ORIE 4630: Spring Term 2019
Homework #10
Due: Not to be submitted

It is realistic to model forward rates by using a function $r(t)$ that varies continuously with time t. If $P(T)$ is the price of a zero-coupon bond having maturity T and par value PAR, then the yield to maturity, under continuously compounded interest, is y_T that satisfies

$$P(T) = \frac{\text{PAR}}{\exp(Ty_T)};$$

thus,

$$y_T = -\frac{1}{T}\log\left\{\frac{P(T)}{\text{PAR}}\right\}.$$

If the yield to maturity is related to the forward rate function $r(t)$ by the equation

$$y_T = \frac{1}{T}\int_0^T r(t)\,dt,$$

then

$$P(T) = \text{PAR} \times \exp\left\{-\int_0^T r(t)\,dt\right\},$$

and

$$r(T) = -\frac{d}{dT}\log P(T).$$

Suppose that the forward rate function $r(t)$ is modeled by a kth-degree polynomial:

$$r(t;\theta) = \theta_0 + \theta_1 t + \cdots + \theta_k t^k,$$

where $\theta_0, \theta_1, \ldots, \theta_k$ are unknown parameters to be estimated from data and $\theta = (\theta_0, \ldots, \theta_k)$. The price $P(T;\theta)$ that results from this parametric form is

$$P(T;\theta) = \text{PAR} \times \exp\left\{-\int_0^T r(t;\theta)\,dt\right\} = \text{PAR} \times \exp\left\{-\left(\theta_0 T + \theta_1\frac{T^2}{2} + \cdots + \theta_k\frac{T^{k+1}}{k+1}\right)\right\},$$

and the yield to maturity y_T is

$$y_T(\theta) = \theta_0 + \theta_1\frac{T}{2} + \cdots + \theta_k\frac{T^k}{k+1}.$$

Suppose that, on a given day, maturities and prices are collected for n zero-coupon bonds; for simplicity, it is assumed that the n zero-coupon bonds have the same par value. Let $T_1, \ldots, T_n$ be the maturities of the bonds, and let the prices be $P_1, \ldots, P_n$; thus, the ith data point is (T_i, P_i). If the data $(T_1, P_1), \ldots, (T_n, P_n)$ yields estimates $\hat{\theta}_0, \hat{\theta}_1, \ldots, \hat{\theta}_k$ of the unknown parameters $\theta_0, \theta_1, \ldots, \theta_k$, and if $\hat{\theta} = (\hat{\theta}_0, \hat{\theta}_1, \ldots, \hat{\theta}_k)$, then the estimate of the price of the ith bond is

$$\hat{P}_i = P(T_i;\hat{\theta}) = \text{PAR} \times \exp\left\{-\left(\hat{\theta}_0 T_i + \hat{\theta}_1\frac{T_i^2}{2} + \cdots + \hat{\theta}_k\frac{T_i^{k+1}}{k+1}\right)\right\}.$$

In terms of this notation, the estimates $\hat{\theta}_0, \hat{\theta}_1, \ldots, \hat{\theta}_k$ are chosen to minimize

$$\sum_{i=1}^{n} \{P_i - \hat{P}_i\}^2 = \sum_{i=1}^{n} \{P_i - P(T_i; \hat{\theta})\}^2.$$

The file `zero_coupon.csv` contains the maturities and prices of zero-coupon bonds having par value PAR= 100. To read the file, print the first six lines of the file, and plot the prices against the maturities, run the following lines:

```
1 zcbonds=read.csv("zero_coupon.csv")
2 head(zcbonds)
3 T=zcbonds$T
4 price=zcbonds$price
5 price=price[order(T)]
6 T=T[order(T)]
7 n=length(T)
8 par=100
9 plot(T,price,type="p",pch=10,cex.lab=1.5,cex.axis=1.5)
```

In line 3, the maturities are stored in the object `T`, and in line 4, the prices are stored in the object `price`. The maturities and prices are re-arranged in lines 5 and 6 so that the maturities are in increasing order. The number of bonds is stored in the variable `n` in line 7, and the par value of the bonds is declared and stored in the variable `par` in line 8. The scatter plot of the pairs (T_i, P_i) $(i = 1, \ldots, n)$ is created in line 9.

