Chapter 5

Sobolev and interpolation inequalities

5.1 Interpolation inequalities

Example 5.1.1

$$\|u\|_{L^2}^2 \leqslant \|u\|_{L^2}\|u'\|_{L^2} \text{ for } u \in C^\infty(\mathbb{R}) \tag{5.1}$$

Proof. Idea: use that $(u^2)' = 2uu'$ and Newton-Leibniz

$$\begin{aligned} u^2(x) &= 2\int_{-\infty}^{x} uu'\,\mathrm{d}y = -2\int_{x}^{\infty} uu'\,\mathrm{d}y \\ &= \int_{-\infty}^{x} uu'\,\mathrm{d}y - \int_{x}^{\infty} uu'\,\mathrm{d}y \\ &\leqslant \int_{-\infty}^{x} |u||u'|\,\mathrm{d}y + \int_{x}^{\infty} |u||u'|\,\mathrm{d}y \\ &= \int_{\mathbb{R}} |u||u'|\,\mathrm{d}y \\ \text{(Hölder's inequality)} \quad &\leqslant \|u\|_{L^2}\|u'\|_{L^2} \end{aligned}$$

□

Question 1

Check that 5.1 is sharp. Namely, that 5.1 becomes equality for $u(x) = e^{-|x|}$ ($u(x)$ is an extremal function for 5.1). Also, 5.1 is shift and scaling invariant, i.e. $u_\alpha(x+h) = e^{-\alpha|x+h|}, h \in \mathbb{R}, \alpha > 0$ -extremals.

Example 5.1.2 (Interpolation inequality)

Ω-domain in $\mathbb{R}^n, u \in L_{p_1}(\Omega) \cap L_{p_2}(\Omega), 1 \leqslant p_1, p_2, < \infty, p_1 < p_2, \theta \in [0,1], \frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}$. Then

$$\|u\|_{L^p} \leqslant \|u\|_{L^{p_1}}^{\theta}\|u\|_{L^{p_2}}^{1-\theta} \tag{5.2}$$

Proof.

$$\int_{\mathbb{R}} |u|^p\,\mathrm{d}x = \int_{\mathbb{R}} |u|^{\theta p}|u|^{(1-\theta)p}\,\mathrm{d}x$$

We apply Hölder's inequality with exponents $P = \frac{p_1}{\theta p}$ and $Q = \frac{p_2}{(1-\theta)p}$ (Note $\frac{1}{P} + \frac{1}{Q} = \frac{\theta p}{p_1} + \frac{(1-\theta)p}{p_2} = 1$). Then

$$
\begin{aligned}
\int_{\mathbb{R}} |u|^{\theta p} |u|^{(1-\theta)p} \, \mathrm{d}x &\leqslant \left(\int_{\mathbb{R}} |u|^{p_1} \, \mathrm{d}x \right)^{\frac{1}{P}} \left(\int_{\mathbb{R}} |u|^{p_2} \, \mathrm{d}x \right)^{\frac{1}{Q}} \\
&= \|u\|_{L^{p_1}}^{\theta} \|u\|_{L^{p_2}}^{1-\theta}
\end{aligned}
$$

□

5.2 Sobolev inequalities

Example 5.2.1 (Sobolev inequality 1D)
$u \in C^\infty([0,1])$, want to prove the embedding $W^{1,1}([0,1]) \subset C([0,1])$, i.e.

$$
\|u\|_{C([0,1])} \leqslant \|u\|_{L^1([0,1])} + \|u'\|_{L^1([0,1])} \tag{5.3}
$$

Proof. By the Newton-Leibniz formula, $u(x) - u(y) = \int_y^x u'(s) \, \mathrm{d}s$. Also,

$$
|u(x)| \leqslant |u(y)| + \int_0^1 |u'(s)| \, \mathrm{d}s \quad \forall x, y \in [0,1]
$$

By integration over $y \in [0,1]$,

$$
|u(x)| \leqslant \int_0^1 |u(s)| \, \mathrm{d}s + \int_0^1 |u'(s)| \, \mathrm{d}s = \|u\|_{W^{1,1}([0,1])}
$$

