Physics 322
Discussion 1
09/10/2024

Problem 1

Calculate the following

(a) $\nabla\cdot(\vec{c}\times\vec{r})$ where $\vec{c}$ is a constant and $\vec{r}$ is the position vector

$\vec{r}=x\hat{x}+y\hat{y}+z\hat{z}$

$\partial_i=\frac{\partial}{\partial x_i}$

$$\nabla\cdot(\vec{c}\times\vec{r})=\sum_i\partial_i(\vec{c}\times\vec{r})_i=\sum_{i,l,m}\partial_i(\epsilon_{ilm}c_l r_m)=\sum_{i,l,m}\epsilon_{ilm}c_l\underbrace{\partial_i x_m}_{\delta_{im}}=\sum_{l,m}\cancel{\epsilon_{ili}}\,c_l$$

0 for $\epsilon_{abc}=0$ if any index is repeated

remember that $(\vec{A}\times\vec{B})_a=\sum_{b,c}\epsilon_{abc}A_bB_c$

Important!

(b) Show that $\nabla\cdot(\nabla\times\vec{A})=0$, i.e., the divergence of the curl is always zero.

$$\nabla\cdot(\nabla\times\vec{A})=\sum_i\partial_i(\nabla\times\vec{A})_i=\sum_{i,j,k}\partial_i(\epsilon_{ijk}\partial_jA_k)=\sum_{ijk}\epsilon_{ijk}\partial_i\partial_jA_k=0$$

The general rule is $\sum_{j,k}\epsilon_{ijk}T_{ij}=0$ if $T_{ij}=T_{ji}$ for $\sum_{j,k}\epsilon_{ijk}T_{ij}=\sum_{j,k}\frac{1}{2}(\epsilon_{ijk}T_{ij}+\underbrace{\epsilon_{jik}T_{ji}}_{=-\epsilon_{ijk}T_{ij}})=0$

Important!

(c) Show that $\nabla\times(\nabla f)=0$, i.e., the curl of the gradient is always zero

$$[\nabla\times(\nabla f)]_i=\sum_{j,k}\epsilon_{ijk}\partial_j(\nabla f)_k=\sum_{j,k}\cancel{\epsilon_{ijk}\partial_j\partial_k}f=0$$

Problem 2

Calculate the line integral of the function $\vec{v}=x^2\hat{x}+2yz\hat{y}+y^2\hat{z}$ from the origin to the point (1,1,1) by a straight line.

How do we parametrize a line? Given an initial point $\vec{a}=(x_a,y_a,z_a)$ and a final point $\vec{b}=(x_b,y_b,z_b)$ we can write

Important!

$$\vec{l}=\vec{b}+t(\vec{a}-\vec{b})=(\underbrace{t(x_a-x_b)+x_b}_{\equiv x(t)},\underbrace{t(y_a-y_b)+y_b}_{\equiv y(t)},\underbrace{t(z_a-z_b)+z_b}_{\equiv z(t)})\Rightarrow d\vec{l}=dt(x_a-x_b,y_a-y_b,z_a-z_b)$$

$t\in[0,1]$

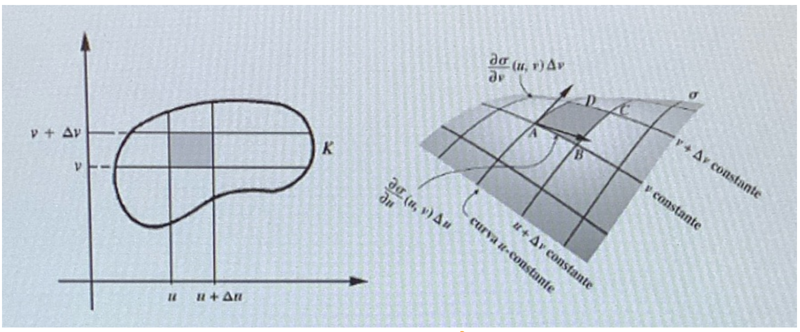
Then,

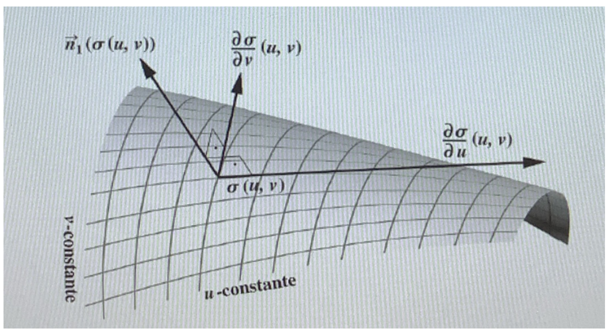
$$\vec{l}=(t,t,t)\ \&\ d\vec{l}=dt(1,1,1)\Rightarrow\int\vec{v}(t)\cdot d\vec{l}(t)=\int_0^1dt\,(x^2(t),2y(t)z(t),y^2(t))\cdot(1,1,1)=\int_0^1dt\,(t^2,2t^2,t^2)\cdot(1,1,1)=\int_0^1dt\,4t^2=\frac{4t^3}{3}\Big|_0^1=\frac{4}{3}$$

Problem 3

Calculate the flux of $\vec{E}=\frac{q}{4\pi\epsilon_0}\frac{\hat{r}}{r^2}$ through a sphere of radius R centered at the origin.

