Physics 322
Discussion 3
09/23/2024

Electrostatics $\Rightarrow \begin{cases} \nabla\cdot\vec{E}=\rho/\epsilon_0 \Rightarrow \nabla^2 V = -\rho/\epsilon_0 \\ \nabla\times\vec{E}=0 \Rightarrow \vec{E}(\vec{r}) = -\nabla V(\vec{r}) \end{cases}$

Potential: $\vec{E}(\vec{r})=-\nabla V(\vec{r}) \Rightarrow V(\vec{r}_a)-V(\vec{r}_b) = -\int_b^a \vec{E}\cdot d\vec{l}$

↳ The zero of the potential is arbitrary. Only ΔV is physical!

↳ Path independent for $\nabla\times\vec{E}=0$. Pick a path with $d\vec{l}\parallel\vec{E}$!

Localized charge distribution $\Rightarrow$ We can pick $V(\infty)=0 \Rightarrow V(\vec{r}) = -\int_\infty^r \vec{E}\cdot d\vec{l} \Rightarrow \begin{cases} V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\sum_j \frac{q_j}{\mathscr{r}_j} & \text{(Discrete)} \\ V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{dq}{\mathscr{r}} & \text{(Continuous)} \end{cases}$

Boundary Conditions: $\begin{cases} \oint_S \vec{E}\cdot d\vec{a} = \frac{Q_{enc}}{\epsilon_0} \Rightarrow (\vec{E}_{out}-\vec{E}_{in})\cdot\hat{n} = \sigma/\epsilon_0 \text{ or } E^{\perp}_{out}-E^{\perp}_{in} = \sigma/\epsilon_0 \text{ or } \left.\frac{\partial V}{\partial n}\right|_{out} - \left.\frac{\partial V}{\partial n}\right|_{in} = -\sigma/\epsilon_0 \\ \oint \vec{E}\cdot d\vec{l} = 0 \Rightarrow (\vec{E}_{out}-\vec{E}_{in})\cdot\vec{t} = 0 \text{ or } \vec{E}^{\parallel}_{out} = \vec{E}^{\parallel}_{in} \text{ and } V_{out}=V_{in} \end{cases}$

↳ normal pointing from "in" to "out"

↳ tangent vector

Force on surface charge: $\vec{f} = \frac{1}{2}(\vec{E}_{above}+\vec{E}_{below})\sigma$

↳ Force/Area

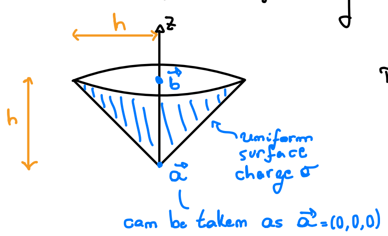
Potential energy in electrostatics: $\begin{cases} U = \frac{1}{2}\sum_i q_i V(\vec{r}_i) & \text{(Discrete)} \\ U = \frac{1}{2}\int \rho(\vec{r})V(\vec{r})\,d\tau \text{ (Continuous) or } U = \int_{\text{All space}} \frac{|\vec{E}|^2\epsilon_0}{2}\,d\tau \end{cases}$

↳ Potential due to all charges but q_i

Conductors: $\begin{cases} \vec{E}=0 \text{ inside it} \\ V \text{ constant inside it} \\ \sigma = -\epsilon_0 \left.\frac{\partial V}{\partial n}\right|_{out} \text{ and } \rho=0 \text{ inside it} \\ E^{\perp}_{out} = \sigma/\epsilon_0 \text{ and } E^{\parallel}_{out}=0 \\ \vec{f} = \frac{\sigma^2}{2\epsilon_0}\hat{n} \text{ and } P = \frac{\sigma^2}{2\epsilon_0} \end{cases}$

Problem 1

Find $V(\vec{a})-V(\vec{b})$ for the following distribution:



