Last time: For differentiable functions f,

① 1st derivative test:

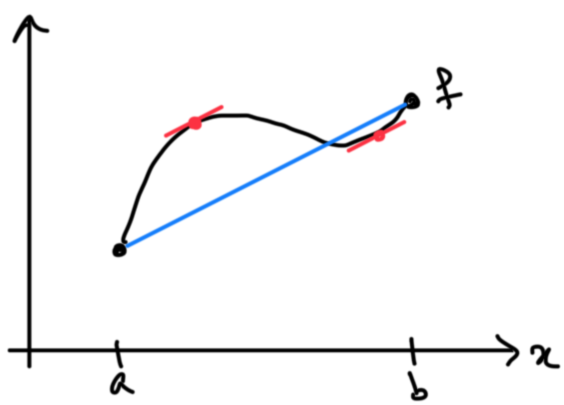
$f(x_0)$ local max/min $\Rightarrow$ $f'(x_0)=0$

② Rolle's theorem:

$f(a)=f(b) \Rightarrow \exists\, c\in(a,b)$ s.t. $f'(c)=0$

Thm (Mean value theorem) If $f:[a,b]\to\mathbb{R}$ is continuous on $[a,b]$ and differentiable on (a,b), then $\exists\, c\in(a,b)$ s.t.

$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$



- This = slope of blue line = mean slope of f
- In order to connect $f(a)$ and $f(b)$, a differentiable function must attain the mean slope.

Pf: Set $g(x) = f(x) - \left[f(a) + \frac{f(b)-f(a)}{b-a}\cdot(x-a)\right]$.

- This is the height of the graph of f above the blue line

As $x \mapsto -f(a) - \frac{f(b)-f(a)}{b-a}\cdot(x-a)$ is a polynomial, it is continuous and differentiable at any $x\in\mathbb{R}$.

So g satisfies:

1. continuous on $[a,b]$
2. differentiable on (a,b)
3. $g(a) = g(b)$: since

$$g(a) = f(a) - [f(a) + 0] = 0$$
$$g(b) = f(b) - \left[f(a) + \frac{f(b)-f(a)}{\cancel{b-a}} \cdot \cancel{(b-a)}\right] = 0$$

Therefore, by Rolle's theorem, $\exists\, c \in (a,b)$ s.t.

$$0 = g'(c) = f'(c) - \left[0 + \frac{f(b)-f(a)}{b-a}\right]$$
$$\Rightarrow f'(c) = \frac{f(b)-f(a)}{b-a}. \qquad \square$$

<u>Cor 1</u> Let $I \subseteq \mathbb{R}$ be an open interval and $f: I \to \mathbb{R}$. If $f'(x) = 0$ $\forall x \in I$, then f is a constant function.

<u>Pf</u>: Let $a, b \in I$ with $a < b$. We will prove that $f(a) = f(b)$.

- This is sufficient. Indeed, f not constant $\Rightarrow$ $\exists\, a, b \in I$ s.t. $f(a) \neq f(b)$.

As f is differentiable at x $\forall x \in I$ and $[a,b] \subseteq I$, then:

1. f is continuous on $[a,b]$
2. f is differentiable on (a,b)

So, by the MVT, $\exists\, c \in (a,b)$ s.t.

$$0 = f'(c) = \frac{f(b)-f(a)}{b-a} \quad \Longrightarrow \quad f(a) = f(b). \qquad \square$$

<u>Cor 2</u> Let $I \subseteq \mathbb{R}$ be an open interval and $f, g : I \to \mathbb{R}$. If $f'(x) = g'(x)$ $\forall x \in I$, then $\exists\, c \in \mathbb{R}$ s.t. $f(x) = g(x) + c$ $\forall x \in I$.

<u>Pf</u>: Let $h : I \to \mathbb{R}$, $h(x) = f(x) - g(x)$. Then $\forall x \in I$ we have

$$h'(x) = f'(x) - g'(x) = 0.$$

So, by Cor 1, we know h is constant: $\exists\, c \in \mathbb{R}$ s.t.

$$c = h(x) = f(x) - g(x) \qquad \forall x \in I$$

$$\Longrightarrow \quad f(x) = g(x) + c \qquad \forall x \in I. \qquad \square$$

<u>Cor 3</u> Let $I \subseteq \mathbb{R}$ be an open interval and $f : I \to \mathbb{R}$.

