Two Improvements in Grover's Algorithm

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Abstract: Grover search algorithm is applied to the current number of objectives in quantum unsorted database, the meaning of each goal difference is not taken into account. When the target is more than a quarter of the total items, the probability of the target falls quickly identify and target increases, and when the target is more than half of the total items, the algorithm will fail. To solve these problems, first of all, based on a weighted index improvement is proposed in which each target in accordance with the right given to the importance of the weight coefficient. These different weight coefficients, we give all the target state, which may make the probability of each target equal weights of the quantum superposition. Secondly, an improved phase matching is proposed the phase rotation in both directions is the same, and the inner product of two states by the target amplitude superposition and initial state of the system is determined. When this inner product is 1/2 or more, when using the improved phase matching, the target may be 100% probability and by only one Grover iteration. Finally, the effectiveness of these measures by a simple search instance validation.

Key Words: Grover algorithm, Targets weighting, Phase matching, Quantum searching, Quantum computing

1 INTRODUCTION

Quantum computing is a rapidly rising leading edge wherein the information processing by quantum mechanics. In the past 30 years, studies have shown that quantum computing is much better than the classical counterpoint. In solving actual problems by quantum parallelism, it has already made a number of quantum algorithms, quantum Grover search algorithm [1] and a very large integer factorization [2] Shor quantum algorithm is the most prominent. Now, the improvement and development of algorithms Grove is a problem in the field of quantum computing popularity and has received many consequents. In Ref. [3], Long et al. first discuss first phase matching and clearing of a misunderstanding of the mark state, along with any phase rotation operation on average inversion Grover's search algorithm can be used to construct a low quantum efficiency of the search algorithm. Since then, they noted that it is impossible to replace two arbitrary phase rotation with the second phase reversal, unless satisfy certain phase matching conditions [4]. In Ref.[5], Long et al. found that systematic errors and random errors in the phase transformation Hadmard-Walsh transformations the state and led to marked reduction in the maximum probability of which affect the efficiency of the algorithm to construct a new approach to ensure the success of a half rate. In Ref. [6], a picture of the generalized Grover's quantum search algorithm, an arbitrary unitary transform and an arbitrary phase rotation, an image is presented, in which the failure to match the requirements of the algorithm and phase mismatch is clearly shown. In the more general case, the phase matching condition is the quantum search concerns not only with the search engine, but also in the initial state. There are two problems in current Grover quantum search algorithm. Firstly, in the search target are treated equally without discrimination, the significance of each goal difference is not taken into account, and found that the probability of each target are equal, it is inappropriate for the situation. Secondly, the lower scores marked increase when the project is greater than 1/4, then the probability of success dramatically marked with the project, and when the score is greater than 1/2 of the marked items, the algorithm will fail. To solve these problems, we propose two improvements. Firstly, each target is given the right to all the weight coefficients and the target state is represented as a quantum superposition, it can make each equal to the probability of the target obtained weight coefficient. Secondly, an improved phase-matching algorithm. Improved use of phase matching, when the inner product and the initial state of the system is superimposed target state is 1/2 or more, the target may have a complete success probability and find Grover iteration by only one.

2 THE GROVER ALGORITHM AND ITS PROBLEMS

2.1 The general Grover algorithm

Assuming that there are N elements in the database. For convenience, we assume that $N = 2^n$, and the search problem has M solutions, with $1 \le M \le N$. The algorithm starts at the state $|0>^{\otimes n}$. The Walsh-Hadamard transform is used to put the state $|0>^{\otimes n}$ in the uniform superposition of states,

$$|\phi> = \frac{1}{\sqrt{N}}(|0> + \cdots + |q_1> + \cdots + |q_M> + \cdots + |N-1>) \quad (1)$$

where $|q_1>, \cdots, |q_M>$ are the marked states. The Grover algorithm then consists of repeated application of a quantum subroutine known as the Grover iteration or Grover operator, which we denote G . Grover iteration can be decomposed into four steps:

(1) The marked state inverse. $|\phi_1> = I_q|\phi>$

$$I_q = I - 2\sum_{m=1}^{M}|q_m><q_m| \quad (2)$$

(2) Applying the Hadamard-Walsh transform. $|\phi_2 = W|\phi_1$

(3) Perform a conditional phase shift. $|\phi_3> = I_n|\phi_2>$

$$I_n = 2|0><0| - I \tag{3}$$

(4) Applying the Hadamard-Walsh transform. $|\phi_4>=W|\phi_3>$.

