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Chapter 2

Sobolev spaces

2.1 Interpolation inequalities

Example 2.1

$$\|u\|_{L^2}^2 \leq \|u\|_{L^2} \|u'\|_{L^2} \text{ for } u \in C^\infty(\mathbb{R}) \quad (2.1)$$

Proof. Idea: use that $(u^2)' = 2uu'$ and Newton-Leibniz

$$\begin{aligned} u^2(x) &= 2 \int_{-\infty}^x uu' \, dy = -2 \int_x^{\infty} uu' \, dy \\ &= \int_{-\infty}^x uu' \, dy - \int_x^{\infty} uu' \, dy \\ &\leq \int_{-\infty}^x |u||u'| \, dy + \int_x^{\infty} |u||u'| \, dy \\ &= \int_{\mathbb{R}} |u||u'| \, dy \\ (\text{Hölder's inequality}) &\leq \|u\|_{L^2} \|u'\|_{L^2} \end{aligned}$$

□

Question 1

Check that 2.1 is sharp. Namely, that 2.1 becomes equality for $u(x) = e^{-|x|}$ ($u(x)$ is an extremal function for 2.1). Also, 2.1 is shift and scaling invariant, i.e. $u_\alpha(x+h) = e^{-\alpha|x+h|}$, $h \in \mathbb{R}$, $\alpha > 0$ -extremals.

Example 2.2 (Interpolation inequality)

Ω -domain in \mathbb{R}^n , $u \in L_{p_1}(\Omega) \cap L_{p_2}(\Omega)$, $1 \leq p_1, p_2, < \infty$, $p_1 < p_2$, $\theta \in [0, 1]$, $\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}$. Then

$$\|u\|_{L^p} \leq \|u\|_{L^{p_1}}^\theta \|u\|_{L^{p_2}}^{1-\theta} \quad (2.2)$$

Proof.

$$\int_{\mathbb{R}} |u|^p \, dx = \int_{\mathbb{R}} |u|^{\theta p} |u|^{(1-\theta)p} \, dx$$

We apply Hölder's inequality with exponents $P = \frac{p_1}{\theta p}$ and $Q = \frac{p_2}{(1-\theta)p}$ (Note $\frac{1}{P} + \frac{1}{Q} = \frac{\theta p}{p_1} + \frac{(1-\theta)p}{p_2} = 1$). Then

$$\begin{aligned} \int_{\mathbb{R}} |u|^{\theta p} |u|^{(1-\theta)p} dx &\leq \left(\int_{\mathbb{R}} |u|^{p_1} dx \right)^{\frac{1}{P}} \left(\int_{\mathbb{R}} |u|^{p_2} dx \right)^{\frac{1}{Q}} \\ &= \|u\|_{L^{p_1}}^{\theta} \|u\|_{L^{p_2}}^{1-\theta} \end{aligned}$$

□

2.2 Sobolev inequalities

Example 2.3 (Sobolev inequality 1D)

$u \in C^\infty([0, 1])$, want to prove the embedding $W^{1,1}([0, 1]) \subset C([0, 1])$, i.e.

$$\|u\|_{C([0,1])} \leq \|u\|_{L^1([0,1])} + \|u'\|_{L^1([0,1])} \quad (2.3)$$

Proof. By the Newton-Leibniz formula, $u(x) - u(y) = \int_y^x u'(s) ds$. Also,

$$|u(x)| \leq |u(y)| + \int_0^1 |u'(s)| ds \quad \forall x, y \in [0, 1]$$

By integration over $y \in [0, 1]$,

$$|u(x)| \leq \int_0^1 |u(s)| ds + \int_0^1 |u'(s)| ds = \|u\|_{W^{1,1}([0,1])}$$

Taking supremum with respect to $x \in [0, 1]$, we obtain $\|u\|_{C([0,1])} \leq \|u\|_{W^{1,1}([0,1])}$

□

Example 2.4 (Sobolev inequality 2D)

$u \in C^\infty([0, 1]^2)$, i.e. $\Omega = [0, 1]^2$, then $W^{1,1}(\Omega) \subset L^2(\Omega) : \|u\|_{L^2} \leq \|u\|_{W^{1,1}(\Omega)}$

Proof. $\int_{\Omega} u^2(x_1, x_2) dx_1 dx_2$ should be estimated. From 2.3, we know that

$$|u(x_1, x_2)| \leq \int_0^1 |u(s, x_2)| + |\partial_{x_1} u(s, x_2)| ds := f(x_2)$$

$$|u(x_1, x_2)| \leq \int_0^1 |u(x_1, s)| + |\partial_{x_2} u(x_1, s)| ds := g(x_1)$$

Then

$$\begin{aligned} \int_{\Omega} u^2 dx &\leq \int_0^1 g(x_1) f(x_2) dx_1 dx_2 \\ &= \int_0^1 f(x_2) dx_2 \int_0^1 g(x_1) dx_1 \\ &= \left(\int_{\Omega} |u(x_1, x_2)| + |\partial_{x_1} u(x_1, x_2)| dx_1 \right) \left(\int_{\Omega} |u(x_1, x_2)| + |\partial_{x_2} u(x_1, x_2)| dx_2 \right) \\ &\leq \|u\|_{W^{1,1}(\Omega)}^2 \end{aligned}$$

□

Question 2: Sobolev inequality 3D

$u \in C^\infty(\bar{\Omega})$, $\Omega = (0, 1)^3$. Prove that $W^{1,1}(\Omega) \subset L^{\frac{3}{2}}(\Omega)$, i.e.

$$\|u\|_{L^{\frac{3}{2}}(\Omega)} \leq \|u\|_{W^{1,1}(\Omega)} \quad (2.4)$$

Hint: first, prove that

$$\int_{\Omega} f(x_1, x_2) g(x_2, x_3) h(x_1, x_3) dx \leq \|f\|_{L^2} \|g\|_{L^2} \|h\|_{L^2}$$

and use 2.3.

