Taylor Polynomials

Objective: To understand how to approximate a differentiable function by polynomials about a point.

Recap: The Newton-Raphson Method:

Solve the equation $e^{-x} = \sin(x)$ to 3 significant figures, using a starting value of $x_1 = 1$.

Warm-Up: Determining Polynomials by Derivatives:

1. Find the unique straight line passing through $(0, 2)$ with derivative -7, by the following steps:
   (a) Let the equation of the line be $y = mx + c$. Then we can find $\frac{\mathrm{d}y}{\mathrm{d}x}$ and set it equal to -7 to find m.
   (b) Substitute $x = 0$, $y = 2$ into the equation, since the line passes through $(0, 2)$, to find c.
2. Find the unique parabola ($y = ax^2 + bx + c$) passing through $(0, 2)$ with derivative -7 and second derivative 4 at $x = 0$. Hint: adapt the strategy from question 1.
3. Find the unique cubic curve passing through $(0, 2)$ with derivative -7, second derivative 4, and third derivative 1 at $x = 0$.

Theory: Taylor Polynomials:

Polynomials are well-behaved and well-understood functions. As such, it can be useful to approximate more complicated functions by polynomials. Suppose we want to study the function $f(x) = e^x$ near $x = 0$.

Find the tangent to $y = f(x)$ at $x = 0$ (the unique line passing through the same point, with the same gradient).

Find the unique quadratic passing through the same point with the same first and second derivatives as $f(x)$ when $x = 0$.

Generalise this; find the unique degree-n polynomial passing through the same point with the same first n derivatives as $f(x)$ when $x = 0$.

Graphs are shown overleaf.

Theory: Taylor Polynomials (cont.):

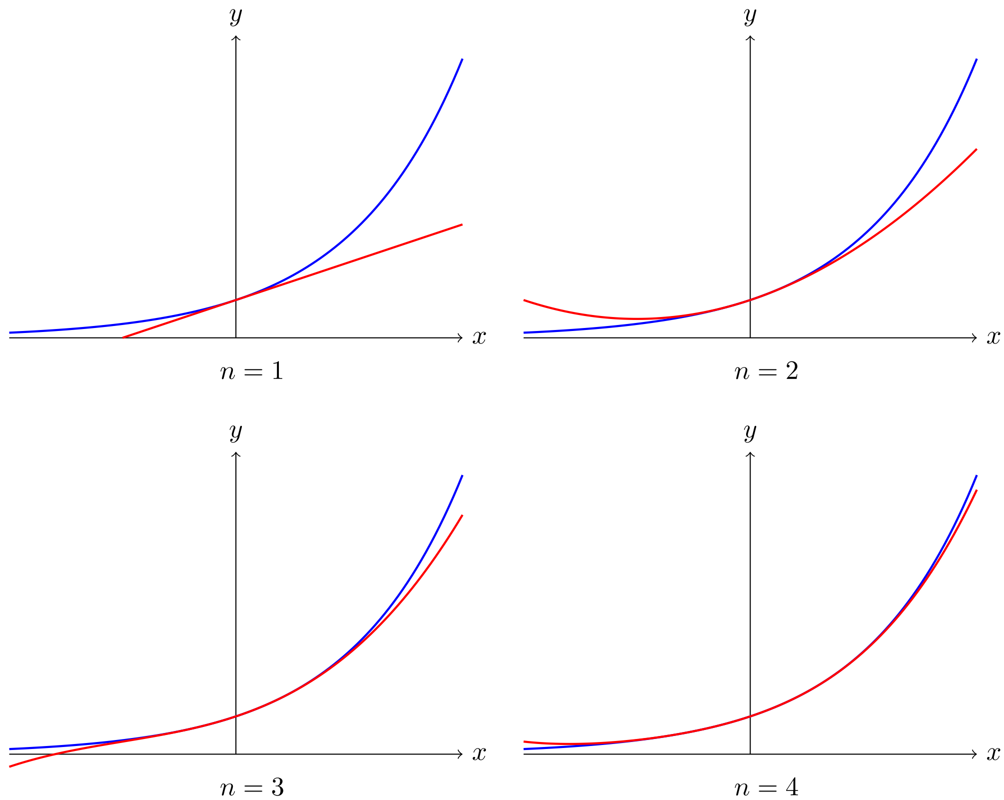


Figure 1: The graph of $y = e^x$ and its degree n Taylor polynomials, for $n = 1, 2, 3, 4$.

Let $f(x)$ be an n-times differentiable function. We define the **Taylor polynomial of degree** n of $f(x)$ around $x = a$ to be the unique polynomial of degree at most n passing through the same point as $y = f(x)$ when $x = a$, and having the same first through to n^{th} derivatives at that point.

