

Magnetostatics

Magnetic Forces on Charge

For a moving charge of velocity \mathbf{v} and charge Q in a magnetic field \mathbf{B} , the magnetic force is found by the Lorentz force law:

$$\mathbf{F}_{\text{mag}} = Q(\mathbf{v} \times \mathbf{B}) \quad [1]$$

Notice that magnetic force does no work. We can see this by considering an infinitesimal displacement on Q , $d\mathbf{l} = \mathbf{v} dt$, then the work done by \mathbf{B} field is

$$dW = \mathbf{F}_{\text{mag}} \cdot d\mathbf{l} = Q \underbrace{(\mathbf{v} \times \mathbf{B}) \cdot \mathbf{v}}_{\text{perpendicular}} dt = 0 \quad [2]$$

Magnetic Force on Current

Line current

Magnetic force on a segment of current carrying wire with $I = \lambda v$ is:

$$\mathbf{F} = \int_l I(d\mathbf{l} \times \mathbf{B}) \quad [3]$$

Surface and volume current density

- Surface current density:

$$\mathbf{K} = \frac{dI}{dl_{\perp}} = \sigma \mathbf{v}; \quad [4]$$

with dl_{\perp} an infinitesimal width running parallel to the current flow. Force on a surface current:

$$\mathbf{F} = \int_A (\mathbf{K} \times \mathbf{B}) da \quad [5]$$

- Volume current density:

$$\mathbf{J} = \frac{dI}{da_{\perp}} = \rho \mathbf{v}; \quad [6]$$

with infinitesimal cross section da_{\perp} running parallel to the flow. Force on a volume current:

$$\mathbf{F} = \int_V (\mathbf{J} \times \mathbf{B}) d\tau \quad [7]$$

Continuity Equation

$$I = \int_s \mathbf{J} da_{\perp} = \int_s \mathbf{J} \cdot d\mathbf{a}. \quad [8]$$

So, noticing a conservation of charge

$$\begin{aligned} \oint \mathbf{J} \cdot d\mathbf{a} &= \int_v \nabla \cdot \mathbf{J} d\tau = -\frac{d}{dt} \int_v \rho d\tau \\ \Rightarrow \nabla \cdot \mathbf{J} &= -\frac{d\rho}{dt} \end{aligned} \quad [9]$$

Magnetostatics: Steady currents

Magnetostatics is the regime where :

$$\frac{\partial \rho}{\partial t} = 0 \Rightarrow \nabla \cdot \mathbf{J} = 0. \quad [10]$$

And the Biot-Savart law is used to find the magnetic field due to steady linear current:

$$\boxed{\mathbf{B}(\mathbf{r}) = \frac{\mu_0 I}{4\pi} \int_l \frac{d\mathbf{l}' \times \hat{\mathbf{r}}}{r'^2}} \quad [11]$$

and for a volume current:

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_V \frac{\mathbf{J}(\vec{\mathbf{r}}') \times \hat{\mathbf{r}}}{r'^2} d\tau' \quad [12]$$

Div of B, curl of B, and Ampere's law

B-S law tells us of the div of B:

$$\nabla \cdot \mathbf{B} = 0 \quad [13]$$

suggesting the inexistence of magnetic monopoles.

We can find curl of B from B-S law too:

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \quad [14]$$

and by Stokes' theorem, we can write:

$$\boxed{\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}}} \quad [15]$$

where the line integral is over an amperean loop.

note: Ampere's law is only useful when \mathbf{B} is constant over the loop, and in which case we can pull it out of the integral. Such symmetry is often found in amperean loops for infinite straight wire, infinite plane, infinite solenoid, and toroid.