Suppose that the forward rate function $r(t; \theta)$ is linear, i.e., $r(t) = \theta_0 + \theta_1 t$. The following lines concern the estimation of θ_0 and θ_1 based on the data stored in `T` and `price`:

```
10 fitlinear=nls(price~par*exp(-theta0*T-theta1*T^2/2),data=zcbonds,
11    start=list(theta0=0,theta1=0),nls.control(tol=1e-6))
12 summary(fitlinear)
13 residuals_linear=summary(fitlinear)$residuals
14 plot(T,residuals_linear,type="p",pch=10,cex.lab=1.5,cex.axis=1.5)
15 coef_linear=summary(fitlinear)$coef[,1]
```

The function `nls()` ("non-linear least squares") is used to perform the estimation in lines 10 and 11, and the results of the estimation are stored in the object `fitlinear`; contents of `fitlinear` are displayed in line 12. In addition to displaying the estimates $\hat{\theta}_0$ and $\hat{\theta}_1$, the "residual standard error," i.e., the value of $\sqrt{\sum_{i=1}^{n} (P_i - \hat{P}_i)^2/(n-2)}$, is given. The residual standard error can be used as a goodness-of-fit statistic. The residuals $P_i - \hat{P}_i$ $(i = 1, \ldots, n)$ are stored in the object `residuals_linear` in line 13, and a residual plot of $P_i - \hat{P}_i$ against T_i is created in line 14. Finally, the estimates $\hat{\theta}_0$ and $\hat{\theta}_1$ are stored in the object `coef_linear` in line 15.

Similar results for a quadratic rate function $r(t; \theta) = \theta_0 + \theta_1 t + \theta_2 t^2$ are obtained by running the following lines:

```
16 fitquad=nls(price~par*exp(-theta0*T-theta1*T^2/2-theta2*T^3/3),
17    data=zcbonds,
18    start=list(theta0=coef_linear[1],theta1=coef_linear[2],theta2=0),
19    nls.control(tol=1e-6))
20 summary(fitquad)
21 residuals_quad=summary(fitquad)$residuals
22 plot(T,residuals_quad,type="p",pch=10,cex.lab=1.5,cex.axis=1.5)
23 coef_quad=summary(fitquad)$coef[,1]
```

The following lines are for fitting a cubic rate function $r(t;\theta) = \theta_0 + \theta_1 t + \theta_2 t^2 + \theta_3 t^3$:

```
fitcubic=nls(price~par*exp(-theta0*T-theta1*T^2/2-theta2*T^3/3-theta3*T^4/4),
   data=zcbonds,
   start=list(theta0=coef_quad[1],theta1=coef_quad[2],theta2=coef_quad[3],theta3=0),
   nls.control(tol=1e-6))
summary(fitcubic)
residuals_cubic=summary(fitcubic)$residuals
plot(T,residuals_cubic,type="p",pch=10,cex.lab=1.5,cex.axis=1.5)
coef_cubic=summary(fitcubic)$coef[,1]
```

Now suppose that the forward rate function $r(t)$ is modeled by a quadratic spline:

$$r(t;\theta) = \theta_0 + \theta_1 t + \theta_2 t^2 + \theta_3 (t - t^*)_+^2 = \begin{cases} \theta_0 + \theta_1 t + \theta_2 t^2, & \text{if } t \le t^*, \\ \theta_0 + \theta_1 t + \theta_2 t^2 + \theta_3 (t - t^*)^2, & \text{if } t > t^*, \end{cases}$$

where t^* is a constant value called a knot. The spline consists of two quadratics functions: one for values of t less than or equal to the knot t^*; and the other for values of t greater than t^*. The two quadratic functions take the same value at $t = t^*$, i.e., there is no jump in the value of $r(t)$ at $t = t^*$; furthermore, the derivatives of the two quadratic functions are the same at $t = t^*$. The price $P(T;\theta)$ that results from this parametric form is

$$P(T;\theta) = \mathrm{PAR} \times \exp\left\{ -\left(\theta_0 T + \theta_1 \frac{T^2}{2} + \theta_2 \frac{T^3}{3} + \theta_3 \frac{(T - t^*)_+^3}{3} \right) \right\},$$

and the yield to maturity $y_T(\theta)$ is

$$y_T(\theta) = \theta_0 + \theta_1 \frac{T}{2} + \theta_2 \frac{T^2}{3} + \theta_3 \frac{(T - t^*)_+^3}{3T}.$$

