

**Math 421, Section 1**  
**Homework 2**  
(Name)

**Problem 1.** Prove that for any  $x, y \in \mathbb{N}$ , if  $x$  is odd and  $y$  is odd then  $x + y$  is even.

**Solution:** Suppose  $x, y \in \mathbb{N}$  are odd, then  $\exists n, m \in \mathbb{N} \cup \{0\}$  s.t.  $x = 2n + 1, y = 2m + 1$ .

$$x + y = 2n + 1 + 2m + 1 = 2(n + m + 1). \quad (1)$$

It is clear that since  $(n + m + 1) \in \mathbb{N}$ ,  $x + y$  is even. □

**Problem 2.** Prove that for any  $x \in \mathbb{N}$ , if  $x$  is odd then  $x^3$  is odd.

**Solution:** Suppose  $x$  is odd, i.e.  $\exists n \in \mathbb{N} \cup \{0\}$  s.t.  $x = 2n + 1$

$$x^3 = (2n + 1)^3 = 8n^3 + 12n^2 + 6n + 1 = 2(4n^3 + 6n^2 + 3n) + 1. \quad (2)$$

It is clear that since  $(4n^3 + 6n^2 + 3n) \in \mathbb{N} \cup \{0\}$ ,  $x^3$  is odd.

□

**Problem 3.** Using induction, prove that for all  $n \in \mathbb{N}$  we have

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2.$$

**Solution:** [Base case]: For  $n = 1$ , we have  $1 = 1$ , which is true.

[Inductive step]: Suppose the statement is true for  $\exists n \in \mathbb{N}$ , i.e.

$$1 + 3 + \cdots + 2n - 1 = n^2 \tag{3}$$

Then for  $n = n + 1$  we have:

$$1 + 3 + \cdots + 2n - 1 + 2(n + 1) - 1 = n^2 + 2n + 1 \tag{4}$$

$$= (n + 1)^2 \tag{5}$$

So the formula is true for  $n + 1$ . Thus, by induction, the statement is true for all  $n \in \mathbb{N}$ .

□

**Problem 4.** Compute the following sum:

$$\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \cdots + \frac{1}{(2n-1)(2n+1)}.$$

Prove that your answer is true for all  $n \in \mathbb{N}$  using induction.

**Solution:** By noticing  $\frac{1}{(2n-1)(2n+1)} = \frac{1}{2}(\frac{1}{2n-1} - \frac{1}{2n+1})$ , a rough calculation suggests that the sum should be  $\frac{1}{2} - \frac{1}{4n+2}$ . It is proved by induction as follows:

[base case]: For  $n = 1$ , we have

$$\frac{1}{1 \cdot 3} = \frac{1}{2} - \frac{1}{6} = \frac{1}{3},$$

which is true.

[Inductive case]: Suppose the statement is true for  $\exists n \in \mathbb{N}$ , i.e.

$$\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \cdots + \frac{1}{(2n-1)(2n+1)} = \frac{1}{2} - \frac{1}{4n+2} \quad (6)$$

Then for  $n = n + 1$ ,

$$\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \cdots + \frac{1}{(2n-1)(2n+1)} + \frac{1}{(2n+1)(2n+3)} = \frac{1}{2} - \frac{1}{4n+2} + \frac{1}{(2n+1)(2n+3)} \quad (7)$$

$$= \frac{1}{2} - \frac{1}{4n+2} + \frac{1}{2}(\frac{1}{2n+1} - \frac{1}{2n+3}) \quad (8)$$

$$= \frac{1}{2} - \frac{1}{4n+6} \quad (9)$$

$$= \frac{1}{2} - \frac{1}{4(n+1)+2} \quad (10)$$

So the formula is true for  $n + 1$ . Thus, by induction, the statement is true for all  $n \in \mathbb{N}$ .

□

**Problem 5.** Prove the following statements for all  $a, b \in \mathbb{R}$ :

- (a)  $-a + (-b) = -(a + b)$ .
- (b) If  $a, b \neq 0$  then  $a^{-1} \cdot b^{-1} = (ab)^{-1}$ .

Carefully justify every step using properties of  $\mathbb{R}$  stated in lecture.

**Solution:** [a]: Consider the original equation,

$$-a + (-b) = -(a + b) \quad (11)$$

Adding  $(a+b)$  to both sides, we can find that it is equivalent to

$$-a + (-b) + (a + b) = -(a + b) + (a + b) \quad (12)$$

Applying inverse addition to the right side, and apply associativity to the left, this is equivalent to

$$-a + a + (-b) + b = 0 \quad (13)$$

Therefore, to prove the original statement is equivalent to prove Equation 13. By the inverse addition property, we have

$$-a + a = 0, \quad -b + b = 0 \quad (14)$$

$$\Rightarrow -a + a + (-b) + b = 0 \quad (15)$$

The statement is thus proved.

[b]: Suppose  $a, b \neq 0$ , we have

$$a^{-1} \cdot b^{-1} \cdot ab \stackrel{\text{commutivity}}{=} a^{-1} \cdot a \cdot b^{-1} \cdot b \stackrel{\text{inverse}}{=} 1 \quad (16)$$

$$\text{also, } (ab)^{-1} \cdot ab \stackrel{\text{inverse}}{=} 1 \quad (17)$$

By transivity, we have

$$a^{-1} \cdot b^{-1} \cdot ab = (ab)^{-1} \cdot ab \quad (18)$$

$$\stackrel{\text{prop.1}}{\Rightarrow} a^{-1} \cdot b^{-1} = (ab)^{-1} \quad (19)$$

The statement is thus proved. □

**Problem 6.** Prove the following statements for all  $a, b, c, d \in \mathbb{R}$ :

- (a) If  $a < b$  and  $c < d$  then  $a + c < b + d$ .
- (b) If  $0 < a < b$  and  $0 < c < d$  then  $ac < bd$ .

**Solution:** [a]: Suppose  $a < b, c < d$ , then by O1,

$$a + c < b + c \tag{20}$$

$$b + c < d + b. \tag{21}$$

By Transitivity,

$$a + c < d + b \tag{22}$$

By commutivity,

$$a + c < b + d \tag{23}$$

Thus proved the inequality. [b]: Suppose  $0 < a < b, 0 < c < d$ . Then by O2,

$$a \cdot c < b \cdot c \tag{24}$$

$$b \cdot c < b \cdot d \tag{25}$$

By transitivity,

$$ac < bd \tag{26}$$

Thus proves the inequality. □