

Problem 1

Let $a > 0$. Define the function $f : \mathbb{R} \rightarrow \mathbb{R}$ by:

$$f(x) = \begin{cases} \frac{\sqrt{x}-\sqrt{a}}{x-a} & (x \neq a, x > 0) \\ 0 & (x = a \text{ or } x \leq 0). \end{cases} \quad (1)$$

Find limit of f as x approaches a , and prove your answer.

When $x \neq a, x > 0$:

$$f(x) = \frac{\sqrt{x}-\sqrt{a}}{x-a} = \frac{\sqrt{x}-\sqrt{a}}{(\sqrt{x}+\sqrt{a})(\sqrt{x}-\sqrt{a})} = \frac{1}{\sqrt{a}+\sqrt{x}}. \quad (2)$$

Claim:

$$\lim_{x \rightarrow a} f(x) = \frac{1}{2\sqrt{a}}. \quad (3)$$

Proof: For any given ε , let $\delta = \min\left(\frac{a}{2}, 2a^{3/2}c^2\varepsilon\right)$, where $c = 1 + \sqrt{2}/2$, s.t. if $|x-a| < \delta$, then $\left|f(x) - \frac{1}{2\sqrt{a}}\right| < \varepsilon$.
Because:

$$\begin{aligned} |x-a| < \delta &\Rightarrow a-\delta < x < a+\delta \\ &\Rightarrow \sqrt{x} + \sqrt{a} > \sqrt{a-\delta} + \sqrt{a} \end{aligned} \quad (4)$$

Since $\delta \leq \frac{a}{2}$,

$$\begin{aligned} \sqrt{x} - \sqrt{a} &> \sqrt{a} + \sqrt{\frac{a}{2}} \\ &= \left(1 + \frac{\sqrt{2}}{2}\right)\sqrt{a} := c\sqrt{a} \end{aligned} \quad (5)$$

It follows that

$$\begin{aligned} 2\sqrt{a}(\sqrt{a} + \sqrt{x})^2 &> 2a^{\frac{3}{2}}c^2 \\ \Rightarrow \frac{|x-a|}{2\sqrt{a}(\sqrt{x} + \sqrt{a})^2} &< \frac{|x-a|}{2a^{\frac{3}{2}}c^2} \\ &= \frac{\delta}{2a^{\frac{3}{2}}c^2} < \varepsilon \end{aligned} \quad (6)$$

So that:

$$\left|f(x) - \frac{1}{2\sqrt{a}}\right| = \left|\frac{1}{\sqrt{a} + \sqrt{x}} - \frac{1}{2\sqrt{a}}\right| = \frac{|\sqrt{a} - \sqrt{x}|}{2\sqrt{a}(\sqrt{a} + \sqrt{x})} = \frac{|x-a|}{2\sqrt{a}(\sqrt{x} + \sqrt{a})^2} < \varepsilon \quad (7)$$

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Problem 2

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function s.t. $\lim_{x \rightarrow 0} f(x) = 0$. Define the function $g : \mathbb{R} \rightarrow \mathbb{R}$ by

$$g(x) = \begin{cases} f(x) \cdot \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases} \quad (8)$$

Prove that $\lim_{x \rightarrow 0} g(x) = 0$

• When $x \neq 0$:

$$|g(x)| = \left| f(x) \cdot \sin\left(\frac{1}{x}\right) \right| \leq |f(x)| \quad (9)$$

Given that $\lim_{x \rightarrow 0} f(x) = 0$, by definition, $\forall \varepsilon > 0, \exists \delta > 0$, s.t. $0 < |x| < \delta \Rightarrow |f(x)| < \varepsilon$

So for $0 < x < \delta$:

$$|g(x)| \leq |f(x)| < \varepsilon. \quad (10)$$

Also notice that $x = 0$:

$$g(0) = 0 < \varepsilon. \quad (11)$$

So $\forall x \in \mathbb{R}, \forall \varepsilon > 0, \exists \delta > 0$ s.t. $0 < |x| < \delta \Rightarrow |g(x)| < \varepsilon$. In other words, $\lim_{x \rightarrow 0} g(x) = 0$.

Problem 3

Suppose that $f, g : \mathbb{R} \rightarrow \mathbb{R}$ are functions s.t. $|f(x)| \leq |g(x)|, \forall x \in \mathbb{R}$. g is continuous at 0, and $g(0) = 0$. Prove that f is continuous at 0.

Given that g is continuous at 0, we know

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } 0 < |x| < \delta \Rightarrow |g(x)| < \varepsilon. \quad (12)$$

So, for the same δ ,

$$0 < |x - 0| < \delta \Rightarrow |f(x) - 0| \leq |g(x)| < \varepsilon \quad (13)$$

Noticing $|f(0)| \leq g(0) = 0 \Rightarrow f(0) = 0$. We have thus proved that

$$\lim_{x \rightarrow 0} f(x) = f(0) \quad (14)$$

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Problem 4

Let $f : \mathbb{R} \rightarrow \mathbb{R}, f(x) = |x|$.