Example 2.5

$u \in C^\infty(\bar{\Omega})$, $\Omega = (0, 1)^3$. Then

$$\|u\|_{L^6(\Omega)} \leq C \|u\|_{W^{1,2}(\Omega)} \quad (2.5)$$

Proof.

$$\begin{aligned} \int_{\Omega} |u|^6 dx &= \int_{\Omega} (|u|^4)^{\frac{3}{2}} dx \\ &\leq C \left(\int_{\Omega} |u|^4 dx + \int_{\Omega} u^3 |\nabla u| dx \right)^{\frac{3}{2}} \\ (\text{by (2.3)}) \quad &\leq C \left(\int_{\Omega} |u|^4 dx \right)^{\frac{3}{2}} + C \left(\int_{\Omega} u^3 |\nabla u| dx \right)^{\frac{3}{2}} \\ &\leq C \|u\|_{L^2}^{\frac{3}{2} \cdot 4} \|u\|_{L^6}^{\frac{3}{2} \cdot (1-\theta) \cdot 4} + C \|u\|_{L^6}^{\frac{3}{2} \cdot 3} \|\nabla u\|_{L^2}^{\frac{3}{2}} \\ \left(\theta = \frac{1}{4} \right) \quad &= C \|u\|_{L^2}^{\frac{3}{2}} \|u\|_{L^6}^{\frac{9}{2}} + C \|u\|_{L^6}^{\frac{9}{2}} \|\nabla u\|_{L^2}^{\frac{3}{2}} \\ \left(\text{Young's inequality with } p = \frac{4}{\varepsilon} \text{ and } q = -4 \right) \quad &\leq \varepsilon \|u\|_{L^6}^6 + C_\varepsilon (\|u\|_{L^2} + \|\nabla u\|_{L^2})^6 \end{aligned}$$

Setting for example, $\varepsilon = \frac{1}{2}$, we obtain

$$\|u\|_{L^6(\Omega)} \leq C \|u\|_{W^{1,2}(\Omega)}$$

□

Theorem 2.1 Sobolev embeddings

- ① $W^{k_1, p_1}(\Omega) \subset W^{k_2, p_2}(\Omega) \iff k_1 \geq k_2$ and $1 \leq p_1, p_2 < \infty, k_1 - \frac{n}{p_1} \geq k_2 - \frac{n}{p_2}, \Omega \subset \mathbb{R}^n$.
- ② $W^{k, p}(\Omega) \subset C^\alpha(\Omega)$ if $\alpha < k - \frac{n}{p}$. If α is not an integer, then the inequality is weak.

Example 2.6

$$H^s(\mathbb{R}^n) \subset C(\mathbb{R}^n) \iff s > \frac{n}{2}$$

Proof. $u(x) = \int_{\mathbb{R}^n} e^{i\xi x} \hat{u}(\xi) d\xi$

$$\begin{aligned}
|u(x)| &\leq \int_{\mathbb{R}^n} |\hat{u}(\xi)| d\xi \\
&= \int_{\mathbb{R}^n} (1 + |\xi|^2)^{-\frac{s}{2}} (1 + |\xi|^2)^{\frac{s}{2}} |\hat{u}(\xi)| d\xi \\
&\stackrel{\text{(Hölder's inequality)}}{\leq} \left(\int_{\mathbb{R}^n} \frac{1}{(1 + |\xi|^2)^s} d\xi \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^n} (1 + |\xi|^2)^s |\hat{u}(\xi)|^2 d\xi \right)^{\frac{1}{2}}
\end{aligned}$$

$\int_{\mathbb{R}^n} \frac{1}{(1 + |\xi|^2)^s} d\xi < \infty \iff s > \frac{n}{2}$. Taking the supremum with respect to $x \in \mathbb{R}^n$, we get

$$\|u\|_{C(\mathbb{R}^n)} \leq C_s \|u\|_{H^s(\mathbb{R}^n)}$$

□

Theorem 2.2 Interpolation inequalities

Let $u \in W^{k_1, p_1}(\Omega) \cap W^{k_2, p_2}(\Omega)$, $\theta \in [0, 1]$, $1 \leq p_1, p_2 \leq \infty$ with $k = \theta k_1 + (1 - \theta)k_2$, $\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}$. Then

$$\|u\|_{W^{k,p}} \leq C \|u\|_{W^{k_1, p_1}}^\theta \|u\|_{W^{k_2, p_2}}^{1-\theta}$$

Corollary 2.1 Particular cases

1. $\|u\|_{H^1} \leq \|u\|_{L^2}^{\frac{1}{2}} \|u\|_{H^2}^{\frac{1}{2}}$
2. $\|u\|_{L^p} \leq \|u\|_{L^p}^\theta \|u\|_{H^2}^{1-\theta}$

2.3 Spaces with zero boundary traces

Definition 2.1

$$W_0^{1,p}(\Omega) := \{u \in W^{1,p}(\Omega), u|_{\partial\Omega} = 0\}$$

An equivalent definition is that the Sobolev spaces $W_0^{1,p}(\Omega)$ for $1 \leq p < \infty$ are defined as the closure of the set of compactly supported test functions $C_0^\infty(\Omega)$ with respect to the $W^{1,p}(\Omega)$ -norm.

Lemma 2.1

These two definitions are equivalent. $u \in \text{"closure"}: u = \lim_{n \rightarrow \infty} \varphi_n, \varphi_n \in C_0^\infty(\Omega) \implies \varphi_n|_{\partial\Omega} = 0$. By continuity, $u|_{\partial\Omega} = 0$. The proof of the converse statement is more technical and is omitted.

2.4 Poincaré's and Friedrich's inequalities

Proposition 2.1 Friedrich's inequality

Let Ω be a bounded domain and $u \in W_0^{1,p}(\Omega)$. Then

$$\|u\|_{L^p} \leq C \|\nabla u\|_{L^p} \tag{2.6}$$

Proof. It is enough to prove 2.6 for $\varphi \in C_0^\infty(\Omega)$. By the Newton-Leibniz formula,

$$u(x_1, x') - u(-L, x') = u(x_1, x') = \int_{-L}^{x_1} \partial_{x_1} u(s, x') \, ds$$

$$\begin{aligned} |u(x_1, x')|^p &\leq \left(\int_{-L}^L |\partial_{x_1} u(s, x')| \, ds \right)^p \\ (\text{Hölder's inequality}) &\leq C_L \int_{-L}^L |\partial_{x_1} u(s, x')|^p \, ds \end{aligned}$$

Integration with respect to x' gives us

$$\int_{\mathbb{R}^{n-1}} |u(x_1, x')|^p \, dx' \leq C_L \|\partial_{x_1} u\|_{L^p}^p$$

Finally, integrating over $x_1 \in [-L, L]$, we obtain

$$\|u\|_{L^p}^p \leq 2LC_L \|\partial_{x_1} u\|_{L^p}^p$$

□

Corollary 2.2 Equivalent norm in $W_0^{1,p}(\Omega)$

Homogeneous norm:

$$\|u\|_{W_0^{1,p}(\Omega)} := \|\nabla u\|_{L^p}$$

Note:-

$u|_{\partial\Omega} = 0$ is important! Otherwise, 2.6 will fail for $u \equiv c$. Since ∇u defines u up to a constant; $u|_{\partial\Omega} = 0$ removes this constant.

Proposition 2.2 Poincaré inequality

Let Ω be a bounded domain with a smooth boundary and $\langle u \rangle := \frac{1}{|\Omega|} \int_{\Omega} u(x) \, dx = 0$. Then

$$\|u\|_{L^p} \leq C \|\nabla u\|_{L^p}$$

2.5 Compactness

Definition 2.2: Sequential compactness

A metric space (X, d) is compact if any sequence $\{x_n\}_{n=1}^\infty \subset X$ has a convergent sub-sequence, i.e. there exists $\{x_{n_k}\}_{k=1}^\infty : \lim_{k \rightarrow \infty} x_{n_k} = x_0 \in X$.

