

## Chapter 5

# Sobolev and interpolation inequalities

### 5.1 Interpolation inequalities

#### Example 5.1.1

$$\|u\|_{L^2}^2 \leq \|u\|_{L^2} \|u'\|_{L^2} \text{ for } u \in C^\infty(\mathbb{R}) \quad (5.1)$$

*Proof.* Idea: use that  $(u^2)' = 2uu'$  and Newton-Leibniz

$$\begin{aligned} u^2(x) &= 2 \int_{-\infty}^x uu' \, dy = -2 \int_x^\infty uu' \, dy \\ &= \int_{-\infty}^x uu' \, dy - \int_x^\infty uu' \, dy \\ &\leq \int_{-\infty}^x |u||u'| \, dy + \int_x^\infty |u||u'| \, dy \\ &= \int_{\mathbb{R}} |u||u'| \, dy \\ (\text{Hölder's inequality}) \quad &\leq \|u\|_{L^2} \|u'\|_{L^2} \end{aligned}$$

□

#### Question 1

Check that 5.1 is sharp. Namely, that 5.1 becomes equality for  $u(x) = e^{-|x|}$  ( $u(x)$  is an extremal function for 5.1). Also, 5.1 is shift and scaling invariant, i.e.  $u_\alpha(x+h) = e^{-\alpha|x+h|}$ ,  $h \in \mathbb{R}$ ,  $\alpha > 0$  -extremals.

#### Example 5.1.2 (Interpolation inequality)

$\Omega$ -domain in  $\mathbb{R}^n$ ,  $u \in L_{p_1}(\Omega) \cap L_{p_2}(\Omega)$ ,  $1 \leq p_1, p_2 < \infty$ ,  $p_1 < p_2$ ,  $\theta \in [0, 1]$ ,  $\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}$ . Then

$$\|u\|_{L^p} \leq \|u\|_{L^{p_1}}^\theta \|u\|_{L^{p_2}}^{1-\theta} \quad (5.2)$$

*Proof.*

$$\int_{\mathbb{R}} |u|^p \, dx = \int_{\mathbb{R}} |u|^{\theta p} |u|^{(1-\theta)p} \, dx$$

We apply Hölder's inequality with exponents  $P = \frac{p_1}{\theta p}$  and  $Q = \frac{p_2}{(1-\theta)p}$  (Note  $\frac{1}{P} + \frac{1}{Q} = \frac{\theta p}{p_1} + \frac{(1-\theta)p}{p_2} = 1$ ). Then

$$\begin{aligned} \int_{\mathbb{R}} |u|^{\theta p} |u|^{(1-\theta)p} dx &\leq \left( \int_{\mathbb{R}} |u|^{p_1} dx \right)^{\frac{1}{P}} \left( \int_{\mathbb{R}} |u|^{p_2} dx \right)^{\frac{1}{Q}} \\ &= \|u\|_{L^{p_1}}^{\theta} \|u\|_{L^{p_2}}^{1-\theta} \end{aligned}$$

□

## 5.2 Sobolev inequalities

### Example 5.2.1 (Sobolev inequality 1D)

$u \in C^\infty([0, 1])$ , want to prove the embedding  $W^{1,1}([0, 1]) \subset C([0, 1])$ , i.e.

$$\|u\|_{C([0,1])} \leq \|u\|_{L^1([0,1])} + \|u'\|_{L^1([0,1])} \quad (5.3)$$

*Proof.* By the Newton-Leibniz formula,  $u(x) - u(y) = \int_y^x u'(s) ds$ . Also,

$$|u(x)| \leq |u(y)| + \int_0^1 |u'(s)| ds \quad \forall x, y \in [0, 1]$$

By integration over  $y \in [0, 1]$ ,

$$|u(x)| \leq \int_0^1 |u(s)| ds + \int_0^1 |u'(s)| ds = \|u\|_{W^{1,1}([0,1])}$$

Taking supremum with respect to  $x \in [0, 1]$ , we obtain  $\|u\|_{C([0,1])} \leq \|u\|_{W^{1,1}([0,1])}$