The quadratic spline rate function $r(t;\theta) = \theta_0 + \theta_1 t + \theta_2 t^2 + \theta_3 (t - t^*)_+^2$ for knot $t^* = 13.58$ is fit by running the following lines:

```
knot=13.58
fitspline=nls(price~par*exp(-theta0*T-theta1*T^2/2-theta2*T^3/3-
   (T>knot)*theta3*(T-knot)^3/3),data=zcbonds,
   start=list(theta0=coef_cubic[1],theta1=coef_cubic[2],
   theta2=coef_cubic[3],theta3=coef_cubic[4]),
   nls.control(tol=1e-6))
summary(fitspline)
residuals_spline=summary(fitspline)$residuals
plot(T,residuals_spline,type="p",pch=10,cex.lab=1.5,cex.axis=1.5)
coef_spline=summary(fitspline)$coef[,1]
```

Finally, consider the Nelson-Siegel model having forward rate function

$$r(t;\theta) = \theta_0 + (\theta_1 + \theta_2 t)\exp(-\theta_3 t);$$

this forward rate function produces produces the price function

$$P(T;\theta) = \mathrm{PAR} \times \exp\left[-\left\{ \theta_0 T + \left(\theta_1 + \frac{\theta_2}{\theta_3} \right) \frac{1 - \exp(-\theta_3 T)}{\theta_3} - \frac{\theta_2}{\theta_3} T \exp(-\theta_3 T) \right\} \right],$$

and the yield to maturity

$$y_T(\theta) = \theta_0 + \left(\theta_1 + \frac{\theta_2}{\theta_3} \right) \frac{1 - \exp(-\theta_3 T)}{\theta_3 T} - \frac{\theta_2}{\theta_3} \exp(-\theta_3 T).$$

The Nelson-Siegel model is fit by running the following lines:

```
nsprice=function(theta){par*exp(-theta[1]*T-
   (theta[2]/theta[4])*(1-exp(-theta[4]*T))-
   (theta[3]/(theta[4])^2)*(1-exp(-theta[4]*T))+
   (theta[3]/theta[4])*T*exp(-theta[4]*T))}
fitns=optim(c(0.01,0.01,-0.001,-0.01),
   fn=function(theta){pricehat=nsprice(theta)
   sum((price-pricehat)^2)},control=list(reltol=1e-10))
pricehat=nsprice(fitns$par)
residuals_ns=price-pricehat
plot(T,residuals_ns,type="p",pch=10,cex.lab=1.5,cex.axis=1.5)
coef_ns=fitns$par
sqrt(sum((price-pricehat)^2)/(n-length(coef_ns)))
```

Since

$$y_T = -\frac{1}{T}\log\left\{\frac{P(T)}{\mathrm{PAR}}\right\},$$

the ith observation (T_i, P_i) gives the empirical yield to maturity

$$\hat{y}_i = -\frac{1}{T_i}\log\left\{\frac{P_i}{\mathrm{PAR}}\right\} \quad (i = 1, \ldots, n).$$

The following lines plot the fitted yield curve $y_T(\hat{\theta})$ for each of the linear, quadratic, cubic, quadratic spline, and Nelson-Siegel forward rate functions $r(t;\theta)$ and superimposes the fitted yield curves on a scatterplot of the empirical yields to maturity $\hat{y}_i$ $(i = 1, \ldots, n)$:

```
empirical_yield=log(par/price)/T
t=seq(from=min(T),to=max(T),length=10000)
yield_linear=coef_linear[1]+coef_linear[2]*t/2
yield_quad=coef_quad[1]+coef_quad[2]*t/2+coef_quad[3]*t^2/3
yield_cubic=coef_cubic[1]+coef_cubic[2]*t/2+coef_cubic[3]*t^2/3+coef_cubic[4]*t^3/4
yield_spline=coef_spline[1]+coef_spline[2]*t/2+coef_spline[3]*t^2/3+
  (t>knot)*coef_spline[4]*(t-knot)^3/(3*t)
yield_ns=coef_ns[1]+(coef_ns[2]+
   coef_ns[3]/coef_ns[4])*(1-exp(-coef_ns[4]*t))/(coef_ns[4]*t)-
   (coef_ns[3]/coef_ns[4])*exp(-coef_ns[4]*t)
plot(T,empirical_yield,pch=10,lwd=1.5,cex.lab=1.5,cex.axis=1.5,ylab="yield")
lines(t,yield_linear,lty=1,lwd=2,col="red")
lines(t,yield_quad,type="l",lty=1,lwd=2,col="blue")
lines(t,yield_cubic,type="l",lty=1,lwd=2,col="green")
lines(t,yield_spline,type="l",lty=1,lwd=2,col="brown")
lines(t,yield_ns,type="l",lty=1,lwd=2,col="grey")
legend("bottomright", c("empirical","linear","quadratic","cubic",
             "spline","N-S"),bty="n",lwd=2,pt.lwd=1,
   pch=c(10,NA,NA,NA,NA,NA),lty=c(NA,1,1,1,1,1),
   col=c("black","red","blue","green","brown","grey"))
```

