

# Chapter 5

## Sobolev and interpolation inequalities

### 5.1 Interpolation inequalities

#### Example 5.1.1

$$\|u\|_{L^2}^2 \leq \|u\|_{L^2} \|u'\|_{L^2} \text{ for } u \in C^\infty(\mathbb{R}) \quad (5.1)$$

*Proof.* Idea: use that  $(u^2)' = 2uu'$  and Newton-Leibniz

$$\begin{aligned} u^2(x) &= 2 \int_{-\infty}^x uu' \, dy = -2 \int_x^{\infty} uu' \, dy \\ &= \int_{-\infty}^x uu' \, dy - \int_x^{\infty} uu' \, dy \\ &\leq \int_{-\infty}^x |u||u'| \, dy + \int_x^{\infty} |u||u'| \, dy \\ &= \int_{\mathbb{R}} |u||u'| \, dy \\ (\text{Hölder's inequality}) \quad &\leq \|u\|_{L^2} \|u'\|_{L^2} \end{aligned}$$

□

#### Question 1

Check that 5.1 is sharp. Namely, that 5.1 becomes equality for  $u(x) = e^{-|x|}$  ( $u(x)$  is an extremal function for 5.1). Also 5.1 is shift and scaling invariant, i.e.  $u_\alpha(x+h) = e^{-\alpha|x+h|}$ ,  $h \in \mathbb{R}, \alpha > 0$  -extremals.

#### Example 5.1.2 (Interpolation inequality)

$\Omega$ -domain in  $\mathbb{R}^n$ ,  $u \in L_{p_1}(\Omega) \cap L_{p_2}(\Omega)$ ,  $1 \leq p_1, p_2, \leq \infty$ ,  $p_1 < p_2$ ,  $\theta \in [0, 1]$ ,  $\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}$ . Then

$$\|u\|_{L^p} \leq \|u\|_{L^{p_1}}^\theta \|u\|_{L^{p_2}}^{1-\theta} \quad (5.2)$$

*Proof.*

$$\int_{\mathbb{R}} |u|^p \, dx = \int_{\mathbb{R}} |u|^{\theta p} |u|^{(1-\theta)p} \, dx$$

We apply Hölder's inequality with exponents  $P = \frac{p_1}{\theta p}$  and  $Q = \frac{p_2}{(1-\theta)p}$  (Note  $\frac{1}{P} + \frac{1}{Q} = \frac{\theta p}{p_1} + \frac{(1-\theta)p}{p_2} = 1$ ). Then

$$\begin{aligned} \int_{\mathbb{R}} |u|^{\theta p} |u|^{(1-\theta)p} dx &\leq \left( \int_{\mathbb{R}} |u|^{p_1} dx \right)^{\frac{1}{P}} \left( \int_{\mathbb{R}} |u|^{p_2} dx \right)^{\frac{1}{Q}} \\ &= \|u\|_{L^{p_1}}^{\theta} \|u\|_{L^{p_2}}^{1-\theta} \end{aligned}$$

□

## 5.2 Sobolev inequalities

### Example 5.2.1 (Sobolev inequality 1D)

$u \in C^\infty([0, 1])$ , want to prove the embedding  $W^{1,1}([0, 1]) \subset C([0, 1])$ , i.e.

$$\|u\|_{C([0,1])} \leq \|u\|_{L^1([0,1])} + \|u'\|_{L^1([0,1])} \quad (5.3)$$

*Proof.* By the Newton-Leibniz formula,  $u(x) - u(y) = \int_y^x u'(s) ds$ . Also,

$$|u(x)| \leq |u(y)| + \int_0^1 |u'(s)| ds \quad \forall x, y \in [0, 1]$$

By integration over  $y \in [0, 1]$ ,

$$|u(x)| \leq \int_0^1 |u(s)| ds + \int_0^1 |u'(s)| ds = \|u\|_{W^{1,1}([0,1])}$$

Taking supremum with respect to  $x \in [0, 1]$ , we obtain  $\|u\|_{C([0,1])} \leq \|u\|_{W^{1,1}([0,1])}$

□

### Example 5.2.2 (Sobolev inequality 2D)

$u \in C^\infty([0, 1]^2)$ , i.e.  $\Omega = [0, 1]^2$ , then  $W^{1,1}(\Omega) \subset L^2(\Omega) : \|u\|_{L^2} \leq \|u\|_{W^{1,1}(\Omega)}$

