Chapter 2: Fundamental Proof Techniques and Number Properties

Proof Based Calculus

Contents

1 Introduction

Building upon the logical foundations established in the previous chapter, this section delves into fundamental proof techniques and essential properties of numbers and real operations. Mastery of these concepts is vital for constructing rigorous mathematical arguments and developing a deeper understanding of calculus. Key topics include the classification of integers as even or odd, direct proofs, mathematical induction, properties of real numbers, and the manipulation of inequalities. The accompanying problem set provides practical applications of these principles, reinforcing theoretical knowledge through systematic problem-solving.

2 Properties of Integers: Even and Odd Numbers

2.1 Definitions

Understanding the nature of integers is foundational in mathematics. Integers can be classified as **even** or **odd** based on their divisibility by 2.

- **Even Numbers**: An integer x is **even** if there exists an integer n such that:

$$x = 2n$$

- **Odd Numbers**: An integer x is **odd** if there exists an integer n such that:

$$x = 2n + 1$$

2.2 Properties of Even and Odd Numbers

1. **Closure Under Addition**:
   - **Even ± Even**: The sum or difference of two even numbers is even.

     $$\text{If } x = 2n \text{ and } y = 2m, \text{ then } x \pm y = 2(n \pm m)$$

   - **Odd ± Odd**: The sum or difference of two odd numbers is even.

     $$\text{If } x = 2n + 1 \text{ and } y = 2m + 1, \text{ then } x \pm y = 2(n + m + 1)$$

   - **Even ± Odd**: The sum or difference of an even and an odd number is odd.

     $$\text{If } x = 2n \text{ and } y = 2m + 1, \text{ then } x \pm y = 2(n + m) + 1$$

2. **Closure Under Multiplication**:
   - **Even × Even**: The product of two even numbers is even.

     $$x \times y = (2n) \times (2m) = 4nm = 2(2nm)$$

   - **Even × Odd**: The product of an even and an odd number is even.

     $$x \times y = (2n) \times (2m + 1) = 4nm + 2n = 2(2nm + n)$$

   - **Odd × Odd**: The product of two odd numbers is odd.

     $$x \times y = (2n + 1) \times (2m + 1) = 4nm + 2n + 2m + 1 = 2(2nm + n + m) + 1$$

2.3 Examples

- **Addition**: Let $x = 3$ (odd) and $y = 5$ (odd). $x + y = 8$ (even).
- **Multiplication**: Let $x = 2$ (even) and $y = 7$ (odd). $x \times y = 14$ (even).

3 Proof Techniques

Effective mathematical reasoning relies on a variety of proof techniques. This section explores two fundamental methods: **Direct Proof** and **Mathematical Induction**.

3.1 Direct Proof

A **direct proof** involves a straightforward chain of logical deductions from known premises to establish the truth of a given statement.

3.1.1 Example: Proving that the sum of two odd integers is even

Statement: For any $x, y \in \mathbb{N}$, if x and y are odd, then $x + y$ is even.

Proof:

Suppose x and y are odd. By definition,

$$x = 2n + 1 \quad \text{and} \quad y = 2m + 1 \quad \text{for some } n, m \in \mathbb{N} \cup \{0\}$$

Adding x and y:

$$x + y = (2n + 1) + (2m + 1) = 2n + 2m + 2 = 2(n + m + 1)$$

Since $n + m + 1$ is an integer, $x + y$ is even.

3.2 Mathematical Induction

Mathematical induction is a powerful technique used to prove statements about natural numbers. It consists of two main steps:

1. **Base Case**: Verify the statement for the initial value (usually $n = 1$).
2. **Inductive Step**: Assume the statement holds for some arbitrary n and then prove it for $n + 1$.

3.2.1 Example: Proving that the sum of the first n odd numbers is n^2

Statement: For all $n \in \mathbb{N}$,

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2$$

Proof:

1. **Base Case** ($n = 1$):

$$1 = 1^2$$

True.

2. **Inductive Step**:

   Inductive Hypothesis: Assume the statement holds for $n = k$:

   $$1 + 3 + 5 + \cdots + (2k - 1) = k^2$$

   To Prove: The statement holds for $n = k + 1$:

   $$1 + 3 + 5 + \cdots + (2k - 1) + [2(k + 1) - 1] = (k + 1)^2$$

   Starting from the inductive hypothesis:

   $$k^2 + [2(k + 1) - 1] = k^2 + 2k + 2 - 1 = k^2 + 2k + 1 = (k + 1)^2$$

   Thus, the statement holds for $n = k + 1$.

