# The Lorenz System

The Lorenz system of differential equations arose from the work of meteorologist/mathematician Edward N. Lorenz, who was studying thermal variations in an air cell underneath a thunderhead. As he was computing numerical solutions to the system of three differential equations that he came up with, he noticed that initial conditions with small differences eventually produced vastly different solutions. What he had observed was sensitivity to initial conditions, a characteristic of chaos. His observation led him to further study of the system, and since that time, about 1963, the Lorenz system has become one of the most widely studied systems of ODEs because of its wide range of behaviors.

The system of differential equations Lorenz used was

$$x' = -ax + ay$$
$$y' = rx - y - xz$$
$$z' = -bz + xy$$
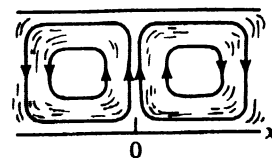
where a, r, and b are positive parameters which denote physical characteristics of air flow. The variable x corresponds to the amplitude of convective currents in the air cell (see picture), y to the temperature difference between rising and falling currents, and z to the deviation of the temperature from the normal temperature in the cell.

Even though a definition of chaos has not been agreed upon by mathematicians, two properties that are generally agreed to characterize it are sensitivity to initial conditions and the presence of period-doubling cycles leading to chaos. The Lorenz system exhibits both of these characteristics. We already mentioned the first, and the second is simply the presence of limit cycles which repeatedly double their period as r is varied in one direction until the orbits begin to wander chaotically. We will explore these dynamics and other behaviors of the Lorenz system.

The graphic was taken from the third source below.
Borrelli, R.L., and Coleman, C.S. *Differential Equations: A Modeling Perspective* (1998: John Wiley & Sons, Inc.)
Borrelli, R.L., Coleman, C.S., and Boyce, W.E. *Differential Equations Laboratory Workbook* (1992: John Wiley & Sons, Inc.)
Jackson, E.A. *Perspectives of Nonlinear Dynamics Vol. 2* (1990: Cambridge University Press)

Exploring the System

Go to the ODE Architect Library and open the file "The Lorenz Equations" in the folder Higher Dimensional Systems. The parameters $a = 10$ and $b = 8/3$ will be set at these values throughout the explorations done here. For Problems 3-5, use the equilibrium feature of ODE Architect to find the equilibria of the system at the values of r you choose to use. Use the Architect to find the eigenvectors and eigenvalues of the Jacobian matrix at each equilibrium, and use the eigenvalues to confirm your observations for probable long-term behavior of the system and for the nature of the equilibrium points (e.g. sinks, sources, attracting, repelling, etc.).

Questions.

1. For $r = 70$ and $r = 130$, plot orbits with initial conditions for x differing by only 0.001 and determine after about what time t the orbits diverge from each other. Repeat for initial conditions for x differing by only 10^{-7}. This is an example of sensitivity to initial conditions. Can you find a value for r for which orbits don't seem to diverge from each other even though the initial conditions for x differs by as much as 1? So sensitivity to initial conditions depends on the values of the parameters.

2. For $r = 70$, pick an initial condition and plot an orbit using 2000 points. Then change the number of points you are using to solve the ODEs to 3000. Plot the orbit again. After what time t do the orbits diverge? Why do they diverge? Is what you are seeing similar to sensitivity to initial conditions?

3. For two different values of r in the range $0 < r < 1$, plot several orbits to conjecture what the long-term behavior of the system is in this range.

4. For $r > 1$, the system has three equilibrium points. For several values of r, $1 < r < 24.74$, try to find orbits that are attracted to the equilibrium points not at the origin, as well as orbits that wander back and forth between neighborhoods of both of them. Plot orbits suggesting the long-term behavior of solutions in this range of r-values. How did any equilibria you found in Problem 3 change? What kind of bifurcation occurred at $r = 1$?

5. For several values of $r > 24.74$, plot orbits that suggest that all equilibrium points you found for the system in previous questions are now unstable. Show chaotic wandering back and forth between neighborhoods of two of the equilibrium points. Also choose initial conditions close to each other and demonstrate sensitivity to initial conditions.

6. For some values of r you can observe attracting limit cycles. As r is changed, the periods of the limit cycles change. One such change is in a period-doubling sequence beginning with the values $r_1 = 160$, $r_2 = 148.5$, $r_3 = 147.5$. Graph orbits at these values and determine the periods of the limit cycles. See if you can find a slightly smaller value of r at which the period doubles again. What happens if r is decreased to 130?