Find the n^{th} Taylor polynomial of $f(x)$ about $x = a$.

Rewrite this by changing variables to $h = x - a$.

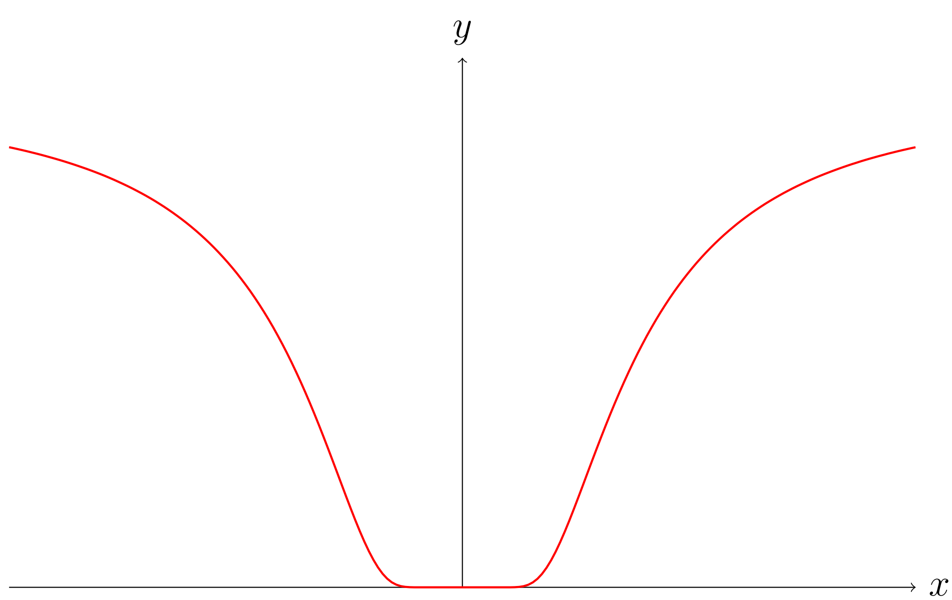
Practice:

1. Compute the degree-8 Taylor polynomial of $\cos(x)$ about $x = 0$. Graphs are shown overleaf.
2. Compute the degree-8 Taylor polynomial of $\sin(x)$ about $x = 0$.
3. Compute the degree-3 Taylor polynomial of $\log_e(x)$ about $x = 1$. Express this in the form $\log_e(1 + x) = \dots$
4. Compute the degree-2 Taylor polynomial of $x^2 e^{-x}$.
5. **Difficult! Cautionary example:** Let $f(x)$ be the function defined by

$$f(x) = \begin{cases} e^{-1/x^2} & : x \neq 0 \\ 0 & : x = 0 \end{cases}$$

Find the degree 1 Taylor polynomial of $f(x)$ about $x = 0$ (you may assume that the derivative exists and is continuous at 0, so you can find the derivative for x *near* 0 and take the limit as $x \to 0$).

It can be shown that $f(x)$ is infinitely differentiable (can be differentiated as many times as you like), and all of its derivatives are 0 at $x = 0$. Therefore every Taylor polynomial of $f(x)$ about $x = 0$ is constantly 0. However, the function is non-zero everywhere except 0; so no matter how high a degree you take, the Taylor polynomial is never a good approximation away from $x = 0$. The graph is shown below.



