Contents

Chapter 2

Sobolev spaces

2.1 Interpolation inequalities

Example 2.1

$$\|u\|_{L^2}^2 \leqslant \|u\|_{L^2}\|u'\|_{L^2} \text{ for } u \in C^\infty(\mathbb{R}) \tag{2.1}$$

Proof. Idea: use that $(u^2)' = 2uu'$ and Newton-Leibniz

$$\begin{aligned}
u^2(x) &= 2\int_{-\infty}^{x} uu' \,\mathrm{d}y = -2\int_{x}^{\infty} uu' \,\mathrm{d}y \\
&= \int_{-\infty}^{x} uu' \,\mathrm{d}y - \int_{x}^{\infty} uu' \,\mathrm{d}y \\
&\leqslant \int_{-\infty}^{x} |u||u'| \,\mathrm{d}y + \int_{x}^{\infty} |u||u'| \,\mathrm{d}y \\
&= \int_{\mathbb{R}} |u||u'| \,\mathrm{d}y \\
\text{(Hölder's inequality)} \quad &\leqslant \|u\|_{L^2}\|u'\|_{L^2}
\end{aligned}$$

□

Question 1

Check that 2.1 is sharp. Namely, that 2.1 becomes equality for $u(x) = e^{-|x|}$ ($u(x)$ is an extremal function for 2.1). Also, 2.1 is shift and scaling invariant, i.e. $u_\alpha(x+h) = e^{-\alpha|x+h|}, h \in \mathbb{R}, \alpha > 0$ -extremals.

Example 2.2 (Interpolation inequality)

Ω-domain in $\mathbb{R}^n, u \in L_{p_1}(\Omega) \cap L_{p_2}(\Omega), 1 \leqslant p_1, p_2, < \infty, p_1 < p_2, \theta \in [0,1], \frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}$. Then

$$\|u\|_{L^p} \leqslant \|u\|_{L^{p_1}}^{\theta}\|u\|_{L^{p_2}}^{1-\theta} \tag{2.2}$$

Proof.

$$\int_{\mathbb{R}} |u|^p \,\mathrm{d}x = \int_{\mathbb{R}} |u|^{\theta p}|u|^{(1-\theta)p} \,\mathrm{d}x$$

We apply Hölder's inequality with exponents $P = \frac{p_1}{\theta p}$ and $Q = \frac{p_2}{(1-\theta)p}$ (Note $\frac{1}{P} + \frac{1}{Q} = \frac{\theta p}{p_1} + \frac{(1-\theta)p}{p_2} = 1$). Then

$$\begin{aligned}\int_{\mathbb{R}} |u|^{\theta p}|u|^{(1-\theta)p}\,\mathrm{d}x &\leqslant \left(\int_{\mathbb{R}} |u|^{p_1}\,\mathrm{d}x\right)^{\frac{1}{P}} \left(\int_{\mathbb{R}} |u|^{p_2}\,\mathrm{d}x\right)^{\frac{1}{Q}} \\ &= \|u\|_{L^{p_1}}^{\theta} \|u\|_{L^{p_2}}^{1-\theta}\end{aligned}$$

□

2.2 Sobolev inequalities

Example 2.3 (Sobolev inequality 1D)
$u \in C^\infty([0,1])$, want to prove the embedding $W^{1,1}([0,1]) \subset C([0,1])$, i.e.

$$\|u\|_{C([0,1])} \leqslant \|u\|_{L^1([0,1])} + \|u'\|_{L^1([0,1])} \tag{2.3}$$

Proof. By the Newton-Leibniz formula, $u(x) - u(y) = \int_y^x u'(s)\,\mathrm{d}s$. Also,

$$|u(x)| \leqslant |u(y)| + \int_0^1 |u'(s)|\,\mathrm{d}s \quad \forall x, y \in [0,1]$$

By integration over $y \in [0,1]$,

$$|u(x)| \leqslant \int_0^1 |u(s)|\,\mathrm{d}s + \int_0^1 |u'(s)|\,\mathrm{d}s = \|u\|_{W^{1,1}([0,1])}$$

Taking supremum with respect to $x \in [0,1]$, we obtain $\|u\|_{C([0,1])} \leqslant \|u\|_{W^{1,1}([0,1])}$ □

Example 2.4 (Sobolev inequality 2D)
$u \in C^\infty([0,1]^2)$, i.e. $\Omega = [0,1]^2$, then $W^{1,1}(\Omega) \subset L^2(\Omega) : \|u\|_{L^2} \leqslant \|u\|_{W^{1,1}(\Omega)}$

Proof. $\int_\Omega u^2(x_1,x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2$ should be estimated. From 2.3, we know that

$$|u(x_1,x_2)| \leqslant \int_0^1 |u(s,x_2)| + |\partial_{x_1} u(s,x_2)|\,\mathrm{d}s \coloneqq f(x_2)$$

$$|u(x_1,x_2)| \leqslant \int_0^1 |u(x_1,s)| + |\partial_{x_2} u(x_1,s)|\,\mathrm{d}s \coloneqq g(x_1)$$

