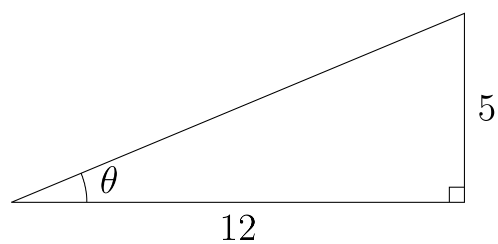
Trigonometric Waveforms

Objective: To understand taking trig functions of any angle and the graphs of trig functions.

Recap of previous material:

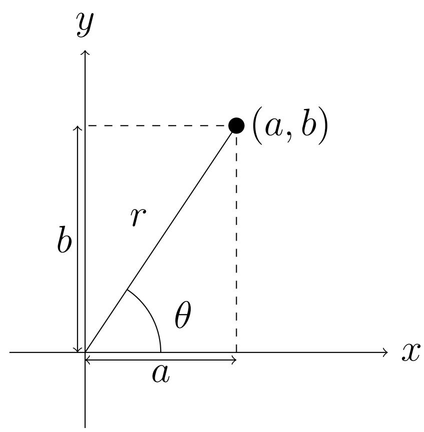
Consider the right-angled triangle below:



1. Find the length of the hypotenuse.
2. Find $\sin(\theta)$.
3. Find $\cos(\theta)$.
4. Find $\tan(\theta)$.
5. For any angle ϕ, what equation links $\sin(\phi)$, $\cos(\phi)$, and $\tan(\phi)$?
6. For any angle ϕ, $1 - \sin^2(\phi) =$

Warm-up:

Consider a point in the first quadrant of the plane with coordinates (a, b):



1. Express the length r in terms of a and b.
2. Express a in terms of r and θ, using a trigonometric function.
3. Express b in terms of r and θ, using a trigonometric function.

Theory - Circles and Extending Trigonometric Functions to Arbitrary Angles:

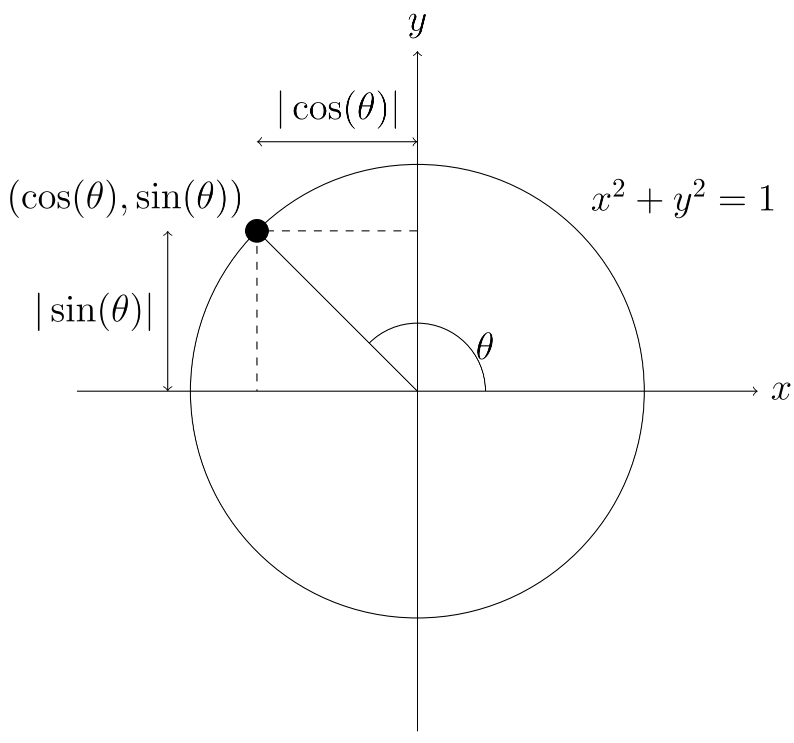
We saw above that the distance of a point (a, b) from the origin is $\sqrt{a^2 + b^2}$. A circle of radius r consists of all points whose distance from the origin is r. Therefore the equation of a circle of radius r centred on the origin is:

$$x^2 + y^2 = r^2$$

We saw also that:

$$a = r\cos(\theta) \qquad\qquad b = r\sin(\theta)$$

for θ the angle to the point (a, b) in the first quadrant. We can use this to define $\sin(\theta)$ and $\cos(\theta)$ for any angle θ, even one too large to appear in a right-angled triangle. In a circle of radius 1 centred on the origin, $r = 1$, so $\cos(\theta)$ is the x-coordinate and $\sin(\theta)$ the y-coordinate of the point on the circle at angle θ:



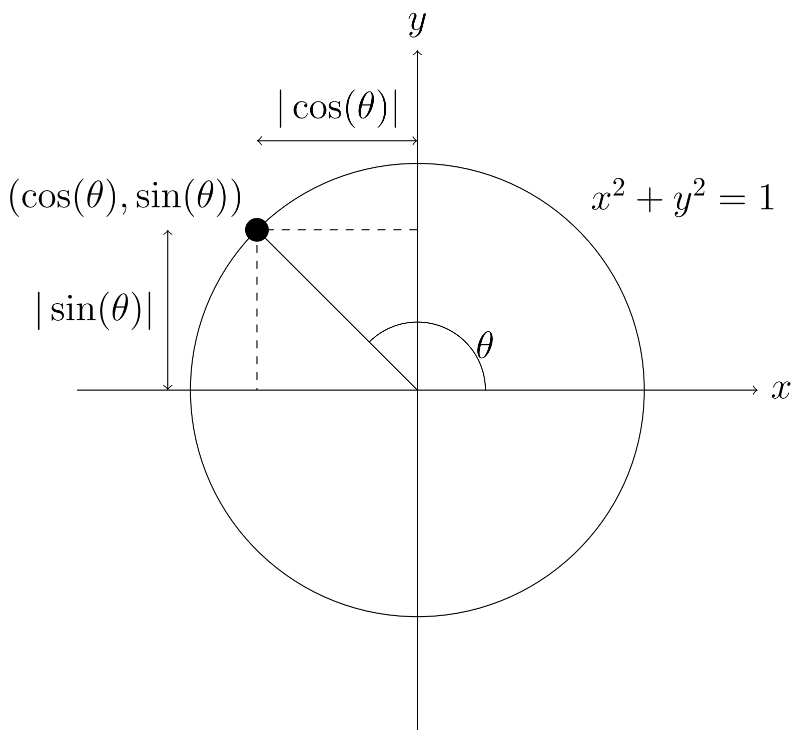
See also:

https://en.wikipedia.org/wiki/Sine#/media/File:Circle_cos_sin.gif

The tangent of an arbitrary angle is then defined by the equation

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$$

Practice:

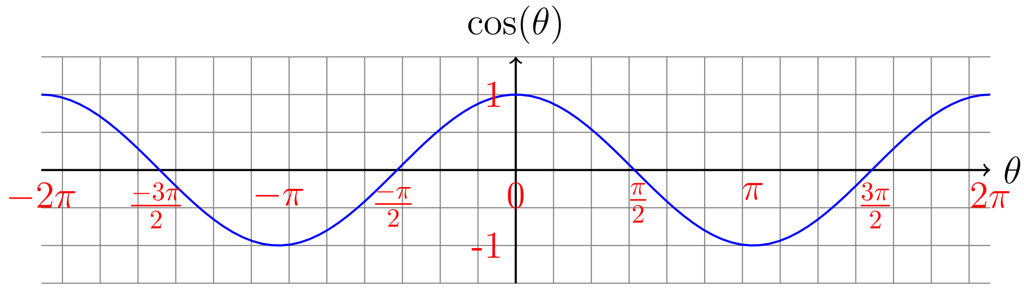


1. In which quadrants of the plane will $\cos(\theta)$ be positive? What ranges of angle does this correspond to?
2. In which quadrants of the plane will $\sin(\theta)$ be positive? What ranges of angles does this correspond to?
3. How are $\sin(\theta)$ and $\sin(-\theta)$ related?
4. How are $\cos(\theta)$ and $\cos(-\theta)$ related?
5. How are $\tan(\theta)$ and $\tan(-\theta)$ related?
6. What is $\tan\left(\frac{\pi}{2}\right)$? Or $\tan\left(\frac{3\pi}{2}\right)$? What about for θ very near these values?

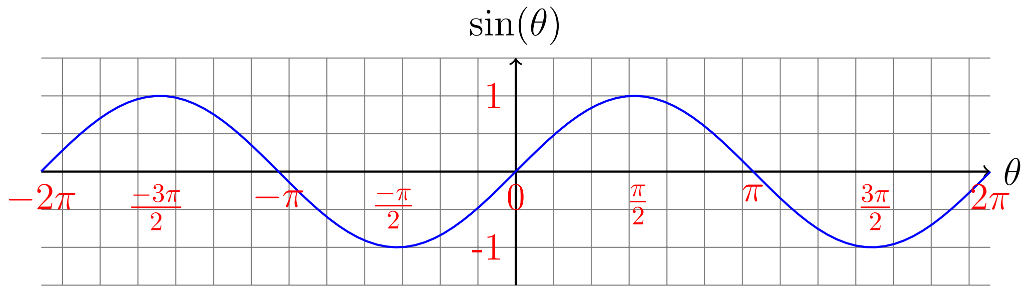
Useful Exercise to Try Yourself:

With some graphing paper (or just draw a grid on some plain paper with a ruler!) and a calculator, draw 3 graphs, one of $\sin(\theta)$, one of $\cos(\theta)$, one of $\tan(\theta)$. Do this by calculating the values of the function to be plotted for several different values of θ, plotting the resulting point, then connecting with a smooth curve. For instance try plotting for θ taking values between -7 and 7, every 0.5.

