# STSCI 5080
# Probability Models and Inference

## Lecture 4: Discrete Random Variables

September 4, 2018

# Notation

For a random variable $X$ (a function on $\Omega$ taking values in the real numbers) and a subset $S$ of the real numbers (e.g. $S = \{0, 1, 2\}$ or $S = [0, 1]$), we use the shorthand notation

$$\{X \in S\} = \{\omega \mid X(\omega) \in S\}.$$

In addition, we also write

$$P(X \in S) = P(\{X \in S\}).$$

# Discrete random variable

**Definition (Discrete random variable)**

A random variable $X$ is discrete if $X$ takes values in a finite or countably infinite set.

# Probability mass function

**Definition**

For a discrete random variable $X$, the probability mass function (pmf) $p(x)$ is a function defined by

$$p(x) = P(X = x)$$

for any real number $x$.

# Properties of pmf

- If $X$ that takes values in $\{x_1, x_2, \ldots\}$ (called the support of $X$), then

$$p(x) = 0 \ \text{ for any } x \notin \{x_1, x_2, \ldots\}.$$

- The pmf $p(x)$ satisfies that $p(x) \geq 0$ for any $x$ and $\sum_x p(x) = 1$ (why?). In addition,

$$P(X \in S) = \sum_{x \in S} p(x).$$

# Example 4.1

## Example

Suppose that we toss a coin three times and all the outcomes occur equally likely. Consider

$$X(\omega) = \text{total number of heads in } \omega.$$

Then the support is $\{0, 1, 2, 3\}$ and the pmf is

$$p(x) = P(X = x) = \begin{cases} \frac{1}{8} & x = 0 \\ \frac{3}{8} & x = 1 \\ \frac{3}{8} & x = 2 \\ \frac{1}{8} & x = 3 \\ 0 & x \notin \{0, 1, 2, 3\} \end{cases}.$$

# Cumulative distribution function

## Definition

For a discrete random variable $X$ with pmf $p(x)$, the cumulative distribution function (cdf) $F(x)$ is defined as

$$F(x) = P(X \leq x) = \sum_{y \leq x} p(y)$$

for any real number $x$.

What is the cdf of the random variable in Example 4.1?

# Example 4.2

## Example (CDF as a measure of risk)

Suppose you consider to invest on Portfolio $\alpha$ or $\beta$.

- $X$: gain of Portfolio $\alpha$ with cdf $F_X$;
- $Y$: gain of Portfolio $\beta$ with cdf $F_Y$.

Now, we know that

$$F_X(-100) = 0.05 \quad \text{and} \quad F_Y(-1000) = 0.05.$$

Which portfolio is more risky?

# Bernoulli random variable

- If a random variable $X$ takes values in $0$ or $1$ with

$$P(X=1)=p \quad \text{and} \quad P(X=0)=1-p,$$

  then $X$ is called a Bernoulli random variable with success probability $p$.

- What is the pmf?

$$p(x)=\begin{cases} p & x=1 \\ 1-p & x=0 \\ 0 & x\notin\{0,1\} \end{cases} = \begin{cases} p^x(1-p)^{1-x} & x=0,1 \\ 0 & x\notin\{0,1\} \end{cases}.$$

- What is the cdf?

# Independent Bernoulli trials

### Definition

Let $X_1, \ldots, X_n$ be Bernoulli random variables with the same success probability $p$ that are independent in the sense that $n$ events

$$\{X_1 = x_1\}, \ldots, \{X_n = x_n\}$$

are independent for any $x_1, \ldots, x_n \in \{0, 1\}$. In this case, we call $X_1, \ldots, X_n$ independent Bernoulli trials with success probability $p$.

# Binomial random variable

## Definition

Let $X_1, \dots, X_n$ be independent Bernoulli trials with success probability $p$. Then the random variable

$$Y = X_1 + \cdots + X_n$$

is called a binomial random variable with parameters $n$ and $p$.

“$Y$ follows the binomial distribution with parameters $n$ and $p$”

$$Y \sim Bin(n, p).$$

# Binomial coefficients

For a positive integer $n$ and $k = 0, 1, \ldots, n$,

$$\binom{n}{k} = \text{number of } k\text{-element subsets of } \{1, \ldots, n\}.$$

For example,

$$\binom{n}{0} = 1, \ \binom{n}{1} = n, \ \binom{n}{n} = 1.$$

In general,

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k(k-1)\cdots 1} = \frac{n!}{(n-k)!k!},$$

where

$$n! = n(n-1)\cdots 1.$$

What are $\binom{n}{2}$ and $\binom{n}{3}$?

# PMF of $Bin(n,p)$

**Theorem**

*The pmf of $Y \sim Bin(n,p)$ is*

$$p(k) = P(Y = k) = \binom{n}{k} p^k (1-p)^{n-k},\ k = 0, 1, \ldots, n.$$

# Poisson random variable

## Definition

Let $\lambda > 0$. $X$ is a Possion random variable with parameter $\lambda$ if its takes values in $\{0, 1, 2, \dots\}$ and its pmf is

$$p(k) = \frac{\lambda^k}{k!} e^{-\lambda}, \; k = 0, 1, 2, \dots .$$

"$X$ follows the Poisson distribution with parameter $\lambda$"

$$X \sim Po(\lambda).$$