*Proof.*  $\int_{\Omega} u^2(x_1, x_2) dx_1 dx_2$  should be estimated. From 5.3, we know that

$$|u(x_1, x_2)| \leq \int_0^1 |u(s, x_2)| + |\partial_{x_1} u(s, x_2)| ds := f(x_2)$$

$$|u(x_1, x_2)| \leq \int_0^1 |u(x_1, s)| + |\partial_{x_2} u(x_1, s)| ds := g(x_1)$$

Then

$$\begin{aligned} \int_{\Omega} u^2 dx &\leq \int_0^1 g(x_1) f(x_2) dx_1 dx_2 \\ &= \int_0^1 f(x_2) dx_2 \int_0^1 g(x_1) dx_1 \\ &= \left( \int_{\Omega} |u(x_1, x_2)| + |\partial_{x_1} u(x_1, x_2)| dx_1 \right) \left( \int_{\Omega} |u(x_1, x_2)| + |\partial_{x_2} u(x_1, x_2)| dx_2 \right) \\ &\leq \|u\|_{W^{1,1}(\Omega)}^2 \end{aligned}$$

□

### Question 2: Sobolev inequality 3D

$u \in C^\infty(\bar{\Omega})$ ,  $\Omega = (0, 1)^3$ . Prove that  $W^{1,1}(\Omega) \subset L^{\frac{3}{2}}(\Omega)$ , i.e.

$$\|u\|_{L^{\frac{3}{2}}(\Omega)} \leq \|u\|_{W^{1,1}(\Omega)} \quad (5.4)$$

Hint: first, prove that

$$\int_{\Omega} f(x_1, x_2) g(x_2, x_3) h(x_1, x_3) \, dx \leq \|f\|_{L^2} \|g\|_{L^2} \|h\|_{L^2}$$

and use 5.3.

### Example 5.2.3

$u \in C^\infty(\bar{\Omega})$ ,  $\Omega = (0, 1)^3$ . Then

$$\|u\|_{L^6(\Omega)} \leq C \|u\|_{W^{1,2}(\Omega)} \quad (5.5)$$

*Proof.*

$$\begin{aligned} \int_{\Omega} |u|^6 \, dx &= \int_{\Omega} (|u|^4)^{\frac{3}{2}} \, dx \\ &\leq C \left( \int_{\Omega} |u|^4 \, dx + \int_{\Omega} u^3 |\nabla u| \, dx \right)^{\frac{3}{2}} \\ (\text{by (5.3)}) \quad &\leq C \left( \int_{\Omega} |u|^4 \, dx \right)^{\frac{3}{2}} + C \left( \int_{\Omega} u^3 |\nabla u| \, dx \right)^{\frac{3}{2}} \\ &\leq C \|u\|_{L^2}^{\frac{3}{2} \cdot \theta \cdot 4} \|u\|_{L^6}^{\frac{3}{2} \cdot (1-\theta) \cdot 4} + C \|u\|_{L^6}^{\frac{3}{2} \cdot 3} \|\nabla u\|_{L^2}^{\frac{3}{2}} \\ \left( \theta = \frac{1}{4} \right) \quad &= C \|u\|_{L^2}^{\frac{3}{2}} \|u\|_{L^6}^{\frac{9}{2}} + C \|u\|_{L^6}^{\frac{9}{2}} \|\nabla u\|_{L^2}^{\frac{3}{2}} \\ \left( \text{Young's inequality with } p = \frac{4}{5} \text{ and } q = -4 \right) \quad &\leq \varepsilon \|u\|_{L^6}^6 + C_\varepsilon (\|u\|_{L^2} + \|\nabla u\|_{L^2})^6 \end{aligned}$$

Setting for example,  $\varepsilon = \frac{1}{2}$ , we obtain

$$\|u\|_{L^6(\Omega)} \leq C \|u\|_{W^{1,2}(\Omega)}$$

□

### Theorem 5.2.1 Sobolev embeddings

- ①  $W^{k_1, p_1}(\Omega) \subset W^{k_2, p_2}(\Omega) \iff k_1 \geq k_2 \text{ and } 1 \leq p_1, p_2 < \infty, k_1 - \frac{n}{p_1} \geq k_2 - \frac{n}{p_2}, \Omega \subset \mathbb{R}^n.$
- ②  $W^{k, p}(\Omega) \subset C^\alpha(\Omega)$  if  $\alpha < k - \frac{n}{p}.$