Assignment 6: Fourier Transform

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Handed out: October 24, 2022
Due: 11:59pm, November 4, 2022
Handed in: 2:15am, November 4, 2022

Important Notes:

- Feel free discuss the homework with the instructor or the TAs.
- Handwritten solutions will not be accepted.
- Turn in a PDF report and .m/.py files through Canvas as a compressed (.zip) file; turn in a hardcopy of PDF printout in class)

Question 1: Fourier transform pairs

Prove the following Fourier and inverse Fourier transforms pairs, where $F\{.\}$ denotes the Fourier transform and $F^{-1}\{.\}$ depicts the inverse Fourier transform. You should use the equations below of the Fourier transform, which is continuous in both time/spatial and frequency domains. Note: no code is required, only written solutions.

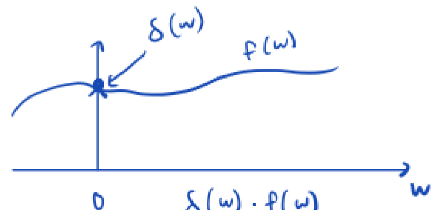
(a)

1. a)

$$\mathcal{F}^{-1}\{\delta(\omega)\} = \frac{1}{2\pi}$$

$$\mathcal{F}^{-1}\{\delta(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega)\, e^{j\omega x}\, d\omega \qquad \delta(\omega) = \begin{cases} \overset{\infty}{\cancel{1}}, & \omega = 0 \\ 0, & \omega \neq 0 \end{cases}$$

$$\int_{-\infty}^{\infty} \delta(\omega)\, d\omega = \boxed{1}$$



$\delta(\omega)\cdot f(\omega)$

only one common point: $\omega = 0$

$$\delta(\omega)\cdot f(\omega) = f(0)$$

$$\mathcal{F}^{-1}\{\delta(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega)\, e^{j\omega x}\, d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega)\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega) \overbrace{e^{j(0)x}}^{1}\, d\omega = \frac{1}{2\pi}(1) = \boxed{\frac{1}{2\pi}}$$

zero everywhere else.

(b)

b) $\mathcal{F}\{\delta(x)\} = 1$

$$\mathcal{F}\{\delta(x)\} = \int_{-\infty}^{\infty} \delta(x) e^{-j\omega x} dx \qquad \delta(\omega) = \begin{cases} 1, & \omega = 0 \\ 0, & \omega \neq 0 \end{cases}$$

$$\int_{-\infty}^{\infty} \delta(\omega)\, d\omega = \boxed{1}$$

$$= \int_{-\infty}^{\infty} \delta(x) \overbrace{e^{-j\omega(0)}}^{\to 1} dx$$

$$= \int_{-\infty}^{\infty} \delta(x)\, dx$$

$$= \boxed{1}$$

(c)

c) $\mathcal{F}\{\cos(x)\} = \delta(w-1) + \delta(w+1)$ $\quad \cos(x) = \dfrac{e^{ix} + e^{i(-x)}}{2}$

$\mathcal{F}\{\cos(x)\} = \int_{-\infty}^{\infty} \cos(x) e^{-jwx}\, dx$ $\quad = \dfrac{e^{jx} + e^{-jx}}{2}$

$$\mathcal{F}\{\cos(x)\} = \int_{-\infty}^{\infty} \frac{e^{jx} + e^{-jx}}{2} e^{-jwx}\, dx$$

$$= \int_{-\infty}^{\infty} \frac{e^{jx - jwx} + e^{-jx - jwx}}{2}\, dx$$

$$= \int_{-\infty}^{\infty} \frac{e^{jx(1-w)} + e^{jx(-1-w)}}{2}\, dx$$

$$= \int_{-\infty}^{\infty} \frac{e^{jx(1-w)}}{2}\, dx + \int_{-\infty}^{\infty} \frac{e^{jx(-1-w)}}{2}\, dx$$

$$= \frac{1}{2}\left(\int_{-\infty}^{\infty} e^{jx(1-w)}\, dx + \int_{-\infty}^{\infty} e^{jx(-1-w)}\, dx \right)$$

$$= \frac{1}{2}\left[\delta(1-w) + \delta(-1-w) \right]$$

$\delta(1-w) = \delta(w-1)$ $\quad \delta(-1-w) = \delta(w--1)$

$\quad = \delta(w+1)$

$$= \boxed{\frac{1}{2}\left[\delta(w-1) + \delta(w+1) \right]}$$

(d)

