10/16/18 Kengo Kato

STSCI 5080 Homework 4

- Due is 10/25 (Th) in class.
- Write your name and NetID at the top of the first page, along with the assignment number.
- Use only the one side of the paper. Attach your pages with a staple at the top left corner.
- There are five problems. Each problem is worth 10 points.

Problems

1. Suppose that a random vector (X, Y) has joint pdf

$$f(x,y) = \begin{cases} 2 & \text{if } 0 \le x \le y \le 1 \\ 0 & \text{otherwise} \end{cases}.$$

   (a) Find the covariance and correlation of X and Y.

   (b) Find $E(X \mid Y)$ and $E(Y \mid X)$.

2. Suppose that we first draw N according to $Po(\lambda)$; toss a fair coin N times and then count the number of heads, which is denoted by Y. Find the mean and variance of Y.

3. Denote by $\mu_k = E(X^k)$ the k-th moment of a random variable X. The k-th *central moment* of X is defined by $\kappa_k = E\{(X-\mu_1)^k\}$. For example, $\kappa_1 = \mu_1 - \mu_1 = 0$ and $\kappa_2 = \mathrm{Var}(X)$. By the binomial theorem, we have

$$\kappa_k = E\{(X-\mu_1)^k\} = E\left\{ \sum_{j=0}^k \binom{k}{j} X^j (-\mu_1)^{k-j} \right\} = \sum_{j=0}^k \binom{k}{j} (-1)^{k-j} \mu_j \mu_1^{k-j}.$$

   Next, the *skewness* and *kurtosis* of X are defined by

$$\beta_1 = \frac{\kappa_3}{\kappa_2^{3/2}} = \frac{E\{(X-\mu_1)^3\}}{\{\mathrm{Var}(X)\}^{3/2}} \quad \text{and} \quad \beta_2 = \frac{\kappa_4}{\kappa_2^2} = \frac{E\{(X-\mu_1)^4\}}{\{\mathrm{Var}(X)\}^2},$$

   respectively. The skewness is a measure of symmetry (of lack of symmetry) of the distribution, and the kurtosis is a measure of the shape of the distribution.[1]

   (a) Show that the skewness and kurtosis do not change if we replace X by $aX + b$.

   (b) Find the skewness and kurtosis of $X \sim N(\mu, \sigma^2)$. (Hint). First calculate the skewness and kurtosis of $Z \sim N(0, 1)$ and then use (a).

[1] See the wikipedia pages. For example, we can test whether the underlying distribution is normal by matching the estimated skewness and kurtosis to the skewness and kurtosis of the normal distribution; cf. Jarque-Bera test.

(c) Find the skewness and kurtosis of $X \sim Ex(\lambda)$. (Hint). First calculate the skewness and kurtosis of $Y \sim Ex(1)$ and then use (a). We note that $Y/\lambda \sim Ex(\lambda)$.

4. Let X have the Laplace density

$$f(x) = \frac{1}{2}e^{-|x|}, \ -\infty < x < \infty.$$

(a) Find the mgf of X by a direct calculation.

(b) Verify that the mgf derived in (a) coincides with that of $Y - Z$ where $Y, Z \sim Ex(1)$ i.i.d.

5. Let $X_1, \ldots, X_n$ be a random sample from a cdf F with mean μ and variance σ^2. Consider

$$Y_n = \frac{2}{n(n+1)} \sum_{j=1}^{n} jX_j.$$

(a) Find the mean and variance of Y_n.

(b) Show that $Y_n \xrightarrow{P} \mu$ as $n \to \infty$.

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Solutions STSCI 5080 Homework 4

1. The marginal pdf of X is

$$f_X(x) = \int_x^1 2dy = 2(1-x) \quad \text{for } 0 \le x \le 1$$

and $f_X(x) = 0$ elsewhere. Likewise, the marginal pdf of Y is

$$f_Y(y) = \int_0^y 2dx = 2y \quad \text{for } 0 \le y \le 1$$

and $f_Y(y) = 0$ elsewhere.

