

Chapter 5

Sobolev and interpolation inequalities

5.1 Interpolation inequalities

Example 5.1.1

$$\|u\|_{L^2}^2 \leq \|u\|_{L^2} \|u'\|_{L^2} \text{ for } u \in C^\infty(\mathbb{R}) \quad (5.1)$$

Proof. Idea: use that $(u^2)' = 2uu'$ and Newton-Leibniz

$$\begin{aligned} u^2(x) &= 2 \int_{-\infty}^x uu' \, dy = -2 \int_x^\infty uu' \, dy \\ &= \int_{-\infty}^x uu' \, dy - \int_x^\infty uu' \, dy \\ &\leq \int_{-\infty}^x |u||u'| \, dy + \int_x^\infty |u||u'| \, dy \\ &= \int_{\mathbb{R}} |u||u'| \, dy \\ (\text{Hölder's inequality}) &\leq \|u\|_{L^2} \|u'\|_{L^2} \end{aligned}$$

□

Question 1

Check that 5.1 is sharp. Namely, that 5.1 becomes equality for $u(x) = e^{-|x|}$ ($u(x)$ is an extremal function for 5.1). Also 5.1 is shift and scaling invariant, i.e. $u_\alpha(x+h) = e^{-\alpha|x+h|}$, $h \in \mathbb{R}, \alpha > 0$ -extremals.

Example 5.1.2 (Interpolation inequality)

Ω -domain in \mathbb{R}^n , $u \in L_{p_1}(\Omega) \cap L_{p_2}(\Omega)$, $1 \leq p_1, p_2, \leq \infty$, $p_1 < p_2$, $\theta \in [0, 1]$, $\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}$. Then

$$\|u\|_{L^p} \leq \|u\|_{L^{p_1}}^\theta \|u\|_{L^{p_2}}^{1-\theta} \quad (5.2)$$

Proof.

$$\int_{\mathbb{R}} |u|^p \, dx = \int_{\mathbb{R}} |u|^{\theta p} |u|^{(1-\theta)p} \, dx$$

We apply Hölder's inequality with exponents $P = \frac{p_1}{\theta p}$ and $Q = \frac{p_2}{(1-\theta)p}$ (Note $\frac{1}{P} + \frac{1}{Q} = \frac{\theta p}{p_1} + \frac{(1-\theta)p}{p_2} = 1$). Then

$$\begin{aligned} \int_{\mathbb{R}} |u|^{\theta p} |u|^{(1-\theta)p} dx &\leq \left(\int_{\mathbb{R}} |u|^{p_1} dx \right)^{\frac{1}{P}} \left(\int_{\mathbb{R}} |u|^{p_2} dx \right)^{\frac{1}{Q}} \\ &= \|u\|_{L^{p_1}}^{\theta} \|u\|_{L^{p_2}}^{1-\theta} \end{aligned}$$

□

5.2 Sobolev inequalities

Example 5.2.1 (Sobolev inequality 1D)

$u \in C^\infty([0, 1])$, want to prove the embedding $W^{1,1}([0, 1]) \subset C([0, 1])$, i.e.

$$\|u\|_{C([0,1])} \leq \|u\|_{L^1([0,1])} + \|u'\|_{L^1([0,1])} \quad (5.3)$$

Proof. By the Newton-Leibniz formula, $u(x) - u(y) = \int_y^x u'(s) ds$. Also,

$$|u(x)| \leq |u(y)| + \int_0^1 |u'(s)| ds \quad \forall x, y \in [0, 1]$$

By integration over $y \in [0, 1]$,

$$|u(x)| \leq \int_0^1 |u(s)| ds + \int_0^1 |u'(s)| ds = \|u\|_{W^{1,1}([0,1])}$$

Taking supremum with respect to $x \in [0, 1]$, we obtain $\|u\|_{C([0,1])} \leq \|u\|_{W^{1,1}([0,1])}$

□

Example 5.2.2 (Sobolev inequality 2D)

$u \in C^\infty([0, 1]^2)$, i.e. $\Omega = [0, 1]^2$, then $W^{1,1}(\Omega) \subset L^2(\Omega) : \|u\|_{L^2} \leq \|u\|_{W^{1,1}(\Omega)}$

Proof. $\int_{\Omega} u^2(x_1, x_2) dx_1 dx_2$ should be estimated. From 5.3, we know that

$$|u(x_1, x_2)| \leq \int_0^1 |u(s, x_2)| + |\partial_{x_1} u(s, x_2)| ds := f(x_2)$$

$$|u(x_1, x_2)| \leq \int_0^1 |u(x_1, s)| + |\partial_{x_2} u(x_1, s)| ds := g(x_1)$$

