Thm 2 If $f:[a,b]\to\mathbb{R}$ is continuous, then f is bounded above.

Pf (of Thm 2): Define

$$A = \{x_0 \in [a,b] : f:[a,x_0]\to\mathbb{R} \text{ is bounded above}\}.$$

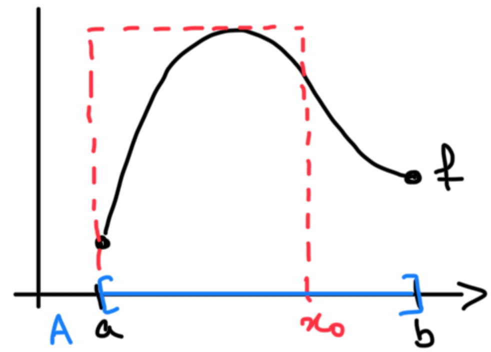
Steps:

① $\sup A$ exists

② $\sup A = b$

Last time

③ $b \in A$ ← Today



Step ③: Want $b\in A$, i.e. $f:[a,b]\to\mathbb{R}$ is bounded above. Recall from Lecture 20:

$$\lim_{x\to b^-} f(x) = f(b) \implies \exists\, \delta>0 \text{ s.t. } f:(b-\delta,b]\to\mathbb{R} \text{ is bounded above}$$

As $b-\delta<b$, then $b-\delta$ cannot be an upper bound for A, so $\exists\, x_0\in A$ s.t. $x_0>b-\delta$. Together:

$$\left.\begin{array}{l} x_0\in A \implies f(x)\le M_1 \quad \forall x\in[a,x_0] \\ \qquad\qquad\;\; f(x)\le M_2 \quad \forall x\in(b-\delta,b] \end{array}\right\} \implies \begin{array}{c} f(x)\le \max\{M_1,M_2\} \\ \forall x\in[a,b] \end{array}$$

So $f:[a,b]\to\mathbb{R}$ is bounded above. $\square$

Thm 3 If $f:[a,b]\to\mathbb{R}$ is continuous, then f has a global maximum.

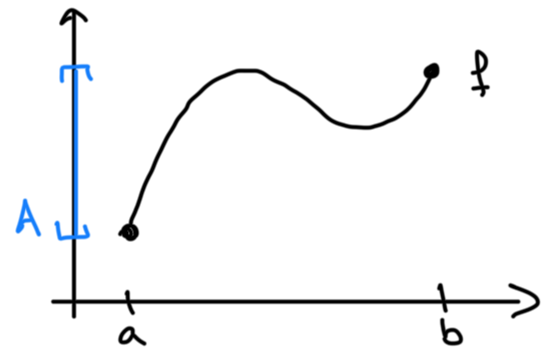
Pf: Define

$$A = f([a,b]) = \{f(x) : x \in [a,b]\}$$

Steps:

① $c = \sup A$ exists

② $c \in A$



Step ①: Want $\exists \sup A \in \mathbb{R}$.

- $A \neq \emptyset$: Note that $f(a) \in A$.
- A is bounded above: By Thm 2,

$f : [a,b] \to \mathbb{R}$ is bounded above

$\Rightarrow \exists M$ s.t. $f(x) \leq M \ \forall x \in [a,b]$

$\Rightarrow M$ is an upper bound for A.

So, by the least upper property, $\exists \sup A \in \mathbb{R}$.

Step ②: Set $c = \sup A$. Want: $\exists x \in [a,b]$ s.t. $f(x) = c$. Suppose not: $f(x) \neq c \ \forall x \in [a,b]$.

Consider $g : [a,b] \to \mathbb{R}$, $g(x) = \dfrac{1}{c - f(x)}$.

Claim: g is continuous on $[a,b]$. Let $x_0 \in [a,b]$.

$f(x)$ is continuous at x_0

$\Rightarrow c - f(x)$ is continuous at x_0 (by Lecture 14)

$\Rightarrow \dfrac{1}{c - f(x)}$ is continuous at x_0, since $c - f(x_0) \neq 0$.

So, by Thm 2, $g:[a,b]\to\mathbb{R}$ is bounded above: $\exists M\in\mathbb{R}$ s.t. $g(x)=\frac{1}{c-f(x)}\le M$ $\forall x\in[a,b]$. Note that $M>0$, since $f(x)<c$ $\forall x\in[a,b]$. So:

$$\frac{1}{c-f(x)}\le M \implies c-f(x)\ge\frac{1}{M} \implies f(x)\le c-\frac{1}{M}$$

