Math 421, Section 1
Homework 3
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Problem 1. Determine whether each of the following functions are injective, surjective, and bijective, and prove your answer.

(a) $f : \mathbb{Z} \to \mathbb{Z}$, $f(x) = 2x$.
(b) $g : \mathbb{R} \to \mathbb{R}$, $g(x) = 2x$.

Solution: (a) Injectivity: Suppose $\exists x_1, x_2 \in \mathbb{Z}, s.t. f(x_1) = f(x_2)$, want to show: $x_1 = x_2$.

$$f(x_1) = f(x_2) \implies 2x_1 = 2x_2 \implies x_1 = x_2. \tag{1}$$

The function is thus injective.

Surjectivity: Want to show $\forall y \in \mathbb{Z}, \exists x \in \mathbb{Z}$ $s.t.$ $f(x) = y$. Suppose $x, y \in \mathbb{Z}$, and let $f(x) = y$. i.e.,

$$2x = y \implies x = \frac{y}{2} \in \mathbb{Z}. \tag{2}$$

However, $\frac{y}{2} \in \mathbb{Z}$ only if y is even. So the above is not true for an arbiturary $y \in \mathbb{Z}$, contradictory to our assumption. Thus, the function is not surjective.

Collecting the above, the function is not bijective.

(b) Injectivity: Suppose $\exists x_1, x_2 \in \mathbb{R}, s.t. g(x_1) = g(x_2)$, want to show: $x_1 = x_2$.

$$g(x_1) = g(x_2) \implies 2x_1 = 2x_2 \implies x_1 = x_2. \tag{3}$$

The function is thus injective.

Surejectivity: Suppose $y \in \mathbb{R}$, we want to find $x \in \mathbb{R}$, $s.t.$ $g(x) = y$.

$$2x = y \implies x = \frac{y}{2} \in \mathbb{R}. \tag{4}$$

So the function is surjective.

Collecting the above, the function $g(x)$ is bijective.

$\square$

Problem 2. Let $f : A \to B$ be a function and $A_1, A_2 \subseteq A$ and $B_1, B_2 \subseteq B$ be subsets. Prove the following statements:

(a) $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$.
(b) $f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2)$.
(c) $f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$.
(d) $f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$.

Solution: (a) *Proof.* $\subseteq$: Let $y \in f(A_1 \cup A_2)$.By definition of image, $\exists x \in A_1 \cup A_2$ s.t. $f(x) = y$.
Hence, $x \in A_1$ or $x \in A_2$. Thus, $y \in f(A_1)$ or $y \in f(A_2)$, implying $y \in f(A_1) \cup f(A_2)$.
Therefore, $f(A_1 \cup A_2) \subseteq f(A_1) \cup f(A_2)$

$\supseteq$: Let $y \in f(A_1) \cup f(A_2)$. Then $y \in f(A_1)$ or $y \in f(A_2)$.
Thus, $\exists x \in A_1$ or $x \in A_2$ s.t. $f(x) = y$.
Therefore, $x \in A_1 \cup A_2$ and $y = f(x) \in f(A_1 \cup A_2)$.
Thus, $f(A_1) \cup f(A_2) \subseteq f(A_1 \cup A_2)$.

Hence, $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$. □

(b) *Proof.* Let $y \in f(A_1 \cap A_2)$. Then $\exists x \in A_1 \cap A_2$ s.t. $f(x) = y$.
Since $x \in A_1$ and $x \in A_2$, $y \in f(A_1)$ and $y \in f(A_2)$. Thus, $y \in f(A_1) \cap f(A_2)$.
Therefore, $f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2)$. □

(c) *Proof.* $\subseteq$: Let $x \in f^{-1}(B_1 \cup B_2)$. Then $f(x) \in B_1 \cup B_2$, so $f(x) \in B_1$ or $f(x) \in B_2$.
Hence, $x \in f^{-1}(B_1)$ or $x \in f^{-1}(B_2)$, implying $x \in f^{-1}(B_1) \cup f^{-1}(B_2)$.
Thus $f^{-1}(B_1 \cup B_2) \subseteq f^{-1}(B_1) \cup f^{-1}(B_2)$

