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STSCI 5080 Practice Midterm Exam 2[1]

Problem 1. Circle the correct choice in each of the following questions.

(1) Let X and Y be lifetimes (in year) of two cars, and suppose that X and Y are independent and each follows the exponential distribution with parameter $\lambda = 0.1$. What is the probability that at least one car will be working for more than 10 years?

a. e^{-1} b. e^{-2} c. $1 - e^{-1}$ $\boxed{\text{d.}}$ $1 - (1 - e^{-1})^2$

(2) Let X and Y be random variables such that $E(X) = 0, E(X^2) = 1, E(Y) = 1, E(Y^2) = 5$, and $\mathrm{Corr}(X, Y) = 0.5$. What is $\mathrm{Var}(X + 2Y)$?

a. 17 b. 18 $\boxed{\text{c.}}$ 21 d. 22

(3) Suppose that we first draw N according to the Poisson distribution with parameter $\lambda = 10$; throw a six-sided die N times and then count the sum of the face values, which is denoted by Y. What is the mean of Y?

a. 3.5 b. 10 c. 30 $\boxed{\text{d.}}$ 35

(4) Find the correct statement. Only one of them is correct.

a. If $X_n \xrightarrow{P} X$ and the expectations are defined, then $E(X_n) \to E(X)$.

b. If X and Y are such that $E(X^k) = E(Y^k)$ for all positive intergers k (assuming that thoese moments exist), then X and Y have the same cdf.

c. If X_n and X are continuous with pdfs f_n and f, respectively, and $X_n \xrightarrow{d} X$, then $f_n(x) \to f(x)$ pointwise.

$\boxed{\text{d.}}$ None of them are correct.

[1] The actual exam is 1-hour long. The instructions of the first midterm exam apply. In the exam, you will be given a scratch sheet and a formula sheet as in the first midterm exam.

Problem 2. Let $X_1, \ldots, X_n$ be a random sample from the uniform distribution on $[0,1]$. Let $X_{(1)} = \min_{1\le i\le n} X_i$ and $X_{(n)} = \max_{1\le i\le n} X_i$.

(a) Derive the pdfs of $X_{(1)}$ and $X_{(n)}$.

(b) Find $E(X_{(n)} - X_{(1)})$.

(a) We note that

$$P(X_{(1)} > x) = P(X_i > x\ \forall i = 1, \ldots, n) = P(X_1 > x) \cdots P(X_n > x) = (1-x)^n$$

for $0 \le x \le 1$, so that

$$P(X_{(1)} \le x) = 1 - (1-x)^n.$$

Hence the pdf of $X_{(1)}$ is

$$f_{X_{(1)}}(x) = \frac{d}{dx} P(X_{(1)} \le x) = n(1-x)^{n-1} \quad \text{for } 0 \le x \le 1.$$

Since $0 \le X_{(1)} \le 1$, we have $f_{X_{(1)}}(x) = 0$ for $x < 0$ or $x > 1$.

Next,

$$P(X_{(n)} \le x) = P(X_i \le x\ \forall i = 1, \ldots, n) = P(X_1 \le x) \cdots P(X_n \le x) = x^n$$

for $0 \le x \le 1$, and so the pdf of $X_{(n)}$ is

$$f_{X_{(n)}}(x) = \frac{d}{dx} P(X_{(n)} \le x) = nx^{n-1} \quad \text{for } 0 \le x \le 1.$$

Since $0 \le X_{(n)} \le 1$, we have $f_{X_{(n)}}(x) = 0$ for $x < 0$ or $x > 1$.

(b) We have

$$\begin{aligned} E(X_{(n)} - X_{(1)}) = E(X_{(n)}) - E(X_{(1)}) &= \int_0^1 nx^n dx - \int_0^1 nx(1-x)^{n-1} dx \\ &= \frac{n}{n+1} - \int_0^1 n(1-y)y^{n-1} dy = \frac{n}{n+1} - 1 + \frac{n}{n+1} \\ &= \frac{n-1}{n+1}. \end{aligned}$$

Problem 3. Let (X, Y) be a continuous random vector with joint pdf

$$f(x,y) = \begin{cases} 6x & \text{if } x, y \geq 0,\ 0 \leq x+y \leq 1 \\ 0 & \text{otherwise} \end{cases}.$$

(a) Find $\mathrm{Cov}(X, Y)$ and $\mathrm{Corr}(X, Y)$.

(b) Find $E(Y \mid X)$ and $\mathrm{Var}(Y \mid X)$.

(a) The marginal pfds of X and Y are

$$f_X(x) = \int_0^{1-x} 6x dy = 6x(1-x) \quad \text{for } 0 \leq x \leq 1,$$
$$f_Y(y) = \int_0^{1-y} 6x dx = 3(1-y)^2 \quad \text{for } 0 \leq y \leq 1.$$

Hence, we have

$$\begin{aligned} E(X) &= \int_0^1 6x^2(1-x)dx = 6\left(\frac{1}{3} - \frac{1}{4}\right) = \frac{1}{2}, \\ E(Y) &= \int_0^1 3y(1-y)^2 dy = 3\int_0^1 z^2(1-z)dz = \frac{1}{4}, \\ E(XY) &= \int_0^1 \int_0^{1-x} 6x^2 y dy dx = 3\int_0^1 x^2(1-x)^2 dx = 3\int_0^1 (x^2 - 2x^3 + x^4)dx \\ &= 3\left(\frac{1}{3} - \frac{1}{2} + \frac{1}{5}\right) = \frac{1}{10}, \end{aligned}$$

so that $\mathrm{Cov}(X, Y) = E(XY) - E(X)E(Y) = 1/10 - 1/8 = -1/40$.

