

1. (Griffiths ed.5, 4.16) Suppose the field inside a large piece of dielectric is  $\mathbf{E}_0$ , so that the electric displacement is  $\mathbf{D}_0 = \epsilon_0 \mathbf{E}_0 + \mathbf{P}$ .
- (a) Now a small spherical cavity (see Fig. 4.19 in the text) is hollowed out of the material. Find the field at the center of the cavity in terms of  $\mathbf{E}_0$  and  $\mathbf{P}$ , and find the displacement at the center of the cavity in terms of  $\mathbf{D}_0$  and  $\mathbf{P}$ . Here and in the rest of this problem assume that the polarization is “frozen in,” so it doesn’t change when the cavity is excavated.
- (b) Do the same for a long needle-shaped cavity running parallel to  $\mathbf{P}$ . (Again, see Fig. 4.19). *Hint:* ignore any asymmetry at the top and the bottom of the needle, which the text picture indicates. Note that “long” means that if  $L$  is the length of the needle and  $A$  is its cross-sectional area, we have  $A/L^2$  sufficiently small such that  $|\mathbf{P}|A/(4\pi\epsilon_0(L/2)^2) \ll |\mathbf{E}_0|$ .
- (c) Do the same for a thin wafer-shaped cavity perpendicular to  $\mathbf{P}$ . (Again, see Fig. 4.19). Assume the cavities are small enough that  $\mathbf{P}$ ,  $\mathbf{E}_0$ , and  $\mathbf{D}_0$  are essentially uniform. *Hint:* carving out a cavity is the same as superimposing an object of the same shape but opposite polarization.

#### Problem 4.16

- (a) Same as  $\mathbf{E}_0$  minus the field at the center of a sphere with uniform polarization  $\mathbf{P}$ . The latter (Eq. 4.14) is  $-\mathbf{P}/3\epsilon_0$ . So  $\boxed{\mathbf{E} = \mathbf{E}_0 + \frac{1}{3\epsilon_0}\mathbf{P}}$ .  $\mathbf{D} = \epsilon_0 \mathbf{E} = \epsilon_0 \mathbf{E}_0 + \frac{1}{3}\mathbf{P} = \mathbf{D}_0 - \mathbf{P} + \frac{1}{3}\mathbf{P}$ , so  $\boxed{\mathbf{D} = \mathbf{D}_0 - \frac{2}{3}\mathbf{P}}$ .
- (b) Same as  $\mathbf{E}_0$  minus the field of  $\pm$  charges at the two ends of the “needle”—but these are small, and far away, so  $\boxed{\mathbf{E} = \mathbf{E}_0}$ .  $\mathbf{D} = \epsilon_0 \mathbf{E} = \epsilon_0 \mathbf{E}_0 = \mathbf{D}_0 - \mathbf{P}$ , so  $\boxed{\mathbf{D} = \mathbf{D}_0 - \mathbf{P}}$ .
- (c) Same as  $\mathbf{E}_0$  minus the field of a parallel-plate capacitor with upper plate at  $\sigma = P$ . The latter is  $-(1/\epsilon_0)P$ , so  $\boxed{\mathbf{E} = \mathbf{E}_0 + \frac{1}{\epsilon_0}\mathbf{P}}$ .  $\mathbf{D} = \epsilon_0 \mathbf{E} = \epsilon_0 \mathbf{E}_0 + \mathbf{P}$ , so  $\boxed{\mathbf{D} = \mathbf{D}_0}$ .

## 2

2. A long cylindrical coaxial capacitor consists of an inner conductor of radius  $a$  and an outer conductor of radius  $b$ . The region between the conductors is filled with a linear and isotropic but inhomogeneous dielectric with relative permittivity given by  $\epsilon_r(s) = \alpha s^{-2}$ , in which  $s$  is the usual cylindrical radial coordinate. The capacitor is charged to a voltage  $V$ .

Determine the electric field between the conductors as a function of the voltage  $V$  and the capacitor geometry for these conditions.

