ORIE 4630: Spring Term 2019
Homework #9
Due: Thursday, April 25, 2019

Students are required to work independently on homework. You should not give or receive help from other students. You should also not receive help from students or former students who took this course in previous years and who may have solutions to similar problems. The solutions you submit should be your own work and not copied from elsewhere.

Homework is due at the end of lecture (12:55pm) on the due date. You will usually have one week to do the assignments. Please don't wait until the homework is nearly due to start. Late homework is not accepted. Also, homework is not accepted by email. You can submit your assignment in lecture or in the drop box in Rhodes Hall.

Please print your name on the front of your homework so that it is legible.

Include your `R` code, output, graphs, and other work with your homework. This will allow the grader to find any errors you make and to give partial credit.

Recall that the development of the Black-Scholes formula begins by dividing the time interval $[0, T]$ into n sub-intervals, each of length T/n, and then assumes a binomial tree model having n steps, in which the price s_j at node j either moves up to

$$s_{2j+1} = s_j \exp\left\{\mu\left(\frac{T}{n}\right) + \sigma\sqrt{\frac{T}{n}}\right\} \quad \text{at node } 2j+1,$$

or moves down to

$$s_{2j} = s_j \exp\left\{\mu\left(\frac{T}{n}\right) - \sigma\sqrt{\frac{T}{n}}\right\} \quad \text{at node } 2j.$$

Here, μ is the drift of the stock price per unit time, and σ is the volatility of the stock price per unit time. Under this price movement, the risk-neutral probability of a move up from node j to node $2j+1$ is

$$q = \frac{\exp(rT/n) - \exp\big(\mu T/n - \sigma\sqrt{T/n}\big)}{\exp\big(\mu T/n + \sigma\sqrt{T/n}\big) - \exp\big(\mu T/n - \sigma\sqrt{T/n}\big)}.$$

The prices at the nodes of the binomial tree for a European call option when the stock price moves according to the Black-Scholes model are calculated by running the following lines:

```
S_0=160; mu=0.07; sigma=0.3; K=170; T=1.5; r=0.012
n=3; call=TRUE; european=TRUE
S_T_values=S_0*exp(mu*T+(2*(n:0)-n)*sigma*sqrt(T/n))
if (call==TRUE) option_values_T=pmax(S_T_values-K,rep(0,n+1)) else
   option_values_T=pmax(K-S_T_values,rep(0,n+1))
option_values=matrix(nrow=n+1,ncol=n+1)
stock_values=matrix(nrow=n+1,ncol=n+1)
option_values[1:(n+1),n+1]=option_values_T
stock_values[1:(n+1),n+1]=S_T_values
q=(exp(r*(T/n))-exp(mu*(T/n)-sigma*sqrt(T/n)))/
```

```
11   (exp(mu*(T/n)+sigma*sqrt(T/n))-exp(mu*(T/n)-sigma*sqrt(T/n)))
12 for (i in n:1)
13 {
14   option_values[1:i,i]=(option_values[1:i,i+1]*q
15                        +option_values[2:(i+1),i+1]*(1-q))/exp(r*(T/n))
16   S_values=S_0*exp(mu*(T*(i-1)/n)+(2*((i-1):0)-(i-1))*sigma*sqrt(T/n))
17   stock_values[1:i,i]=S_values
18   if (european==FALSE)
19   {
20     if (call==TRUE)
21        option_values[1:i,i]=pmax(S_values-K,option_values[1:i,i])
22     else option_values[1:i,i]=pmax(K-S_values,option_values[1:i,i])
23   }
24 }
25 q
26 option_values
27 stock_values
```

If the number of steps n in the Binomial tree model is very large, then it is impractical to show the option prices at all of the nodes. The following program shows only the price of the option at node 0, and it compares the price of the option from the binomial tree model with the price obtained from the Black-Scholes formula:

```
28 S_0=160; mu=0.07; sigma=0.3; K=170; T=1.5; r=0.012
29 n=100; call=TRUE; european=TRUE
30 S_T_values=S_0*exp(mu*T+(2*(n:0)-n)*sigma*sqrt(T/n))
31 if (call==TRUE) option_values_T=pmax(S_T_values-K,rep(0,n+1)) else
32   option_values_T=pmax(K-S_T_values,rep(0,n+1))
33 option_values=option_values_T
34 stock_values=S_T_values
35 q=(exp(r*(T/n))-exp(mu*(T/n)-sigma*sqrt(T/n)))/
36   (exp(mu*(T/n)+sigma*sqrt(T/n))-exp(mu*(T/n)-sigma*sqrt(T/n)))
37 for (i in n:1)
38 {
39   option_values=(option_values[1:i]*q
40                +option_values[2:(i+1)]*(1-q))/exp(r*(T/n))
41   S_values=S_0*exp(mu*(T*(i-1)/n)+(2*((i-1):0)-(i-1))*sigma*sqrt(T/n))
42   if (european==FALSE)
43   {
44     if (call==TRUE) option_values=pmax(S_values-K,option_values) else
45       option_values=pmax(K-S_values,option_values)
46   }
47 }
48 if (call==TRUE) BS_formula=BS_price_formula_call(S_0, sigma, K, T, r) else
49   BS_formula=BS_price_formula_put(S_0, sigma, K, T, r)
50 q
51 option_values
52 BS_formula
53 option_values-BS_formula
```

