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MSc Applied Mathematics

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Stochastic Differential Equation Models for Systemic Risk - Preliminary Report

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1 Introduction and literature review

Maintaining financial stability is an important objective of any central bank. A stable financial system is able to absorb economic shocks and prevent adverse events from disrupting the economy and spreading to other financial systems. Failures in the financial system can have a very negative impact on the economy and on the welfare of individuals. This was most notable in the 2007-2008 financial crisis, where a shock in the form of a collapsing housing bubble in the US resulted in the collapse of key financial institutions. Since the crisis, central banks have put increasing focus on understanding and reducing systemic risk in the financial system. This project analyses interacting particle systems to study systemic risk in the financial system.

We model the financial system as an interconnected system of agents. Systemic risk is the risk that a large number of agents fail simultaneously, leading to a failure in the system as a whole. Interacting particle systems have been used as a simple way to model the financial system and to analyse systemic risk. Garnier et al. (2012) and Fang et al. (2016) study a diffusion process that interact through their empirical mean to study flocking behaviour and systemic risk in heterogeneous mean-field interacting diffusions. In particular, this project studies the impact on systemic risk of imposing a network structure on the interactions between the agents, with policy implications in mind. In 2013, Janet Yellen, Vice Chair of the Board of Governors of the Federal Reserve System gave a speech to highlight the importance of interconnectedness and systemic risk. In her speech, Yellen (2013) mentions that the increased vulnerabilities in the financial system were consequences of the increasing complexity and interconnectedness of aspects of the financial system. Yellen (2013) cites many academic papers which study contagion and systemic risk in a networks of interlinked financial institutions, as understanding this research is important for policymakers when designing policy to safeguard the financial system. For example, Cont et al. (2013) develop an agent-based model for interbank lending and argue that capital requirements should not be uniformly applied across banks, but rather be targeted at systemically important institutions.

2 Model

We build our model incrementally, starting with the simple model of systemic risk used in Garnier et al. (2012), and incrementally layer on extensions. All our models are continuous-time stochastic processes satisfying a system of Itô Stochastic Differential Equations (SDEs) with given initial conditions. As such a system of SDEs is very difficult to solve analytically, we rely on numerical methods to simulate the dynamics of the system, as described in Higham (2001). The dynamics of these models are simulated using SDETools, a MATLAB package for the numerical solution of stochastic differential equations. In particular, the standard Euler-Maruyama method is used for the approximation of these SDEs.

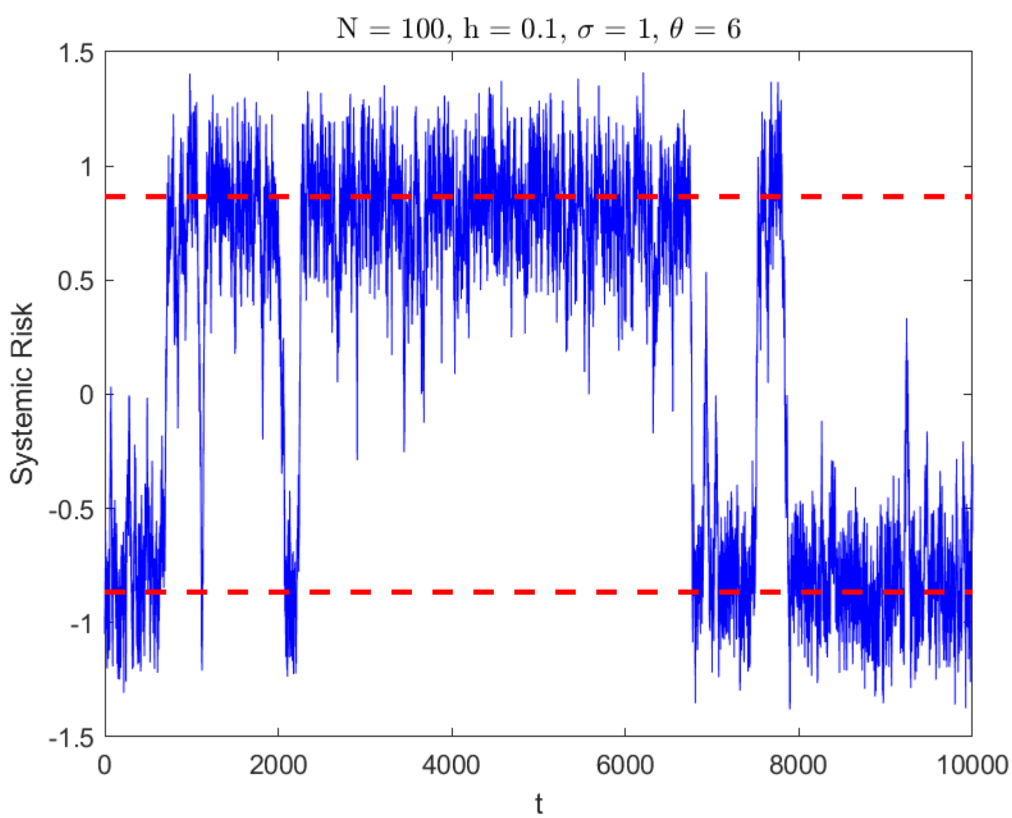
2.1 Systemic risk model

Garnier et al. (2012) considers a model of interacting agents, where each agent can be in one of two states; a normal and failed, and the agent transition between the two states. This is our baseline model for systemic risk, where the failed state can be interpreted as the event where the financial system fails. In particular, let $x_j(t)$ be the state of risk of agent j, taking real values. The dynamics of this state are assumed to have an intrinsic stabilisation mechanism that keeps agents near a normal state. For $j = 1, \ldots, N$, the dynamics of each agent are given by:

$$dx_j(t) = -hU\left(x_j(t)\right)dt + \theta\left(\frac{1}{N}\sum_{i=1}^{N} x_i(t) - x_j(t)\right)dt + \sigma dw_j(t) \tag{1}$$

with initial conditions $x_j(0) = -1$. Here, $-hU(y) = -hV'(y)$ is the restoring force, where V is a potential which is assumed to have two stable states. The parameter h controls the level of intrinsic stabilisation. The empirical mean of the process, $\bar{x}(t) := \frac{1}{N}\sum_{i=1}^{N} x_i(t)$ is taken to be the measure of systemic risk. The bistable-state structure of $V(y)$ determines the normal and failed states of the agents. It is assumed that $U(y) = y^3 - y$, so $V(y) = \frac{1}{4}y^4 - \frac{1}{2}y^2 + c$, and $c = 0$ is assumed. The two stable states are ± 1, where -1 is the normal state and $+1$ the failed state.

Figure 1: Sample path of empirical mean



For a given parametrisation of the model, the empirical mean of the process oscillates between the normal state and the default state which are indicated by the red lines. The financial system is stable if the empirical mean stays in the normal state and the probability of transitioning to the default state is low.

Garnier et al. (2012) paper explores the impact of alternative calibrations of the parameters h, θ, σ, N on systemic risk.

2.2 Network structure model

In the simple model from Garnier et al. (2012), the dynamics of the agents imply that agents exhibit "flocking behaviour", where the sample paths for the agents follow closely the mean behaviour of the system. An interesting question is how different network topologies affect systemic risk, which is explored in this project. These network topologies can be modelled as a graph, Γ. Let $\mathbf{A}$ denote the directed adjacency matrix of Γ, with elements a_{ij} for $i, j = 1, \ldots, N$. Then, the model from 1 can be extended as such to incorporate different network topologies:

$$dx_j(t) = -hU\big(x_j(t)\big)\,dt + \theta\left(\frac{1}{N}\sum_{i=1}^{N} a_{i,j}(t)\big(x_i(t) - x_j(t)\big)\right)dt + \sigma dw_j(t) \tag{2}$$

In general, $\mathbf{A}$ can be a function of t, so the network structure is allowed to adapt over time. To start with, we explore the impact of some common graphs, as described in Chiba et al. (2018). For instance, we consider the impact of some small-world graphs and Erdős-Rényi graphs.

2.2.1 Network homophily graph

Network homophily refers to the theory that similar agents may be more likely to interact with each other than dissimilar ones. Motsch and Tadmor (2013) studies a class of models for self-organised dynamics based on alignment and find that heterophilious dynamics rather than homophilous dynamics enhance consensus. We can model this theory as such:

$$a_{i,j}(t) = \begin{cases} 1, & \text{if} \quad |x_i(t) - x_j(t)| \leq \tau \\ 0, & \text{if} \quad |x_i(t) - x_j(t)| > \tau \end{cases} \quad \text{for} \quad i, j = 1, \ldots, N$$

where τ denotes the radius of interactions - the maximum distance allowed between x_i and x_j for them to continue interacting with each other.

2.2.2 Network heterophily graph

Network heterophily refers to the theory that dissimilar agents may be more likely to interact with each other than similar agents. We can model this theory as such:

$$a_{i,j}(t) = \begin{cases} 1, & \text{if} \quad |x_i(t) - x_j(t)| > \tau \\ 0, & \text{if} \quad |x_i(t) - x_j(t)| \leq \tau \end{cases} \quad \text{for} \quad i, j = 1, \ldots, N$$

where τ denotes the radius of interactions - the minimum distance allowed between x_i and x_j for them to continue interacting with each other.

2.2.3 Erdős-Rényi graph

Erdős-Rényi graphs are the first random graphs that we consider. For any given t, the adjacency matrix is defined by:

$$a_{i,j}(t) = \begin{cases} 1, & \text{with probability } p \\ 0, & \text{with probability } 1-p \end{cases} \quad \text{for} \quad i,j = 1,\dots,N$$

for some $p \in [0,1]$.

2.3 Extensions

Some possible extensions of the model that we may explore are:

- Introduce Markov switching process for the adjacency matrix. If $r(t)$ is a Markov switching process, then dynamics of the network structure are given by $\mathbf{A}(r(t))$.
- Introducing a Hawkes process, which is a self-exciting process that models the impact of fire sales on the financial system.
- Model agent interactions as a game, where each agent solves an optimisation problem to choose their exposures to other banks.

We also plan to do some analysis of the dynamics of the model when the number of agents is very large by taking the mean-field limit of the system. If we model the agent interactions as a game, we can use mean-field game theory to approximate the Nash equilibrium of the game and study how a social planner may control the network structure such that agents optimally choose their exposures over time in a way that minimises systemic risk.

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