Problem 1

Prove the following statements:

1. there exists a number $1 < x < 2$ that solves the equation $x^2 - x - 1 = 0$.
2. There exists a number $x \in \mathbb{R}$ that solves the equation $x^5 - x + 1 = 0$.

Solution: Recall Intermediate Value theorem : if $f : [a, b] \to \mathbb{R}$ is continuous, and $f(a) < 0 < f(b)$, then $\exists c \in [a, b]$ s.t. $f(c) = 0$

1. From lecture, we know that since $f(x) = x^2 - x - 1$ is a polynomial function, it's continuous on $\mathbb{R}$. Noticing that $f(1) = -1$, $f(2) = 1$, by IVT, there exists $x \in [1, 2]$ such that $f(x) = 0$. Specifically, since $f(1) \neq 0$, $f(2) \neq 0, x \in (1, 2)$. ■

2. Similarly, we know that since $f(x) = x^5 - x + 1$ is a polynomial function, it's continuous on $\mathbb{R}$. Noticing that $f(-2) = -29 < 0$, $f(-1) = 1 > 0$, by IVT, there exists $x \in [-2, -1]$ such that $f(x) = 0$. Specifically, since $f(-2) \neq 0$, $f(-1) \neq 0$, $x \in (-2, -1) \in \mathbb{R}$. ■

Problem 2

Let $a < b$ be numbers and $f : [a, b] \to \mathbb{R}$ be a function. We say that $x \in [a, b]$ is a fixed point for f if $f(x) = x$. Prove that if f is continuous and $f(x) \in [a, b]$ for all $x \in [a, b]$, then f has a fixed point.

Solution:

Consider an auxiliary function $g : [a, b] \to \mathbb{R}, g(x) = f(x) - x$. Since f and x (a polynomial function) are both continuous functions, g is also continuous.

Thus the existence of a fixed point for f is equivalent to the existence of a root for g.

Noticing:

$$g(a) = f(a) - a; \quad g(b) = f(b) - b, \tag{1}$$

and that

$$f(x) \in [a, b] \forall x \in [a, b] \quad \Leftrightarrow \quad f(a) \geq a; f(b) \leq b, \tag{2}$$

we have:

$$g(a) \geq 0; \quad g(b) \leq 0. \tag{3}$$

In the cases where $g(a) = 0$ or $g(b) = 0$, we have $f(a) = a$ or $f(b) = b$, respectively.

In the rest of the cases, by IVT, $\exists c \in [a, b]$, $s.t.$ $g(c) = 0$. i.e. $f(c) = c$.

Therefore, $\exists x \in [a, b]$, $s.t.$ $f(x) = x$, and thus proving the existence of a fixed point for f. ■

Problem 3

Let $f : [0, 1] \to \mathbb{R}$ be a continuous function such that $f(0) = f(1)$. Prove that there exists $x \in \left[0, \frac{1}{2}\right]$ such that $f(x) = f\left(x + \frac{1}{2}\right)$. Hint: consider the function $g(x) = f(x) - f\left(x + \frac{1}{2}\right)$. Is it possible for $g(0)$ and $g\left(\frac{1}{2}\right)$ to both be positive?

Solution:

Consider the suggested function $g(x) = f(x) - f\left(x + \frac{1}{2}\right)$. Since f is continuous, $g : [0, 1] \to \mathbb{R}$ is also continuous. Noticing

$$\begin{aligned} g(0) &= f(0) - f\left(\frac{1}{2}\right) = f(1) - f\left(\frac{1}{2}\right), \\ g\left(\frac{1}{2}\right) &= f\left(\frac{1}{2}\right) - f(1) = f\left(\frac{1}{2}\right) - f(0), \end{aligned} \tag{4}$$

and that

$$\begin{aligned} g(0) + g\left(\frac{1}{2}\right) &= f(1) - f\left(\frac{1}{2}\right) + f\left(\frac{1}{2}\right) - f(1) = 0 \\ \Rightarrow g(0) &= -g\left(\frac{1}{2}\right) \end{aligned} \tag{5}$$

So $g(0)$ and $g\left(\frac{1}{2}\right)$ cannot both be positive. i.e.

$$\begin{aligned} g(0) \geq 0, g\left(\frac{1}{2}\right) &\leq 0, \\ \text{or, } g(0) \leq 0, g\left(\frac{1}{2}\right) &\geq 0. \end{aligned} \tag{6}$$

In the special cases where $g(0) = 0$ or $g\left(\frac{1}{2}\right) = 0$, we have $f(0) = f\left(\frac{1}{2}\right)$ or $f\left(\frac{1}{2}\right) = f(1)$, respectively.