Definition 2.3: Compact

A topological space X is compact if any covering of X by open sets has a finite sub-covering.

Note:-

In metric spaces, compactness is equivalent to sequential compactness.
In general topological spaces, they are not related.

Theorem 2.3 Hausdorff

Let (X, d) be a metric space. Then X is compact $\iff X$ is complete and totally bounded.

Definition 2.4: Totally bounded

X is totally bounded if $\forall \varepsilon > 0, \exists$ covering of X by finitely many ε -balls, i.e. $X = \bigcup_{k=1}^N B_\varepsilon(x_k), N = N(\varepsilon)$ and $\{x_k\}$ is an ε -net in X .

Why do we need compactness?

Let X be compact and $f: X \rightarrow Y$ be continuous, then $f(X)$ is compact in Y . How do we solve PDEs of the form (or more general equations)?

$$F(x) = 0 \tag{2.7}$$

1. Construct approximate solutions

$$F(x_n) = g_n, \text{ where } \lim_{n \rightarrow \infty} g_n = 0$$

2. Obtain a priori estimates, i.e. that $\{x_n\}$ is bounded in a proper space
3. If $\{x_n\}$ is pre-compact and F is continuous $\implies x = \lim_{n \rightarrow \infty} x_n$ is a solution of 2.7.

Theorem 2.4 Arzelà-Ascoli

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Then $V \subset C(\bar{\Omega})$ is compact iff:

1. V is closed
2. V is bounded
3. V is equicontinuous = V has a common modulus of continuity

Theorem 2.5 Arzelà-Ascoli for L^p

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, (and $\partial\Omega$ smooth, although not needed), $K \subset L^p(\Omega), 1 \leq p < \infty$. Then K is compact iff:

1. K is closed
2. K is bounded
3. K is equicontinuous in mean (possesses a joint modulus of continuity in L^p).

Definition 2.5: Modulus of continuity

Let $f \in L^p(\Omega), 1 \leq p < \infty, \Omega \subset \mathbb{R}^n$ bounded ($\partial\Omega$ smooth not needed). $\omega: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that $\lim_{z \rightarrow 0} \omega(z) = 0$ is a modulus of continuity of f in $L_p(\Omega)$ if

$$\int_{\Omega} |f(x+h) - f(x)|^p dx \leq \omega(|h|), \quad \forall h \in \mathbb{R}^n,$$

where we used the 0-extension of f outside of Ω .

Corollary 2.3

Let $K = B_1(0) \in W^{1,p}(\Omega); \Omega \subset \mathbb{R}^n$ is bounded, $\partial\Omega$ is smooth, $1 \leq p < \infty$. Then K is pre-compact in $L^p(\Omega)$.

Proof. We need to check equicontinuity, i.e. estimate $\int_{\Omega} |f(x+h) - f(x)|^p dx$.

$$f(x+h) - f(x) = h \int_0^1 \nabla f(x+sh) ds$$

Taking modulus and p -th power of both sides, we get

$$|f(x+h) - f(x)|^p \leq |h|^p \int_0^1 |\nabla f(x+sh)|^p ds$$

Finally, we take an integral over $x \in \Omega$.

$$\begin{aligned} \int_{\Omega} |f(x+h) - f(x)|^p dx &\leq |h|^p \int_0^1 \int_{\Omega} |\nabla f(x+sh)|^p dx ds \\ &\leq C|h|^p \end{aligned}$$

$\omega(z) = cz$ is a joint modulus of continuity. □

Definition 2.6: Compact embedding

Let $V \subset W$ be Banach spaces. Then the embedding is compact if the unit ball of V is pre-compact in W .

Note:-

We proved that $W^{1,p}(\Omega) \subset L^p(\Omega)$ is a compact embedding.

Corollary 2.4

$W^{1,p}(\Omega) \subset L^q(\Omega)$ is a compact embedding if $q < q^*$, where q^* is defined such that $\frac{1}{q^*} = \frac{1}{p} - \frac{1}{n}$ and $\Omega \subset \mathbb{R}^n$, Ω is bounded, $\partial\Omega$ is smooth.

Proof. Let us check equicontinuity.

$$\|f(\cdot+h) - f(\cdot)\|_{L^q} \leq \|f(\cdot+h) - f(\cdot)\|_{L^p}^\theta \|f(\cdot+h) - f(\cdot)\|_{L^{q^*}}^{1-\theta}$$

since $p < q < q^*$ and $0 < \theta < 1$. q^* is a critical exponent in Sobolev embeddings, indeed, $W^{1,p}(\Omega) \subset L^q(\Omega) \implies 1 - \frac{n}{p} \geq -\frac{n}{q}$. Then by corollary 2.3, we have

$$\begin{aligned} \|f(\cdot+h) - f(\cdot)\|_{L^p}^\theta \|f(\cdot+h) - f(\cdot)\|_{L^{q^*}}^{1-\theta} &\leq C|h|^\theta (2\|f\|_{L^{q^*}})^{1-\theta} \\ &\leq C_1|h|^\theta \|f\|_{W^{1,p}}^{1-\theta} \\ &\leq C_1|h|^\theta \end{aligned}$$

□

General fact: $W^{s_1,p_1}(\Omega) \subset W^{s_2,p_2}(\Omega)$, where Ω is bounded, $\partial\Omega$ is smooth. Embedding is compact \iff embedding is not critical.

Dual spaces

Definition 2.7: Dual space

$W^{-s,p}(\Omega) := \left(W_0^{s,q}(\Omega)\right)^*$ is defined as the dual space to $W_0^{s,q}(\Omega)$, i.e. the space of linear continuous functionals on $W_0^{s,q}(\Omega)$, where $\frac{1}{p} + \frac{1}{q} = 1$.

Definition 2.8

$$W^{-s,p}(\Omega) = \left\{ \text{completion of } L^p(\Omega) \text{ w.r.t } \|\ell\|_{W^{-s,p}} := \sup_{\varphi \in \mathcal{D}} \frac{|\langle \ell, \varphi \rangle|}{\|\varphi\|_{W_0^{s,q}}} \right\}$$

Definition 2.9

$$W^{-s,p}(\Omega) = \left\{ \ell \in \mathcal{D}'(\Omega) : \|\ell\|_{W^{-s,p}} := \sup_{\varphi \in \mathcal{D}} \frac{|\langle \ell, \varphi \rangle|}{\|\varphi\|_{W_0^{s,q}}} \right\}$$

Proposition 2.3

Definitions 2.7, 2.8 and 2.9 are equivalent.

Question 3

Suppose $\delta(x) \in W^{-s,p}(\Omega)$, $\Omega \subset \mathbb{R}^n$. How are s, p and n related? We know that $\delta(x)$ is well-defined on continuous functions, so we need $W_0^{s,q}(\Omega) \subset C(\bar{\Omega})$.

