Problem 1

M421 HW5
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Let $a > 0$. Define the function $f : \mathbb{R} \to \mathbb{R}$by:

$$f(x) = \begin{cases} \frac{\sqrt{x}-\sqrt{a}}{x-a} & (x \neq a, x > 0) \\ 0 & (x = a \text{ or } x \leq 0). \end{cases} \tag{1}$$

Find limit of f as x approaches a, and prove your answer.

When $x \neq a, x > 0$:

$$f(x) = \frac{\sqrt{x}-\sqrt{a}}{x-a} = \frac{\sqrt{x}-\sqrt{a}}{\left(\sqrt{x}^2 - \sqrt{a}^2\right)} = \frac{1}{\sqrt{a}+\sqrt{x}}.@() \tag{2}$$

Claim:

$$\lim_{x \to a} f(x) = \frac{1}{2\sqrt{a}}. \tag{3}$$

Proof: For any given ε, let $\delta = \min\left(\frac{a}{2}, 2a^{3/2}c^2\varepsilon\right)$,where $c = 1 + \sqrt{2}/2$, s.t. if $|x-a| < \delta$, then $\left|f(x) - \frac{1}{2\sqrt{a}}\right| < \varepsilon$. Because:

$$\begin{aligned} |x-a| < \delta &\Rightarrow a - \delta < x < a + \delta \\ &\Rightarrow \sqrt{x} + \sqrt{a} > \sqrt{a-\delta} + \sqrt{a} \end{aligned} \tag{4}$$

Since $\delta \leq \frac{a}{2}$,

$$\begin{aligned} \sqrt{x} - \sqrt{a} &> \sqrt{a} + \sqrt{\frac{a}{2}} \\ &= \left(1 + \frac{\sqrt{2}}{2}\right)\sqrt{a} \quad := c\sqrt{a} \end{aligned} \tag{5}$$

It follows that

$$\begin{aligned} & 2\sqrt{a}\left(\sqrt{a}+\sqrt{x}\right)^2 > 2a^{\frac{3}{2}}c^2 \\ \Rightarrow \quad & \frac{|x-a|}{2\sqrt{a}\left(\sqrt{x}+\sqrt{a}\right)^2} < \frac{|x-a|}{2a^{\frac{3}{2}}c^2} \\ & = \frac{\delta}{2a^{\frac{3}{2}}c^2} < \varepsilon \end{aligned} \tag{6}$$

So that:

$$\left|f(x) - \frac{1}{2\sqrt{a}}\right| = \left|\frac{1}{\sqrt{a}+\sqrt{x}} - \frac{1}{2\sqrt{a}}\right| = \frac{\left|\sqrt{a}-\sqrt{x}\right|}{2\sqrt{a}\left(\sqrt{a}+\sqrt{x}\right)} = \frac{|x-a|}{2\sqrt{a}\left(\sqrt{x}+\sqrt{a}\right)^2} < \varepsilon \tag{7}$$

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Problem 2

Let $f : \mathbb{R} \to \mathbb{R}$ be a function s.t. $\lim_{x \to 0} f(x) = 0$. Define the function $g : \mathbb{R} \to \mathbb{R}$ by

$$g(x) = \begin{cases} f(x) \cdot \sin\left(\frac{1}{x}\right) \text{ if } x = 0 \\ 0 \text{ if } x \neq 0. \end{cases} \tag{8}$$

Prove that $\lim_{x \to 0} g(x) = 0$

- When $x \neq 0$:

$$|g(x)| = \left| f(x) \cdot \sin\left(\frac{1}{x}\right) \right| \leq |f(x)| \tag{9}$$

Given that $\lim(x \to 0) f(x) = 0$, by definition, $\forall \varepsilon > 0, \quad \exists \delta > 0$, s.t. $0 < |x| < \delta \quad \Rightarrow |f(x)| < \varepsilon$
So for $0 < x < \delta$:

$$|g(x)| \leq |f(x)| < \varepsilon. \tag{10}$$

Also notice that $x = 0$:

$$g(0) = 0 < \varepsilon. \tag{11}$$

So $\forall x \in \mathbb{R}, \forall \varepsilon > 0$, $\exists \delta > 0$ $s.t.$ $0 < |x| < \delta \Rightarrow |g(x)| < \varepsilon$. In other words, $\lim_{x \to 0} g(x) = 0$.

Problem 3

Suppose that $f, g : \mathbb{R} \to \mathbb{R}$ are functions s.t. $|f(x)| \le |g(x)|, \forall x \in \mathbb{R}$. g is continuous at 0, and $g(0) = 0$. Prove that $f$is continuous at 0.

Given that g is continuous at 0, we know

$$\forall \varepsilon > 0, \exists \delta > 0 \; s.t. \; 0 < |x| < \delta \Rightarrow |g(x)| < \varepsilon. \tag{12}$$

So, for the same δ,

$$0 < |x - 0| < \delta \Rightarrow |f(x) - 0| \le |g(x)| < \varepsilon \tag{13}$$

Noticing $|f(0)| \le g(0) = 0 \quad \Rightarrow f(0) = 0$. We have thus proved that

$$\lim_{x \to 0} f(x) = f(0) \tag{14}$$

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Problem 4

Let $f : \mathbb{R} \to \mathbb{R}, f(x) = |x|$.

