Compound Angle and Other Trig Formulae.

<u>Objective: To be able to apply compound angle formulae and other trig formulae derived from them to solve problems</u>

Recap of previous material. Application—Orbital Motion:

The planet Zorg orbits its star in a circular orbit; its position at time t is given by

$$x(t) = r\cos\left(\frac{2\pi}{T}t\right), \qquad y(t) = r\sin\left(\frac{2\pi}{T}t\right),$$

where T is the orbital period (the length of one year on the planet), and r is the radius of the orbit. The coordinate axes are set up so that the sun is at the origin.

The planet Yarg orbits the same star with radius $\frac{r}{2}$ and period $\frac{T}{2}$, and out of phase. Its position at time t is given by

$$x(t) = \frac{r}{2}\cos\left(\frac{4\pi}{T}t + \frac{\pi}{2}\right), \qquad y(t) = \frac{r}{2}\sin\left(\frac{4\pi}{T}t + \frac{\pi}{2}\right).$$

An astronaut plans to fly from Zorg to Yarg, and wants to do so when the distance between the planets is minimum, to save fuel.

1. Write down an expression for Δx (the difference in x-coordinates of the two planets), and a similar expression for Δy, the difference in y-coordinates.
2. Hence write down an expression for the distance d between the two planets at time t. Simplify this expression.
3. Using the formulae

$$\cos(A)\cos(B) = \frac{1}{2}(\cos(A+B) + \cos(A-B)),$$
$$\sin(A)\sin(B) = \frac{1}{2}(\cos(A-B) - \cos(A+B)),$$

   simplify your expression for the distance further.
4. The distance is a minimum (respectively maximum) precisely when the square of the distance is minimum (respectively maximum), and the square of the

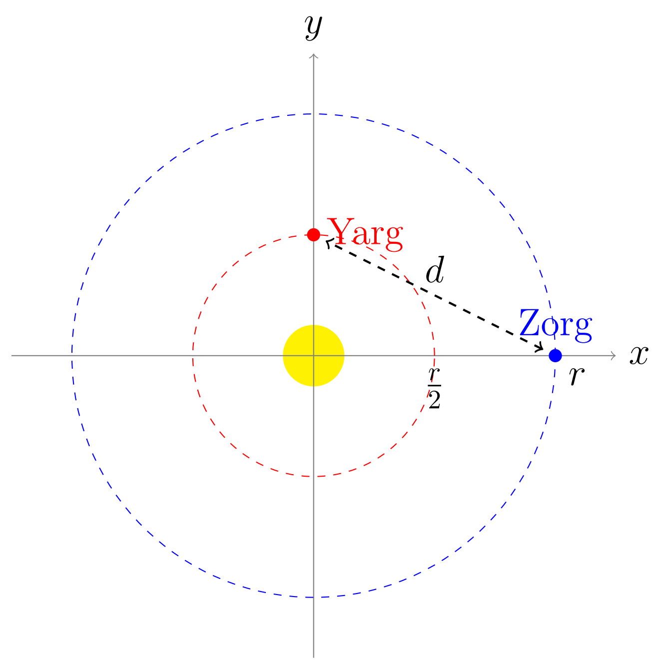
distance is easier to work with. Show that, for $0 \le t < T$ the distance attains its minimum and maximum values when

$$\sin\left(\frac{2\pi}{T}t + \frac{\pi}{2}\right) = 0.$$

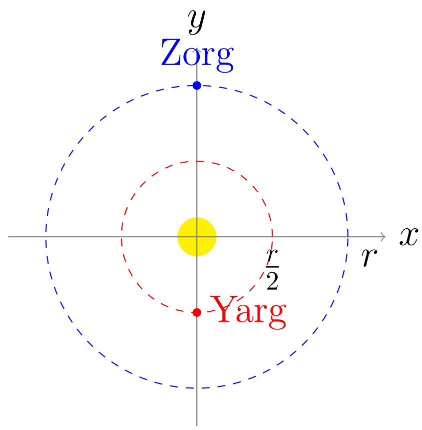
Note: If you haven't yet done differentiating trig functions or maximum/minimum problems, skip this part.

