

---

## Advanced Topics in Partial Differential Equations

---

Lecturer: Anna Kostianko

Transcribed by Harry Li

<https://github.com/HarryLi98/Advanced-PDE-Notes>

harryxcli@gmail.com

March 25, 2023

# Contents

<b>Chapter 2</b>	<b>Sobolev spaces</b>	<b>Page 2</b>
2.1	Interpolation inequalities	2
2.2	Sobolev inequalities	3
2.3	Spaces with zero boundary traces	5
2.4	Poincaré's and Friedrich's inequalities	5
2.5	Compactness	6
2.6	Dual spaces	8
<b>Chapter 3</b>	<b>Linear elliptic problems</b>	<b>Page 10</b>
3.1	Dirichlet and Neumann problems for the Laplacian	10
3.2	More general problems via Lax-Milgram	14
3.3	Introduction to spectral theory	15
3.4	Maximum principle	20

# Chapter 2

## Sobolev spaces

### 2.1 Interpolation inequalities

#### Example 2.1

$$\|u\|_{L^2}^2 \leq \|u\|_{L^2} \|u'\|_{L^2} \text{ for } u \in C^\infty(\mathbb{R}) \quad (2.1)$$

*Proof.* Idea: use that  $(u^2)' = 2uu'$  and Newton-Leibniz

$$\begin{aligned} u^2(x) &= 2 \int_{-\infty}^x uu' \, dy = -2 \int_x^\infty uu' \, dy \\ &= \int_{-\infty}^x uu' \, dy - \int_x^\infty uu' \, dy \\ &\leq \int_{-\infty}^x |u||u'| \, dy + \int_x^\infty |u||u'| \, dy \\ &= \int_{\mathbb{R}} |u||u'| \, dy \\ (\text{Hölder's inequality}) \quad &\leq \|u\|_{L^2} \|u'\|_{L^2} \end{aligned}$$

□

#### Question 1

Check that 2.1 is sharp. Namely, that 2.1 becomes equality for  $u(x) = e^{-|x|}$  ( $u(x)$  is an extremal function for 2.1). Also, 2.1 is shift and scaling invariant, i.e.  $u_\alpha(x+h) = e^{-\alpha|x+h|}$ ,  $h \in \mathbb{R}$ ,  $\alpha > 0$  -extremals.

#### Example 2.2 (Interpolation inequality)

$\Omega$ -domain in  $\mathbb{R}^n$ ,  $u \in L_{p_1}(\Omega) \cap L_{p_2}(\Omega)$ ,  $1 \leq p_1, p_2 < \infty$ ,  $p_1 < p_2$ ,  $\theta \in [0, 1]$ ,  $\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}$ . Then

$$\|u\|_{L^p} \leq \|u\|_{L^{p_1}}^\theta \|u\|_{L^{p_2}}^{1-\theta} \quad (2.2)$$

*Proof.*

$$\int_{\mathbb{R}} |u|^p \, dx = \int_{\mathbb{R}} |u|^{\theta p} |u|^{(1-\theta)p} \, dx$$

We apply Hölder's inequality with exponents  $P = \frac{p_1}{\theta p}$  and  $Q = \frac{p_2}{(1-\theta)p}$  (Note  $\frac{1}{P} + \frac{1}{Q} = \frac{\theta p}{p_1} + \frac{(1-\theta)p}{p_2} = 1$ ). Then

$$\begin{aligned} \int_{\mathbb{R}} |u|^{\theta p} |u|^{(1-\theta)p} dx &\leq \left( \int_{\mathbb{R}} |u|^{p_1} dx \right)^{\frac{1}{P}} \left( \int_{\mathbb{R}} |u|^{p_2} dx \right)^{\frac{1}{Q}} \\ &= \|u\|_{L^{p_1}}^{\theta} \|u\|_{L^{p_2}}^{1-\theta} \end{aligned}$$

□

## 2.2 Sobolev inequalities

**Example 2.3** (Sobolev inequality 1D)

$u \in C^\infty([0, 1])$ , want to prove the embedding  $W^{1,1}([0, 1]) \subset C([0, 1])$ , i.e.

$$\|u\|_{C([0,1])} \leq \|u\|_{L^1([0,1])} + \|u'\|_{L^1([0,1])} \quad (2.3)$$

*Proof.* By the Newton-Leibniz formula,  $u(x) - u(y) = \int_y^x u'(s) ds$ . Also,

$$|u(x)| \leq |u(y)| + \int_0^1 |u'(s)| ds \quad \forall x, y \in [0, 1]$$

By integration over  $y \in [0, 1]$ ,

$$|u(x)| \leq \int_0^1 |u(s)| ds + \int_0^1 |u'(s)| ds = \|u\|_{W^{1,1}([0,1])}$$

Taking supremum with respect to  $x \in [0, 1]$ , we obtain  $\|u\|_{C([0,1])} \leq \|u\|_{W^{1,1}([0,1])}$

□

**Example 2.4** (Sobolev inequality 2D)

$u \in C^\infty([0, 1]^2)$ , i.e.  $\Omega = [0, 1]^2$ , then  $W^{1,1}(\Omega) \subset L^2(\Omega) : \|u\|_{L^2} \leq \|u\|_{W^{1,1}(\Omega)}$

*Proof.*  $\int_{\Omega} u^2(x_1, x_2) dx_1 dx_2$  should be estimated. From 2.3, we know that

$$|u(x_1, x_2)| \leq \int_0^1 |u(s, x_2)| + |\partial_{x_1} u(s, x_2)| ds := f(x_2)$$

$$|u(x_1, x_2)| \leq \int_0^1 |u(x_1, s)| + |\partial_{x_2} u(x_1, s)| ds := g(x_1)$$

Then

$$\begin{aligned} \int_{\Omega} u^2 dx &\leq \int_0^1 g(x_1) f(x_2) dx_1 dx_2 \\ &= \int_0^1 f(x_2) dx_2 \int_0^1 g(x_1) dx_1 \\ &= \left( \int_{\Omega} |u(x_1, x_2)| + |\partial_{x_1} u(x_1, x_2)| dx_1 \right) \left( \int_{\Omega} |u(x_1, x_2)| + |\partial_{x_2} u(x_1, x_2)| dx_2 \right) \\ &\leq \|u\|_{W^{1,1}(\Omega)}^2 \end{aligned}$$

□

### Question 2: Sobolev inequality 3D

$u \in C^\infty(\bar{\Omega})$ ,  $\Omega = (0, 1)^3$ . Prove that  $W^{1,1}(\Omega) \subset L^{\frac{3}{2}}(\Omega)$ , i.e.

$$\|u\|_{L^{\frac{3}{2}}(\Omega)} \leq \|u\|_{W^{1,1}(\Omega)} \quad (2.4)$$

Hint: first, prove that

$$\int_{\Omega} f(x_1, x_2) g(x_2, x_3) h(x_1, x_3) dx \leq \|f\|_{L^2} \|g\|_{L^2} \|h\|_{L^2}$$

and use 2.3.

### Example 2.5

$u \in C^\infty(\bar{\Omega})$ ,  $\Omega = (0, 1)^3$ . Then

$$\|u\|_{L^6(\Omega)} \leq C \|u\|_{W^{1,2}(\Omega)} \quad (2.5)$$

*Proof.*

$$\begin{aligned} \int_{\Omega} |u|^6 dx &= \int_{\Omega} (|u|^4)^{\frac{3}{2}} dx \\ &\leq C \left( \int_{\Omega} |u|^4 dx + \int_{\Omega} u^3 |\nabla u| dx \right)^{\frac{3}{2}} \\ \text{(by (2.3))} \quad &\leq C \left( \int_{\Omega} |u|^4 dx \right)^{\frac{3}{2}} + C \left( \int_{\Omega} u^3 |\nabla u| dx \right)^{\frac{3}{2}} \\ &\leq C \|u\|_{L^2}^{\frac{3}{2} \cdot \theta \cdot 4} \|u\|_{L^6}^{\frac{3}{2} \cdot (1-\theta) \cdot 4} + C \|u\|_{L^6}^{\frac{3}{2} \cdot 3} \|\nabla u\|_{L^2}^{\frac{3}{2}} \\ \left( \theta = \frac{1}{4} \right) \quad &= C \|u\|_{L^2}^{\frac{3}{2}} \|u\|_{L^6}^{\frac{9}{2}} + C \|u\|_{L^6}^{\frac{9}{2}} \|\nabla u\|_{L^2}^{\frac{3}{2}} \\ \left( \text{Young's inequality with } p = \frac{4}{5} \text{ and } q = -4 \right) \quad &\leq \varepsilon \|u\|_{L^6}^6 + C_\varepsilon (\|u\|_{L^2} + \|\nabla u\|_{L^2})^6 \end{aligned}$$

Setting for example,  $\varepsilon = \frac{1}{2}$ , we obtain

$$\|u\|_{L^6(\Omega)} \leq C \|u\|_{W^{1,2}(\Omega)}$$

□

### Theorem 2.1 Sobolev embeddings

- ①  $W^{k_1, p_1}(\Omega) \subset W^{k_2, p_2}(\Omega) \iff k_1 \geq k_2 \text{ and } 1 \leq p_1, p_2 < \infty, k_1 - \frac{n}{p_1} \geq k_2 - \frac{n}{p_2}, \Omega \subset \mathbb{R}^n.$
- ②  $W^{k, p}(\Omega) \subset C^\alpha(\Omega)$  if  $\alpha < k - \frac{n}{p}$ . If  $\alpha$  is not an integer, then the inequality is weak.