Since

$$r(T) = -\frac{d}{dT}\log P(T),$$

the ith observation (T_i, P_i) gives the empirical forward rate

$$\hat{r}_i = -\frac{\log(P_i) - \log(P_{i-1})}{T_i - T_{i-1}} \quad (i = 2, \ldots, n);$$

note that for this definition to make sense, it is imperative that the maturities $T_1, \dots, T_n$ are arranged in increasing order. The following lines plot the fitted forward rate curve $r(t;\hat{\theta})$ for each of the linear, quadratic, cubic, quadratic spline, and Nelson-Siegel forward rate functions $r(t;\theta)$ and superimposes the fitted forward rate curves on a scatterplot of the empirical forward rates $\hat{r}_i$ $(i = 1, \dots, n)$:

```
empirical_rate=diff(-log(price/par))/diff(T)
rate_linear=coef_linear[1]+coef_linear[2]*t
rate_quad=coef_quad[1]+coef_quad[2]*t+coef_quad[3]*t^2
rate_cubic=coef_cubic[1]+coef_cubic[2]*t+coef_cubic[3]*t^2+coef_cubic[4]*t^3
rate_spline=coef_spline[1]+coef_spline[2]*t+coef_spline[3]*t^2+
     (t>knot)*coef_spline[4]*(t-knot)^2
plot(T[-1],empirical_rate,pch=10,lwd=1.5,cex.lab=1.5,cex.axis=1.5,
     ylab="forward rate",xlab="T")
lines(t,rate_linear,lty=1,lwd=2,col="red")
lines(t,rate_quad,type="l",lty=1,lwd=2,col="blue")
lines(t,rate_cubic,type="l",lty=1,lwd=2,col="green")
lines(t,rate_spline,type="l",lty=1,lwd=2,col="brown")
legend("topleft", c("empirical","linear","quadratic","cubic",
            "spline","N-S"),bty="n",lwd=2,pt.lwd=1,
   pch=c(10,NA,NA,NA,NA,NA),lty=c(NA,1,1,1,1,1),
   col=c("black","red","blue","green","brown","grey"))
```

Consider a position of size S in an asset at the outset of some investment period, and suppose that the goal is to estimate the value at risk and expected shortfall of the position over the period to time horizon T. Recall that the value at risk $\mathrm{VaR}(\alpha)$ is the upper α-quantile of the loss distribution over the period under consideration. If L is the loss over the period, then $L = -R$, where R is the revenue over the period, so $\mathrm{VaR}(\alpha)$ is the (lower) α-quantile of the distribution of R. Furthermore, if Ret is the net return over the holding period, then $R = S \times Ret$; it follows that $\mathrm{VaR}(\alpha) = -S \times q(\alpha)$, where $q(\alpha)$ is the (lower) α-quantile of the distribution of Ret. The expected shortfall $\mathrm{ES}(\alpha)$ is the conditional expectation of L given that L is at least $\mathrm{VaR}(\alpha)$, i.e., $\mathrm{ES}(\alpha) = E\{L | L \geq \mathrm{VaR}(\alpha)\}$. In terms of the return Ret, we have $\mathrm{ES}(\alpha) = -S \times E\{Ret | Ret \leq q(\alpha)\}$.

