Last time:

f differentiable at a $\overset{\text{def}}{\iff}$ $f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$ exists

Ex Let $a \in \mathbb{R}$. Are the following functions $f:\mathbb{R}\to\mathbb{R}$ differentiable at a?

(1) $f(x) = 4x + 21$

Yes. Claim: $f'(a) = 4$. For $h \neq 0$,

$$\frac{f(a+h)-f(a)}{h} = \frac{4(\cancel{a}+h)+\cancel{21}-(\cancel{4a}+\cancel{21})}{h} = \frac{4h}{h} = 4$$

- The value of $\frac{f(a+h)-f(a)}{h}$ at $h=0$ does not affect the limit

So
$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0} 4 = 4.$$

(2) $f(x) = x^2$

Yes. Claim: $f'(a) = 2a$. For $h \neq 0$,

$$\frac{f(a+h)-f(a)}{h} = \frac{(a+h)^2-a^2}{h} = \frac{\cancel{a^2}+2ah+h^2-\cancel{a^2}}{h} = 2a+h$$

- We know $\lim_{h\to 0} p(h) = p(0)$ for any polynomial p

So
$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0} (2a+h) = 2a.$$

- In groups:

· In groups.

③ $f(x) = x^3$.

Yes. Claim: $f'(a) = 3a^2$. For $h \neq 0$,

$$\frac{f(a+h)-f(a)}{h} = \frac{(a+h)^3 - a^3}{h} = \frac{\cancel{a^3} + 3a^2h + 3ah^2 + h^3 - \cancel{a^3}}{h}$$

$$= 3a^2 + 3ah + h^2$$

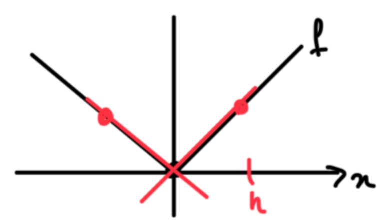
So

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h \to 0} (3a^2 + 3ah + h^2) = 3a^2.$$

④ $f(x) = |x|$ at $a = 0$.

No. Claim: $f'(0)$ does not exist. For $h \neq 0$,

$$\frac{f(0+h)-f(0)}{h} = \frac{|h|}{h} = \begin{cases} 1 & h > 0 \\ -1 & h < 0 \end{cases}$$



So

$$\lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h} = 1, \quad \lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h} = -1$$

$$\Rightarrow \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} \text{ does not exist}$$

<u>Prop</u> If f is differentiable at a, then f is continuous at a.

<u>Pf</u>: We know

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = f'(a), \qquad \lim_{h\to 0} h = 0$$

Therefore, by the limit law for multiplication,

$$\lim_{h\to 0}\left(f(a+h)-f(a)\right) = \lim_{h\to 0}\left(\frac{f(a+h)-f(a)}{h}\cdot h\right)$$

$$= f'(a)\cdot 0 = 0$$

This means: $\forall \varepsilon>0, \ \exists \delta>0$ s.t.

$$0<|h|<\delta \implies |f(a+h)-f(a)|<\varepsilon.$$

Taking h to be $x-a$, we get:

$$0<|x-a|<\delta \implies |f(x)-f(a)|<\varepsilon.$$

So $\lim_{x\to a} f(x) = f(a)$. $\square$

<u>Rmk</u> Continuous $\not\Rightarrow$ differentiable. E.g., $f(x)=|x|$ is continuous and not differentiable at $x=0$.

<u>Ex</u> Let $f:\mathbb{R}\to\mathbb{R}$,

$$f(x) = \begin{cases} x^2 & x\geq 0 \\ -x^2 & x<0 \end{cases}$$

At which points $a\in\mathbb{R}$ is f differentiable?

① Case: $a>0$. Then $f(x)=x^2 \ \forall x\in(a-\delta,a+\delta)$, for $\delta=a$. So the derivative of f at a is the same as for x^2: $f'(a)=2a$.

② Case: $a<0$. Then $f(x)=-x^2 \ \forall x\in(a-\delta,a+\delta)$, for $\delta=|a|$. So the derivative of f at a is the same as for $-x^2$:

$$\lim_{h\to 0}\frac{(a+h)^2-a^2}{h} = 2a$$

$$\Rightarrow \quad f'(a) = \lim_{h\to 0} \frac{-(a+h)^2 - (-a^2)}{h} = -2a.$$

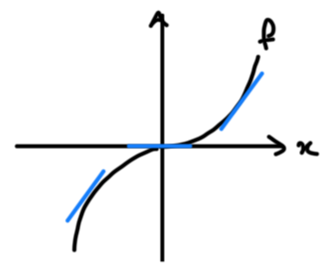
③ Case: $a=0$. For $h \neq 0$,

$$\frac{f(0+h)-f(0)}{h} = \left\{ \begin{array}{ll} \frac{h^2-0}{h} = h & \text{if } h>0 \\ \frac{-h^2-0}{h} = -h & \text{if } h<0 \end{array} \right\} = |h|$$

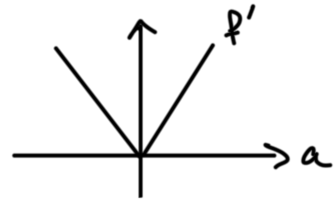
So

$$f'(0) = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = 0.$$

Altogether, f is differentiable at any $a \in \mathbb{R}$, and:



$$f'(a) = \left\{ \begin{array}{ll} 2a & a>0 \\ 0 & a=0 \\ -2a & a<0 \end{array} \right\} = 2|a|$$

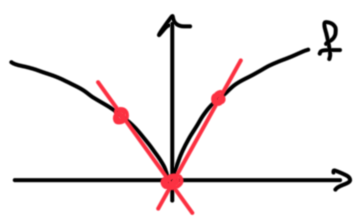


<u>Ex</u> $f(x) = \sqrt{|x|}$. Is f differentiable at $a=0$?

No. We'll prove $f'(0)$ does not exist.

For $h \neq 0$,

$$\frac{f(0+h)-f(0)}{h} = \frac{\sqrt{|h|}}{h} = \left\{ \begin{array}{ll} \frac{1}{\sqrt{h}} & h>0 \\ -\frac{1}{\sqrt{-h}} & h<0 \end{array} \right.$$



Claim: $\lim_{h\to 0^+} \frac{1}{\sqrt{h}}$ does not exist.

Suppose not: $\lim_{h\to 0^+} \frac{1}{\sqrt{h}} = \ell \in \mathbb{R}$.

Consider $\varepsilon = 1$. Then $\exists\, \delta > 0$ s.t.

$$0 < |h| < \delta \implies \left| \frac{1}{\sqrt{h}} - \ell \right| < 1$$

Set $h_0 = \min \left\{ \frac{\delta}{2}, \frac{1}{(|\ell| + 10)^2} \right\}$. Then

$$0 < |h_0| < \delta \qquad \text{but} \qquad \left| \frac{1}{\sqrt{h_0}} - \ell \right| \geq 1$$

since $h_0 > 0$ $\qquad h_0 \leq \frac{\delta}{2}$

$$0 < h_0 \leq \frac{1}{(|\ell| + 10)^2}$$

$$\implies \frac{1}{\sqrt{h_0}} \geq |\ell| + 10$$

$$\implies \frac{1}{\sqrt{h_0}} - \ell \geq \underbrace{|\ell| - \ell}_{\geq 0} + 10 \geq 10$$

which is a contradiction.

Therefore $\lim\limits_{h \to 0} \dfrac{f(0+h) - f(0)}{h}$ does not exist. $\square$