$\supseteq$: Let $x \in f^{-1}(B_1) \cup f^{-1}(B_2)$.
Then $x \in f^{-1}(B_1)$ or $x \in f^{-1}(B_2)$, meaning $f(x) \in B_1$ or $f(x) \in B_2$.
Thus, $f(x) \in B_1 \cup B_2$ and $x \in f^{-1}(B_1 \cup B_2)$.
So $f^{-1}(B_1) \cup f^{-1}(B_2) \subseteq f^{-1}(B_1 \cup B_2)$
Therefore, $f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$. □

(d) *Proof.* $\subseteq$: Let $x \in f^{-1}(B_1 \cap B_2)$. Then $f(x) \in B_1 \cap B_2$, so $f(x) \in B_1$ and $f(x) \in B_2$.
Hence, $x \in f^{-1}(B_1)$ and $x \in f^{-1}(B_2)$, implying $x \in f^{-1}(B_1) \cap f^{-1}(B_2)$.
So, $f^{-1}(B_1 \cap B_2) \subseteq f^{-1}(B_1) \cap f^{-1}(B_2)$

$\supseteq$: Let $x \in f^{-1}(B_1) \cap f^{-1}(B_2)$. Then $f(x) \in B_1$ and $f(x) \in B_2$, so $f(x) \in B_1 \cap B_2$.
Thus, $x \in f^{-1}(B_1 \cap B_2)$.
Therefore, $f^{-1}(B_1) \cap f^{-1}(B_2) \subseteq f^{-1}(B_1 \cap B_2)$.

Therefore, $f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$. □

□

Problem 3. Let $f : A \to B$ be a function. Prove that f is injective if and only if $f(A_1 \cap A_2) = f(A_1) \cap f(A_2)$ for all subsets $A_1, A_2 \subseteq A$.

Solution: We will prove the equivalence by demonstrating both implications.

1. f **is injective** $\Rightarrow \forall A_1, A_2 \subseteq A,\ f(A_1 \cap A_2) = f(A_1) \cap f(A_2)$

Proof. Assume f is injective.

$\subseteq$ Let $y \in f(A_1 \cap A_2)$. Then $\exists x \in A_1 \cap A_2$ such that $f(x) = y$. Since $x \in A_1$ and $x \in A_2$, it follows that $y \in f(A_1)$ and $y \in f(A_2)$. Therefore, $y \in f(A_1) \cap f(A_2)$.

$\supseteq$ Let $y \in f(A_1) \cap f(A_2)$. Then $\exists x_1 \in A_1$ and $\exists x_2 \in A_2$ such that $f(x_1) = y$ and $f(x_2) = y$. Since f is injective, $x_1 = x_2$. Let $x = x_1 = x_2$. Then $x \in A_1 \cap A_2$, and hence $y = f(x) \in f(A_1 \cap A_2)$.

Thus, $f(A_1 \cap A_2) = f(A_1) \cap f(A_2)$ when f is injective. □

2. $\forall A_1, A_2 \subseteq A,\ f(A_1 \cap A_2) = f(A_1) \cap f(A_2) \Rightarrow f$ **is injective**

Proof. Assume $\forall A_1, A_2 \subseteq A,\ f(A_1 \cap A_2) = f(A_1) \cap f(A_2)$. We aim to show that f is injective.

Suppose, for contradiction, that f is not injective. Then $\exists x_1, x_2 \in A$ with $x_1 \neq x_2$ and $f(x_1) = f(x_2) = y$.

Consider the subsets $A_1 = \{x_1\}$ and $A_2 = \{x_2\}$.

Since $x_1 \neq x_2$,

$$A_1 \cap A_2 = \emptyset.$$

Thus,

$$f(A_1 \cap A_2) = f(\emptyset) = \emptyset.$$

$$f(A_1) = \{f(x_1)\} = \{y\}, \quad f(A_2) = \{f(x_2)\} = \{y\},$$

so

$$f(A_1) \cap f(A_2) = \{y\} \cap \{y\} = \{y\}.$$

We have

$$f(A_1 \cap A_2) = \emptyset \neq \{y\} = f(A_1) \cap f(A_2),$$

which contradicts the assumption. Therefore, f must be injective. □

□

Problem 4. Let $f : A \to B$ be a function. Prove that the following two statements are equivalent:

(a) The function f is surjective.
(b) For every set C and for any functions $g : B \to C$ and $h : B \to C$ such that $g \circ f = h \circ f$, we have $g = h$.