Example 2.7

Consider the case where $n = 1$ and $p = 2$. By the Sobolev embedding theorem, $W^{s,2} \subset C(\bar{\Omega})$ if $0 < s - \frac{1}{2}$. Thus we have $\delta(x) \in H^{-s}(\Omega)$ if $s > \frac{1}{2}$.

Chapter 3

Linear elliptic problems

3.1 Dirichlet and Neumann problems for the Laplacian

Example 3.1 (Laplace equation with Dirichlet boundary conditions)

Let $\Omega \in \mathbb{R}^n$ be a bounded domain with $\partial\Omega$ smooth. Consider the Laplace equation with Dirichlet boundary conditions:

$$\begin{cases} \Delta u = f \\ u|_{\partial\Omega} = 0 \end{cases} \quad (3.1)$$

Typical questions:

1. In what space does the solution live?
2. In what sense is the equation understood (classical / weak)?
3. In what sense are the boundary / initial data understood?

In ODEs, we have local existence and uniqueness theorem (for Lipschitz non-linearities), but there is not an equivalent theorem for PDEs. Therefore, we must study particular examples.

Definition 3.1

$u \in W_0^{1,2}(\Omega)$ is a weak solution of 3.1 if $\forall \varphi \in C_0^\infty(\Omega)$,

$$-\int_{\Omega} \nabla u(x) \nabla \varphi(x) \, dx = \int_{\Omega} f(x) \varphi(x) \, dx \quad (3.2)$$

Here, the boundary condition is incorporated into the choice of space $W_0^{1,2}(\Omega) = [C_0^\infty(\Omega)]_{W^{1,2}(\Omega)}$ (the closure of $C_0^\infty(\Omega)$ in the norm of $W^{1,2}(\Omega)$).

3.2 came from the integration by parts formula. Indeed, if $u \in C^2(\Omega) \cap C(\bar{\Omega})$, then $\Delta u = f$ is understood in a classical sense and

$$\int_{\Omega} \Delta u \varphi \, dx = - \int_{\Omega} \nabla u \nabla \varphi \, dx + \int_{\partial\Omega} \partial_n u \varphi \, ds,$$

where the term $\int_{\partial\Omega} \partial_n u \varphi \, ds = 0$ because $\varphi|_{\partial\Omega} = 0$.

Theorem 3.1

Let $f \in H^{-1}(\Omega) := W^{-1,2}(\Omega)$. Then 3.1 has a unique weak solution.

Proof. Application of Riesz representation theorem

$[u, u] := \int_{\Omega} \nabla u \nabla u \, dx$ is an equivalent norm on $W_0^{1,2}(\Omega)$ (due to Friedrich's inequality). Then 3.2 can be rewritten as

$$[u, \varphi] = \int_{\Omega} f(x) \varphi(x) \, dx := \ell(\varphi)$$

Claim: ℓ is a linear continuous functional on $W_0^{1,2}(\Omega)$ (the integral should be understood as duality if we take $f \in H^{-1}(\Omega)$ and if $f \in L^2(\Omega)$, this is a standard Lebesgue integral).

Linearity of ℓ is obvious. ℓ is continuous as it is bounded:

$$|\ell(\varphi)| \leq \|f\|_{H^{-1}} \|\varphi\|_{H^1}$$

But we obtained that 3.2 holds only for $\varphi \in C_0^\infty(\Omega)$, not for $\varphi \in W_0^{1,2}(\Omega)$. However, $W_0^{1,2}(\Omega) = [C_0^\infty(\Omega)]_{W^{1,2}}$. Then approximation arguments give that $\forall \varphi \in H$,

$$[u, \varphi] = \ell(\varphi) \tag{3.3}$$

Then by Riesz representation theorem, there exists a unique $u \in W_0^{1,2}(\Omega)$ which satisfies 3.3. \square

Example 3.2 (Laplace equation with Neumann boundary conditions)

Let $\Omega \in \mathbb{R}^n$ be a bounded domain with $\partial\Omega$ smooth. Consider the Laplace equation with Neumann boundary conditions:

$$\begin{cases} \Delta u = f \\ \partial_n u|_{\partial\Omega} = 0 \end{cases} \tag{3.4}$$

We cannot consider $\varphi \in C_0^\infty(\Omega)$ as test functions, because the information about boundary conditions will be lost. Similarly, considering

$$\varphi \in W_n^{1,2}(\Omega) := \{u \in W^{1,2}(\Omega) : \partial_n u|_{\partial\Omega} = 0\}$$

will not work as well, since $\partial_n u|_{\partial\Omega}$ is not defined for $u \in W^{1,2}(\Omega)$ (since by theorem 2.1, $C^\infty(\Omega) \not\subset W^{1,2}(\Omega)$). Instead, let us take $\varphi \in C^\infty(\bar{\Omega})$ as a test function and assume that u is a classical solution. Then

$$\begin{aligned} \int_{\Omega} f \varphi \, dx &= \int_{\Omega} \Delta u \varphi \, dx \\ &= - \int_{\Omega} \nabla u \nabla \varphi \, dx + \int_{\partial\Omega} \partial_n u \varphi \, ds \\ &= - \int_{\Omega} \nabla u \nabla \varphi \, dx, \end{aligned}$$

as $\int_{\partial\Omega} \partial_n u \varphi \, dx = 0$ due to the boundary conditions. If we take $\varphi(x) = 1$ as a test function, then we get

$$\begin{aligned} \int_{\Omega} f \cdot 1 \, dx &= - \int_{\Omega} \nabla u \nabla 1 \, dx \\ &= 0 \end{aligned}$$

Hence $\langle f \rangle = \frac{1}{|\Omega|} \int_{\Omega} f(x) \, dx = 0$ is a necessary condition for solvability.

Let us notice that all solutions of this problem differs from each other by a constant. Thus, a natural assumption to single out the solution is $\langle u \rangle = 0$.

Definition 3.2

$u \in W^{1,2}(\Omega) \cap \{\langle u \rangle = 0\}$ is a weak solution of 3.4 if $\forall \varphi \in C^\infty(\bar{\Omega})$, we have:

$$\int_{\Omega} \nabla u \nabla \varphi \, dx = - \int_{\Omega} f \varphi \, dx \tag{3.5}$$

Note:-

The boundary conditions are now not in the definition of the space, but in 3.5.

Theorem 3.2

Let $f \in L^2(\Omega) \cap \{\langle f \rangle = 0\}$. Then 3.4 has a unique weak solution.