### Example 2.6

$$H^s(\mathbb{R}^n) \subset C(\mathbb{R}^n) \iff s > \frac{n}{2}$$

*Proof.*  $u(x) = \int_{\mathbb{R}^n} e^{i\xi x} \hat{u}(\xi) d\xi$

$$\begin{aligned}
|u(x)| &\leq \int_{\mathbb{R}^n} |\hat{u}(\xi)| d\xi \\
&= \int_{\mathbb{R}^n} (1 + |\xi|^2)^{-\frac{s}{2}} (1 + |\xi|^2)^{\frac{s}{2}} |\hat{u}(\xi)| d\xi \\
&\stackrel{\text{(Hölder's inequality)}}{\leq} \left( \int_{\mathbb{R}^n} \frac{1}{(1 + |\xi|^2)^s} d\xi \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^n} (1 + |\xi|^2)^s |\hat{u}(\xi)|^2 d\xi \right)^{\frac{1}{2}}
\end{aligned}$$

$\int_{\mathbb{R}^n} \frac{1}{(1 + |\xi|^2)^s} d\xi < \infty \iff s > \frac{n}{2}$ . Taking the supremum with respect to  $x \in \mathbb{R}^n$ , we get

$$\|u\|_{C(\mathbb{R}^n)} \leq C_s \|u\|_{H^s(\mathbb{R}^n)}$$

□

### Theorem 2.2 Interpolation inequalities

Let  $u \in W^{k_1, p_1}(\Omega) \cap W^{k_2, p_2}(\Omega)$ ,  $\theta \in [0, 1]$ ,  $1 \leq p_1, p_2 \leq \infty$  with  $k = \theta k_1 + (1 - \theta)k_2$ ,  $\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}$ . Then

$$\|u\|_{W^{k, p}} \leq C \|u\|_{W^{k_1, p_1}}^\theta \|u\|_{W^{k_2, p_2}}^{1-\theta}$$

### Corollary 2.1 Particular cases

1.  $\|u\|_{H^1} \leq \|u\|_{L^2}^{\frac{1}{2}} \|u\|_{H^2}^{\frac{1}{2}}$
2.  $\|u\|_{L^p} \leq \|u\|_{L^p}^\theta \|u\|_{H^2}^{1-\theta}$

## 2.3 Spaces with zero boundary traces

### Definition 2.1

$$W_0^{1, p}(\Omega) := \{u \in W^{1, p}(\Omega), u|_{\partial\Omega} = 0\}$$

An equivalent definition is that the Sobolev spaces  $W_0^{1, p}(\Omega)$  for  $1 \leq p < \infty$  are defined as the closure of the set of compactly supported test functions  $C_0^\infty(\Omega)$  with respect to the  $W^{1, p}(\Omega)$ -norm.

### Lemma 2.1

These two definitions are equivalent.  $u \in \text{"closure"} : u = \lim_{n \rightarrow \infty} \varphi_n, \varphi_n \in C_0^\infty(\Omega) \implies \varphi_n|_{\partial\Omega} = 0$ . By continuity,  $u|_{\partial\Omega} = 0$ . The proof of the converse statement is more technical and is omitted.

## 2.4 Poincaré's and Friedrich's inequalities

### Proposition 2.1 Friedrich's inequality

Let  $\Omega$  be a bounded domain and  $u \in W_0^{1, p}(\Omega)$ . Then

$$\|u\|_{L^p} \leq C \|\nabla u\|_{L^p} \tag{2.6}$$

*Proof.* It is enough to prove 2.6 for  $\varphi \in C_0^\infty(\Omega)$ . By the Newton-Leibniz formula,

$$u(x_1, x') - u(-L, x') = u(x_1, x') = \int_{-L}^{x_1} \partial_{x_1} u(s, x') ds$$

$$\begin{aligned}
|u(x_1, x')|^p &\leq \left( \int_{-L}^L |\partial_{x_1} u(s, x')| \, ds \right)^p \\
\text{(Hölder's inequality)} \quad &\leq C_L \int_{-L}^L |\partial_{x_1} u(s, x')|^p \, ds
\end{aligned}$$

Integration with respect to  $x'$  gives us

$$\int_{\mathbb{R}^{n-1}} |u(x_1, x')|^p \, dx' \leq C_L \|\partial_{x_1} u\|_{L^p}^p$$

Finally, integrating over  $x_1 \in [-L, L]$ , we obtain

$$\|u\|_{L^p}^p \leq 2LC_L \|\partial_{x_1} u\|_{L^p}^p$$

□

### Corollary 2.2 Equivalent norm in $W_0^{1,p}(\Omega)$

Homogeneous norm:

$$\|u\|_{W_0^{1,p}(\Omega)} := \|\nabla u\|_{L^p}$$

#### Note:-

$u|_{\partial\Omega} = 0$  is important! Otherwise, 2.6 will fail for  $u \equiv c$ . Since  $\nabla u$  defines  $u$  up to a constant;  $u|_{\partial\Omega} = 0$  removes this constant.

### Proposition 2.2 Poincaré inequality

Let  $\Omega$  be a bounded domain with a smooth boundary and  $\langle u \rangle := \frac{1}{|\Omega|} \int_{\Omega} u(x) \, dx = 0$ . Then

$$\|u\|_{L^p} \leq C \|\nabla u\|_{L^p}$$

## 2.5 Compactness

### Definition 2.2: Sequential compactness

A metric space  $(X, d)$  is compact if any sequence  $\{x_n\}_{n=1}^{\infty} \subset X$  has a convergent sub-sequence, i.e. there exists  $\{x_{n_k}\}_{k=1}^{\infty} : \lim_{k \rightarrow \infty} x_{n_k} = x_0 \in X$ .

### Definition 2.3: Compact

A topological space  $X$  is compact if any covering of  $X$  by open sets has a finite sub-covering.

#### Note:-

In metric spaces, compactness is equivalent to sequential compactness.  
In general topological spaces, they are not related.

### Theorem 2.3 Hausdorff

Let  $(X, d)$  be a metric space. Then  $X$  is compact  $\iff X$  is complete and totally bounded.

### Definition 2.4: Totally bounded

$X$  is totally bounded if  $\forall \varepsilon > 0, \exists$  covering of  $X$  by finitely many  $\varepsilon$ -balls, i.e.  $X = \bigcup_{k=1}^N B_{\varepsilon}(x_k), N = N(\varepsilon)$  and  $\{x_k\}$  is an  $\varepsilon$ -net in  $X$ .

## Why do we need compactness?

Let  $X$  be compact and  $f: X \rightarrow Y$  be continuous, then  $f(X)$  is compact in  $Y$ . How do we solve PDEs of the form (or more general equations)?

$$F(x) = 0 \quad (2.7)$$

1. Construct approximate solutions

$$F(x_n) = g_n, \text{ where } \lim_{n \rightarrow \infty} g_n = 0$$

2. Obtain a priori estimates, i.e. that  $\{x_n\}$  is bounded in a proper space
3. If  $\{x_n\}$  is pre-compact and  $F$  is continuous  $\implies x = \lim_{n \rightarrow \infty} x_{n_k}$  is a solution of 2.7.

