Physics 322
Discussion 7
10/28/2024

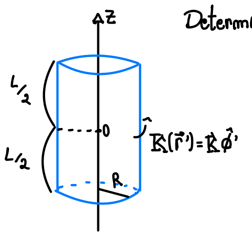
Stationary charges $\left(\frac{\partial \rho}{\partial t}=0\right) \Rightarrow \vec{E}(\vec{r},t)=\vec{E}(\vec{r})$: <u>Electrostatics</u> $\nabla\cdot\vec{E}=\rho/\epsilon_0$ & $\nabla\times\vec{E}=0$

Steady Current $\left(\frac{\partial \vec{J}}{\partial t}=0\right) \Rightarrow \vec{B}(\vec{r},t)=\vec{B}(\vec{r})$: <u>Magnetostatics</u> $\nabla\cdot\vec{B}=0$ & $\nabla\times\vec{B}=\mu_0\vec{J}$

Ampère's Law: $\nabla\times\vec{B}=\mu_0\vec{J} \Rightarrow \oint_{\partial A} d\vec{\ell}\cdot\vec{B}=\mu_0 I_A$ $\therefore$ Biot-Savart Law: $\vec{B}=\frac{\mu_0}{4\pi}\int \underbrace{(\vec{J}(\vec{r}')dV')}^{\text{or } \vec{K}(\vec{r}')da' \text{ or } I d\vec{\ell}} \times \frac{\vec{\mathscr{r}}}{\mathscr{r}^3}$

Magnetic Vector Potential: $\vec{B}=\nabla\times\vec{A}$, $\vec{A}=\frac{\mu_0}{4\pi}\int \frac{\overbrace{(\vec{J}(\vec{r}')dV')}^{\text{or } \vec{K}(\vec{r}')da' \text{ or } I d\vec{\ell}}}{\mathscr{r}}$ in Coulomb's gauge $(\nabla\cdot\vec{A}=0)$

<u>Problem 1.</u>




Determine $\vec{B}$ everwhere

$\vec{r}=\rho\hat{\rho}+z\hat{z}=\rho\cos\phi\,\hat{x}+\rho\sin\phi\,\hat{y}+z\hat{z}$, $\vec{r}'=R\cos\phi'\hat{x}+R\sin\phi'\hat{y}+z'\hat{z} \Rightarrow \vec{\mathscr{r}}=(\rho\cos\phi-R\cos\phi')\hat{x}+(\rho\sin\phi-R\sin\phi')\hat{y}+(z-z')\hat{z}$
& $\mathscr{r}=\rho^2+R^2-2\rho R\cos(\phi-\phi')+(z-z')^2$

$$\vec{K}(\vec{r}')\times\vec{\mathscr{r}}=K\begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ -\sin\phi' & \cos\phi' & 0\\ \rho\cos\phi-R\cos\phi' & \rho\sin\phi-R\sin\phi' & z-z'\end{vmatrix}=\Big\{(z-z')\cos\phi'\hat{x}+(z-z')\sin\phi'\hat{y}-\hat{z}\big[\sin\phi'(\rho\sin\phi-R\sin\phi')+\cos\phi'(\rho\cos\phi-R\cos\phi')\big]\Big\}K$$

Hence,

$$\vec{B}=\frac{\mu_0}{4\pi}\int \vec{K}(\vec{r}')\times\frac{\vec{\mathscr{r}}}{\mathscr{r}^3}\overbrace{da'}^{R\,dz'd\phi'}=\frac{\mu_0 K}{4\pi}R\int_{-\frac{L}{2}}^{\frac{L}{2}}dz'\int_0^{2\pi}d\phi'\frac{\Big\{(z-z')\cos\phi'\hat{x}+(z-z')\sin\phi'\hat{y}-\hat{z}\big[\sin\phi'(\rho\sin\phi-R\sin\phi')+\cos\phi'(\rho\cos\phi-R\cos\phi')\big]\Big\}}{(\rho^2+R^2-2\rho R\cos(\phi-\phi')+(z-z')^2)^{3/2}}$$

