STSCI 5080
Probability Models and Inference

Lecture 17: χ^2 and t Distributions

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Example 17.1

Example

If $X_1, \ldots, X_n \sim Po(\lambda)$ i.i.d., then what is the limiting distribution of $\sqrt{n}(\overline{X}_n - \lambda)$? In addition, what is the limiting distribution of $\sqrt{n}(\sqrt{\overline{X}_n} - \sqrt{\lambda})$?

Chapter 6 Distributions Derived from the Normal Distribution

χ^2 distribution

Definition

Let $Z_1, \ldots, Z_n \sim N(0,1)$ i.i.d. Then $V = Z_1^2 + \cdots + Z_n^2$ is said to follow the χ^2 distribution with n degrees of freedom, $V \sim \chi^2(n)$ in short.

Recall that $Y = Z_1^2$ has pdf

$$g(y) = \begin{cases} \frac{1}{\sqrt{2\pi}} y^{-1/2} e^{-y/2} & \text{if } y > 0 \\ 0 & \text{otherwise} \end{cases},$$

which coincides with the pdf of $Ga(1/2, 2)$. By the regeneration property of the gamma distribution, we have:

Theorem

$\chi^2(n) = Ga(n/2, 2)$. *Hence, the pdf of* $V \sim \chi^2(n)$ *is*

$$f(v) = \frac{1}{2^{n/2}\Gamma(n/2)} v^{n/2-1} e^{-v/2} \quad \text{for } v > 0,$$

and the mgf of V *is*

$$\psi(\theta) = (1 - 2\theta)^{-n/2} \quad \text{for } \theta < 1/2.$$

t distribution

Definition

If $Z \sim N(0,1)$ and $V \sim \chi^2(n)$, and Z and V are independent, then

$$T = \frac{Z}{\sqrt{V/n}}$$

is said to follow the t distribution with n degrees of freedom, $T \sim t(n)$ in short.

Theorem

The pdf of $T \sim t(n)$ *is*

$$f_T(t) = \frac{\Gamma\{(n+1)/2\}}{\sqrt{n\pi}\Gamma(n/2)} \left(1 + \frac{t^2}{n}\right)^{-(n+1)/2}, \quad -\infty < t < \infty.$$

Proof (outline)

The cdf of $U = \sqrt{V/n}$ is

$$P(U \leq u) = P(\sqrt{V/n} \leq u) = P(V \leq nu^2) = F_V(nu^2),$$

so that the pdf of U is

$$f_U(u) = 2nuf_V(nu^2).$$

The pdf of $T = Z/U$ is given by

$$f_T(t) = \int_0^\infty f_U(u)f_Z(ut)du.$$

Properties of t distribution

Denote by $f_n(t)$ the pdf of $t(n)$.

- If $n = 1$, then the pdf is

$$f_1(t) = \frac{1}{\pi(1+t^2)},$$

  which coincides with the Cauchy density.

- If $Y \sim t(n)$, then for any positive integer k,

$$E(|Y|^k) \begin{cases} < \infty & \text{if } k < n \\ = \infty & \text{if } k \geq n \end{cases}.$$

- If $n \to \infty$, then $f_n(t) \to e^{-t^2/2}/\sqrt{2\pi}$ (pdf of $N(0,1)$) pointwise.

Review of Lectures 10–16

Order statistics

$X_1, \ldots, X_n \sim F$ i.i.d. where F has pdf f, and let

$$X_{(1)} = \min_{1 \le i \le n} X_i \quad \text{and} \quad X_{(n)} = \max_{1 \le i \le n} X_i.$$

The cdf and pdf of $X_{(1)}$ are

$$F_{X_{(1)}}(x) = 1 - \{1 - F(x)\}^n \quad \text{and} \quad f_{X_{(1)}}(x) = nf(x)\{1 - F(x)\}^{n-1}.$$

The cdf and pdf of $X_{(n)}$ are

$$F_{X_{(n)}}(x) = \{F(x)\}^n \quad \text{and} \quad f_{X_{(n)}}(x) = nf(x)\{F(x)\}^{n-1}.$$

Expectation

- If $X \geq 0$, then $E(X) \geq 0$.
- $E(aX + bY) = aE(X) + bE(Y)$.
- If X and Y are independent, then
$$E\{g(X)h(Y)\} = E\{g(X)\}E\{h(Y)\}.$$
- Markov inequality:
$$P(|X| > t) \leq \frac{E(|X|)}{t} \quad \text{for any } t > 0.$$

