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STSCI 5080 Midterm Exam 2 Solutions

Problem 1. Circle the correct choice in each of the following questions.

(1) [4 points] Let X and Y be ~~independent~~ random variables such that $E(X) = 1, E(X^2) = 2, E(Y) = 1, E(Y^2) = 5$, and $\mathrm{Corr}(X, Y) = -0.5$. What is $E(2X + 3Y)$?

$\boxed{\text{a.}}$ 5 b. 7 c. 17 d. 19

(2) [4 points] In the previous question, what is $\mathrm{Var}(2X + 3Y)$?

a. 14 $\boxed{\text{b.}}$ 28 c. 40 d. 52

There was a mistake in the setting of this question and I give you full credit on this question for everyone.

(3) [4 points] If X is a Bernoulli random variable with success probability 1/3, that is, $P(X = 1) = 1/3$ and $P(X = 0) = 2/3$, then what is $\mathrm{Var}(X)$?

a. $\frac{1}{9}$ $\boxed{\text{b.}}$ $\frac{2}{9}$ c. $\frac{1}{3}$ d. 1

(4) [4 points] Let $X_1, \dots, X_n$ be a random sample from the Poisson distribution with parameter λ. Let $\overline{X}_n = n^{-1}\sum_{i=1}^n X_i$ denote the sample mean. What is the limiting distribution of $\sqrt{n}\{g(\overline{X}_n) - g(\lambda)\}$ as $n \to \infty$, where $g(\lambda) = \lambda + \lambda^2$?

a. $N(0, \lambda)$ b. $N(0, \lambda + \lambda^2)$ c. $N(0, (1 + 2\lambda)\lambda)$ $\boxed{\text{d.}}$ $N(0, (1 + 2\lambda)^2\lambda)$

(5) [4 points] Find the correct statement. Only one of them is correct.

a. If X and Y are such that $E(X^k) = E(Y^k)$ for all positive integers k (assuming that those moments exist), then X and Y have the same cumulative distribution function.

$\boxed{\text{b.}}$ The moment generating function may not be defined for some random variables.

c. If $\mathrm{Cov}(X, Y) = 0$, then X and Y are independent.

d. None of them are correct.

Problem 2. Let $X_1, \ldots, X_n$ be a random sample from the uniform distribution on $[0,1]$, and let $X_{(n)} = \max_{1\le i\le n} X_i$.

(a) [6 points] Find the cumulative distribution function and probability density function of $X_{(n)}$.

(b) [4 points] Find $\mathrm{Var}(X_{(n)})$. Derive an explicit as possible expression.

(a) We note that

$$P(X_{(n)} \le x) = P(X_i \le x \ \forall i = 1,\ldots,n) = P(X_1 \le x)\cdots P(X_n \le x) = \begin{cases} 0 & \text{if } x < 0 \\ x^n & \text{if } 0 \le x \le 1 \\ 1 & \text{if } x > 1 \end{cases}.$$

The pdf of $X_{(n)}$ is

$$f_{X_{(n)}}(x) = \frac{d}{dx} P(X_{(n)} \le x) = nx^{n-1} \quad \text{for } 0 \le x \le 1$$

and $f_{X_{(n)}}(x) = 0$ elsewhere.

(b) We have

$$E(X_{(n)}) = n\int_0^1 x^n dx = \frac{n}{n+1}$$

and

$$E(X_{(n)}^2) = n\int_0^1 x^{n+1} dx = \frac{n}{n+2},$$

so that

$$\begin{aligned} \mathrm{Var}(X_{(n)}) &= \frac{n}{n+2} - \left(\frac{n}{n+1}\right)^2 = \frac{n(n+1)^2 - n^2(n+2)}{(n+2)(n+1)^2} \\ &= \frac{n^3 + 2n^2 + n - n^3 - 2n^2}{(n+2)(n+1)^2} = \frac{n}{(n+2)(n+1)^2}. \end{aligned}$$

Problem 3. Let (X, Y) have joint probability density function

$$f(x,y) = \begin{cases} e^{-y} & \text{if } 0 \le x \le y < \infty \\ 0 & \text{otherwise} \end{cases}.$$

You may use the following formula:

$$\int_0^\infty x^k e^{-x} dx = k!$$

for $k = 0, 1, 2, \ldots$

(a) [10 points] Find the covariance and correlation between X and Y.