Taking supremum with respect to $x \in [0,1]$, we obtain $\|u\|_{C([0,1])} \leqslant \|u\|_{W^{1,1}([0,1])}$ □

Example 5.2.2 (Sobolev inequality 2D)
$u \in C^\infty([0,1]^2)$, i.e. $\Omega = [0,1]^2$, then $W^{1,1}(\Omega) \subset L^2(\Omega) : \|u\|_{L^2} \leqslant \|u\|_{W^{1,1}(\Omega)}$

Proof. $\int_\Omega u^2(x_1, x_2) \, \mathrm{d}x_1 \, \mathrm{d}x_2$ should be estimated. From 5.3, we know that

$$
|u(x_1, x_2)| \leqslant \int_0^1 |u(s, x_2)| + |\partial_{x_1} u(s, x_2)| \, \mathrm{d}s := f(x_2)
$$

$$
|u(x_1, x_2)| \leqslant \int_0^1 |u(x_1, s)| + |\partial_{x_2} u(x_1, s)| \, \mathrm{d}s := g(x_1)
$$

Then

$$
\begin{aligned}
\int_\Omega u^2 \, \mathrm{d}x &\leqslant \int_0^1 g(x_1) f(x_2) \, \mathrm{d}x_1 \, \mathrm{d}x_2 \\
&= \int_0^1 f(x_2) \, \mathrm{d}x_2 \int_0^1 g(x_1) \, \mathrm{d}x_1 \\
&= \left(\int_\Omega |u(x_1, x_2)| + |\partial_{x_1} u(x_1, x_2)| \, \mathrm{d}x_1 \right) \left(\int_\Omega |u(x_1, x_2)| + |\partial_{x_2} u(x_1, x_2)| \, \mathrm{d}x_2 \right) \\
&\leqslant \|u\|_{W^{1,1}(\Omega)}
\end{aligned}
$$

□

Question 2: Sobolev inequality 3D

$u \in C^\infty(\bar{\Omega}), \Omega = (0,1)^3$. Prove that $W^{1,1}(\Omega) \subset L^{\frac{3}{2}}(\Omega)$, i.e.

$$\|u\|_{L^{\frac{3}{2}}(\Omega)} \leqslant \|u\|_{W^{1,1}(\Omega)} \tag{5.4}$$

Hint: first, prove that

$$\int_\Omega f(x_1,x_2)g(x_2,x_3)h(x_1,x_3)\,\mathrm{d}x \leqslant \|f\|_{L^2}\|g\|_{L^2}\|h\|_{L^2}$$

and use 5.3.

Example 5.2.3
$u \in C^\infty(\bar{\Omega}), \Omega = (0,1)^3$. Then

$$\|u\|_{L^6(\Omega)} \leqslant C\|u\|_{W^{1,2}(\Omega)} \tag{5.5}$$

Proof.

$$\begin{aligned}
\int_\Omega |u|^6\,\mathrm{d}x &= \int_\Omega (|u|^4)^{\frac{3}{2}}\,\mathrm{d}x \\
&\leqslant C\left(\int_\Omega |u|^4\,\mathrm{d}x + \int_\Omega u^3|\nabla u|\,\mathrm{d}x\right)^{\frac{3}{2}} \\
\text{(by (5.3))} \quad &\leqslant C\left(\int_\Omega |u|^4\,\mathrm{d}x\right)^{\frac{3}{2}} + C\left(u^3|\nabla u|\,\mathrm{d}x\right)^{\frac{3}{2}} \\
&\leqslant C\|u\|_{L^2}^{\frac{3}{2}\cdot\theta\cdot 4}\|u\|_{L^6}^{\frac{3}{2}\cdot(1-\theta)\cdot 4} + C\|u\|_{L^6}^{\frac{3}{2}\cdot 3}\|\nabla u\|_{L^2}^{\frac{3}{2}} \\
\left(\theta = \frac{1}{4}\right) \quad &= C\|u\|_{L^2}^{\frac{3}{2}}\|u\|_{L^6}^{\frac{9}{2}} + C\|u\|_{L^6}^{\frac{9}{2}}\|\nabla u\|_{L^2}^{\frac{3}{2}} \\
\left(\text{Young's inequality with } p = \frac{4}{5} \text{ and } q = -4\right) \quad &\leqslant \varepsilon\|u\|_{L^6}^6 + C_\varepsilon(\|u\|_{L^2} + \|\nabla u\|_{L^2})^6
\end{aligned}$$