d)

$$\delta(w - w_0) = e^{i w_0 x}$$

$$\delta(w + w_0) = e^{-i w_0 x}$$

$$\mathcal{F}^{-1}\{\delta(w-2) + \delta(w+2)\} = \cos(2x)$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(e^{j2x} + e^{-j2x}\right) e^{jwx} \, dw$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{jwx + j2x} + e^{jwx - j2x} \, dw$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{jx(w+2)} + e^{jx(w-2)} \, dw$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{e^{jx(w+2)} + e^{jx(w-2)}}{2} \, dw$$

$$= \boxed{\frac{1}{\pi} \cos(2x)}$$

Question 2: Fourier transform properties

Using properties of the Fourier transform listed in Table 5.3 (and also given in the hand-out), find the Fourier transform of the functions below based on the Fourier transforms of the functions in Question #1. Indicate the properties you are using for each step. Note: no code is required, only written solutions.

(a)

a) $\mathcal{F}\{\sin(2x)\} = \frac{1}{2j}\left[\delta(w-2) - \delta(w+2)\right]$

(b)

b) $\mathcal{F}\{\delta(2x-10) - \cos(\frac{x}{7})\} = e^{j\frac{10x}{2}} - \frac{1}{2}\left[\delta(7w-1) + \delta(7w+1)\right]$

(c)

c) $\mathcal{F}\{\cos(x-\theta_0)\sin(4x-\theta_1)\} =$

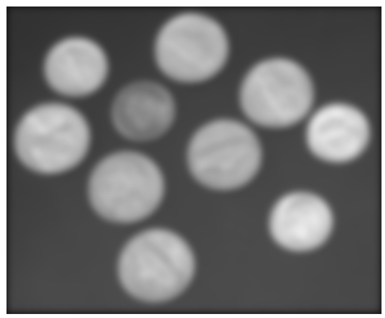
$\frac{1}{2}\left(e^{i\theta}\delta(w-1) + e^{-i\theta}\delta(w+1)\right) \cdot \frac{-i}{2}\left(e^{i\theta}(w-4) + e^{-i\theta}(w+4)\right)$

Question 3: Frequency space filtering

Construct 2D Gaussian filters of sigma = 3, 5, 15, 45 and size 256 x 256 (i.e., rows x columns) such that the Gaussian filter is in the middle of the 2D matrix. Perform the following steps for all filters using an image of your choice. Show the resulting images at each step in the report. You can use built-in functions for this question.

For this question, I'm only displaying the image with the filter with sigma = 3, but all the rest of the images can be found in the output_images folder.

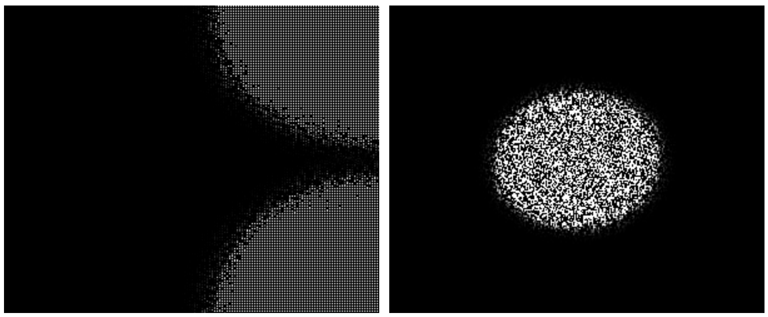
(1) Use 2D convolution to obtain the filtered image.



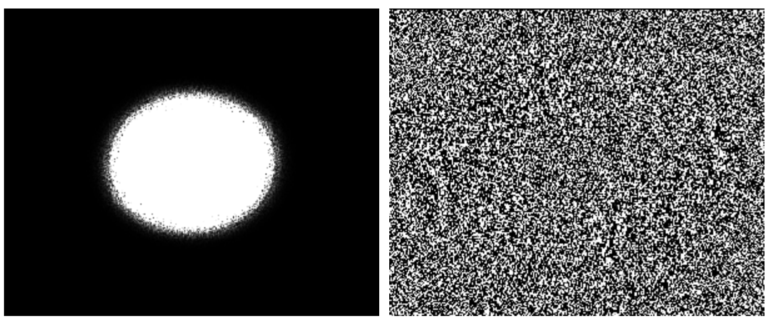
(a) Sigma = 3

(2) Using fft2 (in Python or Matlab), find the Fourier transform of the filter and an image. Center both the fft's using fftshift. Then multiply the two Fourier transforms (this is element wise multiplication not matrix multiplication). Fourier transform of the image is complex, there are several ways to display the FFT. You can show the real and imaginary part or magnitude and phase. Experiment and find which one is best suited for this analysis.

