Math 421, Section 1
Practice Final Exam
Fall 2024

First name: ______________________ **Last name:** ______________________

Instructions:

- This exam contains 8 problems, and there are a total of 60 points available.
- Show all your work in the space provided. You may also use the backs of pages.
- No outside resources are allowed, including notes, calculators, textbooks, etc.

Question	Points	Score
1	6	
2	9	
3	7	
4	8	
5	6	
6	9	
7	6	
8	9	
Total:	60	

1. (6 points) Using induction, prove that for all $n \in \mathbb{N}$ we have

$$1 + 4 + 7 + \cdots + (3n - 2) = \frac{n(3n-1)}{2}.$$

2. Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$.
   (a) (3 points) Find $f((-1, 2))$.
   (b) (3 points) Find $f^{-1}(\{0, 1\})$.
   (c) (3 points) Prove or disprove: The function $f : \mathbb{R} \to \mathbb{R}$ is injective.

3. (a) (3 points) State the definition of the *least upper bound* for $A \subseteq \mathbb{R}$ is $\sup A$.

   (b) (4 points) Find $\sup\{2 - \frac{3}{n} : n \in \mathbb{N}\}$ and prove your answer.

4. (8 points) Define the function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x^2 & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases}$$

Prove that f is differentiable at any $a \in \mathbb{R}$ and find $f'(a)$.

5. (6 points) Let $a, b > 0$ and define the function $f : (0, \infty) \to \mathbb{R}$ by

$$f(x) = \frac{a}{x} + bx.$$

Find the global minimum of f and prove your answer.

6. Suppose $f : [0,1] \to \mathbb{R}$ is a continuous function that satisfies $f(x) \in [0,1]$ for all $x \in [0,1]$.

   (a) (6 points) Prove that there exists a point $x \in [0,1]$ such that $f(x) = 1 - x$.

   (b) (3 points) Suppose that f is also differentiable on $(0,1)$ and $|f'(x)| < 1$ for all $x \in (0,1)$. Prove that there is exactly one point $x \in [0,1]$ such that $f(x) = 1 - x$.

7. (6 points) Prove that the function $f : [-1,1] \to \mathbb{R}$ given by

$$f(x) = \begin{cases} -2 & \text{if } x < 0, \\ 0 & \text{if } x = 0, \\ 2 & \text{if } x > 0 \end{cases}$$

is integrable on $[-1,1]$, and find $\int_{-1}^{1} f$.

8. Let $b > 0$.
   (a) (3 points) Find $\int_0^b \sin x \, dx$ and prove your answer.
   (b) (6 points) Find $\int_0^b \sin^3 x \, dx$ and prove your answer.

$\boxed{\text{Extra paper}}$