2009 Fourth International Conference on Computer Sciences and Convergence Information Technology

A Proposal of a Quantum Search Algorithm

Keisuke Arima*, Noritaka Shigei*, Hiromi Miyajima*
*Graduate School of Science and Engineering,
Kagoshima University
Kagoshima, Japan
Email:k0483174@kadai.jp, {shigei, miya}@eee.kagoshima-u.ac.jp

***Abstract*—For searching an unsorted database with N items, a classical computer requires O(N) steps but Grover's quantum searching algorithm requires only O($\sqrt{N}$) steps. However, it is also known that Grover's algorithm is effective in the case where the initial amplitude distribution of dataset is uniform, but is not always effective in the non-uniform case. In this paper, we will consider the influence of initial amplitude distribution of dataset for Grover's and Ventura's algorithms. Further, in order to improve the performance of quantum searching, we propose a new algorithm and analyze its dynamics.**

***Keywords*-quantum search; Grover's algorithm; Ventura's algorithm; initial amplitude distribution;**

I. Introduction

With the ability of quantum computer, many studies have been made[1], [2]. Shor's and Grover's algorithms on factorization and data searching are well known. Further, Ventura has proposed quantum associative memory by improving Grover's algorithm[3], [5]. Data searching problem is to find desired data effectively from unsorted dataset. For searching an unsorted database with N items, a classical computer requires O(N) steps but Grover's quantum searching algorithm requires only O($\sqrt{N}$) steps [2], [4]. However, it is also known that Grover's algorithm is effective in the case where the initial amplitude distribution of dataset is uniform, but is not always effective in the non-uniform case[3], [6]. Therefore, it is needed to find effective algorithms even in the case where the initial amplitude distribution of dataset is not uniform.

In this paper, we will consider the influence of initial amplitude distribution of dataset for Grover's and Ventura's algorithms. Further, in order to improve the performance of quantum searching, we propose a new algorithm and analyze its dynamics.

II. Preliminaries

The basic unit in quantum computation is a qubit $c_0|0\rangle + c_1|1\rangle$, which is a superposition of two independent states $|0\rangle$ and $|1\rangle$ corresponding to the states 0 and 1, where c_0 and c_1 are complex numbers such that $|c_0|^2 + |c_1|^2 = 1$. We use the Dirac bracket notation, where the ket $|i\rangle$ is analogous to a column vector. Let n be a positive integer and $N = 2^n$. A system with n qubits is described using N independent state $|i\rangle (0 \leq i \leq N-1)$ as follows: $\sum_{i=0}^{N-1} c_i|i\rangle$, where c_i is a complex number, $\sum_{i=0}^{N-1} |c_i|^2 = 1$ and $|c_i|^2$ is the probability of state $|i\rangle$. The direction of c_i on the complex plane is called the phase of state $|i\rangle$ and the absolute value $|c_i|$ is called the amplitude of state $|i\rangle$. In quantum system, starting from any quantum state, the desired state is formed by multiplying column vector of the quantum state by unitary matrix. Finally, we can obtain the desired state with high probability through observation[2]. The problem is how we can find unitary matrix. Grover has proposed the fast data searching algorithm. Let explain the Grover's algorithm shown in Fig.1. Grover has proposed an algorithm for finding one item in an unsorted database. In the conventional computation, if there are N items in the database, it would require $O(N)$ queries to the database. However, Grover has shown how to perform this using the quantum computation with only $O(\sqrt{N})$ queries[2]. Let $Z_N = \{0, 1, \cdots, N-1\}$. Define the following operators:

$$\begin{aligned} I_a &= \text{identity matrix except for} \\ &I(a+1, a+1) = -1, \ a \in Z_N, \end{aligned} \tag{1}$$

which inverts any state $|\psi\rangle$ and

$$\begin{aligned} W(x,y) &= \frac{1}{\sqrt{N}}(-1)^{x_0y_0+\cdots+x_{N-1}y_{N-1}} \\ &\text{for } x = \sum_{i=0}^{N-1} x_i 2^i, \ y = \sum_{i=0}^{N-1} y_i 2^i, \end{aligned} \tag{2}$$

which is called the Walsh or Hadamard transform and performs a special case of discrete Fourier transform. We begin with the $|\bar{0}\rangle$ state and apply W operator, where $|\bar{0}\rangle$ means that all states are 0 and the number of 0's for $\bar{0}$ is N. As a result, all the states have the same amplitude $1/\sqrt{N}$. Next, we apply the I_τ operator, where $|\tau\rangle$ is the searching state. Further, we apply the operator

$$G = -WI_0W. \tag{3}$$

Followed by the I_τ operator $T = (\pi/4)\sqrt{N}$ times and observe the system[2]. G operator has been described as inverting all the state's amplitudes around the average one of all states.

978-0-7695-3896-9/09 $26.00 © 2009 IEEE
DOI 10.1109/ICCIT.2009.126

IEEE computer society

1. Initial state $|\bar{0}\rangle$
2. $|\psi\rangle = W|\bar{0}\rangle = |\bar{1}\rangle$
3. Repeat T times
4. $\quad|\psi\rangle = I_\tau|\psi\rangle$
5. $\quad|\psi\rangle = G|\psi\rangle$
6. Observe the system

Figure 1. Grover's algorithm.

1. $|\psi\rangle = I_\tau|\psi\rangle$
2. $|\psi\rangle = G|\psi\rangle$
3. $|\psi\rangle = I_\rho|\psi\rangle$
4. $|\psi\rangle = G|\psi\rangle$
5. Repeat $\frac{\pi}{4}\sqrt{N}-2$ times
6. $\quad|\psi\rangle = I_\tau|\psi\rangle$
7. $\quad|\psi\rangle = G|\psi\rangle$
8. Observe the system

Figure 2. Ventura's algorithm.

Example 1 [6]:

Let $n = 4$ and $N = 16$. Let searching data $|\tau\rangle = |6\rangle = |0110\rangle$ and the number stored data $m = 16$. Let us apply Grover's algorithm shown in Fig.1.

At step 2,

$$|\psi\rangle = \frac{1}{4}(1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1)^t,$$

where t shows the transposition. Finally, $|\psi\rangle$ is obtained as the following pattern

$$\begin{aligned}|\psi\rangle \quad &= \quad \frac{1}{256}(-13,-13,-13,-13,-13,-13,251,-13,\\ &\quad -13,-13,-13,-13,-13,-13,-13,-13)^t,\end{aligned}$$

where T is defined as the Eq.(4) and Eq.(5) [4].

$$T = \frac{\frac{1}{2}\pi - \arctan\left[\frac{\bar{k}(0)}{\bar{l}(0)}\sqrt{\frac{r}{N-r}}\right]}{\arccos\left[\frac{N-2r}{N}\right]} \tag{4}$$

$$\frac{\bar{k}(0)}{\bar{l}(0)} = \frac{N-1}{m-1} \tag{5}$$

Then, the desired pattern 0110 is obtained with the probability 0.96. We can get the searching data with high probability.

Next, Supposing that stored data are $|0\rangle$, $|3\rangle$, $|6\rangle$, $|9\rangle$, $|12\rangle$ and $|15\rangle$, and searching data is $|6\rangle$. The initial state $|\psi\rangle$ is as follows:

$$|\psi\rangle_i = \begin{cases} 1 & \text{for any } i \text{ of stored data} \\ 0 & \text{otherwise,} \end{cases} \tag{6}$$

At step 2 in Fig.1, supposing that

$$|\psi\rangle = \frac{1}{\sqrt{6}}(1,0,0,1,0,0,1,0,0,1,0,0,1,0,0,1)^t.$$

Continuing the same process as above, we can obtain

$$\begin{aligned}|\psi\rangle \quad &= \quad \frac{1}{8\sqrt{6}}(5,-3,-3,5,-3,-3,13,\\ &\quad -3,-3,5,-3,-3,5,-3,-3,5)^t.\end{aligned}$$

In this case, the desired pattern 0110 is obtained with the probability 0.44. It shows that Grover's algorithm does not always give a good result in the case where the initial amplitude distribution of dataset is not uniform.

On the other hand, Ventura has proposed the generalized algorithm[3], [5]. Let us explain Ventura's algorithm as shown in Fig.2. The differences between Grover's and Ventura's algorithms are some steps, 1, 2, 3 and 4, before the repeating steps and initial state $|\psi\rangle$. Let us define the following operator.

$$\begin{aligned} I_\rho \quad = \quad & \text{identity matrix except for} \\ & I(\rho+1,\rho+1) = -1 \text{ for any } \rho \\ & \text{that is the number for stored data.} \end{aligned} \tag{7}$$

Ventura's idea is to try to make the amplitudes of all the states except for searching data uniform before repeating steps.

