**Physics 322 Honors Problem Set**
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**Problem 1.** Prove Earnshaw's theorem: a charged particle cannot be held in stable equilibrium(in otherwise empty space) by electrostatic forces alone

**Solution:** Consider a charge $q$ in an electrostatic potential $V(\mathbf{r})$. The potential energy of the charge is

$$U(\mathbf{r}) = q\,V(\mathbf{r}).$$

For a stable equilibrium at a point $\mathbf{r}_0$, we require that: 1. $\nabla U(\mathbf{r}_0) = q\,\nabla V(\mathbf{r}_0) = \mathbf{0}$, ensuring equilibrium. 2. The equilibrium must be stable, so $U(\mathbf{r})$ should have a strict local minimum at $\mathbf{r}_0$. Equivalently, $V(\mathbf{r})$ should have a local minimum at $\mathbf{r}_0$.

However, in free space (without charges), the electrostatic potential $V$ satisfies Laplace's equation:

$$\nabla^2 V = 0.$$

We know from PDF that the Harmonnic Functions attain maximum/minimum value only on boundaries; any extremum in the interior must be a saddle point.

Since $V$ cannot have a true local minimum, $U(\mathbf{r})$ cannot either. Thus, no stable equilibrium point for a charged particle can be formed purely by electrostatic potentials in free space. $\square$

**Problem 2.**

**Solution: (a) Choosing an Origin to Eliminate the Dipole Moment**

Consider a charge distribution $\rho(\mathbf{r}')$ with total charge

$$Q = \int \rho(\mathbf{r}')\, d\tau'.$$

Suppose the total charge is nonzero: $Q \neq 0$. The dipole moment about the original origin is defined as

$$\mathbf{p} = \int \rho(\mathbf{r}')\mathbf{r}'\, d\tau'.$$

If we shift our coordinate system by $\mathbf{R}$, defining a new coordinate $\mathbf{r}'' = \mathbf{r}' - \mathbf{R}$, the dipole moment in the new coordinates becomes

$$\mathbf{p}'' = \int \rho(\mathbf{r}')(\mathbf{r}' - \mathbf{R})\, d\tau' = \mathbf{p} - Q\mathbf{R}.$$

Since $Q \neq 0$, we can always choose

$$\mathbf{R} = \frac{\mathbf{p}}{Q},$$

so that

$$\mathbf{p}'' = \mathbf{p} - Q\frac{\mathbf{p}}{Q} = \mathbf{0}.$$

This defines the "center of charge," a choice of origin about which the dipole moment vanishes.

**(b) Multipole Expansion up to the Octupole Term**

Recall that hte POtential due to cahrge distribution is given by

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{(n+1)}}\int_V (r')^n P_n(\cos\alpha)\rho(\mathbf{r}')\, d\tau \tag{1}$$

Let us align the $z$-axis along $\mathbf{r}$ so that $\cos\alpha = \frac{\mathbf{r}\cdot\mathbf{r}'}{rr'}$. Then we can read off the terms:

1. **Monopole term ($n = 0$):**

$$V^{(0)}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r}\int \rho(\mathbf{r}')\, d\tau.$$

2. **Dipole term ($n = 1$):**

$$V^{(1)}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\int r' P_1(\cos\alpha)\rho(\mathbf{r}')\, d\tau.$$

Since $P_1(\cos\alpha) = \cos\alpha$, we have

$$V^{(1)}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\int r'\frac{\mathbf{r}\cdot\mathbf{r}'}{rr'}\rho(\mathbf{r}')\, d\tau = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{r}\cdot\mathbf{p}}{r^3},$$

where $\mathbf{p} = \int \rho(\mathbf{r}')\mathbf{r}'\, d\tau$ is the dipole moment.

