ORIE 4630: Spring Term 2019
Homework #9 Solutions

Question 1. [**25 points**]

i)

```
> option_values
         [,1]      [,2]      [,3]      [,4]       [,5]      [,6]
[1,] 15.96063 27.292619 44.866499 70.172699 103.201146 142.06491
[2,]       NA  6.981901 13.391222 24.898991  44.217151  72.84655
[3,]       NA        NA  1.885072  4.247735   9.571654  21.56833
[4,]       NA        NA        NA  0.000000   0.000000   0.00000
[5,]       NA        NA        NA        NA   0.000000   0.00000
[6,]       NA        NA        NA        NA         NA   0.00000
```

The option price at time $t = 0$ is 15.961.

```
> q
[1] 0.4460073
```

ii)

```
> option_values
         [,1]      [,2]     [,3]      [,4]      [,5]      [,6]
[1,] 15.31239 28.112726 48.87062 79.098833 117.27376 162.86504
[2,]       NA  7.018817 14.69056 29.390819  54.64240  88.25566
[3,]       NA        NA  2.03021  5.141941  13.02306  32.98368
[4,]       NA        NA       NA  0.000000   0.00000   0.00000
[5,]       NA        NA       NA        NA   0.00000   0.00000
[6,]       NA        NA       NA        NA        NA   0.00000
```

The option price at time $t = 0$ is 15.312.

```
> q
[1] 0.3968126
```

Yes, the price of the option at time $t = 0$ changes.
Yes, the value of q changes.

iii)

```
> option_values
         [,1]      [,2]      [,3]      [,4]       [,5]      [,6]
[1,] 15.96063 27.292619 44.866499 70.172699 103.201146 142.06491
[2,]       NA  6.981901 13.391222 24.898991  44.217151  72.84655
[3,]       NA        NA  1.885072  4.247735   9.571654  21.56833
[4,]       NA        NA        NA  0.000000   0.000000   0.00000
[5,]       NA        NA        NA        NA   0.000000   0.00000
[6,]       NA        NA        NA        NA         NA   0.00000
```

The option price at time $t = 0$ is 15.961.

```
> q
[1] 0.4460073
```

No, the value of the option at time $t = 0$ does not change.
No, the value of q does not change.

iv)

```
> option_values
         [,1]      [,2]     [,3]      [,4]      [,5]     [,6]
[1,] 17.87437  8.996149  2.75016  0.000000  0.000000  0.00000
[2,]       NA 25.183747 14.10605  4.989136  0.000000  0.00000
[3,]       NA        NA 34.33002 21.573512  9.050921  0.00000
[4,]       NA        NA       NA 44.910611 31.850370 16.41951
[5,]       NA        NA       NA        NA 55.831472 44.56159
[6,]       NA        NA       NA        NA        NA 65.40976
```

The option price at time $t = 0$ is 17.874.

```
> q
[1] 0.4460073
```

No, the value of q does not change.

v)

```
> option_values
         [,1]      [,2]     [,3]      [,4]      [,5]     [,6]
[1,] 18.22171  9.100571  2.75016  0.000000  0.000000  0.00000
[2,]       NA 25.729808 14.29549  4.989136  0.000000  0.00000
[3,]       NA        NA 35.16813 21.917171  9.050921  0.00000
[4,]       NA        NA       NA 46.154382 32.473810 16.41951
[5,]       NA        NA       NA        NA 56.454912 44.56159
[6,]       NA        NA       NA        NA        NA 65.40976
```

The option price at time $t = 0$ is 18.222.

```
> q
[1] 0.4460073
```

Yes, the price of the option at time $t = 0$ changes.
The price of the option is larger for the American put option than it is for the European put option.

2. [**25 points**]

i)

```
> q
[1] 0.4757915
> option_values
[1] 15.32169
> BS_formula
[1] 15.17049
> option_values-BS_formula
[1] 0.1511967
```

The price of the option at time $t = 0$ is 15.3217.
The value given by the Black-Scholes formula is 15.1705.
The difference between the price and the Black-Scholes value is 0.1512, nearly 15 cents.
The value of the risk-neutral probability is 0.4757915.

ii)

```
> q
[1] 0.4982871
> option_values
[1] 15.16988
> BS_formula
[1] 15.17049
> option_values-BS_formula
[1] -0.000609486
```

The price of the option at time $t = 0$ is 15.1699.
The value given by the Black-Scholes formula is 15.1705.
The difference between the price and the Black-Scholes value is -0.00060, less than one tenth of a cent.
The value of the risk-neutral probability is 0.4982871.

