7.1

Find the energies $E_{311}, E_{222}, E_{321}$ and construct an energy level diagram for the 3d cubic well that includes the 3rd, 4th, 5th excited states. Which of the states on your diagram are degenerate?

Using the formula

$$E_{n_1 n_2 n_3} = \frac{\hbar^2 \pi^2}{2mL^2}(n_1^2 + n_2^2 + n_3^2), \quad E_0 = \frac{\hbar^2 \pi^2}{2mL^2} \tag{1}$$

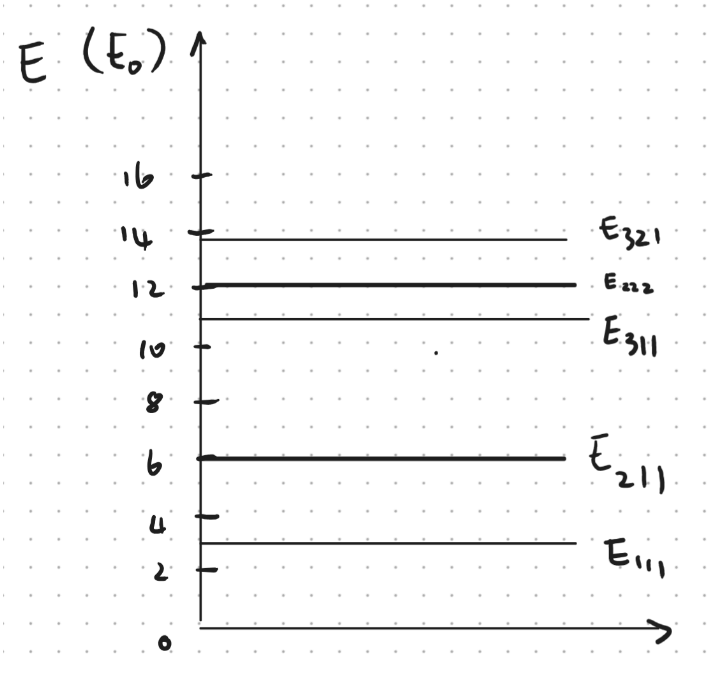
we have:

$$\begin{aligned} E_{311} &= \frac{\hbar^2 \pi^2}{2mL^2}(3^2 + 1^2 + 1^2) = 11E_0 \\ E_{222} &= \frac{\hbar^2 \pi^2}{2mL^2}(2^2 + 2^2 + 2^2) = 12E_0 \\ E_{321} &= \frac{\hbar^2 \pi^2}{2mL^2}(3^2 + 2^2 + 1^2) = 14E_0 \end{aligned} \tag{2}$$

- Degeneracy:

The state (3,1,1) is degenerate with (1,3,1) and (1,1,3), all having energy $11E_0$. The state (3,2,1) is degenerate with all permutations of these numbers: (3,1,2), (2,3,1), (2,1,3), (1,3,2), (1,2,3), all having energy $14E_0$. The state (2,2,2) is non-degenerate.

Therefore, the 1st and 2nd states are degenerate (from textbook), the 3rd excited state corresponds to $E_{\{311\}}$ (and its degenerate states), the 4th excited state is $E_{\{222\}}$, and the 5th excited state is $E_{\{321\}}$ (and its degenerate states).



7.2

A particle is confined to a 3d box that has sides $L_1, L_2 = 2L_1, L_3 = 3L_1$. GIve the sets of quantum numbers n_1, n_2, n_3 that correspond to the lowest 10 energy levels of this box.

From textbook,

$$E_{n_1 n_2 n_3} = \frac{\hbar^2 \pi^2}{2mL_1^2}\left(n_1^2 + \frac{n_2^2}{4} + \frac{n_3^2}{9}\right) \tag{3}$$

So we can write down the sets of quantum numbers corresponding to the lowest 10 energy states in an increasing order:

$$\begin{aligned}
E_{111} &= 1.361E_0 \\
E_{112} &= 1.694E_0 \\
E_{121} &= 2.111E_0 \\
E_{113} &= 2.250E_0 \\
E_{122} &= 2.444E_0 \\
E_{123} &= 3.000E_0 \\
E_{114} &= 3.028E_0 \\
E_{131} &= 3.360E_0 \\
E_{132} &= 3.694E_0 \\
E_{124} &= 3.778E_0
\end{aligned} \tag{4}$$

7.8

Consider a particle moving in a 2d space defined by $V = 0$ for $0 < x < L$ and $0 < y < L$ and $V = \infty$ elsewhere.

- a. write down the wave functions fo rthe particle in this well.
- b. find the expression for the corresponding energies.
- c. what are the sets of quantum numbers for the lowest energy degenerate state?