Solution: (1) implies (2):

Assume f is surjective. Let C be an arbitrary set, and let $g, h : B \to C$ satisfy $g \circ f = h \circ f$.

For any $b \in B$, since f is surjective, there exists $a \in A$ such that $f(a) = b$. Therefore,

$$g(b) = g(f(a)) = (g \circ f)(a) = (h \circ f)(a) = h(f(a)) = h(b).$$

Hence, $g = h$.

(2) implies (1):

Assume statement 2 holds. Suppose, for contradiction, that f is not surjective. Then there exists $b_0 \in B$ such that $b_0 \notin \text{Image}(f)$. We will use this element to construct specific functions g and h that satisfy the premise of statement 2 but are not equal, leading to a contradiction:

Let $C = \{0, 1\}$ and define the functions $g, h : B \to C$ as follows:

$$g(b) = \begin{cases} 0 & \text{if } b = b_0, \\ 1 & \text{otherwise,} \end{cases} \qquad h(b) = 1 \text{ for all } b \in B.$$

Since $b_0 \notin \text{Image}(f)$, for all $a \in A$, $g(f(a)) = 1 = h(f(a))$. Thus, $g \circ f = h \circ f$. However, $g \neq h$ because $g(b_0) = 0$ while $h(b_0) = 1$, which contradicts the uniqueness condition.

Therefore, f must be surjective.

$\square$

Problem 5. Let A be a nonempty set and $f : A \to A$ a function. We call f an *involution* if $(f \circ f)(a) = a$ for all $a \in A$. Prove that if $f : A \to A$ is an involution, then f is bijective. What is the inverse function f^{-1} in terms of f?

Solution: 1. Injectivity

Assume that for some $a_1, a_2 \in A$,

$$f(a_1) = f(a_2).$$

Applying f to both sides of the equation:

$$f(f(a_1)) = f(f(a_2)).$$

Given that f is an involution:

$$(f \circ f)(a_1) = (f \circ f)(a_2) a_1 = a_2.$$

Thus, f is injective.

2. Surjectivity

Take any element $b \in A$. Since f is an involution:

$$f(f(b)) = b.$$

Let $a = f(b)$. Then:

$$f(a) = f(f(b)) = b.$$

Therefore, for every $b \in A$, there exists an $a \in A$ (specifically, $a = f(b)$) such that $f(a) = b$. Therefore f is surjective.

Since f is both injective and surjective, it is bijective.

Inverse Function

By definition, the inverse function f^{-1} satisfies:

$$f^{-1}(f(a)) = a \quad \text{and} \quad f(f^{-1}(a)) = a \quad \text{for all} \quad a \in A.$$

Given that f is an involution:

$$f(f(a)) = a.$$

Comparing the two conditions, we observe that f itself satisfies the properties required of an inverse function. Therefore:

$$f^{-1} = f.$$

$\square$

Problem 6. Prove or disprove the following statements:

(a) The set $\{x \in \mathbb{R} : x \geq 2\}$ is an interval.
(b) The set $\{x \in \mathbb{R} : x \neq 2\}$ is an interval.

(Hint: In order to disprove a statement, you must prove that the negation of the statement is true.)

Solution: (a)

We express the set in interval notation:

$$S = [2, \infty)$$

Let $a, b \in S$ with $a < b$. Since $a \geq 2$ and $b \geq 2$, for any c such that $a < c < b$, it follows that $c \geq 2$. Thus, $c \in S$.

Therefore, S satisfies the definition of an interval.

(b) Consider the negation of the original statement, that the set is not an interval.

Assume, for contradiction, that the set is an interval. Expressing the set in interval notation:

$$T = (-\infty, 2) \cup (2, \infty)$$

This implies that T is the union of two disjoint intervals. However, for T to be an interval, it must be a single continuous set without gaps.

Consider $a = 1 \in (-\infty, 2)$ and $b = 3 \in (2, \infty)$. The point $c = 2$ satisfies $a < c < b$ but $c \notin T$, which contradicts the interval property that all points between a and b must lie within the set.

We have thus proven the validity of the negation of our original statement. Therefore, T is not an interval.

$\square$