Post Fourier transformation:



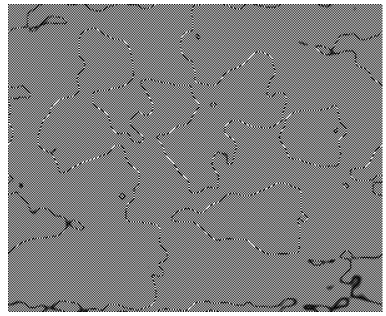
(a) Real (b) Imaginary



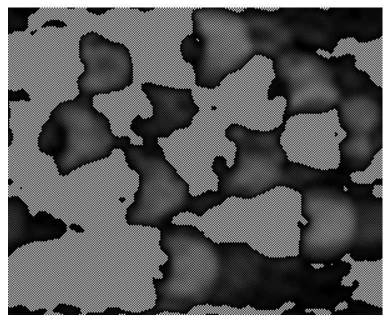
(a) Magnitude (b) Phase

The magnitude and phase part looks the most interesting and best suited for this analysis.

(3) Find the inverse Fourier transform (ifft2, available in both Matlab and Python) of the multiplied image. Inverse transform is also a complex image. Again experiment with real-imaginary or magnitude-phase to find which should be the output image.



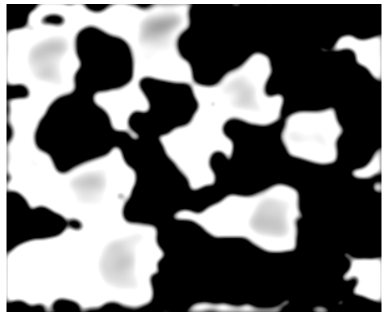
(a) Real (b) Imaginary



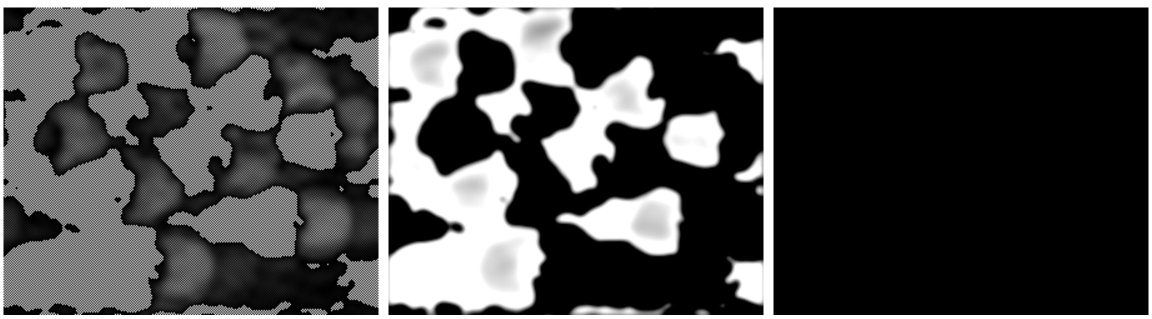
(a) Magnitude (b) Phase

Again, the magnitude and phase part looks the most interesting and best suited for this analysis

(4) Run a 2D convolution filter using the chosen Gaussian filter and obtain the output



(5) Find and display the difference between output from (3) and (4).



(a) Phase (b) Phase with Gauss (c) Abs diff

(a) How does the Fourier transform of filters change with changing in the filter's sigma?

- The image gets noticeably more blurred and details start to blend together into smoother curves.

(b) How do images obtained from step (1) and step (3) differ for different filter sizes?

- More blurred details in last stage. The areas of interest shrink as filter size increases (as seen in the real, imaginary, and magnitude parts).

(c) How do the images from part (3) and (4) differ? Look at the difference image from (5) and comment on the results.

