Last time: Differentiation laws

① $(cf)'(a) = c \cdot f'(a) \quad \forall c \in \mathbb{R}$

② $(f+g)'(a) = f'(a) + g'(a)$

Thm (Product rule) Let $I \subseteq \mathbb{R}$ be an open interval, $f, g : I \to \mathbb{R}$, and $a \in I$. If f and g are differentiable at a, then fg is differentiable at a and

$$(fg)'(a) = f'(a)g(a) + f(a)g'(a).$$

Pf: For $h \neq 0$,

$$\frac{f(a+h)g(a+h) - f(a)g(a)}{h}$$

$$= \frac{f(a+h)g(a+h) - f(a)g(a+h) + f(a)g(a+h) - f(a)g(a)}{h}$$

$$= \frac{f(a+h) - f(a)}{h} \cdot g(a+h) + f(a) \cdot \frac{g(a+h) - g(a)}{h}$$

As f and g are differentiable at a, we know:

① $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = f'(a)$

② $\lim_{h \to 0} \frac{g(a+h) - g(a)}{h} = g'(a)$

③ g continuous at a (by lecture 22)

$\Rightarrow \lim_{x \to a} g(x) = g(a)$

$$\Rightarrow \quad \lim_{h\to 0} g(a+h) = g(a)$$

Therefore, by the limit laws,

$$\lim_{h\to 0} \frac{f(a+h)g(a+h) - f(a)g(a)}{h}$$

$$= \lim_{h\to 0} \left[\frac{f(a+h)-f(a)}{h} \cdot g(a+h) + f(a) \cdot \frac{g(a+h)-g(a)}{h} \right]$$

$$= \quad f'(a) \cdot g(a) + f(a) \cdot g'(a)$$

So $(fg)'(a) = f'(a)g(a) + f(a)g'(a)$. $\square$

<u>Thm</u> (Quotient rule) Let $I \subseteq \mathbb{R}$ be an open interval, $f, g : I \to \mathbb{R}$, and $a \in I$. If f and g are differentiable at a and $g(a) \neq 0$, then $\frac{f}{g}$ is differentiable at a and

$$\left(\frac{f}{g}\right)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{g(a)^2}.$$

<u>Pf</u>: It suffices to show

$$\left(\frac{1}{g}\right)'(a) = -\frac{g'(a)}{g(a)^2}.$$

Indeed, once we prove this, then by the product rule we have

$$\left(f \cdot \frac{1}{g}\right)'(a) = f'(a) \cdot \frac{1}{g(a)} + f(a) \cdot \left(\frac{1}{g}\right)'(a)$$

$$= \frac{f'(a)}{g(a)} - \frac{f(a)g'(a)}{g(a)^2}$$

$$= \frac{f'(a)g(a) - f(a)g'(a)}{g(a)^2}.$$

Want: $\lim_{h\to 0} \frac{\frac{1}{g(a+h)} - \frac{1}{g(a)}}{h} = -\frac{g'(a)}{g(a)^2}$. As g is continuous at a (by Lecture 22) and $g(a) \neq 0$, then $\exists \delta > 0$ s.t. $g(x) \neq 0 \ \forall x \in (a-\delta, a+\delta)$ (by Lecture 15). So for $h \in (-\delta, \delta)$, $\frac{1}{g(a+h)}$ makes sense and

$$\frac{\frac{1}{g(a+h)} - \frac{1}{g(a)}}{h} = \frac{g(a) - g(a+h)}{h \cdot g(a) \cdot g(a+h)}$$

$$= -\frac{g(a+h) - g(a)}{h} \cdot \frac{1}{g(a)g(a+h)}$$

As g is differentiable at a, we know:

① $\lim_{h\to 0} \frac{g(a+h) - g(a)}{h} = g'(a)$

② g continuous at a (by Lecture 22)

$\Rightarrow \lim_{x\to a} g(x) = g(a)$

$\Rightarrow \lim_{h\to 0} g(a+h) = g(a)$ and $g(a) \neq 0$

$\Rightarrow \lim_{h\to 0} \frac{1}{g(a)g(a+h)} = \frac{1}{g(a)^2}$

Together, by the limit laws,

$$\lim_{h\to 0} \frac{\frac{1}{g(a+h)} - \frac{1}{g(a)}}{h} = \lim_{h\to 0} \left[-\frac{g(a+h) - g(a)}{h} \cdot \frac{1}{g(a)g(a+h)} \right]$$

$$= -g'(a) \cdot \frac{1}{g(a)^2}$$

So $\left(\frac{1}{g}\right)'(a) = -\frac{g'(a)}{g(a)^2}$. $\square$

__Ex__ $\forall n \in \mathbb{Z}$, $\frac{d}{dx}(x^n) = nx^{n-1}$.

__Pf__: Case: $n \geq 0$. Already know (Lecture 24).

Case: $n < 0$. Then $x^n = \frac{1}{x^{-n}}$ with $-n > 0$.

Fix $a \neq 0$. We know $g(x) = x^{-n}$ is differentiable with $g'(a) = -na^{-n-1}$ and $g(a) = a^{-n} \neq 0$. So, by the quotient rule,

$$(x^n)'(a) = \left(\frac{1}{g}\right)'(a) = -\frac{g'(a)}{g(a)^2}$$

$$= -\frac{(-n)a^{-n-1}}{(a^{-n})^2} = na^{-n-1+2n} = na^{n-1}. \quad \square$$

__Ex__ Let $f: \mathbb{R} \to \mathbb{R}$, $f(x) = \frac{x}{x^2+1}$. Prove f is differentiable, and find its derivative.

__Pf__: Fix $a \in \mathbb{R}$. Then

$$(x)'(a) = 1, \qquad (x^2+1)'(a) = 2a$$

(These are polynomials, which we already know are differentiable on $\mathbb{R}$.) As $a^2+1 \neq 0$ (since $a^2 \geq 0$), then by the quotient rule we have

$$f'(a) = \frac{1 \cdot (a^2+1) - a \cdot 2a}{(a^2+1)^2} = \frac{-a^2+1}{(a^2+1)^2}. \qquad \square$$

<u>Thm</u> (Chain rule) Let $I, J \subseteq \mathbb{R}$ be open intervals, $f: I \to J$ and $g: J \to \mathbb{R}$ be functions, and $a \in I$. If f is differentiable at a and g is differentiable at $f(a)$, then $g \circ f$ is differentiable at a and

$$(g \circ f)'(a) = g'(f(a)) \cdot f'(a).$$

<u>Recall</u> Differentiation laws

① $(cf)'(a) = cf'(a) \quad \forall c \in \mathbb{R}$

② $(f+g)'(a) = f'(a) + g'(a)$

③ $(fg)'(a) = f'(a)g(a) + f(a)g'(a)$

④ $\left(\frac{f}{g}\right)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{g(a)^2}$

⑤ $(g \circ f)'(a) = g'(f(a)) \cdot f'(a)$

<u>Thm</u> (Chain rule) Let $I, J \subseteq \mathbb{R}$ be open intervals, $f : I \to J$ and $g : J \to \mathbb{R}$ be functions, and $a \in I$. If f is differentiable at a and g is differentiable at $f(a)$, then $g \circ f$ is differentiable at a and

$$(g \circ f)'(a) = g'(f(a)) \cdot f'(a).$$

<u>Idea:</u> For $h \neq 0$,

$$\frac{g(f(a+h)) - g(f(a))}{h} = \underbrace{\frac{g(f(a+h)) - g(f(a))}{f(a+h) - f(a)}}_{= \frac{g(f(a)+k) - g(f(a))}{k}} \cdot \underbrace{\frac{f(a+h) - f(a)}{h}}_{\to f'(a) \text{ as } h \to 0}$$

with $k(h) = f(a+h) - f(a) \to 0$ as $h \to 0$

$\to g'(f(a))$ as $h \to 0$

<u>Problem</u> What if $k = 0$? In fact, we could have $k(h) = 0$ infinitely often near $h = 0$! E.g.

