Problem 1

Prove or disprove the following statements:

1. The set $\{x \in \mathbb{R} : x \geq 2\}$ is open.
2. The set $\{x \in \mathbb{R} : x \neq 2\}$ is open.

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solution:

1. Let $\varepsilon > 0$. Consider $2 \in [2, \infty)$: , and interval $(2-\varepsilon, 2+\varepsilon)$:

   Since $2 - \frac{\varepsilon}{2} \in (2-\varepsilon, 2+\varepsilon)$, but $2 - \frac{\varepsilon}{2} \notin [2, \infty)$,

   it follows that for any $\varepsilon > 0$, the interval $(2-\varepsilon, 2+\varepsilon)$ is not a subset of $\{x \in \mathbb{R} : x \geq 2\}$, so the set $\{x \in \mathbb{R} : x \geq 2\}$ is not open.

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2. Let $\varepsilon > 0, x \in \{x \in \mathbb{R} : x \neq 2\}$. Let $\varepsilon = \left|\frac{x-2}{2}\right|$.

   Then for any $y \in (x-\varepsilon, x+\varepsilon)$, we have

$$\begin{aligned} & y < x + \varepsilon, \quad y > x - \varepsilon \\ & \Rightarrow |y - x| < \varepsilon = \left|\frac{x-2}{2}\right| \end{aligned} \tag{1}$$

   Thus by triangle inequality,

$$\begin{aligned} |y-2| &= |y - x + x - 2| \\ &\geq |x-2| - |y-x| \\ &\geq |x-2| - \left|\frac{x-2}{2}\right| \\ &= \frac{|x-2|}{2} \\ &= \varepsilon > 0 \end{aligned} \tag{2}$$

   Therefore $y \neq 2 \Rightarrow y \in \{x \in \mathbb{R} : x \neq 2\}$. So the set is open.■

Problem 2:

Let $A, B \subseteq \mathbb{R}$ be subsets. Prove the following statements:

1. *(De Morgan's Laws)* $(A \cap B)^c = A^c \cup B^cand(A \cup B)^c = A^c \cap B^c$
2. If A and B are closed then $A \cap B$ and $A \cup B$ are closed.

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solution:

1. - Let $x \in (A \cap B)^c$, then $x \notin (A \cap B) \Rightarrow (x \notin A)$ or $(x \notin B)$

     This is equivalent to $x \in A^c$ or $x \in B^c \Rightarrow x \in (A^c \cup B^c)$.

     So for any $x \in (A \cap B)^c$, $x \in (A^c \cup B^c)$, thus the two sets are equal. ■

   - Let $x \in (A \cup B)^c$, then $x \notin (A \cup B) \Rightarrow x \notin A$ and $x \notin B$.

     So $x \in A^c$ and $x \in B^c \Rightarrow x \in (A^c \cap B^c)$. So for any $x \in (A \cup B)^c$, $x \in (A^c \cap B^c)$, thus the two sets are equal.■

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2. - If A is closed and B is closed, then A^c and B^c are open. Since unions of open sets are open, then $A^c \cup B^c$ is open.

     By De Morgan's Laws, $A^c \cup B^c = (A \cap B)^c$ is closed.

     Thus $A \cap B$ is open.■

   - If A is closed and B is closed, then A^c and $B^c$are open. Since intersections of open sets are open, then $A^c \cap B^c$ is open.

     By De Morgan's Laws, $A^c \cap B^c = (A \cup B)^c$ is open.

     Thus $A \cup B$ is closed.■

Problem 3:

Let $\varepsilon > 0$. For each of the following functions $\mathbb{R} \to \mathbb{R}$ and numbers $l \in \mathbb{R}$, find a δ s.t. $0 < |x - 1| < \delta$ implies $|f(x) - l| < \varepsilon$.

1. $f(x) = x^4$ and $l = 1$
2. $g(x) = \begin{cases} 1/x \text{ if } x \neq 0 \\ 0 \text{ if } x = 0 \end{cases}$, and $l = 1$
3. $h(x) = f(x) + g(x)$ and $l = 2$. hint: in the proof of the corresponding limit laws, we saw how to pick this δ based on our answers for (a) and (b).

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solution:

1.

Problem 4:

let $f, g : \mathbb{R} \to \mathbb{R}$ be functions s.t. $\lim_{x \to a} f(x) = l$ and $\lim_{x \to a} g(x) = m$ for some numbers a,l,m in $\mathbb{R}$. Prove that if $\forall x \in \mathbb{R} f(x) \leq g(x)$, then $l < m$.

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solution:

Problem 5:

Let $f, g : \mathbb{R} \to \mathbb{R}$ be functions and $a \in \mathbb{R}$. Prove or disprove the following statements:

- (a) If $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both do not exist, then $\lim_{x\to a}(f+g)(x)$ does not exist.
- (b) If $\lim_{x\to a} f(x)$ exists and $\lim_{x\to a}(f+g)(x)$ does not exist, then $\lim_{x\to a} g(x)$ does not exist.
- (c) If $\lim_{x\to a} f(x)$ exists and $\lim_{x\to a} g(x)$ does not exist, then $\lim_{x\to a}(f+g)(x)$does not exist.

(hint: Each statement is either an application of the limit law for addition, or it is false. Remember, if the statement is false, then we need to come up with a counterexample.)

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solution: