Notes on Math 322: intro to PDE

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An introduction to partial differential equations, including the heat equation, Poisson's Equation and the wave equation. Fourier series and neccessary background on functional analysis will be covered.

Contents

Notations

The following notations are used in this note:

- Partial Derivatives

$$u_t = \frac{\partial u}{\partial t} = \partial_t u \qquad [1$$

- Laplacian Operator

$$\nabla^2 u = \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} u \qquad [2$$

  - Notedly, Laplacian in Spherical Coordinate:

- Boldface vector:

$$\vec{u} = \boldsymbol{u} \qquad [3$$

1. Heat Equation

For tempreature $u(\boldsymbol{x})$, head conduction or particle diffusion can be described by the head equation:

$$u_t = k\nabla^2 u \qquad [4$$

1.1. Fundamental Solution

The fundamental solution Φ is found by solving the heat equation with a delta function as the initial condition:

$$\begin{cases} \Phi_t = \kappa \nabla^2 \Phi \\ \Phi(\boldsymbol{x}, t = 0) = \delta(\boldsymbol{x}) \end{cases} \qquad [5$$

It is solved to be the Green's function

$$\Phi(\boldsymbol{x}, t) = \frac{1}{(4\pi\kappa t)^{n/2}} \exp\left(-\frac{|\boldsymbol{x}|^2}{4\kappa t}\right) \qquad [6$$

1.2. Initial Value problem

Consider a general initial value $g(\boldsymbol{x})$, heat equation becomes:

$$\begin{cases} u_t = \kappa \nabla^2 u \\ u(\boldsymbol{x}, 0) = g(\boldsymbol{x}) \end{cases} \qquad [7$$

An arguement of linearity and superposition can be made to arrive at the solution:

$$u(\boldsymbol{x}, t) = g \star \Phi \equiv \int_{\mathbb{R}^n} g(\boldsymbol{y}) \Phi(\boldsymbol{x} - \boldsymbol{y}) \, \mathrm{d}v_y \qquad [8$$

- *example*:
  - Useful special functions: Heaviside step function, and error function

    $$\operatorname{erf}(x) \equiv \frac{2}{\sqrt{\pi}} \int_0^x \exp(-z^2) \, \mathrm{d}z \qquad [9$$

  - example statement: consider a long rod heated on the region $[-1, 1]$ at time zero. Mathematically,

    $$\begin{cases} u_t = \kappa u_{xx} \\ u(x, 0) = g(x) = H(x + 1) - H(x - 1) \end{cases} \qquad [10$$

  - *Solution*:

$$
\begin{aligned}
u(x,t) &= g \star \Phi \\
&= \frac{1}{\sqrt{4\pi\kappa t}} \int_{-\infty}^{\infty} g(y) \exp\left(\frac{-(x-y)^2}{4\kappa t}\right) \mathrm{d}y \\
&= \frac{1}{\sqrt{4\pi\kappa t}} \int_{-1}^{1} \exp(-(x-y)^2/4\kappa t) \,\mathrm{d}y
\end{aligned}
\quad [11
$$

let $x - y = z\sqrt{4\kappa t}, z = \dfrac{x-y}{\sqrt{4\pi\kappa t}}$

$$
\begin{aligned}
u &= \frac{-\sqrt{4\pi\kappa t}}{\sqrt{4\pi\kappa t}} \int_{(x+1)/\left(\sqrt{4\kappa t}\right)}^{(x-1)/\left(\sqrt{4\kappa t}\right)} e^{-z^2} \,\mathrm{d}z \\
&= \frac{1}{2}\left(\operatorname{erf}\left(\frac{x+1}{\sqrt{4\kappa t}}\right) - \operatorname{erf}\left(\frac{x-1}{\sqrt{4\kappa t}}\right)\right)
\end{aligned}
\quad [12
$$

Notice that the erf function is an odd function, so we can combine this to be

$$
u = \operatorname{erf}\left(\frac{1}{\sqrt{4\kappa t}}\right) \quad [13
$$

We can study this solution via asympotic analysis

- for small x, talor expansion of erf function to second degree gives

$$
\operatorname{erf}(x) \approx \frac{2x}{\sqrt{\pi}} \quad [14
$$

We are interested in large t, so

$$
\operatorname{erf}\left(\frac{1}{\sqrt{4\kappa t}}\right) \approx \frac{1}{\sqrt{\pi\kappa t}} \sim \frac{1}{\sqrt{t}} \quad [15
$$

1.3. Heat eqn with forcing (heat source/ sink)

Consider the original heat equation without forcing

$$
u_t = \kappa\nabla^2 u \quad [16
$$

Now, consider heat source $f(\boldsymbol{x}, t)$, the heat equation becomes:

$$
\begin{cases} u_t = \kappa\nabla^2 u + f(\boldsymbol{x}, t) \\ u(\boldsymbol{x}, 0) = 0 \end{cases} \quad [17
$$

We can use **Duhamel's Principle** to transform heat source to a collection of heat impulses(initial value problems) over time domain.