(a) We note that

$$E(X) = 2\int_0^1 x(1-x)dx = 2\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{1}{3},$$

and

$$E(Y) = 2\int_0^1 y^2 dy = \frac{2}{3}.$$

In addition,

$$E(XY) = \int_0^1 \int_x^1 (2xy)dydx = \int_0^1 x(1-x^2)dx = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}.$$

Hence, we have

$$\mathrm{Cov}(X,Y) = E(XY) - E(X)E(Y) = \frac{1}{4} - \frac{1}{3}\cdot\frac{2}{3} = \frac{1}{36}.$$

Next, since

$$E(X^2) = 2\int_0^1 x^2(1-x)dx = 2\left(\frac{1}{3} - \frac{1}{4}\right) = \frac{1}{6} \quad \text{and} \quad E(Y^2) = 2\int_0^1 y^3 dy = \frac{1}{2},$$

we have

$$\mathrm{Var}(X) = E(X^2) - \{E(X)\}^2 = \frac{1}{6} - \frac{1}{9} = \frac{1}{18} \quad \text{and} \quad \mathrm{Var}(Y) = E(Y^2) - \{E(Y)\}^2 = \frac{1}{2} - \frac{4}{9} = \frac{1}{18}.$$

Hence, we have

$$\mathrm{Corr}(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)}\sqrt{\mathrm{Var}(Y)}} = \frac{1}{2}.$$

(b) The conditional pdfs are

$$f_{X|Y}(x \mid y) = \begin{cases} \frac{1}{y} & \text{if } 0 \le x \le y \text{ and } 0 < y \le 1 \\ 0 & \text{otherwise} \end{cases} \quad \text{and}$$

$$f_{Y|X}(y \mid x) = \begin{cases} \frac{1}{1-x} & \text{if } x \le y \le 1 \text{ and } 0 \le x < 1 \\ 0 & \text{otherwise} \end{cases},$$

from which we know that $X \sim U[0, Y]$ given Y and $Y \sim U[0, 1-X]$ given X. Hence, we have

$$E(X \mid Y) = \frac{Y}{2} \quad \text{and} \quad E(Y \mid X) = \frac{1-X}{2}.$$

2. From Lecture 12, the mean and variance of Y are $E(Y) = \lambda\mu_F$ and $\mathrm{Var}(Y) = \lambda(\mu_F^2 + \sigma_F^2)$, where F is the cdf of a Bernoulli random variable with success probability $1/2$. Since $\mu_F = 1/2$ and $\sigma_F^2 = 1/4$, we have $E(Y) = \lambda/2$ and $\mathrm{Var}(Y) = \lambda/2$.

3. (a) Let $\beta_1(X)$ and $\beta_2(X)$ denote the skewness and kurtosis of X, respectively. We want to show that $\beta_1(aX+b) = \beta_1(X)$ and $\beta_2(aX+b) = \beta_2(X)$. If we change X to $aX+b$, then $E[\{X - E(X)\}^3]$ changes to

$$E[\{aX + b - E(aX+b)\}^3] = a^3 E[\{X - E(X)\}^3],$$

and $\mathrm{Var}(X)$ changes to $a^2\mathrm{Var}(X)$. Hence,

$$\beta_1(aX+b) = \frac{a^3 E[\{X - E(X)\}^3]}{\{a^2\mathrm{Var}(X)\}^{3/2}} = \frac{E[\{X - E(X)\}^3]}{\{\mathrm{Var}(X)\}^{3/2}} = \beta_1(X).$$

Likewise, we have

$$\beta_2(aX+b) = \frac{a^4 E[\{X - E(X)\}^4]}{\{a^2\mathrm{Var}(X)\}^{2}} = \frac{E[\{X - E(X)\}^4]}{\{\mathrm{Var}(X)\}^{2}} = \beta_2(X).$$

b) By definition, $X = \mu + \sigma Z$ for some $Z \sim N(0,1)$, and by Part (a), we have $\beta_1(X) = \beta_1(Z)$ and $\beta_2(X) = \beta_2(Z)$. We know that $E(Z) = 0$ and $E(Z^2) = 1$, and so

$$\beta_1(Z) = E(Z^3) \quad \text{and} \quad \beta_2(Z) = E(Z^4).$$

The mgf of Z is $\psi_Z(\theta) = e^{\theta^2/2}$, and the successive derivatives of $\psi_Z(\theta)$ are

$$\begin{aligned}
&\psi_Z'(\theta) = \theta e^{\theta^2/2} = \theta\psi_Z(\theta),\\
&\psi_Z''(\theta) = \psi_Z(\theta) + \theta\psi_Z'(\theta) = (1+\theta^2)\psi_Z(\theta),\\
&\psi_Z^{(3)}(\theta) = 2\theta\psi_Z(\theta) + (1+\theta^2)\psi_Z'(\theta) = (3\theta + \theta^3)\psi_Z(\theta),\\
&\psi_Z^{(4)}(\theta) = (3+3\theta^2)\psi_Z(\theta) + (3\theta+\theta^3)\psi_Z'(\theta) = (3 + 6\theta^2 + \theta^4)\psi_Z(\theta).
\end{aligned}$$

Hence, $E(Z^3) = \psi_Z^{(3)}(0) = 0$ and $E(Z^4) = \psi_Z^{(4)}(0) = 3$, and so $\beta_1(Z) = 0$ and $\beta_2(Z) = 3$.

