1

Problem 1. Let $A = \{a_n : n \in \mathbb{N}\} = \{a_1, a_2, \dots\}$ be an infinite subset of $\mathbb{R}$, and suppose that no element of A is listed twice: $a_n \neq a_m$ for all $n \neq m$. Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} \frac{1}{n} & \text{if } x = a_n \text{ for some } n \in \mathbb{N}, \\ 0 & \text{if } x \notin A, \end{cases}$$

Prove that $\lim_{x \to x_0} f(x) = 0$ for any $x_0 \in \mathbb{R}$.

Pf $\quad \forall \varepsilon > 0, \exists \delta > 0$ s.t. $0 < |x - x_0| < \delta \Rightarrow |f(x)| < \varepsilon$.

let $x_0 \in \mathbb{R}$ arbitrary, and let $\varepsilon > 0$ be given.

* By Archemidean's property, $\exists n_0 \in \mathbb{N}$ s.t. $\frac{1}{n_0} < \varepsilon$.

$\Rightarrow$ for any a_n w/ $n > n_0$, $f(a_n) = \frac{1}{n} < \frac{1}{n_0} < \varepsilon$.

So, the only pts making $f(x) \geq \varepsilon$ possible are in set B,

where: finite set $B = \{a_1, a_2, \dots, a_{n_0}\}$

now consider:

① $x_0 \notin A$: let $\delta = \min\{|x_0 - a_i| : a_i \in B\}$

Since B is finite, and $x_0 \in A$, δ exists, and $\delta > 0$.

② $x_0 \in A$, i.e. $x_0 = a_k$ for some $k \in \mathbb{N}$:

let $\delta = \min\{|x_0 - a_i| : a_i \in B, i \neq k\}$.

Since elements in A are distinct, δ exists, and $\delta > 0$

* In either cases, if $0 < |x - x_0| < \delta$, then: $x_0 \notin B$

So either $x \notin A \Rightarrow |f(x) - 0| = 0 < \varepsilon$

or $x = a_n$, $n > n_0 \Rightarrow |f(x) - 0| = |\frac{1}{n}| < |\frac{1}{n_0}| < \varepsilon$

* Collecting above, $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t.

$0 < |x - x_0| < \delta \Rightarrow |f(x) - 0| < \varepsilon$.

$\square$

$\boxed{2}$

Problem 2. Let $A = \{\frac{1}{n} : n \in \mathbb{N}\}$.

(a) Prove that $\sup A = 1$.
(b) Prove that $\inf A = 0$.

Pf (a). • for any $a \in A$: $a = \frac{1}{n}$, $n \in \mathbb{N}$.

Since $n \geq 1 \Rightarrow a \leq 1$

$\Rightarrow 1$ is an upper bound.

• Suppose for contradiction, K is another upper bond, $K < 1$

Consider $\frac{1}{1} \in A$: $1 > K$, contradiction

so $K \geq 1$. $\sup A = 1$.

(b). • for any $a \in A$: $a = \frac{1}{n}$, $n \in \mathbb{N}$.

Since $n \geq 1$, $\frac{1}{n} > 0$.

$\Rightarrow 0$ is lower bound.

• Suppose $M > 0$ is another lower bound.

Since $\exists n \in \mathbb{N}$ s.t. $n > \frac{1}{M}$ (Archemidean)

$\Rightarrow M > \frac{1}{n}$, not lower bond, Contradict

$\Rightarrow M \leq 0$ $\Rightarrow \inf A = 0$

$\boxed{3}$

Problem 3. Let $A \subseteq \mathbb{R}$ be a nonempty set that is bounded below, and define the set

$$-A = \{-a : a \in A\}.$$

Prove that $\sup(-A) = -\inf A$. (Remark: This is why we did not write down a "greatest lower bound property" for $\mathbb{R}$ in lecture: If A is nonempty and bounded below, then $\inf A = -\sup(-A)$ exists by the least upper bound property.)

- A nonempty & bounded below $\Rightarrow \exists \inf A \in \mathbb{R}$
  (least upper bound prop)
- Similarly, $-A$ is nonempty.
  Let m be a lower bound of A,
  then $-m$ is an upper bound of $-A$.
  $\Rightarrow \exists \sup A \in \mathbb{R}$.
- Let $m = \inf A$; $s = \sup(-A)$

① $m \leq a \quad \forall a \in A \quad \Rightarrow -m \geq -a \quad \forall a \in A$

$\Rightarrow -m$ is an upper bond for $-A$

Since $S = \sup(-A)$, $-m \geq S$.

② let ξ be an upper bond of $-A$. then

$-a \leq \xi$

$\Rightarrow \quad a \geq -\xi \quad \Rightarrow \quad -\xi$ is a lower bond for A.

Since $m = \inf A$, $-\xi \leq m \quad \Leftrightarrow \quad -m \leq \xi$

i.e. any upper bound $\geq -m$

$\Rightarrow s \geq -m$

Collecting above, $\begin{cases} -m \geq s \\ s \geq -m \end{cases} \Rightarrow \quad -m = S$

$\Rightarrow -\inf A = \sup(-A)$

$\square$

$\boxed{4}$

Problem 4. Let $A, B \subseteq \mathbb{R}$ be nonempty and bounded sets.