Setting for example, $\varepsilon = \frac{1}{2}$, we obtain

$$\|u\|_{L^6(\Omega)} \leqslant C\|u\|_{W^{1,2}(\Omega)}$$

□

Theorem 5.2.1 Sobolev embeddings

① $W^{k_1,p_1}(\Omega) \subset W^{k_2,p_2}(\Omega) \Longleftrightarrow k_1 \geqslant k_2$ and $1 \leqslant p_1, p_2 < \infty, k_1 - \frac{n}{p_1} \geqslant k_2 - \frac{n}{p_2}, \Omega \subset \mathbb{R}^n$.

② $W^{k,p}(\Omega) \subset C^\alpha(\Omega)$ if $\alpha < k - \frac{n}{p}$. If α is not an integer, then the inequality is weak.

Example 5.2.4
$H^s(\mathbb{R}^n) \subset C(\mathbb{R}^n) \iff s > \frac{n}{2}$

Proof. $u(x) = \int_{\mathbb{R}^n} e^{i\xi x} \hat{u}(\xi)\,\mathrm{d}\xi$

$$\begin{aligned}|u(x)| &\leqslant \int_{\mathbb{R}^n} |\hat{u}(\xi)|\,\mathrm{d}\xi \\ &= \int_{\mathbb{R}^n} \left(1+|\xi|^2\right)^{-\frac{s}{2}} \left(1+|\xi|^2\right)^{\frac{s}{2}} |\hat{u}(\xi)|\,\mathrm{d}\xi \\ \text{(Hölder's inequality)} \quad &\leqslant \left(\int_{\mathbb{R}^n} \frac{1}{\left(1+|\xi|^2\right)^s}\,\mathrm{d}\xi\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^n} \left(1+|\xi|^2\right)^s |\hat{u}(\xi)|^2\,\mathrm{d}\xi\right)^{\frac{1}{2}}\end{aligned}$$

$\int_{\mathbb{R}^n} \frac{1}{(1+|\xi|^2)^s}\,\mathrm{d}\xi < \infty \iff s > \frac{n}{2}$. Taking the supremum with respect to $x \in \mathbb{R}^n$, we get

$$\|u\|_{C(\mathbb{R}^n)} \leqslant C_s \|u\|_{H^s(\mathbb{R}^n)}$$

□

Theorem 5.2.2 Interpolation inequalities

Let $u \in W^{k_1,p_1}(\Omega) \bigcap W^{k_2,p_2}(\Omega)$, $\theta \in [0,1]$, $1 \leqslant p_1, p_2 \leqslant \infty$ with $k = \theta k_1 + (1-\theta)k_2$, $\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}$. Then

$$\|u\|_{W^{k,p}} \leqslant C\|u\|^{\theta}_{W^{k_1,p_1}} \|u\|^{1-\theta}_{W^{k_2,p_2}}$$

Corollary 5.2.1 Particular cases

1. $\|u\|_{H^1} \leqslant \|u\|^{\frac{1}{2}}_{L^2} \|u\|^{\frac{1}{2}}_{H^2}$
2. $\|u\|_{L^p} \leqslant \|u\|^{\theta}_{L^p} \|u\|^{1-\theta}_{H^2}$

5.3 Spaces with zero boundary traces

Definition 5.3.1

$W^{1,p}_0(\Omega) := \left\{u \in W^{1,p}(\Omega),\, u|_{\partial\Omega} = 0\right\}$
Equivalent definition: $W^{1,p}_0(\Omega) =$ "closure of $C^\infty_0(\Omega)$ in $W^{1,p}$ norm."

Lemma 5.3.1

These two definitions are equivalent. $u \in$ "closure": $u = \lim_{n\to\infty} \varphi_n, \varphi_n \in C^\infty_0(\Omega) \implies \varphi_n|_{\partial\Omega} = 0$. By continuity, $u|_{\partial\Omega} = 0$. The proof of the converse statement is more technical and is omitted.

