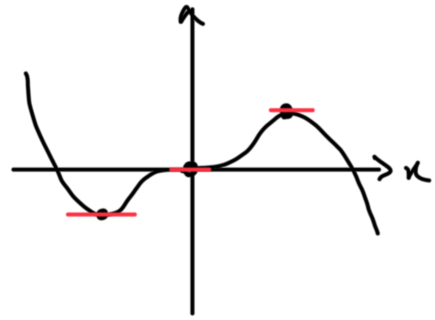
SIGNIFICANCE OF THE DERIVATIVE (Ch. 11)

Def Let $I \subseteq \mathbb{R}$ be an open interval, $f: I \to \mathbb{R}$, and $a \in I$. We say a is a <u>critical point</u> of f if $f'(a) = 0$.

Ex $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^3 - x^5$



$$0 = f'(a) = 3a^2 - 5a^4 = a^2(3 - 5a^2)$$

$$\iff a = 0 \quad \text{or} \quad a^2 = \frac{3}{5}$$

$$\iff a = 0 \quad \text{or} \quad a = \pm\sqrt{\frac{3}{5}}$$

3 critical points.

- What's special about f at these 3 points?
  E.g. f is biggest at $a = \sqrt{\frac{3}{5}}$, but it's not a global max...

Def Let $A \subseteq \mathbb{R}$, $f: A \to \mathbb{R}$, and $a \in A$. We say:

① $f(a)$ is a <u>local maximum</u> of f if $\exists \delta > 0$ s.t. $f(x) \le f(a)$ $\forall x \in A \cap (a - \delta, a + \delta)$

② $f(a)$ is a <u>local minimum</u> of f if $\exists \delta > 0$ s.t. $f(x) \ge f(a)$ $\forall x \in A \cap (a - \delta, a + \delta)$

Rmk Global max/min $\implies$ local max/min

Ex $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^3 - x^5$

$f(\sqrt{\frac{3}{5}})$ is a local max

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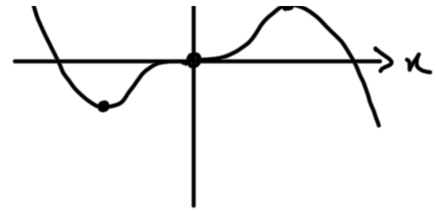


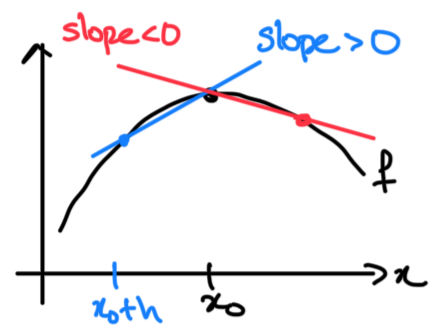
- $f(-\sqrt{\frac{3}{5}})$ is a local min
- $f(0)$ is neither
- f has no global max or min

__Thm__ Let $I \subseteq \mathbb{R}$ be an open interval and $x_0 \in I$. If $f: I \to \mathbb{R}$ is differentiable at x_0 and $f(x_0)$ is a local maximum or minimum, then x_0 is a critical point (i.e. $f'(x_0) = 0$).

__Rmk__ $f(x_0)$ local max/min $\underset{\not\Leftarrow}{\Rightarrow}$ $f'(x_0) = 0$

E.g. $x_0 = 0$ in previous example.

__Pf__: Case: Local max. Suppose $f(x_0)$ is a local max and f is differentiable at x_0. Then $\exists\, \delta > 0$ s.t. for $0 < |h| < \delta$,



$$f(x_0 + h) \le f(x_0)$$

$$\Rightarrow f(x_0+h) - f(x_0) \le 0$$

$$\Rightarrow \begin{cases} \dfrac{f(x_0+h) - f(x_0)}{h} \le 0 & \text{if } h > 0 \\[2ex] \dfrac{f(x_0+h) - f(x_0)}{h} \ge 0 & \text{if } h < 0 \end{cases}$$

As f is differentiable at x_0, we know

$$\lim_{h \to 0^+} \frac{f(x_0+h) - f(x_0)}{h} = f'(x_0) = \lim_{h \to 0^-} \frac{f(x_0+h) - f(x_0)}{h}.$$

Together,

$$f'(x_0) = \lim_{h\to 0^+} \underbrace{\frac{f(x_0+h)-f(x_0)}{h}}_{\leq 0 \ \forall h \in (0,\delta)} \leq 0$$

$$f'(x_0) = \lim_{h\to 0^-} \underbrace{\frac{f(x_0+h)-f(x_0)}{h}}_{\geq 0 \ \forall h \in (-\delta,0)} \geq 0$$

$\Rightarrow f'(x_0) = 0.$

(Recall: HW4#4

- $f(x) \leq g(x) \ \forall x$
- $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist

$\Rightarrow \lim_{x\to a} f(x) \leq \lim_{x\to a} g(x)$

Similar proof works for $\lim_{x\to a^+}$ and $\lim_{x\to a^-}$.)

