Last time: Differentiation laws

① $(cf)'(a) = c \cdot f'(a) \quad \forall c \in \mathbb{R}$

② $(f+g)'(a) = f'(a) + g'(a)$

Thm (Product rule) Let $I \subseteq \mathbb{R}$ be an open interval, $f, g : I \to \mathbb{R}$, and $a \in I$. If f and g are differentiable at a, then fg is differentiable at a and

$$(fg)'(a) = f'(a)g(a) + f(a)g'(a).$$

Pf: For $h \neq 0$,

$$\frac{f(a+h)g(a+h) - f(a)g(a)}{h}$$

$$= \frac{f(a+h)g(a+h) - f(a)g(a+h) + f(a)g(a+h) - f(a)g(a)}{h}$$

$$= \frac{f(a+h) - f(a)}{h} \cdot g(a+h) + f(a) \cdot \frac{g(a+h) - g(a)}{h}$$

As f and g are differentiable at a, we know:

① $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = f'(a)$

② $\lim_{h \to 0} \frac{g(a+h) - g(a)}{h} = g'(a)$

③ g continuous at a (by lecture 22)

$\Rightarrow \lim_{x \to a} g(x) = g(a)$

$$\Rightarrow \lim_{h\to 0} g(a+h) = g(a)$$

Therefore, by the limit laws,

$$\lim_{h\to 0} \frac{f(a+h)g(a+h) - f(a)g(a)}{h}$$

$$= \lim_{h\to 0}\left[\frac{f(a+h)-f(a)}{h}\cdot g(a+h) + f(a)\cdot \frac{g(a+h)-g(a)}{h}\right]$$

$$= f'(a) \cdot g(a) + f(a)\cdot g'(a)$$

So $(fg)'(a) = f'(a)g(a) + f(a)g'(a)$. $\square$

<u>Thm</u> (Quotient rule) Let $I \subseteq \mathbb{R}$ be an open interval, $f, g: I \to \mathbb{R}$, and $a \in I$. If f and g are differentiable at a and $g(a) \neq 0$, then $\frac{f}{g}$ is differentiable at a and

$$\left(\frac{f}{g}\right)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{g(a)^2}.$$

<u>Pf</u>: It suffices to show

$$\left(\frac{1}{g}\right)'(a) = -\frac{g'(a)}{g(a)^2}.$$

Indeed, once we prove this, then by the product rule we have

$$\left(f\cdot\frac{1}{g}\right)'(a) = f'(a)\cdot\frac{1}{g(a)} + f(a)\cdot\left(\frac{1}{g}\right)'(a)$$

$$= \frac{f'(a)}{g(a)} - \frac{f(a)g'(a)}{g(a)^2}$$

$$= \frac{f'(a)g(a) - f(a)g'(a)}{g(a)^2}.$$

Want: $\lim_{h\to 0} \frac{\frac{1}{g(a+h)} - \frac{1}{g(a)}}{h} = -\frac{g'(a)}{g(a)^2}$. As g is continuous at a (by Lecture 22) and $g(a) \neq 0$, then $\exists\, \delta > 0$ s.t. $g(x) \neq 0$ $\forall x \in (a-\delta, a+\delta)$ (by Lecture 15). So for $h \in (-\delta, \delta)$, $\frac{1}{g(a+h)}$ makes sense and

$$\frac{\frac{1}{g(a+h)} - \frac{1}{g(a)}}{h} = \frac{g(a) - g(a+h)}{h \cdot g(a) \cdot g(a+h)}$$

$$= -\frac{g(a+h) - g(a)}{h} \cdot \frac{1}{g(a)g(a+h)}$$

As g is differentiable at a, we know:

① $\lim_{h\to 0} \frac{g(a+h) - g(a)}{h} = g'(a)$

② g continuous at a (by Lecture 22)

$$\Rightarrow \lim_{x\to a} g(x) = g(a)$$

$$\Rightarrow \lim_{h\to 0} g(a+h) = g(a) \quad \text{and } g(a) \neq 0$$

$$\Rightarrow \lim_{h\to 0} \frac{1}{g(a)g(a+h)} = \frac{1}{g(a)^2}$$

Together, by the limit laws,

$$\lim_{h\to 0} \frac{\frac{1}{g(a+h)} - \frac{1}{g(a)}}{h} = \lim_{h\to 0} \left[-\frac{g(a+h) - g(a)}{h} \cdot \frac{1}{g(a)g(a+h)} \right]$$

$$= -g'(a)\cdot\frac{1}{g(a)^2}$$

So $\left(\frac{1}{g}\right)'(a) = -\frac{g'(a)}{g(a)^2}$. $\square$

<u>Ex</u> $\forall n \in \mathbb{Z}$, $\frac{d}{dx}(x^n) = nx^{n-1}$.

<u>Pf</u>: Case: $n \geq 0$. Already know (Lecture 24).

Case: $n < 0$. Then $x^n = \frac{1}{x^{-n}}$ with $-n > 0$.

Fix $a \neq 0$. We know $g(x) = x^{-n}$ is differentiable with $g'(a) = -na^{-n-1}$ and $g(a) = a^{-n} \neq 0$. So, by the quotient rule,

$$(x^n)'(a) = \left(\frac{1}{g}\right)'(a) = -\frac{g'(a)}{g(a)^2}$$

$$= -\frac{(-n)a^{-n-1}}{(a^{-n})^2} = na^{-n-1+2n} = na^{n-1}. \quad \square$$

<u>Ex</u> Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \frac{x}{x^2+1}$. Prove f is differentiable, and find its derivative.

<u>Pf</u>: Fix $a \in \mathbb{R}$. Then

$$(x)'(a) = 1, \qquad (x^2+1)'(a) = 2a$$

(These are polynomials, which we already know are differentiable on $\mathbb{R}$.) As $a^2+1 \neq 0$ (since $a^2 \geq 0$), then by the quotient rule we have

$$f'(a) = \frac{1 \cdot (a^2+1) - a \cdot 2a}{(a^2+1)^2} = \frac{-a^2+1}{(a^2+1)^2}. \qquad \square$$

<u>Thm</u> (Chain rule) Let $I, J \subseteq \mathbb{R}$ be open intervals, $f: I \to J$ and $g: J \to \mathbb{R}$ be functions, and $a \in I$. If f is differentiable at a and g is differentiable at $f(a)$, then $g \circ f$ is differentiable at a and

$$(g \circ f)'(a) = g'(f(a)) \cdot f'(a).$$

Recall Differentiation laws

① $(cf)'(a) = cf'(a) \quad \forall c \in \mathbb{R}$

② $(f+g)'(a) = f'(a) + g'(a)$

③ $(fg)'(a) = f'(a)g(a) + f(a)g'(a)$

④ $\left(\frac{f}{g}\right)'(a) = \dfrac{f'(a)g(a) - f(a)g'(a)}{g(a)^2}$

⑤ $(g \circ f)'(a) = g'(f(a)) \cdot f'(a)$

Thm (Chain rule) Let $I, J \subseteq \mathbb{R}$ be open intervals, $f : I \to J$ and $g : J \to \mathbb{R}$ be functions, and $a \in I$. If f is differentiable at a and g is differentiable at $f(a)$, then $g \circ f$ is differentiable at a and

$$(g \circ f)'(a) = g'(f(a)) \cdot f'(a).$$

Idea: For $h \neq 0$,

$$\frac{g(f(a+h)) - g(f(a))}{h} = \underbrace{\frac{g(f(a+h)) - g(f(a))}{f(a+h) - f(a)}}_{= \frac{g(f(a)+k) - g(f(a))}{k}} \cdot \underbrace{\frac{f(a+h) - f(a)}{h}}_{\to f'(a) \text{ as } h \to 0}$$

with $k(h) = f(a+h) - f(a) \to 0$ as $h \to 0$

$\to g'(f(a))$ as $h \to 0$

Problem What if $k = 0$? In fact, we could have $k(h) = 0$ infinitely often near $h = 0$! E.g.

- $f(x) = \text{constant} \Rightarrow k(h) = f(a+h) - f(a) = 0 \quad \forall h$
  (Well, maybe this isn't really a problem, since the other factor $f'(0)$ is $0 \ldots$)
- $f(x) = x^2 \sin \frac{1}{x}$, $a = 0 \Rightarrow k(h) = h^2 \sin \frac{1}{h}$

Pf: For $h \neq 0$,

$$\frac{g(f(a+h)) - g(f(a))}{h} = \phi(h) \cdot \frac{f(a+h) - f(a)}{h} \qquad (\star)$$

where

$$\phi(h) = \begin{cases} \dfrac{g(f(a+h)) - g(f(a))}{f(a+h) - f(a)} & \text{if } f(a+h) \neq f(a) \\ g'(f(a)) & \text{if } f(a+h) = f(a) \end{cases}$$

Note that in both cases, the equation $(\star)$ holds:

- If $f(a+h) \neq f(a)$: $f(a+h) - f(a)$ cancels out, as before
- If $f(a+h) = f(a)$: $\text{RHS}(\star) = g'(f(a)) \cdot \frac{f(a+h) - f(a)}{h} = 0$

  $\text{LHS}(\star) = \frac{g(f(a+h)) - g(f(a))}{h} = 0$

Claim: $\phi(h)$ is continuous at $h = 0$, i.e. $\lim_{h \to 0} \phi(h) = g'(f(a))$. Fix $\varepsilon > 0$. Want: $\exists \delta > 0$ s.t.

$$|h| < \delta \Rightarrow \underbrace{|\phi(h) - g'(f(a))|}_{} < \varepsilon$$

$$= \begin{cases} \dfrac{g(f(a+h)) - g(f(a))}{f(a+h) - f(a)} - g'(f(a)) \\ g'(f(a)) - g'(f(a)) = 0 \quad \text{already } < \varepsilon! \end{cases}$$

So it suffices to show: $\exists\, \delta > 0$ s.t.

$$|h| < \delta,\ f(a+h) \neq f(a) \implies \left| \frac{g(f(a+h)) - g(f(a))}{f(a+h) - f(a)} - g'(f(a)) \right| < \varepsilon$$

We know:

① g is differentiable at $f(a)$

$$\implies \lim_{k \to 0} \frac{g(f(a)+k) - g(f(a))}{k} = g'(f(a))$$

$\implies \exists\, \delta_1 > 0$ s.t. $|k| < \delta_1$ implies

$$\left| \frac{g(f(a)+k) - g(f(a))}{k} - g'(f(a)) \right| < \varepsilon$$

② f is differentiable at a

$\implies f$ is continuous at a $\implies \lim_{h \to 0} f(a+h) = f(a)$

$\implies \exists\, \delta_2 > 0$ s.t. $|h| < \delta_2$ implies $|f(a+h) - f(a)| < \delta_1$

Altogether,

$$|h| < \delta_2 \implies |\underbrace{f(a+h) - f(a)}_{\text{Take } = k \text{ in ①}}| < \delta_1$$

$$\implies \left| \frac{g(f(a+h)) - g(f(a))}{f(a+h) - f(a)} - g'(f(a)) \right| < \varepsilon$$

as desired. This finishes the claim.

Recall: for $h \neq 0$,

$$\frac{g(f(a+h)) - g(f(a))}{h} = \phi(h) \cdot \frac{f(a+h) - f(a)}{h}$$

By the claim,
$$\lim_{h\to 0} \phi(h) = g'(f(a)).$$
As f is differentiable at a,
$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = f'(a).$$
Therefore, by the limit law for products,
$$\begin{array}{ccc} \displaystyle\lim_{h\to 0} \frac{g(f(a+h))-g(f(a))}{h} & = & \displaystyle\lim_{h\to 0} \left[\phi(h) \cdot \frac{f(a+h)-f(a)}{h} \right] \\ \| & & \| \\ (g\circ f)'(a) & & g'(f(a)) \cdot f'(a) \end{array} \qquad \square$$

<u>Ex</u> (a.) Prove that $f:(-1,1)\to\mathbb{R}$, $f(x)=\sqrt{1-x^2}$ is differentiable, and find its derivative.

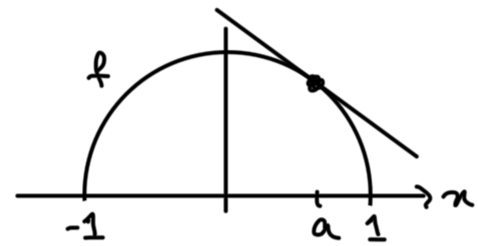
<u>Pf</u>: Write
$$f = g\circ h, \quad \text{where } g(y)=\sqrt{y}, \quad h(x) = 1-x^2$$
Fix $a\in\mathbb{R}$.

- $h:(-1,1)\to(0,\infty)$ is differentiable at a since h is a polynomial, and $h'(a) = -2a$
- $g:(0,\infty)\to\mathbb{R}$ is differentiable at $y=1-a^2>0$ by Hw 8, and $g'(y) = \frac{1}{2\sqrt{y}}$

So, by the chain rule,
$$f'(a) = (g\circ h)'(a) = g'(h(a))\cdot h'(a)$$

$$= \frac{1}{2\sqrt{1-a^2}} \cdot (-2a) = \frac{-a}{\sqrt{1-a^2}} . \qquad \square$$

(b.) Find a formula for the tangent line to the graph of f at $a \in (-1,1)$.



$$L(x) = f(a) + f'(a)(x-a)$$

$$= \sqrt{1-a^2} - \frac{a}{\sqrt{1-a^2}}(x-a)$$

$$= -\frac{a}{\sqrt{1-a^2}}x + \frac{1-a^2+a^2}{\sqrt{1-a^2}} = -\frac{a}{\sqrt{1-a^2}}x + \frac{1}{\sqrt{1-a^2}}$$

(c.) Prove that this tangent line intersects the unit circle $\{(x,y) \in \mathbb{R}^2 : x^2+y^2=1\}$ exactly once.