(b) [5 points] Find the conditional expectation of Y given X.

(c) [5 points] Directly evaluate $E\{E(Y \mid X)\}$ using the result of Part (b), and verify that $E\{E(Y \mid X)\}$ coincides with $E(Y)$.

The marginal pdf of X is

$$f_X(x) = \int_x^\infty e^{-y} dy = e^{-x}$$

for $x \ge 0$ and $f_X(x) = 0$ elsewhere. On the other hand, the marginal pdf of Y is

$$f_Y(y) = \int_0^y e^{-y} dx = ye^{-y}$$

for $y \ge 0$ and $f_Y(y) = 0$ elsewhere.

(a) We have $E(X) = 1, E(Y) = 2, E(X^2) = 2$, and $E(Y^2) = 3! = 6$, so that $\mathrm{Var}(X) = 2 - 1 = 1$ and $\mathrm{Var}(Y) = 6 - 4 = 2$. In addition,

$$E(XY) = \int_0^\infty \int_0^y (xy)e^{-y} dx dy = \frac{1}{2}\int_0^\infty y^3 e^{-y} dy = \frac{3!}{2} = 3.$$

Hence, we have

$$\mathrm{Cov}(X,Y) = E(XY) - E(X)E(Y) = 3 - 2 = 1 \quad \text{and} \quad \mathrm{Corr}(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)}\sqrt{\mathrm{Var}(Y)}} = \frac{1}{\sqrt{2}}.$$

(b) The conditional pdf of Y given X is

$$f_{Y|X}(y \mid x) = e^{x-y}$$

for $0 \le x \le y$ and $f_{Y|X}(y \mid x) = 0$ elsewhere. Hence,

$$E(Y \mid X = x) = e^x \int_x^\infty ye^{-y} dy = e^x \left\{ [-ye^{-y}]_{y=x}^\infty + \int_x^\infty e^{-y} dy \right\} = e^x(xe^{-x} + e^{-x}) = 1 + x$$

for $x \ge 0$, so that $E(Y \mid X) = 1 + X$.

(c) By Part (b), we have

$$E\{E(Y \mid X)\} = E(1 + X) = 1 + E(X) = 2,$$

which coincides with $E(Y) = 2$.

Problem 4. Let $X_1, \ldots, X_n$ be a random sample from a cumulative distribution function F where F has mean μ and variance σ^2. Consider the sample mean

$$\overline{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i.$$

(a) [6 points] Find the mean and variance of $\overline{X}_n$.

(b) [4 points] Prove the law of large numbers: for any $\varepsilon > 0$,

$$\lim_{n \to \infty} P(|\overline{X}_n - \mu| > \varepsilon) = 0.$$

(a) We have

$$E(\overline{X}_n) = \frac{1}{n} E\left(\sum_{i=1}^{n} X_i \right) = \frac{1}{n} \sum_{i=1}^{n} E(X_i) = \frac{1}{n} \sum_{i=1}^{n} \mu = \mu,$$

and by independence,

$$\mathrm{Var}(\overline{X}_n) = \frac{1}{n^2} \mathrm{Var}\left(\sum_{i=1}^{n} X_i \right) = \frac{1}{n^2} \sum_{i=1}^{n} \mathrm{Var}(X_i) = \frac{\sigma^2}{n}.$$

(b) By Chebyshev's inequality,

$$P(|\overline{X}_n - \mu| > \varepsilon) \leq \frac{\mathrm{Var}(\overline{X}_n)}{\varepsilon^2} = \frac{\sigma^2}{n\varepsilon^2}.$$

The right hand side converges to 0 as $n \to \infty$.