3. **Quadrupole term ($n = 2$):**

$$V^{(2)}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\int (r')^2 P_2(\cos\alpha)\rho(\mathbf{r}')\,d\tau.$$

Since $P_2(\cos\alpha) = \frac{3\cos^2\alpha - 1}{2}$, we get

$$V^{(2)}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\int \rho(\mathbf{r}')(r')^2\frac{3(\cos\alpha)^2 - 1}{2}\,d\tau.$$

4. **Octupole term ($n = 3$):** Similarly, $P_3(\cos\alpha) = \frac{5\cos^3\alpha - 3\cos\alpha}{2}$, so:

$$V^{(3)}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^4}\int (r')^3 P_3(\cos\alpha)\rho(\mathbf{r}')\,d\tau.$$

Substitute $P_3(\cos\alpha)$ to get:

$$V^{(3)}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^4}\int \rho(\mathbf{r}')(r')^3\frac{5(\cos\alpha)^3 - 3\cos\alpha}{2}\,d\tau.$$

Thus, up to the octupole term:

$$V(\mathbf{r}) \approx \frac{1}{4\pi\epsilon_0}\left[\frac{Q}{r} + \frac{\mathbf{r}\cdot\mathbf{p}}{r^3} + \frac{1}{r^3}\int \rho(\mathbf{r}')\frac{(r')^2}{2}(3\cos^2\alpha - 1)d\tau + \frac{1}{r^4}\int \rho(\mathbf{r}')\frac{(r')^3}{2}(5\cos^3\alpha - 3\cos\alpha)d\tau\right].$$

**(c) Deriving the Quadrupole Moment Expression**

We now focus on the quadrupole term more explicitly. Starting from the $n = 2$ term:

$$V^{(2)}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\int \rho(\mathbf{r}')(r')^2\frac{3\cos^2\alpha - 1}{2}\,d\tau.$$

Write $\cos\alpha = \frac{\mathbf{r}\cdot\mathbf{r}'}{rr'}$, so:

$$\cos^2\alpha = \frac{(\mathbf{r}\cdot\mathbf{r}')^2}{r^2(r')^2}.$$

Substitute this into the integrand:

$$3\cos^2\alpha - 1 = 3\frac{(\mathbf{r}\cdot\mathbf{r}')^2}{r^2(r')^2} - 1.$$

Multiplying by $(r')^2/2$:

$$\frac{(r')^2}{2}(3\cos^2\alpha - 1) = \frac{3(\mathbf{r}\cdot\mathbf{r}')^2}{2r^2} - \frac{(r')^2}{2}.$$

Thus:

$$V^{(2)}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\int \rho(\mathbf{r}')\left(\frac{3(\mathbf{r}\cdot\mathbf{r}')^2}{2r^2} - \frac{(r')^2}{2}\right)d\tau.$$

Factor out $\frac{1}{2r^3}$:

$$V^{(2)}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{2r^5}\int \rho(\mathbf{r}')[3(\mathbf{r}\cdot\mathbf{r}')^2 - (r')^2 r^2]\, d\tau.$$

Now, note that $(\mathbf{r}\cdot\mathbf{r}')^2 = \sum_{i,j} r_i r_j r'_i r'_j$. We can rewrite the integrand using a symmetric tensor:

$$3(\mathbf{r}\cdot\mathbf{r}')^2 - (r')^2 r^2 = \sum_{i,j}(3r'_i r'_j - (r')^2\delta_{ij})\frac{r_i r_j}{r^2}.$$

This motivates the *definition* of the quadrupole moment tensor $Q_{ij}$:

$$Q_{ij} := \int \rho(\mathbf{r}')\big(3r'_i r'_j - (r')^2\delta_{ij}\big)\, d\tau.$$

Substitute back into $V^{(2)}(\mathbf{r})$:

$$V^{(2)}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{2r^5}\sum_{i,j} Q_{ij} r_i r_j.$$

If the charge distribution is axially symmetric about the $z$-axis, then by symmetry:
- Off-diagonal quadrupole components vanish ($Q_{xy} = Q_{xz} = Q_{yz} = 0$).
- The $x$ and $y$ directions are equivalent, so $Q_{xx} = Q_{yy}$.

Because of the form of $Q_{ij}$, it also follows that $Q_{xx} + Q_{yy} + Q_{zz} = 0$. With $Q_{xx} = Q_{yy}$, we get:

$$2Q_{xx} + Q_{zz} = 0 \implies Q_{xx} = Q_{yy} = -\frac{1}{2}Q_{zz}.$$

Hence the quadrupole tensor simplifies to:

$$Q_{ij} = \begin{pmatrix} -\frac{1}{2}Q_{zz} & 0 & 0 \\ 0 & -\frac{1}{2}Q_{zz} & 0 \\ 0 & 0 & Q_{zz} \end{pmatrix}.$$