Pf: Note that

$L(x)$ intersects the unit circle $\iff$ $(x, L(x))$ satisfies $x^2 + L(x)^2 = 1$.

$$x^2 + L(x)^2 = x^2 + \left(\frac{-ax+1}{\sqrt{1-a^2}}\right)^2$$

$$= x^2 + \frac{a^2x^2 - 2ax + 1}{1-a^2}$$

$$= \frac{(1-\cancel{a^2})x^2 + \cancel{a^2x^2} - 2ax + 1}{1-a^2}$$

So $\quad 1 = x^2 + L(x)^2 = \dfrac{x^2 - 2ax + 1}{1-a^2}$

$$\iff x^2 - 2ax + 1 = 1 - a^2$$

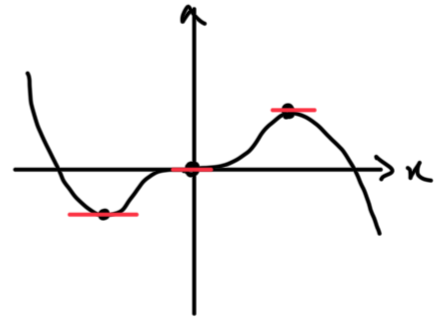
$$\iff 0 = x^2 - 2ax + a^2 = (x-a)^2$$

$$\iff \quad x = a. \qquad \square$$

SIGNIFICANCE OF THE DERIVATIVE (Ch. 11)

<u>Def</u> Let $I \subseteq \mathbb{R}$ be an open interval, $f : I \to \mathbb{R}$, and $a \in I$. We say a is a <u>critical point</u> of f if $f'(a) = 0$.

<u>Ex</u> $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^3 - x^5$



$$0 = f'(a) = 3a^2 - 5a^4 = a^2(3 - 5a^2)$$

$$\iff \quad a = 0 \quad \text{or} \quad a^2 = \frac{3}{5}$$

$$\iff \quad a = 0 \quad \text{or} \quad a = \pm\sqrt{\frac{3}{5}}$$

3 critical points.

- What's special about f at these 3 points?
  E.g. f is biggest at $a = \sqrt{\frac{3}{5}}$, but it's not a global max...

<u>Def</u> Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$, and $a \in A$. We say:

1. $f(a)$ is a <u>local maximum</u> of f if $\exists\, \delta > 0$ s.t. $f(x) \leq f(a)$ $\forall\, x \in A \cap (a - \delta, a + \delta)$

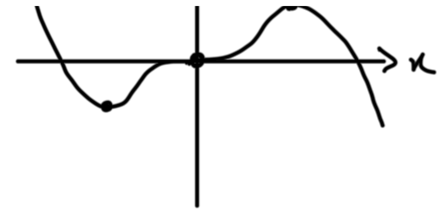
2. $f(a)$ is a <u>local minimum</u> of f if $\exists\, \delta > 0$ s.t. $f(x) \geq f(a)$ $\forall\, x \in A \cap (a - \delta, a + \delta)$

<u>Rmk</u> Global max/min $\Rightarrow$ local max/min

<u>Ex</u> $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^3 - x^5$

$f(\sqrt{\frac{3}{5}})$ is a local max

- $f(\sqrt{\frac{3}{5}})$ is a local max



- $f(-\sqrt{\frac{3}{5}})$ is a local min
- $f(0)$ is neither
- f has no global max or min

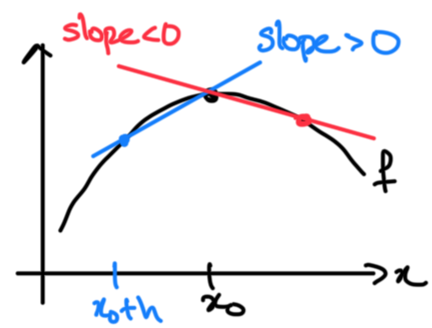
<u>Thm</u> Let $I \subseteq \mathbb{R}$ be an open interval and $x_0 \in I$. If $f: I \to \mathbb{R}$ is differentiable at x_0 and $f(x_0)$ is a local maximum or minimum, then x_0 is a critical point (i.e. $f'(x_0) = 0$).

<u>Rmk</u> $f(x_0)$ local max/min $\overset{\Rightarrow}{\not\Leftarrow}$ $f'(x_0) = 0$

E.g. $x_0 = 0$ in previous example.

<u>Pf</u>: Case: Local max. Suppose $f(x_0)$ is a local max and f is differentiable at x_0. Then $\exists\, \delta > 0$ s.t. for $0 < |h| < \delta$,

$$f(x_0+h) \leq f(x_0)$$



$$\Rightarrow f(x_0+h) - f(x_0) \leq 0$$

$$\Rightarrow \begin{cases} \dfrac{f(x_0+h) - f(x_0)}{h} \leq 0 & \text{if } h > 0 \\ \dfrac{f(x_0+h) - f(x_0)}{h} \geq 0 & \text{if } h < 0 \end{cases}$$

As f is differentiable at x_0, we know

$$\lim_{h \to 0^+} \frac{f(x_0+h) - f(x_0)}{h} = f'(x_0) = \lim_{h \to 0^-} \frac{f(x_0+h) - f(x_0)}{h}.$$

Together,

$$f'(x_0) = \lim_{h\to 0^+} \underbrace{\frac{f(x_0+h)-f(x_0)}{h}}_{\leq 0 \quad \forall h\in(0,\delta)} \leq 0$$

$$f'(x_0) = \lim_{h\to 0^-} \underbrace{\frac{f(x_0+h)-f(x_0)}{h}}_{\geq 0 \quad \forall h\in(-\delta,0)} \geq 0$$

$\Rightarrow f'(x_0) = 0.$

(Recall: Hw4#4

- $f(x) \leq g(x) \ \forall x$
- $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist

$\Rightarrow \lim_{x\to a} f(x) \leq \lim_{x\to a} g(x)$

Similar proof works for $\lim_{x\to a^+}$ and $\lim_{x\to a^-}$.)

Case: Local min. Then $-f$ has a local max. By the previous case we know $-f'(x_0)=0$, so $f'(x_0)=0$. $\square$

Q: How do we find the <u>global</u> max/min of $f:[a,b]\to\mathbb{R}$?

A: By the previous theorem, the global max/min $f(x_0)$ must fall into one of the following cases:

① Critical point: $x_0\in(a,b)$ s.t. $f'(x_0)=0$

② Endpoint: $x_0 = a, b$

③ Points $x_0 \in (a,b)$ where f is not differentiable.

Ex Find all global extrema of $f:[-2,3]\to\mathbb{R}$, $f(x) = 2x^3-3x^2-12x+1$.

Pf: By the EVT, we know $\exists\, x_0, x_1 \in [-2,3]$ s.t. $f(x_0)$ is a global min and $f(x_1)$ is a global max.

Case ①: $x_0, x_1 \in (-2,3)$. As f is a polynomial, we know f is differentiable at x_0, x_1. So $f'(x_0)=0$ and $f'(x_1)=0$ by the previous theorem.

$$0 = f'(x) = 6x^2-6x-12 = 6(x^2-x-2) = 6(x-2)(x+1)$$

$$\Rightarrow \quad x=-1 \quad \text{or} \quad x=2$$

$$\Rightarrow \quad f(x) = f(-1) = 8 \quad \text{or} \quad f(x) = f(2) = -19$$

Case ②: $x_0, x_1 \in \{-2, 3\}$. Note that

$$f(-2) = -3, \qquad f(3) = -8$$

- No case ③, since f is differentiable on $(-2,3)$

Altogether, we have shown

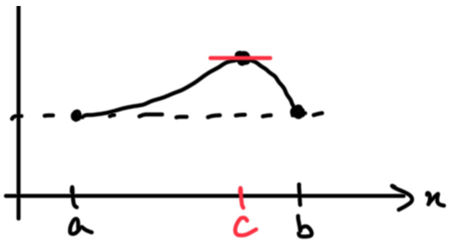
$$x_0, x_1 \in \{-2, -1, 2, 3\}$$

$$f(x) = -3, \ \underline{8}, \ \underline{-19}, \ -8$$

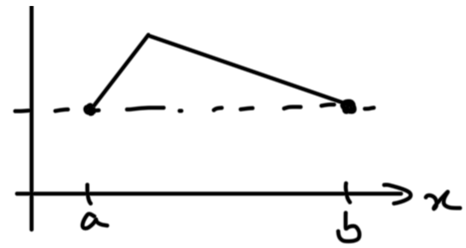
So the global max is $8 = f(-1)$ and the global min is $-19 = f(2)$. $\square$

Thm (Rolle's theorem) If $f:[a,b]\to\mathbb{R}$ is continuous, f is differentiable on (a,b), and $f(a)=f(b)$, then $\exists\, c\in(a,b)$ s.t. $f'(c)=0$.

$\uparrow$ $\uparrow$



- In order to return to $f(b)=f(a)$, a differentiable function must have at least one critical point



- Not differentiable, no critical points

<u>Pf</u>: As f is continuous on $[a,b]$, then by the EVT $\exists\, x_0, x_1 \in [a,b]$ s.t.

$$f(x_0) \leq f(x) \leq f(x_1) \quad \forall x \in [a,b].$$

Case: $x_1 \in (a,b)$. As f is differentiable at x_1, then $f'(x_1)=0$ by the previous theorem. So $c=x_1$ works.

Case: $x_0 \in (a,b)$. Then $c=x_0$ works.

Case: $x_0, x_1 \in \{a,b\}$. As $f(a)=f(b)$, then we have

$$f(a) \leq f(x) \leq f(a) \quad \forall x \in [a,b]$$

$$\Rightarrow \quad f(x) = f(a) \quad \forall x \in [a,b]$$

So f is a constant function. Therefore $f'(x)=0$ $\forall x \in (a,b)$.

In all cases, we found a number $c \in (a,b)$ s.t. $f'(c)=0$. $\square$

Last time: For differentiable functions f,

① 1^{st} derivative test:

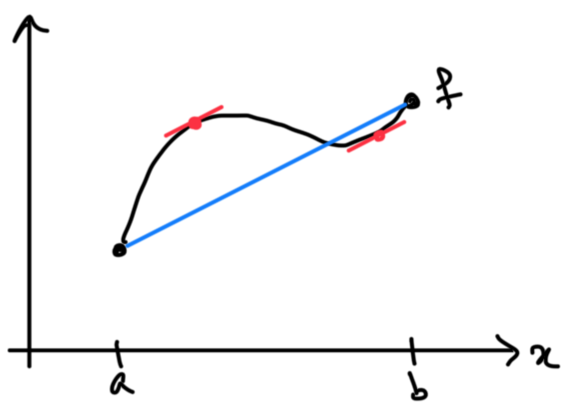
$f(x_0)$ local max/min $\Rightarrow$ $f'(x_0) = 0$

② Rolle's theorem:

$f(a) = f(b) \Rightarrow \exists\, c \in (a,b)$ s.t. $f'(c) = 0$

Thm (Mean value theorem) If $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ and differentiable on (a,b), then $\exists\, c \in (a,b)$ s.t.

$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$



- This = slope of blue line = mean slope of f
- In order to connect $f(a)$ and $f(b)$, a differentiable function must attain the mean slope.

Pf: Set $g(x) = f(x) - \left[f(a) + \frac{f(b)-f(a)}{b-a} \cdot (x-a) \right]$.

- This is the height of the graph of f above the blue line

As $x \mapsto -f(a) - \frac{f(b)-f(a)}{b-a} \cdot (x-a)$ is a polynomial, it is continuous and differentiable at any $x \in \mathbb{R}$.

So g satisfies:

1. continuous on $[a,b]$
2. differentiable on (a,b)
3. $g(a) = g(b)$: since

$$g(a) = f(a) - [f(a) + 0] = 0$$

$$g(b) = f(b) - \left[f(a) + \frac{f(b)-f(a)}{\cancel{b-a}} \cdot \cancel{(b-a)} \right] = 0$$

Therefore, by Rolle's theorem, $\exists\, c \in (a,b)$ s.t.

$$0 = g'(c) = f'(c) - \left[0 + \frac{f(b)-f(a)}{b-a} \right]$$

$$\Rightarrow f'(c) = \frac{f(b)-f(a)}{b-a}. \qquad \square$$

<u>Cor 1</u> Let $I \subseteq \mathbb{R}$ be an open interval and $f: I \to \mathbb{R}$. If $f'(x) = 0$ $\forall x \in I$, then f is a constant function.

<u>Pf</u>: Let $a, b \in I$ with $a < b$. We will prove that $f(a) = f(b)$.

- This is sufficient. Indeed, f not constant $\Rightarrow$ $\exists\, a, b \in I$ s.t. $f(a) \neq f(b)$.

As f is differentiable at x $\forall x \in I$ and $[a,b] \subseteq I$, then:

1. f is continuous on $[a,b]$
2. f is differentiable on (a,b)

So, by the MVT, $\exists\, c \in (a,b)$ s.t.

$$0 = f'(c) = \frac{f(b)-f(a)}{b-a} \implies f(a) = f(b). \qquad \square$$

Cor 2 Let $I \subseteq \mathbb{R}$ be an open interval and $f, g : I \to \mathbb{R}$. If $f'(x) = g'(x)$ $\forall x \in I$, then $\exists\, c \in \mathbb{R}$ s.t. $f(x) = g(x) + c$ $\forall x \in I$.

Pf: Let $h : I \to \mathbb{R}$, $h(x) = f(x) - g(x)$. Then $\forall x \in I$ we have

$$h'(x) = f'(x) - g'(x) = 0.$$

So, by Cor 1, we know h is constant: $\exists\, c \in \mathbb{R}$ s.t.

$$c = h(x) = f(x) - g(x) \quad \forall x \in I$$

$$\implies f(x) = g(x) + c \quad \forall x \in I. \qquad \square$$

Cor 3 Let $I \subseteq \mathbb{R}$ be an open interval and $f : I \to \mathbb{R}$.

1. If $f'(x) > 0$ $\forall x \in I$, then f is strictly increasing: $\forall x, y \in I$, $x < y \implies f(x) < f(y)$.
2. If $f'(x) < 0$ $\forall x \in I$, then f is strictly decreasing: $\forall x, y \in I$, $x < y \implies f(x) > f(y)$.

Pf: 1. Let $a, b \in I$ with $a < b$. Want: $f(a) < f(b)$. As f is differentiable at x $\forall x \in I$, then:

- f is continuous on $[a,b]$
- f is differentiable on (a,b)

So, by the MVT, $\exists\ c \in (a,b)$ s.t.

$$0 < f'(c) = \frac{f(b)-f(a)}{b-a} \quad \Rightarrow \quad f(a) < f(b).$$

(2) Suppose $f'(x) < 0\ \forall x \in I$. Then $-f$ satisfies $-f'(x) > 0\ \forall x \in I$. So, by (1), we know $-f$ is strictly increasing:

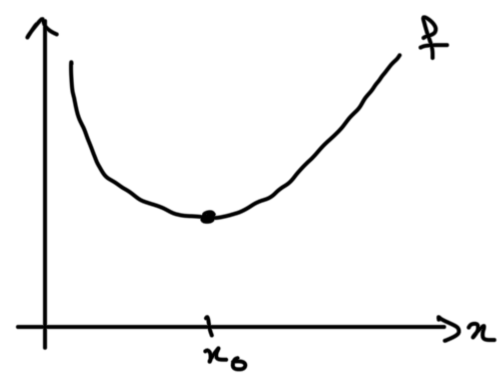
$$\forall a, b \in I, \quad a < b \quad \Rightarrow \quad -f(a) < -f(b)$$
$$\Rightarrow \quad f(a) > f(b)$$

So f is strictly decreasing. $\square$

<u>Ex</u> Fix $a, b > 0$. Find the global minimum of $f : (0,\infty) \to \mathbb{R}$, $f(x) = \frac{a}{x} + bx$ and prove your answer.