Proposition 5.3.1 Friedrich's inequality

Let Ω be a bounded domain and $u \in W^{1,p}_0(\Omega)$. Then

$$\|u\|_{L^p} \leqslant C\|\nabla u\|_{L^p} \tag{5.6}$$

Proof. It is enough to prove 5.6 for $\varphi \in C^\infty_0(\Omega)$. By the Newton-Leibniz formula,

$$u(x_1, x') - u(-L, x') = u(x_1, x') = \int_{-L}^{x_1} \partial_{x_1} u(s, x')\,\mathrm{d}s$$

$$
\begin{aligned}
|u(x_1,x')|^p &\leqslant \left(\int_{-L}^{L} |\partial_{x_1} u(s,x')|\,\mathrm{d}s\right)^p \\
\text{(Hölder's inequality)} \quad &\leqslant C_L \int_{-L}^{L} |\partial_{x_1} u(s,x')|^p\,\mathrm{d}s
\end{aligned}
$$

Integration with respect to x' gives us

$$\int_{\mathbb{R}^{n-1}} |u(x_1,x')|^p\,\mathrm{d}x' \leqslant C_L \|\partial_{x_1} u\|_{L^p}^p$$

Finally, integrating over $x_1 \in [-L,L]$, we obtain

$$\|u\|_{L^p}^p \leqslant 2LC_L \|\partial_{x_1} u\|_{L^p}^p$$

□

Corollary 5.3.1 Equivalent norm in $W_0^{1,p}(\Omega)$

Homogeneous norm:

$$\|u\|_{W_0^{1,p}(\Omega)} := \|\nabla u\|_{L^p}$$

Note:-

$u|_{\partial\Omega} = 0$ is important! Otherwise, 5.6 will fail for $u \equiv c$. Since ∇u defines u up to a constant; $u|_{\partial\Omega} = 0$ removes this constant.

Proposition 5.3.2 Poincaré inequality

Let Ω be a bounded domain with a smooth boundary and $\langle u\rangle := \frac{1}{|\Omega|}\int_\Omega u(x)\,\mathrm{d}x = 0$. Then

$$\|u\|_{L^p} \leqslant C\|\nabla u\|_{L^p}$$

Definition 5.3.2: Sequential compactness

A metric space (X,d) is compact if any sequence $\{x_n\}_{n=1}^\infty \subset X$ has a convergent sub-sequence, i.e. there exists $\{x_{n_k}\}_{k=1}^\infty : \lim\limits_{k\to\infty} x_{n_k} = x_0 \in X$

Definition 5.3.3

A topological space X is compact if any covering of X by open sets has a finite sub-covering

Note:-

In metric spaces, compactness is equivalent to sequential compactness.
In general topological spaces, they are not related.

Theorem 5.3.1 Hausdorff

Let (X,d) be a metric space. Then X is compact $\iff$ X is complete and totally bounded.

Definition 5.3.4

X is totally bounded if $\forall\varepsilon > 0, \exists$ covering of X by finitely many ε-balls, i.e. $X = \bigcup_{k=1}^N B_\varepsilon(x_k), N = N(\varepsilon)$ and $\{x_k\}$ is an ε-net in X.

5.4 Why do we need compactness?

Let X be compact and $f: X \to Y$ be continuous, then $f(X)$ is compact in Y. How do we solve PDEs of the form (or more general equations)?

$$F(x) = 0 \tag{5.7}$$

1. Construct approximate solutions
   $F(x_n) = g_n$, where $\lim_{n\to\infty} g_n = 0$
2. Obtain a priori estimates, i.e. that $\{x_n\}$ is bounded in a proper space
3. If $\{x_n\}$ is pre-compact and F is continuous $\implies x = \lim_{x\to\infty} x_{n_k}$ is a solution of 5.7.

