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Advanced Topics in Partial Differential Equations

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May 14, 2023

Contents

Chapter 2

Sobolev spaces

2.1 Interpolation inequalities

Example 2.1

$$\|u\|_{L^\infty}^2 \leqslant \|u\|_{L^2}\|u'\|_{L^2} \text{ for } u \in C^\infty(\mathbb{R}) \tag{2.1}$$

Proof. Idea: use that $(u^2)' = 2uu'$ and Newton-Leibniz

$$\begin{aligned}
u^2(x) &= 2\int_{-\infty}^{x} uu'\,\mathrm{d}y = -2\int_{x}^{\infty} uu'\,\mathrm{d}y \\
&= \int_{-\infty}^{x} uu'\,\mathrm{d}y - \int_{x}^{\infty} uu'\,\mathrm{d}y \\
&\leqslant \int_{-\infty}^{x} |u||u'|\,\mathrm{d}y + \int_{x}^{\infty} |u||u'|\,\mathrm{d}y \\
&= \int_{\mathbb{R}} |u||u'|\,\mathrm{d}y \\
\text{(Hölder's inequality)} \quad &\leqslant \|u\|_{L^2}\|u'\|_{L^2}
\end{aligned}$$

□

Question 1

Check that 2.1 is sharp. Namely, that 2.1 becomes equality for $u(x) = e^{-|x|}$ ($u(x)$ is an extremal function for 2.1). Also, 2.1 is shift and scaling invariant, i.e. $u_\alpha(x+h) = e^{-\alpha|x+h|}, h \in \mathbb{R}, \alpha > 0$ -extremals.

Example 2.2 (Interpolation inequality)

Ω-domain in $\mathbb{R}^n, u \in L_{p_1}(\Omega) \cap L_{p_2}(\Omega), 1 \leqslant p_1, p_2, < \infty, p_1 < p_2, \theta \in [0,1], \frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}$. Then

$$\|u\|_{L^p} \leqslant \|u\|_{L^{p_1}}^{\theta}\|u\|_{L^{p_2}}^{1-\theta} \tag{2.2}$$

Proof.

$$\int_{\mathbb{R}} |u|^p\,\mathrm{d}x = \int_{\mathbb{R}} |u|^{\theta p}|u|^{(1-\theta)p}\,\mathrm{d}x$$

We apply Hölder's inequality with exponents $P = \frac{p_1}{\theta p}$ and $Q = \frac{p_2}{(1-\theta)p}$ (Note $\frac{1}{P} + \frac{1}{Q} = \frac{\theta p}{p_1} + \frac{(1-\theta)p}{p_2} = 1$). Then

$$\begin{aligned}\int_{\mathbb{R}} |u|^{\theta p}|u|^{(1-\theta)p}\,\mathrm{d}x &\leqslant \left(\int_{\mathbb{R}} |u|^{p_1}\,\mathrm{d}x\right)^{\frac{1}{P}} \left(\int_{\mathbb{R}} |u|^{p_2}\,\mathrm{d}x\right)^{\frac{1}{Q}} \\ &= \|u\|_{L^{p_1}}^{\theta} \|u\|_{L^{p_2}}^{1-\theta}\end{aligned}$$

□

2.2 Sobolev inequalities

Example 2.3 (Sobolev inequality 1D)
$u \in C^\infty([0,1])$, want to prove the embedding $W^{1,1}([0,1]) \subset C([0,1])$, i.e.

$$\|u\|_{C([0,1])} \leqslant \|u\|_{L^1([0,1])} + \|u'\|_{L^1([0,1])} \tag{2.3}$$

Proof. By the Newton-Leibniz formula, $u(x) - u(y) = \int_y^x u'(s)\,\mathrm{d}s$. Also,

$$|u(x)| \leqslant |u(y)| + \int_0^1 |u'(s)|\,\mathrm{d}s \quad \forall x, y \in [0,1]$$

By integration over $y \in [0,1]$,

$$|u(x)| \leqslant \int_0^1 |u(s)|\,\mathrm{d}s + \int_0^1 |u'(s)|\,\mathrm{d}s = \|u\|_{W^{1,1}([0,1])}$$

Taking supremum with respect to $x \in [0,1]$, we obtain $\|u\|_{C([0,1])} \leqslant \|u\|_{W^{1,1}([0,1])}$ □

Example 2.4 (Sobolev inequality 2D)
$u \in C^\infty([0,1]^2)$, i.e. $\Omega = [0,1]^2$, then $W^{1,1}(\Omega) \subset L^2(\Omega) : \|u\|_{L^2} \leqslant \|u\|_{W^{1,1}(\Omega)}$

Proof. $\int_\Omega u^2(x_1,x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2$ should be estimated. From 2.3, we know that

$$|u(x_1,x_2)| \leqslant \int_0^1 |u(s,x_2)| + |\partial_{x_1} u(s,x_2)|\,\mathrm{d}s \coloneqq f(x_2)$$

$$|u(x_1,x_2)| \leqslant \int_0^1 |u(x_1,s)| + |\partial_{x_2} u(x_1,s)|\,\mathrm{d}s \coloneqq g(x_1)$$

Then

$$\begin{aligned}\int_\Omega u^2\,\mathrm{d}x &\leqslant \int_0^1 g(x_1)f(x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2 \\ &= \int_0^1 f(x_2)\,\mathrm{d}x_2 \int_0^1 g(x_1)\,\mathrm{d}x_1 \\ &= \left(\int_\Omega |u(x_1,x_2)| + |\partial_{x_1} u(x_1,x_2)|\,\mathrm{d}x_1\right)\left(\int_\Omega |u(x_1,x_2)| + |\partial_{x_2} u(x_1,x_2)|\,\mathrm{d}x_2\right) \\ &\leqslant \|u\|_{W^{1,1}(\Omega)}^2\end{aligned}$$

□

Question 2: Sobolev inequality 3D

$u \in C^\infty(\bar\Omega), \Omega = (0,1)^3$. Prove that $W^{1,1}(\Omega) \subset L^{\frac{3}{2}}(\Omega)$, i.e.

$$\|u\|_{L^{\frac{3}{2}}(\Omega)} \leqslant \|u\|_{W^{1,1}(\Omega)} \tag{2.4}$$

Hint: first, prove that

$$\int_\Omega f(x_1,x_2)g(x_2,x_3)h(x_1,x_3)\,\mathrm{d}x \leqslant \|f\|_{L^2}\|g\|_{L^2}\|h\|_{L^2}$$

and use 2.3.

Example 2.5

$u \in C^\infty(\bar\Omega), \Omega = (0,1)^3$. Then

$$\|u\|_{L^6(\Omega)} \leqslant C\|u\|_{W^{1,2}(\Omega)} \tag{2.5}$$

Proof.

$$\begin{aligned}
\int_\Omega |u|^6\,\mathrm{d}x &= \int_\Omega (|u|^4)^{\frac{3}{2}}\,\mathrm{d}x \\
&\leqslant C\left(\int_\Omega |u|^4\,\mathrm{d}x + \int_\Omega u^3|\nabla u|\,\mathrm{d}x\right)^{\frac{3}{2}} \\
\text{(by (2.4))} \quad &\leqslant C\left(\int_\Omega |u|^4\,\mathrm{d}x\right)^{\frac{3}{2}} + C\left(\int_\Omega |u|^3|\nabla u|\,\mathrm{d}x\right)^{\frac{3}{2}} \\
&\leqslant C\|u\|_{L^2}^{\frac{3}{2}\cdot\theta\cdot 4}\|u\|_{L^6}^{\frac{3}{2}\cdot(1-\theta)\cdot 4} + C\|u\|_{L^6}^{\frac{3}{2}\cdot 3}\|\nabla u\|_{L^2}^{\frac{3}{2}} \\
\left(\theta = \frac{1}{4}\right) \quad &= C\|u\|_{L^2}^{\frac{3}{2}}\|u\|_{L^6}^{\frac{9}{2}} + C\|u\|_{L^6}^{\frac{9}{2}}\|\nabla u\|_{L^2}^{\frac{3}{2}} \\
\left(\text{Young's inequality with } p = \frac{4}{5} \text{ and } q = -4\right) \quad &\leqslant \varepsilon\|u\|_{L^6}^6 + C_\varepsilon(\|u\|_{L^2} + \|\nabla u\|_{L^2})^6
\end{aligned}$$

Setting for example, $\varepsilon = \frac{1}{2}$, we obtain

$$\|u\|_{L^6(\Omega)} \leqslant C\|u\|_{W^{1,2}(\Omega)}$$

□

Theorem 2.1 Sobolev embeddings

1. $W^{k_1,p_1}(\Omega) \subset W^{k_2,p_2}(\Omega) \iff k_1 \geqslant k_2$ and $1 \leqslant p_1, p_2 < \infty, k_1 - \frac{n}{p_1} \geqslant k_2 - \frac{n}{p_2}, \Omega \subset \mathbb{R}^n$.
2. $W^{k,p}(\Omega) \subset C^\alpha(\Omega)$ if $\alpha < k - \frac{n}{p}$. If α is not an integer, then the inequality is weak.

