# Vector Spaces

**Vector Spaces:**

A vector space is, roughly speaking, somewhere you can add things and multiply them by scalars. More precisely, a vector space is a set $V$ with operations $+ : V \times V \to V$ and $\times : \mathbb{R} \times V \to V$, and a special element $\underline{0} \in V$ such that for any $u, v, w \in V$ and $\lambda, \mu \in \mathbb{R}$:

| **Addition axioms:** | **Multiplication Axioms:** |
|---|---|
| $(u + v) + w = u + (v + w)$ (associativity) | $(\lambda\mu)v = \lambda(\mu v)$ (associativity) |
| $u + v = v + u$ (commutativity) | $(\lambda + \mu)v = \lambda v + \mu v$ (distributivity) |
| $v + \underline{0} = v$ (identity) | $\lambda(u + v) = \lambda u + \lambda v$ (distributivity) |
| $\exists(-v) \in V : v + (-v) = \underline{0}$ (inverses) | $1v = v$ (identity) |

Examples of vector spaces include:

**Subspaces:**

Given a vector space $V$, a subspace is just a subset which is itself a vector space, with the same operations. To check that $U \subseteq V$ is a subspace of $V$, we need to check that:

For each of the examples of a vector space above, try to find an example of a subspace:

**Span:**

Let $V$ be a vector space and $S$ a subset of $V$. Define $\langle S\rangle$ to be the intersection of all subspaces of $V$ which contain $S$:

$$\langle S\rangle = \bigcap\{U \subseteq V \mid U \text{ is a subspace of } V \text{ and } S \subseteq U\}.$$

Prove that $\langle S\rangle$ is a subspace of $V$ and that for any subspace $U$, $S \subseteq U$ if and only if $\langle S\rangle \subseteq U$.

Let $S$ be a finite set, $S = \{v_1, \ldots .v_n\}$. Prove that

$$\langle S\rangle = \left\{\sum_{i=1}^{n} \lambda_i v_i \mid \lambda_1, \ldots, \lambda_n \in \mathbb{R}\right\};$$

that is, prove that $\langle S\rangle$ consists of all linear combinations of $v_1, \ldots, v_n$. Hint: show that the set above is a subspace and contains $S$, then use the property of $\langle S\rangle$ we proved above.

For each of the examples of a subspace $U$ given above, find a set which spans $U$.

**Span and Matrices:**

Consider $\mathbb{R}^3$ and the vectors $v_1 = (3, -4, 7)$ and $v_2 = (-1, 2, 6)$. What is the span of these vectors?

$$\begin{aligned}
\langle v_1, v_2 \rangle &= \{\alpha v_1 + \beta v_2 \mid \alpha, \beta \in \mathbb{R}\} \\
&= \{(3\alpha - \beta, -4\alpha + 2\beta, 7\alpha + 6\beta) \mid \alpha, \beta \in \mathbb{R}\} \\
&= \left\{(\alpha, \beta) \begin{pmatrix} 3 & -4 & 7 \\ -1 & 2 & 6 \end{pmatrix} \mid \alpha, \beta \in \mathbb{R}\right\} \\
&= \left\{v \begin{pmatrix} 3 & -4 & 7 \\ -1 & 2 & 6 \end{pmatrix} \mid v \in \mathbb{R}^2\right\}.
\end{aligned}$$

In general, if $v_1, \ldots, v_n$ are vectors in $\mathbb{R}^m$, then we can form an $n \times m$ matrix $A$ with rows $v_1, \ldots, v_n$; then

$$\langle v_1, \ldots, v_n \rangle = \{uA \mid u \in \mathbb{R}^n\}.$$

Looking at this the other way around, if $A$ is any $n \times m$ matrix, then $\{uA \mid u \in \mathbb{R}^n\}$ is a subspace of $\mathbb{R}^m$ spanned by the rows of $A$. Can you see why?

Given an $n \times m$ matrix $A$, there is a function $f_A : \mathbb{R}^n \to \mathbb{R}^m$ defined by $f_A(u) = uA$. Then the subspace $\{uA \mid u \in \mathbb{R}^n\}$ of $\mathbb{R}^m$ is exactly the image of this function $f_A$. When you see linear maps, you will see that this is actually true for any vector space $V$! Any subspace of $V$ is the image of a linear map from some other vector space, and the image of any linear map is a subspace. So the idea here is that a subspace is the image of another vector space within $V$; in the case of $\mathbb{R}^m$, a subspace is the image of $\mathbb{R}^n$ in $\mathbb{R}^m$ under the linear map defined by a matrix.

Let $A$ be an $n \times m$ matrix and let $f_A : \mathbb{R}^n \to \mathbb{R}^m$ be the function $f_A(u) = uA$. Let $e_i$ be the $i^{\text{th}}$ standard basis vector of $\mathbb{R}^n$ (so $e_i$ has a 1 in the $i^{\text{th}}$ coordinate and 0 everywhere else). Show that $f_A(e_i)$ is the $i^{\text{th}}$ row of $A$.

The upshot of all this is that the map $f_A$ is surjective if and only if the rows of $A$ span $\mathbb{R}^m$.

**Linear Independence:**

What does it mean to say that vectors $v_1, \ldots, v_n$ are linearly independent?

Prove that $v_1, \ldots, v_n$ are linearly independent if and only if no proper subset of $v_1, \ldots, v_n$ spans $\langle v_1, \ldots . v_n \rangle$. In other words, if we remove any $v_i$, the span becomes strictly smaller. Hint: prove that if they are linearly independent and we remove $v_i$, then the span of the remaining vectors does not include $v_i$; then prove that if they are linearly dependent, then there is some $i$ such that $v_i$ is the in the span of the remaining vectors.

Is $\{1 + x, 1 - x, 2 + 2x + x^2\}$ linearly independent in the vector space of polynomials?

**Linear Independence and Matrices:**

We have seen that the span of some vectors in $\mathbb{R}^m$ is the image of the matrix whose rows are those vectors. Now we show a similar result for linear independence. Let $v_1, \ldots, v_n$ be vectors in $\mathbb{R}^m$ and let $A$ be the $n \times m$ matrix whose $i^{\text{th}}$ row is $v_i$. Show that $v_1, \ldots, v_n$ are linearly independent if and only if the matrix equation $uA = 0$ (for $u \in \mathbb{R}^n$) has the unique solution $u = 0$.

Let $f_A : \mathbb{R}^n \to \mathbb{R}^m$ be the function $f_A(u) = uA$. Show that $f_A$ is injective if and only if the rows of $A$ are linearly independent. Hint: first show that $f_A(u) = f_A(v)$ if and only if $f_A(u - v) = 0$.

So we have seen that the rows of $A$ span $\mathbb{R}^m$ if and only if $f_A$ is surjective, and the rows are linearly independent if and only if $f_A$ is injective.

**Bases:**

Define a basis of a vector space $V$.

Show that $v_1, \ldots, v_n$ in $\mathbb{R}^m$ are a basis if and only if the map $f_A : \mathbb{R}^n \to \mathbb{R}^m$ is bijective, where $A$ is the $n \times m$ matrix whose $i^{\text{th}}$ row is $v_i$ and $f_A$ is defined by $f_A(u) = uA$.