Imperial College London

MSc Applied Mathematics

Imperial College London

Department of Mathematics

Stochastic Differential Equation Models for Systemic Risk

Author:
Harry Li
CID: 02178113
Email: hxl21@ic.ac.uk

Professor:
Grigorios Pavliotis

April 23, 2022

Contents

1 Introduction

This project analyses variations of mean-field interacting diffusion models to study systemic risk. An economic interpretation of these agent-based models may be that the agents are individual banks that interact with one another, and the collection of these banks form the financial system. Systemic risk is the risk that a large number of agents fail simultaneously, leading to a failure in the system as a whole.

The mean-field interaction diffusion models are modelled as continuous-time stochastic processes satisfying a system of Itô Stochastic Differential Equations (SDEs) with given initial conditions. Such a system of SDEs is very difficult to solve analytically, and therefore we rely on numerical methods to simulate the dynamics of the system, as described in Higham (2001). The dynamics of these models are simulated using SDETools, a MATLAB package for the numerical solution of stochastic differential equations. In particular, the standard Euler-Maruyama method is used for the approximation of these SDEs.

2 Fang et al. (2016)

To start with, we consider a simple model of mean-field interacting diffusions, as in Fang et al. (2016). Let Y_t^i denote the log-monetary reserve of agent i at time t, taking real values. For $i = 1, \ldots, N$, the dynamics of each agent are given by:

$$dY_t^i = \alpha_i \left(\frac{1}{N} \sum_{j=1}^{N} Y_t^j - Y_t^i \right) dt + \sigma_i dW_t^i \tag{1}$$

with initial condition $Y_0^i = 0$. W_t^i are independent, standard Brownian motions. The dynamics imply that agent i is attracted to the mean level $\bar{Y}_t^i := \sum_{j=1}^{N} Y_t^j$ at time t. The parameters α_i represent the strength of mean-field reversion and σ_i the strength of noise for agent i. The case where $\alpha_i = \alpha$ and $\sigma_i = \sigma$ corresponds to homogeneous case.

Let $\eta < 0$ denote the default level. For a fixed default level η, let $A := \left\{ \min_{0 \le t \le T} \frac{1}{N} \sum_i^N Y_t^i \le \eta \right\}$ denote the event that the system fails as a whole. Fang et al. (2016) uses Laplace asymptotics to characterise the probability of the default event of the overall system in the limit as $N \to \infty$. This probability depends on the parameters α_i and σ_i.

Fang et al. (2016) studies how the default probability is affected by the structure of the system. That is, how different combinations of interactions and volatilities parameters affect the stability of the system.

3 Garnier et al. (2012)

Garnier et al. (2012) considers a model of interacting agents, where each agent can be in one of two states; a normal and failed, and the agent transition between the two states. In particular, let

$x_j(t)$ be the state of risk of agent j, taking real values. The dynamics of this state are assumed to have an intrinsic stabilisation mechanism that keeps agents near a normal state. For $j = 1, \ldots, N$, the dynamics of each agent are given by:

$$dx_j(t) = -hU\big(x_j(t)\big)\,dt + \theta\left(\frac{1}{N}\sum_{i=1}^{N} x_i(t) - x_j(t)\right)dt + \sigma dw_j(t) \tag{2}$$

with initial conditions $x_j(0) = -1$. Here, $-hU(y) = -hV'(y)$ is the restoring force, where V is a potential which is assumed to have two stable states. The parameter h controls the level of intrinsic stabilisation. The empirical mean of the process, $\bar{x} := \frac{1}{N}\sum_{i=1}^{N} x_i(t)$ is taken to be the measure of systemic risk. The bistable-state structure of $V(y)$ determines the normal and failed states of the agents. It is assumed that $U(y) = y^3 - y$, so $V(y) = \frac{1}{4}y^4 - \frac{1}{2}y^2 + c$, and $c = 0$ is assumed. The two stable states are ± 1, where -1 is the normal state and $+1$ the failed state.

Similar to Fang et al. (2016), Garnier et al. (2012) explores the impact of alternative calibrations of the parameters h, θ, σ, N on systemic risk.

