# Fall 2018 STSCI 5080 Discussion 7 (10/12)

## Problems

1. (**Rice 4.7.43**) Show that $\mathrm{Var}(X - Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) - 2\mathrm{Cov}(X, Y)$.

2. (**Rice 4.7.44**) If $X$ and $Y$ are independent random variables with unit variance, find $\mathrm{Cov}(X + Y, X - Y)$.

3. (**Rice 4.7.67**) A random rectangle is formed in the following way: The base, $X$, is chosen to be a uniform $[0, 1]$ random variable and after having generated the base, the height is chosen to be uniform on $[0, X]$. Use the law of total expectation to find the expected circumference and area of the rectangle.

4. (**Rice 4.7.75**) Let $T$ be an exponential random variable with parameter $\lambda$, and conditional on $T$, let $U$ be uniform on $[0, T]$. Find the unconditional mean and variance of $U$.

5. Find the mgf of $Bin(n, p)$ and prove the regeneration property of the binomial distribution:

$$Bin(n, p) * Bin(m, p) = Bin(n + m, p).$$

6. Show that if $X \sim Ga(\alpha, 1)$ then $\beta X \sim Ga(\alpha, \beta)$.

7. (**Rice 4.7.94**) If $X$ is a nonnegative integer-valued random variable, the *probability generating function* of $X$ is defined to be

$$G(s) = \sum_{k=0}^{\infty} s^k p_k,$$

where $p_k = P(X = k)$.

   (a) Show that

$$p_k = \frac{1}{k!} \frac{d^k}{ds^k} G(s)\Big|_{s=0}.$$

   (b) Show that

$$\frac{d}{ds} G(s)\Big|_{s=1} = E(X),$$
$$\frac{d^2}{ds^2} G(s)\Big|_{s=1} = E\{X(X - 1)\}.$$

   (c) Express the probability generating function $G(s)$ in terms of the mgf for $s > 0$.

   (d) Find the probability generating function of the Poisson distribution.

8. (a) Pick any $-\infty < x < \infty$. Show that $P(X > x) \le e^{-\theta x} E(e^{\theta X})$ for any $\theta > 0$. This is called *Chernoff's inequality*.

   (b) For $X \sim N(0, 1)$, show that $P(X > x) \le e^{-x^2/2}$ for any $x > 0$.

# Solutions

1. (**Rice 4.7.43**) We note that

$$\text{Var}(X - Y) = \text{Var}(X) + \text{Var}(-Y) + 2\text{Cov}(X, -Y).$$

It is not difficult to see that $\text{Var}(-Y) = \text{Var}(Y)$ and $\text{Cov}(X, -Y) = -\text{Cov}(X, Y)$.

2. (**Rice 4.7.44**) Let $\tilde{X} = X - E(X)$ and $Y = Y - \tilde{Y}$. We note that

$\text{Cov}(X + Y, X - Y) = E\{(\tilde{X} + \tilde{Y})(\tilde{X} - \tilde{Y})\} = E(\tilde{X}^2 - \tilde{Y}^2) = \text{Var}(X) - \text{Var}(Y) = 1 - 1 = 0.$

3. Let $Y$ denote the height. Conditionally on $X$, $Y \sim U[0, X]$, so that $E(Y \mid X) = X/2$ and hence $E(Y) = E\{E(Y \mid X)\} = 1/4$. The circumference is

$$2(X + Y)$$

so that

$$E\{2(X + Y)\} = 2E(X) + 2E(Y) = 2 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4} = \frac{3}{2}.$$

On the other hand, the area is

$$XY$$

and we want to evaluate $E(XY)$. First, we note that

$$\begin{aligned} E(XY) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (xy) f(x, y) dy dx = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (xy) f_{Y|X}(y \mid x) f_X(x) dy dx \\ &= \int_{-\infty}^{\infty} x \left\{ \int_{-\infty}^{\infty} y f_{Y|X}(y \mid x) dy \right\} f_X(x) dx = E\{X E(Y \mid X)\}. \end{aligned}$$

Second, since $E(Y \mid X) = X/2$, we have

$$E\{X E(Y \mid X)\} = \frac{E(X^2)}{2} = \frac{1}{2} \int_0^1 x^2 dx = \frac{1}{6}.$$

4. Since $U \sim U[0, T]$ conditionally on $T$, we have

$$E(U \mid T) = \frac{T}{2} \quad \text{and} \quad \text{Var}(U \mid T) = \frac{T^2}{12}.$$

