1. (Griffiths ed.5, 4.16) Suppose the field inside a large piece of dielectric is $\boldsymbol{E}_0$, so that the electric displacement is $\boldsymbol{D}_0 = \epsilon_0 \boldsymbol{E}_0 + \boldsymbol{P}$.
   (a) Now a small spherical cavity (see Fig. 4.19 in the text) is hollowed out of the material. Find the field at the center of the cavity in terms of $\boldsymbol{E}_0$ and $\boldsymbol{P}$, and find the displacement at the center of the cavity in terms of $\boldsymbol{D}_0$ and $\boldsymbol{P}$. Here and in the rest of this problem assume that the polarization is "frozen in," so it doesn't change when the cavity is excavated.
   (b) Do the same for a long needle-shaped cavity running parallel to $\boldsymbol{P}$. (Again, see Fig. 4.19). *Hint*: ignore any asymmetry at the top and the bottom of the needle, which the text picture indicates. Note that "long" means that if L is the length of the needle and A is its cross-sectional area, we have A/L^2 sufficiently small such that $|\boldsymbol{P}|A/(4\pi\epsilon_0(L/2)^2) \ll |\boldsymbol{E}_0|$.
   (c) Do the same for a thin wafer-shaped cavity perpendicular to $\boldsymbol{P}$.(Again, see Fig. 4.19).
   Assume the cavities are small enough that $\boldsymbol{P}$, $\boldsymbol{E}_0$, and $\boldsymbol{D}_0$ are essentially uniform. *Hint*: carving out a cavity is the same as superimposing an object of the same shape but opposite polarization.

Problem 4.16

(a) Same as $\mathbf{E}_0$ minus the field at the center of a sphere with uniform polarization $\mathbf{P}$. The latter (Eq. 4.14) is $-\mathbf{P}/3\epsilon_0$. So $\boxed{\mathbf{E} = \mathbf{E}_0 + \frac{1}{3\epsilon_0}\mathbf{P}.}$ $\mathbf{D} = \epsilon_0\mathbf{E} = \epsilon_0\mathbf{E}_0 + \frac{1}{3}\mathbf{P} = \mathbf{D}_0 - \mathbf{P} + \frac{1}{3}\mathbf{P}$, so $\boxed{\mathbf{D} = \mathbf{D}_0 - \frac{2}{3}\mathbf{P}.}$

(b) Same as $\mathbf{E}_0$ minus the field of $\pm$ charges at the two ends of the "needle"—but these are small, and far away, so $\boxed{\mathbf{E} = \mathbf{E}_0.}$ $\mathbf{D} = \epsilon_0\mathbf{E} = \epsilon_0\mathbf{E}_0 = \mathbf{D}_0 - \mathbf{P}$, so $\boxed{\mathbf{D} = \mathbf{D}_0 - \mathbf{P}.}$

(c) Same as $\mathbf{E}_0$ minus the field of a parallel-plate capacitor with upper plate at $\sigma = P$. The latter is $-(1/\epsilon_0)P$, so $\boxed{\mathbf{E} = \mathbf{E}_0 + \frac{1}{\epsilon_0}\mathbf{P}.}$ $\mathbf{D} = \epsilon_0\mathbf{E} = \epsilon_0\mathbf{E}_0 + \mathbf{P}$, so $\boxed{\mathbf{D} = \mathbf{D}_0.}$

2. A long cylindrical coaxial capacitor consists of an inner conductor of radius a and an outer conductor of radius b. The region between the conductors is filled with a linear and isotropic but inhomogeneous dielectric with relative permittivity given by $\epsilon_r(s) = \alpha s^{-2}$, in which s is the usual cylindrical radial coordinate. The capacitor is charged to a voltage V.

   Determine the electric field between the conductors as a function of the voltage V and the capacitor geometry for these conditions.

