Physics 322
Discussion 8
11/04/2024

Relevant Maxwell Equations: $\nabla\cdot\vec{B}=0$ & $\nabla\times\vec{B}=\mu_0(\underbrace{\vec{J}_f}_{\text{free current}}+\underbrace{\vec{J}_b}_{\text{bounded current}})$

Magnetization: $\vec{M}$ (magnetic dipole moment per unit volume)

$\vec{J}_b=\nabla\times\vec{M}$ & $\vec{K}_b=\vec{M}\times\hat{n}$ ← External normal

H-field: $\vec{H}=\frac{1}{\mu_0}\vec{B}-\vec{M}$ ⇒ Maxwell equations: $\nabla\times\vec{H}=\vec{J}_f$ (→ $\oint\vec{H}\cdot d\vec{l}=I_{f,enc}$) & $\nabla\cdot\vec{B}=0$ (or $\nabla\cdot\vec{H}=-\nabla\cdot\vec{M}$)

Boundary Conditions: $\begin{cases}\nabla\times\vec{H}=\vec{J}_f \Rightarrow \vec{H}^{\parallel}_{above}-\vec{H}^{\parallel}_{below}=\vec{K}_f\times\hat{n}\\ \nabla\cdot\vec{H}=-\nabla\cdot\vec{M}\Rightarrow H^{\perp}_{above}-H^{\perp}_{below}=-(M^{\perp}_{above}-M^{\perp}_{below})\end{cases}$

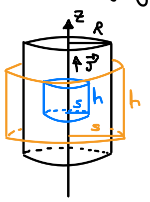
Linear Media: $\vec{M}=\chi_m\vec{H}$ (magnetic susceptibility) & $\vec{B}=\mu\vec{H}$ ⇒ $\vec{J}_b=\chi_m\vec{J}_f$

$\mu=\mu_0(1+\chi_m)$ Permeability

Problem 1.

Let's work P.1 of HW7.

In Coulomb gauge ($\nabla\cdot\vec{A}=0$): $\overbrace{\nabla^2A_i}^{\nabla\cdot(\nabla A_i)}=-\mu_0J_i \Rightarrow \int_V dV\,\nabla\cdot(\nabla A_i)=-\mu_0\int_V dV\,J_i \Rightarrow \int_{\partial V}d\vec{a}\cdot\nabla A_i=-\mu_0\int_V dV\,J_i$ (*)



$$\vec{J}=\begin{cases}\overbrace{J_0(s/R)^2}^{J_z(s)}\hat{z}, & s<R\\ 0, & s\geq R\end{cases}$$

In this question we have Rotational + Translational Symmetry on z-axis: $\vec{A}(s,\phi,z)=\vec{A}(s)\xrightarrow[A_x=0=A_y]{\text{Assuming}}A_z(s)\hat{z}$

(a) With the blue "Gaussian" surface,

$$\int_{\partial V}\underbrace{d\vec{a}}_{da\,\hat{s}}\cdot\overbrace{\nabla A_z}^{\partial_sA_z\hat{s}}=\partial_sA_z\,2\pi s h\,,\quad \int_V\overbrace{dv}^{ds\,dz\,d\phi\, s}\,J_z(s)=2\pi h\int_0^s ds\,J_0\frac{s^3}{R^2}=(2\pi h)\frac{J_0s^4}{4R^2}$$

$$(*)\Rightarrow \partial_sA_z(2\pi h)s=-\mu_0(2\pi h)\frac{J_0s^4}{4R^2}\Rightarrow \partial_sA_z=-\frac{\mu_0J_0}{4R^2}s^3\Rightarrow A_z(s)=-\frac{\mu_0J_0}{16R^2}s^4+C\xrightarrow{A(0)=0}\vec{A}(s)=-\frac{\mu_0J_0s^4}{16R^2}\hat{z}\text{ for } s<R$$ → Check $\nabla\cdot\vec{A}=0$ & $\nabla^2A_z=-\mu_0J_z$

→ $\vec{A}(R)=-\frac{\mu_0J_0R^2}{16}\hat{z}$

(b) With the orange "Gaussian" surface

$$\int_{\partial V}d\vec{a}\cdot\nabla A_z=\partial_sA_z\,2\pi s h,\ \int_V dv\,J_z(s)=2\pi h\int_0^R ds\,J_0\frac{s^3}{R^2}=2\pi h\frac{J_0}{4R^2}R^4=2\pi h\frac{J_0R^2}{4}$$

$$(*)\Rightarrow \partial_sA_z\,2\pi s h=-\mu_0\,2\pi h\frac{J_0R^2}{4}\Rightarrow \partial_sA_z=-\frac{\mu_0J_0}{4}\frac{R^2}{s}\Rightarrow A_z(s)=-\frac{\mu_0J_0}{4}R^2\ln(s)+C'$$