### Theorem 2.4 Arzelà-Ascoli

Let  $\Omega \subset \mathbb{R}^n$  be a bounded domain. Then  $V \subset C(\bar{\Omega})$  is compact iff:

1.  $V$  is closed
2.  $V$  is bounded
3.  $V$  is equicontinuous =  $V$  has a common modulus of continuity

### Theorem 2.5 Arzelà-Ascoli for $L^p$

Let  $\Omega \subset \mathbb{R}^n$  be a bounded domain, (and  $\partial\Omega$  smooth, although not needed),  $K \subset L^p(\Omega)$ ,  $1 \leq p < \infty$ . Then  $K$  is compact iff:

1.  $K$  is closed
2.  $K$  is bounded
3.  $K$  is equicontinuous in mean (possesses a joint modulus of continuity in  $L^p$ ).

### Definition 2.5: Modulus of continuity

Let  $f \in L^p(\Omega)$ ,  $1 \leq p < \infty$ ,  $\Omega \subset \mathbb{R}^n$  bounded ( $\partial\Omega$  smooth not needed).  $\omega: \mathbb{R}^+ \rightarrow \mathbb{R}^+$  such that  $\lim_{z \rightarrow 0} \omega(z) = 0$  is a modulus of continuity of  $f$  in  $L^p(\Omega)$  if

$$\int_{\Omega} |f(x+h) - f(x)|^p dx \leq \omega(|h|), \quad \forall h \in \mathbb{R}^n,$$

where we used the 0-extension of  $f$  outside of  $\Omega$ .

### Corollary 2.3

Let  $K = B_1(0) \in W^{1,p}(\Omega)$ ;  $\Omega \subset \mathbb{R}^n$  is bounded,  $\partial\Omega$  is smooth,  $1 \leq p < \infty$ . Then  $K$  is pre-compact in  $L^p(\Omega)$ .

*Proof.* We need to check equicontinuity, i.e. estimate  $\int_{\Omega} |f(x+h) - f(x)|^p dx$ .

$$f(x+h) - f(x) = h \int_0^1 \nabla f(x+sh) ds$$

Taking modulus and  $p$ -th power of both sides, we get

$$|f(x+h) - f(x)|^p \leq |h|^p \int_0^1 |\nabla f(x+sh)|^p ds$$



Finally, we take an integral over  $x \in \Omega$ .

$$\begin{aligned} \int_{\Omega} |f(x+h) - f(x)|^p dx &\leq |h| \int_0^1 \int_{\Omega} |\nabla f(x+sh)|^p dx ds \\ &\leq C|h| \end{aligned}$$

$\omega(z) = cz$  is a joint modulus of continuity. □

### Definition 2.6: Compact embedding

Let  $V \subset W$  be Banach spaces. Then the embedding is compact if the unit ball of  $V$  is pre-compact in  $W$ .

#### Note:-

We proved that  $W^{1,p}(\Omega) \subset L^p(\Omega)$  is a compact embedding.

### Corollary 2.4

$W^{1,p}(\Omega) \subset L^q(\Omega)$  is a compact embedding if  $q < q^*$ , where  $q^*$  is defined such that  $\frac{1}{q^*} = \frac{1}{p} - \frac{1}{n}$  and  $\Omega \subset \mathbb{R}^n$ ,  $\Omega$  is bounded,  $\partial\Omega$  is smooth.

*Proof.* Let us check equicontinuity.

$$\|f(\cdot+h) - f(\cdot)\|_{L^q} \leq \|f(\cdot+h) - f(\cdot)\|_{L^p}^{\theta} \|f(\cdot+h) - f(\cdot)\|_{L^{q^*}}^{1-\theta}$$

since  $p < q < q^*$  and  $0 < \theta < 1$ .  $q^*$  is a critical exponent in Sobolev embeddings, indeed,  $W^{1,p}(\Omega) \subset L^q(\Omega) \implies 1 - \frac{n}{p} \geq -\frac{1}{q}$ . Then by corollary 2.3, we have

$$\begin{aligned} \|f(\cdot+h) - f(\cdot)\|_{L^p}^{\theta} \|f(\cdot+h) - f(\cdot)\|_{L^{q^*}}^{1-\theta} &\leq C|h|^{\theta} (2\|f\|_{L^{q^*}})^{1-\theta} \\ &\leq C_1|h|^{\theta} \|f\|_{W^{1,p}}^{1-\theta} \\ &\leq C_1|h|^{\theta} \end{aligned}$$

□

General fact:  $W^{s_1,p_1}(\Omega) \subset W^{s_2,p_2}(\Omega)$ , where  $\Omega$  is bounded,  $\partial\Omega$  is smooth. Embedding is compact  $\iff$  embedding is not critical.

## Dual spaces

### Definition 2.7: Dual space

$W^{-s,p}(\Omega) := \left(W_0^{s,q}(\Omega)\right)^*$  is defined as the dual space to  $W_0^{s,q}(\Omega)$ , i.e. the space of linear continuous functionals on  $W_0^{s,q}(\Omega)$ , where  $\frac{1}{p} + \frac{1}{q} = 1$ .

### Definition 2.8

$$W^{-s,p}(\Omega) = \left\{ \text{completion of } L^p(\Omega) \text{ w.r.t } \|\ell\|_{W^{-s,p}} := \sup_{\varphi \in \mathcal{D}} \frac{|\langle \ell, \varphi \rangle|}{\|\varphi\|_{W_0^{s,q}}} \right\}$$

### Definition 2.9

$$W^{-s,p}(\Omega) = \left\{ \ell \in \mathcal{D}'(\Omega) : \|\ell\|_{W^{-s,p}} := \sup_{\varphi \in \mathcal{D}} \frac{|\langle \ell, \varphi \rangle|}{\|\varphi\|_{W_0^{s,q}}} \right\}$$

**Proposition 2.3**

Definitions 2.7, 2.8 and 2.9 are equivalent.

**Question 3**

Suppose  $\delta(x) \in W^{-s,p}(\Omega)$ ,  $\Omega \subset \mathbb{R}^n$ . How are  $s, p$  and  $n$  related? We know that  $\delta(x)$  is well-defined on continuous functions, so we need  $W_0^{s,q}(\Omega) \subset C(\bar{\Omega})$ .

**Example 2.7**

Consider the case where  $n = 1$  and  $p = 2$ . By the Sobolev embedding theorem,  $W^{s,2} \subset C(\bar{\Omega})$  if  $0 < s - \frac{1}{2}$ . Thus we have  $\delta(x) \in H^{-s}(\Omega)$  if  $s > \frac{1}{2}$ .

# Chapter 3

## Linear elliptic problems

### 3.1 Dirichlet and Neumann problems for the Laplacian

**Example 3.1** (Laplace equation with Dirichlet boundary conditions)

Let  $\Omega \in \mathbb{R}^n$  be a bounded domain with  $\partial\Omega$  smooth. Consider the Laplace equation with Dirichlet boundary conditions:

$$\begin{cases} \Delta u = f \\ u|_{\partial\Omega} = 0 \end{cases} \quad (3.1)$$

Typical questions:

1. In what space does the solution live?
2. In what sense is the equation understood (classical / weak)?
3. In what sense are the boundary / initial data understood?

In ODEs, we have local existence and uniqueness theorem (for Lipschitz non-linearities), but there is not an equivalent theorem for PDEs. Therefore, we must study particular examples.

**Definition 3.1**

$u \in W_0^{1,2}(\Omega)$  is a weak solution of 3.1 if  $\forall \varphi \in C_0^\infty(\Omega)$ ,

$$-\int_{\Omega} \nabla u(x) \nabla \varphi(x) \, dx = \int_{\Omega} f(x) \varphi(x) \, dx \quad (3.2)$$

Here, the boundary condition is incorporated into the choice of space  $W_0^{1,2}(\Omega) = [C_0^\infty(\Omega)]_{W^{1,2}(\Omega)}$  (the closure of  $C_0^\infty(\Omega)$  in the norm of  $W^{1,2}(\Omega)$ ).

3.2 came from the integration by parts formula. Indeed, if  $u \in C^2(\Omega) \cap C(\bar{\Omega})$ , then  $\Delta u = f$  is understood in a classical sense and

$$\int_{\Omega} \Delta u \varphi \, dx = - \int_{\Omega} \nabla u \nabla \varphi \, dx + \int_{\partial\Omega} \partial_n u \varphi \, ds,$$

where the term  $\int_{\partial\Omega} \partial_n u \varphi \, ds = 0$  because  $\varphi|_{\partial\Omega} = 0$ .

**Theorem 3.1**

Let  $f \in H^{-1}(\Omega) := W^{-1,2}(\Omega)$ . Then 3.1 has a unique weak solution.

