Fall 2018 STSCI 5080 Discussion 4 (9/21)

Problems

1. (**Rice 2.5.67**) The *Weibull* cdf is

$$F(x) = 1 - e^{-(x/\alpha)^\beta},\ x \geq 0$$

where $\alpha, \beta > 0$.

   (a) Find the pdf of F.

   (b) Show that if W follows a Weibull distribution, then $X = (W/\alpha)^\beta$ follows an exponential distribution.

   (c) How could Weibull random variables be generated from a uniform random number generator?

2. (**Rice 3.8.17**) Let (X, Y) be a random point chosen uniformly on the region $R = \{(x, y) \mid |x| + |y| \leq 1\}$.

   (a) Sketch the region R.

   (b) Find the marginal pdfs of X and Y.

   (c) Find the conditional pdf of Y given X.

Solutions

1. (**Rice 2.5.67**)

   (a) For $x \le 0$, $f(x) = 0$, and for $x > 0$,

   $$f(x) = F'(x) = \frac{\beta}{\alpha}\left(\frac{x}{\alpha}\right)^{\beta-1} e^{-(x/\alpha)^{\beta}}.$$

   (b) The variable X is positive. For $x > 0$,

   $$F_X(x) = P(X \le x) = P((W/\alpha)^{\beta} \le x) = P(W \le \alpha x^{1/\beta}) = F_W(\alpha x^{1/\beta}).$$

   Differentiating both sides, we have

   $$f_X(x) = \frac{\alpha}{\beta} x^{1/\beta - 1} f_W(\alpha x^{1/\beta}) = \cdots = e^{-x}.$$

   Hence, $X \sim Ex(1)$.

   (c) By (b), $W = (X/\alpha)^{1/\beta}$ where $X \sim Ex(1)$ has cdf F. We can generate X as $X = -\log(1-U)$ where $U \sim U[0,1]$, and so

   $$W = \left(\frac{-1}{\alpha}\log(1-U)\right)^{1/\beta}$$

   has cdf F.
   Alternatively, the quantile function F^{-1} is

   $$F^{-1}(u) = \left(\frac{-1}{\alpha}\log(1-u)\right)^{1/\beta}, \ 0 < u < 1,$$

   and so

   $$W = F^{-1}(U) = \left(\frac{-1}{\alpha}\log(1-U)\right)^{1/\beta}$$

   has cdf F.

2. (**Rice 3.8.17**)

   (a) Skip.

   (b) The area of the region R is 2 and so the joint pdf is

   $$f(x,y) = \begin{cases} \frac{1}{2} & \text{if } (x,y) \in R \\ 0 & \text{otherwise} \end{cases}.$$

   For $0 \le x \le 1$, the marginal pdf of X is

   $$f_X(x) = \frac{1}{2}\int_{x-1}^{1-x} dy = 1 - x.$$

For $-1 \le x < 0$, we have

$$f_X(x) = \frac{1}{2}\int_{-x-1}^{1+x} dy = 1 + x.$$

In summary,

$$f_X(x) = \begin{cases} 1 - |x| & \text{if } |x| \le 1 \\ 0 & \text{otherwise} \end{cases}.$$

By symmetry, the marginal pdf of Y is

$$f_Y(y) = \begin{cases} 1 - |y| & \text{if } |y| \le 1 \\ 0 & \text{otherwise} \end{cases}.$$

(c) The conditional pdf of Y given X is

$$f_{Y|X}(y \mid x) = \frac{f(x,y)}{f_X(x)} = \frac{1}{2(1-|x|)}$$

for $|y| \le 1 - |x|$ and $|x| < 1$, and $f_{Y|X}(y \mid x) = 0$ elsewhere.