ORIE 4630: Spring Term 2019
Homework #8
Due: Tuesday, April 16, 2019

Students are required to work independently on homework. You should not give or receive help from other students. You should also not receive help from students or former students who took this course in previous years and who may have solutions to similar problems. The solutions you submit should be your own work and not copied from elsewhere.

Homework is due at the end of lecture (12:55pm) on the due date. You will usually have one week to do the assignments. Please don't wait until the homework is nearly due to start. Late homework is not accepted. Also, homework is not accepted by email. You can submit your assignment in lecture or in the drop box in Rhodes Hall.

Please print your name on the front of your homework so that it is legible.

Include your `R` code, output, graphs, and other work with your homework. This will allow the grader to find any errors you make and to give partial credit.

Suppose that the price of a stock moves according to the Black-Scholes model, so the stock price process is a geometric random walk. Suppose that time is measured in years, and consider a call option with strike price K that is written at time $t = 0$ and has expiry date T; thus, the lifetime of the call option is T years. Let S_0 be the stock price at time $t = 0$; furthermore, let μ be the annual drift of the stock price, and let σ be the annual volatility of the stock price. Under the risk-neutral probability structure, S_T, the price of the stock at the expiry date T, is a random variable having distribution

$$S_T \sim S_0 \exp\big\{(r - \sigma^2/2)T + \sigma B_T\big\},$$

where $B_T \sim N(0, T)$ and r is the continuously compounded annual risk-free rate. The value of the option at expiration is $(S_T - K)_+$, so the expected value of the option under the risk-neutral probability structure is

$$E\big\{(S_T - K)_+\big\} = E\Big(\big[S_0 \exp\big\{(r - \sigma^2/2)T + \sigma\sqrt{T}Z\big\} - K\big]_+\Big),$$

where $Z \sim N(0, 1)$. The price of the option at time $t = 0$ is the discounted version of this expectation, i.e., the price of the option is

$$\exp(-rT)E\Big(\big[S_0 \exp\big\{(r - \sigma^2/2)T + \sigma\sqrt{T}Z\big\} - K\big]_+\Big).$$

The price of the option can be determined by using simulation to estimate the discounted expectation. This approach is undertaken by the function `BS_price_sim()` defined in the following lines:

```
BS_price_sim = function(S_0,  sigma,  K,  T,  r,  n_sim,  seed)
{
  set.seed(seed)
  z = rnorm(n_sim)
  S_T = S_0 * exp(( r - sigma^2 / 2 ) * T + sigma * sqrt(T) * z)
  option_value = ( S_T - K ) * as.numeric(( S_T - K ) >= 0)
  option_value_discounted = exp(-r * T) * option_value
  c(mean(option_value_discounted), sd(option_value_discounted), n_sim)
}
```

For the function `BS_price_sim()`, the argument `n_sim` is the sample size used in the simulation, and the argument `seed` is the seed used for the random number generation.

The expectation can also be determined analytically, and the formula for the discounted expectation is

$$C = \Phi(d_1)S_0 - \Phi(d_2)K\exp(-rT),$$

where

$$d_1 = \frac{\log(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}, \qquad d_2 = d_1 - \sigma\sqrt{T}.$$

This formula is computed by the function `BS_price_formula()` defined in the following lines:

```
BS_price_formula = function(S_0, sigma, K, T, r)
{
  d1 = ( log(S_0 / K) + ( r + sigma^2 / 2 ) * T ) / ( sigma * sqrt(T) )
  d2 = d1 - sigma * sqrt(T)
  option_price = pnorm(d1)  *  S_0 - pnorm(d2)  *  K  *  exp(-r * T)
  option_price
}
```

Although we did not demonstrate the veracity of the analytical formula in class, we can use the functions `BS_price_sim()` and `BS_price_formula()` to show in a specific instance that the analytical formula is indeed equal to the discounted expectation. The following lines considers the specific instance $S_0 = 100$, $\sigma = 0.3$, $K = 98$, $T = 1$, $r = 0.015$, with sample size $100,000,000$ for the simulation, and seed for the random number generation set equal to 2758:

```
17 a = BS_price_sim(100, 0.3, 98, 1, 0.015, 1e8, 2758)
18 a[1]
19 a[2]
20 a[1]  + c(-1, 1) * qnorm(0.995) * a[2] / sqrt(a[3])
21 b = BS_price_formula(100, 0.3, 98, 1, 0.015)
22 b
23 a[1] - b
24 a[1] - b + c(-1, 1) * qnorm(0.995) * a[2] / sqrt(a[3])
```