Then

$$\begin{aligned}\int_\Omega u^2\,\mathrm{d}x &\leqslant \int_0^1 g(x_1) f(x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2 \\ &= \int_0^1 f(x_2)\,\mathrm{d}x_2 \int_0^1 g(x_1)\,\mathrm{d}x_1 \\ &= \left(\int_\Omega |u(x_1,x_2)| + |\partial_{x_1} u(x_1,x_2)|\,\mathrm{d}x_1\right)\left(\int_\Omega |u(x_1,x_2)| + |\partial_{x_2} u(x_1,x_2)|\,\mathrm{d}x_2\right) \\ &\leqslant \|u\|_{W^{1,1}(\Omega)}\end{aligned}$$

□

Question 2: Sobolev inequality 3D

$u \in C^\infty(\bar{\Omega}), \Omega = (0,1)^3$. Prove that $W^{1,1}(\Omega) \subset L^{\frac{3}{2}}(\Omega)$, i.e.

$$\|u\|_{L^{\frac{3}{2}}(\Omega)} \leqslant \|u\|_{W^{1,1}(\Omega)} \tag{2.4}$$

Hint: first, prove that

$$\int_\Omega f(x_1,x_2)g(x_2,x_3)h(x_1,x_3)\,\mathrm{d}x \leqslant \|f\|_{L^2}\|g\|_{L^2}\|h\|_{L^2}$$

and use 2.3.

Example 2.5

$u \in C^\infty(\bar{\Omega}), \Omega = (0,1)^3$. Then

$$\|u\|_{L^6(\Omega)} \leqslant C\|u\|_{W^{1,2}(\Omega)} \tag{2.5}$$

Proof.

$$\begin{aligned}
\int_\Omega |u|^6\,\mathrm{d}x &= \int_\Omega (|u|^4)^{\frac{3}{2}}\,\mathrm{d}x \\
&\leqslant C\left(\int_\Omega |u|^4\,\mathrm{d}x + \int_\Omega u^3|\nabla u|\,\mathrm{d}x\right)^{\frac{3}{2}} \\
\text{(by (2.3))} \quad &\leqslant C\left(\int_\Omega |u|^4\,\mathrm{d}x\right)^{\frac{3}{2}} + C\left(u^3|\nabla u|\,\mathrm{d}x\right)^{\frac{3}{2}} \\
&\leqslant C\|u\|_{L^2}^{\frac{3}{2}\cdot\theta\cdot 4}\|u\|_{L^6}^{\frac{3}{2}\cdot(1-\theta)\cdot 4} + C\|u\|_{L^6}^{\frac{3}{2}\cdot 3}\|\nabla u\|_{L^2}^{\frac{3}{2}} \\
\left(\theta = \frac{1}{4}\right) \quad &= C\|u\|_{L^2}^{\frac{3}{2}}\|u\|_{L^6}^{\frac{9}{2}} + C\|u\|_{L^6}^{\frac{9}{2}}\|\nabla u\|_{L^2}^{\frac{3}{2}} \\
\left(\text{Young's inequality with } p = \frac{4}{5} \text{ and } q = -4\right) \quad &\leqslant \varepsilon\|u\|_{L^6}^6 + C_\varepsilon(\|u\|_{L^2} + \|\nabla u\|_{L^2})^6
\end{aligned}$$

Setting for example, $\varepsilon = \frac{1}{2}$, we obtain

$$\|u\|_{L^6(\Omega)} \leqslant C\|u\|_{W^{1,2}(\Omega)}$$

□

Theorem 2.1 Sobolev embeddings

① $W^{k_1,p_1}(\Omega) \subset W^{k_2,p_2}(\Omega) \iff k_1 \geqslant k_2$ and $1 \leqslant p_1, p_2 < \infty, k_1 - \frac{n}{p_1} \geqslant k_2 - \frac{n}{p_2}, \Omega \subset \mathbb{R}^n$.

② $W^{k,p}(\Omega) \subset C^\alpha(\Omega)$ if $\alpha < k - \frac{n}{p}$. If α is not an integer, then the inequality is weak.

Example 2.6

$H^s(\mathbb{R}^n) \subset C(\mathbb{R}^n) \iff s > \frac{n}{2}$

Proof. $u(x) = \int_{\mathbb{R}^n} e^{i\xi x}\hat{u}(\xi)\,\mathrm{d}\xi$

$$\begin{aligned} |u(x)| &\leqslant \int_{\mathbb{R}^n} |\hat{u}(\xi)|\,\mathrm{d}\xi \\ &= \int_{\mathbb{R}^n} \left(1+|\xi|^2\right)^{-\frac{s}{2}} \left(1+|\xi|^2\right)^{\frac{s}{2}} |\hat{u}(\xi)|\,\mathrm{d}\xi \\ \text{(Hölder's inequality)} \quad &\leqslant \left(\int_{\mathbb{R}^n} \frac{1}{\left(1+|\xi|^2\right)^s}\,\mathrm{d}\xi\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^n} \left(1+|\xi|^2\right)^s |\hat{u}(\xi)|^2\,\mathrm{d}\xi\right)^{\frac{1}{2}} \end{aligned}$$

