

Notes on Math 322: PDE

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Notations

The following notations are used in this note:

- Partial Derivatives

$$\partial_t u = \frac{\partial u}{\partial t} = u_t, \quad \partial_t^2 u = \frac{\partial^2 u}{\partial x^2} = u_{tt} \quad [1]$$

- Laplacian

$$\begin{aligned} \nabla^2 &\equiv \nabla \cdot \nabla \\ \nabla^2 u &= \partial_x^2 u + \partial_y^2 u + \partial_z^2 u = u_{xx} + u_{yy} + u_{zz} \end{aligned} \quad [2]$$

- Boldface vector:

$$\vec{x} = \boldsymbol{x} \quad \text{v.s. scalar } x \quad [3]$$

1. Heat Equation

For temperature $u(\mathbf{x})$, heat conduction or particle diffusion can be described by the heat equation:

$$u_t = \kappa \nabla^2 u \quad [4]$$

1.1. Fundamental Solution

The fundamental solution Φ is found by solving the heat equation with a delta function as the initial condition:

$$\begin{cases} \Phi_t = \kappa \nabla^2 \Phi \\ \Phi(\mathbf{x}, t = 0) = \delta(\mathbf{x}) \end{cases} \quad [5]$$

It is solved to be the Green's function

$$\Phi(\mathbf{x}, t) = \frac{1}{(4\pi\kappa t)^{n/2}} \exp\left(-\frac{|\mathbf{x}|^2}{4\kappa t}\right) \quad [6]$$

1.2. Initial Value problem

Consider a general initial value $g(\mathbf{x})$, heat equation becomes:

$$\begin{cases} u_t = \kappa \nabla^2 u \\ u(\mathbf{x}, 0) = g(\mathbf{x}) \end{cases} \quad [7]$$

An argument of linearity and superposition can be made to arrive at the solution:

$$u(\mathbf{x}, t) = g \star \Phi \equiv \int_{\mathbb{R}^n} g(\mathbf{y}) \Phi(\mathbf{x} - \mathbf{y}) dv_{\mathbf{y}} \quad [8]$$

• *example:*

- Useful special functions: Heaviside step function, and error function

$$\text{erf}(x) \equiv \frac{2}{\sqrt{\pi}} \int_0^x \exp(-z^2) dz \quad [9]$$

- example statement: consider a long rod heated on the region $[-1, 1]$ at time zero. Mathematically,

$$\begin{cases} u_t = \kappa u_{xx} \\ u(x, 0) = g(x) = H(x + 1) - H(x - 1) \end{cases} \quad [10]$$

- *Solution:*

$$\begin{aligned}
u(x, t) &= g \star \Phi \\
&= \frac{1}{\sqrt{4\pi\kappa t}} \int_{-\infty}^{\infty} g(y) \exp\left(\frac{-(x-y)^2}{4\kappa t}\right) dy
\end{aligned} \tag{11}$$

$$\begin{aligned}
&= \frac{1}{\sqrt{4\pi\kappa t}} \int_{-1}^1 \exp(-(x-y)^2/4\kappa t) dy \\
\text{let } x-y &= z\sqrt{4\kappa t}, z = \frac{x-y}{\sqrt{4\pi\kappa t}} \\
u &= \frac{-\sqrt{4\pi\kappa t}}{\sqrt{4\pi\kappa t}} \int_{(x+1)/(\sqrt{4\kappa t})}^{(x-1)/(\sqrt{4\kappa t})} e^{-z^2} dz \\
&= \frac{1}{2} \left(\operatorname{erf}\left(\frac{x+1}{\sqrt{4\kappa t}}\right) - \operatorname{erf}\left(\frac{x-1}{\sqrt{4\kappa t}}\right) \right)
\end{aligned} \tag{12}$$

Notice that the erf function is an odd function, so we can combine this to be

$$u = \operatorname{erf}\left(\frac{1}{\sqrt{4\kappa t}}\right) \tag{13}$$

We can study this solution via asymptotic analysis

- for small x, talor expansion of erf function to second degree gives

$$\operatorname{erf}(x) \approx \frac{2x}{\sqrt{\pi}} \tag{14}$$

We are interested in large t, so

$$\operatorname{erf}\left(\frac{1}{\sqrt{4\kappa t}}\right) \approx \frac{1}{\sqrt{\pi\kappa t}} \sim \frac{1}{\sqrt{t}} \tag{15}$$

1.3. Heat eqn with forcing (heat source/ sink)

Consider the original heat equation without forcing

$$u_t = \kappa \nabla^2 u \tag{16}$$

Now, consider heat source $f(x, t)$, the heat equation becomes:

$$\begin{cases} u_t = \kappa \nabla^2 u + f(x, t) \\ u(x, 0) = 0 \end{cases} \tag{17}$$

We can use **Duhamel's Principle** to transform heat source to a collection of heat impulses(initial value problems) over time domain.