Problem 5. Let X be a normal random variable with mean μ and variance $\sigma^2 > 0$.

(a) [5 points] Find the moment generating function of X.

(b) [5 points] Show that if Y is another random variable independent of X and has the normal distribution with mean ξ and variance τ^2, then show that $X+Y$ has the normal distribution with mean $\mu+\xi$ and variance $\sigma^2+\tau^2$.

(c) [5 points] Find $\mathrm{Var}(X^2)$ when $\mu = 0$.

(d) [5 points] Let Y_n be a χ^2 random variable with n degrees of freedom. Find the limiting distribution of $\sqrt{n}(Y_n/n - 1)$ as $n \to \infty$.

(a) Consider first the mgf of $Z \sim N(0,1)$. Then

$$\psi_Z(\theta) = E(e^{\theta Z}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\theta z^2 - z^2/2} dz.$$

Using the identity

$$\theta z - \frac{z^2}{2} = -\frac{(z-\theta)^2}{2} + \frac{\theta^2}{2},$$

we have

$$\psi_Z(\theta) = e^{\theta^2/2} \times \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(z-\theta)^2} dz = e^{\theta^2/2}.$$

Now, since $X = \mu + \sigma Z$ for some $Z \sim N(0,1)$, the mgf of X is

$$\psi_X(\theta) = E(e^{\theta(\mu+\sigma Z)}) = e^{\mu\theta} E(e^{\theta\sigma Z}) = e^{\theta\mu} \psi_Z(\theta\sigma) = e^{\theta\mu + \theta^2\sigma^2/2}.$$

(b) The mgf of Y is

$$\psi_Y(\theta) = e^{\theta\xi + \theta^2\tau^2/2}.$$

Hence the mgf of $W = X + Y$ is

$$\psi_W(\theta) = \psi_X(\theta)\psi_Y(\theta) = e^{\theta(\mu+\xi) + \theta^2(\sigma^2+\tau^2)/2},$$

which is the mgf of $N(\mu+\xi, \sigma^2+\tau^2)$. Hence $X+Y \sim N(\mu+\xi, \sigma^2+\tau^2)$.

(c) We note that

$$\mathrm{Var}(X^2) = E(X^4) - \{E(X^2)\}^2.$$

Since $\mu = 0$, we have $X = \sigma Z$, so that

$$\mathrm{Var}(X^2) = \sigma^4 E(Z^4) - \sigma^4 \{E(Z^2)\}^2 = \sigma^4\{E(Z^4) - 1\}.$$

Now, we have

$$\begin{aligned}
\psi'_Z(\theta) &= \theta\psi_Z(\theta), \\
\psi''_Z(\theta) &= \psi_Z(\theta) + \theta^2\psi_Z(\theta), \\
\psi^{(3)}_Z(\theta) &= \theta\psi_Z(\theta) + 2\theta\psi_Z(\theta) + \theta^3\psi_Z(\theta) = (3\theta + \theta^3)\psi_Z(\theta), \\
\psi^{(4)}_Z(\theta) &= (3 + 3\theta^2)\psi_Z(\theta) + (3\theta^2 + \theta^4)\psi_Z(\theta) = (3 + 6\theta^2 + \theta^4)\psi_Z(\theta),
\end{aligned}$$

so that $E(Z^4) = 3$. Hence $\mathrm{Var}(X^2) = 2\sigma^4$.

(d) By definition, $Y_n = Z_1^2 + \cdots + Z_n^2$ for some $Z_1, \ldots, Z_n \sim N(0,1)$ i.i.d. We know that $E(Z_1^2) = 1$. By Part (c), $\mathrm{Var}(Z_1^2) = 2$. Hence, by CLT,

$$\sqrt{n}(Y_n/n - 1) \xrightarrow{d} N(0,2).$$