Physics 322
Discussion 2
09/16/2024

For electrostatics (charges not moving $\Rightarrow \rho(\vec{r},t)=\rho(\vec{r})$) Maxwell equations simplify to:

$$\begin{cases}\nabla\cdot\vec{E}=\rho/\epsilon_0\\ \nabla\times\vec{E}=0\end{cases}$$

Which has the general solution

$$\vec{E}(\vec{r})=\frac{1}{4\pi\epsilon_0}\int_V dv'\,\frac{\rho(\vec{r}')}{\mathscr{r}^2}\,\hat{\mathscr{r}} \quad \text{with} \quad \vec{\mathscr{r}}=\vec{r}-\vec{r}'.$$

Indeed,

$$\nabla\cdot\vec{E}=\frac{1}{4\pi\epsilon_0}\int_V dv'\rho(\vec{r}')\,\underbrace{\nabla\cdot\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)}_{4\pi\delta^3(\vec{\mathscr{r}})}=\rho(\vec{r})/\epsilon_0 \quad \text{and} \quad \nabla\times\vec{E}=\frac{1}{4\pi\epsilon_0}\int_V dv'\rho(\vec{r}')\,\underbrace{\nabla\times\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)}_{=\frac{1}{\mathscr{r}^3}\cancel{\nabla\times\vec{\mathscr{r}}}^{\,0}-\vec{\mathscr{r}}\times\nabla\mathscr{r}^{-3}=3\mathscr{r}^{-5}\cancel{\vec{\mathscr{r}}\times\vec{\mathscr{r}}}^{\,0}=0}$$

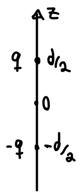
To solve problems we will generally use

$$\vec{E}=\frac{1}{4\pi\epsilon_0}\sum_{i=1}^{N}q_i\frac{\hat{\mathscr{r}}_i}{\mathscr{r}_i^2}\ \text{(Discrete distribution)} \quad \text{or} \quad d\vec{E}=\frac{1}{4\pi\epsilon_0}\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}dq\ \text{(Continuous distribution)}$$

$\rightarrow dq=\lambda dl$ (linear), $\sigma da$ (surface), $\rho dv$ (volume)

## Problem 1.

Consider the charges $-q$ at $(0,0,-d/2)$ and $q$ at $(0,0,d/2)$. Calculate the $\vec{E}$ field along the z-axis (for $z>d$).




Here we have

$\vec{r}=z\hat{z}$, $\vec{r}_+=\frac{d}{2}\hat{z}$, $\vec{r}_-=-\frac{d}{2}\hat{z}$, $\vec{\mathscr{r}}_+=(z-d/2)\hat{z}\Rightarrow\mathscr{r}_+=z-d/2$, $\vec{\mathscr{r}}_-=(z+d/2)\hat{z}\Rightarrow\mathscr{r}_-=z+d/2$.

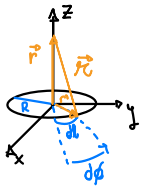
$\vec{p}=q\,d\hat{z}$ (dipole moment vector) — $d\hat{z}$: displacement vector

$$\vec{E}=\frac{1}{4\pi\epsilon_0}\left[\frac{q\,(z-d/2)}{(z-d/2)^3}+\frac{(-q)(z+d/2)}{(z+d/2)^3}\right]\hat{z}=\frac{q}{4\pi\epsilon_0}\frac{2zd\hat{z}}{(z^2-d^2/4)^2}=\frac{1}{4\pi\epsilon_0}\frac{2z\vec{p}}{(z^2-\frac{d^2}{4})^2}\xrightarrow{d\to 0}0$$

$$\xrightarrow[\text{with } \vec{p} \text{ fixed}]{d\to 0}\frac{1}{4\pi\epsilon_0}\frac{2\vec{p}}{z^3}\ \text{(Ideal dipole limit)}$$

## Problem 2

Calculate the $\vec{E}$ field along the z-axis produced by a ring (linear distribution) of radius $R$ with a charge $Q$ (uniformally distributed)



