Fall 2018 STSCI 5080 Discussion 2 (8/31)

Reviews of Lectures 2 and 3

Property D

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

Law of total probability

If $0 < P(B) < 1$, then

$$P(A) = P(A \mid B)P(B) + P(A \mid B^c)P(B^c).$$

Bayes rule

If $P(A) > 0$ and $P(B) > 0$, then

$$P(B \mid A) = \frac{P(A \mid B)P(B)}{P(A)}.$$

Independence of two events

Events A and B are independent if $P(A \cap B) = P(A)P(B)$.

Independence of three events

Events A_1, A_2, A_3 are independent if

$$\begin{cases} P(A_1 \cap A_2) = P(A_1)P(A_2) \\ P(A_1 \cap A_3) = P(A_1)P(A_3) \\ P(A_2 \cap A_3) = P(A_2)P(A_3) \\ P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2)P(A_3) \end{cases}.$$

Problems

1. (**Rice 1.8.6 (b)**) Show the following identity by a formal argument using the axioms of probability and the properties covered in the lectures.

$$\begin{aligned} P(A \cup B \cup C) = P(A) &+ P(B) + P(C) \\ &- P(A \cap B) - P(A \cap C) - P(B \cap C) \\ &+ P(A \cap B \cap C). \end{aligned}$$

2. (**Rice 1.8.7**) Prove Bonferroni's inequality:

$$P(A \cap B) \geq P(A) + P(B) - 1.$$

   Hint: Use Property D.

3. (**Rice 1.8.8**) Prove that

$$P\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} P(A_i).$$

   Hint: Try first the $n = 2$ case. Use Property D.

4. (**Rice 1.8.56**) A couple has two children. What is the probability that both are girls given that the oldest is a girl? What is the probability that both are girls given that at least one of them is a girl?

5. (**Rice 1.8.47**) Urn X has four red, three blue, and two green balls. Urn Y has two red, three blue, and four green balls. A ball is drawn from urn X and put into urn Y, and then a ball is drawn from urn Y.

   (a) What is the probability that a red ball is drawn from urn Y?

   (b) If a red ball is drawn from urn Y, what is the probability that a red ball was drawn from urn X?

6. (**Rice 1.8.64**) If B is an event with $P(B) > 0$, show that the set function $Q(A) = P(A \mid B)$ for $A \subset \Omega$ satisfies the axioms for a probability measure.

7. (**Rice 1.8.65**) Show that if A and B are independent, then A and B^c as well as A^c and B^c are independent.

8. (**Rice 1.8.71**) Show that if A, B, and C are independent, then $A \cap B$ and C are independent and $A \cup B$ and C are independent.

Solutions

1. (**Rice 1.8.6 (b)**) Let $D = B \cup C$, and observe that $A \cup B \cup C = A \cup (B \cup C) = A \cup D$. Then apply Property D to $A \cup D$ to get

$$P(A \cup B \cup C) = P(A \cup D) = P(A) + P(D) - P(A \cap D).$$

Another application of Property D to $P(D) = P(B \cup C)$ leads to

$$P(B \cup C) = P(B) + P(C) - P(B \cap C),$$

and so

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(B \cap C) - P(A \cap D).$$

Furthermore,

$$A \cap D = A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

Applying Property D to the right hand side, we have

$$P((A \cap B) \cup (A \cap C)) = P(A \cap B) + P(A \cap C) - P((A \cap B) \cap (A \cap C)),$$

and

$$(A \cap B) \cap (A \cap C) = A \cap B \cap A \cap C = A \cap A \cap B \cap C = (A \cap A) \cap B \cap C = A \cap B \cap C.$$

Therefore,

$$\begin{aligned} P((A \cap B) \cup (A \cap C)) &= P(A \cap B) + P(A \cap C) - P((A \cap B) \cap (A \cap C)) \\ &= P(A \cap B) + P(A \cap C) - P(A \cap B \cap C). \end{aligned}$$

We are done.

2. (**Rice 1.8.7**) Property D says

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

Property C and Axiom 1 imply that

$$P(A \cup B) \leq P(\Omega) = 1.$$

So

$$1 \geq P(A) + P(B) - P(A \cap B).$$

That is,

$$P(A \cap B) \geq P(A) + P(B) - 1.$$

3. (**Rice 1.8.8**) $\underline{n = 2 \text{ case}}$. Property D says

$$P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2) \leq P(A_1) + P(A_2)$$

and so we are done.

In general, let $B = \bigcup_{i=2}^n A_i$. Then $\bigcup_{i=1}^n A_i = A_1 \cup B$ and

$$P\left(\bigcup_{i=1}^n A_i\right) = P(A_1 \cup B) \le P(A_1) + P(B).$$

Repeat this to $P(B) = P(\bigcup_{i=2}^n A_i)$:

$$P\left(\bigcup_{i=2}^n A_i\right) \le P(A_2) + P\left(\bigcup_{i=3}^n A_i\right) \le \cdots \le P(A_2) + P(A_3) + \cdots + P(A_n).$$

We are done.