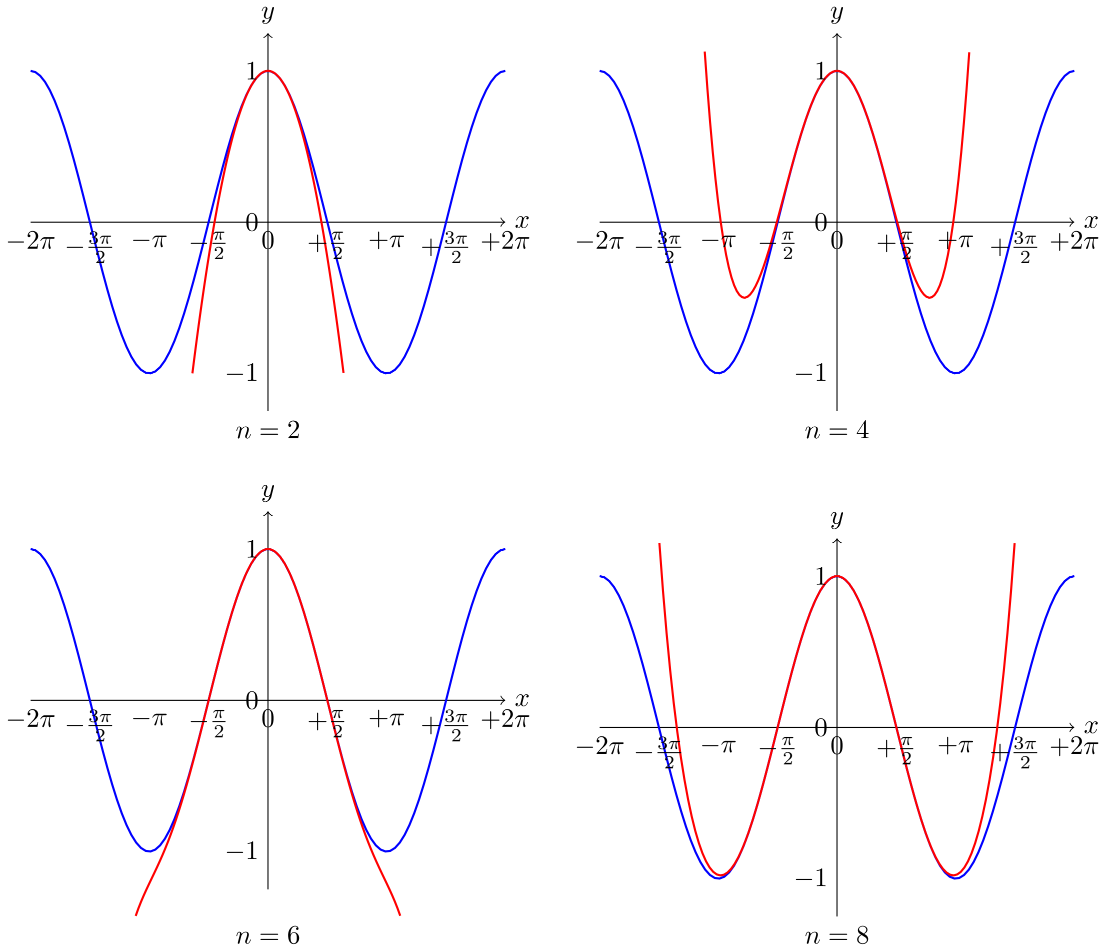


Figure 2: The graph of $y = \cos(x)$, together with its degree-n Taylor polynomials for $n = 2$, 4, 6, and 8.

Note: In the Newton-Raphson method, we take the tangent to $y = f(x)$ at a point x_n, and find the point where that tangent hits the x-axis to be x_{n+1}. That tangent is precisely the degree 1 Taylor polynomial, $f_1(x)$, about $x = x_n$; so in fact the Newton-Raphson method consists of approximating a function by its degree one Taylor polynomial and then solving $f_1(x) = 0$ to get an estimate for the solution of $f(x) = 0$—and then of course iteratively using that estimate as the base point to find a new degree 1 Taylor polynomial, and repeat with that.

We could consider using higher degree Taylor polynomials in a Newton-Raphson-type method; for instance, we could take the degree 2 Taylor polynomial about x_n and solve $f_2(x) = 0$ to find a more accurate x_{n+1}; the problem with this is that quadratics have two roots, so $f_2(x) = 0$ will give us two solutions, one of which is likely to be closer to the root of $f(x)$ than we would have got using $f_1(x)$, but the other of which could be anywhere. So we stick to using the degree 1 Taylor polynomial for Newton-Raphson and similar methods.

Key Points to Remember:

1. A polynomial of degree n can be entirely determined by specifying the value of the polynomial and its first n derivatives at a single point.

2. Given any n-times differentiable function $f(x)$, the **Taylor polynomial of degree** n of $f(x)$ about $x = a$ is $f_n(x)$, the unique polynomial of degree n specified by the value and first n derivatives of $f(x)$ at $x = a$. It is given by

$$\begin{aligned} f_n(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n \\ &= \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i \end{aligned}$$

   or

$$\begin{aligned} f_n(a+h) &= f(a) + f'(a)(h) + \frac{f''(a)}{2!}h^2 + \ldots + \frac{f^{(n)}(a)}{n!}h^n \\ &= \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}h^i. \end{aligned}$$

3. For values of x close to a (*i.e.*, values of h close to 0), the Taylor polynomial is typically a good approximation to the starting function: $f_n(x) \approx f(x)$. The higher the degree n, the further you can go from a and maintain a good approximation. However, how good the approximation is at a certain distance from a and for a given n can vary a lot from function to function.

4. For some functions, the Taylor polynomial fails to give a good approximation once you go too far from a, no matter how high a degree you take.