Lecture 5: Poisson & Laplace equations, Part II

Reading: Stechmann Ch. 3.3-3.4

Last time:

$$-\nabla^2 u = f \quad \text{(Poisson's equation)}$$
$$\nabla^2 u = 0 \quad \text{(Laplace's equation)}$$

Equilibrium heat / concentration ($\cancelto{0}{u_t} = k\nabla^2 u + f(\vec{x})$ or $\cancelto{0}{c_t} = D\nabla^2 c + f(\vec{x})$)

Electrostatics, $\quad \vec{E} = -\nabla\phi : \; -\nabla^2\phi = \frac{\rho(\vec{x})}{\varepsilon_0}$

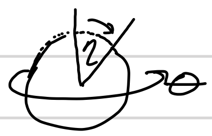
Fundamental solution: $\quad \nabla^2 \Phi = \delta(\vec{x}) \qquad \vec{x} \in \mathbb{R}^n$

$\partial_r \Phi$

$n=2$. Polar coordinates (r,θ): $\quad \nabla^2\Phi = \frac{1}{r}\partial_r(r\Phi_r) + \frac{1}{r^2}\Phi_{\theta\theta}$

Seek $\Phi = \Phi(r)$, Find: $\boxed{\Phi(r) = \frac{1}{2\pi}\log(r)} \qquad (r = |\vec{x}|)$

$n=3$. Spherical coordinates (r,θ,η)



$$\nabla^2\Phi = \frac{1}{r^2}\partial_r(r^2\partial_r\Phi) + \frac{1}{r^2\sin\theta}\partial_\theta(\sin\theta\,\partial_\theta) + \frac{1}{r^2\sin^2\theta}\partial_{\eta\eta}$$

Seek $\Phi = \Phi(r)$, Find: $\boxed{\Phi(r) = \frac{-1}{4\pi r}} \qquad (r = |\vec{x}|)$

(Check! $\quad \nabla^2\left(\frac{-1}{4\pi r}\right) = \frac{1}{r^2}\partial_r\left(r^2\left(\frac{1}{4\pi r^2}\right)\right) = 0 \quad (r>0)$)

Φ solves: Poisson's equation with an impulsive force ($\delta(\vec{x})$)

and/or: Laplace's equation in $\mathbb{R}^n \setminus \{0\}$.

★ Solution to Poisson's equation, $\nabla^2 u = f(\vec{x})$:

$$u(\vec{x}) = f * \Phi = \int_{\mathbb{R}^n} f(\vec{y})\,\Phi(\vec{x}-\vec{y})\,dv_y$$

Why? Formally, $\nabla^2 u = \int_{\mathbb{R}^n} f(\vec{y})\,\nabla^2\Phi(\vec{x}-\vec{y})\,dv_y$

$$= \int_{\mathbb{R}^n} f(\vec{y})\,\delta(\vec{x}-\vec{y})\,dv_y = f(\vec{x})$$

$n=2$: $u(\vec{x}) = \frac{1}{2\pi}\int_{\mathbb{R}^2} f(\vec{y})\log(|\vec{x}-\vec{y}|)\,dv_y$ $\rightarrow dy_1dy_2$

$n=3$: $u(\vec{x}) = \frac{-1}{4\pi}\int_{\mathbb{R}^3} \frac{f(\vec{y})}{|\vec{x}-\vec{y}|}\,dv_y$ $\rightarrow dy_1dy_2dy_3$

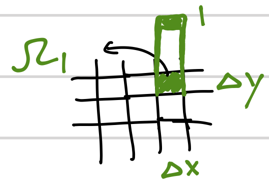
Once again we are simply adding up solutions. $\left(\int \leftrightarrow \Sigma\right)$

★ If a PDE is <u>linear</u> and <u>homogeneous</u> (no forcing),

the sum of two solutions is a solution.

- Let $\nabla^2\Phi_1 = 0$ and $\nabla^2\Phi_2 = 0$, then $\nabla^2(\Phi_1+\Phi_2) = 0$

★ With forcing, it's <u>almost</u> as nice.



$\mathbb{1}$: "indicator function" (1 or 0)

- $\vec{x} \in \mathbb{R}^2$: Let $\nabla^2\Phi_1 = g_1 \mathbb{1}_{\{\vec{x}\in\Omega_1\}}$

or $\nabla^2\Phi_1 = g_1\,\Delta x\Delta y\left(\frac{1}{\Delta x\Delta y}\mathbb{1}_{\{\vec{x}\in\Omega_1\}}\right)$

or $\nabla^2\Phi_1 = g_1\,\Delta x\Delta y\,\delta_\Delta(\vec{x}-\vec{x}_1)$, where $\delta_\Delta(\vec{x}) = \begin{cases} \frac{1}{\Delta x\Delta y} & \vec{x}\in\Omega_1 \\ 0 & \vec{x}\notin\Omega_1 \end{cases}$