<u>Scratch work</u>: Critical points are

$$0 = f'(x) = -\frac{a}{x^2} + b = \frac{bx^2 - a}{x^2}$$
$$\Rightarrow \quad bx^2 - a = 0$$
$$\Rightarrow \quad x_0 = \sqrt{\frac{a}{b}}$$



<u>Solution</u>: We will prove that $f(x_0) = 2\sqrt{ab}$ is the global minimum of f, where $x_0 = \sqrt{\frac{a}{b}}$.

(1) Claim: $f(x) > f(x_0)\ \forall x \in (x_0, \infty)$. Note that

$$x > x_0 = \sqrt{\frac{a}{b}} \quad \Rightarrow \quad x^2 > \frac{a}{b} \quad \Rightarrow \quad bx^2 - a > 0$$

$$\Rightarrow \quad f'(x) = \frac{bx^2 - a}{x^2} > 0$$

Given $x > x_0$, by the MVT there is a point $c \in (x_0, x)$ where

$$0 < f'(c) = \frac{f(x) - f(x_0)}{x - x_0} \quad \Rightarrow \quad f(x) > f(x_0).$$

② Claim: $f(x) > f(x_0) \quad \forall\, x \in (0, x_0)$. Note that

$$0 < x < x_0 = \sqrt{\frac{a}{b}} \quad \Rightarrow \quad x^2 < \frac{a}{b} \quad \Rightarrow \quad bx^2 - a < 0$$

$$\Rightarrow \quad f'(x) = \frac{bx^2 - a}{x^2} < 0$$

Given $x < x_0$, by the MVT there is a point $c \in (x, x_0)$ where

$$0 > f'(c) = \frac{f(x_0) - f(x)}{x_0 - x} \quad \Rightarrow \quad f(x) > f(x_0).$$

Altogether, we conclude that $f(x_0)$ is the (strict) global minimum of $f : (0, \infty) \to \mathbb{R}$. $\square$

<u>Recall</u>: For $f: I \to \mathbb{R}$, $I \subseteq \mathbb{R}$ an open interval:

① $f'(x_0) = 0 \Leftarrow f(x_0)$ local min/max

② $f'(x) > 0 \ \forall x \in I \Rightarrow f$ strictly increasing

③ $f'(x) < 0 \ \forall x \in I \Rightarrow f$ strictly decreasing

- Together, this gives us a good description of f:

<u>Ex</u> $f: \mathbb{R} \to \mathbb{R}$, $f(x) = 2x^3 - 3x^2 - 12x + 1$.

Critical points:

$$0 = f'(x) = 6x^2 - 6x - 12 = 6(x^2 - x - 2) = 6(x-2)(x+1)$$

$$\Rightarrow \quad x = -1, \ 2$$

Note that

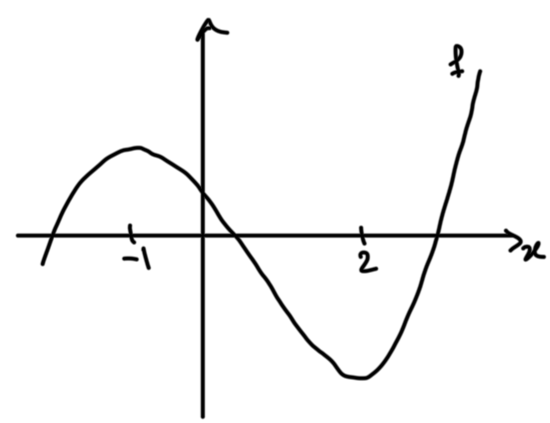
$$0 < f'(x) = 6\underbrace{(x-2)(x+1)}_{} \iff x > 2 \ \text{ or } \ x < -1$$

$\Rightarrow$ $(+)\cdot(+)$

or $(-)\cdot(-)$

So:

- $x < -1$: f increasing
- $x = -1$: critical point, $f(-1) = 8$
- $-1 < x < 2$: f decreasing
- $x = 2$: critical point, $f(2) = -19$
- $x > 2$: f increasing



<u>Thm</u> (2nd derivative test) Suppose $I \subseteq \mathbb{R}$ is an open interval, $f: I \to \mathbb{R}$ is twice differentiable

at $x_0 \in I$, and $f'(x_0) = 0$.

① If $f''(x_0) > 0$, then $f(x_0)$ is a local min.

② If $f''(x_0) < 0$, then $f(x_0)$ is a local max.

<u>Ex</u> $f(x) = 2x^3 - 3x^2 - 12x + 1$

$$0 = f'(x) = 6x^2 - 6x - 12 \implies x_0 = -1, 2$$

$$f''(x) = 12x - 6$$

$f''(-1) = -18 < 0 \implies f(-1)$ local max

$f''(2) = 18 > 0 \implies f(1)$ local min

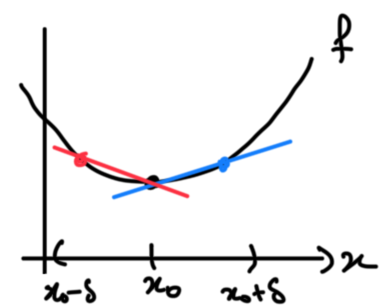
<u>Pf</u>: ① We know

$$\lim_{h \to 0} \frac{f'(x_0+h) - \overset{0}{\cancel{f'(x_0)}}}{h} = f''(x_0)$$

As $f''(x_0) > 0$, then (by Lecture 15) $\exists\, \delta > 0$ s.t. for $0 < |h| < \delta$ we have

$$\frac{f'(x_0+h)}{h} > 0 \implies \begin{cases} f'(x_0+h) > 0 & \text{if } 0 < h < \delta \\ f'(x_0+h) < 0 & \text{if } -\delta < h < 0 \end{cases}$$

$$\implies \begin{cases} f(x_0+h) > f(x_0) & \text{for } 0 < h < \delta \\ f(x_0+h) > f(x_0) & \text{for } -\delta < h < 0 \end{cases}$$



• Just like last example from Lecture 28

Therefore $f(x) \geq f(x_0)$ $\forall\, x \in (x_0 - \delta, x_0 + \delta)$, and so $f(x_0)$ is a local min.

② Suppose $f''(x_0) < 0$. Then $-f$ satisfies $-f''(x_0) > 0$, and so by ① $-f(x_0)$ is a local min of $-f$, and thus $f(x_0)$ is a local max of f. $\square$

Cor Suppose $I \subseteq \mathbb{R}$ is an open interval, and $f: I \to \mathbb{R}$ is twice differentiable at $x_0 \in I$.

① If $f(x_0)$ is a local min, then $f''(x_0) \geq 0$.

② If $f(x_0)$ is a local max, then $f''(x_0) \leq 0$.

Rmk $f''(x_0) > 0 \implies$ local min

$f''(x_0) \geq 0 \impliedby$ local min

E.g.

- $f(x) = x^2$: $f''(0) = 2 > 0$, min
- $f(x) = x^3$: $f''(0) = 0 \geq 0$, not a min
- $f(x) = x^4$: $f''(0) = 0 \geq 0$, min

Pf: ① Assume $f(x_0)$ is a local min. Want: $f''(x_0) \geq 0$. Suppose not: $f''(x_0) < 0$. Then $f(x_0)$ is a local max by the 2nd derivative test. So $\exists \delta > 0$ s.t.

$$\underset{\text{local min}}{f(x_0) \leq f(x)} \underset{\text{local max}}{\leq f(x_0)} \qquad \forall x \in (x_0 - \delta, x_0 + \delta)$$

So $f(x)$ is constant on $(x_0 - \delta, x_0 + \delta)$, and thus $f''(x_0) = 0$ — a contradiction.

② Apply ① to $-f$. $\square$

Recall: MVT says

- f continuous on $[a,b]$ $\Big\{ \implies \exists c \in (a,b)$ s.t.

- f differentiable on (a,b) $\Big\}$ $\qquad f'(c) = \dfrac{f(b)-f(a)}{b-a}$

Rolle's thm: If also $f(a)=f(b)$, then $f'(c)=0$.

<u>Thm</u> (Cauchy's MVT) If $f,g:[a,b]\to\mathbb{R}$ are continuous on $[a,b]$ and differentiable on (a,b), then $\exists\, c\in(a,b)$ s.t.

$$\big(f(b)-f(a)\big)\,g'(c) = \big(g(b)-g(a)\big)\,f'(c).$$

<u>Rmk</u> Taking $g(x)=x$, we get the usual MVT:

$$\big(f(b)-f(a)\big)\cdot 1 = (b-a)\,f'(c)$$

$$\Rightarrow\ f'(c) = \frac{f(b)-f(a)}{b-a}.$$

<u>Pf</u>: Set $h(x) = \big(g(b)-g(a)\big)f(x) - \big(f(b)-f(a)\big)g(x)$.

Then:

- h is continuous on $[a,b]$: since f,g are.
- h is differentiable on (a,b): since f,g are.
- $h(a)=h(b)$:

$$h(a) = \big(g(b)-\cancel{g(a)}\big)f(a) - \big(f(b)-\cancel{f(a)}\big)g(a)$$
$$= f(a)g(b) - f(b)g(a)$$
$$h(b) = \big(\cancel{g(b)}-g(a)\big)f(b) - \big(\cancel{f(b)}-f(a)\big)g(b)$$
$$= -f(b)g(a) + f(a)g(b)$$

So, by Rolle's theorem, $\exists\, c\in(a,b)$ s.t.

$$0 = h'(c) = \big(g(b)-g(a)\big)f'(c) - \big(f(b)-f(a)\big)g'(c)$$

$$\Rightarrow \quad (g(b)-g(a))\, f'(c) = (f(b)-f(a))\, g'(c). \qquad \square$$

<u>Recall</u> Cauchy's MVT: If

- f, g continuous on $[a,b]$
- f, g differentiable on (a,b)

Then: $\exists\, c \in (a,b)$ s.t. $(f(b)-f(a))\,g'(c) = (g(b)-g(a))\,f'(c)$.

<u>Thm</u> (l'Hôpital's rule) Let $I \subseteq \mathbb{R}$ be an open interval and $x_0 \in I$. If:

- $f, g : I \to \mathbb{R}$ are differentiable
- $f(x_0) = 0 = g(x_0)$
- $\displaystyle\lim_{x \to x_0} \frac{f'(x)}{g'(x)}$ exists

then
$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}.$$

- Limit law for division does not apply!
- There are also versions for $x \to \pm\infty$ or $f, g \to \pm\infty$.

<u>Pf</u>: As $\displaystyle\lim_{x \to x_0} \frac{f'(x)}{g'(x)} = \ell$ exists, $\exists\, \delta > 0$ s.t.
$$g'(x) \neq 0 \quad \text{for} \quad 0 < |x - x_0| < \delta$$

Note that
$$g(x) \neq 0 \quad \text{for} \quad 0 < |x - x_0| < \delta$$

as well, since otherwise $g(x) = 0 = g(x_0)$ would imply $\exists\, c$ between x_0 and x where $g'(c) = 0$ by Rolle's theorem.

Given x s.t. $0 < |x - x_0| < \delta$, we apply

Cauchy's MVT to $[x, x_0]$ or $[x_0, x]$ and get:

$\exists\, c_x$ between x_0 and x s.t.

$$(f(x) - \cancelto{0}{f(x_0)})\, g'(c_x) = (g(x) - \cancelto{0}{g(x_0)})\, f'(c_x)$$

$$\Rightarrow \quad \frac{f(x)}{g(x)} = \frac{f'(c_x)}{g'(c_x)}$$

since $g(x) \neq 0$ and $g'(c_x) \neq 0$.

- Ideally: $x \to x_0 \Rightarrow c_x \to x_0 \Rightarrow \dfrac{f'(c_x)}{g'(c_x)} \to \ell$

Want: $\displaystyle\lim_{x \to x_0} \frac{f(x)}{g(x)} = \ell$. Fix $\varepsilon > 0$.

As $\displaystyle\lim_{y \to x_0} \frac{f'(y)}{g'(y)} = \ell$, $\exists\, \delta > 0$ s.t.

$$0 < |y - x_0| < \delta \quad \Rightarrow \quad \left| \frac{f'(y)}{g'(y)} - \ell \right| < \varepsilon$$

Therefore

$$0 < |x - x_0| < \delta \quad \Rightarrow \quad 0 < |\underbrace{c_x}_{\text{Take } = y} - x_0| < |x - x_0| < \delta$$

$$\Rightarrow \left| \frac{f'(c_x)}{g'(c_x)} - \ell \right| < \varepsilon$$

$$\left| \frac{f(x)}{g(x)} - \ell \right| = \left| \frac{f'(c_x)}{g'(c_x)} - \ell \right|$$

So $\displaystyle\lim_{x \to x_0} \frac{f(x)}{g(x)} = \ell$. $\square$

<u>Ex</u> Find $\displaystyle\lim_{x \to 0} \frac{1 - \cos x}{x^2}$ and prove your answer.

Scratch work:

$$\lim_{x \to 0} \frac{1-\cos x}{x^2} \underset{2}{=} \lim_{x \to 0} \frac{\sin x}{2x} \underset{1}{=} \lim_{x \to 0} \frac{\cos x}{2} = \frac{1}{2}$$

Pf: Claim: $\lim_{x \to 0} \frac{\sin x}{2x} = \frac{1}{2}$. Note that

- $\sin x$ and $2x$ are differentiable on $\mathbb{R}$
- At $x = 0$, we have $\sin x = 0$ and $2x = 0$
- $\lim_{x \to 0} \frac{\cos x}{2} = \frac{1}{2}$, since $\frac{1}{2}\cos x$ is continuous at $x = 0$.

So, by l'Hôpital's rule, $\lim_{x \to 0} \frac{\sin x}{2x} = \frac{1}{2}$.