Theorem 5.4.1 Arzelà-Ascoli
Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Then $V \subset C(\bar{\Omega})$ is compact iff:

1. V is closed
2. V is bounded
3. V is equicontinuous = V has a common modulus of continuity

Theorem 5.4.2 Arzelà-Ascoli for L^p
Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, (and $\partial\Omega$ smooth, although not needed), $K \subset L^p(\Omega), 1 \leqslant p < \infty$. Then K is compact iff:

1. K is closed
2. K is bounded
3. K is equicontinuous in mean (possesses a joint modulus of continuity in L^p).

Definition 5.4.1

Let $f \in L^p(\Omega), 1 \leqslant p < \infty, \Omega \subset \mathbb{R}^n$ bounded ($\partial\Omega$ smooth not needed). $\omega: \mathbb{R}^+ \to \mathbb{R}^+$ such that $\lim_{z\to 0} w(z) = 0$ is a modulus of continuity of f in $L_p(\Omega)$ if

$$\int_\Omega |f(x+h) - f(x)|^p \,\mathrm{d}x \leqslant \omega(|h|), \quad \forall h \in \mathbb{R}^n,$$

where we used the 0-extension of f outside of Ω.

Corollary 5.4.1
Let $K = B_1(0) \in W^{1,p}(\Omega); \Omega \subset \mathbb{R}^n$ is bounded, $\partial\Omega$ is smooth, $1 \leqslant p < \infty$. Then K is pre-compact in $L^p(\Omega)$.

Proof. We need to check equicontinuity, i.e. estimate $\int_\Omega |f(x+h) - f(x)|^p \,\mathrm{d}x$.

$$f(x+h) - f(x) = h \int_0^1 \nabla f(x+sh) \,\mathrm{d}s$$

Taking modulus and p-th power of both sides, we get

$$|f(x+h) - f(x)|^p \leqslant |h| \int_0^1 |\nabla f(x+sh)|^p \,\mathrm{d}s$$

Finally, we take an integral over $x \in \Omega$.

$$\int_\Omega |f(x+h) - f(x)|^p \, \mathrm{d}x \leqslant |h| \int_0^1 \int_\Omega |\nabla f(x+sh)|^p \, \mathrm{d}x \, \mathrm{d}s$$
$$\leqslant C|h|$$

$\omega(z) = cz$ is a joint modulus of continuity. ◻

Definition 5.4.2

Let $V \subset W$ be Banach spaces. Then the embedding is compact if the unit ball of V is pre-compact in W.

Note:-

We proved that $W^{1,p}(\Omega) \subset L^p(\Omega)$ is a compact embedding.

Corollary 5.4.2

$W^{1,p}(\Omega) \subset L^q(\Omega)$ is a compact embedding if $q < q^*$, where q^* is defined such that $\frac{1}{q^*} = \frac{1}{p} - \frac{1}{n}$ and $\Omega \subset \mathbb{R}^n$, Ω is bounded, $\partial\Omega$ is smooth.

Proof. Let us check equicontinuity.

$$\|f(\cdot + h) - f(\cdot)\|_{L^q} \leqslant \|f(\cdot + h) - f(\cdot)\|_{L^p}^{\theta} \|f(\cdot + h) - f(\cdot)\|_{L^{q^*}}^{1-\theta}$$

since $p < q < q^*$ and $0 < \theta < 1$. q^* is a critical exponent in Sobolev embeddings, indeed, $W^{1,p}(\Omega) \subset L^q(\Omega) \implies 1 - \frac{n}{p} \geqslant -\frac{1}{q}$. Then by corollary 5.4, we have

$$\begin{aligned}\|f(\cdot + h) - f(\cdot)\|_{L^p}^{\theta} \|f(\cdot + h) - f(\cdot)\|_{L^{q^*}}^{1-\theta} &\leqslant C|h|^{\theta}(2\|f\|_{L^{q^*}})^{1-\theta} \\ &\leqslant C_1|h|^{\theta}\|f\|_{W^{1,p}}^{1-\theta} \\ &\leqslant C_1|h|^{\theta}\end{aligned}$$

◻

General fact: $W^{s_1,p_1}(\Omega) \subset W^{s_2,p_2}(\Omega)$, where Ω is bounded, $\partial\Omega$ is smooth. Embedding is compact $\iff$ embedding is not critical.