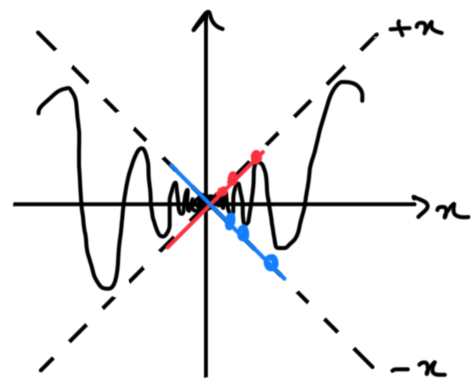
<u>Recall</u> f differentiable at a $\overset{\text{def}}{\iff}$ $f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ exists

<u>Ex</u> Are the following functions differentiable?

① $f(x) = \begin{cases} x \cdot \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$ at $a=0$.



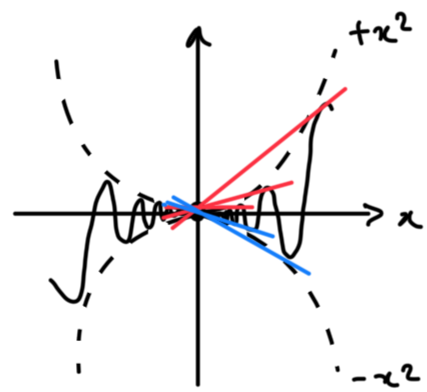
- Previously: f is continuous at 0
- Secant lines can have slope ± 1 arbitrarily close to $x=0$

No. Claim: $f'(0)$ does not exist. For $h \neq 0$,

$$\frac{f(0+h) - f(0)}{h} = \frac{h \cdot \sin \frac{1}{h} - 0}{h} = \sin \frac{1}{h}$$

By Lecture 13, we know $\lim_{h \to 0} \sin \frac{1}{h}$ does not exist.

② $g(x) = \begin{cases} x^2 \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$ at $a=0$.



- Secant lines become horizontal as $h \to 0$:

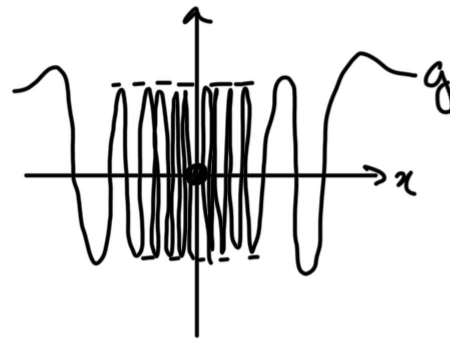
Yes. Claim: $g'(0) = 0$. For $h \neq 0$,

$$\frac{g(0+h)-g(0)}{h} = \frac{h^2 \sin\frac{1}{h} - 0}{h} = h \cdot \sin\frac{1}{h}$$

By Lecture 13, we know $\lim_{h\to 0} h\cdot\sin\frac{1}{h} = 0$.

<u>Rmk</u> This g is differentiable at any $x \in \mathbb{R}$, and

$$g'(x) = \begin{cases} 2x\sin\frac{1}{x} - \cos\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$



But g' is discontinuous at $x=0$!

<u>Def</u> Let $a<b$. We say:

① $f:\mathbb{R}\to\mathbb{R}$ is <u>differentiable</u> if f is differentiable at x $\forall x \in \mathbb{R}$. The function $f':\mathbb{R}\to\mathbb{R}$ is the <u>derivative</u> of f.

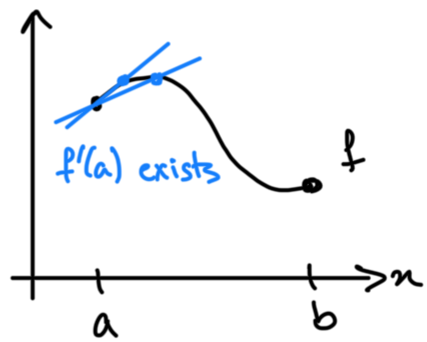
② $f:(a,b)\to\mathbb{R}$ is <u>differentiable</u> if f is differentiable at x $\forall x\in(a,b)$. The function $f':(a,b)\to\mathbb{R}$ is the <u>derivative</u> of f.

③ $f:[a,b]\to\mathbb{R}$ is <u>differentiable</u> if:

- f is differentiable at x $\forall x\in(a,b)$
- $\lim_{h\to 0^+} \frac{f(a+h)-f(a)}{h}$ exists
- $\lim_{h\to 0^-} \frac{f(b+h)-f(b)}{h}$ exists

The function $f':[a,b]\to\mathbb{R}$ is the <u>derivative</u>

of f.



- Differentiable on $[a,b]$
- $f' : [a,b] \to \mathbb{R}$

- Continuous on $[a,b]$
- Not differentiable on $[a,b]$
- $f' : (a,b] \to \mathbb{R}$

<u>Ex</u> • Polynomials $p : \mathbb{R} \to \mathbb{R}$ are differentiable (but we haven't proved this yet!)

- $f(x) = |x|$ is differentiable on $[0, \infty)$ and $(-\infty, 0]$, but not on $\mathbb{R}$ (by lecture 22)
- $f(x) = \sqrt{x}$ is differentiable on $(0, \infty)$, but not on $[0, \infty)$
- $f(x) = \begin{cases} x \cdot \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$ is differentiable on $(0, \infty)$ and $(-\infty, 0)$, but not on $\mathbb{R}$
- $g(x) = \begin{cases} x^2 \cdot \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$ is differentiable on $\mathbb{R}$

<u>Prop</u> $\forall n \in \mathbb{N}$, the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^n$ is differentiable with derivative $f'(x) = n x^{n-1}$.

Pf: Fix $n \in \mathbb{N}$ and $x \in \mathbb{R}$. For $h \neq 0$.

$$\frac{f(a+h)-f(a)}{h} = \frac{(a+h)^n - a^n}{h}$$

$$(a+h)^n = a^n + na^{n-1}h + \frac{n(n-1)}{2}a^{n-2}h^2 + \ldots + nah^{n-1} + h^n$$

$$= \sum_{j=0}^{n} \binom{n}{j} a^{n-j} h^j, \qquad \text{where} \quad \binom{n}{j} = \frac{n!}{j!\,(n-j)!}$$

- "Binomial theorem" — see Ch. 2 problem 3 for a proof

$$\Rightarrow \quad \frac{f(a+h)-f(a)}{h} = \frac{\cancel{a^n} + na^{n-1}h + \frac{n(n-1)}{2}a^{n-2}h^2 + \ldots + h^n - \cancel{a^n}}{h}$$

$$= na^{n-1} + \frac{n(n-1)}{2}a^{n-2}h + \ldots + h^{n-1}$$

This is a polynomial in h. By lecture 14 we know any polynomial is continuous at $h=0$, and so

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = na^{n-1} + 0 + \ldots + 0. \qquad \square$$