By the principle of mathematical induction, the statement is true for all $n \in \mathbb{N}$.

4 Properties of Real Numbers

The real numbers $\mathbb{R}$ possess several fundamental properties that facilitate mathematical operations and proofs. Understanding these properties is crucial for manipulating expressions and establishing equivalences.

4.1 Additive and Multiplicative Inverses

1. **Additive Inverse**: For any $a \in \mathbb{R}$, there exists an element $-a \in \mathbb{R}$ such that:

   $$a + (-a) = 0$$

2. **Multiplicative Inverse**: For any $a \in \mathbb{R}$ with $a \neq 0$, there exists an element $a^{-1} \in \mathbb{R}$ such that:

   $$a \cdot a^{-1} = 1$$

4.2 Associative, Commutative, and Distributive Properties

1. **Associative Property**:

   - **Addition**:

     $$(a + b) + c = a + (b + c)$$

   - **Multiplication**:

     $$(a \cdot b) \cdot c = a \cdot (b \cdot c)$$

2. **Commutative Property**:

   - **Addition**:

     $$a + b = b + a$$

   - **Multiplication**:

     $$a \cdot b = b \cdot a$$

3. **Distributive Property**:

   $$a \cdot (b + c) = a \cdot b + a \cdot c$$

4.3 Exponentiation Properties

4.3.1 Odd Powers Preserve Oddness

Statement: For any $x \in \mathbb{N}$, if x is odd, then x^3 is odd.

Proof:

Suppose x is odd. Then,

$$x = 2n + 1 \quad \text{for some } n \in \mathbb{N} \cup \{0\}$$

Calculating x^3:

$$x^3 = (2n+1)^3 = 8n^3 + 12n^2 + 6n + 1 = 2(4n^3 + 6n^2 + 3n) + 1$$

Since $4n^3 + 6n^2 + 3n$ is an integer, x^3 is odd.

5 Handling Inequalities

Inequalities are fundamental in expressing and solving mathematical relationships. Understanding how inequalities behave under various operations is essential for rigorous proof construction.

5.1 Preservation of Order Under Addition

Property: If $a < b$ and $c < d$, then $a + c < b + d$.

Proof:

Given:

$$a < b \quad \text{and} \quad c < d$$

By the **Order Preservation under Addition**:

$$(a + c) < (b + c) \quad \text{and} \quad (b + c) < (b + d)$$

By **Transitivity**:

$$a + c < b + d$$

5.2 Preservation of Order Under Multiplication

Property: If $0 < a < b$ and $0 < c < d$, then $a \cdot c < b \cdot d$.

Proof:

Given:

$$0 < a < b \quad \text{and} \quad 0 < c < d$$

By the **Order Preservation under Multiplication by Positive Numbers**:

$$a \cdot c < b \cdot c \quad \text{and} \quad b \cdot c < b \cdot d$$

By **Transitivity**:

$$a \cdot c < b \cdot d$$

5.3 Examples

- **Addition of Inequalities**: If $2 < 5$ and $3 < 7$, then $2 + 3 < 5 + 7$, i.e., $5 < 12$.
- **Multiplication of Inequalities**: If $1 < 4$ and $2 < 3$, then $1 \times 2 < 4 \times 3$, i.e., $2 < 12$.

6 Series and Summations

Summations and series are tools for aggregating sequences of numbers. Understanding how to compute and manipulate series is essential in calculus.

6.1 Arithmetic Series

An **arithmetic series** is the sum of the terms of an arithmetic sequence. In the context of the problem set, the sum of the first n odd numbers forms an arithmetic series.

6.1.1 Example: Sum of the First n Odd Numbers

Calculate:

$$1 + 3 + 5 + \cdots + (2n-1) = n^2$$

Proof by Induction: Refer to the induction example in the **Mathematical Induction** section.

6.2 Telescoping Series

A **telescoping series** is a series whose partial sums eventually only have a fixed number of terms after cancellation.

6.2.1 Example: Computing a Telescoping Series

Compute the sum:

$$\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \cdots + \frac{1}{(2n-1)(2n+1)}$$

Solution Strategy:

Notice that each term can be expressed as the difference of two fractions:

$$\frac{1}{(2k-1)(2k+1)} = \frac{1}{2}\left(\frac{1}{2k-1} - \frac{1}{2k+1}\right)$$

Thus, the series becomes:

$$\frac{1}{2}\left(\frac{1}{1} - \frac{1}{3} + \frac{1}{3} - \frac{1}{5} + \cdots + \frac{1}{2n-1} - \frac{1}{2n+1}\right)$$

Most terms cancel out, leaving:

$$\frac{1}{2}\left(1 - \frac{1}{2n+1}\right) = \frac{1}{2} - \frac{1}{4n+2}$$

Proving this result via mathematical induction solidifies its validity for all $n \in \mathbb{N}$.

