1

1. (Griffiths ed.5, 4.16) Suppose the field inside a large piece of dielectric is $\boldsymbol{E}_0$, so that the electric displacement is $\boldsymbol{D}_0 = \epsilon_0 \boldsymbol{E}_0 + \boldsymbol{P}$.
   (a) Now a small spherical cavity (see Fig. 4.19 in the text) is hollowed out of the material. Find the field at the center of the cavity in terms of $\boldsymbol{E}_0$ and $\boldsymbol{P}$, and find the displacement at the center of the cavity in terms of $\boldsymbol{D}_0$ and $\boldsymbol{P}$. Here and in the rest of this problem assume that the polarization is "frozen in," so it doesn't change when the cavity is excavated.
   (b) Do the same for a long needle-shaped cavity running parallel to $\boldsymbol{P}$. (Again, see Fig. 4.19). *Hint*: ignore any asymmetry at the top and the bottom of the needle, which the text picture indicates. Note that "long" means that if L is the length of the needle and A is its cross-sectional area, we have A/L^2 sufficiently small such that $|\boldsymbol{P}|A/(4\pi\epsilon_0(L/2)^2) \ll |\boldsymbol{E}_0|$.
   (c) Do the same for a thin wafer-shaped cavity perpendicular to $\boldsymbol{P}$.(Again, see Fig. 4.19).
   Assume the cavities are small enough that $\boldsymbol{P}$, $\boldsymbol{E}_0$, and $\boldsymbol{D}_0$ are essentially uniform. *Hint*: carving out a cavity is the same as superimposing an object of the same shape but opposite polarization.

---

1. We learn from example 4.2 that the electric field inside sphere due to uniform polarization is

$$\boldsymbol{E}' = -\frac{1}{3\varepsilon_0}\boldsymbol{P}. \tag{1}$$

So the electric field at center due to spherical cavity is

$$\begin{gathered} \boldsymbol{E} = \boldsymbol{E_0} + \frac{1}{3\varepsilon_0}\boldsymbol{P}, \\ \Rightarrow \boldsymbol{D} = \varepsilon_0\left(\boldsymbol{E_0} + \frac{1}{3\varepsilon_0}\boldsymbol{P}\right) = \boldsymbol{D_0} - \frac{2}{3}\boldsymbol{P} \end{gathered} \tag{2}$$

1. As hinted, $|\boldsymbol{P}|A/\left(4\pi\varepsilon_0\left(\frac{L}{2}\right)^2\right) \ll |\boldsymbol{E_0}|$ implies that at the center of cavity,

$$\begin{gathered} \boldsymbol{E} \approx \boldsymbol{E_0} \\ \Rightarrow \boldsymbol{D} = \varepsilon_0\boldsymbol{E_0} = \boldsymbol{D_0} - \boldsymbol{P} \end{gathered} \tag{3}$$

3. We can treat the cavity as a parallel plate capacitor. Its surface charge density is

$$\sigma = \frac{q}{A} = P. \tag{4}$$

So the electric field at center of cavity due to wafer is $\boldsymbol{E}' = -\frac{\boldsymbol{P}}{\varepsilon_0}$, and thus

$$\begin{gathered} \boldsymbol{E} = E_0 + \frac{\boldsymbol{P}}{\varepsilon_0}, \\ \boldsymbol{D} = \varepsilon_0\boldsymbol{E} = D_0 \end{gathered} \tag{5}$$

2. A long cylindrical coaxial capacitor consists of an inner conductor of radius a and an outer conductor of radius b. The region between the conductors is filled with a linear and isotropic but inhomogeneous dielectric with relative permittivity given by $\epsilon_r(s) = \alpha s^{-2}$, in which s is the usual cylindrical radial coordinate. The capacitor is charged to a voltage V.

   Determine the electric field between the conductors as a function of the voltage V and the capacitor geometry for these conditions.

For a dialectic, the absence of free charge gives $\rho_f = 0$. So Gauss law becomes:

$$\begin{aligned} \nabla \cdot \vec{D} &= \rho_f = 0 \\ \frac{1}{s}\partial_s(sD_s) &= 0 \\ D_s &= \frac{C}{s}, \end{aligned} \tag{6}$$

for some constant C. Noticing $\vec{D}(s) = \varepsilon_0 \varepsilon_r \vec{E} = \varepsilon_0 \alpha s^{-2} \vec{E}$, we have

$$\begin{aligned} \vec{E} &= \frac{Cs}{\varepsilon\alpha}\hat{s}, \\ V &= -\int_a^b \vec{E} \cdot \mathrm{d}\vec{l} = -\frac{C}{\varepsilon_0 \alpha}\int_a^b s\,\mathrm{d}s = -\frac{C}{2\varepsilon_0\alpha}(b^2 - a^2). \end{aligned} \tag{7}$$

We find C to be

$$C = \frac{-2\varepsilon\alpha V}{b^2 - a^2}. \tag{8}$$

And thus

$$\vec{E}(s) = \frac{-2\varepsilon\alpha V s}{\varepsilon\alpha(b^2 - a^2)} = \boxed{\frac{2Vs}{a^2 - b^2}\hat{s}.} \tag{9}$$

