

- Welcome to Math 421!
- Textbook: "Calculus" by Spivak, ch. 1-14
- Syllabus is on Canvas
- 2 in-class midterms and a final exam — take note of the dates!
- Weekly homework due on Wednesdays at the beginning of class on Gradescope
- Homework 1 is posted today
- I highly recommend that you type your homework using LaTeX — see Canvas for some resources

Office hours: In the MLC

M 1-2 pm (Edited: Sep. 4)

Tu 10:30 - 11:30 am

Email: laurensl@wisc.edu

- Please ask (and answer) any question concerning the course (e.g. material, logistics) on Piazza

INTRO TO LOGIC

Def A statement is a sentence that is either true or false, but not both.

Ex ① "6 is an even integer" is a (true) statement

- ② "4 is an odd integer" is a (false) statement
- ③ "Today is Tuesday" is a statement

let A and B be statements. We write:

A	if A is true
not A	if A is false
A and B	if both A and B are true
A or B ↑	if A is true, or B is true, or both A and B are true

- Inclusive; not "either A or B"

$A \Rightarrow B$ if (A and B) or (not A)

- We say "A implies B" or "If A then B"

- In this case, B is at least as true as A

Ex "If a number is less than 10, then it's less than 20" A B

Taking number = 5 we get $T \Rightarrow T$
 15 $F \Rightarrow T$
 25 $F \Rightarrow F$

We write

$A \Leftrightarrow B$ if A and B are true together
or false together

- We say "A is equivalent to B" or "A if and only if B"

.. . .

Truth table:

<u>A</u>	<u>B</u>	<u>not A</u>	<u>A and B</u>	<u>A or B</u>	<u>$A \Rightarrow B$</u>	<u>$A \Leftrightarrow B$</u>
T	T	F	T	T	T	T
T	F	F	F	T	F	F
F	T	T	F	T	T	F
F	F	T	F	F	T	T

- Recall: " $A \Rightarrow B$ " means "(A and B) or (not A)"

Ex Using a truth table, show that
 $A \Rightarrow B$ is equivalent to $(\text{not } A) \text{ or } B$.

Solution:

<u>A</u>	<u>B</u>	<u>$A \Rightarrow B$</u>	<u>not A</u>	<u>$(\text{not } A) \text{ or } B$</u>
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

These two statements are true together or false together. \square

Hw Using a truth table, prove De Morgan's laws:

- ① $\text{not } (\text{A and B}) = (\text{not } A) \text{ or } (\text{not } B)$
- ② $\text{not } (\text{A or B}) = (\text{not } A) \text{ and } (\text{not } B)$

Ex Negate the following sentence:

If I speak in front of the class, then I am nervous.

Solution: $\text{not } (A \Rightarrow B) = \text{not } ((\text{not } A) \text{ or } B)$

$$\begin{aligned}
 &\stackrel{\text{↑ by previous ex}}{=} (\text{not } (\text{not } A)) \text{ and } (\text{not } B) \\
 &\stackrel{\text{↑ De Morgan's law}}{=} A \text{ and } (\text{not } B)
 \end{aligned}$$

The negation is:

"I speak in front of the class and I am not nervous". \square

Quantifiers:

- \forall reads "for all" or "for every"
- \exists reads "there exists" or "there is"

Let X be a set, and let $P(x)$ be a statement about elements $x \in X$.

The negation of " $\forall x \in X, P(x)$ is true"
is " $\exists x \in X$ s.t. $P(x)$ is false"

The negation of " $\exists x \in X$ s.t. $P(x)$ is true"
is " $\forall x \in X, P(x)$ is false"

Ex Negate the following:

There is a rectangle that is not a square.

Solution: $x = \text{a rectangle}$ $P(x) = \text{"}x \text{ is not a square"}$
 $X = \text{all rectangles}$

The negation is: "Every rectangle is a square". \square

Ex Negate the following:

Every student had coffee or is late for class.
 $\forall x \in X$ $P(x)$ $Q(x)$

Solution: $\neg (\forall x \in X, P(x) \text{ or } Q(x))$

$$= \exists x \in X \text{ s.t. } \neg (P(x) \text{ or } Q(x))$$

$$= \exists x \in X \text{ s.t. } (\neg P(x)) \text{ or } (\neg Q(x))$$

The negation is:

"There is a student that did not have coffee
and is not late for class." \square

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Tu 10:30 - 11:30 am

Ex Suppose A and B are statements, and we know $(A \text{ or } B)$ is true. Can we conclude $(\text{not } A) \Rightarrow B$?

Solution:

A	B	<u>A or B</u>	<u>not A</u>	<u>$(\text{not } A) \Rightarrow B$</u>
T	T	T	F	T
T	F	T	F	T
F	T	T	T	T
<u>F</u>	<u>F</u>	<u>F</u>		

- Recall: For " $P \Rightarrow Q$ ", the possibilities $\begin{cases} T \Rightarrow T \\ F \Rightarrow T \\ F \Rightarrow F \end{cases}$ are all allowed

Yes, the statement $(\text{not } A) \Rightarrow B$ is true. \square

SETS

A set is a collection of objects.

Let A and B be sets. We write:

$x \in A$ if x is an element of A

$x \notin A$ if x is not an element of A

$A \cap B = \{x : x \in A \text{ and } x \in B\}$
↑ "such that"

$$A \cup B = \{x : x \in A \text{ or } x \in B\}$$

$$A \subseteq B \text{ if } \forall x \in A, x \in B$$

$$A = B \text{ if } A \subseteq B \text{ and } B \subseteq A$$

- One way to describe a set is by listing all of the objects the set contains:

Ex $S = \{1, 2, 3\}$ is a set. We have $3 \in S$, but $4 \notin S$.

Some important sets:

$\emptyset = \{\}$ "Empty set"

$\mathbb{N} = \{1, 2, 3, \dots\}$ "Natural numbers"

$\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$ "Integers"

$\mathbb{Q} = \left\{ \frac{m}{n} : m \in \mathbb{Z}, n \in \mathbb{N} \right\}$ "Rational numbers"

\mathbb{R} "Real numbers"

\mathbb{C} "Complex numbers"

- We'll have more to say about these sets later

Ex • $S = \{x \in \mathbb{N} : x > 0 \text{ and } x < 4\}$

• $\emptyset = \{x \in \mathbb{Z} : x^2 = 2\}$

• $\mathbb{N} = \{x \in \mathbb{Z} : x > 0\}$

• $\mathbb{Z} = \mathbb{N} \cup \{0\} \cup \{-n : n \in \mathbb{N}\}$

- $\mathbb{Q} = \left\{ \frac{m}{n} : m, n \in \mathbb{Z} \text{ with } n \neq 0 \right\}$
- $\mathbb{Q} \subseteq \mathbb{R}$, but $\mathbb{R} \not\subseteq \mathbb{Q}$. E.g., $\pi \in \mathbb{R}$ and $\pi \notin \mathbb{Q}$.
- $\mathbb{C} = \{a+ib : a, b \in \mathbb{R}\}$

INTRO TO PROOFS

"if and only if"

- Def ① $x \in \mathbb{N}$ is even iff $\exists y \in \mathbb{N}$ s.t. $x = 2y$.
- ② $x \in \mathbb{N}$ is odd iff $\exists y \in \mathbb{N} \cup \{0\}$ s.t. $x = 2y + 1$.

Ex Prove that for any $x \in \mathbb{N}$, if x is odd then $x+1$ is even.

Rmk • $5 = 2 \cdot 2 + 1$ is odd, $6 = 2 \cdot 3$ is even
 $421 = 2 \cdot 210 + 1$ is odd, $422 = 2 \cdot 211$ is even

• A proof is an explanation which convinces the reader why a statement is always true

Solution: Let $x \in \mathbb{N}$.

- This is a "for all" statement, so we start with any natural number. We want to show that the statement " x odd $\Rightarrow x+1$ even" is true regardless of which $x \in \mathbb{N}$ we start with.

Suppose x is odd.

- If x isn't odd, then the statement is automatically true

Then $\exists y \in \mathbb{N} \cup \{0\}$ s.t. $x = 2y + 1$.

- This is "the definition of x being odd."
- We want to show $x+1$ is even, i.e. $\exists z \in \mathbb{N}$ s.t. $x+1 = 2z$.

Adding 1 to both sides of this equation, we get $x+1 = 2y+2 = 2 \cdot (y+1)$.

- So $z=y+1$ works, and $z \in \mathbb{N}$.

Set $z=y+1$. Then $z \in \mathbb{N}$ and $x+1=2z$. Therefore $x+1$ is even. \square

Ex Prove that $\forall x, y \in \mathbb{N}$, if x is odd and y is odd then xy is odd.

Solution: Let $x, y \in \mathbb{N}$. Suppose x and y are odd. Then $\exists m \in \mathbb{N} \cup \{0\}$ s.t. $x=2m+1$ and $\exists n \in \mathbb{N} \cup \{0\}$ s.t. $y=2n+1$. Therefore

$$\begin{aligned} xy &= (2m+1)(2n+1) \\ &= 4mn + 2m + 2n + 1 \\ &= 2(2mn+m+n) + 1 \end{aligned}$$

Set $z=2mn+m+n$. Then $z \in \mathbb{N} \cup \{0\}$ and $xy=2z+1$, and so xy is odd. \square

Ex Prove that $\forall x \in \mathbb{N}$, x is even if and only if $x+1$ is odd.

Rmk " $A \Leftrightarrow B$ " is equivalent to " $A \Rightarrow B$ and $B \Rightarrow A$ ".

Solution: Let $x \in \mathbb{N}$.

\Rightarrow : Suppose x is even. Then $\exists y \in \mathbb{N}$ s.t. $x = 2y$. Therefore $x+1 = 2y+1$ with $y \in \mathbb{N} \cup \{0\}$, so $x+1$ is odd.

\Leftarrow : Suppose $x+1$ is odd. Then $\exists y \in \mathbb{N} \cup \{0\}$ s.t. $x+1 = 2y+1$. Therefore $x = 2y$ with $y \in \mathbb{N}$. (Note that $y \neq 0$, since otherwise $y=0 \Rightarrow x=0 \Rightarrow x \notin \mathbb{N}$.) So x is even. \square

- Last time: "Prove that $\forall n \in \mathbb{N}$, $P(n)$ is true"
- Those were direct proofs: we fixed an arbitrary $n \in \mathbb{N}$, and were able to prove $P(n)$.
- Sometimes that's not enough! Here's another useful technique for proving statements about natural numbers:

Mathematical induction:

Suppose we want to prove that a statement $P(n)$ about $n \in \mathbb{N}$ is true $\forall n \in \mathbb{N}$. Then it suffices to verify two steps:

① Base case: $P(1)$ is true

② Inductive step: If $P(n)$ is true for some $n \in \mathbb{N}$, then $P(n+1)$ is true.

