Math 421, Section 1
Homework 2
Harry Luo

Problem 1. Prove that for any $x, y \in \mathbb{N}$, if x is odd and y is odd then $x + y$ is even.

Solution: Suppose $x, y \in \mathbb{N}$ are odd, then $\exists n, m \in \mathbb{N} \cup \{0\} \;\; s.t. \; x = 2n + 1, y = 2m + 1$.

$$x + y = 2n + 1 + 2m + 1 = 2(n + m + 1). \tag{1}$$

Since $(n + m + 1) \in \mathbb{N}, x + y$ is even. $\square$

Problem 2. Prove that for any $x \in \mathbb{N}$, if x is odd then x^3 is odd.

Solution: Suppose x is odd, i.e. $\exists n \in \mathbb{N} \cup \{0\}\; s.t.\; x = 2n + 1$

$$x^3 = (2n+1)^3 = 8n^3 + 12n^2 + 6n + 1 = 2(4n^3 + 6n^2 + 3n) + 1. \tag{2}$$

Trivially, since $(4n^3 + 6n^2 + 3n) \in \mathbb{N} \cup \{0\}, x^3$ is odd.

$\square$

Problem 3. Using induction, prove that for all $n \in \mathbb{N}$ we have

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2.$$

Solution: [Base case]: For $n = 1$, we have $1 = 1$. Upon careful reflection, this holds true.
[Inductive step]: Suppose the statement is true for $\exists n \in \mathbb{N}$, i.e.

$$1 + 3 + \cdots + 2n - 1 = n^2 \tag{3}$$

Then for $n = n + 1$ we have:

$$\begin{align} 1 + 3 + \cdots + 2n - 1 + 2(n+1) - 1 &= n^2 + 2n + 1 \tag{4} \\ &= (n+1)^2 \tag{5} \end{align}$$

So the formula is true for $n + 1$. Thus, by induction, the statement is true for all $n \in \mathbb{N}$.

$\square$

Problem 4. Compute the following sum:

$$\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\cdots+\frac{1}{(2n-1)(2n+1)}.$$

Prove that your answer is true for all $n \in \mathbb{N}$ using induction.

Solution: By noticing $\frac{1}{(2n-1)(2n+1)} = \frac{1}{2}(\frac{1}{2n-1} - \frac{1}{2n+1})$, a rough calculation suggests that the sum should be $\frac{1}{2} - \frac{1}{4n+2}$. It is proved by induction as follows:

[base case]: For $n = 1$, we have

$$\frac{1}{1\cdot 3} = \frac{1}{2} - \frac{1}{6} = \frac{1}{3},$$

which is true.

[Inductive case]: Suppose the statement is true for $\exists n \in \mathbb{N}$, i.e.

$$\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\cdots+\frac{1}{(2n-1)(2n+1)} = \frac{1}{2} - \frac{1}{4n+2} \tag{6}$$

Then for $n = n+1$,

$$\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\cdots+\frac{1}{(2n-1)(2n+1)}+\frac{1}{(2n+1)(2n+3)} = \frac{1}{2} - \frac{1}{4n+2} + \frac{1}{(2n+1)(2n+3)} \tag{7}$$

$$= \frac{1}{2} - \frac{1}{4n+2} + \frac{1}{2}(\frac{1}{2n+1} - \frac{1}{2n+3}) \tag{8}$$

$$= \frac{1}{2} - \frac{1}{4n+6} \tag{9}$$

$$= \frac{1}{2} - \frac{1}{4(n+1)+2} \tag{10}$$

So the formula is true for $n + 1$. Thus, by induction, the statement is true for all $n \in \mathbb{N}$.

$\square$

Problem 5. Prove the following statements for all $a, b \in \mathbb{R}$:

(a) $-a + (-b) = -(a + b)$.
(b) If $a, b \neq 0$ then $a^{-1} \cdot b^{-1} = (ab)^{-1}$.

Carefully justify every step using properties of $\mathbb{R}$ stated in lecture.

Solution: [a]: Consider the original equation,

$$-a + (-b) = -(a + b) \tag{11}$$

Adding (a+b) to both sides, we can find that it is equivalent to

$$-a + (-b) + (a + b) = -(a + b) + (a + b) \tag{12}$$

Applying inverse addition to the right side, and apply associativity to the left, this is equivalent to

$$-a + a + (-b) + b = 0 \tag{13}$$

Therefore, to prove the original statement, it is suffice to prove its equivalence, i.e. Equation 13. By the inverse addition property, we have

$$-a + a = 0, \ -b + b = 0 \tag{14}$$

$$\Rightarrow -a + a + (-b) + b = 0 \tag{15}$$

The statement is thus proved.

[b]: Suppose $a, b \neq 0$, we have

$$a^{-1} \cdot b^{-1} \cdot ab \overset{\text{commutivity}}{=} a^{-1} \cdot a \cdot b^{-1} \cdot b \overset{\text{inverse}}{=} 1 \tag{16}$$

$$\text{also, } (ab)^{-1} \cdot ab \overset{inverse}{=} 1 \tag{17}$$

By transivity, we have

$$a^{-1} \cdot b^{-1} \cdot ab = (ab)^{-1} \cdot ab \tag{18}$$

$$\overset{\text{prop.1}}{\Rightarrow} a^{-1} \cdot b^{-1} = (ab)^{-1} \tag{19}$$

As desired. $\square$

Problem 6. Prove the following statements for all $a, b, c, d \in \mathbb{R}$:

(a) If $a < b$ and $c < d$ then $a + c < b + d$.
(b) If $0 < a < b$ and $0 < c < d$ then $ac < bd$.

Solution: [a]: Suppose $a < b, c < d$, then by O1,

$$a + c < b + c \tag{20}$$
$$b + c < d + b. \tag{21}$$

By Transitivity,

$$a + c < d + b \tag{22}$$

By commutivity,

$$a + c < b + d \tag{23}$$

Thus proves the inequality.

[b]: Suppose $0 < a < b$, $0 < c < d$. Then by O2,

$$a \cdot c < b \cdot c \tag{24}$$
$$b \cdot c < b \cdot d \tag{25}$$

By transitivity,

$$ac < bd \tag{26}$$

Thus proves the inequality. $\square$