Physics 322, Midterm #1 Formula Sheet

Lorentz Force on infinitessimal charge element dq (electrostatic fields only):

$$d\boldsymbol{F} = dq\boldsymbol{E}$$

Gauss's Law:

$$\oint \boldsymbol{E} \cdot d\boldsymbol{a} = \frac{q_{\text{enclosed}}}{\epsilon_0}, \qquad \nabla \cdot \boldsymbol{E} = \frac{\rho}{\epsilon_0}$$

Curl of $\boldsymbol{E}$ (electrostatics):

$$\nabla \times \boldsymbol{E} = 0$$

Electric field of a static charge distribution:

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\boldsymbol{r}')\hat{\boldsymbol{\imath}}}{\imath^2} d\tau' = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\boldsymbol{r}')(\boldsymbol{r} - \boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|^3} d\tau'.$$

Electrostatic potential difference:

$$\boldsymbol{E}(\boldsymbol{r}) = -\nabla V(\boldsymbol{r}), \qquad V(\boldsymbol{b}) - V(\boldsymbol{a}) = -\int_a^b \boldsymbol{E} \cdot d\boldsymbol{l}$$

Potential for a static charge distribution (setting $V(r \to \infty) = 0$):

$$V(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\boldsymbol{r}')}{\imath} d\tau' = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|} d\tau'$$

Electrostatics boundary conditions: ($\hat{\boldsymbol{n}}$ perpendicular to boundary, $\hat{\boldsymbol{t}}$ tangent to boundary, $+/-$ just above/below boundary:

$$(\boldsymbol{E} \cdot \hat{\boldsymbol{n}})_+ - (\boldsymbol{E} \cdot \hat{\boldsymbol{n}})_- = \frac{\sigma}{\epsilon_0}, \qquad (\boldsymbol{E} \cdot \hat{\boldsymbol{t}})_+ = (\boldsymbol{E} \cdot \hat{\boldsymbol{t}})_-, \qquad V_+ = V_-$$

Potential energy of point charge configuration:

$$W = \frac{1}{4\pi\epsilon_0} \sum_i \sum_{j>i} \frac{q_i q_j}{\imath_{ij}} = \frac{1}{4\pi\epsilon_0} \sum_i \sum_{j>i} \frac{q_i q_j}{|\boldsymbol{r}_i - \boldsymbol{r}_j|}$$

Potential energy (no point charge singularities):

$$W = \frac{1}{2} \int \rho(\boldsymbol{r})V(\boldsymbol{r})d\tau, \qquad W = \frac{\epsilon_0}{2} \int_{\text{all space}} |\boldsymbol{E}(\boldsymbol{r})|^2 d\tau$$

Coefficients of potential, capacitance:

$$V_i = \sum_j p_{ij} Q_j, \qquad C = \frac{Q}{V_1 - V_2} = \frac{1}{p_{11} + p_{22} - 2p_{12}}$$

VECTOR DERIVATIVES

Cartesian. $\quad d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}; \quad d\tau = dx\,dy\,dz$

Gradient: $\quad \nabla t = \dfrac{\partial t}{\partial x}\hat{\mathbf{x}} + \dfrac{\partial t}{\partial y}\hat{\mathbf{y}} + \dfrac{\partial t}{\partial z}\hat{\mathbf{z}}$

Divergence: $\nabla \cdot \mathbf{v} = \dfrac{\partial v_x}{\partial x} + \dfrac{\partial v_y}{\partial y} + \dfrac{\partial v_z}{\partial z}$

Curl: $\quad \nabla \times \mathbf{v} = \left(\dfrac{\partial v_z}{\partial y} - \dfrac{\partial v_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\dfrac{\partial v_x}{\partial z} - \dfrac{\partial v_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\dfrac{\partial v_y}{\partial x} - \dfrac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}}$

Laplacian: $\quad \nabla^2 t = \dfrac{\partial^2 t}{\partial x^2} + \dfrac{\partial^2 t}{\partial y^2} + \dfrac{\partial^2 t}{\partial z^2}$