(In other words, " $\forall n \in \mathbb{N}, P(n) \Rightarrow P(n+1)$ ".)

$$\bullet \quad \begin{array}{c} P(1) \\ P(1) \Rightarrow P(2) \end{array} \left\{ \begin{array}{c} \Rightarrow P(2) \\ P(2) \Rightarrow P(3) \end{array} \right\} \Rightarrow \begin{array}{c} P(3) \\ P(3) \Rightarrow P(4) \end{array} \left\{ \begin{array}{c} \Rightarrow P(4) \\ \vdots \end{array} \right\} \Rightarrow \dots$$

In this way, $P(n)$ is true $\forall n \in \mathbb{N}$

• Induction is really a theorem. It's true, but we won't prove it in this class (because we would have to construct \mathbb{N} from scratch)

Ex Prove that $\forall n \in \mathbb{N}$,

$$1+2+\dots+n = \frac{n(n+1)}{2}.$$

Solution: We will argue by induction.

- Here, $P(n)$ is the statement $1+2+\dots+n = \frac{n(n+1)}{2}$.

Base case: For $n=1$, the statement $1 = \frac{1 \cdot 2}{2}$ is true.

Inductive step: Assume $1+2+\dots+n = \frac{n(n+1)}{2}$ is true for some $n \in \mathbb{N}$.

- We want to show $P(n+1)$, i.e. $1+2+\dots+(n+1) = \frac{(n+1)(n+2)}{2}$
- How do we go from $1+2+\dots+n$ to $1+2+\dots+(n+1)$?

Adding $n+1$ to both sides, we get

$$\begin{aligned} 1+2+\dots+n+(n+1) &= \frac{n(n+1)}{2} + (n+1) \\ &= \frac{n(n+1) + 2(n+1)}{2} \\ &= \frac{(n+1)(n+2)}{2} \end{aligned}$$

So the formula is true for $n+1$.

By induction, we conclude that the formula holds $\forall n \in \mathbb{N}$. \square

Ex Prove that $\forall n \in \mathbb{N}$,

$$1^2 + 2^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

Solution: We'll use induction.

$$\text{...} - . . . \cdot 2 \cdot 3 \cdot \dots$$

Base case: For $n=1$, $1 = \frac{1}{6}$ is true.

Inductive step: Assume

$$1^2 + 2^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

for some $n \in \mathbb{N}$. Adding $(n+1)^2$ to both sides, we get

$$\begin{aligned} 1^2 + 2^2 + \dots + (n+1)^2 &= \frac{n(n+1)(2n+1)}{6} + (n+1)^2 \\ &= \frac{n+1}{6} [n(2n+1) + 6(n+1)] \\ &= \frac{n+1}{6} (2n^2 + 7n + 6) \\ &= \frac{n+1}{6} (n+2)(2n+3) \\ &= \frac{(n+1)(n+2)(2n+3)}{6} \end{aligned}$$

So the formula is true for $n+1$.

By induction, we conclude that the formula holds $\forall n \in \mathbb{N}$. \square

• In groups:

Ex Prove that $\forall n \in \mathbb{N}$,

$$1 + 4 + 7 + \dots + (3n+1) = \frac{(n+1)(3n+2)}{2}.$$

Ex Compute the following sum for $n \in \mathbb{N}$, and prove your answer:

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \dots + \frac{1}{n(n+1)}$$

$$\text{Scratch work: } \frac{1}{n} - \frac{1}{n+1} = \frac{(n+1) - n}{n(n+1)} = \frac{1}{n(n+1)}$$

$$\begin{aligned} &\Rightarrow \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \dots + \frac{1}{n(n+1)} \\ &= \cancel{\frac{1}{1}} - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} + \dots + \cancel{\frac{1}{n}} - \cancel{\frac{1}{n+1}} \\ &= 1 - \frac{1}{n+1} = \frac{n}{n+1} \end{aligned}$$

Now that we know the answer, we can prove it

Solution: We will prove that $\forall n \in \mathbb{N}$,

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \dots + \frac{1}{n(n+1)} = \frac{n}{n+1}$$

by induction.

Base case: For $n=1$, $\frac{1}{1 \cdot 2} = \frac{1}{1+1}$ is true.

Inductive step: Assume

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \dots + \frac{1}{n(n+1)} = \frac{n}{n+1}$$

is true for some $n \in \mathbb{N}$. Adding $\frac{1}{(n+1)(n+2)}$ to both sides, we get

$$\begin{aligned} \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \dots + \frac{1}{n(n+1)} + \frac{1}{(n+1)(n+2)} &= \frac{n}{n+1} + \frac{1}{(n+1)(n+2)} \\ &= \frac{n(n+2) + 1}{(n+1)(n+2)} \\ &= \frac{n^2 + 2n + 1}{(n+1)(n+2)} \\ &= \frac{(n+1)^2}{(n+1)(n+2)} \\ &= \frac{n+1}{n+2} \end{aligned}$$

$- n+2$

So the formula is true for $n+1$.

By induction, we conclude that the formula holds $\forall n \in \mathbb{N}$. \square

PROPERTIES OF NUMBERS (Ch. 1)

The real numbers \mathbb{R} satisfy:

(A1) Closure: if $a, b \in \mathbb{R}$ then $a+b \in \mathbb{R}$

(A2) Commutativity: $\forall a, b \in \mathbb{R}$, $a+b = b+a$

(A3) Associativity: $\forall a, b, c \in \mathbb{R}$, $(a+b)+c = a+(b+c)$

(A4) Identity: $\exists 0 \in \mathbb{R}$ s.t. $a+0=0+a=a \quad \forall a \in \mathbb{R}$

(A5) Inverse: $\forall a \in \mathbb{R}$, $\exists -a \in \mathbb{R}$ s.t.

$$a+(-a) = (-a)+a = 0$$

(M1) Closure: if $a, b \in \mathbb{R}$ then $a \cdot b \in \mathbb{R}$

(M2) Commutativity: $\forall a, b \in \mathbb{R}$, $a \cdot b = b \cdot a$

(M3) Associativity: $\forall a, b, c \in \mathbb{R}$, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$

(M4) Identity: $\exists 1 \in \mathbb{R}$ s.t. $a \cdot 1 = 1 \cdot a = a \quad \forall a \in \mathbb{R}$

(M5) Inverse: $\forall a \in \mathbb{R}$ with $a \neq 0$, $\exists a^{-1} \in \mathbb{R}$
s.t. $a \cdot a^{-1} = a^{-1} \cdot a = 1$

(D) Distributivity: $\forall a, b, c \in \mathbb{R}$, $(a+b) \cdot c = a \cdot c + b \cdot c$

Rmk A set F with addition and multiplication operations satisfying these 11 properties is called a field. E.g.:

- \mathbb{N} is not a field, (A4) fails
- \mathbb{Z} is not a field, (M5) fails

- \mathbb{Q} , \mathbb{R} , and \mathbb{C} are fields
- These properties are all we need in order to use $+$ and \cdot .

Proposition 1 If $a, b, c \in \mathbb{R}$ s.t. $a+b=a+c$, then $b=c$.

Rmk In particular, taking $c=0$ we get:
 $a+b=a \Rightarrow b=0$.

Proof: Assume $a, b, c \in \mathbb{R}$ s.t. $a+b=a+c$. By (A5), $\exists -a \in \mathbb{R}$. Adding this to the left, we get:

$$-a + (a+b) = -a + (a+c)$$

By (A3), this implies

$$(-a+a) + b = (-a+a) + c$$

By (A5),

$$0 + b = 0 + c$$

By (A4),

$$b=c.$$

□

Prop 2 If $a, b, c \in \mathbb{R}$ s.t. $a \cdot b = a \cdot c$ and $a \neq 0$, then $b=c$.

Rmk In particular: $a \cdot b = a$ and $a \neq 0 \Rightarrow b=1$.

Pf: Suppose $a, b, c \in \mathbb{R}$ s.t. $a \cdot b = a \cdot c$ and $a \neq 0$.

As $a \neq 0$, $\exists a^{-1} \in R$ by (M5). Together,
 $a^{-1} \cdot (a \cdot b) = a^{-1} \cdot (a \cdot c)$

By (M3), $(a^{-1} \cdot a) \cdot b = (a^{-1} \cdot a) \cdot c$

By (M5), $1 \cdot b = 1 \cdot c$

By (M4), $b = c.$ □

Prop 3 $\forall a \in R, a \cdot 0 = 0 = 0 \cdot a.$

Pf: Let $a \in R.$ Then

$$a \cdot 0 = a \cdot (0 + 0) = a \cdot 0 + a \cdot 0$$

\uparrow \uparrow
 (A4) (D)

By Prop 1, we may cancel $a \cdot 0$ from both sides and conclude $a \cdot 0 = 0.$

By (M2), $0 \cdot a = a \cdot 0 = 0.$ □

Prop 4 If $a, b \in R$ and $a \cdot b = 0,$ then $a=0$ or $b=0.$

Pf: Suppose $a \cdot b = 0.$ If $a=0$ then we're done, so assume $a \neq 0.$ We want to show $b=0.$

As $a \neq 0,$ then $\exists a^{-1} \in R$ by (M5).

Multiplying $a \cdot b = 0$ by a^{-1} on the left ... not

... playing around with a in the left, we get

$$a^{-1} \cdot (a \cdot b) = a^{-1} \cdot 0 = 0$$

\uparrow
Prop 3

On the other hand,

$$a^{-1} \cdot (a \cdot b) = (a^{-1} \cdot a) \cdot b = 1 \cdot b = b$$

\uparrow (M3) \uparrow (M5) \uparrow (M4)

Altogether, we get $b = a^{-1} \cdot (a \cdot b) = 0$, as desired. \square

Prop 5 $\forall a, b \in \mathbb{R}$, $(-a) \cdot b = - (a \cdot b) = a \cdot (-b)$.

Pf: let $a, b \in \mathbb{R}$. Then

$$(-a) \cdot b + a \cdot b = (-a+a) \cdot b = 0 \cdot b = 0$$

\uparrow (D) \uparrow (AS) \uparrow Prop 3

Adding $-(a \cdot b)$ to both sides, we conclude

$$(-a) \cdot b = - (a \cdot b).$$

Similarly,

$$\begin{aligned} a \cdot (-b) + a \cdot b &= (-b) \cdot a + b \cdot a = (-b+b) \cdot a \\ &\quad \uparrow \qquad \qquad \uparrow \\ &\quad (\text{M2}) \qquad (\text{D}) \\ &= 0 \cdot a = 0 \\ &\quad \uparrow \qquad \uparrow \\ &\quad (\text{AS}) \qquad \text{Prop 3} \end{aligned}$$

Adding $-(a \cdot b)$ to both sides, we conclude

$$a \cdot (-b) = - (a \cdot b).$$

□

Last time: $\forall a, b \in \mathbb{R}, (-a) \cdot b = - (a \cdot b) = a \cdot (-b)$.

Idea: ① $(-a) \cdot b + a \cdot b = \dots = 0$

② $(-a) \cdot b = - (a \cdot b)$

• Using a similar idea, we can also prove:

Prop 6 $\forall a, b \in \mathbb{R}, (-a) \cdot (-b) = a \cdot b$.

