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STSCI 5080 Midterm Exam 1 Solutions

Problem 1. Circle the correct choice in each of the following questions.

(1) [5 points] Two six-sided dice are thrown sequentially, and the face values that come up are recorded. If all the outcomes occur equally likely, what is the probability that the sum of two face values is 6?

$\boxed{\text{a.}}$ $\frac{5}{36}$ b. $\frac{1}{6}$ c. $\frac{7}{36}$ d. $\frac{2}{9}$

(2) [5 points] Suppose that two events A and B are independent and such that $P(A) = 1/3$ and $P(B) = 3/8$. What is the probability of $B \cap A^c$?

a. $\frac{1}{8}$ $\boxed{\text{b.}}$ $\frac{1}{4}$ c. $\frac{3}{8}$ d. $\frac{1}{2}$

(3) [5 points] Suppose that two events A and B are such that $P(B) = 2/3$ and $P(A \cap B) = 1/9$. What is the conditional probability $P(A \mid B)$?

a. $\frac{1}{18}$ $\boxed{\text{b.}}$ $\frac{1}{6}$ c. $\frac{1}{3}$ d. $\frac{1}{2}$

(4) [5 points] Suppose that the number of typos on a single page of a certain book follows the Poisson distribution with parameter $\lambda = 1/2$. What is the probability that there is at least one error on this page?

$\boxed{\text{a.}}$ $1 - e^{-1/2}$ b. $e^{-1/2}$ c. $1 - e^{-1}$ d. e^{-1}

(5) [5 points] Find the correct statement. Only one of them is correct and the other three are incorrect.

a. If f and g are probability density functions, then $2f - g$ is a also a probability density function.

b. If f and g are probability density functions, then their product fg is a also a probability density function.

c. A vector of two continuous random variables is continuous as a random vector.

$\boxed{\text{d.}}$ None of a, b, and c are correct.

Problem 2. Urn X has three red and six blue balls. Urn Y has six red and three blue balls. First, we choose urn X with probability 3/4 and urn Y with probability 1/4, and then draw a ball from the chosen urn.

(a) [5 points] What is the probability that a red ball is drawn?

(b) [5 points] If a red ball is drawn, what is the probability that urn Y was chosen?

(a) Define events

$$\begin{aligned} A &= \text{a red ball is drawn}, \\ B &= \text{urn X is chosen}. \end{aligned}$$

We know that $P(B) = 3/4, P(A \mid B) = 3/9 = 1/3$, and $P(A \mid B^c) = 6/9 = 2/3$. Hence, the law of total probability yields that

$$P(A) = P(A \mid B)P(B) + P(A \mid B^c)P(B^c) = \frac{1}{3} \cdot \frac{3}{4} + \frac{2}{3} \cdot \frac{1}{4} = \frac{1}{4} + \frac{1}{6} = \frac{5}{12}.$$

(b) By the Bayes rule,

$$P(B^c \mid A) = \frac{P(A \mid B^c)P(B^c)}{P(A)} = \frac{\frac{1}{6}}{\frac{5}{12}} = \frac{2}{5}.$$

Problem 3. Define a probability density function f by

$$f(x) = \begin{cases} 2xe^{-x^2} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}.$$

(a) [5 points] Compute the corresponding cumulative distribution function F.

(b) [5 points] Compute the quantile function F^{-1} of F.

(c) [5 points] Explain how to generate a random variable with cumulative distribution function F from a uniform random variable U on $[0, 1]$.

(a) Since $f(x) = 0$ for $x < 0$, we have $F(x) = 0$ for $x < 0$. For $x \geq 0$, we have

$$F(x) = \int_0^x f(y)dy = \int_0^x 2ye^{-y^2} dy = \left[-e^{-y^2}\right]_0^x = 1 - e^{-x^2},$$

where we have used the fact that $(-e^{-y^2})' = 2ye^{-y^2}$.

(b) For $u \in (0, 1)$, we need to solve $1 - e^{-x^2} = u$ w.r.t. $x \geq 0$. Since

$$\begin{aligned} & 1 - e^{-x^2} = u \\ \Leftrightarrow\ & e^{-x^2} = 1 - u \\ \Leftrightarrow\ & -x^2 = \log(1 - u) \\ \Leftrightarrow\ & x = \sqrt{-\log(1 - u)}, \end{aligned}$$

we have $F^{-1}(u) = \sqrt{-\log(1 - u)}$ for $u \in (0, 1)$.

(c) $Y = F^{-1}(U) = \sqrt{-\log(1 - U)}$ has cdf F.

Problem 4. Let X and Y be independent exponential random variables with parameters $1/2$ and 1, respectively.

(a) [5 points] Find the joint probability density function of (X, Y).

(b) [5 points] Calculate $P(X > 2Y)$.

(a) The marginal pdfs of X and Y are

$$f_X(x) = \begin{cases} \frac{1}{2}e^{-x/2} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}, \quad f_Y(y) = \begin{cases} e^{-y} & \text{if } y \geq 0 \\ 0 & \text{otherwise} \end{cases}.$$

Since X and Y are independent, the joint pdf is

$$f(x, y) = f_X(x) f_Y(y) = \frac{1}{2} e^{-\frac{x}{2} - y}, \ x, y \geq 0$$

and $f(x, y) = 0$ elsewhere.

(b) We have

$$\begin{aligned} P(X > 2Y) &= \int_0^\infty \int_0^{x/2} f(x, y) dy dx = \int_0^\infty \frac{1}{2} e^{-x/2} \left\{ \int_0^{x/2} e^{-y} dy \right\} dx \\ &= \int_0^\infty \frac{1}{2} e^{-x/2} (1 - e^{-x/2}) dx = 1 - \frac{1}{2} \int_0^\infty e^{-x} dx \\ &= 1 - \frac{1}{2} = \frac{1}{2}. \end{aligned}$$

Problem 5. Let (X, Y) be a continuous random vector with joint probability density function

$$f(x,y) = \begin{cases} c(x-y) & \text{if} \quad 0 \le y \le x \le 1 \\ 0 & \text{otherwise} \end{cases},$$

where c is a positive constant.

(a) [5 points] Sketch the region on which the function f is positive.

(b) [5 points] Find the value of the constant c.

(c) [5 points] Find the marginal probability density function of X.

(d) [5 points] Find the conditional probability density function of Y given X.

(a) Skip.

(b) We have

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y) dy dx = c \int_0^1 \int_0^x (x-y) dy dx = c \int_0^1 \left[xy - \frac{y^2}{2} \right]_{y=0}^{y=x} dx = \frac{c}{2} \int_0^1 x^2 dx = \frac{c}{6}.$$

We need the above integral to be 1, so that $c = 6$.

(c) The marginal pdf of X is

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y) dy = \int_0^x f(x,y) dy = 6 \int_0^x (x-y) dy = 3x^2$$

for $0 \le x \le 1$, and $f_X(x) = 0$ elsewhere.

(d) The conditional pdf of Y given X is

$$f_{Y|X}(y \mid x) = \frac{f(x,y)}{f_X(x)} = \frac{2(x-y)}{x^2}$$

for $0 \le y \le x \le 1$ and $x > 0$, and $f_{Y|X}(y \mid x) = 0$ elsewhere.