$$\underbrace{=}_{\substack{z'-z\to z'\\ \text{axial symmetry}\\ \text{take } \phi=0}}\frac{\mu_0 K}{4\pi}R\int_{-\frac{L}{2}-z}^{\frac{L}{2}-z}dz'\int_0^{2\pi}d\phi'\frac{-z'\hat{\rho}(\phi')-\hat{z}(-R+\rho\cos\phi')}{(\rho^2+R^2-2\rho R\cos\phi'+z'^2)^{3/2}}=\frac{\mu_0 K}{4\pi}R\Bigg\{\underbrace{\int_{-\frac{L}{2}-z}^{\frac{L}{2}-z}dz'\int_0^{2\pi}d\phi'\frac{-z'\hat{\rho}(\phi')}{(\rho^2+R^2-2\rho R\cos\phi'+z'^2)^{3/2}}}_{I_1}+\hat{z}\underbrace{\int_{-\frac{L}{2}-z}^{L/2-z}dz'\int_0^{2\pi}d\phi'\frac{R-\rho\cos\phi'}{(\rho^2+R^2-2\rho R\cos\phi'+z'^2)^{3/2}}}_{I_2}\Bigg\}$$

Note that

$$I_1=\int_{-\frac{L}{2}-z}^{L/2-z}dz'\int_0^{2\pi}d\phi'\frac{-z'\hat{\rho}(\phi')}{(\rho^2+R^2-2\rho R\cos\phi'+z'^2)^{3/2}}=\int_0^{2\pi}d\phi'\frac{\hat{\rho}(\phi')}{(\rho^2+R^2-2\rho R\cos\phi'+z'^2)^{1/2}}\Bigg|_{-\frac{L}{2}-z}^{L/2-z}=\int_0^{2\pi}d\phi'\frac{(\frac{L}{2}-z)\hat{\rho}(\phi')}{(\rho^2+R^2-2\rho R\cos\phi'+(\frac{L}{2}-z)^2)^{1/2}}+\int_0^{2\pi}d\phi'\frac{(\frac{L}{2}+z)\hat{\rho}(\phi')}{(\rho^2+R^2-2\rho R\cos\phi'+(\frac{L}{2}+z)^2)^{1/2}}$$

at $\phi=0$ we expect non-zero contribution only for the x component.

$$I_1=\int_0^{2\pi}d\phi'\frac{(\frac{L}{2}-z)\cos\phi'\hat{x}}{(\rho^2+R^2-2\rho R\cos\phi'+(\frac{L}{2}-z)^2)^{1/2}}+\int_0^{2\pi}d\phi'\frac{(\frac{L}{2}+z)\cos\phi'\hat{x}}{(\rho^2+R^2-2\rho R\cos\phi'+(\frac{L}{2}+z)^2)^{1/2}}=\left[\frac{(\frac{L}{2}-z)}{(\rho^2+R^2+(\frac{L}{2}-z)^2)^{1/2}}\overbrace{f}^{\text{Defined in HW 6}}\left(\frac{2\rho R}{\rho^2+R^2+(\frac{L}{2}-z)^2}\right)+\frac{(\frac{L}{2}+z)}{(\rho^2+R^2+(\frac{L}{2}-z)^2)^{1/2}}f\left(\frac{2\rho R}{\rho^2+R^2+(\frac{L}{2}+z)^2}\right)\right]\overbrace{\hat{\rho}}^{\text{generalize back to }\hat{\rho}}$$

We can the write

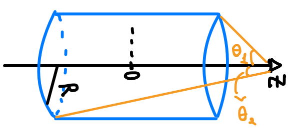
$$\vec{B}=B_\rho\hat{\rho}+B_z\hat{z}, \text{ with } \overbrace{B_\rho(\rho,z)}^{\text{Edge effects}}=\frac{\mu_0 K}{4\pi}R\left[\frac{(\frac{L}{2}-z)}{(\rho^2+R^2+(\frac{L}{2}-z)^2)^{1/2}}f\left(\frac{2\rho R}{\rho^2+R^2+(\frac{L}{2}-z)^2}\right)+\frac{(\frac{L}{2}+z)}{(\rho^2+R^2+(\frac{L}{2}-z)^2)^{1/2}}f\left(\frac{2\rho R}{\rho^2+R^2+(\frac{L}{2}+z)^2}\right)\right] \text{ and } B_z=\frac{\mu_0 K}{4\pi}R\int_{-\frac{L}{2}-z}^{L/2-z}dz'\int_0^{2\pi}d\phi'\frac{R-\rho\cos\phi'}{(\rho^2+R^2-2\rho R\cos\phi'+z'^2)^{3/2}}$$