$\int_{\mathbb{R}^n} \frac{1}{(1+|\xi|^2)^s}\,\mathrm{d}\xi < \infty \iff s > \frac{n}{2}$. Taking the supremum with respect to $x \in \mathbb{R}^n$, we get

$$\|u\|_{C(\mathbb{R}^n)} \leqslant C_s \|u\|_{H^s(\mathbb{R}^n)}$$

□

Theorem 2.2 Interpolation inequalities

Let $u \in W^{k_1,p_1}(\Omega) \cap W^{k_2,p_2}(\Omega)$, $\theta \in [0,1]$, $1 \leqslant p_1, p_2 \leqslant \infty$ with $k = \theta k_1 + (1-\theta)k_2$, $\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}$. Then

$$\|u\|_{W^{k,p}} \leqslant C\|u\|_{W^{k_1,p_1}}^{\theta} \|u\|_{W^{k_2,p_2}}^{1-\theta}$$

Corollary 2.1 Particular cases

1. $\|u\|_{H^1} \leqslant \|u\|_{L^2}^{\frac{1}{2}} \|u\|_{H^2}^{\frac{1}{2}}$
2. $\|u\|_{L^p} \leqslant \|u\|_{L^p}^{\theta} \|u\|_{H^2}^{1-\theta}$

2.3 Spaces with zero boundary traces

Definition 2.1

$W_0^{1,p}(\Omega) \coloneqq \left\{u \in W^{1,p}(\Omega),\ u|_{\partial\Omega} = 0\right\}$
An equivalent definition is that the Sobolev spaces $W_0^{1,p}(\Omega)$ for $1 \leqslant p < \infty$ are defined as the closure of the set of compactly supported test functions $C_0^\infty(\Omega)$ with respect to the $W^{1,p}(\Omega)$-norm.

Lemma 2.1

These two definitions are equivalent. $u \in$ "closure": $u = \lim_{n\to\infty} \varphi_n, \varphi_n \in C_0^\infty(\Omega) \implies \varphi_n|_{\partial\Omega} = 0$. By continuity, $u|_{\partial\Omega} = 0$. The proof of the converse statement is more technical and is omitted.

2.4 Poincaré's and Friedrich's inequalities

Proposition 2.1 Friedrich's inequality

Let Ω be a bounded domain and $u \in W_0^{1,p}(\Omega)$. Then

$$\|u\|_{L^p} \leqslant C\|\nabla u\|_{L^p} \tag{2.6}$$

Proof. It is enough to prove 2.6 for $\varphi \in C_0^\infty(\Omega)$. By the Newton-Leibniz formula,

$$u(x_1, x') - u(-L, x') = u(x_1, x') = \int_{-L}^{x_1} \partial_{x_1} u(s, x')\, \mathrm{d}s$$

$$\begin{aligned} |u(x_1, x')|^p &\leqslant \left(\int_{-L}^{L} |\partial_{x_1} u(s, x')|\, \mathrm{d}s \right)^p \\ \text{(Hölder's inequality)} \quad &\leqslant C_L \int_{-L}^{L} |\partial_{x_1} u(s, x')|^p\, \mathrm{d}s \end{aligned}$$

Integration with respect to x' gives us

$$\int_{\mathbb{R}^{n-1}} |u(x_1, x')|^p\, \mathrm{d}x' \leqslant C_L \|\partial_{x_1} u\|_{L^p}^p$$

Finally, integrating over $x_1 \in [-L, L]$, we obtain

$$\|u\|_{L^p}^p \leqslant 2LC_L \|\partial_{x_1} u\|_{L^p}^p$$

□

Corollary 2.2 Equivalent norm in $W_0^{1,p}(\Omega)$

Homogeneous norm:

$$\|u\|_{W_0^{1,p}(\Omega)} := \|\nabla u\|_{L^p}$$

Note:-

$u|_{\partial\Omega} = 0$ is important! Otherwise, 2.6 will fail for $u \equiv c$. Since ∇u defines u up to a constant; $u|_{\partial\Omega} = 0$ removes this constant.

Proposition 2.2 Poincaré inequality

Let Ω be a bounded domain with a smooth boundary and $\langle u \rangle := \frac{1}{|\Omega|} \int_\Omega u(x)\, \mathrm{d}x = 0$. Then

$$\|u\|_{L^p} \leqslant C \|\nabla u\|_{L^p}$$

2.5 Compactness

Definition 2.2: Sequential compactness

A metric space (X, d) is compact if any sequence $\{x_n\}_{n=1}^\infty \subset X$ has a convergent sub-sequence, i.e. there exists $\{x_{n_k}\}_{k=1}^\infty$: $\lim_{k\to\infty} x_{n_k} = x_0 \in X$

Definition 2.3

A topological space X is compact if any covering of X by open sets has a finite sub-covering

Note:-

In metric spaces, compactness is equivalent to sequential compactness.
In general topological spaces, they are not related.