□

### Example 5.2.2 (Sobolev inequality 2D)

$u \in C^\infty([0, 1]^2)$ , i.e.  $\Omega = [0, 1]^2$ , then  $W^{1,1}(\Omega) \subset L^2(\Omega) : \|u\|_{L^2} \leq \|u\|_{W^{1,1}(\Omega)}$

*Proof.*  $\int_{\Omega} u^2(x_1, x_2) dx_1 dx_2$  should be estimated. From 5.3, we know that

$$|u(x_1, x_2)| \leq \int_0^1 |u(s, x_2)| + |\partial_{x_1} u(s, x_2)| ds := f(x_2)$$

$$|u(x_1, x_2)| \leq \int_0^1 |u(x_1, s)| + |\partial_{x_2} u(x_1, s)| ds := g(x_1)$$

Then

$$\begin{aligned} \int_{\Omega} u^2 dx &\leq \int_0^1 g(x_1) f(x_2) dx_1 dx_2 \\ &= \int_0^1 f(x_2) dx_2 \int_0^1 g(x_1) dx_1 \\ &= \left( \int_{\Omega} |u(x_1, x_2)| + |\partial_{x_1} u(x_1, x_2)| dx_1 \right) \left( \int_{\Omega} |u(x_1, x_2)| + |\partial_{x_2} u(x_1, x_2)| dx_2 \right) \\ &\leq \|u\|_{W^{1,1}(\Omega)}^2 \end{aligned}$$

□

### Question 2: Sobolev inequality 3D

$u \in C^\infty(\bar{\Omega})$ ,  $\Omega = (0, 1)^3$ . Prove that  $W^{1,1}(\Omega) \subset L^{\frac{3}{2}}(\Omega)$ , i.e.

$$\|u\|_{L^{\frac{3}{2}}(\Omega)} \leq \|u\|_{W^{1,1}(\Omega)} \quad (5.4)$$

Hint: first, prove that

$$\int_{\Omega} f(x_1, x_2) g(x_2, x_3) h(x_1, x_3) \, dx \leq \|f\|_{L^2} \|g\|_{L^2} \|h\|_{L^2}$$

and use 5.3.

### Example 5.2.3

$u \in C^\infty(\bar{\Omega})$ ,  $\Omega = (0, 1)^3$ . Then

$$\|u\|_{L^6(\Omega)} \leq C \|u\|_{W^{1,2}(\Omega)} \quad (5.5)$$

*Proof.*

$$\begin{aligned} \int_{\Omega} |u|^6 \, dx &= \int_{\Omega} (|u|^4)^{\frac{3}{2}} \, dx \\ &\leq C \left( \int_{\Omega} |u|^4 \, dx + \int_{\Omega} u^3 |\nabla u| \, dx \right)^{\frac{3}{2}} \\ \text{(by (5.3))} \quad &\leq C \left( \int_{\Omega} |u|^4 \, dx \right)^{\frac{3}{2}} + C \left( \int_{\Omega} u^3 |\nabla u| \, dx \right)^{\frac{3}{2}} \\ &\leq C \|u\|_{L^2}^{\frac{3}{2} \cdot \theta \cdot 4} \|u\|_{L^6}^{\frac{3}{2} \cdot (1-\theta) \cdot 4} + C \|u\|_{L^6}^{\frac{3}{2} \cdot 3} \|\nabla u\|_{L^2}^{\frac{3}{2}} \\ \left( \theta = \frac{1}{4} \right) \quad &= C \|u\|_{L^2}^{\frac{3}{2}} \|u\|_{L^6}^{\frac{9}{2}} + C \|u\|_{L^6}^{\frac{9}{2}} \|\nabla u\|_{L^2}^{\frac{3}{2}} \\ \left( \text{Young's inequality with } p = \frac{4}{5} \text{ and } q = -4 \right) \quad &\leq \varepsilon \|u\|_{L^6}^6 + C_\varepsilon (\|u\|_{L^2} + \|\nabla u\|_{L^2})^6 \end{aligned}$$