Variance/Covariance/Correlation

- $\mathrm{Var}(X) = E[\{X - E(X)\}^2] = E(X^2) - \{E(X)\}^2$.
- $\mathrm{Cov}(X, Y) = E[\{X - E(X)\}\{Y - E(Y)\}] = E(XY) - E(X)E(Y)$.
- $$\mathrm{Corr}(X, Y) = \frac{\mathrm{Cov}(X, Y)}{\sqrt{\mathrm{Var}(X)}\sqrt{\mathrm{Var}(Y)}}.$$
- Cauchy-Schwarz inequality:
  $$|\mathrm{Cov}(X, Y)| \le \sqrt{\mathrm{Var}(X)}\sqrt{\mathrm{Var}(Y)}.$$
- $\mathrm{Var}(aX + bY) = a^2\mathrm{Var}(X) + 2ab\mathrm{Cov}(X, Y) + b^2\mathrm{Var}(Y)$.

- Independence and covariance:

$$X \text{ and } Y \text{ are independent} \Rightarrow \mathrm{Cov}(X, Y) = 0.$$
$$\mathrm{Cov}(X, Y) = 0 \not\Rightarrow X \text{ and } Y \text{ are independent}.$$

- If $X_1, \ldots, X_n$ are independent, then

$$\mathrm{Var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \mathrm{Var}(X_i).$$

Conditional expectation and variance

- Understand their definitions.
- Law of total expectation:

$$E\{E(Y \mid X)\} = E(Y).$$

- Computing unconditional variance using conditional expectation and variance.

$$\mathrm{Var}(Y) = E\{\mathrm{Var}(Y \mid X)\} + \mathrm{Var}\{E(Y \mid X)\}.$$

- Compound Poisson random variable and its mean and variance.

MGF

- Understand the definition of the mgf.
- If X has mgf $\psi(\theta)$, then $E(|X|^k) < \infty$ for any positive integer k, and

  $$E(X^k) = \psi^{(k)}(0).$$

  Does a Cauchy random variable have an mgf?
- Consequence of the uniqueness theorem: If X and Y are independent with mgfs $\psi_X(\theta)$ and $\psi_Y(\theta)$, then the mgf of $Z = X + Y$ is

  $$\psi_Z(\theta) = \psi_X(\theta)\psi_Y(\theta). \tag{*}$$

  If (*) coincides with the mgf of a known cdf F, then $Z = X + Y \sim F$.

MGFs of some standard distributions

- $Bin(n,p)$:
$$\psi(\theta) = \{1 + p(e^\theta - 1)\}^n, \ -\infty < \theta < \infty.$$

- $Po(\lambda)$:
$$\psi(\theta) = e^{\lambda(e^\theta - 1)}, \ -\infty < \theta < \infty.$$

- $N(\mu, \sigma^2)$:
$$\psi(\theta) = e^{\theta\mu + \theta^2\sigma^2/2}, \ -\infty < \theta < \infty.$$

- $Ga(\alpha, \beta)$:
$$\psi(\theta) = (1 - \beta\theta)^{-\alpha}, \ \theta < 1/\beta.$$

- $Ex(\lambda) = Ga(1, 1/\lambda)$:
$$\psi(\theta) = (1 - \theta/\lambda)^{-1}, \ \theta < \lambda.$$

Regeneration property

We have

$$
\begin{aligned}
&Bin(n,p) * Bin(m,p) = Bin(n+m,p),\\
&Po(\lambda) * Po(\mu) = Po(\lambda+\mu),\\
&N(\mu_1,\sigma_1^2) * N(\mu_2,\sigma_2^2) = N(\mu_1+\mu_2,\sigma_1^2+\sigma_2^2),\\
&Ga(\alpha_1,\beta) * Ga(\alpha_2,\beta) = Ga(\alpha_1+\alpha_2,\beta).
\end{aligned}
$$

Limit theorems

- Understand LLN.
- Chebyshev inequality:

$$P\{|X - E(X)| > t\} \leq \frac{\mathrm{Var}(X)}{t^2}, \ t > 0.$$

- Understand the continuity theorem for the mgf.

CLT and delta method

- $X_1, \ldots, X_n \sim F$ i.i.d. where F has mean μ and variance $\sigma^2 > 0$.
- The sample mean is
$$\overline{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i.$$
- CLT:
$$\frac{\sqrt{n}(\overline{X}_n - \mu)}{\sigma} \xrightarrow{d} N(0, 1),$$
or equivalently
$$\sqrt{n}(\overline{X}_n - \mu) \xrightarrow{d} N(0, \sigma^2).$$
- Delta method: If $g(x)$ is differentiable at $x = \mu$, then
$$\sqrt{n}\{g(\overline{X}_n) - g(\mu)\} \xrightarrow{d} N(0, \{g'(\mu)\}^2 \sigma^2).$$