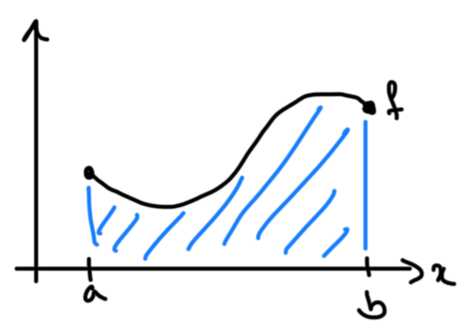
Claim: $\lim_{x \to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}$. Note that

- $1 - \cos x$ and x^2 are differentiable on $\mathbb{R}$
- At $x = 0$, we have $1 - \cos x = 0$ and $x^2 = 0$
- $\lim_{x \to 0} \frac{\sin x}{2x} = \frac{1}{2}$, by the previous claim.

So, by l'Hôpital's rule, $\lim_{x \to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}$. $\square$

INTEGRALS (Ch. 13)

Q: Given $f : [a,b] \to \mathbb{R}$, how do we find the area under the graph of f?



Def Let $a<b$. A *partition* of $[a,b]$ is a set $P = \{t_0, t_1, \ldots, t_n\}$ where $a = t_0 < t_1 < \ldots < t_n = b$.

Def Let $f:[a,b] \to \mathbb{R}$ be a bounded function and P a partition of $[a,b]$. We define the *upper* and *lower sums*

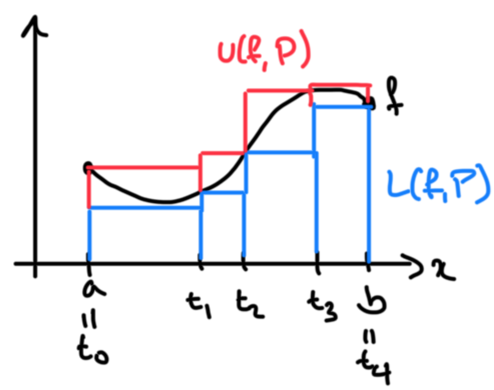
$$U(f,P) = \sum_{i=1}^{n} \sup\{f(x) : x \in [t_{i-1}, t_i]\} \cdot (t_i - t_{i-1})$$

$$L(f,P) = \sum_{i=1}^{n} \inf\{f(x) : x \in [t_{i-1}, t_i]\} \cdot (t_i - t_{i-1})$$

Rmk ① The sup/inf exists, because $\{f(x) : x \in [t_{i-1}, t_i]\}$ is:

- nonempty: since $t_{i-1} < t_i$
- bounded: since f is bounded on $[a,b]$

② $\inf\{\ldots\}(t_i - t_{i-1})$ is the area of the ith blue rectangle: The upper/lower sum is an over-/underestimate of the total area.



③ $L(f,P) \leq U(f,P)$ $\forall$ partition P (by HW7#4)

④ This is *not* an upper/lower Riemann sum — we account for *any* partition, not just regular partitions.

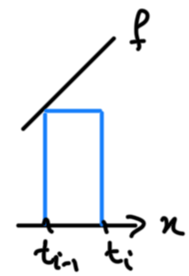
Ex $f:[1,3] \to \mathbb{R}$, $f(x) = x$

Fix $n \in \mathbb{N}$. Then

$$P_n = \left\{ t_0 = 1,\ t_1 = 1 + \frac{2}{n},\ \ldots,\ t_i = 1 + \frac{2i}{n},\ \ldots,\ t_n = 1 + \frac{2n}{n} = 3 \right\}$$

is a partition of $[1,3]$. We compute:

$$L(f, P_n) = \sum_{i=1}^{n} \inf\{f(x) : x \in [t_{i-1}, t_i]\} \cdot (t_i - t_{i-1})$$



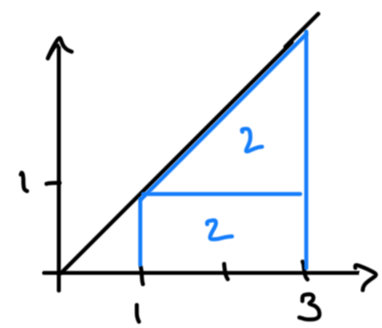
$$= f(1) \cdot \frac{2}{n} + f\left(1 + \frac{2}{n}\right) \cdot \frac{2}{n} + \ldots + f\left(1 + \frac{2(n-1)}{n}\right) \cdot \frac{2}{n}$$

$$= \frac{2}{n}\left[1 + \left(1 + \frac{2}{n}\right) + \ldots + \left(1 + \frac{2(n-1)}{n}\right)\right]$$

$$= \frac{2}{n}\left[n \cdot 1 + \frac{2}{n}\underbrace{\left(1 + 2 + \ldots + (n-1)\right)}_{\frac{(n-1)n}{2}}\right]$$

$$= \frac{2}{n}\left[n + (n-1)\right] = 4 - \frac{2}{n}$$

- ≈ 4 for n large. Makes sense:



Similarly,

$$U(f, P_n) = \ldots = 4 + \frac{2}{n}.$$

Last time: $\cdot$ $f:[a,b]\to\mathbb{R}$ bounded

- Partition of $[a,b]$: $P=\{t_0,t_1,\dots,t_n\}$, $a=t_0<t_1<\dots<t_n=b$
- Upper/lower sums:

$$U(f,P)=\sum_{i=1}^{n}\sup\{f(x): x\in[t_{i-1},t_i]\}\cdot(t_i-t_{i-1})$$

$$L(f,P)=\sum_{i=1}^{n}\inf\{f(x): x\in[t_{i-1},t_i]\}\cdot(t_i-t_{i-1})$$

Ex Fix $b>0$. Consider $f:[0,b]\to\mathbb{R}$, $f(x)=x^2$.

Let $n\in\mathbb{N}$. Then

$$P_n=\left\{t_0=0,\ t_1=\frac{b}{n},\ \dots,\ t_i=\frac{bi}{n},\ \dots,\ t_n=\frac{bn}{n}=b\right\}$$

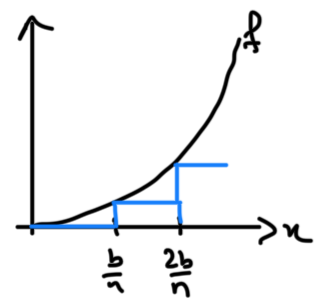
is a partition of $[0,b]$. We compute:

$$L(f,P_n)=\sum_{i=1}^{n}\inf\{f(x): x\in[t_{i-1},t_i]\}\cdot(t_i-t_{i-1})$$

$$=f(0)\cdot\frac{b}{n}+f(1)\cdot\frac{b}{n}+\dots+f\left(\frac{b(n-1)}{n}\right)\cdot\frac{b}{n}$$

$$=\frac{b}{n}\left[0^2+\left(\frac{b}{n}\right)^2+\dots+\left(\frac{b(n-1)}{n}\right)^2\right]$$

$$=\frac{b^3}{n^3}\Big[\underbrace{1^2+2^2+\dots+(n-1)^2}_{\frac{(n-1)n(2n-1)}{6}}\Big]$$



(by lecture 3)

$$=\frac{b^3}{6}\cdot\frac{(n-1)(2n-1)}{n^2}=\frac{b^3}{3}\left(1-\frac{3n-1}{2n^2}\right)$$

- Makes sense: $\int_0^b x^2\,dx=\frac{b^3}{3}$

Similarly,

$$U(f, P_n) = \dots = \frac{b^3}{3}\left(1 + \frac{3n+1}{2n^2}\right).$$

- Refining the partition always yields a better estimate:

<u>Lem</u> Let P, Q be partitions of $[a,b]$. If $P \subseteq Q$ then

$$L(f,P) \leq L(f,Q) \quad \text{and} \quad U(f,P) \geq U(f,Q).$$

<u>Pf</u>: Special case: Suppose

$$P = \{t_0, t_1, \dots, t_n\}, \qquad Q = P \cup \{s\}$$

with $t_{j-1} < s < t_j$ for some $j \in \{1, \dots, n\}$. Then

$$\sup\{f(x) : x \in [t_{j-1}, t_j]\} \underset{\uparrow}{\geq} \begin{cases} \sup\{f(x) : x \in [t_{j-1}, s]\} \\ \sup\{f(x) : x \in [s, t_j]\} \end{cases}$$

by HW7#4

$$\begin{aligned}
\Rightarrow \quad & \sup\{f(x) : x \in [t_{j-1}, t_j]\}(t_j - t_{j-1}) \\
&= \sup\{f(x) : x \in [t_{j-1}, t_j]\}(t_j - s + s - t_{j-1}) \\
&\geq \sup\{f(x) : x \in [t_{j-1}, s]\}(s - t_{j-1}) \\
&\quad + \sup\{f(x) : x \in [s, t_j]\}(t_j - s)
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \quad U(f,P) &= \sum_{i=1}^{n} \sup\{f(x) : x \in [t_{i-1}, t_i]\} \cdot (t_i - t_{i-1}) \\
&\geq U(f,Q)
\end{aligned}$$

Similarly,

$$\inf\{f(x) : x \in [t_{j-1}, t_j]\} \le \begin{cases} \inf\{\ \dots \quad [t_{j-1}, s]\} \\ \inf\{\ \dots \quad [s, t_j]\} \end{cases}$$

$$\Rightarrow \dots \Rightarrow L(f,P) \le L(f,Q).$$

General case: Write $P \subseteq P_1 \subseteq P_2 \subseteq \dots \subseteq P_k = Q$ where each P_i contains exactly one more point than P_{i-1}. By the special case,

$$L(f,P) \le L(f,P_1) \le L(f,P_2) \le \dots \le L(f,Q),$$
$$U(f,P) \ge U(f,P_1) \ge U(f,P_2) \ge \dots \ge U(f,Q). \qquad \square$$

<u>Prop</u> $\forall$ partitions P, Q of $[a,b]$,

$$L(f,P) \le U(f,Q).$$

- Any lower sum is less than any other upper sum
- We know this is true for $P = Q$.

<u>Pf</u>: Set $R = P \cup Q$. Then R is a partition of $[a,b]$, and

$$L(f,P) \underset{\substack{\uparrow \\ \text{Lemma}}}{\le} L(f,R) \le U(f,R) \underset{\substack{\uparrow \\ \text{Lemma}}}{\le} U(f,Q).$$

$\square$

<u>Def</u> Let $a<b$ and $f : [a,b] \to \mathbb{R}$ be a bounded function. We write

$$L(f) = \sup\{L(f,P) : P \text{ is a partition of } [a,b]\}$$
$$U(f) = \inf\{U(f,P) : P \text{ is a partition of } [a,b]\}$$

We say f is (Darboux) <u>integrable</u> if $L(f) = U(f)$.
When this is true, we call $L(f) = U(f)$ the <u>integral</u> of f on $[a,b]$ and we denote it by $\int_a^b f$ or $\int_a^b f(x)\,dx$.

<u>Rmk</u> $\forall$ partition P, we have

$$L(f,P) \le L(f) \le U(f) \le U(f,P)$$

- Immediate
- By the Prop, $U(f,P)$ upper bound for $\{L(f,Q) : Q \text{ partition}\}$

$$\Rightarrow U(f,P) \ge L(f), \quad \forall P$$

$$\Rightarrow U(f) \ge L(f).$$

<u>Rmk</u> The definition of "Riemann integrable" is a little different. However, it turns out that:

f is Darboux integrable $\Leftrightarrow$ f is Riemann integrable

So, we can call the integral $\int_a^b f$ defined above the "Darboux integral" or the "Riemann integral".

<u>Ex</u> Fix $b > 0$. Consider $f : [0,b] \to \mathbb{R}$, $f(x) = x^2$.

Claim: $\int_0^b f = \frac{b^3}{3}$. Recall that $\forall n \in \mathbb{N}$, $\exists$ a partition P_n s.t.

$$L(f,P_n) = \frac{b^3}{3}\left(1 - \frac{3n-1}{2n^2}\right), \qquad U(f,P_n) = \frac{b^3}{3}\left(1 + \frac{3n+1}{2n^2}\right).$$

So

$$\frac{b^3}{3}\left(1-\frac{3n-1}{2n^2}\right) \le L(f) \le U(f) \le \frac{b^3}{3}\left(1+\frac{3n+1}{2n^2}\right) \quad \forall n \in \mathbb{N}.$$

In particular, given $\varepsilon > 0$, we can pick $n \in \mathbb{N}$ so that

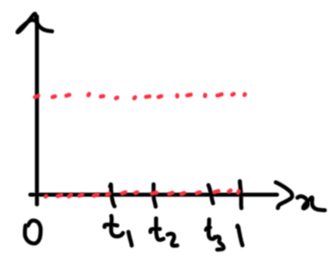
$$\frac{1}{n} < \frac{\varepsilon}{2} \quad \Rightarrow \quad \frac{3n-1}{2n^2} \le \frac{3n+1}{2n^2} \le \frac{3n+n}{2n^2} = \frac{2}{n} < \varepsilon$$

$$\Rightarrow \quad \frac{b^3}{3}(1-\varepsilon) \le L(f) \le U(f) \le \frac{b^3}{3}(1+\varepsilon)$$

Therefore

$$L(f) = \frac{b^3}{3} = U(f).$$

<u>Ex</u> $f : [0,1] \to \mathbb{R}$, $f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$



Then

$$\sup\{f(x) : x \in [t_{i-1}, t_i]\} = 1$$
$$\inf\{f(x) : x \in [t_{i-1}, t_i]\} = 0$$

$$\Rightarrow \begin{cases} U(f,P) = \sum_{i=1}^{n} 1 \cdot (t_i - t_{i-1}) = t_n - t_0 = 1 \\ L(f,P) = \sum_{i=1}^{n} 0 \cdot (t_i - t_{i-1}) = 0 \end{cases}$$

for any partition P. So

$$L(f) = 0 \ne 1 = U(f).$$

① Hw 11 due in 2 weeks

② Class and office hours as usual Monday – Wednesday next week

<u>Last time</u>: $U(f) = \inf\{U(f,P) : P \text{ partition}\}$

$L(f) = \sup\{L(f,P) : P \text{ partition}\}$

- In general, $L(f,P) \le L(f) \le U(f) \le U(f,Q) \quad \forall P, Q$
- f is integrable $\overset{\text{def}}{\iff} L(f) = U(f) = \int_a^b f$

<u>Ex</u> $f : [0,1] \to \mathbb{R}$, $\quad f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$

Then $L(f) = 0 \neq 1 = U(f)$, so $\int_0^1 f$ does not exist.

