Physics 322
Discussion 5
10/17/2024

Laplace equation: $\nabla^2 V = 0$

· In cartesian coordinates

$$\frac{\partial^2}{\partial x^2}V(x,y,z)+\frac{\partial^2}{\partial y^2}V(x,y,z)+\frac{\partial^2}{\partial z^2}V(x,y,z)=0 \;\therefore\; \text{Separation of variables } V(x,y,z)=X(x)Y(y)Z(z)$$

$$\Rightarrow \frac{1}{X(x)}\frac{d^2X}{dx^2}=\overset{-\text{ for periodic},\,+\text{ for non-periodic}}{\pm\underbrace{\alpha^2}_{\alpha^2>0}},\quad \frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2}=\pm\underbrace{\beta^2}_{\beta^2>0},\quad \frac{1}{Z(z)}\frac{d^2Z(z)}{dz^2}=\pm\underbrace{\gamma^2}_{\gamma^2>0},\quad \text{with}\quad (\pm\alpha^2)+(\pm\beta^2)+(\pm\gamma^2)=0$$

Note that

$$\frac{d^2\psi_0(x)}{dx^2}\Rightarrow \psi_0(x)=A_0+B_0x,\quad \frac{d^2\psi_k(x)}{dx^2}=-\overset{K^2>0}{K^2}\psi_k(x)\Rightarrow \psi_k(x)=A_k\cos(Kx)+B_k\sin(Kx),\quad \frac{d^2\psi_k(x)}{dx^2}=K^2\psi_k(x)\Rightarrow \psi_k(x)=A_ke^{Kx}+B_ke^{-Kx}\text{ or } \psi_k(x)=A_k\cosh(kx)+B_k\sinh(kx)$$

· In Spherical coordinates

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V(r,\theta,\phi)}{\partial r}\right)+\frac{1}{r^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V(r,\theta,\phi)}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2V(r,\theta,\phi)}{\partial\phi^2}\right]=0 \;\therefore\; \text{Azimuthal symmetry (invariance by rotations about } z\text{-axis} \Rightarrow \text{no } \phi \text{ dependence)}$$

$V(r,\theta,\phi)=V(r,\theta)$. Separation of variables $V(r,\theta)=R(r)\Theta(\theta)$

$$\Rightarrow \frac{1}{R(r)}\frac{d}{dr}\left[r^2\frac{dR(r)}{dr}\right]=\ell(\ell+1)\text{ with solution } R(r)=A_\ell r^\ell+B_\ell r^{-(\ell+1)},\quad \frac{1}{\sin\theta}\frac{d}{d\theta}\left[\sin\theta\frac{d\Theta(\theta)}{d\theta}\right]=-\ell(\ell+1)\Theta(\theta)\text{ with solution } P_\ell(\cos\theta)\text{ and } \ell\in\{0,1,\dots\}$$

$$\int_{-1}^{1}dx\,P_{\ell'}(x)P_\ell(x)=\frac{2}{2\ell+1}\delta_{\ell\ell'}$$

$$P_0(x)=1,\; P_1(x)=x,\; P_2(x)=\frac{3x^2-1}{2}$$

The general solution reads

$$V(r,\theta)=\sum_{\ell=0}^{\infty}\left(A_\ell r^\ell+B_\ell r^{-(\ell+1)}\right)P_\ell(\cos\theta)$$

· In Cylindrical coordinates

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V(r,\phi,z)}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2V(r,\phi,z)}{\partial\phi^2}+\frac{\partial^2V(r,\phi,z)}{\partial z^2}=0 \;\therefore\; \text{Translational symmetry on } z\text{-axis } V(r,\phi,z)=V(r,\phi).\Rightarrow r\frac{\partial}{\partial r}\left[r\frac{\partial V(r,\phi)}{\partial r}\right]+\frac{\partial^2V(r,\phi)}{\partial\phi^2}=0$$

Problem 1.

Let's solve problem 5 of HW4.