7 Conclusion

Chapter 2 has introduced fundamental proof techniques and explored essential properties of integers and real numbers. By understanding the classification of integers as even or odd, employing direct proofs and mathematical induction, and manipulating inequalities, students are equipped with the necessary tools to construct rigorous mathematical arguments. The properties of real numbers, including additive and multiplicative inverses as well as associative, commutative, and distributive laws, provide a solid foundation for further exploration in calculus and higher mathematics. Mastery of these topics ensures that students can approach complex problems with confidence and precision, paving the way for advanced study and research.

8 Functions: Injective, Surjective, and Bijective Functions

Problem 1. Determine whether each of the following functions are injective, surjective, and bijective, and prove your answer.

1. $f : \mathbb{Z} \to \mathbb{Z}, f(x) = 2x$.
2. $g : \mathbb{R} \to \mathbb{R}, g(x) = 2x$.

Solution:

1. **Injectivity**: Suppose $\exists x_1, x_2 \in \mathbb{Z}$ such that $f(x_1) = f(x_2)$. We want to show that $x_1 = x_2$.
$$f(x_1) = f(x_2) \implies 2x_1 = 2x_2 \implies x_1 = x_2.$$
Thus, the function f is injective.

   Surjectivity: We need to show that $\forall y \in \mathbb{Z}, \exists x \in \mathbb{Z}$ such that $f(x) = y$. Suppose $y \in \mathbb{Z}$ and let $x = \frac{y}{2}$.
$$f(x) = y \implies 2x = y \implies x = \frac{y}{2}.$$
However, $x \in \mathbb{Z}$ only if y is even. Therefore, f is surjective only onto the even integers, not all of $\mathbb{Z}$. Thus, f is not surjective.

   Bijectivity: Since f is injective but not surjective, it is not bijective.

2. **Injectivity**: Suppose $\exists x_1, x_2 \in \mathbb{R}$ such that $g(x_1) = g(x_2)$. We want to show that $x_1 = x_2$.
$$g(x_1) = g(x_2) \implies 2x_1 = 2x_2 \implies x_1 = x_2.$$
Thus, the function g is injective.

   Surjectivity: We need to show that $\forall y \in \mathbb{R}, \exists x \in \mathbb{R}$ such that $g(x) = y$. Let $x = \frac{y}{2}$.
$$g(x) = y \implies 2x = y \implies x = \frac{y}{2}.$$
Since $x \in \mathbb{R}$ for any $y \in \mathbb{R}$, g is surjective.

   Bijectivity: Since g is both injective and surjective, it is bijective.

□

Problem 2. Let $f : A \to B$ be a function and $A_1, A_2 \subseteq A$ and $B_1, B_2 \subseteq B$ be subsets. Prove the following statements:

1. $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$.
2. $f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2)$.
3. $f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$.
4. $f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$.

Solution:

1. *Proof.* $\subseteq$: Let $y \in f(A_1 \cup A_2)$. By definition of image, $\exists x \in A_1 \cup A_2$ such that $f(x) = y$. Hence, $x \in A_1$ or $x \in A_2$. Thus, $y \in f(A_1)$ or $y \in f(A_2)$, implying $y \in f(A_1) \cup f(A_2)$.

   $\supseteq$: Let $y \in f(A_1) \cup f(A_2)$. Then $y \in f(A_1)$ or $y \in f(A_2)$. Thus, $\exists x \in A_1$ or $x \in A_2$ such that $f(x) = y$. Therefore, $x \in A_1 \cup A_2$ and $y = f(x) \in f(A_1 \cup A_2)$.

   Thus, $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$. □

2. *Proof.* Let $y \in f(A_1 \cap A_2)$. Then $\exists x \in A_1 \cap A_2$ such that $f(x) = y$. Since $x \in A_1$ and $x \in A_2$, it follows that $y \in f(A_1)$ and $y \in f(A_2)$. Thus, $y \in f(A_1) \cap f(A_2)$.

