Lecture 8: The wave equation: Applications

Reading: Stechmann Ch. 4.1–4.2
(Haberman 4.1–4.2)

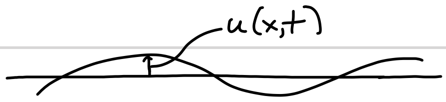
$$u_{tt} = c^2 \nabla^2 u \qquad \text{Second-order, linear.}$$

c: wave propagation speed.

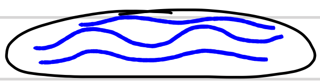
Examples:

1. Vibrating strings (1D) and drums (2D)

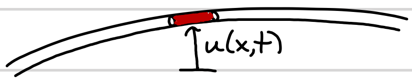
1D: $u_{tt} = c^2 u_{xx}$



2D: $u_{tt} = c^2 (u_{xx} + u_{yy})$

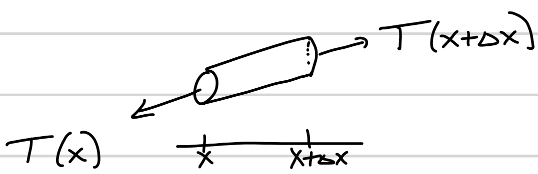


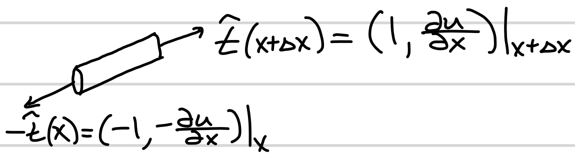
String:



Let $u(x,t)$ be the vertical "displacement" at position x and time t.

The tensile forces acting on a small segment of length Δx are





Tension T [Force]

Force balance: $m\vec{a} = \sum \vec{F}$.

The segment has mass $m = \rho \Delta x$ (ρ: 1D density $\left[\frac{\text{mass}}{\text{length}}\right]$)

Horizontal force balance: $0 = \left[-T(x) + T(x+\Delta x)\right] \quad (*_1)$

Vertical: $\rho \Delta x \frac{d^2 u}{dt^2} = -\rho g \Delta x + \left[-T(x)\frac{\partial u}{\partial x}(x) + T(x+\Delta x)\frac{\partial u}{\partial x}(x+\Delta x)\right] \quad (*_2)$

IF $\Delta x \ll 1$, Taylor expansion yields:

$$(*_1) \qquad -T(x) + \left(T(x) + T'(x)\Delta x + O(\Delta x^2)\right) = 0$$

So $\quad T'(x)\Delta x + O(\Delta x^2) = 0 \qquad \left(= \underset{\uparrow}{\rho \Delta x}\, \hat{x}\cdot\vec{a}, \text{ assumed } 0\right)$

$\bigstar_1$ $T'(x) = 0$. (Tension equilibrates through the string): $T(x) = T$.

$\bigstar_2$ $\rho u_{tt} = -\rho g + \left(-T(x)u_x(x) + [T(x)u_x(x)] + \Delta x \frac{\partial}{\partial x}[T(x)u_x(x)] + O(\Delta x^2)\right)$

Since $\frac{\partial}{\partial x}[T(x)u_x(x)] = \underset{\substack{\uparrow \\ 0}}{T'(x)}\underset{\substack{\uparrow \\ u'}}{u_x} + T(x)u_{xx}(x) = T u_{xx}$

As $\Delta x \to 0$ we find:

$$u_{tt} = \frac{T}{\rho} u_{xx} - g \qquad \left[\frac{L}{T^2}\right] = \left[\frac{\text{Force}}{\text{Mass}}\right]$$

IF gravity is small compared to elastic forces, $u_{tt} = c^2 u_{xx}$ $\left(c = \sqrt{\frac{T}{\rho}}\right)$

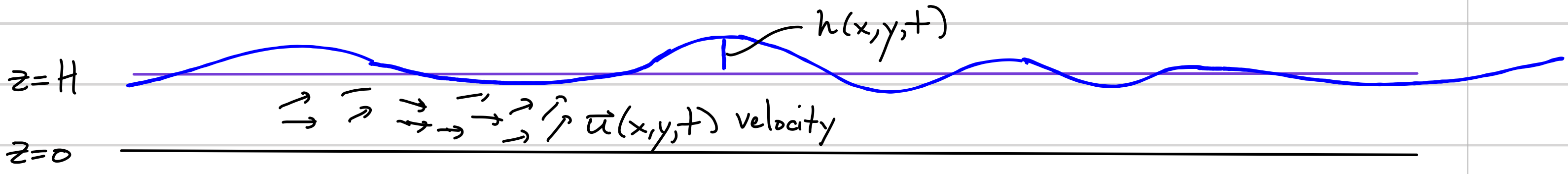
2D: same idea.

This model assumes the string can stretch, and that $u(x,t)$ is small for all (x,t).

