Physics 322

Discussion 4

09/30/2024

Electrostatics: $\nabla\times\vec{E}=0$ & $\nabla\cdot\vec{E}=0 \Rightarrow \nabla^2 V=-\rho/\epsilon_0$ Poisson's Equation

Theorem: $\nabla^2 V=-\rho/\epsilon_0$ has a unique solution in $\mathcal{V}$ if:

(i) $\rho(\vec{x})\ \forall \vec{x}\in\mathcal{V}$ is known

(ii) $V(\vec{x})\ \forall \vec{x}\in\partial\mathcal{V}$ is known ∎

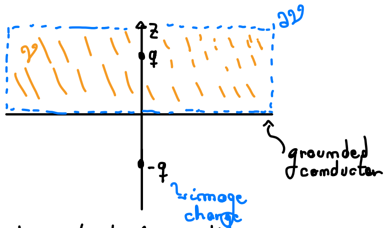
The image method utilizes this theorem: Given $\rho(\vec{x})$ we can calculate $V_\rho(\vec{x})$ by the methods we have learned thus far. This will satisfy $\nabla^2 V_\rho=-\rho/\epsilon_0$ but not necessarily (ii). We can then add a set of image charges outside $\mathcal{V}$ such that $V(\vec{x})=V_\rho(\vec{x})+V_{\rho_{im}}(\vec{x})$ satisfies (ii) and, since $\rho_{im}$ is outside $\mathcal{V}$ $\nabla^2 V=\nabla^2 V_\rho+\cancel{\nabla^2 V_{\rho_{im}}}^{0\text{ in }\mathcal{V}}=-\rho/\epsilon_0$ and $V(\vec{x})=V_\rho(\vec{x})+V_{\rho_{im}}(\vec{x})$ is the desired solution.

## Problem 1

A charge $q$ is a distance $z_q$ above a infinite grounded conducting plane (located at $z=0$). Determine

(a) The potential and electric field everywhere.

We can take $\mathcal{V}=\{\vec{x}=(x,y,z)\,|\,z\geq 0\}\Rightarrow\partial\mathcal{V}$ is the $z=0$ plane plus the region $|\vec{x}|\to\infty$ with $z\geq 0$. We have that:




$V(x,y,0)=0$ ↪ grounded conductor and $V(|\vec{x}|\to\infty \text{ with } z\geq 0)\to 0$ ↪ Potential far away from localized charges

$$V_q=\frac{1}{4\pi\epsilon_0}\frac{q}{[x^2+y^2+(z-z_q)^2]^{1/2}} \quad\text{and}\quad V_{q_{im}}=\frac{1}{4\pi\epsilon_0}\frac{-q}{[x^2+y^2+(z+z_q)^2]^{1/2}}$$

The potential is then

$$V(x,y,z)=\frac{q}{4\pi\epsilon_0}\left\{\frac{1}{[x^2+y^2+(z-z_q)^2]^{1/2}}-\frac{1}{[x^2+y^2+(z+z_q)^2]^{1/2}}\right\} \text{ for } z\geq 0 \Rightarrow V(x,y,0)=0 \ \ \&\ \ V(x,y,z)\to 0 \text{ for } |\vec{x}|\to\infty$$

The potential everywhere is

$$V(x,y,z)=\begin{cases}\dfrac{q}{4\pi\epsilon_0}\left\{\dfrac{1}{[x^2+y^2+(z-z_q)^2]^{1/2}}-\dfrac{1}{[x^2+y^2+(z+z_q)^2]^{1/2}}\right\}, & \text{for } z\geq 0\\ 0 & \text{for } z\leq 0.\end{cases} \Rightarrow \vec{E}(x,y,z)=\begin{cases}\overbrace{\dfrac{q}{4\pi\epsilon_0}\left\{\dfrac{(x,y,z-z_q)}{[x^2+y^2+(z-z_q)^2]^{3/2}}-\dfrac{(x,y,z+z_q)}{[x^2+y^2+(z+z_q)^2]^{3/2}}\right\}}^{=\vec{E}_q+\vec{E}_{q_{im}}} & \text{for } z\geq 0\\ 0 & \text{for } z\leq 0\end{cases}$$

(b) The induced surface charge and the total charge at the conductor

$$\sigma=-\epsilon_0\frac{\partial V}{\partial z}\bigg|_{z=0^+}=\frac{q}{4\pi}\left\{\frac{-z_q}{[x^2+y^2+z_q^2]^{3/2}}-\frac{z_q}{[x^2+y^2+z_q^2]^{3/2}}\right\}\Rightarrow\sigma(r)=-\left(\frac{q}{2\pi}\right)\frac{z_q}{(z_q^2+r^2)^{3/2}}\Rightarrow Q_{induced}=\int \underbrace{dq}_{\sigma da}=\int_0^\infty dr\,r\int_0^{2\pi}d\phi\left[-\left(\frac{q}{2\pi}\right)\frac{z_q}{(z_q^2+r^2)^{3/2}}\right]=-q\int_0^\infty dr\frac{r}{(1+r^2)^{3/2}}=-q\left[-(1+r^2)^{-1/2}\right]_0^\infty=-q$$

Makes Sense

(c) Calculate the force of the plane on the charge ($\vec{F}_q$) and the force of the charge on the plane ($\vec{F}_{Plane}$)

$$\vec{F}_q=q\vec{E}_{q_{im}}(0,0,z_q)=q\cdot\left(\frac{-q}{4\pi\epsilon_0}\right)\frac{2z_q}{(2z_q)^3}\hat{z}=\frac{-q^2}{16\pi\epsilon_0 z_q^2}\hat{z} \quad\text{and}\quad \vec{F}_{Plane}=\int_0^\infty dr\,r\int_0^{2\pi}d\phi\frac{\sigma^2}{2\epsilon_0}\hat{z}=+\frac{q^2\hat{z}}{4\pi\epsilon_0}\int_0^\infty dr\,r\frac{z_q^2}{(z_q^2+r^2)^3}=\frac{q^2\hat{z}}{4\pi\epsilon_0 z_q^2}\int_0^\infty dr\frac{r}{(1+r^2)^3}=\frac{q^2}{16\pi\epsilon_0 z_q^2}\hat{z}$$

$\Rightarrow \vec{F}_q+\vec{F}_{Plane}=0$ ↪ O.K. (Newton's third law)

(d) If the charge $q$ has mass $m$ and is released from rest a distance $d$ above the plane how long does it take to reach the plane?

$$\vec{F}_q=m\ddot{z}_q\hat{z}=\frac{-q^2}{16\pi\epsilon_0 z_q^2}\hat{z}\Rightarrow\ddot{z}_q=-Az_q^{-2} \text{ with } A=\frac{q^2}{16\pi\epsilon_0 m}\Rightarrow\frac{dv_q}{dt}=-Az_q^{-2}\Rightarrow v_q\,dv_q=-Az_q^{-2}v_q\,dt=-Az_q^{-2}dz_q\Rightarrow\frac{v_q^2}{2}(z_q)=\frac{A}{z_q}+C$$

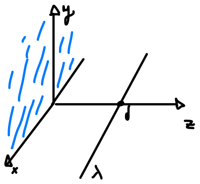
↪ Pick the "−" for the particle is moving down

$$v_q(z_q)=\pm\sqrt{2A}\sqrt{\frac{1}{z_q}-\frac{1}{d}}\Rightarrow\sqrt{\frac{z_q}{d-z_q}}\,dz_q=-\sqrt{\frac{2A}{d}}\,dt\Rightarrow\int_d^0\sqrt{\frac{z_q}{d-z_q}}\,dz_q=-\sqrt{\frac{2A}{d}}\int_0^t dt'\Rightarrow d\int_1^0\sqrt{\frac{x}{1-x}}\,dx=-\sqrt{\frac{2A}{d}}\,t\Rightarrow t=\sqrt{\frac{2\pi^3 d^3\epsilon_0 m}{q^2}}$$

## Problem 2

A uniform line charge $\lambda$ is placed on an infinite straight wire, a distance $d$ above a grounded conducting plane.

(a) Find the potential everywhere




We can take $\overset{\text{This is essencially a 2d problem}}{\mathcal{V}=\{\vec{x}=(y,z)\mid z\geq 0\}}\Rightarrow \partial\mathcal{V}$ is the $z=0$ point plus the $\overset{z\gg d}{z\to\infty}$ point. The boundary conditions are

$$V(y,z=0)=0 \quad \text{and} \quad V(y,z\gg d)\to 0$$

We then expect

$$V_\lambda(y,z)=\frac{-\lambda}{2\pi\epsilon_0}\ln\left(\frac{[y^2+(z-d)^2]^{1/2}}{\underbrace{a}_{\text{some reference point}}}\right)=\frac{-\lambda}{4\pi\epsilon_0}\ln\left[\frac{y^2+(z-d)^2}{a^2}\right] \quad \text{and} \quad V_{\lambda_{im}}=\frac{\lambda}{4\pi\epsilon_0}\ln\left[\frac{y^2+(z+d)^2}{a^2}\right]$$