File Edit Search Equations Solutions Tools Help

```
/*  THE LORENZ SYSTEM: I
The nonlinear Lorenz system
models temperature changes below
a thunderhead:            */
x' = -a*x + a*y
y' = r*x - y - x*z
z' = -b*z + x*y
a=10;b=8/3;r=28;x0=1;y0=1;z0=1
/*One feature of chaos is sensi-
tive dependence on initial data.
A small change in initial data
eventually leads to big changes in
solutions.  In Lorenz I, II, III
r=28, but r=260 & 222 in IV.    */
```

Enter

Type of sweep
Single   Dual (Linear)   Dual (Matrix)

Sweep 1
y   Start 1.01   Stop 1   #Points 2

Sweep 2
<None>   Start   Stop   #Points 2

IC Sweep Solver Equilibrium

Sweep Solve Extend

Equations successfully parsed

Lorenz system sensitive dependence on initial data

x0=y0=z0=1 (dim)
x0=z0=1,y0=1.01 (dark)
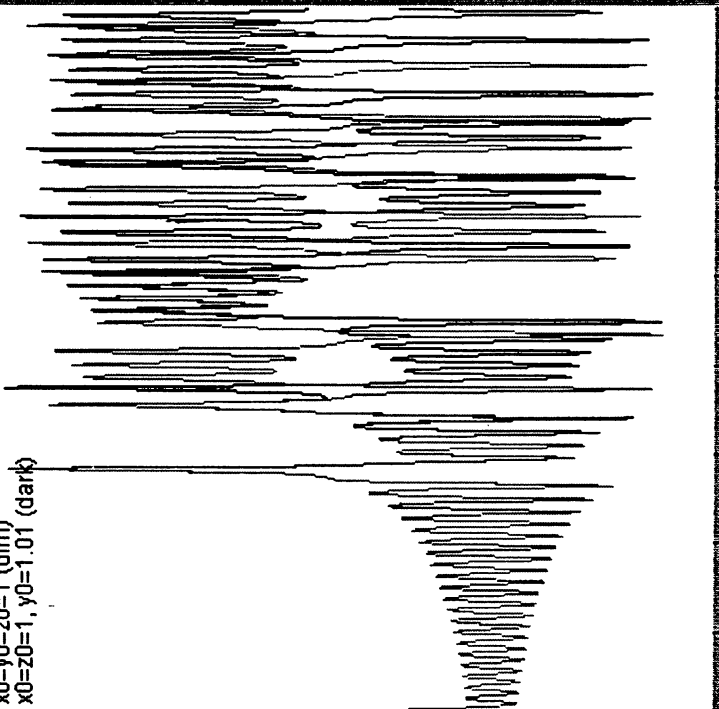
x-component of solution

t-x t-y t-z x-y 2D 3D Equil Data Model

ODE Architect Tool

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```
/*      THE LORENZ SYSTEM: II
The nonlinear Lorenz system is:      */
    x' = -a*x + a*y
    y' = r*x - y - x*z
    z' = -b*z + x*y
    a=10;b=8/3;r=28;x0=1;y0=1;z0=1
/*Guess where an equilibrium point is.
Enter coordinates under the equili-
tab below.  Then use "Equil" tab at
lower right for location.  Use "Eigen-
values" option on the window edit menu
to find eigenvalues of Jacobian matrix
at the equilibrium point.  Is the
point stable?  Other equil. points? */
```
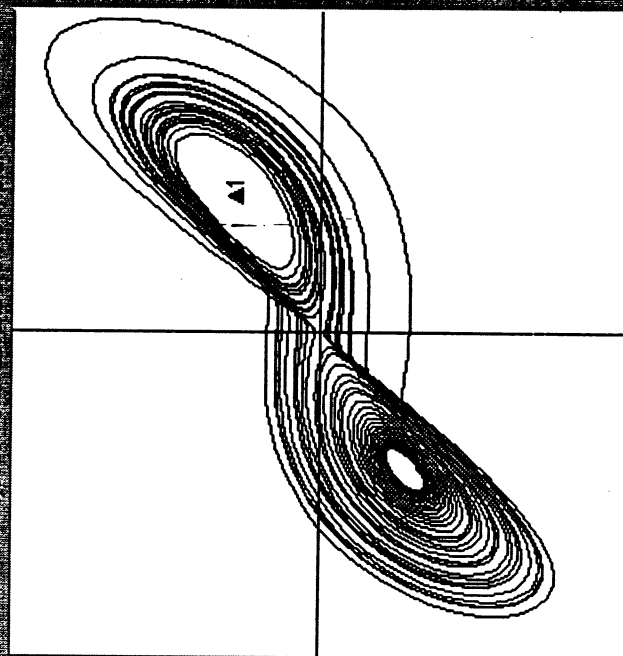
Enter | Solve | Sweep | Extend

Exclude EQ

8	
6	
25	

IC | Sweep | Solver

Projection | Linearization | Find orbits |

| t-x | t-y | t-z | x-y | 2D | 3D | Equil | Data | Model |

| 8.485 | 8.485 | 27 | Successful |

| t-x | t-y | t-z | x-y | 2D | 3D | Equil | Data | Model |

ODE Architect Tool

ODE Architect Tool - LORENZC.ODX

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```
/*      THE LORENZ SYSTEM: III       */
The nonlinear Lorenz system is :
    x' = -a*x + a*y
    y' = r*x - y - x*z
    z' = -b*z + x*y
    a=10;b=8/3;r=28;   x0=1;y0=1;z0=1
/*Orbits seem to wander back and
forth chaotically between two
"surfaces" that are in different planes
in xyz-space.  It is nearly impossible
to predict when an orbit will move
from one surface to the other and how
many times it will oscillate near one
surface before moving away - chaos! */
```
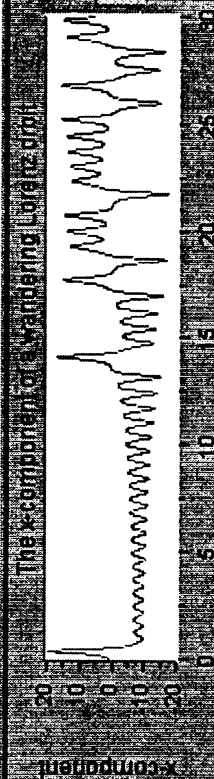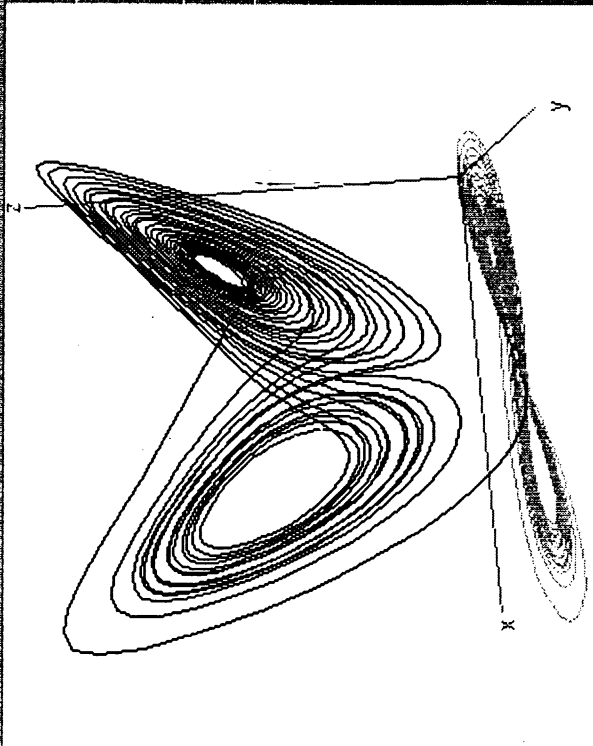
The x-component for a wandering Lorenz orbit

