Last time $\forall n \in \mathbb{N}$, $f(x) = x^n$ is differentiable on $\mathbb{R}$, and $f'(x) = nx^{n-1}$.

Def Let $I \subseteq \mathbb{R}$ be an open interval, $f : I \to \mathbb{R}$, and $a \in I$. We say:

① f is twice differentiable at a if:

- $f : I \to \mathbb{R}$ is differentiable at x $\forall x \in I$, and
- $f' : I \to \mathbb{R}$ is differentiable at a.

When this is true, we call $(f')'(a)$ the second derivative of f at a and we denote it by $f''(a)$ or $\frac{d^2 f}{dx^2}(a)$.

② f is n-times differentiable at a if $f^{(n-1)} : I \to \mathbb{R}$ exists and is differentiable at a. When this is true, we call $(f^{(n-1)})'(a)$ the nth derivative of f at a and we denote it by $f^{(n)}(a)$ or $\frac{d^n f}{dx^n}(a)$.

Ex ① Polynomials $p(x)$ are k-times differentiable at any $a \in \mathbb{R}$, $\forall k \in \mathbb{N}$. E.g.,

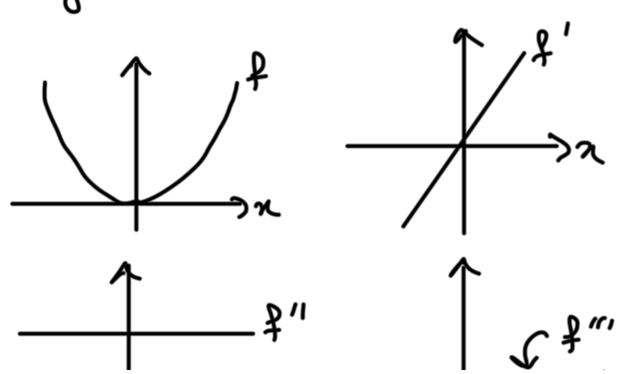
$f(x) = x^2$

$\Rightarrow f'(x) = 2x$

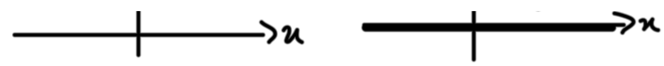
$\Rightarrow f''(x) = 2$

$\Rightarrow f'''(x) = 0$

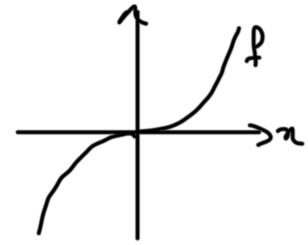
$\Rightarrow f^{(k)}(x) = 0 \quad \forall k > 3$



$\Rightarrow f'(x) = 0 \quad \forall x \neq 0$

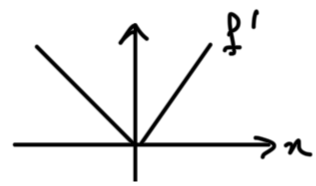


② $f : \mathbb{R} \to \mathbb{R}, \quad f(x) = \begin{cases} x^2 & x \geq 0 \\ -x^2 & x < 0 \end{cases}$

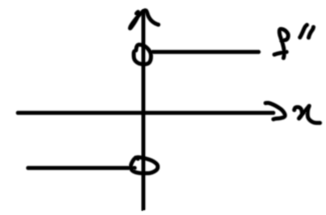


Then:

- f is differentiable on $\mathbb{R}$, $f'(x) = 2|x|$ (by lecture 22)



- f' is not differentiable at 0



So f is not twice differentiable at $x = 0$.

