Math 421, Section 1
Homework 1
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Problem 1 (De Morgan's laws)**.** Let A and B be statements. Use a truth table to prove the following:

(a) "Not (A and B)" is equivalent to "(not A) or (not B)".
(b) "Not (A or B)" is equivalent to "(not A) and (not B)".

Solution:

(a):

A	B	not (A and B)	not A	not B	(not A) or (not B)
T	T	F	F	F	F
T	F	T	F	T	T
F	T	T	T	F	T
F	F	T	T	T	T

We have shown that the columns for "not (A and B)" and "(not A) or (not B)" are the same, so the two statements are equivalent.

(b):

A	B	not (A or B)	not A	not B	(not A) and (not B)
T	T	F	F	F	F
T	F	F	F	T	F
F	T	F	T	F	F
F	F	T	T	T	T

We have shown that the columns for "not (A or B)" and "(not A) and (not B)" are the same, so the two statements are equivalent.

$\square$

Problem 2 (The distributive property)**.** Let A, B, and C be statements. Use a truth table to prove the following:

(a) "A and (B or C)" is equivalent to "(A and B) or (A and C)".
(b) "A or (B and C)" is equivalent to "(A or B) and (A or C)".

Solution:

(a):

A	B	C	A and (B or C)	A and B	A and C	(A and B) or (A and C)
T	T	T	T	T	T	T
T	T	F	T	T	T	T
T	F	T	T	F	T	T
T	F	F	F	F	F	F
F	T	T	F	F	F	F
F	T	F	F	F	F	F
F	F	T	F	F	F	F
F	F	F	F	F	F	F

We have shown that the columns for "A and (B or C)" and "(A and B) or (A and C)" are the same, so the two statements are equivalent.

(b):

A	B	C	A or (B and C)	A or B	A or C	(A or B) and (A or C)
T	T	T	T	T	T	T
T	T	F	T	T	T	T
T	F	T	T	T	T	T
T	F	F	T	T	T	T
F	T	T	T	T	T	T
F	T	F	F	T	F	F
F	F	T	F	F	T	F
F	F	F	F	F	F	F

We have shown that the columns for "A or (B and C)" and "(A or B) and (A or C)" are the same, so the two statements are equivalent.

□

Problem 3. Let A and B be statements. If we know that A implies B, which one of the following can we conclude?

(a) A cannot be false.
(b) A and B are both true.
(c) If A is false, then B is false.
(d) B cannot be false.
(e) If B is false, then A is false.
(f) If B is true, then A is true.
(g) At least one of A and B is true.

Solution:

(e) is the correct conclusion. For an implication, the only way for it to be true while its consequent is false, is to construct a false antecedent. Therefore, if B is false, then A must be false.

□

Problem 4. Negate the following sentences:

(a) If there is a job worth doing, then it is worth doing well.
(b) Every cloud has a silver lining.
(c) For every complex problem, there is an answer that is clear, simple, and wrong.

Solution:

(a): We denote: A as "there is a job worth doing" and B as "it is worth doing well". The original sentence can be written as $A \implies B$. The negation of this sentence is:

$$\text{not}(A \implies B) = A \text{ and not} B.$$

In English this is $\boxed{\text{"There is a job worth doing, and it is not worth doing well."}}$

(b): The negation of a universal statement is to find an exsistential counterexample. Thus the negation of "Every cloud has a silver lining" is $\boxed{\text{There is a cloud without a silver lining.}}$

(c): We denote:

- $x =$ a complex problem.
- $X =$ set of all complex problems.
- $y =$ an answer.
- $Y =$ set of all answers.
- $C(y)= y$ is clear.
- $S(y) = y$ is simple.
- $W(y) = y$ is wrong.

The statement can be translated as:

$$\forall x \in X,\ \exists y \in Y \text{ s.t.}(C(y) \text{ and} S(y) \text{ and } W(y)))$$

Its negation is:

$$\exists x \in X,\ \forall y \in Y \text{ s.t.}(\text{not}\{C(y) \text{ and} S(y) \text{ and } W(y)\}) \tag{1}$$
$$= \exists x \in X,\ \forall y \in Y \text{ s.t.}((\text{not } C(y)) \text{ and(not } S(y)) \text{ and (not} W(y))) \tag{2}$$

In English, this reads:

$\boxed{\text{"There is at least one complex problem that doesn't have any answer that is simultaneously clear, simple, and wrong."}}$

□

Problem 5. Let A, B, and C be statements. Negate the following sentences:

(a) At least one of A and B are true.
(b) Both A and B are false.
(c) At least two of A, B, and C are false.

Solution:

(a)

$$\text{translation:} \qquad A \text{ or } B \tag{3}$$
$$\text{negation:} \qquad \text{not}(A \text{ or } B) = \text{not}A \text{ and not}B \tag{4}$$

(b)

$$\text{translation:} \qquad \text{not}A \text{ and not}B \tag{5}$$
$$\text{negation:} \qquad \text{not}(\text{not}A \text{ and not}B) = A \text{ or } B \tag{6}$$

(c)

$$\text{translation: } (\text{not } A \text{ and not } B) \text{ or } (\text{not } A \text{ and } \text{ not } C) \text{ or } (\text{ not } B \text{ and } \text{ not } C) \tag{7}$$
$$\text{negation: } \text{not}((\text{not } A \text{ and not } B) \text{ or } (\text{not } A \text{ and } \text{ not } C) \text{ or } (\text{ not } B \text{ and } \text{ not } C)) \tag{8}$$
$$= (A \text{ or } B) \text{ and } (A \text{ or } C) \text{ and } (B \text{ or } C) \tag{9}$$

In English, the negation is $\boxed{\text{at least two of A, B, and C are true.}}$ □

Problem 6. Let X be a set, and let $P(x)$ be a statement about elements x in X. Negate the following sentences:

(a) For every x in X, there is a y in X not equal to x, for which $P(y)$ is true.
(b) If $P(x)$ and $P(y)$ are both true, then $x = y$.

Solution:

(a)translation:
$$\forall x \in X, \exists y \in X \ \text{ s.t.} \,(y \neq x \,\text{and}\, P(y))$$
Negation:
$$\exists x \in X, \forall y \in X \ \text{ s.t.} \,(y = x \,\text{or}\, \text{not} P(y))$$
(b) translation:
$$\forall x, y \in X, \ \text{ s.t.} \,((P(x) \ \text{and} \ \ P(y)) \implies (x = y))$$
Negation:
$$\exists x, y \in X \ \text{ s.t.} \quad \text{not} \,(\ \text{not}\,(P(x) \ \text{and}\, P(y)) \ \text{or}\, (x = y)) \tag{10}$$
$$= \exists x, y \in X \ \text{ s.t.} \,(P(x) \ \text{and}\, P(y) \ \text{and}\, (x \neq y)) \tag{11}$$

$\square$