STSCI 5080
Probability Models and Inference

Lecture 24: Testing

November 29, 2018

Two sided alternative hypothesis

- Consider the testing problem:

$$H_0 : \theta = \theta_0 \quad \text{vs.} \quad H_1 : \theta \neq \theta_0.$$

Suppose that the MLE $\widehat{\theta}$ is such that

$$\sqrt{n}(\widehat{\theta} - \theta) \xrightarrow{d} N(0, \sigma^2(\theta))$$

for any value of θ.

- Consider the statistic

$$T_n = \frac{\sqrt{n}(\widehat{\theta} - \theta_0)}{\sigma(\theta_0)}.$$

Note: θ_0 and not θ!

Recap

For the testing problem,

$$H_0 : \theta = \theta_0 \quad \text{vs.} \quad H_1 : \theta \neq \theta_0,$$

a test with asymptotic level α is given by

$$\left| \frac{\sqrt{n}(\widehat{\theta} - \theta_0)}{\sigma(\theta_0)} \right| > z_{\alpha/2} \Rightarrow \text{reject } H_0,$$

where $z_{\alpha/2} = \Phi^{-1}(1 - \alpha/2)$ and Φ is the cdf of the $N(0, 1)$-distribution.
If $\alpha = 0.05$, we can choose $z_{\alpha/2} = 1.96$.

Why does this test have asymptotic level α?
If $\theta = \theta_0$,

$$\frac{\sqrt{n}(\widehat{\theta} - \theta_0)}{\sigma(\theta_0)} \xrightarrow{d} Z \sim N(0, 1),$$

and so

$$P_{\theta=\theta_0} \left\{ \left| \frac{\sqrt{n}(\widehat{\theta} - \theta_0)}{\sigma(\theta_0)} \right| > z_{\alpha/2} \right\} \approx P(|Z| > z_{\alpha/2}) = \alpha.$$

One-sided alternative hypothesis

- Consider the testing problem:

$$H_0 : \theta = \theta_0 \quad \text{vs.} \quad H_1 : \theta > \theta_0.$$

Suppose that the MLE $\widehat{\theta}$ is such that

$$\sqrt{n}(\widehat{\theta} - \theta) \xrightarrow{d} N(0, \sigma^2(\theta))$$

for any value of θ.

- Again consider the statistic

$$T_n = \frac{\sqrt{n}(\widehat{\theta} - \theta_0)}{\sigma(\theta_0)}.$$

Note: θ_0 and not θ!

If $\theta = \theta_0$, then

$$T_n \xrightarrow{d} N(0, 1).$$

On the other hand, if $\theta > \theta_0$, then

$$T_n \approx \frac{\sqrt{n}(\theta - \theta_0)}{\sigma(\theta_0)} \to \infty.$$

So, a reasonable test will be

$$T_n > c \Rightarrow \text{ reject } H_0.$$

Note: T_n and NOT $|T_n|$.

The threshold c is chosen in such a way that

$$\lim_{n\to\infty} P_{\theta=\theta_0}(T_n > c) = \alpha.$$

If $\theta = \theta_0$, then $T_n \xrightarrow{d} Z \sim N(0,1)$, and so

$$P_{\theta=\theta_0}(T_n > c) \approx P(Z > c) = 1 - \underbrace{P(Z \leq c)}_{=\Phi(c)}.$$

Solving

$$1 - \Phi(c) = \alpha, \text{ i.e., } \Phi(c) = 1 - \alpha,$$

we have

$$c = \Phi^{-1}(1-\alpha) = z_\alpha.$$

Note: z_α and NOT $z_{\alpha/2}$.

Typical values of z_α

$$z_\alpha \approx \begin{cases} 1.645 & \text{if } \alpha = 0.05 \\ 2.33 & \text{if } \alpha = 0.01 \end{cases}.$$

Recap

For the testing problem

$$H_0 : \theta = \theta_0 \quad \text{vs.} \quad H_1 : \theta > \theta_0,$$

a test with asymptotic level α is given by

$$\frac{\sqrt{n}(\widehat{\theta} - \theta_0)}{\sigma(\theta_0)} > z_\alpha \Rightarrow \text{reject } H_0.$$

Example 24.1

Let

$$X \sim Bin(n, p)$$

where $0 < p < 1$ is unknown, and consider the testing problem

$$H_0 : p = p_0 \quad \text{vs.} \quad H_1 : p > p_0.$$

The MLE is $\widehat{p} = X/n$ and $\sqrt{n}(\widehat{p} - p) \xrightarrow{d} N(0, p(1-p))$. Hence, a test with asymptotic level α is given by

$$\frac{\sqrt{n}(\widehat{p} - p_0)}{\sqrt{p_0(1-p_0)}} > z_\alpha \Rightarrow \text{reject } H_0.$$

Baaaaaaack to the very first example

- There is a theory that people can postpone their death until after an important event.
- To test the theory, Phillips and Smith[1] (1990) collected data on deaths around some (important!) festival for a certain group of people.
- Of 103 deaths, 33 died the week before the festival and 70 died the week after.

[1] D.P. Phillips and D.G. Smith. (1990). "Postponement of death until symbolically meaningful occasions". *JAMA* **263** 1947-1951.

