9/18/18 Kengo Kato

STSCI 5080 Practice Midterm Exam 1 Solutions

Problem 1. Circle the correct choice in each of the following questions.

(1) [5 points] Suppose that you toss a coin three times and all the outcomes occur equally likely. What is the probability that at least two heads appear?

a. $\frac{1}{8}$ b. $\frac{1}{4}$ c. $\frac{3}{8}$ $\boxed{\text{d.}}$ $\frac{1}{2}$

Comment:

$$P(\{\text{at least two heads appear}\}) = \frac{4}{8} = \frac{1}{2}.$$

(2) [5 points] Suppose that two events A and B are independent and such that $P(A) = 1/3$ and $P(B) = 1/6$. What is the probability of $A \cup B$?

a. $\frac{2}{9}$ b. $\frac{1}{3}$ $\boxed{\text{c.}}$ $\frac{4}{9}$ d. $\frac{1}{2}$

Comment: By the independence of A and B, we have $P(A \cap B) = P(A)P(B)$. Hence, $P(A \cup B) = P(A) + P(B) - P(A \cap B) = P(A) + P(B) - P(A)P(B) = \frac{1}{3} + \frac{1}{6} - \frac{1}{3} \cdot \frac{1}{6} = \frac{4}{9}$.

(3) [5 points] Suppose that two events A and B are such that $P(B) = 1/3$ and $P(A \cap B) = 1/6$. What is the conditional probability $P(A \mid B)$?

a. $\frac{1}{18}$ b. $\frac{1}{6}$ c. $\frac{1}{3}$ $\boxed{\text{d.}}$ $\frac{1}{2}$

Comment: $P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{1}{2}$.

(4) [5 points] Suppose that the number of typos on a single page of a certain book follows the Poisson distribution with parameter $\lambda = 1$. What is the probability that there is at least one error on this page?

a. $1 - e^{-1/2}$ b. $e^{-1/2}$ $\boxed{\text{c.}}$ $1 - e^{-1}$ d. e^{-1}

Comment: If $X \sim Po(1)$, then $P(X \geq 1) = 1 - P(X = 0) = 1 - e^{-1}$.

(5) [5 points] Find the correct statement about a probability density function (pdf). Only one of them is correct and the other three are incorrect.

a. If f is a pdf, then $-f$ is also a pdf.

b. If f is a pdf, then $2f$ is also a pdf.

$\boxed{\text{c.}}$ If f and g are pdfs, then $\frac{1}{2}f + \frac{1}{2}g$ is also a pdf.

b. If f and g are pdfs, then their product fg is also a pdf.

Comment: See the slides of Lecture 6.

Problem 2. Suppose that the student has to solve a multiple-choice question with four alternatives, and she knows the correct answer with probability 60% and guesses with probability 40%. In case she guesses, she randomly chooses each alternative with probability 25%.

(a) [5 points] What is the probability that she chooses the correct answer?

(b) [5 points] What is the probability that she knew the correct answer given that she chooses the correct answer?

(a) Define events

$$A = \text{she chooses the correct answer,}$$
$$B = \text{she knows the correct answer.}$$

We know that $P(B) = 0.6, P(A \mid B) = 1$, and $P(A \mid B^c) = 0.25$. Hence, the law of total probability yields that

$$P(A) = P(A \mid B)P(B) + P(A \mid B^c)P(B^c) = 0.6 + 0.25 \times 0.4 = 0.7.$$

(b) By the Bayes rule,

$$P(B \mid A) = \frac{P(A \mid B)P(B)}{P(A)} = \frac{0.6}{0.7} = \frac{6}{7}.$$

Problem 3. Define a pdf f by

$$f(x) = \begin{cases} \frac{2}{x^2} & \text{if } x \geq 2 \\ 0 & \text{otherwise} \end{cases}.$$

(a) [5 points] Compute the cdf F that corresponds to pdf f.

(b) [5 points] Compute the quantile function F^{-1} of F.

(c) [5 points] Explain how to generate a random variable with cdf F from a uniform random variable U on $[0, 1]$.

(a) Since $f(x) = 0$ for $x < 2$, $F(x) = 0$ for $x < 0$. For $x \geq 2$,

$$F(x) = \int_2^x \frac{2}{y^2} dy = \left[-\frac{2}{y} \right]_{y=2}^{y=x} = 1 - \frac{2}{x}.$$

(b) For $u \in (0, 1)$, solving

$$1 - \frac{2}{x} = u,$$

we have $x = \frac{2}{1-u}$. Hence,

$$F^{-1}(u) = \frac{2}{1-u}, \quad u \in (0, 1).$$

(c) $Y = F^{-1}(U) = \frac{2}{1-U}$ has cdf F.

Problem 4. Let X and Y be independent exponential random variables with parameters α and β, respectively.

(a) [5 points] Find the joint pdf of (X, Y).

(b) [5 points] Calculate $P(X > Y)$.

(a) The variables X and Y have pdfs $f_X(x) = \alpha e^{-\alpha x}$ and $f_Y(y) = \beta e^{-\beta y}$ for $x, y \geq 0$, respectively. Since they are independent, the joint pdf is

$$f(x, y) = f_X(x) f_Y(y) = \alpha\beta e^{-\alpha x - \beta y}, \ \ x, y \geq 0$$

and $f(x, y) = 0$ elsewhere.

(b) We have

$$\begin{aligned} P(X > Y) &= \int_0^\infty \alpha e^{-\alpha x} \left\{ \int_0^x \beta e^{-\beta y} dy \right\} dx \\ &= \int_0^\infty \alpha e^{-\alpha x}(1 - e^{-\beta x}) dx \\ &= 1 - \alpha \int_0^\infty e^{-(\alpha+\beta)x} dx \\ &= 1 - \frac{\alpha}{\alpha + \beta}. \end{aligned}$$

Problem 5. Let (X, Y) be a uniform random vector on the set $A = \{(x, y) \in \mathbb{R}^2 \mid 0 \le y \le 1 - x^2, 0 \le x \le 1\}$.

(a) [5 points] Find the joint pdf of (X, Y). (Hint). Draw the set A. Find the area of A.

(b) [10 points] Find the marginal pdfs of X and Y.

(c) [5 points] Find the conditional pdf of X given Y.

(a) The area of A is

$$|A| = \int_0^1 (1 - x^2) dx = 1 - \frac{1}{3} = \frac{2}{3}.$$

Hence, the joint pdf of (X, Y) is

$$f(x, y) = \begin{cases} \frac{3}{2} & \text{if } (x, y) \in A \\ 0 & \text{otherwise} \end{cases}.$$

(b) For $0 \le x \le 1$, the point (x, y) is in A as long as $0 \le y \le 1 - x^2$. So, the marginal pdf of X is

$$f_X(x) = \int_0^{1-x^2} \frac{3}{2} dy = \frac{3}{2}(1 - x^2)$$

for $0 \le x \le 1$ and $f_X(x) = 0$ elsewhere. Similarly, the marginal pdf of Y is

$$f_Y(y) = \frac{3}{2}\sqrt{1 - y}$$

for $0 \le y \le 1$ and $f_Y(y) = 0$ elsewhere.

(c) The conditional pdf of X given Y is

$$f_{X|Y}(x \mid y) = \frac{f(x, y)}{f_Y(y)} = \frac{1}{\sqrt{1 - y}}$$

for $0 \le x \le \sqrt{1 - y}$ and $0 \le y < 1$, and $f_{X|Y}(x \mid y) = 0$ elsewhere.