Pf: Let $a, b \in \mathbb{R}$. Then

$$\begin{aligned} (-a) \cdot (-b) + (- (a \cdot b)) &= \xleftarrow{\text{Prop 5}} (-a) \cdot (-b) + a \cdot (-b) \\ &\xrightarrow{(D)} = (-a+a) \cdot (-b) \\ &= 0 \cdot (-b) = 0 \\ &\quad \uparrow \quad \uparrow \\ &\quad (\text{AS}) \quad \text{Prop 3} \end{aligned}$$

Adding $a \cdot b$ to both sides, we conclude

$$(-a) \cdot (-b) = a \cdot b.$$

□

• What else do we know about \mathbb{R} ?

Inequalities " $<$ " on \mathbb{R} satisfy:

• trichotomy: For each $a, b \in \mathbb{R}$, one and only one of the following statements holds:

$$a < b, \quad a = b, \quad \text{or} \quad b < a$$

• transitivity: $\forall a, b, c \in \mathbb{R} \quad \text{if } a < b \text{ and }$

.....: $a, b, c \in \mathbb{R}$, if and only if
 $b < c$ then $a < c$.

Addition and multiplication satisfy:

- (O1) $\forall a, b, c \in \mathbb{R}$, if $a < b$ then $a + c < b + c$
(O2) $\forall a, b, c \in \mathbb{R}$, if $a < b$ and $0 < c$
then $a \cdot c < b \cdot c$

Rmk ① A relation " $<$ " on a set S satisfying trichotomy and transitivity is called an order relation. E.g., inequalities " $<$ " on \mathbb{Z} is an order relation.

② A field F with an order relation " $<$ " that also satisfies (O1) and (O2) is called an ordered field. E.g.,

- \mathbb{Q} and \mathbb{R} are ordered fields
- \mathbb{C} is not an ordered field

Notation We write

$$\begin{array}{ll} a > b & \text{if } b < a \\ a \leq b & \text{if } a < b \text{ or } a = b \\ a \geq b & \text{if } b \leq a \end{array}$$

- I.e.: Once you have " $<$ ", you also know what " $>$ ", " \leq ", and " \geq " mean.

Prop 7 $\forall a \in \mathbb{R}, a > 0 \iff -a < 0.$

Pf: Let $a \in \mathbb{R}$.

\Rightarrow : Suppose $a > 0$. Then:

$$a > 0 \stackrel{(O1)}{\Rightarrow} a + (-a) > 0 + (-a)$$
$$\begin{array}{ccc} & \parallel & \parallel \\ & 0 & -a \\ \text{by (A5)} & & \text{by (A4)} \end{array}$$

$$\Rightarrow 0 < -a.$$

\Leftarrow : Suppose $-a < 0$. Then:

$$-a < 0 \stackrel{(O1)}{\Rightarrow} -a + a < 0 + a$$
$$\begin{array}{ccc} & \parallel & \parallel \\ & 0 & a \\ \text{by (A5)} & & \text{by (A4)} \end{array}$$

$$\Rightarrow 0 < a.$$

□

Prop 8 $\forall a, b, c \in \mathbb{R}, \text{ if } a < b \text{ and } c < 0$

then $a \cdot c > b \cdot c$.

Pf: Suppose $a < b$ and $c < 0$. By Prop 7 we know $-c > 0$, and so (O2) implies

$$a \cdot (-c) < b \cdot (-c)$$

$$\begin{array}{ccc} & \parallel & \parallel \\ -(a \cdot c) & & -(b \cdot c) \\ & \text{by Prop 5.} & \end{array}$$

Using (O1), we add $ac+bc$ to both sides:

$$-ac + (ac+bc) < -bc + (ac+bc)$$

Now,

$$-ac + (ac+bc) = (-ac+ac) + bc = 0 + bc = bc$$

$\uparrow \quad \uparrow \quad \uparrow$
(A3) (A5) (A4)

and

$$\begin{aligned} -bc + (ac+bc) &= -bc + (bc+ac) = (-bc+bc) + ac \\ &\quad \uparrow \quad \uparrow \\ &= 0 + ac = ac \\ &\quad \uparrow \quad \uparrow \\ &= (A2) \quad (A3) \\ &= (A5) \quad (A4) \end{aligned}$$

Altogether, we conclude $bc < ac$. □

Prop 9 $\forall a, b \in \mathbb{R}$,

① if $a > 0$ and $b > 0$ then $a \cdot b > 0$.

② if $a < 0$ and $b < 0$ then $a \cdot b > 0$.

③ if $a \neq 0$ then $a^2 > 0$.

Rmk In particular, $1 = 1^2 > 0$.

Pf: ① As $b > 0$,

$$a > 0 \xrightarrow{(O2)} a \cdot b > 0 \cdot b$$

$\stackrel{||}{>} 0$ by Prop 3

② Suppose $a < 0$ and $b < 0$. Then $-a > 0$

Suppose $a > 0$. Then $-a < 0$
and $-b > 0$ by Prop 7. Applying ① to
 $-a$ and $-b$, we get

$$(-a) \cdot (-b) > 0$$

$$\stackrel{||}{ab} \text{ by Prop 6.}$$

③ Suppose $a \neq 0$. By trichotomy either $a > 0$ or $a < 0$.

- Case $a > 0$: By ①, $a \cdot a > 0$.
- Case $a < 0$: By ②, $a \cdot a > 0$.

□

Prop 10

① $\forall a \in \mathbb{R}$, if $a > 0$ then $a^{-1} > 0$.

② $\forall a, b \in \mathbb{R}$, if $0 < a < b$ then $0 < b^{-1} < a^{-1}$.

Pf: ① Suppose this statement were false.

Then $\exists a \in \mathbb{R}$ s.t. $a > 0$ but $a^{-1} \leq 0$. Note that $a^{-1} \neq 0$, since otherwise

$$a^{-1} = 0 \Rightarrow a \cdot a^{-1} = 0$$

but we already know $a \cdot a^{-1} = 1 \neq 0$. So $a^{-1} < 0$.

Then:

$$a > 0 \xrightarrow{\text{Prop 8}} a \cdot a^{-1} < 0 \xrightarrow{(\text{M5})} 1 < 0$$

This contradicts that $1 > 0$. Therefore the statement ① was actually true.

(2) Suppose $0 < a < b$. Then $a^{-1} > 0$ and $b^{-1} > 0$ by ①, and so $a^{-1} \cdot b^{-1} > 0$ by Prop 9.

Using (O2), we multiply $0 < a < b$ by $a^{-1} \cdot b^{-1}$:

$$0 \cdot (a^{-1} \cdot b^{-1}) < a \cdot (a^{-1} \cdot b^{-1}) < b \cdot (a^{-1} \cdot b^{-1})$$

Now we simplify each of these terms:

$$0 \cdot (a^{-1} \cdot b^{-1}) = 0$$

\uparrow
Prop 3

$$a \cdot (a^{-1} \cdot b^{-1}) = (a \cdot a^{-1}) \cdot b^{-1} = 1 \cdot b^{-1} = b^{-1}$$

\uparrow
(M3) \uparrow
(M5) \uparrow
(M4)

$$b \cdot (a^{-1} \cdot b^{-1}) = b \cdot (b^{-1} \cdot a^{-1}) = (b \cdot b^{-1}) \cdot a^{-1} = 1 \cdot a^{-1} = a^{-1}$$

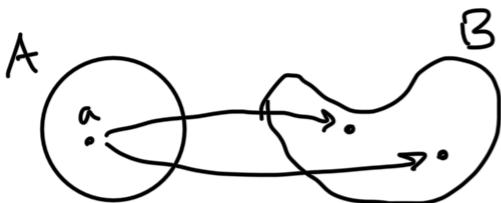
\uparrow
(M2) \uparrow
(M3) \uparrow
(M5) \uparrow
(M4)

Altogether, we conclude $0 < b^{-1} < a^{-1}$.

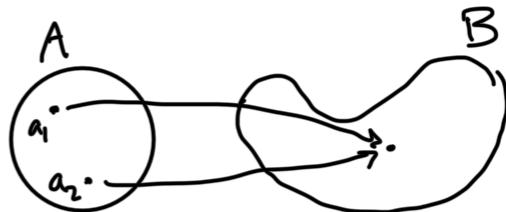
□

FUNCTIONS (Ch. 3)

Def Let A, B be nonempty sets. A function $f: A \rightarrow B$ is a way of associating to each element $a \in A$ exactly one element in B , denoted $f(a)$.



Not a function



OK

A is called the domain of f .

B is called the range of f .

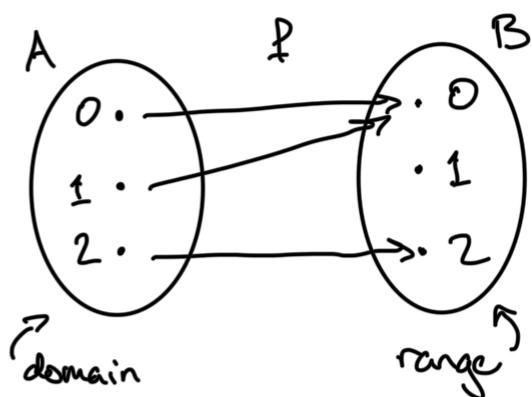
Ex let $A = \{0, 1, 2\} = B$. Consider $f(x) = x^2 - x$.
(or "x ↦ x² - x")

Is $f: A \rightarrow B$ a function? Yes.

$$0 \mapsto 0$$

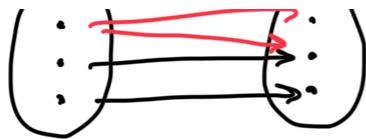
$$1 \mapsto 0$$

$$2 \mapsto 2$$



Ex For the same sets A, B , are the following functions from A to B ?



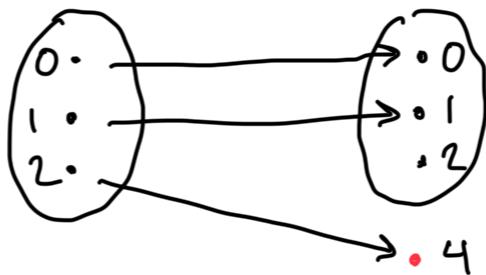


No.



No.

- ③ $x \mapsto x^2$
 $0 \mapsto 0$
 $1 \mapsto 1$
 $2 \mapsto 4$



No.

Def let $f: A \rightarrow B$ be a function.

- ① If $X \subseteq A$, the image of X under f is

$$f(X) = \{f(a) : a \in X\}.$$

- ② The image of f is $f(A)$.

- ③ If $Y \subseteq B$, the preimage of Y under f is

$$f^{-1}(Y) = \{a \in A : f(a) \in Y\}.$$

Rank Warning: Here, " f^{-1} " is not the inverse function of f ! In particular:

- $f^{-1}(Y)$ always makes sense, even when f does not have an inverse.
- f and f^{-1} do not cancel in general.