Theorem 2.3 Hausdorff

Let (X, d) be a metric space. Then X is compact $\iff$ X is complete and totally bounded.

Definition 2.4

X is totally bounded if $\forall \varepsilon > 0, \exists$ covering of X by finitely many ε-balls, i.e. $X = \bigcup_{k=1}^{N} B_\varepsilon(x_k), N = N(\varepsilon)$ and $\{x_k\}$ is an ε-net in X.

Why do we need compactness?

Let X be compact and $f : X \to Y$ be continuous, then $f(X)$ is compact in Y. How do we solve PDEs of the form (or more general equations)?

$$F(x) = 0 \tag{2.7}$$

1. Construct approximate solutions
   $F(x_n) = g_n$, where $\lim_{n\to\infty} g_n = 0$
2. Obtain a priori estimates, i.e. that $\{x_n\}$ is bounded in a proper space
3. If $\{x_n\}$ is pre-compact and F is continuous $\implies x = \lim_{x\to\infty} x_{n_k}$ is a solution of 2.7.

Theorem 2.4 Arzelà-Ascoli

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Then $V \subset C(\bar{\Omega})$ is compact iff:

1. V is closed
2. V is bounded
3. V is equicontinuous = V has a common modulus of continuity

Theorem 2.5 Arzelà-Ascoli for L^p

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, (and $\partial\Omega$ smooth, although not needed), $K \subset L^p(\Omega), 1 \leqslant p < \infty$. Then K is compact iff:

1. K is closed
2. K is bounded
3. K is equicontinuous in mean (possesses a joint modulus of continuity in L^p).

Definition 2.5

Let $f \in L^p(\Omega), 1 \leqslant p < \infty, \Omega \subset \mathbb{R}^n$ bounded ($\partial\Omega$ smooth not needed). $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\lim_{z\to 0} w(z) = 0$ is a modulus of continuity of f in $L_p(\Omega)$ if

$$\int_\Omega |f(x+h) - f(x)|^p \, \mathrm{d}x \leqslant \omega(|h|), \quad \forall h \in \mathbb{R}^n,$$

where we used the 0-extension of f outside of Ω.

Corollary 2.3

Let $K = B_1(0) \in W^{1,p}(\Omega); \Omega \subset \mathbb{R}^n$ is bounded, $\partial\Omega$ is smooth, $1 \leqslant p < \infty$. Then K is pre-compact in $L^p(\Omega)$.

Proof. We need to check equicontinuity, i.e. estimate $\int_\Omega |f(x+h) - f(x)|^p \, dx$.

$$f(x+h) - f(x) = h \int_0^1 \nabla f(x + sh) \, ds$$

Taking modulus and p-th power of both sides, we get

$$|f(x+h) - f(x)|^p \leqslant |h| \int_0^1 |\nabla f(x + sh)|^p \, ds$$

Finally, we take an integral over $x \in \Omega$.

$$\begin{aligned} \int_\Omega |f(x+h) - f(x)|^p \, dx &\leqslant |h| \int_0^1 \int_\Omega |\nabla f(x+sh)|^p \, dx \, ds \\ &\leqslant C|h| \end{aligned}$$

$\omega(z) = cz$ is a joint modulus of continuity. □

Definition 2.6

Let $V \subset W$ be Banach spaces. Then the embedding is compact if the unit ball of V is pre-compact in W.

Note:-

We proved that $W^{1,p}(\Omega) \subset L^p(\Omega)$ is a compact embedding.

Corollary 2.4

$W^{1,p}(\Omega) \subset L^q(\Omega)$ is a compact embedding if $q < q^*$, where q^* is defined such that $\frac{1}{q^*} = \frac{1}{p} - \frac{1}{n}$ and $\Omega \subset \mathbb{R}^n$, Ω is bounded, $\partial\Omega$ is smooth.

Proof. Let us check equicontinuity.

$$\|f(\cdot + h) - f(\cdot)\|_{L^q} \leqslant \|f(\cdot + h) - f(\cdot)\|_{L^p}^\theta \|f(\cdot + h) - f(\cdot)\|_{L^{q^*}}^{1-\theta}$$

since $p < q < q^*$ and $0 < \theta < 1$. q^* is a critical exponent in Sobolev embeddings, indeed, $W^{1,p}(\Omega) \subset L^q(\Omega) \implies 1 - \frac{n}{p} \geqslant -\frac{1}{q}$. Then by corollary 2.3, we have

$$\begin{aligned} \|f(\cdot + h) - f(\cdot)\|_{L^p}^\theta \|f(\cdot + h) - f(\cdot)\|_{L^{q^*}}^{1-\theta} &\leqslant C|h|^\theta (2\|f\|_{L^{q^*}})^{1-\theta} \\ &\leqslant C_1 |h|^\theta \|f\|_{W^{1,p}}^{1-\theta} \\ &\leqslant C_1 |h|^\theta \end{aligned}$$

□

General fact: $W^{s_1,p_1}(\Omega) \subset W^{s_2,p_2}(\Omega)$, where Ω is bounded, $\partial\Omega$ is smooth. Embedding is compact $\iff$ embedding is not critical.