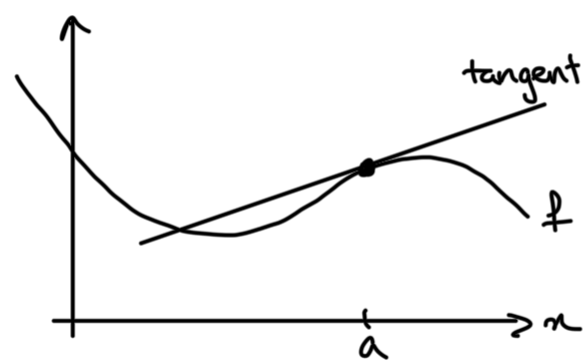
for all $x\in[a,b]$. So $c-\frac{1}{M}$ is an upper bound for A. This contradicts that $c=\sup A$ is the least upper bound. $\square$

CALCULUS

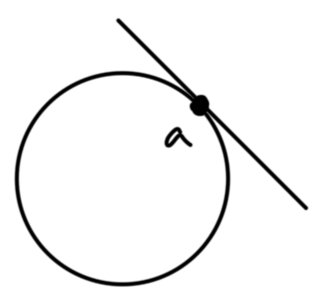
- Calculus is centered around 2 fundamental problems

Q: For an arbitrary function $f:\mathbb{R}\to\mathbb{R}$,

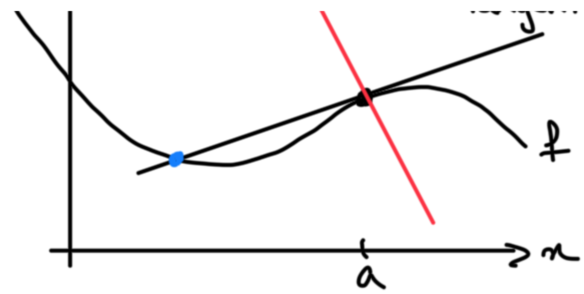
① How do we find the line tangent to the graph of f at a point $x=a$?



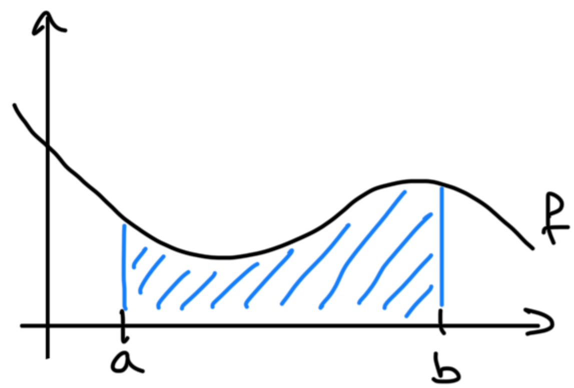
- Well, for a circle, it's the line through a that intersects the circle only once. (There's only 1 such line.)



• ... But this doesn't work for arbitrary curves:



(2) How do we find the graph of $f(x)$ for $a \le x \le b$?



• Well, for a circle, the area is πr^2.

• ... But this doesn't work for arbitrary curves.

• Surprisingly, these 2 seemingly independent questions are closely related

A: (1) Derivatives (Ch. 9–12)
(2) Integrals (Ch. 13–14)

DERIVATIVES (Ch. 9)

Def Let $I \subseteq \mathbb{R}$ be an open interval, $f : I \to \mathbb{R}$ a function, and $a \in I$. We say f is differentiable at a if

$$\lim \frac{f(a+h)-f(a)}{} \quad \text{exists.}$$

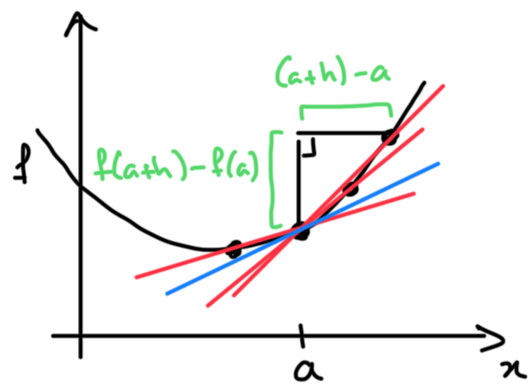
$h \to 0$ $\qquad h$

When this is true, we call the limit the <u>derivative of f at a</u> and we denote it by $f'(a)$ or $\frac{df}{dx}(a)$.

<u>Rmk</u> "I is an open interval" accounts for all of the cases $I = (b,c), (-\infty, b), (b, \infty), \mathbb{R}$ simultaneously. In all of these cases, $\exists\, \delta > 0$ s.t. $(a-\delta, a+\delta) \subseteq I$, so the limit makes sense.

<u>Rmk</u> $f'(a)$ is...

① The slope of the tangent line to the graph of f at $(a, f(a))$.



② The instantaneous rate of change of $f(x)$ near a:

$$f'(a) \approx \frac{f(a+h) - f(a)}{h} \quad \Longrightarrow \quad f(a+h) \approx f(a) + f'(a) \cdot h$$