Example 2 [6]:

Let us apply Ventura's algorithm to the latter in Example 1.

$$I_\tau|\psi\rangle = \frac{1}{\sqrt{6}}(1,0,0,1,0,0,-1,0,0,1,0,0,1,0,0,1)^t$$

Finally, we obtain

$$\begin{aligned}|\psi\rangle \quad &= \quad \frac{1}{16\sqrt{6}}(-1,-1,-1,-1,-1,-1,39,-1,\\ &\quad -1,-1,-1,-1,-1,-1,-1,-1)^t,\end{aligned}$$

where T is obtained by Eq.(4) and Eq.(8).

$$\frac{\bar{k}(0)}{\bar{l}(0)} = \frac{[8(m-2)(N-m)+N^2](N-1)}{4(m-2)(N-m)(N-2)-N^2(m-1)} \tag{8}$$

In this case, the desired pattern 0110 is obtained with the probability 0.99. We can get the searching data with high probability.

Next, supposing that searching data is $|6\rangle$ and stored data are as follows:

$$|\psi\rangle = \frac{1}{16}(1,1,1,1,0,1,1,1,0,1,1,1,0,1,1,0)^t.$$

After applying I_τ, G, I_ρ and G, the following state is obtained:

$$|\psi\rangle = \frac{1}{8\sqrt{3}}(1,1,1,1,-5,1,9,1,-5,1,1,1,-5,1,1,-5)^t.$$

The desired pattern $|6\rangle$ is obtained with the probability 0.441. It shows that Ventura's algorithm does not always give a good result in any case.

III. The proposed algorithm and its dynamics

Grover's algorithm is effective in the case where initial amplitude distribution of dataset is uniform($N = m$). However, it is not always effective in the case where initial amplitude distribution of dataset is not uniform($N \neq m$) as shown in Example 1. Hence, Ventura has proposed an algorithm as generalized Grover's one and insisted that it is effective in all cases[3], [5]. However, As shown in Example 2, it is not always true. Then, let us consider the reason why it happens.

First, let us review the dynamics of Grover's and Ventura's algorithms. Let the amplitude of desired data at a time step t of Grover's algorithm be denoted by $\alpha(t)$, the amplitude of stored data by $\beta_i(t)$ and the average amplitude of stored data by $\bar{\beta}(t)$ as follows:

$$\bar{\beta}(t) = \frac{1}{N-1}\sum_{j=2}^{N}\beta_j(t). \tag{9}$$

Then, Biron has shown the following result[4]:

$$\begin{aligned}
\begin{bmatrix} \alpha(t) \\ \bar{\beta}(t) \end{bmatrix} &= \begin{bmatrix} a\,\sin(\omega t+\phi) \\ b\,\cos(\omega t+\phi) \end{bmatrix} && (10)\\
a &= \sqrt{\alpha(0)^2+\bar{\beta}(0)^2(N-1)} && (11)\\
b &= \sqrt{\bar{\beta}(0)^2+\alpha(0)^2\frac{1}{N-1}} && (12)\\
\omega &= \arccos\left(\frac{N-2}{N}\right) && (13)\\
\phi &= \arctan\left(\frac{\alpha(0)}{\bar{\beta}(0)}\sqrt{\frac{1}{N-1}}\right) && (14)\\
\alpha(0) &= \frac{1}{\sqrt{m}} && (15)\\
\bar{\beta}(0) &= \frac{1}{\sqrt{m}}\frac{m-1}{N-1}, && (16)
\end{aligned}$$

where $\alpha(0)$ and $\bar{\beta}(0)$ mean the average amplitude of $\alpha(t)$ and $\beta_i(t)$ before repetition step, respectively. As a is the function of m, let us compute the maximum amplitude of the function $a(m)$, for $0 < m \leq N$. Let us differentiate the function $a(m)$ with respect to m. As a result, the function $a(m)$ has an absolute minimum at $m = \sqrt{N}$. Therefore, the function is decreasing monotonically for $0 < m \leq \sqrt{N}$ and is increasing monotonically for $\sqrt{N} < m \leq N$. Grover's algorithm has the maximal amplitude at $m = N$. Likewise, let us consider the Ventura's algorithm. In this case, the equations of its dynamics are the same as Grover's one except for the Eq.(15) and (16). The Eq.(15) and (16) are substituted for the Eq.(17) and (18):

$$\begin{aligned}
\alpha(0) &= \frac{8(m-2)(N-m)+N^2}{N^2\sqrt{m}} && (17)\\
\bar{\beta}(0) &= \frac{4(m-2)(N-m)(N-2)-N^2(m-1)}{(N-1)N^2\sqrt{m}} && (18)
\end{aligned}$$