   Therefore, $f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2)$. □

3. *Proof.* $\subseteq$: Let $x \in f^{-1}(B_1 \cup B_2)$. Then $f(x) \in B_1 \cup B_2$, so $f(x) \in B_1$ or $f(x) \in B_2$. Hence, $x \in f^{-1}(B_1)$ or $x \in f^{-1}(B_2)$, implying $x \in f^{-1}(B_1) \cup f^{-1}(B_2)$. Thus, $f^{-1}(B_1 \cup B_2) \subseteq f^{-1}(B_1) \cup f^{-1}(B_2)$.

   $\supseteq$: Let $x \in f^{-1}(B_1) \cup f^{-1}(B_2)$. Then $x \in f^{-1}(B_1)$ or $x \in f^{-1}(B_2)$, meaning $f(x) \in B_1$ or $f(x) \in B_2$. Thus, $f(x) \in B_1 \cup B_2$ and $x \in f^{-1}(B_1 \cup B_2)$.

   Therefore, $f^{-1}(B_1) \cup f^{-1}(B_2) \subseteq f^{-1}(B_1 \cup B_2)$.

   Hence, $f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$. □

4. *Proof.* $\subseteq$: Let $x \in f^{-1}(B_1 \cap B_2)$. Then $f(x) \in B_1 \cap B_2$, so $f(x) \in B_1$ and $f(x) \in B_2$. Hence, $x \in f^{-1}(B_1)$ and $x \in f^{-1}(B_2)$, implying $x \in f^{-1}(B_1) \cap f^{-1}(B_2)$.

   $\supseteq$: Let $x \in f^{-1}(B_1) \cap f^{-1}(B_2)$. Then $f(x) \in B_1$ and $f(x) \in B_2$, so $f(x) \in B_1 \cap B_2$. Thus, $x \in f^{-1}(B_1 \cap B_2)$.

   Therefore, $f^{-1}(B_1) \cap f^{-1}(B_2) \subseteq f^{-1}(B_1 \cap B_2)$.

   Hence, $f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$. □

□

Problem 3. Let $f : A \to B$ be a function. Prove that f is injective if and only if $f(A_1 \cap A_2) = f(A_1) \cap f(A_2)$ for all subsets $A_1, A_2 \subseteq A$.

Solution: We will prove the equivalence by demonstrating both implications.

1. f **is injective** $\Rightarrow \forall A_1, A_2 \subseteq A,\ f(A_1 \cap A_2) = f(A_1) \cap f(A_2)$

Proof. Assume f is injective.

$\subseteq$ Let $y \in f(A_1 \cap A_2)$. Then $\exists x \in A_1 \cap A_2$ such that $f(x) = y$. Since $x \in A_1$ and $x \in A_2$, it follows that $y \in f(A_1)$ and $y \in f(A_2)$. Thus, $y \in f(A_1) \cap f(A_2)$.

$\supseteq$ Let $y \in f(A_1) \cap f(A_2)$. Then $y \in f(A_1)$ or $y \in f(A_2)$. Thus, $\exists x_1 \in A_1$ and $\exists x_2 \in A_2$ such that $f(x_1) = y$ and $f(x_2) = y$. Since f is injective, $x_1 = x_2$. Let $x = x_1 = x_2$. Then $x \in A_1 \cap A_2$, and hence $y = f(x) \in f(A_1 \cap A_2)$.

Thus, $f(A_1 \cap A_2) = f(A_1) \cap f(A_2)$ when f is injective. □

2. $\forall A_1, A_2 \subseteq A,\ f(A_1 \cap A_2) = f(A_1) \cap f(A_2) \Rightarrow f$ **is injective**

Proof. Assume $\forall A_1, A_2 \subseteq A$, $f(A_1 \cap A_2) = f(A_1) \cap f(A_2)$. We aim to show that f is injective.

Suppose, for contradiction, that f is not injective. Then $\exists x_1, x_2 \in A$ with $x_1 \neq x_2$ and $f(x_1) = f(x_2) = y$.

Consider the subsets $A_1 = \{x_1\}$ and $A_2 = \{x_2\}$.

Since $x_1 \neq x_2$,

$$A_1 \cap A_2 = \emptyset.$$

Thus,

$$f(A_1 \cap A_2) = f(\emptyset) = \emptyset.$$

Meanwhile,

$$f(A_1) = \{f(x_1)\} = \{y\}, \quad f(A_2) = \{f(x_2)\} = \{y\},$$

so

$$f(A_1) \cap f(A_2) = \{y\} \cap \{y\} = \{y\}.$$

We have

$$f(A_1 \cap A_2) = \emptyset \neq \{y\} = f(A_1) \cap f(A_2),$$

which contradicts the assumption.

