STSCI 5080
Probability Models and Inference

Lecture 7: Joint Distributions

September 13, 2018

Example 7.1

Example

Let X be a continuous random variable with pdf

$$f(x) = \begin{cases} 3x^2 & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}.$$

Calculate the cdf and quantile function of X.

Example 7.1

Example

Let X be a continuous random variable with pdf

$$f(x) = \begin{cases} 3x^2 & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}.$$

Calculate the cdf and quantile function of X.

For $x < 0$, $F(x) = 0$, for $0 \leq x \leq 1$,

$$F(x) = \int_0^x 3y^2 dy = x^3,$$

and for $x > 1$, $F(x) = 1$. Next, for given $u \in (0, 1)$, solving

$$x^3 = u$$

w.r.t. x leads to $x = u^{1/3}$. So $F^{-1}(u) = u^{1/3}$.

Chapter 3 Joint Distributions

Random vector

Definition

For two random variables X, Y defined on the same sample space Ω, the vector (X, Y) is called a random vector.

Discrete random vector

- If X and Y are discrete with supports $\{x_1, x_2, \dots\}$ and $\{y_1, y_2, \dots\}$, then the vector (X, Y) takes values in $\{(x_i, y_j) : i, j = 1, 2, \dots\}$.
- Some pairs (x_i, y_j) may be given 0 probability.
- But anyway the vector (X, Y) takes values in a finite or countably infinite set.

Discrete random vector (cont.)

Definition

A random vector (X, Y) is discrete if X and Y are discrete.

Joint pmf

Definition

For a discrete random vector (X, Y), the joint probability mass function (joint pmf) is defined by

$$p(x, y) = P(X = x, Y = y) = P(\{X = x\} \cap \{Y = y\})$$

for any x and y.

From joint pmf to marginal pmf

- How to calculate the pmf of Y from the joint pmf of (X,Y)?
- The pmf of Y is given by

$$p_Y(y) = \sum_x p(x,y).$$

Why?

$$\sum_x p(x,y) = \sum_i p(x_i,y) = \sum_i P(X = x_i, Y = y)$$
$$= \sum_i P(\{X = x_i\} \cap \{Y = y\}) = P\left(\bigcup_i (\{X = x_i\} \cap \{Y = y\})\right)$$
$$= P\left(\left(\bigcup_i \{X = x_i\}\right) \cap \{Y = y\}\right) = P(\Omega \cap \{Y = y\}) = P(Y = y).$$

Marginal pmf

Theorem

The pmf of Y is given by

$$p_Y(y) = \sum_x p(x,y).$$

We call $p_Y(y)$ the marginal pmf of Y. Similarly, the pmf of X is given by

$$p_X(x) = \sum_y p(x,y).$$

Some properties of joint pmf

- The joint pmf $p(x, y)$ satisfies that $p(x, y) \geq 0$ for any x and y, and

$$\sum_x \sum_y p(x, y) = 1.$$

- For any subset $B \subset \mathbb{R}^2$,

$$P((X, Y) \in B) = P(\{(X, Y) \in B\}) = \sum_{(x,y) \in B} p(x, y).$$

- The joint cdf of (X, Y) is defined by

$$F(x, y) = P(X \leq x, Y \leq y) = \sum_{u \leq x} \sum_{v \leq y} p(u, v).$$

Example 7.2

Example

Suppose you toss a coin three times, and let

$$
\begin{aligned}
X &= \text{the total number of heads in the first toss,} \\
Y &= \text{the total number of heads.}
\end{aligned}
$$

What is the joint pmf of (X, Y)?

Continuous random vector

Definition (PDF on $\mathbb{R}^2$)

A function $f(x,y)$ on $\mathbb{R}^2$ is called a pdf on $\mathbb{R}^2$ if $f(x,y) \geq 0$ for any $(x,y) \in \mathbb{R}^2$ and

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy = 1.$$

Definition (Continuous random vector)

A random vector (X,Y) is continuous if there exists a pdf $f(x,y)$ on $\mathbb{R}^2$ such that

$$P((X,Y) \in B) = \iint_B f(x,y)dxdy$$

for any subset $B \subset \mathbb{R}^2$. We say that (X,Y) has joint pdf $f(x,y)$.

Marginal pdf

Theorem

The pdf of Y is given by

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)dx.$$

We call $f_Y(y)$ the marginal pdf of Y. Similarly, the pdf of X is given by

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy.$$

Joint cdf

Definition

Let (X, Y) be a continuous random vector with joint pdf $f(x, y)$. Then the joint cdf of (X, Y) is defined by

$$F(x, y) = P(X \leq x, Y \leq y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(u, v) du dv.$$

From joint cdf to joint pdf

From the fundamental theorem in multivariate calculus,

$$f(x,y) = \frac{\partial^2}{\partial x \partial y} F(x,y),$$

whenever the derivative is well defined.

Caution

- If the vector (X, Y) is continuous, X and Y are continuous individually.

- Even if two random variables X and Y are continuous individually, the vector (X, Y) need *not* be continuous.

- Example: Let $X \sim U[0,1]$ and $Y = X$. Then (X, Y) concentrates on the diagonal line

$$D = \{(x,y) \mid 0 \leq x \leq 1, x = y\}$$

in the sense that $P((X,Y) \in D) = 1$. However, since D has area 0, if there were a joint pdf $f(x,y)$ for (X,Y), then

$$P((X,Y) \in D) = \iint_D f(x,y)dxdy = 0,$$

a contradiction!

Example 7.3

Example

Consider a pdf on $\mathbb{R}^2$ defined by

$$f(x,y) = \begin{cases} \frac{12}{7}(x^2 + xy) & \text{if } 0 \le x, y \le 1 \\ 0 & \text{otherwise} \end{cases}.$$

What is $P(X > Y)$? What are the marginal pdfs of X and Y?

Uniform random vector on a 2D set

Definition

Let $A \subset \mathbb{R}^2$ as set with positive and finite area: $0 < |A| < \infty$, where

$$|A| = \iint_A dxdy.$$

Then a function of the form

$$f(x,y) = \begin{cases} \frac{1}{|A|} & \text{if } (x,y) \in A \\ 0 & \text{otherwise} \end{cases}$$

is a pdf on $\mathbb{R}^2$. A random vector (X,Y) with pdf f is called a uniform random vector on A.

"(X,Y) is a point randomly chosen from A"

Example 7.4

Example

Let $A = \{(x, y) \mid x, y \geq 0, x + y \leq 1\}$. Define a pdf f on $\mathbb{R}^2$ by

$$f(x,y) = \begin{cases} c & \text{if } (x,y) \in A \\ 0 & \text{otherwise} \end{cases}.$$

Find the value of c. Calculate the marginal pdfs of X and Y.