**Math 421, Section 1**
**Homework 3**
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**Problem 1.** Determine whether each of the following functions are injective, surjective, and bijective, and prove your answer.

(a) $f : \mathbb{Z} \to \mathbb{Z}$, $f(x) = 2x$.
(b) $g : \mathbb{R} \to \mathbb{R}$, $g(x) = 2x$.

**Solution:** (a) Injectivity: Suppose $\exists x_1, x_2 \in \mathbb{Z}, s.t. f(x_1) = f(x_2)$ , want to show: $x_1 = x_2$.

$$f(x_1) = f(x_2) \implies 2x_1 = 2x_2 \implies x_1 = x_2. \tag{1}$$

The function is thus injective.

Surjectivity: Want to show $\forall y \in \mathbb{Z}, \exists x \in \mathbb{Z}$ $s.t.$ $f(x) = y$. Suppose $x, y \in \mathbb{Z}$, and let $f(x) = y$. i.e.,

$$2x = y \implies x = \frac{y}{2} \in \mathbb{Z}. \tag{2}$$

However, $\frac{y}{2} \in \mathbb{Z}$ only if $y$ is even. So the above is not true for an arbiturary $y \in \mathbb{Z}$, contradictory to our assumption. Thus, the function is not surjective.

Collecting the above, the function is not bijective.

(b) Injectivity: Suppose $\exists x_1, x_2 \in \mathbb{R}, s.t. g(x_1) = g(x_2)$ , want to show: $x_1 = x_2$.

$$g(x_1) = g(x_2) \implies 2x_1 = 2x_2 \implies x_1 = x_2. \tag{3}$$

The function is thus injective.

Surejectivity: Suppose $y \in \mathbb{R}$, we want to find $x \in \mathbb{R}$, $s.t.$ $g(x) = y$.

$$2x = y \implies x = \frac{y}{2} \in \mathbb{R}. \tag{4}$$

So the function is surjective.

Collecting the above, the function $g(x)$ is bijective.

□

**Problem 2.** Let $f : A \to B$ be a function and $A_1, A_2 \subseteq A$ and $B_1, B_2 \subseteq B$ be subsets. Prove the following statements:

(a) $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$.
(b) $f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2)$.
(c) $f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$.
(d) $f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$.

**Solution:** (a) *Proof.* $\subseteq$: Let $y \in f(A_1 \cup A_2)$.By definition of image, $\exists x \in A_1 \cup A_2$ s.t. $f(x) = y$.
Hence, $x \in A_1$ or $x \in A_2$. Thus, $y \in f(A_1)$ or $y \in f(A_2)$, implying $y \in f(A_1) \cup f(A_2)$.
Therefore, $f(A_1 \cup A_2) \subseteq f(A_1) \cup f(A_2)$

$\supseteq$: Let $y \in f(A_1) \cup f(A_2)$. Then $y \in f(A_1)$ or $y \in f(A_2)$.
Thus, $\exists x \in A_1$ or $x \in A_2$ s.t. $f(x) = y$.
Therefore, $x \in A_1 \cup A_2$ and $y = f(x) \in f(A_1 \cup A_2)$.
Thus, $f(A_1) \cup f(A_2) \subseteq f(A_1 \cup A_2)$.

Hence, $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$. □

(b) *Proof.* Let $y \in f(A_1 \cap A_2)$. Then $\exists x \in A_1 \cap A_2$ s.t. $f(x) = y$.
Since $x \in A_1$ and $x \in A_2$, $y \in f(A_1)$ and $y \in f(A_2)$. Thus, $y \in f(A_1) \cap f(A_2)$.
Therefore, $f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2)$. □

(c) *Proof.* $\subseteq$: Let $x \in f^{-1}(B_1 \cup B_2)$. Then $f(x) \in B_1 \cup B_2$, so $f(x) \in B_1$ or $f(x) \in B_2$.
Hence, $x \in f^{-1}(B_1)$ or $x \in f^{-1}(B_2)$, implying $x \in f^{-1}(B_1) \cup f^{-1}(B_2)$.
Thus $f^{-1}(B_1 \cup B_2) \subseteq f^{-1}(B_1) \cup f^{-1}(B_2)$

$\supseteq$: Let $x \in f^{-1}(B_1) \cup f^{-1}(B_2)$.
Then $x \in f^{-1}(B_1)$ or $x \in f^{-1}(B_2)$, meaning $f(x) \in B_1$ or $f(x) \in B_2$.
Thus, $f(x) \in B_1 \cup B_2$ and $x \in f^{-1}(B_1 \cup B_2)$.
So $f^{-1}(B_1) \cup f^{-1}(B_2) \subseteq f^{-1}(B_1 \cup B_2)$
Therefore, $f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$. □

(d) *Proof.* $\subseteq$: Let $x \in f^{-1}(B_1 \cap B_2)$. Then $f(x) \in B_1 \cap B_2$, so $f(x) \in B_1$ and $f(x) \in B_2$.
Hence, $x \in f^{-1}(B_1)$ and $x \in f^{-1}(B_2)$, implying $x \in f^{-1}(B_1) \cap f^{-1}(B_2)$.
So, $f^{-1}(B_1 \cap B_2) \subseteq f^{-1}(B_1) \cap f^{-1}(B_2)$

$\supseteq$: Let $x \in f^{-1}(B_1) \cap f^{-1}(B_2)$. Then $f(x) \in B_1$ and $f(x) \in B_2$, so $f(x) \in B_1 \cap B_2$.
Thus, $x \in f^{-1}(B_1 \cap B_2)$.
Therefore, $f^{-1}(B_1) \cap f^{-1}(B_2) \subseteq f^{-1}(B_1 \cap B_2)$.