The surface can be parametrized as $\vec{S}(r,\phi) = (r\cos\phi, r\sin\phi, r)$ w/ $0<r<h$ & $0<\theta<2\pi$

$$\frac{\partial\vec{S}}{\partial r} = (\cos\phi,\sin\phi,1) \ \& \ \frac{\partial\vec{S}}{\partial\phi} = r(-\sin\phi,\cos\phi,0) \Rightarrow \frac{\partial\vec{S}}{\partial r}\times\frac{\partial\vec{S}}{\partial\phi} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \cos\phi & \sin\phi & 1 \\ -r\sin\phi & r\cos\phi & 0 \end{vmatrix} = (-r\cos\phi,-r\sin\phi,r)$$

$$da = |(-r\cos\phi,-r\sin\phi,r)|\,dr\,d\phi = \sqrt{2}\,r\,dr\,d\phi$$

Then $\vec{r}_a=(0,0,0)$, $\vec{r}_b=(0,0,h)$, $\vec{r}'=(r\cos\phi,r\sin\phi,r) \Rightarrow \vec{\mathscr{r}}_a = -\vec{r}' \Rightarrow \mathscr{r}_a = \sqrt{2}\,r$ and $\vec{\mathscr{r}}_b = (-r\cos\phi,-r\sin\phi,h-r) \Rightarrow \mathscr{r}_b = (r^2+(h-r)^2)^{1/2}$. Thus,

$$V(\vec{r}_a) = \frac{1}{4\pi\epsilon_0}\int_0^{2\pi}\int_0^h \frac{\sigma\sqrt{2}\,r\,dr\,d\phi}{\sqrt{2}\,r} \Rightarrow V(\vec{r}_a) = \frac{\sigma h}{2\epsilon_0}$$

and

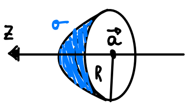
$$V(\vec{b}) = \frac{1}{4\pi\epsilon_0}\int_0^{2\pi}\int_0^h \frac{\sigma\sqrt{2}\,r\,dr\,d\phi}{(r^2+(h-r)^2)^{1/2}} = \frac{\sqrt{2}\,\sigma}{2\epsilon_0}\int_0^h \frac{r\,dr}{(r^2+(h-r)^2)^{1/2}} \overset{x=r/h}{=} \frac{\sigma h}{\sqrt{2}\,\epsilon_0}\underbrace{\int_0^1 \frac{x\,dx}{(x^2+(1-x)^2)^{1/2}}}_{\text{Dimensionless integral - Always good to do}} = \frac{\sigma h}{\sqrt{2}\,\epsilon_0}\frac{1}{\sqrt{2}}\left(\ln(1+\sqrt{2})\right)$$

$$\Rightarrow V(\vec{r}_b)-V(\vec{r}_a) = \frac{\sigma h}{2\epsilon_0}\left[1-\ln(1+\sqrt{2})\right]$$

Problem 2

A half-sphere shell of radius R has a surface charge σ.

(a) Determine $V(\vec{a})$ ($V(\infty)=0$)



$dq = \sigma\,da = \sigma R^2\,d\Omega$, $\vec{r}=(0,0,0)$ and $\vec{r}' = R\hat{r}' \Rightarrow \mathscr{r}=R$

$d\Omega$: $\sin\theta\,d\theta\,d\phi$

$$V(\vec{a}) = \frac{1}{4\pi\epsilon_0}\int \frac{\sigma R^2\,d\Omega}{R} = \frac{\sigma R}{4\pi\epsilon_0}\underbrace{\int d\Omega}_{2\pi \text{ (half of a full sphere)}} = \frac{\sigma R}{2\epsilon_0}$$

(b) A particle of mass m and charge q ($q>0$) is placed at rest at $\vec{a}$. Determine the speed of the particle after it moves far away from $\vec{a}$.