Line 18 gives the simulation-based estimate of the discounted expectation, i.e. the discounted value of the option at expiration, and line 19 gives the simulation-based estimate of the standard deviation of the discounted expectation; line 20 gives a 99% confidence interval for the discounted expectation. The value obtained from the analytical formula is given in line 22. If the analytical formula is correct, then the value it produces should be close to the estimate of the discounted expectation produced by the simulation, and the value should be contained in the 99% confidence interval derived from the simulation. Line 23 shows the difference between estimate of the discounted expectation derived from the simulation and the value obtained from the analytical formula. This difference should be small if the analytical formula is correct; line 24 gives a 99% confidence interval for the difference between the discounted expectation and the value obtained from the analytical formula. If the analytical formula is correct, then this confidence interval for the difference should contain the value 0.

Recall that in the Black-Scholes framework, the stock price process is assumed to be geometric Brownian motion. To determine the price of options, the risk-neutral probability structure is assumed, under which the stock price process is

$$S_t \sim S_0 \exp\big\{(r - \sigma^2/2)t + \sigma B_t\big\},$$

where B_t is standard Brownian motion and r is the continuously compounded annual risk-free rate. This process for the stock price is for calculating the price of options only and is not the true process that describes the movements of the stock price. If the stock price process is actually geometric Brownian motion with annual drift μ and annual volatility σ, then the stock price process is

$$S_t \sim S_0 \exp\big\{\mu t + \sigma B_t\big\}.$$

In particular, if t is a fixed point in time, then $B_t \sim N(0, t)$, so the distribution of the stock price at fixed time t is

$$S_t \sim S_0 \exp\big\{\mu t + \sigma\sqrt{t}Z\big\},$$

where $Z \sim N(0, 1)$. Now consider a call option written at time $t = 0$ with strike price K and expiry date T, and let t be a time $0 \leq t \leq T$. Let C_t be the price of the call option at time t. Of course, this price depends on the stock price S_t at time t, and is given by

$$C_t = \Phi(d_1)S_t - \Phi(d_2)K \exp\{-r(T - t)\},$$

where

$$d_1 = \frac{\log(S_t/K) + (r + \sigma^2/2)(T - t)}{\sigma\sqrt{T - t}}, \qquad d_2 = d_1 - \sigma\sqrt{T - t},$$

and r be the continuously compounded annual risk-free rate. By the put-call parity, the price of a European put option at time t is

$$P_t = C_t + \exp\{-r(T - t))\} - S_t.$$

The following program `BS_price_t_sim()` simulates the price of the call option at time t; `n_sim` is the simulation size, and `seed` is the seed used for random number generation in the simulation.

```
BS_price_t_sim = function(t, S_0, mu, sigma, K, T, r, n_sim, seed)
{
  set.seed(seed)
  z = rnorm(n_sim)
  S_t = S_0 * exp( mu * t + sigma * sqrt(t) * z)
  option_price_t = BS_price_formula(S_t, sigma, K, T-t, r)
}
```

The simulation program `BS_price_t_sim()` can be used to investigate the distribution of the price of a call option at a fixed time t, where $0 \leq t \leq T$. The following lines consider, by simulation, the price of the call at $t = 0.6$ in the specific instance $S_0 = 100$, $\mu = 0.06$, $\sigma = 0.3$, $K = 98$, $T = 1$, $r = 0.015$, where the sample size is $100,000,000$ for the simulation, and the seed for random number generation is set equal to 2758:

```
C = BS_price_formula(100, 0.3, 98, 1, 0.015)
C
C_t = BS_price_t_sim(0.6, 100, 0.06, 0.3, 98, 1, 0.015, 1e08, 2758)
par(lwd = 2); par(cex = 2)
hist(C_t / C)
```

```
hist(log(C_t / C))
par(lwd = 1); par(cex = 1)
mean(C_t / C); sd(C_t / C)
mean(log(C_t / C)); sd(log(C_t / C))
ind = as.numeric(C_t <= C)
mean(ind)
```