For a dielectric, the absence of free charge gives  $\rho_f = 0$ . So Gauss law becomes:

$$\begin{aligned}\nabla \cdot \vec{D} &= \rho_f = 0 \\ \frac{1}{s} \partial_s (s D_s) &= 0 \\ D_s &= \frac{C}{s},\end{aligned}\tag{1}$$

for some constant  $C$ . Noticing  $\vec{D}(s) = \epsilon_0 \epsilon_r \vec{E} = \epsilon_0 \alpha s^{-2} \vec{E}$ , we have

$$\begin{aligned}\vec{E} &= \frac{Cs}{\epsilon_0 \alpha} \hat{s}, \\ V &= - \int_a^b \vec{E} \cdot d\vec{l} = - \frac{C}{\epsilon_0 \alpha} \int_a^b s \, ds = - \frac{C}{2\epsilon_0 \alpha} (b^2 - a^2).\end{aligned}\tag{2}$$

We find  $C$  to be

$$C = \frac{-2\epsilon_0 \alpha V}{b^2 - a^2}.\tag{3}$$

And thus

$$\vec{E}(s) = \frac{-2\epsilon_0 \alpha V s}{\epsilon_0 \alpha (b^2 - a^2)} = \boxed{\frac{2Vs}{a^2 - b^2} \hat{s}}.\tag{4}$$

### 3

3. (Griffiths ed.5, 4.25) Suppose the entire region below  $z = 0$  is filled with a uniform linear dielectric material of susceptibility  $\chi_e$  and the region above  $z = 0$  is filled with a uniform linear dielectric material with a different susceptibility  $\chi'_e$ . A point charge  $q$  is at the location  $(x = 0, y = 0, z = d)$ , where  $d > 0$  (see Fig. 4.28 in the text). Find the potential in both regions ( $z > 0$  and  $z < 0$ ).

$$\begin{aligned}
 V^>(r, z) &= \frac{1}{4\pi\epsilon'} \left( \frac{q}{(r^2 + (z - d)^2)^{1/2}} + \frac{q'}{(r^2 + (z + d)^2)^{1/2}} \right) \\
 \Rightarrow V^>(r, 0) &= \frac{1}{4\pi\epsilon'} \frac{q + q'}{(r^2 + d^2)^{1/2}}, \quad \frac{\partial V^>}{\partial z}(r, 0) = \frac{1}{4\pi\epsilon'} \frac{d(q - q')}{(r^2 + d^2)^{3/2}}.
 \end{aligned} \tag{5}$$

and,

$$\begin{aligned}
 V^<(r, z) &= \frac{1}{4} \pi \epsilon \frac{q''}{(r^2 + (z - d)^2)^{1/2}} \\
 \Rightarrow V^<(r, 0) &= \frac{1}{4\pi\epsilon} \frac{q''}{(r^2 + d^2)^{1/2}}, \quad \frac{\partial V^<}{\partial z}(r, 0) = \frac{1}{4\pi\epsilon} \frac{q''}{(r^2 + d^2)^{3/2}},
 \end{aligned} \tag{6}$$

where  $\epsilon' = \epsilon_0(1 + \chi'_e)$ ,  $\epsilon = \epsilon_0(1 + \chi_e)$ .

Boundary condition gives

$$\begin{aligned}
 \frac{1}{\epsilon'}(q + q') &= \frac{q''}{\epsilon}, \quad (q - q') = q'' \\
 \Rightarrow q'' &= \frac{2\epsilon}{\epsilon + \epsilon'} q, \quad q' = \frac{\epsilon' - \epsilon}{\epsilon' + \epsilon} q
 \end{aligned} \tag{7}$$

Hence,

$$\begin{aligned}
 V^>(r, z) &= \frac{q}{4\pi\epsilon'} \left( \frac{1}{(r^2 + (z - d)^2)^{1/2}} + \left( \frac{\epsilon' - \epsilon}{\epsilon' + \epsilon} \right) \frac{1}{(r^2 + (z + d)^2)^{1/2}} \right), \quad (z \geq 0); \\
 V^<(r, z) &= \frac{2q}{4\pi(\epsilon' + \epsilon)} \frac{1}{(r^2 + (z + d)^2)^{1/2}}, \quad (z \leq 0.)
 \end{aligned} \tag{8}$$

#### 4

4. A ideal dipole  $\mathbf{p} = p\hat{z}$  is in the center of a spherical uniform dielectric shell with inner radius  $a$  and outer radius  $b$ . Find the potential in all regions.