- $f(x) = \text{constant} \Rightarrow k(h) = f(a+h) - f(a) = 0 \quad \forall h$

(Well, maybe this isn't really a problem, since the other factor $f'(0)$ is $0 \ldots$)

- $f(x) = x^2 \sin\frac{1}{x}$, $a=0 \Rightarrow k(h) = h^2 \sin\frac{1}{h}$

Pf: For $h \neq 0$,

$$\frac{g(f(a+h)) - g(f(a))}{h} = \phi(h) \cdot \frac{f(a+h) - f(a)}{h} \qquad (\star)$$

where

$$\phi(h) = \begin{cases} \dfrac{g(f(a+h)) - g(f(a))}{f(a+h) - f(a)} & \text{if } f(a+h) \neq f(a) \\ g'(f(a)) & \text{if } f(a+h) = f(a) \end{cases}$$

Note that in both cases, the equation $(\star)$ holds:

- If $f(a+h) \neq f(a)$: $f(a+h) - f(a)$ cancels out, as before
- If $f(a+h) = f(a)$: $\quad \text{RHS}(\star) = g'(f(a)) \cdot \dfrac{f(a+h) - f(a)}{h} = 0$

$\quad \text{LHS}(\star) = \dfrac{g(f(a+h)) - g(f(a))}{h} = 0$

Claim: $\phi(h)$ is continuous at $h=0$, i.e.

$\lim_{h \to 0} \phi(h) = g'(f(a))$. Fix $\varepsilon > 0$. Want: $\exists \delta > 0$ s.t.

$$|h| < \delta \Rightarrow \underbrace{|\phi(h) - g'(f(a))|}_{} < \varepsilon$$

$$= \begin{cases} \dfrac{g(f(a+h)) - g(f(a))}{f(a+h) - f(a)} - g'(f(a)) \\ g'(f(a)) - g'(f(a)) = 0 \quad \text{already } < \varepsilon! \end{cases}$$

So it suffices to show: $\exists\, \delta > 0$ s.t.

$$|h| < \delta,\ f(a+h) \neq f(a) \implies \left| \frac{g(f(a+h)) - g(f(a))}{f(a+h) - f(a)} - g'(f(a)) \right| < \varepsilon$$

We know:

① g is differentiable at $f(a)$

$$\implies \lim_{k \to 0} \frac{g(f(a)+k) - g(f(a))}{k} = g'(f(a))$$

$\implies \exists\, \delta_1 > 0$ s.t. $|k| < \delta_1$ implies

$$\left| \frac{g(f(a)+k) - g(f(a))}{k} - g'(f(a)) \right| < \varepsilon$$

② f is differentiable at a

$\implies f$ is continuous at a $\implies \lim_{h \to 0} f(a+h) = f(a)$

$\implies \exists\, \delta_2 > 0$ s.t. $|h| < \delta_2$ implies $|f(a+h) - f(a)| < \delta_1$

Altogether,

$$|h| < \delta_2 \implies |\underbrace{f(a+h) - f(a)}_{\text{Take } = k \text{ in } ①}| < \delta_1$$

$$\implies \left| \frac{g(f(a+h)) - g(f(a))}{f(a+h) - f(a)} - g'(f(a)) \right| < \varepsilon$$

as desired. This finishes the claim.

Recall: for $h \neq 0$,

$$\frac{g(f(a+h)) - g(f(a))}{h} = \phi(h) \cdot \frac{f(a+h) - f(a)}{h}$$

By the claim,

$$\lim_{h\to 0} \phi(h) = g'(f(a)).$$

As f is differentiable at a,

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = f'(a).$$

Therefore, by the limit law for products,

$$\begin{array}{ccc} \displaystyle\lim_{h\to 0} \frac{g(f(a+h))-g(f(a))}{h} & = & \displaystyle\lim_{h\to 0} \left[\phi(h) \cdot \frac{f(a+h)-f(a)}{h} \right] \\ \| & & \| \\ (g\circ f)'(a) & & g'(f(a)) \cdot f'(a) \end{array} \qquad \square$$

<u>Ex</u> (a.) Prove that $f:(-1,1)\to\mathbb{R}$, $f(x)=\sqrt{1-x^2}$ is differentiable, and find its derivative.