Therefore, $f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$. □

□

**Problem 3.** Let $f : A \to B$ be a function. Prove that $f$ is injective if and only if $f(A_1 \cap A_2) = f(A_1) \cap f(A_2)$ for all subsets $A_1, A_2 \subseteq A$.

**Solution:** We will prove the equivalence by demonstrating both implications.

**1.** $f$ **is injective** $\Rightarrow \forall A_1, A_2 \subseteq A,\ f(A_1 \cap A_2) = f(A_1) \cap f(A_2)$

*Proof.* Assume $f$ is injective.

$\subseteq$ Let $y \in f(A_1 \cap A_2)$. Then $\exists x \in A_1 \cap A_2$ such that $f(x) = y$. Since $x \in A_1$ and $x \in A_2$, it follows that $y \in f(A_1)$ and $y \in f(A_2)$. Therefore, $y \in f(A_1) \cap f(A_2)$.

$\supseteq$ Let $y \in f(A_1) \cap f(A_2)$. Then $\exists x_1 \in A_1$ and $\exists x_2 \in A_2$ such that $f(x_1) = y$ and $f(x_2) = y$. Since $f$ is injective, $x_1 = x_2$. Let $x = x_1 = x_2$. Then $x \in A_1 \cap A_2$, and hence $y = f(x) \in f(A_1 \cap A_2)$.

Thus, $f(A_1 \cap A_2) = f(A_1) \cap f(A_2)$ when $f$ is injective. □

**2.** $\forall A_1, A_2 \subseteq A,\ f(A_1 \cap A_2) = f(A_1) \cap f(A_2) \Rightarrow f$ **is injective**

*Proof.* Assume $\forall A_1, A_2 \subseteq A,\ f(A_1 \cap A_2) = f(A_1) \cap f(A_2)$. We aim to show that $f$ is injective.

Suppose, for contradiction, that $f$ is not injective. Then $\exists x_1, x_2 \in A$ with $x_1 \neq x_2$ and $f(x_1) = f(x_2) = y$.

Consider the subsets $A_1 = \{x_1\}$ and $A_2 = \{x_2\}$.

Since $x_1 \neq x_2$,

$$A_1 \cap A_2 = \emptyset.$$

Thus,

$$f(A_1 \cap A_2) = f(\emptyset) = \emptyset.$$

$$f(A_1) = \{f(x_1)\} = \{y\}, \quad f(A_2) = \{f(x_2)\} = \{y\},$$

so

$$f(A_1) \cap f(A_2) = \{y\} \cap \{y\} = \{y\}.$$

We have

$$f(A_1 \cap A_2) = \emptyset \neq \{y\} = f(A_1) \cap f(A_2),$$

which contradicts the assumption. Therefore, $f$ must be injective. □

□

**Problem 4.** Let $f : A \to B$ be a function. Prove that the following two statements are equivalent:

(a) The function $f$ is surjective.
(b) For every set $C$ and for any functions $g : B \to C$ and $h : B \to C$ such that $g \circ f = h \circ f$, we have $g = h$.

**Solution: (1) implies (2):**

Assume $f$ is surjective. Let $C$ be an arbitrary set, and let $g, h : B \to C$ satisfy $g \circ f = h \circ f$.

For any $b \in B$, since $f$ is surjective, there exists $a \in A$ such that $f(a) = b$. Therefore,

$$g(b) = g(f(a)) = (g \circ f)(a) = (h \circ f)(a) = h(f(a)) = h(b).$$

Hence, $g = h$.

**(2) implies (1):**

Assume statement 2 holds. Suppose, for contradiction, that $f$ is not surjective. Then there exists $b_0 \in B$ such that $b_0 \notin \mathrm{Image}(f)$. We will use this element to construct specific functions $g$ and $h$ that satisfy the premise of statement 2 but are not equal, leading to a contradiction:

Let $C = \{0, 1\}$ and define the functions $g, h : B \to C$ as follows:

$$g(b) = \begin{cases} 0 & \text{if } b = b_0, \\ 1 & \text{otherwise,} \end{cases} \qquad h(b) = 1 \text{ for all } b \in B.$$

Since $b_0 \notin \mathrm{Image}(f)$, for all $a \in A$, $g(f(a)) = 1 = h(f(a))$. Thus, $g \circ f = h \circ f$. However, $g \neq h$ because $g(b_0) = 0$ while $h(b_0) = 1$, which contradicts the uniqueness condition.