1. Initial state $|\psi\rangle$
2. Repeat T times
3. $|\psi\rangle = I_\tau|\psi\rangle$
4. $|\psi\rangle = G|\psi\rangle$
5. $|\psi\rangle = I_\rho|\psi\rangle$
6. $|\psi\rangle = G|\psi\rangle$
7. Observe the system

Figure 3. A proposed algorithm

As a result, the function $a(m)$ has two extreme values. The first value is the maximum at $m = \frac{N}{4}+2$ and the second value is the minimum near $m = \frac{3}{4}N$. Therefore, the function $a(m)$ is increasing monotonically for $0 < m \leq \frac{N}{4}+2$, is decreasing monotonically for $\frac{N}{4}+2 \leq m < m_*$ and is increasing monotonically for $m_* \leq m \leq N$, where $m_* \approx \frac{3}{4}N$. Ventura's algorithm has the maximum amplitudes at $m = \frac{N}{4}+2$ and $m = N$.

Fig. 4 shows the relation between Grover's and Ventura's algorithms for the number m of stored data at $N = 2^{10}$, that is, initial amplitude distribution for each data is $\frac{1}{\sqrt{m}}$ for m data and 0 otherwise. It shows that, Grover's and Ventura's algorithms are effective in the case where m is large ($m = N$) or small ($m \approx \frac{N}{4}$), respectively. In order to cover the another part, we propose a new algorithm as shown in Fig. 3. Let us show an example for the proposed algorithm.

Example 3:

Let $n = 4$, $N = 16$. Let searching data $|\tau\rangle = |0110\rangle$ and the number of stored data $m = 8$.

$$|\psi\rangle = \frac{1}{2\sqrt{2}}(1,0,1,0,1,0,1,0,1,0,1,0,1,0,1,0)^t$$

The following result is obtained after applying I_τ, G, I_ρ, G:

$$\begin{aligned}
|\psi\rangle \;=\; & \frac{1}{8\sqrt{2}}(-1,0,-1,0,-1,0,11,0,-1,0,-1,0,\\
& -1,0,-1,0)^t
\end{aligned}$$

The desired pattern $|6\rangle$ is obtained with the probability $(\frac{11}{8\sqrt{2}})^2 \approx 0.95$.

Next, in order to find the repeating number T which gives the maximum amplitude of the desired data, let us analyze the dynamics of the proposed algorithm by using the same method as Biron[4].

A. The recursion equation of the proposed algorithm

Let the amplitude of desired(searching) data at time t be denoted by $\alpha(t)$, the amplitude of stored data by $\beta_i(t)$, ($i = 2, 3, \cdots, m$), and the amplitude of other data by $\gamma_j(t)$, ($j = m+1, \cdots, N$).

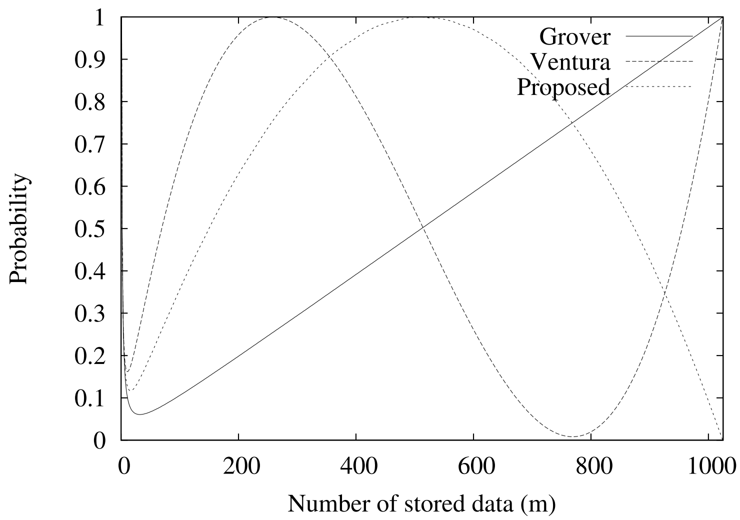

Figure 4. The relation between the observed probability of searching data and the number of stored data for $N = 1024$.

