Machine Learning for Data Science (CS4786)
Lecture 5

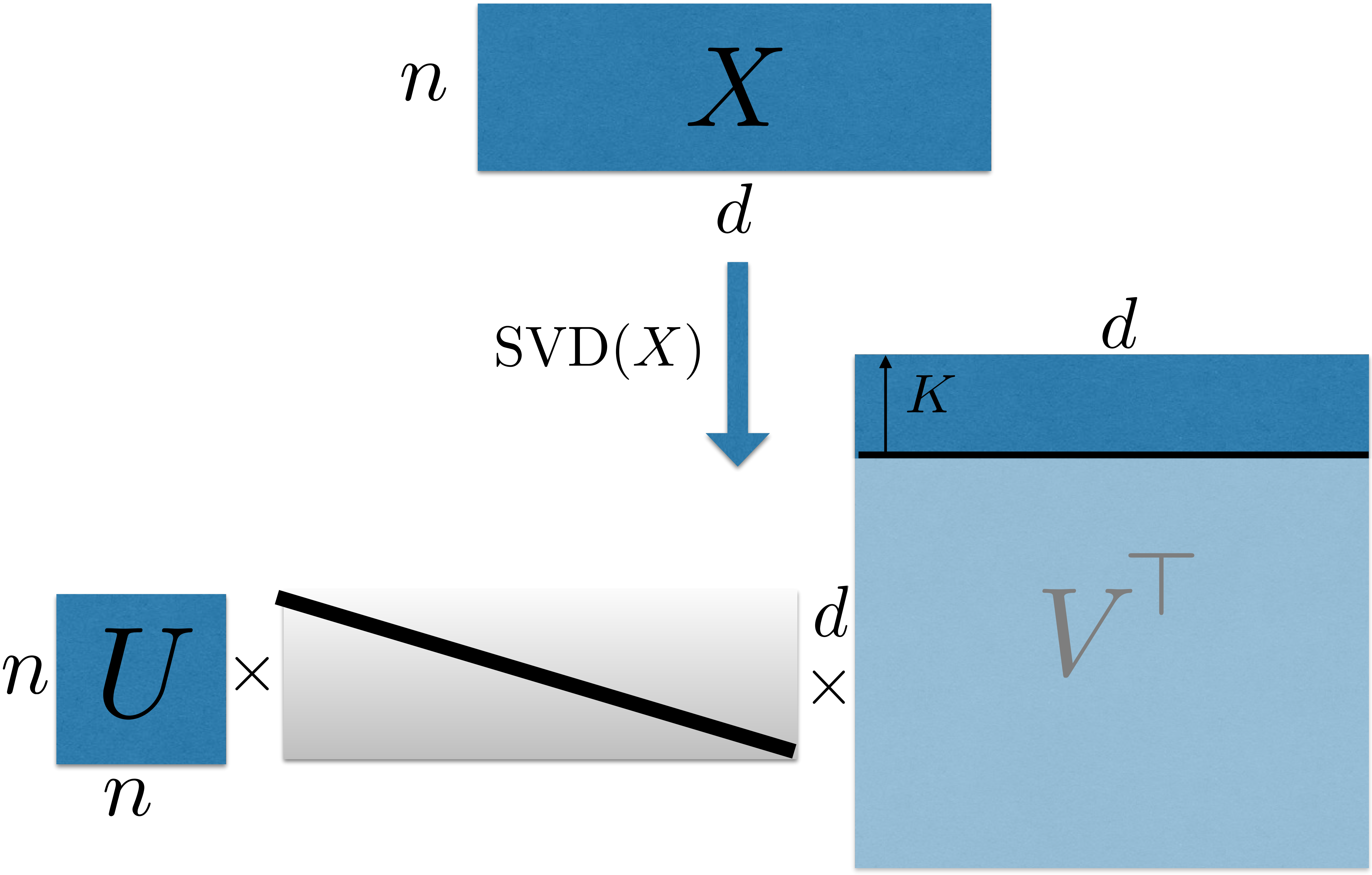
Random Projections & Canonical Correlation Analysis