Example 2.6

$H^s(\mathbb{R}^n) \subset C(\mathbb{R}^n) \iff s > \frac{n}{2}$

Proof. $u(x) = \int_{\mathbb{R}^n} e^{i\xi x} \hat{u}(\xi)\,\mathrm{d}\xi$

$$\begin{aligned}
|u(x)| &\leqslant \int_{\mathbb{R}^n} |\hat{u}(\xi)|\,\mathrm{d}\xi \\
&= \int_{\mathbb{R}^n} \left(1+|\xi|^2\right)^{-\frac{s}{2}} \left(1+|\xi|^2\right)^{\frac{s}{2}} |\hat{u}(\xi)|\,\mathrm{d}\xi \\
\text{(Hölder's inequality)} \quad &\leqslant \left(\int_{\mathbb{R}^n} \frac{1}{\left(1+|\xi|^2\right)^s}\,\mathrm{d}\xi\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^n} \left(1+|\xi|^2\right)^s |\hat{u}(\xi)|^2\,\mathrm{d}\xi\right)^{\frac{1}{2}}
\end{aligned}$$

$\int_{\mathbb{R}^n} \frac{1}{(1+|\xi|^2)^s}\,\mathrm{d}\xi < \infty \iff s > \frac{n}{2}$. Taking the supremum with respect to $x \in \mathbb{R}^n$, we get

$$\|u\|_{C(\mathbb{R}^n)} \leqslant C_s \|u\|_{H^s(\mathbb{R}^n)}$$

□

Theorem 2.2 Interpolation inequalities

Let $u \in W^{k_1,p_1}(\Omega) \cap W^{k_2,p_2}(\Omega)$, $\theta \in [0,1]$, $1 \leqslant p_1, p_2 \leqslant \infty$ with $k = \theta k_1 + (1-\theta)k_2$, $\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}$. Then

$$\|u\|_{W^{k,p}} \leqslant C\|u\|^{\theta}_{W^{k_1,p_1}} \|u\|^{1-\theta}_{W^{k_2,p_2}}$$

Corollary 2.1 Particular cases

1. $\|u\|_{H^1} \leqslant \|u\|^{\frac{1}{2}}_{L^2} \|u\|^{\frac{1}{2}}_{H^2}$
2. $\|u\|_{L^p} \leqslant \|u\|^{\theta}_{L^p} \|u\|^{1-\theta}_{H^2}$

2.3 Spaces with zero boundary traces

Definition 2.1

$W^{1,p}_0(\Omega) := \left\{u \in W^{1,p}(\Omega),\, u|_{\partial\Omega} = 0\right\}$
An equivalent definition is that the Sobolev spaces $W^{1,p}_0(\Omega)$ for $1 \leqslant p < \infty$ are defined as the closure of the set of compactly supported test functions $C^\infty_0(\Omega)$ with respect to the $W^{1,p}(\Omega)$-norm.

Lemma 2.1

These two definitions are equivalent. $u \in$ "closure": $u = \lim_{n\to\infty} \varphi_n, \varphi_n \in C^\infty_0(\Omega) \implies \varphi_n|_{\partial\Omega} = 0$. By continuity, $u|_{\partial\Omega} = 0$. The proof of the converse statement is more technical and is omitted.

2.4 Poincaré's and Friedrich's inequalities

Proposition 2.1 Friedrich's inequality

Let Ω be a bounded domain and $u \in W^{1,p}_0(\Omega)$. Then

$$\|u\|_{L^p} \leqslant C\|\nabla u\|_{L^p} \tag{2.6}$$

Proof. It is enough to prove 2.6 for $\varphi \in C_0^\infty(\Omega)$. By the Newton-Leibniz formula,

$$u(x_1, x') - u(-L, x') = u(x_1, x') = \int_{-L}^{x_1} \partial_{x_1} u(s, x')\, \mathrm{d}s$$

$$\begin{aligned} |u(x_1, x')|^p &\leqslant \left(\int_{-L}^{L} |\partial_{x_1} u(s, x')|\, \mathrm{d}s \right)^p \\ \text{(Hölder's inequality)} \quad &\leqslant C_L \int_{-L}^{L} |\partial_{x_1} u(s, x')|^p\, \mathrm{d}s \end{aligned}$$

Integration with respect to x' gives us

$$\int_{\mathbb{R}^{n-1}} |u(x_1, x')|^p\, \mathrm{d}x' \leqslant C_L \|\partial_{x_1} u\|_{L^p}^p$$

Finally, integrating over $x_1 \in [-L, L]$, we obtain

$$\|u\|_{L^p}^p \leqslant 2LC_L \|\partial_{x_1} u\|_{L^p}^p$$

□

Corollary 2.2 Equivalent norm in $W_0^{1,p}(\Omega)$

Homogeneous norm:

$$\|u\|_{W_0^{1,p}(\Omega)} := \|\nabla u\|_{L^p}$$

Note:-

$u|_{\partial\Omega} = 0$ is important! Otherwise, 2.6 will fail for $u \equiv c$. Since ∇u defines u up to a constant; $u|_{\partial\Omega} = 0$ removes this constant.

Proposition 2.2 Poincaré inequality

Let Ω be a bounded domain with a smooth boundary and $\langle u \rangle := \frac{1}{|\Omega|} \int_\Omega u(x)\, \mathrm{d}x = 0$. Then

$$\|u\|_{L^p} \leqslant C \|\nabla u\|_{L^p}$$

2.5 Compactness

Definition 2.2: Sequential compactness

A metric space (X, d) is compact if any sequence $\{x_n\}_{n=1}^\infty \subset X$ has a convergent sub-sequence, i.e. there exists $\{x_{n_k}\}_{k=1}^\infty : \lim_{k\to\infty} x_{n_k} = x_0 \in X$.

Definition 2.3: Compact

A topological space X is compact if any covering of X by open sets has a finite sub-covering.

Note:-

In metric spaces, compactness is equivalent to sequential compactness.
In general topological spaces, they are not related.

Theorem 2.3 Hausdorff
Let (X, d) be a metric space. Then X is compact $\iff$ X is complete and totally bounded.

Definition 2.4: Totally bounded

X is totally bounded if $\forall \varepsilon > 0, \exists$ covering of X by finitely many ε-balls, i.e. $X = \bigcup_{k=1}^{N} B_\varepsilon(x_k), N = N(\varepsilon)$ and $\{x_k\}$ is an ε-net in X.

Why do we need compactness?

Let X be compact and $f : X \to Y$ be continuous, then $f(X)$ is compact in Y. How do we solve PDEs of the form (or more general equations)?

$$F(x) = 0 \tag{2.7}$$

1. Construct approximate solutions
   $F(x_n) = g_n$, where $\lim_{n\to\infty} g_n = 0$
2. Obtain a priori estimates, i.e. that $\{x_n\}$ is bounded in a proper space
3. If $\{x_n\}$ is pre-compact and F is continuous $\implies$ $x = \lim_{x\to\infty} x_{n_k}$ is a solution of 2.7.