We denote the averages of the amplitudes by

$$\bar{\alpha}(t) = \alpha(t), \tag{19}$$

$$\bar{\beta}(t) = \frac{1}{m-1}\sum_{i=2}^{m}\beta_i(t), \tag{20}$$

$$\bar{\gamma}(t) = \frac{1}{N-m}\sum_{j=m+1}^{N}\gamma_j(t). \tag{21}$$

Let $C(t)$ be the weighted average over states

$$\begin{aligned} C(t) &= \frac{1}{N}\left[\sum_{j=m+1}^{N}\gamma_j(t) + \sum_{i=2}^{m}\beta_i(t) - \alpha(t)\right] \\ &= \frac{1}{N}\left[(N-m)\bar{\gamma}(t) + (m-1)\bar{\beta}(t) - \bar{\alpha}(t)\right]. \end{aligned} \tag{22}$$

After the step 6, the following relation holds:

$$\bar{\alpha}(t+1) = 2C(t') + \bar{\alpha}(t') \tag{23}$$

$$\bar{\beta}(t+1) = 2C(t') + \bar{\beta}(t') \tag{24}$$

$$\bar{\gamma}(t+1) = 2C(t') - \bar{\gamma}(t') \tag{25}$$

B. Solution of the recursion equation

From the Eq.(23), (24) and (25), the Eq.(26) is obtained. Formally, let

$$\mathbf{v}(t) = \begin{bmatrix} \bar{\alpha}(t) \\ \bar{\beta}(t) \\ \bar{\gamma}(t) \end{bmatrix},$$

$$\mathbf{A} = \begin{bmatrix} \frac{N^2-8N+8m}{N^2} & \frac{8(m-1)(N-m)}{N^2} & \frac{4(N-m)(N-2m)}{N^2} \\ -\frac{8(N-m)}{N^2} & \frac{8(m-1)(N-m)-N^2}{N^2} & \frac{4(N-m)(N-2m)}{N^2} \\ -\frac{4(N-2m)}{N^2} & \frac{4(m-1)(N-2m)}{N^2} & \frac{-8m(N-m)+N^2}{N^2} \end{bmatrix}. \tag{27}$$

Then, the following relation holds for the Eq.(26):

$$\mathbf{v}(t+1) = \mathbf{A}\mathbf{v}(t) \tag{28}$$

By transforming the matrix A into the diagonal matrix using singular matrix S, the following relation holds:

$$\mathbf{A}^D = \mathbf{S}^{-1}\mathbf{A}\mathbf{S} \tag{29}$$

Then, let $\mathbf{w}(t) = \mathbf{S}^{-1}\mathbf{v}(t)$, we can get the Eq.(30).

$$\begin{aligned} \mathbf{S}\mathbf{w}(t+1) &= \mathbf{A}\mathbf{S}\mathbf{w}(t) \\ \mathbf{S}^{-1}\mathbf{S}\mathbf{w}(t+1) &= \mathbf{S}^{-1}\mathbf{A}\mathbf{S}\mathbf{w}(t) \\ \mathbf{w}(t+1) &= \mathbf{A}^D\mathbf{w}(t) \end{aligned} \tag{30}$$

By computing the eigenvalue of A, the diagonal matrix A^D is obtained as follow:

$$\mathbf{A}^D = \begin{bmatrix} \frac{16m-16N+2N^2-jX}{2N^2} & 0 & 0 \\ 0 & \frac{16m-16N+2N^2+jX}{2N^2} & 0 \\ 0 & 0 & -1 \end{bmatrix} \tag{31}$$

that is

$$\mathbf{A}^D = \begin{bmatrix} \delta e^{-j\omega} & 0 & 0 \\ 0 & \delta e^{j\omega} & 0 \\ 0 & 0 & e^{j\pi} \end{bmatrix}. \tag{32}$$

From the Fig.5, we can get the Eq.(33) and (34).