Recall that in the Black-Scholes framework, there is no advantage to early exercise of an American call option; thus, the price of an American call option should be the same as the price of a European call option. It follows that for any t, the Black-Scholes price of a call option should be at least as large as the exercise value, i.e., $C_t \geq (S_t - K)_+$. The following lines consider, by simulation, the difference $C_t - (S_t - K)_+$ for $t = 0.6$ in the specific instance $S_0 = 100$, $\mu = 0.06$, $\sigma = 0.3$, $K = 98$, $T = 1$, $r = 0.015$, where the sample size is $100,000,000$ for the simulation, and the seed for random number generation is set equal to 2758:

```
compare_t_sim = function(t, S_0, mu, sigma, K, T, r, n_sim, seed)
{
  set.seed(seed)
  z = rnorm(n_sim)
  S_t = S_0 * exp( mu * t + sigma * sqrt(t) * z)
  option_price_t = BS_price_formula(S_t, sigma, K, T-t, r)
  execute_value_t = (S_t - K) * as.numeric((S_t - K) >= 0)
  option_price_t - execute_value_t
}
diff_t = compare_t_sim(0.6, 100, 0.06, 0.3, 98, 1, 0.015, 1e07, 2758)
par(lwd = 2); par(cex = 2)
hist(diff_t)
mean(diff_t)
sd(diff_t)
min(diff_t)
```

The following lines simulate the processes S_t, C_t, and P_t (for a European put option) at 400 equally spaced values of t for $0 < t \leq T$, and plots them in the specific instance $S_0 = 100$, $\mu = 0$, $\sigma = 0.1$, $K = 100$, $T = 1$, $r = 0.015$, with the seed for random number generation set equal to 2758:

```
par(mfcol = c(3,1))
S_0 = 100; sigma = 0.1; K = 100; T = 1; r = 0.015; seed = 2758
set.seed(seed)
t = (1 : 400) / 400; t = t * T
z = rnorm(400, mean = 0, sd=sigma * sqrt(T / 400))
t = c(0, t)
z = c(0, z)
b = cumsum(z)
S_t = S_0 * exp(b)
plot(t , S_t, type = "l", lwd = 2, ylim = c(min(S_t), max(S_t)), xlab = "time", ylab = "S_t")
lines(c(0, T), c(S_0, S_0))
C_t = BS_price_formula(S_t, sigma, K, T - t, r)
plot(t, C_t, type = "l", lwd = 2, ylim = c(0, max(C_t)), xlab = "time", ylab = "C_t")
C = C_t[1]
lines(c(0, T), c(C, C))
P_t = C_t + exp(-r * (T - t)) * K - S_t
plot(t, P_t, type = "l", lwd = 2, ylim = c(0, max(P_t)), xlab = "time", ylab = "P_t")
```

```
75 P = P_t[1]
76 lines(c(0, T), c(P, P))
```

Consider a call option, written at time $t = 0$, that has strike price K and expiry date T. Let C_t be the price of the option at time t, where $0 \leq t \leq T$. Then we can regard C_t as a function of the stock price S, the expiry date T, the time t, the strike price K, the stock volatility σ, and the continuously compounded annual interest rate r, i.e., $C_t = C(S, T, t, K, \sigma, r)$. Recall that "delta" is defined by $\Delta = \partial C/\partial S$. Furthermore, if P_t is the price of a European put, we can similarly regard P_t as a function of S, T, t, K, σ, and r, i.e., $P_t = P(S, T, t, K, \sigma, r)$. The "delta" for the put option is $\partial P/\partial S = \Delta - 1$. The following lines calculate the price of the call option, the price of the put option, "delta" for the call option, and "delta" for the put option:

```
77 delta = function(S, T, t, K, sigma, r)
78 {
79   d1 = ( log(S / K) + ( r + sigma^2 / 2 ) * ( T - t ) )/ ( sigma * sqrt(T - t) )
80   d2 = d1 - sigma * sqrt(T - t)
81   C = pnorm(d1) * S - pnorm(d2) * K * exp(-r * (T - t))
82   P = C + exp(-r * (T - t)) * K - S
83   call_delta = pnorm(d1)
84   put_delta = call_delta - 1
85   a=c("call price", "put price","call delta","put delta")
86   b=round(c(C, P, call_delta, put_delta), 5)
87   rbind(a,b)
88 }
```

Questions:

1. [24 points] Run lines 1 to 9 to define the function `BS_price_sim()`, and run lines 10 to 16 to define `BS_price_formula`. Adapt and run lines 17 to 24 for the case where the stock has initial price $S_0 = 80$ at time $t = 0$ with annual drift $\mu = 0.055$ and volatility $\sigma = 0.2$; the call option has strike price $K = 75$ and lifetime $T = 1.25$ years; and the continuously compounded risk-free rate is $r = 0.03$. Use the sample size $10,000,000$, set the seed equal to 4630 for random number generation, and use confidence level 90% for the confidence intervals. Submit the output.

i) Report the simulation-based estimate of the discounted risk-neutral expectation of the value of the call option at expiration.

ii) Report the simulation-based estimate of the standard deviation of the discounted risk-neutral value of the call option at expiration.

