Fall 2018 STSCI 5080 Discussion 5 (9/28)

Problems

1. (**Rice 3.8.67**) A card contains n chips and has an error-correcting mechanism such that the card still functions if a single chip fails but does not function if two or more chips fail. If each chip has a lifetime that is an independent exponential with parameter λ, find the density function of the card's lifetime.

2. (**Rice 3.8.69**) Find the density of the minimum of n independent Weibull random variables, each of which has the density

$$f(t) = \beta\alpha^{-\beta}t^{\beta-1}e^{-(t/\alpha)^\beta},\ t \geq 0.$$

3. (**Rice 3.8.72**) Let $X_1, \ldots, X_n$ be independent continuous random variables each with cumulative distribution function F. Show that the joint cdf of $X_{(1)} = \min_{1\leq i\leq n} X_i$ and $X_{(n)} = \max_{1\leq i\leq n} X_i$ is

$$F(x, y) = F(y)^n - \{F(y) - F(x)\}^n,\ x \leq y.$$

4. (**Rice 4.7.2**) If X is a discrete random variable such that $P(X = 1/k) = 1/n$ for $k = 1, \ldots, n$, then find $E(X)$.

5. (**Rice 4.7.5**) Let X be a continuous random variable with pdf

$$f(x) = \frac{1 + \alpha x}{2},\ -1 \leq x \leq 1,$$

where $-1 \leq \alpha \leq 1$. Find $E(X)$.

6. (**Rice 4.7.8**) Show that if X is a discrete random variable taking values in the positive integers, then $E(X) = \sum_{k=1}^{\infty} P(X \geq k)$.

Solutions

1. (**Rice 3.8.67**) Let T_i denote the lifetime of the i-th chip. Then the card's lifetime is $T_{(2)}$ (second minimum among $X_1, \dots, X_n$), whose pdf is given by

$$f_{T_{(2)}}(t) = \frac{n!}{(n-2)!} f_T(t) F_T(t) \{1 - F_T(t)\}^{n-2}.$$

Since $f_T(t) = \lambda e^{-\lambda t}$ and $F_T(t) = 1 - e^{-\lambda t}$ for $t \geq 0$, we conclude that

$$f_{T_{(2)}}(t) = n(n-1)\lambda e^{-(n-1)\lambda t}(1 - e^{-\lambda t}), \ t \geq 0.$$

2. (**Rice 3.8.69**) Let $T_1, \dots, T_n \sim f$ i.i.d. Then the pdf of $X_{(1)}$ is

$$f_{T_{(1)}}(t) = nf(t)\{1 - F(t)\}^{n-1}.$$

Since $F(t) = 1 - e^{-(t/\alpha)^\beta}$ for $t \geq 0$, we conclude that

$$f_{T_{(1)}}(t) = n\beta\alpha^{-\beta} t^{\beta-1} e^{-n(t/\alpha)^\beta}, \ t \geq 0.$$

3. (**Rice 3.8.72**) Fix $x \leq y$. We will first evaluate

$$P(X_{(1)} > x, X_{(n)} \leq y).$$

We note that

$$X_{(1)} > x \text{ and } X_{(n)} \leq y \Leftrightarrow x < X_i \leq y, \text{ for all } i = 1, \dots, n,$$

so that

$$\begin{aligned} P(X_{(1)} > x, X_{(n)} \leq y) &= P(x < X_i \leq y, \text{ for all } i = 1, \dots, n) \\ &= \prod_{i=1}^n P(x < X_i \leq y) = \{F(y) - F(x)\}^n, \end{aligned}$$

where the second equality follows from independence of $X_1, \dots, X_n$.
Next, recall the decomposition

$$B = (B \cap A) \cup (B \cap A^c),$$

where the two events on RHS are disjoint, so that

$$P(B) = P(B \cap A) + P(B \cap A^c).$$

Setting $A = \{X_{(1)} \leq x\}$ and $B = \{X_{(n)} \leq y\}$, we have

$$P(X_{(n)} \leq y) = P(X_{(1)} \leq x, X_{(n)} \leq y) + P(X_{(1)} > x, X_{(n)} \leq y).$$

Since $P(X_{(n)} \leq y) = F(y)^n$, we conclude that

$$F(x, y) = P(X_{(n)} \leq y) - P(X_{(1)} \leq x, X_{(n)} \leq y) = F(y)^n - \{F(y) - F(x)\}^n.$$

4. (**Rice 4.7.2**) By definition,

$$E(X) = \sum_{k=1}^{n} kP(X = k) = \frac{1}{n}\sum_{k=1}^{n} k = \frac{1}{n} \cdot \frac{n(n+1)}{2} = \frac{n+1}{2}.$$

5. (**Rice 4.7.5**) By definition,

$$E(X) = \int_{-\infty}^{\infty} xf(x)dx = \int_{-1}^{1} xf(x)dx = \frac{1}{2}\int_{-1}^{1}(x + \alpha x^2)dx = \frac{1}{2}\left[\frac{x^2}{2} + \alpha\frac{x^3}{3}\right]_{-1}^{1} = \frac{\alpha}{3}.$$

6. (**Rice 4.7.8**) Since X takes values in the positive integers,

$$P(X \geq k) = \sum_{j=k}^{\infty} p(j),$$

so that

$$\sum_{k=1}^{\infty} P(X \geq k) = \sum_{k=1}^{\infty}\sum_{j=k}^{\infty} p(j) = \sum_{j=1}^{\infty}\sum_{k=1}^{j} p(j) = \sum_{j=1}^{\infty} jp(j) = E(X).$$

See also the following:

$$\begin{array}{lllll} k=1: & p(1) & p(2) & p(3) & \cdots \\ k=2: & & p(2) & p(3) & \cdots \\ k=3: & & & p(3) & \cdots \\ \quad\vdots & & & & \end{array}$$