Let's now analyze some interesting cases:

(i) Field at z-axis $(\rho=0)$

In this case, since $f(0)=0$, we get

$$B_\rho=0 \text{ \& } B_z=\frac{\mu_0 K}{4\pi}R^2\int_{-\frac{L}{2}-z}^{L/2-z}dz'\int_0^{2\pi}d\phi'\frac{1}{(R^2+z'^2)^{3/2}}=\frac{\mu_0 K}{2}\frac{z'}{\sqrt{R^2+z'^2}}\Bigg|_{-\frac{L}{2}-z}^{\frac{L}{2}-z}=\frac{\mu_0 K}{2}\left[\overbrace{\frac{(\frac{L}{2}-z)}{\sqrt{R^2+(\frac{L}{2}-z)^2}}}^{-\cos\theta_1}+\overbrace{\frac{(\frac{L}{2}+z)}{\sqrt{R^2+(\frac{L}{2}+z)^2}}}^{\cos\theta_2}\right]=\frac{\mu_0 K}{2}(\cos\theta_2-\cos\theta_1)$$




$\vec{B}(z)=\frac{\mu_0 K}{2}(\cos\theta_2-\cos\theta_1)\hat{z}$ → Infinite solenoid in the limit $\theta_2=0$ & $\theta_1=\pi$.
with $K=nI$ we get P.2 of HW 6

(ii) Infinite Solenoid Limit $(L \gg R, \rho, z)$

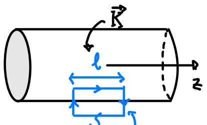
In this case $f(0)=0 \Rightarrow B_\rho = 0$. The $z$-integral gives

$$B_z = \frac{\mu_0 K}{4\pi} R \int_{-\infty}^{\infty} dz' \int_0^{2\pi} d\phi' \frac{R-\rho\cos\phi'}{(\underbrace{\rho^2+R^2-2\rho R\cos\phi'}_{d^2}+z'^2)^{3/2}} = \frac{\mu_0 K}{4\pi} R \int_0^{2\pi} d\phi' (R-\rho\cos\phi') \int_{-\infty}^{\infty} dz' \frac{dz'}{(d^2+z'^2)^{3/2}} = \frac{\mu_0 K}{4\pi} R \int_0^{2\pi} d\phi' (R-\rho\cos\phi') \overbrace{\frac{1}{d^2} \frac{z'}{(d^2+z'^2)^{1/2}} \Bigg|_{-\infty}^{\infty}}^{2/d^2} = \frac{\mu_0 K R}{2\pi} \int_0^{2\pi} d\phi' \frac{\overbrace{(R-\rho\cos\phi')}^{\frac{1}{2R}\left[(R^2-\rho^2)+(R^2+\rho^2-2R\rho\cos\phi')\right]}}{\rho^2+R^2-2\rho R\cos\phi'}$$

$$= \frac{\mu_0 K}{4\pi}\left[2\pi + (R^2-\rho^2) \underbrace{\int_0^{2\pi} d\phi' \frac{1}{\rho^2+R^2-2\rho R\cos\phi'}}_{\frac{2\pi}{\sqrt{(R^2-\rho^2)^2}}}\right] = \frac{\mu_0 K}{2}\left[1 + \frac{(R^2-\rho^2)}{|R^2-\rho^2|}\right]$$

$$\Rightarrow \vec{B} = \begin{cases} \mu_0 K \hat{z}, & \text{if } \rho < R \\ 0, & \text{if } \rho > R \end{cases}$$

which could be obtained via Ampère's law.




$$\oint \vec{B} \cdot d\vec{\ell} = B\ell = \mu_0 I_{enc} = \mu_0 K \ell \Rightarrow \vec{B} = \begin{cases} \mu_0 K \hat{z}, & \text{inside} \\ 0, & \text{outside} \end{cases}$$

take sides to zero

null field outside (must be constant and go to zero at $\infty$)