(a) Using separation of variables $V(r,\phi)=R(r)\Phi(\phi)$,

$$\frac{r}{R(r)}\frac{d}{dr}\left[r\frac{dR(r)}{dr}\right]+\underbrace{\frac{1}{\Phi(\phi)}\frac{d^2\Phi(\phi)}{d\phi^2}}_{-m^2}=0$$

$-m^2$ (must be periodic for it's a angular variable) with $m\in\{0,1,\dots\}$

$$\Rightarrow r^2\frac{d^2R(r)}{dr^2}+r\frac{dR(r)}{dr}-m^2R(r)=0\quad\text{and}\quad \frac{d^2\Phi}{d\phi^2}=-m^2\Phi(\phi)\Rightarrow\Phi(\phi)=A_m\cos(m\phi)+B_m\sin(m\phi)$$

$R(r)=r^\lambda$

$$\lambda(\lambda-1)+\lambda-m^2=0\Rightarrow\lambda=\pm m\Rightarrow R(r)=C_mr^m+D_mr^{-m}\text{ for } m\neq0.\quad\text{For } m=0 \text{ we have } \frac{d}{dr}\left[r\frac{dR(r)}{dr}\right]=0\Rightarrow R(r)=A_0+B_0\ln r$$

Thus,

$$V(r,\phi)=a_0+b_0\ln r+\sum_{m=1}^{\infty}r^m\left(a_m\cos(m\phi)+b_m\sin(m\phi)\right)+r^{-m}\left(c_m\cos(m\phi)+d_m\sin(m\phi)\right)$$

(b) We can write the general solutions

(Solution must be finite for $r\to\infty$)

$$V^>(r,\phi)=a_0^>+\cancelto{0}{b_0^>}\ln r+\sum_{m=1}^{\infty}r^m\left[\cancelto{0}{a_m^>}\cos(m\phi)+\cancelto{0}{b_m^>}\sin(m\phi)\right]+r^{-m}\left[c_m^>\cos(m\phi)+d_m^>\sin(m\phi)\right]\quad\text{for } r\geq R$$

(Solution must be finite for $r\to0$)

$$V^<(r,\phi)=a_0^<+\cancelto{0}{b_0^<}\ln r+\sum_{m=1}^{\infty}r^m\left[a_m^<\cos(m\phi)+b_m^<\sin(m\phi)\right]+r^{-m}\left[\cancelto{0}{c_m^<}\cos(m\phi)+\cancelto{0}{d_m^<}\sin(m\phi)\right]\quad\text{for } r\leq R$$

Then,

$$V^{>}(r,\phi)=a_0^{>}+\sum_{m=1}^{\infty} r^{-m}\left[c_m^{>}\cos(m\phi)+d_m^{>}\sin(m\phi)\right] \quad \text{for } r\geq R$$

$$V^{<}(r,\phi)=a_0^{<}+\sum_{m=1}^{\infty} r^{m}\left[a_m^{<}\cos(m\phi)+b_m^{<}\sin(m\phi)\right] \quad \text{for } r\leq R$$

Now,

$$V^{>}(R,\phi)=V^{<}(R,\phi)=V_0(\phi)=K(\cos\phi)^2=\frac{K}{2}+\frac{K}{2}\cos(2\phi) \Rightarrow a_0^{<}=\frac{K}{2},\ R^2a_2^{<}=\frac{K}{2},\ a_0^{>}=\frac{K}{2},\ R^{-2}c_2^{>}=\frac{K}{2}.$$ All other coefficients must be zero.

Hence,

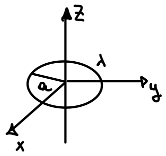
$$V^{>}(r,\phi)=\frac{K}{2}\left[1+\left(\frac{r}{R}\right)^2\cos(2\phi)\right]$$

$$V^{<}(r,\phi)=\frac{K}{2}\left[1+\left(\frac{R}{r}\right)^2\cos(2\phi)\right]$$

Problem 2.

Let's workout problem 4 of HW 4.