1. Inside the cavity, ( $r < a$ ): Separation of Variables with dipole gives

$$V_I(r, \theta) = \frac{1}{4\pi\epsilon_0} \left( \frac{p \cos \theta}{r^2} + \sum_{l=0}^{\infty} A_{1,l} r^l P_l(\cos \theta) \right) \quad (9)$$

where  $p$  is the dipole moment. Dipole expansion gives

$$V_I(r, \theta) = \frac{p}{4\pi\epsilon_0} \left( \frac{1}{r^2} + \frac{r}{a^3} \left( \frac{1}{\epsilon_r} (A_2 + B_2) - 1 \right) \right) \cos \theta \quad (10)$$

$$\frac{\partial V_I}{\partial r} = \frac{p \cos \theta}{4\pi\epsilon_0 a^3} \left( \frac{1}{\epsilon_r} (A_2 + B_2) - 3 \right) \quad (11)$$

1. Within shell, ( $a < r < b$ ), **assuming a dielectric constant  $\epsilon_r$** , we can jot down the following by using results from Separation of Variables:

$$V_{II}(r, \theta) = \frac{1}{4\pi\epsilon} \sum_{l=0}^{\infty} \left( A_{2,l} r^l + \frac{B_{2,l}}{r^{l+1}} \right) P_l(\cos \theta) \quad (12)$$

Dipole expansion, taking  $l = 1$  gives:

$$V_{II}(r, \theta) = \frac{p}{4\pi\epsilon} \left( A_2 \frac{r}{a^3} + \frac{B_2}{r^2} \right) \cos \theta. \quad (13)$$

$$\frac{\partial V_{II}(a, \theta)}{\partial r} = \frac{p \cos \theta}{4\pi\epsilon a^3} (A_2 - 2B_2); \quad \frac{\partial V_{II}(b, \theta)}{\partial r} = \frac{p}{4\pi\epsilon b^3} \left( A_2 \left( \frac{b}{a} \right)^3 - 2B_2 \right) \quad (14)$$

1. Outside the shell, ( $r > b$ ):

$$V_{III}(r, \theta) = \frac{1}{4\pi\epsilon_0} \sum_{l=0}^{\infty} \frac{B_{3,l}}{r^{l+1}} P_l(\cos \theta). \quad (15)$$

Taking  $l = 1$  gives:

$$V_{III}(r, \theta) = \frac{p}{4\pi\epsilon r^2} \left( A_2 \left( \frac{b}{a} \right)^3 + B_2 \right). \quad (16)$$

$$\frac{\partial V_{III}(b, \theta)}{\partial r} = \frac{p}{4\pi\epsilon b^3} (-2) \left( A_2 \left( \frac{b}{a} \right)^3 + B_2 \right) \quad (17)$$

Boundary Conditions:

$$\begin{aligned} \epsilon \frac{\partial V_{II}(a, \theta)}{\partial r} &= \epsilon_0 \frac{\partial V_I(a, \theta)}{\partial r}, \\ \epsilon_0 \frac{\partial V_{III}(b, \theta)}{\partial r} &= \epsilon \frac{\partial V_{II}(b, \theta)}{\partial r} \end{aligned} \quad (18)$$

Painful algebra gives:

$$\begin{aligned}
A_2 &= \left(\frac{b}{a}\right)^{-3} \frac{B_2(2\varepsilon_r - 2)}{2 + \varepsilon_r}, \\
B_2 &= \frac{3\varepsilon_r}{(1 + 2\varepsilon_r) - 2\left(\frac{a}{b}\right)^3 \frac{(1 - \varepsilon_r)^2}{2 + \varepsilon_r}} \equiv f(\varepsilon_r, a, b)
\end{aligned} \tag{19}$$