Then

$$\begin{aligned} \int_{\Omega} u^2 dx &\leq \int_0^1 g(x_1) f(x_2) dx_1 dx_2 \\ &= \int_0^1 f(x_2) dx_2 \int_0^1 g(x_1) dx_1 \\ &= \left(\int_{\Omega} |u(x_1, x_2)| + |\partial_{x_1} u(x_1, x_2)| dx_1 \right) \left(\int_{\Omega} |u(x_1, x_2)| + |\partial_{x_2} u(x_1, x_2)| dx_2 \right) \\ &\leq \|u\|_{W^{1,1}(\Omega)}^2 \end{aligned}$$

□

Question 2: Sobolev inequality 3D

$u \in C^\infty(\bar{\Omega})$, $\Omega = (0, 1)^3$. Prove that $W^{1,1}(\Omega) \subset L^{\frac{3}{2}}(\Omega)$, i.e.

$$\|u\|_{L^{\frac{3}{2}}(\Omega)} \leq \|u\|_{W^{1,1}(\Omega)} \quad (5.4)$$

Hint: first, prove that

$$\int_{\Omega} f(x_1, x_2) g(x_2, x_3) h(x_1, x_3) \, dx \leq \|f\|_{L^2} \|g\|_{L^2} \|h\|_{L^2}$$

and use 5.3.

Example 5.2.3

$u \in C^\infty(\bar{\Omega})$, $\Omega = (0, 1)^3$. Then

$$\|u\|_{L^6(\Omega)} \leq C \|u\|_{W^{1,2}(\Omega)} \quad (5.5)$$

Proof.

$$\begin{aligned} \int_{\Omega} |u|^6 \, dx &= \int_{\Omega} (|u|^4)^{\frac{3}{2}} \, dx \\ &\leq C \left(\int_{\Omega} |u|^4 \, dx + \int_{\Omega} u^3 |\nabla u| \, dx \right)^{\frac{3}{2}} \\ \text{(by (5.3))} \quad &\leq C \left(\int_{\Omega} |u|^4 \, dx \right)^{\frac{3}{2}} + C \left(\int_{\Omega} u^3 |\nabla u| \, dx \right)^{\frac{3}{2}} \\ &\leq C \|u\|_{L^2}^{\frac{3}{2} \cdot 4} \|u\|_{L^6}^{\frac{3}{2} \cdot (1-\theta) \cdot 4} + C \|u\|_{L^6}^{\frac{3}{2} \cdot 3} \|\nabla u\|_{L^2}^{\frac{3}{2}} \\ \left(\theta = \frac{1}{4} \right) \quad &= C \|u\|_{L^2}^{\frac{3}{2}} \|u\|_{L^6}^{\frac{9}{2}} + C \|u\|_{L^6}^{\frac{9}{2}} \|\nabla u\|_{L^2}^{\frac{3}{2}} \\ \left(\text{Young's inequality with } p = \frac{4}{5} \text{ and } q = -4 \right) \quad &\leq \varepsilon \|u\|_{L^6}^6 + C_\varepsilon (\|u\|_{L^2} + \|\nabla u\|_{L^2})^6 \end{aligned}$$

Setting for example, $\varepsilon = \frac{1}{2}$, we obtain

$$\|u\|_{L^6(\Omega)} \leq C \|u\|_{W^{1,2}(\Omega)}$$

□

Theorem 5.2.1 Sobolev embeddings

- ① $W^{k_1, p_1}(\Omega) \subset W^{k_2, p_2}(\Omega) \iff k_1 \geq k_2 \text{ and } 1 \leq p_1, p_2 < \infty, k_1 - \frac{n}{p_1} \geq k_2 - \frac{n}{p_2}, \Omega \subset \mathbb{R}^n.$
- ② $W^{k, p}(\Omega) \subset C^\alpha(\Omega) \text{ if } \alpha < k - \frac{n}{p}.$