Proof. The proof is analogous to the problem with Dirichlet boundary conditions, but instead of applying Friedrich's inequality, we should apply Poincaré's inequality and use density of $C^\infty(\Omega) \in W^{1,2}(\Omega)$. \square

Example 3.3 (Non-homogeneous Neumann boundary conditions)

Let $\Omega \in \mathbb{R}^n$ be a bounded domain with $\partial\Omega$ smooth. Consider the Laplace equation with non-homogeneous Neumann boundary conditions:

$$\begin{cases} \Delta u = f \\ \partial_n u|_{\partial\Omega} = g \end{cases} \quad (3.6)$$

Definition 3.3

$u \in W^{1,2}(\Omega) \cap \{\langle u \rangle = 0\}$ is a weak solution of 3.6 if $\forall \varphi \in C^\infty(\bar{\Omega})$, we have:

$$\int_{\Omega} \nabla u \nabla \varphi \, dx = - \int_{\Omega} f \varphi \, dx + \int_{\partial\Omega} g \varphi \, ds \quad (3.7)$$

Note that if $\varphi \equiv 1$, then a necessary condition for solvability is

$$- \int_{\Omega} f \, dx + \int_{\partial\Omega} g \, ds = 0$$

Theorem 3.3

Let $f \in L^2(\Omega)$, $g \in W^{-\frac{1}{2},2}(\partial\Omega)$ be such that $\int_{\Omega} f \, dx = \int_{\partial\Omega} g \, ds$. Then 3.6 has a unique weak solution.

Proof. $[u, u] := \int_{\Omega} \nabla u \nabla u \, dx$ is an equivalent norm on $u \in W^{1,2}(\Omega) \cap \{\langle u \rangle = 0\}$ due to the Poincaré inequality. Then 3.7 can be rewritten as

$$[u, \varphi] = \ell(\varphi) := - \int_{\Omega} f \varphi \, dx + \int_{\partial\Omega} g \varphi \, ds$$

We claim that ℓ is a linear continuous functional on $W^{1,2}(\Omega) \cap \{\langle u \rangle = 0\}$. Indeed, linearity is obvious. To show ℓ is continuous, we have

$$\left| - \int_{\Omega} f \varphi \, dx + \int_{\partial\Omega} g \varphi \, ds \right| \leq \|f\|_{L^2} \|\varphi\|_{L^2} + \|g\|_{H^{-\frac{1}{2}}(\partial\Omega)} \|\varphi\|_{H^{\frac{1}{2}}(\partial\Omega)}$$

$$(\text{By the trace theorem and Poincaré's inequality}) \leq \|f\|_{L^2} \|\varphi\|_{W^{1,2}(\Omega)} + \|g\|_{H^{-\frac{1}{2}}(\partial\Omega)} \|\varphi\|_{W^{1,2}(\Omega)}$$

Then by Riesz representation theorem, there exists a unique $u \in W^{1,2}(\Omega) \cap \{\langle u \rangle = 0\}$ that is a weak solution of 3.6. \square

Example 3.4 (Non-homogeneous Dirichlet boundary conditions)

Let $\Omega \in \mathbb{R}^n$ be a bounded domain with $\partial\Omega$ smooth. Consider the Laplace equation with non-homogeneous

Dirichlet boundary conditions:

$$\begin{cases} \Delta u = 0 \\ u|_{\partial\Omega} = g \end{cases} \quad (3.8)$$

Let us take $g \in W^{\frac{1}{2},2}(\partial\Omega)$. Then there exists $v \in W^{1,2}(\Omega)$ such that $v|_{\partial\Omega} = g$ (by the trace theorem). We look for the solution of 3.8 in the form $u = v + w$, where $w \in W_0^{1,2}(\Omega)$.

Definition 3.4

$u = v + w$ is a weak solution of 3.8 if $v|_{\partial\Omega} = g$, where $g \in W^{\frac{1}{2},2}(\partial\Omega)$, $w \in W_0^{1,2}(\Omega)$ and $\forall \varphi \in C^\infty(\bar{\Omega})$, we have

$$\int_{\Omega} \nabla(v + w) \nabla \varphi \, dx = 0 \quad (3.9)$$

Theorem 3.4

Let $g \in W^{\frac{1}{2},2}(\partial\Omega)$. Then 3.8 has a unique weak solution.

Proof. We can rearrange 3.9 to get

$$\ell(\varphi) := -[v, \varphi] = \int_{\Omega} \nabla w \nabla \varphi \, dx = [w, \varphi],$$

and the functional ℓ can be shown to be linear and continuous. By the Riesz representation theorem, there exists a unique $w \in W_0^{1,2}(\Omega)$ such that 3.9 is satisfied. Note that this w depends on the choice of v . But $u = v + w$ does not depend on the choice of v . Indeed, let u_1 and u_2 be two solutions of 3.8. Then $u = u_1 - u_2$ solves

$$\begin{cases} \Delta u = 0 \\ u|_{\partial\Omega} = 0 \end{cases}$$

We have previously shown that the weak solution of this problem is unique. Therefore, $u_1 = u_2$. \square

Note:-

There is no universal choice of the space of test functions. Even for Dirichlet and Neumann boundary conditions, we need to consider different spaces. $\varphi \in C_0^\infty(\Omega)$ corresponds to the standard theory of distributions, while $\varphi \in C^\infty(\bar{\Omega})$ corresponds to “non-standard” distributions.

Example 3.5

Let $\Omega \in \mathbb{R}^n$ be a bounded domain with $\partial\Omega$ smooth. Consider

$$\begin{cases} \sum_{i,j} \partial_{x_i} (a_{ij}(x) \partial_{x_j} u) = g \\ u|_{\partial\Omega} = 0 \end{cases} \quad (3.10)$$

Where we make the following assumptions on the matrix $a(x) := \{a_{ij}(x)\}_{i,j}$:

1. $a(x)$ is a symmetric matrix for every x :

$$a_{ij}(x) = a_{ji}(x)$$

2. $a(x)$ is uniformly elliptic. That is, for all $\xi \in \mathbb{R}^n$, there exists $\mu, M > 0$ which are independent of x such that

$$\mu |\xi|^2 \leq \sum_{i,j} a_{ij} \xi_i \xi_j \leq M |\xi|^2$$

Definition 3.5

$u \in W^{1,2}(\Omega)$ is a weak solution to 3.10 $\iff \forall \varphi \in C_0^\infty(\Omega)$, we have

$$\sum_{i,j} \int_{\Omega} a_{ij} \partial_{x_j} u \partial_{x_i} \varphi \, dx = - \int_{\Omega} g \varphi \, dx$$

Theorem 3.5

Let $a(x)$ be symmetric and uniformly elliptic. Then 3.10 has a unique weak solution.

Proof. Let us denote

$$[u, \varphi]_a = \int_{\Omega} \sum_{i,j} a_{ij}(x) \partial_{x_j} u(x) \partial_{x_i} \varphi(x) \, dx.$$

Then since $a(x)$ is symmetric, the bilinear form $[u, v]_a$ is also symmetric, i.e. $[u, v]_a = [v, u]_a$. Since $a(x)$ is uniformly elliptic, there exist $\mu, M > 0$ such that

$$\mu[u, u] \leq [u, u]_a \leq M[u, u].$$

Therefore, $(W_0^{1,2}(\Omega), [\cdot, \cdot]_a)$ is a Hilbert space with the norm equivalent to the standard $W_0^{1,2}(\Omega)$ norm.