5. Solve the equation in part 4 for $0 \le t < T$.

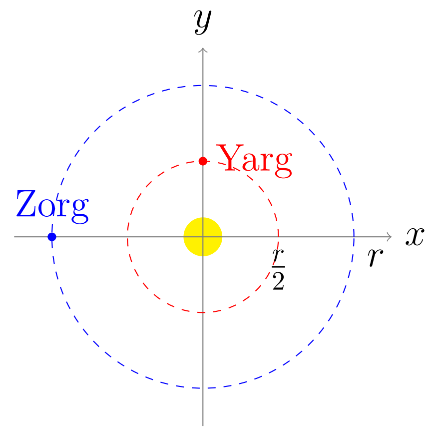
6. Hence say when the astronaut should make their flight.



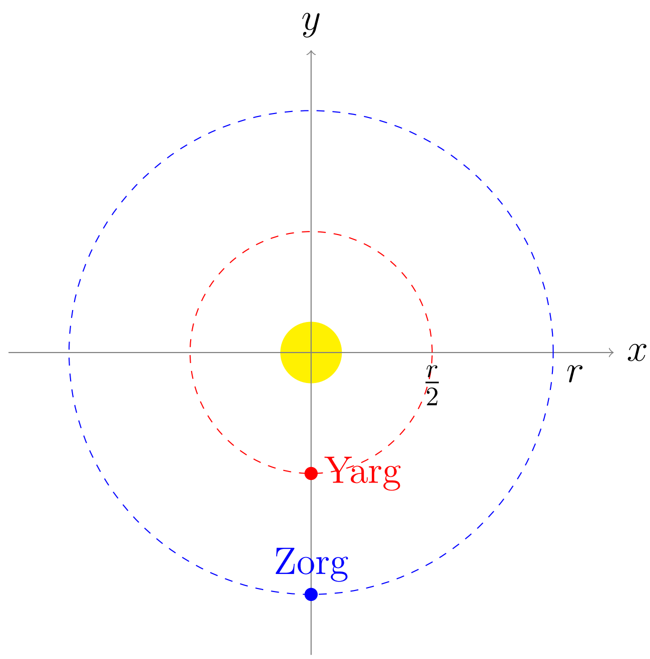
$t = 0$



$t = \frac{T}{4}$



$t = \frac{T}{2}$



$t = \frac{3T}{4}$

Practice:

Recall the **compound angle formulae:**

$$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$$
$$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$
$$\tan(\alpha+\beta) = \frac{\tan(\alpha)+\tan(\beta)}{1-\tan(\alpha)\tan(\beta)}$$

1. (a) Write down a formula for $\sin(2\theta)$ in terms of $\sin(\theta)$ and $\cos(\theta)$. Hint: $2\theta = \theta + \theta$.
   (b) Hence, given that $\sin(12°) \approx 0.208$ and $\cos(12°) \approx 0.978$, find $\sin(24°)$.
2. (a) Write down a formula for $\cos(2\theta)$ in terms of $\sin(\theta)$ and $\cos(\theta)$.
   (b) Apply the formula $\cos^2(\theta) + \sin^2(\theta) = 1$ to eliminate $\sin(\theta)$ from this formula.
   (c) Apply the Pythagorean formula again in a different way to your formula from part (a) to eliminate $\cos(\theta)$.
3. The **half angle formulae**:
   (a) Using your answer to question 2(b), express $\cos(\theta)$ in terms of $\cos(2\theta)$.
   (b) Given that $\cos(324°) \approx 0.809$, find $\cos(162°)$.
   (c) Using your answer to question 2(c), express $\sin\left(\frac{\theta}{2}\right)$ in terms of $\cos(\theta)$.
   (d) Given that $\sin(2.8) \approx 0.335$, find $\sin(1.4)$.
4. Write down a formula for $\tan(2\theta)$ in terms of $\tan(\theta)$. Hence, given that $\tan\left(\frac{\pi}{3}\right) = \sqrt{3}$, find $\tan\left(\frac{2\pi}{3}\right)$.
5. (a) Express $\sin(3\theta)$ in terms of $\sin(\theta)$, $\sin(2\theta)$, $\cos(\theta)$, and $\cos(2\theta)$. Hint: $3\theta = 2\theta + \theta$.
   (b) Hence express $\sin(3\theta)$ in terms of $\sin(\theta)$ and $\cos(\theta)$ only. Hint: use your answers to questions 1 and 2.
   (c) Eliminate $\cos(\theta)$ to obtain an expression for $\sin(3\theta)$ in terms of $\sin(\theta)$ only.
6. Apply the compound angle formula for cosine to the expression
   $$\cos(\alpha+\beta) + \cos(\alpha-\beta).$$
   Hence write $\cos(5x)\cos(10x)$ as a sum of two cosines.
7. Apply the compound angle formula for cosine to the expression
   $$\cos(\alpha-\beta) - \cos(\alpha+\beta).$$
   Hence write $\sin(12\pi t)\sin(2\pi t)$ as a sum of cosines.