3

3. (Griffiths ed.5, 4.25) Suppose the entire region below $z = 0$ is filled with a uniform linear dielectric material of susceptibility χ_e and the region above $z = 0$ is filled with a uniform linear dielectric material with a different susceptibility χ_e' A point charge q is at the location $(x = 0, y = 0, z = d)$, where $d > 0$ (see Fig. 4.28 in the text). Find the potential in both regions ($z > 0$ and $z < 0$).

$$\begin{aligned} V^{>}(r,z) &= \frac{1}{4\pi\varepsilon'}\left(\frac{q}{(r^2+(z-d)^2)^{1/2}} + \frac{q'}{(r^2+(z+d)^2)^{1/2}}\right) \\ \Rightarrow V^{>}(r,0) &= \frac{1}{4\pi\varepsilon'}\frac{q+q'}{(r^2+d^2)^{1/2}}, \quad \frac{\partial V^{>}}{\partial z}(r,0) = \frac{1}{4\pi\varepsilon'}\frac{d(q-q')}{(r^2+d^2)^{3/2}}. \end{aligned} \tag{10}$$

and,

$$\begin{aligned} V^{<}(r,z) &= \frac{1}{4}\pi\varepsilon\frac{q''}{(r^2+(z-d)^2)^{1/2}} \\ \Rightarrow V^{<}(r,0) &= \frac{1}{4\pi\varepsilon}\frac{q''}{(r^2+d^2)^{1/2}}, \quad \frac{\partial V^{<}}{\partial z}(r,0) = \frac{1}{4\pi\varepsilon}\frac{q''}{(r^2+d^2)^{3/2}}, \end{aligned} \tag{11}$$

where $\varepsilon' = \varepsilon_0(1+\chi_e'), \varepsilon = \varepsilon_0(1+\chi_e)$.

Boundary condition gives

$$\begin{aligned} \frac{1}{\varepsilon'}(q+q') &= \frac{q''}{\varepsilon}, \quad (q-q') = q'' \\ \Rightarrow q'' &= \frac{2\varepsilon}{\varepsilon+\varepsilon'}q, \quad q' = \frac{\varepsilon'-\varepsilon}{\varepsilon'+\varepsilon}q \end{aligned} \tag{12}$$

Hence,

$$\begin{aligned} V^{>}(r,z) &= \frac{q}{4\pi\varepsilon'}\left(\frac{1}{(r^2+(z-d)^2)^{\frac{1}{2}}} + \left(\frac{\varepsilon'-\varepsilon}{\varepsilon'+\varepsilon}\right)\frac{1}{(r^2+(z+d)^2)^{\frac{1}{2}}}\right), \quad (z \geq 0); \\ V^{<}(r,z) &= \frac{2q}{4\pi(\varepsilon'+\varepsilon)}\frac{1}{(r^2+(z+d)^2)^{\frac{1}{2}}}, \quad (z \leq 0.) \end{aligned} \tag{13}$$

4. A ideal dipole $\boldsymbol{p} = p\hat{z}$ is in the center of a spherical uniform dielectric shell with inner radius a and outer radius b. Find the potential in all regions.

- Inside the cavity, $(r < a)$: Separation of Variables with dipole gives

$$V_I(r,\theta) = \frac{1}{4\pi\varepsilon_0}\left(\frac{p\cos\theta}{r^2} + \sum_{l=0}^{\infty} A_{1,l} r^l P_l(\cos\theta)\right) \tag{14}$$

where p is the dipole moment. Dipole expansion gives

$$V_I(r,\theta) = \frac{p}{4\pi\varepsilon_0}\left(\frac{1}{r^2} + \frac{r}{a^3}\left(\frac{1}{\varepsilon_r}(A_2 + B_2) - 1\right)\right)\cos\theta \tag{15}$$

$$\frac{\partial V_I}{\partial r} = \frac{p\cos\theta}{4\pi\varepsilon 0 a^3}\left(\frac{1}{\varepsilon_r}(A_2 + B_2) - 3\right) \tag{16}$$

- Within shell, $(a < r < b)$, **assuming a dialectic constant ε_r**, we can jot down the following by using results from Separation of Variables:

$$V_{\mathrm{II}}(r,\theta) = \frac{1}{4\pi\varepsilon}\sum_{l=0}^{\infty}\left(A_{2,l} r^l + \frac{B_{2,l}}{r^{l+1}}\right)P_l(\cos\theta)) \tag{17}$$

Dipole expansion, taking $l = 1$ gives:

$$V_{\mathrm{II}}(r,\theta) = \frac{p}{4\pi\varepsilon}\left(A_2\frac{r}{a^3} + \frac{B_2}{r^2}\right)\cos\theta. \tag{18}$$

$$\frac{\partial V_{\mathrm{II}}(a,\theta)}{\partial r} = \frac{p\cos\theta}{4\pi\varepsilon a^3}(A_2 - 2B_2); \quad \frac{\partial V_{\mathrm{II}}(b,\theta)}{\partial r} = \frac{p}{4\pi\varepsilon b^3}\left(A_2\left(\frac{b}{a}\right)^3 - 2B_2\right) \tag{19}$$