Ex $A = \{0, 1, 2\} = B$, $f: A \rightarrow B$

$$0 \mapsto 0$$

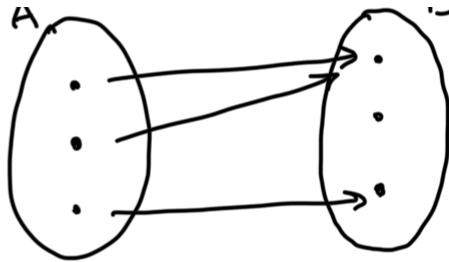
$$\begin{aligned} \cdot f(\{0, 1\}) &= \{f(x) : x \in \{0, 1\}\} \\ &= \{0\} \end{aligned}$$

$$1 \mapsto 0$$

$$2 \mapsto 2$$

B

- image of f
 $= f(A) = \{0, 2\}$



- $f^{-1}(\{1, 2\}) = \{x \in A : f(x) \in \{1, 2\}\}$
 $= \{2\}$

- $f(\{1, 2\}) = \{0, 2\}$

- $f(f^{-1}(\{1, 2\})) = f(\{2\}) = \{2\} \neq \{1, 2\}$

- $f^{-1}(f(\{1, 2\})) = f^{-1}(\{0, 2\}) = \{0, 1, 2\} \neq \{1, 2\}$
↑ ↑

- Notice how f and f^{-1} don't cancel!

Prop let $f: A \rightarrow B$ be a function.

① $\forall Y \subseteq B, f(f^{-1}(Y)) \subseteq Y$.

② $\forall X \subseteq A, f^{-1}(f(X)) \supseteq X$.

Pf: ① Let $Y \subseteq B$. Want to show: $\forall b \in f(f^{-1}(Y))$, we have $b \in Y$. Let $b \in f(f^{-1}(Y))$. By definition of image,

$$b \in f(f^{-1}(Y)) = \{f(x) : x \in f^{-1}(Y)\},$$

so $\exists a \in f^{-1}(Y)$ s.t. $f(a) = b$. By definition of preimage,

$$a \in f^{-1}(Y) = \{x \in A : f(x) \in Y\},$$

so $f(a) \in Y$. Altogether, we conclude $b = f(a) \in Y$, as desired.

② Let $X \subseteq A$. Want: $\forall a \in X, a \in f^{-1}(f(X))$.

Let $a \in X$. Then

$$f(a) \in f(X) = \{f(x) : x \in X\}$$

by definition of image. So

$$a \in f^{-1}(f(X)) = \{x \in A : f(x) \in f(X)\}$$

by definition of preimage. \square

Def Let $f: A \rightarrow B$ be a function.

① f is surjective (or onto) iff $f(A) = B$.

In other words: $\forall b \in B, \exists a \in A$ s.t. $f(a) = b$.

② f is injective (or one-to-one) iff it satisfies: if $a_1, a_2 \in A$ s.t. $f(a_1) = f(a_2)$ then $a_1 = a_2$.

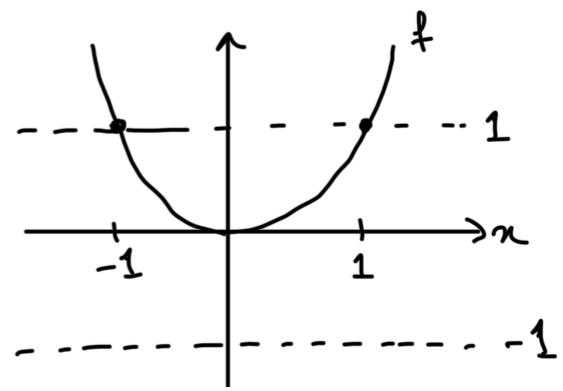
③ f is bijection iff it is injective and surjective.

Rmk These properties depend not only on the function f , but also on the domain and range.

Ex $f(x) = x^2$

① $f: \mathbb{R} \rightarrow \mathbb{R}$ is...

- not injective: $-1, 1 \in \mathbb{R}$,
 $f(-1) = f(1)$, and $-1 \neq 1$
- not surjective: $-1 \in \mathbb{R}$,
but $f(x) \neq -1 \quad \forall x \in \mathbb{R}$



② $A = \{x \in \mathbb{R} : x \geq 0\}$.

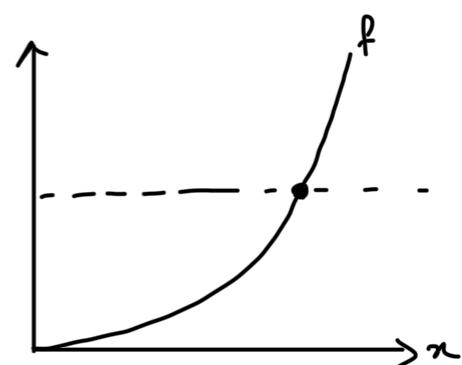
$f: A \rightarrow A$ is...

• injective:

$$a_1, a_2 \geq 0, \quad a_1^2 = a_2^2 \Rightarrow a_1 = a_2$$

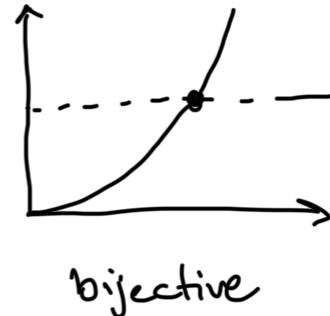
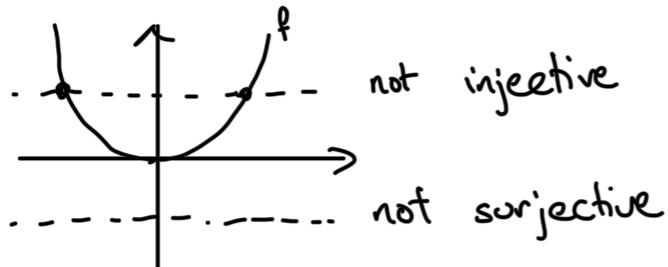
• surjective: $\forall b \geq 0,$

$$\exists a \geq 0 \text{ s.t. } a^2 = b$$



Last time $f: A \rightarrow B$

- f injective: $a_1, a_2 \in A$ s.t. $f(a_1) = f(a_2) \Rightarrow a_1 = a_2$
- f surjective: $\forall b \in B, \exists a \in A$ s.t. $f(a) = b$



Lemma If $f: A \rightarrow B$ is bijective, then:
 $\forall b \in B, \exists$ exactly one $a \in A$ s.t. $f(a) = b$.

Pf: let $b \in B$. By definition of surjective,
 \exists at least one $a \in A$ s.t. $f(a) = b$.

Claim: There is at most one $a \in A$ s.t.
 $f(a) = b$. Suppose $a_1, a_2 \in A$ s.t. $f(a_1) = b = f(a_2)$.
By definition of injective, $a_1 = a_2$. \square

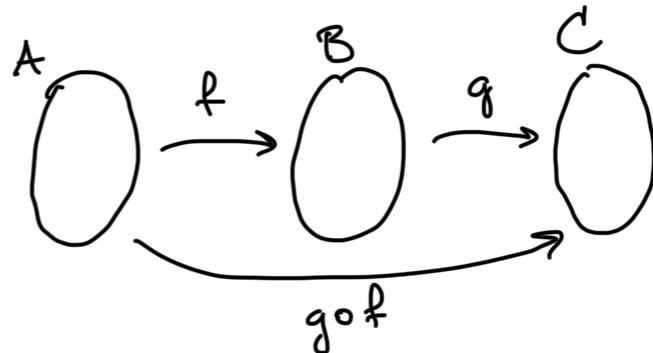
• So, for a bijective function, $f^{-1}: B \rightarrow A$ makes sense as a function:

Def Let $f: A \rightarrow B$ be bijective. The inverse of f is the function $f^{-1}: B \rightarrow A$, $f^{-1}(b) = a$, where $a \in A$ is the unique element in A s.t. $f(a) = b$.

Ques If f is bijective then $f^{-1}(f(a)) = a$

Prop f^{-1} is injective, i.e.,
 $\forall a \in A$ and $f(f^{-1}(b)) = b \quad \forall b \in B$.

Def Let A, B, C be nonempty sets and let
 $f: A \rightarrow B, g: B \rightarrow C$ be functions. The composition
of g with f is the function $g \circ f: A \rightarrow C$,
 $(g \circ f)(a) = g(f(a))$.



Ex let $f, g: \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = 2x$, $g(x) = x+1$.

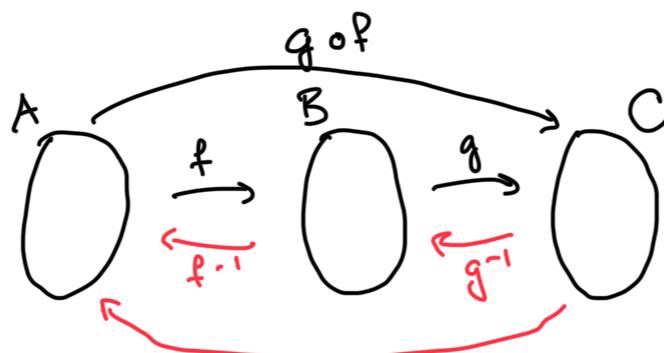
Then

$$(g \circ f)(x) = g(f(x)) = g(2x) = 2x+1 \quad \leftarrow \text{Not equal!}$$

$$(f \circ g)(x) = f(g(x)) = f(x+1) = 2(x+1)$$

Rmk Composition of functions is not commutative.

Prop If $f: A \rightarrow B$ and $g: B \rightarrow C$ are bijective,
then $g \circ f: A \rightarrow C$ is bijective and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.



$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}$$

Pf: Injective: let $a_1, a_2 \in A$ s.t. $(g \circ f)(a_1) = (g \circ f)(a_2)$.

Want: $a_1 = a_2$. We know

$$g(f(a_1)) = (g \circ f)(a_1) = (g \circ f)(a_2) = g(f(a_2))$$

As g is injective,

$$f(a_1), f(a_2) \in B, \quad g(f(a_1)) = g(f(a_2)) \Rightarrow f(a_1) = f(a_2).$$

Likewise, as f is injective,

$$a_1, a_2 \in A, \quad f(a_1) = f(a_2) \Rightarrow a_1 = a_2.$$

So $a_1 = a_2$, as desired.

Surjective: let $c \in C$. Want: $\exists a \in A$ s.t.

$(g \circ f)(a) = c$. As $c \in C$ and g is surjective, then

$\exists b \in B$ s.t. $g(b) = c$. Similarly, as $b \in B$ and

f is surjective, then $\exists a \in A$ s.t. $f(a) = b$. Altogether;

$$(g \circ f)(a) = g(f(a)) = g(b) = c,$$

as desired.

Claim: $(g \circ f)^{-1}(c) = (f^{-1} \circ g^{-1})(c) \quad \forall c \in C$.