Note that this program uses the function `BS_price_formula_call()` in line 48 and the function `BS_price_formula_put()` in line 49. These functions are defined in the following lines:

```
BS_price_formula_call = function(S_0, sigma, K, T, r)
{
  d1 = ( log(S_0 / K) + ( r + sigma^2 / 2 ) * T ) / ( sigma * sqrt(T) )
  d2 = d1 - sigma * sqrt(T)
  option_price = pnorm(d1) * S_0 - pnorm(d2) * K * exp(-r*T)
  option_price
}
```

```
BS_price_formula_put = function(S_0, sigma, K, T, r)
{
  C=BS_price_formula_call(S_0, sigma, K, T, r)
  option_price = C + K * exp(-r*T) - S_0
  option_price
}
```

There are other types of options besides European and American options. For instance, an Asian call option has value $(\bar{S}-K)_+$ at expiry, where $\bar{S}$ is the average price of the stock over the time period $[0,T]$: thus, $\bar{S}=\dfrac{1}{T}\displaystyle\int_0^T S_t dt$, where S_t is the price of the stock at time t $(0 \le t \le T)$. The following program obtains the price of an Asian call option by simulation:

```
S_0=160; mu=0.07; sigma=0.3; K=170; T=1.5; r=0.012
n=20000; n_sim=20000; seed=1574; set.seed(seed)
q=(exp(r*(T/n))-exp(mu*(T/n)-sigma*sqrt(T/n)))/
  (exp(mu*(T/n)+sigma*sqrt(T/n))-exp(mu*(T/n)-sigma*sqrt(T/n)))
option_price=matrix(ncol=n_sim)
for (i in 1:n_sim)
{
  b=rbinom(n,1,q)
  S_values=S_0*exp(mu*(T/n)+cumsum(2*b-1)*sigma*sqrt(T/n))
  S_values=c(S_0,S_values)
  S_mean=mean(S_values)
  option_value_T=max(c(S_mean-K,0))
  option_price[i]=option_value_T/exp(r*T)
}
a=mean(option_price)
b=sd(option_price)
c=rbind(c("mean","standard deviation"),round(c(a,b),7)); c
a+c(-1,1)*qnorm(0.975)*b/sqrt(n_sim)
```

Suppose that each year is divided into m equally spaced intervals, and consider a zero-coupon bond that matures in T years, i.e., in $m \times T$ intervals, with par value P. Suppose the present value of the bond is $P(T)$; then the spot rate y_T satisfies the equation

$$P(T)=\frac{P}{(1+y_T)^{mT}}, \quad \text{which yields} \quad y_T=\left\{\frac{P}{P(T)}\right\}^{1/(mT)}-1.$$

The resulting rate y_T is a per-interval rate; the annual rate is $m \times y_T$.

The following lines calculate per-interval spot rates from zero-coupon bond prices. For each zero-coupon bond, there is associated a par value P, a maturity T measured in numbers of years, and a present value $P(T)$.

```
m=2
maturities=c(.5,1,1.5,2)
```

```
present_values=c(98.71,95.92,92.33,89.82)
par_values=c(100,100,100,100)
spot_rates=function(par_values,present_values,maturities)
{
  spot_rates=(par_values/present_values)^(1/(m*maturities))-1
}
spot_rates=spot_rates(par_values,present_values,maturities)
annual_spot_rates=m*spot_rates
spot_rates
annual_spot_rates
```

Consider a coupon bond having maturity T years and par value P, where coupon payment C occurs at m intervals per year. If the per-interval spot rates over the life of the bond are $y_1, y_2, \ldots, y_{mT}$, then the price of the bond is

$$\frac{C}{(1+y_1)} + \frac{C}{(1+y_2)^2} + \cdots + \frac{C}{(1+y_{mT-1})^{mT-1}} + \frac{C+P}{(1+y_{mT})^{mT}}.$$

The following lines calculate the price of a coupon bond:

```
present_value_coupon_bond=function(C,P,T,m,spot_rates)
{
  intervals=seq(from=1,to=m*T)
  payments=C/(1+spot_rates)^intervals
  payments=c(payments,P/(1+spot_rates[m*T])^(m*T))
  sum(payments)
}
C=1.5; P=100; T=2; m=2
spot_rates=c(0.0131, 0.0210, 0.0270, 0.0272)
present_value_coupon_bond(C,P,T,m,spot_rates)
```

Consider a coupon bond having maturity T years and par value P, where coupon payment C occurs at m intervals per year. If the present value of the bond is V, then the yield to maturity of the bond is the value of y satisfying

$$\frac{C}{(1+y)} + \frac{C}{(1+y)^2} + \cdots + \frac{C}{(1+y)^{mT-1}} + \frac{C+P}{(1+y)^{mT}} = V.$$

