

Contents

Chapter 2	Sobolev spaces	Page 2
2.1	Interpolation inequalities	2
2.2	Sobolev inequalities	3
2.3	Spaces with zero boundary traces	5
2.4	Poincare's and Friedrich's inequalities	5
2.5	Compactness	6
	Why do we need compactness? — 7	
2.6	Dual spaces	8
Chapter 3	Linear elliptic problems	Page 10
3.1	Dirichlet and Neumann problems for the Laplacian	10

Chapter 2

Sobolev spaces

2.1 Interpolation inequalities

Example 2.1

$$\|u\|_{L^2}^2 \leq \|u\|_{L^2} \|u'\|_{L^2} \text{ for } u \in C^\infty(\mathbb{R}) \quad (2.1)$$

Proof. Idea: use that $(u^2)' = 2uu'$ and Newton-Leibniz

$$\begin{aligned} u^2(x) &= 2 \int_{-\infty}^x uu' \, dy = -2 \int_x^{\infty} uu' \, dy \\ &= \int_{-\infty}^x uu' \, dy - \int_x^{\infty} uu' \, dy \\ &\leq \int_{-\infty}^x |u||u'| \, dy + \int_x^{\infty} |u||u'| \, dy \\ &= \int_{\mathbb{R}} |u||u'| \, dy \\ (\text{Hölder's inequality}) &\leq \|u\|_{L^2} \|u'\|_{L^2} \end{aligned}$$

□

Question 1

Check that 2.1 is sharp. Namely, that 2.1 becomes equality for $u(x) = e^{-|x|}$ ($u(x)$ is an extremal function for 2.1). Also, 2.1 is shift and scaling invariant, i.e. $u_\alpha(x+h) = e^{-\alpha|x+h|}$, $h \in \mathbb{R}$, $\alpha > 0$ -extremals.

Example 2.2 (Interpolation inequality)

Ω -domain in \mathbb{R}^n , $u \in L_{p_1}(\Omega) \cap L_{p_2}(\Omega)$, $1 \leq p_1, p_2, < \infty$, $p_1 < p_2$, $\theta \in [0, 1]$, $\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}$. Then

$$\|u\|_{L^p} \leq \|u\|_{L^{p_1}}^\theta \|u\|_{L^{p_2}}^{1-\theta} \quad (2.2)$$

Proof.

$$\int_{\mathbb{R}} |u|^p \, dx = \int_{\mathbb{R}} |u|^{\theta p} |u|^{(1-\theta)p} \, dx$$

We apply Hölder's inequality with exponents $P = \frac{p_1}{\theta p}$ and $Q = \frac{p_2}{(1-\theta)p}$ (Note $\frac{1}{P} + \frac{1}{Q} = \frac{\theta p}{p_1} + \frac{(1-\theta)p}{p_2} = 1$). Then

$$\begin{aligned} \int_{\mathbb{R}} |u|^{\theta p} |u|^{(1-\theta)p} dx &\leq \left(\int_{\mathbb{R}} |u|^{p_1} dx \right)^{\frac{1}{P}} \left(\int_{\mathbb{R}} |u|^{p_2} dx \right)^{\frac{1}{Q}} \\ &= \|u\|_{L^{p_1}}^{\theta} \|u\|_{L^{p_2}}^{1-\theta} \end{aligned}$$

□

2.2 Sobolev inequalities

Example 2.3 (Sobolev inequality 1D)

$u \in C^\infty([0, 1])$, want to prove the embedding $W^{1,1}([0, 1]) \subset C([0, 1])$, i.e.

$$\|u\|_{C([0,1])} \leq \|u\|_{L^1([0,1])} + \|u'\|_{L^1([0,1])} \quad (2.3)$$

Proof. By the Newton-Leibniz formula, $u(x) - u(y) = \int_y^x u'(s) ds$. Also,

$$|u(x)| \leq |u(y)| + \int_0^1 |u'(s)| ds \quad \forall x, y \in [0, 1]$$

By integration over $y \in [0, 1]$,

$$|u(x)| \leq \int_0^1 |u(s)| ds + \int_0^1 |u'(s)| ds = \|u\|_{W^{1,1}([0,1])}$$