Also, let $\nabla^2\Phi_2 = g_2\,\Delta x\Delta y\,\delta_\Delta(\vec{x}-\vec{x}_2)$

Then: $\nabla^2(\Phi_1+\Phi_2) = \left(g_1\,\delta_\Delta(\vec{x}-\vec{x}_1) + g_2\,\delta_\Delta(\vec{x}-\vec{x}_2)\right)\Delta x\Delta y$

Adding more, as $\Delta x, \Delta y \to 0$, $\nabla^2(\Phi_1+\Phi_2+\dots) = \int_{\mathbb{R}^2} g(\vec{y})\,\delta(\vec{x}-\vec{y})\,dxdy = g(\vec{x})$

Examples:

Electrostatic point charge. $\rho(\vec{x}) = -\delta(\vec{x})$ (negatively charged particle)

$\vec{x} \in \mathbb{R}^3$. $\ominus$ ϕ?

$$\nabla^2 \phi = \frac{-\rho(\vec{x})}{\varepsilon_0} = \frac{1}{\varepsilon_0}\delta(\vec{x})$$

So $\phi(\vec{x}) = \frac{1}{\varepsilon_0}\Phi(\vec{x}) = \frac{-1}{4\pi\varepsilon_0|\vec{x}|}$ in 3D.

(use $\nabla^2\Phi = \delta(\vec{x})$ and linearity $(\frac{1}{\varepsilon_0})\times$)

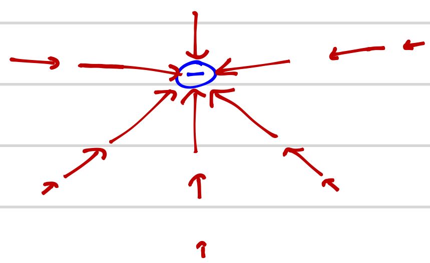
(or $\phi(\vec{x}) = \int_{\mathbb{R}^3} \left(\frac{1}{\varepsilon_0}\delta(\vec{y})\right) \frac{-1}{4\pi|\vec{x}-\vec{y}|}\, dv_y = \frac{-1}{4\pi\varepsilon_0|\vec{x}|}$)

So $\phi(\vec{x})$ grows as $\frac{-1}{4\pi\varepsilon_0 r}$ (potential energy)

$\vec{E}$-field? $\vec{E} = -\nabla\phi$

① $\vec{E} = -\left(\hat{r}\partial_r + \frac{1}{r}\hat{\theta}\partial_\theta + \frac{1}{r\sin\eta}\hat{\eta}\partial_\eta\right)\left(\frac{-1}{4\pi\varepsilon_0 r}\right)$

$= \hat{r}\left(-\frac{1}{4\pi\varepsilon_0 r^2}\right)$



($\frac{1}{r^2}$ decay)

② $r = |\vec{x}| = \sqrt{x^2+y^2+z^2}$,

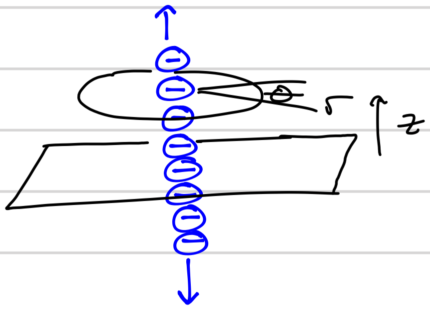
$\vec{E} = -\nabla\phi = -\left(\hat{x}\partial_x + \hat{y}\partial_y + \hat{z}\partial_z\right)\left(\frac{-1}{4\pi\varepsilon|\vec{x}|}\right)$

Note: $\partial_x\left(\frac{1}{|\vec{x}|}\right) = \partial_x\left(x^2+y^2+z^2\right)^{-1/2} = -\frac{1}{2}\left(x^2+y^2+z^2\right)^{-3/2}\cdot 2x = \frac{-x}{|\vec{x}|^3}$

$$\vec{E} = \hat{x}\left(\frac{-x}{4\pi\varepsilon_0|\vec{x}|^3}\right) + \hat{y}\left(\frac{-y}{4\pi\varepsilon_0|\vec{x}|^3}\right) + \hat{z}\left(\frac{-z}{4\pi\varepsilon_0|\vec{x}|^3}\right)$$

$$= \frac{-1}{4\pi\varepsilon_0|\vec{x}|^3}\underbrace{\left(x\hat{x}+y\hat{y}+z\hat{z}\right)}_{|\vec{x}|\hat{r}}$$

• Line of charge:



Every cross-section is identical.

$$\nabla^2\phi = -\frac{\rho(\vec{x})}{\varepsilon_0}, \quad \rho(\vec{x}) = -\tilde{\delta}(r) \qquad \left(\int_0^{2\pi}\int_0^R \tilde{\delta}(r)\, r\, dr\, d\theta = 1\right)$$

