STSCI 5080 Homework 6 (optional)

- Due is 12/4 (Tu) in class, but you can also submit HW 6 on 11/29 (Th) in class.

- Write your name and NetID at the top of the first page, along with the assignment number.

- Use only the one side of the paper. Attach your pages with a staple at the top left corner.

- This is an extra credit work. You can earn extra 2% from this homework.

- Please complete the course evaluation when you have time. Constructive comments welcome!

Problems

1. Let $X \sim Bin(n, p)$. We know that the MLE is $\widehat{p} = X/n$ and $\sqrt{n}(\widehat{p} - p) \xrightarrow{d} N(0, p(1 - p))$.

   (a) (5 points) The arcsine function $\arcsin y$ is the inverse function of the sine function $\sin x$ on $-\pi/2 \leq x \leq \pi/2$, i.e., $\sin(\arcsin y) = y$ for $|y| \leq 1$. Verify that the derivative of the arsine function is
   $$(\arcsin y)' = \frac{1}{\sqrt{1 - y^2}}, \quad |y| < 1.$$

   (Hint). Differentiate both sides of $\sin(\arcsin y) = y$ w.r.t. y.

   (b) (5 points) Consider the function
   $$g(y) = 2\arcsin(\sqrt{y}).$$

   Verify that $g(y)$ is the variance stabilizing transformation for $\widehat{p}$.

2. Let
   $$X_1, \ldots, X_n \sim Ex(\lambda) \text{ i.i.d.}$$

   for some $\lambda > 0$. We know that the MLE is $\widehat{\lambda} = 1/\overline{X}$ and $\sqrt{n}(\widehat{\lambda} - \lambda) \xrightarrow{d} N(0, \lambda^2)$ as $n \to \infty$. Consider the following testing problem:
   $$H_0 : \lambda = 0.5 \quad \text{vs.} \quad H_1 : \lambda \neq 0.5.$$

   (a) (5 points) Find a test based on the MLE that has asymptotic level 5%. You don't need any derivation in this question and so you only have to state your test.

   (b) (5 points) Justify that your test in Part (a) has asymptotic level 5%.