First, consider nonparametric estimation of $\mathrm{VaR}(\alpha)$ and $\mathrm{ES}(\alpha)$. Based on historical returns $Ret_1, \dots, Ret_n$, the value at risk $\mathrm{VaR}(\alpha)$ is estimated by $\widehat{\mathrm{VaR}}(\alpha) = -S \times \hat{q}(\alpha)$, where $\hat{q}(\alpha)$ is the sample α-quantile of $Ret_1, \dots, Ret_n$. The expected shortfall $\mathrm{ES}(\alpha)$ is estimated by

$$\widehat{\mathrm{ES}}(\alpha) = -S \times \frac{\sum_{i=1}^{n} Ret_i \cdot I\{Ret_i \leq \hat{q}(\alpha)\}}{\sum_{i=1}^{n} I\{Ret_i \leq \hat{q}(\alpha)\}},$$

where the indicator $I\{Ret_i \leq \hat{q}(\alpha)\}$ is defined by

$$I\{Ret_i \leq \hat{q}(\alpha)\} = \begin{cases} 1, & \text{if } Ret_i \leq \hat{q}(\alpha), \\ 0, & \text{if } Ret_i > \hat{q}(\alpha). \end{cases}$$

Next consider parametric estimation of $\mathrm{VaR}(\alpha)$ and $\mathrm{ES}(\alpha)$ under the assumption that Ret is normally distributed with mean μ and standard deviation σ, i.e., $Ret \sim N(\mu, \sigma^2)$. Then $q(\alpha) = \mu + \sigma\Phi^{-1}(\alpha)$, where $\Phi(\cdot)$ is the c.d.f. of the standard normal distribution, so it follows that $\mathrm{VaR}(\alpha) = -S \times \{\mu + \sigma\Phi^{-1}(\alpha)\}$, which is estimated from historical data $Ret_1, \dots, Ret_n$ by $\widehat{\mathrm{VaR}}(\alpha) = -S \times \{\hat{\mu} + \hat{\sigma}\Phi^{-1}(\alpha)\}$, where $\hat{\mu}$ and $\hat{\sigma}$ are the sample mean and sample standard deviation of $Ret_1, \dots, Ret_n$, respectively. Under the normal model for Ret, the expected shortfall is $\text{ES}(\alpha) = -S \times \big(\mu - \sigma\big[\phi\{\Phi^{-1}(\alpha)\}\big]/\alpha\big)$, where $\phi(\cdot)$ is the p.d.f. of the standard normal distribution; the estimate of $\text{ES}(\alpha)$ is $\widehat{\text{ES}}(\alpha) = -S \times \big(\hat{\mu} - \hat{\sigma}\big[\phi\{\Phi^{-1}(\alpha)\}\big]/\alpha\big)$.

Suppose the goal is to estimate the value at risk and expected shortfall for a $100 position in Apple (AAPL) over a single trading day. Recall that the file `returns.csv` contains historical daily log gross returns for various stocks and indexes from Jan 4, 2006 to Aug 18, 2017. Thus, there are 2927 days of log gross returns.To read and convert the log gross returns to net returns, run the following lines:

```
90 Returns=read.csv("returns.csv")
91 names(Returns)
92 Ret=Returns[ , 2:35]
93 class(Ret)
94 netRet=exp(Ret)-1
95 head(netRet)
96 AAPL_netRet=netRet$AAPL
97 n=length(AAPL_netRet)
98 AAPL_netRet=AAPL_netRet[(n-999):n]
```

In line 98, the most recent 1000 daily Apple (AAPL) net returns are stored in the object `AAPL_netRet`.

Based on the 1000 historical daily net returns, nonparametric estimates of VaR(0.05) and ES(0.05) over a one-day period for a $100 position in Apple (AAPL) can be obtained by running the following lines:

```
99 S=100
100 alpha=0.05
101 netRet_quantile=as.numeric(quantile(AAPL_netRet,alpha))
102 VaR=-S*netRet_quantile
103 ind=(AAPL_netRet<=netRet_quantile)
104 ES=-S*sum(AAPL_netRet*ind)/sum(ind)
105 VaR
106 ES
```

Based on the 1000 historical daily net returns, parametric estimates of VaR(0.05) and ES(0.05) over a one-day period for a $100 position in Apple (AAPL) can be obtained under the assumption that daily returns are normally distributed by running the following lines:

```
107 S=100
108 alpha=0.05
109 VaR=-S*(mean(AAPL_netRet)+qnorm(alpha)*sd(AAPL_netRet))
110 ES=-S*(mean(AAPL_netRet)-dnorm(qnorm(alpha))*sd(AAPL_netRet)/alpha)
111 VaR
112 ES
```

