11/6/18

# STSCI 5080 Homework 5

- Due is 11/15 (Th) in class.
- Write your name and NetID at the top of the first page, along with the assignment number.
- Use only the one side of the paper. Attach your pages with a staple at the top left corner.
- There are four problems. The total number of points is 50. Each small question is worth 5 points.
- $P_\theta, E_\theta$, and $\mathrm{Var}_\theta$ mean that the probability, expectation, and variance are taken under the parameter $\theta$. For example, if $X \sim N(\theta, 1)$, then $E_\theta(X) = \theta$.

## Problems

1. Let
$$X_1, \ldots, X_n \sim N(0, \sigma^2) \text{ i.i.d.}$$
where $\sigma^2 > 0$ is unknown.
   (a) Find the log likelihood function for $\sigma^2$.
   (b) Find the FOC for the MLE of $\sigma^2$.
   (c) Verify that the MLE of $\sigma^2$ is $\widehat{\sigma}^2 = n^{-1} \sum_{i=1}^n X_i^2$.

   (Hint). The parameter of interest is $\sigma^2$ and not $\sigma$. If you are uncomfortable in working with $\sigma^2$, then you can put $\theta = \sigma^2$ and work with $\theta$.

2. Work with the setting of Problem 1.
   (a) Find the distribution of $n\widehat{\sigma}^2/\sigma^2$. (Hint). $X_i/\sigma \sim N(0,1)$.
   (b) Find the mean and variance of $\widehat{\sigma}^2$. You may use the fact that the mean and variance of $\chi^2(n)$ are $n$ and $2n$, respectively.

3. Work with the setting of Problem 1.
   (a) Find the limiting distribution of $\sqrt{n}(\widehat{\sigma}^2 - \sigma^2)$. (Hint). Apply CLT. You may use the fact that $E(Z^4) = 3$ for $Z \sim N(0,1)$ and so $E_{\sigma^2}(X_1^4) = 3\sigma^4$.
   (b) Let $\sigma = \sqrt{\sigma^2}$. The MLE of $\sigma$ is $\widehat{\sigma} = \sqrt{\widehat{\sigma}^2}$. Find the limiting distribution of $\sqrt{n}(\widehat{\sigma} - \sigma)$. (Hint). The delta method.

   (Hint). Again if you are uncomfortable in working with $\sigma^2$, then you can put $\theta = \sigma^2$ and work with $\theta$.

4. You observe the numbers of failures occurring in machines in a factory in a year. Denote by $X_i$ the number of failures of the $i$-th machine, and suppose that

$$X_1, \dots, X_n \sim Po(\lambda) \text{ i.i.d.}$$

where $n$ is the number of machines in the factory. We know that the MLE of $\lambda$ is $\widehat{\lambda} = \overline{X} = n^{-1} \sum_{i=1}^{n} X_i$.

(a) Find the explicit expression of $P_\lambda(X_1 \geq 2)$ as a function of $\lambda$. Note that $P_\lambda(X_1 \geq 2)$ is the probability that at least two failures occur in a randomly chosen machine.

(b) Let $\theta = P_\lambda(X_1 \geq 2)$. Find the MLE of $\theta$.

(c) Denote by $\widehat{\theta}$ the MLE of $\theta$ obtained in Part (b). Now, suppose that $\widehat{\lambda} = 0.4$. Find an approximate numerical value of $\widehat{\theta}$ up to three decimal places.

**Solutions**

1. (a) The pdf of $N(0,\sigma^2)$ is
$$f_{\sigma^2}(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-x^2/(2\sigma^2)},$$
and so the joint pdf is
$$\prod_{i=1}^n f_{\sigma^2}(x_i) = \frac{1}{(2\pi\sigma^2)^{n/2}} e^{-\sum_{i=1}^n x_i^2/(2\sigma^2)}.$$
The likelihood function is
$$L_n(\sigma^2) = \frac{1}{(2\pi\sigma^2)^{n/2}} e^{-\sum_{i=1}^n X_i^2/(2\sigma^2)}.$$
The log likelihood function is
$$\ell_n(\sigma^2) = \log L_n(\sigma^2) = -\frac{n}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^n X_i^2.$$

   (b) We note that
$$\ell_n'(\sigma^2) = -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4}\sum_{i=1}^n X_i^2.$$
So the FOC is
$$-\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4}\sum_{i=1}^n X_i^2 = 0.$$

   (c) This can be verified by solving the first order condition.

2. (a) $n\widehat{\sigma}^2/\sigma^2 = \sum_{i=1}^n (X_i/\sigma)^2 \sim \chi^2(n)$.

   (b) The mean and variance of $\chi^2(n)$ are $n$ and $2n$, respectively, and so $E_{\sigma^2}(n\widehat{\sigma}^2/\sigma^2) = n$ and $\mathrm{Var}_{\sigma^2}(n\widehat{\sigma}^2/\sigma^2) = 2n$. Since $E_{\sigma^2}(n\widehat{\sigma}^2/\sigma^2) = nE_{\sigma^2}(\widehat{\sigma}^2)/\sigma^2$ and $\mathrm{Var}_{\sigma^2}(n\widehat{\sigma}^2/\sigma^2) = n^2\mathrm{Var}_{\sigma^2}(\widehat{\sigma}^2)/\sigma^4$, we have
$$E_{\sigma^2}(\widehat{\sigma}^2) = \sigma^2 \quad \text{and} \quad \mathrm{Var}_{\sigma^2}(\widehat{\sigma}^2) = \frac{2\sigma^4}{n}.$$

3. (a) Since $\widehat{\sigma}^2 = n^{-1}\sum_{i=1}^n X_i^2$ and $E_{\sigma^2}(X_i^2) = \sigma^2$ and $\mathrm{Var}_{\sigma^2}(X_i^2) = 2\sigma^4$, we have
$$\sqrt{n}(\widehat{\sigma}^2 - \sigma^2) \xrightarrow{d} N(0, 2\sigma^4)$$
by CLT.

   (b) We apply the delta method with $g(\sigma^2) = \sqrt{\sigma^2}$. Since $g'(\sigma^2) = 1/(2\sqrt{\sigma^2}) = 1/(2\sigma)$, we have
$$\sqrt{n}(\widehat{\sigma} - \sigma) \xrightarrow{d} N(0, \{g'(\sigma^2)\}^2 2\sigma^4) = N(0, \sigma^2/2).$$

4. (a)

$$P_\lambda(X_1 \geq 2) = 1 - P_\lambda(X_1 \leq 1) = 1 - e^{-\lambda} - e^{-\lambda}\lambda.$$

(b) The MLE of $\theta = 1 - e^{-\lambda} - e^{-\lambda}\lambda$ is

$$\widehat{\theta} = 1 - e^{-\widehat{\lambda}} - e^{-\widehat{\lambda}}\widehat{\lambda}.$$

(c)

$$\widehat{\theta} = 1 - e^{-0.4} - e^{-0.4} \cdot 0.4 \approx 0.061 \quad \text{or} \quad 0.062.$$