Let $c \in C$. By definition, $(g \circ f)^{-1}(c)$ is equal to the unique $a \in A$ s.t. $(g \circ f)(a) = c$. We will show that $a_2 = (f^{-1} \circ g^{-1})(c)$ satisfies this property:

$$(g \circ f)(a_2) = g(f(f^{-1}(g^{-1}(c))))$$

$$= g^{-1}(c), \text{ since } f^{-1} \text{ is the inverse function of } f$$

$$= g(g^{-1}(c)) = c$$

\uparrow since g^{-1} is the inverse function of g .

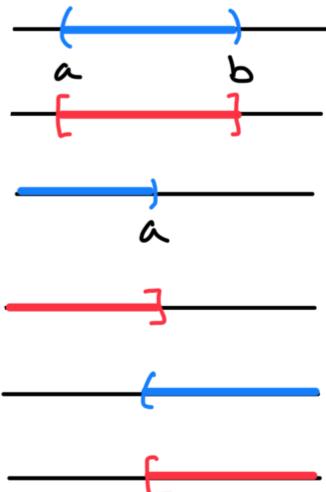
So $a_2 = a_1$, and thus $(g \circ f)^{-1}(c) = (f^{-1} \circ g^{-1})(c)$. \square

SUBSETS OF \mathbb{R}

Def A set $I \subseteq \mathbb{R}$ is an interval iff $\forall x, y, z \in \mathbb{R}$, $x, z \in I$ and $x < y < z \Rightarrow y \in I$.

Ex Given $a, b \in \mathbb{R}$ with $a < b$,

- $(a, b) = \{x \in \mathbb{R} : a < x < b\}$
- $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$
- $(-\infty, a) = \{x \in \mathbb{R} : x < a\}$
- $(-\infty, a] = \dots$
- (a, ∞)
- $[a, \infty)$



- These are all examples of intervals
- let's prove this in one case:

lem $\forall a, b \in \mathbb{R}$ with $a < b$, (a, b) is an interval.

Pf: Let $a < b$. Suppose $x, z \in (a, b)$ and $x < y < z$.
We want to show $y \in (a, b)$, i.e. $a < y < b$:

$$x \in (a, b) \Rightarrow \left. \begin{array}{l} a < x < b \\ x < y \end{array} \right\} \Rightarrow a < y$$

$$z \in (a, b) \Rightarrow \left. \begin{array}{l} a < z < b \\ y < z \end{array} \right\} \Rightarrow y < b$$

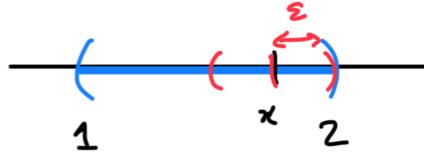
So $a < y < b$, as desired. □

Def A set $U \subseteq \mathbb{R}$ is open iff $\forall x \in U$,
 $\exists \varepsilon > 0$ s.t. $(x - \varepsilon, x + \varepsilon) \subseteq U$.

Ex ① $(1, 2)$ is open:

let $x \in (1, 2)$. Then $1 < x < 2$.

Set $\varepsilon = \min\left\{\frac{2-x}{2}, \frac{x-1}{2}\right\}$.



Then $\varepsilon > 0$. Moreover, we have $(x - \varepsilon, x + \varepsilon) \subseteq (1, 2)$:

$$y \in (x - \varepsilon, x + \varepsilon) \Rightarrow \left\{ \begin{array}{l} y < x + \varepsilon \leq x + (2 - x) = 2 \\ y > x - \varepsilon \geq x - (x - 1) = 1 \end{array} \right\} \Rightarrow y \in (1, 2)$$

$\uparrow \varepsilon \leq 2 - x$
 $\downarrow \varepsilon \leq x - 1$

- Note that ε is equal to whichever number is smaller, and so ε is \leq both numbers.

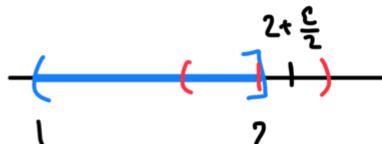
② $(1, 2]$ is not open:

- $\text{not(open)} = \text{not } (\forall x \in U, \exists \varepsilon > 0 \text{ s.t. } (x - \varepsilon, x + \varepsilon) \subseteq U)$
 $= \exists x \in U \text{ s.t. } \forall \varepsilon > 0, (x - \varepsilon, x + \varepsilon) \not\subseteq U$.

Consider $2 \in (1, 2]$. Let $\varepsilon > 0$.

Then $(2 - \varepsilon, 2 + \varepsilon) \not\subseteq (1, 2]$, since

$2 + \frac{\varepsilon}{2} \in (2 - \varepsilon, 2 + \varepsilon)$ but $2 + \frac{\varepsilon}{2} \notin (1, 2]$.



③ \mathbb{R} is open: Let $x \in \mathbb{R}$. Set $\varepsilon = 10$.

Then $(x-\varepsilon, x+\varepsilon) \subseteq \mathbb{R}$.

④ \emptyset is open: The definition " $\forall x \in \emptyset, \dots$ " is vacuously true.

- There are no such x , so there is nothing to check!

Def A set $F \subseteq \mathbb{R}$ is closed iff

$F^c = \{x \in \mathbb{R} : x \notin F\}$ is open.

" F complement"

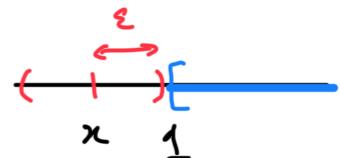
Ex ① $[1, 2]$ is closed: Fix $x \in [1, 2]^c$.

Then $x \notin [1, 2]$, so $x < 1$ or $x > 2$.

Note: $\text{not } (x \geq 1 \text{ and } x \leq 2) = x < 1 \text{ or } x > 2$.

Case: $x < 1$. Set $\varepsilon = 1 - x$.

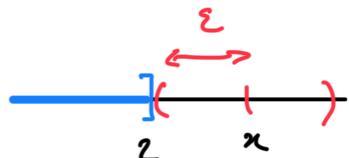
Then $\varepsilon > 0$ and $(x-\varepsilon, x+\varepsilon) \subseteq [1, 2]^c$.



$$\begin{aligned} y \in (x-\varepsilon, x+\varepsilon) &\Rightarrow y < x+\varepsilon = x+(1-x)=1 \\ &\Rightarrow y \notin [1, 2]. \end{aligned}$$

Case: $x > 2$. Set $\varepsilon = x-2$.

Then $\varepsilon > 0$ and $(x-\varepsilon, x+\varepsilon) \subseteq [1, 2]^c$.

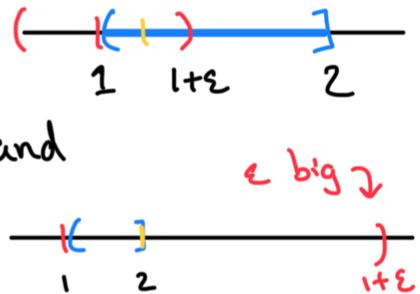


$$\begin{aligned} y \in (x-\varepsilon, x+\varepsilon) &\Rightarrow y > x-\varepsilon = x-(x-2)=2 \\ &\Rightarrow y \notin [1, 2]. \end{aligned}$$

Altogether, $[1, 2]^c$ is open, and so $[1, 2]$ is closed.

② $(1, 2]$ is not closed: I.e., $(1, 2]^c$ is not open. Consider $1 \in (1, 2]^c$. $\leftarrow \varepsilon \text{ small}$

Let $\varepsilon > 0$. Then $(1-\varepsilon, 1+\varepsilon) \not\subseteq (1, 2]^c$, since $1 + \min\{\frac{\varepsilon}{2}, 1\} \in (1-\varepsilon, 1+\varepsilon)$ and $\leftarrow \varepsilon \text{ big} \downarrow$
 $1 + \min\{\frac{\varepsilon}{2}, 1\} \in (1, 2]$.



③ \mathbb{R} is closed: $\mathbb{R}^c = \emptyset$ is open.

④ \emptyset is closed: $\emptyset^c = \mathbb{R}$ is open.

Rmk Warning: "not open" \neq "closed".

A set can be...

- open: e.g. $(1, 2)$
- closed: e.g. $[1, 2]$
- not open and not closed: e.g. $(1, 2]$
- both open and closed: e.g. \mathbb{R}, \emptyset .

Q: A, B open $\Rightarrow A \cap B$ open?
 $A \cup B$ open?

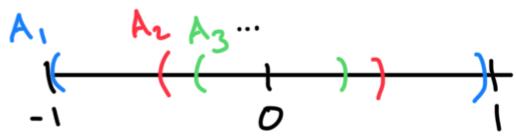
- Yes, and you can use more than 2 sets:

Prop 1 $\forall n \in \mathbb{N}$, if $A_1, \dots, A_n \subseteq \mathbb{R}$ are open

then $A_1 \cap \dots \cap A_n$ is open.

- Any intersection of finitely many open sets is open.

Ex $A_n = (-\frac{1}{n}, \frac{1}{n})$



- $A_1 \cap A_2 \cap \dots \cap A_n = (-\frac{1}{n}, \frac{1}{n})$ is open
- $A_1 \cap A_2 \cap \dots = \{x \in \mathbb{R} : x \in A_n \ \forall n \in \mathbb{N}\}$
 $= \{0\}$ is not open
- "Finitely many" is the best we can do.

Last time: A open $\stackrel{\text{def}}{\iff} \forall x \in A, \exists \varepsilon > 0$ s.t.
 $(x - \varepsilon, x + \varepsilon) \subseteq A$.

E.g. $(1, 2)$.

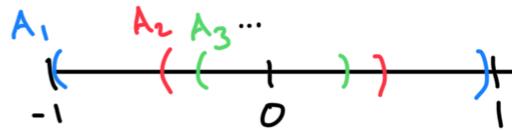
Q: A, B open $\Rightarrow A \cap B$ open?
 $A \cup B$ open?

- Yes, and you can use more than 2 sets:

Prop 1 $\forall n \in \mathbb{N}$, if $A_1, \dots, A_n \subseteq \mathbb{R}$ are open
then $A_1 \cap \dots \cap A_n$ is open.

- Any intersection of **finitely** many open sets is open.

Ex $A_n = (-\frac{1}{n}, \frac{1}{n})$



- $A_1 \cap A_2 \cap \dots \cap A_n = (-\frac{1}{n}, \frac{1}{n})$ is open
- $A_1 \cap A_2 \cap \dots = \{x \in \mathbb{R} : x \in A_n \ \forall n \in \mathbb{N}\}$
 $= \{0\}$ is not open

- "Finitely many" is the best we can do.

Pf: Suppose $A_1, \dots, A_n \subseteq \mathbb{R}$ are open. Let $x \in A_1 \cap \dots \cap A_n$. Want: $\exists \varepsilon > 0$ s.t. $(x - \varepsilon, x + \varepsilon) \subseteq A_1 \cap \dots \cap A_n$.