Theorem 2.4 Arzelà-Ascoli
Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Then $V \subset C(\bar{\Omega})$ is compact iff:

1. V is closed
2. V is bounded
3. V is equicontinuous = V has a common modulus of continuity

Theorem 2.5 Arzelà-Ascoli for L^p
Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, (and $\partial\Omega$ smooth, although not needed), $K \subset L^p(\Omega), 1 \leqslant p < \infty$. Then K is compact iff:

1. K is closed
2. K is bounded
3. K is equicontinuous in mean (possesses a joint modulus of continuity in L^p).

Definition 2.5: Modulus of continuity

Let $f \in L^p(\Omega), 1 \leqslant p < \infty, \Omega \subset \mathbb{R}^n$ bounded ($\partial\Omega$ smooth not needed). $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\lim_{z\to 0} w(z) = 0$ is a modulus of continuity of f in $L_p(\Omega)$ if

$$\int_\Omega |f(x+h) - f(x)|^p \,\mathrm{d}x \leqslant \omega(|h|), \quad \forall h \in \mathbb{R}^n,$$

where we used the 0-extension of f outside of Ω.

Corollary 2.3
Let $K = B_1(0) \in W^{1,p}(\Omega); \Omega \subset \mathbb{R}^n$ is bounded, $\partial\Omega$ is smooth, $1 \leqslant p < \infty$. Then K is pre-compact in $L^p(\Omega)$.

Proof. We need to check equicontinuity, i.e. estimate $\int_\Omega |f(x+h) - f(x)|^p \,\mathrm{d}x$.

$$f(x+h) - f(x) = h\int_0^1 \nabla f(x+sh)\,\mathrm{d}s$$

Taking modulus and p-th power of both sides, we get

$$|f(x+h) - f(x)|^p \leqslant |h| \int_0^1 |\nabla f(x+sh)|^p \,\mathrm{d}s$$

Finally, we take an integral over $x \in \Omega$.

$$\begin{aligned}\int_\Omega |f(x+h) - f(x)|^p \,\mathrm{d}x &\leqslant |h| \int_0^1 \int_\Omega |\nabla f(x+sh)|^p \,\mathrm{d}x\,\mathrm{d}s \\ &\leqslant C|h|\end{aligned}$$

$\omega(z) = cz$ is a joint modulus of continuity. □

Definition 2.6: Compact embedding

Let $V \subset W$ be Banach spaces. Then the embedding is compact if the unit ball of V is pre-compact in W.

Note:-

We proved that $W^{1,p}(\Omega) \subset L^p(\Omega)$ is a compact embedding.

Corollary 2.4

$W^{1,p}(\Omega) \subset L^q(\Omega)$ is a compact embedding if $q < q^*$, where q^* is defined such that $\frac{1}{q^*} = \frac{1}{p} - \frac{1}{n}$ and $\Omega \subset \mathbb{R}^n$, Ω is bounded, $\partial\Omega$ is smooth.

Proof. Let us check equicontinuity.

$$\|f(\cdot+h) - f(\cdot)\|_{L^q} \leqslant \|f(\cdot+h) - f(\cdot)\|_{L^p}^{\theta} \|f(\cdot+h) - f(\cdot)\|_{L^{q^*}}^{1-\theta}$$

since $p < q < q^*$ and $0 < \theta < 1$. q^* is a critical exponent in Sobolev embeddings, indeed, $W^{1,p}(\Omega) \subset L^q(\Omega) \implies 1 - \frac{n}{p} \geqslant -\frac{1}{q}$. Then by corollary 2.3, we have

$$\begin{aligned}\|f(\cdot+h) - f(\cdot)\|_{L^p}^{\theta} \|f(\cdot+h) - f(\cdot)\|_{L^{q^*}}^{1-\theta} &\leqslant C|h|^\theta (2\|f\|_{L^{q^*}})^{1-\theta} \\ &\leqslant C_1|h|^\theta \|f\|_{W^{1,p}}^{1-\theta} \\ &\leqslant C_1|h|^\theta\end{aligned}$$

□

General fact: $W^{s_1,p_1}(\Omega) \subset W^{s_2,p_2}(\Omega)$, where Ω is bounded, $\partial\Omega$ is smooth. Embedding is compact $\iff$ embedding is not critical.

2.6 Dual spaces

Definition 2.7: Dual space

$W^{-s,p}(\Omega) := \left(W_0^{s,q}(\Omega)\right)^*$ is defined as the dual space to $W_0^{s,q}(\Omega)$ =, i.e. the space of linear continuous functionals on $W_0^{s,q}(\Omega)$, where $\frac{1}{p} + \frac{1}{q} = 1$.

Definition 2.8

$$W^{-s,p}(\Omega) = \left\{ \text{completion of } L^p(\Omega) \text{ w.r.t } \|\ell\|_{W^{-s,p}} := \sup_{\varphi \in \mathcal{D}} \frac{|(\ell,\varphi)|}{\|\varphi\|_{W_0^{s,q}}} \right\}$$

Definition 2.9

$$W^{-s,p}(\Omega) = \left\{ \ell \in \mathcal{D}'(\Omega) : \|\ell\|_{W^{-s,p}} := \sup_{\varphi \in \mathcal{D}} \frac{|\langle \ell,\varphi \rangle|}{\|\varphi\|_{W_0^{s,q}}} \right\}$$

Proposition 2.3

Definitions 2.7, 2.8 and 2.9 are equivalent.

Question 3

Suppose $\delta(x) \in W^{-s,p}(\Omega), \Omega \subset \mathbb{R}^n$. How are s, p and n related? We know that $\delta(x)$ is well-defined on continuous functions, so we need $W_0^{s,q}(\Omega) \subset C(\bar{\Omega})$.

Example 2.7

Consider the case where $n = 1$ and $p = 2$. By the Sobolev embedding theorem, $W^{s,2} \subset C(\bar{\Omega})$ if $0 < s - \frac{1}{2}$. Thus we have $\delta(x) \in H^{-s}(\Omega)$ if $s > \frac{1}{2}$.

Chapter 3

Linear elliptic problems

3.1 Dirichlet and Neumann problems for the Laplacian

Example 3.1 (Laplace equation with Dirichlet boundary conditions)
Let $\Omega \in \mathbb{R}^n$ be a bounded domain with $\partial\Omega$ smooth. Consider the Laplace equation with Dirichlet boundary conditions:

$$\begin{cases} \Delta u = f \\ u|_{\partial\Omega} = 0 \end{cases} \tag{3.1}$$

Typical questions:

1. In what space does the solution live?
2. In what sense is the equation understood (classical / weak)?
3. In what sense are the boundary / initial data understood?

In ODEs, we have local existence and uniqueness theorem (for Lipschitz non-linearities), but there is not an equivalent theorem for PDEs. Therefore, we must study particular examples.

Definition 3.1

$u \in W_0^{1,2}(\Omega)$ is a weak solution of 3.1 if $\forall \varphi \in C_0^\infty(\Omega)$,

$$-\int_\Omega \nabla u(x)\nabla\varphi(x)\,\mathrm{d}x = \int_\Omega f(x)\varphi(x)\,\mathrm{d}x \tag{3.2}$$

Here, the boundary condition is incorporated into the choice of space $W_0^{1,2}(\Omega) = [C_0^\infty(\Omega)]_{W^{1,2}(\Omega)}$ (the closure of $C_0^\infty(\Omega)$ in the norm of $W^{1,2}(\Omega)$).
3.2 came from the integration by parts formula. Indeed, if $u \in C^2(\Omega) \cap C(\bar{\Omega})$, then $\Delta u = f$ is understood in a classical sense and

$$\int_\Omega \Delta u\varphi\,\mathrm{d}x = -\int_\Omega \nabla u\nabla\varphi\,\mathrm{d}x + \int_{\partial\Omega} \partial_n u\varphi\,\mathrm{d}s,$$

where the term $\int_{\partial\Omega} \partial_n u\varphi\,\mathrm{d}s = 0$ because $\varphi|_{\partial\Omega} = 0$.

Theorem 3.1
Let $f \in H^{-1}(\Omega) := W^{-1,2}(\Omega)$. Then 3.1 has a unique weak solution.