Therefore, $f$ must be surjective.

$\square$

**Problem 5.** Let $A$ be a nonempty set and $f : A \to A$ a function. We call $f$ an *involution* if $(f \circ f)(a) = a$ for all $a \in A$. Prove that if $f : A \to A$ is an involution, then $f$ is bijective. What is the inverse function $f^{-1}$ in terms of $f$?

**Solution: 1. Injectivity**

Assume that for some $a_1, a_2 \in A$,

$$f(a_1) = f(a_2).$$

Applying $f$ to both sides of the equation:

$$f(f(a_1)) = f(f(a_2)).$$

Given that $f$ is an involution:

$$(f \circ f)(a_1) = (f \circ f)(a_2)a_1 = a_2.$$

Thus, $f$ is injective.

**2. Surjectivity**

Take any element $b \in A$. Since $f$ is an involution:

$$f(f(b)) = b.$$

Let $a = f(b)$. Then:

$$f(a) = f(f(b)) = b.$$

Therefore, for every $b \in A$, there exists an $a \in A$ (specifically, $a = f(b)$) such that $f(a) = b$. Therefore $f$ is surjective.

Since $f$ is both injective and surjective, it is bijective.

**Inverse Function**

By definition, the inverse function $f^{-1}$ satisfies:

$$f^{-1}(f(a)) = a \quad \text{and} \quad f(f^{-1}(a)) = a \quad \text{for all} \quad a \in A.$$

Given that $f$ is an involution:

$$f(f(a)) = a.$$

Comparing the two conditions, we observe that $f$ itself satisfies the properties required of an inverse function. Therefore:

$$f^{-1} = f.$$

$\square$

**Problem 6.** Prove or disprove the following statements:

(a) The set $\{x \in \mathbb{R} : x \geq 2\}$ is an interval.
(b) The set $\{x \in \mathbb{R} : x \neq 2\}$ is an interval.

(Hint: In order to disprove a statement, you must prove that the negation of the statement is true.)

**1. The Set $\{x \in \mathbb{R} \mid x \geq 2\}$ is an Interval** **Proof**: To confirm that $S = [2, \infty)$ is indeed an interval, we verify the interval definition.

**Solution:** (a) **Take any two elements** $a, b \in S$ **with** $a < b$:

$$a, b \geq 2 \quad \text{and} \quad a < b.$$

(b) **Consider any** $c \in \mathbb{R}$ **such that** $a < c < b$:

$$a < c < b \quad \text{and} \quad a \geq 2 \quad \Rightarrow \quad c > a \geq 2 \quad \Rightarrow \quad c \geq 2.$$

(c) **Thus**, $c \in S$ since $c \geq 2$.

Since every number between any two elements of $S$ is also contained within $S$, $S$ satisfies the definition of an interval.

**Conclusion**: The set $\{x \in \mathbb{R} \mid x \geq 2\}$ is an interval, specifically the closed and unbounded interval $[2, \infty)$.

## 2. The Set $\{x \in \mathbb{R} \mid x \neq 2\}$ is an Interval

**Statement**: The set $T = \{x \in \mathbb{R} \mid x \neq 2\}$ is an interval.

**Proof**:

We aim to determine whether the set $T = \mathbb{R} \setminus \{2\}$ satisfies the definition of an interval.

**Assumption for Contradiction**: Suppose $T$ is an interval.

**Analysis**:

1. **Structure of $T$**:

$$T = (-\infty, 2) \cup (2, \infty)$$

This is the real line with the single point $x = 2$ removed.

2. **Interval Properties**: - For $T$ to be an interval, it must be connected; that is, there should be no "gaps" in $T$. - However, $T$ explicitly excludes the point $x = 2$, creating a discontinuity.

3. **Violation of Interval Definition**: - Consider two points $a = 1$ and $b = 3$ in $T$, with $a < 2 < b$. - According to the interval definition, every $c$ such that $a < c < b$ must be in $T$. - Take $c = 2$, which satisfies $a < c < b$, but $c = 2 \notin T$. - This contradicts the requirement that all intermediate points must be included in the interval.

**Conclusion**: The set $T = \{x \in \mathbb{R} \mid x \neq 2\}$ is not an interval because it fails to include all real numbers between certain pairs of its elements, specifically excluding the point $x = 2$.

**Summary**

(a) **True**: The set $\{x \in \mathbb{R} \mid x \geq 2\}$ is an interval, precisely the closed and unbounded interval $[2, \infty)$.
(b) **False**: The set $\{x \in \mathbb{R} \mid x \neq 2\}$ is not an interval, as it excludes the point $x = 2$, resulting in a disconnected set.

**Final Answer**:

(a) **True**. The set $\{x \in \mathbb{R} \mid x \geq 2\}$ is the interval $[2, \infty)$.
(b) **False**. The set $\{x \in \mathbb{R} \mid x \neq 2\}$ is not an interval.

□