As $x \in A_1$ and A_1 is open, $\exists \varepsilon_1 > 0$ s.t.
 $(x - \varepsilon_1, x + \varepsilon_1) \subseteq A_1$. Likewise, as $x \in A_2$ and A_2 is

open, $\exists \varepsilon_2 > 0$ s.t. $(x - \varepsilon_2, x + \varepsilon_2) \subseteq A_2$. Similarly, for each $i=1, 2, \dots, n$, we see $\exists \varepsilon_i > 0$ s.t. $(x - \varepsilon_i, x + \varepsilon_i) \subseteq A_i$.

Set $\varepsilon = \min\{\varepsilon_1, \dots, \varepsilon_n\}$. Then $\varepsilon > 0$ and $\varepsilon \leq \varepsilon_i \quad \forall i=1, \dots, n$. So

$$(x - \varepsilon, x + \varepsilon) \subseteq (x - \varepsilon_i, x + \varepsilon_i) \subseteq A_i \quad \text{for } i=1, \dots, n$$

$$\Rightarrow (x - \varepsilon, x + \varepsilon) \subseteq A_1 \cap A_2 \cap \dots \cap A_n$$

as desired. \square

Prop 2 A set I , if $A_i \subseteq \mathbb{R}$ is open $\forall i \in I$, then $\bigcup_{i \in I} A_i = \{x \in \mathbb{R} : \exists i \in I \text{ s.t. } x \in A_i\}$ is open

- Any union of open sets is open, even if there are infinitely many

Ex In particular, taking $I = \{1, 2, \dots, n\}$ we get: A_1, \dots, A_n open $\Rightarrow A_1 \cup \dots \cup A_n$ open.

Ex $I = (0, 1)$, $A_j = (-j, j)$ for $j \in I$.

$$\begin{aligned} \bigcup_{j \in I} A_j &= \{x \in \mathbb{R} : \exists j \in (0, 1) \text{ s.t. } -j < x < j\} \\ &= (-1, 1) \end{aligned}$$

is open.

Pf: Suppose $A_i \subseteq \mathbb{R}$ is open $\forall i \in I$. Let $x \in \bigcup_{i \in I} A_i$. Then $\exists j \in I$ s.t. $x \in A_j$. As A_j is open, $\exists \varepsilon > 0$ s.t. $(x - \varepsilon, x + \varepsilon) \subseteq A_j$. Then $(x - \varepsilon, x + \varepsilon) \subseteq A_j$, since:

$$y \in (x - \varepsilon, x + \varepsilon) \Rightarrow y \in A_j \Rightarrow y \in \bigcup_{i \in I} A_i.$$

Therefore $\bigcup_{i \in I} A_i$ is open. \square

Defn A set X with a collection of subsets satisfying Props 1 and 2 (and containing \emptyset and X) is called a topological space.

LIMITS (Ch. 5)

Recall that for $a \in \mathbb{R}$, we define

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases}$$

Lem (Triangle inequality) $\forall a, b \in \mathbb{R}$, $|a+b| \leq |a| + |b|$.

Idea: 4 cases, based on the signs of a, b .

See Ch. 1 for a proof.

Cor (Reverse triangle inequality) $\forall a, b \in \mathbb{R}$,
 $|a+b| \geq |a| - |b|$.

--

$$\underline{\text{Pf}}: |a| = |a+b + (-b)| \leq |a+b| + |-b|. \quad \square$$

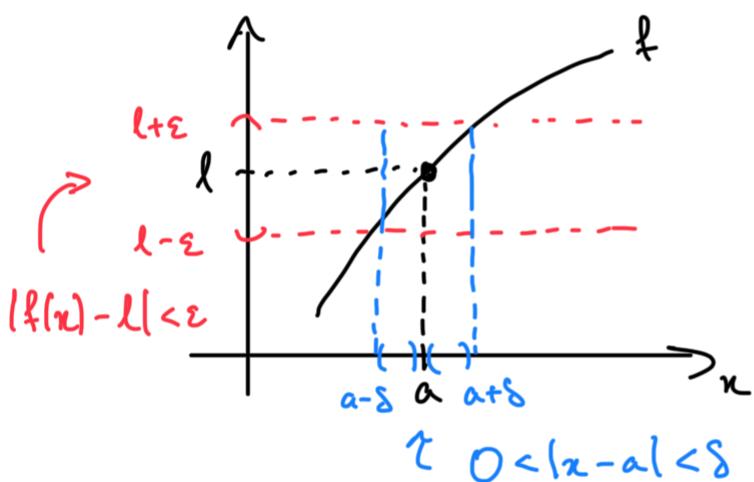
↑
lem

Def let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a function and $a, l \in \mathbb{R}$. We say that the limit of $f(x)$ as x approaches a is l if: $\forall \varepsilon > 0, \exists \delta > 0$ s.t.

$$x \in \mathbb{R} \text{ and } 0 < |x-a| < \delta \Rightarrow |f(x) - l| < \varepsilon.$$

When this is true, we write " $\lim_{x \rightarrow a} f(x) = l$ "

or " $f(x) \rightarrow l$ as $x \rightarrow a$ ".



- Heuristically: Can we make $f(x)$ as close to l as we like by requiring that x is sufficiently close (but not equal!) to a

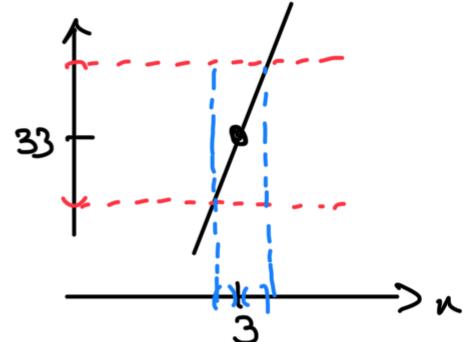
- ① Midterm 1 is next Friday (Oct 4), in class
 - ② It will be based on Homeworks 1-4
 - ③ I will post a practice midterm later this week.
-

Last time: $\lim_{n \rightarrow a} f(n) = l \stackrel{\text{def}}{\iff} \forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } 0 < |n - a| < \delta \text{ implies } |f(n) - l| < \varepsilon.$

Ex Do the following limits exist?

① $f(x) = 4x + 21$ as $x \rightarrow 3$

Claim: $\lim_{x \rightarrow 3} f(x)$ exists and is equal to 33. let $\varepsilon > 0$.



Scratch work: Want $\delta > 0$ s.t.

$$0 < |x - 3| < \delta \Rightarrow |4x + 21 - 33| < \varepsilon$$

$$\begin{aligned} |4x - 12| &= 4|x - 3| \\ \Rightarrow |x - 3| &< \frac{\varepsilon}{4} \quad \leftarrow \text{Set } \delta = \frac{\varepsilon}{4} \end{aligned}$$

Set $\delta = \frac{\varepsilon}{4}$. Then $\delta > 0$ and

$$\begin{aligned} 0 < |x - 3| < \delta &\Rightarrow |f(x) - 33| = |4x + 21 - 33| \\ &= 4|x - 3| < 4\delta = \varepsilon. \end{aligned}$$

So $\lim_{x \rightarrow 3} f(x) = 33$. □

(2) $f(x) = x^2$ as $x \rightarrow 2$.

Claim: $\lim_{x \rightarrow 2} f(x) = 4$. Let $\varepsilon > 0$.

Scratch work:

$$\begin{aligned} 0 < |x-2| < \delta &\Rightarrow |x^2 - 4| < \varepsilon \\ &\quad || \\ &\Rightarrow |x+2| \cdot |x-2| < \varepsilon \\ &\Rightarrow |x-2| < \frac{\varepsilon}{|x+2|} \quad \text{Not good enough...} \end{aligned}$$

• $\delta > 0$ needs to be a constant, independent of x

As long as we choose $\underline{\delta \leq 1}$, then

$$\begin{aligned} |x-2| < \delta \leq 1 &\Rightarrow -1 < x-2 < 1 \\ &\Rightarrow 3 < x+2 < 5 \Rightarrow |x+2| < 5 \end{aligned}$$

So

$$\begin{aligned} |x^2 - 4| &= |x+2| \cdot |x-2| < 5 |x-2| < \varepsilon \\ &\Rightarrow |x-2| < \frac{\varepsilon}{5} \quad \text{Need } \underline{\delta \leq \frac{\varepsilon}{5}} \end{aligned}$$

Set $\delta = \min \{1, \frac{\varepsilon}{5}\}$. Then $\delta > 0$ and

$$0 < |x-2| < \delta \Rightarrow \begin{cases} |x+2| \leq |x-2| + 4 < \delta + 4 \leq 5 \\ |x^2 - 4| = |x+2| \cdot |x-2| < 5\delta \leq \varepsilon. \end{cases}$$

So $\lim_{x \rightarrow 2} x^2 = 4$.

$$\delta \leq \frac{\varepsilon}{5}$$

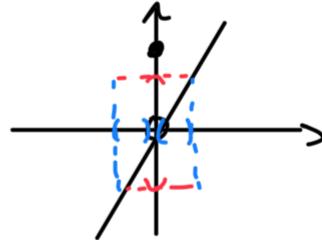
$x \rightarrow 2$

- In groups:

$$\textcircled{3} \quad f(x) = \begin{cases} 2x & x \neq 0 \\ 1 & x = 0 \end{cases} \quad \text{as } x \rightarrow 0.$$

- The value of $\lim_{x \rightarrow a} f(x)$ does not depend on $f(a)$!

Claim: $\lim_{x \rightarrow 0} f(x) = 0$.



let $\epsilon > 0$. Set $\delta = \frac{\epsilon}{2}$.

Then $\delta > 0$ and

$$0 < |x - 0| < \delta \Rightarrow |f(x) - 0| = 2|x| < 2\delta = \epsilon.$$

$\hookrightarrow x \neq 0 \quad \Rightarrow = 2x$

□

- We say "the" limit... why can't there be 2?

Prop (Uniqueness of limits) Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a function and $a, l, m \in \mathbb{R}$. If $\lim_{x \rightarrow a} f(x) = l$ and $\lim_{x \rightarrow a} f(x) = m$, then $l = m$.

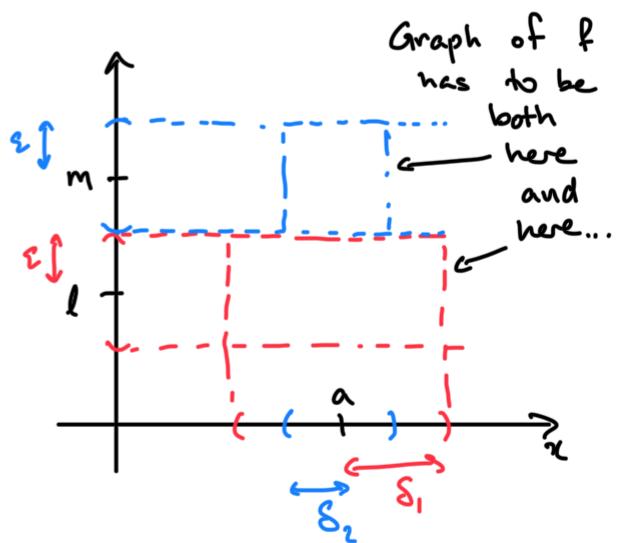
Pf: Suppose not: $l \neq m$.