*Proof.* Application of Riesz representation theorem

$[u, u] := \int_{\Omega} \nabla u \nabla u \, dx$  is an equivalent norm on  $W_0^{1,2}(\Omega)$  (due to Friedrich's inequality). Then 3.2 can be rewritten as

$$[u, \varphi] = \int_{\Omega} f(x) \varphi(x) \, dx := \ell(\varphi)$$

Claim:  $\ell$  is a linear continuous functional on  $W_0^{1,2}(\Omega)$  (the integral should be understood as duality if we take  $f \in H^{-1}(\Omega)$  and if  $f \in L^2(\Omega)$ , this is a standard Lebesgue integral).

Linearity of  $\ell$  is obvious.  $\ell$  is continuous as it is bounded:

$$|\ell(\varphi)| \leq \|f\|_{H^{-1}} \|\varphi\|_{H^1}$$

But we obtained that 3.2 holds only for  $\varphi \in C_0^\infty(\Omega)$ , not for  $\varphi \in W_0^{1,2}(\Omega)$ . However,  $W_0^{1,2}(\Omega) = [C_0^\infty(\Omega)]_{W^{1,2}}$ . Then approximation arguments give that  $\forall \varphi \in H$ ,

$$[u, \varphi] = \ell(\varphi) \tag{3.3}$$

Then by Riesz representation theorem, there exists a unique  $u \in W_0^{1,2}(\Omega)$  which satisfies 3.3.  $\square$

### Example 3.2 (Laplace equation with Neumann boundary conditions)

Let  $\Omega \in \mathbb{R}^n$  be a bounded domain with  $\partial\Omega$  smooth. Consider the Laplace equation with Neumann boundary conditions:

$$\begin{cases} \Delta u = f \\ \partial_n u|_{\partial\Omega} = 0 \end{cases} \tag{3.4}$$

We cannot consider  $\varphi \in C_0^\infty(\Omega)$  as test functions, because the information about boundary conditions will be lost. Similarly, considering

$$\varphi \in W_n^{1,2}(\Omega) := \{u \in W^{1,2}(\Omega) : \partial_n u|_{\partial\Omega} = 0\}$$

will not work as well, since  $\partial_n u|_{\partial\Omega}$  is not defined for  $u \in W^{1,2}(\Omega)$  (since by theorem 2.1,  $C^\infty(\Omega) \not\subset W^{1,2}(\Omega)$ ). Instead, let us take  $\varphi \in C^\infty(\bar{\Omega})$  as a test function and assume that  $u$  is a classical solution. Then

$$\begin{aligned} \int_{\Omega} f \varphi \, dx &= \int_{\Omega} \Delta u \varphi \, dx \\ &= - \int_{\Omega} \nabla u \nabla \varphi \, dx + \int_{\partial\Omega} \partial_n u \varphi \, ds \\ &= - \int_{\Omega} \nabla u \nabla \varphi \, dx, \end{aligned}$$

as  $\int_{\partial\Omega} \partial_n u \varphi \, dx = 0$  due to the boundary conditions. If we take  $\varphi(x) = 1$  as a test function, then we get

$$\begin{aligned} \int_{\Omega} f \cdot 1 \, dx &= - \int_{\Omega} \nabla u \nabla 1 \, dx \\ &= 0 \end{aligned}$$

Hence  $\langle f \rangle = \frac{1}{|\Omega|} \int_{\Omega} f(x) \, dx = 0$  is a necessary condition for solvability.

Let us notice that all solutions of this problem differs from each other by a constant. Thus, a natural assumption to single out the solution is  $\langle u \rangle = 0$ .

### Definition 3.2

$u \in W^{1,2}(\Omega) \cap \{\langle u \rangle = 0\}$  is a weak solution of 3.4 if  $\forall \varphi \in C^\infty(\bar{\Omega})$ , we have:

$$\int_{\Omega} \nabla u \nabla \varphi \, dx = - \int_{\Omega} f \varphi \, dx \tag{3.5}$$

**Note:-**

The boundary conditions are now not in the definition of the space, but in 3.5.

**Theorem 3.2**

Let  $f \in L^2(\Omega) \cap \{\langle f \rangle = 0\}$ . Then 3.4 has a unique weak solution.

*Proof.* The proof is analogous to the problem with Dirichlet boundary conditions, but instead of applying Friedrich's inequality, we should apply Poincaré's inequality and use density of  $C^\infty(\Omega) \in W^{1,2}(\Omega)$ .  $\square$

**Example 3.3 (Non-homogeneous Neumann boundary conditions)**

Let  $\Omega \in \mathbb{R}^n$  be a bounded domain with  $\partial\Omega$  smooth. Consider the Laplace equation with non-homogeneous Neumann boundary conditions:

$$\begin{cases} \Delta u = f \\ \partial_n u|_{\partial\Omega} = g \end{cases} \quad (3.6)$$

**Definition 3.3**

$u \in W^{1,2}(\Omega) \cap \{\langle u \rangle = 0\}$  is a weak solution of 3.6 if  $\forall \varphi \in C^\infty(\bar{\Omega})$ , we have:

$$\int_{\Omega} \nabla u \nabla \varphi \, dx = - \int_{\Omega} f \varphi \, dx + \int_{\partial\Omega} g \varphi \, ds \quad (3.7)$$

Note that if  $\varphi \equiv 1$ , then a necessary condition for solvability is

$$- \int_{\Omega} f \, dx + \int_{\partial\Omega} g \, ds = 0$$

**Theorem 3.3**

Let  $f \in L^2(\Omega)$ ,  $g \in W^{-\frac{1}{2},2}(\partial\Omega)$  be such that  $\int_{\Omega} f \, dx = \int_{\partial\Omega} g \, ds$ . Then 3.6 has a unique weak solution.

*Proof.*  $[u, u] := \int_{\Omega} \nabla u \nabla u \, dx$  is an equivalent norm on  $u \in W^{1,2}(\Omega) \cap \{\langle u \rangle = 0\}$  due to the Poincaré inequality. Then 3.7 can be rewritten as

$$[u, \varphi] = \ell(\varphi) := - \int_{\Omega} f \varphi \, dx + \int_{\partial\Omega} g \varphi \, ds$$

We claim that  $\ell$  is a linear continuous functional on  $W^{1,2}(\Omega) \cap \{\langle u \rangle = 0\}$ . Indeed, linearity is obvious. To show  $\ell$  is continuous, we have

$$\left| - \int_{\Omega} f \varphi \, dx + \int_{\partial\Omega} g \varphi \, ds \right| \leq \|f\|_{L^2} \|\varphi\|_{L^2} + \|g\|_{H^{-\frac{1}{2}}(\partial\Omega)} \|\varphi\|_{H^{\frac{1}{2}}(\partial\Omega)}$$

$$(\text{By the trace theorem and Poincaré's inequality}) \leq \|f\|_{L^2} \|\varphi\|_{W^{1,2}(\Omega)} + \|g\|_{H^{-\frac{1}{2}}(\partial\Omega)} \|\varphi\|_{W^{1,2}(\Omega)}$$

Then by Riesz representation theorem, there exists a unique  $u \in W^{1,2}(\Omega) \cap \{\langle u \rangle = 0\}$  that is a weak solution of 3.6.  $\square$

**Example 3.4 (Non-homogeneous Dirichlet boundary conditions)**

Let  $\Omega \in \mathbb{R}^n$  be a bounded domain with  $\partial\Omega$  smooth. Consider the Laplace equation with non-homogeneous Dirichlet boundary conditions:

$$\begin{cases} \Delta u = 0 \\ u|_{\partial\Omega} = g \end{cases} \quad (3.8)$$

Let us take  $g \in W^{\frac{1}{2},2}(\partial\Omega)$ . Then there exists  $v \in W^{1,2}(\Omega)$  such that  $v|_{\partial\Omega} = g$  (by the trace theorem). We

look for the solution of 3.8 in the form  $u = v + w$ , where  $w \in W_0^{1,2}(\Omega)$ .

#### Definition 3.4

$u = v + w$  is a weak solution of 3.8 if  $v|_{\partial\Omega} = g$ , where  $g \in W^{\frac{1}{2},2}(\partial\Omega)$ ,  $w \in W_0^{1,2}(\Omega)$  and  $\forall \varphi \in C^\infty(\bar{\Omega})$ , we have

$$\int_{\Omega} \nabla(v + w) \nabla \varphi \, dx = 0 \quad (3.9)$$

#### Theorem 3.4

Let  $g \in W^{\frac{1}{2},2}(\partial\Omega)$ . Then 3.8 has a unique weak solution.