1. Prove that $f$is continuous.
2. Let $g : \mathbb{R} \to \mathbb{R}$be a continuous function. Prove that $|g|$ is also a continuous function.

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1. Consider arbitrary point $c \in \mathbb{R}$. To prove that f is continuous over $\mathbb{R}$, we need to show that

   $$\forall \varepsilon > 0, \exists \delta > 0, s.t.\ |x - c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon. \tag{15}$$

   Let $\varepsilon > 0$ be given, choosing $\delta = \min\{\varepsilon, |c|\}$ when $c \neq 0$, then

   - c > 0 :

   $$|f(x) - f(c)| = ||x| - c| = |x - c| < \delta = \varepsilon. \tag{16}$$

   - c < 0:

   $$|f(x) - f(c)| = ||x| - |c|| = |-x + c| = |x - c| < \delta = \varepsilon. \tag{17}$$

   - c = 0: choose $\delta = \varepsilon$.

   $$|f(x) - f(c)| = ||x| - |0|| = |x| < \delta = \varepsilon. \tag{18}$$

   So in all cases and for all $\varepsilon > 0$, there exists a $\delta > 0$ s.t. $|x - c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon$. Therefore, f is continuous over $\mathbb{R}$.

2. Given that g is a continuous function, and, from part 1, that $|x|$ is continuous, we can construct their composite function $|g| = f \circ g$.

   From lecture, we know that the composition of two continuous functions is also continuous. Therefore, $|g|$ is a continuous function.

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Problem 5

Let $a, b, c \in \mathbb{R}$ with $a < b < c$. Suppose that $f : [a, b] \to \mathbb{R}; g : [b, c] \to \mathbb{R}$ are both continuous, and $f(b) = g(b)$. Prove that the fucntion $h : [a, c] \to \mathbb{R}$ defined by

$$h(x) = \begin{cases} f(x) & \text{if } x \leq b \\ g(x) & \text{if } x > b \end{cases} \tag{19}$$

is continuous.

- Continuity on $[a, b)$: when $x \in [a, b), h(x) = f(x)$. Since f is continuous on $[a, b]$, h is continuous on $[a, b)$.
- Continuity on $(b, c]$: when $x \in (b, c], h(x) = g(x)$. Since g is continuous on $[b, c]$, h is continuous on $(b, c]$.
- Continuity at b : Given that $f(x)$ is continuous on $[a, b]$, and $g(x)$ is continuous on $[b, c]$, we have the following:

  $$\lim_{x \to b-} f(x) = f(b), \quad \lim_{x \to b+} g(x) = g(b) \tag{20}$$

  Since $f(b) = g(b)$, it follows that

  $$\lim_{x \to b-} h(x) = \lim_{x \to b-} f(x) = f(b) = g(b) = \lim_{x \to b+} g(x) = \lim_{x \to b+} h(x). \tag{21}$$

  Using the fact that

  $$\lim_{x \to b} h(x) = l \Leftrightarrow \lim_{x \to b^-} h(x) = l = \lim_{x \to b^+} h(x), \tag{22}$$

  and $h(b) = f(b)$, We have shown that $\lim_{x \to b} h(x) = h(b)$, and thus h is continuous at b.
- Collecting the above, we have shown that h is continuous on $[a, c]$.

Problem 6

Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a continuous function. Prove that for any open set $U \subseteq \mathbb{R}$, the set $f^{-1}(U)$ is also open.

Take an arbitrary point $x_0 \in f^{-1}(U)$. By definition of preimage, $f(x_0) \in U$. Since $U$is an open set, then by definition

$$\exists \varepsilon > 0, s.t.\ (f(x_0) - \varepsilon, f(x_0) + \varepsilon) \subseteq U. \tag{23}$$

Considering the continuity of f at $x_0 \in \mathbb{R}$, we have

$$\begin{aligned} \forall \varepsilon > 0, \exists \delta > 0, s.t.\ |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon. \\ \Rightarrow \quad x_0 - \delta < x < x_0 + \delta; \quad f(x_0) - \varepsilon < f(x) < f(x_0) + \varepsilon. \end{aligned} \tag{24}$$

It follows that

$$f(x) \in (f(x_0) - \varepsilon, f(x_0) + \varepsilon) \subseteq U. \tag{25}$$

So by definition of preimage,

$$x \in f^{-1}(U). \tag{26}$$

Since x_0 is arbitrary, we have shown that

$$\forall x_0 \in f^{-1}(U), \exists \delta > 0, s.t.\ (x_0 - \delta, x_0 + \delta) \subseteq f^{-1}(U). \tag{27}$$

Which satisfies the definition of an open set. Therefore, $f^{-1}(U)$ is open. ■