<u>Ex</u> $\int_0^b x^2\,dx = \frac{b^3}{3}$

Idea: · Pick a regular partition P_n, compute

· Get $\frac{b^3}{3}\left(1 - \frac{3n-1}{2n^2}\right) \le L(f) \le U(f) \le \frac{b^3}{3}\left(1 + \frac{3n+1}{2n^2}\right) \quad \forall n \in \mathbb{N}$

$\Rightarrow \quad \frac{b^3}{3}(1-\varepsilon) \le L(f) \le U(f) \le \frac{b^3}{3}(1+\varepsilon) \quad \forall \varepsilon > 0$

$\Rightarrow \quad L(f) = U(f) = \frac{b^3}{3}$.

<u>Prop</u> Let $f : [a,b] \to \mathbb{R}$ be bounded. Then:

f is integrable $\iff$ $\forall \varepsilon > 0$, $\exists$ a partition P of $[a,b]$ s.t. $U(f,P) - L(f,P) < \varepsilon$.

<u>Pf</u>: $\boxed{\Rightarrow}$: Fix $\varepsilon > 0$. As $U(f) + \frac{\varepsilon}{2} > U(f)$, then

$\exists P_1$ s.t.

$$U(f,P_1) < U(f) + \frac{\varepsilon}{2}.$$

Similarly, as $L(f) - \frac{\varepsilon}{2} < L(f)$, then $\exists P_2$ s.t.

$$L(f,P_2) > L(f) - \frac{\varepsilon}{2}.$$

Set $P = P_1 \cup P_2$. Then, by lecture 31,

$$U(f,P) \leq U(f,P_1) < U(f) + \frac{\varepsilon}{2}$$
$$L(f,P) \geq L(f,P_2) > L(f) - \frac{\varepsilon}{2}$$

$$\Rightarrow U(f,P) - L(f,P) < \cancel{U(f)} + \frac{\varepsilon}{2} - \left(\cancel{L(f)} - \frac{\varepsilon}{2}\right) = \varepsilon$$

$U(f) = L(f)$ since f is integrable

$\Leftarrow$): Given $\varepsilon > 0$, we know $\exists P$ s.t.

$$U(f,P) - L(f,P) < \varepsilon.$$

Note that

$$L(f,P) \leq L(f) \leq U(f) \leq U(f,P)$$

$$\Rightarrow 0 \leq U(f) - L(f) \leq U(f,P) - L(f,P) < \varepsilon.$$

As $\varepsilon > 0$ was arbitrary, we must have $U(f) - L(f) = 0$, and so f is integrable. $\square$

<u>Thm</u> If $f : [a,b] \to \mathbb{R}$ is integrable, then cf is integrable on $[a,b]$ $\forall c \in \mathbb{R}$, and

$$\int_a^b cf = c \cdot \int_a^b f.$$

<u>Pf</u>: <u>Case: $c = 0$.</u> Then

$$L(cf,P) = 0 = U(cf,P) \quad \forall \text{ partitions } P$$

$$\Rightarrow \quad L(cf) = 0 = U(cf)$$

$$\Rightarrow \quad \int_a^b cf = 0 = 0 \cdot \int_a^b f$$

<u>Case: $c > 0$</u>. Fix $\varepsilon > 0$. As f is integrable, $\exists$ a partition P s.t.

$$U(f,P) - L(f,P) < \frac{\varepsilon}{c}.$$

Then

$$L(cf, P) = \sum_{i=1}^{n} \underbrace{\inf\{cf(x) : x \in [t_{i-1}, t_i]\}}_{= c \cdot \inf\{f(x) : x \in [t_{i-1}, t_i]\}} \cdot (t_i - t_{i-1})$$

$$= c \cdot L(f, P)$$

(Fact: If $c > 0$, then $\inf(2A) = 2 \cdot \inf A$ $\forall$ set A.)

- You proved this for $c = 2$ on Midterm 2. Same method works for any $c > 0$.

Similarly,

$$U(cf, P) = \dots = c \cdot U(f, P).$$

Altogether, we get:

$$U(cf, P) - L(cf, P) = c\,[U(f,P) - L(f,P)]$$

$$< c \cdot \frac{\varepsilon}{c} = \varepsilon.$$

As $\varepsilon > 0$ was arbitrary, we conclude cf is integrable. Moreover:

① cf is integrable, so

$$c \cdot L(f,P) = L(cf,P) \leq \int_a^b cf \leq U(cf,P) = c \cdot U(f,P)$$

② f is integrable, so

$$c \cdot L(f,P) \leq c \cdot \int_a^b f \leq c \cdot U(f,P)$$

- We just showed that there can only be 1 number between $L(cf,P)$ and $U(cf,P)$.

Therefore $\int_a^b cf = c \int_a^b f$.

<u>Case: $c<0$</u>. Hw11. $\square$

<u>Thm</u> If $f, g : [a,b] \to \mathbb{R}$ are integrable, then $f+g$ is integrable on $[a,b]$ and

$$\int_a^b (f+g) = \int_a^b f + \int_a^b g.$$

<u>Pf</u>: Fix $\varepsilon > 0$. As f, g are integrable, $\exists$ partitions P_1, P_2 s.t.

$$U(f,P_1) - L(f,P_1) < \frac{\varepsilon}{2}, \quad U(f,P_2) - L(f,P_2) < \frac{\varepsilon}{2}.$$

Set $P = P_1 \cup P_2$. Note that:

$$\sup\{f(x)+g(x) : x \in [t_{i-1}, t_i]\}$$
$$\leq \sup\{f(x) : x \in [t_{i-1}, t_i]\} + \sup\{g(x) : x \in [t_{i-1}, t_i]\}$$

$$\Rightarrow U(f+g, P) \leq U(f,P) + U(g,P)$$
$$\leq U(f,P_1) + U(g,P_2)$$

↑ by Lecture 31, since $P_1, P_2 \subseteq P$

Similarly,

$$L(f+g,P) \geq L(f,P) + L(g,P) \geq L(f,P_1) + L(g,P_2).$$

Altogether, we get:

$$\begin{aligned} &U(f+g,P) - L(f+g,P) \\ &\quad \leq U(f,P_1) - L(f,P_1) + U(g,P_2) - L(g,P_2) \\ &\quad < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

As $\varepsilon > 0$ was arbitrary, we conclude $f+g$ is integrable. Moreover:

$$L(f,P) + L(g,P) \leq L(f+g,P) \leq \int_a^b f+g \leq U(f+g,P) \leq U(f,P) + U(g,P)$$

$$L(f,P) + L(g,P) \leq \int_a^b f + \int_a^b g \leq U(f,P) + U(g,P)$$

Therefore $\int_a^b f+g = \int_a^b f + \int_a^b g$. $\square$

Properties of integrals:

① $\int_a^b cf = c \cdot \int_a^b f \quad \forall c \in \mathbb{R}$

② $\int_a^b (f+g) = \int_a^b f + \int_a^b g$

Last time

③ $\int_a^c f = \int_a^b f + \int_b^c f$ $\leftarrow$ Today

<u>Thm</u> Let $a<b<c$ and $f:[a,c] \to \mathbb{R}$ be bounded.
Then:

f is integrable on $[a,c]$ $\iff$ f is integrable on $[a,b]$ and $[b,c]$.

In both cases, we have

$$\int_a^c f = \int_a^b f + \int_b^c f.$$

<u>Ex</u> Recall: $\forall b>0$, x^2 is integrable on $[0,b]$ and $\int_0^b x^2 dx = \frac{b^3}{3}$ (by Lecture 31).

By the Theorem, x^2 is also integrable on $[a,b]$ $\forall\, 0<a<b$ and

$$\int_0^b x^2\, dx = \int_0^a x^2\, dx + \int_a^b x^2\, dx$$

$$\Rightarrow \int_a^b x^2 dx = \int_0^b x^2 dx - \int_0^a x^2 dx = \frac{b^3}{3} - \frac{a^3}{3}.$$

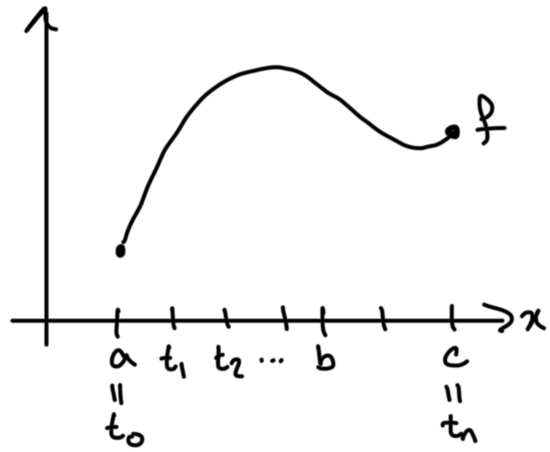
<u>Pf</u>: <u>$\Rightarrow$</u>|: Suppose f is integrable on $[a,c]$.
Fix $\varepsilon>0$. Then $\exists$ a partion P of $[a,c]$ s.t.

$U(f,P) - L(f,P) < \varepsilon$.

Set $Q = P \cup \{b\}$. Then

$$U(f,Q) \le U(f,P)$$
$$L(f,Q) \ge L(f,P)$$

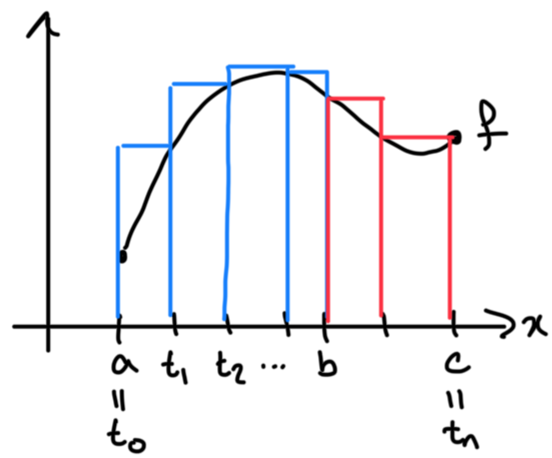
$$\Rightarrow \quad U(f,Q) - L(f,Q) \le U(f,P) - L(f,P) < \varepsilon.$$



Write $Q = Q_1 \cup Q_2$, where Q_1 is a partition of $[a,b]$ and Q_2 is a partition of $[b,c]$. Then:

$$U(f,Q) = U(f,Q_1) + U(f,Q_2)$$
$$L(f,Q) = L(f,Q_1) + L(f,Q_2)$$

- Total area = blue + red rectangles



Altogether,

$$\underbrace{U(f,Q_1) - L(f,Q_1)}_{\ge 0} + \underbrace{U(f,Q_2) - L(f,Q_2)}_{\ge 0}$$
$$= U(f,Q) - L(f,Q) < \varepsilon$$

$$\Rightarrow \quad U(f,Q_1) - L(f,Q_1) < \varepsilon, \qquad U(f,Q_2) - L(f,Q_2) < \varepsilon$$

As $\varepsilon > 0$ was arbitrary, we conclude f is integrable on $[a,b]$ and $[b,c]$. Moreover,

$$L(f,Q) = L(f,Q_1) + L(f,Q_2) \le \int_a^b f + \int_b^c f$$
$$\le U(f,Q_1) + U(f,Q_2) = U(f,Q)$$

$$L(f,Q) \le \int_a^c f \le U(f,Q)$$

Therefore $\int_a^b f + \int_b^c f = \int_a^c f$.

$\underline{\Leftarrow}$: Suppose f is integrable on $[a,b]$ and $[b,c]$. Fix $\varepsilon > 0$. Then $\exists$ partitions Q_1 of $[a,b]$ and Q_2 of $[b,c]$ s.t.

$$U(f,Q_1) - L(f,Q_1) < \frac{\varepsilon}{2}, \qquad U(f,Q_2) - L(f,Q_2) < \frac{\varepsilon}{2}.$$

Set $Q = Q_1 \cup Q_2$. Then Q is a partition of $[a,c]$ and

$$\begin{aligned} &U(f,Q) - L(f,Q) \\ &= U(f,Q_1) - L(f,Q_1) + U(f,Q_2) - U(f,Q_2) \\ &< \quad \frac{\varepsilon}{2} \quad + \quad \frac{\varepsilon}{2} \quad = \varepsilon \end{aligned}$$

As $\varepsilon > 0$ was arbitrary, we conclude f is integrable on $[a,c]$. Moreover,

$$L(f,Q_1) + L(f,Q_2) = L(f,Q) \le \int_a^c f \le U(f,Q) = U(f,Q_1) + U(f,Q_2)$$

$$L(f,Q_1) + L(f,Q_2) \le \int_a^b f + \int_b^c f \le U(f,Q_1) + U(f,Q_2)$$

Therefore $\int_a^c f = \int_a^b f + \int_b^c f$. $\square$

<u>Thm</u> If $f : [a,b] \to \mathbb{R}$ is continuous, then f is integrable on $[a,b]$.

- We'll prove this later

<u>Lem</u> If $f : [a,b] \to \mathbb{R}$ is integrable and

$$m \le f(x) \le M \quad \forall x \in [a,b],$$

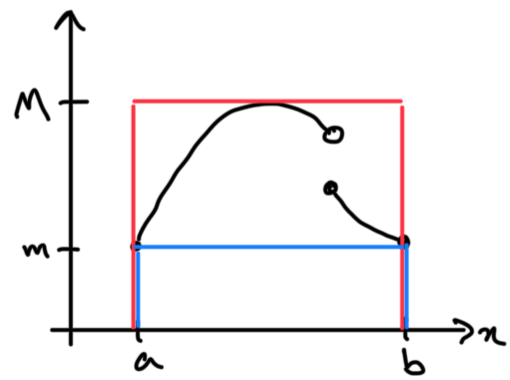
then

$$m(b-a) \le \int_a^b f \le M(b-a).$$

Pf: Consider the partition $P = \{a, b\}$.

Then

$$U(f,P) = \sup\{f(x) : x \in [a,b]\} \cdot (b-a)$$
$$\le M \cdot (b-a)$$
$$L(f,P) = \inf\{f(x) : x \in [a,b]\} \cdot (b-a)$$
$$\ge m \cdot (b-a)$$



So

$$m(b-a) \le L(f,P) \le \int_a^b f \le U(f,P) \le M(b-a). \qquad \square$$

Thm (MVT for integrals) If $f : [a,b] \to \mathbb{R}$ is continuous, then $\exists\, c \in [a,b]$ s.t. ($\Rightarrow$ also integrable!)