Let $\lambda = M/N$, and $CI(x)$ denote the integer closest to the real number x, where we round halves down by convention. Then repeat the Grover iteration

$$R = CI\left(\frac{\arccos(\sqrt{\lambda})}{2\arcsin(\sqrt{\lambda})}\right) \tag{4}$$

times rotates $|\phi>$ to within an angle $\arcsin\sqrt{\lambda} \le \pi/4$ of a superposition of marked states. Observation of the state in the computational basis then yields a solution to the search problem with probability at least one–half.

2.2 The successful probability of Grover algorithm

In fact, Grover iteration can be considered by the starting rotation vector and the solution to the problem of uniformity across the superposition state of two-dimensional search space. So that the sum represented by the normalized state in all of this is not to solve the problem of searching, and represents the state of the normalized sum of all x in which the search is to solve the problem. Simple algebra shows that the initial state can be re-expressed as

$$|\phi>=\cos(t)|\alpha>+\sin(t)|\beta> \tag{5}$$

where $t = \arcsin\sqrt{\lambda}$. After R steps of Grover iteration, the initial state is taken to

$$G^R|\phi>=\cos((2R+1)\arcsin\sqrt{\lambda})|\alpha>+ \sin((2R+1)\arcsin\sqrt{\lambda})|\beta> \tag{6}$$

Hence, the successful searching probability is

$$P = \sin^2((2R+1)\arcsin\sqrt{\lambda}) \tag{7}$$

2.3 The problems of the general Grover algorithm

Firstly, it can be seen from Eq.(7), for each individual target significance difference is not taken into account and the probability of acquiring each tag item is equal, which is not suitable for some cases. Secondly, from Eq.(4) and Eq.(7), when it is easy to find that decreases rapidly when $\lambda > 0.25$, $R = 1$, algorithm will fail. Therefore, the general Grover's algorithm is no longer useful when $\lambda > 0.25$.

3 THE IMPROVEMENT BASED ON THE WEIGHTED TARGETS

3.1 The quantum superposition of all target states

Assuming $|q_1>,|q_2>,\cdots,|q_M>$ are all searched target states, $\Omega=\{q_1, q_2, \cdots, q_M\}$, the real $w_{q_1}, w_{q_2}, \cdots, w_{q_M}$ are a group of real weight coefficient indicating that satisfy $w_i > 0$, $\sum_{i\in\Omega} w_i = 1$, and represent the degree of significance of each target. All quantum superposition state of the search target can be constructed as follows

$$|q>=\sum_{i=1}^{N-1} b_i|i>=\begin{cases}\sum\sqrt{w_i}\,|i> & i\in\Omega\\ 0|i> & i\notin\Omega\end{cases} \tag{8}$$

3.2 The construction of iterative operators

Based on Eq.(8), the tagged state inverse operator may be constructed as follows

$$I_q = I - 2|q><q| \tag{9}$$

The conditional phase-shifting operator can be constructed as follows

$$I_0 = I - 2|0><0| \tag{10}$$

Take the following search engine

$$G = -WI_0WI_q \tag{11}$$

3.3 The iterative equation of algorithm

Assuming for some iteration t the state of the system is described by the superposition $|\phi>^{(t)}=\sum_{i=0}^{N-1} a_i^{(t)}|i>$, after the transformation I_q (first sub-step of an iteration), the superposition becomes

$$|\phi>^{(t+1/2)}=\sum_{i=0}^{N-1}(a_i^{(t)} - 2<q|\phi>^{(t)}\, b_i)|i> \tag{12}$$

