Sums of Sinusoids "Cheat Sheet"

A step-by-step guide to converting a sum of two sinusoids with the same frequency into a single sinusoid, with a running example of expressing

$$3\cos\left((2t+\frac{\pi}{3})\right)-\sin(2t)$$

in the form $R\sin(2t+\alpha)$:

Step-by-step instructions:

1. Expand out any phase shifts using compound angle formulae.
2. Expand the desired expression using the compound angle formula.
3. Compare the expressions obtained by the last two steps.
4. Deduce two equations relating R and α to known quantities.
5. Square these two equations and add.
6. Use the pythagorean identity to simplify and get a value for R^2.
7. Take the positive square root (or negative, but positive makes step 9 slightly easier).
8. Going back to the equations from step 4, divide the sin equation by the cos equation to find $\tan(\alpha)$. Simplify if possible.
9. Using the equations from step 4 (and the fact that R is positive), work out which quadrant α is in.
10. Use the previous two steps to find α, taking arctan and possibly adding π.
11. Put the values for R and α back into the desired expression, which is now equal to the original expression.
12. If desired, plug values into a calculator for a numerical answer.

Running example:

$$3\cos\left((2t+\frac{\pi}{3})\right)-\sin(2t)$$

1. We have a phase-shifted term, $\cos\left(2t+\frac{\pi}{3}\right)$, which expands to

$$\cos(2t)\cos\left(\frac{\pi}{3}\right)-\sin(2t)\sin\left(\frac{\pi}{3}\right)=\frac{1}{2}\cos(2t)-\frac{\sqrt{3}}{2}\sin(2t)$$

So our expression becomes:

$$\frac{3}{2}\cos(2t)-\frac{3\sqrt{3}}{2}\sin(2t)-\sin(2t)=\frac{3}{2}\cos(2t)-\frac{3\sqrt{3}-2}{2}\sin(2t).$$

2. Our desired expression is

$$R\sin(2t+\alpha)=R\sin(\alpha)\cos(2t)+R\cos(\alpha)\sin(2t).$$

3. Compare

$$\frac{3}{2}\cos(2t)-\frac{3\sqrt{3}-2}{2}\sin(2t) \qquad \text{and} \qquad R\sin(\alpha)\cos(2t)+R\cos(\alpha)\sin(2t).$$

4. Hence

$$R\sin(\alpha)=\frac{3}{2} \qquad \text{and} \qquad R\cos(\alpha)=\frac{2-3\sqrt{3}}{2}.$$

5.

$$R^2\sin^2(\alpha)+R^2\cos^2(\alpha)=\left(\frac{3}{2}\right)^2+\left(\frac{2-3\sqrt{3}}{2}\right)^2=\frac{40-12\sqrt{3}}{4}.$$

6.

$$R^2\sin^2(\alpha)+R^2\cos^2(\alpha)=R^2(\sin^2(\alpha)+\cos^2(\alpha))=R^2$$

$$\text{so} \qquad R^2=\frac{40-12\sqrt{3}}{4}.$$

7.

$$R=\frac{\sqrt{40-12\sqrt{3}}}{2}.$$

8.

$$\tan(\alpha)=\frac{\sin(\alpha)}{\cos(\alpha)}=\frac{R\sin(\alpha)}{R\cos(\alpha)}=\frac{3/2}{(2-3\sqrt{3})/2}=\frac{3}{2-3\sqrt{3}}.$$

9. Since $R\sin(\alpha) = \frac{3}{2}$, $\sin(\alpha)$ is positive. Since $R\cos(\alpha) = \frac{2-3\sqrt{3}}{2}$ and $3\sqrt{3} > 2$, $\cos(\alpha)$ is negative. So α is in the top left quadrant.

10. Since α is in the top left quadrant, arctan will not give us the correct value, so we need to add π. So

$$\alpha = \tan^{-1}\left(\frac{3}{2-3\sqrt{3}}\right) + \pi.$$

11. So we have

$$3\cos\left((2t+\frac{\pi}{3})\right) - \sin(2t) = \frac{\sqrt{40-12\sqrt{3}}}{2}\sin\left(2t + \tan^{-1}\left(\frac{3}{2-3\sqrt{3}}\right) + \pi\right).$$

12.

$$3\cos\left((2t+\frac{\pi}{3})\right) - \sin(2t) \approx 2.192\sin(2t + 2.388).$$