11/27/18

STSCI 5080 Homework 6 (optional)

- Due is 12/4 (Tu) in class, but you can also submit HW 6 on 11/29 (Th) in class.
- Write your name and NetID at the top of the first page, along with the assignment number.
- Use only the one side of the paper. Attach your pages with a staple at the top left corner.
- This is an extra credit work. You can earn extra 2% from this homework.
- Please complete the course evaluation when you have time. Constructive comments welcome!

Problems

1. Let $X \sim Bin(n, p)$. We know that the MLE is $\widehat{p} = X/n$ and $\sqrt{n}(\widehat{p} - p) \xrightarrow{d} N(0, p(1-p))$.

   (a) (5 points) The arcsine function $\arcsin y$ is the inverse function of the sine function $\sin x$ on $-\pi/2 \le x \le \pi/2$, i.e., $\sin(\arcsin y) = y$ for $|y| \le 1$. Verify that the derivative of the arsine function is
   $$(\arcsin y)' = \frac{1}{\sqrt{1-y^2}}, \ |y| < 1.$$
   (Hint). Differentiate both sides of $\sin(\arcsin y) = y$ w.r.t. y.

   (b) (5 points) Consider the function
   $$g(y) = 2\arcsin(\sqrt{y}).$$
   Verify that $g(y)$ is the variance stabilizing transformation for $\widehat{p}$.

2. Let
   $$X_1, \dots, X_n \sim Ex(\lambda) \text{ i.i.d.}$$
   for some $\lambda > 0$. We know that the MLE is $\widehat{\lambda} = 1/\overline{X}$ and $\sqrt{n}(\widehat{\lambda} - \lambda) \xrightarrow{d} N(0, \lambda^2)$ as $n \to \infty$. Consider the following testing problem:
   $$H_0 : \lambda = 0.5 \quad \text{vs.} \quad H_1 : \lambda \ne 0.5.$$

   (a) (5 points) Find a test based on the MLE that has asymptotic level 5%. You don't need any derivation in this question and so you only have to state your test.

   (b) (5 points) Justify that your test in Part (a) has asymptotic level 5%.

Solutions

1. (a) Differentiating both sides of $\sin(\arcsin y) = y$ w.r.t. y, we have

$$(\arcsin y)' \cos(\arcsin y) = 1$$

by the chain rule. Since $\cos(\arcsin y) = \sqrt{1 - \sin^2(\arcsin y)} = \sqrt{1-y^2}$, we have

$$(\arcsin y)' = \frac{1}{\sqrt{1-y^2}}.$$

(b) By the chain rule, we have

$$g'(y) = 2(\sqrt{y})' \frac{1}{\sqrt{1-(\sqrt{y})^2}} = \frac{1}{\sqrt{y}\sqrt{1-y}}.$$

Hence,

$$\sqrt{n}\{g(\widehat{p}) - g(p)\} \xrightarrow{d} N(0, \{g'(p)\}^2 p(1-p)) = N(0,1)$$

by the delta method, which shows that $g(p)$ is the variance stabilizing transformation.

2. (a) A desired test is given by

$$\left| \frac{\sqrt{n}(\widehat{\lambda} - 0.5)}{0.5} \right| > 1.96 \Rightarrow \text{reject } H_0.$$

(b) If $\lambda = 0.5$, then

$$\frac{\sqrt{n}(\widehat{\lambda} - 0.5)}{0.5} \xrightarrow{d} Z \sim N(0,1),$$

so that

$$P_{\lambda=0.5} \left\{ \left| \frac{\sqrt{n}(\widehat{\lambda} - 0.5)}{0.5} \right| > 1.96 \right\} \approx P(|Z| > 1.96) = 0.05.$$

This implies that the test has asymptotic level 5%.