Problem 1

Prove or disprove the following statements:

1. The set $\{x \in \mathbb{R} : x \geq 2\}$ is open.
2. The set $\{x \in \mathbb{R} : x \neq 2\}$ is open.

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solution:

1. Let $\varepsilon > 0$. Consider $2 \in [2, \infty)$: , and interval $(2-\varepsilon, 2+\varepsilon)$:

   Since $2 - \frac{\varepsilon}{2} \in (2-\varepsilon, 2+\varepsilon)$, but $2 - \frac{\varepsilon}{2} \notin [2, \infty)$,

   it follows that for any $\varepsilon > 0$, the interval $(2-\varepsilon, 2+\varepsilon)$ is not a subset of $\{x \in \mathbb{R} : x \geq 2\}$, so the set $\{x \in \mathbb{R} : x \geq 2\}$ is not open. So we have **disproved** the statement. ■

2. Let $\varepsilon > 0, x \in \{x \in \mathbb{R} : x \neq 2\}$. Let $\varepsilon = \left|\frac{x-2}{2}\right|$.

   Then for any $y \in (x-\varepsilon, x+\varepsilon)$, we have

$$\begin{aligned} y < x + \varepsilon, \quad y > x - \varepsilon \\ \Rightarrow |y - x| < \varepsilon = \left|\frac{x-2}{2}\right| \end{aligned} \tag{1}$$

   Thus by triangle inequality,

$$\begin{aligned} |y-2| &= |y - x + x - 2| \\ &\geq |x-2| - |y-x| \\ &\geq |x-2| - \left|\frac{x-2}{2}\right| \\ &= \frac{|x-2|}{2} \\ &= \varepsilon > 0 \end{aligned} \tag{2}$$

   Therefore $y \neq 2 \Rightarrow y \in \{x \in \mathbb{R} : x \neq 2\}$. So the set is open. The statement is thus **proved** ■

Problem 2:

Let $A, B \subseteq \mathbb{R}$ be subsets. Prove the following statements:

1. *(De Morgan's Laws)* $(A \cap B)^c = A^c \cup B^c \text{and} (A \cup B)^c = A^c \cap B^c$
2. If A and B are closed then $A \cap B$ and $A \cup B$ are closed.

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solution:

1. - Let $x \in (A \cap B)^c$, then $x \notin (A \cap B) \Rightarrow (x \notin A)$ or $(x \notin B)$

     This is equivalent to $x \in A^c$ or $x \in B^c \Rightarrow x \in (A^c \cup B^c)$.

     So for any $x \in (A \cap B)^c$, $x \in (A^c \cup B^c)$, thus the two sets are equal.

   - Let $x \in (A \cup B)^c$, then $x \notin (A \cup B) \Rightarrow x \notin A$ and $x \notin B$.

     So $x \in A^c$ and $x \in B^c \Rightarrow x \in (A^c \cap B^c)$. So for any $x \in (A \cup B)^c$, $x \in (A^c \cap B^c)$, thus the two sets are equal. ■

2. - If A is closed and B is closed, then A^c and B^c are open. Since unions of open sets are open, then $A^c \cup B^c$ is open.

     By De Morgan's Laws, $A^c \cup B^c = (A \cap B)^c$ is open.

     Thus $A \cap B$ is closed.

   - If A is closed and B is closed, then A^c and $B^c$are open. Since finite intersections of open sets are open, $A^c \cap B^c$ is open.

     By De Morgan's Laws, $A^c \cap B^c = (A \cup B)^c$ is open.

     Thus $A \cup B$ is closed. ■

Problem 3:

Let $\varepsilon > 0$. For each of the following functions $\mathbb{R} \to \mathbb{R}$ and numbers $l \in \mathbb{R}$, find a δ s.t. $0 < |x - 1| < \delta$ implies $|f(x) - l| < \varepsilon$.

1. $f(x) = x^4$ and $l = 1$
2. $g(x) = \begin{cases} 1/x \text{ if } x \neq 0 \\ 0 \text{ if } x = 0 \end{cases}$, and $l = 1$
3. $h(x) = f(x) + g(x)$ and $l = 2$. hint: in the proof of the corresponding limit laws, we saw how to pick this δ based on our answers for (a) and (b).

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solution:

1. For any arbitrary ε , there exists a $\delta = \min\{1, \varepsilon/15\}$, s.t. $0 < |x - 1| < \delta$, so

$$0 < |x - 1| < 1 \Rightarrow \begin{cases} |x + 1| < 3 \\ |x^2 + 1| < 5 \end{cases} \tag{3}$$

and

$$\begin{aligned} |f(x) - l| = \left|x^4 - 1\right| &= |x - 1||x + 1|\left|x^2 + 1\right| \\ &< \delta * 3 * 5 \\ &= 15\delta = \varepsilon \end{aligned} \tag{4}$$

