10/2/18 (Correction: 10/3/18) Kengo Kato

STSCI 5080 Homework 3

- Due is 10/11 (Th) in class.
- Write your name and NetID at the top of the first page, along with the assignment number.
- Use only the one side of the paper. Attach your pages with a staple at the top left corner.
- There are five problems. Each problem is worth 10 points.

Problems

1. Let X and Y be independent and continuous random variables with pdfs f_X and f_Y, respectively.

   (a) Verify that the pdf of $Z = -Y$ is $f_Z(z) = f_Y(-z)$. (Hint). Compute the cdf of Z and then differentiate it.

   (b) Derive the pdf of $W = X - Y$ using the pdfs of X and Y. (Hint). The convolution formula.

   (c) If $X, Y \sim Ex(1)$ i.i.d., then verify that the pdf of $W = X - Y$ is

   $$f_W(w) = \frac{1}{2}e^{-|w|}, \ -\infty < w < \infty.$$

   This is called the *Laplace* density or double exponential density.

2. Let $X_1, \dots, X_n$ be independent and continuous random variables with common pdf f.

   (a) Using Rice 3.8.72, derive the joint pdf of $(X_{(1)}, X_{(n)})$.

   (b) Find the joint pdf of $(X_{(1)}, X_{(n)})$ if the common distribution is the uniform distribution on $[0, 1]$.

   (c) Suppose that the common distribution of $X_1, \dots, X_n$ is the uniform distribution on $[0, 1]$. In addition, let U be a uniform random variable on $[0, 1]$ independent of $(X_{(1)}, X_{(n)})$ (in the sense that the joint pdf is $f_{U,X_{(1)},X_{(n)}}(u, x, y) = f_U(u) f_{X_{(1)},X_{(n)}}(x, y)$ where f_U is the marginal pdf of U and $f_{X_{(1)},X_{(n)}}$ is the joint pdf of $(X_{(1)}, X_{(n)})$). Then compute $P(X_{(1)} < U < X_{(n)})$.

3. Find the mean and variance of the Laplace density given in Problem 1 (c).

4. Let X be a random variable, and let g, h be non-decreasing functions on $\mathbb{R}$ (a function g on $\mathbb{R}$ is non-decreasing if whenever $x < y$, we have $g(x) \le g(y)$). Suppose that $E\{|g(X)|\} < \infty, E\{|h(X)|\} < \infty$, and $E\{|g(X)h(X)|\} < \infty$. Show that

   $$E\{g(X)h(X)\} \ge E\{g(X)\}E\{h(X)\}.$$

   This is called *Chebyshev's association inequality*.

   (Hint). Let Y be a random variable independent of X and has the same pmf/pdf as X; consider the sign of

   $$\{g(X) - g(Y)\}\{h(X) - h(Y)\}.$$

5. Let (X, Y) be a uniform random vector on the disk $A = \{(x, y) \mid x^2 + y^2 \leq 1\}$.

   (a) Show that X and Y are not independent.

   (b) Show that $\mathrm{Cov}(X, Y) = 0$.