Exercise Sheet 4

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Exercise 1 (Büchi Automata):

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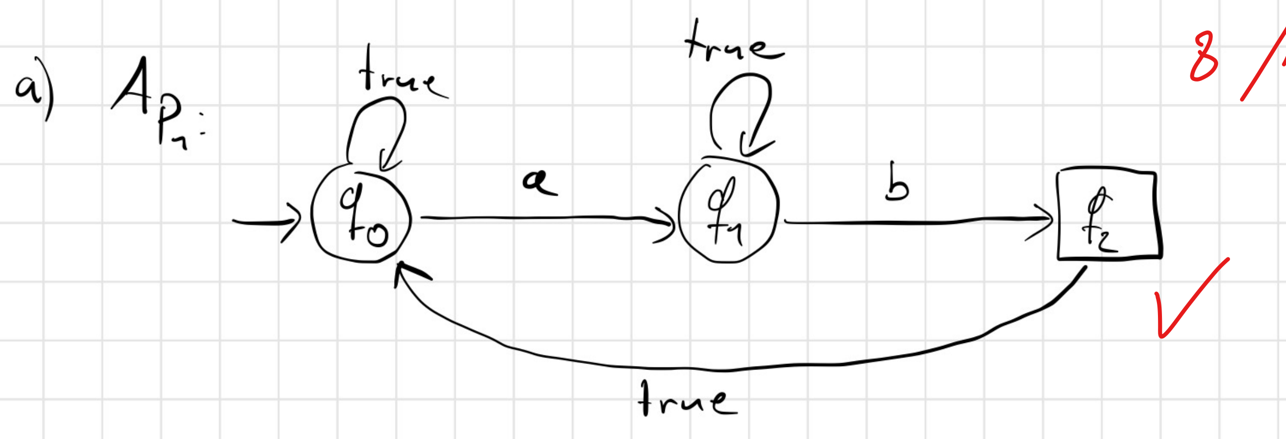
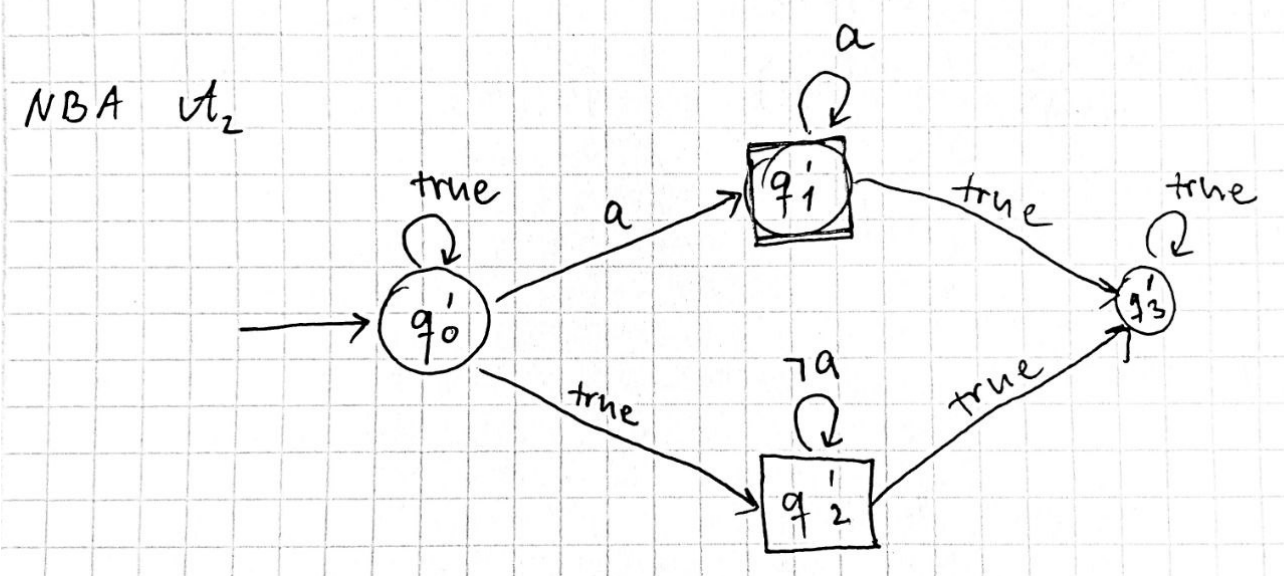


recognizes the complement of P_1.

a)