Proof. Application of Riesz representation theorem

$[u,u] \coloneqq \int_\Omega \nabla u \nabla u \,\mathrm{d}x$ is an equivalent norm on $W_0^{1,2}(\Omega)$ (due to Friedrich's inequality). Then 3.2 can be rewritten as

$$[u,\varphi] = -\int_\Omega f(x)\varphi(x)\,\mathrm{d}x \coloneqq \ell(\varphi)$$

Claim: ℓ is a linear continuous functional on $W_0^{1,2}(\Omega)$ (the integral should be understood as duality if we take $f \in H^{-1}(\Omega)$ and if $f \in L^2(\Omega)$, this is a standard Lebesgue integral).
Linearity of ℓ is obvious. ℓ is continuous as it is bounded:

$$|\ell(\varphi)| \leqslant \|f\|_{H^{-1}} \|\varphi\|_{H^1}$$

But we obtained that 3.2 holds only for $\varphi \in C_0^\infty(\Omega)$, not for $\varphi \in W_0^{1,2}(\Omega)$. However, $W_0^{1,2}(\Omega) = [C_0^\infty(\Omega)]_{W^{1,2}}$. Then approximation arguments give that $\forall \varphi \in H$,

$$[u,\varphi] = \ell(\varphi) \tag{3.3}$$

Then by Riesz representation theorem, there exists a unique $u \in W_0^{1,2}(\Omega)$ which satisfies 3.3. □

Example 3.2 (Laplace equation with Neumann boundary conditions)
Let $\Omega \in \mathbb{R}^n$ be a bounded domain with $\partial\Omega$ smooth. Consider the Laplace equation with Neumann boundary conditions:

$$\begin{cases} \Delta u = f \\ \partial_n u|_{\partial\Omega} = 0 \end{cases} \tag{3.4}$$

We cannot consider $\varphi \in C_0^\infty(\Omega)$ as test functions, because the information about boundary conditions will be lost. Similarly, considering

$$\varphi \in W_n^{1,2}(\Omega) \coloneqq \{u \in W^{1,2}(\Omega) \colon\ \partial_n u|_{\partial\Omega} = 0\}$$

will not work as well, since $\partial_n u|_{\partial\Omega}$ is not defined for $u \in W^{1,2}(\Omega)$ (since by theorem 2.1, $C^\infty(\Omega) \not\subset W^{1,2}(\Omega)$).
Instead, let us take $\varphi \in C^\infty(\bar{\Omega})$ as a test function and assume that u is a classical solution. Then

$$\begin{aligned} \int_\Omega f\varphi\,\mathrm{d}x &= \int_\Omega \Delta u \varphi\,\mathrm{d}x \\ &= -\int_\Omega \nabla u \nabla\varphi\,\mathrm{d}x + \int_{\partial\Omega} \partial_n u \varphi\,\mathrm{d}s \\ &= -\int_\Omega \nabla u \nabla \varphi\,\mathrm{d}x, \end{aligned}$$

as $\int_{\partial\Omega} \partial_n u\varphi\,\mathrm{d}x = 0$ due to the boundary conditions. If we take $\varphi(x) = 1$ as a test function, then we get

$$\begin{aligned} \int_\Omega f \cdot 1\,\mathrm{d}x &= -\int_\Omega \nabla u \nabla 1\,\mathrm{d}x \\ &= 0 \end{aligned}$$

Hence $\langle f \rangle = \frac{1}{|\Omega|}\int_\Omega f(x)\,\mathrm{d}x = 0$ is a necessary condition for solvability.
Let us notice that all solutions of this problem differs from each other by a constant. Thus, a natural assumption to single out the solution is $\langle u \rangle = 0$.

Definition 3.2

$u \in W^{1,2}(\Omega) \cap \{\langle u \rangle = 0\}$ is a weak solution of 3.4 if $\forall \varphi \in C^\infty(\bar{\Omega})$, we have:

$$\int_\Omega \nabla u \nabla \varphi\,\mathrm{d}x = -\int_\Omega f\varphi\,\mathrm{d}x \tag{3.5}$$

Note:-

The boundary conditions are now not in the definition of the space, but in 3.5.

Theorem 3.2

Let $f \in L^2(\Omega) \cap \{\langle f \rangle = 0\}$. Then 3.4 has a unique weak solution.

Proof. The proof is analogous to the problem with Dirichlet boundary conditions, but instead of applying Friedrich's inequality, we should apply Poincaré's inequality and use density of $C^\infty(\Omega) \in W^{1,2}(\Omega)$. □

Example 3.3 (Non-homogeneous Neumann boundary conditions)

Let $\Omega \in \mathbb{R}^n$ be a bounded domain with $\partial\Omega$ smooth. Consider the Laplace equation with non-homogeneous Neumann boundary conditions:

$$\begin{cases} \Delta u = f \\ \partial_n u|_{\partial\Omega} = g \end{cases} \tag{3.6}$$

Definition 3.3

$u \in W^{1,2}(\Omega) \cap \{\langle u \rangle = 0\}$ is a weak solution of 3.6 if $\forall \varphi \in C^\infty(\bar{\Omega})$, we have:

$$\int_\Omega \nabla u \nabla \varphi \,\mathrm{d}x = -\int_\Omega f\varphi \,\mathrm{d}x + \int_{\partial\Omega} g\varphi \,\mathrm{d}s \tag{3.7}$$

Note that if $\varphi \equiv 1$, then a necessary condition for solvability is

$$-\int_\Omega f \,\mathrm{d}x + \int_{\partial\Omega} g \,\mathrm{d}s = 0$$

Theorem 3.3

Let $f \in L^2(\Omega), g \in W^{-\frac{1}{2},2}(\partial\Omega)$ be such that $\int_\Omega f \,\mathrm{d}x = \int_{\partial\Omega} g \,\mathrm{d}s$. Then 3.6 has a unique weak solution.

Proof. $[u, u] \coloneqq \int_\Omega \nabla u \nabla u ds$ is an equivalent norm on $u \in W^{1,2}(\Omega) \cap \{\langle u \rangle = 0\}$ due to the Poincaré inequality. Then 3.7 can be rewritten as

$$[u, \varphi] = \ell(\varphi) \coloneqq -\int_\Omega f\varphi \,\mathrm{d}x + \int_{\partial\Omega} g\varphi \,\mathrm{d}s$$

We claim that ℓ is a linear continuous functional on $W^{1,2}(\Omega) \cap \{\langle u \rangle = 0\}$. Indeed, linearity is obvious. To show ℓ is continuous, we have

$$\begin{aligned} \left| -\int_\Omega f\varphi \,\mathrm{d}x + \int_{\partial\Omega} g\varphi \,\mathrm{d}s \right| &\leqslant \|f\|_{L^2}\|\varphi\|_{L^2} + \|g\|_{H^{-\frac{1}{2}}(\partial\Omega)}\|\varphi\|_{H^{\frac{1}{2}}(\partial\Omega)} \\ \text{(By the trace theorem and Poincaré's inequality)} &\leqslant \|f\|_{L^2}\|\varphi\|_{W^{1,2}(\Omega)} + \|g\|_{H^{-\frac{1}{2}}(\partial\Omega)}\|\varphi\|_{W^{1,2}(\Omega)} \end{aligned}$$

Then by Riesz representation theorem, there exists a unique $u \in W^{1,2}(\Omega) \cap \{\langle u \rangle = 0\}$ that is a weak solution of 3.6. □

Example 3.4 (Non-homogeneous Dirichlet boundary conditions)

Let $\Omega \in \mathbb{R}^n$ be a bounded domain with $\partial\Omega$ smooth. Consider the Laplace equation with non-homogeneous

Dirichlet boundary conditions:

$$\begin{cases}\Delta u = 0 \\ u|_{\partial\Omega} = g\end{cases} \tag{3.8}$$

Let us take $g \in W^{\frac{1}{2},2}(\partial\Omega)$. Then there exists $v \in W^{1,2}(\Omega)$ such that $v|_{\partial\Omega} = g$ (by the trace theorem). We look for the solution of 3.8 in the form $u = v + w$, where $w \in W_0^{1,2}(\Omega)$.