*Proof.* We can rearrange 3.9 to get

$$\ell(\varphi) := -[v, \varphi] = \int_{\Omega} \nabla w \nabla \varphi \, dx = [w, \varphi],$$

and the functional  $\ell$  can be shown to be linear and continuous. By the Riesz representation theorem, there exists a unique  $w \in W_0^{1,2}(\Omega)$  such that 3.9 is satisfied. Note that this  $w$  depends on the choice of  $v$ . But  $u = v + w$  does not depend on the choice of  $v$ . Indeed, let  $u_1$  and  $u_2$  be two solutions of 3.8. Then  $u = u_1 - u_2$  solves

$$\begin{cases} \Delta u = 0 \\ u|_{\partial\Omega} = 0 \end{cases}$$

We have previously shown that the weak solution of this problem is unique. Therefore,  $u_1 = u_2$ .  $\square$

#### Note:-

There is no universal choice of the space of test functions. Even for Dirichlet and Neumann boundary conditions, we need to consider different spaces.  $\varphi \in C_0^\infty(\Omega)$  corresponds to the standard theory of distributions, while  $\varphi \in C^\infty(\bar{\Omega})$  corresponds to “non-standard” distributions.

#### Example 3.5

Let  $\Omega \in \mathbb{R}^n$  be a bounded domain with  $\partial\Omega$  smooth. Consider

$$\begin{cases} \sum_{i,j} \partial_{x_i}(a_{ij}(x) \partial_{x_j} u) = g \\ u|_{\partial\Omega} = 0 \end{cases} \quad (3.10)$$

Where we make the following assumptions on the matrix  $a(x) := \{a_{ij}(x)\}_{i,j}$ :

1.  $a(x)$  is a symmetric matrix for every  $x$ :

$$a_{ij}(x) = a_{ji}(x)$$

2.  $a(x)$  is uniformly elliptic. That is, for all  $\xi \in \mathbb{R}^n$ , there exists  $\mu, M > 0$  which are independent of  $x$  such that

$$\mu |\xi|^2 \leq \sum_{i,j} a_{ij} \xi_i \xi_j \leq M |\xi|^2$$

#### Definition 3.5

$u \in W^{1,2}(\Omega)$  is a weak solution to 3.10  $\iff \forall \varphi \in C_0^\infty(\Omega)$ , we have

$$\sum_{i,j} \int_{\Omega} a_{ij} \partial_{x_j} u \partial_{x_i} \varphi \, dx = - \int_{\Omega} g \varphi \, dx$$

**Theorem 3.5**

Let  $a(x)$  be symmetric and uniformly elliptic. Then 3.10 has a unique weak solution.

*Proof.* Let us denote

$$[u, \varphi]_a = \int_{\Omega} \sum_{i,j} a_{ij}(x) \partial_{x_j} u(x) \partial_{x_i} \varphi(x) dx.$$

Then since  $a(x)$  is symmetric, the bilinear form  $[u, v]_a$  is also symmetric, i.e.  $[u, v]_a = [v, u]_a$ . Since  $a(x)$  is uniformly elliptic, there exist  $\mu, M > 0$  such that

$$\mu[u, u] \leq [u, u]_a \leq M[u, u].$$

Therefore,  $(W_0^{1,2}(\Omega), [\cdot, \cdot]_a)$  is a Hilbert space with the norm equivalent to the standard  $W_0^{1,2}(\Omega)$  norm.

By the Riesz representation theorem, there exists a unique weak solution to 3.10.  $\square$

## 3.2 More general problems via Lax-Milgram

By the Riesz representation theorem, for any linear continuous functional,  $\ell$  on a Hilbert space  $H$ , there exists a unique  $x \in H$  such that  $\forall \varphi \in H$ , we have  $(x, \varphi) = \ell(\varphi)$ .

If we want  $a(x, y)$  to be an equivalent inner product on  $H$ , then  $a(x, y)$  must be symmetric.

We now consider the case where  $a(x, y)$  is not assumed to be symmetric.

**Definition 3.6: Bilinear form**

A bilinear form  $a(\cdot, \cdot): H \times H \rightarrow \mathbb{R}$  is bounded if

$$|a(x, y)| \leq C \|x\| \|y\|$$

**Definition 3.7: Coercive**

A bilinear form  $a(\cdot, \cdot)$  is coercive if  $\exists \alpha > 0$  such that  $a(x, x) \geq \alpha \|x\|^2$ .

**Theorem 3.6**

Let  $a(x, y)$  be a bounded and coercive bilinear form on  $H$ . Then any linear continuous functional  $\ell: H \rightarrow \mathbb{R}$  can be represented in the form

$$a(x, y) = \ell(\varphi), \quad \forall \varphi \in H. \quad (3.11)$$

i.e.  $\forall \ell \in H^*$ , there exists a unique  $x = x(\ell) \in H$  such that 3.11 is satisfied.

**Example 3.6**

Let  $\Omega \in \mathbb{R}^n$  be a bounded domain with  $\partial\Omega$  smooth. Consider the problem

$$\begin{cases} \sum_{i,j} \partial_{x_i} (a_{ij}(x) \partial_{x_j} u) + \sum_i b_i(x) \partial_{x_i} u = g(x) \\ u|_{\partial\Omega} = 0 \end{cases} \quad (3.12)$$

**Definition 3.8**

$u \in W_0^{1,2}(\Omega)$  is a weak solution of 3.12 if  $\forall \varphi \in C_0^\infty(\Omega)$ , we have

$$A(u, \varphi) := \sum_{i,j} \int_{\Omega} a_{ij} \partial_{x_j} u \partial_{x_i} \varphi dx - \sum_i \int_{\Omega} b_i(x) \partial_{x_i} u \varphi dx = \ell(\varphi) := - \int_{\Omega} g(x) \varphi(x) dx$$

**Theorem 3.7**

Let  $\{a_{ij}\} \in L^\infty(\Omega)$  be a uniformly elliptic matrix,  $b_i(x)$  be a smooth divergent free vector field and  $g(x) \in H^{-1}(\Omega)$ . Then 3.12 has a unique weak solution.

*Proof.* We use the Lax-Milgram theorem. We know that  $\ell(\varphi)$  is a linear continuous functional on  $W_0^{1,2}(\Omega)$ . Furthermore,  $A(u, \varphi)$  is bilinear and bounded. Indeed, by Friedrich's inequality, we have

$$|A(u, \varphi)| \leq C_1 \|\nabla u\|_{L^2} \|\nabla \varphi\|_{L^2} + C_2 \|\nabla u\|_{L^2} \|\varphi\|_{L^2} \leq C \|u\|_{W_0^{1,2}(\Omega)} \|\varphi\|_{W_0^{1,2}(\Omega)}$$

$A(u, \varphi)$  is coercive since

$$\begin{aligned} A(u, u) &= \sum_{i,j} \int_{\Omega} a_{ij} \partial_{x_j} u \partial_{x_i} u \, dx - \sum_i \int_{\Omega} b_i(x) \partial_{x_i} (u) u \, dx \\ &\geq \alpha \|\nabla u\|_{L^2}^2 - \frac{1}{2} \int_{\Omega} \sum_i b_i(x) \partial_{x_i} (u^2) \, dx \\ &= \alpha \|\nabla u\|_{L^2}^2 + \frac{1}{2} \int_{\Omega} \operatorname{div} b \cdot u^2(x) \, dx \\ &= \alpha \|\nabla u\|_{L^2}^2 \end{aligned}$$

By the Lax-Milgram theorem, there exists a unique weak solution of 3.12. □

### 3.3 Introduction to spectral theory

$H$  is a Hilbert space.  $\mathcal{L}(H)$  is a space of linear continuous operators.

**Lemma 3.1**

$A$  is continuous  $\iff A$  is bounded, i.e.

$$\|A\| := \sup_{x \in H} \frac{\|Ax\|}{\|x\|} < \infty$$

**Lemma 3.2**

$(\mathcal{L}(H), \|\cdot\|)$  is a Banach space.

**Definition 3.9: Invertible operator**

$A$  is invertible  $\iff \exists A^{-1} \in \mathcal{L}(H)$  such that

$$AA^{-1} = A^{-1}A = I \tag{3.13}$$

**Definition 3.10: Spectrum**

$\lambda \in \sigma(A)$  (the spectrum of  $A$ )  $\iff \lambda I - A$  is not invertible.

