Exponentials and Logarithms

Theory:

- In an expression b^a, b is called the **base** and a the **exponent** or **power**.
- The exponential function of base b is the function b^x.
- The logarithm function to base b is the function $\log_b(x)$.
- The most common bases used are 10 and e; the logarithm to base e is called the **natural logarithm** and denoted $\ln(x)$.
- The logarithm and exponential functions are inverses, so:

$$b^{\log_b(x)} = x = \log_b(b^x).$$

- Laws of exponents:

$$b^{n+m} = b^n b^m, \qquad b^{-n} = \frac{1}{b^n}, \qquad (b^n)^m = b^{nm}.$$

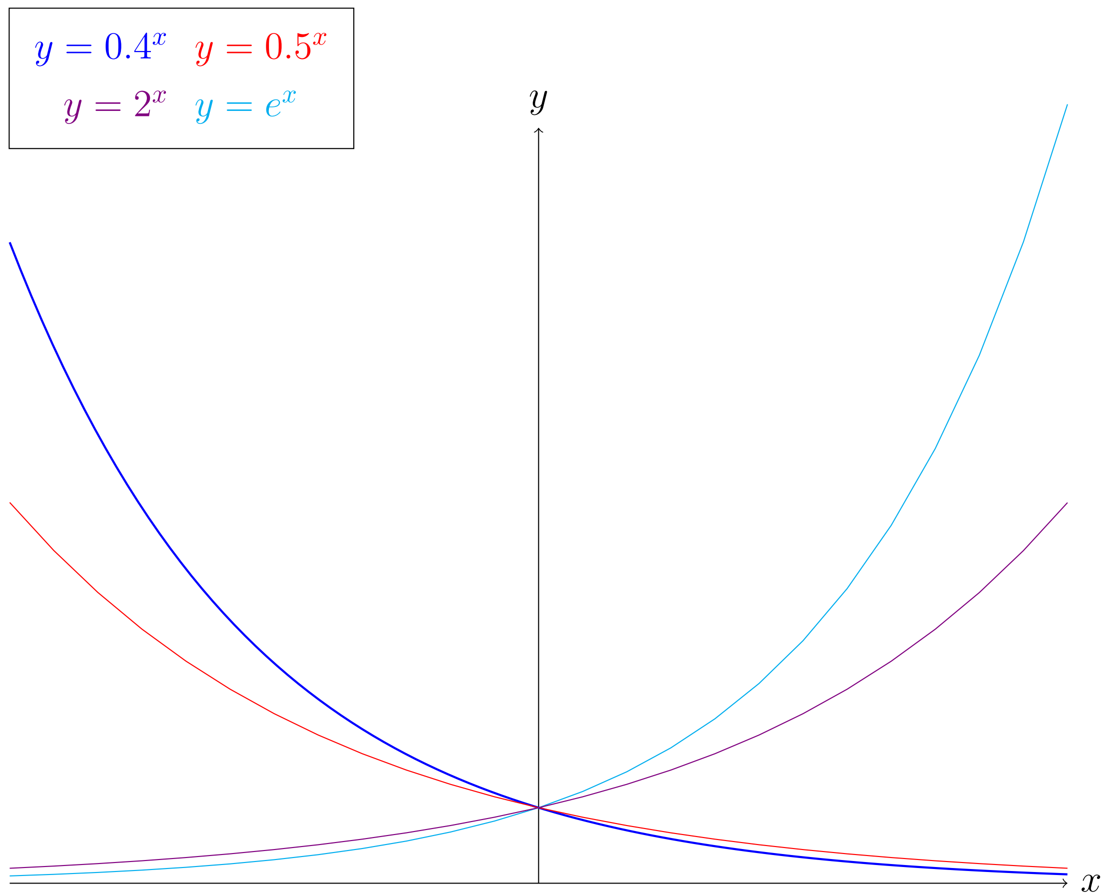
- Laws of logarithms:

$$\log_b(nm) = \log_b(n) + \log_b(m), \qquad \log_b(n^m) = m \log_b(n).$$

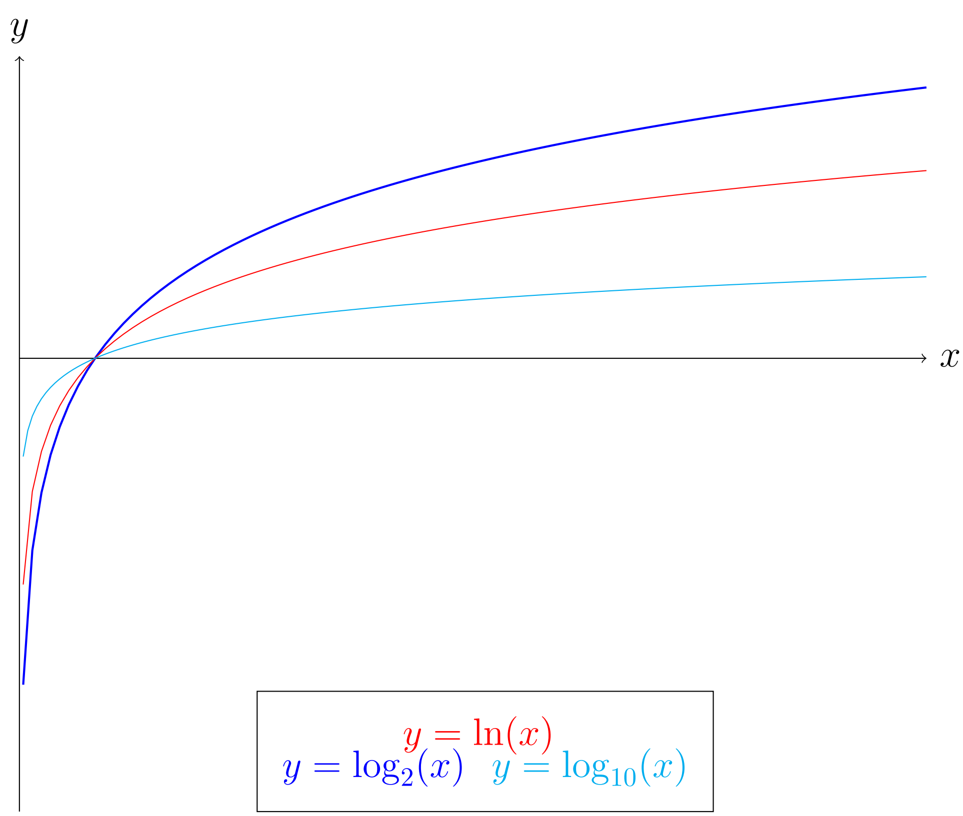
- The change-of-base rule for logarithms:

$$\log_a(x) = \frac{\log_b(x)}{\log_b(a)}.$$

- The graphs of the log and exponential functions for a selection of bases are shown below.



$y = 0.4^x$ $y = 0.5^x$
$y = 2^x$ $y = e^x$
y
x



y
x
$y = \ln(x)$
$y = \log_2(x)$ $y = \log_{10}(x)$

Practice:

1. Sketch the graph of $y = e^x$.
2. Sketch the graph of $y = \left(\frac{1}{2}\right)^x$.
3. Sketch the graph of $y = \log(x)$.
4. Solve $e^x = 2$.
5. Solve $\log_{10}(x) = 4$.
6. Solve $\log_3(3x) = 4$.
7. Solve $10^{3+x} = 19000$.
8. Solve $e^{3x+2} = 5$.
9. Solve $\ln(5x - 1) = 2$.
10. Solve $10^{3x}10^{x-1} = 100$.
11. Solve $\ln(5x) - \ln(2) = 4$.
12. Solve $2\ln(x) + \ln(5) = 2$.
13. We will derive the change-of-base rule. Let $y = a^x$ for some $a > 0$, $a \neq 1$.
    (a) Write down $\log_a(y)$.
    (b) Let b be any positive number different from 1. Write down $\log_b(y)$.
    (c) Substituting your answer from part (a) into part (b), write an equation linking $\log_b(x)$ and $\log_a(x)$.
14. Using the change-of-base rule, solve $4^x = 3$.
15. Solve $2^{3x-1} = 7$.
16. Suppose $x = ae^{kt}$ for some parameters a and k. Suppose also that the initial value of x is 7, and $x(1) = 7e^2$. Find a and k.
17. Suppose $x = ab^t$ for some parameters a and b. Given that $x(1) = 3$ and $x(3) = 10$, plot the graph of $\ln(x)$ against t. Use your graph to find the values of a and b.
18. Suppose $y = ax^n$. Given that $y(1) = 5$ and $y(5) = 100$, plot the graph of $\log(y)$ against $\log(x)$, and hence find the values of a and n.

19. Suppose there is an island with a plentiful supply of vegetation and no predators. Ten rabbits are introduced to this island. The rabbit population P at time t years is modelled by $P = P_0 e^{\lambda t}$.

    (a) What is the value of P_0?

    (b) Suppose that after 1 year the rabbit population is 100. What is the value of λ?

    (c) Find the rabbit population after 2 years.

    (d) What does the rabbit population do as t gets very large? Does this seem realistic? What assumptions might need to be revised in light of this?

20. After a patient takes some medication, the concentration x of that medication in their bloodstream gradually decreases according to $x = ae^{\lambda t}$, where t is the time in hours.

    (a) Would you expect λ to be positive or negative?

    (b) What does the parameter a represent?

    (c) Find an expression for the time t at which the concentration is half of its initial value. This time is called the **half-life** of the medication.

    (d) What proportion of the initial concentration remains after 5 half-lives?

    (e) After how many half-lives is the concentration down to 10% of its initial value?

    (f) Suppose that the half-life of a particular medicine is 6 hours. What is the value of λ?

    (g) Suppose that the initial concentration of medicine in the bloodstream is $10\,\mathrm{mg\,mol^{-1}}$ (milligrams per mole), and use the value of λ from the previous part. After how long will the concentration of medicine be $1\,\mathrm{mg\,mol^{-1}}$?