Approximate confidence intervals having nominal coverage level $100(1-\gamma)\%$ can be obtained for $\text{VaR}(\alpha)$ and $\text{ES}(\alpha)$ by employing the bootstrap method. Both nonparametric and parametric confidence intervals are available. In each case, B bootstrap samples are generated from the estimated distribution for Ret; let the bth bootstrap sample be denoted by $Ret_1^{*b}, \ldots, Ret_n^{*b}$ $(b = 1, \ldots, B)$. In the nonparametric case, $Ret_1^{*b}, \ldots, Ret_n^{*b}$ is a sample drawn at random with replacement from the original data values $Ret_1, \ldots, Ret_n$. In the parametric case, $Ret_1^{*b}, \ldots, Ret_n^{*b}$ is a sample of size n drawn from the $N(\hat{\mu}, \hat{\sigma}^2)$ distribution, where $\hat{\mu}$ and $\hat{\sigma}$ are the sample mean and sample standard deviation of $Ret_1, \ldots, Ret_n$, respectively. Let $\widehat{\text{VaR}}(\alpha)^{*b}$ and $\widehat{\text{ES}}(\alpha)^{*b}$ be the estimates of $\text{VaR}(\alpha)$ and $\text{ES}(\alpha)$ based on the bth bootstrap sample $Ret_1^{*b}, \ldots, Ret_n^{*b}$ ($b = 1, \ldots, B$). The $100(1-\gamma)\%$ bootstrap confidence interval for $\mathrm{VaR}(\alpha)$ has endpoints that are the lower and upper $\gamma/2$-quantiles of $\widehat{\mathrm{VaR}}(\alpha)^{*1}, \ldots, \widehat{\mathrm{VaR}}(\alpha)^{*B}$; similarly, the the $100(1-\gamma)\%$ bootstrap confidence interval for $\mathrm{ES}(\alpha)$ has endpoints that are the lower and upper $\gamma/2$-quantiles of $\widehat{\mathrm{ES}}(\alpha)^{*1}, \ldots, \widehat{\mathrm{ES}}(\alpha)^{*B}$.

The 95% nonparametric bootstrap confidence intervals based on 100000 bootstrap samples for VaR(0.05) and ES(0.05) over a one-day period for a $100 position in Apple (AAPL) can be obtained by running the following lines:

```
B=100000
VaR_bootstrap=(1:B)
ES_bootstrap=(1:B)
seed=5498
set.seed(seed)
for(i in 1:B){
   bootstrap_sample=AAPL_netRet[sample(1:1000,replace=TRUE)]
   bootstrap_quantile=quantile(bootstrap_sample,alpha)
   VaR_bootstrap[i]=-S*as.numeric(bootstrap_quantile)
   ind_bootstrap=(bootstrap_sample<=bootstrap_quantile)
   ES_bootstrap[i]=-S*sum(bootstrap_sample*ind_bootstrap)/sum(ind_bootstrap)
}
VaR_confidence_interval=quantile(VaR_bootstrap,probs=c(0.025,0.975))
ES_confidence_interval=quantile(ES_bootstrap,probs=c(0.025,0.975))
VaR_confidence_interval
ES_confidence_interval
```

The 95% parametric bootstrap confidence intervals based on 100000 bootstrap samples for VaR(0.05) and ES(0.05) over a one-day period for a $100 position in Apple (AAPL) under the assumption that daily returns are normally distributed can be obtained by running the following lines:

```
B=100000
VaR_bootstrap=(1:B)
ES_bootstrap=(1:B)
seed=5498
set.seed(seed)
n=length(AAPL_netRet)
for(i in 1:B){
   bootstrap_sample=rnorm(n,mean=mean(AAPL_netRet),sd=sd(AAPL_netRet))
   VaR_bootstrap[i]=-S*(mean(bootstrap_sample)+
   qnorm(alpha)*sd(bootstrap_sample))
   ES_bootstrap[i]=-S*(mean(bootstrap_sample)-dnorm(qnorm(alpha))*
   sd(bootstrap_sample)/alpha)
}
VaR_confidence_interval=quantile(VaR_bootstrap,probs=c(0.025,0.975))
ES_confidence_interval=quantile(ES_bootstrap,probs=c(0.025,0.975))
VaR_confidence_interval
ES_confidence_interval
```

Questions:

1. [70 points]

The file `strips_dec95.csv` contains the maturities in years and prices on December 31, 1995 of strip bonds having par value PAR= 100. A strip bonds is a type of zero-coupon bond created by stripping a coupon bond of its coupons, which are sold separately from the bond; these data are from the U.S. Treasury.