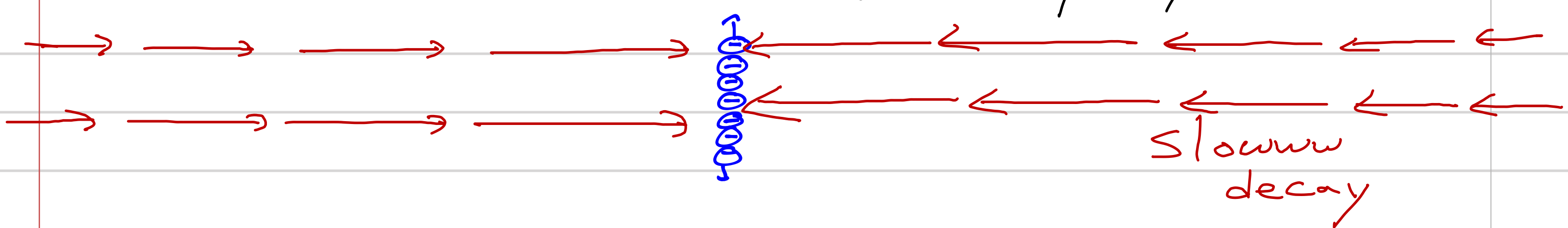
Seek a 2D solution, $\phi(r,\theta,z) = \phi(r,\theta) = \phi(r)$

Polar coordinates : $\frac{1}{r}\partial_r\left(\frac{1}{r}\phi_r\right) = \frac{1}{\varepsilon_0}\tilde{\delta}(r)$

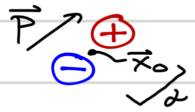
$$\phi = \frac{1}{2\pi\varepsilon_0}\log(r)$$

$$\vec{E} = -\nabla\phi = -\left(\hat{r}\partial_r + \frac{1}{r}\hat{\theta}\partial_\theta + \hat{z}\partial_z\right)\phi$$

$$= -\hat{r}\left(\frac{1}{2\pi\varepsilon_0 r}\right)$$ Only decays like $\frac{1}{r}$.



☆ Dipoles

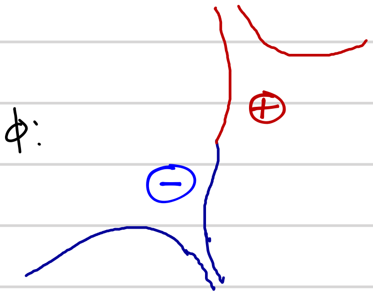


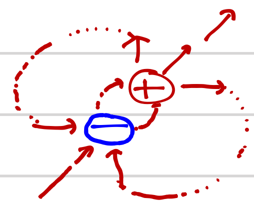
$|\vec{p}|=1$

$\vec{x} \in \mathbb{R}^3$: $\quad \rho(\vec{x}) = \frac{1}{\alpha}\left(\delta\left(\vec{x} - \left[\vec{x}_0 + \frac{\alpha}{2}\vec{p}\right]\right) - \delta\left(\vec{x} - \left[\vec{x}_0 - \frac{\alpha}{2}\vec{p}\right]\right) \right)$

$$\phi = \frac{1/\alpha}{4\pi\varepsilon_0 |\vec{x} - (\vec{x}_0 + \frac{\alpha}{2}\vec{p})|} - \frac{1/\alpha}{4\pi\varepsilon_0 |\vec{x} - (\vec{x}_0 - \frac{\alpha}{2}\vec{p})|}$$

ϕ:





$(\vec{E} = -\nabla\phi)$

as $\alpha \to 0$, Taylor expand:

$$\phi = \frac{1/\alpha}{4\pi\varepsilon_0}\left[\frac{1}{|\vec{x}-\vec{x}_0|} + \frac{\alpha}{2}\vec{p}\cdot\nabla_0\left(\frac{1}{|\vec{x}-\vec{x}_0|}\right) + O(\alpha^2) \right]$$

$$- \frac{1}{4\pi\varepsilon_0}\left[\frac{1}{|\vec{x}-\vec{x}_0|} - \frac{\alpha}{2}\vec{p}\cdot\nabla_0\left(\frac{1}{|\vec{x}-\vec{x}_0|}\right) + O(\alpha^2) \right]$$

$$\xrightarrow{\alpha\to 0} \frac{1}{4\pi\varepsilon_0}\vec{p}\cdot\nabla_0\left(\frac{1}{|\vec{x}-\vec{x}_0|}\right) = \frac{1}{4\pi\varepsilon_0}\frac{\vec{p}\cdot(\vec{x}-\vec{x}_0)}{|\vec{x}-\vec{x}_0|^3}$$

$$\left(\sim \frac{1}{r^2}\right)$$