Problem 1

Prove the following statements:

1. there exists a number $1 < x < 2$ that solves the equation $x^2 - x - 1 = 0$.
2. There exists a number $x \in \mathbb{R}$ that solbes the equation $x^5 - x + 1 = 0$.

Solution:

Problem 2

Let $a < b$ be numbers and $f : [a, b] \to \mathbb{R}$ be a function. We say that $x \in [a, b]$ is a fixed point for f if $f(x) = x$. Prove that if f is continuous and $f(x) \in [a, b]$ for all $x \in [a, b]$, then f has a fixed point.

Solution:

Problem 3

Let $f : [0, 1] \to \mathbb{R}$ be a continuous function such that $f(0) = f(1)$. Prove that there exists $x \in \left[0, \frac{1}{2}\right]$ such that $f(x) = f\left(x + \frac{1}{2}\right)$. Hint: consider the function $g(x) = f(x) - f\left(x + \frac{1}{2}\right)$. Is it possible for $g(0)$ and $g\left(\frac{1}{2}\right)$ to both be positive?

Solution:

Problem 4

For each of the following functions $f : [-1, 1] \to \mathbb{R}$, find all global extrema and find the points $x \in [-1, 1]$ at which f attains these extrema.

1.
$$f(x) = \begin{cases} 1 - x & \text{if } x \geq 0 \\ 1 + x & \text{if } x < 0. \end{cases} \tag{1}$$

2.
$$f(x) = \begin{cases} 1 - x & \text{if } x \geq 0 \\ -1 - x & \text{if } x < 0. \end{cases} \tag{2}$$

3.
$$f(x) = \begin{cases} 1 - x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases} \tag{3}$$

Problem 5

Let $h > 0$. Prove that there is a point on the parabola

$$\{(x, x^2) \in \mathbb{R}^2 : -10 \leq x \leq 10\}, \tag{4}$$

that is closest to the point $(0, h)$.

Solution:

Problem 6

Let $a < b$ be numbers and $f, g, h : [a, b] \to \mathbb{R}$ be functions.

1. Prove that if f is continuous, then $|f|$ has a global maximum. Given a continuous function f we define $\|f\|$ to be equal to this value. (i.e. the global maximum of $|f|$).
2. Prove that if g is continuous, then $\|cg\| = |c| \cdot \|g\|$ for any $c \in \mathbb{R}$.
3. Prove that if g and h are continuous, then $\|g + h\| \leq \|g\| + \|h\|$.