Fall 2018 STSCI 5080 Discussion 11 (11/9)

Problems

1. It is not known what proportion p of the purchases of a certain brand of breakfast cereal are made by women and what proportion are made by men. In a random sample of 70 purchases of this cereal, it was found that 58 were made by women and 12 were made by men. Find the MLE of p.

2. (Rice 8.10.4 modified) Let f_θ be a pmf of the form

$$f_\theta(x) = \begin{cases} \frac{2}{3}\theta & \text{if } x = 0 \\ \frac{1}{3}\theta & \text{if } x = 1 \\ \frac{2}{3}(1-\theta) & \text{if } x = 2 \\ \frac{1}{3}(1-\theta) & \text{if } x = 3 \\ 0 & \text{elsewhere} \end{cases},$$

where $0 < \theta < 1$. We observe a random sample

$$(X_1, X_2, \ldots, X_{10}) = (3, 0, 2, 1, 3, 2, 1, 0, 2, 1) \tag{*}$$

from such a pmf.

   (a) Find the joint pmf of $(X_1, \ldots, X_n)$ at (*).
   (c) Find the MLE of θ.

3. (Rice 8.10.16 modified) Let $X_1, \ldots, X_n$ be a random sample from the pdf

$$f_\sigma(x) = \frac{1}{2\sigma} e^{-|x|/\sigma}, \ -\infty < x < \infty.$$

   (a) Find the log likelihood function for σ.
   (b) Find the FOC for the MLE of σ.
   (c) Find the MLE $\widehat{\sigma}$ of σ.
   (d) Find the limiting distribution of $\sqrt{n}(\widehat{\sigma} - \sigma)$. You may use the fact that $E_\sigma(|X_1|) = \sigma$ and $E_\sigma(|X_1|^2) = 2\sigma^2$.

4. (Rice 8.10.48 modified) Let

$$X_1, \ldots, X_n \sim Po(\lambda) \text{ i.i.d.}$$

for some $\lambda > 0$. We know that the MLE of λ is $\widehat{\lambda} = \overline{X}$. Next, consider

$$\theta = P_\lambda(X_1 = 0) = e^{-\lambda}.$$

If we define

$$Y_i = \begin{cases} 1 & \text{if } X_1 = 0 \\ 0 & \text{otherwise} \end{cases},$$

then $Y_1, \ldots, Y_n$ are independent Bernoulli trials with success probability θ. So we can estimate θ by

$$\theta = \overline{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i.$$

Since $\lambda = -\log\theta$, we can also consider the following estimator for λ:

$$\widetilde{\lambda} = -\log(\overline{Y}).$$

(a) Find the limiting distribution of $\sqrt{n}(\overline{Y} - \theta)$ as $n \to \infty$.

(b) Find the limiting distribution of $\sqrt{n}(\widetilde{\lambda} - \lambda)$.

(c) We know that $\sqrt{n}(\widehat{\lambda} - \lambda) \xrightarrow{d} N(0, \lambda)$ by CLT. Verify that λ is smaller than or equal to the variance of the limiting normal distribution for $\sqrt{n}(\widetilde{\lambda} - \lambda)$ given in Part (b).

Solutions

1. Let X denote the number of purchases of this cereal were made by women; then $X \sim Bin(n = 70, p)$. The MLE is
$$\widehat{p} = X/n = 58/70 \approx 0.83.$$

2. (a) The joint pmf at (*) is
$$f_\theta(3)f_\theta(0)\cdots f_\theta(1) = \{f_\theta(0)\}^2\{f_\theta(1)\}^3\{f_\theta(2)\}^3\{f_\theta(3)\}^2 = \frac{2^5}{3^{10}}\theta^5(1-\theta)^5.$$

   (b) The log likelihood function is
$$\ell(\theta) = \log\frac{2^5}{3^{10}}\theta^5(1-\theta)^5 = \log\frac{2^5}{3^{10}} + 5\{\log\theta + \log(1-\theta)\}.$$
   The FOC is
$$\ell'(\theta) = 5\left(\frac{1}{\theta} - \frac{1}{1-\theta}\right) = 0.$$
   Solving the FOC, we obtain the MLE (maximum likelihood estimate)
$$\widehat{\theta} = \frac{1}{2}.$$

3. (a) The joint pdf is
$$\frac{1}{(2\sigma)^n}e^{-\sum_{i=1}^n |x_i|/\sigma}.$$
   The likelihood function is
$$L_n(\sigma) = \frac{1}{(2\sigma)^n}e^{-\sum_{i=1}^n |X_i|/\sigma}.$$
   The log likelihood function is
$$\ell_n(\theta) = \log L_n(\theta) = -n\log(2\sigma) - \frac{1}{\sigma}\sum_{i=1}^n |X_i|.$$

   (b) We note that
$$\ell_n'(\sigma) = -\frac{n}{\sigma} + \frac{1}{\sigma^2}\sum_{i=1}^n |X_i|.$$
   The FOC is
$$-\frac{n}{\sigma} + \frac{1}{\sigma^2}\sum_{i=1}^n |X_i| = 0.$$

   (c) Solving the FOC, we have
$$\widehat{\sigma} = \frac{1}{n}\sum_{i=1}^n |X_i|.$$

(d) $\text{Var}_\sigma(|X_1|) = E_\sigma(X_1^2) - \{E_\sigma(|X_1|)\}^2 = \sigma^2$. By CLT, we have $\sqrt{n}(\widehat{\sigma} - \sigma) \xrightarrow{d} N(0, \sigma^2)$.

4. (a) By CLT,

$$\sqrt{n}(\overline{Y} - \theta) \xrightarrow{d} N(0, \theta(1-\theta)) = N(0, e^{-\lambda}(1 - e^{-\lambda})).$$

(b) Let $g(\theta) = -\log\theta$. Since $g'(\theta) = -1/\theta$, we have

$$\sqrt{n}(\widetilde{\lambda} - \lambda) = \sqrt{n}(g(\widehat{\theta}) - g(\theta)) \xrightarrow{d} N(0, (1-\theta)/\theta) = N(0, (1 - e^{-\lambda})/e^{-\lambda}).$$

(c) We note that

$$(1 - e^{-\lambda})/e^{-\lambda} = e^{\lambda} - 1.$$

Now,

$$\frac{e^{\lambda} - 1}{\lambda} = \frac{\lambda + \frac{\lambda^2}{2} + \cdots}{\lambda} > 1.$$