(a) Prove that if $A \subseteq B$, then $\inf B \leq \inf A \leq \sup A \leq \sup B$.
(b) Prove that $\sup(A \cup B) = \max\{\sup A, \sup B\}$. (Hint: A is a subset of $A \cup B$, so we can apply part (a) to these two sets.)

(a). $A \subseteq B \Rightarrow \forall a \in A,\ a \in B$.

- Since A, B non empty & bounded,
  $\exists \inf A, \inf B;\ \sup A, \sup B \in \mathbb{R}$ (By prob. 3)
- Any lower bond of B is a lower bond of A.
  $$\Rightarrow \inf B \leq \inf A.$$
- Any upper bound of B is an upper bound of A.
  $$\Rightarrow \sup B \geq \sup A.$$
- By def, $\inf A \leq \sup A$.

Collecting above,
$$\inf B \leq \inf A \leq \sup A \leq \sup B.$$

(b). noticing $A \subseteq A \cup B$; $B \subseteq A \cup B$.

Applying part (a):
$$\begin{cases} \sup(A \cup B) \geq \sup A \\ \sup(A \cup B) \geq \sup B \end{cases}$$
$$\Rightarrow \sup(A \cup B) \geq \max\{\sup A, \sup B\}.$$

- Any upper bound for both A and B is an upper bond for $A \cup B$. Let $M = \max\{\sup A, \sup B\}$
  Then, $\sup(A \cup B) \leq M$

$$\Rightarrow \begin{cases} \sup(A \cup B) \leq \max\{\sup A, \sup B\} \\ \sup(A \cup B) \geq \max\{\sup A, \sup B\} \end{cases}$$
$$\Rightarrow \sup(A \cup B) = \max(\sup A, \sup B) \quad \square$$

5

Problem 5. Let $a_n, b_n \in \mathbb{R}$ be numbers such that $a_n \leq b_n$ for each $n \in \mathbb{N}$. Suppose that $[a_{n+1}, b_{n+1}] \subseteq [a_n, b_n]$ for all $n \in \mathbb{N}$. Prove that the set

$$\bigcap_{n\in\mathbb{N}} [a_n, b_n] = \{x \in \mathbb{R} : x \in [a_n, b_n] \text{ for all } n \in \mathbb{N}\}$$

is nonempty. (Hint: Consider the set $A = \{a_n : n \in \mathbb{N}\}$.)

As suggested, consider $A = \{a_n : n \in \mathbb{N}\}$

for the nested intervals, each has lower bounds of:

$a_{n+1} > a_n$. $\forall n \in \mathbb{N}$.

So the sequence a_n is non-decreasing

so A is bounded below, and non-empty.

By Least upper Bound prop,

$\exists \sup A \in \mathbb{R}$. denote as c.

* for each $n \in \mathbb{N}$:

$a_n \leq c \quad \forall n \in \mathbb{N}$ (least upper bond)

Since $[a_{n+1}, b_{n+1}] \subseteq [a_n, b_n]$

$\Rightarrow c \leq b_n \; \forall n \in \mathbb{N}$, since b_n is an upper bound.

$\Rightarrow a_n \leq c \leq b_n$, or: $c \in [a_n, b_n]$

since c exists for each $n \in \mathbb{N}$, c exists in every $[a_n, b_n]$

$$\Rightarrow c \in \bigcap_{n=1}^{\infty} [a_n, b_n]$$

So $\bigcap_{n=1}^{\infty} [a_n, b_n]$ is non-empty.

6

Problem 6. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Suppose that f is *increasing*: for any $x, y \in \mathbb{R}$, if $x \le y$ then $f(x) \le f(y)$. Prove that $\lim_{x\to a^-} f(x)$ exists for any $a \in \mathbb{R}$. (Hint: Consider the set $A = \{f(x) : x < a\}$.)

- As suggested, consider: $A = \{f(x) : x < a\}$ $\forall a \in \mathbb{R}$

  Since $f: \mathbb{R} \to \mathbb{R}$, A is non-empty.

  Since f increasing, $x < a \Rightarrow f(x) < f(a)$

  A is bounded above.

  $\Rightarrow \exists \sup A \in \mathbb{R}$. Denote as L.

- To show that $\lim_{x\to a^-} f(x)$ exists, we prove that

  $\lim_{x\to a^-} f(x) = L$, or $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $0 < |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon$.

- By def, $f(x) \le \sup A = L$

  $f(x) - L \le 0 \Rightarrow |f(x) - L| = L - f(x)$.

  So $|f(x) - L| < \varepsilon \iff f(x) > L - \varepsilon$.

- By def of sup, $\forall \varepsilon > 0$, $\exists x$ s.t. $f(x) > L - \varepsilon$.

  specifically, $\exists \delta > 0$ s.t. $0 < |x - a| < \delta \Rightarrow f(x) > L - \varepsilon$

- So, for all x where $0 < |x - a| < \delta$:

  $L - \varepsilon < f(x) \le L$.

  $\Rightarrow |f(x) - L| < \varepsilon$.

$\Rightarrow \forall \varepsilon > 0, \exists \delta > 0$ s.t. $0 < |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon$.

$\Rightarrow \lim_{x\to a^-} f(x) = L = \sup A$.