$\sigma = (\{a\}\emptyset)^\omega$ is not accepted but $\sigma \in P_1$

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b) $TS \otimes A_{P_1}$:

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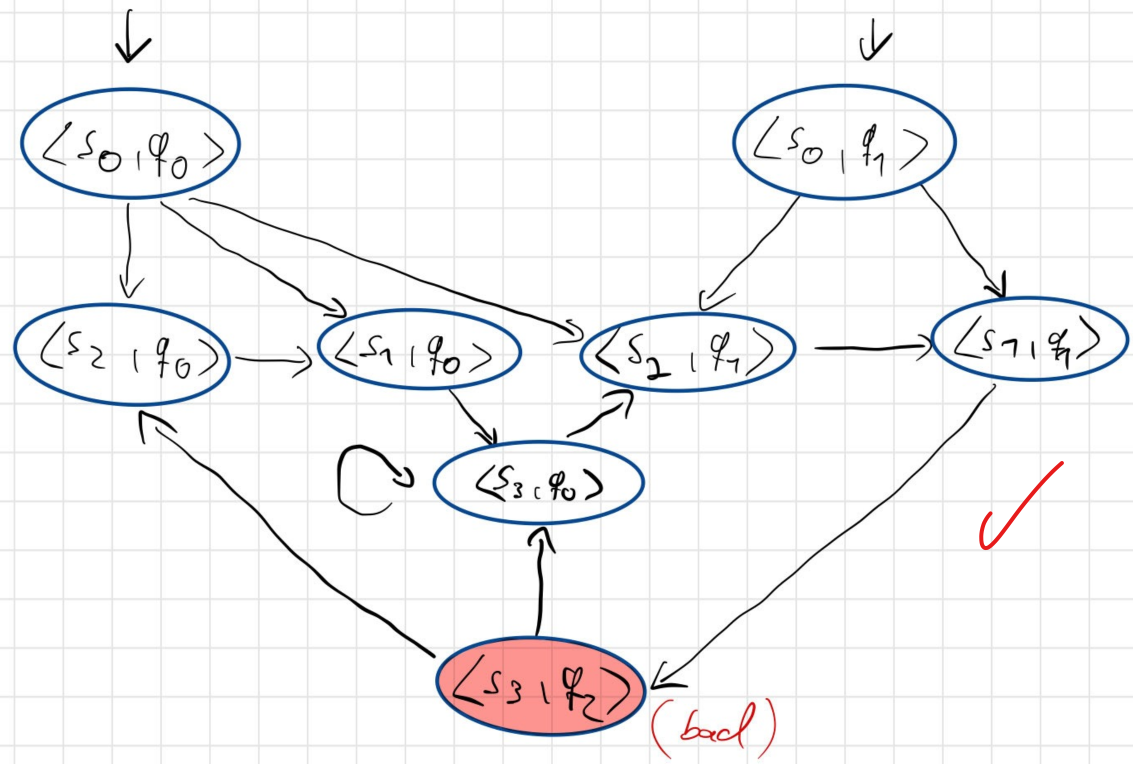
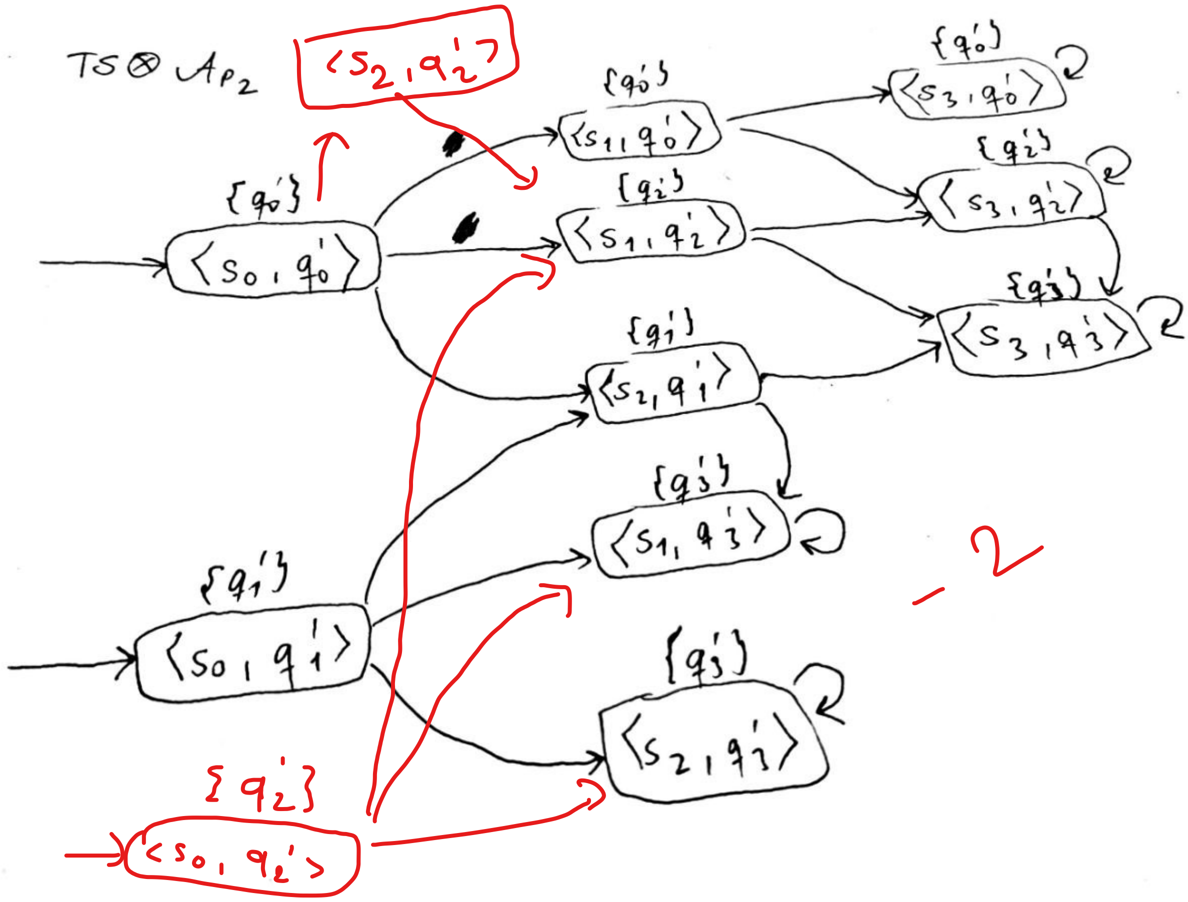