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2. For any arbitrary ε there exists $\delta = \min\left\{\frac{1}{2}, \varepsilon/2\right\}$, s.t. $0 < |x - 1| < \delta$, so

$$1 - \delta < x < \delta + 1 \Rightarrow \frac{1}{2} < \frac{1}{x} < 2. \tag{5}$$

and

$$|g(x) - 1| = \left|\frac{1}{x} - 1\right| = \frac{|x - 1|}{x} < 2|x - 1| = 2\delta = \varepsilon. \tag{6}$$

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3. $$|h(x) - 2| = |f(x) - 1 + g(x) - 1| < |f(x) - 1| + |g(x) - 1| \tag{7}$$

From the previous two parts, we know that we can choose $\delta_1 = \min\left\{1, \frac{\varepsilon_1}{15}\right\}$ and $\delta_2 = \min\left\{\frac{1}{2}, \frac{\varepsilon_2}{2}\right\}$. To ensure Equation 7 is smaller than ε , we choose

$$\delta = \min\left\{\frac{1}{2}, 1, \frac{\varepsilon}{2}, \frac{\varepsilon}{15}\right\} = \min\left\{\frac{1}{2}, \frac{\varepsilon}{15}.\right\} \tag{8}$$

Therefore,

$$|h(x) - 2| < \varepsilon. \tag{9}$$

■

Problem 4:

let $f, g : \mathbb{R} \to \mathbb{R}$ be functions s.t. $\lim_{x\to a} f(x) = l$ and $\lim_{x\to a} g(x) = m$ for some numbers a, $l, m \in \mathbb{R}$. Prove that if $\forall x \in \mathbb{R}, f(x) \leq g(x)$, then $l \leq m$.

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solution:

Given that $\lim_{x\to a} f(x) = l$ and $\lim_{x\to a} g(x) = m$, we know that

$$\begin{gathered} \forall \varepsilon > 0, \exists \delta_1 > 0, s.t.\ 0 < |x - a| < \delta_1 \Rightarrow |f(x) - l| < \varepsilon \\ \Rightarrow l - \varepsilon < f(x) < \varepsilon + l \\ \forall \varepsilon > 0, \exists \delta_2 > 0, s.t.\ 0 < |x - a| < \delta_2 \Rightarrow |g(x) - m| < \varepsilon \\ \Rightarrow m - \varepsilon < g(x) < \varepsilon + m \end{gathered} \tag{10}$$

If $\forall x \in \mathbb{R}, f(x) \leq g(x)$, then

$$\begin{gathered} l - \varepsilon < f(x) \leq g(x) < \varepsilon + m \\ \Rightarrow \quad l - \varepsilon < \varepsilon + m \\ \Rightarrow l < m + 2\varepsilon \end{gathered} \tag{11}$$

Since $\varepsilon > 0$ is arbitrary, the above inequality can be reduced to:

$$l \leq m. \tag{12}$$

The statement is thus proved. ■

Problem 5:

Let $f, g : \mathbb{R} \to \mathbb{R}$ be functions and $a \in \mathbb{R}$. Prove or disprove the following statements:

- (a) If $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both do not exist, then $\lim_{x\to a}(f+g)(x)$ does not exist.
- (b) If $\lim_{x\to a} f(x)$ exists and $\lim_{x\to a}(f+g)(x)$ does not exist, then $\lim_{x\to a} g(x)$ does not exist.
- (c) If $\lim_{x\to a} f(x)$ exists and $\lim_{x\to a} g(x)$ does not exist, then $\lim_{x\to a}(f+g)(x)$does not exist.

(hint: Each statement is either an application of the limit law for addition, or it is false. Remember, if the statement is false, then we need to come up with a counterexample.)

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solution:

- (a) Suppose $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both does not exist, but consider a special case where $f(x) = -g(x)$, then

  $$\lim_{x\to a}(f+g)(x) = \lim_{x\to a} 0 = 0, \tag{13}$$

  which is a well-defined limit. So the statement is negated, i.e. **false**, by counterexample.

- (b) Suppose $\lim_{x\to a} f(x) = m$, and $\lim_{x\to a}(f+g)(x)$ does not exist, but $\lim_{x\to a} g(x) = l$, for some $l, m \in \mathbb{R}$. Then by limit law for addition,

  $$\lim_{x\to a}(f+g) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x) = l + m \in \mathbb{R}. \tag{14}$$

  This contradicts with the assumption that $\lim_{x\to a}(f+g)(x)$ does not exist. So our assumption is false, and the original statement is **true**.

- (c) Assume that $\lim_{x\to a} f(x) = l$ exists and $\lim_{x\to a} g(x)$ does not exist, but $\lim_{x\to a}(f+g)(x) = m$, exists, for some $l, m \in \mathbb{R}$. Then, by the limit law for addtion, we have,

  $$\begin{aligned} &\lim_{x\to a}(f+g)(x) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x) = m \\ \Rightarrow \quad &\lim_{x\to a} g(x) = m - \lim_{x\to a} f(x) = m - l \in \mathbb{R}. \end{aligned} \tag{15}$$

  We found that $\lim(x \to a)g(x)$ is well defined, which contradicts to our assumption. So our assumption that $\lim(x \to a)(f+g)(x)$ exists is false, and the original statement is **true** by contradiction.

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