Physics 322
Discussion 5
10/21/2024

General Dielectric: ↳ Polarization

$\rho = \rho_b + \rho_f$, $\rho_b = -\nabla\cdot\vec{P}$ and $\sigma_b = \vec{P}\cdot\hat{n}$ ↳ Electric displacement

$\nabla\cdot\vec{D} = \rho_f$ and $\oint \vec{D}\cdot d\vec{a} = Q_{f enc}$ with $\vec{D} = \epsilon_0\vec{E} + \vec{P}$

B.C.: $D^{\perp}_{above} - D^{\perp}_{below} = \sigma_f$ and $D^{\parallel}_{above} - D^{\parallel}_{below} = P^{\parallel}_{above} - P^{\parallel}_{below}$

Linear Dielectric: $\vec{P} = \epsilon_0 \chi_e \vec{E}$ ↳ Electric susceptibility

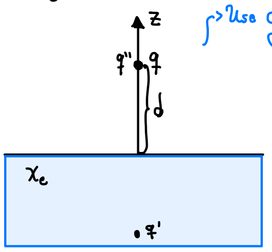
$\vec{D} = \underbrace{\epsilon_0(1+\chi_e)}_{\text{Permitivity } \epsilon}\vec{E}$, $\epsilon_r = 1 + \chi_e = \frac{\epsilon}{\epsilon_0}$ ↳ relative permitivity (dielectric constant)

$\rho_b = -\left(\frac{\chi_e}{1+\chi_e}\right)\rho_f \; (\rho_f = 0 \Rightarrow \nabla^2 V = 0)$

B.C.: $\epsilon_{above}\frac{\partial V_{above}}{\partial n} - \epsilon_{below}\frac{\partial V_{below}}{\partial n} = -\sigma_f$ $\quad V_{above} = V_{below}$

Problem 1

Suppose the entire region below the plane $z=0$ is filled with uniform dielectric material of susceptibility χ_e. Calculate the force on a point charge q situated a distance d above the origin.



↳ Use cylindrical coordinates

Maxwell equations read
$$\begin{cases} \nabla^2 V^{>} = -\frac{q}{\epsilon_0}\delta^{(3)}(\vec{r} - d\hat{z}) & \text{for } z>0 \\ \nabla^2 V^{<} = 0 & \text{for } z<0 \end{cases}$$

with B.C. $\left[\epsilon_0\frac{\partial V^{>}}{\partial z} - \epsilon\frac{\partial V^{<}}{\partial z}\right]_{z=0} = 0$ (No free charge) and $\left[V^{>} - V^{<}\right]_{z=0} = 0$

Now, we can use the method of image to solve for the potential. For $z>0$ we can place a image charge q' at $z=-d$.

$$V^{>}(r,z) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{[r^2+(z-d)^2]^{1/2}} + \frac{q'}{[r^2+(z+d)^2]^{1/2}}\right] \Rightarrow \frac{\partial V^{>}}{\partial z} = -\frac{1}{4\pi\epsilon_0}\left[\frac{(z-d)q}{[r^2+(z-d)^2]^{3/2}} + \frac{(z+d)q'}{[r^2+(z+d)^2]^{3/2}}\right]$$

$$\Rightarrow V^{>}(r,0) = \frac{1}{4\pi\epsilon_0}\frac{(q+q')}{[r^2+d^2]^{1/2}} \quad \& \quad \frac{\partial V^{>}(r,0)}{\partial z} = \frac{-1}{4\pi\epsilon_0}\frac{(q'-q)d}{[r^2+d^2]^{3/2}}$$

For $z<0$, we can place an image charge q'' at $z=d$.

$$V^{<}(r,z) = \frac{1}{4\pi\epsilon}\left[\frac{(q+q'')}{[r^2+(z-d)^2]^{1/2}}\right] \Rightarrow \frac{\partial V^{<}}{\partial z} = \frac{-1}{4\pi\epsilon}\left[\frac{(z-d)(q+q'')}{[r^2+(z-d)^2]^{3/2}}\right]$$

$$\Rightarrow V^{<}(r,0) = \frac{1}{4\pi\epsilon}\frac{(q+q'')}{[r^2+d^2]^{1/2}} \quad \& \quad \frac{\partial V^{<}(r,0)}{\partial z} = \frac{1}{4\pi\epsilon}\frac{(q+q'')d}{[r^2+d^2]^{3/2}}$$