b)



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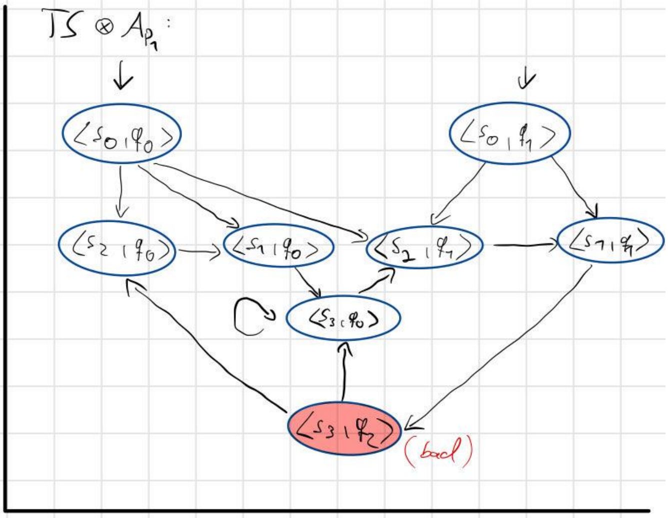
c) Check whether $T \otimes A_{\varphi_1} \models$ "eventually forever $\neg \varphi_2$"



Nested DFS:

Initially: $U := \emptyset$, $\pi := \emptyset$
$V := \emptyset$, $\xi := \emptyset$