Taking supremum with respect to $x \in [0, 1]$, we obtain $\|u\|_{C([0,1])} \leq \|u\|_{W^{1,1}([0,1])}$

□

Example 2.4 (Sobolev inequality 2D)

$u \in C^\infty([0, 1]^2)$, i.e. $\Omega = [0, 1]^2$, then $W^{1,1}(\Omega) \subset L^2(\Omega) : \|u\|_{L^2} \leq \|u\|_{W^{1,1}(\Omega)}$

Proof. $\int_{\Omega} u^2(x_1, x_2) dx_1 dx_2$ should be estimated. From 2.3, we know that

$$|u(x_1, x_2)| \leq \int_0^1 |u(s, x_2)| + |\partial_{x_1} u(s, x_2)| ds := f(x_2)$$

$$|u(x_1, x_2)| \leq \int_0^1 |u(x_1, s)| + |\partial_{x_2} u(x_1, s)| ds := g(x_1)$$

Then

$$\begin{aligned} \int_{\Omega} u^2 dx &\leq \int_0^1 g(x_1) f(x_2) dx_1 dx_2 \\ &= \int_0^1 f(x_2) dx_2 \int_0^1 g(x_1) dx_1 \\ &= \left(\int_{\Omega} |u(x_1, x_2)| + |\partial_{x_1} u(x_1, x_2)| dx_1 \right) \left(\int_{\Omega} |u(x_1, x_2)| + |\partial_{x_2} u(x_1, x_2)| dx_2 \right) \\ &\leq \|u\|_{W^{1,1}(\Omega)}^2 \end{aligned}$$

□

Question 2: Sobolev inequality 3D

$u \in C^\infty(\bar{\Omega})$, $\Omega = (0, 1)^3$. Prove that $W^{1,1}(\Omega) \subset L^{\frac{3}{2}}(\Omega)$, i.e.

$$\|u\|_{L^{\frac{3}{2}}(\Omega)} \leq \|u\|_{W^{1,1}(\Omega)} \quad (2.4)$$

Hint: first, prove that

$$\int_{\Omega} f(x_1, x_2) g(x_2, x_3) h(x_1, x_3) dx \leq \|f\|_{L^2} \|g\|_{L^2} \|h\|_{L^2}$$

and use 2.3.

Example 2.5

$u \in C^\infty(\bar{\Omega})$, $\Omega = (0, 1)^3$. Then

$$\|u\|_{L^6(\Omega)} \leq C \|u\|_{W^{1,2}(\Omega)} \quad (2.5)$$

Proof.

$$\begin{aligned} \int_{\Omega} |u|^6 dx &= \int_{\Omega} (|u|^4)^{\frac{3}{2}} dx \\ &\leq C \left(\int_{\Omega} |u|^4 dx + \int_{\Omega} u^3 |\nabla u| dx \right)^{\frac{3}{2}} \\ \text{(by (2.3))} \quad &\leq C \left(\int_{\Omega} |u|^4 dx \right)^{\frac{3}{2}} + C \left(\int_{\Omega} u^3 |\nabla u| dx \right)^{\frac{3}{2}} \\ &\leq C \|u\|_{L^2}^{\frac{3}{2} \cdot 4} \|u\|_{L^6}^{\frac{3}{2} \cdot (1-\theta) \cdot 4} + C \|u\|_{L^6}^{\frac{3}{2} \cdot 3} \|\nabla u\|_{L^2}^{\frac{3}{2}} \\ \left(\theta = \frac{1}{4} \right) \quad &= C \|u\|_{L^2}^{\frac{3}{2}} \|u\|_{L^6}^{\frac{9}{2}} + C \|u\|_{L^6}^{\frac{9}{2}} \|\nabla u\|_{L^2}^{\frac{3}{2}} \\ \left(\text{Young's inequality with } p = \frac{4}{\varepsilon} \text{ and } q = -4 \right) \quad &\leq \varepsilon \|u\|_{L^6}^6 + C_\varepsilon (\|u\|_{L^2} + \|\nabla u\|_{L^2})^6 \end{aligned}$$