In the rest of the cases, IVT implies that there exists $x \in \left[0, \frac{1}{2}\right]$ such that $g(x) = 0$, i.e. $f(x) = f\left(x + \frac{1}{2}\right)$. ■

Problem 4

For each of the following functions $f : [-1, 1] \to \mathbb{R}$, find all global extrema and find the points $x \in [-1, 1]$ at which f attains these extrema. No proof is required.

1.
$$f(x) = \begin{cases} 1 - x & \text{if } x \geq 0 \\ 1 + x & \text{if } x < 0. \end{cases} \tag{7}$$

$\max f = 1, x = 0; \quad \min f = 0, x = 1 \text{ or } -1.$

1.
$$f(x) = \begin{cases} 1 - x & \text{if } x \geq 0 \\ -1 - x & \text{if } x < 0. \end{cases} \tag{8}$$

$\max f = 1, x = 0; \quad \min f$ Does not exist.

1.
$$f(x) = \begin{cases} 1 - x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases} \tag{9}$$

$\max f = 1, x = 0 \text{ or } -1; \quad \min f = 0, x = 1$

Problem 5

Let $h > 0$. Prove that there is a point on the parabola

$$\{(x, x^2) \in \mathbb{R}^2 : -10 \leq x \leq 10\}, \tag{10}$$

that is closest to the point $(0, h)$.

Solution:

We construct a distance function $g : [-10, 10] \to \mathbb{R}$ that

$$g(x) = \sqrt{x^2 + (x^2 - h)^2} = \sqrt{x^4 + (1 - 2h)x^2 + h^2}. \tag{11}$$

$g(x)$ is continuous on $[-10, 10]$, since it's a composition of $x^4 + (1 - 2h)x^2 + h^2$, a continuous function since it's a polynomial function, and $\sqrt{x}$, a continuous function.

So by EVT, $g(x)$ has a global minimum at some $x \in [-10, 10]$.

In other words, there exists a point on the parabola that is closest to the point $(0, h)$. ■

Problem 6

Let $a < b$ be numbers and $f, g, h : [a, b] \to \mathbb{R}$ be functions.

1. Prove that if f is continuous, then $|f|$ has a global maximum. Given a continuous function f we define $\|f\|$ to be equal to this value. (i.e. the global maximum of $|f|$).
2. Prove that if g is continuous, then $\|cg\| = |c| \cdot \|g\|$ for any $c \in \mathbb{R}$.
3. Prove that if g and h are continuous, then $\|g + h\| \leq \|g\| + \|h\|$.

Solution:

1. Since $|f|$ is the composition of f and $|x|$, both continuous functions, $|f|$ is continuous on $[a, b]$. By EVT, $|f|$ has a global maximum on $[a, b]$.

2. By definition, $\|f\| \Leftrightarrow \max(|f|)$.

   So, by noticing that $|c|$ is a positive constant, $\forall c \in \mathbb{R}$:

$$\|cg\| = \max(|cg|) = \max(|c||g|) = |c| \cdot \max(|g|) = |c| \cdot \|g\|. \tag{12}$$

3. Similarly,

$$\|g + h\| = \max(|g + h|). \tag{13}$$

Since

$$\begin{gathered} |g + h| \leq |g| + |h| \\ \Rightarrow \max(|g + h|) \leq \max(|g| + |h|), \end{gathered} \tag{14}$$

and also

$$\begin{gathered} |g| \leq \|g\| \quad , |h| \leq \|h\| \\ \Rightarrow |g| + |h| \leq \|g\| + \|h\| \\ \Rightarrow \max(|g| + |h|) \leq \|g\| + \|h\|, \end{gathered} \tag{15}$$

we can find that

$$\|g + h\| = \max(|g + h|) \leq \max(|g| + |h|) \leq \|g\| + \|h\|, \tag{16}$$

as wanted.