The Tall, the Fat and the Ugly

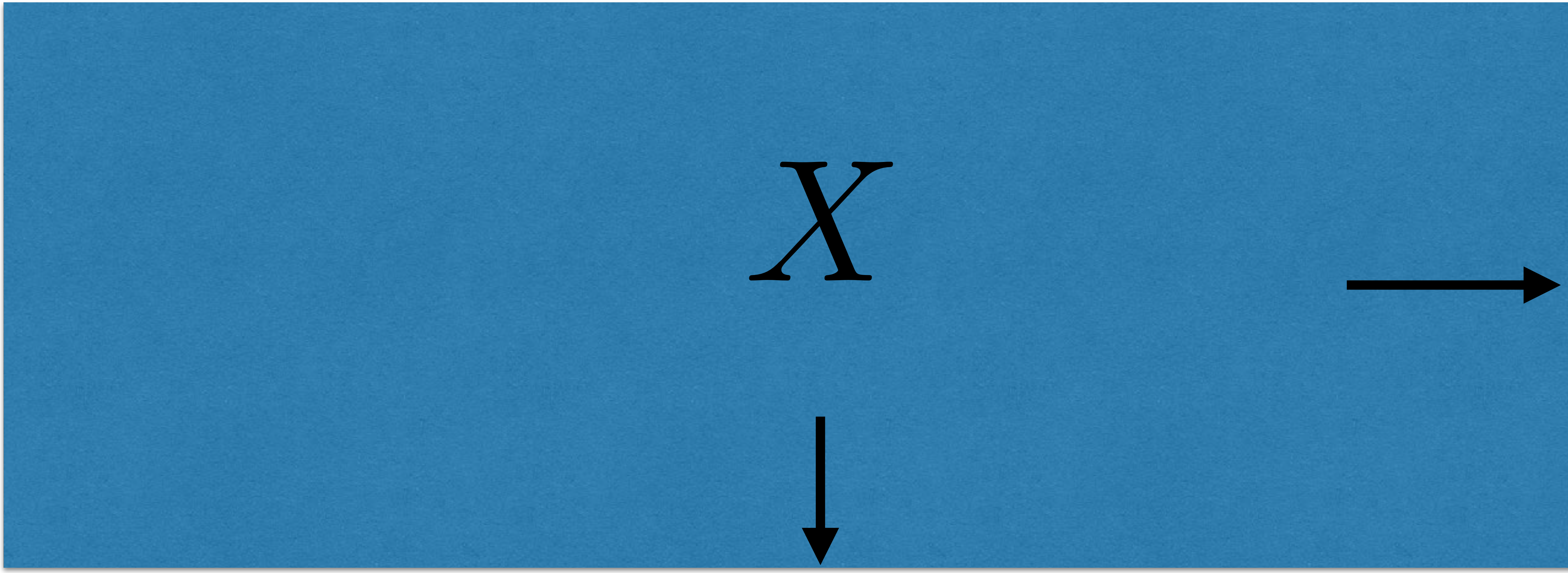
$$ {}_{d}\underset{n}{X^\top} \times {}_{n}\underset{d}{X} \Big/ n = {}_{d}\overset{d}{\Sigma} $$

$$ {}_{d}\underset{K}{W} = \mathrm{Eigs}(\Sigma, K) $$

THE TALL, **the Fat** AND THE UGLY



The Tall, the Fat and **the Ugly**



- d and n so large we can't even store in memory
- Only have time to be linear in $\text{size}(X) = n \times d$

I there any hope?

Pick a Random W

$$Y = X \times \begin{array}{c} \left[\begin{array}{ccc} +1 & \ldots & -1 \\ -1 & \ldots & +1 \\ +1 & \ldots & -1 \\ & \vdots & \\ & \vdots & \\ +1 & \ldots & -1 \end{array}\right] d \Bigg/ \sqrt{K} \\ K \end{array}$$

Why should Random Projections even work?!

Random Projection

- What does "it works" even mean?

  Distances between all pairs of data-points in low dim. projection is roughly the same as their distances in the high dim. space.

  That is, when K is "large enough", with "high probability", for all pairs of data points $i, j \in \{1, \ldots, n\}$,

$$(1-\epsilon)\left\|\mathbf{y}_i - \mathbf{y}_j\right\|_2 \le \left\|\mathbf{x}_i - \mathbf{x}_j\right\|_2 \le (1+\epsilon)\left\|\mathbf{y}_i - \mathbf{y}_j\right\|_2$$

Why should Random Projections even work?!

- Lets start with a one dimensional projection (K = 1)

$$y_t = \mathbf{x}_t^\top \mathbf{u} \quad \text{where each } \mathbf{u}[i] = \text{ random} \pm 1$$

- What is the expected value of:

$$1. \qquad y_t - y_s?$$

$$2. \qquad (y_t - y_s)^2?$$

Why should Random Projections even work?!

Hence for any $s, t \in \{1, \ldots, n\}$,

$$\mathbb{E}\left[|\mathbf{y}_s - \mathbf{y}_t|^2\right] = \|\mathbf{x}_s - \mathbf{x}_t\|_2^2$$

Lets try ...

Law of large numbers says that average over multiple draws is close to expectation

Pick a Random W

$$Y = X \times \left.\begin{array}{c} \left[\begin{array}{ccc} +1 & \ldots & -1 \\ -1 & \ldots & +1 \\ +1 & \ldots & -1 \\ & \vdots & \\ & \vdots & \\ +1 & \ldots & -1 \end{array}\right] \\ K \end{array}\right. d \Bigg/ \sqrt{K}$$

WHY SHOULD RANDOM PROJECTIONS EVEN WORK?!

