Quantum, Standard, and Contextual Fisher Information in J.Jae et. al.

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1 Introduction

In quantum metrology, the precision of parameter estimation is fundamentally limited by the Fisher Information (FI) associated with the measurement outcomes. Different forms of FI, namely Quantum Fisher Information (QFI), standard Fisher Information (FI), and Contextual Fisher Information (coFI), offer varying insights into the estimation process. QFI represents the maximum FI attainable by optimizing over all possible quantum measurements, while standard FI pertains to a specific measurement strategy. Contextual Fisher Information (coFI) incorporates the concept of contextuality, a non-classical feature of quantum mechanics, to potentially surpass classical and standard quantum limits.

This note aims to:

- Differentiate between QFI, standard FI, and coFI.
- Calculate the FI based on two distinct sets of probabilities derived from a quantum state undergoing phase shifts.
- Demonstrate how QFI contributes to the enhancement of coFI.

2 Definitions and Preliminaries

2.1 Quantum Fisher Information (QFI)

For a pure quantum state $\psi(\theta)$, the Quantum Fisher Information (QFI) is given by:

$$F_Q = 4\left[\partial_\theta\psi|\partial_\theta\psi - |\psi|\partial_\theta\psi|^2\right]$$

where $\partial_\theta\psi = \frac{\partial}{\partial\theta}\psi(\theta)$.

2.2 Standard Fisher Information (FI)

Given a set of probabilities $\{p_i(\theta)\}$ associated with measurement outcomes, the standard Fisher Information (FI) is defined as:

$$F(\theta) = \sum_i \frac{1}{p_i(\theta)}\left(\frac{\partial p_i(\theta)}{\partial\theta}\right)^2$$

2.3 Contextual Fisher Information (coFI)

Contextual Fisher Information (coFI) extends the standard FI by incorporating quasiprobabilities that capture contextuality. It is defined as:

$$F_{\text{co}}(\theta) = \sum_{a,b} w(a,b|\theta)\left[\partial_\theta \ln w(a,b|\theta)\right]^2$$

where $w(a,b|\theta)$ are the operational quasiprobabilities derived from joint and marginal probabilities.

3 Calculation of Quantum Fisher Information (QFI)

For the pure state ψ_{out}, the QFI is calculated as:

$$F_Q = 4\left[\partial_\theta\psi|\partial_\theta\psi - |\psi|\partial_\theta\psi|^2\right]$$

We found in prior discussion that

$$F_Q = 1$$

4 Calculation of Standard Fisher Information (FI)

We separately calculate the Fisher Information for two groups of probabilities:

1. Equations (1)–(4) (Joint Probabilities)
2. Equations (17)–(18) (Marginal Probabilities)

4.1 Joint Probabilities FI

The joint probabilities are given by:

$$p(H,D|\mathcal{A},\mathcal{B},\theta) = \frac{1}{4}\left(1+\cos(\theta+\theta_0)\right) \tag{1}$$
$$p(H,A|\mathcal{A},\mathcal{B},\theta) = \frac{1}{4}\left(1+\cos(\theta+\theta_0)\right) \tag{2}$$
$$p(V,D|\mathcal{A},\mathcal{B},\theta) = \frac{1}{4}\left(1-\cos(\theta+\theta_0)\right) \tag{3}$$
$$p(V,A|\mathcal{A},\mathcal{B},\theta) = \frac{1}{4}\left(1-\cos(\theta+\theta_0)\right) \tag{4}$$

The FI for joint probabilities is:

$$F_{\text{joint}}(\theta) = \sum_{a\in\{H,V\},b\in\{D,A\}} \frac{1}{p(a,b|\mathcal{A},\mathcal{B},\theta)}\left(\frac{\partial p(a,b|\mathcal{A},\mathcal{B},\theta)}{\partial\theta}\right)^2$$

