10/2/18 (Correction: 10/3/18) Kengo Kato

STSCI 5080 Homework 3

- Due is 10/11 (Th) in class.
- Write your name and NetID at the top of the first page, along with the assignment number.
- Use only the one side of the paper. Attach your pages with a staple at the top left corner.
- There are five problems. Each problem is worth 10 points.

Problems

1. Let X and Y be independent and continuous random variables with pdfs f_X and f_Y, respectively.
   (a) Verify that the pdf of $Z = -Y$ is $f_Z(z) = f_Y(-z)$. (Hint). Compute the cdf of Z and then differentiate it.
   (b) Derive the pdf of $W = X - Y$ using the pdfs of X and Y. (Hint). The convolution formula.
   (c) If $X, Y \sim Ex(1)$ i.i.d., then verify that the pdf of $W = X - Y$ is
   $$f_W(w) = \frac{1}{2}e^{-|w|}, \ -\infty < w < \infty.$$
   This is called the *Laplace* density or double exponential density.

2. Let $X_1, \dots, X_n$ be independent and continuous random variables with common pdf f.
   (a) Using Rice 3.8.72, derive the joint pdf of $(X_{(1)}, X_{(n)})$.
   (b) Find the joint pdf of $(X_{(1)}, X_{(n)})$ if the common distribution is the uniform distribution on $[0, 1]$.
   (c) Suppose that the common distribution of $X_1, \dots, X_n$ is the uniform distribution on $[0, 1]$. In addition, let U be a uniform random variable on $[0, 1]$ independent of $(X_{(1)}, X_{(n)})$ (in the sense that the joint pdf is $f_{U,X_{(1)},X_{(n)}}(u, x, y) = f_U(u) f_{X_{(1)},X_{(n)}}(x, y)$ where f_U is the marginal pdf of U and $f_{X_{(1)},X_{(n)}}$ is the joint pdf of $(X_{(1)}, X_{(n)})$). Then compute $P(X_{(1)} < U < X_{(n)})$.

3. Find the mean and variance of the Laplace density given in Problem 1 (c).

4. Let X be a random variable, and let g, h be non-decreasing functions on $\mathbb{R}$ (a function g on $\mathbb{R}$ is non-decreasing if whenever $x < y$, we have $g(x) \le g(y)$). Suppose that $E\{|g(X)|\} < \infty, E\{|h(X)|\} < \infty$, and $E\{|g(X)h(X)|\} < \infty$. Show that
   $$E\{g(X)h(X)\} \ge E\{g(X)\}E\{h(X)\}.$$
   This is called *Chebyshev's association inequality*.

   (Hint). Let Y be a random variable independent of X and has the same pmf/pdf as X; consider the sign of
   $$\{g(X) - g(Y)\}\{h(X) - h(Y)\}.$$

5. Let (X, Y) be a uniform random vector on the disk $A = \{(x, y) \mid x^2 + y^2 \leq 1\}$.

   (a) Show that X and Y are not independent.

   (b) Show that $\mathrm{Cov}(X, Y) = 0$.

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Solutions STSCI 5080 Homework 3

1. (a) The cdf of Z is

$$F_Z(z) = P(-Y \le z) = P(Y \ge -z) = 1 - P(Y < -z) = 1 - F_Y(-z),$$

where we have used the fact that $P(Y < -z) = P(Y \le z)$ because of continuity of Y. Differentiating both sides, we have $f_Z(z) = f_Y(-z)$.

(b) The pdf of W is

$$f_W(w) = \int_{-\infty}^{\infty} f_X(x) f_Z(w-x)dx = \int_{-\infty}^{\infty} f_X(x) f_Y(x-w)dx.$$

(c) We consider the cases where $w \ge 0$ and $w < 0$ separately. If $w < 0$, then as long as $x \ge 0$, $x - w > 0$ and so $f_Y(x-w) = e^{-(x-w)}$. Hence,

$$f_W(w) = \int_0^{\infty} e^{-x} e^{-(x-w)} dx = e^w \int_0^{\infty} e^{-2x} dx = \frac{1}{2} e^w.$$

Next, consider the case where $w \ge 0$. In this case, $f_X(x) f_W(x-w)$ is positive only if $x > w$, and so

$$f_W(w) = \int_w^{\infty} e^{-x} e^{-(x-w)} dx = e^w \underbrace{\int_w^{\infty} e^{-2x} dx}_{=\frac{1}{2}e^{-2w}} = \frac{1}{2} e^{-w}.$$