Setting for example,  $\varepsilon = \frac{1}{2}$ , we obtain

$$\|u\|_{L^6(\Omega)} \leq C \|u\|_{W^{1,2}(\Omega)}$$

□

### Theorem 5.2.1 Sobolev embeddings

- ①  $W^{k_1, p_1}(\Omega) \subset W^{k_2, p_2}(\Omega) \iff k_1 \geq k_2 \text{ and } 1 \leq p_1, p_2 < \infty, k_1 - \frac{n}{p_1} \geq k_2 - \frac{n}{p_2}, \Omega \subset \mathbb{R}^n.$
- ②  $W^{k, p}(\Omega) \subset C^\alpha(\Omega)$  if  $\alpha < k - \frac{n}{p}$ . If  $\alpha$  is not an integer, then the inequality is weak.

### Example 5.2.4

$$H^s(\mathbb{R}^n) \subset C(\mathbb{R}^n) \iff s > \frac{n}{2}$$

*Proof.*  $u(x) = \int_{\mathbb{R}^n} e^{i\xi x} \hat{u}(\xi) d\xi$

$$\begin{aligned}
|u(x)| &\leq \int_{\mathbb{R}^n} |\hat{u}(\xi)| d\xi \\
&= \int_{\mathbb{R}^n} (1 + |\xi|^2)^{-\frac{s}{2}} (1 + |\xi|^2)^{\frac{s}{2}} |\hat{u}(\xi)| d\xi \\
(\text{H\"older's inequality}) &\leq \left( \int_{\mathbb{R}^n} \frac{1}{(1 + |\xi|^2)^s} d\xi \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^n} (1 + |\xi|^2)^s |\hat{u}(\xi)|^2 d\xi \right)^{\frac{1}{2}}
\end{aligned}$$

$\int_{\mathbb{R}^n} \frac{1}{(1 + |\xi|^2)^s} d\xi < \infty \iff s > \frac{n}{2}$ . Taking the supremum with respect to  $x \in \mathbb{R}^n$ , we get

$$\|u\|_{C(\mathbb{R}^n)} \leq C_s \|u\|_{H^s(\mathbb{R}^n)}$$

□

### Theorem 5.2.2 Interpolation inequalities

Let  $u \in W^{k_1, p_1}(\Omega) \cap W^{k_2, p_2}(\Omega)$ ,  $\theta \in [0, 1]$ ,  $1 \leq p_1, p_2 \leq \infty$  with  $k = \theta k_1 + (1 - \theta)k_2$ ,  $\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}$ . Then

$$\|u\|_{W^{k, p}} \leq C \|u\|_{W^{k_1, p_1}}^\theta \|u\|_{W^{k_2, p_2}}^{1-\theta}$$

### Corollary 5.2.1 Particular cases

1.  $\|u\|_{H^1} \leq \|u\|_{L^2}^{\frac{1}{2}} \|u\|_{H^2}^{\frac{1}{2}}$
2.  $\|u\|_{L^p} \leq \|u\|_{L^p}^\theta \|u\|_{H^2}^{1-\theta}$

## 5.3 Spaces with zero boundary traces

### Definition 5.3.1

$$W_0^{1, p}(\Omega) := \{u \in W^{1, p}(\Omega), u|_{\partial\Omega} = 0\}$$

Equivalent definition:  $W_0^{1, p}(\Omega)$  = “closure of  $C_0^\infty(\Omega)$  in  $W^{1, p}$  norm.”

### Lemma 5.3.1

These two definitions are equivalent.  $u \in \text{“closure”} : u = \lim_{n \rightarrow \infty} \varphi_n, \varphi_n \in C_0^\infty(\Omega) \implies \varphi_n|_{\partial\Omega} = 0$ . By continuity,  $u|_{\partial\Omega} = 0$ . The proof of the converse statement is more technical and is omitted.

