Fall 2018 STSCI 5080 Discussion 6 (10/5)

Problems

1. Verify that if $X \sim U[a,b]$, then $E(X) = (a+b)/2$ and $\mathrm{Var}(X) = (b-a)^2/12$.

2. Let $X, Y \sim U[-1/2, 1/2]$ i.i.d., and let $Z = X + Y$.

   (a) Verify that the pdf of Z is

   $$f_Z(z) = \begin{cases} 1+z & \text{if } -1 \le z < 0 \\ 1-z & \text{if } 0 \le z \le 1 \\ 0 & \text{otherwise} \end{cases}.$$

   This is called the *triangle density*.

   (b) Find the mean and variance of Z.

3. Let X and Y be random variables such that $E(|X|) < \infty$ and $E(|Y|) < \infty$. Show that $E[\max\{X, Y\}] \ge \max\{E(X), E(Y)\}$ and $E[\min\{X, Y\}] \le \min\{E(X), E(Y)\}$.

4. (**Rice 4.7.13**) If X is a nonnegative continuous random variable, show that

   $$E(X) = \int_0^\infty \{1 - F(x)\} dx.$$

   Apply this result to find the mean of the exponential distribution.

   (Hint). Use the fact that

   $$x = \int_0^x dy = \int_0^\infty I(x,y) dy,$$

   where $I(x,y)$ is the indicator function

   $$I(x,y) = \begin{cases} 1 & \text{if } 0 \le y \le x \\ 0 & \text{otherwise} \end{cases}.$$

5. Generalize Problem 4 to

   $$E(X^k) = k \int_0^\infty x^{k-1} P(X > x) dx,$$

   where X is a nonnegative continuous random variable and k is a positive integer.

6. Let X and Y be random variables with cdfs F and G, respectively. If $F(x) \le G(x)$ for all $x \in \mathbb{R}$, then X is said to *stochastically dominate* Y. Suppose that X stochastically dominates Y, and F and G have quantile functions F^{-1} and G^{-1}, respectively[1].

   (a) Show that $F^{-1}(u) \ge G^{-1}(u)$ for all $u \in (0,1)$. (Hint). Apply $x = G^{-1}(u)$ to $F(x) \le G(x)$.

[1] *Stochastic dominance* plays an important role in economic theory. See the wikipedia page as a reference.

(b) Show that $E(X) \geq E(Y)$ provided that $E(|X|) < \infty$ and $E(|Y|) < \infty$. (Hint). Use the fact that $F^{-1}(U)$ has cdf F for $U \sim U[0,1]$.

7. (**Rice 4.7.70**) If X and Y are independent, show that $E(X \mid Y) = E(X)$ with probability one.

8. (**Rice 4.7.77**) Let X and Y have the joint density

$$f(x,y) = e^{-y}, \quad 0 \leq x \leq y.$$

(a) Find $\mathrm{Cov}(X,Y)$ and $\mathrm{Corr}(X,Y)$.

(b) Find the conditional exectation of X given Y and Y given X.

(c) Find the density functions of the random variables $E(X \mid Y)$ and $E(Y \mid X)$.

Solutions

1. The pdf of $U[a,b]$ is
$$f(x) = \frac{1}{b-a} \text{ if } a \le x \le b$$
and $f(x) = 0$ elsewhere. So we have
$$E(X) = \frac{1}{b-a}\int_a^b x dx = \frac{1}{(b-a)}\left[\frac{x^2}{2}\right]_a^b = \frac{b^2-a^2}{2(b-a)} = \frac{a+b}{2}.$$
In addition, the second moment is
$$E(X^2) = \frac{1}{b-a}\int_a^b x^2 dx = \frac{1}{b-a}\left[\frac{x^3}{3}\right]_a^b = \frac{b^3-a^3}{3(b-a)} = \frac{a^2+ab+b^2}{3}.$$
Therefore, the variance is
$$\mathrm{Var}(X) = E(X^2) - \{E(X)\}^2 = \frac{a^2+ab+b^2}{3} - \frac{a^2+2ab+b^2}{4} = \frac{a^2-2ab+b^2}{12} = \frac{(b-a)^2}{12}.$$

