Solving ODEs: Separation of Variables

Objective: To be able to identify and solve separable, first-order ODEs.

Warm-up: Radioactive Decay:

Consider a sample of a radioactive element, and let N be the number of atoms in the sample that have not yet decayed. Each atom decays at some point *at random*, each atom having the same probability p of decaying within a given second (or other short time interval). The number of atoms you expect to decay in a given second is equal to the probability of any given atom decaying, p, times the number of atoms, N.

Therefore in a small period of time, you expect N to decrease by pN. In other words, the rate at which N changes is equal to minus pN, so N obeys the **first-order ordinary differential equation**

$$\frac{\mathrm{d}N}{\mathrm{d}t} = -pN.$$

We shall solve this equation, and study its solution somewhat.

1. Use the substitution formula for integration (in reverse!) to show that
$$\int \frac{1}{N}\frac{\mathrm{d}N}{\mathrm{d}t}\,\mathrm{d}t = \int \frac{1}{N}\,\mathrm{d}N$$
and hence evaluate this integral.

2. Divide both sides of the ODE by N and integrate with respect to t to show that
$$\ln(N) = -pt + c,$$
where c is an unknown constant.

3. Let N_0 be the number of undecayed atoms at time $t = 0$. Hence find c.

4. Hence show that
$$N = N_0 e^{-pt}$$

5. Find an expression in terms of p for the time after which eactly half of the original number of atoms has decayed.

Theory: Separation of Variables:

We shall generalise the idea we employed in the warm-up to develop a general technique called **separation of variables** for solving certain types of first-order ODE.

A first-order ODE is called **separable** if it can be written in the form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x)g(t)$$

for some functions f and g. Then we can divide through by f and integrate with respect to t to get

$$\int \frac{1}{f(x)} \frac{\mathrm{d}x}{\mathrm{d}t} \, \mathrm{d}t = \int g(t) \, \mathrm{d}t. \tag{$\star$}$$

Now we use the substitution formula for integrals, but in reverse from how we usually use it. Imagine we wanted to find

$$\int \frac{1}{f(x)} \, \mathrm{d}x;$$

we could introduce a new variable t, vary x according to some function of t, and have by substitution:

$$\int \frac{1}{f(x)} \, \mathrm{d}x = \int \frac{1}{f(x)} \frac{\mathrm{d}x}{\mathrm{d}t} \, \mathrm{d}t.$$

But the right-hand side of this is precisely the left-hand side of equation ($\star$)! Therefore we can put this into equation ($\star$) to get

$$\int \frac{1}{f(x)} \, \mathrm{d}x = \int g(t) \, \mathrm{d}t.$$

As long as we can integrate $\frac{1}{f(x)}$ and g, then, we can find an equation linking x and t, which we can then hope to rearrange to get x as a function of t and thereby have a solution to our original ODE.

Solve

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x^2 e^{-t}$$

with the **initial condition** $x(0) = 1$.

Find the general solution to

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x \tan(y).$$

Practice:

1. Find the particular solution to
$$\frac{\mathrm{d}y}{\mathrm{d}x} = y\sin(x)$$
satisfying the boundary condition that $y = 2$ when $x = \frac{\pi}{2}$.

2. Find the general solution to
$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{te^{t^2}}{y}.$$

3. A capacitor discharges from voltage V_0 at time 0 through a resistor.

   (a) Using the equations $Q = CV_{\mathrm{C}}$ (charge is capacitance times capacitor voltage) and $V_{\mathrm{R}} = IR$ (Ohm's Law: resistor voltage is current times resistance), set up and solve a differential equation describing the voltage V_{C} on the capacitor.

   (b) Find (in terms of R and C) the time after which the voltage has decayed to $\frac{V_0}{e}$. This is called the **time constant** of the RC circuit. Note that it does not depend on the value of V_0!

   (c) Assume that $C = 200\mu\mathrm{F}$, $R = 1\mathrm{k}\Omega$, and $V_0 = 10\mathrm{V}$. After how long will the voltage across the capacitor be 1V?

Application: Population Growth:

1. A simple model for the growth of a population (of humans, animals, plants, bacteria, whatever...) says that the population grows at a rate proportional to its current size; so if P is the number of organisms in the population, then
$$\frac{\mathrm{d}P}{\mathrm{d}t} = \lambda P,$$
where λ is the growth rate, which measures how many more births than deaths there are. Solve this differential equation subject to the condition that the initial population is P_0.

2. In your solution, what happens as $t \to \infty$ (assuming λ is positive)? Does this seem realistic?

3. A more sophisticated model of population growth assumes that the habitat has a **carrying capacity** $P_{\max}$: a maximum population that it can support. If P is small, we should have roughly the growth we had before, but when P gets close to $P_{\max}$, the growth rate should drop towards 0. The following ODE has these properties:
$$\frac{\mathrm{d}P}{\mathrm{d}t} = \lambda P \left(1 - \frac{P}{P_{\max}}\right).$$
Consider the value of $\left(1 - \frac{P}{P_{\max}}\right)$ for different values of P—much smaller than, slightly smaller than, equal to, and bigger than the carrying capacity—to convince yourself that this ODE has the properties we want.

4. By separating variables in this ODE, show that
$$\int \frac{P_{\max}}{P(P_{\max} - P)}\,\mathrm{d}P = \int \lambda\,\mathrm{d}t.$$

5. Show that
$$\frac{1}{P} + \frac{1}{P_{\max} - P} = \frac{P_{\max}}{P(P_{\max} - P)}.$$

6. Hence solve the ODE (with initial population P_0) to show that
$$\frac{P}{P_{\max} - P} = \frac{P_0}{P_{\max} - P_0} e^{\lambda t}.$$

7. Multiply both sides of this equation by $(P_{\max} - P)$ and rearrange to show that
$$P = \frac{P_{\max} P_0}{P_0 + (P_{\max} - P_0)e^{-\lambda t}}.$$

8. What happens to the population as $t \to \infty$?

Key Points to Remember:

1. A first-order ODE is called **separable** if it can be written in the form
$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x)g(t)$$
for some functions f and g.

2. Given a separable ODE as above, we can solve by dividing through by $f(x)$ and integrating with respect to t to obtain
$$\int \frac{1}{f(x)}\,\mathrm{d}x = \int \frac{1}{f(x)}\frac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t = \int g(t)\,\mathrm{d}t + c.$$
This process is called **separation of variables**.

3. Initial conditions can be used to find the value of the constant of integration.

4. A very common type of separable equation has the simple form
$$\frac{\mathrm{d}x}{\mathrm{d}t} = \lambda x$$
for some constant λ. The solution to this is
$$x = x_0 e^{\lambda t},$$
where x_0 is the value of x at $t = 0$. If $\lambda > 0$, the solution grows exponentially (*e.g.*, population growth when the population is far smaller than the habitat can support), if $\lambda < 0$, the solution decays exponentially towards 0 (*e.g.*, voltage on a discharging capacitor, radioactive decay, metabolism of a drug in the body, attenuation of radiation as it passes through a material).

5. In exponential growth/decay, $e^{\lambda t}$, the constant λ is called the growth rate (if positive) or the decay rate (if negative). If $\lambda < 0$, then $\frac{-1}{\lambda}$ is called the **time constant**; it is the time taken for the function to decay to $\frac{1}{e}$ of its prior value.