After $-WI_0W$ (second sub-step of an iterative), the superposition becomes

$$|\phi>^{(t+1)}=\sum_{i=0}^{N-1}(2<\phi^{(t+1/2)}>-a_i^{(t+1/2)})|i>$$
$$=\sum_{i=0}^{N-1}(2<a_i^{(t)}-2<q|\phi>^{(t)}\,b_i>-a_i^{(t)}+2<q|\phi>^{(t)}\,b_i)|i> \tag{13}$$

From Eq.(8) and Eq.(13),

$$<q|\phi>^{(t+1)}+<q|\phi>^{(t-1)}=2<q|\phi>^{(t)}\,(1-2N<q>^2) \tag{14}$$

To get the expression of $<\phi|q>^{(t)}$, suppose

$$<\phi|q>^{(t)}=A\sin(\omega t+\varphi) \tag{15}$$

Inserting this expression into Eq.(14) and using some trigonometric functions, three parameters can be obtained in the expression

$$A = 1 \tag{16}$$

$$\omega = 2\arcsin(<q|\phi>) \tag{17}$$

$$\varphi = \arcsin(<q|\phi>) \tag{18}$$

3.4 The successful probability of algorithm

By Eq.(16), Eq.(15) can be represented as follows

$$<q|\phi>^{(t)}=\sin(\omega t+\varphi) \tag{19}$$

Therefore, it is necessary to study the relationship between the probability of success and the inner product. For this relationship, we have to make such a theorem is as follows.

Theorem1 By t steps of Grover iteration, the probability of finding all the targets to meet this relationship is as follows

$$P^{(t)} \ge (<\phi|q>^{(t)})^2 \tag{20}$$

Proof $P^{(t)} = \sum_{i\in\Omega}(a_i^{(t)})^2 = \left(\sum_{i\in\Omega}(a_i^{(t)})^2\right)\left(\sum_{i\in\Omega} b_i^2\right)$

$$\geq\left(\sum_{i\in\Omega}a_i^{(t)}b_i\right)^2=\left(\sum_{i=0}^{N-1}a_i^{(t)}b_i\right)^2=(<\phi\,|\,q>^{(t)})^2$$

where $\Omega=\{q_1,q_2,\cdots,q_M\}$. (Proof end)

Therefore, it is sufficient that the probability represented by the square of the inner product.

Noting that when $t=(\pi/2-\varphi)/\omega$, $(<\phi\,|\,q>^{(t)})^2=1$, and t is usually not an integer, then the following steps to get iteration

$$t_0=CI\left(\frac{\pi/2-\varphi}{\omega}\right)=CI\left(\frac{\arccos(<q\,|\,\phi>)}{2\arcsin(<q\,|\,\phi>)}\right)$$

Let $\lambda=<q\,|\,\phi>^2$, $(0<\lambda<1)$, then,

$$t_0=CI\left(\frac{\arccos(\sqrt{\lambda})}{2\arcsin(\sqrt{\lambda})}\right) \tag{21}$$

where the meaning of the $CI(x)$ is same as Eq.(4).

3.5 The Improvement Based on Phase Matching

Two phase rotation operators defined by Eq.(9) and Eq.(10) can be summarized as follow

$$I_q=I-(1-e^{i\alpha})\,|\,q><q\,| \tag{22}$$

$$I_0=I-(1-e^{i\beta})\,|\,0><0\,| \tag{23}$$

For the matching of α and β, we propose an adaptive phase matching described by Theorem 2.

Theorem 2 Let $\lambda=<q\,|\,\phi>^2$, when $1/4\leq\lambda\leq1$ and $\alpha=\beta=f(\lambda)=\arccos(1-1/2\lambda)$, The goals can be found by probability $P=1$ after the only one Grover iteration.