Definition 3.4

$u = v + w$ is a weak solution of 3.8 if $v|_{\partial\Omega} = g$, where $g \in W^{\frac{1}{2},2}(\partial\Omega), w \in W_0^{1,2}(\Omega)$ and $\forall \varphi \in C^\infty(\bar{\Omega})$, we have

$$\int_\Omega \nabla(v+w)\nabla\varphi \,\mathrm{d}x = 0 \tag{3.9}$$

Theorem 3.4

Let $g \in W^{\frac{1}{2},2}(\partial\Omega)$. Then 3.8 has a unique weak solution.

Proof. We can rearrange 3.9 to get

$$\ell(\varphi) := -[v,\varphi] = \int_\Omega \nabla w \nabla\varphi \,\mathrm{d}x = [w,\varphi],$$

and the functional ℓ can be shown to be linear and continuous. By the Riesz representation theorem, there exists a unique $w \in W^{1,2}(\Omega)$ such that 3.9 is satisfied. Note that this w depends on the choice of v. But $u = v + w$ does not depend on the choice of v. Indeed, let u_1 and u_2 be two solutions of 3.8. Then $u = u_1 - u_2$ solves

$$\begin{cases}\Delta u = 0 \\ u|_{\partial\Omega} = 0\end{cases}$$

We have previously shown that the weak solution of this problem is unique. Therefore, $u_1 = u_2$. □

Note:-

There is no universal choice of the space of test functions. Even for Dirichlet and Neumann boundary conditions, we need to consider different spaces. $\varphi \in C_0^\infty(\Omega)$ corresponds to the standard theory of distributions, while $\varphi \in C^\infty(\bar{\Omega})$ corresponds to "non-standard" distributions.

Example 3.5

Let $\Omega \in \mathbb{R}^n$ be a bounded domain with $\partial\Omega$ smooth. Consider

$$\begin{cases}\sum_{i,j} \partial_{x_i}(a_{ij}(x)\partial_{x_j}u) = g \\ u|_{\partial\Omega} = 0\end{cases} \tag{3.10}$$

Where we make the following assumptions on the matrix $a(x) := \{a_{ij}(x)\}_{i,j}$:

1. $a(x)$ is a symmetric matrix for every x:

$$a_{ij}(x) = a_{ji}(x)$$

2. $a(x)$ is uniformly elliptic. That is, for all $\xi \in \mathbb{R}^n$, there exists $\mu, M > 0$ which are independent of x such that

$$\mu|\xi^2| \leqslant \sum_{i,j} a_{ij}\xi_i\xi_j \leqslant M|\xi^2|$$

Definition 3.5

$u \in W^{1,2}(\Omega)$ is a weak solution to 3.10 $\iff$ $\forall \varphi \in C_0^\infty(\Omega)$, we have

$$\sum_{i,j} \int_\Omega a_{ij} \partial_{x_j} u \partial_{x_i} \varphi \, \mathrm{d}x = -\int_\Omega g\varphi \, \mathrm{d}x$$

Theorem 3.5

Let $a(x)$ be symmetric and uniformly elliptic. Then 3.10 has a unique weak solution.

Proof. Let us denote

$$[u, \varphi]_a = \int_\Omega \sum_{i,j} a_{ij}(x) \partial_{x_j} u(x) \partial_{x_i} \varphi(x) \, \mathrm{d}x.$$

Then since $a(x)$ is symmetric, the bilinear form $[u, v]_a$ is also symmetric, i.e. $[u, v]_a = [v, u]_a$. Since $a(x)$ is uniformly elliptic, there exist $\mu, M > 0$ such that

$$\mu[u, u] \leqslant [u, u]_a \leqslant M[u, u].$$

Therefore, $\left(W_0^{1,2}(\Omega), [\cdot, \cdot]_a\right)$ is a Hilbert space with the norm equivalent to the standard $W_0^{1,2}(\Omega)$ norm.
By the Riesz representation theorem, there exists a unique weak solution to 3.10. □

3.2 More general problems via Lax-Milgram

By the Riesz representation theorem, for any linear continuous functional, ℓ on a Hilbert space H, there exists a unique $x \in H$ such that $\forall \varphi \in H$, we have $(x, \varphi) = \ell(\varphi)$.
If we want $a(x, y)$ to be an equivalent inner product on H, then $a(x, y)$ must be symmetric.
We now consider the case where $a(x, y)$ is not assumed to be symmetric.

Definition 3.6: Bilinear form

A bilinear form $a(\cdot, \cdot) \colon H \times H \to \mathbb{R}$ is bounded if

$$|a(x, y)| \leqslant C\|x\|\|y\|$$

Definition 3.7: Coercive

A bilinear form $a(\cdot, \cdot)$ is coercive if $\exists \alpha > 0$ such that $a(x, x) \geqslant \alpha\|x\|^2$.

Theorem 3.6

Let $a(x, y)$ be a bounded and coercive bilinear form on H. Then any linear continuous functional $\ell \colon H \to \mathbb{R}$ can be represented in the form

$$a(x, y) = \ell(\varphi), \quad \forall \varphi \in H. \tag{3.11}$$

i.e. $\forall \ell \in H^*$, there exists a unique $x = x(\ell) \in H$ such that 3.11 is satisfied.

Example 3.6

Let $\Omega \in \mathbb{R}^n$ be a bounded domain with $\partial\Omega$ smooth. Consider the problem

$$\begin{cases} \sum_{i,j} \partial_{x_i}(a_{ij}(x) \partial_{x_j} u) + \sum_i b_i(x) \partial_{x_i} u = g(x) \\ u|_{\partial\Omega} = 0 \end{cases} \tag{3.12}$$

Definition 3.8

$u \in W_0^{1,2}(\Omega)$ is a weak solution of 3.12 if $\forall \varphi \in C_0^\infty(\Omega)$, we have

$$A(u,\varphi) := \sum_{i,j} \int_\Omega a_{ij}\partial_{x_j} u \partial_{x_i}\varphi \, dx - \sum_i \int_\Omega b_i(x)\partial_{x_i} u\varphi \, dx = \ell(\varphi) := -\int_\Omega g(x)\varphi(x)\, dx$$

Theorem 3.7

Let $\{a_{ij}\} \in L^\infty(\Omega)$ be a uniformly elliptic matrix, $b_i(x)$ be a smooth divergent free vector field and $g(x) \in H^{-1}(\Omega)$. Then 3.12 has a unique weak solution.

Proof. We use the Lax-Milgram theorem. We know that $\ell(\varphi)$ is a linear continuous functional on $W_0^{1,2}(\Omega)$. Furthermore, $A(u,\varphi)$ is bilinear and bounded. Indeed, by Friedrich's inequality, we have

$$|A(u,\varphi)| \leqslant C_1\|\nabla u\|_{L^2}\|\nabla\varphi\|_{L^2} + C_2\|\nabla u\|_{L^2}\|\varphi\|_{L^2} \leqslant C\|u\|_{W_0^{1,2}(\Omega)}\|\varphi\|_{W_0^{1,2}(\Omega)}$$

$A(u,\varphi)$ is coercive since

$$\begin{aligned} A(u,u) &= \sum_{i,j}\int_\Omega a_{ij}\partial_{x_j} u\partial_{x_i} u \, dx - \sum_i \int_\Omega b_i(x)\partial_{x_i}(u)u \, dx \\ &\geqslant \alpha\|\nabla u\|_{L^2}^2 - \frac{1}{2}\int_\Omega \sum_i b_i(x)\partial_{x_i}(u^2)\, dx \\ &= \alpha\|\nabla u\|_{L^2}^2 + \frac{1}{2}\int_\Omega \operatorname{div} b \cdot u^2(x)\, dx \\ &= \alpha\|\nabla u\|_{L^2}^2 \end{aligned}$$

By the Lax-Milgram theorem, there exists a unique weak solution of 3.12. □

3.3 Introduction to spectral theory

H is a Hilbert space. $\mathcal{L}(H)$ is a space of linear continuous operators.

Lemma 3.1

A is continuous $\iff$ A is bounded, i.e.

$$\|A\| := \sup_{x\in H} \frac{\|Ax\|}{\|x\|} < \infty$$

Lemma 3.2

$(\mathcal{L}(H), \|\cdot\|)$ is a Banach space.