For a dialectic, the absence of free charge gives $\rho_f = 0$. So Gauss law becomes:

$$\begin{aligned} \boldsymbol{\nabla} \cdot \vec{D} &= \rho_f = 0 \\ \frac{1}{s}\partial_s(sD_s) &= 0 \\ D_s &= \frac{C}{s}, \end{aligned} \tag{1}$$

for some constant C. Noticing $\vec{D}(s) = \varepsilon_0 \varepsilon_r \vec{E} = \varepsilon_0 \alpha s^{-2} \vec{E}$, we have

$$\begin{aligned} \vec{E} &= \frac{Cs}{\varepsilon\alpha}\hat{s}, \\ V &= -\int_a^b \vec{E} \cdot \mathrm{d}\vec{l} = -\frac{C}{\varepsilon_0\alpha}\int_a^b s\,\mathrm{d}s = -\frac{C}{2\varepsilon_0\alpha}(b^2 - a^2). \end{aligned} \tag{2}$$

We find C to be

$$C = \frac{-2\varepsilon\alpha V}{b^2 - a^2}. \tag{3}$$

And thus

$$\vec{E}(s) = \frac{-2\varepsilon\alpha V s}{\varepsilon\alpha(b^2 - a^2)} = \boxed{\frac{2Vs}{a^2 - b^2}\hat{s}.} \tag{4}$$

3

3. (Griffiths ed.5, 4.25) Suppose the entire region below $z = 0$ is filled with a uniform linear dielectric material of susceptibility χ_e and the region above $z = 0$ is filled with a uniform linear dielectric material with a different susceptibility χ_e' A point charge q is at the location $(x = 0, y = 0, z = d)$, where $d > 0$ (see Fig. 4.28 in the text). Find the potential in both regions ($z > 0$ and $z < 0$).

$$
\begin{aligned}
V^{>}(r,z) &= \frac{1}{4\pi\varepsilon'}\left(\frac{q}{\left(r^2+(z-d)^2\right)^{1/2}} + \frac{q'}{\left(r^2+(z+d)^2\right)^{1/2}}\right) \\
\Rightarrow V^{>}(r,0) &= \frac{1}{4\pi\varepsilon'}\frac{q+q'}{\left(r^2+d^2\right)^{1/2}}, \quad \frac{\partial V^{>}}{\partial z}(r,0) = \frac{1}{4\pi\varepsilon'}\frac{d(q-q')}{\left(r^2+d^2\right)^{3/2}}.
\end{aligned}
\tag{5}
$$

and,

$$
\begin{aligned}
V^{<}(r,z) &= \frac{1}{4}\pi\varepsilon\frac{q''}{\left(r^2+(z-d)^2\right)^{1/2}} \\
\Rightarrow V^{<}(r,0) &= \frac{1}{4\pi\varepsilon}\frac{q''}{\left(r^2+d^2\right)^{1/2}}, \quad \frac{\partial V^{<}}{\partial z}(r,0) = \frac{1}{4\pi\varepsilon}\frac{q''}{\left(r^2+d^2\right)^{3/2}},
\end{aligned}
\tag{6}
$$

where $\varepsilon' = \varepsilon_0(1+\chi_e'), \varepsilon = \varepsilon_0(1+\chi_e)$.

Boundary condition gives

$$
\begin{aligned}
&\frac{1}{\varepsilon'}(q+q') = \frac{q''}{\varepsilon}, \quad (q-q') = q'' \\
&\Rightarrow q'' = \frac{2\varepsilon}{\varepsilon+\varepsilon'}q, \quad q' = \frac{\varepsilon'-\varepsilon}{\varepsilon'+\varepsilon}q
\end{aligned}
\tag{7}
$$

Hence,

$$
\begin{aligned}
V^{>}(r,z) &= \frac{q}{4\pi\varepsilon'}\left(\frac{1}{\left(r^2+(z-d)^2\right)^{\frac{1}{2}}} + \left(\frac{\varepsilon'-\varepsilon}{\varepsilon'+\varepsilon}\right)\frac{1}{\left(r^2+(z+d)^2\right)^{\frac{1}{2}}}\right), \quad (z \geq 0); \\
V^{<}(r,z) &= \frac{2q}{4\pi(\varepsilon'+\varepsilon)}\frac{1}{\left(r^2+(z+d)^2\right)^{\frac{1}{2}}}, \quad (z \leq 0.)
\end{aligned}
\tag{8}
$$

4. A ideal dipole $\boldsymbol{p} = p\hat{z}$ is in the center of a spherical uniform dielectric shell with inner radius a and outer radius b. Find the potential in all regions.