By the Riesz representation theorem, there exists a unique weak solution to 3.10. \square

3.2 More general problems via Lax-Milgram

By the Riesz representation theorem, for any linear continuous functional, ℓ on a Hilbert space H , there exists a unique $x \in H$ such that $\forall \varphi \in H$, we have $(x, \varphi) = \ell(\varphi)$.

If we want $a(x, y)$ to be an equivalent inner product on H , then $a(x, y)$ must be symmetric.

We now consider the case where $a(x, y)$ is not assumed to be symmetric.

Definition 3.6: Bilinear form

A bilinear form $a(\cdot, \cdot): H \times H \rightarrow \mathbb{R}$ is bounded if

$$|a(x, y)| \leq C \|x\| \|y\|$$

Definition 3.7: Coercive

A bilinear form $a(\cdot, \cdot)$ is coercive if $\exists \alpha > 0$ such that $a(x, x) \geq \alpha \|x\|^2$.

Theorem 3.6

Let $a(x, y)$ be a bounded and coercive bilinear form on H . Then any linear continuous functional $\ell: H \rightarrow \mathbb{R}$ can be represented in the form

$$a(x, y) = \ell(\varphi), \quad \forall \varphi \in H. \quad (3.11)$$

i.e. $\forall \ell \in H^*$, there exists a unique $x = x(\ell) \in H$ such that 3.11 is satisfied.

Example 3.6

Let $\Omega \in \mathbb{R}^n$ be a bounded domain with $\partial\Omega$ smooth. Consider the problem

$$\begin{cases} \sum_{i,j} \partial_{x_i} (a_{ij}(x) \partial_{x_j} u) + \sum_i b_i(x) \partial_{x_i} u = g(x) \\ u|_{\partial\Omega} = 0 \end{cases} \quad (3.12)$$

Definition 3.8

$u \in W_0^{1,2}(\Omega)$ is a weak solution of 3.12 if $\forall \varphi \in C_0^\infty(\Omega)$, we have

$$A(u, \varphi) := \sum_{i,j} \int_{\Omega} a_{ij} \partial_{x_j} u \partial_{x_i} \varphi \, dx - \sum_i \int_{\Omega} b_i(x) \partial_{x_i} u \varphi \, dx = \ell(\varphi) := - \int_{\Omega} g(x) \varphi(x) \, dx$$

Theorem 3.7

Let $\{a_{ij}\} \in L^\infty(\Omega)$ be a uniformly elliptic matrix, $b_i(x)$ be a smooth divergent free vector field and $g(x) \in H^{-1}(\Omega)$. Then 3.12 has a unique weak solution.

Proof. We use the Lax-Milgram theorem. We know that $\ell(\varphi)$ is a linear continuous functional on $W_0^{1,2}(\Omega)$. Furthermore, $A(u, \varphi)$ is bilinear and bounded. Indeed, by Friedrich's inequality, we have

$$|A(u, \varphi)| \leq C_1 \|\nabla u\|_{L^2} \|\nabla \varphi\|_{L^2} + C_2 \|\nabla u\|_{L^2} \|\varphi\|_{L^2} \leq C \|u\|_{W_0^{1,2}(\Omega)} \|\varphi\|_{W_0^{1,2}(\Omega)}$$

$A(u, \varphi)$ is coercive since

$$\begin{aligned} A(u, u) &= \sum_{i,j} \int_{\Omega} a_{ij} \partial_{x_j} u \partial_{x_i} u \, dx - \sum_i \int_{\Omega} b_i(x) \partial_{x_i} (u^2) \, dx \\ &\geq \alpha \|\nabla u\|_{L^2}^2 - \frac{1}{2} \int_{\Omega} \sum_i b_i(x) \partial_{x_i} (u^2) \, dx \\ &= \alpha \|\nabla u\|_{L^2}^2 + \frac{1}{2} \int_{\Omega} \operatorname{div} b \cdot u^2(x) \, dx \\ &= \alpha \|\nabla u\|_{L^2}^2 \end{aligned}$$

By the Lax-Milgram theorem, there exists a unique weak solution of 3.12. □

3.3 Introduction to spectral theory

H is a Hilbert space. $\mathcal{L}(H)$ is a space of linear continuous operators.

Lemma 3.1

A is continuous $\iff A$ is bounded, i.e.

$$\|A\| := \sup_{x \in H} \frac{\|Ax\|}{\|x\|} < \infty$$

Lemma 3.2

$(\mathcal{L}(H), \|\cdot\|)$ is a Banach space.

Definition 3.9: Invertible operator

A is invertible $\iff \exists A^{-1} \in \mathcal{L}(H)$ such that

$$AA^{-1} = A^{-1}A = I \tag{3.13}$$

Definition 3.10: Spectrum

$\lambda \in \sigma(A)$ (the spectrum of A) $\iff \lambda I - A$ is not invertible.

In other words, $\lambda \notin \sigma(A)$, (λ is in the resolvent set) iff the equation $\lambda u - Au = f$ has a unique solution for all $f \in H$.

Note:-

If $\dim H = n < \infty$, then $\mathcal{L}(H) = M(n \times n)$ ($n \times n$ matrices) and

1. All linear operators are continuous,
2. All $\lambda \in \sigma(A)$ correspond to eigenvalues

$$A\rho_\lambda = \lambda\rho_\lambda \implies \sigma(A) = \sigma_p(A) \text{ (point spectrum).}$$

3. $\lambda \in \sigma(A) \iff \det(\lambda I - A) = 0$
4. Only one of two equalities from 3.13 holding is enough.

All of the statements may fail when $\dim H = \infty$.

Example 3.7

Let $H = \ell_2$, the space of square summable sequences. Let $T_r, T_l \in \mathcal{L}(H)$ denote the right and left shift operators, respectively;

$$\begin{aligned} T_r(x_1, x_2, x_3, \dots) &= (0, x_1, x_2, x_3, \dots) \\ T_l(x_1, x_2, x_3, \dots) &= (x_2, x_3, \dots) \end{aligned}$$

Then $T_l \circ T_r = I$, but $T_r \circ T_l(x) = (0, x_2, x_3, \dots)$. Hence T_r has a left inverse, but not a right inverse. $\ker(T_r) = \{0\}$, i.e. it is injective (no eigenvalues), but the range of $T_r \neq H$, since $T_r(x) \perp e_1$ for any $x \in H$. So T_r is not invertible since $T_r(H)$ is a proper closed subspace of H . Therefore, a new type of spectrum appeared - the residual spectrum $\sigma_R(A)$, which is impossible in finite-dimensional spaces.

Also, the determinant does not exist in infinite-dimensional spaces. Indeed, if it existed, then

$$1 = \det(T_l \circ T_r) = \det(T_l) \det(T_r),$$

which implies that both T_l and T_r are invertible, but this is not true.

