

**Math 421, Section 1**  
**Homework 8**

**Problem 1.** For each of the following functions  $f : (0, \infty) \rightarrow \mathbb{R}$ , prove that  $f$  is differentiable at any point  $a > 0$  and find  $f'(a)$ .

- (a)  $f(x) = \frac{1}{x}$
- (b)  $f(x) = \sqrt{x}$

**Problem 2.** Consider the function  $f : \mathbb{R} \rightarrow \mathbb{R}$  given by  $f(x) = \max\{0, x\}$ . For each  $a \in \mathbb{R}$ , determine if  $f$  is differentiable at  $a$  and prove your answer.

**Problem 3.** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a function, and suppose that  $f$  is differentiable at  $a$  for any  $a \in \mathbb{R}$ .

- (a) Prove that for any constant  $c \in \mathbb{R}$ , the function  $g : \mathbb{R} \rightarrow \mathbb{R}$ ,  $g(x) = f(x) + c$  is differentiable at any  $a \in \mathbb{R}$  with  $g'(a) = f'(a)$ .
- (b) Prove that for any constant  $c \in \mathbb{R}$ , the function  $g : \mathbb{R} \rightarrow \mathbb{R}$ ,  $g(x) = f(x + c)$  is differentiable at any  $a \in \mathbb{R}$  with  $g'(a) = f'(a + c)$ .

**Problem 4.** Suppose that  $f : \mathbb{R} \rightarrow \mathbb{R}$  is a function that satisfies  $f(0) = 0$  and  $f'(0) = 0$ . Define the function  $g : \mathbb{R} \rightarrow \mathbb{R}$  by

$$g(x) = \begin{cases} f(x) \cdot \sin \frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Prove that  $g$  is differentiable at 0 and  $g'(0) = 0$ .

**Problem 5.** Prove that the function  $f : \mathbb{R} \rightarrow \mathbb{R}$ ,  $f(x) = |x|^3$  is twice differentiable at any point  $a \in \mathbb{R}$ , but is not three-times differentiable at 0.