1.) Base case: For $n=1$, $1 = \frac{1 \cdot 2}{2}$ is true.

Inductive step: Suppose

$$1+4+\dots+(3n-2) = \frac{n(3n-1)}{2}$$

is true for some $n \in \mathbb{N}$. Adding $3n+1$ to both sides,

$$\begin{aligned} 1+4+\dots+(3n-2)+(3n+1) &= \frac{n(3n-1)}{2} + (3n+1) \\ &= \frac{3n^2 - n + 6n + 2}{2} \\ &= \frac{(n+1)(3n+2)}{2} \end{aligned}$$

So the statement is true for $n+1$.

Together, we conclude that the statement is true $\forall n \in \mathbb{N}$ by induction.

2.) (a.) $f((-1,2)) = [0,4)$

(b.) $f^{-1}(\{0,1\}) = \{-1, 0, 1\}$

(c) Disprove: $f(-1) = 1 = f(1)$

3.) (a.) • $\sup A$ is an upper bound for A

and

• L is an upper bound for $A \Rightarrow L \geq \sup A$

(b.) Claim: $\sup\{2 - \frac{3}{n} : n \in \mathbb{N}\} = 2$.

• 2 is an upper bound, since

$$n \in \mathbb{N} \Rightarrow n > 0 \Rightarrow \frac{1}{n} > 0 \Rightarrow 2 - \frac{3}{n} < 2.$$

- Suppose L is an upper bound. Fix $\varepsilon > 0$. Then $\exists\, n \in \mathbb{N}$ s.t. $\frac{1}{n} < \frac{\varepsilon}{3}$, and so

$$L \geq 2 - \frac{3}{n} > 2 - 3 \cdot \frac{\varepsilon}{3} = 2 - \varepsilon.$$

As $\varepsilon > 0$ was arbitrary, then $L \geq 2$.

4.) Case: $a > 0$. Then $f(x) = x^2$ on $(a-\delta, a+\delta)$ for $\delta = a$. We know x^2 is differentiable at a since it's a polynomial, and so $f'(a) = 2a$.

Case: $a < 0$. Then $f(x) = 0$ on $(a-\delta, a+\delta)$ for $\delta = |a|$. We know 0 is differentiable at a, and so $f'(a) = 0$.

Case: $a = 0$. For $h \neq 0$,

$$\frac{f(h) - f(0)}{h} = \begin{cases} \frac{h^2 - 0}{h} = h & h > 0 \\ 0 & h < 0 \end{cases}$$

Fix $\varepsilon > 0$. Set $\delta = \varepsilon$. Then

$$0 < |h| < \delta \Rightarrow \left| \frac{f(h) - f(0)}{h} - 0 \right| = \begin{cases} |h| < \delta = \varepsilon \\ 0 < \varepsilon \end{cases}$$

So $f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h} = 0$.

Altogether,

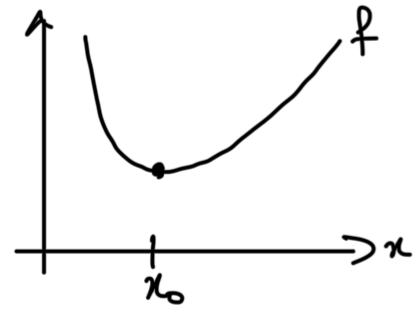
$$f'(a) = \begin{cases} 2a & a > 0 \\ 0 & a \leq 0 \end{cases} = 2\max\{0, a\}.$$

5.) Scratch work: Critical points are

$$0 = f'(x) = -\frac{a}{x^2} + b = \frac{bx^2 - a}{x^2}$$

$$\Rightarrow bx^2 - a = 0$$

$$\Rightarrow x_0 = \sqrt{\frac{a}{b}}$$



Solution: We will prove that $f(x_0) = 2\sqrt{ab}$ is the global minimum of f, where $x_0 = \sqrt{\frac{a}{b}}$.

Claim: $f(x) > f(x_0)$ $\forall x \in (x_0, \infty)$. Note that

$$x > x_0 = \sqrt{\frac{a}{b}} \quad \Rightarrow \quad x^2 > \frac{a}{b} \quad \Rightarrow \quad bx^2 - a > 0$$

$$\Rightarrow \quad f'(x) = \frac{bx^2 - a}{x^2} > 0$$

Given $x > x_0$, by the MVT there is a point $c \in (x_0, x)$ where

$$0 < f'(c) = \frac{f(x) - f(x_0)}{x - x_0} \quad \Rightarrow \quad f(x) > f(x_0).$$

Claim: $f(x) > f(x_0)$ $\forall x \in (0, x_0)$. Note that

$$0 < x < x_0 = \sqrt{\frac{a}{b}} \quad \Rightarrow \quad x^2 < \frac{a}{b} \quad \Rightarrow \quad bx^2 - a < 0$$

$$\Rightarrow \quad f'(x) = \frac{bx^2 - a}{x^2} < 0$$

Given $x < x_0$, by the MVT there is a point $c \in (x, x_0)$ where

$$0 > f'(c) = \frac{f(x_0) - f(x)}{x_0 - x} \quad \Rightarrow \quad f(x) > f(x_0).$$

6.) (a.) Consider $g: [0,1] \to \mathbb{R}$, $g(x) = f(x) - 1 + x$.