2. (a) The common pdf is
$$f(x) = \begin{cases} 1 & \text{if} -\frac{1}{2} \le x \le \frac{1}{2} \\ 0 & \text{otherwise} \end{cases},$$
and the pdf of Z is
$$f_Z(z) = \int_{-\infty}^{\infty} f(x)f(x-z)dx = \int_{-1/2}^{1/2} f(x-z)dx.$$
Now, $f(x-z) = 1$ if and only if
$$-\frac{1}{2} \le x - z \le \frac{1}{2}, \text{ i.e., } z - \frac{1}{2} \le x \le z + \frac{1}{2},$$
and so
$$\begin{aligned}\int_{-1/2}^{1/2} f(x-z)dx &= \int_{[-1/2,1/2]\cap[z-1/2,z+1/2]} dx \\ &= (\text{length of } [-1/2,1/2]\cap[z-1/2,z+1/2]) \\ &= \begin{cases} z+1/2-(-1/2) = z+1 & \text{if } -1 \le z < 0 \\ 1/2-(z-1/2) = 1-z & \text{if } 0 \le z \le 1 \\ 0 & \text{otherwise} \end{cases}.\end{aligned}$$

(b) We know that the mean and variance of $U[a,b]$ is $(a+b)/2$ and $(b-a)^2/12$, and so the mean and variance of $U[-1/2,1/2]$ is 0 and $1/12$. Hence,
$$E(Z) = E(X) + E(Y) = 0, \ \mathrm{Var}(Z) = \mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) = \frac{1}{6},$$
where we have used independence of X and Y.

3. Since $\max\{X,Y\} \geq X$, we have $E[\max\{X,Y\}] \geq E(X)$. Similarly, we have $E[\max\{X,Y\}] \geq E(Y)$, so that $E[\max\{X,Y\}] \geq \max\{E(X),E(Y)\}$. Likewise, we have $E[\min\{X,Y\}] \leq \min\{E(X),E(Y)\}$.

4. Using the hint, we have

$$E(X) = \int_0^\infty xf(x)dx = \int_0^\infty \int_0^\infty I(x,y)dyf(x)dx = \int_0^\infty \left\{ \int_0^\infty I(x,y)f(x)dx \right\} dy.$$

For a fixed y, $I(x,y) = 1$ only if $x \geq y$, so that

$$\int_0^\infty I(x,y)f(x)dx = \int_y^\infty f(x)dx = 1 - F(y).$$

Hence, we have

$$E(X) = \int_0^\infty \{1 - F(y)\}dy.$$

If $X \sim Ex(\lambda)$, then $F(x) = 1 - e^{-\lambda x}$ for $x \geq 0$, so that

$$E(X) = \int_0^\infty e^{-\lambda x}dx = \frac{1}{\lambda}.$$

5. We first note that

$$x^k = k\int_0^x y^{k-1}dy = k\int_0^\infty y^{k-1}I(x,y)dy.$$

Hence,

$$\begin{aligned} E(X^k) &= k\int_0^\infty \int_0^\infty y^{k-1}I(x,y)dyf(x)dx \\ &= k\int_0^\infty y^{k-1}\left\{ \int_0^\infty I(x,y)f(x)dx \right\} dy \\ &= k\int_0^\infty y^{k-1}P(X > y)dy. \end{aligned}$$

6. (a) Since $F(x) \leq G(x)$ for any x, taking $x = G^{-1}(u)$, we have $F(G^{-1}(u)) \leq G(G^{-1}(u)) = u$. Since F^{-1} is non-decreasing, we have $G^{-1}(u) = F^{-1}(F(G^{-1}(u))) \leq F^{-1}(u)$.

   (b) For $U \sim U[0,1]$, $F^{-1}(U)$ has cdf F and $G^{-1}(U)$ has cdf G, and so

   $$E\{F^{-1}(U)\} = E(X) \quad \text{and} \quad E\{G^{-1}(U)\} = E(Y).$$

   However, since $F^{-1}(U) \geq G^{-1}(U)$, we have

   $$E\{F^{-1}(U)\} \geq E\{G^{-1}(U)\},$$

   which implies that $E(X) \geq E(Y)$.

