MT 2

1) spherical shell, $\sigma = \sigma_0(1+\cos\theta)$, radius a

total charge $Q = \int \sigma\, da = \sigma_0 \int (1+\cos\theta)\, a^2\, d(\cos\theta)\, d\phi$

$$= 2\pi a^2 \sigma_0 \int_{-1}^{1} (1+x)\, dx = 2\pi a^2 \sigma_0 \left(x + \frac{x^2}{2}\right)\Big|_{-1}^{1} = 4\pi a^2 \sigma_0$$

potential

$$V_{r<a} = \sum_{l=0}^{\infty} A_l \left(\frac{r}{a}\right)^l P_l$$

$$V_{r>a} = \sum_{l=0}^{\infty} A_l \left(\frac{a}{r}\right)^{l+1} P_l$$

$$\left(\frac{-\partial V_{r>a}}{\partial r} - \left(\frac{-\partial V_{r<a}}{\partial r}\right)\right)\Bigg|_{r=a} = \sigma/\epsilon_0$$

$$\left(\sum_{l=0}^{\infty} -\left(-(l+1)\frac{a^{l+1}}{r^{l+2}} A_l P_l\right) + \sum_{l=0}^{\infty} A_l\, l \frac{r^{l-1}}{a^l} P_l\right)\Bigg|_{r=a} = \frac{\sigma_0}{\epsilon_0}(1+\cos\theta)$$

$$\sum_{l=0}^{\infty}\left(\frac{(l+1)A_l P_l}{a} + l\frac{A_l}{a}P_l\right) = \sigma_0/\epsilon_0\, (P_0 + P_1)$$

$$= \sum_l \frac{2l+1}{a} A_l P_l = \sigma_0/\epsilon_0\, (P_0 + P_1)$$

$$\frac{1}{a} A_0 = \sigma_0/\epsilon_0 \qquad \frac{3}{a} A_1 = \sigma_0/\epsilon_0$$

$$A_0 = \sigma_0 a/\epsilon_0 \qquad A_1 = \sigma_0 a/3\epsilon_0$$

$\Rightarrow$

$$V_{r<a} = \frac{\sigma_0 a}{\epsilon_0} + \frac{\sigma_0 a}{3\epsilon_0}\frac{r}{a}P_1 = \frac{\sigma_0 a}{\epsilon_0} + \frac{\sigma_0 r}{3\epsilon_0}P_1 = \frac{\sigma_0}{\epsilon_0}\left(a + \frac{r}{3}P_1\right)$$

$$V_{r>a} = \frac{\sigma_0 a}{\epsilon_0}\frac{a}{r} + \frac{\sigma_0 a}{3\epsilon_0}\frac{a^2}{r^2}P_1 = \frac{\sigma_0}{\epsilon_0}\frac{a^2}{r} + \frac{\sigma_0}{3\epsilon_0}\frac{a^3}{r^2}P_1 = \frac{\sigma_0}{\epsilon_0}\left(\frac{a^2}{r} + \frac{a^3}{3r^2}P_1\right)$$

in terms of total charge on the sphere

$$V_{r<a} = \frac{Q}{4\pi a^2\epsilon_0}\left(a + \frac{r}{3}P_1\right) = \frac{Q}{4\pi a\epsilon_0} + \frac{Qr}{4\pi a^2\, 3\epsilon_0}P_1$$

$$V_{r>a} = \frac{Q}{4\pi a^2\epsilon_0}\left(\frac{a^2}{r} + \frac{a^3}{3r^2}P_1\right) = \frac{Q}{4\pi\epsilon_0 r} + \underbrace{\frac{Qa^3}{4\pi a^2(3r^2)\epsilon_0}P_1}_{\frac{Qa}{4\pi\epsilon_0(3r^2)}P_1}$$

check:
$$\left(-\frac{\partial V_{r>a}}{\partial r} + \frac{\partial V_{r<a}}{\partial r}\right)\Bigg|_a =$$

$$\left(-\frac{\sigma_0}{\epsilon_0}\left(-\frac{a^2}{r^2} - \frac{2a^3}{3r^3}P_1\right) + \frac{\sigma_0}{\epsilon_0}\left(\frac{1}{3}\right)P_1\right)\Bigg|_{r=a} = \frac{\sigma_0}{\epsilon_0}\left(1 + \left(\frac{2}{3}+\frac{1}{3}\right)P_1\right) = \frac{\sigma_0}{\epsilon_0}(P_0 + P_1) \checkmark$$

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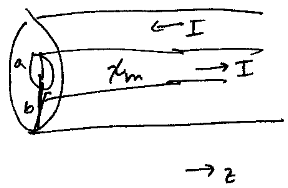
2)



a) $\oint \vec{H}\cdot d\vec{l} = I_{f\,\text{enclosed}}$ $\qquad \vec{H} = H_\phi(s)\,\hat{\phi}$

$s<a$

$$H_\phi 2\pi s = \frac{I \pi s^2}{2\pi a^2} \Rightarrow H_{\phi\, s<a} = \frac{Is}{2\pi a^2}$$

$a<s<b$

$$H_\phi 2\pi s = I \Rightarrow H_{\phi\, a<s<b} = \frac{I}{2\pi s}$$

$s>b \qquad \vec{H} = 0$

$$\Rightarrow s<a: \quad \vec{H} = \frac{Is}{2\pi a^2}\hat{\phi} \quad \vec{B} = \mu\vec{H} = \mu_0(1+\chi_m)\frac{Is}{2\pi a^2}\hat{\phi} \quad \vec{M} = \chi_m\vec{H} = \frac{\chi_m I s}{2\pi a^2}\hat{\phi}$$

$$a<s<b \quad \vec{H} = \frac{I}{2\pi s}\hat{\phi} \quad \vec{B} = \mu_0\vec{H} = \frac{\mu_0 I}{2\pi s}\hat{\phi} \quad \vec{M} = 0$$

$$s>b \quad \vec{H} = 0, \ \vec{B} = 0, \ \vec{M} = 0$$

b) $\vec{J}_{b\,s<a} = \nabla\times\vec{M} = \frac{1}{s}\frac{\partial}{\partial s}(sM_\phi)\,\hat{z} = \frac{\chi_m I}{\pi a^2}\hat{z} = \chi_m \vec{J}_f \qquad (\nabla\times\vec{M} = \nabla\times\chi_m\vec{H} = \chi_m\nabla\times\vec{H} = \chi_m\vec{J}_f)$

$$K_{b\,s=a} = (\vec{M}\times\hat{n})\Big|_{s=a} = \vec{M}(s=a)\times\hat{s} = \frac{\chi_m I}{2\pi a}(-\hat{z})$$

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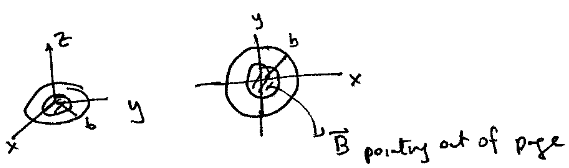
3)



$\vec{B} = B(t)\,\hat{z}$

a) $\oint \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\int \vec{B}\cdot d\vec{a} \qquad \vec{E} = E_\phi(s)\,\hat{\phi}$

$$E_\phi 2\pi s = -\frac{dB}{dt}\pi a^2$$

$$\Rightarrow \vec{E}(s=b) = -dB/dt\,\frac{a^2}{2b}\hat{\phi}$$

b) $\vec{J} = \sigma\vec{E}$

$$I = \int \vec{J}\cdot d\vec{a} = \sigma\int\vec{E}\cdot d\vec{a} = -\sigma(dB/dt)\frac{a^2}{2b}A$$