Want: $\exists\, x \in [0,1]$ s.t. $g(x) = 0$. Note that:

$$g(0) = f(0) - 1 \leq 0, \qquad g(1) = f(1) \geq 0$$

Case: $g(0) < 0 < g(1)$. As f and $-1+x$ are continuous on $[0,1]$, then so is g. So, by the Intermediate Value Theorem, $\exists\, x \in (0,1)$ s.t. $g(x) = 0$.

Case: $g(0) = 0$. Then $x = 0$ works.

Case: $g(1) = 0$. Then $x = 1$ works.

(b.) Suppose not: $\exists\, x_1 < x_2$ in $[0,1]$ s.t. $f(x_1) = 1 - x_1$ and $f(x_2) = 1 - x_2$. As f is differentiable on $(0,1)$, it is also differentiable on (x_1, x_2). So, by the Mean Value Theorem, $\exists\, c \in (x_1, x_2)$ s.t.

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{(1 - x_2) - (1 - x_1)}{x_2 - x_1} = -1 .$$

This contradicts $|f'(x)| < 1 \;\; \forall\, x \in (0,1)$.

7.) For $n \in \mathbb{N}$, set $P_n = \{-1, -\frac{1}{n}, \frac{1}{n}, 1\}$. Then

$$U(f, P_n) = -2 \cdot \left(-\tfrac{1}{n} - (-1)\right) + 2 \cdot \left(\tfrac{1}{n} - \left(-\tfrac{1}{n}\right)\right) + 2 \cdot \left(1 - \tfrac{1}{n}\right)$$
$$= \frac{4}{n}$$

$$L(f, P_n) = -2 \cdot \left(-\tfrac{1}{n} - (-1)\right) + (-2) \cdot \left(\tfrac{1}{n} - \left(-\tfrac{1}{n}\right)\right) + 2 \cdot \left(1 - \tfrac{1}{n}\right)$$
$$= -\frac{4}{n}$$

Together,

$$-\frac{4}{n} = L(f, P_n) \le L(f) \le U(f) \le U(f, P_n) = \frac{4}{n}.$$

Fix $\varepsilon > 0$. Then $\exists\, n \in \mathbb{N}$ s.t. $\frac{1}{n} < \frac{\varepsilon}{4}$, and so

$$-\varepsilon < -\frac{4}{n} \le L(f) \le U(f) \le \frac{4}{n} < \varepsilon.$$

As $\varepsilon > 0$ was arbitrary, we conclude

$$L(f) = 0 = U(f).$$

So f is integrable on $[-1, 1]$ and $\int_{-1}^{1} f = 0$.

8.) (a.) We know:

- $-\cos x$ is differentiable on $[0, b]$
- $\sin x$ is integrable on $[0, b]$: since it's continuous.

So, by the Fundamental Theorem of Calculus,

$$\int_0^b \sin x \, dx = -\cos(b) + \cos(0) = 1 - \cos b.$$

(b.) We will apply integration by parts to $f(x) = \sin^2 x$ and $g(x) = -\cos x$. We know:

- f, g are differentiable on $[0, b]$
- $f'(x) = 2\sin x \cos x$ and $g'(x) = \sin x$ are continuous on $[0, b]$

So,

$$\int_0^b \sin^3 x \, dx = \int_0^b f g' = f(b)g(b) - f(0)g(0) - \int_0^b f' g$$

$$= -\sin^2 b \cos b + \int_0^b 2 \sin x \underbrace{\cos^2 x}_{= 1 - \sin^2 x} \, dx$$

$$= -\sin^2 b \cos b + 2\int_0^b \sin x \, dx$$
$$- 2\int_0^b \sin^3 x \, dx$$

$$\Rightarrow \quad 3\int_0^b \sin^3 x \, dx = -\sin^2 b \cos b + 2\int_0^b \sin x \, dx$$
$$= -\sin^2 b \cos b + 2(1-\cos b)$$

$$\Rightarrow \quad \int_0^b \sin^3 x \, dx = -\frac{1}{3}\sin^2 b \cos b + \frac{2}{3}(1-\cos b).$$