Theory—Sums of Sinusoids of Equal Frequency:

The compound angle formulae split a single sine or cosine into a combination of sines and cosines:

$$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta),$$

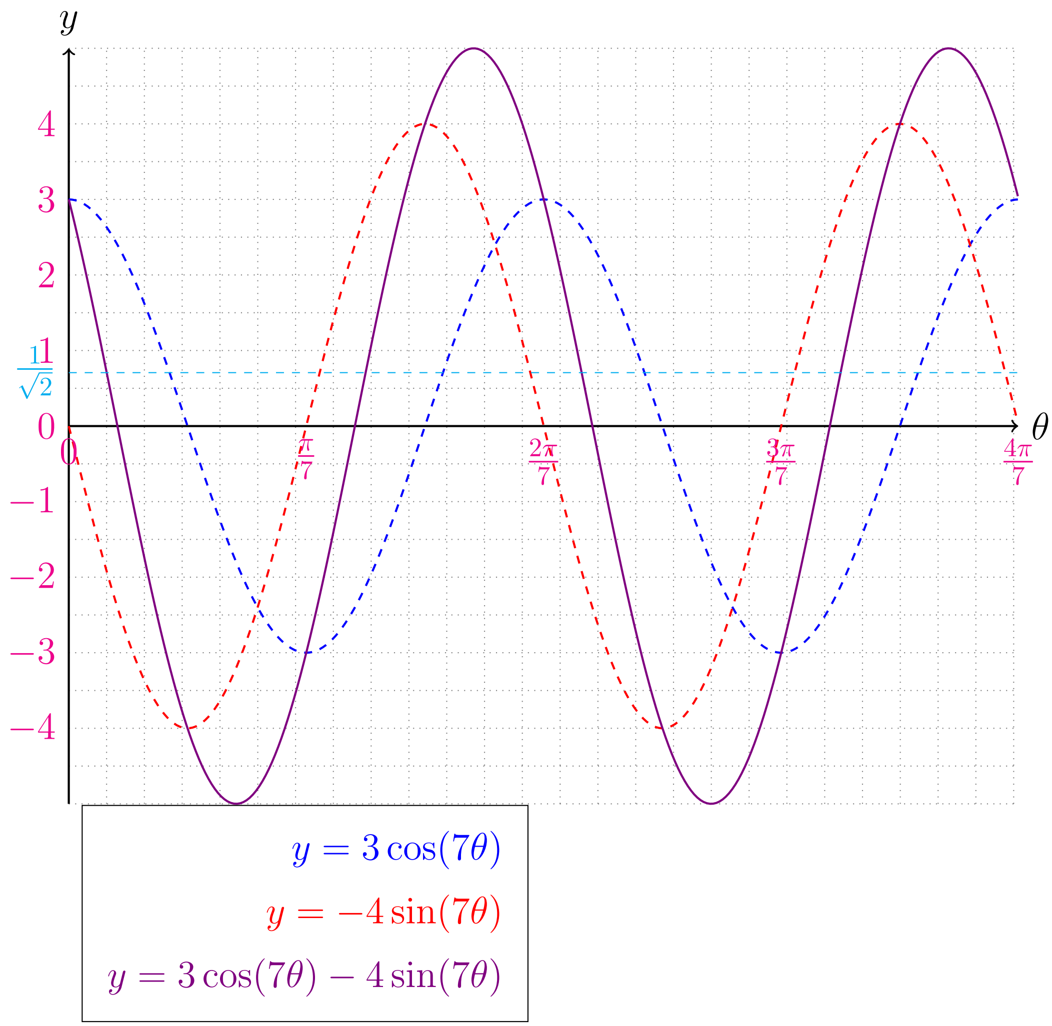
$$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta).$$