Now, imposing the B.C.

$$\frac{1}{4\pi}\frac{1}{[r^2+d^2]^{1/2}}\left(\frac{q+q''}{\epsilon} - \frac{q-q'}{\epsilon_0}\right) = 0 \quad \& \quad \frac{1}{4\pi}\frac{d}{[r^2+d^2]^{3/2}}\left((q'+q) - (q+q'')\right) = 0$$

$$\Rightarrow q' = q'' \;\&\; \epsilon_0(q-q') - \epsilon(q+q') = 0 \Rightarrow q' = \frac{(\epsilon_0-\epsilon)}{(\epsilon+\epsilon_0)}q \;\&\; q+q'' = q\left(1 - \frac{(\epsilon_0-\epsilon)}{\epsilon+\epsilon_0}\right) = \frac{2\epsilon}{\epsilon+\epsilon_0}q$$

Thus, we have

$$V^{>}(r,z) = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{r^2+(z-d)^2}} + \left(\frac{\epsilon_0-\epsilon}{\epsilon_0+\epsilon}\right)\frac{1}{\sqrt{r^2+(z+d)^2}}\right] \Rightarrow \vec{E}^{>} = \frac{q}{4\pi\epsilon_0}\left\{\left[\frac{1}{[r^2+(z-d)^2]^{3/2}} + \left(\frac{\epsilon_0-\epsilon}{\epsilon_0+\epsilon}\right)\frac{1}{[r^2+(z+d)^2]^{3/2}}\right]r\hat{r} + \left[\frac{(z-d)}{[r^2+(z-d)^2]^{3/2}} + \left(\frac{\epsilon_0-\epsilon}{\epsilon_0+\epsilon}\right)\frac{(z+d)}{[r^2+(z+d)^2]^{3/2}}\right]\hat{z}\right\} \quad \text{for } z>0$$

$$V^{<}(r,z) = \frac{2q}{4\pi(\epsilon+\epsilon_0)}\frac{1}{\sqrt{r^2+(z-d)^2}} \Rightarrow \vec{E}^{<} = \frac{2q}{4\pi(\epsilon+\epsilon_0)}\frac{1}{[r^2+(z-d)^2]^{3/2}}\left(r\hat{r} + (z-d)\hat{z}\right) = \frac{2q}{4\pi(\epsilon+\epsilon_0)}\frac{\vec{\mathscr{r}}}{\mathscr{r}^3} \text{ with } \vec{\mathscr{r}} = r\hat{r} + (z-d)\hat{z} \quad \text{for } z<0$$

From this we can get the polarization

↳ $\vec{P} = (\epsilon - \epsilon_0)\vec{E}$

$$\vec{P} = \frac{2q}{4\pi}\frac{(\epsilon-\epsilon_0)}{(\epsilon+\epsilon_0)}\frac{\vec{\mathscr{r}}}{\mathscr{r}^3} \Rightarrow \rho_b = -\nabla\cdot\vec{P} = -\frac{2q}{4\pi}\frac{(\epsilon-\epsilon_0)}{(\epsilon+\epsilon_0)}\nabla\cdot\left(\frac{\vec{\mathscr{r}}}{\mathscr{r}^3}\right) = -\frac{2q}{4\pi}\frac{(\epsilon-\epsilon_0)}{(\epsilon+\epsilon_0)}\underbrace{\delta^{(3)}(\vec{\mathscr{r}})}_{0 \text{ for } z<0} = 0 \quad \text{and} \quad \sigma_b = \vec{P}\cdot\hat{z}\Big|_{z=0} = -\frac{2q}{4\pi}\frac{(\epsilon-\epsilon_0)}{(\epsilon+\epsilon_0)}\frac{d}{(r^2+d^2)^{3/2}}$$

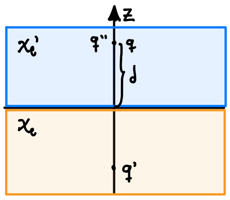
Finally, the force on q is

↳ subtracting the field of q

$$\vec{F} = q\vec{E}^{>}(0,d) = \frac{q^2}{16\pi\epsilon_0 d^2}\left(\frac{\epsilon_0-\epsilon}{\epsilon_0+\epsilon}\right)$$

Problem 2

Let's workout problem 3 of HW5.