ODE Architect Tool - LORENZD.ODX

```
//        THE LORENZ SYSTEM: IV
x' = -a*x + a*y
y' = r1*x - y - x*z
z' = -b*z + x*y
a = 10; b = 8/3; r1 = 260; r2 = 222
x1' = -a*x1 + a*y1          //A copy of the
y1' = r2*x1 - y1 - x1*z1    //Lorenz system
z1' = -b*z1 + x1*y1         //(different r).
/*Period-doubling sequences of cycles for
sequences of parameter values (r here)
are thought to characterize the onset of
chaos.  The sequence, r1=260, r2=222,......
leads to a sequence of attracting period-
doubling cycles [scroll down for more].*/
```
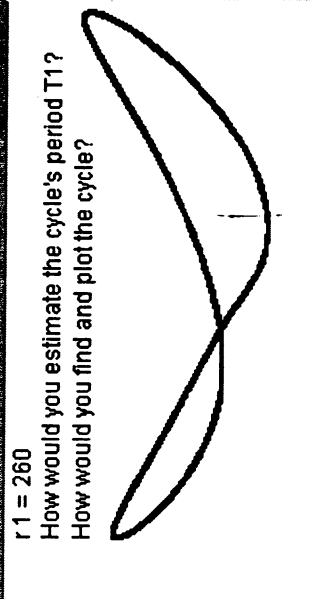
Integrator
Nsteps   5
Points 1000

	0
	-39.6306
	34.4407
	-47.5621
	53.5345
	304.493

Sweep / Solver / Equilibrium

r1 = 260
How would you estimate the cycle's period T1?
How would you find and plot the cycle?

3D / Equil / Data / Model
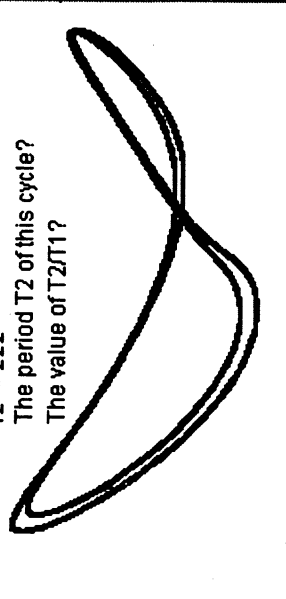
r2 = 222
The period T2 of this cycle?
The value of T2/T1?

3D / Equil / Data / Model

ODE Architect Tool