The initial energy is $E_i = qV(\vec{a}) = \frac{q\sigma R}{2\epsilon_0}$. The final energy is $E_f = \frac{mV_\infty^2}{2}$. By conservation of energy

$$E_f = E_i \Rightarrow V_\infty = \sqrt{\frac{q\sigma R}{m\epsilon_0}}$$

Problem 3.

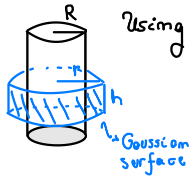
Consider a conducting cylinder of radius R and uniformally distributed charge Q.

$\hookrightarrow$ Length $L \gg R$

(a) $\vec{E}(\vec{r})$ for $r<R$?

The cylinder is a conductor $\Rightarrow \vec{E}=0$ for $r<R$

(b) $\vec{E}(\vec{r})$ for $r>R$?



Using Gauss law,

$$\oint_S \vec{E}\cdot d\vec{a} = \frac{Q_{enc}}{\epsilon_0} \Rightarrow E(r)\,2\pi r\cdot h = \frac{\underbrace{\lambda}_{Q/L} h}{\epsilon_0} \Rightarrow \vec{E}(r) = \frac{\lambda}{2\pi\epsilon_0 r}\hat{r} \text{ for } r>R$$

(c) Calculate $V(r)$ everywhere (letting $V(R)=0$)

In the $L\to\infty$ limit this is not a localized distribution, so we must use $V(\vec{r}_a)-V(\vec{r}_b) = -\int_b^a \vec{E}\cdot d\vec{l}$. Then,

$$V(r)-\cancelto{0}{V(R)} = -\int_R^r \left(\frac{\lambda}{2\pi\epsilon_0 r'}\right)\hat{r}'\cdot\hat{r}'dr' = -\frac{\lambda}{2\pi\epsilon_0}\int_R^r \frac{dr'}{r'} = -\frac{\lambda}{2\pi\epsilon_0}\ln\left(\frac{r}{R}\right) \text{ for } r>R$$

Since a conductor is an equipotential $V(R)=0 \Rightarrow V(r)=0$ for $<R$. Then,

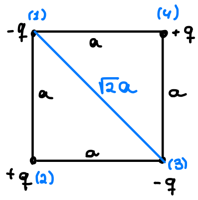
$$V(r) = \begin{cases} 0, & \text{for } r\leq R \\ -\frac{\lambda}{2\pi\epsilon_0}\ln\left(\frac{r}{R}\right), & \text{for } r\geq R \end{cases}$$

(d) Calculate σ using the boundary conditions.

$$\sigma = -\epsilon_0 \frac{\partial V}{\partial n}\Big|_{out} = -\epsilon_0 \frac{\partial}{\partial r}\left(-\frac{\lambda}{2\pi\epsilon_0}\ln\left(\frac{r}{R}\right)\right)\Big|_{r=R} = \frac{\lambda}{2\pi R} \Rightarrow \sigma = \frac{Q}{2\pi R\cdot L}$$ $\rightsquigarrow$ Makes sense!

Problem 4

Calculate the work to assemble the following configuration

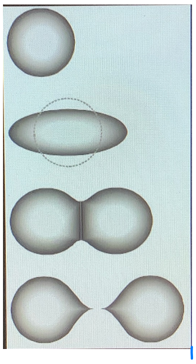


$$W = \frac{1}{2}\sum_{i=1,2,3,4} q_i V(\vec{r}_i) = \frac{1}{4\pi\epsilon_0}\frac{1}{\cancel{2}}\Big\{ \overset{\text{by Symmetry}}{\cancel{2}}(-q)\underbrace{\left[\frac{q}{a}+\frac{q}{a}-\frac{q}{\sqrt{2}a}\right]}_{\frac{q}{a}\left(2-\frac{1}{\sqrt{2}}\right)} + \cancel{2}(q)\overbrace{\left[-\frac{q}{a}-\frac{q}{a}+\frac{q}{\sqrt{2}a}\right]}^{-\frac{q}{a}\left(2-\frac{1}{\sqrt{2}}\right)}\Big\} = \frac{1}{4\pi\epsilon_0}\frac{2q^2}{a}\left(-2+\frac{1}{\sqrt{2}}\right)$$

Problem 5

A liquid droplet of radius R with a uniformally distributed charge Q is divided into two of same radius and charge

(a) What is the energy released in this process?