Proof $|\,\phi>^{(1/2)}=I_q\,|\,\phi>=[I-(1-e^{i\alpha})\,|\,q><q\,|]\,|\,\phi>$

$$=|\,\phi>-(1-e^{i\alpha})<q\,|\,\phi>|\,q>$$

$$|\,\phi>^{(1)}=-W\,I_0W\,|\,\phi>^{(1/2)}$$

$$=[(1-e^{i\beta})\,|\,\phi><\phi\,|-I\,]\,|\,\phi>^{(1/2)}$$

$$=-e^{i\beta}\,|\,\phi>-(1-e^{i\alpha})(1-e^{i\beta})<q\,|\,\phi>^2|\,\phi>$$

$$+(1-e^{i\alpha})<q\,|\,\phi>|\,q>$$

$$<q\,|\,\phi>^{(1)}=-(1-e^{i\alpha})(1-e^{i\beta})<q\,|\,\phi>^3$$

$$+(1-e^{i\alpha}-e^{i\beta})<q\,|\,\phi>$$

Let $\alpha=\beta$, Using $\lambda=<q\,|\,\phi>^2$ produces

$$|<q\,|\,\phi>^{(1)}|^2=(4\lambda^3-4\lambda^2)\cos^2\alpha$$

$$-(8\lambda^3-12\lambda^2+4\lambda)\cos\alpha+(4\lambda^3-8\lambda^2+5\lambda)$$

When $\cos\alpha=1-1/2\lambda$, $\alpha=\beta=\arccos(1-1/2\lambda)$, $|<q\,|\,\phi>^{(1)}|^2$ obtains the maximum as follows

$$|<q\,|\,\phi>^{(1)}|^2_{\max}=4\lambda^3-8\lambda^2+5\lambda-\frac{(8\lambda^3-12\lambda^2+4\lambda)^2}{4(4\lambda^3-4\lambda^2)}=1$$

By theorem1, the probability of finding the targets $P\max=\left|<q\,|\,\phi>^1\right|^2=1$. From $-1\leq\cos\alpha\leq1$, the range $1/4\leq\lambda\leq1$ is obtained. When $0<\lambda<1/4$, the original phase matching $\alpha=\beta=\pi$ may be used. (proof end)

4 SEARCHING EXAMPLE

There are 32 students in a class whose sequence number is from 0-31. The searched targets are the students whose sequence number satisfies $n=[(5k+3)/3]$, where $k=0,1,\cdots,18$; $[\,x\,]$ denote the integer closest to the real number x, where by convention we rounded down half. The target serial numbers and marked states are shown in Table 1.In present example, $N=32$, $M=19$, using 5 qubits can store all sequence numbers. As $M/N=19/32>0.5$, the general Grover algorithm is invalidated. With two improvements proposed in this paper, the targets can be used to find the probability of 100% and by the only one Grover iteration.

Table1. Target serial numbers and marked states

k	Serial number	Marked state	k	Serial number	Marked state
0	1	|00001>	10	18	|10010>
1	3	|00011>	11	19	|10011>
2	4	|00100>	12	21	|10101>
3	6	|00110>	13	23	|10111>
4	8	|01000>	14	24	|11000>
5	9	|01001>	15	26	|11010>
6	11	|01011>	16	28	|11100>
7	13	|01101>	17	29	|11101>
8	14	|01110>	18	31	|11111>
9	16	|10000>			

Assuming that all of the target state is a quantum superposition.

$$|\,q>=\sqrt{\frac{10}{19\times6}}\left(|1>+|3>+|4>+|6>+|8>+|9>\right)$$

$$+\sqrt{\frac{6}{19\times7}}\left(|11>+|13>+|14>+|16>+|18>+|19>+|21>\right)$$

$$+\sqrt{\frac{3}{19\times6}}\left(|23>+|24>+|26>+|28>+|29>+|31>\right)$$

Now, $\lambda=(<q\,|\,\phi>)^2=\left(\frac{3\sqrt{10}}{4\sqrt{57}}+\frac{7\sqrt{6}}{4\sqrt{266}}+\frac{3\sqrt{3}}{4\sqrt{57}}\right)^2$,

$\alpha=\beta=\arccos(1-1/2\lambda)=\arccos(0.108809699256823)$.

