Fall 2018 STSCI 5080 Supplemental Material 3 (9/27)

More general definition of expectation

In this supplemental material, we will define the expectation in a more rigorous way. First, we consider the discrete case.

Definition 1 (Expectation: discrete case)**.** Let X be a discrete random variable with pmf $p(x)$. If

$$\sum_{x>0} xp(x) < \infty \quad \text{or} \quad \sum_{x<0} |x|p(x) < \infty,$$

then, we define

$$E(X) = \sum_{x>0} xp(x) - \sum_{x<0} |x|p(x).$$

On the other hand, if

$$\sum_{x>0} xp(x) = \infty \quad \text{and} \quad \sum_{x<0} |x|p(x) = \infty,$$

then $E(X)$ is not defined.

Consider the following three cases: (i) $\sum_{x>0} xp(x) = \infty$ but $\sum_{x<0} |x|p(x) < \infty$; (ii) $\sum_{x>0} xp(x) < \infty$ but $\sum_{x<0} |x|p(x) = \infty$; and (iii) $\sum_{x>0} xp(x) < \infty$ and $\sum_{x<0} |x|p(x) < \infty$.

Case (i): $\sum_{x>0} xp(x) = \infty$ but $\sum_{x<0} |x|p(x) < \infty$. In this case, $E(X) = \infty$.

Case (ii): $\sum_{x>0} xp(x) < \infty$ but $\sum_{x<0} |x|p(x) = \infty$. In this case, $E(X) = -\infty$.

Case (iii): $\sum_{x>0} xp(x) < \infty$ and $\sum_{x<0} |x|p(x) < \infty$. In this case,

$$E(X) = \sum_{x>0} xp(x) + \sum_{x<0} xp(x) = \sum_{x} xp(x).$$

We note that

$$\sum_{x} |x|p(x) = \sum_{x>0} xp(x) + \sum_{x<0} |x|p(x)$$

and so

$$\begin{aligned}\sum_{x} |x|p(x) < \infty &\Leftrightarrow \sum_{x>0} xp(x) < \infty \text{ and } \sum_{x<0} |x|p(x) < \infty \\ &\Leftrightarrow E(X) \text{ is defined and finite.}\end{aligned}$$

The continuous case is completely analogous.

Definition 2 (Expectation: continuous case)**.** Let X be a continuous random variable with pdf $f(x)$. If

$$\int_0^\infty xf(x)dx < \infty \quad \text{or} \quad \int_{-\infty}^0 |x|f(x)dx < \infty,$$

then, we define

$$E(X) = \int_0^\infty xf(x)dx - \int_{-\infty}^0 |x|f(x)dx.$$

On the other hand, if

$$\int_0^\infty xf(x)dx < \infty = \infty \quad \text{and} \quad \int_{-\infty}^0 |x|f(x)dx = \infty,$$

then $E(X)$ is not defined.

Consider the following three cases: (i) $\int_0^\infty xf(x)dx = \infty$ but $\int_{-\infty}^0 |x|f(x)dx < \infty$; (ii) $\int_0^\infty xf(x)dx < \infty$ but $\int_{-\infty}^0 |x|f(x)dx = \infty$; and (iii) $\int_0^\infty xf(x)dx < \infty$ and $\int_{-\infty}^0 |x|f(x)dx < \infty$.

Case (i): $\int_0^\infty xf(x)dx = \infty$ but $\int_{-\infty}^0 |x|f(x)dx < \infty$. In this case, $E(X) = \infty$.

Case (ii): $\int_0^\infty xf(x)dx < \infty$ but $\int_{-\infty}^0 |x|f(x)dx = \infty$. In this case, $E(X) = -\infty$.

Case (iii): $\int_0^\infty xf(x)dx < \infty$ and $\int_{-\infty}^0 |x|f(x)dx < \infty$. In this case,

$$E(X) = \int_0^\infty xf(x)dx + \int_{-\infty}^0 xf(x)dx = \int_{-\infty}^\infty xf(x)dx.$$

We note that

$$\int_{-\infty}^\infty |x|f(x)dx = \int_0^\infty xf(x)dx + \int_{-\infty}^0 |x|f(x)dx$$

and so

$$\int_{-\infty}^\infty |x|f(x)dx < \infty \Leftrightarrow \int_0^\infty xf(x)dx < \infty \text{ and } \int_{-\infty}^0 |x|f(x)dx < \infty$$
$$\Leftrightarrow E(X) \text{ is defined and finite.}$$

Example 1. Let X be a discrete random variable with $P(X = 2^k) = 1/2^{k+1}$ for $k = 0, 1, \ldots$. In this case, X is positive and so $E(X)$ is defined (as $\sum_{x<0} |x|p(x) = 0$). However,

$$\sum_{x>0} xp(x) = \sum_{k=0}^\infty 2^k \frac{1}{2^{k+1}} = \infty,$$

so that $E(X)$ is defined but $E(X) = \infty$.

Example 2. Let X be a continuous random variable with the Cauchy density

$$f(x) = \frac{1}{\pi(1+x^2)}, \ -\infty < x < \infty.$$

Then,

$$\int_{-\infty}^0 |x|f(x)dx = \int_0^\infty xf(x)dx = \frac{1}{\pi}\int_0^\infty \frac{x}{1+x^2}dx = \frac{1}{2\pi}\left[\log(1+x^2)\right]_0^\infty = \infty,$$

so that $E(X)$ is not defined.