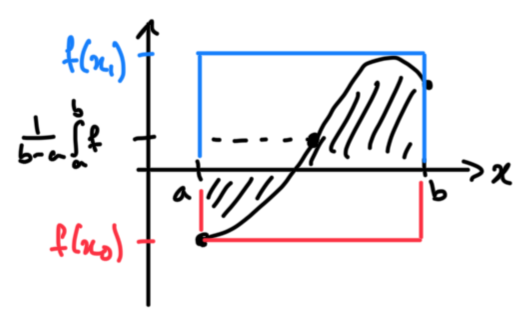
$$f(c) = \frac{1}{b-a} \int_a^b f.$$

Pf: As f is continuous on $[a,b]$, then by the EVT $\exists\, x_0, x_1 \in [a,b]$ s.t.

$$f(x_0) \le f(x) \le f(x_1) \quad \forall x \in [a,b]$$

$$\Rightarrow \quad f(x_0)(b-a) \le \int_a^b f \le f(x_1)(b-a)$$

$$\Rightarrow \quad f(x_0) \le \frac{1}{b-a} \int_a^b f \le f(x_1)$$



Case: $f(x_0) < \frac{1}{b-a} \int_a^b f < f(x_1)$. As f is continous on $[x_0, x_1]$, then by the IVT $\exists\, c \in (a,b)$ s.t. $f(c) = \frac{1}{b-a}\int_a^b f$.

Case: $\frac{1}{b-a}\int_a^b f = f(x_0)$. Then $c = x_0$ works.

Case: $\frac{1}{b-a}\int_a^b f = f(x_1)$. Then $c = x_1$ works. $\square$

Last time:

$$m \le f(x) \le M \quad \forall x \in [a,b] \quad \Rightarrow \quad m(b-a) \le \int_a^b f \le M(b-a).$$

Thm If $f:[a,b] \to \mathbb{R}$ is integrable, then the function $F:[a,b] \to \mathbb{R}$ given by

$$F(x) = \int_a^x f$$

is continous on $[a,b]$.

Ex For $f:[0,b] \to \mathbb{R}$, $f(x) = x^2$, we have

$$F(x) = \int_0^x f = \frac{x^3}{3}.$$

Pf: As f is integrable on $[a,b]$, it is also integrable on $[a,x]$ $\forall x \in (a,b)$ by Lecture 33, so $F(x)$ makes sense.

Want:

- $\lim_{x \to x_0} F(x) = F(x_0) \quad \forall x_0 \in (a,b)$
- $\lim_{x \to a^+} F(x) = F(a)$
- $\lim_{x \to b^-} F(x) = F(b)$

So it suffices to show:

① $\lim_{x \to x_0^+} F(x) = F(x_0) \quad \forall x_0 \in [a,b)$

② $\lim_{x \to x_0^-} F(x) = F(x_0) \quad \forall x_0 \in (a,b]$

Step ①: Fix $x_0 \in [a,b)$. Want: $\lim_{h \to 0^+} F(x_0+h) = F(x_0)$.

Let $h > 0$. Then

$$F(x_0+h) - F(x_0) = \int_a^{x_0+h} f - \int_a^{x_0} f = \int_{x_0}^{x_0+h} f$$

by lecture 33. As f is integrable on $[a,b]$, we know f is bounded (by definition of 'integrable'), so $\exists M > 0$ s.t.

$$-M \leq f(x) \leq M \quad \forall x \in [a,b]$$

$$\Rightarrow \underbrace{-M(x_0+h-x_0)}_{= -Mh} \leq \int_{x_0}^{x_0+h} f \leq \underbrace{M(x_0+h-x_0)}_{= Mh}$$

$$\Rightarrow \left| \int_{x_0}^{x_0+h} f \right| \leq Mh$$

Fix $\varepsilon > 0$. Set $\delta = \min\{\frac{\varepsilon}{M}, b - x_0\}$. Then for $0 < h < \delta$ we have $x_0 + h \in [a,b]$ and

$$|F(x_0+h) - F(x_0)| = \left| \int_{x_0}^{x_0+h} f \right| \leq M \cdot h < M\delta \leq \varepsilon.$$

So $\lim_{h \to 0^+} F(x_0+h) = F(x_0)$.

Step 2: Fix $x_0 \in (a,b]$. Want: $\lim_{h \to 0^-} F(x_0+h) = F(x_0)$.

For $h < 0$ we have

$$F(x_0+h) - F(x_0) = -\int_{x_0+h}^{x_0} f$$

$$\underbrace{-M(x_0-(x_0+h))}_{= Mh = -M|h|} \leq \int_{x_0+h}^{x_0} f \leq \underbrace{M(x_0-(x_0+h))}_{= -Mh = M|h|}$$

$$\Rightarrow \left| \int_{x_0+h}^{x_0} f \right| \leq M|h|$$

Fix $\varepsilon > 0$. Set $\delta = \min\{\frac{\varepsilon}{M}, x_0 - a\}$. Then for $-\delta < h < 0$ we have $x_0 + h \in [a,b]$ and

$$\left|F(x_0+h) - F(x_0)\right| = \left|\int_{x_0+h}^{x_0} f\right| \le M|h| < M\cdot\delta \le \varepsilon.$$

So $\lim_{h\to 0^-} F(x_0+h) = F(x_0)$. $\square$

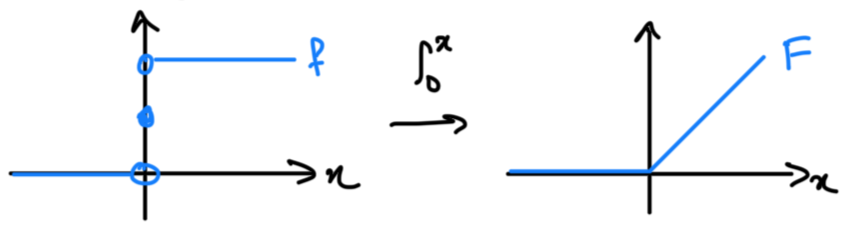
<u>Rmk</u> In fact,

$$\text{continuous on } [a,b] \;\underset{\not\Leftarrow}{\Rightarrow}\; \text{integrable on } [a,b]$$

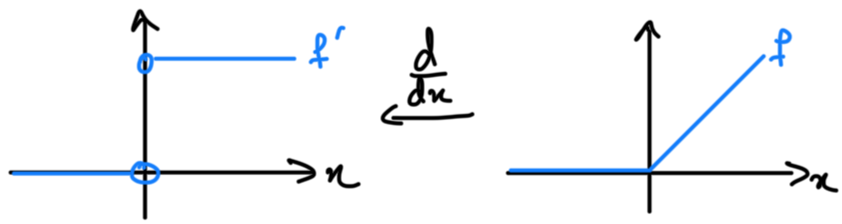
E.g. Hw11 #2

So F is "smoother" than f is.

$$\text{integrable} \supseteq \text{continuous}$$



By comparison, the derivative f' may be "rougher" than f. E.g.



FUNDAMENTAL THEOREM OF CALCULUS (Ch. 14)

<u>Thm</u> (FTC I) Let $f: [a,b] \to \mathbb{R}$ be integrable and $x_0 \in [a,b]$. If f is continuous at x_0, then

the function $F : [a,b] \to \mathbb{R}$,

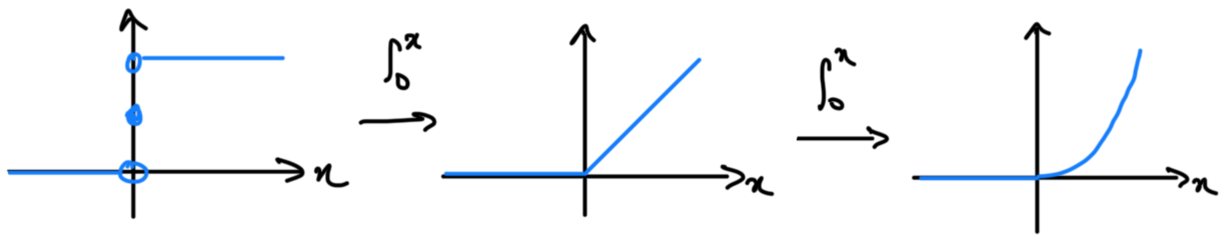
$$F(x) = \int_a^x f \ ,$$

is differentiable at x_0 and $F'(x_0) = f(x_0)$.

<u>Rmk</u> If $x_0 = a$ or b, then we mean the one-sided derivative of F.

<u>Rmk</u> So:

integrable $\supseteq$ continuous $\supseteq$ differentiable $\supseteq \ldots$



<u>Pf</u>: Case: $x_0 \in (a,b)$. Fix $\varepsilon > 0$. Want: $\exists\, \delta > 0$ s.t.

$$0 < |h| < \delta \implies \left| \frac{F(x_0+h) - F(x_0)}{h} - f(x_0) \right| < \varepsilon .$$

As f is continuous at x_0, $\exists\, \delta > 0$ s.t.

$$|x - x_0| < \delta \implies |f(x) - f(x_0)| < \frac{\varepsilon}{2}$$

In other words,

$$f(x_0) - \frac{\varepsilon}{2} < f(x) < f(x_0) + \frac{\varepsilon}{2} \qquad \forall x \in (x_0 - \delta, x_0 + \delta)$$

So, for $0 < h < \delta$, we have

$$\left(f(x_0) - \frac{\varepsilon}{2} \right) (\cancel{x_0} + h - \cancel{x_0}) \le \underbrace{\int_{x_0}^{x_0+h} f}_{F(x_0+h) - F(x_0)} \le \left(f(x_0) + \frac{\varepsilon}{2} \right) (\cancel{x_0} + h - \cancel{x_0})$$

$$\implies f(x_0) - \frac{\varepsilon}{2} \le \frac{F(x_0+h) - F(x_0)}{h} \le f(x_0) + \frac{\varepsilon}{2}$$

$$\Rightarrow \quad \left| \frac{F(x_0+h) - F(x_0)}{h} - f(x_0) \right| \leq \frac{\varepsilon}{2} < \varepsilon .$$

Similarly,

$$-\delta < h < 0 \Rightarrow \quad \dots \quad \Rightarrow \left| \frac{F(x_0+h) - F(x_0)}{h} - f(x_0) \right| < \varepsilon .$$

As $\varepsilon > 0$ was arbitrary, we conclude

$$F'(x_0) = \lim_{h \to 0} \frac{F(x_0+h) - F(x_0)}{h} = f(x_0).$$

Case: $x_0 = a$. Only need $h > 0$.

Case: $x_0 = b$. Only need $h < 0$. $\square$

Next 2 weeks:

1. No office hours
2. Class in person December 2–9
3. Class on December 11 will be pre-recorded and posted on Canvas
4. Hw 12 will not be graded

<u>Last time</u>: FTC I

f continuous (at x_0) $\Rightarrow$ $\frac{d}{dx}\int_a^x f = f$ (at x_0)

<u>Thm</u> (FTC II) If $g:[a,b]\to\mathbb{R}$ is differentiable and $g':[a,b]\to\mathbb{R}$ is integrable, then

$$\int_a^b g' = g(b)-g(a).$$

- In order to compute $\int_a^b f$, we just need to find an antiderivative g

<u>Ex</u> (1) Claim: $\int_a^b x^2\,dx = \frac{b^3}{3} - \frac{a^3}{3}$ $\forall a<b$.

We know:

- $\frac{1}{3}x^3$ is differentiable on $[a,b]$ and $\frac{d}{dx}\left(\frac{1}{3}x^3\right)=x^2$: since $\frac{1}{3}x^3$ is a polynomial
- x^2 is integrable: since x^2 is continuous on $[a,b]$

So FTC II applies.

- Much faster than using partitions. (Cf. Lecture 31.)

② Claim: $\int_a^b \sqrt{x}\, dx = \frac{2}{3}b^{3/2} - \frac{2}{3}a^{3/2} \quad \forall\, 0<a<b.$

We know:

- $\frac{2}{3}x^{3/2}$ is differentiable on $[a,b]$: since $\frac{2}{3}x$ and $x^{1/2}$ are (by lecture 23 and HW8, respectively)
- $\sqrt{x}$ is integrable: since $\sqrt{x}$ is continuous on $[a,b]$

So FTC II applies.

- This would be very difficult with partitions. E.g., what's $\sqrt{1}+\sqrt{2}+\ldots+\sqrt{n} = \ldots$?

③ $\int_a^b |x|\, dx = g(b)-g(a)$, where $g(x) = \begin{cases} \frac{1}{2}x^2 & x\geq 0 \\ -\frac{1}{2}x^2 & x<0 \end{cases}$

We know:

- $g'(x) = |x|$: by lecture 22
- $|x|$ is integrable: since it's continuous on $[a,b]$

So FTC II applies.

<u>Cor</u> $\forall n\in\mathbb{Z}$ with $n\neq -1$, we have

$$\int_a^b x^n\, dx = \frac{b^{n+1}}{n+1} - \frac{a^{n+1}}{n+1}$$

for any $a<b$ in $\begin{cases} \mathbb{R} & \text{if } n\geq 0, \\ (0,\infty) & \text{if } n\leq -2. \end{cases}$

<u>Pf</u>: Fix n and $a<b$ as above. Set $g(x) = \frac{x^{n+1}}{n+1}$. We know g is differentiable on $[a,b]$ and $g'(x) = x^n$ by lecture 25. We also know

x^n is integrable, since it is continuous on $[a,b]$ (and $[a,b] \subseteq (0,\infty)$ if $n \le -2$). So, by FTC II,

$$\int_a^b x^n \, dx = g(b) - g(a) = \frac{b^{n+1}}{n+1} - \frac{a^{n+1}}{n+1}. \qquad \square$$

<u>Cor</u> If $p(x) = c_0 + c_1 x + \dots + c_N x^N$ is a polynomial, then

$$\int_a^b p = c_0(b-a) + \frac{c_1}{2}(b^2 - a^2) + \dots + \frac{c_N}{N+1}(b^{N+1} - a^{N+1})$$

for any $a < b$.