$$\begin{bmatrix} \bar{\alpha}(t+1) \\ \bar{\beta}(t+1) \\ \bar{\gamma}(t+1) \end{bmatrix} = \begin{bmatrix} \frac{N^2-8N+8m}{N^2} & \frac{8(m-1)(N-m)}{N^2} & \frac{4(N-m)(N-2m)}{N^2} \\ -\frac{8(N-m)}{N^2} & \frac{8(m-1)(N-m)-N^2}{N^2} & \frac{4(N-m)(N-2m)}{N^2} \\ -\frac{4(N-2m)}{N^2} & \frac{4(m-1)(N-2m)}{N^2} & \frac{-8m(N-m)+N^2}{N^2} \end{bmatrix} \begin{bmatrix} \bar{\alpha}(t) \\ \bar{\beta}(t) \\ \bar{\gamma}(t) \end{bmatrix} \tag{26}$$

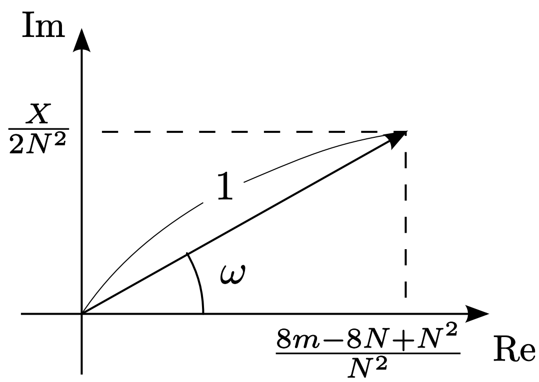

Figure 5. Complex plane of expression (31)

$$\omega = \arccos\Big(\frac{8m-8N+N^2}{N^2}\Big) \tag{33}$$

$$\delta = 1. \tag{34}$$

Let

$$\mathbf{w}(0) = [w_1(0), w_2(0), w_3(0)]^t. \tag{35}$$

From the Eq.(30) and (32), we can get the Eq.(36).

$$\begin{aligned} \mathbf{w}(1) &= \begin{bmatrix} e^{-j\omega} & 0 & 0 \\ 0 & e^{j\omega} & 0 \\ 0 & 0 & e^{j\pi} \end{bmatrix} \mathbf{w}(0) = \begin{bmatrix} e^{-j\omega} w_1(0) \\ e^{j\omega} w_2(0) \\ e^{j\pi} w_3(0) \end{bmatrix} \\ \mathbf{w}(2) &= \begin{bmatrix} e^{-j\omega} & 0 & 0 \\ 0 & e^{j\omega} & 0 \\ 0 & 0 & e^{j\pi} \end{bmatrix} \mathbf{w}(1) = \begin{bmatrix} e^{-j2\omega} w_1(0) \\ e^{j2\omega} w_2(0) \\ e^{j2\pi} w_3(0) \end{bmatrix} \\ &\vdots \\ \mathbf{w}(t) &= \begin{bmatrix} e^{-jt\omega} w_1(0) \\ e^{jt\omega} w_2(0) \\ e^{jt\pi} w_3(0) \end{bmatrix} \end{aligned} \tag{36}$$

From the relation between $\mathbf{w}(0)$ and $\mathbf{v}(0)$, we can get the Eq.(37).

$$\mathbf{w}(0) = \begin{bmatrix} -j\frac{8(N-m)}{X}\bar{\alpha}(0) + \Big\{\frac{1}{2} - \frac{(N-2m)^2}{2Y}\Big\}\bar{\beta}(0) \\ j\frac{8(N-m)}{X}\bar{\alpha}(0) + \Big\{\frac{1}{2} - \frac{(N-2m)^2}{2Y}\Big\}\bar{\beta}(0) \\ -\frac{2(m-1)(N-2m)}{Y}\bar{\beta}(0) \end{bmatrix}, \tag{37}$$

where

$$X = 8\sqrt{(N-m)(N^2-4N+4m)} \tag{38}$$

$$Y = N^2 - 4N + 4m. \tag{39}$$

From the equation $\mathbf{v}(t) = \mathbf{S}\mathbf{w}(t)$, the following relation holds:

$$\mathbf{v}(t) = \mathbf{S} \begin{bmatrix} \Big(-j\frac{8(N-m)}{X}\bar{\alpha}(0) + \Big\{\frac{1}{2} - \frac{(N-2m)^2}{2Y}\Big\}\bar{\beta}(0)\Big)e^{-j\omega t} \\ \Big(j\frac{8(N-m)}{X}\bar{\alpha}(0) + \Big\{\frac{1}{2} - \frac{(N-2m)^2}{2Y}\Big\}\bar{\beta}(0)\Big)e^{j\omega t} \\ -\frac{2(m-1)(N-2m)}{Y}\bar{\beta}(0)e^{j\pi t} \end{bmatrix} \tag{40}$$

As a result, the Eq.(41) hold.