We thus conclude:

$$\begin{aligned}
V_I(r, \theta) &= \frac{p \cos \theta}{4\pi\varepsilon r^2} \left( 1 + \left(\frac{r}{a}\right)^3 \left( \frac{f(\varepsilon_r, a, b)}{\varepsilon_r} \right) \left( 1 + \left(\frac{a}{b}\right)^3 \frac{2(\varepsilon_r - 1)}{2 + \varepsilon_r} \right) - 1 \right), \\
V_{II}(r, \theta) &= \frac{p \cos \theta}{4\pi\varepsilon r^2} f(\varepsilon_r, a, b) \left( 1 + \frac{2(\varepsilon_r - 1)}{2 + \varepsilon_r} \left(\frac{r}{b}\right)^3 \right), \\
V_{III}(r, \theta) &= \frac{p \cos \theta}{4\pi\varepsilon r^2} \frac{3f(\varepsilon_r, a, b)}{(\varepsilon_r + 2)}
\end{aligned} \tag{20}$$

## 5

5. (Griffiths ed.5, 4.26) A spherical conductor of radius  $a$  carries a charge  $Q$ . It is surrounded by linear (isotropic, homogenous) dielectric material of electric susceptibility  $\chi_e$ , out to radius  $b$ . (See Fig. 4.32 in the text.) Find the energy of this configuration.

### Problem 4.5

Field of  $\mathbf{p}_1$  at  $\mathbf{p}_2$  ( $\theta = \pi/2$  in Eq. 3.103):  $\mathbf{E}_1 = \frac{p_1}{4\pi\epsilon_0 r^3} \hat{\theta}$  (points *down*).

Torque on  $\mathbf{p}_2$ :  $\mathbf{N}_2 = \mathbf{p}_2 \times \mathbf{E}_1 = p_2 E_1 \sin 90^\circ = p_2 E_1 = \boxed{\frac{p_1 p_2}{4\pi\epsilon_0 r^3}}$  (points *into* the page).

Field of  $\mathbf{p}_2$  at  $\mathbf{p}_1$  ( $\theta = \pi$  in Eq. 3.103):  $\mathbf{E}_2 = \frac{p_2}{4\pi\epsilon_0 r^3} (-2\hat{\mathbf{r}})$  (points to the *right*).

Torque on  $\mathbf{p}_1$ :  $\mathbf{N}_1 = \mathbf{p}_1 \times \mathbf{E}_2 = \boxed{\frac{2p_1 p_2}{4\pi\epsilon_0 r^3}}$  (points *into* the page).

### Problem 4.26

From Ex. 4.5:

$$\mathbf{D} = \begin{cases} 0, & (r < a) \\ \frac{Q}{4\pi r^2} \hat{\mathbf{r}}, & (r > a) \end{cases}, \quad \mathbf{E} = \begin{cases} 0, & (r < a) \\ \frac{Q}{4\pi\epsilon r^2} \hat{\mathbf{r}}, & (a < r < b) \\ \frac{Q}{4\pi\epsilon_0 r^2} \hat{\mathbf{r}}, & (r > b) \end{cases}.$$

$$\begin{aligned} W &= \frac{1}{2} \int \mathbf{D} \cdot \mathbf{E} d\tau = \frac{1}{2} \frac{Q^2}{(4\pi)^2} 4\pi \left\{ \frac{1}{\epsilon} \int_a^b \frac{1}{r^2} \frac{1}{r^2} r^2 dr + \frac{1}{\epsilon_0} \int_b^\infty \frac{1}{r^2} dr \right\} = \frac{Q^2}{8\pi} \left\{ \frac{1}{\epsilon} \left( \frac{-1}{r} \right) \Big|_a^b + \frac{1}{\epsilon_0} \left( \frac{-1}{r} \right) \Big|_b^\infty \right\} \\ &= \frac{Q^2}{8\pi\epsilon_0} \left\{ \frac{1}{(1+\chi_e)} \left( \frac{1}{a} - \frac{1}{b} \right) + \frac{1}{b} \right\} = \boxed{\frac{Q^2}{8\pi\epsilon_0(1+\chi_e)} \left( \frac{1}{a} + \frac{\chi_e}{b} \right)}. \end{aligned}$$