In summary, we have

$$f_W(w) = \frac{1}{2} e^{-|w|}.$$

2. (a) By Rice 3.8.72, the joint cdf of $(X_{(1)}, X_{(n)})$ is

$$F(x,y) = F(y)^n - \{F(y) - F(x)\}^n, \ x \le y.$$

So the joint pdf of $(X_{(1)}, X_{(n)})$ is

$$f(x,y) = \frac{\partial^2}{\partial x \partial y} F(x,y) = n(n-1) f(x) f(y) \{F(y) - F(x)\}^{n-2}, \ x \le y.$$

(b) If the common distribution is the uniform distribution on $[0,1]$, we have $f(x) = 1$ and $F(x) = x$ for $0 \le x \le 1$, and so

$$f(x,y) = n(n-1)(y-x)^{n-2}, \ 0 \le x \le y \le 1.$$

(c) We have

$$\begin{aligned} P(X_{(1)} < U < X_{(n)}) &= \int_0^1 \int_0^y \int_x^y f(x,y) du dx dy = n(n-1) \int_0^1 \int_0^y (y-x)^{n-1} dx dy \\ &= (n-1) \int_0^1 y^n dy = \frac{n-1}{n+1}. \end{aligned}$$

3. Let X have the Laplace density

$$f(x) = \frac{1}{2}e^{-|x|}, \ -\infty < x < \infty.$$

Since $xe^{-|x|}$ is an odd function, we have

$$E(X) = \frac{1}{2}\int_{-\infty}^{\infty} xe^{-|x|}dx = 0.$$

Next, since $x^2e^{-|x|}$ is an even function, we have

$$\mathrm{Var}(X) = E(X^2) = \int_0^{\infty} x^2e^{-x}dx = \cdots = 2.$$

4. Since g and h are both non-decreasing, we have

$$\{g(X) - g(Y)\}\{h(X) - h(Y)\} \geq 0.$$

Taking expectation, we have

$$E[\{g(X) - g(Y)\}\{h(X) - h(Y)\}] \geq 0.$$

Because of independence of X and Y, the left hand side is

$$E\{g(X)h(X)\} - E\{g(X)\}E\{h(Y)\} - E\{g(Y)\}E\{h(X)\} + E\{g(Y)h(Y)\}.$$

Since X and Y have the same pmf/pdf, we have $E\{g(X)\} = E\{g(Y)\}, E\{h(X)\} = E\{h(Y)\}$, and $E\{g(X)h(X)\} = E\{g(Y)h(Y)\}$, so that we have

$$2E\{g(X)h(X)\} - 2E\{g(X)\}E\{h(X)\} \geq 0,$$

namely,

$$E\{g(X)h(X)\} \geq E\{g(X)\}E\{h(X)\}.$$

5. (a) Since the area of the disk is π, the joint pdf of (X, Y) is

$$f(x, y) = \frac{1}{\pi} \text{ if } x^2 + y^2 \leq 1$$

and $f(x, y) = 0$ elsewhere. The marginal pdf of X is

$$f_X(x) = \frac{1}{\pi}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy = \frac{2}{\pi}\sqrt{1-x^2}$$

for $|x| \leq 1$, and $f_X(x) = 0$ elsewhere. By symmetry, $f_Y(y) = (2/\pi)\sqrt{1-y^2}$ for $|y| \leq 1$ and $f_Y(y) = 0$ elsewhere. Since $f(x, y) \neq f_X(x)f_Y(y)$, X and Y are not independent.

(b) First, since $xf_X(x)$ is an odd function, we have

$$E(X) = \int_{-\infty}^{\infty} xf_X(x)dx = \int_{-1}^{1} xf_X(x)dx = 0.$$

By symmetry, we have $E(Y) = 0$. Next, we have

$$\begin{aligned} E(XY) &= \frac{1}{\pi} \iint_A (xy)dxdy = \frac{1}{\pi} \int_{-1}^{1} x \left\{ \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} ydy \right\} dx \\ &= \frac{1}{\pi} \int_{-1}^{1} x(1-x^2)dx = 0. \end{aligned}$$

Hence, we have $\mathrm{Cov}(X, Y) = E(XY) - \{E(X)\}\{E(Y)\} = 0$.