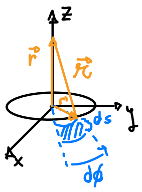

$$dq=\lambda\underbrace{dl}_{R d\phi}=\frac{Q}{2\pi R}R\,d\phi=\frac{Q}{2\pi}d\phi,\quad \lambda=Q/(2\pi R),\quad \vec{r}=z\hat{z},\ \vec{r}'=R\hat{s}=R\cos\phi\,\hat{x}+R\sin\phi\,\hat{y}\Rightarrow\vec{\mathscr{r}}=z\hat{z}-R(\cos\phi\,\hat{x}+\sin\phi\,\hat{y})\ \&\ \mathscr{r}=(z^2+R^2)^{1/2}$$

$$d\vec{E}=\frac{1}{4\pi\epsilon_0}\frac{(z\hat{z}-R\cos\phi\,\hat{x}-R\sin\phi\,\hat{y})}{(z^2+R^2)^{3/2}}\frac{Q}{2\pi}d\phi\Rightarrow\vec{E}=\int_0^{2\pi}d\phi\,\frac{1}{4\pi\epsilon_0}\frac{(z\hat{z}-R\cos\phi\,\hat{x}-R\sin\phi\,\hat{y})}{(z^2+R^2)^{3/2}}\frac{Q}{2\pi}\Rightarrow\vec{E}(z)=\frac{Q}{4\pi\epsilon_0}\frac{z\hat{z}}{(z^2+R^2)^{3/2}}\xrightarrow{z\gg R}\vec{E}(z)=\frac{Q}{4\pi\epsilon_0}\frac{\hat{z}}{z^2}$$

Point charge! O.K.!

## Problem 3.

Calculate the $\vec{E}$ field along the z-axis produced by a disk of radius $R$ with a charge $Q$ (uniformally distributed)

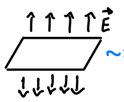



$$dq=\underbrace{\sigma}_{Q/(\pi R^2)}\underbrace{da}_{s\,ds\,d\phi}=\frac{Q}{\pi R^2}s\,ds\,d\phi,\quad \vec{r}=z\hat{z},\ \vec{r}'=s\hat{s}=s\cos\phi\,\hat{x}+s\sin\phi\,\hat{y}\Rightarrow\vec{\mathscr{r}}=z\hat{z}-s\cos\phi\,\hat{x}-s\sin\phi\,\hat{y}\ \&\ \mathscr{r}=(z^2+s^2)^{1/2}$$

$$d\vec{E}=\frac{1}{4\pi\epsilon_0}\frac{(z\hat{z}-s\cos\phi\,\hat{x}-s\sin\phi\,\hat{y})}{(z^2+s^2)^{3/2}}\sigma\,s\,ds\,d\phi\Rightarrow\vec{E}=\frac{1}{4\pi\epsilon_0}\sigma\int_0^R ds\,s\,\frac{1}{(z^2+s^2)^{3/2}}\int_0^{2\pi}d\phi\,(z\hat{z}-s\cos\phi\,\hat{x}-s\sin\phi\,\hat{y})=\frac{1}{4\pi\epsilon_0}\sigma\,2\pi z\hat{z}\int_0^R ds\,s\,\frac{1}{(z^2+s^2)^{3/2}}$$

$$\vec{E}(z)=(\sigma/2\epsilon_0)\,z\hat{z}\int_0^R ds\,s\,(z^2+s^2)^{-3/2}=(\sigma/2\epsilon_0)\,\text{sign}(z)\hat{z}\left[1-\frac{|z|}{(z^2+R^2)^{1/2}}\right]\Rightarrow\vec{E}(z)=\left(\frac{\sigma}{2\epsilon_0}\right)\text{sign}(z)\hat{z}\left[1-\frac{|z|}{(z^2+R^2)^{1/2}}\right]$$

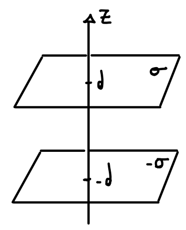
In the $R\to\infty$ limit we get $\vec{E}=\frac{\sigma}{2\epsilon_0}\text{sign}(z)\hat{z}$ (Charged plane solution)




$\rightsquigarrow$ Constant and pointing away from the plane (if $\sigma>0$)

## Problem 4.

Consider a plane at $z=d$ with surface charge $\sigma$ and another one at $z=-d$ with surface charge $-\sigma$. Calculate the $\vec{E}$ field everywhere.




We have that $\vec{E}_+ = \begin{cases} \frac{\sigma}{2\epsilon_0}\hat{z} & \text{for } z>d \\ -\frac{\sigma}{2\epsilon_0}\hat{z} & \text{for } z<d \end{cases}$ and $\vec{E}_- = \begin{cases} -\frac{\sigma}{2\epsilon_0}\hat{z} & \text{for } z>-d \\ \frac{\sigma}{2\epsilon_0}\hat{z} & \text{for } z<-d \end{cases}$

Then, $\vec{E} = \vec{E}_+ + \vec{E}_- = \begin{cases} 0 & \text{for } z>d \\ -\frac{\sigma}{\epsilon_0}\hat{z} & \text{for } -d<z<d \\ 0 & \text{for } z<d \end{cases}$ ⇝ This is the setup of parallel plate capacitors!