Definition 3.9: Invertible operator

A is invertible $\iff \exists A^{-1} \in \mathcal{L}(H)$ such that

$$AA^{-1} = A^{-1}A = I \tag{3.13}$$

Definition 3.10: Spectrum

$\lambda \in \sigma(A)$ (the spectrum of A) $\iff$ $\lambda I - A$ is not invertible.
In other words, $\lambda \notin \sigma(A)$, (λ is in the resolvent set) iff the equation $\lambda u - Au = f$ has a unique solution for all $f \in H$.

Note:-

If $\dim H = n < \infty$, then $\mathcal{L}(H) = M(n \times n)$ ($n \times n$ matrices) and

1. All linear operators are continuous,
2. All $\lambda \in \sigma(A)$ correspond to eigenvalues
$$A\rho_\lambda = \lambda\rho_\lambda \implies \sigma(A) = \sigma_p(A) \text{ (point spectrum).}$$
3. $\lambda \in \sigma(A) \iff \det(\lambda I - A) = 0$
4. Only one of two equalities from 3.13 holding is enough.

All of the statements may fail when $\dim H = \infty$.

Example 3.7

Let $H = \ell_2$, the space of square summable sequences. Let $T_r, T_l \in \mathcal{L}(H)$ denote the right and left shift operators, respectively;

$$T_r(x_1, x_2, x_3, \dots) = (0, x_1, x_2, x_3, \dots)$$
$$T_l(x_1, x_2, x_3, \dots) = (x_2, x_3, \dots)$$

Then $T_l \circ T_r = I$, but $T_r \circ T_l(x) = (0, x_2, x_3, \dots)$. Hence T_r has a left inverse, but not a right inverse. $\ker(T_r) = \{0\}$, i.e. it is injective (no eigenvalues), but the range of $T_r \neq H$, since $T_r(x) \perp e_1$ for any $x \in H$. So T_r is not invertible since $T_r(H)$ is a proper closed subspace of H. Therefore, a new type of spectrum appeared - the residual spectrum $\sigma_R(A)$, which is impossible in finite-dimensional spaces.
Also, the determinant does not exist in infinite-dimensional spaces. Indeed, if it existed, then

$$1 = \det(T_l \circ T_r) = \det(T_l)\det(T_r),$$

which implies that both T_l and T_r are invertible, but this is not true.

Definition 3.11: Approximate point spectrum

$\lambda \in \sigma_{\text{app}}(A)$ (approximate point spectrum) $\iff$ $\exists x_n \in H, \|x_n\| = 1$ and $\lim\limits_{n\to\infty}(Ax_n - \lambda x_n) = 0$.
It can be proved that $\lambda \in \sigma_{\text{app}}(A) \iff$ the image of $(\lambda I - H)$ is not closed.

Example 3.8

$H = L^2(0,1), Af(x) := xf(x)$.

$$\lambda u - Au = f \iff (\lambda - x)u(x) = f(x),$$

then $u(x) = \frac{f(x)}{\lambda - x}$ and $(A - \lambda I)$ is not invertible $\iff$ $\lambda \in [0,1]$. Hence, $\sigma(A) = [0,1]$.
We can check that $\text{Range}(\lambda I - A)$ is not closed if $\lambda \in [0,1]$.

The next theorem shows that these are all the possible obstacles to invert the operator.

Theorem 3.8 Weyl's theorem
$A \in \mathcal{L}(H)$. Then $\sigma(A) = \sigma_p(A) \cup \sigma_{\text{app}}(A) \cup \sigma_R(A)$.

Definition 3.12: Resolvent

$R_A(\lambda) := (\lambda I - A)^{-1}$ is called the resolvent of $A \in \mathcal{L}(H)$.

More standard facts:

1. If $|\lambda| > \|A\|$, then $\lambda \in \sigma(A)$. Indeed,

$$\frac{1}{\lambda - A} = \frac{1}{\lambda}\frac{1}{1 - \frac{A}{\lambda}} = \frac{1}{\lambda}\sum_{n=0}^{\infty}\left(\frac{A}{n}\right)^n$$

   is an absolutely convergent series if $|\lambda| > \|A\|$.

2. Resolvent identity:

$$R_A(\lambda) - R_A(\mu) = -(\lambda - \mu)R_A(\lambda)R_A(\mu), \tag{3.14}$$

   which follows from

$$\frac{1}{\lambda - x} - \frac{1}{\lambda - \mu} = (\mu - \lambda)\frac{1}{\lambda - x}\frac{1}{\mu - x},$$

   where we substitute in $x = A$. From 3.14, by taking the limit as $\mu \to \lambda$, we get

$$\frac{d}{d\lambda}R_A(\lambda) = -R_A(\lambda)^2.$$

   Hence $R_A(\lambda)$ is an analytic function of λ.

3. By Liouville's theorem applied to $R_A(\lambda), \sigma(A) \neq \emptyset$

Definition 3.13: Compact operator

$A \in \mathcal{L}(H)$ is compact $\iff$ $AB_1(0)$ is a precompact set in H.

Definition 3.14: Adjoint operator

Let $A \in \mathcal{L}(H)$. The adjoint operator $A^* \in \mathcal{L}(H)$ is defined via $(Ax, y) = (x, A^*y), \quad \forall x, y \in H$. It exists due to Riesz representation theorem. In the finite dimensional case, the adjoint operator coincides with the transpose operator.

Definition 3.15: Self-adjoint operator

$A \in \mathcal{L}(H)$ is self-adjoint if $A = A^*$

Definition 3.16: Fredholm

$A \in \mathcal{L}(H)$ is Fredholm if $\text{Range}(A)$ and $\text{Range}(A^*)$ are closed and $\ker(A)$ and $\ker(A^*)$ are both finite dimensional.
Then the index of A is defined by

$$\text{ind}(A) := \dim\ker(A) - \dim\ker(A^*)$$

Theorem 3.9 Key theorem of Fredholm operators theorem
$\text{ind}(A)$ is a topological invariant. Namely, if $A(t), t \in [0,1]$ is a continuous curve of Fredholm operators,

then

$$\operatorname{ind}(A(0)) = \operatorname{ind}(A(1)).$$

Definition 3.17: Essential spectrum

$\lambda \in \sigma_{\text{ess}}(A)$ if $\lambda I - A$ is not Fredholm.

Properties:

1. K is compact $\iff$ K^* is compact.
2. $A \in \mathcal{L}(H), K$ compact $\implies$ AK and KA are compact.
3. A is Fredholm $\iff$ A^* is Fredholm.
4. Fredholm alternative: Let A be Fredholm. Then:

$$H = \operatorname{Range}(A) \oplus \ker(A^*)$$
$$H = \operatorname{Range}(A^*) \oplus \ker(A)$$

5. A is Fredholm $\iff$ it is invertible by modulus of compact operators, i.e. $\exists B : AB = I + K_1, BA = I + K_2$, with K_1, K_2 compact.
6. Let K be a compact operator. Then $\sigma_{\text{ess}}(K) = 0$ and for any $\varepsilon > 0, \sigma(K) \setminus B_\varepsilon(0)$ consists of finitely many eigenvalues of finite multiplicity.
7. If $A = A^*$, then $\sigma(A)$ is real and $\sigma_R(A) = \emptyset$.

Thus, the simplest case is the case of positive, compact and self-adjoint operators.

Theorem 3.10 Hilbert-Schmidt

Let $A \in \mathcal{L}(H)$ be a compact, self-adjoint and positive ($(Ax, x) > 0$ if $x \neq 0$) operator. Then there exists a sequence of non-zero real eigenvalues $\lambda_i \in \sigma_p(A)$ such that $|\lambda_i|$ is monotonically non-increasing

$$\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant \dots,$$

and the corresponding eigenvectors $\{e_n\}_{n=1}^{\infty}(Ae_n = \lambda e_n)$ form the orthonormal basis in H.
Moreover, any $x \in H$ can be written as

$$x = \sum_{n=1}^{\infty} \langle x, e_n \rangle e_n,$$

and A can be written as

$$Ax = \sum_{n=1}^{\infty} \lambda_n \langle x, e_n \rangle e_n.$$

Applications

Example 3.9 (Spectrum of the Laplacian with Dirichlet boundary conditions)

Let $\Omega \in \mathbb{R}^n$ be a bounded domain with $\partial\Omega$ smooth. Consider the Laplace equation with Dirichlet boundary conditions:

$$\begin{cases} -\Delta u = f, \qquad f \in L^2(\Omega) \\ u|_{\partial\Omega} = 0 \end{cases}$$

$-\Delta$ is not a bounded operator in $H := L^2(\Omega)$, so we cannot apply directly apply the Hilbert-Schmidt theorem.

