Magnetostatics

Magnetic Forces on Charge

For a moving charge of velocity $\boldsymbol{v}$ and charge Q in a magnetic field $\boldsymbol{B}$, the magnetic force is found by the Lorentz force law:

$$F_{\text{mag}} = Q(\boldsymbol{v} \times \boldsymbol{B}) \quad [1$$

Notice that magnetic force does no work. We can see this by considering an infinetesmal displacement on Q, $\mathrm{d}\boldsymbol{l} = \boldsymbol{v}\,\mathrm{d}t$, then the work done by B field is

$$\mathrm{d}W = F_{\text{mag}} \cdot \mathrm{d}\boldsymbol{l} = Q\underbrace{(\boldsymbol{v} \times \boldsymbol{B}) \cdot \boldsymbol{v}}_{\text{perpendicular}}\,\mathrm{d}t = 0 \quad [2$$

Magnetic Force on Current

Line current

Magnetic force on a segment of current carrying wire with $I = \lambda \boldsymbol{v}$ is:

$$\boldsymbol{F} = \int_l I(\mathrm{d}\boldsymbol{l} \times \boldsymbol{B}) \quad [3$$

Surface and volume current density

- Surface current density:

$$\boldsymbol{K} = \frac{\mathrm{d}\boldsymbol{I}}{\mathrm{d}l_\perp} = \sigma \boldsymbol{v}; \quad [4$$

  with $\mathrm{d}l_\perp$ an infinitesimal width running parallel to the current flow. Force on a surface current:

$$\boldsymbol{F} = \int_A (\boldsymbol{K} \times \boldsymbol{B})\,\mathrm{d}a \quad [5$$

- Volume current density:

$$\boldsymbol{J} = \frac{\mathrm{d}\boldsymbol{I}}{\mathrm{d}a_\perp} = \rho \boldsymbol{v}; \quad [6$$

  with infinitesimal cross section $\mathrm{d}a_\perp$ running parallel to the flow. Force on a volume current:

$$\boldsymbol{F} = \int_V (\boldsymbol{J} \times \boldsymbol{B})\,\mathrm{d}\tau \quad [7$$

Continuity Equation

$$I = \int_s \boldsymbol{J}\,\mathrm{d}a_\perp = \int_s \boldsymbol{J} \cdot \mathrm{d}a. \quad [8$$

So, noticing a conservation of charge

$$\begin{gathered} \oint \boldsymbol{J}\,\mathrm{d}a = \int_v \boldsymbol{\nabla} \cdot \boldsymbol{J}\,\mathrm{d}\tau = -\frac{\mathrm{d}}{\mathrm{d}t}\int_v \rho\,\mathrm{d}\tau \\ \Rightarrow \boldsymbol{\nabla} \cdot \boldsymbol{J} = -\frac{\mathrm{d}\rho}{\mathrm{d}t} \end{gathered} \quad [9$$

Magnetostatics: Steady currents

Magnetostatics is the regime where :

$$\frac{\partial \rho}{\partial t} = 0 \Rightarrow \boldsymbol{\nabla} \cdot \boldsymbol{J} = 0. \quad [10$$

And the Biot-Savart law is used to find the magnetic field due to steady linear current:

$$\boxed{\boldsymbol{B}(r) = \frac{\mu_0 I}{4\pi} \int_l \frac{\mathrm{d}\boldsymbol{l}' \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}} \quad [11$$

and for a volume current:

$$\boldsymbol{B}(r) = \frac{\mu_0}{4\pi} \int_V \frac{\boldsymbol{J}(\vec{r}') \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}) \,\mathrm{d}\tau' \quad [12$$

Div of B, curl of B, and Ampere's law

B-S law tells us of the div of B:

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0 \quad [13$$

suggesting the inexistence of magnetic monopoles.

We can find curl of B from B-S law too:

$$\boldsymbol{\nabla} \times \boldsymbol{B} = \mu_0 \boldsymbol{J} \quad [14$$

and by Stokes' theorem, we can write:

$$\boxed{\oint \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{l} = \mu_0 I_{\text{enc}}} \quad [15$$

where the line integal is over an amperean loop.

note: Ampere's law is only useful when $\boldsymbol{B}$ is constant over the loop, and in whcih case we can pull it out of the integral. Such symmetry is often found in amperean loops for infinite straight wire, infinite plane, infinite solunoid, and toroid.