Magnetic Vector Potential

By helmholtz theorem, $\nabla \cdot \mathbf{B} = 0$ invites the notion of a vector potential \mathbf{A} such that

$$\mathbf{B} = \nabla \times \mathbf{A} \quad [16]$$

. We choose $\nabla \cdot \mathbf{A} = 0$, then ampere's law gives:

$$\begin{aligned} \nabla \times \mathbf{B} &= \nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu_0 \mathbf{J} \\ \Rightarrow \quad &\boxed{\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}} \end{aligned} \quad [17]$$

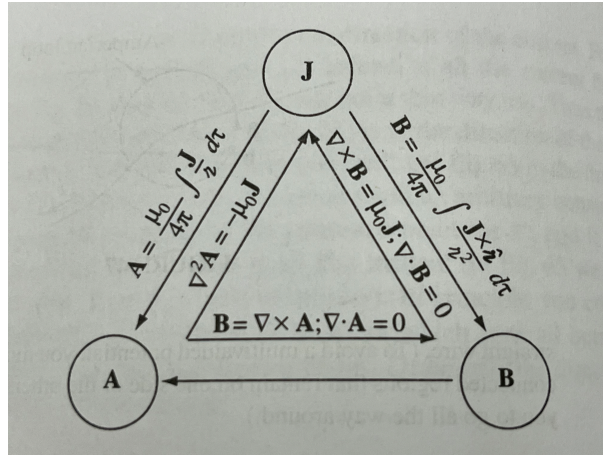
which is poisson's equation for \mathbf{A} . The solution for some local charge distribution (volume, linear, surface) can be read off as:

$$\mathbf{A}(\vec{\mathbf{r}}) = \frac{\mu_0}{4\pi} \int_V \frac{\mathbf{J}(\vec{\mathbf{r}}')}{r'} d\tau, \quad [18]$$

$$\mathbf{A} = \frac{\mu_0 I}{4\pi} \int_l \frac{1}{r} d\mathbf{l}, \quad [19]$$

$$\mathbf{A} = \frac{\mu_0}{4\pi} \int_S \frac{\mathbf{K}}{r'} da' \quad [20]$$

Relations between A, J, B that describes magnetostatis is summarized as follows:



Boundary conditions and Discontinuities

For a surface charge, and a perpendicular amperean loop:

$$\oint B \cdot dl = \mu_0 K l \Rightarrow B_+ - B_- = \mu_0 K, \quad [21]$$

so that

$$B_+ - B_- = \mu_0(\mathbf{K} \times \hat{n}),$$

$$\boxed{\frac{\partial A_+}{\partial n} - \frac{\partial A_-}{\partial n} = -\mu_0 K} \quad [22]$$

and notedly,

$$A_+ = A_-.$$
 [23]

Multipole Expansion of vector potential

$$A(\vec{r}) = \frac{\mu_0 I}{4\pi} \oint \frac{1}{r'} d\mathbf{l}' = \frac{\mu_0 I}{4\pi} \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \oint (r')^n P_{n(\cos\alpha)} d\mathbf{l}' \quad [24]$$

Given that $\oint dl' = 0$, $\mathbf{A}_{\text{mono}} = 0$, and with some vector algebra,

$$\boxed{A_{\text{dipole}} = \frac{\mu_0 I}{4\pi} \frac{\mathbf{m} \times \hat{\mathbf{r}}}{r^2}} \quad [25]$$

$$m \equiv Ia$$

- putting dipole at origin,

$$\mathbf{m} \times \hat{r} = m \sin \theta, \quad [26]$$

So

$$\begin{aligned} \mathbf{A}_{\text{dipole}} &= \frac{\mu_0 m \sin \theta}{4\pi r^2} \hat{\phi}, \\ \mathbf{B}_{\text{dip}} &= \nabla \times \mathbf{A} = \frac{\mu_0 m}{4\pi r^3} (2 \cos \theta \hat{r} + \sin \theta \hat{\theta}) \end{aligned} \quad [27]$$

Magnetic Fields in Matter

Torque and Force on a magnetic dipole

For a magnetic dipole square, $m = Iab$. Torque in a magnetic field is:

$$\mathbf{N} = \mathbf{m} \times \mathbf{B} \quad [28]$$

For an infinitesimal loop, with dipole moment \mathbf{m} , in a field \mathbf{B} , the force is

$$\mathbf{F} = \nabla(\mathbf{m} \cdot \mathbf{B}) \quad [29]$$

Such torque is the main cause behind paramagnetism.