<u>Pf</u>: Recall from Lecture 32 that

$$\int_a^b cf = c \cdot \int_a^b f, \qquad \int_a^b (f+g) = \int_a^b f + \int_a^b g. \qquad \square$$

<u>Pf</u> (of FTC II): Suppose $g : [a,b] \to \mathbb{R}$ is differentiable and $g' : [a,b] \to \mathbb{R}$ is integrable. So $\int_a^b g'$ exists and is the unique number that satisfies

$$L(g', P) \le \int_a^b g' \le U(g', P) \quad \forall \text{ partition } P.$$

Want: $\int_a^b g' = g(b) - g(a)$. So it suffices to show

$$L(g', P) \le g(b) - g(a) \le U(g', P) \quad \forall P.$$

Let $P = \{ a = t_0 < t_1 < \dots < t_n = b \}$ be a partition of $[a,b]$. For each $i = 1, 2, \dots, n$, the MVT says $\exists\, x_i \in (t_{i-1}, t_i)$ s.t.

$$g'(x_i) = \frac{g(t_i) - g(t_{i-1})}{t_i - t_{i-1}}$$

$$\Rightarrow \quad g(t_i) - g(t_{i-1}) = g'(x_i) \cdot (t_i - t_{i-1})$$

$$\leq \sup \{ g'(x) : x \in [t_{i-1}, t_i] \} (t_i - t_{i-1})$$

$$\Rightarrow \quad \sum_{i=1}^{n} g(t_i) - g(t_{i-1}) \leq \sum_{i=1}^{n} \sup \{ \text{"} \} (t_i - t_{i-1}) = U(g', P)$$

$$= \cancel{g(t_1)} - g(t_0) + \cancel{g(t_2)} - \cancel{g(t_1)} + \cancel{\ldots} + g(t_n) - \cancel{g(t_{n-1})}$$

$$= g(b) - g(a)$$

$$\Rightarrow \quad g(b) - g(a) \leq U(g', P).$$

Similarly,

$$g(t_i) - g(t_{i-1}) = g'(x_i) \cdot (t_i - t_{i-1})$$

$$\geq \inf \{ g'(x) : x \in [t_{i-1}, t_i] \} \cdot (t_i - t_{i-1})$$

$$\Rightarrow \quad \ldots \quad \Rightarrow \quad g(b) - g(a) \geq L(g', P)$$

Together, we obtain

$$L(g', P) \leq g(b) - g(a) \leq U(g', P)$$

for any P. As $\int_a^b g'$ is the unique number that satisfies this inequality we conclude

$$\int_a^b g' = g(b) - g(a). \qquad \square$$

- Together, FTC I and II allow us to differentiate and integrate many functions.

Ex Fix $a \in \mathbb{R}$, and set $f(x) = \int_a^{x^3} \frac{1}{1+\sin^2 t}\, dt$.

Prove f is differentiable on $\mathbb{R}$ and find $f'(x)$.

Write

$$f = g \circ h \quad \text{where} \quad g(y) = \int_a^y \frac{1}{1+\sin^2 t}\, dt, \quad h(x) = x^3.$$

Then:

- $h : \mathbb{R} \to \mathbb{R}$ is differentiable and $h'(x) = 3x^2$: since it's a polynomial
- $g : \mathbb{R} \to \mathbb{R}$ is differentiable and $g'(y) = \frac{1}{1+\sin^2 y}$: by FTC I, since $\frac{1}{1+\sin^2 t}$ is continuous at any $t \in \mathbb{R}$

So, by the chain rule,

$$f'(x) = (g \circ h)'(x) = g'(h(x)) \cdot h'(x)$$

$$= \frac{1}{1+\sin^2(x^3)} \cdot 3x^2. \qquad \square$$

Last time:

① FTC I: f continuous (at x_0) $\Rightarrow$ $\frac{d}{dx}\int_a^x f = f$ (at x_0)

② FTC II: g' integrable $\Rightarrow$ $\int_a^b \frac{d}{dx} g = g(b) - g(a)$

Ex: $\frac{d}{dx}\int_a^{x^3} \frac{1}{1+\sin^2 t}\,dt = \frac{1}{1+\sin^2(x^3)} \cdot 3x^2$

Thm (Leibniz integral rule) If:

- $f, g : [a,b] \to \mathbb{R}$ are differentiable
- $f(x) < g(x) \quad \forall x \in [a,b]$
- $h : \mathbb{R} \to \mathbb{R}$ is continuous

then the function $F : [a,b] \to \mathbb{R}$,

$$F(x) = \int_{f(x)}^{g(x)} h(t)\,dt$$

is differentiable and

$$F'(x) = h(g(x)) \cdot g'(x) - h(f(x)) \cdot f'(x).$$

Ex $\frac{d}{dx}\int_a^{x^3} \frac{1}{1+\sin^2 t}\,dt$

Here,

$$f(x) = a, \qquad g(x) = x^3, \qquad h(t) = \frac{1}{1+\sin^2 t}$$

$$\Rightarrow F'(x) = h(g(x)) \cdot g'(x) - h(f(x)) \cdot f'(x)$$

$$= \frac{1}{1+\sin^2(x^3)} \cdot 3x^2 - \frac{1}{1+\sin^2 a} \cdot 0$$

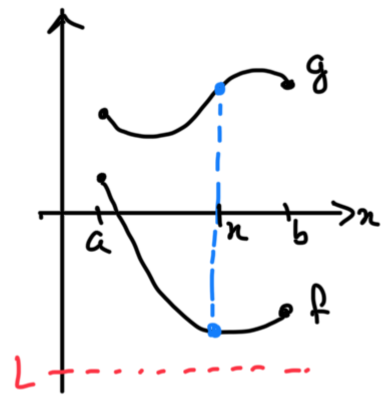
This agrees with our previous answer.

Pf: Write

$$F(x) = \int_L^{g(x)} h(t)\,dt - \int_L^{f(x)} h(t)\,dt$$
$$= (H \circ g)(x) - (H \circ f)(x)$$

where:

- $L \in \mathbb{R}$ s.t. $f(x) > L \ \forall x \in [a,b]$. (Such an L exists by the EVT.)
- $H(y) = \int_L^y h(t)\,dt$



Then:

- $f, g : [a,b] \to (L, \infty)$
- $H : (L, \infty) \to \mathbb{R}$ is differentiable and $H'(y) = h(y)$ by FTC I, since h is continuous at any $y \in \mathbb{R}$

So, by the chain rule,

$$F'(x) = (H \circ g)'(x) - (H \circ f)'(x)$$
$$= H'(g(x)) \cdot g'(x) - H'(f(x)) \cdot f'(x)$$
$$= h(g(x)) \cdot g'(x) - h(f(x)) \cdot f'(x). \qquad \square$$

- By FTC II, we know how to integrate any function that has an antiderivative.

Q: Are $\frac{1}{x}$, $\frac{1}{x^2+1}$, e^{-x^2} integrable?

Thm If $f : [a,b] \to \mathbb{R}$ is continuous, then f is integrable on $[a,b]$.

<u>Ex</u> We know $\frac{1}{x}$, $\frac{1}{x^2+1}$, e^{-x^2} are continuous, so the functions

$$\log x = \int_1^x \frac{1}{t}\,dt$$
$$\tan^{-1} x = \int_0^x \frac{1}{1+t^2}\,dt$$
$$\operatorname{erf} x = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\,dt$$

make sense. In fact, by FTC I, we also know that these functions are differentiable:

$$\frac{d}{dx}\log x = \frac{1}{x}, \qquad \frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}, \qquad \frac{d}{dx}\operatorname{erf} x = \frac{2}{\sqrt{\pi}} e^{-x^2}$$

<u>Outline of proof</u>:

$$\begin{aligned} f:[a,b]\to\mathbb{R} \text{ continuous } &\Rightarrow f \text{ uniformly continuous} \\ &\Rightarrow f \text{ integrable} \end{aligned}$$

<u>Def</u> Let $I \subseteq \mathbb{R}$ be an open interval and $f: I \to \mathbb{R}$ a function. We say f is <u>uniformly continuous</u> if $\forall \varepsilon > 0\ \exists \delta > 0$ s.t. $\forall x, a \in I$,

$$|x-a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon.$$

<u>Rmk</u> The order of quantifiers matters!

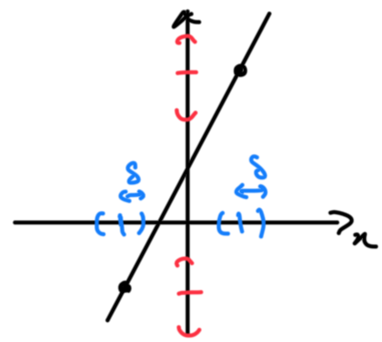
- For 'continuous', need: $\forall a \in I$, $\forall \varepsilon > 0\ \exists \delta > 0 \ldots$ So $\delta = \delta(\varepsilon, a)$.
- For 'uniformly continuous', $\delta = \delta(\varepsilon)$. I.e, δ can be chosen uniformly in a.

$\underline{\text{Ex}}$ $f:\mathbb{R}\to\mathbb{R}$, $f(x) = 4x+21$.

lecture 10: $\lim_{x\to 3} f(x) = f(3)$, $\delta = \frac{\varepsilon}{4}$ works

Claim: f is uniformly continuous.

Fix $\varepsilon > 0$. Set $\delta = \frac{\varepsilon}{4}$. Then:



$$|x-a| < \delta \implies |f(x)-f(a)|$$
$$= |4x+21-(4a+21)|$$
$$= 4|x-a| < 4\delta = \varepsilon.$$

<u>Last time</u>: f uniformly continuous $\Leftrightarrow$ $\forall \varepsilon>0$, $\exists \delta>0$ s.t. $\forall x, y$, $|x-y|<\delta$ implies $|f(x)-f(y)|<\varepsilon$.

<u>Ex</u> $f:\mathbb{R}\to\mathbb{R}$, $f(x)=x^2$.

Claim: f is continuous. Fix $a\in\mathbb{R}$ and $\varepsilon>0$. Set $\delta=\min\{1, \frac{\varepsilon}{2|a|+1}\}$. Then

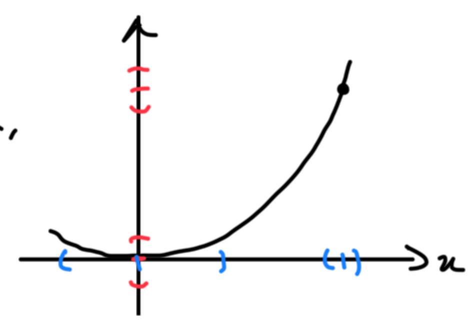
$$|x-a|<\delta \implies |x^2-a^2| = |x+a|\cdot|x-a|$$
$$< (|x-a|+|2a|)\cdot\delta$$
$$\le (1+2|a|)\cdot\delta \le \varepsilon.$$

- Not good enough for 'uniformly continuous'...

Claim: f is not uniformly continuous.

Suppose f were uniformly continuous. Set $\varepsilon=1$. Then $\exists\delta>0$ s.t. $\forall x, a\in\mathbb{R}$,

$$|x-a|<\delta \implies |x^2-a^2|<1.$$



Take $a=\frac{1}{\delta}$, $x=\frac{1}{\delta}+\frac{\delta}{2}$:

$$|x-a|=\frac{\delta}{2}<\delta, \quad \text{but} \quad |x^2-a^2| = \left|\frac{1}{\delta^2}+1+\frac{\delta^2}{4}-\frac{1}{\delta^2}\right|$$
$$= 1+\frac{\delta^2}{4} \ge 1$$

a contradiction.

<u>Rmk</u> In general,

$$f:I\to\mathbb{R} \text{ continuous } \underset{\Leftarrow}{\not\Rightarrow} f \text{ uniformly continuous}$$

<u>Thm</u> If $f:[a,b]\to\mathbb{R}$ is continuous, then f is

uniformly continuous.

<u>Pf</u>: Fix $\varepsilon > 0$. Set

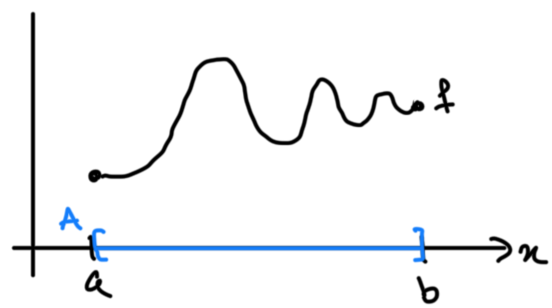
$$A = \{ x_0 \in [a,b] : \exists \delta > 0 \text{ s.t. } \forall x, y \in [a, x_0], \; |x-y| < \delta \text{ implies } |f(x)-f(y)| < \varepsilon \}.$$

Steps:

① sup A exists

② sup A = b

③ $b \in A$



Step ①: Note that

- $A \neq \emptyset$: We know $a \in A$, since $x, y \in [a,a] = \{a\}$ implies $x = y = a$ and $|f(a) - f(a)| = 0 < \varepsilon$.
- A is bounded above: by b

So, by the least upper bound property, $\exists \sup A \in \mathbb{R}$.

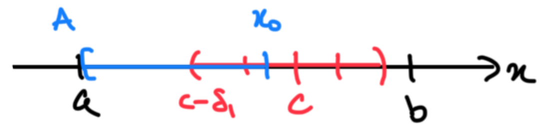
Step ②: Set $c = \sup A$. We know $c \leq b$, since b is an upper bound for A. Want: $c = b$. Suppose not: $c < b$. As f is continuous at c, $\exists \delta_1 > 0$ s.t.

$$|x - c| < \delta_1 \Rightarrow |f(x) - f(c)| < \frac{\varepsilon}{2}.$$

As $c - \frac{\delta_1}{2} < c = \sup A$, then $\exists x_0 \in A$ s.t. $x_0 > c - \frac{\delta_1}{2}$. By definition of $x_0 \in A$, $\exists \delta_2 > 0$ s.t.

$$x, y \in [a, x_0], \quad |x - y| < \delta_2 \Rightarrow |f(x) - f(y)| < \varepsilon.$$

Set $\delta = \min\{\frac{\delta_1}{2}, \delta_2\}$. We will show that $c + \frac{\delta_1}{2} \in A$.



For $x, y \in [a, c+\frac{\delta_1}{2}] = [a, x_0] \cup [c - \frac{\delta_1}{2}, c + \frac{\delta_1}{2}]$:

- Case: $x, y \in [a, x_0]$. We have

$$|x-y| < \delta \leq \delta_2 \implies |f(x) - f(y)| < \varepsilon.$$

- Case: $x, y \in [c - \frac{\delta_1}{2}, c + \frac{\delta_1}{2}] \implies |x-c| < \delta_1, \ |y-c| < \delta_1$

$$\implies |f(x) - f(y)| \leq |f(x) - f(c)| + |f(c) - f(y)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

- Case: $x \in [a, x_0]$ and $y \in [c - \frac{\delta_1}{2}, c + \frac{\delta_1}{2}]$ (or vice versa).

$$|x-y| < \delta \leq \frac{\delta_1}{2} \implies |x-c| \leq |x-y| + |y-c| < \frac{\delta_1}{2} + \frac{\delta_1}{2} = \delta_1$$

$$\implies |f(x) - f(c)| < \frac{\varepsilon}{2}$$

$$\implies |f(x) - f(y)| \leq |f(x) - f(c)| + |f(c) - f(y)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

In all cases, we have $|x-y| < \delta \implies |f(x) - f(y)| < \varepsilon$. Therefore $c + \frac{\delta_1}{2} \in A$. But this contradicts that $c = \sup A$, so we conclude $c = b$.