In other words,  $\lambda \notin \sigma(A)$ , ( $\lambda$  is in the resolvent set) iff the equation  $\lambda u - Au = f$  has a unique solution for all  $f \in H$ .

**Note:-**

If  $\dim H = n < \infty$ , then  $\mathcal{L}(H) = M(n \times n)$  ( $n \times n$  matrices) and

1. All linear operators are continuous,



2. All  $\lambda \in \sigma(A)$  correspond to eigenvalues

$$A\rho_\lambda = \lambda\rho_\lambda \implies \sigma(A) = \sigma_p(A) \text{ (point spectrum).}$$

3.  $\lambda \in \sigma(A) \iff \det(\lambda I - A) = 0$

4. Only one of two equalities from 3.13 holding is enough.

All of the statements may fail when  $\dim H = \infty$ .

### Example 3.7

Let  $H = \ell_2$ , the space of square summable sequences. Let  $T_r, T_l \in \mathcal{L}(H)$  denote the right and left shift operators, respectively;

$$T_r(x_1, x_2, x_3, \dots) = (0, x_1, x_2, x_3, \dots)$$

$$T_l(x_1, x_2, x_3, \dots) = (x_2, x_3, \dots)$$

Then  $T_l \circ T_r = I$ , but  $T_r \circ T_l(x) = (0, x_2, x_3, \dots)$ . Hence  $T_r$  has a left inverse, but not a right inverse.  $\ker(T_r) = \{0\}$ , i.e. it is injective (no eigenvalues), but the range of  $T_r \neq H$ , since  $T_r(x) \perp e_1$  for any  $x \in H$ . So  $T_r$  is not invertible since  $T_r(H)$  is a proper closed subspace of  $H$ . Therefore, a new type of spectrum appeared - the residual spectrum  $\sigma_R(A)$ , which is impossible in finite-dimensional spaces.

Also, the determinant does not exist in infinite-dimensional spaces. Indeed, if it existed, then

$$1 = \det(T_l \circ T_r) = \det(T_l) \det(T_r),$$

which implies that both  $T_l$  and  $T_r$  are invertible, but this is not true.

### Definition 3.11: Approximate point spectrum

$\lambda \in \sigma_{\text{app}}(A)$  (approximate point spectrum)  $\iff \exists x_n \in H, \|x_n\| = 1$  and  $\lim_{n \rightarrow \infty} (Ax_n - \lambda x_n) = 0$ .

It can be proved that  $\lambda \in \sigma_{\text{app}}(A) \iff$  the image of  $(\lambda I - H)$  is not closed.

### Example 3.8

$H = L^2(0, 1), Af(x) := xf(x)$ .

$$\lambda u - Au = f \iff (\lambda - x)u(x) = f(x),$$

then  $u(x) = \frac{f(x)}{\lambda - x}$  and  $(A - \lambda I)$  is not invertible  $\iff \lambda \in [0, 1]$ . Hence,  $\sigma(A) = [0, 1]$ .

We can check that  $\text{Range}(\lambda I - A)$  is not closed if  $\lambda \in [0, 1]$ .

The next theorem shows that these are all the possible obstacles to invert the operator.

### Theorem 3.8 Weyl's theorem

$A \in \mathcal{L}(H)$ . Then  $\sigma(A) = \sigma_p(A) \cup \sigma_{\text{app}}(A) \cup \sigma_R(A)$ .

### Definition 3.12: Resolvent

$R_A(\lambda) := (\lambda I - A)^{-1}$  is called the resolvent of  $A \in \mathcal{L}(H)$ .

More standard facts:

1. If  $|\lambda| > \|A\|$ , then  $\lambda \in \sigma(A)$ . Indeed,

$$\frac{1}{\lambda - A} = \frac{1}{\lambda} \frac{1}{1 - \frac{A}{\lambda}} = \frac{1}{\lambda} \sum_{n=0}^{\infty} \left(\frac{A}{\lambda}\right)^n$$

is an absolutely convergent series if  $|\lambda| > \|A\|$ .

2. Resolvent identity:

$$R_A(\lambda) - R_A(\mu) = -(\lambda - \mu)R_A(\lambda)R_A(\mu), \quad (3.14)$$

which follows from

$$\frac{1}{\lambda - x} - \frac{1}{\lambda - \mu} = (\mu - \lambda) \frac{1}{\lambda - x} \frac{1}{\mu - x},$$

where we substitute in  $x = A$ . From 3.14, by taking the limit as  $\mu \rightarrow \lambda$ , we get

$$\frac{d}{d\lambda} R_A(\lambda) = -R_A(\lambda)^2.$$

Hence  $R_A(\lambda)$  is an analytic function of  $\lambda$ .

3. By Liouville's theorem applied to  $R_A(\lambda)$ ,  $\sigma(A) \neq \emptyset$

#### Definition 3.13: Compact operator

$A \in \mathcal{L}(H)$  is compact  $\iff AB_1(0)$  is a precompact set in  $H$ .

#### Definition 3.14: Adjoint operator

Let  $A \in \mathcal{L}(H)$ . The adjoint operator  $A^* \in \mathcal{L}(H)$  is defined via  $(Ax, y) = (x, A^*y)$ ,  $\forall x, y \in H$ . It exists due to Riesz representation theorem. In the finite dimensional case, the adjoint operator coincides with the transpose operator.

#### Definition 3.15: Self-adjoint operator

$A \in \mathcal{L}(H)$  is self-adjoint if  $A = A^*$

#### Definition 3.16: Fredholm

$A \in \mathcal{L}(H)$  is Fredholm if  $\text{Range}(A)$  and  $\text{Range}(A^*)$  are closed and  $\ker(A)$  and  $\ker(A^*)$  are both finite dimensional.

Then the index of  $A$  is defined by

$$\text{ind}(A) := \dim \ker(A) - \dim \ker(A^*)$$

#### Theorem 3.9 Key theorem of Fredholm operators theorem

$\text{ind}(A)$  is a topological invariant. Namely, if  $A(t), t \in [0, 1]$  is a continuous curve of Fredholm operators, then

$$\text{ind}(A(0)) = \text{ind}(A(1)).$$

#### Definition 3.17: Essential spectrum

$\lambda \in \sigma_{\text{ess}}(A)$  if  $\lambda I - A$  is not Fredholm.

Properties:

1.  $K$  is compact  $\iff K^*$  is compact.

2.  $A \in \mathcal{L}(H), K$  compact  $\implies AK$  and  $KA$  are compact.
3.  $A$  is Fredholm  $\iff A^*$  is Fredholm.
4. Fredholm alternative: Let  $A$  be Fredholm. Then:

$$H = \text{Range}(A) \oplus \ker(A^*)$$

$$H = \text{Range}(A^*) \oplus \ker(A)$$

5.  $A$  is Fredholm  $\iff$  it is invertible by modulus of compact operators, i.e.  $\exists B : AB = I + K_1, BA = I + K_2$ , with  $K_1, K_2$  compact.
6. Let  $K$  be a compact operator. Then  $\sigma_{\text{ess}}(K) = 0$  and for any  $\varepsilon > 0, \sigma(K) \setminus B_\varepsilon(0)$  consists of finitely many eigenvalues of finite multiplicity.
7. If  $A = A^*$ , then  $\sigma(A)$  is real and  $\sigma_R(A) = \emptyset$ .

Thus, the simplest case is the case of positive, compact and self-adjoint operators.

### Theorem 3.10 Hilbert-Schmidt

Let  $A \in \mathcal{L}(H)$  be a compact, self-adjoint and positive ( $\langle Ax, x \rangle > 0$  if  $x \neq 0$ ) operator. Then there exists a sequence of non-zero real eigenvalues  $\lambda_i \in \sigma_p(A)$  such that  $|\lambda_i|$  is monotonically non-increasing

$$\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \dots,$$

and the corresponding eigenvectors  $\{e_n\}_{n=1}^\infty$  ( $Ae_n = \lambda e_n$ ) form the orthonormal basis in  $H$ . Moreover, any  $x \in H$  can be written as

$$x = \sum_{n=1}^{\infty} \langle x, e_n \rangle e_n,$$

and  $A$  can be written as

$$Ax = \sum_{n=1}^{\infty} \lambda_n \langle x, e_n \rangle e_n.$$

## Applications

### Example 3.9 (Spectrum of the Laplacian with Dirichlet boundary conditions)

Let  $\Omega \in \mathbb{R}^n$  be a bounded domain with  $\partial\Omega$  smooth. Consider the Laplace equation with Dirichlet boundary conditions:

$$\begin{cases} -\Delta u = f, & f \in L^2(\Omega) \\ u|_{\partial\Omega} = 0 \end{cases}$$

$-\Delta$  is not a bounded operator in  $H := L^2(\Omega)$ , so we cannot apply directly the Hilbert-Schmidt theorem.