<u>Pf</u>: Write

$$f = g\circ h, \qquad \text{where} \quad g(y)=\sqrt{y}, \quad h(x) = 1-x^2$$

Fix $a\in\mathbb{R}$.

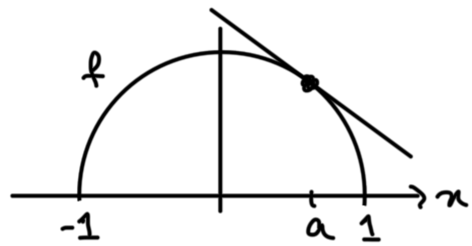
- $h:(-1,1)\to(0,\infty)$ is differentiable at a since h is a polynomial, and $h'(a) = -2a$
- $g:(0,\infty)\to\mathbb{R}$ is differentiable at $y = 1-a^2>0$ by Hw 8, and $g'(y) = \frac{1}{2\sqrt{y}}$

So, by the chain rule,

$$f'(a) = (g\circ h)'(a) = g'(h(a))\cdot h'(a)$$

$$= \frac{1}{2\sqrt{1-a^2}} \cdot (-2a) = \frac{-a}{\sqrt{1-a^2}}. \qquad \square$$

(b.) Find a formula for the tangent line to the graph of f at $a \in (-1,1)$.



$$L(x) = f(a) + f'(a)(x-a)$$

$$= \sqrt{1-a^2} - \frac{a}{\sqrt{1-a^2}}(x-a)$$

$$= -\frac{a}{\sqrt{1-a^2}}x + \frac{1-a^2+a^2}{\sqrt{1-a^2}} = -\frac{a}{\sqrt{1-a^2}}x + \frac{1}{\sqrt{1-a^2}}$$

(c.) Prove that this tangent line intersects the unit circle $\{(x,y) \in \mathbb{R}^2 : x^2+y^2=1\}$ exactly once.

<u>Pf</u>: Note that

$L(x)$ intersects the unit circle $\iff$ $(x, L(x))$ satisfies $x^2 + L(x)^2 = 1$.

$$x^2 + L(x)^2 = x^2 + \left(\frac{-ax+1}{\sqrt{1-a^2}}\right)^2$$

$$= x^2 + \frac{a^2x^2 - 2ax + 1}{1-a^2}$$

$$= \frac{(1-\cancel{a^2})x^2 + \cancel{a^2x^2} - 2ax + 1}{1-a^2}$$

So $$1 = x^2 + L(x)^2 = \frac{x^2 - 2ax + 1}{1-a^2}$$

$$\iff x^2 - 2ax + 1 = 1 - a^2$$

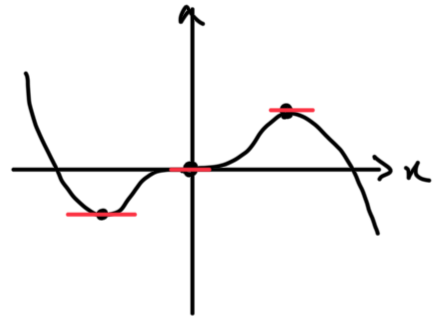
$$\iff 0 = x^2 - 2ax + a^2 = (x-a)^2$$

$\iff \quad x = a. \qquad \square$

SIGNIFICANCE OF THE DERIVATIVE (Ch. 11)

<u>Def</u> Let $I \subseteq \mathbb{R}$ be an open interval, $f: I \to \mathbb{R}$, and $a \in I$. We say a is a <u>critical point</u> of f if $f'(a) = 0$.

<u>Ex</u> $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^3 - x^5$



$0 = f'(a) = 3a^2 - 5a^4 = a^2(3 - 5a^2)$

$\iff \quad a = 0 \quad \text{or} \quad a^2 = \frac{3}{5}$

$\iff \quad a = 0 \quad \text{or} \quad a = \pm\sqrt{\frac{3}{5}}$

3 critical points.

- What's special about f at these 3 points? E.g. f is biggest at $a = \sqrt{\frac{3}{5}}$, but it's not a global max...