Definition 3.11: Approximate point spectrum

$\lambda \in \sigma_{\text{app}}(A)$ (approximate point spectrum) $\iff \exists x_n \in H, \|x_n\| = 1$ and $\lim_{n \rightarrow \infty} (Ax_n - \lambda x_n) = 0$.

It can be proved that $\lambda \in \sigma_{\text{app}}(A) \iff$ the image of $(\lambda I - H)$ is not closed.

Example 3.8

$H = L^2(0, 1), Af(x) := xf(x)$.

$$\lambda u - Au = f \iff (\lambda - x)u(x) = f(x),$$

then $u(x) = \frac{f(x)}{\lambda - x}$ and $(A - \lambda I)$ is not invertible $\iff \lambda \in [0, 1]$. Hence, $\sigma(A) = [0, 1]$.

We can check that $\text{Range}(\lambda I - A)$ is not closed if $\lambda \in [0, 1]$.

The next theorem shows that these are all the possible obstacles to invert the operator.

Theorem 3.8 Weyl's theorem

$A \in \mathcal{L}(H)$. Then $\sigma(A) = \sigma_p(A) \cup \sigma_{\text{app}}(A) \cup \sigma_R(A)$.

Definition 3.12: Resolvent

$R_A(\lambda) := (\lambda I - A)^{-1}$ is called the resolvent of $A \in \mathcal{L}(H)$.

More standard facts:

1. If $|\lambda| > \|A\|$, then $\lambda \in \sigma(A)$. Indeed,

$$\frac{1}{\lambda - A} = \frac{1}{\lambda} \frac{1}{1 - \frac{A}{\lambda}} = \frac{1}{\lambda} \sum_{n=0}^{\infty} \left(\frac{A}{\lambda} \right)^n$$

is an absolutely convergent series if $|\lambda| > \|A\|$.

2. Resolvent identity:

$$R_A(\lambda) - R_A(\mu) = -(\lambda - \mu)R_A(\lambda)R_A(\mu), \quad (3.14)$$

which follows from

$$\frac{1}{\lambda - x} - \frac{1}{\lambda - \mu} = (\mu - \lambda) \frac{1}{\lambda - x} \frac{1}{\mu - x},$$

where we substitute in $x = A$. From 3.14, by taking the limit as $\mu \rightarrow \lambda$, we get

$$\frac{d}{d\lambda} R_A(\lambda) = -R_A(\lambda)^2.$$

Hence $R_A(\lambda)$ is an analytic function of λ .

3. By Liouville's theorem applied to $R_A(\lambda)$, $\sigma(A) \neq \emptyset$

Definition 3.13: Compact operator

$A \in \mathcal{L}(H)$ is compact $\iff AB_1(0)$ is a precompact set in H .

Definition 3.14: Adjoint operator

Let $A \in \mathcal{L}(H)$. The adjoint operator $A^* \in \mathcal{L}(H)$ is defined via $(Ax, y) = (x, A^*y)$, $\forall x, y \in H$. It exists due to Riesz representation theorem. In the finite dimensional case, the adjoint operator coincides with the transpose operator.

Definition 3.15: Self-adjoint operator

$A \in \mathcal{L}(H)$ is self-adjoint if $A = A^*$

Definition 3.16: Fredholm

$A \in \mathcal{L}(H)$ is Fredholm if $\text{Range}(A)$ and $\text{Range}(A^*)$ are closed and $\ker(A)$ and $\ker(A^*)$ are both finite dimensional.

Then the index of A is defined by

$$\text{ind}(A) := \dim \ker(A) - \dim \ker(A^*)$$

Theorem 3.9 Key theorem of Fredholm operators theorem

$\text{ind}(A)$ is a topological invariant. Namely, if $A(t), t \in [0, 1]$ is a continuous curve of Fredholm operators,

then

$$\text{ind}(A(0)) = \text{ind}(A(1)).$$

Definition 3.17: Essential spectrum

$\lambda \in \sigma_{\text{ess}}(A)$ if $\lambda I - A$ is not Fredholm.

Properties:

1. K is compact $\iff K^*$ is compact.
2. $A \in \mathcal{L}(H), K$ compact $\implies AK$ and KA are compact.
3. A is Fredholm $\iff A^*$ is Fredholm.
4. Fredholm alternative: Let A be Fredholm. Then:

$$H = \text{Range}(A) \oplus \ker(A^*)$$

$$H = \text{Range}(A^*) \oplus \ker(A)$$

5. A is Fredholm \iff it is invertible by modulus of compact operators, i.e. $\exists B : AB = I + K_1, BA = I + K_2$, with K_1, K_2 compact.
6. Let K be a compact operator. Then $\sigma_{\text{ess}}(K) = 0$ and for any $\varepsilon > 0, \sigma(K) \setminus B_\varepsilon(0)$ consists of finitely many eigenvalues of finite multiplicity.
7. If $A = A^*$, then $\sigma(A)$ is real and $\sigma_R(A) = \emptyset$.

Thus, the simplest case is the case of positive, compact and self-adjoint operators.

Theorem 3.10 Hilbert-Schmidt

Let $A \in \mathcal{L}(H)$ be a compact, self-adjoint and positive ($\langle Ax, x \rangle > 0$ if $x \neq 0$) operator. Then there exists a sequence of non-zero real eigenvalues $\lambda_i \in \sigma_p(A)$ such that $|\lambda_i|$ is monotonically non-increasing

$$\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \dots,$$

and the corresponding eigenvectors $\{e_n\}_{n=1}^\infty$ ($Ae_n = \lambda e_n$) form the orthonormal basis in H . Moreover, any $x \in H$ can be written as

$$x = \sum_{n=1}^{\infty} \langle x, e_n \rangle e_n,$$

and A can be written as

$$Ax = \sum_{n=1}^{\infty} \lambda_n \langle x, e_n \rangle e_n.$$

Applications

Example 3.9 (Spectrum of the Laplacian with Dirichlet boundary conditions)

Let $\Omega \in \mathbb{R}^n$ be a bounded domain with $\partial\Omega$ smooth. Consider the Laplace equation with Dirichlet boundary conditions:

$$\begin{cases} -\Delta u = f, & f \in L^2(\Omega) \\ u|_{\partial\Omega} = 0 \end{cases}$$

$-\Delta$ is not a bounded operator in $H := L^2(\Omega)$, so we cannot directly apply the Hilbert-Schmidt theorem.