Let's use the same technique we used to solve the previous problem. Here:

$$V^{>}(r,z)=\frac{1}{4\pi\epsilon'}\left\{\frac{q}{[r^2+(z-d)^2]^{1/2}}+\frac{q'}{[r^2+(z+d)^2]^{1/2}}\right\}\Rightarrow V^{>}(r,0)=\frac{1}{4\pi\epsilon'}\frac{(q+q')}{[r^2+d^2]^{1/2}}\ \text{and}\ \frac{\partial V^{>}(r,0)}{\partial z}=\frac{1}{4\pi\epsilon'}\frac{d(q-q')}{[r^2+d^2]^{3/2}}$$

$\hookrightarrow \epsilon'=\epsilon_0(1+\chi_e')$

$$V^{<}(r,z)=\frac{1}{4\pi\epsilon}\frac{q''}{[r^2+(z-d)^2]^{1/2}}\Rightarrow V^{<}(r,0)=\frac{1}{4\pi\epsilon}\frac{q''}{[r^2+d^2]^{1/2}}\ \text{and}\ \frac{\partial V^{<}(r,0)}{\partial z}=\frac{1}{4\pi\epsilon}\frac{q''}{[r^2+d^2]^{3/2}}$$

$\hookrightarrow \epsilon=\epsilon_0(1+\chi_e)$

The B.C. now give

$$\frac{1}{\epsilon'}(q+q')=\frac{q''}{\epsilon}\ \text{and}\ (q-q')=q''\Rightarrow 2q=q''\left(\frac{\epsilon+\epsilon'}{\epsilon}\right)\Rightarrow q''=\left(\frac{2\epsilon}{\epsilon+\epsilon'}\right)q\ \text{and}\ q'=\left(\frac{\epsilon'-\epsilon}{\epsilon'+\epsilon}\right)q$$

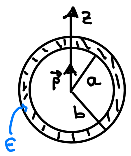
Hence,

$$V^{>}(r,z)=\frac{q}{4\pi\epsilon'}\left\{\frac{1}{[r^2+(z-d)^2]^{1/2}}+\left(\frac{\epsilon'-\epsilon}{\epsilon'+\epsilon}\right)\frac{1}{[r^2+(z+d)^2]^{1/2}}\right\}\quad \text{for } z\geqslant 0$$

$$V^{<}(r,z)=\frac{2q}{4\pi(\epsilon'+\epsilon)}\frac{1}{[r^2+(z+d)^2]^{1/2}}\quad \text{for } z\leqslant 0$$

Problem 3

Let's workout problem 4 of HW5.



Define three regions: (I) $r<a$, (II) $a<r<b$, (III) $r>c$. We then have:

$$V^{I}(r,\theta)=\frac{1}{4\pi\epsilon_0}\left[\frac{P\overset{P_1(\cos\theta)}{\cos\theta}}{r^2}+\sum_{\ell=0}^{\infty}A_\ell^{I}r^{\ell}P_\ell(\cos\theta)\right]$$

$$V^{II}(r,\theta)=\frac{1}{4\pi\epsilon}\sum_{\ell=0}^{\infty}\left[A_\ell^{II}r^{\ell}+B_\ell^{II}r^{-(\ell+1)}\right]P_\ell(\cos\theta)$$

$$V^{III}(r,\theta)=\frac{1}{4\pi\epsilon_0}\sum_{\ell=0}^{\infty}B_\ell^{III}r^{-(\ell+1)}P_\ell(\cos\theta)$$

