Fall 2018 STSCI 5080 Discussion 11 (11/9)

Problems

1. Let $X_1, \ldots, X_n \sim N(\mu, \sigma^2)$ i.i.d., and consider the sample variance

$$S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \overline{X})^2.$$

   We know that $E(S^2) = \sigma^2$. Find $\mathrm{Var}(S^2)$.

2. Show that if $X \sim f$ (a pdf), then $Y = \mu + \sigma X \sim \sigma^{-1} f((x - \mu)/\sigma)$.

3. Let f be a pdf on $\mathbb{R}$ that is symmetric, i.e., $f(x) = f(-x)$ for any x. Show that if $\int_0^\infty x f(x) dx < \infty$, then $f(x - \mu)$ has mean μ.

4. (optional) If $Y_n \sim \chi^2(n)$, show that $\sqrt{n}(Y_n/n - 1)$ converges in distribution to $N(0, 2)$ by directly showing that the mgf of $\sqrt{n}(Y_n/n - 1)$ converges to that of $N(0, 2)$. (Hint). Use the expansion $\log(1 - x) = -x - x^2/2 - x^2 R(x)$ where $\lim_{x \to 0} R(x) = 0$.

Solutions

1. Recall that
$$(n-1)S^2/\sigma^2 \sim \chi^2(n-1),$$
and a $\chi^2(n-1)$ random variable has variance $2(n-1)$. Hence,
$$\mathrm{Var}\{(n-1)S^2/\sigma^2\} = 2(n-1).$$
But the left hand side is $(n-1)^2\mathrm{Var}(S^2)/\sigma^4$, so that
$$\mathrm{Var}(S^2) = \frac{2\sigma^4}{n-1}.$$

2. $$P(Y \le y) = P(\mu + \sigma X \le y) = P(X \le (y-\mu)/\sigma).$$
The pdf of Y is
$$f_Y(y) = \frac{d}{dy}P(Y \le y) = \frac{1}{\sigma}f((y-\mu)/\sigma).$$

3. The mean of $f(x-\mu)$ is
$$\int_{-\infty}^{\infty} xf(x-\mu)dx = \int_{-\infty}^{\infty}(y+\mu)f(y)dy = \int_{-\infty}^{\infty} yf(y)dy + \mu\int_{-\infty}^{\infty} f(y)dy.$$
The latter integral is 1 because f is a pdf; the former integral is zero because $yf(y)$ is an odd function.

4. Let $Z_n = \sqrt{n}(Y_n/n - 1)$. The mgf of Z_n is
$$\psi_{Z_n}(\theta) = E(e^{\theta\sqrt{n}(Y_n/n-1)}) = e^{-\sqrt{n}\theta}E(e^{\theta Y_n/\sqrt{n}}) = e^{-\sqrt{n}\theta}(1-2\theta/\sqrt{n})^{-n/2}.$$
Taking the log, we have
$$\log\psi_{Z_n}(\theta) = -\sqrt{n}\theta - \frac{n}{2}\log(1-2\theta/\sqrt{n}).$$
Using the expansion $\log(1-x) = -x - x^2/2 - x^2R(x)$, we have
$$\log\psi_{Z_n}(\theta) = \theta^2 + 2\theta^2R(2\theta/\sqrt{n}).$$
Since $\lim_{n\to\infty} R(2\theta/\sqrt{n}) = 0$, we have
$$\lim_{n\to\infty}\psi_{Z_n}(\theta) = \lim_{n\to\infty} e^{\log\psi_{Z_n}(\theta)} = e^{\theta^2},$$
which coincides with the mgf of $N(0,2)$.