- Like repeating the experiment K times and averaging

$$\mathbf{y}_t[k] = \mathbf{x}_t^\top \mathbf{u}_k / \sqrt{K} \quad \text{where each } \mathbf{u}_k[i] = \text{ random} \pm 1$$

$$(\mathbf{y}_s[k] - \mathbf{y}_t[k])^2 = \left(\mathbf{x}_t^\top \mathbf{u}_k - \mathbf{x}_s^\top \mathbf{u}_k\right)^2 / K$$

$$\|\mathbf{y}_t - \mathbf{y}_s\|_2^2 = \sum_{k=1}^{K} (\mathbf{y}_s[k] - \mathbf{y}_t[k])^2 = \frac{1}{K} \sum_{k=1}^{K} \left(\mathbf{x}_t^\top \mathbf{u}_k - \mathbf{x}_s^\top \mathbf{u}_k\right)^2$$

This is an average over K trials

Why should Random Projections even work?!

For any $\epsilon > 0$, if $K \approx \log(n/\delta)/\epsilon^2$, with probability $1-\delta$ over draw of W, for all pairs of data points $i, j \in \{1, \ldots, n\}$,

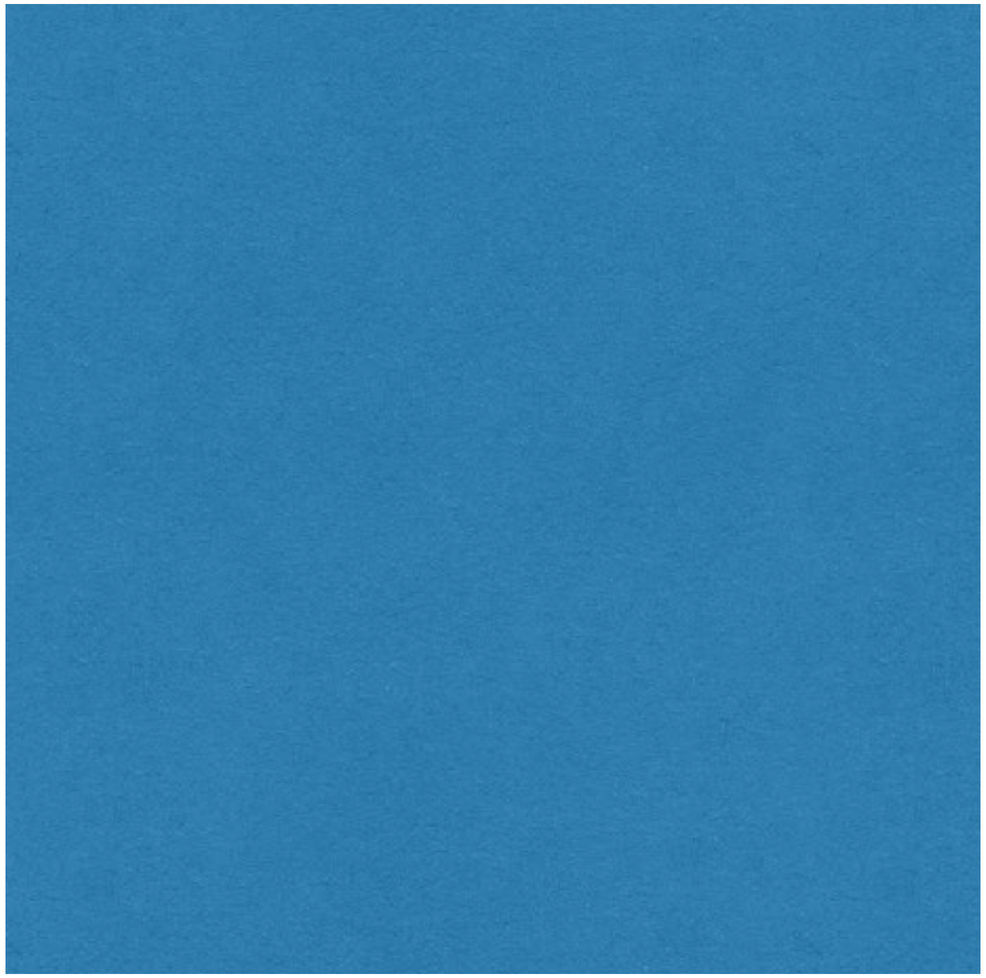
$$(1-\epsilon)\left\|\mathbf{y}_i - \mathbf{y}_j\right\|_2^2 \leq \left\|\mathbf{x}_i - \mathbf{x}_j\right\|_2 \leq (1+\epsilon)\left\|\mathbf{y}_i - \mathbf{y}_j\right\|_2^2$$

Lets try ...

This is called the Johnson-Lindenstrauss lemma or JL lemma for short.

Why is this so Ridiculously Magical?

n=
1000



d = 1000

If we take $K = 69.1/\epsilon^2$, with probability 0.99 distances are preserved to accuracy ϵ

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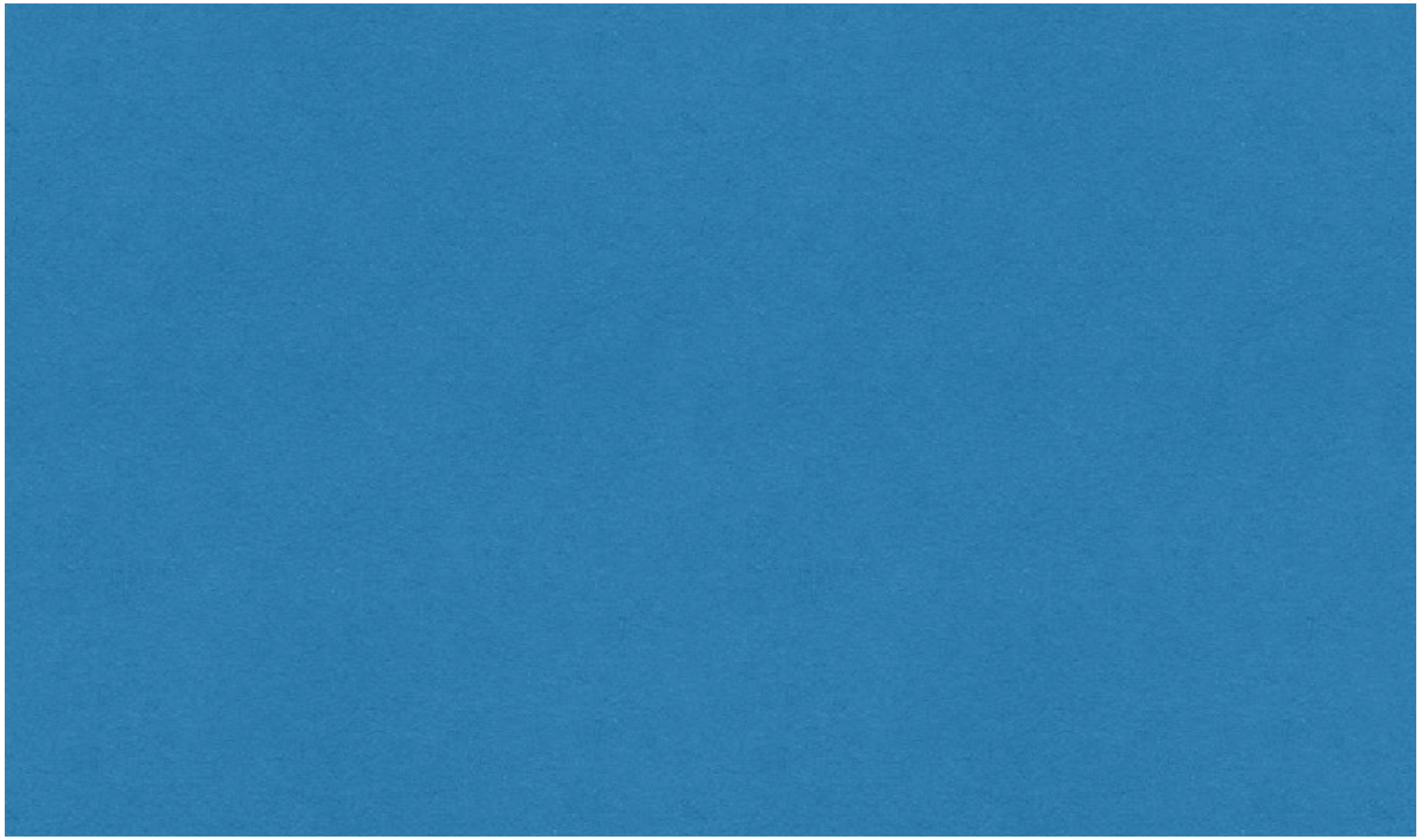

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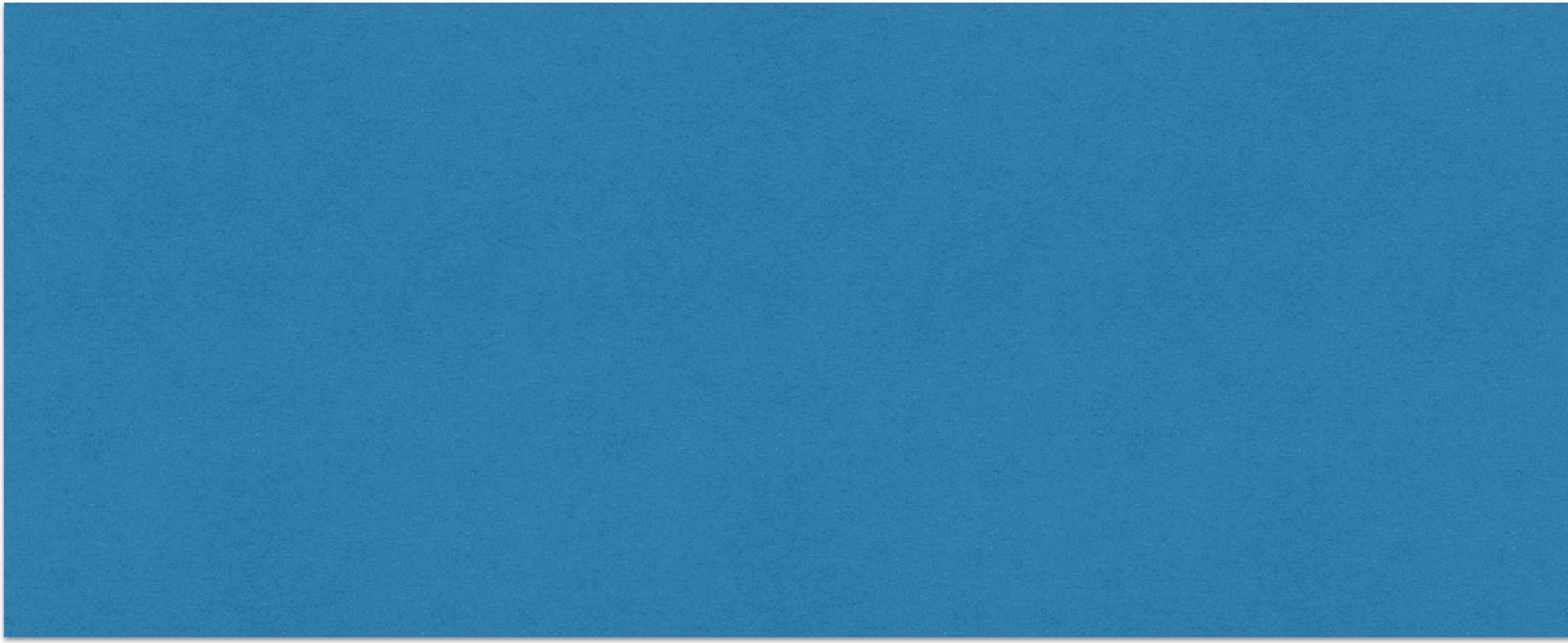

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Two View Dimensionality Reduction

- Data comes in pairs $(\mathbf{x}_1, \mathbf{x}'_1), \ldots, (\mathbf{x}_n, \mathbf{x}'_n)$ where $\mathbf{x}_t$'s are d dimensional and $\mathbf{x}'_t$'s are d' dimensional

- Goal: Compress say view one into $\mathbf{y}_1, \ldots, \mathbf{y}_n$, that are K dimensional vectors