## Problem 5.

Given the electric potential $V(r) = A\frac{e^{-\lambda r}}{r}$ determine $\vec{E}(r)$, $\rho(r)$, and $Q$.

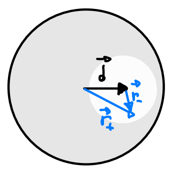
$$\vec{E} = -\nabla V = -\nabla\left(A\frac{e^{-\lambda r}}{r}\right) = -A\overbrace{\nabla\left(\frac{e^{-\lambda r}}{r}\right)}^{\nabla f(r) = \frac{\partial f}{\partial r}\hat{r}} = -Ae^{-\lambda r}\left[-\frac{1}{r^2} - \frac{\lambda}{r}\right]\hat{r} = A(1+\lambda r)\frac{e^{-\lambda r}}{r^2}\hat{r} \Rightarrow \vec{E}(r) = A(1+\lambda r)\frac{e^{-\lambda r}}{r^2}\hat{r}$$

$$\nabla\cdot\vec{E} = \rho/\epsilon_0 \Rightarrow \rho(r) = \epsilon_0(\nabla\cdot\vec{E}) = \epsilon_0 A\,\nabla\cdot\left((1+\lambda r)\frac{e^{-\lambda r}}{r^2}\hat{r}\right) = \epsilon_0 A\left[\underbrace{\nabla\left((1+\lambda r)e^{-\lambda r}\right)}_{\substack{(\lambda+(1+\lambda r)(-\lambda))e^{-\lambda r}\hat{r} \\ = -\lambda^2 r e^{-\lambda r}\hat{r}}}\cdot\frac{\hat{r}}{r^2} + \overbrace{(1+\lambda r)e^{-\lambda r}\underbrace{\nabla\cdot\left(\frac{\hat{r}}{r^2}\right)}_{4\pi\delta^3(\vec{r})}}^{f(r)\delta^3(\vec{r}) = f(0)\delta^3(\vec{r})}\right] \Rightarrow \rho(r) = \epsilon_0 A\left[4\pi\delta^3(\vec{r}) - \lambda^2\frac{e^{-\lambda r}}{r}\right]$$

$$\Rightarrow Q = \int_{\mathbb{R}^3} dV\rho(r) = \int_{\mathbb{R}^3} dV\,\epsilon_0 A\left[4\pi\delta^3(\vec{r}) - \lambda^2\frac{e^{-\lambda r}}{r}\right] = \epsilon_0 A\left[4\pi - \lambda^2\int_0^\infty dr\,4\pi r^2\frac{e^{-\lambda r}}{r}\right] = 4\pi\epsilon_0 A\left[1 - \lambda^2\underbrace{\int_0^\infty dr\,r e^{-\lambda r}}_{=\partial_\lambda\int_0^\infty dr(-1)e^{-\lambda r} = -\partial_\lambda\left(-\frac{e^{-\lambda r}}{\lambda}\Big|_0^\infty\right) = -\partial_\lambda(\lambda^{-1}) = \lambda^{-2}}\right] \Rightarrow Q = 0$$

## Problem 6.

A uniform charged sphere with charge density $\rho$ has in its interior a spherical cavity. Calculate $\vec{E}$ inside the cavity ($\vec{d}$ is a vector connecting the center of the two spheres)




First, for a point inside a sphere ($r<R$) Gauss law gives

$$\int\vec{E}\cdot d\vec{a} = \frac{1}{\epsilon_0}\int_0^r dr'\,4\pi r'^2\rho \Rightarrow 4\pi r^2 E(r) = \frac{1}{\epsilon_0}\frac{4}{3}\pi r^3\rho \Rightarrow \vec{E}(r) = \frac{\rho}{3\epsilon_0}\overbrace{r\hat{r}}^{\vec{r}}$$

We can then, by the principle of superposition, consider this as the superposition of a sphere (with no cavity) with charge density $\rho$, such that

$$\vec{E}_+ = \frac{\rho}{3\epsilon_0}\vec{r}_+$$

Plus a smaller sphere with charge density $-\rho$, such that, in the cavity,

$$\vec{E}_- = -\frac{\rho}{3\epsilon_0}\vec{r}_- \Rightarrow \vec{E} = \vec{E}_+ + \vec{E}_- = \frac{\rho}{3\epsilon_0}\overbrace{(\vec{r}_+ - \vec{r}_-)}^{=\vec{d}} \Rightarrow \vec{E} = \frac{\rho\vec{d}}{3\epsilon_0}$$ inside the cavity - And it's constant!