STSCI 5080
Probability Models and Inference

Lecture 8: Independence of RVs and Conditional Distributions

September 18, 2018

Independence of random variables

Definition

Random variables X and Y are independent if

$$P(X \in A, Y \in B) = P(X \in A)P(Y \in B)$$

for any subsets $A \subset \mathbb{R}$ and $B \subset \mathbb{R}$.

Theorem

If X and Y are independent, their functions $g(X)$ and $h(Y)$ are also independent.

Independence: discrete case

Theorem

Suppose that the vector (X, Y) is discrete with joint pmf $p(x, y)$. Then X and Y are independent if and only if

$$p(x, y) = p_X(x)p_Y(y)$$

for any x and y.

Proof?

Independence: continuous case

Theorem

Suppose that the vector (X,Y) is continuous with joint pdf $f(x,y)$. Then X and Y are independent if and only if

$$f(x,y) = f_X(x) f_Y(y)$$

for any x and y.

Theorem

If random variables X and Y are independent and continuous with pdfs f_X and f_Y, respectively, then the vector (X,Y) is continuous with joint pdf

$$f(x,y) = f_X(x) f_Y(y)$$

for any x and y.

Recap

- The vector of continuous random variables need not be continuous (need not have a joint pdf).
- The vector of independent continuous random variables is continuous, and has a joint pdf $f(x,y) = f_X(x)f_Y(y)$.

Example 8.1

Example

Let (X, Y) be a continuous random vector with joint pdf

$$f(x,y) = xe^{-x(y+1)}, \ x, y \geq 0.$$

Are X and Y independent?

Example 8.1

Example

Let (X, Y) be a continuous random vector with joint pdf

$$f(x, y) = xe^{-x(y+1)}, \ x, y \geq 0.$$

Are X and Y independent?

The marginal pdfs are

$$f_X(x) = \int_0^\infty f(x, y)dy = \left[-e^{-x(y+1)}\right]_{y=0}^{y=\infty} = e^{-x},$$

$$f_Y(y) = \int_0^\infty f(x, y)dx = \left[-\frac{xe^{-x(y+1)}}{y+1}\right]_{x=0}^{x=\infty} + \int_0^\infty \frac{e^{-x(y+1)}}{y+1}dy$$

$$= 0 + \left[-\frac{e^{-x(y+1)}}{(y+1)^2}\right]_{x=0}^{x=\infty} = \frac{1}{(y+1)^2}$$

for $x, y \geq 0$. X and Y are not independent.

Example 8.2

Example

Let X and Y be independent random variables with the common pdf

$$g(x) = \begin{cases} 2x & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}.$$

Find the joint pdf of (X, Y) and calculate $P(X > 2Y)$.

Example 8.2

Example

Let X and Y be independent random variables with the common pdf

$$g(x) = \begin{cases} 2x & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}.$$

Find the joint pdf of (X, Y) and calculate $P(X > 2Y)$.

The joint pdf is

$$f(x, y) = \begin{cases} 4xy & \text{if } 0 \leq x, y \leq 1 \\ 0 & \text{otherwise} \end{cases}.$$

Next,

$$P(X > 2Y) = \int_0^1 \int_0^{x/2} 4xy \, dydx = \int_0^1 \frac{x^3}{2} dx = \frac{1}{8}.$$

Conditional pmf

Definition

Let (X, Y) be discrete with joint pmf $p(x, y)$. For every y such that $p_Y(y) > 0$,

$$p_{X|Y}(x \mid y) = \frac{p(x, y)}{p_Y(y)} \quad \text{for any } x$$

is called the conditional pmf of X given $Y = y$. Formally, we set $p_{X|Y}(x \mid y) = 0$ when $p_Y(y) = 0$.

Some properties of joint pmf

- By definition,

$$p_{X|Y}(x \mid y) = \frac{P(X = x, Y = y)}{P(Y = y)} = P(X = x \mid Y = y).$$

- The conditional pmf is a pmf as a function of x:

$$\sum_x p_{X|Y}(x \mid y) = \frac{\sum_x p(x, y)}{p_Y(y)} = \frac{p_Y(y)}{p_Y(y)} = 1.$$

- For any subset $B \subset \mathbb{R}$,

$$P(X \in B \mid Y = y) = \sum_{x \in B} p_{X|Y}(x \mid y).$$

- From conditional pmf to joint pmf:

$$p(x, y) = p_{X|Y}(x \mid y) p_Y(y).$$

Example 8.3

Example

Professor K. Kat0 (not me!) often makes incorrect statements and answers each of his students' questions incorrectly with probability $1/4$, independently of other questions. In each lecture, Professor Kat0 is asked $0, 1$, and 2 questions with probability $1/3$. Let X and Y be the number of questions he answers wrong and the number of questions he is asked, respectively. Find the joint pmf of (X, Y).