Exc Let $n \in \mathbb{N}$. The function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^n$ is k-times differentiable on $\mathbb{R}$ $\forall k \in \mathbb{N}$, and

$$f^{(k)}(a) = \begin{cases} \frac{n!}{(n-k)!} a^{n-k} & \text{if } k \leq n, \\ 0 & \text{if } k > n. \end{cases}$$

Idea: $f'(x) = \underbrace{n}_{\frac{n!}{(n-1)!}} x^{n-1}$

$$\Rightarrow f''(x) = n \frac{d}{dx}(x^{n-1}) = \underbrace{n(n-1)}_{\frac{n!}{(n-2)!}} x^{n-2}$$

$$\Rightarrow f'''(x) = n(n-1) \frac{d}{dx}(x^{n-2}) = \underbrace{n(n-1)(n-2)}_{\frac{n!}{(n-3)!}} x^{n-3}$$

$\vdots$

$$\Rightarrow f^{(n+1)}(x) = n(n-1) \cdots 2 \cdot 1 \, \frac{d}{dx}(1) = 0$$

$f^{(n+2)}(x) = 0$

$$\Rightarrow f^{(n+1)}(x) = 0$$

$\vdots$

Induction on $k \in \mathbb{N}$.

DIFFERENTIATION LAWS (Ch. 10)

Thm Let $I \subseteq \mathbb{R}$ be an open interval, $f, g : I \to \mathbb{R}$, and $a \in I$. If f and g are differentiable at a, then:

① cf is differentiable at a $\forall c \in \mathbb{R}$, and
$$(cf)'(a) = c \cdot f'(a).$$

② $f+g$ is differentiable at a, and
$$(f+g)'(a) = f'(a) + g'(a).$$

Pf: ① Fix $c \in \mathbb{R}$. As f is differentiable at a,
$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = f'(a).$$

So, by the limit law for multiplication by a constant,
$$\lim_{h \to 0} \left[c \cdot \frac{f(a+h) - f(a)}{h} \right] = c \cdot f'(a)$$
$$\parallel$$
$$\lim_{h \to 0} \frac{cf(a+h) - cf(a)}{h}$$

So $(cf)'(a) = c \cdot f'(a)$.

② As f and g are differentiable at a,

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = f'(a), \qquad \lim_{h\to 0} \frac{g(a+h)-g(a)}{h} = g'(a).$$

So, by the limit law for addition,

$$\lim_{h\to 0}\left[\frac{f(a+h)-f(a)}{h} + \frac{g(a+h)-g(a)}{h}\right] = f'(a)+g'(a)$$

$$\shortparallel$$

$$\lim_{h\to 0} \frac{f(a+h)+g(a+h)-(f(a)+g(a))}{h}$$

So $(f+g)'(a) = f'(a)+g'(a)$. $\square$

<u>Cor</u> If $p:\mathbb{R}\to\mathbb{R}$ is a polynomial, then p is differentiable.

<u>Pf</u>: Let

$$p(x) = c_0 + c_1 x + c_2 x^2 + \ldots + c_N x^N$$

be a polynomial, for some $N\geq 0$ and $c_0, c_1, \ldots, c_N \in \mathbb{R}$. Fix $a\in\mathbb{R}$. We will prove

$$p'(a) = c_1 + 2c_2 a + \ldots + N c_N a^{N-1}.$$

Recall that $x\mapsto x^n$ is differentiable at a and $(x^n)'(a) = na^{n-1}$, $\forall n\in\mathbb{N}$ (by Lecture 23). So, by part ① of the previous theorem, $c_n x^n$ is differentiable at a and $(c_n x^n)'(a) = nc_n a^{n-1}$, for each $n=1,2,\ldots,N$. Additionally, the constant function $x\mapsto c_0$ is differentiable at a with derivative

$$(c_0)'(a) = \lim_{h\to 0} \frac{c_0 - c_0}{h} = \lim_{h\to 0} 0 = 0.$$

So, by part 2 of the previous theorem, $p(x) = c_0 + c_1 x + \dots + c_N x^N$ is differentiable at a and

$$p'(a) = 0 + c_1 + 2c_2 a + \dots + N c_N a^{N-1}. \qquad \square$$