Set $\epsilon = \frac{|l-m|}{2} > 0$. As

$\lim_{x \rightarrow a} f(x) = l$, $\exists \delta_1 > 0$ s.t.

$$0 < |x - a| < \delta_1 \Rightarrow |f(x) - l| < \epsilon.$$

As $\lim_{x \rightarrow a} f(x) = m$, $\exists \delta_2 > 0$ s.t.



$$0 < |x - a| < \delta_2 \Rightarrow |f(x) - m| < \varepsilon.$$

Set $\delta = \min\{\delta_1, \delta_2\}$. Then $\delta > 0$, and for any $0 < |x - a| < \delta$ we have

$$\begin{aligned} 2\varepsilon &= |l - m| = |l - f(x) + f(x) - m| \\ &\leq |l - f(x)| + |f(x) - m| < \varepsilon + \varepsilon = 2\varepsilon \\ &\Rightarrow 2\varepsilon < 2\varepsilon \end{aligned}$$

which is a contradiction. \square

Recall $\lim_{x \rightarrow a} f(x) = l \stackrel{\text{def}}{\iff} \forall \varepsilon > 0, \exists \delta > 0 \text{ s.t.}$
 $0 < |x-a| < \delta \text{ implies } |f(x) - l| < \varepsilon$

- How can we prove a limit doesn't exist?

Ex $f(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases} \text{ as } x \rightarrow 0.$

Claim: $\lim_{x \rightarrow 0} f(x)$ does not exist. Suppose not:

$\lim_{x \rightarrow 0} f(x)$ exists and is equal to l . Want:

$$\text{not} \left(\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } (x \in \mathbb{R}, 0 < |x| < \delta \Rightarrow |f(x) - l| < \varepsilon) \right)$$

$$= \exists \varepsilon > 0 \text{ s.t. } \forall \delta > 0, (x \in \mathbb{R}, 0 < |x| < \delta, \text{ and } |f(x) - l| \geq \varepsilon)$$

Set $\varepsilon = \frac{1}{2}$. Let $\delta > 0$.

- Case: $l \geq \frac{1}{2}$. Then $x = -\frac{\delta}{2}$ satisfies

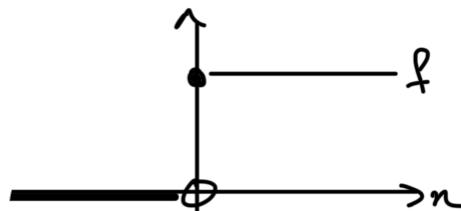
$$0 < |x| < \delta \text{ and } |f(x) - l| = l \geq \frac{1}{2}.$$

- Case: $l < \frac{1}{2}$. Then $x = +\frac{\delta}{2}$ satisfies

$$0 < |x| < \delta \text{ and } |f(x) - l| = 1 - l > \frac{1}{2}.$$

A contradiction — so no such l can exist. \square

Theorem (limit laws) Let $f, g: \mathbb{R} \rightarrow \mathbb{R}$ be functions



Lemma: If $f: \mathbb{R} \rightarrow \mathbb{R}$, $g: \mathbb{R} \rightarrow \mathbb{R}$ and $a \in \mathbb{R}$. If $\lim_{n \rightarrow a} f(n) = l$ and $\lim_{n \rightarrow a} g(n) = m$, then:

- ① For any $c \in \mathbb{R}$, $\lim_{n \rightarrow a} (cf)(n) = cl$.
- ② $\lim_{n \rightarrow a} (f+g)(n) = l+m$
- ③ $\lim_{n \rightarrow a} (f \cdot g)(n) = l \cdot m$.

Rank: If $f, g: \mathbb{R} \rightarrow \mathbb{R}$ are functions, then

- $cf: \mathbb{R} \rightarrow \mathbb{R}$, $(cf)(n) = c \cdot f(n)$
- $f+g: \mathbb{R} \rightarrow \mathbb{R}$, $(f+g)(n) = f(n) + g(n)$
- $f \cdot g: \mathbb{R} \rightarrow \mathbb{R}$, $(f \cdot g)(n) = f(n) \cdot g(n)$

are all functions.

Pf: ① Let $\varepsilon > 0$. As $\lim_{n \rightarrow a} f(n) = l$, we know $\exists \delta > 0$ s.t.

$$0 < |n-a| < \delta \Rightarrow |f(n)-l| < \frac{\varepsilon}{|c|+10} \quad \text{← } c \text{ could be zero!}$$

So, for $0 < |n-a| < \delta$ we have

$$|c \cdot f(n) - c \cdot l| = |c| \cdot |f(n) - l| \leq |c| \cdot \frac{\varepsilon}{|c|+10} < \varepsilon$$

Therefore $\lim_{n \rightarrow a} (c \cdot f)(n) = c \cdot l$.

• $cf(n) - c \cdot l$ is small because it is $c \cdot (\text{small})$

② Let $\varepsilon > 0$. As $\lim_{n \rightarrow a} f(n) = l$, $\exists \delta_1 > 0$ s.t.

$$0 < |x-a| < \delta_1 \Rightarrow |f(x) - l| < \frac{\varepsilon}{2}.$$

Likewise, as $\lim_{x \rightarrow a} g(x) = m$, $\exists \delta_2 > 0$ s.t.

$$0 < |x-a| < \delta_2 \Rightarrow |g(x) - m| < \frac{\varepsilon}{2}.$$

Set $\delta = \min\{\delta_1, \delta_2\} > 0$. Then for $0 < |x-a| < \delta$,

$$\begin{aligned} |f(x) + g(x) - (l+m)| &\leq |f(x) - l| + |g(x) - m| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

So $\lim_{x \rightarrow a} (f+g)(x) = l+m$.

- $f(x) + g(x) - (l+m)$ is small because it is equal to $(f(x) - l) + (g(x) - m) = \text{small} + \text{small}$.

③ Let $\varepsilon > 0$. As $\lim_{x \rightarrow a} f(x) = l$, $\exists \delta_1 > 0$ s.t.

$$0 < |x-a| < \delta_1 \Rightarrow |f(x) - l| < \frac{\varepsilon}{2(|m|+10)}$$

Likewise, as $\lim_{x \rightarrow a} g(x) = m$, $\exists \delta_2 > 0$ s.t.

$$0 < |x-a| < \delta_2 \Rightarrow |g(x) - m| < \min\left\{\frac{\varepsilon}{2(|l|+10)}, 1\right\}$$

Set $\delta = \min\{\delta_1, \delta_2\} > 0$. Then, for $0 < |x-a| < \delta$,

$$\begin{aligned} |f(x) \cdot g(x) - l \cdot m| &= |f(x)g(x) - lg(x) + lg(x) - lm| \\ &\leq |f(x)g(x) - lg(x)| + |lg(x) - lm| \\ &= \underbrace{|g(x)|}_{\text{red}} \cdot |f(x) - l| + |l| \cdot |g(x) - m| \end{aligned}$$

$$\leq |g(x)-ml| + |ml| \leq l + |ml|$$

$$\begin{aligned} &< (|ml|+l) \cdot \frac{\varepsilon}{2(|ml|+l)} + |l| \cdot \frac{\varepsilon}{2(|l|+1)} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

So $\lim_{x \rightarrow a} (f \cdot g)(x) = l \cdot m.$

□

Ex Prove $\lim_{x \rightarrow a} (4x^3 + 2|x|) = 4a^3 + 2|a| \quad \forall a \in \mathbb{R}.$

- This would be rather cumbersome to prove using ε and δ !

Pf: let $a \in \mathbb{R}$. Note that $\lim_{x \rightarrow a} x = a$. Indeed, given $\varepsilon > 0$, we can take $\delta = \varepsilon$ and $0 < |x-a| < \delta \Rightarrow |x-a| < \delta = \varepsilon$.

So, by the limit laws, we have

- $\lim_{x \rightarrow a} 2|x| \stackrel{(1)}{=} 2|a|$
- $\lim_{x \rightarrow a} x^2 = \lim_{x \rightarrow a} (x \cdot x) \stackrel{(3)}{=} a \cdot a = a^2$
- $\lim_{x \rightarrow a} x^3 = \lim_{x \rightarrow a} (x \cdot x^2) \stackrel{(3)}{=} a \cdot a^2 = a^3$
- $\lim_{x \rightarrow a} 4x^3 \stackrel{(1)}{=} 4a^3$
- $\lim_{x \rightarrow a} (4x^3 + 2|x|) \stackrel{(1)}{=} 4a^3 + 2|a|.$

□

Recall $\lim_{x \rightarrow a} f(x) = l \stackrel{\text{def}}{\iff} \forall \varepsilon > 0, \exists \delta > 0 \text{ s.t.}$
 $0 < |x - a| < \delta \text{ implies } |f(x) - l| < \varepsilon$

Ex $\forall a > 0, \lim_{x \rightarrow a} \sqrt{x} = \sqrt{a}$.

Let $a > 0$ and $\varepsilon > 0$.

Scratch work:

$$|\sqrt{x} - \sqrt{a}| = \frac{|x - a|}{|\sqrt{x} + \sqrt{a}|} \begin{matrix} \leftarrow \text{Good, we know } |x - a| < \delta \\ \leftarrow \text{Not good enough.} \end{matrix}$$

Want $\frac{1}{|\sqrt{x} + \sqrt{a}|} < \text{constant.}$

As long as we pick $\delta \leq a$, then

$$\begin{aligned} 0 < |x - a| < \delta \leq a &\Rightarrow -a < x - a < a \\ &\Rightarrow 0 < x \Rightarrow 0 < \sqrt{x} \\ &\Rightarrow \sqrt{a} < \sqrt{x} + \sqrt{a} \\ &\Rightarrow 0 < \frac{1}{\sqrt{x} + \sqrt{a}} < \frac{1}{\sqrt{a}} \end{aligned}$$

So

$$|\sqrt{x} - \sqrt{a}| = \frac{|x - a|}{|\sqrt{x} + \sqrt{a}|} < \frac{\delta}{\sqrt{a}} \leq \varepsilon \Rightarrow \text{Need } \delta \leq \sqrt{a} \cdot \varepsilon.$$

L

Set $\delta = \min \{a, \sqrt{a} \cdot \varepsilon\}$. Then $\delta > 0$ and

$$0 < |x - a| < \delta \Rightarrow \begin{cases} x - a > -\delta \stackrel{\delta \leq a}{\geq} -a \Rightarrow x > 0 \Rightarrow \sqrt{x} > 0 \\ |\sqrt{x} - \sqrt{a}| = \frac{|x - a|}{\sqrt{x} + \sqrt{a}} < \frac{\delta}{\sqrt{a}} \leq \varepsilon \end{cases}$$

$\delta \leq \sqrt{a} \cdot \varepsilon$

$$\text{So } \lim_{x \rightarrow a} \sqrt{x} = \sqrt{a}.$$

□

Thm (Limit law for division) If $\lim_{x \rightarrow a} f(x) = l$

and $\lim_{x \rightarrow a} g(x) = m \neq 0$, then $\lim_{x \rightarrow a} \left(\frac{f}{g}\right)(x) = \frac{l}{m}$.