Setting for example, $\varepsilon = \frac{1}{2}$, we obtain

$$\|u\|_{L^6(\Omega)} \leq C \|u\|_{W^{1,2}(\Omega)}$$

□

Theorem 2.1 Sobolev embeddings

- ① $W^{k_1, p_1}(\Omega) \subset W^{k_2, p_2}(\Omega) \iff k_1 \geq k_2$ and $1 \leq p_1, p_2 < \infty, k_1 - \frac{n}{p_1} \geq k_2 - \frac{n}{p_2}, \Omega \subset \mathbb{R}^n$.
- ② $W^{k, p}(\Omega) \subset C^\alpha(\Omega)$ if $\alpha < k - \frac{n}{p}$. If α is not an integer, then the inequality is weak.

Example 2.6

$$H^s(\mathbb{R}^n) \subset C(\mathbb{R}^n) \iff s > \frac{n}{2}$$

Proof. $u(x) = \int_{\mathbb{R}^n} e^{i\xi x} \hat{u}(\xi) d\xi$

$$\begin{aligned}
|u(x)| &\leq \int_{\mathbb{R}^n} |\hat{u}(\xi)| d\xi \\
&= \int_{\mathbb{R}^n} (1 + |\xi|^2)^{-\frac{s}{2}} (1 + |\xi|^2)^{\frac{s}{2}} |\hat{u}(\xi)| d\xi \\
&\stackrel{\text{(Hölder's inequality)}}{\leq} \left(\int_{\mathbb{R}^n} \frac{1}{(1 + |\xi|^2)^s} d\xi \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^n} (1 + |\xi|^2)^s |\hat{u}(\xi)|^2 d\xi \right)^{\frac{1}{2}}
\end{aligned}$$

$\int_{\mathbb{R}^n} \frac{1}{(1 + |\xi|^2)^s} d\xi < \infty \iff s > \frac{n}{2}$. Taking the supremum with respect to $x \in \mathbb{R}^n$, we get

$$\|u\|_{C(\mathbb{R}^n)} \leq C_s \|u\|_{H^s(\mathbb{R}^n)}$$

□

Theorem 2.2 Interpolation inequalities

Let $u \in W^{k_1, p_1}(\Omega) \cap W^{k_2, p_2}(\Omega)$, $\theta \in [0, 1]$, $1 \leq p_1, p_2 \leq \infty$ with $k = \theta k_1 + (1 - \theta)k_2$, $\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}$. Then

$$\|u\|_{W^{k, p}} \leq C \|u\|_{W^{k_1, p_1}}^\theta \|u\|_{W^{k_2, p_2}}^{1-\theta}$$

Corollary 2.1 Particular cases

1. $\|u\|_{H^1} \leq \|u\|_{L^2}^{\frac{1}{2}} \|u\|_{H^2}^{\frac{1}{2}}$
2. $\|u\|_{L^p} \leq \|u\|_{L^p}^\theta \|u\|_{H^2}^{1-\theta}$

2.3 Spaces with zero boundary traces

Definition 2.1

$$W_0^{1, p}(\Omega) := \{u \in W^{1, p}(\Omega), u|_{\partial\Omega} = 0\}$$

Equivalent definition: $W_0^{1, p}(\Omega) = \text{“closure of } C_0^\infty(\Omega) \text{ in } W^{1, p} \text{ norm.”}$

Lemma 2.1

These two definitions are equivalent. $u \in \text{“closure”} : u = \lim_{n \rightarrow \infty} \varphi_n, \varphi_n \in C_0^\infty(\Omega) \implies \varphi_n|_{\partial\Omega} = 0$. By continuity, $u|_{\partial\Omega} = 0$. The proof of the converse statement is more technical and is omitted.