How do we parametrize a surface? For a sphere of Radius R we have $x^2+y^2+z^2=R^2$.





important! check that $x^2+y^2+z^2=R^2$

We now take $x=R\sin\theta\cos\phi$, $y=R\sin\theta\sin\phi$, and $z=R\cos\theta$, with $0\leq\theta<\pi$ & $0\leq\phi<2\pi$. We can then write $\sigma(\theta,\phi)=R(\sin\theta\cos\phi,\sin\theta\sin\phi,\cos\theta)$

$$\frac{\partial\sigma}{\partial\theta}=R(\cos\theta\cos\phi,\cos\theta\sin\phi,-\sin\theta),\ \frac{\partial\sigma}{\partial\phi}=R(-\sin\theta\sin\phi,\sin\theta\cos\phi,0),\ \frac{\partial\sigma}{\partial\theta}\times\frac{\partial\sigma}{\partial\phi}=\begin{vmatrix}\hat{x}&\hat{y}&\hat{z}\\R\cos\theta\cos\phi&R\cos\theta\sin\phi&-R\sin\theta\\-R\sin\theta\sin\phi&R\sin\theta\cos\phi&0\end{vmatrix}=R^2\sin\theta(\sin\theta\cos\phi,\sin\theta\sin\phi,\cos\theta)$$

The area element reads

$$da = \overbrace{\left|\frac{\partial \sigma}{\partial \theta} \times \frac{\partial \sigma}{\partial \phi}\right|}^{R^2 \sin\theta} d\theta\, d\phi = R^2 \sin\theta \, d\theta\, d\phi$$

The normal is

$$\hat{n} = \frac{\frac{\partial \sigma}{\partial \theta} \times \frac{\partial \sigma}{\partial \phi}}{\left|\frac{\partial \sigma}{\partial \theta} \times \frac{\partial \sigma}{\partial \phi}\right|} = (\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta)$$ ⇝ Always check if the normal is the external one!

↳ at $\theta = 0$ we have $\hat{n} = \hat{z}$ ⇝ O.K.!

The oriented area element reads

$$d\vec{a} = \hat{n}\, da = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)\, R^2 \sin\theta \, d\theta\, d\phi$$

Hence,

$$\frac{\hat{r}\cdot\hat{n}}{r^2} = \frac{1}{R^3} R\, \hat{n}\cdot\hat{n} = \frac{1}{R^2}$$

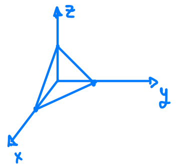
$$\int \vec{E}\cdot d\vec{a} = \frac{q}{4\pi\epsilon_0} \int \frac{\hat{r}\cdot\hat{n}}{r^2}\, da = \frac{q}{4\pi\epsilon_0} \underbrace{\int_0^{\pi} d\theta \int_0^{2\pi} d\phi \sin\theta}_{} = \frac{q}{\epsilon_0}$$

This is usually written as $\int_{S^2} d\omega = 4\pi$

Problem 4

Calculate the volume integral of the function $T = z^2$ over the tetrahedron with corners at $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$.

First, what is the integration zone? We can see that



How should we parametrize this region? The region K is $0 \le x \le 1$ and $0 \le y \le 1-x$. The upper surface is $h(x,y) = z = 1-x-y$ and the lower one is $g(x,y) = z = 0$.

Hence,

$$\int_V T\, dV = \iint_K \left[\underbrace{\int^{1-x-y} z^2 dz}_{\frac{z^3}{3}\Big|_0^{1-x-y} = \frac{(1-x-y)^3}{3}} \right] dx\,dy = \int_0^1 dx \int_0^{1-x} dy \frac{(-1)^3}{3} \underbrace{(y-(1-x))^3}_{y' = y-(1-x)}$$

$$= \int_0^1 dx \int_0^{x-1} dy \frac{1}{3} y^3 = \int_0^1 dx \frac{y^4}{12}\Big|_0^{x-1} = \frac{1}{12}\int_0^1 dx \underbrace{(1-x)^4}_{x' = 1-x} = \frac{1}{12}\int_0^1 dx\, x^4 = \frac{1}{60}$$