Fall 2018 STSCI 5080 Discussion 12 (11/15)

Problems

1. Suppose that the heights of men in a certain population follow the normal distribution with mean μ and variance 9. If the sample size $n = 36$ and the sample mean $\overline{X}$ are given, then find a confidence interval for μ with level 95%.

2. (Rice 8.10.21 modified) Let
$$X_1, \dots, X_n \sim f_\theta \text{ i.i.d.}$$
for some $-\infty < \theta < \infty$, where f_θ is a pdf defined by
$$f_\theta(x) = \begin{cases} e^{-(x-\theta)} & \text{if } x \geq \theta \\ 0 & \text{otherwise} \end{cases}.$$
   (a) Find the likelihood function for θ.
   (b) Find the MLE of θ. (Hint). Be careful, and don't differentiate before thinking. For what values of θ is the likelihood positive?

3. (Rice 8.10.50 modified) Let
$$X_1, \dots, X_n \sim f_\theta$$
for some $\theta > 0$, where f_θ is a pdf defined by
$$f_\theta(x) = \frac{x}{\theta^2} e^{-x^2/(2\theta^2)}, \quad x \geq 0.$$
   (a) Find the log likelihood function for θ.
   (b) Verify that the MLE of θ is
$$\widehat{\theta} = \sqrt{\frac{1}{2n} \sum_{i=1}^{n} X_i^2}.$$
   (c) Find the limiting distribution of $\sqrt{n}(\widehat{\theta} - \theta)$ as $n \to \infty$. You may use the fact that $E_\theta(X_1^2) = 2\theta^2$ and $E_\theta(X_1^4) = 24\theta^4$ without derivations.
   (d) Use Part (b) to find a confidence interval for θ with asymptotic level 95%.

4. Let
$$X_1, \dots, X_n \sim N(0, \theta) \text{ i.i.d.}$$
for some $\theta > 0$. We know that the MLE of θ is
$$\widehat{\theta} = \frac{1}{n} \sum_{i=1}^{n} X_i^2,$$
and
$$\sqrt{n}(\widehat{\theta} - \theta) \xrightarrow{d} N(0, 2\theta^2) \qquad (*)$$
as $n \to \infty$.

(a) Find a confidence interval for θ with asymptotic level 95% by estimating the asymptotic variance in (*).

(b) Verify that the function

$$g(\theta) = \frac{1}{\sqrt{2}} \log \theta$$

is a variance stabilizing transformation of $\widehat{\theta}$.

(c) Find another confidence interval for θ with asymptotic level 95% using the variance stabilizing transformation in Part (b).

Solutions

1. A CI for μ with level 95% is

$$\left[\overline{X} - \frac{1.96\sigma_0}{\sqrt{n}}, \overline{X} + \frac{1.96\sigma_0}{\sqrt{n}}\right]$$

where $\sigma_0 = \sqrt{9} = 3$ and $\sqrt{n} = \sqrt{36} = 6$. Hence

$$[\overline{X} - 0.98, \overline{X} + 0.98]$$

is a desired CI.

2. (a) The joint pdf is

$$\prod_{i=1}^n f_\theta(x_i) = \begin{cases} e^{-\sum_{i=1}^n (x_i - \theta)} & \text{if } x_i \geq \theta \text{ for all } i = 1, \ldots, n \\ 0 & \text{otherwise} \end{cases}.$$

Since

$$x_i \geq \theta \text{ for all } i = 1, \ldots, n \Leftrightarrow \theta \leq x_{(1)},$$

the likelihood function is

$$L_n(\theta) = \begin{cases} e^{-\sum_{i=1}^n (X_i - \theta)} & \text{if } \theta \leq X_{(1)} \\ 0 & \text{otherwise} \end{cases}.$$

(b) The likelihood function is increasing in θ as long as $\theta \leq X_{(1)}$, and so the MLE is $\widehat{\theta} = X_{(1)}$.

3. (a) The joint pdf is

$$\prod_{i=1}^n f_\theta(x_i) = \frac{\prod_{i=1}^n x_i}{\theta^{2n}} e^{-\sum_{i=1}^n x_i^2/(2\theta^2)}.$$

The likelihood function is

$$L_n(\theta) = \frac{\prod_{i=1}^n X_i}{\theta^{2n}} e^{-\sum_{i=1}^n X_i^2/(2\theta^2)}.$$

The log likelihood function is

$$\ell_n(\theta) = \log L_n(\theta) = \log \prod_{i=1}^n X_i - 2n \log \theta - \frac{1}{2\theta^2} \sum_{i=1}^n X_i^2.$$

(b) Since

$$\ell_n'(\theta) = -\frac{2n}{\theta} + \frac{1}{\theta^3} \sum_{i=1}^n X_i^2,$$

solving the FOC

$$\ell_n'(\theta) = 0,$$

we obtain the MLE

$$\widehat{\theta} = \sqrt{\frac{1}{2n} \sum_{i=1}^n X_i^2}.$$

(c) Let $Y_i = X_i^2/2$ for $i = 1, \ldots, n$. We note that $E_\theta(Y_1) = \theta^2$ and $\mathrm{Var}_\theta(Y_1) = \mathrm{Var}_\theta(X_1)/4 = 5\theta^4$. So

$$\sqrt{n}(\overline{Y} - \theta^2) \xrightarrow{d} N(0, 5\theta^4).$$

Consider the function $g(y) = \sqrt{y}$. Since $g'(y) = 1/(2\sqrt{y})$, we have

$$\sqrt{n}(\widehat{\theta} - \theta) = \sqrt{n}(g(\overline{Y}) - g(\theta^2)) \xrightarrow{d} N(0, \{g'(\theta^2)\}^2 \cdot 5\theta^4) = N(0, 5\theta^2/4).$$

(d)

$$\left[\widehat{\theta} - \frac{1.96\widehat{\theta}\sqrt{5/4}}{\sqrt{n}}, \widehat{\theta} + \frac{1.96\widehat{\theta}\sqrt{5/4}}{\sqrt{n}}\right]$$

is a desired CI.

4. (a)

$$\left[\widehat{\theta} - \frac{1.96\widehat{\theta}\sqrt{2}}{\sqrt{n}}, \widehat{\theta} + \frac{1.96\widehat{\theta}\sqrt{2}}{\sqrt{n}}\right]$$

is a desired CI.

(b) Since $g'(\theta) = 1/(\sqrt{2}\theta)$, we have

$$\sqrt{n}\{g(\widehat{\theta}) - g(\theta)\} \xrightarrow{d} N(0, \{g'(\theta)\}^2 2\theta^2) = N(0, 1)$$

by the delta method.

(c) Since $g^{-1}(y) = e^{\sqrt{2}y}$,

$$\begin{aligned}&\left[g^{-1}\left(g(\widehat{\theta}) - 1.96/\sqrt{n}\right), g^{-1}\left(g(\widehat{\theta}) - 1.96/\sqrt{n}\right)\right]\\ &= \left[\exp(\log\widehat{\theta} - 1.96\sqrt{2}/\sqrt{n}), \exp(\log\widehat{\theta} + 1.96\sqrt{2}/\sqrt{n})\right]\end{aligned}$$

is a desired CI.