- It seems to be inverted with all fine details from (3) being lost and blended in (4).

(d) Now remove the fftshift from the above pipeline and run the whole pipeline again? What do you observe? What difference did that make? Is fftshift required in the above pipeline?

- With the ffshift removed, the last stage of the image is completely unrecognizable with wave artifacts present after transformations and the areas of interest moving to the corners. The results are strange.

Question 4: Frequency space filtering

Pick 2 images* and do the following steps. Save images at each step.

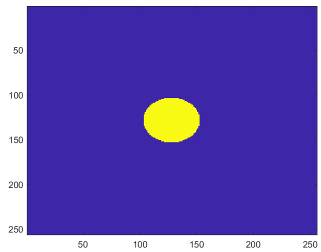
*I'm only showing one. The results for the other image is in the output_images folder.

(1) Find the Fourier transform of the image. Use fftshift to center the image's Fourier transform.

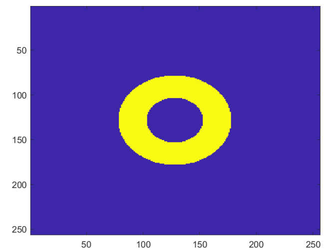
(a) cameraman fft

It looks like there's nothing there, but the fft of the image is just white.

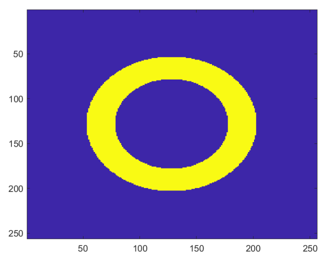
(2) Then declare a filter of size 256 x 256 as shown in the image below. Below the image is created with an inner radius of 10 pixels and outer radius of 20 pixel from the center. Create 4 filters with inner radius and outer radius as [0,25], [25,50], [50,75], [75,100].



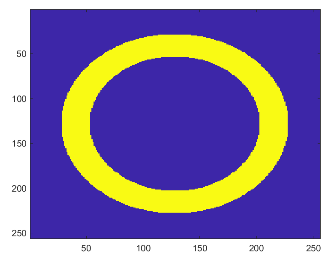
(a) filter[0 25]



(b) filter[25 50]

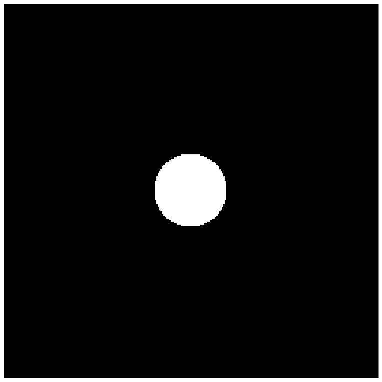


(a) filter[50 75]

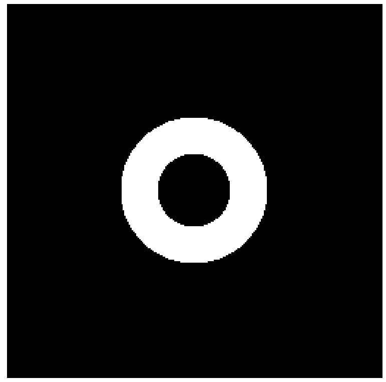


(b) filter[75 100]

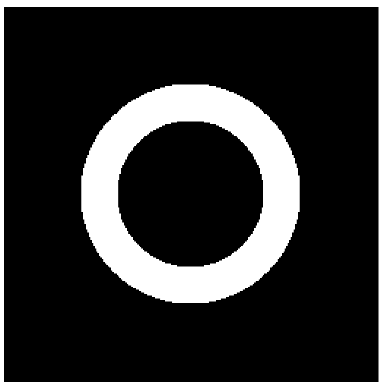
(3) Now multiply the filters above with the Fourier transform of the image obtained in step (1).