2.6 Dual spaces

Definition 2.7

$W^{-s,p}(\Omega) := \left(W_0^{s,q}(\Omega)\right)^*$ is defined as the dual space to $W_0^{s,q}(\Omega)$ =, i.e. the space of linear continuous functionals on $W_0^{s,q}(\Omega)$, where $\frac{1}{p} + \frac{1}{q} = 1$.

Definition 2.8

$$W^{-s,p}(\Omega) = \left\{ \text{completion of } L^p(\Omega) \text{ w.r.t } \|\ell\|_{W^{-s,p}} := \sup_{\varphi \in \mathcal{D}} \frac{|(\ell,\varphi)|}{\|\varphi\|_{W_0^{s,q}}} \right\}$$

Definition 2.9

$$W^{-s,p}(\Omega) = \left\{ \ell \in \mathcal{D}'(\Omega) : \|\ell\|_{W^{-s,p}} := \sup_{\varphi \in \mathcal{D}} \frac{|\langle \ell,\varphi \rangle|}{\|\varphi\|_{W_0^{s,q}}} \right\}$$

Proposition 2.3

Definitions 2.7, 2.8 and 2.9 are equivalent.

Question 3

Suppose $\delta(x) \in W^{-s,p}(\Omega), \Omega \subset \mathbb{R}^n$. How are s, p and n related? We know that $\delta(x)$ is well-defined on continuous functions, so we need $W_0^{s,q}(\Omega) \subset C(\bar{\Omega})$.

Example 2.7

Consider the case where $n = 1$ and $p = 2$. By the Sobolev embedding theorem, $W^{s,2} \subset C(\bar{\Omega})$ if $0 < s - \frac{1}{2}$. Thus we have $\delta(x) \in H^{-s}(\Omega)$ if $s > \frac{1}{2}$.

Chapter 3

Linear elliptic problems

3.1 Dirichlet and Neumann problems for the Laplacian

Example 3.1 (Laplace equation with Dirichlet boundary conditions)
Let $\Omega \in \mathbb{R}^n$ be a bounded domain with $\partial\Omega$ smooth. Consider the Laplace equation with Dirichlet boundary conditions:

$$\begin{cases} \Delta u = f \\ u|_{\partial\Omega} = 0 \end{cases} \tag{3.1}$$

Typical questions:

1. In what space does the solution live?
2. In what sense is the equation understood (classical / weak)?
3. In what sense are the boundary / initial data understood?

In ODEs, we have local existence and uniqueness theorem (for Lipschitz non-linearities), but there is not an equivalent theorem for PDEs. Therefore, we must study particular examples.

Definition 3.1

$u \in W_0^{1,2}(\Omega)$ is a weak solution of 3.1 if $\forall \varphi \in C_0^\infty(\Omega)$,

$$-\int_\Omega \nabla u(x)\nabla\varphi(x)\,\mathrm{d}x = \int_\Omega f(x)\varphi(x)\,\mathrm{d}x \tag{3.2}$$

Here, the boundary condition is incorporated into the choice of space $W_0^{1,2}(\Omega) = [C_0^\infty(\Omega)]_{W^{1,2}(\Omega)}$ (the closure of $C_0^\infty(\Omega)$ in the norm of $W^{1,2}(\Omega)$).
3.2 came from the integration by parts formula. Indeed, if $u \in C^2(\Omega) \cap C(\bar{\Omega})$, then $\Delta u = f$ is understood in a classical sense and

$$\int_\Omega \Delta u\varphi\,\mathrm{d}x = -\int_\Omega \nabla u\nabla\varphi\,\mathrm{d}x + \int_{\partial\Omega} \partial_n u\varphi\,\mathrm{d}s,$$

where the term $\int_{\partial\Omega} \partial_n u\varphi\,\mathrm{d}s = 0$ because $\varphi|_{\partial\Omega} = 0$.

Theorem 3.1
Let $f \in H^{-1}(\Omega) := W^{-1,2}(\Omega)$. Then 3.1 has a unique weak solution.

Proof. Application of Riesz representation theorem

$[u,u] := \int_\Omega \nabla u \nabla u \, dx$ is an equivalent norm on $W_0^{1,2}(\Omega)$ (due to Friedrich's inequality). Then 3.2 can be rewritten as

$$[u,\varphi] = \int_\Omega f(x)\varphi(x)\, dx := \ell(\varphi)$$

Claim: ℓ is a linear continuous functional on $W_0^{1,2}(\Omega)$ (the integral should be understood as duality if we take $f \in H^{-1}(\Omega)$ and if $f \in L^2(\Omega)$, this is a standard Lebesgue integral).
Linearity of ℓ is obvious. ℓ is continuous as it is bounded:

$$|\ell(\varphi)| \leqslant \|f\|_{H^{-1}} \|\varphi\|_{H^1}$$

But we obtained that 3.2 holds only for $\varphi \in C_0^\infty(\Omega)$, not for $\varphi \in W_0^{1,2}(\Omega)$. However, $W_0^{1,2}(\Omega) = [C_0^\infty(\Omega)]_{W^{1,2}}$. Then approximation arguments give that $\forall \varphi \in H$,

$$[u,\varphi] = \ell(\varphi) \tag{3.3}$$

Then by Riesz representation theorem, there exists a unique $u \in W_0^{1,2}(\Omega)$ which satisfies 3.3. ◻

