11/6/18

STSCI 5080 Homework 5

- Due is 11/15 (Th) in class.
- Write your name and NetID at the top of the first page, along with the assignment number.
- Use only the one side of the paper. Attach your pages with a staple at the top left corner.
- There are four problems. The total number of points is 50. Each small question is worth 5 points.
- P_θ, E_θ, and Var_θ mean that the probability, expectation, and variance are taken under the parameter θ. For example, if $X \sim N(\theta, 1)$, then $E_\theta(X) = \theta$.

Problems

1. Let
$$X_1, \dots, X_n \sim N(0, \sigma^2) \text{ i.i.d.}$$
where $\sigma^2 > 0$ is unknown.
   (a) Find the log likelihood function for σ^2.
   (b) Find the FOC for the MLE of σ^2.
   (c) Verify that the MLE of σ^2 is $\widehat{\sigma}^2 = n^{-1} \sum_{i=1}^n X_i^2$.

   (Hint). The parameter of interest is σ^2 and not σ. If you are uncomfortable in working with σ^2, then you can put $\theta = \sigma^2$ and work with θ.

2. Work with the setting of Problem 1.
   (a) Find the distribution of $n\widehat{\sigma}^2/\sigma^2$. (Hint). $X_i/\sigma \sim N(0, 1)$.
   (b) Find the mean and variance of $\widehat{\sigma}^2$. You may use the fact that the mean and variance of $\chi^2(n)$ are n and $2n$, respectively.

3. Work with the setting of Problem 1.
   (a) Find the limiting distribution of $\sqrt{n}(\widehat{\sigma}^2 - \sigma^2)$. (Hint). Apply CLT. You may use the fact that $E(Z^4) = 3$ for $Z \sim N(0, 1)$ and so $E_{\sigma^2}(X_1^4) = 3\sigma^4$.
   (b) Let $\sigma = \sqrt{\sigma^2}$. The MLE of σ is $\widehat{\sigma} = \sqrt{\widehat{\sigma}^2}$. Find the limiting distribution of $\sqrt{n}(\widehat{\sigma} - \sigma)$. (Hint). The delta method.

   (Hint). Again if you are uncomfortable in working with σ^2, then you can put $\theta = \sigma^2$ and work with θ.

4. You observe the numbers of failures occurring in machines in a factory in a year. Denote by X_i the number of failures of the i-th machine, and suppose that

$$X_1, \dots, X_n \sim Po(\lambda) \text{ i.i.d.}$$

where n is the number of machines in the factory. We know that the MLE of λ is $\widehat{\lambda} = \overline{X} = n^{-1} \sum_{i=1}^{n} X_i$.

(a) Find the explicit expression of $P_\lambda(X_1 \geq 2)$ as a function of λ. Note that $P_\lambda(X_1 \geq 2)$ is the probability that at least two failures occur in a randomly chosen machine.

(b) Let $\theta = P_\lambda(X_1 \geq 2)$. Find the MLE of θ.

(c) Denote by $\widehat{\theta}$ the MLE of θ obtained in Part (b). Now, suppose that $\widehat{\lambda} = 0.4$. Find an approximate numerical value of $\widehat{\theta}$ up to three decimal places.