<u>Def</u> Let $A \subseteq \mathbb{R}$, $f: A \to \mathbb{R}$, and $a \in A$. We say:

① $f(a)$ is a <u>local maximum</u> of f if $\exists\, \delta > 0$ s.t. $f(x) \leq f(a) \quad \forall\, x \in A \cap (a - \delta, a + \delta)$

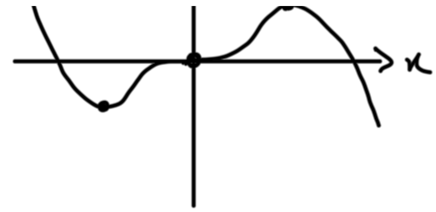
② $f(a)$ is a <u>local minimum</u> of f if $\exists\, \delta > 0$ s.t. $f(x) \geq f(a) \quad \forall\, x \in A \cap (a - \delta, a + \delta)$

<u>Rmk</u> Global max/min $\Rightarrow$ local max/min

<u>Ex</u> $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^3 - x^5$

$f\left(\sqrt{\frac{3}{5}}\right)$ is a local max

- $f(\sqrt{\frac{3}{5}})$ is a local max
- $f(-\sqrt{\frac{3}{5}})$ is a local min
- $f(0)$ is neither
- f has no global max or min



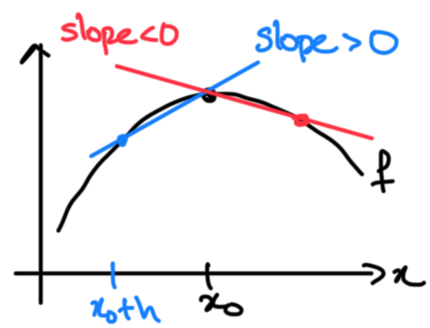
<u>Thm</u> Let $I \subseteq \mathbb{R}$ be an open interval and $x_0 \in I$. If $f: I \to \mathbb{R}$ is differentiable at x_0 and $f(x_0)$ is a local maximum or minimum, then x_0 is a critical point (i.e. $f'(x_0) = 0$).

<u>Rmk</u> $f(x_0)$ local max/min $\overset{\Longrightarrow}{\not\Longleftarrow}$ $f'(x_0) = 0$

E.g. $x_0 = 0$ in previous example.

<u>Pf</u>: Case: Local max. Suppose $f(x_0)$ is a local max and f is differentiable at x_0. Then $\exists\, \delta > 0$ s.t. for $0 < |h| < \delta$,

$$f(x_0 + h) \le f(x_0)$$



$$\Rightarrow f(x_0+h) - f(x_0) \le 0$$

$$\Rightarrow \begin{cases} \dfrac{f(x_0+h) - f(x_0)}{h} \le 0 & \text{if } h > 0 \\ \dfrac{f(x_0+h) - f(x_0)}{h} \ge 0 & \text{if } h < 0 \end{cases}$$

As f is differentiable at x_0, we know

$$\lim_{h \to 0^+} \frac{f(x_0+h) - f(x_0)}{h} = f'(x_0) = \lim_{h \to 0^-} \frac{f(x_0+h) - f(x_0)}{h}.$$

Together,

$$\left.\begin{aligned} f'(x_0) &= \lim_{h\to 0^+} \underbrace{\frac{f(x_0+h)-f(x_0)}{h}}_{\le 0 \quad \forall h\in(0,\delta)} \le 0 \\ f'(x_0) &= \lim_{h\to 0^-} \underbrace{\frac{f(x_0+h)-f(x_0)}{h}}_{\ge 0 \quad \forall h\in(-\delta,0)} \ge 0 \end{aligned}\right\} \Rightarrow f'(x_0)=0.$$

(Recall: Hw4#4

$$\left.\begin{aligned} &\bullet\ f(x)\le g(x)\ \forall x \\ &\bullet\ \lim_{x\to a} f(x) \text{ and } \lim_{x\to a} g(x) \text{ exist} \end{aligned}\right\} \Rightarrow \lim_{x\to a} f(x) \le \lim_{x\to a} g(x)$$

Similar proof works for $\lim_{x\to a^+}$ and $\lim_{x\to a^-}$.)

