Math 421, Section 1
Homework 3
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Problem 1. Determine whether each of the following functions are injective, surjective, and bijective, and prove your answer.

(a) $f : \mathbb{Z} \to \mathbb{Z}$, $f(x) = 2x$.
(b) $g : \mathbb{R} \to \mathbb{R}$, $g(x) = 2x$.

Solution: (a) Injectivity: Suppose $\exists x_1, x_2 \in \mathbb{Z}, s.t. f(x_1) = f(x_2)$, want to show: $x_1 = x_2$.

$$f(x_1) = f(x_2) \implies 2x_1 = 2x_2 \implies x_1 = x_2. \tag{1}$$

The function is thus injective.

Surjectivity: Want to show $\forall y \in \mathbb{Z}, \exists x \in \mathbb{Z}$ $s.t.$ $f(x) = y$. Suppose $x, y \in \mathbb{Z}$, and let $f(x) = y$. i.e.,

$$2x = y \implies x = \frac{y}{2} \in \mathbb{Z}. \tag{2}$$

However, $\frac{y}{2} \in \mathbb{Z}$ only if y is even. So the above is not true for an arbiturary $y \in \mathbb{Z}$, contradictory to our assumption. Thus, the function is not surjective.

Collecting the above, the function is not bijective.

(b) Injectivity: Suppose $\exists x_1, x_2 \in \mathbb{R}, s.t. g(x_1) = g(x_2)$, want to show: $x_1 = x_2$.

$$g(x_1) = g(x_2) \implies 2x_1 = 2x_2 \implies x_1 = x_2. \tag{3}$$

The function is thus injective.

Surejectivity: Suppose $y \in \mathbb{R}$, we want to find $x \in \mathbb{R}$, $s.t.$ $g(x) = y$.

$$2x = y \implies x = \frac{y}{2} \in \mathbb{R}. \tag{4}$$

So the function is surjective.

Collecting the above, the function $g(x)$ is bijective.

$\square$

Problem 2. Let $f : A \to B$ be a function and $A_1, A_2 \subseteq A$ and $B_1, B_2 \subseteq B$ be subsets. Prove the following statements:

(a) $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$.
(b) $f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2)$.
(c) $f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$.
(d) $f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$.

Solution: (a) $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$

Proof. $\subseteq$: Let $y \in f(A_1 \cup A_2)$.By definition of image, $\exists x \in A_1 \cup A_2$ s.t. $f(x) = y$. Hence, $x \in A_1$ or $x \in A_2$. Thus, $y \in f(A_1)$ or $y \in f(A_2)$, implying $y \in f(A_1) \cup f(A_2)$. Therefore, $f(A_1 \cup A_2) \subseteq f(A_1) \cup f(A_2)$

$\supseteq$: Let $y \in f(A_1) \cup f(A_2)$. Then $y \in f(A_1)$ or $y \in f(A_2)$.
Thus, $\exists x \in A_1$ or $x \in A_2$ s.t. $f(x) = y$.
Therefore, $x \in A_1 \cup A_2$ and $y = f(x) \in f(A_1 \cup A_2)$.
Thus, $f(A_1) \cup f(A_2) \subseteq f(A_1 \cup A_2)$.

Hence, $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$. □

(b) $f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2)$

Proof. Let $y \in f(A_1 \cap A_2)$. Then $\exists x \in A_1 \cap A_2$ s.t. $f(x) = y$.
Since $x \in A_1$ and $x \in A_2$, $y \in f(A_1)$ and $y \in f(A_2)$. Thus, $y \in f(A_1) \cap f(A_2)$.
Therefore, $f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2)$. □

(c) $f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$

Proof. $\subseteq$: Let $x \in f^{-1}(B_1 \cup B_2)$. Then $f(x) \in B_1 \cup B_2$, so $f(x) \in B_1$ or $f(x) \in B_2$.
Hence, $x \in f^{-1}(B_1)$ or $x \in f^{-1}(B_2)$, implying $x \in f^{-1}(B_1) \cup f^{-1}(B_2)$.

$\supseteq$: Let $x \in f^{-1}(B_1) \cup f^{-1}(B_2)$.
Then $x \in f^{-1}(B_1)$ or $x \in f^{-1}(B_2)$, meaning $f(x) \in B_1$ or $f(x) \in B_2$.
Thus, $f(x) \in B_1 \cup B_2$ and $x \in f^{-1}(B_1 \cup B_2)$.
Therefore, $f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$. □

(d) $f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$

Proof. $\subseteq$: Let $x \in f^{-1}(B_1 \cap B_2)$. Then $f(x) \in B_1 \cap B_2$, so $f(x) \in B_1$ and $f(x) \in B_2$.
Hence, $x \in f^{-1}(B_1)$ and $x \in f^{-1}(B_2)$, implying $x \in f^{-1}(B_1) \cap f^{-1}(B_2)$.

$\supseteq$: Let $x \in f^{-1}(B_1) \cap f^{-1}(B_2)$. Then $f(x) \in B_1$ and $f(x) \in B_2$, so $f(x) \in B_1 \cap B_2$.
Thus, $x \in f^{-1}(B_1 \cap B_2)$.
Therefore, $f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$. □

□

Problem 3. Let $f : A \to B$ be a function. Prove that f is injective if and only if $f(A_1 \cap A_2) = f(A_1) \cap f(A_2)$ for all subsets $A_1, A_2 \subseteq A$.

Solution: $\square$

Problem 4. Let $f : A \to B$ be a function. Prove that the following two statements are equivalent:

(a) The function f is surjective.
(b) For every set C and for any functions $g : B \to C$ and $h : B \to C$ such that $g \circ f = h \circ f$, we have $g = h$.

Solution: (Type your solution to problem 4 here.) $\square$

Problem 5. Let A be a nonempty set and $f : A \to A$ a function. We call f an *involution* if $(f \circ f)(a) = a$ for all $a \in A$. Prove that if $f : A \to A$ is an involution, then f is bijective. What is the inverse function f^{-1} in terms of f?

Solution: (Type your solution to problem 5 here.) $\square$

Problem 6. Prove or disprove the following statements:

(a) The set $\{x \in \mathbb{R} : x \geq 2\}$ is an interval.
(b) The set $\{x \in \mathbb{R} : x \neq 2\}$ is an interval.

(Hint: In order to disprove a statement, you must prove that the negation of the statement is true.)

Solution: (Type your solution to problem 6 here.) $\square$