- Outside the shell, $(r > b)$:

$$V_{\mathrm{III}}(r,\theta) = \frac{1}{4\pi\varepsilon_0}\sum_{l=0}^{\infty}\frac{B_{3,l}}{r^{l+1}}P_l(\cos\theta). \tag{20}$$

Taking $l = 1$ gives:

$$V_{\mathrm{III}}(r,\theta) = \frac{p}{4\pi\varepsilon r^2}\left(A_2\left(\frac{b}{a}\right)^3 + B_2\right). \tag{21}$$

$$\frac{\partial V_{\mathrm{III}}(b,\theta)}{\partial r} = \frac{p}{4\pi\varepsilon b^3}(-2)\left(A_2\left(\frac{b}{a}\right)^3 + B_2\right) \tag{22}$$

Boundary Conditions:

$$\begin{aligned} \varepsilon\frac{\partial V_{\mathrm{II}}(a,\theta)}{\partial r} &= \varepsilon_0\frac{\partial V_I(a,\theta)}{\partial r}, \\ \varepsilon_0\frac{\partial V_{\mathrm{III}}(b,\theta)}{\partial r} &= \varepsilon\frac{\partial V_{\mathrm{II}}}{\partial r}(b,\theta) \end{aligned} \tag{23}$$

Painful algebra gives:

$$
\begin{aligned}
A_2 &= \left(\frac{b}{a}\right)^{-3} \frac{B_2(2\varepsilon_r - 2)}{2+\varepsilon_r}, \\
B_2 &= \frac{3\varepsilon_r}{(1+2\varepsilon_r) - 2\left(\frac{a}{b}\right)^3 \frac{(1-\varepsilon_r)^2}{2+\varepsilon_r}} \equiv f(\varepsilon_r, a, b)
\end{aligned} \tag{24}
$$

We thus conclude:

$$
\begin{aligned}
V_I(r,\theta) &= \frac{p\cos\theta}{4\pi\varepsilon r^2}\left(1 + \left(\frac{r}{a}\right)^3 \left(\frac{f(\varepsilon_r, a, b)}{\varepsilon_r}\right)\left(1 + \left(\frac{a}{b}\right)^3 \frac{2(\varepsilon_r - 1)}{2+\varepsilon_r}\right) - 1\right), \\
V_{\mathrm{II}}(r,\theta) &= \frac{p\cos\theta}{4\pi\varepsilon r^2} f(\varepsilon_r, a, b)\left(1 + \frac{2(\varepsilon_r - 1)}{2+\varepsilon_r}\left(\frac{r}{b}\right)^3\right), \\
V_{\mathrm{III}}(r,\theta) &= \frac{p\cos\theta}{4\pi\varepsilon r^2}\,\frac{3f(\varepsilon_r, a, b)}{(\varepsilon_r + 2)}
\end{aligned} \tag{25}
$$

5

5. (Griffiths ed.5, 4.26) A spherical conductor of radius a carries a charge Q. It is surrounded by linear (isotropic, homogenous) dielectric material of electric susceptibility χ_e, out to radius b. (See Fig. 4.32 in the text.) Find the energy of this configuration.

Identical set up, with dielectric material of permittivity ε, can be found in Example 4.5 of textbook. Citing the result, we have:

$$\boldsymbol{D} = \begin{cases} \frac{Q}{4\pi r^2}\hat{r} & (r > a) \\ 0 & (r < a) \end{cases}, \quad \boldsymbol{E} = \begin{cases} 0 & (r < a) \\ \frac{Q}{4\pi\varepsilon r^2}\hat{r} & (r \in a, b) \\ \frac{Q}{4\pi\varepsilon_0 r^2}\hat{r} & (r > b) \end{cases}. \tag{26}$$

Using $W = \frac{1}{2}\int_V \boldsymbol{D} \cdot \boldsymbol{E}\,\mathrm{d}\tau$, with $\varepsilon = \varepsilon_0(1 + \chi_e)$, we have:

$$\begin{aligned} W &= \frac{1}{2}\int \boldsymbol{D} \cdot \boldsymbol{E}\,\mathrm{d}\tau \\ &= \frac{Q^2}{8\pi}\left(\frac{1}{\varepsilon}\int_a^b \frac{1}{r^2}\frac{1}{r^2}r^2\,\mathrm{d}r \quad + \frac{1}{\varepsilon_0}\int_b^\infty \frac{1}{r^2}\,\mathrm{d}r\right) \\ &= \frac{Q^2}{8\pi\varepsilon_0}\left(\left(\frac{1}{a} - \frac{1}{b}\right)\left(\frac{1}{1 + \chi_e}\right) + \frac{1}{b}\right) \\ &= \left(\frac{1}{a} + \frac{\chi_e}{b}\right)\frac{Q^2}{8\pi\varepsilon_0(1 + \chi_e)} \end{aligned} \tag{27}$$