Example 3.2 (Laplace equation with Neumann boundary conditions)
Let $\Omega \in \mathbb{R}^n$ be a bounded domain with $\partial\Omega$ smooth. Consider the Laplace equation with Neumann boundary conditions:

$$\begin{cases} \Delta u = f \\ \partial_n u|_{\partial\Omega} = 0 \end{cases} \tag{3.4}$$

We cannot consider $\varphi \in C_0^\infty(\Omega)$ as test functions, because the information about boundary conditions will be lost. Similarly, considering

$$\varphi \in W_n^{1,2}(\Omega) := \{u \in W^{1,2}(\Omega) \colon \ \partial_n u|_{\partial\Omega} = 0\}$$

will not work as well, since $\partial_n u|_{\partial\Omega}$ is not defined for $u \in W^{1,2}(\Omega)$ (since by theorem 2.1, $C^\infty(\Omega) \not\subset W^{1,2}(\Omega)$). Instead, let us take $\varphi \in C^\infty(\bar{\Omega})$ as a test function and assume that u is a classical solution. Then

$$\begin{aligned} \int_\Omega f\varphi\, dx &= \int_\Omega \Delta u \varphi\, dx \\ &= -\int_\Omega \nabla u \nabla\varphi\, dx + \int_{\partial\Omega} \partial_n u \varphi\, ds \\ &= -\int_\Omega \nabla u \nabla \varphi\, dx, \end{aligned}$$

as $\int_{\partial\Omega} \partial_n u\varphi\, dx = 0$ due to the boundary conditions. If we take $\varphi(x) = 1$ as a test function, then we get

$$\begin{aligned} \int_\Omega f \cdot 1\, dx &= -\int_\Omega \nabla u \nabla 1\, dx \\ &= 0 \end{aligned}$$

Hence $\langle f \rangle = \frac{1}{|\Omega|}\int_\Omega f(x)\, dx = 0$ is a necessary condition for solvability.
Let us notice that all solutions of this problem differs from each other by a constant. Thus, a natural assumption to single out the solution is $\langle u \rangle = 0$.

Definition 3.2

$u \in W^{1,2}(\Omega) \cap \{\langle u \rangle = 0\}$ is a weak solution of 3.4 if $\forall \varphi \in C^\infty(\bar{\Omega})$, we have:

$$\int_\Omega \nabla u \nabla \varphi\, dx = -\int_\Omega f\varphi\, dx \tag{3.5}$$

Note:-

The boundary conditions are now not in the definition of the space, but in 3.5.

Theorem 3.2

Let $f \in L^2(\Omega) \cap \{\langle f \rangle = 0\}$. Then 3.4 has a unique weak solution.

Proof. The proof is analogous to the problem with Dirichlet boundary conditions, but instead of applying Friedrich's inequality, we should apply Poincaré's inequality and use density of $C^\infty(\Omega) \in W^{1,2}(\Omega)$. ☐

Example 3.3 (Non-homogeneous Neumann boundary conditions)

Let $\Omega \in \mathbb{R}^n$ be a bounded domain with $\partial\Omega$ smooth. Consider the Laplace equation with non-homogeneous Neumann boundary conditions:

$$\begin{cases} \Delta u = f \\ \partial_n u|_{\partial\Omega} = g \end{cases} \tag{3.6}$$

Definition 3.3

$u \in W^{1,2}(\Omega) \cap \{\langle u \rangle = 0\}$ is a weak solution of 3.6 if $\forall \varphi \in C^\infty(\bar{\Omega})$, we have:

$$\int_\Omega \nabla u \nabla \varphi \,\mathrm{d}x = -\int_\Omega f\varphi \,\mathrm{d}x + \int_{\partial\Omega} g\varphi \,\mathrm{d}s \tag{3.7}$$

Note that if $\varphi \equiv 1$, then a necessary condition for solvability is

$$-\int_\Omega f \,\mathrm{d}x + \int_{\partial\Omega} g \,\mathrm{d}s = 0$$

Theorem 3.3

Let $f \in L^2(\Omega), g \in W^{-\frac{1}{2},2}(\partial\Omega)$ be such that $\int_\Omega f \,\mathrm{d}x = \int_{\partial\Omega} g \,\mathrm{d}s$. Then 3.6 has a unique weak solution.

