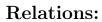
Ordered Pairs, Relations, and Functions

Objective: To understand how the ZF axioms allow us to define relations and functions, and work with these.

Ordered Pairs:

Recall that $\langle a, b \rangle$ is defined to be $\{\{a\}, \{a, b\}\}.$

- 1. Prove that $\langle a, b \rangle = \langle c, d \rangle$ if and only if a = c and b = d.
- 2. Let's define [a, b] to be $\{a, \{a, b\}\}$.
 - (a) Prove that [a, b] = [c, d] if and only if a = c and b = d.
 - (b) Recall that we define the natural numbers by $0 = \emptyset$ and $n + 1 = n \cup \{n\}$. What is the difference between 2 and [0, 0]?
- 3. Suggest a definition for the ordered triple $\langle a, b, c \rangle$. Prove that this satisfies the property you would expect of an ordered triple.
- 4. Prove that if A and B are sets, then $A \times B = \{\langle a, b \rangle \mid a \in A \land b \in B\}$ is a set.



Recall that a **binary relation** on a set A is a subset of $A \times A$.

Draw a picture of the relation \leq on $\{0, 1, 2, 3, 4\}$.

Assume $\mathbb R$ is a set with all the familiar properties. Draw a picture of the relation \sim on $\mathbb R$ defined by $x\sim y$ if and only if $x^2\leq y\wedge y<2x+4$.

Exercises:

- 1. Let R be a relation on A. Define what it means for R to be:
 - (a) Reflexive
 - (b) Symmetric
 - (c) Transitive
 - (d) Antisymmetric
 - (e) A partial order
 - (f) A total order (linear order)
 - (g) A well-order
- 2. Let (P, \leq) be a poset and define a new relation \sim on P by $a \sim b$ if and only if $b \leq a$. Prove that \sim is a partial order, and is linear if and only if \leq is linear.
- 3. Let A be a set and \equiv an equivalence relation (a symmetric, transitive, reflexive relation) on A. For each $a \in A$, let [a] be the equivalence class of A—the class of b in A such that $a \equiv b$. Prove that:
 - (a) For each $a \in A$, [a] is indeed a set.
 - (b) For any a and b in A, [a] = [b] or $[a] \cap [b] = \emptyset$, where $[a] \cap [b] = \{x \in A \mid x \in [a] \land x \in [b]\}$.
 - (c) Prove that A/\equiv , the collection of all equivalence classes of \equiv in A, is a set.
- 4. Let A be a set and \sim a relation on A which is transitive and reflexive (we call such a relation a pre-order). Prove that:
 - (a) The relation \equiv on A defined by $a \equiv b$ if and only if $a \sim b \wedge b \sim a$ is an equivalence relation.
 - (b) The relation \leq on A/\equiv defined by $[a] \leq [b]$ if and only if $a \sim b$ is well-defined and a partial order.

Functions:

Let A and B be sets. Give the definition of a **function** from A to B.

Let (A, \leq) and (B, \sim) be posets. Define what it means for $f: A \to B$ to be **order-preserving**.

Define what it means for a function $f:A\to B$ to be **injective**, surjective, and **bijective**.

Exercises:

- 1. Let (P, \leq) be a poset and Q a subset of P. Prove that the inclusion function $i: Q \to P: q \mapsto q$ is order-preserving with respect to the restricted order on Q.
- 2. Let $f:A\to B$ and $g:B\to C$ be functions. Prove that $g\circ f:A\to C$ is a function.
- 3. Let A and B be non-empty sets and $f: A \to B$ a function. Say that f is a **section** if there is a function $g: B \to A$ such that $g \circ f$ is the identity function on A, and say that f is a **retraction** if there is a function $h: B \to A$ such that $f \circ h$ is the identity function on B.
 - (a) Prove that if f is a section, then f is injective.
 - (b) Prove the converse: if f is injective, then f is a section.
 - (c) Prove that if f is a retraction, then f is surjective.
 - (d) Attempt to prove that if f is surjective, then f is a retraction. What goes wrong?