IF $\rho = \rho(x)$, $\quad u_{tt} = c(x)^2 u_{xx}$,

where $c(x) = \sqrt{\frac{T}{\rho(x)}}$ is the <u>local</u> wavespeed.

2. Shallow water waves (linearized)



(1) $\frac{\partial h}{\partial t} + H\nabla\cdot\vec{u} = 0$ (Mass conservation)

(2) $\frac{\partial \vec{u}}{\partial t} = -g\nabla h$ ($\vec{a} = \frac{\vec{F}}{m}$; Euler equations, potential flow...)

$$\partial_t(1) = \frac{\partial^2 h}{\partial t^2} + H\nabla\cdot\vec{u}_t = 0$$

$$\frac{\partial^2 h}{\partial t^2} + H\nabla\cdot(-g\nabla h) = 0$$

$$h_{tt} = c^2\nabla^2 h, \quad c = \sqrt{gH} \rightarrow \text{Larger } H \Rightarrow \text{Larger speed.}$$

Deep water? ______ (Nonlinear Schrödinger eqn!)

3. Maxwell's equations of electromagnetism

(a) $\frac{\partial \vec{E}}{\partial t} - \frac{1}{\mu_0\varepsilon_0}\nabla\times\vec{B} = \vec{0}$, $\nabla\cdot\vec{B} = 0$

(b) $\frac{\partial \vec{B}}{\partial t} + \nabla\times\vec{E} = \vec{0}$, $\nabla\cdot\vec{E} = \cancel{\frac{\rho(\vec{x})}{\varepsilon_0}}\, 0$ no charges right now.

ε_0: vaccum permittivity, μ_0: vaccum permeability

$$\partial_t(a) = \frac{\partial^2 \vec{E}}{\partial t^2} - \frac{1}{\mu_0\varepsilon_0}\nabla\times\vec{B}_t = \vec{0}$$

$$\nabla\times(b) = \nabla\times\vec{B}_t + \underbrace{\nabla\times\nabla\times\vec{E}}_{\nabla(\nabla\cdot\vec{E}) - \nabla^2\vec{E}} = \vec{0}$$

So $\frac{\partial^2 \vec{E}}{\partial t^2} - \frac{1}{\mu_0\varepsilon_0}\nabla^2\vec{E} = \vec{0}$, or $\vec{E}_{tt} = c^2\nabla^2\vec{E}$, $c = \sqrt{\frac{1}{\mu_0\varepsilon_0}}$

(3 wave equations...)

also, $\vec{B}_{tt} = c^2\nabla^2\vec{B}$ (similar derivation)

4. Pressure waves (acoustic/sound waves)

Let $\rho(\vec{x},t)$ be a perturbation density around its equilibrium value, ρ_0.

and $\vec{u}(\vec{x},t)$ be the gas velocity

For small disturbances,

$$\frac{\partial \rho}{\partial t} + \rho_0 \nabla\cdot\vec{u} = 0 \quad \text{(Mass conservation)}$$

$$\rho_0 \frac{\partial \vec{u}}{\partial t} + \frac{\partial p}{\partial \rho}\nabla\rho = 0 \qquad \left(\frac{m\vec{a}}{\text{Volume}} = \frac{\vec{F}}{\text{Volume}}\right)$$

(p: pressure)

$$\Rightarrow \rho_{tt} = -\rho_0 \nabla\cdot\vec{u}_t = \underbrace{\left.\frac{\partial p}{\partial \rho}\right|_{\rho_0}}_{c^2} \nabla^2 \rho$$

Energy conservation

Let $\mathcal{E} = \int_{\mathbb{R}^n} \frac{1}{2}\left(u_t^2 + c^2|\nabla u|^2\right) dV$

Note: $\frac{d}{dt}|\nabla u|^2 = \frac{d}{dt}\nabla u\cdot\nabla u = \nabla u_t\cdot\nabla u + \nabla u\cdot\nabla u_t = 2\nabla u\cdot\nabla u_t$

and: $\nabla\cdot(u_t\nabla u) = \nabla u_t\cdot\nabla u + u_t\nabla^2 u$

So: $\frac{d}{dt}\mathcal{E} = \int_{\mathbb{R}^n} \frac{1}{2}\left(2u_t u_{tt} + c^2(2\nabla u\cdot\nabla u_t)\right) dV$

$$= \int_{\mathbb{R}^n} u_t u_{tt} + c^2\left(\nabla\cdot(u_t\nabla u) - u_t\nabla^2 u\right) dV$$

$$= \int_{\mathbb{R}^n} u_t\left(u_{tt} - c^2\nabla^2 u\right) dV + \underbrace{0}_{\substack{\text{Boundary term,}\\ \text{assuming a decaying}\\ \text{wave signal as } |\vec{x}|\to\infty.}}$$