Fall 2018 STSCI 5080 Supplemental Material 4 (10/11)

1 Characteristic and moment generating functions

The purpose of this supplementary material is to prove that the characteristic and moment generating functions uniquely determine the cdf. First of all, we recall their definitions.

Definition 1 (Moment generating function)**.** Let X be a random variable such that $E(e^{\theta X}) < \infty$ for all $|\theta| < a$ for some $a > 0$. Then the function

$$\psi(\theta) = E(e^{\theta X}), \ |\theta| < a$$

is called the *moment generating function* (mgf) of X. The mgf is determined uniquely by the pmf/pdf or equivalently by the cdf.

In what follows, $i = \sqrt{-1}$ denotes the pure imaginary number. Recall the Euler formula:

$$e^{ix} = \cos x + i \sin x$$

for real x.

Definition 2 (Characteristic function)**.** The *characteristic function* (chf) of a random variable X is defined by

$$\varphi(t) = E(e^{itX}) = E\{\cos(tX)\} + iE\{\sin(tX)\}, \ -\infty < t < \infty.$$

The chf is determined by the pmf/pdf or equivalently by the cdf.

Remark 1. The mgf may not exit for some random variables (e,g. a Cauchy random variable) but the chf is defined for any random variable since the cosine and sine functions are bounded by 1. But if the mgf $\psi(\theta)$ exists, then the chf $\varphi(t)$ is obtained by formally setting $\theta = it$ in $\psi(\theta)$: $\varphi(t) = \psi(it)$.

Example 1 (Bernoulli random variable)**.** If X is a Bernoulli random variable with success probability p, then the mgf is $\psi(\theta) = pe^{\theta} + (1 - p)$ and the chf is $\varphi(t) = pe^{it} + (1 - p)$.

Example 2 ($N(0, 1)$)**.** If $X \sim N(0, 1)$, then the mgf is $\psi(\theta) = e^{\theta^2/2}$ and the chf is $\psi(t) = e^{-t^2/2}$.

The goal of this supplementary material is to prove the following theorems:

Theorem 1 (Uniqueness theorem for mgf)**.** *Suppose that $X \sim F$ and $Y \sim G$ have chfs φ_F and φ_G, respectively. If*

$$\varphi_F(t) = \varphi_G(t) \text{ for all } -\infty < t < \infty,$$

then $F \equiv G$.

Theorem 2 (Uniqueness theorem for chf)**.** *Suppose that $X \sim F$ and $Y \sim G$ have mgfs $\psi_F(\theta)$ and $\psi_G(\theta)$ in an open interval I containing the origin, respectively. If*

$$\psi_F(\theta) = \psi_G(\theta) \text{ for all } \theta \in I,$$

then $F \equiv G$.

2 Proof of Theorem 1 for integer-valued random variables

We first prove Theorem 1 for integer-valued random variables because then the proof is elementary. In fact, if X is integer-valued, we can exactly recover the pmf from the chf.

Theorem 3 (Inversion formula for pmf)**.** *Suppose that X is integer-valued with pmf $p(k)$ and chf $\varphi(t)$. Then*

$$p(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi(t)e^{-itk}dt$$

for any integer k. So the pmf $p(k)$ can be recovered from the chf $\varphi(t)$.

Proof of Theorem 3. By definition, the chf is

$$\varphi(t) = \sum_j e^{itj}p(j).$$

Since

$$\int_{-\pi}^{\pi} e^{itj}dt = \begin{cases} 2\pi & \text{if } j = 0 \\ 0 & \text{if } j \neq 0 \end{cases}$$

for any integer j, we have

$$\int_{-\pi}^{\pi} \varphi(t)e^{-itk}dt = \int_{-\pi}^{\pi} \left\{ \sum_j e^{it(j-k)}p(k) \right\} dt = \sum_j \left\{ \int_{-\pi}^{\pi} e^{it(j-k)}dt \right\} p(j) = (2\pi)p(k).$$

This completes the proof. $\square$

3 Proof of Theorem 1 in the general case

The proof of Theorem 1 in the general case requires some effort. First we define the indicator function.