7. (**Rice 4.7.70**) We focus on the case where (X, Y) is discrete. Then $p(x, y) = p_{X|Y}(x \mid y)p_Y(y)$ for all (x, y) (including the case where $p_Y(y) = 0$), and if X and Y are independent, then $p(x, y) = p_X(x)p_Y(y)$ for all (x, y), so that $p_X(x)p_Y(y) = p_{X|Y}(x \mid y)p_Y(y)$. So if $p_Y(y) > 0$, we have $p_{X|Y}(x \mid y) = p_X(x)$ for all x, so that $E(X \mid Y = y) = E(X)$. However,

$$P(p_Y(Y) > 0) = \sum_{y:p_Y(y)>0} p_Y(y) = 1,$$

so that $E(X \mid Y) = E(X)$ with probability one.

8. (**Rice 4.7.77**) The marginal pdf of X is

$$f_X(x) = \int_x^\infty e^{-y} dy = e^{-x}$$

for $x \geq 0$ and $f_X(x) = 0$ elsewhere. On the other hand, the marginal pdf of Y is

$$f_Y(y) = \int_0^y e^{-y} dx = ye^{-y}$$

for $y \geq 0$ and $f_Y(y) = 0$ elsewhere. We will use the following formula:

$$\int_0^\infty x^k e^{-x} dx = k!$$

for $k = 0, 1, 2, \ldots$

(a) We have $E(X) = 1, E(Y) = 2, E(X^2) = 2$, and $E(Y^2) = 3! = 6$, so that $\mathrm{Var}(X) = 2 - 1 = 1$ and $\mathrm{Var}(Y) = 6 - 4 = 2$. In addition,

$$E(XY) = \int_0^\infty \int_0^y (xy)e^{-y} dx dy = \frac{1}{2} \int_0^\infty y^3 e^{-y} dy = \frac{3!}{2} = 3.$$

Hence, we have

$$\mathrm{Cov}(X, Y) = E(XY) - E(X)E(Y) = 3 - 2 = 1 \quad \text{and} \quad \mathrm{Corr}(X, Y) = \frac{\mathrm{Cov}(X, Y)}{\sqrt{\mathrm{Var}(X)}\sqrt{\mathrm{Var}(Y)}} = \frac{1}{\sqrt{2}}.$$

(b) The conditional pdf of X given Y is

$$f_{X|Y}(x \mid y) = \frac{f(x, y)}{f_X(y)} = \frac{1}{y}$$

for $0 \leq x \leq y$ and $y > 0$, and $f_{X|Y}(x \mid y) = 0$ elsewhere. This implies that given Y, $X \sim U[0, Y]$, so that $E(X \mid Y) = Y/2$.
On the other hand, the conditional pdf of Y given X is

$$f_{Y|X}(y \mid x) = e^{x-y}$$

for $0 \leq x \leq y$ and $f_{Y|X}(y \mid x) = 0$ elsewhere. Hence,

$$E(Y \mid X = x) = e^x \int_x^\infty ye^{-y} dy = e^x \left\{ [-ye^{-y}]_{y=x}^\infty + \int_x^\infty e^{-y} dy \right\} = e^x(xe^{-x} + e^{-x}) = 1 + x$$

for $x \geq 0$, so that $E(Y \mid X) = 1 + X$.

(c) $Z = E(X \mid Y) = Y/2$ and $W = E(Y \mid X) = 1 + X$. Then the cdf of Z is

$$F_Z(z) = P(Z \leq z) = P(Y \leq 2z) = F_Y(2z),$$

so that

$$f_Z(z) = \frac{d}{dz} F_Y(2z) = 2 f_Y(2y) = \begin{cases} 4ye^{-2y} & \text{if } y \geq 0 \\ 0 & \text{otherwise} \end{cases}.$$

On the other hand,

$$F_W(w) = P(W \leq w) = P(X \leq w - 1) = F_X(w - 1),$$

so that

$$f_W(w) = \frac{d}{dw} F_X(w - 1) = f_W(w - 1) = \begin{cases} e^{-(w-1)} & \text{if } w \geq 1 \\ 0 & \text{otherwise} \end{cases}.$$