- Suppose that each person dies after the festival with probability p.
- The total number of deaths after the festival X follows $Bin(n, p)$ where $n = 103$.
- In this example, $X = 70$, and so the MLE of p is

$$\widehat{p} = \frac{X}{n} = \frac{70}{103} = 0.68...$$

- If they can postpone their deaths, $p > 0.5$; otherwise $p = 0.5$.
- We want to test:

$$H_0 : p = 0.5 \quad \text{vs.} \quad H_1 : p > 0.5.$$

- The value of the test statistic is

$$\frac{\sqrt{n}(\widehat{p} - 0.5)}{\sqrt{0.5 \cdot 0,5}} = \frac{\sqrt{103}(0.68 - 0.5)}{\sqrt{0.5 \cdot 0.5}} = 3.65...$$

Large enough to reject H_0 even if $\alpha = 0.01$ at which $z_\alpha = 2.33$.

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Summary of the course

- Probability Part 1: Probability space, random variable, pmf/pdf, conditional pmf/pdf.
- Probability Part 2: Order statistics, expectation, variance, covariance, correlation, conditional expectation, mgf, LLN, CLT.
- Statistics Part: Sampling distributions derived from a normal distribution (χ^2 and t-distributions), estimation, confidence interval, and testing based on the method of maximum likelihood.

Topics that could have been covered

- Multivariate distributions (multinomial distribution and multivariate normal distribution).
- Sufficient statistics, unbiased estimation, exponential family, etc.
- Bayesian methods (posterior, posterior mean, credible interval).
- Bootstrap (alternative way to construct CIs).
- Optimization (how to find MLEs in complicated models?).

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I highly recommend you to study Bayesian methods/bootstrap/optimization. They are extremely important in modern statistics!

Practice problems[2]

[2] May or may not be useful in the final.

Problem 1

Let f_θ be a pmf of the form

$$f_\theta(x) = \begin{cases} \frac{1}{6}\theta & \text{if } x = 1 \\ \frac{1}{3}\theta & \text{if } x = 2 \\ \frac{1}{2}(1-\theta) & \text{if } x = 3 \\ 0 & \text{otherwise} \end{cases},$$

where $0 < \theta < 1$ is unknown. Your data set is

$$(X_1, \ldots, X_9) = (3, 3, 1, 2, 3, 2, 3, 3, 2).$$

What is the MLE (maximum likelihood estimate) of θ?

$$(X_1, \ldots, X_9) = (3, 3, 1, 2, 3, 2, 3, 3, 2).$$

The joint pmf is

$$\begin{aligned} f_\theta(X_1) \cdots f_\theta(X_9) &= f_\theta(3) \cdots f_\theta(2) \\ &= \frac{1}{6} \cdot \frac{1}{3^3} \cdot \frac{1}{2^5} \theta^4 (1-\theta)^5. \end{aligned}$$

The log likelihood function is

$$\ell_n(\theta) = \log f_\theta(X_1) \cdots f_\theta(X_9) = -\log(6 \cdot 3^3 \cdot 2^5) + 4 \log \theta + 5 \log(1-\theta).$$

The FOC is

$$\ell'(\theta) = 0 \Leftrightarrow \frac{4}{\theta} - \frac{5}{1-\theta} = 0.$$

The MLE is

$$\widehat{\theta} = \frac{4}{9}.$$

Delta method

Theorem

Suppose that $\sqrt{n}(Y_n - \mu) \xrightarrow{d} N(0, \sigma^2)$ *as* $n \to \infty$ *for some* $-\infty < \mu < \infty$ *and* $\sigma^2 > 0$, *and* $g(y)$ *is differentiable at* $y = \mu$. *Then*

$$\sqrt{n}\{g(Y_n) - g(\mu)\} \xrightarrow{d} N(0, \{g'(\mu)\}^2\sigma^2).$$

Problem 2

Let

$$X_1, \ldots, X_n \sim Po(\lambda) \text{ i.i.d.}$$

where $\lambda > 0$ is unknown. The MLE is $\widehat{\lambda} = \overline{X}$.
The mean and variance of $Po(\lambda)$ is λ. By CLT,

$$\sqrt{n}(\widehat{\lambda} - \lambda) = \sqrt{n}(\overline{X} - \lambda) \xrightarrow{d} N(0, \lambda).$$

Now, we want to estimate $\lambda^{-1/2}$ (skewness of $Po(\lambda)$). The MLE of $\lambda^{-1/2}$ is $\widehat{\lambda}^{-1/2}$.

Question

What is the limiting distribution of $\sqrt{n}(\widehat{\lambda}^{-1/2} - \lambda^{-1/2})$?

Recall that

$$(x^{\alpha})' = \alpha x^{\alpha-1}.$$

Let $g(\lambda) = \lambda^{-1/2}$. Since $g'(\lambda) = -\frac{1}{2}\lambda^{-3/2}$, we have

$$\begin{aligned}
\sqrt{n}(\widehat{\lambda}^{-1/2} - \lambda^{-1/2}) &= \sqrt{n}\{g(\widehat{\lambda}) - g(\lambda)\} \\
&\xrightarrow{d} N(0, \{g'(\lambda)\}^2 \lambda) \\
&= N(0, \lambda^{-2}/4).
\end{aligned}$$