2.4 Poincaré's and Friedrich's inequalities

Proposition 2.1 Friedrich's inequality

Let Ω be a bounded domain and $u \in W_0^{1, p}(\Omega)$. Then

$$\|u\|_{L^p} \leq C \|\nabla u\|_{L^p} \tag{2.6}$$

Proof. It is enough to prove 2.6 for $\varphi \in C_0^\infty(\Omega)$. By the Newton-Leibniz formula,

$$u(x_1, x') - u(-L, x') = u(x_1, x') = \int_{-L}^{x_1} \partial_{x_1} u(s, x') \, ds$$

$$\begin{aligned} |u(x_1, x')|^p &\leq \left(\int_{-L}^L |\partial_{x_1} u(s, x')| \, ds \right)^p \\ (\text{Hölder's inequality}) &\leq C_L \int_{-L}^L |\partial_{x_1} u(s, x')|^p \, ds \end{aligned}$$

Integration with respect to x' gives us

$$\int_{\mathbb{R}^{n-1}} |u(x_1, x')|^p \, dx' \leq C_L \|\partial_{x_1} u\|_{L^p}^p$$

Finally, integrating over $x_1 \in [-L, L]$, we obtain

$$\|u\|_{L^p}^p \leq 2LC_L \|\partial_{x_1} u\|_{L^p}^p$$

□

Corollary 2.2 Equivalent norm in $W_0^{1,p}(\Omega)$

Homogeneous norm:

$$\|u\|_{W_0^{1,p}(\Omega)} := \|\nabla u\|_{L^p}$$

Note:-

$u|_{\partial\Omega} = 0$ is important! Otherwise, 2.6 will fail for $u \equiv c$. Since ∇u defines u up to a constant; $u|_{\partial\Omega} = 0$ removes this constant.

Proposition 2.2 Poincaré inequality

Let Ω be a bounded domain with a smooth boundary and $\langle u \rangle := \frac{1}{|\Omega|} \int_{\Omega} u(x) \, dx = 0$. Then

$$\|u\|_{L^p} \leq C \|\nabla u\|_{L^p}$$

2.5 Compactness

Definition 2.2: Sequential compactness

A metric space (X, d) is compact if any sequence $\{x_n\}_{n=1}^\infty \subset X$ has a convergent sub-sequence, i.e. there exists $\{x_{n_k}\}_{k=1}^\infty : \lim_{k \rightarrow \infty} x_{n_k} = x_0 \in X$

Definition 2.3

A topological space X is compact if any covering of X by open sets has a finite sub-covering

Note:-

In metric spaces, compactness is equivalent to sequential compactness.
In general topological spaces, they are not related.

Theorem 2.3 Hausdorff

Let (X, d) be a metric space. Then X is compact $\iff X$ is complete and totally bounded.

Definition 2.4

X is totally bounded if $\forall \varepsilon > 0, \exists$ covering of X by finitely many ε -balls, i.e. $X = \bigcup_{k=1}^N B_\varepsilon(x_k), N = N(\varepsilon)$ and $\{x_k\}$ is an ε -net in X .

2.5.1 Why do we need compactness?

Let X be compact and $f: X \rightarrow Y$ be continuous, then $f(X)$ is compact in Y . How do we solve PDEs of the form (or more general equations)?

$$F(x) = 0 \tag{2.7}$$

1. Construct approximate solutions

$$F(x_n) = g_n, \text{ where } \lim_{n \rightarrow \infty} g_n = 0$$

2. Obtain a priori estimates, i.e. that $\{x_n\}$ is bounded in a proper space
3. If $\{x_n\}$ is pre-compact and F is continuous $\implies x = \lim_{n \rightarrow \infty} x_n$ is a solution of 2.7.