We know that



$\vec{r}=(0,0,z)$, $\vec{r}\,'=(a\cos\phi', a\sin\phi', 0)$

$$V(0,0,z)=\frac{1}{4\pi\epsilon_0}\int\frac{\lambda a\,d\phi'}{(a^2+z^2)^{1/2}}=\frac{\overbrace{(2\pi a\lambda)}^{q}}{4\pi\epsilon_0}\frac{1}{(z^2+a^2)^{1/2}} \Rightarrow V(z)=\frac{q}{4\pi\epsilon_0}\frac{1}{|\vec{r}-\vec{r}\,'|}$$

From class, we know that

angle between $\vec{r}$ and $\vec{r}\,'$. In this case $\gamma=\pi/2$

$$\frac{1}{|\vec{r}-\vec{r}\,'|}=\sum_{\ell=0}^{\infty}\frac{r_<^{\ell}}{r_>^{\ell+1}}P_\ell(\cos\gamma) \Rightarrow \frac{1}{|\vec{r}-\vec{r}\,'|}=\sum_{\ell=0}^{\infty}\frac{z^\ell}{a^{\ell+1}}P_\ell(0) \ \text{for } z<a \quad \text{and} \quad \frac{1}{|\vec{r}-\vec{r}\,'|}=\sum_{\ell=0}^{\infty}\frac{a^\ell}{z^{\ell+1}}P_\ell(0) \ \text{for } z>a$$

Therefore,

$$V^{>}(z)=\sum_{\ell=0}^{\infty}\overbrace{\left(\frac{q}{4\pi\epsilon_0}a^{\ell}P_\ell(0)\right)}^{B_\ell^{>}}\frac{1}{z^{\ell+1}} \quad \text{for } z>a$$

$$V^{<}(z)=\sum_{\ell=0}^{\infty}\underbrace{\left(\frac{q}{4\pi\epsilon_0}a^{-(\ell+1)}P_\ell(0)\right)}_{A_\ell^{<}}z^{\ell} \quad \text{for } z<a$$

Going off the axis, we get

$$V^{>}(r,\theta)=\frac{q}{4\pi a\epsilon_0}\sum_{\ell=0}^{\infty}\left(\frac{a}{r}\right)^{\ell+1}P_\ell(0)P_\ell(\cos\theta) \quad \text{for } r>a$$

$$V^{<}(r,\theta)=\frac{q}{4\pi a\epsilon_0}\sum_{\ell=0}^{\infty}\left(\frac{r}{a}\right)^{\ell}P_\ell(0)P_\ell(\cos\theta) \quad \text{for } r<a$$

Problem 3.

A specified charge density $\sigma_0(\theta)$ is glued over the surface of a spherical shell of radius R. Find the resulting potential inside and outside the sphere.

The general solution reads

$$V^{>}(r,\theta)=\sum_{\ell=0}^{\infty}\left(\underbrace{\cancel{A_\ell^{>}r^{\ell}}}_{0 \text{ (Solution must go to 0 for } r\to\infty)}+B_\ell^{>}r^{-(\ell+1)}\right)P_\ell(\cos\theta) \quad \text{for } r>R$$

$$V^{<}(r,\theta)=\sum_{\ell=0}^{\infty}\left(A_\ell^{<}r^{\ell}+\underbrace{\cancel{B_\ell^{<}r^{-(\ell+1)}}}_{0 \text{ (Solution must be finite for } r\to 0)}\right)P_\ell(\cos\theta) \quad \text{for } r<R$$

Requiring the continuity of V at $r=R$,

$$\sum_{\ell=0}^{\infty} B_\ell^{>} R^{-(\ell+1)} P_\ell(\cos\theta) = \sum_{\ell=0}^{\infty} A_\ell^{<} R^{\ell} P_\ell(\cos\theta) \Rightarrow B_\ell^{>} = A_\ell^{<} R^{2\ell+1}$$