In Coulomb's gauge $\vec{A}$ is continuous at boundaries: $-\frac{\mu_0J_0}{4}R^2\ln R+C'=-\frac{\mu_0J_0R^2}{16}\Rightarrow C'=\frac{\mu_0J_0R^2}{4}\left[-\frac{1}{4}+\ln R\right]$

$$\Rightarrow\vec{A}(s)=-\frac{\mu_0J_0R^2}{4}\left[\ln\left(\frac{s}{R}\right)+\frac{1}{4}\right]\hat{z}\text{ for } s\geq R$$ → Check $\nabla\cdot\vec{A}=0$ & $\nabla^2A_z=-\mu_0J_z$

(c) For the magnetic field we get

$$\vec{B}=\nabla\times\vec{A}=-\partial_sA_z\hat{\phi}=\frac{\mu_0J_0s^3}{4R^2}\hat{\phi}\Rightarrow\vec{B}(s)=\frac{\mu_0J_0R}{4}\left(\frac{s}{R}\right)^3\hat{\phi}\text{ for } s<R$$ → Same result can be obtained by Ampère's law

Problem 2.

Let's work P.5 of HW7.

$$\vec{J}_f = 0 \Rightarrow \nabla\times\vec{H} = 0 \Rightarrow \vec{H} = -\nabla W \Rightarrow \overbrace{\nabla\cdot\vec{H}}^{-\nabla\cdot\vec{M}} = -\nabla^2 W \Rightarrow \nabla^2 W = \nabla\cdot\vec{M}$$

For a uniformly magnetized sphere, $\nabla\cdot\vec{M}=0$ except at $r=R$. Thus, we must solve $\nabla^2 W = 0$ for $r \gtrless R$. Given the azimuthal symmetry,

$$W^> = \sum_{\ell=0}^{\infty}\left(\cancelto{0}{A_\ell^> r^\ell} + B_\ell^> r^{-(\ell+1)}\right)P_\ell(\cos\theta) \quad \text{for } r \geq R$$

$$W^< = \sum_{\ell=0}^{\infty}\left(A_\ell^< r^\ell + \cancelto{0}{B_\ell^< r^{-(\ell+1)}}\right)P_\ell(\cos\theta) \quad \text{for } r \leq R$$

W, like V, is continuous at boundaries. Then, we can write

$$W^> = \sum_{\ell=0}^{\infty} A_\ell\left(\frac{R}{r}\right)^{\ell+1} P_\ell(\cos\theta)$$

$$W^< = \sum_{\ell=0}^{\infty} A_\ell\left(\frac{r}{R}\right)^{\ell} P_\ell(\cos\theta)$$

We can now impose the B.C.

$$\underbrace{H^\perp_{above} - H^\perp_{below}}_{H^\perp = -\nabla W\cdot\hat{r} = -\frac{\partial W}{\partial r}} = -(\cancelto{0}{M^\perp_{above}} - M^\perp_{below}) = M^\perp_{below} \Rightarrow \frac{\partial W^>}{\partial r} - \frac{\partial W^<}{\partial r}\bigg|_{r=R} = -M^\perp_{below} = -\overbrace{\vec{M}}^{\vec{M}=M\hat{z}}\cdot\hat{r} = -M\cos\theta = -M P_1(\cos\theta)$$

We then expect only $\ell=1$ terms,

$$W^> = A_1\frac{R^2}{r^2}\cos\theta \;\;\&\;\; W^< = A_1\frac{r}{R}\cos\theta \Rightarrow \frac{\partial W^>}{\partial r}\bigg|_{r=R} = -2\frac{A_1}{R}\cos\theta \;\;\&\;\; \frac{\partial W^<}{\partial r}\bigg|_{r=R} = \frac{A_1}{R}\cos\theta$$

B.C. $\Rightarrow \; -3\frac{A_1}{R}\cos\theta = -M\cos\theta \Rightarrow A_1 = \frac{MR}{3}$

Therefore,

$$W^> = \frac{MR}{3}\left(\frac{R}{r}\right)^2\cos\theta \quad \text{for } r \geq R \Rightarrow \vec{H}^> = -\nabla W^> = -\frac{MR^3}{3}\nabla\left(\frac{\cos\theta}{r^2}\right) = -\frac{MR^3}{3}\left(-\frac{2}{r^3}\hat{r}\cos\theta - \frac{\sin\theta\,\hat{\theta}}{r^3}\right) = \frac{M}{3}\left(\frac{R}{r}\right)^3\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)$$

$$W^< = \frac{MR}{3}\left(\frac{r}{R}\right)\cos\theta \quad \text{for } r \leq R \Rightarrow \vec{H}^< = -\nabla W^< = -\frac{M}{3}\nabla(r\cos\theta) = -\frac{M}{3}\left(\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}\right) = -\frac{\vec{M}}{3}$$

$$\vec{B}^> = \frac{\mu_0 M}{3}\left(\frac{R}{r}\right)^3\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)$$

$$\vec{B}^< = \mu_0\left(\vec{H}^< + \vec{M}\right) = \frac{2}{3}\mu_0\vec{M}$$