STSCI 5080
Probability Models and Inference
Lecture 9: Generalizations and Functions of Random Vectors

September 20, 2018

Random vector

Definition

A vector of n random variables $(X_1, \ldots, X_n)$ is called an n-dimensional random vector.

Discrete random vector

- If $X_1, \ldots, X_n$ are discrete individually, then the vector $(X_1, \ldots, X_n)$ is said to be discrete.

- The joint pmf of $(X_1, \ldots .X_n)$ is defined by

$$p(x_1, \ldots, x_n) = P(X_1 = x_1, \ldots, X_n = x_n).$$

- From joint pmf to marginal pmf:

$$p_{X_1}(x_1) = \sum_{x_2} \cdots \sum_{x_n} p(x_1, x_2, \ldots, x_n).$$

Similarly, we can calculate the joint pmf of (X_1, X_2) as

$$p_{X_1,X_2}(x_1, x_2) = \sum_{x_3} \cdots \sum_{x_n} p(x_1, x_2, x_3, \ldots, x_n).$$

- For any subset $B \subset \mathbb{R}^n$,

$$P((X_1, \ldots, X_n) \in B) = \sum_{(x_1,\ldots,x_n) \in B} p(x_1, \ldots, x_n).$$

- The joint cdf of $(X_1, \ldots, X_n)$ is defined by

$$\begin{aligned} F(x_1, \ldots, x_n) &= P(X_1 \leq x_1, \ldots, X_n \leq x_n) \\ &= \sum_{y_1 \leq x_1} \cdots \sum_{y_n \leq x_n} p(y_1, \ldots, y_n). \end{aligned}$$

Continuous random vector

- A function $f(x_1, \ldots, x_n)$ on $\mathbb{R}^n$ is a pdf on $\mathbb{R}^n$ if $f(x_1, \ldots, x_n) \geq 0$ for all $(x_1, \ldots, x_n) \in \mathbb{R}^n$ and

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x_1, \ldots, x_n) dx_1 \cdots dx_n = 1.$$

- A random vector $(X_1, \ldots, X_n)$ is continuous if there exists a pdf $f(x_1, \ldots, x_n)$ on $\mathbb{R}^n$ such that

$$P((X_1, \ldots, X_n) \in B) = \int \cdots \int_B f(x_1, \ldots, x_n) dx_1 \cdots dx_n$$

for any subset B of $\mathbb{R}^n$. The function $f(x_1, \ldots, x_n)$ is the joint pdf of $(X_1, \ldots, X_n)$.

- From joint pdf to marginal pdf:

$$f_{X_1}(x_1) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x_1, x_2, \ldots, x_n) dx_2 \cdots dx_n.$$

Similarly, we can calculate the joint pdf of (X_1, X_2) as

$$f_{X_1,X_2}(x_1, x_2) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x_1, x_2, x_3, \ldots, x_n) dx_3 \cdots dx_n.$$

Joint cdf

- The joint cdf of $(X_1, \ldots, X_n)$ is defined by

$$\begin{aligned} F(x_1, \ldots, x_n) &= P(X_1 \leq x_1, \ldots, X_n \leq x_n) \\ &= \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_n} f(y_1, \ldots, y_n) dy_1 \cdots dy_n. \end{aligned}$$

- From joint cdf to joint pdf:

$$f(x_1, \ldots, x_n) = \frac{\partial^n}{\partial x_1 \cdots \partial x_n} F(x_1, \ldots, x_n).$$

Independence

Definition

Random variables $X_1, \ldots, X_n$ are independent if

$$P(X_1 \in A_1, \ldots, X_n \in A_n) = P(X_1 \in A_1) \cdots P(X_n \in A_n)$$

for any subsets $A_1, \ldots, A_n \subset \mathbb{R}$.

Definition

If random variables $X_1, \ldots, X_n$ are independent with common cdf F (on $\mathbb{R}$), then they are called a random sample from F.

$$\text{“}X_1, \ldots, X_n \sim F \text{ i.i.d.”},$$

where “i.i.d.” means “independent and identically distributed”. E.g.,

$$X_1, \ldots, X_n \sim N(\mu, \sigma^2) \text{ i.i.d.}$$

Types of data

- Cross section data $X_1, \ldots, X_n$: Taken from different individuals/households at one period.
- Time series data $X_1, \ldots, X_T$: Taken from the same individual for different time periods.
- Panel (or longitudinal) data $X_{it}, i = 1, \ldots, n; t = 1, \ldots, T$: Taken from different individuals for different time periods.

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- Spatial data, spatio-temporal data etc.