Outer DFS: V and ξ remain unchanged

1. $U = \{\langle s_0, \varphi_0\rangle\}$, $\pi = \langle s_0, \varphi_0\rangle$
2. $U = \{\langle s_0, \varphi_0\rangle, \langle s_1, \varphi_0\rangle\}$, $\pi = \langle s_0, \varphi_0\rangle, \langle s_1, \varphi_0\rangle$
3. $U = \{\langle s_0, \varphi_0\rangle, \langle s_1, \varphi_0\rangle, \langle s_3, \varphi_0\rangle\}$
   $\pi = \langle s_0, \varphi_0\rangle, \langle s_1, \varphi_0\rangle, \langle s_3, \varphi_0\rangle$
4. $U = \{\langle s_0, \varphi_0\rangle, \langle s_1, \varphi_0\rangle, \langle s_3, \varphi_0\rangle, \langle s_2, \varphi_1\rangle\}$
   $\pi = \langle s_0, \varphi_0\rangle, \langle s_1, \varphi_0\rangle, \langle s_3, \varphi_0\rangle, \langle s_2, \varphi_1\rangle$
5. $U = \{\langle s_0, \varphi_0\rangle, \langle s_1, \varphi_0\rangle, \langle s_3, \varphi_0\rangle, \langle s_2, \varphi_1\rangle, \langle s_1, \varphi_1\rangle\}$
   $\pi = \langle s_0, \varphi_0\rangle, \langle s_1, \varphi_0\rangle, \langle s_3, \varphi_0\rangle, \langle s_2, \varphi_1\rangle, \langle s_1, \varphi_1\rangle$
6. $U = \{\langle s_0, \varphi_0\rangle, \langle s_1, \varphi_0\rangle, \langle s_3, \varphi_0\rangle, \langle s_2, \varphi_1\rangle, \langle s_1, \varphi_1\rangle, \langle s_3, \varphi_2\rangle\}$
   $\pi = \langle s_0, \varphi_0\rangle, \langle s_1, \varphi_0\rangle, \langle s_3, \varphi_0\rangle, \langle s_2, \varphi_1\rangle, \langle s_1, \varphi_1\rangle, \langle s_3, \varphi_2\rangle$
7. $U = \{\langle s_0, \varphi_0\rangle, \langle s_1, \varphi_0\rangle, \langle s_3, \varphi_0\rangle, \langle s_2, \varphi_1\rangle, \langle s_1, \varphi_1\rangle, \langle s_3, \varphi_2\rangle, \langle s_2, \varphi_0\rangle\}$
   $\pi = \langle s_0, \varphi_0\rangle, \langle s_1, \varphi_0\rangle, \langle s_3, \varphi_0\rangle, \langle s_2, \varphi_1\rangle, \langle s_1, \varphi_1\rangle, \langle s_3, \varphi_2\rangle, \langle s_2, \varphi_0\rangle$
8. $U = \{\langle s_0, \varphi_0\rangle, \langle s_1, \varphi_0\rangle, \langle s_3, \varphi_0\rangle, \langle s_2, \varphi_1\rangle, \langle s_1, \varphi_1\rangle, \langle s_3, \varphi_2\rangle, \langle s_2, \varphi_0\rangle\}$
   $\pi = \langle s_0, \varphi_0\rangle, \langle s_1, \varphi_0\rangle, \langle s_3, \varphi_0\rangle, \langle s_2, \varphi_1\rangle, \langle s_1, \varphi_1\rangle, \langle s_3, \varphi_2\rangle$
9. $U = \{\langle s_0, \varphi_0\rangle, \langle s_1, \varphi_0\rangle, \langle s_3, \varphi_0\rangle, \langle s_2, \varphi_1\rangle, \langle s_1, \varphi_1\rangle, \langle s_3, \varphi_2\rangle, \langle s_2, \varphi_0\rangle\}$
   $\pi = \langle s_0, \varphi_0\rangle, \langle s_1, \varphi_0\rangle, \langle s_3, \varphi_0\rangle, \langle s_2, \varphi_1\rangle, \langle s_1, \varphi_1\rangle$

inner DFS: U and π remain unchanged

Cycle_Check($\langle s_3, \varphi_2\rangle$, $V = \emptyset$, $\xi = \emptyset$):

1. $V = \{\langle s_3, \varphi_2\rangle\}$, $\xi = \langle s_3, \varphi_2\rangle$
2. $V = \{\langle s_3, \varphi_2\rangle, \langle s_2, \varphi_0\rangle\}$
   $\xi = \langle s_3, \varphi_2\rangle, \langle s_2, \varphi_0\rangle$
3. $V = \{\langle s_3, \varphi_2\rangle, \langle s_2, \varphi_0\rangle, \langle s_1, \varphi_0\rangle\}$
   $\xi = \langle s_3, \varphi_2\rangle, \langle s_2, \varphi_0\rangle, \langle s_1, \varphi_0\rangle$
4. $V = \{\langle s_3, \varphi_2\rangle, \langle s_2, \varphi_0\rangle, \langle s_1, \varphi_0\rangle, \langle s_3, \varphi_0\rangle\}$
   $\xi = \langle s_3, \varphi_2\rangle, \langle s_2, \varphi_0\rangle, \langle s_1, \varphi_0\rangle, \langle s_3, \varphi_0\rangle$

c)

