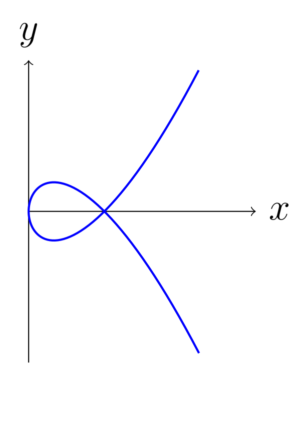
2. The curve C is given by $y = \cos^{-1}\left(e^{-x}\right)$ from $x = 0$ to $x = \ln\left(\sqrt{2}\right)$. Show that the arc length of C is given by

   $$s = \int_0^{\pi/4} \sec(y)\,\mathrm{d}y.$$

   Hint: rewrite for x as a function of y. Now substitute $u = \sec(y) + \tan(y)$ to solve this integral and find the arc length of C.

3. The "alpha curve" plotted below has parametric equations $x(t) = t^2$, $y(t) = t^3 - t$ for t from -1.5 to 1.5. Find the arc length of this curve. Note that this curve *cannot* be expressed by giving y as a function of x or x as a function of y! Some curves can only be given parametrically!

   Hint for evaluating the integral: complete the square and make a linear substitution to turn the square root into $\sqrt{u^2+1}$; then compare with question 1.



Key Points to Remember:

1. A **parametrisation** of a curve C is a pair of functions $x(t)$ and $y(t)$ and two times, a and b, such that the point $(x(t), y(t))$ traces out the curve C as t varies from a to b—*i.e.*, it is instructions for a walk along C. The variable t is called the **parameter**.

2. The arc length s of a parametrised curve is given by

$$\begin{aligned} s &= \int_a^b \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\ \mathrm{d}t \\ &= \int_{x(a)}^{x(b)} \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\ \mathrm{d}x \\ &= \int_{y(a)}^{y(b)} \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2 + 1}\ \mathrm{d}y. \end{aligned}$$