Let us consider the inverse operator $A = (-\Delta)^{-1}$ constructed via weak solutions, namely $u = (-\Delta)^{-1} f = Af$ solves $[u, \varphi] = (f, \varphi)$. That is, $\forall \varphi \in C_0^\infty(\Omega)$ (or H_0^1) and $u \in H_0^1(\Omega)$, we have

$$\int_\Omega \nabla u \nabla \varphi \,\mathrm{d}x = \int_\Omega f \varphi \,\mathrm{d}x$$

1. A is a bounded operator from H to $H_0^1(\Omega)$. Indeed, let $\varphi = u$. Then

$$\begin{aligned} \|u\|_{H_0^1}^2 &= (f, u) \\ &\leqslant \|f\|_{L^2} \|u\|_{L^2} \\ &\leqslant C \|f\|_{L^2} \|u\|_{H_0^1} \end{aligned}$$

   Hence

$$\|u\|_{H_0^1} \leqslant C \|f\|_{L^2} \implies \frac{\|Af\|_{H_0^1}}{\|f\|_{L^2}} \leqslant C$$

2. A is compact since the embedding $H_0^1 \subset H$ is compact.

3. A is self-adjoint. Let $f, g \in L^2(\Omega), u = (-\Delta)^{-1} f, v = (-\Delta)^{-1} g$. Take $\varphi = v$ in variational formulation for u, and $\varphi = u$ in variational formulation for v:

$$\begin{aligned} &[u, v] = (f, v) \\ &[u, v] = (g, u) \\ \implies &(f, v) = (g, u) \implies (f, Ag) = (g, A^* f) \implies A = A^* \end{aligned}$$

4. A is positive.

$$0 < [u, u] = (f, u) = (f, Af)$$

By the Hilbert-Schmidt theorem, there exists a complete orthonormal system $\{e_n\}$ of eigenvectors of A with $Ae_n = \lambda e_n$.
e_n by definition solves $\lambda_n [e_n, \varphi_n] = (e_n, \varphi)$. Indeed, for $u_n = Ae_n$, we have $[u_n, \varphi] = (e_n, \varphi)$. Since $u_n = \lambda_n e_n$, this implies that

$$[e_n, \varphi] = \lambda_n^{-1} (e_n, \varphi). \tag{3.15}$$

Hence e_n is a weak solution of

$$\begin{cases} -\Delta e_n = \lambda_n^{-1} e_n \\ e_n|_{\partial\Omega} = 0 \end{cases}$$

Consider the minimisation problem:

$$\min_{u \in H_0^1(\Omega)} \|\nabla u\|_{L^2}^2$$

under the constraint $\|u\|_{L^2}^2 = 1$. Let u_n be a minimising sequence with $\|u_n\|_{L^2}^2 = 1$ and $\|\nabla u_n\|_{L^2}^2 \to \lambda_0$. We want to prove that λ_0 is the minimum of u. We need to prove the existence of the minimiser and to find λ_0.

1. u_n is bounded in $H_0^1(\Omega)$. $H_0^1(\Omega) \Subset L^2(\Omega)$ is compactly embedded, so it converges strongly to u_0 in $L^2(\Omega)$. Also, since ∇u is bounded, by Banach-Alaoglu theorem and reflexivity, $u_n \rightharpoonup u_0$ weakly in H_0^1 (up to a subsequence) and $\|\nabla u_0\|_{L^2} \leqslant \lim_{n\to\infty} \inf \|\nabla u_n\|_{L^2} = \lambda_0$. Since λ_0 is an infimum, then $\|\nabla u_0\|_{L^2} = \lambda_0$. Thus u_0 is a minimiser.

2. Let use the Euler-Lagrange method.

$$L_\lambda(u) = \|\nabla u\|_{L^2}^2 + \lambda \|u\|_{L^2}$$

   Let φ be an arbitrary (smooth) function and define $R_\varphi(\varepsilon) := L_\lambda(u + \varepsilon\varphi), \varepsilon \in \mathbb{R}$. Then the necessary condition for u to be a minimum of L is

$$\frac{d}{d\varepsilon} R_\varphi(\varepsilon)\Big|_{\varepsilon=0} = 0, \qquad \forall \varphi.$$

$$
\begin{aligned}
L_\lambda(u+\varepsilon\varphi) &= \|\nabla(u+\varepsilon\varphi)\|_{L^2}^2 + \lambda\|u+\varepsilon\varphi\|_{L^2}^2 \\
&= \|\nabla u\|_{L^2}^2 + 2\varepsilon(\nabla u, \nabla\varphi) + \varepsilon^2\|\nabla\varphi\|_{L^2}^2 + \lambda\|u\|_{L^2}^2 + 2\varepsilon\lambda(u,\varphi) + \varepsilon^2\|\varphi\|_{L^2}^2
\end{aligned}
$$

Hence u is a minimiser of L if $\forall\varphi \in H_0^1$, we have

$$(\nabla u, \nabla\varphi) + \lambda(u,\varphi) = 0.$$

Comparing this equality with 3.15, we see that $u = u_0$ is the eigenvector of $(-\Delta)^{-1}$ and λ^{-1} is the corresponding eigenvalue (in this notation, $-\lambda > 0$). Equivalently, u_0 is the eigenvector of $-\Delta$ and $-\lambda$ is the corresponding eigenvalue. From the Hilbert-Schmidt theorem, we know that

$$-\frac{1}{\lambda_1} \geqslant -\frac{1}{\lambda_2} \geqslant -\frac{1}{\lambda_3} \geqslant \cdots - \frac{1}{\lambda_i}, \tag{3.16}$$

where λ_i are eigenvalues of $(-\Delta)^{-1}$. Taking $\varphi = u$, we get $\|\nabla u\|_{L^2}^2 + \lambda\|u\|_{L^2}^2 = 0$, and if $u = e_1$, then

$$\|\nabla e_1\|_{L^2}^2 - \lambda_1\|e_1\|_{L^2}^2 = 0$$

and for any other u,

$$\|\nabla u\|_{L^2}^2 - \lambda_1\|u\|_{L^2}^2 > 0$$

Hence for all $u \in H_0^1(\Omega)$,

$$\|u\|_{L^2}^2 \leqslant \frac{1}{\lambda_1}\|\nabla u\|_{L^2}^2.$$

3.4 Maximum principle

Theorem 3.11 Classical maximum principle

Let $u \in C^2(\Omega) \cap C(\bar{\Omega})$, where $\Omega \subset \mathbb{R}^n$ is a bounded domain, and $\nabla u = 0$. Then

$$\max_{x\in\partial\Omega} u(x) \geqslant u(x) \geqslant \min_{x\in\partial\Omega} u(x)$$

In other words, the maximum and minimum of a harmonic function is attained on the boundary.

Proof. Let us prove the max-inequality. Let $v = u + \varepsilon e^{x_1}$. Then $\nabla v = \varepsilon e^{x_1}, \varepsilon > 0$. Assume that the maximum is attained in the interior point $x_0 \in \Omega$. Then $\nabla u(x_0) = 0$, and the matrix of second derivatives must be non-positive. In particular, $\mathrm{tr}(D^2 u(x_0)) = \Delta v(x_0) \leqslant 0$, which implies that $\varepsilon e^{x_1} < 0$, a contradiction. Therefore, the maximum v is attained on the boundary. Taking the limit as $\varepsilon \longrightarrow 0$, the same is true for u (but due to the limit, the inequalities become non-strict). The proof is analogous for the min-inequality. □

This formulation does not exclude that the maximum u may be attained inside Ω. However, the strong-maximum principle claims that if the maximum or minimum of u is attained inside Ω, then u is constant.