Then,

$$V^{>}(r,\theta) = \sum_{\ell=0}^{\infty} A_\ell^{<} R^{2\ell+1} r^{-(\ell+1)} P_\ell(\cos\theta) \Rightarrow \left.\frac{\partial V^{>}(r,\theta)}{\partial r}\right|_{r=R} = \sum_{\ell=0}^{\infty} A_\ell^{<} (-1)(\ell+1) R^{\ell-1} P_\ell(\cos\theta)$$

$$V^{<}(r,\theta) = \sum_{\ell=0}^{\infty} A_\ell^{<} r^{\ell} P_\ell(\cos\theta) \Rightarrow \left.\frac{\partial V^{<}(r,\theta)}{\partial r}\right|_{r=R} = \sum_{\ell=0}^{\infty} A_\ell^{<} \ell R^{\ell-1} P_\ell(\cos\theta)$$

Now,

$$\left.\left(\frac{\partial V^{>}}{\partial r} - \frac{\partial V^{<}}{\partial r}\right)\right|_{r=R} = -\frac{\sigma_0(\theta)}{\epsilon_0} \Rightarrow \sum_{\ell=0}^{\infty} (2\ell+1) A_\ell^{<} R^{\ell-1} P_\ell(\cos\theta) = \frac{\sigma_0(\theta)}{\epsilon_0} \Rightarrow (2\ell+1) A_\ell^{<} R^{\ell-1} \frac{2}{2\ell+1} \epsilon_0 = \int_0^{\pi} d\theta \sin\theta\, P_\ell(\cos\theta)\,\sigma_0(\theta) \Rightarrow A_\ell^{<} = \frac{1}{2\epsilon_0 R^{\ell-1}} \int_0^{\pi} d\theta \sin\theta\, \sigma_0(\theta) P_\ell(\cos\theta)$$

Hence,

$$V^{>}(r,\theta) = \left(\frac{R}{2\epsilon_0}\right) \sum_{\ell=0}^{\infty} I_\ell \left(\frac{R}{r}\right)^{\ell+1} P_\ell(\cos\theta), \quad r>R$$

$$V^{<}(r,\theta) = \left(\frac{R}{2\epsilon_0}\right) \sum_{\ell=0}^{\infty} I_\ell \left(\frac{r}{R}\right)^{\ell} P_\ell(\cos\theta), \quad r<R$$

with $I_\ell = \int_0^{\pi} d\theta \sin\theta\, \sigma_0(\theta) P_\ell(\cos\theta)$. For $\sigma_0(\theta) = K\cos\theta = K P_1(\cos\theta)$, $I_\ell = K \int_0^{\pi} d\theta\, P_\ell(\cos\theta) P_1(\cos\theta) \sin\theta = \frac{2}{3} K \delta_{\ell 1}$. Hence,

$$V^{>}(r,\theta) = \left(\frac{KR}{3\epsilon_0}\right)\left(\frac{R}{r}\right)^2 \cos\theta \quad \text{for } r>R$$

$$V^{<}(r,\theta) = \left(\frac{KR}{3\epsilon_0}\right)\left(\frac{r}{R}\right) \cos\theta \quad \text{for } r<R$$

Note that, for this $\sigma_0(\theta)$,

$$q = \int_0^{\pi} d\theta \sin\theta\, 2\pi R^2 \sigma_0(\theta) = 0$$

$$\vec{p} = \int_0^{\pi} d\theta \sin\theta \int_0^{2\pi} d\phi\, (\cos\phi\sin\theta, \sin\phi\sin\theta, \cos\theta) K P_1(\cos\theta) R^3 = 2\pi K R^3 \int_0^{\pi} d\theta \sin\theta\, P_1(\cos\theta) P_1(\cos\theta)\,\hat{z} = 2\pi K R^3 \frac{2}{3}\hat{z} = \frac{4\pi R^3}{3} K \hat{z}$$

The potential outside can then be written as

$$V^{>}(r,\theta) = \frac{1}{4\pi\epsilon_0} \frac{\hat{r}\cdot\vec{p}}{r^2} \quad \text{with } \vec{p} = \left(\frac{4}{3}\pi R^3\right) K \hat{z}$$