    - Retain information redundant between the two views

    - Eliminate "noise" specific to only one of the views

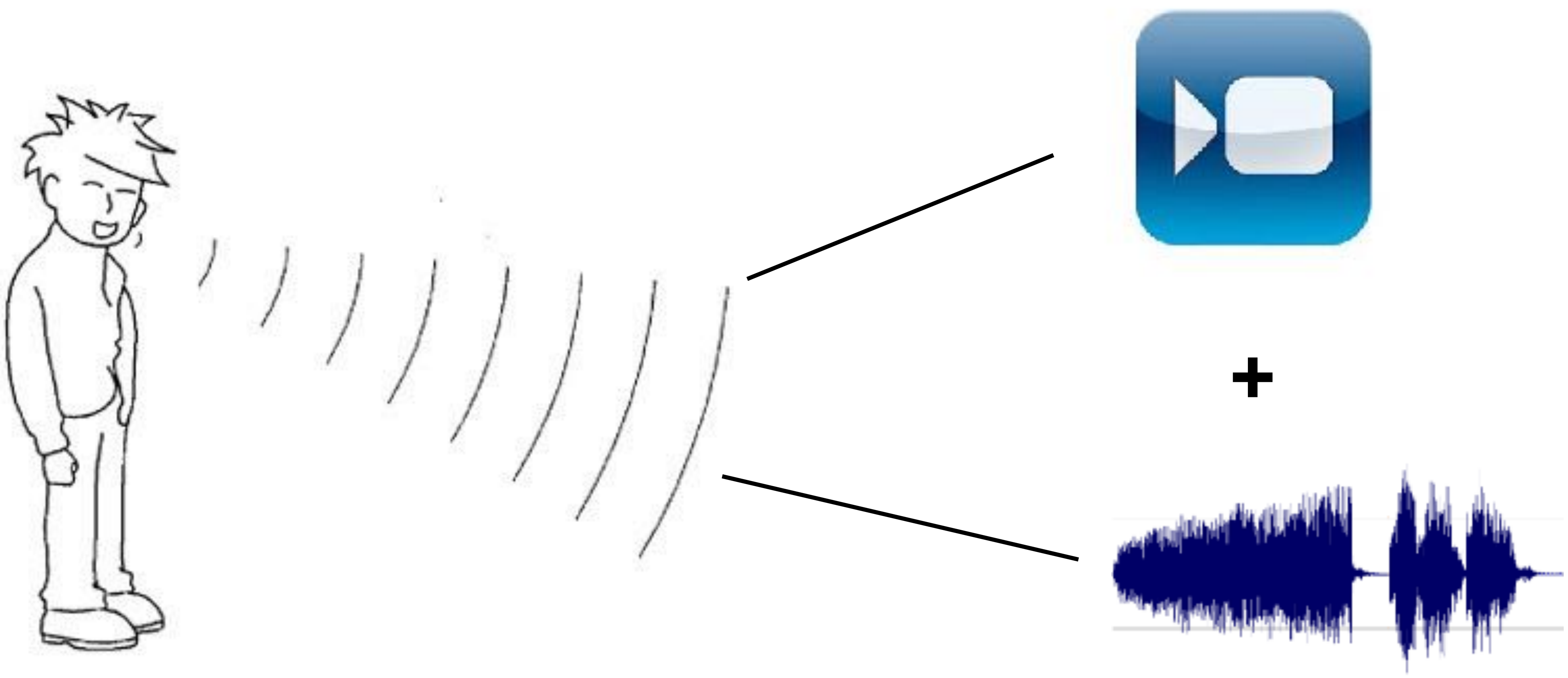
Canonical Correlation Analysis

Age

\+ Gender

Candies per week

Example I: Speech Recognition



- Audio might have background sounds uncorrelated with video
- Video might have lighting changes uncorrelated with audio
- Redundant information between two views: the speech

Example II: Combining Feature Extractions

- Method A and Method B are both equally good feature extraction techniques
- Concatenating the two features blindly yields large dimensional feature vector with redundancy
- Applying techniques like CCA extracts the key information between the two methods
- Removes extra unwanted information

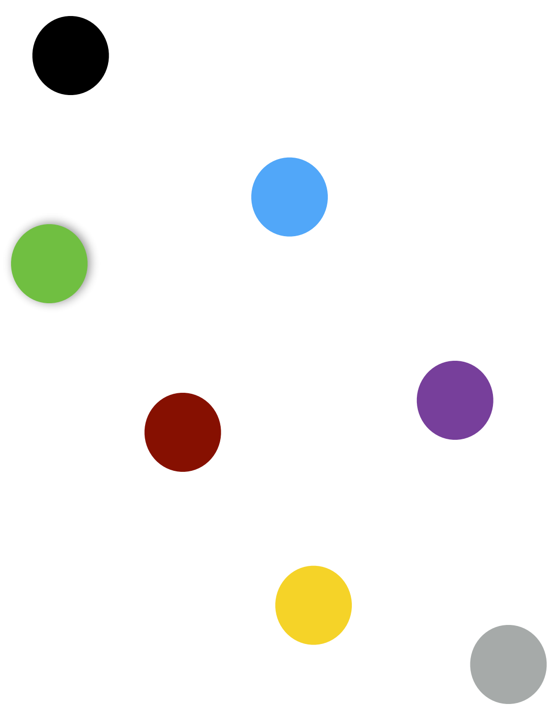
How do we get the right direction? (say K = 1)

