Math 421, Section 1
Homework 6

Problem 1. Prove the following statements:

(a) There exists a number $1 < x < 2$ that solves the equation $x^2 - x - 1 = 0$. (It turns out that there is one such number, and it is called the *golden ratio*.)
(b) There exists a number $x \in \mathbb{R}$ that solves the equation $x^5 - x + 1 = 0$. (Unlike the golden ratio, it turns out that this number cannot be expressed in terms of addition, multiplication, and radicals! This number demonstrates that there is no analogue of the quadratic formula for quintic polynomials.)

Problem 2. Let $a < b$ be numbers and $f : [a, b] \to \mathbb{R}$ be a function. We say that $x \in [a, b]$ is a *fixed point* for f if $f(x) = x$. Prove that if f is continuous and $f(x) \in [a, b]$ for all $x \in [a, b]$, then f has a fixed point.

Problem 3. Let $f : [0, 1] \to \mathbb{R}$ be a continuous function such that $f(0) = f(1)$. Prove that there exists $x \in [0, \frac{1}{2}]$ such that $f(x) = f(x + \frac{1}{2})$. (Hint: Consider the function $g(x) = f(x) - f(x + \frac{1}{2})$. Is it possible for $g(0)$ and $g(\frac{1}{2})$ to both be positive?)

Problem 4. For each of the following functions $f : [-1, 1] \to \mathbb{R}$, find all global extrema and find the points $x \in [-1, 1]$ at which f attains these extrema. (You do not need to prove your answer.)

(a) $f(x) = \begin{cases} 1 - x & \text{if } x \geq 0, \\ 1 + x & \text{if } x < 0. \end{cases}$
(b) $f(x) = \begin{cases} 1 - x & \text{if } x \geq 0, \\ -1 - x & \text{if } x < 0. \end{cases}$
(c) $f(x) = \begin{cases} 1 - x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$

Problem 5. Let $h > 0$. Prove that there is a point on the parabola

$$\{(x, x^2) \in \mathbb{R}^2 : -10 \leq x \leq 10\}$$

that is closest to the point $(0, h)$.

Problem 6. Let $a < b$ be numbers and $f, g, h : [a, b] \to \mathbb{R}$ be functions.

(a) Prove that if f is continuous, then $|f|$ has a global maximum. Given a continuous function f, we define $\|f\|$ to be equal to this value (i.e. the global maximum of $|f|$).
(b) Prove that if g is continuous, then $\|cg\| = |c| \cdot \|g\|$ for any $c \in \mathbb{R}$.
(c) Prove that if g and h are continuous, then $\|g + h\| \leq \|g\| + \|h\|$.