$\boxed{1}$

Problem 1. For each of the following functions $f : (0, \infty) \to \mathbb{R}$, prove that f is differentiable at any point $a > 0$ and find $f'(a)$.

(a) $f(x) = \frac{1}{x}$
(b) $f(x) = \sqrt{x}$

(a) Claim: $f'(a) = -\frac{1}{a^2}$

Pf: for $h \neq 0$:

$$\frac{f(a+h) - f(a)}{h} = \frac{\frac{1}{a+h} - \frac{1}{a}}{h} = \frac{a-a-h}{a(a+h)h} = \frac{-1}{a(a+h)} = \frac{-1}{a} \cdot \frac{1}{a+h}$$

- Using limit law of multiplication,

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \left(-\frac{1}{a} \cdot \frac{1}{a+h}\right) \overset{!}{=} \lim_{h \to 0} \left(-\frac{1}{a}\right) \cdot \lim_{h \to 0} \frac{1}{a+h}$$

$$\overset{!}{=} -\frac{1}{a} \cdot \frac{1}{a} = -\frac{1}{a^2}$$

$\square$

(b) Claim: $f'(a) = \frac{1}{2} \frac{1}{\sqrt{a}}$

Pf: for $h \neq 0$:

$$\frac{f(a+h) - f(a)}{h} = \frac{\sqrt{a+h} - \sqrt{a}}{h} \overset{!}{=} \frac{(\sqrt{a+h} - \sqrt{a})(\sqrt{a+h} + \sqrt{a})}{h(\sqrt{a+h} + \sqrt{a})}$$

$$\overset{!}{=} \frac{a+h-a}{h(\sqrt{a+h} + \sqrt{a})} = \frac{1}{\sqrt{a+h} + \sqrt{a}}$$

$$\Rightarrow \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{1}{\sqrt{a+h} + \sqrt{a}} = \frac{1}{\sqrt{a} + \sqrt{a}} = \frac{1}{2\sqrt{a}}$$

as wanted

$\square$

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Problem 2. Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \max\{0, x\}$. For each $a \in \mathbb{R}$, determine if f is differentiable at a and prove your answer.

$$f(x) = \begin{cases} 0, & x \leq 0 \\ x, & x > 0 \end{cases}$$

① $\underline{a<0}$ $\quad f'(a) = \lim\limits_{h\to 0} \dfrac{f(a+h)-f(a)}{h} = \lim\limits_{h\to 0} \dfrac{0}{h} = 0$

② $\underline{a>0}$ $\quad f'(a) = \lim\limits_{h\to 0} \dfrac{f(a+h)-f(a)}{h} = \lim\limits_{h\to 0} \dfrac{a+h-a}{h} = 1$

③ $\underline{a=0}$ $\quad f(x)$ changes definition, so need to examine if

$$f'(0^-) = f'(0^+)$$

- $f'(0^-) = \lim\limits_{h\to 0^-} \dfrac{f(0+h)-f(0)}{h} = \lim\limits_{h\to 0^-} \dfrac{0}{h} = 0$
- $f'(0^+) = \lim\limits_{h\to 0^+} \dfrac{f(0+h)-f(0)}{h} = \lim\limits_{h\to 0^+} \dfrac{0+h-0}{h} = 1$

$\Rightarrow f'(0^-) \neq f'(0^+)$.

$\Rightarrow f$ is differentiable for $a>0$, $a<0$.

but <u>not</u> at $a = 0$. $\square$

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Problem 3. Let $f : \mathbb{R} \to \mathbb{R}$ be a function, and suppose that f is differentiable at a for any $a \in \mathbb{R}$.

(a) Prove that for any constant $c \in \mathbb{R}$, the function $g : \mathbb{R} \to \mathbb{R}$, $g(x) = f(x) + c$ is differentiable at any $a \in \mathbb{R}$ with $g'(a) = f'(a)$.
(b) Prove that for any constant $c \in \mathbb{R}$, the function $g : \mathbb{R} \to \mathbb{R}$, $g(x) = f(x + c)$ is differentiable at any $a \in \mathbb{R}$ with $g'(a) = f'(a + c)$.