4. (**Rice 1.8.56**) The sample space is $\Omega = \{bb, bg, gb, gg\}$. Let

$$\begin{aligned} A &= \text{both children are girls} = \{gg\}, \\ B &= \text{the oldest child is a girl} = \{gb, gg\}, \\ C &= \text{at least one of the children is a girl} = \{bg, gb, gg\}. \end{aligned}$$

Then

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{1}{2} \quad \text{and} \quad P(A \mid C) = \frac{P(A \cap C)}{P(C)} = \frac{1}{3}.$$

5. (**Rice 1.8.47**) (a). Let

$$\begin{aligned} A &= \text{a red ball is drawn from urn Y}, \\ B &= \text{a red ball is drawn from urn X} \end{aligned}$$

We want to compute $P(A)$. To this end, we will use the law of total probability:

$$P(A) = P(A \mid B)P(B) + P(A \mid B^c)P(B^c).$$

We know that $P(B) = 4/9$ and $P(B^c) = 1 - P(B) = 5/9$. Next, if a red ball is drawn from urn X, then urn Y contains three red balls and seven non-red balls, so that $P(A \mid B) = 3/10$. On the other hand, if a non-red ball is drawn from urn X, then urn Y contains two red balls and eight non-red balls, so that $P(A \mid B^c) = 1/5$. Hence,

$$P(A) = \frac{3}{10} \cdot \frac{4}{9} + \frac{1}{5} \cdot \frac{5}{9} = \frac{11}{45}.$$

(b). We want to compute $P(B \mid A)$. To this end, we use the Bayes rule:

$$P(B \mid A) = \frac{P(A \mid B)P(B)}{P(A)} = \cdots = \frac{6}{11}.$$

6. (**Rice 1.8.64**) Check Axioms 1–3 one by one.

   Axiom 1: $P(\Omega \mid B) = P(\Omega \cap B)/P(B) = P(B)/P(B) = 1$.

   Axiom 2: $P(A \mid B) \ge 0$ for any event A is trivial.

Axiom 3: Let A_1, A_2 be disjoint, i.e., $A_1 \cap A_2 = \emptyset$. Then

$$B \cap (A_1 \cup A_2) = (B \cap A_1) \cup (B \cap A_2)$$

and the two sets on the RHS are disjoint. So

$$P(B \cap (A_1 \cup A_2)) = P((B \cap A_1) \cup (B \cap A_2)) = P(B \cap A_1) + P(B \cap A_2).$$

Divide both sides by $P(B)$ and get

$$P(A_1 \cup A_2 \mid B) = P(A_1 \mid B) + P(A_2 \mid B).$$

7. (**Rice 1.8.65**) We first show that A and B^c are independent. Recall that A can be partitioned as

$$A = (A \cap B) \cup (A \cap B^c),$$

and $A \cap B$ and $A \cap B^c$ are disjoint. So $P(A) = P(A \cap B) + P(A \cap B^c)$. By the independence of A and B, we have $P(A \cap B) = P(A)P(B)$. Hence,

$$P(A \cap B^c) = P(A) - P(A \cap B) = P(A) - P(A)P(B) = P(A)\{1 - P(B)\} = P(A)P(B^c),$$

which implies that A and B^c are independent.

Next, we show that A^c and B^c are independent. Partition B^c as

$$B^c = (A \cap B^c) \cup (A^c \cap B^c),$$

where $A \cap B^c$ and $A^c \cap B^c$ are disjoint. Since A and B^c are independent, we have

$$P(B^c) = P(A \cap B^c) + P(A^c \cap B^c) = P(A)P(B^c) + P(A^c \cap B^c),$$

namely,

$$P(A^c \cap B^c) = P(B^c) - P(A)P(B^c) = P(A^c)P(B^c),$$

which implies that A^c and B^c are independent.

8. The independence of A, B, and C implies that $P(A \cap B \cap C) = P(A)P(B)P(C)$ and $P(A \cap B) = P(A)P(B)$. So, $P((A \cap B) \cap C) = P(A \cap B \cap C) = P(A \cap B)P(C)$, which implies that $A \cap B$ and C are independent.

Next, we know that

$$(A \cup B) \cap C = (A \cap C) \cup (B \cap C).$$

Applying Property D, we have

$$P((A \cup B) \cap C) = P((A \cap C) \cup (B \cap C)) = P(A \cap C) + P(B \cap C) - P(A \cap B \cap C).$$

Now, the independence of A, B, and C implies that $P(A \cap C) = P(A)P(C), P(B \cap C) = P(B)P(C)$, and $P(A \cap B \cap C) = P(A)P(B)P(C)$, so that

$$\begin{aligned} P((A \cup B) \cap C) &= P(A)P(C) + P(B)P(C) - P(A)P(B)P(C) \\ &= P(C)\{P(A) + P(B) - P(A)P(B)\} = P(C)\{P(A) + P(B) - P(A \cap B)\} = P(C)P(A \cup B), \end{aligned}$$

which implies that $A \cup B$ and C are independent.