Theorem 2.4 Arzelà-Ascoli

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Then $V \subset C(\bar{\Omega})$ is compact iff:

1. V is closed
2. V is bounded
3. V is equicontinuous = V has a common modulus of continuity

Theorem 2.5 Arzelà-Ascoli for L^p

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, (and $\partial\Omega$ smooth, although not needed), $K \subset L^p(\Omega), 1 \leq p < \infty$. Then K is compact iff:

1. K is closed
2. K is bounded
3. K is equicontinuous in mean (possesses a joint modulus of continuity in L^p).

Definition 2.5

Let $f \in L^p(\Omega), 1 \leq p < \infty, \Omega \subset \mathbb{R}^n$ bounded ($\partial\Omega$ smooth not needed). $\omega: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that $\lim_{z \rightarrow 0} \omega(z) = 0$ is a modulus of continuity of f in $L_p(\Omega)$ if

$$\int_{\Omega} |f(x+h) - f(x)|^p dx \leq \omega(|h|), \quad \forall h \in \mathbb{R}^n,$$

where we used the 0-extension of f outside of Ω .

Corollary 2.3

Let $K = B_1(0) \in W^{1,p}(\Omega); \Omega \subset \mathbb{R}^n$ is bounded, $\partial\Omega$ is smooth, $1 \leq p < \infty$. Then K is pre-compact in $L^p(\Omega)$.

Proof. We need to check equicontinuity, i.e. estimate $\int_{\Omega} |f(x+h) - f(x)|^p dx$.

$$f(x+h) - f(x) = h \int_0^1 \nabla f(x+sh) ds$$

Taking modulus and p -th power of both sides, we get

$$|f(x+h) - f(x)|^p \leq |h|^p \int_0^1 |\nabla f(x+sh)|^p ds$$

Finally, we take an integral over $x \in \Omega$.

$$\begin{aligned} \int_{\Omega} |f(x+h) - f(x)|^p dx &\leq |h|^p \int_0^1 \int_{\Omega} |\nabla f(x+sh)|^p dx ds \\ &\leq C|h|^p \end{aligned}$$

$\omega(z) = cz$ is a joint modulus of continuity. □

Definition 2.6

Let $V \subset W$ be Banach spaces. Then the embedding is compact if the unit ball of V is pre-compact in W .

Note:-

We proved that $W^{1,p}(\Omega) \subset L^p(\Omega)$ is a compact embedding.

Corollary 2.4

$W^{1,p}(\Omega) \subset L^q(\Omega)$ is a compact embedding if $q < q^*$, where q^* is defined such that $\frac{1}{q^*} = \frac{1}{p} - \frac{1}{n}$ and $\Omega \subset \mathbb{R}^n$, Ω is bounded, $\partial\Omega$ is smooth.

Proof. Let us check equicontinuity.

$$\|f(\cdot+h) - f(\cdot)\|_{L^q} \leq \|f(\cdot+h) - f(\cdot)\|_{L^p}^{\theta} \|f(\cdot+h) - f(\cdot)\|_{L^{q^*}}^{1-\theta}$$

since $p < q < q^*$ and $0 < \theta < 1$. q^* is a critical exponent in Sobolev embeddings, indeed, $W^{1,p}(\Omega) \subset L^q(\Omega) \implies 1 - \frac{n}{p} \geq -\frac{1}{q}$. Then by corollary 2.3, we have

$$\begin{aligned} \|f(\cdot+h) - f(\cdot)\|_{L^p}^{\theta} \|f(\cdot+h) - f(\cdot)\|_{L^{q^*}}^{1-\theta} &\leq C|h|^{\theta} (2\|f\|_{L^{q^*}})^{1-\theta} \\ &\leq C_1|h|^{\theta} \|f\|_{W^{1,p}}^{1-\theta} \\ &\leq C_1|h|^{\theta} \end{aligned}$$

□

General fact: $W^{s_1,p_1}(\Omega) \subset W^{s_2,p_2}(\Omega)$, where Ω is bounded, $\partial\Omega$ is smooth. Embedding is compact \iff embedding is not critical.