The Eq.(41) is represented as the Eq.(42):

$$\begin{bmatrix} \bar{\alpha}(t) \\ \bar{\beta}(t) \\ \bar{\gamma}(t) \end{bmatrix} = \begin{bmatrix} a\ \sin(\omega t + \phi) \\ b\ \cos(\omega t + \phi) + K \\ c\ \cos(\omega t + \phi) + L \end{bmatrix} \tag{42}$$

where

$$\tan\phi = \frac{\bar{\alpha}(0)}{\bar{\beta}(0)} \frac{16(N-m)Y}{X\{Y-(N-2m)^2\}}, \tag{43}$$

$$a^2 = \bar{\alpha}(0)^2 + \Big[\frac{X}{8(N-m)}\Big\{\frac{1}{2} - \frac{(N-2m)^2}{2Y}\Big\}\Big]^2 \bar{\beta}(0)^2, \tag{44}$$

$$b^2 = \Big\{1 - \frac{(N-2m)^2}{Y}\Big\}^2 \bar{\beta}(0)^2 + \Big\{\frac{16(N-m)}{X}\Big\}^2 \bar{\alpha}(0)^2, \tag{45}$$

$$c^2 = \Big[\frac{N-2m}{2(N-m)}\Big\{1 - \frac{(N-2m)^2}{Y}\Big\}\Big]^2 \bar{\beta}(0)^2 + \Big\{\frac{8(N-2m)}{X}\Big\}^2 \bar{\alpha}(0)^2, \tag{46}$$

$$K = \frac{(N-2m)^2}{Y}\bar{\beta}(0), \tag{47}$$

$$L = -\frac{2(m-1)(N-2m)}{Y}\bar{\beta}(0). \tag{48}$$

From the Eq.(42), the case of $\cos(\omega t + \phi) = 0$ is not always maximal one for the Eq.(42), because $\bar{\beta}(t)$ is not minimum amplitude. But, let us consider the case where K and L are sufficiently small. Then, $\cos(\omega t + \phi) = 0$ holds:
that is,

$$\omega t + \phi = (i + 1/2)\pi \quad (i = 0, 1, 2, ...). \tag{49}$$

As $i = 0$, the minimal repeating number T for $\bar{\alpha}(t)$ is as following:

$$\begin{aligned} T &= \frac{(1/2)\pi - \phi}{\omega} \\ &= \frac{(1/2)\pi - \arctan\Big[\frac{\bar{\alpha}(0)}{\bar{\beta}(0)} \frac{\sqrt{N^2-4N+4m}}{2(m-1)\sqrt{N-m}}\Big]}{\arccos\Big(\frac{8m-8N+N^2}{N^2}\Big)} \end{aligned} \tag{50}$$

$$
\begin{bmatrix} \bar{\alpha}(t) \\ \bar{\beta}(t) \\ \bar{\gamma}(t) \end{bmatrix} = \begin{bmatrix} \bar{\alpha}(0)\cos\omega t + \frac{X}{8(N-m)}\left\{\frac{1}{2} - \frac{(N-2m)^2}{2Y}\right\}\bar{\beta}(0)\sin\omega t \\ \left\{1 - \frac{(N-2m)^2}{Y}\right\}\bar{\beta}(0)\cos\omega t - \frac{16(N-m)}{X}\bar{\alpha}(0)\sin\omega t + \frac{(N-2m)^2}{Y}\bar{\beta}(0) \\ \frac{N-2m}{2(N-m)}\left\{1 - \frac{(N-2m)^2}{Y}\right\}\bar{\beta}(0)\cos\omega t - \frac{8(N-2m)}{X}\bar{\alpha}(0)\sin\omega t - \frac{2(m-1)(N-2m)}{Y}\bar{\beta}(0) \end{bmatrix}, \tag{41}
$$