Search process can be described as follows

$\hat{}\quad |\,\hat{\phi}>=I_q\,|\,\phi>=(I-(1-e^{i\alpha})\,|\,q><q\,|)\,|\,\phi>$

$$=\frac{1}{\sqrt{32}}\sum_{j_0=1}^{13}|q_{j_0}>+\left(\frac{1}{\sqrt{32}}-\frac{\sqrt{10}}{\sqrt{19\times 6}}<q|\phi>(1-e^{i\alpha})\right)\sum_{j_1=1}^{6}|q_{j_1}>$$
$$+\left(\frac{1}{\sqrt{32}}-\frac{\sqrt{6}}{\sqrt{19\times 7}}<q|\phi>(1-e^{i\alpha})\right)\sum_{j_2=1}^{7}|q_{j_2}>$$
$$+\left(\frac{1}{\sqrt{32}}-\frac{\sqrt{3}}{\sqrt{19\times 6}}<q|\phi>(1-e^{i\alpha})\right)\sum_{j_3=1}^{6}|q_{j_3}>$$

ˆ $|\widetilde{\phi}>=-WI_0W|\hat{\phi}>=((1-e^{i\beta})|\phi><\phi|-I)|\hat{\phi}>$
$=(abab\ baba\ bbac\ acca\ cacc\ acad\ dada\ ddad)$

where $a=0$;

$$b=\frac{1}{\sqrt{32}}+\left(\frac{\sqrt{10}}{\sqrt{19\times 6}}-\frac{<q|\phi>}{2\sqrt{2}}\right)<q|\phi>(1-\cos\alpha)$$
$$+i\frac{\sqrt{10}}{\sqrt{19\times 6}}<q|\phi>\sin\alpha;$$
$$c=\frac{1}{\sqrt{32}}+\left(\frac{\sqrt{6}}{\sqrt{19\times 7}}-\frac{<q|\phi>}{2\sqrt{2}}\right)<q|\phi>(1-\cos\alpha)$$
$$+i\frac{\sqrt{6}}{\sqrt{19\times 7}}<q|\phi>\sin\alpha;$$
$$d=\frac{1}{\sqrt{32}}+\left(\frac{\sqrt{3}}{\sqrt{19\times 6}}-\frac{<q|\phi>}{2\sqrt{2}}\right)<q|\phi>(1-\cos\alpha)$$
$$+i\frac{\sqrt{3}}{\sqrt{19\times 6}}<q|\phi>\sin\alpha.$$

ˆ The probability of finding all targets is calculated as follows

The marked states: $(|1>+|3>+|4>+|6>+|8>+|9>)$,

$P_1=6|b|^2=0.526315789473684$;

The marked states: $(|11>+|13>+|14>+|16>+|18>+|19>+|21>)$,

$P_2=7|c|^2=0.3157894736\ 84211$;

The marked states: $(|23>+|24>+|26>+|28>+|29>+|31>)$,

$P_3=6|d|^2=0.1578947368\ 42105$.

The probability of finding all targets is

$P=P_1+P_2+P_3=1$.

5 CONCLUSION

In order to solve the current differences in Grover algorithm ignores all targets sense, we found that the probability of each target and the target to increase the problem of falling, firstly, based on the weighted target Grover algorithms, which makes the probability of each target is equal to its weight coefficient. Secondly, an improved weighted Grover algorithm to obtain the phase matching condition. The inner product of these improvements, when the target state and the initial state of the system overlay is 1/2 or more, in two issues of general Grover algorithm to solve.

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