Dual spaces

Definition 2.7

$W^{-s,p}(\Omega) := \left(W_0^{s,q}(\Omega)\right)^*$ is defined as the dual space to $W_0^{s,q}(\Omega)$, i.e. the space of linear continuous functionals on $W_0^{s,q}(\Omega)$, where $\frac{1}{p} + \frac{1}{q} = 1$.

Definition 2.8

$$W^{-s,p}(\Omega) = \left\{ \text{completion of } L^p(\Omega) \text{ w.r.t } \|\ell\|_{W^{-s,p}} := \sup_{\varphi \in \mathcal{D}} \frac{|\langle \ell, \varphi \rangle|}{\|\varphi\|_{W_0^{s,q}}} \right\}$$

Definition 2.9

$$W^{-s,p}(\Omega) = \left\{ \ell \in \mathcal{D}'(\Omega) : \|\ell\|_{W^{-s,p}} := \sup_{\varphi \in \mathcal{D}} \frac{|\langle \ell, \varphi \rangle|}{\|\varphi\|_{W_0^{s,q}}} \right\}$$

Proposition 2.3

Definitions 2.7, 2.8 and 2.9 are equivalent.

Example 2.7

$\delta(x) \in W^{-s,p}(\Omega), \Omega \in \mathbb{R}^n$. Find s, p, n . $\delta(x)$ is well-defined on continuous functions, so we need $W_0^{s,q}(\Omega) \subset C(\bar{\Omega})$. For example, $n = 1, p = 2$, then $\delta(x) \in H^{-s}(\Omega)$ for $s > \frac{1}{2}$.

Chapter 3

Linear elliptic problems

3.1 Dirichlet and Neumann problems for the Laplacian

Example 3.1 (Laplace equation with Dirichlet boundary conditions)

Let $\Omega \in \mathbb{R}^n$ be a bounded domain with $\partial\Omega$ smooth.

$$\begin{cases} \Delta u = f \\ u|_{\partial\Omega} = 0 \end{cases} \quad (3.1)$$

Typical questions:

1. In what space does the solution live?
2. In what sense is the understood (classical / weak)?
3. In what sense are the boundary / initial data understood?

In ODEs, we have local existence and uniqueness theorem (for Lipschitz non-linearities), but there is not an equivalent theorem for PDEs. Therefore, we must study particular examples.

Definition 3.1

$u \in W_0^{1,2}(\Omega)$ is a weak solution of 3.1 if $\forall \varphi \in C_0^\infty(\Omega)$

$$-\int_{\Omega} \nabla u(x) \nabla \varphi(x) \, dx = -\int_{\Omega} f(x) \varphi(x) \, dx \quad (3.2)$$

Here, the boundary condition is incorporated into the choice of space $W_0^{1,2}(\Omega) = [C_0^\infty(\Omega)]_{W^{1,2}(\Omega)}$ (the closure of $C_0^\infty(\Omega)$ in the norm of $W^{1,2}(\Omega)$).

3.2 came from the integration by parts formula. Indeed, if $u \in C^2(\Omega) \cap C(\bar{\Omega})$, then $\Delta u = f$ is understood in a classical sense and

$$\int_{\Omega} \Delta u \varphi \, dx = -\int_{\Omega} \nabla u \nabla \varphi \, dx + \int_{\partial\Omega} \partial_n u \varphi \, ds,$$

where the term $\int_{\partial\Omega} \partial_n u \varphi \, ds = 0$ because $\varphi|_{\partial\Omega} = 0$.

Theorem 3.1

Let $f \in H^{-1}(\Omega) := W^{-1,2}(\Omega)$. Then 3.1 has a unique weak solution.