iii)

```
> q
[1] 0.4982871
> option_values
[1] 17.08362
> BS_formula
[1] 17.08423
> option_values-BS_formula
[1] -0.000609486
```

The price of the option at time $t = 0$ is 17.0836.
The value given by the Black-Scholes formula is 17.0842.
The difference between the price and the Black-Scholes value is -0.000609486, less than one tenth of a cent.
The value of the risk-neutral probability is 0.4982871.

iv)

```
> q
[1] 0.4982871
> option_values
[1] 17.4267
> BS_formula
[1] 17.08423
> option_values-BS_formula
[1] 0.3424724
```

The price of the option at time $t = 0$ is 17.4267.
The value given by the Black-Scholes formula is 17.0842.
The difference between the price and the Black-Scholes value is 0.3425, nearly 35 cents.
The value of the risk-neutral probability is 0.4982871.

v)

```
> q
[1] 0.4994583
> option_values
[1] 17.42702
> BS_formula
[1] 17.08423
> option_values-BS_formula
[1] 0.3427923
```

The price of the option at time $t = 0$ is 17.4270.
The value given by the Black-Scholes formula is 17.0842.
The difference between the price and the Black-Scholes value is 0.3428, nearly 35 cents.
The value of the risk-neutral probability is 0.4994583.
The difference between the option price derived by using $n = 50000$ and the price derived by using $n = 5000$ is $17.4270 - 17.4267 = 0.0003$, or three one-hundredths of a cent. It would appear that using $n = 5000$ suffices to price the option; using $n = 50000$ is unwarranted.

3. [**20 points**]

i)

```
> c=rbind(c("mean","standard deviation"),round(c(a,b),7)); c
     [,1]        [,2]
[1,] "mean"      "standard deviation"
[2,] "5.7057688" "12.3025779"
> a+c(-1,1)*qnorm(0.975)*b/sqrt(n_sim)
[1] 4.943261 6.468276
```

The estimate of the price is 5.7058.
The 95% confidence interval for the price is $(4.9433, 6.4683)$
Since we likely want the confidence interval to have width less than a few cents to properly price the option, clearly a larger sample size is required.

ii)

```
> c=rbind(c("mean","standard deviation"),round(c(a,b),7)); c
     [,1]        [,2]
[1,] "mean"      "standard deviation"
[2,] "6.1104239" "12.1769317"
> a+c(-1,1)*qnorm(0.975)*b/sqrt(n_sim)
[1] 5.355704 6.865144
```

The estimate of the price is 6.1104.
The 95% confidence interval for the price is $(5.3557, 6.8651)$
Since we likely want the confidence interval to have width less than a few cents to properly price the option, clearly a larger sample size is required.

iii)

```
> c=rbind(c("mean","standard deviation"),round(c(a,b),7)); c
     [,1]        [,2]
[1,] "mean"      "standard deviation"
[2,] "6.3721436" "12.9914631"
> a+c(-1,1)*qnorm(0.975)*b/sqrt(n_sim)
[1] 6.336134 6.408153
```

The estimate of the price is 6.3721.
The 95% confidence interval for the price is $(6.3361, 6.4082)$
Since we likely want the confidence interval to have width less than a few cents to properly price the option, a larger sample size might still be required.

iv)
$\dfrac{2(1.960)(12.99)}{\sqrt{n}} \leq 0.01$ yields $n \geq 25930000$; this n is extremely large!

4. [10 points]
The output is

```
> spot_rates
[1] 0.004520342 0.005139350 0.005834109 0.006194149
> annual_spot_rates
[1] 0.01808137 0.02055740 0.02333644 0.02477659
```

i) The 2-year spot rate expressed as an annual rate is 2.478%.
ii) The 1-year spot rate expressed as a quarterly rate is 0.5834%.

5. [10 points]
The output is

```
> C=1.1; P=100; T=1; m=4
> spot_rates=c(0.008, 0.009, 0.011, 0.012)
> present_value_coupon_bond(C,P,T,m,spot_rates)
[1] 99.62558
```

The price of the bond is 99.6256.

6. [10 points]
The output is

```
> yield=yield_to_maturity$root
> yield
[1] 0.008446984
> m*yield
[1] 0.03378794
```

The yield to maturity expressed as a quarterly rate is 0.8447%.
The yield to maturity expressed as an annual rate is 3.379%.