Case: Local min. Then $-f$ has a local max. By the previous case we know $-f'(x_0)=0$, so $f'(x_0)=0$. □

Q: How do we find the global max/min of $f:[a,b]\to\mathbb{R}$?

A: By the previous theorem, the global max/min $f(x_0)$ must fall into one of the following cases:

① Critical point: $x_0\in(a,b)$ s.t. $f'(x_0)=0$

② Endpoint: $x_0=a,b$

③ Points $x_0\in(a,b)$ where f is not differentiable.

Ex Find all global extrema of $f:[-2,3]\to\mathbb{R}$, $f(x)=2x^3-3x^2-12x+1$.

Pf: By the EVT, we know $\exists\, x_0, x_1 \in [-2,3]$ s.t. $f(x_0)$ is a global min and $f(x_1)$ is a global max.

Case ①: $x_0, x_1 \in (-2,3)$. As f is a polynomial, we know f is differentiable at x_0, x_1. So $f'(x_0)=0$ and $f'(x_1)=0$ by the previous theorem.

$$0 = f'(x) = 6x^2 - 6x - 12 = 6(x^2 - x - 2) = 6(x-2)(x+1)$$

$$\Rightarrow \quad x=-1 \quad \text{or} \quad x=2$$

$$\Rightarrow \quad f(x) = f(-1) = 8 \quad \text{or} \quad f(x) = f(2) = -19$$

Case ②: $x_0, x_1 \in \{-2, 3\}$. Note that

$$f(-2) = -3, \qquad f(3) = -8$$

- No case ③, since f is differentiable on $(-2,3)$

Altogether, we have shown

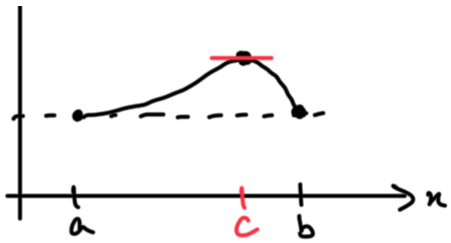
$$x_0, x_1 \in \{-2, -1, 2, 3\}$$

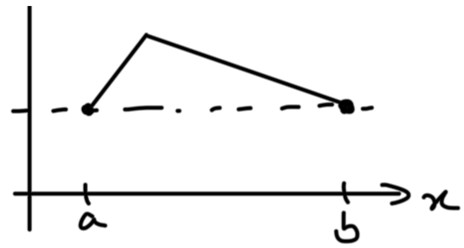
$$f(x) = -3, \ \underline{8}, \ \underline{-19}, \ -8$$

So the global max is $8 = f(-1)$ and the global min is $-19 = f(2)$. $\square$

Thm (Rolle's theorem) If $f: [a,b] \to \mathbb{R}$ is continuous, f is differentiable on (a,b), and $f(a) = f(b)$, then $\exists\, c \in (a,b)$ s.t. $f'(c) = 0$.

↑ ↑





- In order to return to $f(b)=f(a)$, a differentiable function must have at least one critical point

- Not differentiable, no critical points

Pf: As f is continuous on $[a,b]$, then by the EVT $\exists\, x_0, x_1 \in [a,b]$ s.t.

$$f(x_0) \le f(x) \le f(x_1) \quad \forall x \in [a,b].$$

Case: $x_1 \in (a,b)$. As f is differentiable at x_1, then $f'(x_1)=0$ by the previous theorem. So $c=x_1$ works.

Case: $x_0 \in (a,b)$. Then $c=x_0$ works.

Case: $x_0, x_1 \in \{a,b\}$. As $f(a)=f(b)$, then we have

$$f(a) \le f(x) \le f(a) \quad \forall x \in [a,b]$$

$$\Rightarrow \quad f(x) = f(a) \quad \forall x \in [a,b]$$

So f is a constant function. Therefore $f'(x)=0$ $\forall x \in (a,b)$.