Then,

$$V(x,y,z)=\frac{\lambda}{4\pi\epsilon_0}\ln\left[\frac{y^2+(z+d)^2}{y^2+(z-d)^2}\right] \text{ for } z\geq 0 \overset{\text{Checking}}{\Rightarrow} V(y,0)=0 \text{ and } V(y,z\gg d)\to 0$$

The potential is then

$$V(x,y,z)=\begin{cases}\dfrac{\lambda}{4\pi\epsilon_0}\ln\left[\dfrac{y^2+(z+d)^2}{y^2+(z-d)^2}\right] & \text{for } z\geq 0\\[2ex] 0 & \text{for } z\leq 0\end{cases}$$

(b) The induced charge and surface charge

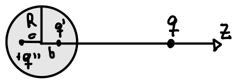
$$\sigma=-\epsilon_0\frac{\partial V}{\partial z}\bigg|_{0^+}=-\epsilon_0\frac{\lambda}{4\pi\epsilon_0}\left\{\frac{y^2+(z-d)^2}{y^2+(z+d)^2}\left[\frac{2(z+d)}{y^2+(z-d)^2}-\frac{y^2+(z+d)^2}{[y^2+(z-d)^2]^2}2(z-d)\right]\right\}_{z=0}=\frac{-\lambda}{4\pi}\left\{\frac{2d}{y^2+d^2}+\frac{2d}{y^2+d^2}\right\}\Rightarrow\sigma(y)=-\frac{\lambda d}{\pi(y^2+d^2)}$$

$$Q_{ind}=\underbrace{\int_{-\infty}^{\infty}dx}_{\equiv L}\int_{-\infty}^{\infty}dy\left(-\frac{\lambda d}{\pi(y^2+d^2)}\right)\Rightarrow\lambda_{ind}=\frac{Q_{ind}}{L}=-\frac{\lambda}{\pi}\underbrace{\int_{-\infty}^{\infty}dy\,\frac{1}{(y^2+1)}}_{\pi}\Rightarrow\lambda_{ind}=-\lambda \rightsquigarrow \text{O.K.!}$$

## Problem 3

Consider a $\overset{\text{charge } Q}{\text{charged}}$ conducting $\overset{\text{Radius } R}{\text{sphere}}$ at a potential $V_0$. A charge $q$ is placed a distance '$a$' from the center of the sphere.

(a) Determine the potential everywhere




We can take $\mathcal{V}=\{\vec{x}\mid R\leq r<\infty\}$. The boundary conditions are

$$V(R)=V_0 \quad \text{and} \quad V(\infty)\to 0$$

We can then write

$$V_q=\frac{q}{4\pi\epsilon_0}\frac{1}{\sqrt{r^2+a^2-2ra\cos\theta}},\quad V_{q'}=\frac{\overbrace{\left(-\frac{R}{a}q\right)}^{q'}}{4\pi\epsilon_0}\frac{1}{\sqrt{r^2+\underbrace{b^2}_{b=\frac{R^2}{a}}-2rb\cos\theta}},\quad \text{and} \quad V_{q''}=\overset{\text{We must have } q'+q''=Q}{\frac{q''}{4\pi\epsilon_0}\frac{1}{\sqrt{r^2+c^2-2rc\cos\theta}}}$$

At $r=R$ we have

$$V_q=\frac{q}{4\pi\epsilon_0}\frac{1}{(R^2+a^2-2aR\cos\theta)^{1/2}},\quad V_{q'}=-\frac{q}{4\pi\epsilon_0}\frac{1}{(a^2+R^2-2Ra\cos\theta)^{1/2}}=-V_q,\quad \text{and} \quad V_{q''}=\frac{q''}{4\pi\epsilon_0}\frac{1}{(R^2+c^2-2Rc\cos\theta)^{1/2}}\overset{\text{Pick } c=0 \text{ to kill the } \cos\theta \text{ dependence}}{=}\frac{q''}{4\pi\epsilon_0}\frac{1}{R}=\frac{q''}{4\pi\epsilon_0}\frac{1}{a}=V_0\Rightarrow q''=(4\pi\epsilon_0 a)V_0$$