Next, we note that

$$E(X^2) = \int_0^1 6x^3(1-x)dx = 6\left(\frac{1}{4} - \frac{1}{5}\right) = \frac{3}{10} \quad \text{and} \quad E(Y^2) = 3\int_0^1 y^2(1-y)^2 dy = \frac{1}{10},$$

so that $\mathrm{Var}(X) = 3/10 - 1/8 = 3/40$ and $\mathrm{Var}(Y) = 1/10 - 1/16 = 1/40$. Hence,

$$\mathrm{Corr}(X, Y) = \frac{-1/40}{\sqrt{3/1600}} = -\frac{1}{\sqrt{3}}.$$

(b) The conditional pdf of Y given X is

$$f_{Y|X}(y \mid x) = \frac{f(x,y)}{f_X(x)} = \frac{1}{1-x} \quad \text{for } 0 \leq y \leq 1-x, 0 \leq x < 1.$$

This shows that $Y \sim U[0, 1-X]$ given X, so that

$$E(Y \mid X) = \frac{1-X}{2} \quad \text{and} \quad \mathrm{Var}(Y \mid X) = \frac{(1-X)^2}{12}.$$

You can also directly calculate $E(Y \mid X)$ and $\mathrm{Var}(Y \mid X)$.

Problem 4. Let X be a Poisson random variable with parameter λ.

(a) Find the mgf of X.

(b) Find the skewness of X, which is defined by

$$\beta_1 = \frac{E[\{X - E(X)\}^3]}{\{\mathrm{Var}(X)\}^{3/2}}.$$

You may use the following identity: $E[\{X - E(X)\}^3] = E(X^3) - 3E(X)E(X^2) + 2\{E(X)\}^3$.

(c) If Y is a Poisson random variable with parameter κ and Y is independent of X, then show that $X + Y$ follows the Poisson distribution with parameter $\lambda + \kappa$.

(a) The pmf of X is

$$p(x) = \frac{\lambda^x}{x!} e^{-\lambda}, \ x = 0, 1, 2, \ldots$$

The mgf of X is

$$\psi_X(\theta) = e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda e^\theta)^x}{x!} = e^{\lambda(e^\theta - 1)}$$

for $-\infty < \theta < \infty$.

(b) The successive derivatives of $\psi_X(\theta)$ are

$$\begin{aligned}
\psi'_X(\theta) &= \lambda e^\theta \psi_X(\theta), \\
\psi''_X(\theta) &= \lambda e^\theta \psi_X(\theta) + \lambda^2 e^{2\theta} \psi_X(\theta), \\
\psi'''_X(\theta) &= \lambda e^\theta \psi_X(\theta) + \lambda^2 e^{2\theta} \psi_X(\theta) + 2\lambda^2 e^{2\theta} \psi_X(\theta) + \lambda^3 e^{3\theta} \psi_X(\theta) \\
&= \lambda e^\theta \psi_X(\theta) + 3\lambda^2 e^{2\theta} \psi_X(\theta) + \lambda^3 e^{3\theta} \psi_X(\theta).
\end{aligned}$$

Hence $E(X) = \lambda, E(X^2) = \lambda + \lambda^2$, and $E(X^3) = \lambda + 3\lambda^2 + \lambda^3$, so that

$$\beta_1 = \frac{\lambda + 3\lambda^2 + \lambda^3 - 3\lambda(\lambda + \lambda^2) + \lambda^3}{\lambda^{3/2}} = \lambda^{-1/2}.$$

(c) The mgf of Y is

$$\psi_Y(\theta) = e^{\kappa(e^\theta - 1)},$$

and so the mgf of $Z = X + Y$ is

$$\psi_Z(\theta) = \psi_X(\theta)\psi_Y(\theta) = e^{(\lambda+\kappa)(e^\theta - 1)},$$

which coincides with the mgf of $Po(\lambda + \kappa)$. Hence, $Z = X + Y \sim Po(\lambda + \kappa)$.

Problem 5. Let Y_n denote a binomial random variable with parameters n and p where $0 < p < 1$.

(a) Derive the limiting distribution of $\sqrt{n}(Y_n/n - p)$ as $n \to \infty$.

(b) Suppose that we want to estimate $g(p) = p(1-p)$ which is the variance of the corresponding Bernoulli trial. Find the limiting distribution of $\sqrt{n}\{g(Y_n/n) - g(p)\}$.

(a) Since $Y_n = X_1 + \cdots + X_n$ for independent Bernoulli trials $X_1, \ldots, X_n$ with success probability p, and the mean and variance of X_i are p and $p(1-p)$, respectively, we have

$$\sqrt{n}(Y_n/n - p) = \sqrt{n}(\overline{X}_n - p) \xrightarrow{d} N(0, p(1-p))$$

by CLT.

(b) Since $g'(p) = 1 - 2p$, we have

$$\sqrt{n}\{g(Y_n/n) - g(p)\} \xrightarrow{d} N(0, (1-2p)^2 p(1-p))$$

by the delta method.