Therefore, f must be injective. □

□

Problem 4. Let $f : A \to B$ be a function. Prove that the following two statements are equivalent:

1. The function f is surjective.
2. For every set C and for any functions $g : B \to C$ and $h : B \to C$ such that $g \circ f = h \circ f$, we have $g = h$.

Solution: (1) implies (2):

Assume f is surjective. Let C be an arbitrary set, and let $g, h : B \to C$ satisfy $g \circ f = h \circ f$. For any $b \in B$, since f is surjective, there exists $a \in A$ such that $f(a) = b$. Therefore,

$$g(b) = g(f(a)) = (g \circ f)(a) = (h \circ f)(a) = h(f(a)) = h(b).$$

Hence, $g = h$.

(2) implies (1):

Assume statement 2 holds. Suppose, for contradiction, that f is not surjective. Then there exists $b_0 \in B$ such that $b_0 \notin \text{Image}(f)$. We will use this element to construct specific functions g and h that satisfy the premise of statement 2 but are not equal, leading to a contradiction:

Let $C = \{0, 1\}$ and define the functions $g, h : B \to C$ as follows:

$$g(b) = \begin{cases} 0 & \text{if } b = b_0, \\ 1 & \text{otherwise,} \end{cases} \qquad h(b) = 1 \text{ for all } b \in B.$$

Since $b_0 \notin \text{Image}(f)$, for all $a \in A$, $g(f(a)) = 1 = h(f(a))$. Thus, $g \circ f = h \circ f$. However, $g \neq h$ because $g(b_0) = 0$ while $h(b_0) = 1$, which contradicts the uniqueness condition.

Therefore, f must be surjective.

$\square$

Problem 5. Let A be a nonempty set and $f : A \to A$ a function. We call f an *involution* if $(f \circ f)(a) = a$ for all $a \in A$. Prove that if $f : A \to A$ is an involution, then f is bijective. What is the inverse function f^{-1} in terms of f?

Solution: 1. Injectivity

Assume that for some $a_1, a_2 \in A$,

$$f(a_1) = f(a_2).$$

Applying f to both sides of the equation:

$$f(f(a_1)) = f(f(a_2)).$$

Given that f is an involution:

$$(f \circ f)(a_1) = (f \circ f)(a_2)a_1 = a_2.$$

Thus, f is injective.

2. Surjectivity

Take any element $b \in A$. Since f is an involution:

$$f(f(b)) = b.$$

Let $a = f(b)$. Then:

$$f(a) = f(f(b)) = b.$$

Therefore, for every $b \in A$, there exists an $a \in A$ (specifically, $a = f(b)$) such that $f(a) = b$. Hence, f is surjective.

Since f is both injective and surjective, it is bijective.

Inverse Function

By definition, the inverse function f^{-1} satisfies:

$$f^{-1}(f(a)) = a \quad \text{and} \quad f(f^{-1}(a)) = a \quad \text{for all} \quad a \in A.$$

Given that f is an involution:

$$f(f(a)) = a.$$

Comparing the two conditions, we observe that f itself satisfies the properties required of an inverse function. Therefore:

$$f^{-1} = f.$$

□

Problem 6. Prove or disprove the following statements:

1. The set $\{x \in \mathbb{R} : x \geq 2\}$ is an interval.
2. The set $\{x \in \mathbb{R} : x \neq 2\}$ is an interval.

(Hint: In order to disprove a statement, you must prove that the negation of the statement is true.)

Solution: (a)
We express the set in interval notation:

$$S = [2, \infty)$$

Let $a, b \in S$ with $a < b$. Since $a \geq 2$ and $b \geq 2$, for any c such that $a < c < b$, it follows that $c \geq 2$. Thus, $c \in S$.

Therefore, S satisfies the definition of an interval.

(b) Consider the negation of the original statement, that the set is not an interval.

Assume, for contradiction, that the set is an interval. Expressing the set in interval notation:

$$T = (-\infty, 2) \cup (2, \infty)$$

This implies that T is the union of two disjoint intervals. However, for T to be an interval, it must be a single continuous set without gaps.

Consider $a = 1 \in (-\infty, 2)$ and $b = 3 \in (2, \infty)$. The point $c = 2$ satisfies $a < c < b$ but $c \notin T$, which contradicts the interval property that all points between a and b must lie within the set.

We have thus proven the validity of the negation of our original statement. Therefore, T is not an interval. $\square$