Chapter 1
Introduction and Overview

The main topic of this book is *partial differential equations (PDEs)*, and a good starting point is a comparison with *ordinary differential equations (ODEs)*. You should already be familiar with ODEs; some examples are

$$\frac{\mathrm{d}u}{\mathrm{d}x} = u, \quad \frac{\mathrm{d}^2u}{\mathrm{d}x^2} = u, \quad \text{and} \quad \left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2 = u. \tag{1.1}$$

The common feature of these ODEs is that they involve *ordinary derivatives*, such as the first derivative $\mathrm{d}u/\mathrm{d}x$ and the second derivative $\mathrm{d}^2u/\mathrm{d}x^2$. The function of interest is $u(x)$ and it depends on *one variable*, x. In contrast, PDEs involve *partial derivatives* of functions of several variables, such as $u(x,t)$ which depends on the two variables x and t.

In remainder of this chapter, the goal is to give an overview of the topics of this book. As it is an *overview*, this chapter will often focus on *big-picture ideas* rather than details.

1.1 Example PDEs and Solutions

To begin our exploration of PDEs, it will be helpful to consider some example PDEs and solutions. The following are some examples of PDEs:

$$\partial_t u + \partial_x u = 0 \qquad \text{(advection equation)} \tag{1.2}$$
$$\partial_t u + u\partial_x u = 0 \qquad \text{(Burgers' equation, inviscid)} \tag{1.3}$$
$$\partial_t u = \partial_x^2 u \qquad \text{(heat equation or diffusion equation)} \tag{1.4}$$
$$\partial_x^2 u + \partial_y^2 u = 0 \qquad \text{(Laplace's equation)} \tag{1.5}$$
$$\partial_t^2 u - \partial_x^2 u = 0 \qquad \text{(wave equation)} \tag{1.6}$$
$$\partial_t u = \partial_x^2 u + u^3 \qquad \text{(reaction–diffusion equation)} \tag{1.7}$$

In writing these PDEs, the notation $\partial_t u$ has been used for the partial derivative of $u(x,t)$ with respect to t, and similarly for $\partial_x u$. Alternative notation will sometimes be used for partial derivatives, so that (1.2) could be written instead as

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0 \quad \text{or} \quad u_t + u_x = 0. \tag{1.2$'$}$$

For a second derivative, the notation could be, for instance, $\partial_x^2 u$, $\partial^2 u/\partial x^2$, or u_{xx}.

The examples in (1.2)–(1.7) illustrate some of the variety of different types of PDEs. For example, (1.2)–(1.3) are first-order PDEs, and the rest are second-order PDEs, where this classification is based on the highest-order derivative that appears in the PDE. Two of them—(1.3) and (1.7)—are nonlinear PDEs, and the rest are linear PDEs, as will be discussed further below, in Sec. 1.3. Notice that some of these PDEs look very similar, such as the heat equation and wave equation, which differ by only a change from ∂_t to ∂_t^2; while these differences may appear small, they are very important and can fundamentally change the nature of the solutions.

Next, let's consider some example *solutions* to PDEs. For the moment, we are not concerned with *how* the solution was found. Later we will address the important topic of general methods for *finding solutions* to PDEs. Our first objective is more basic: you should understand how to *verify that a function is a solution to a PDE*.

Example 1.1. Verify that

$$u(x,t) = e^{-(x-t)^2} \tag{1.8}$$

is a solution to the advection equation,

$$u_t + u_x = 0. \tag{1.9}$$

To do so, calculate the derivatives that appear in the PDE:

$$u_t = 2(x-t)e^{-(x-t)^2}, \tag{1.10}$$

$$u_x = -2(x-t)e^{-(x-t)^2}. \tag{1.11}$$

Adding together these formulas for u_t and u_x, we see that $u_t + u_x = 0$, so (1.8) is indeed a solution to the PDE.

Example 1.2. Verify that, in fact, any function of the form

$$u(x,t) = g(x-t), \tag{1.12}$$

for any function g, is a solution to the advection equation,

$$u_t + u_x = 0. \tag{1.13}$$

To verify, calculate the derivatives that appear in the PDE:

$$u_t = -g'(x-t), \tag{1.14}$$
$$u_x = g'(x-t), \tag{1.15}$$

where $g'(y)$ is the derivative of $g(y)$. Adding together these formulas for u_t and u_x, we see that $u_t + u_x = 0$, so (1.12) is indeed a solution to the PDE.