5. $V = \{ \langle s_3, q_2 \rangle, \langle s_2, q_0 \rangle, \langle s_1, q_0 \rangle, \langle s_3, q_0 \rangle, \langle s_2, q_1 \rangle$
$\xi = \langle s_3, q_2 \rangle, \langle s_2, q_0 \rangle, \langle s_1, q_0 \rangle, \langle s_3, q_0 \rangle, \langle s_2, q_1 \rangle$
6. $V = \{ \langle s_3, q_2 \rangle, \langle s_2, q_0 \rangle, \langle s_1, q_0 \rangle, \langle s_3, q_0 \rangle, \langle s_2, q_1 \rangle, \langle s_1, q_1 \rangle$
$\xi = \langle s_3, q_2 \rangle, \langle s_2, q_0 \rangle, \langle s_1, q_0 \rangle, \langle s_3, q_0 \rangle, \langle s_2, q_1 \rangle, \langle s_1, q_1 \rangle$
7. $V = \{ \langle s_3, q_2 \rangle, \langle s_2, q_0 \rangle, \langle s_1, q_0 \rangle, \langle s_3, q_0 \rangle, \langle s_2, q_1 \rangle, \langle s_1, q_1 \rangle$
$\xi = \langle s_3, q_2 \rangle, \langle s_2, q_0 \rangle, \langle s_1, q_0 \rangle, \langle s_3, q_0 \rangle, \langle s_2, q_1 \rangle, \langle s_1, q_1 \rangle, \langle s_3, q_2 \rangle$
return "true"

return "no" + counterexample:

$\rightarrow$ the conterexample consists of the SCC:
$\langle s_3, q_2 \rangle, \langle s_2, q_0 \rangle, \langle s_1, q_0 \rangle, \langle s_3, q_0 \rangle, \langle s_2, q_1 \rangle, \langle s_1, q_1 \rangle, \langle s_3, q_2 \rangle$
which is reachable via the path:
$\langle s_0, q_0 \rangle, \langle s_1, q_0 \rangle, \langle s_3, q_0 \rangle, \langle s_2, q_1 \rangle, \langle s_1, q_1 \rangle$

d) 0/5

Exercise 2 (LTL Formulae):

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(i) $\Diamond$`winter`

(ii) $\Box$`awesome`

(iii) $\texttt{here} \wedge \Diamond(\neg\texttt{here} \wedge \Diamond(\texttt{here}))$

(iv) $(\texttt{live} \wedge \texttt{hero}) \wedge ((\texttt{live} \wedge \texttt{hero}) U (\texttt{hero} \oplus \texttt{live}))$

(v) $\Diamond\Box(\neg\texttt{in\_debt})$

(vi) `true`

(vii) First we define auxiliary formulas:

- $\varphi_l := \texttt{legen} \wedge \neg\texttt{wait\_for\_it} \wedge \neg\texttt{dary}$
- $\varphi_w := \neg\texttt{legen} \wedge \texttt{wait\_for\_it} \wedge \neg\texttt{dary}$
- $\varphi_d := \neg\texttt{legen} \wedge \neg\texttt{wait\_for\_it} \wedge \texttt{dary}$

And finally: $\varphi_7 := \varphi_l \wedge \bigcirc(\varphi_w \wedge \varphi_w U \varphi_d)$

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Aufgabe 3 (Equivalence of LTL Formulae)

(a) It holds that: $\Diamond\Box\varphi \Rightarrow \Box\Diamond\varphi$

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Beweis. Let $\sigma \models \Diamond\Box\varphi$ be a trace. Then we know that

$$\exists n. \forall k((k \geq n) \rightarrow \sigma[k..] \models \varphi) \tag{1}$$

Let $a \in \mathbb{N}$. We need to show that $\exists b \geq a. \sigma[b..] \models \varphi$

– **Case** $a \geq n$: chose $b := a + 1$. Because of 1 we know that $b > a \geq n \Rightarrow \sigma[b..] \models \varphi$

- **Case** $a < n$: chose $b := n$. We know that $n > a$ and because of 1 we can conclude that $\sigma[n..] \models \varphi$

To see that these formulas are not equivalent we can look at the trace

$$\sigma := (ab)^\omega \text{ and the formula } \varphi := a$$

Its easy to see that $\sigma \not\models \diamond\Box\varphi$ but $\sigma \models \Box \diamond \varphi$ ☐