Age

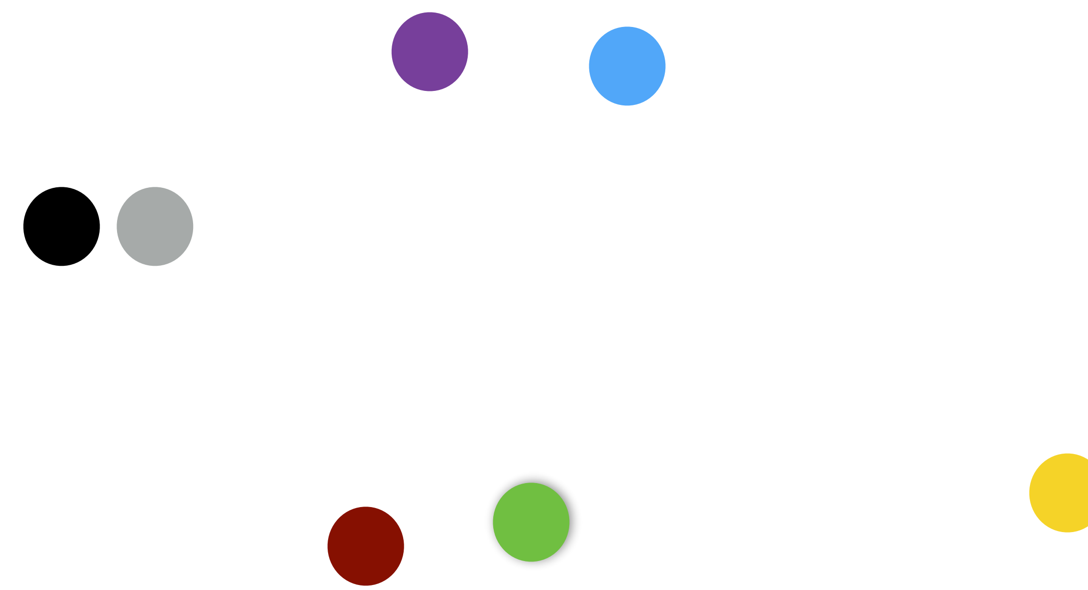
\+ Gender

Candies per week

Which Direction to Pick?