$\square$

**Problem 3.**

**Solution: (a)** Consider a neutral but polarizable atom in an electric field $\mathbf{E}$. The induced dipole moment is given by $\mathbf{p} = \alpha\mathbf{E}$, where $\alpha$ is the polarizability. The force on a dipole is:

$$\mathbf{F} = (\mathbf{p} \cdot \nabla)\mathbf{E}.$$

Substituting $\mathbf{p} = \alpha\mathbf{E}$, we have:

$$\mathbf{F} = \alpha(\mathbf{E} \cdot \nabla)\mathbf{E}.$$

To connect this with $\nabla(E^2)$, note that:

$$E^2 = \mathbf{E} \cdot \mathbf{E}.$$

Taking the gradient:

$$\nabla E^2 = \nabla(\mathbf{E} \cdot \mathbf{E}) = 2(\mathbf{E} \cdot \nabla)\mathbf{E} + 2\mathbf{E} \times (\nabla \times \mathbf{E}).$$

In electrostatics, $\nabla \times \mathbf{E} = 0$. Therefore:

$$\nabla E^2 = 2(\mathbf{E} \cdot \nabla)\mathbf{E}.$$

Substitute this back into the expression for $\mathbf{F}$:

$$\mathbf{F} = \alpha(\mathbf{E} \cdot \nabla)\mathbf{E} = \frac{\alpha}{2}\nabla(E^2).$$

Hence, the force on the atom is:

$$\mathbf{F} = \frac{1}{2}\alpha\nabla(E^2).$$

This result shows that a neutral polarizable atom is pulled toward regions of stronger electric field.

**(b)** Now we consider whether such a force can trap the atom at a stable equilibrium. For a stable equilibrium, we would need a local maximum of $E^2$, since the force $\mathbf{F} = \frac{1}{2}\alpha\nabla(E^2)$ points towards increasing $E^2$.

Suppose, for the sake of contradiction, that $E^2$ does have a local maximum at some point $P$ in a charge-free region. Then we can draw a sphere of radius $R$ around $P$ such that for every point $P'$ on the spherical surface:

$$E^2(P') < E^2(P).$$

This implies:

$$|\mathbf{E}(P')| < |\mathbf{E}(P)|.$$

However, from Problem 3.4 in Griffiths, we know that if no charge is enclosed within the sphere, the electric field at the center $P$ is equal to the average of the field over the spherical surface:

$$\frac{1}{4\pi R^2}\int E\, da = E(P).$$

If we align the $z$-axis along $\mathbf{E}(P)$, we get:

$$\frac{1}{4\pi R^2}\int E_z\, da = E(P).$$

But since $|\mathbf{E}(P')| < |\mathbf{E}(P)|$ everywhere on the surface, we have:

$$\int E_z\, da \leq \int |\mathbf{E}(P')|\, da < \int |\mathbf{E}(P)|\, da = 4\pi R^2 E(P).$$

This implies:

$$E(P) < E(P),$$

a contradiction.

The contradiction arises from the assumption that $E^2$ could have a local maximum in free space. Therefore, $E^2$ cannot have a local maximum in a region without charge. While it can have a local minimum (for instance, at a point where the field is zero, such as the midpoint between two equal charges), a local maximum is forbidden.

$\square$

**Problem 4.**

**Solution: (a)**

From Snell's law:

$$\sin\theta_T = \frac{n_1}{n_2}\sin\theta_I.$$

For $\theta_I > \theta_c$, we have $\sin\theta_T > 1$. Define:

$$\cos\theta_T = \sqrt{1-\sin^2\theta_T} = i\sqrt{\sin^2\theta_T - 1}.$$

The transmitted wavevector in medium 2 is:

$$\mathbf{k}_T = \frac{\omega n_2}{c}(\sin\theta_T\,\hat{x} + \cos\theta_T\,\hat{z}).$$

Since $\sin\theta_T = (n_1/n_2)\sin\theta_I$, we rewrite:

$$\mathbf{k}_T = \frac{\omega n_2}{c}\left(\frac{n_1}{n_2}\sin\theta_I\,\hat{x} + i\sqrt{\left(\frac{n_1}{n_2}\sin\theta_I\right)^2 - 1\,\hat{z}}\right).$$

Set:

$$k = \frac{\omega n_1}{c}\sin\theta_I, \quad \kappa = \frac{\omega}{c}\sqrt{(n_1\sin\theta_I)^2 - n_2^2}.$$

The transmitted electric field can then be expressed as:

$$\tilde{E}_T(r,t) = \tilde{E}_0 T e^{i(kx-\omega t)}e^{-\kappa z}.$$

## (b)

For parallel polarization, the Fresnel reflection coefficient $R_\parallel$ can be expressed in terms of $\alpha = (\cos\theta_T)/(\cos\theta_I)$. Under total internal reflection, $\alpha$ is purely imaginary. If we write $\alpha = ia$ with $a$ real, the magnitude of the reflection coefficient simplifies to:

$$|R_\parallel|^2 = 1.$$

Thus, there is total reflection for parallel polarization.

## (c)

A similar argument applies for perpendicular polarization. The corresponding Fresnel coefficient also simplifies to:

$$|R_\perp|^2 = 1.$$

Hence, total internal reflection is complete for both polarizations.

## (d)

Choosing phases so that the transmitted field is real, the evanescent electric field for perpendicular polarization (with the field along $\hat{y}$) is:

$$\mathbf{E}(r,t) = E_0 e^{-\kappa z}\cos(kx-\omega t)\hat{y}.$$

From Maxwell's equations, the corresponding magnetic field is:

$$\mathbf{B}(r,t) = \frac{E_0}{\omega}e^{-\kappa z}\big[\kappa\sin(kx-\omega t)\hat{x} + k\cos(kx-\omega t)\hat{z}\big].$$

## (e)

Substitute **E** and **B** into Maxwell's equations. One finds:

$$\nabla\cdot\mathbf{E} = 0, \quad \nabla\cdot\mathbf{B} = 0, \quad \nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}, \quad \nabla\times\mathbf{B} = \mu_2\epsilon_2\frac{\partial\mathbf{E}}{\partial t}.$$

All are satisfied due to the chosen form of $k$, $\kappa$, and the material relations.

## (f)

The time-averaged Poynting vector determines the energy flow associated with the evanescent wave. Given the electric and magnetic fields derived in part (d):

$$\mathbf{E}(r,t) = E_0 e^{-\kappa z}\cos(kx-\omega t)\,\hat{\mathbf{y}},$$

$$\mathbf{B}(r,t) = \frac{E_0}{\omega}e^{-\kappa z}\left[\kappa\sin(kx-\omega t)\,\hat{\mathbf{x}} + k\cos(kx-\omega t)\,\hat{\mathbf{z}}\right].$$

The Poynting vector is defined as:

$$\mathbf{S} = \frac{1}{\mu_2}(\mathbf{E}\times\mathbf{B}).$$

Substitute the components of **E** and **B**:

$$\mathbf{E} = (0, E_0 e^{-\kappa z}\cos(kx-\omega t), 0), \quad \mathbf{B} = \left(\frac{E_0\kappa}{\omega}e^{-\kappa z}\sin(kx-\omega t), 0, \frac{E_0 k}{\omega}e^{-\kappa z}\cos(kx-\omega t)\right).$$

The cross product $\mathbf{E}\times\mathbf{B}$ is:

$$\mathbf{E}\times\mathbf{B} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 0 & E_0 e^{-\kappa z}\cos(kx-\omega t) & 0 \\ \frac{E_0\kappa}{\omega}e^{-\kappa z}\sin(kx-\omega t) & 0 & \frac{E_0 k}{\omega}e^{-\kappa z}\cos(kx-\omega t) \end{vmatrix}.$$

Expanding the determinant:

$$\mathbf{E}\times\mathbf{B} = E_0^2 e^{-2\kappa z}\left[\frac{k}{\omega}\cos^2(kx-\omega t)\,\hat{\mathbf{x}} - \frac{\kappa}{\omega}\sin(kx-\omega t)\cos(kx-\omega t)\,\hat{\mathbf{z}}\right].$$

The Poynting vector becomes:

$$\mathbf{S} = \frac{1}{\mu_2}(\mathbf{E} \times \mathbf{B}) = \frac{E_0^2}{\mu_2 \omega} e^{-2\kappa z} \left[ k \cos^2(kx - \omega t)\, \hat{\mathbf{x}} - \kappa \sin(kx - \omega t) \cos(kx - \omega t)\, \hat{\mathbf{z}} \right].$$

The time averages of the relevant trigonometric functions are:

$$\langle \cos^2(kx - \omega t) \rangle = \frac{1}{2}, \quad \langle \sin(kx - \omega t) \cos(kx - \omega t) \rangle = 0.$$

Substituting these into the expression for $\mathbf{S}$:

$$\langle \mathbf{S} \rangle = \frac{E_0^2}{\mu_2 \omega} e^{-2\kappa z} \left[ \frac{k}{2}\, \hat{\mathbf{x}} + 0 \cdot \hat{\mathbf{z}} \right].$$

Simplifying:

$$\langle \mathbf{S} \rangle = \frac{E_0^2 k}{2 \mu_2 \omega} e^{-2\kappa z} \hat{\mathbf{x}}.$$

$\square$

**Problem 5.**

**Solution:** By definition of TM modes, we have a nonzero longitudinal electric field $E_z(x, y, z, t)$, and from Maxwell's equations in the phasor form (assuming time-dependence $e^{-i\omega t}$), the fields satisfy the wave equation.
**Wave Equation for TM Modes:** Since the fields vary as $e^{i(kz-\omega t)}$ along $z$, let

$$E_z(x, y, z) = \mathcal{E}(x, y)e^{i(kz)}.$$

Inside the waveguide (with no charge), Maxwell's equations lead to the scalar Helmholtz equation for $E_z$:

$$\nabla^2 E_z + \frac{\omega^2}{c^2}E_z = 0.$$

Because $E_z$ depends on $z$ as $e^{ikz}$, we write:

$$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + \left(\frac{\omega^2}{c^2} - k^2\right) E_z = 0.$$

Define $\mathcal{E}(x, y)$ via $E_z(x, y, z) = \mathcal{E}(x, y)e^{i(kz)}$. Then:

$$\frac{\partial^2 \mathcal{E}}{\partial x^2} + \frac{\partial^2 \mathcal{E}}{\partial y^2} + \left(\frac{\omega^2}{c^2} - k^2\right) \mathcal{E} = 0.$$

Set

$$k_x^2 + k_y^2 = \frac{\omega^2}{c^2} - k^2.$$

We separate variables:

$$\mathcal{E}(x, y) = X(x)Y(y).$$

This gives:

$$\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = -(k_x^2 + k_y^2).$$

From this, we obtain two separate ODEs:

$$X''(x) + k_x^2 X(x) = 0, \quad Y''(y) + k_y^2 Y(y) = 0.$$

**Boundary Conditions:** The waveguide walls are perfect conductors. On these walls, the tangential electric field must vanish. Since $E_z$ is tangential at $x = 0, a$ and $y = 0, b$, we require:

$$E_z(0, y, z) = 0, \quad E_z(a, y, z) = 0, \quad E_z(x, 0, z) = 0, \quad E_z(x, b, z) = 0.$$

These conditions imply:

$$X(0) = X(a) = 0, \quad Y(0) = Y(b) = 0.$$

The functions that vanish at both ends are sine functions. Thus:

$$X(x) = \sin\left(\frac{m\pi x}{a}\right), \quad m = 1, 2, 3, \ldots$$

$$Y(y) = \sin\left(\frac{n\pi y}{b}\right), \quad n = 1, 2, 3, \ldots$$

(For TM modes, both $m, n \geq 1$ to avoid trivial solutions.)
**Field Form:** Thus the longitudinal electric field is:

$$E_z(x, y, z) = E_0 \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) e^{i(kz - \omega t)}.$$

**Cutoff Frequencies:** From the separation constants, we know:

$$k_x = \frac{m\pi}{a}, \quad k_y = \frac{n\pi}{b}.$$

The dispersion relation is:

$$k^2 + k_x^2 + k_y^2 = \frac{\omega^2}{c^2}.$$

This gives:

$$k = \sqrt{\frac{\omega^2}{c^2} - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}.$$

Define the cutoff frequency $\omega_{mn}$:

$$\omega_{mn} = c\pi\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}.$$

For propagation, we need $\omega > \omega_{mn}$. If $\omega < \omega_{mn}$, $k$ becomes imaginary and the mode does not propagate (it is evanescent).
**Transverse Fields:** Once $E_z$ is known, the transverse electric and magnetic fields $(E_x, E_y, B_x, B_y)$ can be obtained using Maxwell's equations. For TM modes:

$$B_z = 0.$$

Using $\nabla \times \mathbf{E} = -i\omega\mu_0\mathbf{B}$ and $\nabla \times \mathbf{B} = i\omega\epsilon_0\mathbf{E}$, one can show:

$$E_x, E_y \sim \frac{\partial E_z}{\partial x}, \frac{\partial E_z}{\partial y}, \quad B_x, B_y \sim \frac{k}{\omega\mu_0} E_z.$$

All transverse fields are determined by spatial derivatives of the known $E_z$.
**Phase and Group Velocities:** Rewrite $k$ as:

$$k = \frac{1}{c}\sqrt{\omega^2 - \omega_{mn}^2}.$$

The phase velocity is:

$$v_{\text{phase}} = \frac{\omega}{k} = \frac{c}{\sqrt{1 - (\omega_{mn}/\omega)^2}} > c.$$

The group velocity, which represents the energy transport speed, is:

$$v_{\text{group}} = \frac{d\omega}{dk} = c\sqrt{1 - (\omega_{mn}/\omega)^2} < c.$$

□

**Problem 6.**

**Solution:** (a) *Coordinate-free expressions:*

Start with the given dipole potential for electric dipole radiation along the $z$-axis:

$$V(r,\theta,t) = -\frac{p_0\omega}{4\pi\epsilon_0 c}\frac{\cos\theta}{r}\sin[\omega(t-r/c)].$$

Replace $p_0\cos\theta$ by $\mathbf{p}_0\cdot\hat{\mathbf{r}}$:

$$V(r,t) = -\frac{\omega}{4\pi\epsilon_0 c}\frac{\mathbf{p}_0\cdot\hat{\mathbf{r}}}{r}\sin[\omega(t-r/c)].$$

Similarly, the vector potential (Eq. 11.17) in a coordinate-free form:

$$\mathbf{A}(r,t) = \frac{\mu_0\omega}{4\pi}\frac{\mathbf{p}_0}{r}\sin[\omega(t-r/c)].$$

For the fields (Eqs. 11.18 and 11.19), use vector identities:

$$\mathbf{E}(r,t) = \frac{\mu_0\omega^2}{4\pi}\frac{\hat{\mathbf{r}}\times(\mathbf{p}_0\times\hat{\mathbf{r}})}{r}\cos[\omega(t-r/c)],$$

$$\mathbf{B}(r,t) = \frac{\mu_0\omega^2}{4\pi c}\frac{\mathbf{p}_0\times\hat{\mathbf{r}}}{r}\cos[\omega(t-r/c)].$$

Finally, the time-averaged Poynting vector (Eq. 11.21):

$$\langle\mathbf{S}\rangle = \frac{\mu_0\omega^4}{32\pi^2 c}\frac{(\mathbf{p}_0\times\hat{\mathbf{r}})^2}{r^2}\hat{\mathbf{r}}.$$

(b) *Rotating dipole:*

Consider:

$$\mathbf{p}(t) = p_0[\cos(\omega t)\hat{\mathbf{x}} + \sin(\omega t)\hat{\mathbf{y}}].$$

This can be viewed as the superposition of two oscillating dipoles: one along $x$ and one along $y$, with a 90° phase shift. Using linearity and the known fields for each component dipole (as in part (a), but oriented along $x$ and $y$ respectively), we sum the contributions:

$$\mathbf{E}(r,t) = \mathbf{E}_x(r,t) + \mathbf{E}_y(r,t), \quad \mathbf{B}(r,t) = \mathbf{B}_x(r,t) + \mathbf{B}_y(r,t).$$

After inserting the appropriate phase factors, we find that the cross terms vanish on time averaging due to orthogonality and phase difference. The final time-averaged Poynting vector then simplifies to:

$$\langle\mathbf{S}\rangle = \langle\mathbf{S}_x\rangle + \langle\mathbf{S}_y\rangle,$$

since cross terms average to zero.

Each component dipole radiates:

$$P_{\text{single}} = \frac{\mu_0 p_0^2\omega^4}{12\pi c}.$$

The total power for the two perpendicular dipoles (one along $x$ and one along $y$ with a 90° phase shift) is:

$$P_{\text{total}} = 2\times\frac{\mu_0 p_0^2\omega^4}{12\pi c} = \frac{\mu_0 p_0^2\omega^4}{6\pi c}.$$

□