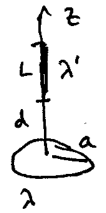
1)



a) $\vec{F}$ on line charge due to ring

$$d\vec{F} = \underset{\substack{\uparrow \\ \text{line charge}}}{dq}\ \vec{E}_{ring}$$

$$\vec{E}_{ring} = \frac{\lambda}{4\pi\epsilon_0}\int \frac{(\vec{r}-\vec{r}')\,\overset{a\,d\phi}{d\ell}}{|\vec{r}-\vec{r}'|^3} = \frac{\lambda a}{4\pi\epsilon_0}\int_0^{2\pi} \frac{(z\hat{z} - a\cos\phi\,\hat{x} - a\sin\phi\,\hat{y})\,d\phi}{(z^2+a^2)^{3/2}}$$

$\vec{r} = z\hat{z}$

$\vec{r}' = a\cos\phi\,\hat{x} + a\sin\phi\,\hat{y}$

$\hat{x}$ + $\hat{y}$ component integrate to 0

$$\int_0^{2\pi}\cos\phi\,d\phi = 0 \qquad \int_0^{2\pi}\sin\phi\,d\phi = 0$$

$$\vec{E}_{ring} = \frac{\lambda a z}{2\epsilon_0(z^2+a^2)^{3/2}}\,\hat{z}$$

$$d\vec{F} = \lambda' dz\,\vec{E}_{ring}(z)$$

$$\vec{F} = \lambda'\hat{z}\int_d^{d+L} dz\,\frac{\lambda a z}{2\epsilon_0(z^2+a^2)^{3/2}} = \frac{\hat{z}\lambda\lambda' a}{2\epsilon_0}\,\frac{1}{2}\int_{d^2+a^2}^{(d+L)^2+a^2} du\,u^{-3/2}$$

$u = z^2+a^2$

$z\,dz = du/2$

$$= \frac{\hat{z}\lambda\lambda' a}{2\epsilon_0}\left(-u^{-1/2}\Big|_{d^2+a^2}^{(d+L)^2+a^2}\right)$$

$$\vec{F} = \frac{\hat{z}\lambda\lambda' a}{2\epsilon_0}\left(\frac{1}{\sqrt{d^2+a^2}} - \frac{1}{\sqrt{(d+L)^2+a^2}}\right)$$

b) $d \gg L, a$

$$\frac{1}{d(1+a^2/d^2)^{1/2}} - \frac{1}{d\sqrt{\underbrace{(1+L/d)^2 + a^2/d^2}_{1+2L/d+L^2/d^2+a^2/d^2}}}$$

$$= \frac{1}{d}\left(1 - \frac{a^2}{2d^2} + \dots\right) - \frac{1}{d}\left(1 - \frac{1}{2}\left(\frac{2L}{d} + L^2/d^2 + a^2/d^2\right) + \dots\right)$$

$$= \cancel{\frac{1}{d}} - \cancel{\frac{a^2}{2d^3}} + \dots - \cancel{\frac{1}{d}} + \frac{L}{d^2} + \frac{L^2}{2d^3} + \cancel{\frac{a^2}{2d^3}} + \dots$$

$$= L/d^2 + \dots$$

(need to go to next order in the expansion to get full set of terms

$$\Rightarrow \vec{F} = \frac{\hat{z}\lambda\lambda' a}{2\epsilon_0}\frac{L}{d^2} + \dots$$

total charge on ring:

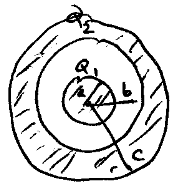
$$q_{ring} = \lambda 2\pi a$$

total charge on line charge:

$$q_{line} = \lambda' L$$

$$\Rightarrow \vec{F} = \hat{z}\frac{q_{ring}\,q_{line}}{4\pi\epsilon_0 d^2} + \dots \quad \text{like point charges}$$

2)



Q_1 on inner conductor, Q_2 on outer

a) E in all regions. Use Gauss' law : $\vec{E} = E(r)\,\hat{r}$

$$\oint \vec{E}\cdot d\vec{a} = E\,4\pi r^2 = q_{enclosed}/\epsilon_0$$

$r<a \qquad \vec{E} = 0$ (conductor)

$a<r<b \qquad \vec{E} = \dfrac{Q_1}{4\pi r^2 \epsilon_0}\,\hat{r}$

$b<r<c \qquad \vec{E} = 0$ (conductor)

$r>c \qquad \vec{E} = \dfrac{Q_1+Q_2}{4\pi r^2 \epsilon_0}\,\hat{r}$

b) Total energy:

$W = \frac{1}{2}\int \rho V\, d\tau$ is one way, but easiest is $\displaystyle W = \frac{1}{2}\int_{\text{all space}} \epsilon_0 |\vec{E}|^2\, d\tau$

$$W = \frac{\epsilon_0}{2}\left(\int_a^b \frac{Q_1^2}{(4\pi r^2\epsilon_0)^2}\, 4\pi r^2\, dr + \int_c^\infty \frac{(Q_1+Q_2)^2}{(4\pi r^2 \epsilon_0)^2}\, 4\pi r^2\, dr \right)$$

$$= \frac{\epsilon_0}{2}\left\{ \frac{Q_1^2}{4\pi\epsilon_0^2}\int_a^b \frac{1}{r^2}\,dr + \frac{(Q_1+Q_2)^2}{4\pi\epsilon_0^2}\int_c^\infty \frac{1}{r^2}\,dr \right\}$$

$$= \frac{1}{2}\,\frac{1}{4\pi\epsilon_0}\left\{ Q_1^2\left(\frac{1}{a}-\frac{1}{b}\right) + (Q_1+Q_2)^2\,\frac{1}{c} \right\}$$

$$= \frac{1}{8\pi\epsilon_0}\left\{ Q_1^2\left(\frac{1}{a}-\frac{1}{b}+\frac{1}{c}\right) + \frac{2Q_1Q_2}{c} + \frac{Q_2^2}{c} \right\}$$

$W = \frac{1}{2}\int \rho V\, d\tau \;\rightarrow$ calculate V everywhere:

$V(r=\infty) = 0$ ref

$$V_{r>c} = -\int_\infty^r \frac{(Q_1+Q_2)}{4\pi\epsilon_0 r^2}\,dr = \frac{Q_1+Q_2}{4\pi\epsilon_0 r}$$

$$V_{b<r<c} = -\int_\infty^c \frac{(Q_1+Q_2)}{4\pi\epsilon_0 r^2}\,dr - \int_c^r 0\,dr = \frac{Q_1+Q_2}{4\pi\epsilon_0 c}$$

$$V_{a<r<b} = -\int_\infty^c \frac{(Q_1+Q_2)}{4\pi\epsilon_0 r^2}\,dr - \int_c^b 0\,dr - \int_b^r \frac{Q_1}{4\pi\epsilon_0 r^2}\,dr = \frac{Q_1+Q_2}{4\pi\epsilon_0 c} + \frac{Q_1}{4\pi\epsilon_0 r} - \frac{Q_1}{4\pi\epsilon_0 b}$$

$$V_{r<a} = -\int_\infty^c \frac{(Q_1+Q_2)}{4\pi\epsilon_0 r^2}\,dr - \int_c^b 0\,dr - \int_b^a \frac{Q_1}{4\pi\epsilon_0 r^2}\,dr - \int_c^r 0\,dr = \frac{Q_1+Q_2}{4\pi\epsilon_0 c} + \frac{Q_1}{4\pi\epsilon_0 a} - \frac{Q_1}{4\pi\epsilon_0 b}$$

$W = \frac{1}{2}\int \rho V d\tau$

needed to make $\vec{E}=0$ for $b<r<c$

$\rho = Q_1 \delta(r-a) - Q_1 \delta(r-b) + (Q_1+Q_2)\delta(r-c)$

$\rightarrow W = \frac{1}{2} Q_1 V(r=a) - \frac{1}{2} Q_1 V(r=b) + (Q_1+Q_2) V(r=c)$

$= \frac{1}{2} Q_1 \left(\frac{Q_1+Q_2}{4\pi\epsilon_0 c} + \frac{Q_1}{4\pi\epsilon_0 a} - \frac{Q_1}{4\pi\epsilon_0 b} \right) - \frac{1}{2} Q_1 \left(\frac{Q_1+Q_2}{4\pi\epsilon_0 c} \right) + \frac{1}{2}(Q_1+Q_2)\frac{(Q_1+Q_2)}{4\pi\epsilon_0 C}$

$= \frac{1}{8\pi\epsilon_0} \left\{ \cancel{\frac{Q_1(Q_1+Q_2)}{c}} + \frac{Q_1^2}{a} - \frac{Q_1^2}{b} - \cancel{\frac{Q_1(Q_1+Q_2)}{c}} + \frac{Q_1^2+2Q_1Q_2+Q_2^2}{c} \right\}$