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(b) It holds that: $\diamond\Box\varphi \wedge \diamond\Box\psi \;\Leftrightarrow\; \diamond(\Box\varphi \wedge \Box\psi)$

Beweis. Let σ be a trace such that $\sigma \models \diamond\Box\varphi \wedge \diamond\Box\psi$. This entails that we can find n_1, n_2 such that for all $k_1 \geq n_1$ and $k_2 \geq n_2$ it holds that $\sigma[k_1..] \models \varphi \wedge \sigma[k_2..] \models \cancel{\varphi}$ ψ

Now let $m := \max\{n_1, n_2\}$. We know that for all:

- $k_1 \geq m \Rightarrow k_1 \geq n_1 \Rightarrow \sigma[k_1..] \models \varphi$
- $k_2 \geq m \Rightarrow k_2 \geq n_2 \Rightarrow \sigma[k_2..] \models \psi$.

Thus $\sigma \models \diamond(\Box\varphi \wedge \Box\psi)$.

Its easy to see the other direction. Assume $\sigma \models \diamond(\Box\varphi \wedge \Box\psi)$. We know that

$$\exists n.(\forall(k_1 \geq n).\sigma[k_1..] \models \varphi \wedge \forall(k_2 \geq n).\sigma[k_2..] \models \psi)$$

which implies that

$$\exists n_1.(\forall(k_1 \geq n).\sigma[k_1..] \models \varphi) \wedge \exists n_2.(\forall(k_2 \geq n).\sigma[k_2..] \models \psi)$$

(simply choose $n_1 = n_2 = n$). ☐

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(c) It holds that: $\varphi \wedge \Box(\varphi \rightarrow \bigcirc \diamond \varphi) \;\Rightarrow\; \Box \diamond \varphi$

Beweis. Let σ be a trace such that

$$\sigma \models \varphi \wedge \Box(\varphi \rightarrow \bigcirc \diamond \varphi) \tag{2}$$

Let $M := \{n \in \mathbb{N} | \sigma[n..] \models \varphi\}$. Now we proof that M is infinite by showing that

- (IB) $0 \in M$: this follows directly from the first part of 2.
- (IS) $x \in M \Rightarrow \exists x' \in M.x' > x$: follows from the second part of 2 because for each position x where φ holds we know that there eventually is a position x' after the succesor of x (which is thus strictly greater than x) that also satisfies φ.

From the lecture we know that $\Box \diamond \varphi$ is equivalent to: $\exists^\infty x.(\sigma[x..] \models \varphi)$. Because M is infinite we know that σ satisfies this.

To show that the formulas are not equivalent we consider the trace:

$$\sigma := ab^\omega \text{ and the formula } \varphi := b$$

$\sigma[0] \not\models \varphi$ and thus $\sigma \not\models \varphi \wedge \Box(\varphi \rightarrow \bigcirc \diamond \varphi)$ however $\sigma \models \Box \diamond \varphi$. ☐

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(d) These two formulas are incomparable.

Beweis. Let

- $\beta_1 := (\varphi U \psi) U \pi$ and $\beta_2 := \varphi U (\psi U \pi)$
- $AP := \{a, b, c\}$
- $\varphi := a$, $\psi := b$ and $\pi := c$
- $\sigma_1 := \{a\}\{b\}\{a\}\{b\}\{c\}^\omega$ and $\sigma_2 := \{a\}\{c\}\emptyset^\omega$

It holds that:

- $\sigma_1 \models \beta_1$ and $\sigma_1 \not\models \beta_2$
- $\sigma_2 \models \beta_2$ and $\sigma_2 \not\models \beta_1$

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Aufgabe 4 (Positive Normal Form)

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a)

$$\begin{aligned}\theta &= (a \Rightarrow \neg \bigcirc b) \cup (a \wedge b) \\ &\equiv (\neg a \vee \neg \bigcirc b) \cup (a \wedge b) \\ &\equiv (\neg a \vee \bigcirc \neg b) \cup (a \wedge b)\end{aligned}$$

b)

$$\begin{aligned}\neg\theta &\equiv \neg((a \Rightarrow \neg \bigcirc b) \cup (a \wedge b)) \\ &\equiv \neg(\neg a \vee \neg \bigcirc b)\, R \,\neg(a \wedge b) \\ &\equiv (a \wedge \bigcirc b)\, R \,(\neg a \vee \neg b)\end{aligned}$$

c)

d)