this means that $Q=(4\pi\epsilon_0 a)V_0-\frac{Rq}{a}$

The boundary at $r\to\infty$ is trivially satisfied. The potential is then

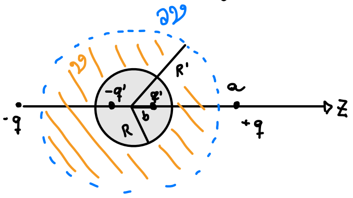
$$V(r,\theta)=\begin{cases}\dfrac{1}{4\pi\epsilon_0}\left[\dfrac{q}{(r^2+a^2-2ra\cos\theta)^{1/2}}+\dfrac{q'}{(r^2+b^2-2rb\cos\theta)^{1/2}}+\dfrac{q''}{r}\right] & \text{for } r\geq R \quad \left(q'=-\frac{R}{a}q,\ q''=(4\pi\epsilon_0 a)V_0, \text{ and } b=\frac{R^2}{a}\right)\\[2ex] V_0, & \text{for } r\leq R\end{cases}$$

Problem 4

can be taken to be at $V=0$

A uncharged metal sphere is placed in an otherwise uniform electric field $\vec{E}=E_0\hat{z}$. Determine

(a) The potential everywhere.




First, we note the boundary conditions

$$V(R)=0 \quad \text{and} \quad V(R\to\infty)=-E_0 z$$

Take $\mathcal{V}$ to be the region $\mathcal{V}=\{\vec{r} \text{ such that } R\leq r\leq R' \ll a \text{ with } R'\gg R\}$ If we place the image charge $\pm q$ at $z=\pm a$ then we would get the images $\pm q'$ at $z=\pm b$ (with $q'=-\frac{qR}{a}$ and $b=\frac{R^2}{a}$). This gives

$$\vec{r}=r\hat{r},\ \vec{r}_{\pm q}=\pm a\hat{z} \text{ and } \vec{r}_{\pm q'}=\pm b\hat{z} \Rightarrow \mathscr{r}_{\pm q}=|\vec{r}-\vec{r}_{\pm q}|=\sqrt{r^2+a^2\mp ra\cos\theta} \text{ and } \mathscr{r}_{\pm q'}=|\vec{r}-\vec{r}_{\pm q'}|=\sqrt{r^2+b^2\mp rb\cos\theta}$$

Hence,

$$V=\frac{1}{4\pi\epsilon_0}\left[q\underbrace{\left(\frac{1}{\mathscr{r}_q}-\frac{1}{\mathscr{r}_{-q}}\right)}_{\approx\frac{2r}{a^2}\cos\theta \text{ for } r\ll a}+q'\underbrace{\left(\frac{1}{\mathscr{r}_{q'}}-\frac{1}{\mathscr{r}_{-q'}}\right)}_{\approx\frac{2b\cos\theta}{r^2}=\frac{2R^2}{ar^2}\cos\theta}\right]\approx\frac{1}{4\pi\epsilon_0}\left[\frac{2qr}{a^2}\cos\theta-\frac{qR}{a}\frac{2R^2}{ar^2}\cos\theta\right]=\frac{1}{4\pi\epsilon_0}\left(\frac{2q}{a^2}\right)\left(r-\frac{R^3}{r^2}\right)\cos\theta=\frac{1}{4\pi\epsilon_0}\left(\frac{2q}{a^2}\right)\left(1-\frac{R^3}{r^3}\right)\overbrace{r\cos\theta}^{z}$$

Let's now check the boundary conditions:

$$V(R)=0 \text{ and } V(r\gg R)=\frac{1}{4\pi\epsilon_0}\left(\frac{2q}{a^2}\right)z=-E_0 z \Rightarrow \text{If we pick } q/a^2 \text{ such that } q/a^2=-(4\pi\epsilon_0)\frac{E_0}{2} \quad V(r\gg R)=-E_0 z$$

Thus, the potential everywhere reads

$$V(r,\theta)=\begin{cases}-E_0\left(1-\frac{R^3}{r^3}\right)r\cos\theta & \text{for } r\geq R\\ 0 & \text{for } r\leq R\end{cases}$$

(b) The induced surface charge and the total charge

$$\sigma=-\epsilon_0\frac{\partial V}{\partial r}\Big|_{R^+}=\epsilon_0 E_0\left[\frac{3R^3}{r^3}+1-\frac{R^3}{r^3}\right]_{r=R}\cos\theta=3\epsilon_0 E_0\cos\theta \quad \text{and} \quad Q_{in}=\int_0^{\pi}d\theta\sin\theta\int_0^{2\pi}d\phi\ 3\epsilon_0 E_0\cos\theta=0$$