Spherical. $\quad d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}; \quad d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi$

Gradient: $\quad \nabla t = \dfrac{\partial t}{\partial r}\hat{\mathbf{r}} + \dfrac{1}{r}\dfrac{\partial t}{\partial \theta}\hat{\boldsymbol{\theta}} + \dfrac{1}{r\sin\theta}\dfrac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}}$

Divergence: $\nabla \cdot \mathbf{v} = \dfrac{1}{r^2}\dfrac{\partial}{\partial r}(r^2 v_r) + \dfrac{1}{r\sin\theta}\dfrac{\partial}{\partial \theta}(\sin\theta\,v_\theta) + \dfrac{1}{r\sin\theta}\dfrac{\partial v_\phi}{\partial \phi}$

Curl: $\quad \nabla \times \mathbf{v} = \dfrac{1}{r\sin\theta}\left[\dfrac{\partial}{\partial \theta}(\sin\theta\,v_\phi) - \dfrac{\partial v_\theta}{\partial \phi}\right]\hat{\mathbf{r}}$

$$+ \dfrac{1}{r}\left[\dfrac{1}{\sin\theta}\dfrac{\partial v_r}{\partial \phi} - \dfrac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \dfrac{1}{r}\left[\dfrac{\partial}{\partial r}(r v_\theta) - \dfrac{\partial v_r}{\partial \theta}\right]\hat{\boldsymbol{\phi}}$$

Laplacian: $\quad \nabla^2 t = \dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2\dfrac{\partial t}{\partial r}\right) + \dfrac{1}{r^2\sin\theta}\dfrac{\partial}{\partial \theta}\left(\sin\theta\dfrac{\partial t}{\partial \theta}\right) + \dfrac{1}{r^2\sin^2\theta}\dfrac{\partial^2 t}{\partial \phi^2}$

Cylindrical. $\quad d\mathbf{l} = ds\,\hat{\mathbf{s}} + s\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}; \quad d\tau = s\,ds\,d\phi\,dz$

Gradient: $\quad \nabla t = \dfrac{\partial t}{\partial s}\hat{\mathbf{s}} + \dfrac{1}{s}\dfrac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}} + \dfrac{\partial t}{\partial z}\hat{\mathbf{z}}$

Divergence: $\nabla \cdot \mathbf{v} = \dfrac{1}{s}\dfrac{\partial}{\partial s}(s v_s) + \dfrac{1}{s}\dfrac{\partial v_\phi}{\partial \phi} + \dfrac{\partial v_z}{\partial z}$

Curl: $\quad \nabla \times \mathbf{v} = \left[\dfrac{1}{s}\dfrac{\partial v_z}{\partial \phi} - \dfrac{\partial v_\phi}{\partial z}\right]\hat{\mathbf{s}} + \left[\dfrac{\partial v_s}{\partial z} - \dfrac{\partial v_z}{\partial s}\right]\hat{\boldsymbol{\phi}} + \dfrac{1}{s}\left[\dfrac{\partial}{\partial s}(s v_\phi) - \dfrac{\partial v_s}{\partial \phi}\right]\hat{\mathbf{z}}$

Laplacian: $\quad \nabla^2 t = \dfrac{1}{s}\dfrac{\partial}{\partial s}\left(s\dfrac{\partial t}{\partial s}\right) + \dfrac{1}{s^2}\dfrac{\partial^2 t}{\partial \phi^2} + \dfrac{\partial^2 t}{\partial z^2}$

VECTOR IDENTITIES

Triple Products

(1) $\quad \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$

(2) $\quad \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$

Product Rules

(3) $\quad \nabla(fg) = f(\nabla g) + g(\nabla f)$

(4) $\quad \nabla(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A}$

(5) $\quad \nabla \cdot (f\mathbf{A}) = f(\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot (\nabla f)$

(6) $\quad \nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$

(7) $\quad \nabla \times (f\mathbf{A}) = f(\nabla \times \mathbf{A}) - \mathbf{A} \times (\nabla f)$