We can use this in reverse to combine sines and cosines **of the same frequency** into a single sine or cosine.

Write $3\cos(7\theta) - 4\sin(7\theta)$ in the form $R\cos(7\theta+\alpha)$ for some R and α.

Hence solve $3\cos(7\theta) - 4\sin(7\theta) = \frac{1}{\sqrt{2}}$ for $0 \le \theta < \pi$.

Here we show the graphs of the functions from the last page:



Practice:

1. Express $\sin(3t) - \sqrt{3}\cos(3t)$ in the form $R\sin(3t - \alpha)$.
2. Solve the equation $6\cos(4x) + 8\sin(4x) = -0.5$ for $0 \leq x < \frac{\pi}{2}$.
3. The voltage of mains electricity varies with time by the function $230\sqrt{2}\sin(100\pi t)$. A drive coil on a so-called "3-phase" electric motor receives two alternating voltages, with a time offset between them:

$$A = 230\sqrt{2}\sin(100\pi t), \qquad B = 230\sqrt{2}\sin\left(100\pi t + \frac{2\pi}{3}\right).$$

   The resulting voltage on the drive coil is $A - B$, the difference between these two voltages. Express the resulting voltage on the drive coil as a single sine function of time; *i.e.*, write $A - B$ in the form $R\sin(100\pi t + \alpha)$. Hence suggest why for high-power applications 3-phase motors are used instead of single phase motors (with just one mains voltage applied).

4. Sound is a pressure wave in the air; for a single pure note at frequency f and amplitude A, the pressure P_1 at a point in the air varies with time according to $P_1 = A\sin(2\pi ft)$. When two sounds are played, with pressure waves P_1 and P_2, the overall effect on air pressure is $P_1 + P_2$.

   Suppose an additional sound is played, with $P_2 = A\sin((2\pi + 1)ft)$. The overall result of these two pressure waves is that the total air pressure is $P_{\text{total}} = P_1 + P_2$. Apply the compound angle formula for sine to P_2 and hence find P_{total}. Describe the sound which results when both these sounds are played together, and hence suggest how noise-cancelling headphones work.

Key Points to Remember:

1. The **compound angle formulae** for sine and cosine are:

$$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$
$$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$$

2. We can decompose the product of two sines or of two cosines:

$$\cos(\alpha)\cos(\beta) = \frac{1}{2}\left(\cos(\alpha+\beta) + \cos(\alpha-\beta)\right)$$
$$\sin(\alpha)\sin(\beta) = \frac{1}{2}\left(\cos(\alpha-\beta) - \cos(\alpha+\beta)\right)$$

3. We can express a sum of sinusoids **of the same frequency** (and any phase and amplitude) as a single sinusoid:

$$A\sin(\omega t+\phi) + B\cos(\omega t+\psi) = R\sin(\omega t+\alpha) \text{ or } S\cos(\omega t+\beta)$$

   where R and α (or S and β) are found by expanding the right-hand side with a compound angle formula and comparing coefficients of sin and of cos.

4. The sum of sinusoids **of the same frequency** is a sinusoid. The product of sinusoids, or the sum of sinusoids of different frequencies, is a more complicated waveform.