

Math 421, Section 1
Homework 2
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Problem 1. Prove that for any $x, y \in \mathbb{N}$, if x is odd and y is odd then $x + y$ is even.

Solution: Suppose $x, y \in \mathbb{N}$ are odd, then $\exists n, m \in \mathbb{N} \cup \{0\}$ s.t. $x = 2n + 1, y = 2m + 1$.

$$x + y = 2n + 1 + 2m + 1 = 2(n + m + 1). \tag{1}$$

Since $(n + m + 1) \in \mathbb{N}$, $x + y$ is even.

□

Problem 2. Prove that for any $x \in \mathbb{N}$, if x is odd then x^3 is odd.

Solution: Suppose x is odd, i.e. $\exists n \in \mathbb{N} \cup \{0\}$ s.t. $x = 2n + 1$

$$x^3 = (2n + 1)^3 = 8n^3 + 12n^2 + 6n + 1 = 2(4n^3 + 6n^2 + 3n) + 1. \quad (2)$$

Trivially, since $(4n^3 + 6n^2 + 3n) \in \mathbb{N} \cup \{0\}$, x^3 is odd.

□

Problem 3. Using induction, prove that for all $n \in \mathbb{N}$ we have

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2.$$

Solution: [Base case]: For $n = 1$, we have $1 = 1$. Upon careful reflection, this holds true.

[Inductive step]: Suppose the statement is true for $\exists n \in \mathbb{N}$, i.e.

$$1 + 3 + \cdots + 2n - 1 = n^2 \tag{3}$$

Then for $n = n + 1$ we have:

$$1 + 3 + \cdots + 2n - 1 + 2(n + 1) - 1 = n^2 + 2n + 1 \tag{4}$$

$$= (n + 1)^2 \tag{5}$$

So the formula is true for $n + 1$. Thus, by induction, the statement is true for all $n \in \mathbb{N}$.

□

Problem 4. Compute the following sum:

$$\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \cdots + \frac{1}{(2n-1)(2n+1)}.$$

Prove that your answer is true for all $n \in \mathbb{N}$ using induction.

Solution: By noticing $\frac{1}{(2n-1)(2n+1)} = \frac{1}{2}(\frac{1}{2n-1} - \frac{1}{2n+1})$, a rough calculation suggests that the sum should be $\frac{1}{2} - \frac{1}{4n+2}$. It is proved by induction as follows:

[base case]: For $n = 1$, we have

$$\frac{1}{1 \cdot 3} = \frac{1}{2} - \frac{1}{6} = \frac{1}{3},$$

which is true.

[Inductive case]: Suppose the statement is true for $\exists n \in \mathbb{N}$, i.e.

$$\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \cdots + \frac{1}{(2n-1)(2n+1)} = \frac{1}{2} - \frac{1}{4n+2} \quad (6)$$

Then for $n = n + 1$,

$$\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \cdots + \frac{1}{(2n-1)(2n+1)} + \frac{1}{(2n+1)(2n+3)} = \frac{1}{2} - \frac{1}{4n+2} + \frac{1}{(2n+1)(2n+3)} \quad (7)$$

$$= \frac{1}{2} - \frac{1}{4n+2} + \frac{1}{2}(\frac{1}{2n+1} - \frac{1}{2n+3}) \quad (8)$$

$$= \frac{1}{2} - \frac{1}{4n+6} \quad (9)$$

$$= \frac{1}{2} - \frac{1}{4(n+1)+2} \quad (10)$$

So the formula is true for $n + 1$. Thus, by induction, the statement is true for all $n \in \mathbb{N}$.

□

Problem 5. Prove the following statements for all $a, b \in \mathbb{R}$:

- (a) $-a + (-b) = -(a + b)$.
- (b) If $a, b \neq 0$ then $a^{-1} \cdot b^{-1} = (ab)^{-1}$.

Carefully justify every step using properties of \mathbb{R} stated in lecture.

Solution: [a]: Consider the original equation,

$$-a + (-b) = -(a + b) \quad (11)$$

Adding $(a+b)$ to both sides, we can find that it is equivalent to

$$-a + (-b) + (a + b) = -(a + b) + (a + b) \quad (12)$$

Applying inverse addition to the right side, and apply associativity to the left, this is equivalent to

$$-a + a + (-b) + b = 0 \quad (13)$$

Therefore, to prove the original statement, it is suffice to prove its equivalence, i.e. Equation 13. By the inverse addition property, we have

$$-a + a = 0, \quad -b + b = 0 \quad (14)$$

$$\Rightarrow -a + a + (-b) + b = 0 \quad (15)$$

The statement is thus proved.

[b]: Suppose $a, b \neq 0$, we have

$$a^{-1} \cdot b^{-1} \cdot ab \stackrel{\text{commutivity}}{=} a^{-1} \cdot a \cdot b^{-1} \cdot b \stackrel{\text{inverse}}{=} 1 \quad (16)$$

$$\text{also, } (ab)^{-1} \cdot ab \stackrel{\text{inverse}}{=} 1 \quad (17)$$

By transivity, we have

$$a^{-1} \cdot b^{-1} \cdot ab = (ab)^{-1} \cdot ab \quad (18)$$

$$\stackrel{\text{prop.1}}{\Rightarrow} a^{-1} \cdot b^{-1} = (ab)^{-1} \quad (19)$$

As desired. □

Problem 6. Prove the following statements for all $a, b, c, d \in \mathbb{R}$:

- (a) If $a < b$ and $c < d$ then $a + c < b + d$.
- (b) If $0 < a < b$ and $0 < c < d$ then $ac < bd$.

Solution: [a]: Suppose $a < b, c < d$, then by O1,

$$a + c < b + c \tag{20}$$

$$b + c < d + b. \tag{21}$$

By Transitivity,

$$a + c < d + b \tag{22}$$

By commutivity,

$$a + c < b + d \tag{23}$$

Thus proves the inequality.

[b]: Suppose $0 < a < b, 0 < c < d$. Then by O2,

$$a \cdot c < b \cdot c \tag{24}$$

$$b \cdot c < b \cdot d \tag{25}$$

By transitivity,

$$ac < bd \tag{26}$$

Thus proves the inequality.

□