Case: Local min. Then $-f$ has a local max. By the previous case we know $-f'(x_0)=0$, so $f'(x_0)=0$. $\square$

Q: How do we find the global max/min of $f:[a,b]\to\mathbb{R}$?

A: By the previous theorem, the global max/min $f(x_0)$ must fall into one of the following cases:

① Critical point: $x_0 \in (a,b)$ s.t. $f'(x_0)=0$

② Endpoint: $x_0 = a, b$

③ Points $x_0 \in (a,b)$ where f is not differentiable.

Ex Find all global extrema of $f:[-2,3]\to\mathbb{R}$, $f(x) = 2x^3 - 3x^2 - 12x + 1$.

Pf: By the EVT, we know $\exists\, x_0, x_1 \in [-2,3]$ s.t. $f(x_0)$ is a global min and $f(x_1)$ is a global max.

Case ①: $x_0, x_1 \in (-2,3)$. As f is a polynomial, we know f is differentiable at x_0, x_1. So $f'(x_0)=0$ and $f'(x_1)=0$ by the previous theorem.

$$0 = f'(x) = 6x^2 - 6x - 12 = 6(x^2 - x - 2) = 6(x-2)(x+1)$$

$$\Rightarrow \quad x=-1 \quad \text{or} \quad x=2$$

$$\Rightarrow \quad f(x) = f(-1) = 8 \quad \text{or} \quad f(x) = f(2) = -19$$

Case ②: $x_0, x_1 \in \{-2, 3\}$. Note that

$$f(-2) = -3, \qquad f(3) = -8$$

- No case ③, since f is differentiable on $(-2,3)$

Altogether, we have shown

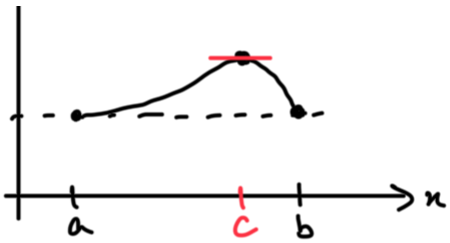
$$x_0, x_1 \in \{-2, -1, 2, 3\}$$

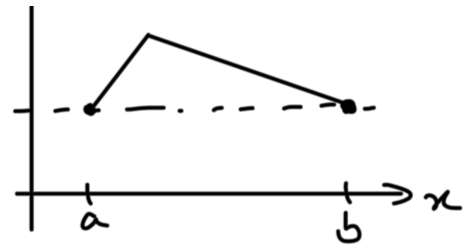
$$f(x) = -3,\ \underline{8},\ \underline{-19},\ -8$$

So the global max is $8 = f(-1)$ and the global min is $-19 = f(2)$. $\square$

<u>Thm</u> (Rolle's theorem) If $f:[a,b] \to \mathbb{R}$ is continuous, f is differentiable on (a,b), and $f(a) = f(b)$, then $\exists\, c \in (a,b)$ s.t. $f'(c) = 0$.

$\uparrow$ $\uparrow$





- In order to return to $f(b) = f(a)$, a differentiable function must have at least one critical point
- Not differentiable, no critical points

<u>Pf</u>: As f is continuous on $[a,b]$, then by the EVT $\exists\, x_0, x_1 \in [a,b]$ s.t.

$$f(x_0) \le f(x) \le f(x_1) \quad \forall x \in [a,b].$$

Case: $x_1 \in (a,b)$. As f is differentiable at x_1, then $f'(x_1) = 0$ by the previous theorem. So $c = x_1$ works.

Case: $x_0 \in (a,b)$. Then $c = x_0$ works.

Case: $x_0, x_1 \in \{a,b\}$. As $f(a) = f(b)$, then we have

$$f(a) \le f(x) \le f(a) \quad \forall x \in [a,b]$$

$$\Rightarrow \quad f(x) = f(a) \quad \forall x \in [a,b]$$

So f is a constant function. Therefore $f'(x) = 0$ $\forall x \in (a,b)$.

In all cases, we found a number $c \in (a,b)$ s.t. $f'(c) = 0$. $\square$