Let us consider the inverse operator  $A = (-\Delta)^{-1}$  constructed via weak solutions, namely  $u = (-\Delta)^{-1}f = Af$  solves  $[u, \varphi] = (f, \varphi)$ . That is,  $\forall \varphi \in C_0^\infty(\Omega)$  (or  $H_0^1$ ) and  $u \in H_0^1(\Omega)$ , we have

$$\int_{\Omega} \nabla u \nabla \varphi \, dx = \int_{\Omega} f \varphi \, dx$$

1.  $A$  is a bounded operator from  $H$  to  $H_0^1(\Omega)$ . Indeed, let  $\varphi = u$ . Then

$$\begin{aligned} \|u\|_{H_0^1}^2 &= (f, u) \\ &\leq \|f\|_{L^2} \|u\|_{L^2} \\ &\leq C \|f\|_{L^2} \|u\|_{H_0^1} \end{aligned}$$

Hence

$$\|u\|_{H_0^1} \leq C \|f\|_{L^2} \implies \frac{\|Af\|_{H_0^1}}{\|f\|_{L^2}} \leq C$$

2.  $A$  is compact since the embedding  $H_0^1 \subset H$  is compact.

3.  $A$  is self-adjoint. Let  $f, g \in L^2(\Omega)$ ,  $u = (-\Delta)^{-1}f$ ,  $v = (-\Delta)^{-1}g$ . Take  $\varphi = v$  in variational formulation for  $u$ , and  $\varphi = u$  in variational formulation for  $v$ :

$$\begin{aligned} [u, v] &= (f, v) \\ [u, v] &= (g, u) \\ \implies (f, v) &= (g, u) \implies (f, Ag) = (g, A^*f) \implies A = A^* \end{aligned}$$

4.  $A$  is positive.

$$0 < [u, u] = (f, u) = (f, Af)$$

By the Hilbert-Schmidt theorem, there exists a complete orthonormal system  $\{e_n\}$  of eigenvectors of  $A$  with  $Ae_n = \lambda e_n$ .

$e_n$  by definition solves  $\lambda_n[e_n, \varphi_n] = (e_n, \varphi)$ . Indeed, for  $u_n = Ae_n$ , we have  $[u_n, \varphi] = (e_n, \varphi)$ . Since  $u_n = \lambda_n e_n$ , this implies that

$$[e_n, \varphi] = \lambda_n^{-1}(e_n, \varphi). \quad (3.15)$$

Hence  $e_n$  is a weak solution of

$$\begin{cases} -\Delta e_n = \lambda_n^{-1} e_n \\ e_n|_{\partial\Omega} = 0 \end{cases}$$

Consider the minimisation problem:

$$\min_{u \in H_0^1(\Omega)} \|\nabla u\|_{L^2}^2$$

under the constraint  $\|u\|_{L^2}^2 = 1$ . Let  $u_n$  be a minimising sequence with  $\|u_n\|_{L^2}^2 = 1$  and  $\|\nabla u_n\|_{L^2}^2 \rightarrow \lambda_0$ . We want to prove that  $\lambda_0$  is the minimum of  $u$ . We need to prove the existence of the minimiser and to find  $\lambda_0$ .

1.  $u_n$  is bounded in  $H_0^1(\Omega)$ .  $H_0^1(\Omega) \Subset L^2(\Omega)$  is compactly embedded, so it converges strongly to  $u_0$  in  $L^2(\Omega)$ . Also, since  $\nabla u$  is bounded, by Banach-Alaoglu theorem and reflexivity,  $u_n \rightharpoonup u_0$  weakly in  $H_0^1$  (up to a subsequence) and  $\|\nabla u_0\|_{L^2} \leq \liminf_{n \rightarrow \infty} \|\nabla u_n\|_{L^2} = \lambda_0$ . Since  $\lambda_0$  is an infimum, then  $\|\nabla u_0\|_{L^2} = \lambda_0$ . Thus  $u_0$  is a minimiser.
2. Let use the Euler-Lagrange method.

$$L_\lambda(u) = \|\nabla u\|_{L^2}^2 + \lambda \|u\|_{L^2}^2$$

Let  $\varphi$  be an arbitrary (smooth) function and define  $R_\varphi(\varepsilon) := L_\lambda(u + \varepsilon\varphi)$ ,  $\varepsilon \in \mathbb{R}$ . Then the necessary condition for  $u$  to be a minimum of  $L$  is

$$\frac{d}{d\varepsilon} R_\varphi(\varepsilon)|_{\varepsilon=0} = 0, \quad \forall \varphi.$$

$$\begin{aligned} L_\lambda(u + \varepsilon\varphi) &= \|\nabla(u + \varepsilon\varphi)\|_{L^2}^2 + \lambda \|u + \varepsilon\varphi\|_{L^2}^2 \\ &= \|\nabla u\|_{L^2}^2 + 2\varepsilon(\nabla u, \nabla \varphi) + \varepsilon^2 \|\nabla \varphi\|_{L^2}^2 + \lambda \|u\|_{L^2}^2 + 2\varepsilon\lambda(u, \varphi) + \varepsilon^2 \|\varphi\|_{L^2}^2 \end{aligned}$$

Hence  $u$  is a minimiser of  $L$  if  $\forall \varphi \in H_0^1$ , we have

$$(\nabla u, \nabla \varphi) + \lambda(u, \varphi) = 0.$$

Comparing this equality with 3.15, we see that  $u = u_0$  is the eigenvector of  $(-\Delta)^{-1}$  and  $\lambda^{-1}$  is the corresponding eigenvalue (in this notation,  $-\lambda > 0$ ). Equivalently,  $u_0$  is the eigenvector of  $-\Delta$  and  $-\lambda$  is the corresponding eigenvalue. From the Hilbert-Schmidt theorem, we know that

$$-\frac{1}{\lambda_1} \geq -\frac{1}{\lambda_2} \geq -\frac{1}{\lambda_3} \geq \dots -\frac{1}{\lambda_i}, \quad (3.16)$$

where  $\lambda_i$  are eigenvalues of  $(-\Delta)^{-1}$ . Taking  $\varphi = u$ , we get  $\|\nabla u\|_{L^2}^2 + \lambda \|u\|_{L^2}^2 = 0$ , and if  $u = e_1$ , then

$$\|\nabla e_1\|_{L^2}^2 - \lambda_1 \|e_1\|_{L^2}^2 = 0$$

and for any other  $u$ ,

$$\|\nabla u\|_{L^2}^2 - \lambda_1 \|u\|_{L^2}^2 > 0$$

Hence for all  $u \in H_0^1(\Omega)$ ,

$$\|u\|_{L^2}^2 \leq \frac{1}{\lambda_1} \|\nabla u\|_{L^2}^2.$$

### 3.4 Maximum principle

#### Theorem 3.11 Classical maximum principle

Let  $u \in C^2(\Omega) \cap C(\bar{\Omega})$ , where  $\Omega \subset \mathbb{R}^n$  is a bounded domain, and  $\nabla u = 0$ . Then

$$\max_{x \in \partial\Omega} u(x) \geq u(x) \geq \min_{x \in \partial\Omega} u(x)$$

In other words, the maximum and minimum of a harmonic function is attained on the boundary.

*Proof.* Let us prove the max-inequality. Let  $v = u + \varepsilon e^{x_1}$ . Then  $\nabla v = \varepsilon e^{x_1}$ ,  $\varepsilon > 0$ . Assume that the maximum is attained in the interior point  $x_0 \in \Omega$ . Then  $\nabla u(x_0) = 0$ , and the matrix of second derivatives must be non-positive. In particular,  $\text{tr}(D^2 u(x_0)) = \Delta v(x_0) \leq 0$ , which implies that  $\varepsilon e^{x_1} < 0$ , a contradiction. Therefore, the maximum  $v$  is attained on the boundary. Taking the limit as  $\varepsilon \rightarrow 0$ , the same is true for  $u$  (but due to the limit, the inequalities become non-strict). The proof is analogous for the min-inequality.  $\square$

This formulation does not exclude that the maximum  $u$  may be attained inside  $\Omega$ . However, the strong-maximum principle claims that if the maximum or minimum of  $u$  is attained inside  $\Omega$ , then  $u$  is constant.