Hence, we have

$$E(U) = E\{E(U \mid T)\} = \frac{1}{2\lambda}$$

and

$$\text{Var}(U) = \text{Var}\{E(U \mid T)\} + E\{\text{Var}(U \mid T)\} = \frac{\text{Var}(T)}{4} + \frac{E(T^2)}{12}.$$

Since $Ex(\lambda) = Ga(1, 1/\lambda)$, the mgf of $T$ is

$$\psi_T(\theta) = (1 - \theta/\lambda)^{-1}, \; \theta < \lambda,$$

so that

$$E(T) = \psi'(0) = \frac{1}{\lambda} \quad \text{and} \quad E(T^2) = \psi''(0) = \frac{2}{\lambda^2}.$$

Therefore, we have

$$\text{Var}(U) = \frac{1}{4\lambda^2} + \frac{1}{6\lambda^2} = \frac{5}{12\lambda^2}.$$

5. If $Y \sim Bin(n,p)$, then $Y = X_1 + \cdots + X_n$ for independent Bernoulli trials $X_1, \ldots, X_n$ with success probability $p$. Each $X_i$ has mgf

$$\psi_X(\theta) = E(e^{\theta X_i}) = e^\theta p + 1 \cdot (1-p) = 1 + p(e^\theta - 1),$$

and so the mgf of $Y$ is

$$\psi_Y(\theta) = E\{e^{\theta(X_1+\cdots+X_n)}\} = E(e^{\theta X_1}) \cdots E(e^{\theta X_n}) = \{\psi_X(\theta)\}^n = \{1 + p(e^\theta - 1)\}^n.$$

If $Y_1 \sim Bin(n,p)$ and $Y_2 \sim Bin(m,p)$ are independent, them the mgf of $Z = Y_1 + Y_2$ is

$$\psi_Z(\theta) = \psi_{Y_1}(\theta)\psi_{Y_2}(\theta) = \{1 + p(e^\theta - 1)\}^{n+m},$$

which is the mgf of $Bin(n+m,p)$. Hence, we have $Z \sim Bin(n+m,p)$.

6. The mgf of $X$ is

$$\psi_X(\theta) = (1-\theta)^{-\alpha},\ \theta < 1,$$

and the mgf of $Y = \beta X$ is

$$\psi_Y(\theta) = E(e^{\theta\beta X}) = \psi_X(\beta\theta) = (1-\beta\theta)^{-\alpha},\ \theta < 1/\beta,$$

which is the mgf of $Ga(\alpha,\beta)$. Hence, we have $Y \sim Ga(\alpha,\beta)$.

7. (**Rice 4.7.94**)

   (a) We note that

$$G^{(k)}(s) = \sum_{j=k}^{\infty} j(j-1)\cdots(j-k+1)s^{j-k}p_j = E\{X(X-1)\cdots(X-k+1)s^{X-k}\}. \quad (*)$$

   Plugging in $s = 0$, we have

$$G^{(k)}(0) = k!p_k, \quad \text{i.e.,} \quad p_k = G^{(k)}(0)/k!.$$

   (b) From (*), we have

$$G'(1) = E(X) \quad \text{and} \quad G''(0) = E\{X(X-1)\}.$$

   (c) Since $G(s) = E(s^X)$, we have $G(s) = \psi(\log s)$.

   (d) If $X \sim Po(\lambda)$, then

$$p_k = \frac{\lambda^k}{k!}e^{-\lambda},$$

   so that

$$G(s) = \sum_{k=0}^{\infty} \frac{(s\lambda)^k}{k!}e^{-\lambda} = e^{(s-1)\lambda}.$$

8. (a) Because of the equivalence $X > x \Leftrightarrow e^{\theta X} > e^{\theta x}$, we have

$$P(X > x) = P(e^{\theta X} > e^{\theta x}).$$

   Applying Markov's inequality, we have

$$P(e^{\theta X} > e^{\theta x}) \le e^{-\theta x}E(e^{\theta X}).$$

(b) If $X \sim N(0,1)$, we know that $E(e^{\theta X}) = e^{\theta^2/2}$, and so

$$P(X > x) \leq e^{-\theta x + \theta^2/2}$$

for any $\theta > 0$. The right hand side is minimized at $\theta = x$, and so

$$P(X > x) \leq e^{-x^2/2}.$$