9/11/18 Kengo Kato

STSCI 5080 Homework 2

- Due is 9/20 (Th) in class.
- Write your name and NetID at the top of the first page, along with the assignment number.
- Use only the one side of the paper. Attach your pages with a staple at the top left corner.
- There are five problems. Each problem is worth 10 points.

Problems

1. Suppose that you have four urns U_1, U_2, U_3, U_4, and for each $k = 1, 2, 3, 4$, urn U_k contains k red balls and $10 - k$ blue balls. Now, you first choose an urn with probability $1/4$ and then draw a ball from the chosen urn.
   (a) Calculate the probability that you draw a red ball. (Hint). The generalized law of total probability.
   (b) Calculate the probability that the chosen urn was U_4 given that you draw a red ball. (Hint). The Bayes rule.

2. Let X be a discrete random variable taking values in $\{0, 1, 2\}$ with $P(X = 0) = p, P(X = 1) = q$, and $P(X = 2) = 1 - p - q$, where p, q satisfy that $0 < p, q < 1$ and $p + q < 1$. Find the cdf of X and draw its graph. (Hint). The cdf of a discrete random variable is a step function.

3. Define a function g by
$$g(x) = \begin{cases} \sqrt{1 - x^2} & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}.$$
   (a) Draw the graph of g.
   (b) Define a function f by $f(x) = cg(x)$ for any real x where $c > 0$ is a constant. If f is a pdf, find the value of c. (Hint). The area of the unit circle is...

4. (**Rice 2.5.45**) Suppose that the lifetime of an electronic component follows an exponential distribution with $\lambda = 0.1$.
   (a) Find the probability that the lifetime is less than 10.
   (b) Find the probability that the lifetime is between 5 and 15.
   (c) Find t such that the probability that the lifetime is greater than t is 0.01.

5. (**Rice 2.5.60**) Find the pdf of $Y = e^X$ where $X \sim N(0, 1)$. This is called the *lognormal density*[1].

[1] You can google the lognormal density and find the answer but I want you to *derive* the lognormal density.

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Solutions STSCI 5080 Homework 2

1. (a) Define the following events:

$$
\begin{aligned}
A &= \text{you draw a red ball,}\\
B_k &= \text{you choose urn } U_k,\ k = 1, \dots, 4.
\end{aligned}
$$

Events $B_1, \dots, B_4$ form a partition of Ω, and $P(B_k) = 1/4$ for $k = 1, \dots, 4$. In addition, $P(A \mid B_k) = k/10$, so that the generalized law of total probability yields that

$$
P(A) = \sum_{k=1}^{4} P(A \mid B_k) P(B_k) = \sum_{k=1}^{4} \frac{k}{10} \cdot \frac{1}{4} = \frac{1}{4}.
$$

(b) We want to compute $P(B_4 \mid A)$. But the Bayes rule yields that

$$
P(B_4 \mid A) = \frac{P(A \mid B_4) P(B_4)}{P(A)} = \frac{\frac{4}{10} \cdot \frac{1}{4}}{\frac{1}{4}} = \frac{2}{5}.
$$

2. The cdf is

$$
F(x) = \begin{cases} 0 & \text{if } x < 0 \\ p & \text{if } 0 \le x < 1 \\ p + q & \text{if } 1 \le x < 2 \\ 1 & \text{if } x \ge 2 \end{cases}.
$$

3. (a) Skip.

(b) The integral $\int_{-\infty}^{\infty} g(x)dx = \int_0^1 g(x)dx$ coincides with $1/4$ of the area of the unit circle, and so

$$
\int_0^1 g(x)dx = \frac{\pi}{4}.
$$

Hence, we have $c = 4/\pi$.

Alternatively, we can directly compute the integral $\int_0^1 g(x)dx$ by using the change of variables $x = \sin\theta$ with $dx = \cos\theta d\theta$:

$$
\int_0^1 \sqrt{1 - x^2} dx = \int_0^{\pi/2} (\cos\theta)^2 d\theta = \int_0^{\pi/2} \frac{1 + \cos 2\theta}{2} d\theta = \frac{\pi}{4}.
$$

4. Let T be the lifetime. We know that $T \sim Exp(\lambda)$ with $\lambda = 0.1$. The cdf of T is

$$
F(t) = P(T \le t) = \lambda \int_0^t e^{-\lambda s} ds = 1 - e^{-\lambda t}
$$

for $t \ge 0$.

(a) $P(T \le 10) = F(10) = 1 - e^{-10\lambda} = 1 - e^{-1} \approx 0.63.$

(b) $P(5 \le T \le 15) = F(15) - F(5) = e^{-5\lambda} - e^{-15\lambda} = e^{-5\lambda}(1 - e^{-10\lambda}) = e^{-1/2}(1 - e^{-1}) \approx 0.38$.

(c) $P(T > t) = 1 - F(t) = e^{-\lambda t}$, and so solving

$$e^{-\lambda t} = 0.01,$$

we have

$$t = \frac{-\log(0.01)}{\lambda} = 10 \log(100) \approx 46.05.$$

5. Let f_X and f_Y denote pdfs of X and Y, respectively. We know that $f_X(x) = (2\pi)^{-1/2} e^{-x^2/2}$. Since Y is positive, $f_Y(y) = 0$ for $y \le 0$. For $y > 0$,

$$F_Y(y) = P(Y \le y) = P(e^X \le y) = P(X \le \log y) = F_X(\log y).$$

Differentiating both sides w.r.t. y, we have

$$f_Y(y) = (\log y)' f_X(\log y) = \frac{1}{\sqrt{2\pi} y} e^{-(\log y)^2/2}.$$