Independence: discrete case

Theorem

Suppose that $X_1, \ldots, X_n$ *are discrete. Then* $X_1, \ldots, X_n$ *are independent if and only if*

$$p(x_1, \ldots, x_n) = p_{X_1}(x_1) \cdots p_{X_n}(x_n)$$

for any $x_1, \ldots, x_n$.

Example

Independent Bernoulli trials $X_1, \ldots, X_n$ (cf. Lecture 4) are independent.

Example 9.1

Example

If $X_1, \ldots, X_n \sim Po(\lambda)$ i.i.d., then find their joint pmf.

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If $X_1, \ldots, X_n \sim Po(\lambda)$ i.i.d., then find their joint pmf.

The marginal pmf of X_i is

$$p_{X_i}(x_i) = \frac{\lambda^{x_i}}{x_i!} e^{-\lambda}, \; x_i = 0, 1, 2, \ldots,$$

so that

$$p(x_1, \ldots, x_n) = \prod_{i=1}^{n} \frac{\lambda^{x_i}}{x_i!} e^{-\lambda} = \frac{\lambda^{\sum_{i=1}^{n} x_i}}{\prod_{i=1}^{n} x_i!} e^{-n\lambda}.$$

Independence: continuous case

Theorem

Suppose that the vector $(X_1, \ldots, X_n)$ *is continuous with joint pdf* $f(x_1, \ldots, x_n)$*. Then* $X_1, \ldots, X_n$ *are independent if and only if*

$$f(x_1, \ldots, x_n) = f_{X_1}(x_1) \cdots f_{X_n}(x_n)$$

for any $x_1, \ldots, x_n$*.*

Theorem

If random variables $X_1, \ldots, X_n$ *are independent and continuous with pdfs* $f_{X_1}, \ldots, f_{X_n}$*, respectively, then the vector* $(X_1, \ldots, X_n)$ *is continuous with joint pdf*

$$f(x_1, \ldots, x_n) = f_{X_1}(x_1) \cdots f_{X_n}(x_n)$$

for any $x_1, \ldots, x_n$*.*

Functions of random vectors

Example

If X and Y are independent, then find the pmf/pdf of $Z = X + Y$.

Discrete case

Theorem

If X and Y are independent and discrete with pmfs p_X and p_Y, respectively, then the pmf of $Z = X + Y$ is

$$p_Z(z) = \sum_x p_X(x) p_Y(z - x)$$

for any z. This is called the convolution of p_X and p_Y.

Example 9.2

Example

If $X, Y \sim Po(\lambda)$ i.i.d., then $X + Y \sim Po(2\lambda)$.

Example 9.2

Example

If $X, Y \sim Po(\lambda)$ i.i.d., then $X + Y \sim Po(2\lambda)$.

$$\begin{aligned}
\sum_x p_X(x) p_Y(z-x) &= \sum_{x=0}^{z} \frac{\lambda^x}{x!} e^{-\lambda} \frac{\lambda^{z-x}}{(z-x)!} e^{-\lambda} \\
&= \frac{\lambda^z e^{-2\lambda}}{z!} \sum_{x=0}^{z} \underbrace{\frac{z!}{x!(z-x)!}}_{=\binom{z}{x}} \\
&= \frac{(2\lambda)^z}{z!} e^{-2\lambda},
\end{aligned}$$

where we have used the binomial theorem:

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}.$$

Continuous case

Theorem

If X and Y are independent and continuous with pdfs f_X and f_Y, respectively, then the pdf of $Z = X + Y$ is

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z - x) dx$$

for any z. This is called the convolution of f_X and f_Y.

Proof?

Example 9.3

Example

If $X, Y \sim Ex(\lambda)$ i.i.d., then find the pdf of $X + Y$.

Example 9.3

Example

If $X, Y \sim Ex(\lambda)$ i.i.d., then find the pdf of $X + Y$.

The pdf of $Z = X + Y$ is

$$f_Z(z) = \begin{cases} \lambda^2 z e^{-\lambda z} & \text{if } z \geq 0 \\ 0 & \text{otherwise} \end{cases} .$$

Quotient

Example

If X and Y are independent and continuous, find the pdf of Y/X.

Theorem

If X and Y are independent and continuous with pdfs f_X and f_Y, respectively, then the pdf of $Z = Y/X$ is

$$f_Z(z) = \int_{-\infty}^{\infty} |x| f_X(x) f_Y(xz) dx.$$

Rice p. 98.

Example 9.4

Example

If $X, Y \sim N(0,1)$ i.i.d., then find the pdf of $Z = Y/X$.

Example 9.4

Example

If $X, Y \sim N(0, 1)$ i.i.d., then find the pdf of $Z = Y/X$.

The pdf of Z is

$$f_Z(z) = \frac{1}{\pi(1 + z^2)}, \ -\infty < z < \infty.$$

This is called the Cauchy density.