Example 8.3

- Whether his answer is incorrect is a Bernoulli random variable with success probability $1/4$.

- Given that $Y = y \in \{1, 2\}$, $X \sim Bin(y, 1/4)$, so that

$$p_{X|Y}(x \mid y) = \binom{y}{x} \left(\frac{1}{4}\right)^{x} \left(\frac{3}{4}\right)^{y-x}, \ x = 0, \ldots, y.$$

  Given that $Y = 0$, we have $X = 0$.

- We know that $p_Y(y) = 1/3$ for $y = 0, \ldots, 2$.

Conditional pdf

Definition

Let (X, Y) be continuous with joint pdf $f(x, y)$. For every y such that $f_Y(y) > 0$,

$$f_{X|Y}(x \mid y) = \frac{f(x, y)}{f_Y(y)} \quad \text{for any } x$$

is called the conditional pdf of X given $Y = y$. Formally, we set $f_{X|Y}(x \mid y) = 0$ when $f_Y(y) = 0$.

Some properties

- Since any single point is assigned 0 probability for a continuous random variable, $f_{X|Y}(x \mid y)$ is not $P(X = x \mid Y = y)$.

- For $\delta_1 > 0$ and $\delta_2 > 0$ small,

$$
\begin{aligned}
& P(x \leq X \leq x + \delta_1 \mid y \leq Y \leq y + \delta_2) \\
& = \frac{P(x \leq X \leq x + \delta_1, y \leq Y \leq y + \delta_2)}{P(y \leq Y \leq y + \delta_2)} \\
& \approx \frac{f(x, y)\delta_1\delta_2}{f_Y(y)\delta_2} \\
& = f_{X|Y}(x \mid y)\delta_1 .
\end{aligned}
$$

- The conditional pdf is a pdf as a function of x:

$$\int_{-\infty}^{\infty} f_{X|Y}(x \mid y)dx = \frac{\int_{-\infty}^{\infty} f(x,y)dx}{f_Y(y)} = \frac{f_Y(y)}{f_Y(y)} = 1.$$

- From conditional pdf to joint pdf:

$$f(x,y) = f_{X|Y}(x \mid y) f_Y(y).$$

Example 8.4

Example

Let $A = \{(x,y) \mid x,y \geq 0, x+y \leq 1\}$. Define a pdf

$$f(x,y) = \begin{cases} 2 & \text{if } (x,y) \in A \\ 0 & \text{otherwise} \end{cases}.$$

Find the conditional pdf of X given Y.

Example 8.4

Example

Let $A = \{(x,y) \mid x,y \geq 0, x+y \leq 1\}$. Define a pdf

$$f(x,y) = \begin{cases} 2 & \text{if } (x,y) \in A \\ 0 & \text{otherwise} \end{cases}.$$

Find the conditional pdf of X given Y.

The marginal pdf of Y is

$$f_Y(y) = 2(1-y), \quad \text{for } 0 \leq y \leq 1.$$

So the conditional pdf is

$$f_{X|Y}(x \mid y) = \frac{2}{2(1-y)} = \frac{1}{1-y}$$

for $0 \leq x \leq 1-y$ and $0 \leq y < 1$, and $f_{X|Y}(x \mid y) = 0$ elsewhere.

Example 8.5

Example

Let (X, Y) be a continuous random vector with joint pdf

$$f(x,y) = xe^{-x(y+1)}, \; x, y \geq 0$$

Find the conditional pdf of Y given X.

Example 8.5

Example

Let (X, Y) be a continuous random vector with joint pdf

$$f(x,y) = xe^{-x(y+1)}, \ x, y \geq 0$$

Find the conditional pdf of Y given X.

Since $f_X(x) = e^{-x}$ for $x \geq 0$,

$$f_{Y|X}(y \mid x) = xe^{-xy}, \ y \geq 0.$$

Given X, $Y \sim Ex(X)$.