Magnetic Vector Potential

By helmholtz theorem, $\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$ invites the notion of a vector potential $\boldsymbol{A}$ such that

$$\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A} \quad [16$$

. We choose $\boldsymbol{\nabla} \cdot A = 0$, then ampere's law gives:

$$\begin{gathered} \boldsymbol{\nabla} \times \boldsymbol{B} = \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{A}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{A}) - \nabla^2 \boldsymbol{A} = \mu_0 \boldsymbol{J} \\ \Rightarrow \boxed{\nabla^2 \boldsymbol{A} = -\mu_0 \boldsymbol{J}} \end{gathered} \quad [17$$

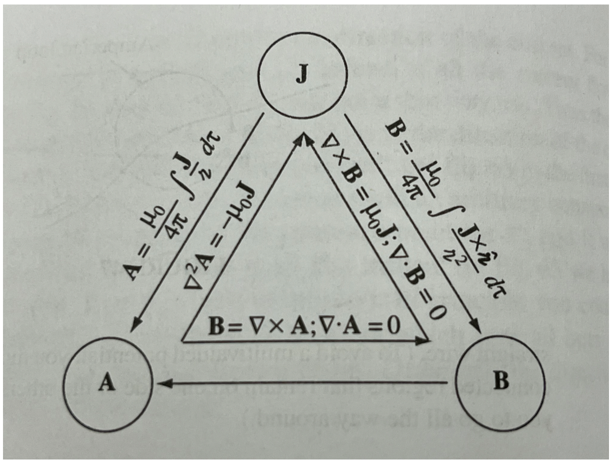
which is poisson's equation for $\boldsymbol{A}$. The solution for some local charge distribution (volume, linear, surface) can be read off as:

$$\boldsymbol{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int_V \frac{\boldsymbol{J}(\vec{r})}{\mathscr{r}} \,\mathrm{d}\tau, \quad [18$$

$$\boldsymbol{A} = \frac{\mu_0 I}{4\pi} \int_l \frac{1}{\mathscr{r}} \, \mathrm{d}\boldsymbol{l},$$ [19

$$\boldsymbol{A} = \frac{\mu_0}{4\pi} \int_S \frac{\boldsymbol{K}}{\mathscr{r}} \, \mathrm{d}a'$$ [20

Relations between $\boldsymbol{A}, \boldsymbol{J}, \boldsymbol{B}$ that decalibes magnetostatis is summarized as follows:



Boundary conditions and Discontinuities

For a surface charge, and a perpendicular amperean loop:

$$\oint \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{l} = \mu_0 \boldsymbol{K} l \Rightarrow \boldsymbol{B}_+ - \boldsymbol{B}_- = \mu_0 K,$$ [21

so that

$$\boldsymbol{B}_+ - \boldsymbol{B}_- = \mu_0 (\boldsymbol{K} \times \hat{n}),$$
$$\boxed{\frac{\partial \boldsymbol{A}_+}{\partial n} - \frac{\partial \boldsymbol{A}_-}{\partial n} = -\mu_0 \boldsymbol{K}}$$ [22

and notedly,

$$\boldsymbol{A}_+ = \boldsymbol{A}_-.$$ [23

Multipole Expansion of vector potential

$$\boldsymbol{A}(\vec{r}) = \frac{\mu_0 I}{4\pi} \oint \frac{1}{\mathscr{r}} \, \mathrm{d}\boldsymbol{l}' = \frac{\mu_0 I}{4\pi} \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \oint (r')^n P_{n(\cos\alpha)} \, \mathrm{d}\boldsymbol{l}'$$ [24

Given that $\oint \mathrm{d}\boldsymbol{l}' = 0$, $\boldsymbol{A}_{\text{mono}} = 0$, and with some vector algebra,

$$\boxed{\boldsymbol{A}_{\text{dipole}} = \frac{\mu_0 I}{4\pi} \frac{\boldsymbol{m} \times \hat{r}}{r^2}}$$
$$\boxed{\boldsymbol{m} \equiv I \boldsymbol{a}}$$ [25

- putting dipole at origin,

$$\boldsymbol{m} \times \hat{r} = m \sin\theta,$$ [26

  So

$$\boxed{\boldsymbol{A}_{\text{dipole}} = \frac{\mu_0 m \sin\theta}{4\pi r^2}\hat{\varphi}}$$
$$\boldsymbol{B}_{\text{dip}} = \boldsymbol{\nabla} \times \boldsymbol{A} = \frac{\mu_0 m}{4\pi r^3}\left(2\cos\theta\hat{r} + \sin\theta\hat{\theta}\right) \tag{27}$$

Magnetic Fields in Matter

Torque and Force on a magnetic dipole

For a magnetic dipole square, $m = Iab$. Torque in a magnetic field is:

$$\boldsymbol{N} = \boldsymbol{m} \times \boldsymbol{B} \tag{28}$$

For an infinitesimal loop, with dipole moment $\boldsymbol{m}$, in a field $\boldsymbol{B}$, the force is

$$\boldsymbol{F} = \boldsymbol{\nabla}(\boldsymbol{m} \cdot \boldsymbol{B}) \tag{29}$$

Such torque is the main cause behind paramagnetism.

