

Math 421, Section 1
Homework 1
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Problem 1 (De Morgan's laws). Let A and B be statements. Use a truth table to prove the following:

- (a) "Not (A and B)" is equivalent to " $(\text{not } A) \text{ or } (\text{not } B)$ ".
- (b) "Not (A or B)" is equivalent to " $(\text{not } A) \text{ and } (\text{not } B)$ ".

Solution:

(a):

A	B	not (A and B)	not A	not B	$(\text{not } A) \text{ or } (\text{not } B)$
T	T	F	F	F	F
T	F	T	F	T	T
F	T	T	T	F	T
F	F	T	T	T	T

We have shown that the columns for " $\text{not } (A \text{ and } B)$ " and " $(\text{not } A) \text{ or } (\text{not } B)$ " are the same, so the two statements are equivalent.

(b):

A	B	not (A or B)	not A	not B	$(\text{not } A) \text{ and } (\text{not } B)$
T	T	F	F	F	F
T	F	F	F	T	F
F	T	F	T	F	F
F	F	T	T	T	T

We have shown that the columns for " $\text{not } (A \text{ or } B)$ " and " $(\text{not } A) \text{ and } (\text{not } B)$ " are the same, so the two statements are equivalent.

□

Problem 2 (The distributive property). Let A, B, and C be statements. Use a truth table to prove the following:

- (a) “A and (B or C)” is equivalent to “(A and B) or (A and C)”.
- (b) “A or (B and C)” is equivalent to “(A or B) and (A or C)”.

Solution:

(a):

A	B	C	A and (B or C)	A and B	A and C	(A and B) or (A and C)
T	T	T	T	T	T	T
T	T	F	T	T	T	T
T	F	T	T	F	T	T
T	F	F	F	F	F	F
F	T	T	F	F	F	F
F	T	F	F	F	F	F
F	F	T	F	F	F	F
F	F	F	F	F	F	F

We have shown that the columns for “A and (B or C)” and “(A and B) or (A and C)” are the same, so the two statements are equivalent.

(b):

A	B	C	A or (B and C)	A or B	A or C	(A or B) and (A or C)
T	T	T	T	T	T	T
T	T	F	T	T	T	T
T	F	T	T	T	T	T
T	F	F	T	T	T	T
F	T	T	T	T	T	T
F	T	F	F	T	F	F
F	F	T	F	F	T	F
F	F	F	F	F	F	F

We have shown that the columns for “A or (B and C)” and “(A or B) and (A or C)” are the same, so the two statements are equivalent.

□

Problem 3. Let A and B be statements. If we know that A implies B, which one of the following can we conclude?

- (a) A cannot be false.
- (b) A and B are both true.
- (c) If A is false, then B is false.
- (d) B cannot be false.
- (e) If B is false, then A is false.
- (f) If B is true, then A is true.
- (g) At least one of A and B is true.

Solution:

(e) is the correct conclusion. For an implication, the only way for it to be true while its consequent is false, is to construct a false antecedent. Therefore, if B is false, then A must be false.

□

Problem 4. Negate the following sentences:

- (a) If there is a job worth doing, then it is worth doing well.
- (b) Every cloud has a silver lining.
- (c) For every complex problem, there is an answer that is clear, simple, and wrong.

Solution:

(a): We denote: A as "there is a job worth doing" and B as "it is worth doing well". The original sentence can be written as $A \implies B$. The negation of this sentence is:

$$\text{not}(A \implies B) = A \text{ and not } B.$$

In English this is "There is a job worth doing, and it is not worth doing well."

(b): The negation of a universal statement is to find an existential counterexample. Thus the negation of "Every cloud has a silver lining" is There is a cloud without a silver lining.

(c): We denote:

- x = a complex problem.
- X = set of all complex problems.
- y = an answer.
- Y = set of all answers.
- $A(x, y) = y$ is an answer to the problem x .
- $C(y) = y$ is clear.
- $S(y) = y$ is simple.
- $W(y) = y$ is wrong.

The statement can be translated as:

$$\forall x \in X \exists y \in Y \text{ s.t. } (A(x, y) \text{ and } C(y) \text{ and } S(y) \text{ and } W(y))$$

Its negation is:

$$\exists x \in X \forall y \in Y \text{ s.t. } (\text{not}\{A(x, y) \text{ and } C(y) \text{ and } S(y) \text{ and } W(y)\})$$

In English, this reads:

"There is at least one complex problem that doesn't have any answer that is simultaneously clear, simple, and wrong."

□

Problem 5. Let A, B, and C be statements. Negate the following sentences:

- (a) At least one of A and B are true.
- (b) Both A and B are false.
- (c) At least two of A, B, and C are false.

Solution:

(a)

translation: $A \text{ or } B$ (1)

negation: $\text{not}(A \text{ or } B) = \text{not}A \text{ and } \text{not}B$ (2)

(b)

translation: $\text{not}A \text{ and } \text{not}B$ (3)

negation: $\text{not}(\text{not}A \text{ and } \text{not}B) = A \text{ or } B$ (4)

(c)

translation: $(\text{not } A \text{ and } \text{not } B) \text{ or } (\text{not } A \text{ and } \text{not } C) \text{ or } (\text{not } B \text{ and } \text{not } C)$ (5)

negation: $\text{not}((\text{not } A \text{ and } \text{not } B) \text{ or } (\text{not } A \text{ and } \text{not } C) \text{ or } (\text{not } B \text{ and } \text{not } C))$ (6)

$= (A \text{ or } B) \text{ and } (A \text{ or } C) \text{ and } (B \text{ or } C)$ (7)

In English, the negation is at least two of A, B, and C are true. □

Problem 6. Let X be a set, and let $P(x)$ be a statement about elements x in X . Negate the following sentences:

- (a) For every x in X , there is a y in X not equal to x , for which $P(y)$ is true.
- (b) If $P(x)$ and $P(y)$ are both true, then $x = y$.

Solution: (Type your solution to problem 6 here.)

(You may want to use the following symbols: ‘for all’ \forall , ‘there exists’ \exists , and ‘in’ \in .) \square