Proof. $[u, u] := \int_\Omega \nabla u \nabla u ds$ is an equivalent norm on $u \in W^{1,2}(\Omega) \cap \{\langle u \rangle = 0\}$ due to the Poincaré inequality. Then 3.7 can be rewritten as

$$[u, \varphi] = \ell(\varphi) := -\int_\Omega f\varphi \,\mathrm{d}x + \int_{\partial\Omega} g\varphi \,\mathrm{d}s$$

We claim that ℓ is a linear continuous functional on $W^{1,2}(\Omega) \cap \{\langle u \rangle = 0\}$. Indeed, linearity is obvious. To show ℓ is continuous, we have

$$\begin{aligned} \left| -\int_\Omega f\varphi \,\mathrm{d}x + \int_{\partial\Omega} g\varphi \,\mathrm{d}s \right| &\leqslant \|f\|_{L^2} \|\varphi\|_{L^2} + \|g\|_{H^{-\frac{1}{2}}(\partial\Omega)} \|\varphi\|_{H^{\frac{1}{2}}(\partial\Omega)} \\ \text{(By the trace theorem and Poincaré's inequality)} &\leqslant \|f\|_{L^2} \|\varphi\|_{W^{1,2}(\Omega)} + \|g\|_{H^{-\frac{1}{2}}(\partial\Omega)} \|\varphi\|_{W^{1,2}(\Omega)} \end{aligned}$$

Then by Riesz representation theorem, there exists a unique $u \in W^{1,2}(\Omega) \cap \{\langle u \rangle = 0\}$ that is a weak solution of 3.6. ☐

Example 3.4 (Non-homogeneous Dirichlet boundary conditions)

Let $\Omega \in \mathbb{R}^n$ be a bounded domain with $\partial\Omega$ smooth. Consider the Laplace equation with non-homogeneous

Dirichlet boundary conditions:

$$\begin{cases} \Delta u = 0 \\ u|_{\partial\Omega} = g \end{cases} \tag{3.8}$$

Let us take $g \in W^{\frac{1}{2},2}(\partial\Omega)$. Then there exists $v \in W^{1,2}(\Omega)$ such that $v|_{\partial\Omega} = g$ (by the trace theorem). We look for the solution of 3.8 in the form $u = v + w$, where $w \in W_0^{1,2}(\Omega)$.

Definition 3.4

$u = v + w$ is a weak solution of 3.8 if $v|_{\partial\Omega} = g$, where $g \in W^{\frac{1}{2},2}(\partial\Omega), w \in W_0^{1,2}(\Omega)$ and $\forall \varphi \in C^\infty(\bar{\Omega})$, we have

$$\int_\Omega \nabla(v + w)\nabla\varphi \,\mathrm{d}x = 0 \tag{3.9}$$

Theorem 3.4

Let $g \in W^{\frac{1}{2},2}(\partial\Omega)$. Then 3.8 has a unique weak solution.

Proof. We can rearrange 3.9 to get

$$\ell(\varphi) := -[v, \varphi] = \int_\Omega \nabla w \nabla\varphi \,\mathrm{d}x = [w, \varphi],$$

and the functional ℓ can be shown to be linear and continuous. By the Riesz representation theorem, there exists a unique $w \in W^{1,2}(\Omega)$ such that 3.9 is satisfied. Note that this w depends on the choice of v. But $u = v + w$ does not depend on the choice of v. Indeed, let u_1 and u_2 be two solutions of 3.8. Then $u = u_1 - u_2$ solves

$$\begin{cases} \Delta u = 0 \\ u|_{\partial\Omega} = 0 \end{cases}$$

We have previously shown that the weak solution of this problem is unique. Therefore, $u_1 = u_2$. □

Note:-

There is no universal choice of the space of test functions. Even for Dirichlet and Neumann boundary conditions, we need to consider different spaces. $\varphi \in C_0^\infty(\Omega)$ corresponds to the standard theory of distributions, while $\varphi \in C^\infty(\bar{\Omega})$ corresponds to "non-standard" distributions.

Example 3.5

Let $\Omega \in \mathbb{R}^n$ be a bounded domain with $\partial\Omega$ smooth. Consider

$$\begin{cases} \sum_{i,j} \partial_{x_i}(a_{ij}(x)\partial_{x_j} u) = g \\ u|_{\partial\Omega} = 0 \end{cases} \tag{3.10}$$

Where we make the following assumptions on the matrix $a(x) := \{a_{ij}(x)\}_{i,j}$:

1. $a(x)$ is a symmetric matrix for every x:

$$a_{ij}(x) = a_{ji}(x)$$

2. $a(x)$ is uniformly elliptic. That is, for all $\xi \in \mathbb{R}^n$, there exists $\mu, M > 0$ which are independent of x such that

$$\mu|\xi^2| \leqslant \sum_{i,j} a_{ij}\xi_i\xi_j \leqslant M|\xi^2|$$

Definition 3.5

$u \in W^{1,2}(\Omega)$ is a weak solution to 3.10 $\iff$ $\forall \varphi \in C_0^\infty(\Omega)$, we have

$$\sum_{i,j} \int_\Omega a_{ij} \partial_{x_j} u \partial_{x_i} \varphi \, \mathrm{d}x = -\int_\Omega g\varphi \, \mathrm{d}x$$

Theorem 3.5

Let $a(x)$ be symmetric and uniformly elliptic. Then 3.10 has a unique weak solution.