i) Read the data from the `strips_dec95.csv` file into `R` by adapting and running lines 1 to 9. How many bond prices are included in the dataset? Submit the plot of the prices versus maturity.

ii) Fit the linear forward rate function to the data by adapting and running lines 10 to 15. Submit the plot of the residuals. Report the smallest and largest residuals. Report the estimates of the coefficients θ_0 and θ_1. Report the residual standard error.

iii) Fit the quadratic forward rate function to the data by adapting and running lines 16 to 23. Submit the plot of the residuals. Report the smallest and largest residuals. Report the estimates of the coefficients θ_0, θ_1, and θ_2. Report the residual standard error.

iv) Fit the cubic forward rate function to the data by adapting and running lines 24 to 31. Submit the plot of the residuals. Report the smallest and largest residuals. Report the estimates of the coefficients θ_0, θ_1, θ_2, and θ_3. Report the residual standard error.

v) Fit the quadratic spline forward rate function to the data by adapting and running lines 32 to 41. Use knot 15.8740. Submit the plot of the residuals. Report the smallest and largest residuals. Report the estimates of the coefficients θ_0, θ_1, θ_2, and θ_3. Report the residual standard error.

vi) Fit the quadratic spline forward rate function to the data by adapting and running lines 32 to 41. Use knot 3.6219. Submit the plot of the residuals. Report the smallest and largest residuals. Report the residual standard error. Which knot gives better fit to the data: 15.8740 or 3.6219?

vii) Fit the Nelson-Siegel forward rate function to the data by adapting and running lines 42 to 53. Submit the plot of the residuals. Report the smallest and largest residuals. Report the estimates of the coefficients θ_0, θ_1, θ_2, and θ_3. Report the residual standard error.

viii) Which of the five forms for the forward rate function (linear, quadratic, cubic, spline with knot 15.8740, Nelson-Siegel) gives the best fit to the data? Justify your answer briefly.

ix) By adapting and running lines 54 to 73, obtain two scatterplots of the empirical yields to maturity with five fitted yield to maturity curves superimposed, where the five fitted yield to maturity curves are those based on the linear, quadratic, cubic, quadratic spline, and Nelson-Siegel forward rate functions. In the first plot, use knot 15.8740; in the second plot, use knot 3.6219.

x) Based on the five fitted yield to maturity curves, what are the prices of a zero-coupon bond having par value 1000 and maturity 2 years? What are the prices of a zero-coupon bond having par value 1000 and maturity 23 years? For the quadratic spline method, use knot 15.8740.

xi) By adapting and running lines 74 to 89, obtain two scatterplots of the empirical forward rates with five fitted forward rate curves superimposed, where the five fitted forward rate curves are those based on the linear, quadratic, cubic, quadratic spline, and Nelson-Siegel forward rate functions. In the first plot, use knot 15.8740; in the second plot, use knot 3.6219.

2. [30 points]

Suppose the goal is to estimate the value at risk and expected shortfall for a \$1000 position in Disney (DIS) over a single trading day. Recall that the file `returns.csv` contains historical daily log gross returns for various stocks and indexes from Jan 4, 2006 to Aug 18, 2017. By adapting and running lines 90 to 98, to read and convert the log gross returns to net returns, and store the most recent 500 daily Disney (DIS) net returns in the object `DIS_netRet`.

i) Submit the first six lines of `DIS_netRet`.

ii) Adapt and run lines 99 to 106 to obtain nonparametric estimates of VaR(0.1) and ES(0.1). Submit the portion of the output showing the estimates.

iii) Adapt and run lines 107 to 112 to obtain parametric estimates of VaR(0.1) and ES(0.1) under the assumption of normally distributed net returns. Submit the portion of the output showing the estimates.

iv) Adapt and run lines 113 to 128 to obtain 90% nonparametric bootstrap confidence intervals for VaR(0.1) and ES(0.1). Use five values for the number of bootstrap samples: $B = 500$, $B = 2000$, $B = 10000$, $B = 50000$, and $B = 100000$. For each value of B, submit the portion of the output showing the confidence intervals.

v) Adapt and run lines 129 to 144 to obtain 90% parametric bootstrap confidence intervals for VaR(0.1) and ES(0.1) under the assumption of normally distributed net returns. Use five values for the number of bootstrap samples: $B = 500$, $B = 2000$, $B = 10000$, $B = 50000$, and $B = 100000$. For each value of B, submit the portion of the output showing the confidence intervals.