Fall 2018 STSCI 5080 Discussion 9 (10/26)

Example 16.4

Let $X_1, \dots, X_n \sim Ex(\lambda)$ i.i.d. Find the limiting distribution of $\sqrt{n}(1/\overline{X}_n - \lambda)$.

First we have

$$\sqrt{n}(\overline{X}_n - 1/\lambda) \xrightarrow{d} N(0, 1/\lambda^2).$$

We apply the delta method with $g(x) = 1/x, \mu = 1/\lambda$, and $\sigma^2 = 1/\lambda^2$. Since $g'(x) = -1/x^2$, we have

$$g'(1/\lambda) = -\frac{1}{1/\lambda^2} = -\lambda^2,$$

so that by the delta method,

$$\sqrt{n}(1/\overline{X}_n - \lambda) = \sqrt{n}(g(\overline{X}_n) - g(1/\lambda)) \xrightarrow{d} N(0, \{g'(\mu)\}^2\sigma^2) = N(0, \lambda^2).$$

Problems

1. In Example 16.4, what is the limiting distribution of $\sqrt{n}(-\log \overline{X}_n - \log \lambda)$?
2. Find the mean and variance of $V \sim \chi^2(n)$.

Solutions

1. We apply the delta method with $g(x) = -\log x, \mu = 1/\lambda$, and $\sigma^2 = 1/\lambda^2$. Since $g'(x) = -1/x$, we have

$$g'(1/\lambda) = -\lambda,$$

so that

$$\sqrt{n}(-\log \overline{X}_n - \log \lambda) \xrightarrow{d} N(0, 1).$$

2. By definition, $V = Z_1^2 + \cdots + Z_n^2$, where $Z_1, \ldots, Z_n \sim N(0, 1)$ i.i.d., so that

$$E(V) = \sum_{i=1}^{n} E(Z_i^2) = n.$$

In addition, since $\mathrm{Var}(Z_1^2) = E(Z_1^4) - \{E(Z_1^2)\}^2 = 3 - 1 = 2$, we have

$$\mathrm{Var}(V) = \sum_{i=1}^{n} \mathrm{Var}(Z_i^2) = 2n.$$