First, compute the derivatives:

$$\frac{\partial p(H,D|\mathcal{A},\mathcal{B},\theta)}{\partial\theta} = \frac{1}{4}\left(-\sin(\theta+\theta_0)\right) \tag{5}$$
$$\frac{\partial p(H,A|\mathcal{A},\mathcal{B},\theta)}{\partial\theta} = \frac{1}{4}\left(-\sin(\theta+\theta_0)\right) \tag{6}$$
$$\frac{\partial p(V,D|\mathcal{A},\mathcal{B},\theta)}{\partial\theta} = \frac{1}{4}\left(\sin(\theta+\theta_0)\right) \tag{7}$$
$$\frac{\partial p(V,A|\mathcal{A},\mathcal{B},\theta)}{\partial\theta} = \frac{1}{4}\left(\sin(\theta+\theta_0)\right) \tag{8}$$

Substituting into the FI expression:

$$\begin{aligned}
F_{\text{joint}}(\theta) &= \frac{\left(\frac{-\sin(\theta+\theta_0)}{4}\right)^2}{\frac{1}{4}\left(1+\cos(\theta+\theta_0)\right)} + \frac{\left(\frac{-\sin(\theta+\theta_0)}{4}\right)^2}{\frac{1}{4}\left(1+\cos(\theta+\theta_0)\right)} && (9)\\
&\quad + \frac{\left(\frac{\sin(\theta+\theta_0)}{4}\right)^2}{\frac{1}{4}\left(1-\cos(\theta+\theta_0)\right)} + \frac{\left(\frac{\sin(\theta+\theta_0)}{4}\right)^2}{\frac{1}{4}\left(1-\cos(\theta+\theta_0)\right)} && (10)\\
&= \frac{\frac{\sin^2(\theta+\theta_0)}{16}}{\frac{1+\cos(\theta+\theta_0)}{4}} + \frac{\frac{\sin^2(\theta+\theta_0)}{16}}{\frac{1+\cos(\theta+\theta_0)}{4}} && (11)\\
&\quad + \frac{\frac{\sin^2(\theta+\theta_0)}{16}}{\frac{1-\cos(\theta+\theta_0)}{4}} + \frac{\frac{\sin^2(\theta+\theta_0)}{16}}{\frac{1-\cos(\theta+\theta_0)}{4}} && (12)\\
&= \frac{\sin^2(\theta+\theta_0)}{4(1+\cos(\theta+\theta_0))} + \frac{\sin^2(\theta+\theta_0)}{4(1+\cos(\theta+\theta_0))} && (13)\\
&\quad + \frac{\sin^2(\theta+\theta_0)}{4(1-\cos(\theta+\theta_0))} + \frac{\sin^2(\theta+\theta_0)}{4(1-\cos(\theta+\theta_0))} && (14)\\
&= \frac{\sin^2(\theta+\theta_0)}{2(1+\cos(\theta+\theta_0))} + \frac{\sin^2(\theta+\theta_0)}{2(1-\cos(\theta+\theta_0))} && (15)\\
&= 2\cos^2\left(\frac{\theta+\theta_0}{2}\right) && (16)
\end{aligned}$$

where the trigonometric identity used is:

$$\frac{\sin^2 x}{1+\cos x} + \frac{\sin^2 x}{1-\cos x} = 2\cos^2\left(\frac{x}{2}\right)$$

4.2 Marginal Probabilities FI

The marginal probabilities are:

$$p(D|\mathcal{B},\theta) = \frac{1}{2}\left(1+\sin(\theta+\theta_0)\cos\phi\right) \tag{17}$$
$$p(A|\mathcal{B},\theta) = \frac{1}{2}\left(1-\sin(\theta+\theta_0)\cos\phi\right) \tag{18}$$

Their derivatives are:

$$\frac{\partial p(D|\mathcal{B},\theta)}{\partial\theta} = \frac{1}{2}\cos(\theta+\theta_0)\cos\phi \tag{19}$$
$$\frac{\partial p(A|\mathcal{B},\theta)}{\partial\theta} = -\frac{1}{2}\cos(\theta+\theta_0)\cos\phi \tag{20}$$

The FI for marginal probabilities is:

$$F_{\text{marginal}}(\theta) = \sum_{b\in\{D,A\}} \frac{1}{p(b|\mathcal{B},\theta)} \left(\frac{\partial p(b|\mathcal{B},\theta)}{\partial \theta}\right)^2$$