---

1. Inside the cavity, $(r < a)$: Separation of Variables with dipole gives

$$V_I(r,\theta) = \frac{1}{4\pi\varepsilon_0}\left(\frac{p\cos\theta}{r^2} + \sum_{l=0}^{\infty} A_{1,l} r^l P_l(\cos\theta)\right) \tag{9}$$

where p is the dipole moment. Dipole expansion gives

$$V_I(r,\theta) = \frac{p}{4\pi\varepsilon_0}\left(\frac{1}{r^2} + \frac{r}{a^3}\left(\frac{1}{\varepsilon_r}(A_2 + B_2) - 1\right)\right)\cos\theta \tag{10}$$

$$\frac{\partial V_I}{\partial r} = \frac{p\cos\theta}{4\pi\varepsilon 0 a^3}\left(\frac{1}{\varepsilon_r}(A_2 + B_2) - 3\right) \tag{11}$$

1. Within shell, $(a < r < b$), **assuming a dialectic constant** $\boldsymbol{\varepsilon_r}$, we can jot down the following by using results from Separation of Variables:

$$V_{\text{II}}(r,\theta) = \frac{1}{4\pi\varepsilon}\sum_{l=0}^{\infty}\left(A_{2,l} r^l + \frac{B_{2,l}}{r^{l+1}}\right)P_l(\cos\theta)) \tag{12}$$

Dipole expansion, taking $l = 1$ gives:

$$V_{\text{II}}(r,\theta) = \frac{p}{4\pi\varepsilon}\left(A_2\frac{r}{a^3} + \frac{B_2}{r^2}\right)\cos\theta. \tag{13}$$

$$\frac{\partial V_{\text{II}}(a,\theta)}{\partial r} = \frac{p\cos\theta}{4\pi\varepsilon a^3}(A_2 - 2B_2); \quad \frac{\partial V_{\text{II}}(b,\theta)}{\partial r} = \frac{p}{4\pi\varepsilon b^3}\left(A_2\left(\frac{b}{a}\right)^3 - 2B_2\right) \tag{14}$$

1. Outside the shell, $(r > b)$:

$$V_{\text{III}}(r,\theta) = \frac{1}{4\pi\varepsilon_0}\sum_{l=0}^{\infty}\frac{B_{3,l}}{r^{l+1}}P_l(\cos\theta). \tag{15}$$

Taking $l = 1$ gives:

$$V_{\text{III}}(r,\theta) = \frac{p}{4\pi\varepsilon r^2}\left(A_2\left(\frac{b}{a}\right)^3 + B_2\right). \tag{16}$$

$$\frac{\partial V_{\text{III}}(b,\theta)}{\partial r} = \frac{p}{4\pi\varepsilon b^3}(-2)\left(A_2\left(\frac{b}{a}\right)^3 + B_2\right) \tag{17}$$

Boundary Conditions:

$$\begin{aligned} \varepsilon\frac{\partial V_{\text{II}}(a,\theta)}{\partial r} &= \varepsilon_0\frac{\partial V_I(a,\theta)}{\partial r}, \\ \varepsilon_0\frac{\partial V_{\text{III}}(b,\theta)}{\partial r} &= \varepsilon\frac{\partial V_{\text{II}}}{\partial r}(b,\theta) \end{aligned} \tag{18}$$

Painful algebra gives:

$$
\begin{aligned}
A_2 &= \left(\frac{b}{a}\right)^{-3} \frac{B_2(2\varepsilon_r - 2)}{2+\varepsilon_r}, \\
B_2 &= \frac{3\varepsilon_r}{(1+2\varepsilon_r) - 2\left(\frac{a}{b}\right)^3 \frac{(1-\varepsilon_r)^2}{2+\varepsilon_r}} \equiv f(\varepsilon_r, a, b)
\end{aligned}
\tag{19}
$$

