# STSCI 5080
# Probability Models and Inference
## Lecture 6: Continuous Random Variables (cont.)

September 11, 2018

# Announcement

- Homework 2 will be posted on Blackboard after the lecture.
- Due is Sep. 20 (Th) in class.

# Applications of distributions in Statistics

Discrete distributions

- Binomial distribution $\rightarrow$ sum of $\{0, 1\}$ variables.
- Poisson distribution $\rightarrow$ count data with values in $\{0, 1, 2, \ldots\}$.

Continuous distributions

- Exponential distribution $\rightarrow$ positive data.
- Normal distribution $\rightarrow$ data with values in both positive and negative numbers.

Other continuous distributions: gamma and beta distributions. Rice 2.2.2 and 2.2.4.

# Example 6.1

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$$P(X \geq 1) = \sum_{k \geq 1} p(k) = \sum_{k=0}^{\infty} p(k) - p(0) = 1 - p(0) = 1 - P(X = 0).$$

In addition,

$$P(X = 0) = e^{-\lambda} = e^{-1/2},$$

so that $P(X \geq 1) = 1 - e^{-1/2} \approx 0.393$.

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Since $Y = 100 + 15X$ for $X \sim N(0, 1)$, we have

$$120 < Y < 130 \Leftrightarrow \frac{4}{3} < X < 2,$$

so that

$$\begin{aligned} P(120 < Y < 130) &= F_X(2) - F_X(4/3) \\ &\approx 0.977 - 0.908 \\ &= 0.069. \end{aligned}$$

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# Functions of a random variable

## Question

Let $X$ be a continuous random variable with pdf $f_X$, and consider a new random variable $Y = g(X)$ for some function $g$. Find the pdf of $Y$.

General strategy: Evaluate the cdf of $Y$ using the cdf of $X$, and differentiate the cdf of $Y$.

# Location-scale transformation

## Example

Let $Y = aX + b$ for some $a > 0$ and $-\infty < b < \infty$. Then $Y$ has pdf

$$f_Y(y) = \frac{1}{a} f_X\left(\frac{y-b}{a}\right).$$

# Location-scale transformation

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Let $Y = aX + b$ for some $a > 0$ and $-\infty < b < \infty$. Then $Y$ has pdf

$$f_Y(y) = \frac{1}{a} f_X\left(\frac{y-b}{a}\right).$$

$$F_Y(y) = P(Y \le y) = P\left(X \le \frac{y-b}{a}\right) = F_X\left(\frac{y-b}{a}\right).$$

Differentiating both sides w.r.t. $y$, we have

$$f_Y(y) = \frac{1}{a} f_X\left(\frac{y-b}{a}\right).$$

# Exponential transformation

## Example

Let $Y = e^X$. Then $Y$ has pdf

$$f_Y(y) = \begin{cases} 0 & y \leq 0 \\ \frac{1}{y} f_X(\log y) & y > 0 \end{cases}.$$

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$$f_Y(y) = \begin{cases} 0 & y \leq 0 \\ \frac{1}{y} f_X(\log y) & y > 0 \end{cases}.$$

Since $Y > 0$, $f_Y(y) = 0$ for $y \leq 0$, For $y > 0$,

$$F_Y(y) = P(Y \leq y) = P(X \leq \log y) = F_X(\log y).$$

Differentiating both sides w.r.t. $y$, we have

$$f_Y(y) = (\log y)' f_X(\log y) = \frac{1}{y} f_X(\log y).$$

For the general monotone transformation case, see Rice Proposition B in p. 62.

# Square transformation

## Example

Let $Y = X^2$. Then $Y$ has pdf

$$f_Y(y) = \begin{cases} 0 & y \leq 0 \\ \frac{1}{2\sqrt{y}} \left\{ f_X(\sqrt{y}) + f_X(-\sqrt{y}) \right\} & y > 0 \end{cases}.$$

# Square transformation

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Let $Y = X^2$. Then $Y$ has pdf

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Since $Y > 0$, $f_Y(y) = 0$ for $y \leq 0$. For $y > 0$,

$$F_Y(y) = P(Y \leq y) = P(-\sqrt{y} \leq X \leq \sqrt{y}) = F_X(\sqrt{y}) - F_X(-\sqrt{y}).$$

Differentiating both sides w.r.t. $y$, we have

$$f_Y(y) = (\sqrt{y})' f_X(\sqrt{y}) - (-\sqrt{y})' f_Y(y) = \frac{1}{2\sqrt{y}} \left\{ f_X(\sqrt{y}) + f_X(-\sqrt{y}) \right\},$$

# Chi-square random variable with $1$ degree of freedom

## Definition

If $X \sim N(0,1)$, then $Y = X^2$ is called a chi-square random variable with 1 degree of freedom, $Y \sim \chi^2(1)$ in short. The pdf of $Y$ is

$$f_Y(y) = \begin{cases} 0 & y \le 0 \\ \frac{1}{\sqrt{2\pi}} y^{-1/2} e^{-y/2} & y > 0 \end{cases}.$$

# Quantile function

**Definition**

Let $X$ be a continuous random variable with values in an interval $I$ and cdf $F$ that is strictly increasing on $I$. Then the inverse function of $F$, i.e., $F^{-1}$, defined on $(0, 1)$ is called the quantile function of $X$ or $F$.

For $u \in (0, 1)$, the value of $F^{-1}(u)$ is the solution of

$$F(x) = u$$

w.r.t. $x$.

**Definition**

$F^{-1}(0.5)$ is called the median of $X$ or $F$.

# Fundamental theorem of random number generation

**Theorem**

*For* $U \sim U[0,1]$, *let* $Y = F^{-1}(U)$. *Then* $Y$ *has cdf* $F$.

# Example 6.2

### Example

Let $F$ be the cdf of $Exp(\lambda)$, i.e., $F(x) = 1 - e^{-\lambda x}$ for $x \geq 0$. The inverse function is

$$F^{-1}(u) = -\frac{1}{\lambda}\log(1-u),$$

so that for $U \sim U[0,1]$,

$$Y = -\frac{1}{\lambda}\log(1-U) \sim Exp(\lambda).$$