4 Impact of different network topologies on systemic risk

In the simple model from Fang et al. (2016) and Garnier et al. (2012), the dynamics of the agents imply that agents exhibit “flocking behaviour”, where the sample paths for the agents follow closely the mean behaviour of the system. An interesting question is how different network topologies affect systemic risk, which is explored in this project. These network topologies can be modelled as a graph, Γ. Let $\mathbf{A}$ denote the adjacency matrix of Γ, with elements a_{ij} for $i, j = 1, \ldots, N$. Then, the model from 2 can be extended as such to incorporate different network topologies:

$$dx_j(t) = -hU\big(x_j(t)\big)\,dt + \theta\left(\frac{1}{N}\sum_{i=1}^{N} a_{i,j}\big(x_i(t) - x_j(t)\big)\right)dt + \sigma dw_j(t) \tag{3}$$

In general, $\mathbf{A}$ can be a function of t, so the network structure is allowed to adapt over time. To start with, we explore the impact of some common graphs, as described in Chiba et al. (2018). For instance, we consider the impact of some small-world graphs, bipartite graphs, K-nearest neighbours graphs and Erdős-Rényi graphs.

4.1 Network homophily

Network homophily refers to the theory that similar agents may be more likely to interact with each other than dissimilar ones. Motsch and Tadmor (2013) studies a class of models for self-organised dynamics based on alignment and find that heterophilious dynamics rather than homophilous dynamics enhance consensus. We can model this theory as such:

$$a_{i,j} = \begin{cases} 1, & \text{if} \quad |x_i - x_j| \le \tau \\ 0, & \text{if} \quad |x_i - x_j| > \tau \end{cases} \quad \text{for} \quad i, j = 1, \ldots, N$$

where τ denotes the radius of interactions - the maximum distance allowed between x_i and x_j for them to continue interacting with each other.

4.2 Network heterophily

Network heterophily refers to the theory that dissimilar agents may be more likely to interact with each other than similar agents. We can model this theory as such:

$$a_{i,j} = \begin{cases} 1, & \text{if} \quad |x_i - x_j| > \tau \\ 0, & \text{if} \quad |x_i - x_j| \leq \tau \end{cases} \quad \text{for} \quad i, j = 1, \ldots, N$$

where τ denotes the radius of interactions - the minimum distance allowed between x_i and x_j for them to continue interacting with each other.

4.3 Bipartite / Disconnected

The bipartite graph described in Chiba et al. (2018) is adapted for our model as such:

$$a_{i,j} = \begin{cases} 1, & \text{sign}(x_i) \neq \text{sign}(x_j) \\ 0, & \text{else} \end{cases} \quad \text{for} \quad i, j = 1, \ldots, N$$

The dynamics of this model are similar to the model with the heterophilious graph.

4.4 k-nearest neighbours

Let K_i denote the set of the k-nearest neighbours of $x_j(t)$ at time t. We assume that the adjacency matrix is of the form:

$$a_{i,j} = \begin{cases} 1, & \text{if} \quad x_j \in K_i \\ 0, & \text{if} \quad x_j \notin K_i \end{cases} \quad \text{for} \quad i, j = 1, \ldots, N$$

That is, agents only interact with their k-nearest neighbours. Note that the adjacency matrix is not necessarily symmetric.

4.5 Erdős-Rényi model

Erdős-Rényi graphs are the first random graphs that we consider. The adjacency matrix is defined by:

$$a_{i,j} = \begin{cases} 1, & \text{with probability } p \\ 0, & \text{with probability } 1 - p \end{cases} \quad \text{for} \quad i, j = 1, \ldots, N$$

for some $p \in [0, 1]$.

References

Chiba, Hayato, Georgi S. Medvedev, and Matthew S. Mizuhara (2018), "Bifurcations in the kuramoto model on graphs." *Chaos: An Interdisciplinary Journal of Nonlinear Science*, 28, 073109, URL https://doi.org/10.1063%2F1.5039609.

Fang, Fei, Yiwei Sun, and Konstantinos Spiliopoulos (2016), "The effect of heterogeneity on flocking behavior and systemic risk." URL https://arxiv.org/abs/1607.08287.

Garnier, Josselin, George Papanicolaou, and Tzu-Wei Yang (2012), "Large deviations for a mean field model of systemic risk." URL https://arxiv.org/abs/1204.3536.

Higham, D. J. (2001), "An algorithmic introduction to numerical simulation of stochastic differential equations." *SIAM Review*, 43, 525–546, URL http://dx.doi.org/10.1137/S0036144500378302.

MATLAB (2022), *version 9.12.0.1884302 (R2022a)*. The MathWorks Inc., Natick, Massachusetts.

Motsch, Sebastien and Eitan Tadmor (2013), "Heterophilious dynamics enhances consensus." URL https://arxiv.org/abs/1301.4123.

SDETools (2022), "Sdetools." URL https://github.com/horchler/SDETools. Andrew Horchler.