ORIE 4630: Fall Term 2017
Homework #7
Due: Thursday, March 28, 2017

Students are required to work independently on homework. You should not give or receive help from other students. You should also not receive help from students or former students who took this course in previous years and who may have solutions to similar problems. The solutions you submit should be your own work and not copied from elsewhere.

Homework is due at the end of lecture (12:55pm) on the due date. You will usually have one week to do the assignments. Please don't wait until the homework is nearly due to start. Late homework is not accepted. Also, homework is not accepted by email. You can submit your assignment in lecture or in the drop box in Rhodes Hall.

Please print your name on the front of your homework so that it is legible.

Include your `R` code, output, graphs, and other work with your homework. This will allow the grader to find any errors you make and to give partial credit.

1. [**5 points**] An investor puts \$200 in a money market account that pays interest at an annual rate of 6% compounded daily. How much money does the investor have in the account after two years, with 365 days per year? What would your answer be if the interest was compounded annually?

2. [**5 points**] An heiress knows that she will be receiving \$10,000,000 from her uncle's estate at the end of two years. Suppose that interest is compounded annually and that the annual rate of interest is 3% for the first year and 5% for the second year. What is the current, i.e., discounted value, of the heiress' inheritance?

3. [**5 points**] The stock of a certain company currently sells for \$1,200 per share. A buyer and a seller enter a contract in which the buyer agrees to purchase one share of the stock from the seller for an amount P 200 days from now. What should P be if the risk-free annual interest rate is 4% compounded continuously?

4. [**15 points**] Consider a one-period binomial tree model with nodes 1, 2, and 3. Suppose that the stock price at the start of the period is $s_1 = \$100$; at the end of the period the stock price either decreases to $s_2 = \$85$ or increases to $s_3 = \$110$. Now consider a call option with strike price $K = \$90$ on one share of stock that expires at the end of the period. Assume that the risk-free interest rate over the period is 7%.

i) What is $f(2)$, the value of the option at node 2?

ii) What is $f(3)$, the value of the option at node 3?

iii) What is the hedge ratio?

iv) Describe a synthetic portfolio that replicates the call option.

v) What is the fair price, i.e., the arbitrage-free price, of the option?

vi) Derive the risk-neutral probability q that the stock price increases to \$110 at the end of the period.

vii) What is the expected value of the option at the end of the period under the risk-neutral probability?

viii) Show that the expected value of the option at the end of the period when discounted to the beginning to the period is the same as the fair price you derived from the synthetic portfolio in part v).

5. [15 points] Consider a one-period binomial tree model with nodes 1, 2, and 3. Suppose that the stock price at the start of the period is $s_1 = \$50$; at the end of the period the stock price either decreases to $s_2 = \$40$ or increases to $s_3 = \$60$. Consider a put option with strike price $K = \$48$ on one share of stock that expires at the end of the period. Recall that a put option gives the holder the right to sell the stock for the strike price at expiry. Assume that the risk-free interest rate over the period is 4%.

i) What is $f(2)$, the value of the option at node 2?

ii) What is $f(3)$, the value of the option at node 3?

iii) What is the hedge ratio?

iv) Describe a synthetic portfolio that replicates the put option.

v) What is the price of the option?

vi) Derive the risk-neutral probability q that the stock price moves to \$60 at the end of the period.

vii) What is the expected value of the option at the end of the period under the risk-neutral probability?

viii) Show that the expected value of the option at the end of the period when discounted to the beginning to the period is the same as the fair price you derived from the synthetic portfolio in part v).

6. [15 points] Consider a two-period binomial tree model so that the binomial tree has nodes $1, 2, \ldots, 7$. The price of the stock at the seven nodes are $s_1 = 100$, $s_2 = 90$, $s_3 = 110$, $s_4 = 70$, $s_5 = 95$, $s_6 = 100$, and $s_7 = 125$. Suppose, for convenience, that the risk-free rate per period is 0%.

i) Find the risk-neutral probabilities q_1, q_2, and q_3 for nodes 1, 2, and 3.

ii) Consider a European call option with strike price \$90 on one share of stock that expires at the end of the two periods. Find the value of the option at nodes $1, 2, \ldots, 7$. What is the price of the option at the start of the two periods?

iii) Consider a European call option with strike price \$97 on one share of stock that expires at the end of the two periods. What is the price of the option at the start of the two periods?

iv) Consider a European call option with strike price \$114 on one share of stock that expires at the end of the two periods. What is the price of the option at the start of the two periods?

7. [15 points] Consider the same two-period binomial tree model described in Question 6.

i) Consider a European put option with strike price \$114 on one share of stock that expires at the end of the two periods. Find the value of the option at nodes $1, 2, \ldots, 7$. What is the price of the option at the start of the two periods?

ii) Consider a European put option with strike price \$97 on one share of stock that expires at the end of the two periods. What is the price of the option at the start of the two periods?

iii) Consider a European put option with strike price \$90 on one share of stock that expires at the end of the two periods. What is the price of the option at the start of the two periods?

iv) Show that, in each of the cases i), ii), and iii), $P = C + K - s_1$, where P is the price of the European put option with strike price K at the start of the two periods, C is the price of the European call option with strike price K at the start of the two periods, and s_1 is the price of the stock at the beginning of the two periods.

8. [15 points] Consider a two-period binomial tree model where the stock prices starts at \$100, and at each node, the price either moves up by 15% or moves down by 10%. Suppose that the risk-free interest rate is 5% per period. Consider a put option with strike price \$105 on one share of stock that expires at the end of the two periods.

i) Describe the risk-neutral probabilities for the binomial tree.

ii) Suppose the put option is a European option. What is the price of the option at the start of the two periods?

iii) Suppose that the put option is an American option. What is the price of the option at the start of the two periods?

9. [10 points] Consider an American call option on one share of stock in a binomial tree model where the risk-free interest rate per period is r. Recall that node j leads to nodes $2j$ and $2j+1$ and that the risk-neutral probability q_j for node j is given by $q_j = \dfrac{(1+r)s_j - s_{2j}}{s_{2j+1} - s_{2j}}$. Suppose that the option is not exercised at node $2j+1$ or at node $2j$, i.e., $f(2j+1) \geq s_{2j+1} - K$ and $f(2j) \geq s_{2j} - K$. The discounted expected value at node j under the risk-neutral probability q_j is $f(j) = \dfrac{1}{1+r}\big\{q_j f(2j+1) + (1-q_j)f(2j)\big\}$. Show that $f(j) \geq s_j - K$, i.e., show that the value of the option under the risk-neutral probability is at least as large as the value of the option if the option is exercised.