Notes on Math 421: Calculus, proof based

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Contents

1. Numbers

1.1. Even and Odd number

Even: $x \in \mathbb{N}$ is even iff $\exists y \in \mathbb{N}\ s.t.\ x = 2y$.

Odd: $x \in \mathbb{N}$ is odd iff $\exists y \in \mathbb{N} \cup \{0\}\ s.t.\ x = 2y + 1$.

1.2. Mathematical Induction

To prove some statement $P(n)$ is true for all $n \in \mathbb{N}$, we need to prove two things:

1. [Base Case] $P(n)$ is true.
2. [Inductive Step]: $\forall n \in \mathbb{N}, \quad P(n) \Rightarrow P(n+1)$

- Note that, it can often be useful to use formulas for fractions such like

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1} \quad ; \quad \frac{1}{(2n-1)(2n+1)} = \frac{1}{2}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right)$$ [1]

1.3. Properties of $\mathbb{R}$

- Addition
  - closure: $\forall x, y \in \mathbb{R}, x + y \in \mathbb{R}$
  - commutative: $\forall x, y \in \mathbb{R}, x + y = y + x$
  - associative: $\forall x, y, z \in \mathbb{R}, x + (y + z) = (x + y) + z$
  - identity: $\forall x \in \mathbb{R}, x + 0 = x$
  - inverse: $\forall x \in \mathbb{R}, \exists -x \in \mathbb{R}\ s.t. x + (-x) = 0$
- Multiplication
  - closure: $\forall x, y \in \mathbb{R}, x \cdot y \in \mathbb{R}$
  - comuutative: $\forall x, y \in \mathbb{R}, x \cdot y = y \cdot x$
  - associative: $\forall x, y, z \in \mathbb{R}, x \cdot (y \cdot z) = (x \cdot y) \cdot z$
  - identity: $\forall x \in \mathbb{R}, x \cdot 1 = x$
  - inverse: $\forall x \in \mathbb{R}, x \neq 0, \exists x^{-1} \in \mathbb{R}\ s.t.\ x \cdot x^{-1} = 1$
- Distributivity: $\forall a, b, c \in \mathbb{R}, (a + b) \cdot c = a \cdot c + b \cdot c$

1.3.1. Propositions

- if $a, b, c \in \mathbb{R}\ s.t.\ a + b = a + c \quad \Rightarrow b = c$.
- If $a, c, b \in \mathbb{R}\ s.t.\ a \cdot b = a \cdot c, a \neq 0 \quad \Rightarrow b = c$
- $\forall a \in \mathbb{R}, \quad a \cdot 0 = 0 = 0 \cdot a$
- if $a, b \in \mathbb{R}, a \cdot b = 0$, then $a = 0$or $b = 0$
- $\forall a, b \in \mathbb{R}, (-a) \cdot b = -(a \cdot b) = a \cdot (-b)$
- $\forall a, b \in \mathbb{R}, \quad (-a) \cdot (-b) = ab$

1.4. Properties of Inequalities

- Trichotomy: for each $a, b \in \mathbb{R}$, only one of the following is ture: $a < b, \quad a = b, \quad b < a$.
- Transitivity: $\forall a, b, c \in \mathbb{R}, a < b \quad \text{and} \quad b < c \quad \Rightarrow a < c$

- Addition: $\forall a, b, c \in \mathbb{R}, a < b \quad \Rightarrow a + c < b + c$
- Multiplication: $\forall a, b, c \in \mathbb{R}, a < b \quad \text{and} \quad c > 0 \quad \Rightarrow ac < bc$
- Reciprocal: $\forall a, b \in \mathbb{R}, a < b \quad \text{and} \quad c < 0 \quad \Rightarrow ac > bc$
- flip sign: $\forall a, b \in \mathbb{R}, a < b \quad \Rightarrow -b < -a$

2. Functions and Sets

2.1. Image and Preimage

- *def*: Let $f : A \to B$ be a function:

  If $X \subset A$, the **image** of X under f is

$$f(X) = \{f(a) : a \in X\} \quad [2$$

- The image of f is $f(A)$

If $Y \subset B$ the **preimage** of Y under f is

$$f^{-1}(Y) = \{a \in A : f(a) \in Y\} \quad [3$$

2.2. Surjective, Injective,Bijective

- *def*: Let $f : A \to B$ be a function:
- **Surjective**: f is surjective iff $f(A) = B$. i.e

$$\forall b \in B, \exists a \in A \quad s.t. \quad f(a) = b \quad [4$$

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- **Injective**: f is injective iff $f(a) = f(b) \Rightarrow a = b$. i.e

$$\forall a, b \in A, f(a) = f(b) \quad \Rightarrow \quad a = b \quad [5$$

Bijective: both surjective and injective

2.3. Interval

- *def*: A set $I \in \mathbb{R}$ is an **interval** iff

$$(\forall x, y, x \in \mathbb{R}\ , x, z \in I\ , x < y < z) \Rightarrow y \in I \quad [6$$

- *Lemma*: $\forall a, b \in \mathbb{R}$, $a < b$, $\Rightarrow$ (a, b) is an interval.

2.3.1. Definition of Open and Closed Intervals

- *def*: A set $U \subseteq \mathbb{R}$ is **open** iff

$$\forall x \in U, \exists\, \varepsilon > 0 \quad s.t. \quad (x - \varepsilon, x + \varepsilon) \subseteq U \quad [7$$

- *def*: A set $F \subseteq \mathbb{R}$ is **closed** iff $F^c = \{x \in \mathbb{R} : x \notin F\}$ is open.
- Lemma: Union of open sets is open.
- Lemma: Intersections of finitely many open sets is open.

3. Limits

3.1. Definition of Limit via epsilon-delta

$$\lim_{x \to a} f(x) = l$$
$$\Leftrightarrow \forall \varepsilon > 0, \exists \delta > 0 \; s.t. \; 0 < |x - a| < \delta \quad \Rightarrow |f(x) - l| < \varepsilon$$ [8

3.2. Limit Operation laws

- *Theorem*: Let $f, g : \mathbb{R} \to \mathbb{R}$ be functions and $a \in \mathbb{R}$ be a limit point. If $\lim_{x \to a} f(x) = l$ and $\lim_{x \to a} g(x) = m$, then:
  - $$\lim_{x \to a} (f(x) + g(x)) = l + m$$ [9
  - $$\lim_{x \to a} (f(x) - g(x)) = l - m$$ [10
  - $$\lim_{x \to a} (f(x) \cdot g(x)) = l \cdot m$$ [11
  - $$\lim_{x \to a} \left(\frac{f(x)}{g(x)} \right) = \frac{l}{m} \quad , \text{if } m \neq 0$$ [12

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