In all cases, we found a number $c \in (a,b)$ s.t. $f'(c)=0$. $\square$

Last time: For differentiable functions f,

① 1^{st} derivative test:

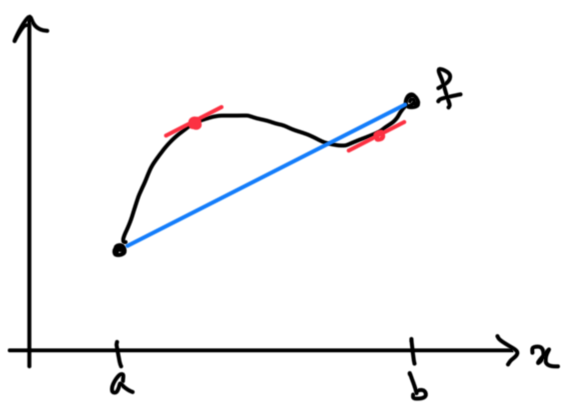
$$f(x_0) \text{ local max/min} \Rightarrow f'(x_0) = 0$$

② Rolle's theorem:

$$f(a) = f(b) \Rightarrow \exists\, c \in (a,b) \text{ s.t. } f'(c) = 0$$

Thm (Mean value theorem) If $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ and differentiable on (a,b), then $\exists\, c \in (a,b)$ s.t.

$$f'(c) = \frac{f(b) - f(a)}{b-a}.$$



- This = slope of blue line = mean slope of f
- In order to connect $f(a)$ and $f(b)$, a differentiable function must attain the mean slope.

Pf: Set $g(x) = f(x) - \left[f(a) + \frac{f(b)-f(a)}{b-a} \cdot (x-a) \right]$.

- This is the height of the graph of f above the blue line

As $x \mapsto -f(a) - \frac{f(b)-f(a)}{b-a} \cdot (x-a)$ is a polynomial, it is continuous and differentiable at any $x \in \mathbb{R}$.

So g satisfies:

① continuous on $[a,b]$

② differentiable on (a,b)

③ $g(a) = g(b)$: since

$$g(a) = f(a) - [f(a) + 0] = 0$$

$$g(b) = f(b) - \left[f(a) + \frac{f(b)-f(a)}{\cancel{b-a}} \cdot \cancel{(b-a)} \right] = 0$$

Therefore, by Rolle's theorem, $\exists\, c \in (a,b)$ s.t.

$$0 = g'(c) = f'(c) - \left[0 + \frac{f(b)-f(a)}{b-a} \right]$$

$$\Rightarrow f'(c) = \frac{f(b)-f(a)}{b-a}. \qquad \square$$

<u>Cor 1</u> Let $I \subseteq \mathbb{R}$ be an open interval and $f: I \to \mathbb{R}$. If $f'(x) = 0$ $\forall x \in I$, then f is a constant function.

<u>Pf</u>: Let $a, b \in I$ with $a<b$. We will prove that $f(a) = f(b)$.

- This is sufficient. Indeed, f not constant $\Rightarrow$ $\exists\, a, b \in I$ s.t. $f(a) \neq f(b)$.

As f is differentiable at x $\forall x \in I$ and $[a,b] \subseteq I$, then:

① f is continuous on $[a,b]$

② f is differentiable on (a,b)

So, by the MVT, $\exists\, c \in (a,b)$ s.t.

$$0 = f'(c) = \frac{f(b)-f(a)}{b-a} \implies f(a) = f(b). \qquad \square$$

<u>Cor 2</u> Let $I \subseteq \mathbb{R}$ be an open interval and $f, g : I \to \mathbb{R}$. If $f'(x) = g'(x)$ $\forall x \in I$, then $\exists\, c \in \mathbb{R}$ s.t. $f(x) = g(x) + c$ $\forall x \in I$.

<u>Pf</u>: Let $h : I \to \mathbb{R}$, $h(x) = f(x) - g(x)$. Then $\forall x \in I$ we have

$$h'(x) = f'(x) - g'(x) = 0.$$

So, by Cor 1, we know h is constant: $\exists\, c \in \mathbb{R}$ s.t.

$$c = h(x) = f(x) - g(x) \quad \forall x \in I$$

$$\implies f(x) = g(x) + c \quad \forall x \in I. \qquad \square$$

<u>Cor 3</u> Let $I \subseteq \mathbb{R}$ be an open interval and $f : I \to \mathbb{R}$.