### Proposition 5.3.1 Friedrich's inequality

Let  $\Omega$  be a bounded domain and  $u \in W_0^{1, p}(\Omega)$ . Then

$$\|u\|_{L^p} \leq C \|\nabla u\|_{L^p} \quad (5.6)$$

*Proof.* It is enough to prove 5.6 for  $\varphi \in C_0^\infty(\Omega)$ . By the Newton-Leibniz formula,

$$u(x_1, x') - u(-L, x') = u(x_1, x') = \int_{-L}^{x_1} \partial_{x_1} u(s, x') ds$$

$$\begin{aligned}
|u(x_1, x')|^p &\leq \left( \int_{-L}^L |\partial_{x_1} u(s, x')| \, ds \right)^p \\
\text{(Hölder's inequality)} \quad &\leq C_L \int_{-L}^L |\partial_{x_1} u(s, x')|^p \, ds
\end{aligned}$$

Integration with respect to  $x'$  gives us

$$\int_{\mathbb{R}^{n-1}} |u(x_1, x')|^p \, dx' \leq C_L \|\partial_{x_1} u\|_{L^p}^p$$

Finally, integrating over  $x_1 \in [-L, L]$ , we obtain

$$\|u\|_{L^p}^p \leq 2LC_L \|\partial_{x_1} u\|_{L^p}^p$$

□

### Corollary 5.3.1 Equivalent norm in $W_0^{1,p}(\Omega)$

Homogeneous norm:

$$\|u\|_{W_0^{1,p}(\Omega)} := \|\nabla u\|_{L^p}$$

#### Note:-

$u|_{\partial\Omega} = 0$  is important! Otherwise, 5.6 will fail for  $u \equiv c$ . Since  $\nabla u$  defines  $u$  up to a constant;  $u|_{\partial\Omega} = 0$  removes this constant.

### Proposition 5.3.2 Poincaré inequality

Let  $\Omega$  be a bounded domain with a smooth boundary and  $\langle u \rangle := \frac{1}{|\Omega|} \int_{\Omega} u(x) \, dx = 0$ . Then

$$\|u\|_{L^p} \leq C \|\nabla u\|_{L^p}$$

### Definition 5.3.2: Sequential compactness

A metric space  $(X, d)$  is compact if any sequence  $\{x_n\}_{n=1}^{\infty} \subset X$  has a convergent sub-sequence, i.e. there exists  $\{x_{n_k}\}_{k=1}^{\infty} : \lim_{k \rightarrow \infty} x_{n_k} = x_0 \in X$

### Definition 5.3.3

A topological space  $X$  is compact if any covering of  $X$  by open sets has a finite sub-covering

#### Note:-

In metric spaces, compactness is equivalent to sequential compactness.  
In general topological spaces, they are not related.

### Theorem 5.3.1 Hausdorff

Let  $(X, d)$  be a metric space. Then  $X$  is compact  $\iff X$  is complete and totally bounded.

### Definition 5.3.4

$X$  is totally bounded if  $\forall \varepsilon > 0, \exists$  covering of  $X$  by finitely many  $\varepsilon$ -balls, i.e.  $X = \bigcup_{k=1}^N B_{\varepsilon}(x_k), N = N(\varepsilon)$  and  $\{x_k\}$  is an  $\varepsilon$ -net in  $X$ .

## 5.4 Why do we need compactness?

Let  $X$  be compact and  $f: X \rightarrow Y$  be continuous, then  $f(X)$  is compact in  $Y$ . How do we solve PDEs of the form (or more general equations)?

$$F(x) = 0 \tag{5.7}$$

1. Construct approximate solutions

$$F(x_n) = g_n, \text{ where } \lim_{n \rightarrow \infty} g_n = 0$$

2. Obtain a priori estimates, i.e. that  $\{x_n\}$  is bounded in a proper space
3. If  $\{x_n\}$  is pre-compact and  $F$  is continuous  $\implies x = \lim_{n \rightarrow \infty} x_{n_k}$  is a solution of 5.7.

### Theorem 5.4.1 Arzelà-Ascoli

Let  $\Omega \subset \mathbb{R}^n$  be a bounded domain. Then  $V \subset C(\bar{\Omega})$  is compact iff:

1.  $V$  is closed
2.  $V$  is bounded
3.  $V$  is equicontinuous =  $V$  has a common modulus of continuity

### Theorem 5.4.2 Arzelà-Ascoli for $L^p$

Let  $\Omega \subset \mathbb{R}^n$  be a bounded domain, (and  $\partial\Omega$  smooth, although not needed),  $K \subset L^p(\Omega)$ ,  $1 \leq p \leq \infty$ . Then  $K$  is compact iff:

1.  $K$  is closed
2.  $K$  is bounded
3.  $K$  is equicontinuous in mean (possesses a joint modulus of continuity in  $L^p$ ).