Diamagnetism: revolving electron in $\boldsymbol{B}$

For a revolving electronaround neucleus,

$$\begin{cases} I = -\dfrac{ev}{2\pi R} \\ \boldsymbol{m} = -\dfrac{1}{2}evR\hat{z} \end{cases} \tag{30}$$

In the presence of a magnetic field, and by newton's second law,

$$\frac{1}{4\pi\varepsilon_0}\frac{e^2}{R^2} + evB = m\frac{v^2}{R^2}$$
$$\Rightarrow \Delta v = \frac{eRB}{2m_e}, \quad \Delta\boldsymbol{m} = -\frac{e^2R^2}{4m_e}\boldsymbol{B} \tag{31}$$

Magnetization

We denote magneitzation

$$\boxed{\boldsymbol{M} = \frac{\boldsymbol{m}}{V},} \tag{32}$$

magnetic moment per unit volume.

Field due to Magnetized Object

Recall $\boldsymbol{A}$ due to magnetic dipole with dipole moment $\boldsymbol{m}$:

$$\boldsymbol{A} = \frac{\mu_0}{4\pi}\frac{\boldsymbol{m} \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}. \tag{33}$$

For a magnetized object, each volume element carries $\boldsymbol{m} = \boldsymbol{M}\,\mathrm{d}\tau$, so $\boldsymbol{A}$ due to a magnetized object with magnetization $\boldsymbol{M}$is :

$$\boldsymbol{A} = \frac{\mu_0}{4\pi}\int_V \frac{\boldsymbol{M}(\vec{r}) \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\,\mathrm{d}\tau. \tag{34}$$

If we define a volume current density

$$\boxed{\boldsymbol{J_b} = \boldsymbol{\nabla} \times \boldsymbol{M}}$$ [35

and a surface current density

$$\boxed{\boldsymbol{K_b} = \boldsymbol{M} \times \hat{n}}$$ [36

then we can write

$$\boldsymbol{A}(\vec{r}) = \underbrace{\frac{\mu_0}{4\pi} \int_V \frac{\boldsymbol{J_b}(r')}{\mathscr{r}} \, \mathrm{d}\tau}_{\text{potential of Vol. current}} + \underbrace{\frac{\mu_0}{4\pi} \oint_S \frac{\boldsymbol{K_b}(r')}{\mathscr{r}} \, \mathrm{d}a'}_{\text{potential of Surf. current}}$$ [37

Auxiliary Field H

In a material, we can take the total current to be

$$\boldsymbol{J} = \boldsymbol{J}_b + \boldsymbol{J}_f$$ [38

and by Amepere's law,

$$\boldsymbol{\nabla} \times \left(\frac{1}{\mu_0} \boldsymbol{B} - \boldsymbol{M} \right) = \boldsymbol{J}_f.$$ [39

We are thus motivated to define an Auxiliary Field H:

$$\boxed{\boldsymbol{H} \equiv \frac{1}{\mu_0} \boldsymbol{B} - \boldsymbol{M}.}$$ [40

From which we can write the Amepere's Law in Magnetized material as

$$\boldsymbol{\nabla} \times \boldsymbol{H} = \boldsymbol{J}_f \quad \Leftrightarrow \quad \oint \boldsymbol{H} \cdot \mathrm{d}\boldsymbol{l} = I_{\text{f\_enc}}$$ [41

Boundary Conditions for H and B

Divergence of H gives:

$$\boldsymbol{\nabla} \cdot \boldsymbol{H} = -\boldsymbol{\nabla} \cdot \boldsymbol{M}.$$ [42

And from fundemental theorem of divergence,

$$H^{\perp}_{\text{above}} - H^{\perp}_{\text{below}} = -\left| \left(M^{\perp}_{\text{above}} - M^{\perp}_{\text{below}} \right) \right|.$$ [43

While curl of H gives

$$\boldsymbol{H}^{\parallel}_{\text{above}} - \boldsymbol{H}^{\parallel}_{\text{below}} = K_f \times \hat{n}.$$ [44