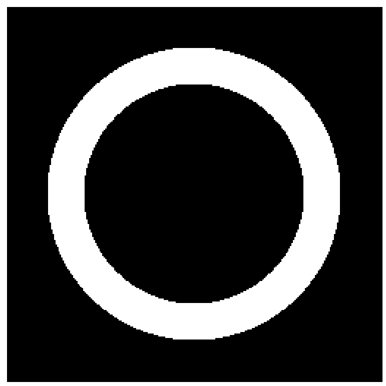
(a) filter[0 25]



(b) filter[25 50]

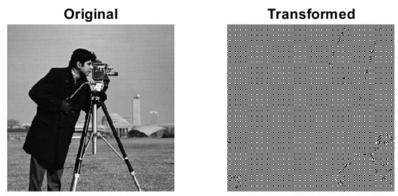


(a) filter[50 75]

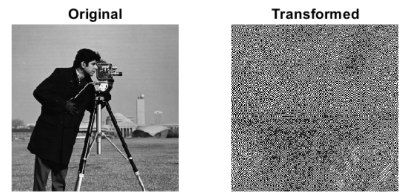


(b) filter[75 100]

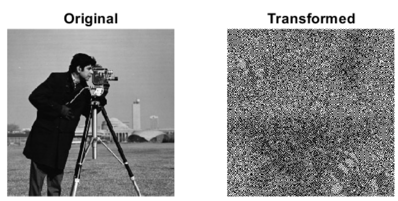
(4) Find the inverse Fourier transform of the result of part (3). Compare the output with your starting image.



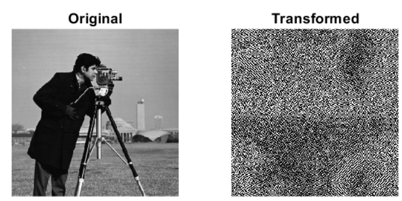
(a) filter[0 25]



(b) filter[25 50]

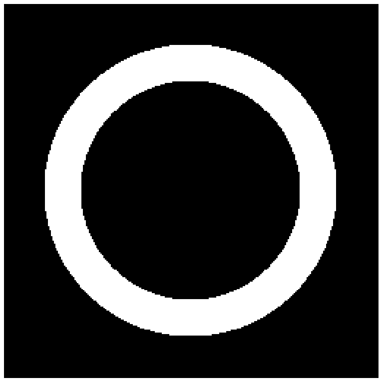


(a) filter[50 75]

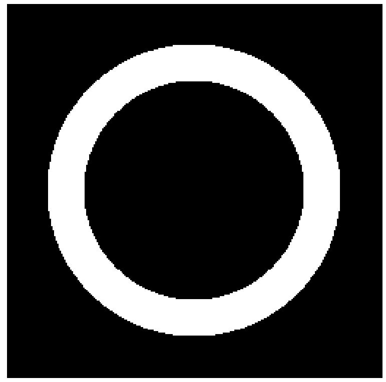


(b) filter[75 100]

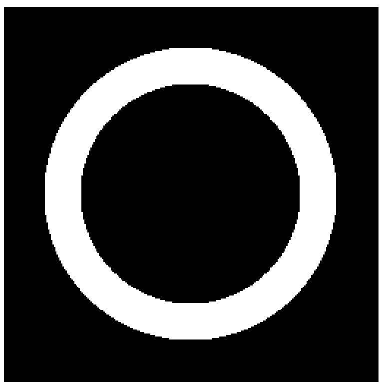
(5) Take the inverse Fourier transform of the filter. Display the magnitude of the filters.



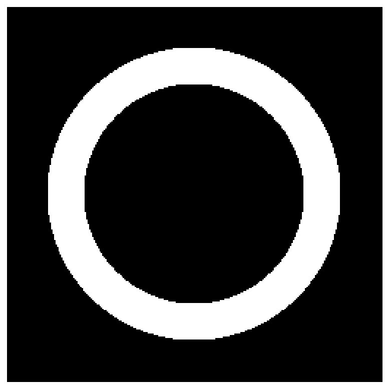
(a) filter[0 25]



(b) filter[25 50]



(a) filter[50 75]



(b) filter[75 100]

(a) What do you observe in the resulting images? Explain the differences in the outputs.

- Lots of static artifacts. I also might have done this wrong somehow.

(b) Compare output results with input and comment on the differences.

- Still looks like lots of static. Some kind of weird detail as the filter radius increases.

(c) From the outputs can you say something about the behavior of the 4 filters you created.

- Not really except that some weird detail becomes more distinguished as the filter radius increases (anvil looking shape becomes more prominent).

(d) Comment on the results obtained from step (5).

- The magnitudes of the different filters all seem to be the same shape for some reason. Maybe it has something to do with the circular shape?