1. Prove that f is continuous.
 2. Let $g : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function. Prove that $|g|$ is also a continuous function.
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1. Consider arbitrary point $c \in \mathbb{R}$. To prove that f is continuous over \mathbb{R} , we need to show that

$$\forall \varepsilon > 0, \exists \delta > 0, \text{ s.t. } |x - c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon. \quad (15)$$

Let $\varepsilon > 0$ be given, choosing $\delta = \min\{\varepsilon, |c|\}$ when $c \neq 0$, then

- $c > 0$:

$$|f(x) - f(c)| = ||x| - |c|| = |x - c| < \delta = \varepsilon. \quad (16)$$

- $c < 0$:

$$|f(x) - f(c)| = ||x| - |c|| = |-x + c| = |x - c| < \delta = \varepsilon. \quad (17)$$

- $c = 0$: choose $\delta = \varepsilon$.

$$|f(x) - f(c)| = ||x| - |0|| = |x| < \delta = \varepsilon. \quad (18)$$

So in all cases and for all $\varepsilon > 0$, there exists a $\delta > 0$ s.t. $|x - c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon$. Therefore, f is continuous over \mathbb{R} .

2. Given that g is a continuous function, and, from part 1, that $|x|$ is continuous, we can construct their composite function $|g| = f \circ g$.

From lecture, we know that the composition of two continuous functions is also continuous. Therefore, $|g|$ is a continuous function.

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Problem 5

Let $a, b, c \in \mathbb{R}$ with $a < b < c$. Suppose that $f : [a, b] \rightarrow \mathbb{R}$; $g : [b, c] \rightarrow \mathbb{R}$ are both continuous, and $f(b) = g(b)$. Prove that the function $h : [a, c] \rightarrow \mathbb{R}$ defined by

$$h(x) = \begin{cases} f(x) & \text{if } x \leq b \\ g(x) & \text{if } x > b \end{cases} \quad (19)$$

is continuous.

- Continuity on $[a, b)$: when $x \in [a, b)$, $h(x) = f(x)$. Since f is continuous on $[a, b]$, h is continuous on $[a, b)$.
- Continuity on $(b, c]$: when $x \in (b, c]$, $h(x) = g(x)$. Since g is continuous on $[b, c]$, h is continuous on $(b, c]$.
- Continuity at b : Given that $f(x)$ is continuous on $[a, b]$, and $g(x)$ is continuous on $[b, c]$, we have the following:

$$\lim_{x \rightarrow b^-} f(x) = f(b), \quad \lim_{x \rightarrow b^+} g(x) = g(b) \quad (20)$$

Since $f(b) = g(b)$, it follows that

$$\lim_{x \rightarrow b^-} h(x) = \lim_{x \rightarrow b^-} f(x) = f(b) = g(b) = \lim_{x \rightarrow b^+} g(x) = \lim_{x \rightarrow b^+} h(x). \quad (21)$$

Using the fact that

$$\lim_{x \rightarrow b} h(x) = l \Leftrightarrow \lim_{x \rightarrow b^-} h(x) = l = \lim_{x \rightarrow b^+} h(x), \quad (22)$$

and $h(b) = f(b)$, We have shown that $\lim_{x \rightarrow b} h(x) = h(b)$, and thus h is continuous at b .

- Collecting the above, we have shown that h is continuous on $[a, c]$.

Problem 6

Suppose that $f : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function. Prove that for any open set $U \subseteq \mathbb{R}$, the set $f^{-1}(U)$ is also open.

Take an arbitrary point $x_0 \in f^{-1}(U)$. By definition of preimage, $f(x_0) \in U$. Since U is an open set, then by definition

$$\exists \varepsilon > 0, \text{ s.t. } (f(x_0) - \varepsilon, f(x_0) + \varepsilon) \subseteq U. \quad (23)$$

Considering the continuity of f at $x_0 \in \mathbb{R}$, we have

$$\begin{aligned} \forall \varepsilon > 0, \exists \delta > 0, \text{ s.t. } |x - x_0| < \delta &\Rightarrow |f(x) - f(x_0)| < \varepsilon. \\ \Rightarrow x_0 - \delta < x < x_0 + \delta; \quad f(x_0) - \varepsilon < f(x) < f(x_0) + \varepsilon. \end{aligned} \quad (24)$$

It follows that

$$f(x) \in (f(x_0) - \varepsilon, f(x_0) + \varepsilon) \subseteq U. \quad (25)$$

So by definition of preimage,

$$x \in f^{-1}(U). \quad (26)$$

Since x_0 is arbitrary, we have shown that

$$\forall x_0 \in f^{-1}(U), \exists \delta > 0, \text{ s.t. } (x_0 - \delta, x_0 + \delta) \subseteq f^{-1}(U). \quad (27)$$

Which satisfies the definition of an open set. Therefore, $f^{-1}(U)$ is open. ■