Bondary conditions for B are:

$$\begin{gathered} B^{\perp}_{\text{above}} - B^{\perp}_{\text{below}} = 0, \\ \boldsymbol{B}^{\parallel}_{\text{above}} - \boldsymbol{B}^{\parallel}_{\text{below}} = \mu_0 \left(\boldsymbol{K}_f \times \hat{n} \right). \end{gathered}$$ [45

Linear and Nonlinear MEdia

- For linear media, magnetization is proportional to the Auxiliary Field H:

$$\begin{aligned} \boldsymbol{M} &= \chi_m \boldsymbol{H}, \\ \boldsymbol{B} &= \mu \boldsymbol{H}. \end{aligned} \tag{46}$$

Where $\mu \equiv \mu_0(1 + \chi_m)$

Electrodynamics

Ohm's law

For a material with conductivity σ, the current density is:

$$\boldsymbol{J} = \sigma \boldsymbol{E}, \tag{47}$$

and from which we can find

$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = \frac{1}{\sigma}\,\mathrm{d}\boldsymbol{J} = 0, \tag{48}$$

so we are in such a regime where there's no charge accumulation (incompressible field).

Equation 47 contains information on current density, and from which we can retrieve the current flowing through a materical as

$$\boxed{I = \int_V J\,\mathrm{d}\boldsymbol{a}} \tag{49}$$

By noticing a proportionality between the total currnet flowing from one electrode to the otehr and the potential difference that set them aside, we arrive at a more familiar form of Ohm's law:

$$V = IR. \tag{50}$$

EMF

For a circuit, there are two driving factors for the current, and we decalibe them using $\boldsymbol{f}$: force per unit charge.

$$\boldsymbol{f} = \boldsymbol{f}_s + \boldsymbol{E}. \tag{51}$$

So for a loop circuit,

$$\oint \boldsymbol{f}\,\mathrm{d}\boldsymbol{l} = \oint \boldsymbol{f}_s \cdot \mathrm{d}\boldsymbol{l} + \underbrace{\oint \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{l}}_{=0} \equiv \mathcal{E} \tag{52}$$

Motional EMF and Faraday's Law

Initially, the theory is such that when f_s is due to magnetic force, then there is EMF generated in a loop. But then Faraday found through various experiments that:

> Whenever (and for whatever reason) the magnetic flux throught a loop changes, an emf
>
> $$\boxed{\mathcal{E} = -\frac{\mathrm{d}\Phi_B}{\mathrm{d}t}} \tag{53}$$
>
> will appear in the loop, where
>
> $$\Phi_B = \int_A \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{a} \tag{54}$$

Maybe it's worth keep in mind WHY this is the case. We now know the how's, but it was hinted by GPT that the why is coming from Special Relativity. Stay tuned– we need to revisit this with an reletivistic eye! (though it may be many years ahead...)

Electromagnetic Induction, Induced Electric Field

From Faraday's law:

$$\mathcal{E} = \oint \boldsymbol{E}\,\mathrm{d}\boldsymbol{l} = -\frac{\mathrm{d}\Phi}{\mathrm{d}t}$$
$$\Rightarrow \boxed{\oint \boldsymbol{E}\,\mathrm{d}\boldsymbol{l} = -\int_A \frac{\partial \boldsymbol{B}}{\partial t}\,\mathrm{d}\boldsymbol{a}}, \qquad [55$$

and by Stoke's theorem, we can write:

$$\boxed{\boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}} \qquad [56$$

When finding the direction of emf, we can use Lenz's law: **Nature abhors a change in flux.**

When symmetry permits, we can use all tricks from Ampere's law (particually, amperean loop) on Equation 55 to find the induced electric field.

If $\boldsymbol{E}$ is a pure Faraday field, that is, if E is due exclusively to a changing B, with $\rho = 0$, then:

$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = 0 \qquad [57$$

Notice an analogy between Equation 56, Equation 57 and magnetostatics,

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0, \boldsymbol{\nabla} \times \boldsymbol{B} = \mu_0 \boldsymbol{J}. \qquad [58$$

Further exploit this analogy, we can use Biot-Savar's law to find induced electric field:

$$\boldsymbol{E} = -\frac{1}{4\pi}\frac{\partial}{\partial t}\int_V \frac{\boldsymbol{B} \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\,\mathrm{d}\tau \qquad [59$$

Quasistatic Approximation

When finding the change in magnetic flux, we can use *magnetostatics* for an appriciable approximation on $\Phi = \int_A \boldsymbol{B}\,\mathrm{d}\boldsymbol{a}$