(a) By differential law for addition (Lec)

$$g'(a) = f'(a) + c'$$

while $c' = \lim_{h \to 0} \frac{c - c}{h} = 0$

$$\Rightarrow g'(a) = f'(a).$$

(b). By chain rule (Lec)

$$g'(a) = f'(a+c) \cdot x'\big|_{x=a}$$

while, $x'\big|_{x=a} = \lim_{h \to 0} \frac{a+h-a}{h} = 1$

$$\Rightarrow g'(a) = f'(a+c).$$ as wanted.

$\square$

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Problem 4. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a function that satisfies $f(0) = 0$ and $f'(0) = 0$. Define the function $g : \mathbb{R} \to \mathbb{R}$ by

$$g(x) = \begin{cases} f(x) \cdot \sin \frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Prove that g is differentiable at 0 and $g'(0) = 0$.

for $h \neq 0$. $$\frac{g(0+h) - g(0)}{h} = \frac{f(h) \sin \frac{1}{h}}{h} \qquad (*)$$

given that: $\begin{cases} f(0) = 0 \\ f'(0) = 0 \end{cases} \Rightarrow \lim\limits_{h \to 0} \frac{f(h) - f(0)}{h} = \lim\limits_{h \to 0} \frac{f(h)}{h} = 0$

So, combined with limit law for multiplication,

$$(*) \Rightarrow \lim_{h \to 0} \frac{g(0+h) - g(0)}{h} = \lim_{h \to 0} \frac{f(h)}{h} \cdot \lim_{h \to 0} \sin \frac{1}{h} = 0.$$

i.e. $g'(0) = 0$ as wanted $\square$

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Problem 5. Prove that the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = |x|^3$ is twice differentiable at any point $a \in \mathbb{R}$, but is not three-times differentiable at 0.

$$f(x) = \begin{cases} x^3 & x > 0 \\ -x^3 & x < 0 \end{cases}$$

① $a > 0$: for $h \neq 0$ $\dfrac{f(a+h) - f(a)}{h} = \dfrac{(a+h)^3 - a^3}{h} = 3a^2 + 3ah + h^2$

so $f'(a) = \lim\limits_{h \to 0} 3a^2 + 3ah + h^2 = 3a^2$

$f''(a) = \lim\limits_{h \to 0} \dfrac{3(a+h)^2 - 3a^2}{h} = \lim\limits_{h \to 0} 6a + 3h = 6a$

$\Rightarrow$ twice differentiable for $a > 0$

② $a < 0$: for $h \neq 0$:

$f'(a) = \lim\limits_{h \to 0} \dfrac{-(a+h)^3 - (-a^3)}{h} = \lim\limits_{h \to 0} \dfrac{-3ah^2 - 3a^2h - h^3}{h} = -3a^2$

$f''(a) = \lim\limits_{h \to 0} \dfrac{-3(a+h)^2 + 3a^2}{h} = \lim\limits_{h \to 0} \dfrac{-6ah - 3h^2}{h} = -6a$

$\Rightarrow$ twice differentiable for $a < 0$

③ $a = 0$: for $h \neq 0$:

- $\dfrac{f(0+h) - f(0)}{h} = \begin{cases} \dfrac{h^3 - 0}{h} = h^2 & h > 0 \\ \dfrac{-h^3 - 0}{h} = -h^2 & h < 0 \end{cases}$

$\Rightarrow f'(0) = \lim\limits_{h \to 0} \pm h^2 = 0$

- $\dfrac{f'(0+h) - f'(0)}{h} = 0 \Rightarrow f''(0) = 0$

$\Rightarrow$ twice differentiable at $a = 0$

$\Rightarrow \boxed{f''(a) = |6a| \text{ for } a \in \mathbb{R}}$

* But we saw from lec that $g'(0)$ DNE for $g(x) = |x|$

so, by differentiation Law, $f'''(0) = 6 \cdot g'(0)$, DNE $\square$