Note that y is a per-interval rate; the yield to maturity expressed as an annual rate is $m \times y$. The following lines calculate the yield to maturity of a coupon bond:

```
C=1.5; P=100; T=2; m=2
present_value_from_yield=function(yield,C,P,T,m)
{
  intervals=seq(from=1,to=m*T)
  payments=C/(1+yield)^intervals
  payments=c(payments,P/(1+yield)^(m*T))
  sum(payments)
}
present_value=95
yield_to_maturity=uniroot(function(yield)
  {present_value_from_yield(yield,C,P,T,m)-present_value},
                          lower=0.001,upper=0.5,tol=1e-9)
yield=yield_to_maturity$root
yield
m*yield
```

Questions:

1. [25 points]

i) Suppose that the price of a stock is initially $S_0 = 120$, with drift $\mu = 0.04$ per year and volatility $\sigma = 0.3$ per year. Consider a European call option with expiry in 1.25 years and strike price 125. Adapt the given programs to calculate the option prices along the binomial tree for the Black-Scholes model with $n = 5$ steps, where the interest rate is assumed to be $r = 0.02$ per year compounded continuously. Submit the output that shows the option prices. What is the price of the option at time $t = 0$? What is the value of the risk-neutral probability q?

ii) Repeat part i), but this time suppose that the drift of the stock price is $\mu = 0.1$. Does the price of the option at time $t = 0$ change from the price in part i)? Does the value of q change from the value in part i)?

iii) Repeat part i), but this time suppose that the option is an American call option. Does the price of the option at time $t = 0$ change from the price in part i)? Does the value of q change from the value in part i)?

iv) Repeat part i), but this time suppose that the option is a European put option. Does the value of q change from the value in part i)?

v) Repeat part i), but this time suppose that the option is an American put option. Does the price of the option at time $t = 0$ change from the price in part iv)? Does the price increase or decrease from the price in part iv)?

2. [25 points]

i) Suppose that the price of a stock is initially $S_0 = 120$, with drift $\mu = 0.04$ per year and volatility $\sigma = 0.3$ per year. Consider a European call option with expiry in 1.25 years and strike price 125. Adapt the given programs to calculate the option price at $t = 0$ for the Black-Scholes model with $n = 25$ steps, where the interest rate is assumed to be $r = 0.02$ per year compounded continuously. Submit the output from the appropriate program. What is the price of the option at time $t = 0$? What is the value given by the Black-Scholes formula? How does the price of the option compare to the value from the Black-Scholes formula? What is the value of the risk-neutral probability q?

ii) Repeat part i), but this time use $n = 5000$ steps.

iii) Repeat part i), but this time consider a European put option and use $n = 5000$.

iv) Repeat part i), but this time consider an American put option and use $n = 5000$.

v) Repeat part iv), but this time use $n = 50000$. Does the price derived by using $n = 50000$ differ from the price derived by using $n = 10000$? The purpose of this question is to show how simulation can be used to deduce the price of an American put option. Is using $n = 50000$ steps warranted, or does $n = 10000$ steps suffice for pricing the option?

3. [20 points]

i) Suppose that the price of a stock is initially $S_0 = 120$, with drift $\mu = 0.04$ per year and volatility $\sigma = 0.3$ per year. Consider an Asian call option with expiry in 1.25 years and strike price 125. Adapt the given programs to estimate by simulation the option price at $t = 0$ for the Black-Scholes model with $n = 50$ steps, where the interest rate is assumed to be $r = 0.02$ per year compounded continuously. Use a sample size 1000 and set the seed to 4630 for the simulation. Submit the output from the appropriate program. What is the estimate of the price of the option at time $t = 0$? Give a 95% confidence interval for the option price. Is the simulation size 1000 sufficiently large?

ii) Repeat part i), but this time take the number of steps to be $n = 10000$.

iii) Repeat part i), but, in addition to taking the number of steps to be $n = 10000$, take the sample size to be 500000.

iv) How large a sample size for the simulation would be required for the confidence interval to have width less than one cent, i.e., width less than 0.01?

4. [10 points] Suppose that the prices of zero-coupon bonds having par value 100 and 13-week, 26-week, 1-year, and 2-year maturities are 99.55, 98.98, 97.70, and 95.18, respectively. Adapt the given programs to find the 13-week, 26-week, 52-week and 2-year spot rates from these prices. Consider quarterly compounding. Submit the output from the appropriate program.

i) What is the 2-year spot rate expressed as an annual rate?

ii) What is the 1-year spot rate expressed as a quarterly rate?

5. [10 points] Consider a coupon bond having maturity 1 year and par value 100, where the coupon payment 1.1 occurs quarterly. Suppose the quarterly spot rates over the life of the bond are $y_1 = 0.008$, $y_2 = 0.009$, $y_3 = 0.011$, and $y_4 = 0.012$. Adapt the given programs to find the present value of the bond. Submit the output from the appropriate program. What is the price of the bond?

6. [10 points] Consider a coupon bond having maturity 1 year and par value 100, where the coupon payment 1.1 occurs quarterly. Suppose the price of the bond is 101. Adapt the given programs to find the yield to maturity of the bond. Submit the output from the appropriate program. What is the yield to maturity expressed as a quarterly rate? What is the yield to maturity expressed as an annual rate?