We thus conclude:

$$
\begin{aligned}
V_I(r,\theta) &= \frac{p\cos\theta}{4\pi\varepsilon r^2}\left(1 + \left(\frac{r}{a}\right)^3 \left(\frac{f(\varepsilon_r, a, b)}{\varepsilon_r}\right)\left(1 + \left(\frac{a}{b}\right)^3 \frac{2(\varepsilon_r - 1)}{2+\varepsilon_r}\right) - 1\right), \\
V_{\mathrm{II}}(r,\theta) &= \frac{p\cos\theta}{4\pi\varepsilon r^2} f(\varepsilon_r, a, b)\left(1 + \frac{2(\varepsilon_r - 1)}{2+\varepsilon_r}\left(\frac{r}{b}\right)^3\right), \\
V_{\mathrm{III}}(r,\theta) &= \frac{p\cos\theta}{4\pi\varepsilon r^2}\,\frac{3f(\varepsilon_r, a, b)}{(\varepsilon_r + 2)}
\end{aligned}
\tag{20}
$$

5

5. (Griffiths ed.5, 4.26) A spherical conductor of radius a carries a charge Q. It is surrounded by linear (isotropic, homogenous) dielectric material of electric susceptibility χ_e, out to radius b. (See Fig. 4.32 in the text.) Find the energy of this configuration.

Problem 4.5

Field of $\mathbf{p}_1$ at $\mathbf{p}_2$ ($\theta = \pi/2$ in Eq. 3.103): $\mathbf{E}_1 = \dfrac{p_1}{4\pi\epsilon_0 r^3}\,\hat{\boldsymbol{\theta}}$ (points *down*).

Torque on $\mathbf{p}_2$: $\mathbf{N}_2 = \mathbf{p}_2 \times \mathbf{E}_1 = p_2 E_1 \sin 90° = p_2 E_1 = \boxed{\dfrac{p_1 p_2}{4\pi\epsilon_0 r^3}}$ (points *into* the page).

Field of $\mathbf{p}_2$ at $\mathbf{p}_1$ ($\theta = \pi$ in Eq. 3.103): $\mathbf{E}_2 = \dfrac{p_2}{4\pi\epsilon_0 r^3}(-2\,\hat{\mathbf{r}})$ (points to the *right*).

Torque on $\mathbf{p}_1$: $\mathbf{N}_1 = \mathbf{p}_1 \times \mathbf{E}_2 = \boxed{\dfrac{2p_1 p_2}{4\pi\epsilon_0 r^3}}$ (points *into* the page).

Problem 4.26

From Ex. 4.5:

$$\mathbf{D} = \left\{ \begin{array}{ll} 0, & (r < a) \\ \dfrac{Q}{4\pi r^2}\,\hat{\mathbf{r}}, & (r > a) \end{array} \right\}, \quad \mathbf{E} = \left\{ \begin{array}{ll} 0, & (r < a) \\ \dfrac{Q}{4\pi\epsilon r^2}\,\hat{\mathbf{r}}, & (a < r < b) \\ \dfrac{Q}{4\pi\epsilon_0 r^2}\,\hat{\mathbf{r}}, & (r > b) \end{array} \right\}.$$

$$\begin{aligned} W &= \frac{1}{2}\int \mathbf{D}\cdot\mathbf{E}\,d\tau = \frac{1}{2}\frac{Q^2}{(4\pi)^2}4\pi\left\{\frac{1}{\epsilon}\int_a^b \frac{1}{r^2}\frac{1}{r^2}r^2\,dr + \frac{1}{\epsilon_0}\int_b^\infty \frac{1}{r^2}\,dr\right\} = \frac{Q^2}{8\pi}\left\{\frac{1}{\epsilon}\left(\frac{-1}{r}\right)\Big|_a^b + \frac{1}{\epsilon_0}\left(\frac{-1}{r}\right)\Big|_b^\infty\right\} \\ &= \frac{Q^2}{8\pi\epsilon_0}\left\{\frac{1}{(1+\chi_e)}\left(\frac{1}{a}-\frac{1}{b}\right)+\frac{1}{b}\right\} = \boxed{\frac{Q^2}{8\pi\epsilon_0(1+\chi_e)}\left(\frac{1}{a}+\frac{\chi_e}{b}\right)}. \end{aligned}$$