### Definition 5.4.1

Let  $f \in L^p(\Omega)$ ,  $1 \leq p \leq \infty$ ,  $\Omega \subset \mathbb{R}^n$  bounded ( $\partial\Omega$  smooth not needed).  $\omega: \mathbb{R}^+ \rightarrow \mathbb{R}^+$  such that  $\lim_{z \rightarrow 0} \omega(z) = 0$  is a modulus of continuity of  $f$  in  $L_p(\Omega)$  if

$$\int_{\Omega} |f(x+h) - f(x)|^p dx \leq \omega(|h|), \quad \forall h \in \mathbb{R}^n,$$

where we used to 0-extension of  $f$  outside of  $\Omega$ .

### Corollary 5.4.1

Let  $K = B_1(0) \in W^{1,p}(\Omega)$ ;  $\Omega \subset \mathbb{R}^n$  is bounded,  $\partial\Omega$  is smooth,  $1 \leq p < \infty$ . Then  $K$  is pre-compact in  $L^p(\Omega)$ .

*Proof.* We need to check equicontinuity, i.e. estimate  $\int_{\Omega} |f(x+h) - f(x)|^p dx$ .

$$f(x+h) - f(x) = h \int_0^1 \nabla f(x+sh) ds$$

Taking modulus and  $p$ -th power of both sides, we get

$$|f(x+h) - f(x)|^p \leq |h|^p \int_0^1 |\nabla f(x+sh)|^p ds$$

Finally, we take an integral over  $x \in \Omega$ .

$$\begin{aligned} \int_{\Omega} |f(x+h) - f(x)|^p dx &\leq |h| \int_0^1 \int_{\Omega} |\nabla f(x+sh)|^p dx ds \\ &\leq C|h| \end{aligned}$$

$\omega(z) = cz$  is a joint modulus of continuity. □

#### Definition 5.4.2

Let  $V \subset W$  be Banach spaces. Then the embedding is compact if the unit ball of  $V$  is pre-compact in  $W$ .

#### Note:-

We proved that  $W^{1,p}(\Omega) \subset L^p(\Omega)$  is a compact embedding.

#### Corollary 5.4.2

$W^{1,p}(\Omega) \subset L^q(\Omega)$  is a compact embedding if  $q < q^*$ , where  $q^*$  is defined such that  $\frac{1}{q^*} = \frac{1}{p} - \frac{1}{n}$  and  $\Omega \subset \mathbb{R}^n$ ,  $\Omega$  is bounded,  $\partial\Omega$  is smooth.

*Proof.* Let us check equicontinuity.

$$\|f(\cdot+h) - f(\cdot)\|_{L^q} \leq \|f(\cdot+h) - f(\cdot)\|_{L^p}^\theta \|f(\cdot+h) - f(\cdot)\|_{L^{q^*}}^{1-\theta}$$

since  $p < q < q^*$  and  $0 < \theta < 1$ .  $q^*$  is a critical exponent in Sobolev embeddings, indeed,  $W^{1,p}(\Omega) \subset L^q(\Omega) \implies 1 - \frac{n}{p} \geq -\frac{1}{q}$ . Then by corollary 5.4, we have

$$\begin{aligned} \|f(\cdot+h) - f(\cdot)\|_{L^p}^\theta \|f(\cdot+h) - f(\cdot)\|_{L^{q^*}}^{1-\theta} &\leq C|h|^\theta (2\|f\|_{L^{q^*}})^{1-\theta} \\ &\leq C_1|h|^\theta \|f\|_{W^{1,p}}^{1-\theta} \\ &\leq C_1|h|^\theta \end{aligned}$$

□

General fact:  $W^{s_1,p_1}(\Omega) \subset W^{s_2,p_2}(\Omega)$ , where  $\Omega$  is bounded,  $\partial\Omega$  is smooth. Embedding is compact  $\iff$  embedding is not critical.