Definition 3 (Indicator function)**.** For a subset $A \subset \mathbb{R}$, define the indicator function I_A by

$$I_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}.$$

We note that

$$E\{I_A(X)\} = P(X \in A).$$

For example, if X has pmf $p(x)$, then

$$E\{I_A(X)\} = \sum_x I_A(x)p(x) = \sum_{x \in A} p(x) = P(X \in A).$$

We also need the following technical lemma.

Lemma 1. *Let X be a random variable with chf φ, and for $\sigma > 0$, let $Z^\sigma \sim N(0, \sigma^2)$ independent of X. Then for any continuous function $g : \mathbb{R} \to \mathbb{R}$ that is zero outside of a bounded interval, we have*

$$E\{g(X + Z^\sigma)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{g}(t) e^{-\sigma^2 t^2/2} \varphi(-t) dt,$$

where

$$\widehat{g}(t) = \int_{-\infty}^{\infty} g(x) e^{itx} dx.$$

Proof of Lemma 1. We have

$$E\{g(x + Z^\sigma)\} = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} g(x+z) e^{-z^2/(2\sigma^2)} dz = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} g(u) e^{-(u-x)^2/(2\sigma^2)} du.$$

Since the chf of the $N(0,1)$ distribution is $e^{-t^2/2}$, we have

$$e^{-t^2/2} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2/2 + ity} dy. \tag{*}$$

Setting $t = (u - x)/\sigma$ in (*), we have

$$e^{-(u-x)^2/(2\sigma^2)} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2/2 + iy(u-x)/\sigma} dy = \frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t^2\sigma^2/2 + it(u-x)} dt.$$

Now, we have

$$\begin{aligned} E\{g(x + Z^\sigma)\} &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(u) e^{-t^2\sigma^2/2 + it(u-x)} dt du \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(u) e^{-t^2\sigma^2/2 + it(u-x)} du dt \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} g(u) e^{itu} du \right\} e^{-t^2\sigma^2/2 - itx} dt \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{g}(t) e^{-t^2\sigma^2/2 - itx} dt. \end{aligned}$$

Because of independence between X and Z^σ, we have

$$\begin{aligned} E\{g(X + Z^\sigma)\} &= E\left[E\{g(x + Z^\sigma)\}|_{x=X} \right] \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{g}(t) e^{-t^2\sigma^2/2} E(e^{-itX}) dt \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{g}(t) e^{-t^2\sigma^2/2} \varphi(-t) dt. \end{aligned}$$

This completes the proof. □

We are now in position to prove Theorem 1.

Proof of Theorem 1. For $\sigma > 0$, let $Z^\sigma \sim N(0, \sigma^2)$ independent of (X, Y). For any $a < b$ and $\varepsilon > 0$, define the function

$$g_{a,b,\varepsilon}(x) = \begin{cases} 0 & \text{if } x < a - \varepsilon \\ \text{linear} & \text{if } a - \varepsilon \leq x < a \\ 1 & \text{if } a \leq x \leq b \\ \text{linear} & \text{if } b < x \leq b + \varepsilon \\ 0 & \text{if } x > b + \varepsilon \end{cases}.$$

(Draw the graph of the function!) We note that

$$I_{(a,b]}(x) \leq g_{a,b,\varepsilon}(x) \leq I_{(a-\varepsilon,b+\varepsilon]}(x) \quad \text{and} \quad |g_{a,b,\varepsilon}(x) - g_{a,b,\varepsilon}(y)| \leq \frac{|x-y|}{\varepsilon}.$$

Hence, we have

$$0 \leq E\{g_{a,b,\varepsilon}(X)\} - P(a < X \leq b) \leq P(a - \varepsilon < X \leq a) + P(b < X \leq b + \varepsilon),$$

and in addition

$$|E\{g_{a,b,\varepsilon}(X)\} - E\{g_{a,b,\varepsilon}(X + Z^\sigma)\}| \leq E\left[|g_{a,b,\varepsilon}(X) - g_{a,b,\varepsilon}(X + Z^\sigma)|\right] \leq \frac{E(|Z^\sigma|)}{\varepsilon} \leq \sqrt{\frac{2}{\pi}} \cdot \frac{\sigma}{\varepsilon},$$

where we have used the fact that $E(|Z^\sigma|) = \sigma\sqrt{2/\pi}$. The same inequalities hold if we replace X by Y.