Example 5.2.4

$$H^s(\mathbb{R}^n) \subset C(\mathbb{R}^n) \iff s > \frac{n}{2}$$

Proof. $u(x) = \int_{\mathbb{R}^n} e^{i\xi x} \hat{u}(\xi) d\xi$

$$\begin{aligned}
|u(x)| &\leq \int_{\mathbb{R}^n} |\hat{u}(\xi)| d\xi \\
&= \int_{\mathbb{R}^n} (1 + |\xi|^2)^{-\frac{s}{2}} (1 + |\xi|^2)^{\frac{s}{2}} |\hat{u}(\xi)| d\xi \\
&\stackrel{\text{(Hölder's inequality)}}{\leq} \left(\int_{\mathbb{R}^n} \frac{1}{(1 + |\xi|^2)^s} d\xi \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^n} (1 + |\xi|^2)^s |\hat{u}(\xi)|^2 d\xi \right)^{\frac{1}{2}} \\
&\int_{\mathbb{R}^n} \frac{1}{(1 + |\xi|^2)^s} d\xi < \infty \iff s > \frac{n}{2}. \text{ Taking the supremum with respect to } x \in \mathbb{R}^n, \text{ we get}
\end{aligned}$$

$$\|u\|_{C(\mathbb{R}^n)} \leq C_s \|u\|_{H^s(\mathbb{R}^n)}$$

□

Theorem 5.2.2 Interpolation inequalities

Let $u \in W^{k_1, p_1}(\Omega) \cap W^{k_2, p_2}(\Omega)$, $\theta \in [0, 1]$, $1 \leq p_1, p_2 \leq \infty$ with $k = \theta k_1 + (1 - \theta)k_2$, $\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}$. Then

$$\|u\|_{W^{k, p}} \leq C \|u\|_{W^{k_1, p_1}}^\theta \|u\|_{W^{k_2, p_2}}^{1-\theta}$$

Corollary 5.2.1 Particular cases

1. $\|u\|_{H^1} \leq \|u\|_{L^2}^{\frac{1}{2}} \|u\|_{H^2}^{\frac{1}{2}}$
2. $\|u\|_{L^p} \leq \|u\|_{L^p}^\theta \|u\|_{H^2}^{1-\theta}$

5.3 Spaces with zero boundary traces

Definition 5.3.1

$$W_0^{1, p}(\Omega) := \{u \in W^{1, p}(\Omega), u|_{\partial\Omega} = 0\}$$

Equivalent definition: $W_0^{1, p}(\Omega) = \text{“closure of } C_0^\infty(\Omega) \text{ in } W^{1, p} \text{ norm.”}$

Lemma 5.3.1

These two definitions are equivalent. $u \in \text{“closure”} : u = \lim_{n \rightarrow \infty} \varphi_n, \varphi_n \in C_0^\infty(\Omega) \implies \varphi_n|_{\partial\Omega} = 0$. By continuity, $u|_{\partial\Omega} = 0$. The proof of the converse statement is more technical and is omitted.

Proposition 5.3.1 Friedrich's inequality

Let Ω be a bounded domain and $u \in W_0^{1, p}(\Omega)$. Then

$$\|u\|_{L^p} \leq C \|\nabla u\|_{L^p} \tag{5.6}$$

Proof. It is enough to prove 5.6 for $\varphi \in C_0^\infty(\Omega)$. By the Newton-Leibniz formula,

$$u(x_1, x') - u(-L, x') = u(x_1, x') = \int_{-L}^{x_1} \partial_{x_1} u(s, x') ds$$

$$\begin{aligned}
|u(x_1, x')|^p &\leq \left(\int_{-L}^L |\partial_{x_1} u(s, x')| \, ds \right)^p \\
\text{(Hölder's inequality)} \quad &\leq C_L \int_{-L}^L |\partial_{x_1} u(s, x')|^p \, ds
\end{aligned}$$

Integration with respect to x' gives us

$$\int_{\mathbb{R}^{n-1}} |u(x_1, x')|^p \, dx' \leq C_L \|\partial_{x_1} u\|_{L^p}^p$$

Finally, integrating over $x_1 \in [-L, L]$, we obtain

$$\|u\|_{L^p}^p \leq 2LC_L \|\partial_{x_1} u\|_{L^p}^p$$

□

Corollary 5.3.1 Equivalent norm in $W_0^{1,p}(\Omega)$

Homogeneous norm:

$$\|u\|_{W_0^{1,p}(\Omega)} := \|\nabla u\|_{L^p}$$

Note:-

$u|_{\partial\Omega} = 0$ is important! Otherwise 5.6 will fail for $u \equiv c$. Since ∇u defines u up to a constant; $u|_{\partial\Omega} = 0$ removes this constant.

Proposition 5.3.2 Poincaré inequality

Let Ω be a bounded domain with a smooth boundary and $\langle u \rangle := \frac{1}{|\Omega|} \int_{\Omega} u(x) \, dx = 0$. Then

$$\|u\|_{L^p} \leq C \|\nabla u\|_{L^p}$$