From HW 2 we know that $U_i = \frac{3}{5}\frac{Q^2}{4\pi\epsilon_0 R}$. We expect each droplet to have half the original charge and the volume to be conserved:

$$V_f = V_i \Rightarrow 2R_f^3 = R^3 \Rightarrow R_f = 2^{-1/3}R. \text{ Hence, } U_f = 2\,\frac{3}{5}\frac{(Q/2)^2}{4\pi\epsilon_0 2^{-1/3}R} = 2^{1-2+1/3}U_i = 2^{-2/3}U_i \Rightarrow \Delta U = U_i - U_f = (1-2^{-2/3})\frac{3}{5}\frac{Q^2}{4\pi\epsilon_0 R}$$

(b) We can use this to model U^{235} fission (Toy model of the Bohr-Wheeler liquid drop model). Assuming U^{235} could fission in this way, what would be the released energy in MeV? The average radius of a nucleus can be approximated as $R \approx 1.3\cdot A^{1/3}F$ with $1F(\text{fermi}) = 10^{-13}$cm and A the mass number (n° of protons + n° of neutrons)

$\hookrightarrow$ Densely packed sphere of nucleons approx.

U^{235}_{92} has $Q = \underbrace{Z}_{92}e$ and $A=235$. Then,

$$\Delta U = (1-2^{-2/3})\frac{3}{5}\frac{Q^2}{4\pi\epsilon_0 R} = (1-2^{-2/3})\frac{3}{5}Z^2\underbrace{\left(\frac{e^2}{4\pi\epsilon_0\hbar c}\right)}_{=\alpha \text{ (fine structure constant)}}\left(\frac{\hbar c}{R}\right) = (1-2^{-2/3})\frac{3}{5}Z^2\alpha\left(\frac{200\,MeV\cdot 10^{-15}m}{1.3\cdot(235)^{1/3}10^{-15}m}\right) = \frac{3}{5}(1-2^{-2/3})Z^2\alpha\left(\frac{200}{1.3(235)^{1/3}}\right)$$

$\Rightarrow \Delta U \approx 342\,MeV$ $\rightsquigarrow$ The actual value is closer to 203MeV That is not too far off given the simplicity of this toy model!

Problem 6

Calculate the repulsive force between two hemisheres of a metal sphere of radius R and charge Q.

In this case, let's calculate the force on the upper hemisphere

$$\vec{f} = \frac{\sigma^2}{2\epsilon_0}\hat{r} \text{ with } \sigma = \frac{Q}{4\pi R^2} \Rightarrow \vec{F} = \int \vec{f}\, da = \int_0^{\pi/2} d\theta \sin\theta \int_0^{2\pi} d\phi \frac{\sigma^2}{2\epsilon_0} R^2 (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) = \frac{\sigma^2}{2\epsilon_0} 2\pi R^2 \underline{\int_0^{\pi/2} d\theta \sin\theta \cos\theta}\, \hat{z} = \frac{\pi R^2}{2\epsilon_0} \frac{Q^2}{16\pi^2 R^4} \hat{z}$$

$$\frac{1}{2}\int_0^{\pi/2} d\theta \sin(2\theta) = \frac{1}{4}(-\cos(2\theta))\Big|_0^{\pi/2} = -\frac{1}{4}\left(-1-1\right) = \frac{1}{2}$$

$\Rightarrow$ There is a repulsive force of $\frac{Q^2}{32\pi R^2 \epsilon_0}$ between the two hemispheres