Now, by the previous lemma together with the assumption that $\varphi_F \equiv \varphi_G$, we have

$$\begin{aligned}
\mathbb{E}\{g_{a,b,\varepsilon}(X + Z^\sigma)\} &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{g}(t) e^{-\sigma^2 t^2/2} \varphi_F(-t)dt \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{g}(t) e^{-\sigma^2 t^2/2} \varphi_G(-t)dt \\
&= \mathbb{E}\{g_{a,b,\varepsilon}(Y + Z^\sigma)\},
\end{aligned}$$

so that we have

$$\begin{aligned}
|P(a < X \leq b) - P(a < Y \leq b)| &\leq P(a - \varepsilon < X \leq a) + P(b < X \leq b + \varepsilon) \\
&\quad + P(a - \varepsilon < Y \leq a) + P(b < Y \leq b + \varepsilon) + 2\sqrt{\frac{2}{\pi}} \cdot \frac{\sigma}{\varepsilon}.
\end{aligned}$$

Taking $a \to -\infty, \sigma \to 0$, and then $\varepsilon \to 0$, we have $F(b) = G(b)$ [here we have used the fact that the cdf is right continuous]. Since b is arbitrary, we conclude that $F \equiv G$. $\square$

4 Proof of Theorem 2

The proof of Theorem 2 requires some knowledge of complex analysis. We first extend ψ_F to the complex region $D = \{\theta + it \mid \theta \in I, -\infty < t < \infty\}$:

$$\widetilde{\psi}_F(z) = \widetilde{\psi}_F(\theta, t) = E(e^{(\theta+it)X}),\ z = \theta + it,\ \theta \in I,\ -\infty < t < \infty.$$

Since $E(e^{\theta X}) < \infty$ for all $\theta \in I$, the function $\widetilde{\psi}_F$ is well-defined in D. We will verify that $\widetilde{\psi}_F$ is holomorphic in D. To this end, we have to verify that $\widetilde{\psi}_F(\theta, t)$ is continuously differentiable in (θ, t) and satisfies the Cauchy-Riemann equation:

$$\frac{\partial \widetilde{\psi}_F}{\partial \theta}(\theta, t) + i\frac{\partial \widetilde{\psi}_F}{\partial t}(\theta, t) = 0.$$

Now, because

$$\frac{\partial}{\partial \theta} e^{(\theta+it)X} = Xe^{(\theta+it)X} \quad \text{and} \quad \frac{\partial}{\partial t} e^{(\theta+it)X} = iXe^{(\theta+it)X},$$

we have

$$\frac{\partial \widetilde{\psi}_F}{\partial \theta}(\theta, t) = E\{Xe^{(\theta+it)X}\} \quad \text{and} \quad \frac{\partial \widetilde{\psi}_F}{\partial t}(\theta, t) = iE\{Xe^{(\theta+it)X}\}.$$

Hence, the Cauchy-Riemann equation is satisfied. To be precise, to guarantee the interchange between derivative and expectation, we have used the dominated convergence theorem in measure theory. The continuity of the map $(\theta, t) \mapsto E\{Xe^{(\theta+it)X}\}$ also follows from the dominated convergence theorem, and so we have verified that the function $\widetilde{\psi}_F$ is holomorphic in D.

Likewise, we can extend ψ_G to a holomorphic function $\widetilde{\psi}_G$ in D, but since $\widetilde{\psi}_F$ and $\widetilde{\psi}_G$ coincide on the line segment I, we have $\widetilde{\psi}_F(z) = \widetilde{\psi}_G(z)$ for all $z \in D$ by the identity theorem for holomorphic functions. In particular, we have

$$E(e^{itX}) = \widetilde{\psi}_F(it) = \widetilde{\psi}_G(it) = E(e^{itY})$$

for all $-\infty < t < \infty$, so that we have $F \equiv G$ by Theorem 1. □