(8) $\quad \nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A})$

Second Derivatives

(9) $\quad \nabla \cdot (\nabla \times \mathbf{A}) = 0$

(10) $\quad \nabla \times (\nabla f) = 0$

(11) $\quad \nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A}$

FUNDAMENTAL THEOREMS

Gradient Theorem: $\quad \int_{\mathbf{a}}^{\mathbf{b}}(\nabla f) \cdot d\mathbf{l} = f(\mathbf{b}) - f(\mathbf{a})$

Divergence Theorem: $\quad \int(\nabla \cdot \mathbf{A})\,d\tau = \oint \mathbf{A} \cdot d\mathbf{a}$

Curl Theorem: $\quad \int(\nabla \times \mathbf{A}) \cdot d\mathbf{a} = \oint \mathbf{A} \cdot d\mathbf{l}$

FUNDAMENTAL CONSTANTS

$\epsilon_0 = 8.85 \times 10^{-12} \, \text{C}^2/\text{Nm}^2$ (permittivity of free space)

$\mu_0 = 4\pi \times 10^{-7} \, \text{N}/\text{A}^2$ (permeability of free space)

$c = 3.00 \times 10^8 \, \text{m}/s$ (speed of light)

$e = 1.60 \times 10^{-19} \, \text{C}$ (charge of the electron)

$m = 9.11 \times 10^{-31} \, \text{kg}$ (mass of the electron)

SPHERICAL AND CYLINDRICAL COORDINATES

Spherical

$$\begin{cases} x = r \sin\theta \cos\phi \\ y = r \sin\theta \sin\phi \\ z = r \cos\theta \end{cases}$$

$$\begin{cases} \hat{\mathbf{x}} = \sin\theta \cos\phi \, \hat{\mathbf{r}} + \cos\theta \cos\phi \, \hat{\boldsymbol{\theta}} - \sin\phi \, \hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin\theta \sin\phi \, \hat{\mathbf{r}} + \cos\theta \sin\phi \, \hat{\boldsymbol{\theta}} + \cos\phi \, \hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \cos\theta \, \hat{\mathbf{r}} - \sin\theta \, \hat{\boldsymbol{\theta}} \end{cases}$$

$$\begin{cases} r = \sqrt{x^2 + y^2 + z^2} \\ \theta = \tan^{-1}\left(\sqrt{x^2 + y^2}/z\right) \\ \phi = \tan^{-1}(y/x) \end{cases}$$

$$\begin{cases} \hat{\mathbf{r}} = \sin\theta \cos\phi \, \hat{\mathbf{x}} + \sin\theta \sin\phi \, \hat{\mathbf{y}} + \cos\theta \, \hat{\mathbf{z}} \\ \hat{\boldsymbol{\theta}} = \cos\theta \cos\phi \, \hat{\mathbf{x}} + \cos\theta \sin\phi \, \hat{\mathbf{y}} - \sin\theta \, \hat{\mathbf{z}} \\ \hat{\boldsymbol{\phi}} = -\sin\phi \, \hat{\mathbf{x}} + \cos\phi \, \hat{\mathbf{y}} \end{cases}$$

Cylindrical

$$\begin{cases} x = s \cos\phi \\ y = s \sin\phi \\ z = z \end{cases}$$

$$\begin{cases} \hat{\mathbf{x}} = \cos\phi \, \hat{\mathbf{s}} - \sin\phi \, \hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin\phi \, \hat{\mathbf{s}} + \cos\phi \, \hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}$$

$$\begin{cases} s = \sqrt{x^2 + y^2} \\ \phi = \tan^{-1}(y/x) \\ z = z \end{cases}$$

$$\begin{cases} \hat{\mathbf{s}} = \cos\phi \, \hat{\mathbf{x}} + \sin\phi \, \hat{\mathbf{y}} \\ \hat{\boldsymbol{\phi}} = -\sin\phi \, \hat{\mathbf{x}} + \cos\phi \, \hat{\mathbf{y}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}$$