$= \frac{1}{8\pi\epsilon_0} \left\{ Q_1^2 \left(\frac{1}{a} - \frac{1}{b} + \frac{1}{c} \right) + \frac{2Q_1Q_2}{c} + \frac{Q_2^2}{c} \right\}$ ✓

c) Capacitance for $b \approx a$

Should expect parallel plate capacitor limit, with separation $d \rightarrow b-a$

$Q_1 = Q$, $-Q$ on inner surface ($r=b$), $Q_2 = 0$

$\vec{E}_{r<a} = 0 \quad \vec{E}_{a<r<b} = \frac{Q}{4\pi\epsilon_0 r^2}\hat{r} \quad \vec{E}_{r>b} = 0$

$a<r<b$: $V_Q = -\int_\infty^r E dr = \frac{Q}{4\pi\epsilon_0 r} - \frac{Q}{4\pi\epsilon_0 b}$

$r<a$: $\frac{Q}{4\pi\epsilon_0 a} - \frac{Q}{4\pi\epsilon_0 b}$

$C = \frac{Q}{V_Q(r=a) - V_Q(r=b)} = \frac{Q}{Q\left(\frac{1}{4\pi\epsilon_0 a} - \frac{1}{4\pi\epsilon_0 b}\right)}$

$C = \frac{4\pi\epsilon_0 ab}{b-a} \rightarrow \approx \epsilon_0 \frac{(4\pi a^2)}{d} = \frac{\epsilon_0 A}{d}$ ✓

3)

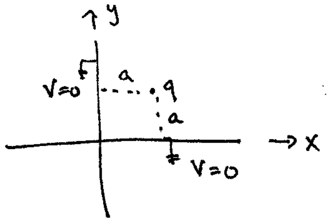
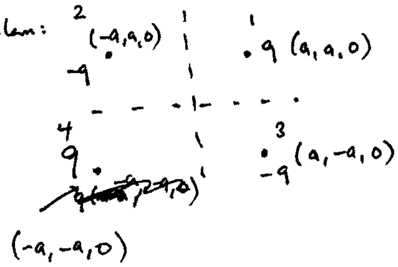


Image problem:



$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$

$\vec{r}_1 = a\hat{x} + a\hat{y}$

$\vec{r}_2 = -a\hat{x} + a\hat{y}$

$\vec{r}_3 = a\hat{x} - a\hat{y}$

$\vec{r}_4 = -a\hat{x} - a\hat{y}$

a)
$$V(x,y,z) = \frac{1}{4\pi\epsilon_0} q \left\{ \frac{1}{((x-a)^2+(y-a)^2+z^2)^{1/2}} - \frac{1}{((x+a)^2+(y-a)^2+z^2)^{1/2}} - \frac{1}{((x-a)^2+(y+a)^2+z^2)^{1/2}} + \frac{1}{((x+a)^2+(y+a)^2+z^2)^{1/2}} \right\}$$

Note $V=0$ for $x=0$ and $V=0$ for $y=0$, as needed

b) Surface charge density:

for plane with $x=0$:
$$\sigma(x=0,y,z) = -\epsilon_0 \frac{\partial V}{\partial x}\Big|_{x=0} = \frac{-q\epsilon_0}{4\pi\epsilon_0} \left\{ \frac{-(x-a)}{((x-a)^2+(y-a)^2+z^2)^{3/2}} + \frac{(x+a)}{((x+a)^2+(y-a)^2+z^2)^{3/2}} + \frac{(x-a)}{((x-a)^2+(y+a)^2+z^2)^{3/2}} - \frac{(x+a)}{((x+a)^2+(y+a)^2+z^2)^{3/2}} \right\}\Bigg|_{x=0}$$

$$= \frac{-q}{4\pi} \left\{ \frac{a}{(a^2+(y-a)^2+z^2)^{3/2}} + \frac{a}{(a^2+(y-a)^2+z^2)^{3/2}} - \frac{a}{(a^2+(y+a)^2+z^2)^{3/2}} - \frac{a}{(a^2+(y+a)^2+z^2)^{3/2}} \right\}$$

$$= \frac{-q}{2\pi} \left\{ \frac{a}{(a^2+(y-a)^2+z^2)^{3/2}} - \frac{a}{(a^2+(y+a)^2+z^2)^{3/2}} \right\}$$

Similarly, for plane with $y=0$:
$$\sigma(x,y=0,z) = \frac{-q}{2\pi} \left\{ \frac{a}{((x-a)^2+a^2+z^2)^{3/2}} - \frac{a}{((x+a)^2+a^2+z^2)^{3/2}} \right\}$$