Table I
THE NUMBER OF STORED DATA WITH THE MAXIMUM AMPLITUDE FOR N

n	N	$m = \frac{N}{2}$ amplitude	m_s amplitude
4	16	8	9
		0.945313	0.970276
5	32	16	18
		0.961319	0.977245
6	64	32	32
		0.999182	0.999182
7	128	64	66
		0.996586	0.997949
8	256	128	124
		0.995620	0.997199
9	512	256	255
		0.999947	0.999967
10	1024	512	506
		0.999448	0.999654

$$
\frac{\bar{\alpha}(0)}{\bar{\beta}(0)} = 1 \tag{51}
$$

Thus, we can get approximately the repeating number T which gives the maximum amplitude $\bar{\alpha}(t)$ of the desired data. Let us approximate the Eq.(50) when N and m are large numbers.

$$
\begin{aligned}
T &= \frac{\pi}{8}\sqrt{\frac{N^2}{N-m} - \frac{\bar{\alpha}(0)}{\bar{\beta}(0)}\frac{N^2}{8m(N-m)}} \\
&\quad + \left(\frac{\bar{\alpha}(0)}{\bar{\beta}(0)}\right)^3 \frac{N^4}{96m^3(N-m)^2} \\
&\cong \frac{\pi}{8}\sqrt{2N} - \frac{\bar{\alpha}(0)}{\bar{\beta}(0)}\frac{1}{2} + \left(\frac{\bar{\alpha}(0)}{\bar{\beta}(0)}\right)^3 \frac{1}{3N}
\end{aligned} \tag{52}
$$

The result shows that the number of steps is $O(\sqrt{N})$ as the same as the Grover's and Ventura's algorithms. Lastly, we will get the maximal value of $\bar{\alpha}(t)$ of the proposed algorithm as Grover's and Ventura's ones. With the Eq.(42) and (44), a is differentiated by m. Then, we can get the maximum at $m = \frac{N}{2}$ for $0 < m \leq N$.

C. Numerical simulation

Let us compute the number m_s which gives the maximum amplitude by using T of the Eq.(50). For each N, let us compute the maximal repeating number T for m, $0 < m \leq N$. Then, we can get the number m_s with the maximum amplitude of $\bar{\alpha}(t)$. Table Ishows the result of m_s for N. In order to compare m_s and $m = \frac{N}{2}$, the result of $m = \frac{N}{2}$ is also shown in Table I.

The difference of the amplitude between them causes from K and L, but it is small as N increases. Likewise, Fig.4 shows the maximal amplitude of $\bar{\alpha}(t)$ for each m at $N = 1024$. It shows that the proposed algorithm has the maximal amplitude at the half of N.

IV. CONCLUSIONS AND FUTURE WORK

In this paper, it is shown that Grover's and Ventura's algorithms are effective in the case where the initial amplitude distribution of dataset is uniform but is not always effective in the non-uniform case. Further, in order to improve the performance of quantum searching, we have proposed a new algorithm and analyzed its dynamics. It seems that parallel processing of three algorithms is effective in the almost all cases. Our future works are to show the exact repeating number of the proposed algorithm and propose more effective algorithms and analyze them.

REFERENCES

[1] P. Shor, "Polynomial-Time Algorithms for Prime Factorization and Discrete Logarithms on a Quantum Computer", SIAM Journal of Computing, vol.26, no.5, pp.1484-1509, 1997.

[2] L. Grover, "A Fast Quantum Mechanical Algorithm for Database Search, Proc. of 28th ACM Symp. on Theory of Computing, pp.212-219, 1996.

[3] D. Ventura and T. Martinez, "Quantum Associative Memory", Information Science, vol.124, pp.273-296, 2000.

[4] D. Biron, O. Biham, E. Biham, M. Grassl and D.A. Lidar, "Generalized Grover Search Algorithm for Arbitrary Initial Amplitude Distribution", Lecture Notes In Computer Science, Vol.1509, pp.140-147, 1998.

[5] D. Ventura and T. Martinez, "Initializing the Amplitude Distribution of A Quantum State", Foundations of Physics Letters, Vol.12, No.6, pp.547-559, 1999.

[6] K. Arima, H. Miyajima, N.Shigei and M. Maeda, "Some Properties of Quantum Data Search Algorithms", The 23rd Int. Technical Conf. on Circuits/Systems, Computers and Communications, no.P1-42, pp.1169-1172, 2008.