Substituting the values:

$$\begin{aligned}
F_{\text{marginal}}(\theta) = & \frac{\left(\frac{\cos(\theta+\theta_0)\cos\phi}{2}\right)^2}{\frac{1}{2}\left(1+\sin(\theta+\theta_0)\cos\phi\right)} && (21)\\
& + \frac{\left(\frac{-\cos(\theta+\theta_0)\cos\phi}{2}\right)^2}{\frac{1}{2}\left(1-\sin(\theta+\theta_0)\cos\phi\right)} && (22)\\
= & \frac{\frac{\cos^2(\theta+\theta_0)\cos^2\phi}{4}}{\frac{1+\sin(\theta+\theta_0)\cos\phi}{2}} + \frac{\frac{\cos^2(\theta+\theta_0)\cos^2\phi}{4}}{\frac{1-\sin(\theta+\theta_0)\cos\phi}{2}} && (23)\\
= & \frac{\cos^2(\theta+\theta_0)\cos^2\phi}{2(1+\sin(\theta+\theta_0)\cos\phi)} + \frac{\cos^2(\theta+\theta_0)\cos^2\phi}{2(1-\sin(\theta+\theta_0)\cos\phi)} && (24)\\
= & \frac{\cos^2(\theta+\theta_0)\cos^2\phi}{2}\left(\frac{1}{1+\sin(\theta+\theta_0)\cos\phi} + \frac{1}{1-\sin(\theta+\theta_0)\cos\phi}\right) && (25)\\
= & \frac{\cos^2(\theta+\theta_0)\cos^2\phi}{2}\left(\frac{2}{1-\sin^2(\theta+\theta_0)\cos^2\phi}\right) && (26)\\
= & \frac{\cos^2(\theta+\theta_0)\cos^2\phi}{1-\sin^2(\theta+\theta_0)\cos^2\phi} && (27)
\end{aligned}$$

where the identity used is:

$$\frac{1}{1+x} + \frac{1}{1-x} = \frac{2}{1-x^2}$$

4.3 Total Standard Fisher Information

Combining both contributions:

$$\begin{aligned}
F_{\text{standard}}(\theta) =& F_{\text{joint}}(\theta) + F_{\text{marginal}}(\theta) && (28)\\
=& 2\cos^2\left(\frac{\theta+\theta_0}{2}\right) + \frac{\cos^2(\theta+\theta_0)\cos^2\phi}{1-\sin^2(\theta+\theta_0)\cos^2\phi} && (29)
\end{aligned}$$

5 Calculation of Contextual Fisher Information (coFI)

Contextual Fisher Information (coFI) leverages quasiprobabilities derived from the joint and marginal probabilities to incorporate contextuality into the Fisher Information framework. It was calculated in our previous note as:

$$
\begin{aligned}
F_{\text{co}} = {} & \frac{\left[-\cos\phi\cos(\theta+\theta_0)+\sin(\theta+\theta_0)\right]^2}{4\left[1+\cos(\theta+\theta_0)+\sin(\theta+\theta_0)\cos\phi\right]} \\
& + \frac{\left[\cos\phi\cos(\theta+\theta_0)+\sin(\theta+\theta_0)\right]^2}{4\left[1+\cos(\theta+\theta_0)-\sin(\theta+\theta_0)\cos\phi\right]} \\
& + \frac{\left[\cos\phi\cos(\theta+\theta_0)+\sin(\theta+\theta_0)\right]^2}{4\left[1-\cos(\theta+\theta_0)+\sin(\theta+\theta_0)\cos\phi\right]} \\
& + \frac{\left[-\cos\phi\cos(\theta+\theta_0)+\sin(\theta+\theta_0)\right]^2}{4\left[1-\cos(\theta+\theta_0)-\sin(\theta+\theta_0)\cos\phi\right]}.
\end{aligned}
$$

References

[1] M. Barbieri, *Quantum Fisher Information in Metrology*, 2022.

[2] J. Jae, et al., *Contextual Quantum Metrology*, 2024.