iii) Report the simulation-based 90% confidence interval for the discounted risk-neutral expectation of the value of the call option at expiration.

iv) Report the price of the call option derived from the Black-Scholes formula.

v) Report the simulation-based estimate of the difference between the discounted risk-neutral expectation of the value of the call option at expiration and the price of the call option as derived from the Black-Scholes formula.

vi) Report the simulation-based 90% confidence interval for the difference between the discounted risk-neutral expectation of the value of the call option at expiration and the price of the call option as derived from the Black-Scholes formula. Does the confidence interval provide evidence against the veracity of the Black-Scholes formula? Justify your answer briefly.

vii) Calculate the price of a European put option, written at the same time as the call option, having strike price $K = 75$ and lifetime $T = 1.25$ years.

2. [22 points] Run lines 25 to 31 to define the function `BS_price_t_sim()`. Adapt and run lines 32 to 42 to study the price of a call option having strike price $K = 75$ and expiry date $T = 1.25$ years. The underlying stock has initial price $S_0 = 80$ at time $t = 0$ when the option is written; the stock is assumed to have drift $\mu = 0.055$ per year and volatility $\sigma = 0.2$ per year; and the continuously compounded risk-free rate is $r = 0.03$ per year. Use the sample size $10,000,000$, and set the seed equal to 4630 for random number generation. Suppose that an investor decides to buy the option at time $t = 0$, and she plans to sell the option at time $t = 0.75$ year.

i) Report the price of the call option when the option is written.

ii) Submit the histogram of the simulated gross returns for the investor's strategy.

iii) Submit the histogram of the simulated log gross returns for the investor's strategy.

iv) Report the simulation-based estimate of the expected gross return for the investor's strategy.

v) Report the simulation-based estimate of the standard deviation of the gross return for the investor's strategy.

vi) Report the simulation-based estimate of the expected log gross return for the investor's strategy.

vii) Report the simulation-based estimate of the standard deviation of the log gross return for the investor's strategy.

viii) Report the simulation-based estimate of the probability that the investor makes a profit from her investment.

3. [15 points] Adapt and run lines 43 to 57 to study the price of a call option having strike price $K = 75$ and expiry date $T = 1.25$ years. The underlying stock has initial price $S_0 = 80$ at time $t = 0$ when the option is written; the stock is assumed to have drift $\mu = 0.055$ per year and volatility $\sigma = 0.2$ per year; and the continuously compounded risk-free rate is $r = 0.03$ per year. Use the sample size $10,000,000$, and set the seed equal to 4630 for the random number generation. Consider the difference $C_t - (S_t - K)_+$ for $t = 0.75$, where C_t is the price of the option at time t and $(S_t - K)_+$ is the exercise value of the option at time t.

i) Submit the histogram of the simulated differences $C_t - (S_t - K)_+$ for $t = 0.75$.

ii) Report the simulation-based estimate of the expectation of the difference $C_t - (S_t - K)_+$ for $t = 0.75$.

iii) Report the simulation-based estimate of the standard deviation of the difference $C_t - (S_t - K)_+$ for $t = 0.75$.

iv) Report the minimum of the simulated values of the difference $C_t - (S_t - K)_+$ for $t = 1$. In the simulation, does $(S_t - K)_+$, the exercise value of the option, ever exceed C_t, the price of the option, for $t = 0.75$?

4. [15 points] Adapt and run lines 58 to 76 to simulate the processes S_t, C_t, and P_t and plot them in the specific instance $S_0 = 80$, $\mu = 0$, $\sigma = 0.2$, $K = 75$, $T = 1.25$, $r = 0.03$, with the seed for random number generation set equal to 8938 and 8957. In both cases, report the values of S_t, C_t, and P_t for $t = 1.25$.

5. [24 points] Run lines 77 to 88 to define the function `delta()`. Run the function `delta()` for the case $S = 80$, $T = 1.25$, $t = 0$, $K = 75$, $\sigma = 0.2$, and $r = 0.03$. Submit the output.

i) Report the price of the call option.

ii) Report delta for the call option.

iii) Compute the leverage for the call option.

iv) Compute the number of call options that would be purchased to replicate the change in price of 100 shares of stock. How much would the options cost? How much would the shares cost?

v) Compute the number of call options that would be purchased to replicate the change in price of $250,000 worth of stock. How much would the options cost?

vi) Compute the number of put options that would be purchased to hedge the change in price of 100 shares of stock. How much would the options cost? How much would the shares cost?

vii) Compute the number of put options that would be purchased to hedge the change in price of $250,000 work of stock. How much would the options cost?