- a.By separation of variables (set $k_3 = 0$), the wavefunction in 2d takes the form:

$$\psi(x, y) = A \sin k_1 x \sin k_2 y = A \sin \frac{n_1 \pi x}{L} \sin \frac{n_2 \pi y}{L} \tag{5}$$

- b. The energy is given by

$$E_{n_1 n_2} = \frac{\hbar^2 \pi^2}{2mL^2} (n_1^2 + n_2^2) \tag{6}$$

- c.the lowest energy degenerate state corresponds to permutations of $\{n_1, n_2\} = \{1, 2\}$, i.e. $n_1 = 1, n_2 = 2$ and $n_1 = 2, n_2 = 1$

7.9

If $n = 3$,

- a. what are the possible values of l ?
- b. for each value of l in (a), list the possible values of m.
- c. using the fac that there tow quantum states for each combiniation of values of l and m because of electron spin, find the total number of electron states with $n = 3$.

- a. $l = 0, 1, 2, ..., (n - 1) = 0, 1, 2$.
- b. $m = -l, (-l + 1), ..., l$.

$$\begin{gathered}
l = 0, m = 0; \\
l = 1, m = 0, \pm 1; \\
l = 2, m = 0, \pm 1, \pm 2.
\end{gathered} \tag{7}$$

- c. since there are 9 different values for m, the total number of electron states with $n = 3$ is $2 * 9 = 18$.

7.16

What are the possible values of n and m if

- a. l = 3,
- b. l = 4,
- c. l = 0?
- d. compute the minimum possible energy for each case.

- a. since the $l_{\max} = n - 1$, n can take $4, 5, 6, \ldots$; $m = \pm 3, \pm 2, \pm 1, 0$
- b. $n = 5, 6, 7, \ldots$; $m = \pm 4, \pm 3, \pm 2, \pm 1, 0$
- c. $n = 1, m = 0$.
- d.minimum energy is given by $E_n = -\frac{E_1}{n^2}$, where $E_1 = \frac{1}{2}\left(k\frac{e^2}{\hbar}\right)^2 \mu = 13.6 \text{ eV}$. Therefore,

$$\begin{aligned} E_4 &= -\frac{13.6}{4^2} \text{ eV} = -0.85 \text{ eV} \\ E_5 &= -\frac{13.6}{5^2} \text{ eV} = -0.54 \text{ eV} \\ E_1 &= -13.6 \text{ eV} \end{aligned} \tag{8}$$

Add-on problem

A free particle of mass m with wave number k_1 is traveling to the right. at $x = 0$, the potential jumps from zero to V_0 and remains at this value for positive x.

- a. If the total energy is equal to $2V_0$, what is the wave number in the region $x > 0$? Express your answer in terms of k_1 and V_0 .

Energy in each region is found as

$$\begin{aligned} E_1 = K_1 + V_1 &= \frac{\hbar^2 k_1^2}{2m} & (x < 0), \\ E_2 = K_2 + V_2 &= \frac{\hbar^2 k_2^2}{2m} + V_0 & (x > 0), \end{aligned} \tag{9}$$

where k_1, k_2 are the wavenumbers in region $x < 0, x > 0$ respectively.

Given Total energy $E = 2V_0$, we have:

$$\begin{aligned} 2V_0 &= \frac{\hbar^2 k_1^2}{2m} & \Rightarrow k_1^2 = \frac{4mV_0}{\hbar^2} \\ 2V_0 &= \frac{\hbar^2 k_2^2}{2m} + V_0 & \Rightarrow k_2^2 = \frac{2mV_0}{\hbar^2} \end{aligned} \tag{10}$$

Therefore, algebra gives

$$k_2 = \frac{k_1}{\sqrt{2}}. \tag{11}$$

- b. Calculate the reflection coefficient R and the transmission coefficient T at the potential step. Note that $1 + R = T$ (typo?) .

From lecture,

$$R = \left(\frac{k_1 - k_2}{k_1 + k_2}\right)^2, \quad T = \frac{4k_1 k_2}{(k_1 + k_2)^2}, \quad R + T = 1 \tag{12}$$

(it was hinted in problem that $1 + R = T$, but is it a typo? I think it should be $R + T = 1$ as per lecture slides.)

Since we found the relationship between k_1, k_2, we can write

$$\begin{aligned} R &= \left(\frac{k_1 - k_1/\sqrt{2}}{k_1 + k_1/\sqrt{2}}\right)^2 = \left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)^2 \approx 0.02943 \\ T &= \frac{4k_1 \cdot k_1/\sqrt{2}}{\left(k_1 + k_1/\sqrt{2}\right)^2} = \frac{4/\sqrt{2}}{\left(1+1/\sqrt{2}\right)^2} \approx 0.9705 \end{aligned} \tag{13}$$

It can be verified that $R + T = 1$.

- c. If one million particles with wave number k_1 are incident upon the potential step, how many particles are expected to continue along the positive x direction? How does this compare with the classical prediction (which says that all particles go through if their energy is above that of the potential step?)

Number of transmitted particles:

$$N_T = TN_0 = 0.9705 * 10^6 \approx 9.705 * 10^5 \tag{14}$$

Number of reflected particles

$$N_R = RN_0 = 0.02943 * 10^6 \approx 2.943 * 10^4. \tag{15}$$

However, classically, since each particles carries same energy $2V_0$, all particles should go through.