Step ③: Want $b \in A$. Repeating the previous step with $b = \sup A$ in place of c, we conclude $b \in A$. $\square$

<u>Thm</u> If $f:[a,b]\to\mathbb{R}$ is continuous, then f is integrable.

<u>Recall</u>: f integrable $\iff$ $\begin{cases} \cdot\ f \text{ bounded} \\ \cdot\ \forall \varepsilon>0\ \exists P \text{ s.t.} \\ \quad U(f,P)-L(f,P)<\varepsilon \end{cases}$

<u>Pf</u>: We know f is bounded by the EVT. Fix $\varepsilon>0$. As f is uniformly continuous, $\exists \delta>0$ s.t.

$$|x-y|<\delta \implies |f(x)-f(y)|<\frac{\varepsilon}{b-a}.$$

Fix $n\in\mathbb{N}$ s.t. $\frac{b-a}{n}<\delta$, and set

$$P=\left\{t_0=a,\ t_1=a+\frac{b-a}{n},\ t_2=a+2\frac{b-a}{n},\ \dots,\ t_n=b\right\}.$$

Then P is a partition of $[a,b]$. For each $i=1,2,\dots,n$, f is continuous on $[t_{i-1},t_i]$, and so by the EVT $\exists\ x_i,y_i\in[t_{i-1},t_i]$ s.t.

$$\sup\{f(x):\ x\in[t_{i-1},t_i]\}=f(x_i)$$
$$\inf\{f(x):\ x\in[t_{i-1},t_i]\}=f(y_i)$$

$$\implies \sup\{\dots\}-\inf\{\dots\}=f(x_i)-f(y_i)<\frac{\varepsilon}{b-a}$$

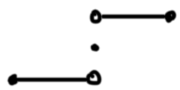
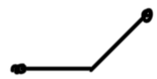
since $|x_i-y_i|\le\frac{b-a}{n}<\delta$

$$\implies U(f,P)-L(f,P)<\sum_{i=1}^{n}\frac{\varepsilon}{b-a}\cdot\underbrace{(t_i-t_{i-1})}_{\frac{b-a}{n}}$$

$$=n\cdot\frac{\varepsilon}{b-a}\cdot\frac{b-a}{n}=\varepsilon.$$

As $\varepsilon>0$ was arbitrary, we conclude that f is integrable. $\square$

<u>Rmk</u> Integrable on $[a,b]$ $\overset{\text{Thm}}{\supsetneq}$ Continuous on $[a,b]$ $\overset{\text{lecture 22}}{\supsetneq}$ Differentiable on $[a,b]$ $\supsetneq \cdots$

E.g.,   

<u>Ex</u> Fix $b \geq 1$. As $\frac{1}{x}$ is continuous on $(0,\infty)$, we now know

$$F(a) = \int_1^a \frac{1}{t}\,dt$$
$$G(a) = \int_b^{ab} \frac{1}{t}\,dt$$

exist for any $a \geq 1$. Moreover, by FTC I, these functions are differentiable:

$$F'(a) = \frac{1}{a}, \qquad G'(a) = \frac{1}{ab} \cdot b = \frac{1}{a}.$$

So $F'(a) = G'(a)$ $\forall a \in (1,\infty)$.

(To be continued...)

Announcements:

- Class on Wednesday (December 11) will be review, pre-recorded and posted on Canvas

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Last time: For $a, b \geq 1$,

$$F(a) = \int_1^a \frac{1}{t}\,dt \quad , \qquad G(a) = \int_b^{ab} \frac{1}{t}\,dt$$

$$\Rightarrow \quad F'(a) = G'(a) \quad \forall\, a \in (1, \infty)$$

So $F(a) = G(a) + c$ for some constant $c \in \mathbb{R}$ (by Lecture 28). But at $a = 1$,

$$F(1) = 0 = G(1).$$

Therefore $c = 0$, and so

$$\int_1^a \frac{1}{t}\,dt = F(a) = G(a) = \int_b^{ab} \frac{1}{t}\,dt$$

$$\Rightarrow \int_1^{ab} \frac{1}{t}\,dt = \int_b^{ab} \frac{1}{t}\,dt + \int_1^b \frac{1}{t}\,dt$$

$$= \int_1^a \frac{1}{t}\,dt + \int_1^b \frac{1}{t}\,dt$$

Defining $\ln x = \int_1^x \frac{1}{t}\,dt$, we conclude

$$\ln(ab) = \ln a + \ln b \qquad \forall\, a, b \geq 1.$$

Def ① $\int_b^a f = -\int_a^b f \quad \forall\, a < b$.

② $\int_a^a f = 0 \qquad \forall\, a \in \mathbb{R}$.

Ex As $\frac{1}{x}$ is continuous on $(0, \infty)$, then

$$\ln x \overset{\text{def}}{=} \int_1^x \frac{1}{t}\,dt$$

makes sense for any $x \in (0,\infty)$ and $\frac{d}{dx}\ln x = \frac{1}{x}$.

<u>Thm</u> (Integration by parts) If $f, g : [a,b] \to \mathbb{R}$ are differentiable and $f', g' : [a,b] \to \mathbb{R}$ are continuous, then

$$\int_a^b fg' = f(b)g(b) - f(a)g(a) - \int_a^b f'g.$$

<u>Pf</u>: By the product rule,

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x) \quad \forall x \in [a,b].$$

On the other hand, $(fg)'$ is integrable on $[a,b]$ (since it's continuous) and so by FTC II,

$$\int_a^b (fg)' = f(b)g(b) - f(a)g(a).$$

Together,

$$\int_a^b f'g + \int_a^b fg' = \int_a^b (fg)' = f(b)g(b) - f(a)g(a)$$

$$\Rightarrow \int_a^b fg' = f(b)g(b) - f(a)g(a) - \int_a^b f'g. \qquad \square$$

<u>Thm</u> (u substitution) Let $I \subseteq \mathbb{R}$ be an interval. If $f : I \to \mathbb{R}$ is continuous, $u : [a,b] \to I$ is differentiable, and u' is continuous on $[a,b]$, then

$$\int_{u(a)}^{u(b)} f = \int_a^b (f \circ u) \cdot u'.$$

Pf: As u is continuous on $[a,b]$, we know $\exists L \in \mathbb{R}$ s.t. $u(x) > L \ \forall x \in [a,b]$ by the EVT. Set

$$F(x) = \int_L^x f .$$

Then:

- $F(x)$ makes sense $\forall x$: since f is continuous.
- $F : I \to \mathbb{R}$ is differentiable and $F' = f$: by FTC I, since f is continuous.
- $u : [a,b] \to I$ is differentiable

So, by the chain rule,

$$(F \circ u)'(x) = (F' \circ u)(x) \cdot u'(x)$$
$$= (f \circ u)(x) \cdot u'(x)$$

$$\Rightarrow \int_a^b (F \circ u)' = \int_a^b (f \circ u) \cdot u'$$

$$\parallel$$

$$F(u(b)) - F(u(a)) = \int_{u(a)}^{u(b)} f .$$

(by FTC II) $\square$

Ex Find $\int_a^b \frac{1}{x \ln x} \, dx$ for $1 < a < b$.

Write $\int_a^b \underbrace{\frac{1}{\ln x}}_{f \circ u} \cdot \underbrace{\frac{1}{x}}_{u'} \, dx$, where $u(x) = \ln x$

$$\Rightarrow f(y) = \frac{1}{y}$$

We know:

- $f : (0, \infty) \to \mathbb{R}$ is continuous
- $u : [a, b] \to \underline{(0, \infty)}$ is differentiable

  since $a > 1$
- u' is continuous on $[a, b]$

So, by u substitution,

$$\int_a^b \frac{1}{x \ln x} \, dx = \int_a^b (f \circ u) \cdot u'$$

$$= \int_{u(a)}^{u(b)} f$$

$$= \int_{\ln a}^{\ln b} \frac{1}{x} \, dx$$

$$= \ln(\ln b) - \ln(\ln a).$$

Final exam: Tuesday, December 17, 7:45–9:45 am
Vilas 4028

INTRO TO LOGIC

- Truth tables
- Negation
- Intro to proofs
- Induction: $P(n)$ is true $\forall n \in \mathbb{N}$
  - Base case: $P(1)$ is true
  - Inductive step: $P(n) \Rightarrow P(n+1)$

SETS

- Properties of arithmetic
- Functions $f: A \to B$
  - Images and preimages $f^{-1}(Y)$, $Y \subseteq B$
  - Injective and surjective
  - Bijective, inverse function $f^{-1}: B \to A$
- Examples of intervals
- Open and closed sets
  - $\mathcal{U}$ open $\Leftrightarrow$ $\forall x \in \mathcal{U}$ $\exists \varepsilon > 0$ s.t. $(x-\varepsilon, x+\varepsilon) \subseteq \mathcal{U}$
  - F closed $\Leftrightarrow$ F^c open

LIMITS

- Examples of limits:

$$\lim_{x \to x_0} f(x) = \ell \iff \forall \varepsilon > 0 \ \exists \delta > 0 \text{ s.t. } 0 < |x - x_0| < \delta \text{ implies } |f(x) - \ell| < \varepsilon$$

- Limit laws: If $\lim f(x)$ and $\lim g(x)$ exist, then
  - $\lim cf$ exists $\forall c \in \mathbb{R}$
  - $\lim (f+g)$ exists
  - $\lim (fg)$ exists
  - If also $\lim g \neq 0$: $\lim \frac{f}{g}$ exists
  - If also $g(\lim f) = \lim g$: $\lim g \circ f$ exists

CONTINUITY

- f is continuous at x_0:

  f continuous at $x_0 \iff \lim_{x \to x_0} f(x) = f(x_0)$

  $\iff \forall \varepsilon > 0 \ \exists \delta > 0$ s.t. $|x - x_0| < \delta$ implies $|f(x) - f(x_0)| < \varepsilon$

- f is continuous on $[a,b]$:
  - f is continuous on (a,b): $\lim_{x \to x_0} f(x) = f(x_0) \ \forall x_0 \in (a,b)$
  - $\lim_{x \to a^+} f(x) = f(a)$
  - $\lim_{x \to b^-} f(x) = f(b)$

- IVT: Existence of a solution ($\exists$ at least one x...)

$$\left.\begin{array}{l} - f : [a,b] \to \mathbb{R} \text{ continuous} \\ - f(a) > y > f(b) \text{ or } f(a) < y < f(b) \end{array}\right\} \Rightarrow \begin{array}{c} \exists c \in (a,b) \text{ s.t.} \\ f(c) = y \end{array}$$

- EVT: Existence of a global max/min

  $f : [a,b] \to \mathbb{R}$ continuous $\Rightarrow \exists$ a global max/min

- Definition of global max/min
- Sups (and infs):
  - $a \le \sup A \quad \forall a \in A$
  - $a \le L \quad \forall a \in A \implies \sup A \le L$

DIFFERENTIATION

- f is differentiable at $x_0 \iff \lim_{h \to 0} \frac{f(x_0+h)-f(x_0)}{h}$ exists
- f is differentiable on $[a,b]$:
  - f is differentiable on (a,b): $\lim_{h \to 0} \frac{f(x_0+h)-f(x_0)}{h}$ exists $\forall x_0 \in (a,b)$
  - $\lim_{h \to 0^+} \frac{f(a+h)-f(a)}{h}$ exists
  - $\lim_{h \to 0^-} \frac{f(b+h)-f(b)}{h}$
- n-times differentiable
- Laws of differentiation: If f and g are differentiable, then
  - cf is differentiable $\forall c \in \mathbb{R}$
  - $f+g$ is differentiable
  - Product rule: fg is differentiable
  - Quotient rule: If also $g(x_0) \neq 0$, then $\frac{f}{g}$ is differentiable
  - Chain rule: If also $f: I \to J$, $g: J \to \mathbb{R}$ then $g \circ f$ is differentiable

- Critical points
  - "Find the global max/min of $f:[a,b]\to\mathbb{R}$"
  - "Find the global max/min of $f:(0,\infty)\to\mathbb{R}$"
- Strictly increasing (or decreasing):
  $a<b \Rightarrow f(a)<f(b)$
- MVT: Uniqueness of a solution ($\exists$ at most one x...)

  $$\left.\begin{array}{l} - f:[a,b]\to\mathbb{R} \text{ continuous} \\ - \text{differentiable on } (a,b) \end{array}\right\} \Rightarrow \exists\, c\in(a,b) \text{ s.t. } f'(c)=\frac{f(b)-f(a)}{b-a}$$
- l'Hôpital's rule

INTEGRATION

- Examples:

  f is integrable on $[a,b]$ $\iff$ $L(f)=U(f)$

  $\iff \forall \varepsilon>0\ \exists P$ s.t. $U(f,P)-L(f,P)<\varepsilon$
- "Prove f is integrable and $\int_a^b f = \ldots$"
  - $\forall\varepsilon>0\ \exists P$ s.t. $U(f,P)-L(f,P)<\varepsilon$
  - "Moreover, $\ldots = L(f,P)\le\int_a^b f\le U(f,P)=\ldots$"
- $L(f,P)\le\int_a^b f\le U(f,P)\quad\forall P$

  E.g. $P=\{a,b\}$: $\quad m(b-a)\le\int_a^b f\le M(b-a)$

FUNDAMENTAL THEOREM OF CALCULUS

- FTC I: f continuous (at x_0) $\Rightarrow$ $\frac{d}{dx}\int_a^x f = f$ (at x_0)

- FTC II: g' integrable on $[a,b]$ $\Rightarrow$ $\int_a^b g' = g(b) - g(a)$
- Continuous $\Rightarrow$ integrable
- Uniformly continuous $\Leftrightarrow$ $\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. $|x-y| < \delta$ implies $|f(x) - f(y)| < \varepsilon$

  (I.e. $\delta = \delta(\varepsilon)$.)
- Leibniz integral rule
- Integration by parts
- u substitution