Since the dipole gives a prefered $\overset{\ell=1}{\ell}$, let's assume all the solutions only involve this ℓ. Then, using the continuity we get

$$V^{I}(r,\theta)=\frac{P}{4\pi\epsilon_0}\left\{\frac{1}{r^2}+\frac{r}{a^3}\left[\frac{1}{\epsilon_r}\left(A^{II}+B^{II}\right)-1\right]\right\}\cos\theta\Rightarrow\frac{\partial V^{I}(a,\theta)}{\partial r}=\frac{P}{4\pi\epsilon_0 a^3}\left[\frac{1}{\epsilon_r}(A^{II}+B^{II})-3\right]\cos\theta$$

$$V^{II}(r,\theta)=\frac{P}{4\pi\epsilon}\left(\frac{A^{II}r}{a^3}+\frac{B^{II}}{r^2}\right)\cos\theta\Rightarrow\frac{\partial V^{II}(a,\theta)}{\partial r}=\frac{P}{4\pi\epsilon a^3}\left[A^{II}-2B^{II}\right]\cos\theta\ \&\ \frac{\partial V^{II}(b,\theta)}{\partial r}=\frac{P}{4\pi\epsilon b^3}\left[A^{II}\left(\frac{b}{a}\right)^3-2B^{II}\right]$$

$$V^{III}(r,\theta)=\frac{P}{4\pi\epsilon r^2}\left[A^{II}\left(\frac{b}{a}\right)^3+B^{II}\right]\cos\theta\Rightarrow\frac{\partial V^{III}(b,\theta)}{\partial r}=\frac{P}{4\pi\epsilon b^3}(-2)\left[A^{II}\left(\frac{b}{a}\right)^3+B^{II}\right]$$

The B.C. give

$$\epsilon\frac{\partial V^{II}(a,\theta)}{\partial r}-\epsilon_0\frac{\partial V^{I}(a,\theta)}{\partial r}=0\Rightarrow\frac{(A^{II}+B^{II})}{\epsilon_r}-3=A^{II}-2B^{II}\Rightarrow A^{II}(1-\epsilon_r)+B^{II}(1+2\epsilon_r)=3\epsilon_r\Longrightarrow B^{II}\left[(1+2\epsilon_r)+\left(\frac{a}{b}\right)^3\frac{(2\epsilon_r-2)(1-\epsilon_r)}{(2+\epsilon_r)}\right]=3\epsilon_r\Rightarrow B^{II}=\frac{3\epsilon_r}{\left[(1+2\epsilon_r)-2\left(\frac{a}{b}\right)^3\frac{(1-\epsilon_r)^2}{2+\epsilon_r}\right]}$$

$$\epsilon_0\frac{\partial V^{III}(b,\theta)}{\partial r}-\epsilon\frac{\partial V^{II}(b,\theta)}{\partial r}=0\Rightarrow\frac{(-2)}{\epsilon_r}\left[A^{II}\left(\frac{b}{a}\right)^3+B^{II}\right]=A^{II}\left(\frac{b}{a}\right)^3-2B^{II}\Rightarrow A^{II}\left(\frac{b}{a}\right)^3\left(1+\frac{2}{\epsilon_r}\right)=2B^{II}\left(1-\frac{1}{\epsilon_r}\right)\Rightarrow A^{II}\left(\frac{b}{a}\right)^3\frac{(2+\epsilon_r)}{2(\epsilon_r-1)}=B^{II}\Rightarrow A^{II}\left(\frac{b}{a}\right)^3=\frac{(2\epsilon_r-2)}{2+\epsilon_r}B^{II}$$

Then,

$$V^{I}(r,\theta)=\frac{P\cos\theta}{4\pi\epsilon_0 r^2}\left\{1+\left(\frac{r}{a}\right)^3\left[\frac{f(\epsilon_r,b,a)}{\epsilon_r}\left(1+\left(\frac{a}{b}\right)^3\frac{2(\epsilon_r-1)}{2+\epsilon_r}\right)-1\right]\right\}$$

$$V^{II}=\frac{P\cos\theta}{4\pi\epsilon r^2}f(\epsilon_r,a,b)\left[1+\frac{2(\epsilon_r-1)}{2+\epsilon_r}\left(\frac{r}{b}\right)^3\right]$$

with $f(\epsilon_r,a,b)=\dfrac{3\epsilon_r}{\left[(1+2\epsilon_r)-2\left(\frac{a}{b}\right)^3\frac{(1-\epsilon_r)^2}{2+\epsilon_r}\right]}$. Note that the limit $a\to 0$ is subtle.

$$V^{III}(r,\theta)=\frac{P\cos\theta}{4\pi\epsilon_0 r^2}\frac{3}{(\epsilon_r+2)}f(\epsilon_r,a,b)$$