**Physics 322**, Honors Assignment

1. (Griffiths ed.5, 3.2) Prove Earnshaw's theorem: a charged particle cannot be held in stable equilibrium (in otherwise empty space) by electrostatic forces alone.

2. In the multipole expansion of the electrostatic scalar potential in class, we carried out the expansion of $1/|\boldsymbol{r} - \boldsymbol{r}'|$ up to the quadrupole term, but focused primarily on the monopole and dipole terms. Here we will expand upon this further, including considering some higher order effects.
   (a) If there is a charge distribution which has a monopole moment that is nonzero, show that it is possible to find an origin of coordinates such that the dipole moment is zero. (This point is called the "center of charge" of the distribution.)
   (b) Carry out the expansion of $1/|\boldsymbol{r} - \boldsymbol{r}'|$ up to and including the octupole ($n = 3$, $V_{\text{octupole}} \sim 1/r^4$) term.
   (c) Derive an expression for the quadrupole moment in the case that the charge distribution $\rho(\boldsymbol{r})$ has axial symmetry with respect to the $z$ axis (*i.e.*, the charge distribution is invariant with respect to rotations about the $z$ axis).

3. (Griffiths ed.5, 4.33) Earnshaw's theorem (see the problem above) says that you cannot trap a charged particle in an electrostatic field. Question: could you trap a neutral (but polarizable) atom in an electrostatic field?
   (a) Show that the force on the atom is $\boldsymbol{F} = \frac{1}{2}\alpha\nabla(E^2)$. Note that the dipole is drawn into a region of stronger field.
   (b) The question becomes, therefore, whether it is possible for $E^2$ to have a local maximum (in a charge-free region). In that case the force would push the atom back to its equilibrium position. Show that the answer is no (you can use the result, from the text's problem 3.4, that the average electric field over a spherical surface, due to charges outside the sphere, is the same as the field at the center).

4. (Griffiths ed.5, 9.41) According to Snell's law, when light passes through an optically dense medium into a less dense one ($n_1 > n_2$), the propagation vector $\boldsymbol{k}$ bends away from the normal (see the text's Fig. 9.28). In particular, if the light is incident at the *critical angle*, $\theta_c \equiv \sin^{-1}(n_2/n_1)$, then $\theta_T = 90°$, and the transmitted ray just grazes the surface. If $\theta_I$ exceeds $\theta_c$, there is no refracted ray at all, only a reflected one (this is the phenomenon of total internal reflection). The fields are not zero in medium 2; what we get is a so-called *evanescent wave*, which is rapidly attenuated and transports no energy into medium 2. A quick way to construct the evanescent way is to use the results for oblique incidence, with $k_T = \omega n_2/c$ and $\boldsymbol{k}_T = k_T(\sin\theta_T\hat{x} + \cos\theta_T\hat{z})$, with the only change that

   $$\sin\theta_T = \frac{n_1}{n_2}\sin\theta_I > 1,$$

   and

   $$\cos\theta_T = \sqrt{1 - \sin^2\theta_I} = i\sqrt{\sin^2\theta_T - 1}.$$

   (a) Show that

   $$\tilde{\boldsymbol{E}}_T(\boldsymbol{r}, t) = \tilde{\boldsymbol{E}}_{0T}e^{-\kappa z}e^{i(kx-\omega t)},$$

   where

   $$\kappa \equiv \frac{\omega}{c}\sqrt{(n_1\sin\theta_I)^2 - n_2^2}, \qquad k \equiv \frac{\omega n_1}{c}\sin\theta_I.$$

   This is a wave propagating in the $x$ direction (parallel to the interface) and attenuated in the $z$ direction.

(b) Noting that $\alpha = (\cos\theta_T)/(\cos\theta_I)$ is now imaginary, calculate the reflection coefficient for polarization parallel to the plane of incidence. [Notice that you get 100 percent reflection, which is better than at a conducting surface.]
(c) Do the same for polarization perpendicular to the plane of incidence.
(d) In the case of polarization perpendicular to the plane of incidence, show that the (real) evanescent fields are

$$\boldsymbol{E}(\boldsymbol{r}, t) = E_0 e^{-\kappa z}\cos(kx - \omega t)\hat{y}, \quad \boldsymbol{B}(\boldsymbol{r}, t) = \frac{E_0}{\omega} e^{-\kappa z}\left(\kappa \sin(kx - \omega t)\hat{x} + k\cos(kx - \omega t)\hat{z}\right).$$

(e) Check that the fields in (d) satisfy Maxwell's equations.
(f) For the fields in (d), construct the Poynting vector, and show that, on average, no energy is transmitted in the $z$ direction.

5. For the rectangular waveguide as discussed in Griffiths 9.5.2 and Lecture 36, work out the theory of the TM modes (all field components, the cutoff frequencies, and the wave and group velocities).

6. (Griffiths ed.5, 11.2-4, modified) (a) For electric dipole radiation, the result in Eq. 11.14

$$V(r, \theta, t) = -\frac{p_0\omega}{4\pi\epsilon_0 c}\left(\frac{\cos\theta}{r}\right)\sin[\omega(t - r/c)],$$

can be expressed in "coordinate-free" form by writing $p_0\cos\theta = \boldsymbol{p}_0 \cdot \hat{\boldsymbol{r}}$. Do this for $V(r, \theta, t)$, and likewise for the expressions for $\boldsymbol{A}(r, \theta, t)$, $\boldsymbol{E}(r, \theta, t)$, $\boldsymbol{E}(r, \theta, t)$, and $\langle \boldsymbol{S} \rangle$ (Eqs. 11.17, 11.18, 11.19, and 11.21).
(b) Now consider a rotating electric dipole can be thought of as the superposition of two oscillating dipoles, one along the $x$-axis and the other along the $y$-axis (see Fig. 11.7 in the text), with the latter out of phase by $90°$:

$$\boldsymbol{p} = p_0(\cos(\omega t)\hat{x} + \sin(\omega t)\hat{y}).$$

Using the principle of superposition and Eqs. 11.18-11.19 in the text (perhaps in the form suggested in the previous problem), find the fields of a rotating dipole. Next, calculate the Poynting vector and the intensity of the radiation, and determine the total power radiated.