① If $f'(x) > 0$ $\forall x \in I$, then f is <u>strictly increasing</u>: $\forall x, y \in I$, $x < y \Rightarrow f(x) < f(y)$.

② If $f'(x) < 0$ $\forall x \in I$, then f is <u>strictly decreasing</u>: $\forall x, y \in I$, $x < y \Rightarrow f(x) > f(y)$.

<u>Pf</u>: ① Let $a, b \in I$ with $a < b$. Want: $f(a) < f(b)$. As f is differentiable at x $\forall x \in I$, then:

- f is continuous on $[a,b]$
- f is differentiable on (a,b)

So, by the MVT, $\exists\ c \in (a,b)$ s.t.

$$0 < f'(c) = \frac{f(b)-f(a)}{b-a} \implies f(a) < f(b).$$

② Suppose $f'(x) < 0\ \forall x \in I$. Then $-f$ satisfies $-f'(x) > 0\ \forall x \in I$. So, by ①, we know $-f$ is strictly increasing:

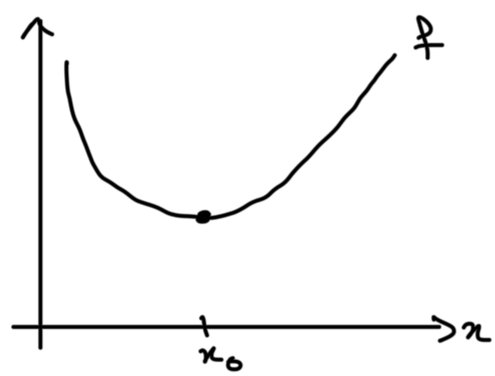
$$\forall a, b \in I, \quad a < b \implies -f(a) < -f(b)$$
$$\implies f(a) > f(b)$$

So f is strictly decreasing. $\square$

<u>Ex</u> Fix $a, b > 0$. Find the global minimum of $f : (0,\infty) \to \mathbb{R}$, $f(x) = \frac{a}{x} + bx$ and prove your answer.

<u>Scratch work</u>: Critical points are

$$0 = f'(x) = -\frac{a}{x^2} + b = \frac{bx^2 - a}{x^2}$$
$$\implies bx^2 - a = 0$$
$$\implies x_0 = \sqrt{\frac{a}{b}}$$



<u>Solution</u>: We will prove that $f(x_0) = 2\sqrt{ab}$ is the global minimum of f, where $x_0 = \sqrt{\frac{a}{b}}$.

① Claim: $f(x) > f(x_0)\ \forall x \in (x_0, \infty)$. Note that

$$x > x_0 = \sqrt{\frac{a}{b}} \implies x^2 > \frac{a}{b} \implies bx^2 - a > 0$$

$$\Rightarrow \quad f'(x) = \frac{bx^2 - a}{x^2} > 0$$

Given $x > x_0$, by the MVT there is a point $c \in (x_0, x)$ where

$$0 < f'(c) = \frac{f(x) - f(x_0)}{x - x_0} \quad \Rightarrow \quad f(x) > f(x_0).$$

(2) Claim: $f(x) > f(x_0) \ \forall\, x \in (0, x_0)$. Note that

$$0 < x < x_0 = \sqrt{\frac{a}{b}} \quad \Rightarrow \quad x^2 < \frac{a}{b} \quad \Rightarrow \quad bx^2 - a < 0$$

$$\Rightarrow \quad f'(x) = \frac{bx^2 - a}{x^2} < 0$$

Given $x < x_0$, by the MVT there is a point $c \in (x, x_0)$ where

$$0 > f'(c) = \frac{f(x_0) - f(x)}{x_0 - x} \quad \Rightarrow \quad f(x) > f(x_0).$$

Altogether, we conclude that $f(x_0)$ is the (strict) global minimum of $f : (0, \infty) \to \mathbb{R}$. $\square$