Diamagnetism: revolving electron in \mathbf{B}

For a revolving electron around nucleus,

$$\begin{cases} I = -\frac{ev}{2\pi R} \\ \mathbf{m} = -\frac{1}{2}evR\hat{z} \end{cases} \quad [30]$$

In the presence of a magnetic field, and by newton's second law,

$$\begin{aligned} \frac{1}{4\pi\epsilon_0} \frac{e^2}{R^2} + evB &= m \frac{v^2}{R^2} \\ \Rightarrow \Delta v &= \frac{eRB}{2m_e}, \quad \Delta \mathbf{m} = -\frac{e^2 R^2}{4m_e} \mathbf{B} \end{aligned} \quad [31]$$

Magnetization

We denote magnetization

$$\boxed{\mathbf{M} = \frac{\mathbf{m}}{V}}, \quad [32]$$

magnetic moment per unit volume.

Field due to Magnetized Object

Recall \mathbf{A} due to magnetic dipole with dipole moment \mathbf{m} :

$$\mathbf{A} = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \hat{\mathbf{r}}}{r^2}. \quad [33]$$

For a magnetized object, each volume element carries $\mathbf{m} = \mathbf{M} d\tau$, so \mathbf{A} due to a magnetized object with magnetization \mathbf{M} is :

$$\mathbf{A} = \frac{\mu_0}{4\pi} \int_V \frac{\mathbf{M}(\vec{r}) \times \hat{\mathbf{r}}}{r^2} d\tau. \quad [34]$$

If we define a volume current density

$$\mathbf{J}_b = \nabla \times \mathbf{M}, \quad [35]$$

and a surface current density

$$\mathbf{K}_b = \mathbf{M} \times \hat{n}, \quad [36]$$

then we can write

$$\mathbf{A}(\vec{r}) = \underbrace{\frac{\mu_0}{4\pi} \int_V \frac{\mathbf{J}_b(r')}{r'} d\tau}_{\text{potential of Vol. current}} + \underbrace{\frac{\mu_0}{4\pi} \oint_S \frac{\mathbf{K}_b(r')}{r'} da'}_{\text{potential of Surf. current}} \quad [37]$$

Auxiliary Field \mathbf{H}

In a material, we can take the total current to be

$$\mathbf{J} = \mathbf{J}_b + \mathbf{J}_f \quad [38]$$

and by Amepere's law,

$$\nabla \times \left(\frac{1}{\mu_0} \mathbf{B} - \mathbf{M} \right) = \mathbf{J}_f. \quad [39]$$

We are thus motivated to define an Auxiliary Field \mathbf{H} :

$$\mathbf{H} \equiv \frac{1}{\mu_0} \mathbf{B} - \mathbf{M}. \quad [40]$$

From which we can write the Amepere's Law in Magnetized material as

$$\nabla \times \mathbf{H} = \mathbf{J}_f \quad \Leftrightarrow \quad \oint \mathbf{H} \cdot d\mathbf{l} = I_{f_{\text{enc}}} \quad [41]$$

Boundary Conditions for \mathbf{H} and \mathbf{B}

Divergence of \mathbf{H} gives:

$$\nabla \cdot \mathbf{H} = -\nabla \cdot \mathbf{M}. \quad [42]$$

And from fundamental theorem of divergence,

$$H_{\text{above}}^{\perp} - H_{\text{below}}^{\perp} = -|(M_{\text{above}}^{\perp} - M_{\text{below}}^{\perp})|. \quad [43]$$

While curl of \mathbf{H} gives

$$\mathbf{H}_{\text{above}}^{\parallel} - \mathbf{H}_{\text{below}}^{\parallel} = \mathbf{K}_f \times \hat{n}. \quad [44]$$

Bondary conditions for \mathbf{B} are:

$$\begin{aligned} B_{\text{above}}^{\perp} - B_{\text{below}}^{\perp} &= 0, \\ B_{\text{above}}^{\parallel} - B_{\text{below}}^{\parallel} &= \mu_0 (\mathbf{K}_f \times \hat{n}). \end{aligned} \quad [45]$$

Linear and Nonlinear Media

- For linear media,