(c) Let $Y \sim Ex(1)$. We first note that $Y/\lambda \sim Ex(\lambda)$. This follows from the fact that the mgf of Y is $\psi_Y(\theta) = (1-\theta)^{-1}$ for $\theta < 1$ and so the mgf of Y/λ is

$$E(e^{\theta Y/\lambda}) = \psi_Y(\theta/\lambda) = (1 - \theta/\lambda)^{-1}$$

for $\theta < \lambda$, which coincides with the mgf of $Ex(\lambda)$. Hence, by Part (a), we only have to compute $\beta_1(Y)$ and $\beta_2(Y)$.

Recall the Taylor expansion of $1/(1-x)$:

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots$$

for $|x| < 1$, so that

$$\left(\frac{1}{1-x}\right)^{(k)}\Big|_{x=0} = k!$$

This yields that $E(Y^k) = \psi_Y^{(k)}(0) = k!$ In particular, $\text{Var}(Y) = E(Y^2) - \{E(Y)\}^2 = 2 - 1 = 1$.

Now, we note that

$$\kappa_3 = \mu_3 - 3\nu_2\mu_1 + 2\mu_1^3 \quad \text{and} \quad \kappa_4 = \mu_4 - 4\mu_3\mu_1 + 6\mu_2\mu_1^2 - 3\mu_1^4.$$

Hence, we have

$$\beta_1(Y) = 3! - 3\cdot 2 + 2 = 2 \quad \text{and} \quad \beta_2(Y) = 4! - 4\cdot 3! + 6\cdot 2 - 3 = 24 - 24 + 12 - 3 = 9.$$

4. (a) The mgf of X is

$$\begin{aligned}
\psi_X(\theta) = E(e^{\theta X}) &= \frac{1}{2}\int_0^\infty e^{-(1-\theta)x}dx + \frac{1}{2}\int_{-\infty}^0 e^{(1+\theta)x}dx \\
&= \frac{1}{2}\left[\frac{-e^{-(1-\theta)x}}{1-\theta}\right]_{x=0}^\infty + \frac{1}{2}\left[\frac{e^{(1+\theta)x}}{1+\theta}\right]_{x=-\infty}^0 \\
&= \frac{1}{2}\left(\frac{1}{1-\theta} + \frac{1}{1+\theta}\right) = \frac{1}{1-\theta^2}
\end{aligned}$$

for $|\theta| < 1$.

(b) The mgfs of Y and Z are $\psi_Y(\theta) = \psi_Z(\theta) = 1/(1-\theta)$ for $\theta < 1$. Hence, the mgf of $W = Y - Z$ is

$$\begin{aligned}
\psi_W(\theta) &= E(e^{\theta(Y-Z)}) = E(e^{\theta Y}e^{-\theta Z}) \\
&= E(e^{\theta Y})E(e^{-\theta Z}) = \psi_Y(\theta)\psi_Z(-\theta) \\
&= \frac{1}{1-\theta}\cdot\frac{1}{1+\theta} = \frac{1}{1-\theta^2}
\end{aligned}$$

for $|\theta| < 1$, which coincides with the mgf of X.

5. (a) The mean of Y_n is

$$E(Y_n) = \frac{2}{n(n+1)}\sum_{j=1}^n E(jX_j) = \frac{2}{n(n+1)}\sum_{j=1}^n jE(X_j) = \frac{2\mu}{n(n+1)}\underbrace{\sum_{j=1}^n j}_{=n(n+1)/2} = \mu.$$

Next, because of independent of $X_1, \ldots, X_n$, we have

$$\text{Var}(Y_n) = \frac{4}{n^2(n+1)^2}\sum_{j=1}^n \text{Var}(jX_j) = \frac{4\sigma^2}{n^2(n+1)^2}\underbrace{\sum_{j=1}^n j^2}_{=n(n+1)(2n+1)/6} = \frac{2\sigma^2(2n+1)}{3n(n+1)}.$$

(b) By Chebyshev's inequality, for any $\varepsilon > 0$,

$$P(|Y_n - \mu| > \varepsilon) \leq \frac{\mathrm{Var}(Y_n)}{\varepsilon^2}.$$

Since $\lim_{n\to\infty} \mathrm{Var}(Y_n) = 0$, we conclude that $Y_n \xrightarrow{P} \mu$.