Maximum principle for weak solutions

Chain rule in Sobolev spaces

For smooth functions f and u, we know that

$$\nabla f(u(x)) = f'(u(x))\nabla u \tag{3.17}$$

Let $f \in C^1(\Omega)$ and ∇f be bounded, and $u \in W^{1,p}(\Omega)$. Then the LHS and RHS of 3.17 are well-defined and by approximation (Ω is regular enough to have density of $C^\infty(\Omega) \in W^{1,p}(\Omega)$), 3.17 holds for such functions. However, we want 3.17 to be satisfied for any globally Lipschitz continuous function $f \in W^{1,\infty}(\Omega)$. We know that Lipschitz functions are differentiable almost everywhere, so $f'(z)$ is well-defined. But what is $f'(u(x))$? The set $K \coloneqq \{z \in \mathbb{R} : f'(z) \text{ does not exist}\}$ has zero measure, but $u^{-1}(K) = V$ may have positive measure, implying $f'(u(x))$ is not defined.

Lemma 3.3
Let $u \in W^{1,p}(\Omega), K \subset \mathbb{R}$ with $\text{meas}(K) = 0$ and $v = u^{-1}(K)$. Then $\nabla u(x) = 0$ a.e. on V (without proof).

Then $f'(u(x))\nabla u$ is well-defined.

Theorem 3.12
Let $u \in W^{1,p}(\Omega)$ and f be globally Lipschitz. Then $f(u) \in W^{1,p}(\Omega)$ and 3.17 holds.

Proof. We will present the proof only for the case where

$$f(z) = \begin{cases} z, & z > 0 \\ 0, & z \leqslant 0, \end{cases}$$

which is crucial for the maximum principle. We denote

$$u_+(x) = \max\{u(x), 0\}.$$

We expect that $\nabla u_+ = \begin{cases} \nabla u, & u > 0 \\ 0, & u \leqslant 0 \end{cases}$.
Such $\nabla u_+ \in W^{1,p}(\Omega)$, so we only need to check the integration by parts formula.

$$\int_\Omega u_+(x)\text{div}\varphi(x)\,\text{d}x = \int_{x:u>0} \nabla u(x)\varphi(x)\,\text{d}x \qquad \forall \varphi \in C_0^\infty. \tag{3.18}$$

To do this, we introduce the following C^1-approximations of f such that:

1. f_ε' are uniformly bounded
2. $f_\varepsilon = f$ if $x \notin (0, \varepsilon)$

Then

$$\int_\Omega f_\varepsilon(u(x))\text{div}\varphi(x)\,\text{d}x = -\int_\Omega f_\varepsilon'(u(x))\nabla u(x)\varphi(x)\,\text{d}x \qquad \forall \varphi \in C_0^\infty.$$

Obviously the LHS of the above equality tends to the LHS of the classical formula 3.18. We need to check the convergence of the RHS.

$$\begin{aligned} \left|\int_\Omega (f_\varepsilon'\nabla u - f\nabla u)\varphi\,\text{d}x\right| &\leqslant \int_{0<u(x)<\varepsilon} C(1 + |f_\varepsilon'(u(x))|)|\nabla u|\,\text{d}x \\ &\leqslant C_1 \int_{0<u(x)<\varepsilon} |\nabla u|\,\text{d}x \end{aligned}$$

Let $K_\varepsilon := \{x\colon 0 < u(x) < \varepsilon\}$. Then K_ε are nested and $\cap_{n=1}^\infty K_{\frac{1}{n}} = \emptyset$ by σ-additivity of the Lebesgue measure. Then $\text{meas}(K_\varepsilon) \to 0$ as $\varepsilon \to 0$. By absolute continuity of the Lebesgue integral,

$$\int_{0<u(x)<\varepsilon} |\nabla u(x)|\,\text{d}x \to 0 \text{ as } \varepsilon \to 0$$

and the integration by parts formula 3.18 is proved. Thus, $u_+ \in W^{1,p}(\Omega)$ and indeed

$$\nabla u_+ = \begin{cases} \nabla u, & u > 0 \\ 0, & u \leqslant 0 \end{cases}$$

$\square$

Corollary 3.1

Let $u_-(x) = -\min\{u(x), 0\}$. Then $u \in W^{1,p}(\Omega)$ and $\nabla u_-(x) = \begin{cases} -\nabla u, & u < 0 \\ 0, & u \geqslant 0 \end{cases}$. Then $|u(x)| = u_+(x) + u_-(x)$ is also in $W^{1,p}(\Omega)$.

Corollary 3.2

$\nabla u(x) = 0$ a.e. on the set K where $u(x) = 0$.

Proof. Indeed, $u = u_+ + u_-, \nabla u = \nabla u_+ - \nabla u_-$, and

$$\int_\Omega u \operatorname{div}\varphi \, dx = -\int_\Omega \nabla u \varphi \, dx$$
$$\int_\Omega u_+ \operatorname{div}\varphi - \int_\Omega u_- \operatorname{div}\varphi = -\int_{u>0} \nabla u \varphi \, dx - \int_{u=0} \nabla u \varphi \, dx - \int_{u<0} \nabla u \varphi \, dx$$

Hence

$$\int_{u=0} \nabla u \varphi \, dx = 0 \qquad \forall \varphi \in C_0^\infty \implies \nabla u(x) = 0 \text{ a.e. on } u(x) = 0.$$

□

Corollary 3.3

$\nabla u_+ \cdot \nabla u_- = 0$ a.e. (because the supports of u_+ and u_- are disjoint).

$$\|\nabla u\|_{L^p}^p = \|\nabla u_+\|_{L^p}^p + \|\nabla u_-\|_{L^p}^p = \|\nabla |u|\|_{L^p}$$

$$\|u\|_{L^p}^p = \|u_+\|_{L^p}^p + \|u_-\|_{L^p}^p = \||u|\|_{L^p}$$

We return to the maximum principle.

Proposition 3.1

Let $u_1, u_2 \in W^{1,2}(\Omega)$ be weak solutions of $\begin{cases} -\Delta u_1 = f_1 \\ u_1|_{\partial\Omega} = u_1^0 \end{cases}$ and $\begin{cases} -\Delta u_2 = f_2 \\ u_2|_{\partial\Omega} = u_2^0 \end{cases}$, and let $u_1^0 \leqslant u_2^0$ a.e. and $f_1 \leqslant f_2$ in distributions. Then $u_1(x) \leqslant u_2(x)$ a.e. in Ω.

Proof. Let $v = u_1 - u_2$. It satisfies
$\begin{cases} -\Delta v = 0 \\ v|_{\partial\Omega} = v_0 \end{cases}$, where $f = f_1 - f_2 \leqslant 0$ and $v_0 = u_1^0 - u_2^0 \leqslant 0$.
It is sufficient to prove that $v_+(x) = 0$ a.e. We multiply the equation with the test function $\varphi = v_+(x) \in H_0^1(\Omega)$.

$$\begin{aligned}(\nabla v, \nabla v_+) &= (\nabla v_+ - \nabla v_-, \nabla v_+) \\ &= \|\nabla v_+\|_{L^2}^2 \\ &= (f, v_+) \leqslant 0\end{aligned}$$

Hence $\|\nabla v_+\|_{L^2}^2 = 0$, and by Friedrich's inequality, this implies that $v_+ = 0$ a.e. □

Note:-

$\ell \in D'(\Omega)$ is non-negative if and only if $\langle l, \varphi \rangle \geqslant 0 \quad \forall \varphi \in D(\Omega), \varphi \geqslant 0$.

Corollary 3.4

The classical maximum principle follows from proposition 3.1 by taking $f = f_1 = 0$, $u_1 = u$ and $u_2 = \max_{x \in \partial\Omega} u(x) = \text{const}$.

Note:-

From our exposition, it looks like u_+, u_- approach is more general than the classical one. This is not true! It does not cover general operators $L = \sum_{i,j} a_{ij}(x)\partial_{x_i}\partial_{x_j} + \sum_i b_i(x)\partial_{x_i} + c(x)$, but the classical theory does.