### Maximum principle for weak solutions

#### Chain rule in Sobolev spaces

For smooth functions  $f$  and  $u$ , we know that

$$\nabla f(u(x)) = f'(u(x)) \nabla u \tag{3.17}$$

Let  $f \in C^1(\Omega)$  and  $\nabla f$  be bounded, and  $u \in W^{1,p}(\Omega)$ . Then the LHS and RHS of 3.17 are well-defined and by approximation ( $\Omega$  is regular enough to have density of  $C^\infty(\Omega) \in W^{1,p}(\Omega)$ ), 3.17 holds for such functions. However, we want 3.17 to be satisfied for any globally Lipschitz continuous function  $f \in W^{1,\infty}(\Omega)$ . We know that Lipschitz functions are differentiable almost everywhere, so  $f'(z)$  is well-defined. But what is  $f'(u(x))$ ? The set  $K := \{z \in \mathbb{R} : f'(z) \text{ does not exist}\}$  has zero measure, but  $u^{-1}(K) = V$  may have positive measure, implying  $f'(u(x))$  is not defined.

#### Lemma 3.3

Let  $u \in W^{1,p}(\Omega)$ ,  $K \subset \mathbb{R}$  with  $\text{meas}(K) = 0$  and  $v = u^{-1}(K)$ . Then  $\nabla u(x) = 0$  a.e. on  $V$  (without proof).

Then  $f'(u(x)) \nabla u$  is well-defined.

#### Theorem 3.12

Let  $u \in W^{1,p}(\Omega)$  and  $f$  be globally Lipschitz. Then  $f(u) \in W^{1,p}(\Omega)$  and 3.17 holds.

*Proof.* We will present the proof only for the case where

$$f(z) = \begin{cases} z, & z > 0 \\ 0, & z \leq 0, \end{cases}$$

which is crucial for the maximum principle. We denote

$$u_+(x) = \max\{u(x), 0\}.$$

We expect that  $\nabla u_+ = \begin{cases} \nabla u, & u > 0 \\ 0, & u \leq 0 \end{cases}$ .

Such  $\nabla u_+ \in W^{1,p}(\Omega)$ , so we only need to check the integration by parts formula.

$$\int_{\Omega} u_+(x) \operatorname{div} \varphi(x) \, dx = \int_{x: u > 0} \nabla u(x) \varphi(x) \, dx \quad \forall \varphi \in C_0^\infty. \quad (3.18)$$

To do this, we introduce the following  $C^1$ -approximations of  $f$  such that:

1.  $f'_\varepsilon$  are uniformly bounded
2.  $f_\varepsilon = f$  if  $x \notin (0, \varepsilon)$

Then

$$\int_{\Omega} f_\varepsilon(u(x)) \operatorname{div} \varphi(x) \, dx = - \int_{\Omega} f'_\varepsilon(u(x)) \nabla u(x) \varphi(x) \, dx \quad \forall \varphi \in C_0^\infty.$$

Obviously the LHS of the above equality tends to the LHS of the classical formula 3.18. We need to check the convergence of the RHS.

$$\begin{aligned} \left| \int_{\Omega} (f'_\varepsilon \nabla u - f \nabla u) \varphi \, dx \right| &\leq \int_{0 < u(x) < \varepsilon} C(1 + |f'_\varepsilon(u(x))|) |\nabla u| \, dx \\ &\leq C_1 \int_{0 < u(x) < \varepsilon} |\nabla u| \, dx \end{aligned}$$

Let  $K_\varepsilon := \{x : 0 < u(x) < \varepsilon\}$ . Then  $K_\varepsilon$  are nested and  $\cap_{n=1}^\infty K_{\frac{1}{n}} = \emptyset$  by  $\sigma$ -additivity of the Lebesgue measure. Then  $\operatorname{meas}(K_\varepsilon) \rightarrow 0$  as  $\varepsilon \rightarrow 0$ . By absolute continuity of the Lebesgue integral,

$$\int_{0 < u(x) < \varepsilon} |\nabla u(x)| \, dx \rightarrow 0 \text{ as } \varepsilon \rightarrow 0$$

and the integration by parts formula 3.18 is proved. Thus,  $u_+ \in W^{1,p}(\Omega)$  and indeed

$$\nabla u_+ = \begin{cases} \nabla u, & u > 0 \\ 0, & u \leq 0 \end{cases}$$

□

### Corollary 3.1

Let  $u_-(x) = -\min\{u(x), 0\}$ . Then  $u \in W^{1,p}(\Omega)$  and  $\nabla u_-(x) = \begin{cases} -\nabla u, & u < 0 \\ 0, & u \geq 0 \end{cases}$ . Then  $|u(x)| = u_+(x) + u_-(x)$  is also in  $W^{1,p}(\Omega)$ .

### Corollary 3.2

$\nabla u(x) = 0$  a.e. on the set  $K$  where  $u(x) = 0$ .

*Proof.* Indeed,  $u = u_+ + u_-$ ,  $\nabla u = \nabla u_+ - \nabla u_-$ , and

$$\begin{aligned} \int_{\Omega} u \operatorname{div} \varphi \, dx &= - \int_{\Omega} \nabla u \varphi \, dx \\ \int_{\Omega} u_+ \operatorname{div} \varphi - \int_{\Omega} u_- \operatorname{div} \varphi &= - \int_{u > 0} \nabla u \varphi \, dx - \int_{u=0} \nabla u \varphi \, dx - \int_{u < 0} \nabla u \varphi \, dx \end{aligned}$$

Hence

$$\int_{u=0} \nabla u \varphi \, dx = 0 \quad \forall \varphi \in C_0^\infty \implies \nabla u(x) = 0 \text{ a.e. on } u(x) = 0.$$

□

**Corollary 3.3**

$\nabla u_+ \cdot \nabla u_- = 0$  a.e. (because the supports of  $u_+$  and  $u_-$  are disjoint).

$$\|\nabla u\|_{L^p}^p = \|\nabla u_+\|_{L^p}^p + \|\nabla u_-\|_{L^p}^p = \|\nabla |u|\|_{L^p}^p$$

$$\|u\|_{L^p}^p = \|u_+\|_{L^p}^p + \|u_-\|_{L^p}^p = \||u|\|_{L^p}^p$$

We return to the maximum principle.

**Proposition 3.1**

Let  $u_1, u_2 \in W^{1,2}(\Omega)$  be weak solutions of  $\begin{cases} -\Delta u_1 = f_1 \\ u_1|_{\partial\Omega} = u_1^0 \end{cases}$  and  $\begin{cases} -\Delta u_2 = f_2 \\ u_2|_{\partial\Omega} = u_2^0 \end{cases}$ , and let  $u_1^0 \leq u_2^0$  a.e. and  $f_1 \leq f_2$  in distributions. Then  $u_1(x) \leq u_2(x)$  a.e. in  $\Omega$ .

*Proof.* Let  $v = u_1 - u_2$ . It satisfies

$$\begin{cases} -\Delta v = 0 \\ v|_{\partial\Omega} = v_0 \end{cases}, \text{ where } f = f_1 - f_2 \leq 0 \text{ and } v_0 = u_1^0 - u_2^0 \leq 0.$$

It is sufficient to prove that  $v_+(x) = 0$  a.e. We multiply the equation with the test function  $\varphi = v_+(x) \in H_0^1(\Omega)$ .

$$\begin{aligned} (\nabla v, \nabla v_+) &= (\nabla v_+ - \nabla v_-, \nabla v_+) \\ &= \|\nabla v_+\|_{L^2}^2 \\ &= (f, v_+) \leq 0 \end{aligned}$$

Hence  $\|\nabla v_+\|_{L^2}^2 = 0$ , and by Friedrich's inequality, this implies that  $v_+ = 0$  a.e. □

**Note:-**

$\ell \in D'(\Omega)$  is non-negative if and only if  $\langle \ell, \varphi \rangle \geq 0 \quad \forall \varphi \in D(\Omega), \varphi \geq 0$ .

**Corollary 3.4**

The classical maximum principle follows from proposition 3.1 by taking  $f = f_1 = 0$ ,  $u_1 = u$  and  $u_2 = \max_{x \in \partial\Omega} u(x) = \text{const.}$

**Note:-**

From our exposition, it looks like  $u_+, u_-$  approach is more general than the classical one. This is not true! It does not cover general operators  $L = \sum_{i,j} a_{ij}(x) \partial_{x_i} \partial_{x_j} + \sum_i b_i(x) \partial_{x_i} + c(x)$ , but the classical theory does.