Let us consider the inverse operator $A = (-\Delta)^{-1}$ constructed via weak solutions, namely $u = (-\Delta)^{-1}f = Af$ solves $[u, \varphi] = (f, \varphi)$. That is, $\forall \varphi \in C_0^\infty(\Omega)$ (or H_0^1) and $u \in H_0^1(\Omega)$, we have

$$\int_{\Omega} \nabla u \nabla \varphi \, dx = \int_{\Omega} f \varphi \, dx$$

1. A is a bounded operator from H to $H_0^1(\Omega)$. Indeed, let $\varphi = u$. Then

$$\begin{aligned} \|u\|_{H_0^1}^2 &= (f, u) \\ &\leq \|f\|_{L^2} \|u\|_{L^2} \\ &\leq C \|f\|_{L^2} \|u\|_{H_0^1} \end{aligned}$$

Hence

$$\|u\|_{H_0^1} \leq C \|f\|_{L^2} \implies \frac{\|Af\|_{H_0^1}}{\|f\|_{L^2}} \leq C$$

2. A is compact since the embedding $H_0^1 \subset H$ is compact.
3. A is self-adjoint. Let $f, g \in L^2(\Omega)$, $u = (-\Delta)^{-1}f$, $v = (-\Delta)^{-1}g$. Take $\varphi = v$ in variational formulation for u , and $\varphi = u$ in variational formulation for v :

$$\begin{aligned} [u, v] &= (f, v) \\ [u, v] &= (g, u) \\ \implies (f, v) &= (g, u) \implies (f, Ag) = (g, A^*f) \implies A = A^* \end{aligned}$$

4. A is positive.

$$0 < [u, u] = (f, u) = (f, Af)$$

By the Hilbert-Schmidt theorem, there exists a complete orthonormal system $\{e_n\}$ of eigenvectors of A with $Ae_n = \lambda e_n$.

e_n by definition solves $\lambda_n[e_n, \varphi_n] = (e_n, \varphi)$. Indeed, for $u_n = Ae_n$, we have $[u_n, \varphi] = (e_n, \varphi)$. Since $u_n = \lambda_n e_n$, this implies that

$$[e_n, \varphi] = \lambda_n^{-1}(e_n, \varphi). \quad (3.15)$$

Hence e_n is a weak solution of

$$\begin{cases} -\Delta e_n = \lambda_n^{-1} e_n \\ e_n|_{\partial\Omega} = 0 \end{cases}$$

Consider the minimisation problem:

$$\min_{u \in H_0^1(\Omega)} \|\nabla u\|_{L^2}^2$$

under the constraint $\|u\|_{L^2}^2 = 1$. Let u_n be a minimising sequence with $\|u_n\|_{L^2}^2 = 1$ and $\|\nabla u_n\|_{L^2}^2 \rightarrow \lambda_0$. We want to prove that λ_0 is the minimum of u . We need to prove the existence of the minimiser and to find λ_0 .

1. u_n is bounded in $H_0^1(\Omega)$. $H_0^1(\Omega) \Subset L^2(\Omega)$ is compactly embedded, so it converges strongly to u_0 in $L^2(\Omega)$. Also, since ∇u is bounded, by Banach-Alaoglu theorem and reflexivity, $u_n \rightharpoonup u_0$ weakly in H_0^1 (up to a subsequence) and $\|\nabla u_0\|_{L^2} \leq \liminf_{n \rightarrow \infty} \|\nabla u_n\|_{L^2} = \lambda_0$. Since λ_0 is an infimum, then $\|\nabla u_0\|_{L^2} = \lambda_0$. Thus u_0 is a minimiser.
2. Let use the Euler-Lagrange method.

$$L_\lambda(u) = \|\nabla u\|_{L^2}^2 + \lambda \|u\|_{L^2}^2$$

Let φ be an arbitrary (smooth) function and define $R_\varphi(\varepsilon) := L_\lambda(u + \varepsilon\varphi)$, $\varepsilon \in \mathbb{R}$. Then the necessary condition for u to be a minimum of L is

$$\frac{d}{d\varepsilon} R_\varphi(\varepsilon) \Big|_{\varepsilon=0} = 0, \quad \forall \varphi.$$

$$\begin{aligned}
L_\lambda(u + \varepsilon\varphi) &= \|\nabla(u + \varepsilon\varphi)\|_{L^2}^2 + \lambda\|u + \varepsilon\varphi\|_{L^2}^2 \\
&= \|\nabla u\|_{L^2}^2 + 2\varepsilon(\nabla u, \nabla\varphi) + \varepsilon^2\|\nabla\varphi\|_{L^2}^2 + \lambda\|u\|_{L^2}^2 + 2\varepsilon\lambda(u, \varphi) + \varepsilon^2\|\varphi\|_{L^2}^2
\end{aligned}$$

Hence u is a minimiser of L if $\forall \varphi \in H_0^1$, we have

$$(\nabla u, \nabla\varphi) + \lambda(u, \varphi) = 0.$$

Comparing this equality with 3.15, we see that $u = u_0$ is the eigenvector of $(-\Delta)^{-1}$ and λ^{-1} is the corresponding eigenvalue (in this notation, $-\lambda > 0$). Equivalently, u_0 is the eigenvector of $-\Delta$ and $-\lambda$ is the corresponding eigenvalue. From the Hilbert-Schmidt theorem, we know that

$$-\frac{1}{\lambda_1} \geq -\frac{1}{\lambda_2} \geq -\frac{1}{\lambda_3} \geq \dots -\frac{1}{\lambda_i}, \quad (3.16)$$

where λ_i are eigenvalues of $(-\Delta)^{-1}$. Taking $\varphi = u$, we get $\|\nabla u\|_{L^2}^2 + \lambda\|u\|_{L^2}^2 = 0$, and if $u = e_1$, then

$$\|\nabla e_1\|_{L^2}^2 - \lambda_1\|e_1\|_{L^2}^2 = 0$$

and for any other u ,

$$\|\nabla u\|_{L^2}^2 - \lambda_1\|u\|_{L^2}^2 > 0$$

Hence for all $u \in H_0^1(\Omega)$,

$$\|u\|_{L^2}^2 \leq \frac{1}{\lambda_1}\|\nabla u\|_{L^2}^2.$$

3.4 Maximum principle

Theorem 3.11 Classical maximum principle

Let $u \in C^2(\Omega) \cap C(\bar{\Omega})$, where $\Omega \subset \mathbb{R}^n$ is a bounded domain, and $\nabla u = 0$. Then

$$\max_{x \in \partial\Omega} u(x) \geq u(x) \geq \min_{x \in \partial\Omega} u(x)$$

In other words, the maximum and minimum of a harmonic function is attained on the boundary.

Proof. Let us prove the max-inequality. Let $v = u + \varepsilon e^{x_1}$. Then $\nabla v = \varepsilon e^{x_1}$, $\varepsilon > 0$. Assume that the maximum is attained in the interior point $x_0 \in \Omega$. Then $\nabla u(x_0) = 0$, and the matrix of second derivatives must be non-positive. In particular, $\text{tr}(D^2u(x_0)) = \Delta v(x_0) \leq 0$, which implies that $\varepsilon e^{x_1} < 0$, a contradiction. Therefore, the maximum v is attained on the boundary. Taking the limit as $\varepsilon \rightarrow 0$, the same is true for u (but due to the limit, the inequalities become non-strict). The proof is analogous for the min-inequality. \square

This formulation does not exclude that the maximum u may be attained inside Ω . However, the strong-maximum principle claims that if the maximum or minimum of u is attained inside Ω , then u is constant.