Proof. Application of Riesz representation theorem

$[u, u] := \int_{\Omega} \nabla u \nabla u \, dx$ is an equivalent norm on $W_0^{1,2}(\Omega)$ (due to Friedrich's inequality). Then 3.2 can be rewritten as

$$[u, \varphi] = \int_{\Omega} f(x) \varphi(x) \, dx := \ell(\varphi)$$

Claim: ℓ is a linear continuous functional on $W_0^{1,2}(\Omega)$ (the integral should be understood as duality if we take $f \in H^{-1}(\Omega)$ and if $f \in L^2(\Omega)$, this is a standard Lebesgue integral).

Linearity of ℓ is obvious. ℓ is continuous as it is bounded:

$$|\ell(\varphi)| \leq \|f\|_{H^{-1}} \|\varphi\|_{H^1}$$

But we obtained that 3.2 holds only for $\varphi \in C_0^\infty$, not for $\varphi \in W_0^{1,2}(\Omega)$. However, $W_0^{1,2}(\Omega) = [C_0^\infty(\Omega)]_{W^{1,2}}$. Then approximation arguments give that $\forall \varphi \in H$,

$$[u, \varphi] = \ell(\varphi) \quad (3.3)$$

Then by Riesz representation theorem, there exists a unique $u \in W_0^{1,2}(\Omega)$ which satisfies 3.3. \square

Example 3.2 (Laplace equation with Neumann boundary conditions)

Let $\Omega \in \mathbb{R}^n$ be a bounded domain with $\partial\Omega$ smooth.

$$\begin{cases} \Delta u = f \\ \partial_n u|_{\partial\Omega} = 0 \end{cases} \quad (3.4)$$

We cannot consider $\varphi \in C_0^\infty(\Omega)$ as test functions, because the information about boundary conditions will be lost. Similarly,

$$\varphi \in W_n^{1,2}(\Omega) := \{u \in W^{1,2}(\Omega) : \partial_n u|_{\partial\Omega} = 0\}$$

will not work as well, since $\partial_n u|_{\partial\Omega}$ is not defined for $u \in W^{1,2}(\Omega)$ (since by theorem 2.1, $C^\infty(\Omega) \not\subset W^{1,2}(\Omega)$). Instead, let us take $\varphi \in C^\infty(\bar{\Omega})$ as a test function and assume that u is a classical solution. Then

$$\begin{aligned} \int_{\Omega} f \varphi \, dx &= \int_{\Omega} \Delta u \varphi \, dx \\ &= - \int_{\Omega} \nabla u \nabla \varphi \, dx + \int_{\partial\Omega} \partial_n u \varphi \, ds \\ &= - \int_{\Omega} \nabla u \nabla \varphi \, dx, \end{aligned}$$

as $\int_{\partial\Omega} \partial_n u \varphi \, dx = 0$ due to the boundary conditions. If we take $\varphi(x) = 1$ as a test function, then we get

$$\begin{aligned} \int_{\Omega} f \cdot 1 \, dx &= - \int_{\Omega} \nabla u \nabla 1 \, dx \\ &= 0 \end{aligned}$$

Hence $\langle f \rangle = \frac{1}{|\Omega|} \int_{\Omega} f(x) \, dx = 0$ is a necessary condition for solvability.

Let us notice that all solutions of this problem differs from each other by a constant. Thus, a natural assumption to single out the solution is $\langle u \rangle = 0$.

Definition 3.2

$u \in W^{1,2}(\Omega) \cap \{\langle u \rangle = 0\}$ is a weak solution of 3.4 if $\forall \varphi \in C^\infty(\bar{\Omega})$, we have:

$$\int_{\Omega} \nabla u \nabla \varphi \, dx = - \int_{\Omega} f \varphi \, dx \quad (3.5)$$

Note:-

The boundary conditions are now not in the definition of the space, but in 3.5.

Theorem 3.2

Let $f \in L^2(\Omega) \cap \{\langle f \rangle = 0\}$. Then 3.4 has a unique weak solution.

Proof. The proof is analogous to the problem with Dirichlet boundary conditions, but instead of apply Friedrich's inequality, we should apply Poincaré's inequality and use density of $C^\infty(\Omega) \in W^{1,2}(\Omega)$. \square