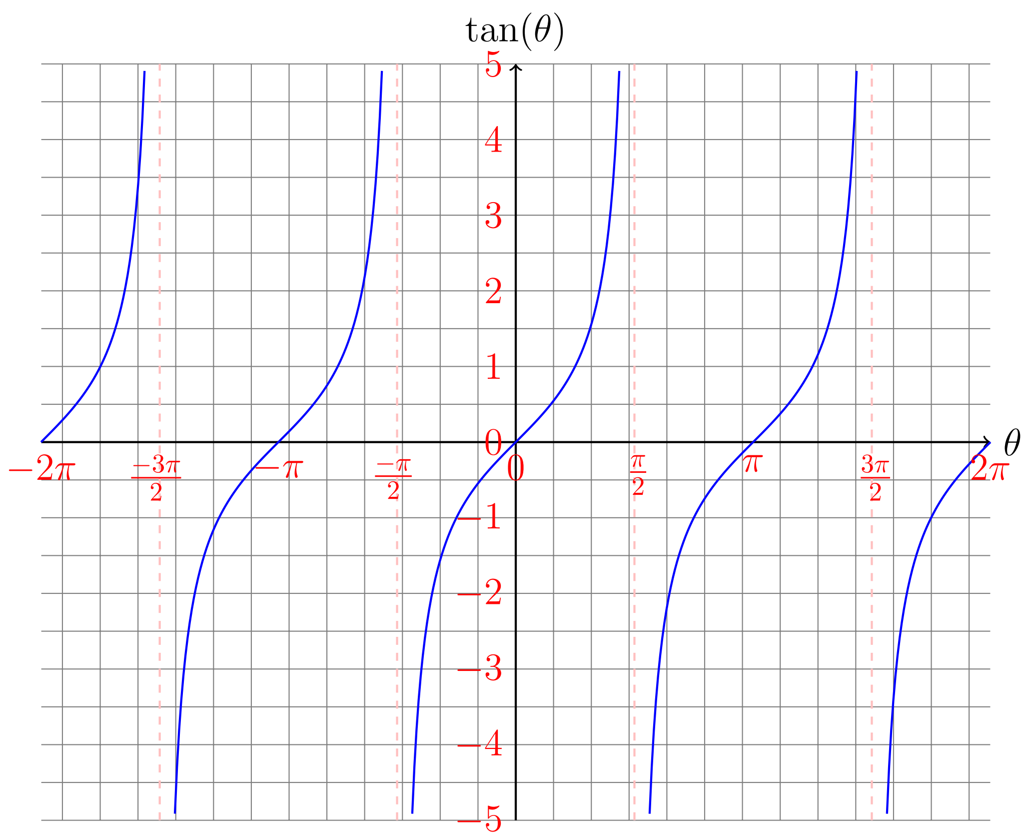
The graphs you should get from this are shown overleaf, with important values of θ marked.



$\cos(\theta)$
1
-1
θ
-2π
$\frac{-3\pi}{2}$
$-\pi$
$\frac{-\pi}{2}$
0
$\frac{\pi}{2}$
π
$\frac{3\pi}{2}$
2π



$\sin(\theta)$
1
-1
θ
-2π
$\frac{-3\pi}{2}$
$-\pi$
$\frac{-\pi}{2}$
0
$\frac{\pi}{2}$
π
$\frac{3\pi}{2}$
2π



$\tan(\theta)$
5
4
3
2
1
0
-1
-2
-3
-4
-5
θ
-2π
$\frac{-3\pi}{2}$
$-\pi$
$\frac{-\pi}{2}$
0
$\frac{\pi}{2}$
π
$\frac{3\pi}{2}$
2π

Key Points to Remember:

1. The distance from the origin to the point (x, y) is $\sqrt{x^2 + y^2}$.
2. The equation of the circle of radius r, centred on the origin, is $x^2 + y^2 = r^2$.
3. For any angle θ, $\cos(\theta)$ (respectively $\sin(\theta)$) is defined to be the x-coordinate (respectively, the y-coordinate) of the point on the unit circle (radius 1, centre the origin) which makes an angle θ going anticlockwise from the positive x-axis. When θ is an acute angle, this agree with the right-angled triangle definition.
4. For any angle θ, $\tan(\theta)$ is defined to be $\frac{\sin(\theta)}{\cos(\theta)}$.
5. $\tan(\theta)$ is not defined at angles where $\cos(\theta) = 0$; *i.e.*, at $\theta = \ldots\frac{-3\pi}{2}, \frac{-\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$.