Heat Equation

For tempreature $u(\boldsymbol{x})$, head conduction or particle diffusion can be described by the head equation:

$$u_t = k\nabla^2 u$$

Fundamental Solution

The fundamental solution Φ is found by solving the heat equation with a delta function as the initial condition:

$$\begin{cases} \Phi_t = \kappa\nabla^2\Phi \\ \Phi(\boldsymbol{x}, t = 0) = \delta(\boldsymbol{x}) \end{cases}$$

It is solved to be the Green's function

$$\Phi(\boldsymbol{x}, t) = \frac{1}{(4\pi\kappa t)^{n/2}} \exp\left(-\frac{|\boldsymbol{x}|^2}{4\kappa t}\right)$$

Initial Value problem

Consider a general initial value $g(\boldsymbol{x})$, heat equation becomes:

$$\begin{cases} u_t = \kappa\nabla^2 u \\ u(\boldsymbol{x}, 0) = g(\boldsymbol{x}) \end{cases}$$

An arguement of linearity and superposition can be made to arrive at the solution:

$$u(\boldsymbol{x}, t) = g \star \Phi \equiv \int_{\mathbb{R}^n} g(\boldsymbol{y})\Phi(\boldsymbol{x} - \boldsymbol{y})\,\mathrm{d}v_y$$

- *example*:
  - Useful special functions: Heaviside step function, and error function

    $$\operatorname{erf}(x) \equiv \frac{2}{\sqrt{\pi}} \int_0^x \exp(-z^2)\,\mathrm{d}z$$

  - example statement: consider a long rod heated on the region $[-1, 1]$ at time zero. Mathematically,

    $$\begin{cases} u_t = \kappa u_{xx} \\ u(x, 0) = g(x) = H(x + 1) - H(x - 1) \end{cases}$$

  - *Solution*:

    $$\begin{aligned} u(x, t) &= g \star \Phi \\ &= \frac{1}{\sqrt{4\pi\kappa t}} \int_{-\infty}^{\infty} g(y) \exp\left(\frac{-(x - y)^2}{4\kappa t}\right) \mathrm{d}y \\ &= \frac{1}{\sqrt{4\pi\kappa t}} \int_{-1}^{1} \exp\left(-(x - y)^2/4\kappa t\right) \mathrm{d}y \end{aligned}$$

  let $x - y = z\sqrt{4\kappa t}$, $z = \dfrac{x - y}{\sqrt{4\pi\kappa t}}$

$$
\begin{aligned}
u &= \frac{-\sqrt{4\pi\kappa t}}{\sqrt{4\pi\kappa t}} \int_{(x+1)/\left(\sqrt{4\kappa t}\right)}^{(x-1)/\left(\sqrt{4\kappa t}\right)} e^{-z^2}\,\mathrm{d}z \\
&= \frac{1}{2}\left(\operatorname{erf}\left(\frac{x+1}{\sqrt{4\kappa t}}\right) - \operatorname{erf}\left(\frac{x-1}{\sqrt{4\kappa t}}\right)\right)
\end{aligned}
$$

Notice that the erf function is an odd function, so we can combine this to be

$$
u = \operatorname{erf}\left(\frac{1}{\sqrt{4\kappa t}}\right)
$$

We can study this solution via asympotic analysis

- for small x, talor expansion of erf function to second degree gives

$$
\operatorname{erf}(x) \approx \frac{2x}{\sqrt{\pi}}
$$

  We are interested in large t, so

$$
\operatorname{erf}\left(\frac{1}{\sqrt{4\kappa t}}\right) \approx \frac{1}{\sqrt{\pi\kappa t}} \sim \frac{1}{\sqrt{t}}
$$

Heat eqn with forcing (heat source/ sink)

Consider the original heat equation without forcing

$$
u_t = \kappa \nabla^2 u
$$

Now, consider heat source $f(\boldsymbol{x}, t)$, the heat equation becomes:

$$
\begin{cases}
u_t = \kappa \nabla^2 u + f(\boldsymbol{x}, t) \\
u(\boldsymbol{x}, 0) = 0
\end{cases}
$$

We can use **Duhamel's Principle** to transform heat source to a collection of heat impulses(initial value problems) over time domain.