Proof. Let us denote

$$[u, \varphi]_a = \int_\Omega \sum_{i,j} a_{ij}(x) \partial_{x_j} u(x) \partial_{x_i} \varphi(x) \, \mathrm{d}x.$$

Then since $a(x)$ is symmetric, the bilinear form $[u, v]_a$ is also symmetric, i.e. $[u, v]_a = [v, u]_a$. Since $a(x)$ is uniformly elliptic, there exist $\mu, M > 0$ such that

$$\mu[u, u] \leqslant [u, u]_a \leqslant M[u, u].$$

Therefore, $\left(W_0^{1,2}(\Omega), [\cdot, \cdot]_a\right)$ is a Hilbert space with the norm equivalent to the standard $W_0^{1,2}(\Omega)$ norm.
By the Riesz representation theorem, there exists a unique weak solution to 3.10. □

3.2 More general problems via Lax-Milgram

By the Riesz representation theorem, for any linear continuous functional, ℓ on a Hilbert space H, there exists a unique $x \in H$ such that $\forall \varphi \in H$, we have $(x, \varphi) = \ell(\varphi)$.
If we want $a(x, y)$ to be an equivalent inner product on H, then $a(x, y)$ must be symmetric.
We now consider the case where $a(x, y)$ is not assumed to be symmetric.

Definition 3.6: Bilinear form

A bilinear form $a(\cdot, \cdot) \colon H \times H \to \mathbb{R}$ is bounded if

$$|a(x, y)| \leqslant C\|x\|\|y\|$$

Definition 3.7: Coercive

A bilinear form $a(\cdot, \cdot)$ is coercive if $\exists \alpha > 0$ such that $a(x, x) \geqslant \alpha \|x\|^2$.

Theorem 3.6

Let $a(x, y)$ be a bounded and coercive bilinear form on H. Then any linear continuous functional $\ell \colon H \to \mathbb{R}$ can be represented in the form

$$a(x, y) = \ell(\varphi), \quad \forall \varphi \in H. \tag{3.11}$$

i.e. $\forall \ell \in H^*$, there exists a unique $x = x(\ell) \in H$ such that 3.11 is satisfied.

Example 3.6

Let $\Omega \in \mathbb{R}^n$ be a bounded domain with $\partial\Omega$ smooth. Consider the problem

$$\begin{cases} \sum_{i,j} \partial_{x_i}(a_{ij}(x) \partial_{x_j} u) + \sum_i b_i(x) \partial_{x_i} u = g(x) \\ u|_{\partial\Omega} = 0 \end{cases} \tag{3.12}$$

Definition 3.8

$u \in W_0^{1,2}(\Omega)$ is a weak solution of 3.12 if $\forall \varphi \in C_0^\infty(\Omega)$, we have

$$A(u,\varphi) := \sum_{i,j} \int_\Omega a_{ij} \partial_{x_j} u \partial_{x_i} \varphi \, \mathrm{d}x - \sum_i \int_\Omega b_i(x) \partial_{x_i} u \varphi \, \mathrm{d}x = \ell(\varphi) := -\int_\Omega g(x)\varphi(x) \, \mathrm{d}x$$

Theorem 3.7

Let $\{a_{ij}\} \in L^\infty(\Omega)$ be a uniformly elliptic matrix, $b_i(x)$ be a smooth divergent free vector field and $g(x) \in H^{-1}(\Omega)$. Then 3.12 has a unique weak solution.

Proof. We use the Lax-Milgram theorem. We know that $\ell(\varphi)$ is a linear continuous functional on $W_0^{1,2}(\Omega)$. Furthermore, $A(u,\varphi)$ is bilinear and bounded. Indeed, by Friedrich's inequality, we have

$$|A(u,\varphi)| \leqslant C_1 \|\nabla u\|_{L^2} \|\nabla \varphi\|_{L^2} + C_2 \|\nabla u\|_{L^2} \|\varphi\|_{L^2} \leqslant C \|u\|_{W_0^{1,2}(\Omega)} \|\varphi\|_{W_0^{1,2}(\Omega)}$$

$A(u,\varphi)$ is coercive since

$$\begin{aligned} A(u,u) &= \sum_{i,j} \int_\Omega a_{ij} \partial_{x_j} u \partial_{x_i} u \, \mathrm{d}x - \sum_i \int_\Omega b_i(x) \partial_{x_i}(u) u \, \mathrm{d}x \\ &\geqslant \alpha \|\nabla u\|_{L^2}^2 - \frac{1}{2} \int_\Omega \sum_i b_i(x) \partial_{x_i}(u^2) \, \mathrm{d}x \\ &= \alpha \|\nabla u\|_{L^2}^2 + \frac{1}{2} \int_\Omega \mathrm{div} b \cdot u^2(x) \, \mathrm{d}x \\ &= \alpha \|\nabla u\|_{L^2}^2 \end{aligned}$$

By the Lax-Milgram theorem, there exists a unique weak solution of 3.12. □