① If $f'(x) > 0$ $\forall x \in I$, then f is <u>strictly increasing</u>: $\forall x, y \in I$, $x < y \implies f(x) < f(y)$.

② If $f'(x) < 0$ $\forall x \in I$, then f is <u>strictly decreasing</u>: $\forall x, y \in I$, $x < y \implies f(x) > f(y)$.

<u>Pf</u>: ① Let $a, b \in I$ with $a < b$. Want: $f(a) < f(b)$. As f is differentiable at x $\forall x \in I$, then:

- f is continuous on $[a,b]$
- f is differentiable on (a,b)

So, by the MVT, $\exists\ c \in (a,b)$ s.t.

$$0 < f'(c) = \frac{f(b)-f(a)}{b-a} \implies f(a) < f(b).$$

② Suppose $f'(x) < 0\ \forall x \in I$. Then $-f$ satisfies $-f'(x) > 0\ \forall x \in I$. So, by ①, we know $-f$ is strictly increasing:

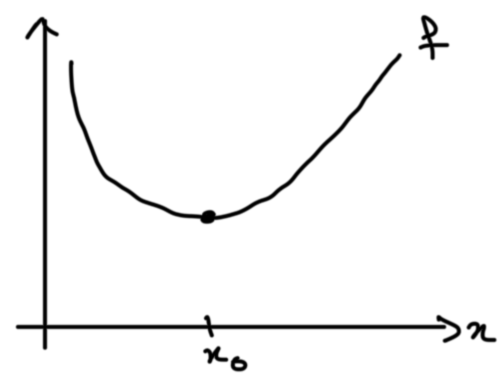
$$\forall a,b \in I, \quad a<b \implies -f(a) < -f(b)$$
$$\implies f(a) > f(b)$$

So f is strictly decreasing. $\square$

Ex Fix $a,b>0$. Find the global minimum of $f:(0,\infty)\to\mathbb{R}$, $f(x) = \frac{a}{x} + bx$ and prove your answer.

Scratch work: Critical points are

$$0 = f'(x) = -\frac{a}{x^2} + b = \frac{bx^2-a}{x^2}$$
$$\implies bx^2 - a = 0$$
$$\implies x_0 = \sqrt{\frac{a}{b}}$$



Solution: We will prove that $f(x_0) = 2\sqrt{ab}$ is the global minimum of f, where $x_0 = \sqrt{\frac{a}{b}}$.

① Claim: $f(x) > f(x_0)\ \forall x \in (x_0, \infty)$. Note that

$$x > x_0 = \sqrt{\frac{a}{b}} \implies x^2 > \frac{a}{b} \implies bx^2 - a > 0$$

$$\Rightarrow \quad f'(x) = \frac{bx^2 - a}{x^2} > 0$$

Given $x > x_0$, by the MVT there is a point $c \in (x_0, x)$ where

$$0 < f'(c) = \frac{f(x) - f(x_0)}{x - x_0} \quad \Rightarrow \quad f(x) > f(x_0).$$

② Claim: $f(x) > f(x_0) \;\; \forall\, x \in (0, x_0)$. Note that

$$0 < x < x_0 = \sqrt{\tfrac{a}{b}} \quad \Rightarrow \quad x^2 < \frac{a}{b} \quad \Rightarrow \quad bx^2 - a < 0$$

$$\Rightarrow \quad f'(x) = \frac{bx^2 - a}{x^2} < 0$$

Given $x < x_0$, by the MVT there is a point $c \in (x, x_0)$ where

$$0 > f'(c) = \frac{f(x_0) - f(x)}{x_0 - x} \quad \Rightarrow \quad f(x) > f(x_0).$$

Altogether, we conclude that $f(x_0)$ is the (strict) global minimum of $f : (0, \infty) \to \mathbb{R}$. $\square$