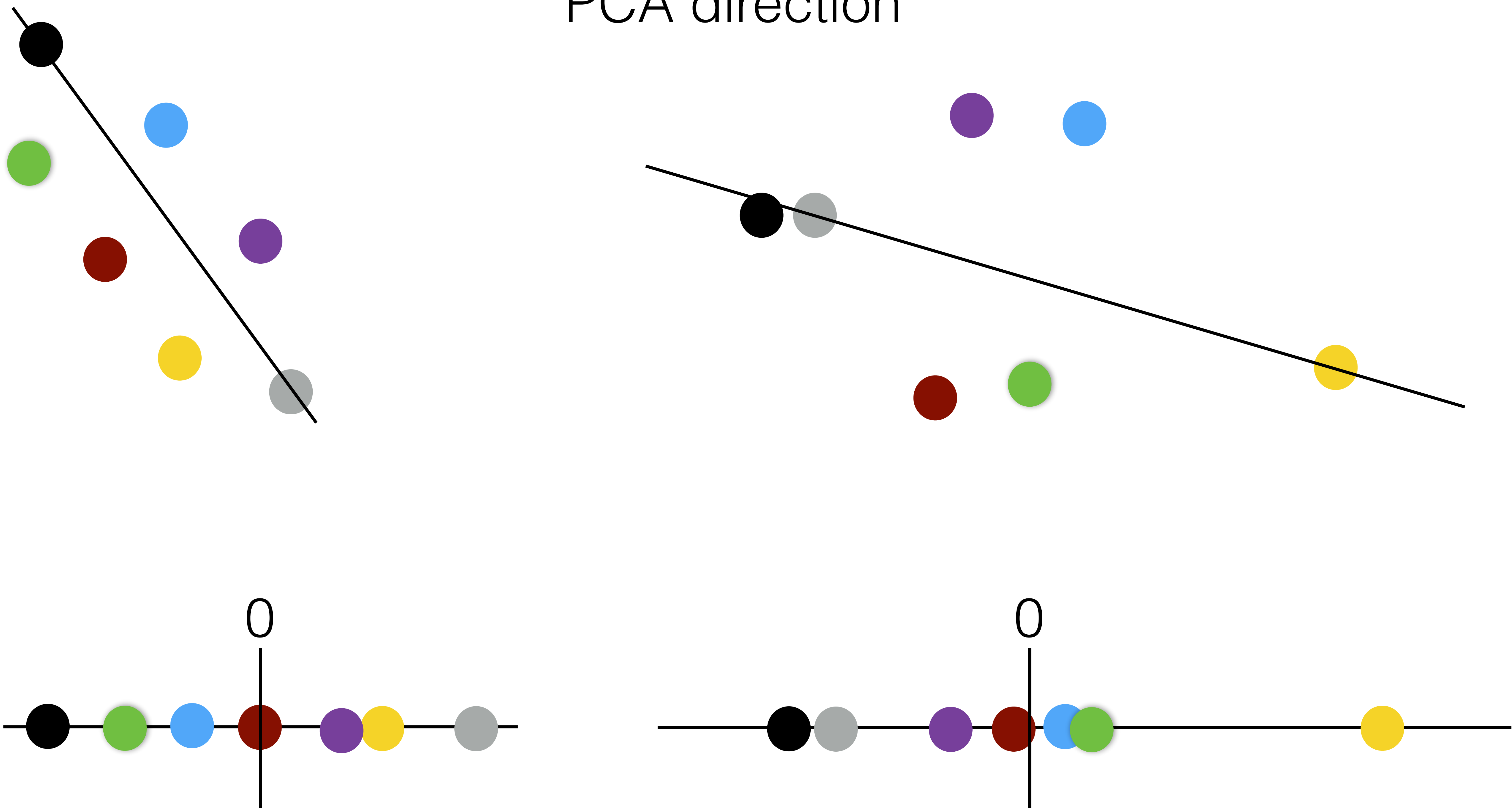
View I



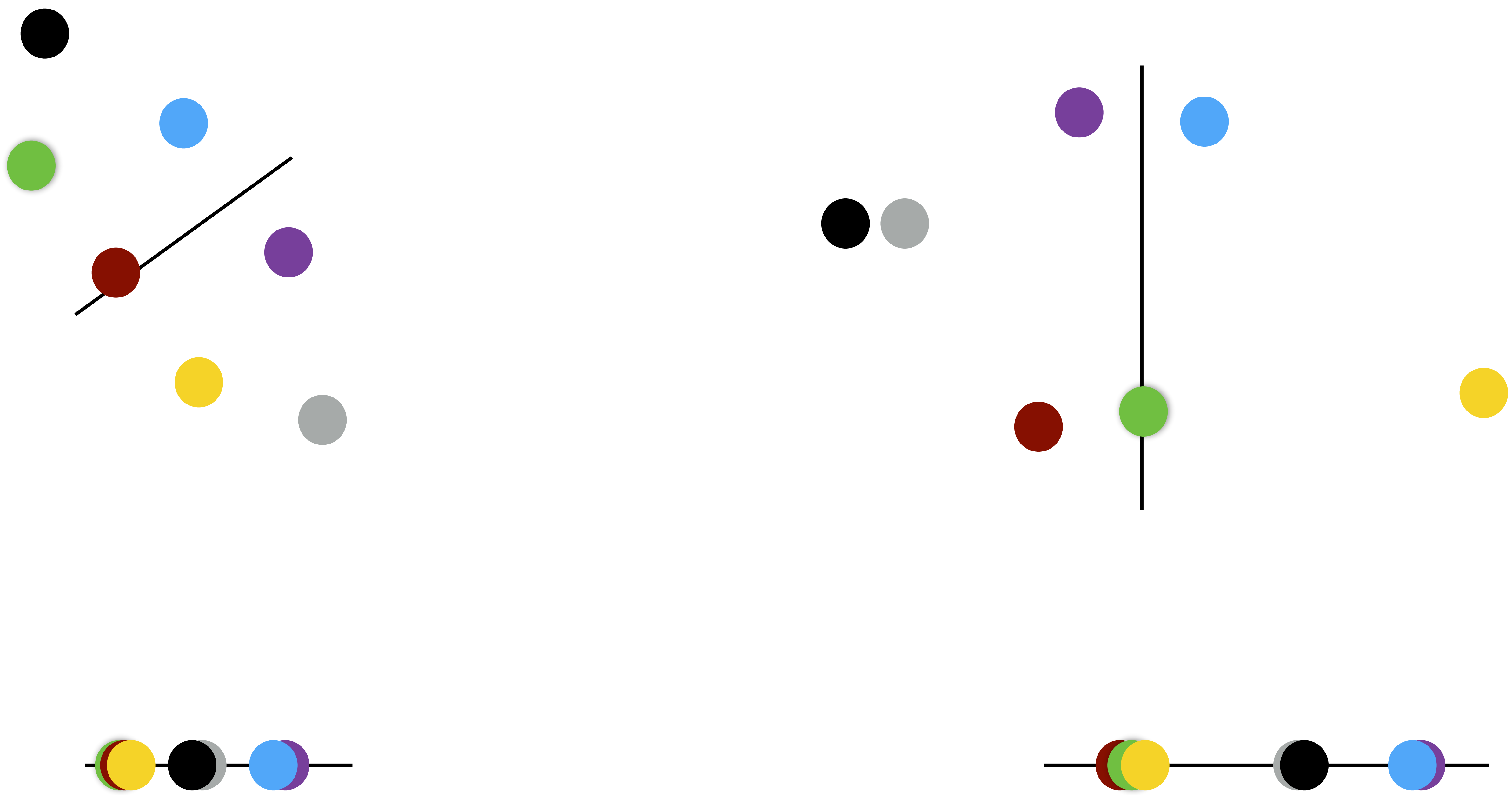
View II

Which Direction to Pick?

PCA direction



Which Direction to Pick?



Direction has large covariance

How do we pick the right direction to project to?