Pf: It suffices to show $\lim_{x \rightarrow a} \left(\frac{1}{g}\right)(x) = \frac{1}{m}$.

Indeed, once we prove this, then by the limit law for multiplication (lecture 11) we'll have

$$\lim_{x \rightarrow a} \left(f \cdot \frac{1}{g}\right)(x) = l \cdot \frac{1}{m}.$$

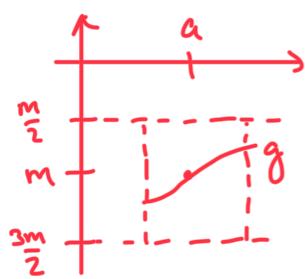
Let $\varepsilon > 0$. As $\lim_{x \rightarrow a} g(x) = m$ and $m \neq 0$,

$\exists \delta > 0$ s.t.

$$0 < |x - a| < \delta \implies |g(x) - m| < \min\left\{\frac{|m|}{2}, \frac{|m|^2 \varepsilon}{2}\right\}$$

Then, for $0 < |x - a| < \delta$ we have

$$\left| \frac{1}{g(x)} - \frac{1}{m} \right| = \frac{|g(x) - m|}{|m| \cdot |g(x)|} < \frac{|g(x) - m|}{|m| \cdot \frac{|m|}{2}} < \varepsilon$$



$$\begin{aligned} |g(x) - m| &< \frac{|m|}{2} \\ \Rightarrow -\frac{|m|}{2} &< g(x) - m < \frac{|m|}{2} \\ \Rightarrow \begin{cases} m > 0: & g(x) > m - \frac{|m|}{2} = \frac{m}{2} \\ m < 0: & g(x) < m + \frac{|m|}{2} = \frac{m}{2} \end{cases} \\ \Rightarrow |g(x)| &> \frac{|m|}{2} \Rightarrow \frac{1}{|g(x)|} < \frac{1}{|m|/2} \end{aligned}$$

$$\text{So } \lim_{x \rightarrow a} \frac{1}{g(x)} = \frac{1}{m}.$$

□

Ex Prove that $\lim_{x \rightarrow a} \frac{x^3 + 4x}{x^2 + 21} = \frac{a^3 + 4a}{a^2 + 21}$ for $a \in \mathbb{R}$.

Pf: Let $a \in \mathbb{R}$.

Claim: $\lim_{x \rightarrow a} (x^3 + 4x) = a^3 + 4a$ and $\lim_{x \rightarrow a} (x^2 + 21) = a^2 + 21$.

Exercise. (Just like lecture 11.)

Note that $a^2 + 21 \neq 0$ since $a^2 \geq 0$. So, by the limit law for division, $\lim_{x \rightarrow a} \frac{x^3 + 4x}{x^2 + 21} = \frac{a^3 + 4a}{a^2 + 21}$.

Thm (Limit law for composition) If $\lim_{x \rightarrow a} f(x) = l$,

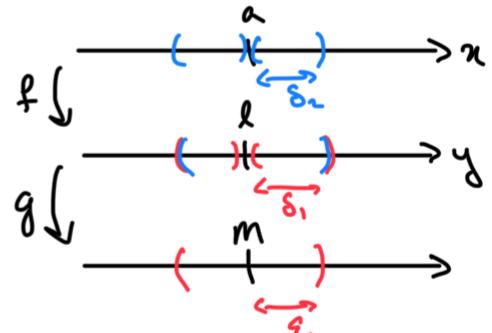
$\lim_{y \rightarrow l} g(y) = m$, and $g(l) = m$, then $\lim_{x \rightarrow a} (g \circ f)(x) = m$.

Pf: Fix $\varepsilon > 0$. As $\lim_{y \rightarrow l} g(y) = m$, $\exists \delta_1 > 0$ s.t.

$$0 < |y - l| < \delta_1 \Rightarrow |g(y) - m| < \varepsilon.$$

In fact, since $g(l) = m$, we have

$$|y - l| < \delta_1 \stackrel{\textcircled{1}}{\Rightarrow} |g(y) - m| < \varepsilon.$$



As $\lim_{x \rightarrow a} f(x) = l$, $\exists \delta_2 > 0$ s.t.

$$0 < |x - a| < \delta_2 \stackrel{\textcircled{2}}{\Rightarrow} |f(x) - l| < \delta_1.$$

Altogether:

$$0 < |x - a| < \delta_2 \stackrel{\textcircled{2}}{\Rightarrow} |f(x) - l| < \delta_1,$$

$$\stackrel{\textcircled{1}}{\Rightarrow} |g(f(x)) - m| < \varepsilon.$$

30 $\lim_{x \rightarrow a} (g \circ f)(x) = m.$

u

Ex Prove that $\lim_{x \rightarrow a} \sqrt{x^2 + 2l} = \sqrt{a^2 + 2l} \quad \forall a \in \mathbb{R}.$

Pf: Note that $\sqrt{x^2 + 2l} = (g \circ f)(x)$ for
 $f(x) = x^2 + 2l$ and $g(y) = \sqrt{y}.$

We already saw $\lim_{x \rightarrow a} f(x) = a^2 + 2l$ (in the previous example). As $l = a^2 + 2l > 0$, then $\lim_{y \rightarrow l} g(y) = \sqrt{l}$
 $= \sqrt{a^2 + 2l}$ (by the first example today). So, by
the limit law for composition, $\lim_{x \rightarrow a} \sqrt{x^2 + 2l} = \sqrt{a^2 + 2l}.$

- ① No office hours next week.
 - ② Homework 5 will be due 2 weeks from today.
 - ③ Class will be held as usual next week.
-

Def let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a function and $a, l \in \mathbb{R}$.

We say:

$$\textcircled{1} \quad \lim_{x \rightarrow a^+} f(x) = l \quad (\text{or} \quad \lim_{x \downarrow a} f(x) = l) \quad \text{if:}$$

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } 0 < x - a < \delta \Rightarrow |f(x) - l| < \varepsilon.$$

$$\textcircled{2} \quad \lim_{x \rightarrow a^-} f(x) = l \quad (\text{or} \quad \lim_{x \uparrow a} f(x) = l) \quad \text{if:}$$

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } -\delta < x - a < 0 \Rightarrow |f(x) - l| < \varepsilon.$$

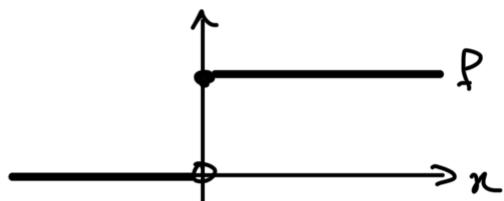
Rmk For any $f: \mathbb{R} \rightarrow \mathbb{R}$ and $a, l \in \mathbb{R}$, we have

$$\lim_{x \rightarrow a} f(x) = l \iff \lim_{x \rightarrow a^+} f(x) = l = \lim_{x \rightarrow a^-} f(x).$$

Indeed,

$$0 < |x - a| < \delta \iff 0 < x - a < \delta \text{ or } -\delta < x - a < 0.$$

Ex $f(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$



$$\textcircled{1} \quad \lim_{x \rightarrow 0^+} f(x) = 1: \text{ let } \varepsilon > 0. \text{ Set } \delta = 10.$$

Then $0 < x - 0 < \delta \Rightarrow |f(x) - 1| = 0 < \varepsilon.$

=1

② $\lim_{x \rightarrow 0^-} f(x) = 0$: let $\varepsilon > 0$. Set $\delta = 10$.

Then $-\delta < x - 0 < 0 \Rightarrow \underline{\underline{|f(x)-0| = 0}} < \varepsilon$.

③ $\lim_{x \rightarrow 0} f(x)$ does not exist: since

$$\lim_{x \rightarrow 0^-} f(x) = 0 \neq 1 = \lim_{x \rightarrow 0^+} f(x).$$

□

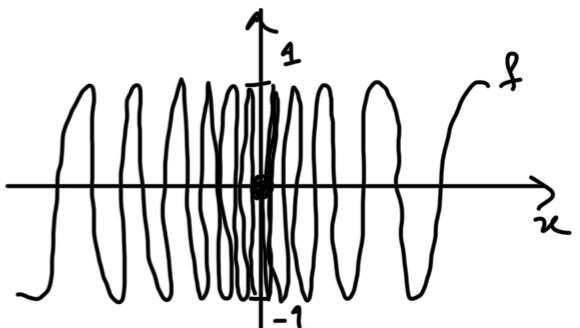
- How else can $\lim_{x \rightarrow a} f(x)$ fail to exist?

Ex $f(x) = \begin{cases} \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$

Claim: $\lim_{x \rightarrow 0} f(x)$ does not exist. Suppose not: $\exists l \in \mathbb{R}$

s.t. $\lim_{x \rightarrow 0} f(x) = l$. Note that for any $n \in \mathbb{N}$,

$$f\left(\frac{2}{(2n+1)\pi}\right) = \sin \frac{(2n+1)\pi}{2} = \begin{cases} 1 & n \text{ even} \\ -1 & n \text{ odd} \end{cases}$$



Set $\varepsilon = 1$. let $\delta > 0$.

Case: $l \geq 0$. let $n \in \mathbb{N}$ be an odd number large enough so that $\frac{2}{(2n+1)\pi} < \delta$. Then $x = \frac{2}{(2n+1)\pi}$ satisfies $0 < |x| < \delta$ and $\underline{\underline{|f(x)-l| = l+1}} > 1 = \varepsilon$.

Case: $l < 0$. let $n \in \mathbb{N}$ be an even number large

enough so that $\frac{2}{(2n+1)\pi} < \delta$. Then $x = \frac{2}{(2n+1)\pi}$ satisfies

$$0 < |x| < \delta \quad \text{and} \quad |f(x) - l| = \underbrace{|1 - l|}_{=1} \geq 1 = \varepsilon.$$

In both cases, we have a contradiction. \square

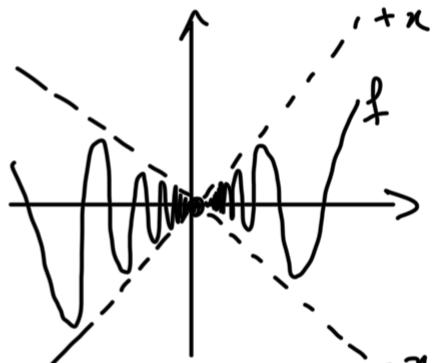
Ex $f(x) = \begin{cases} x \sin \frac{1}{x} & x \neq 0 \\ 0 & x=0 \end{cases}$

Claim: $\lim_{x \rightarrow 0} f(x) = 0$.

let $\varepsilon > 0$. Set $\delta = \varepsilon$. Then

$\delta > 0$ and

$$0 < |x-0| < \delta \Rightarrow |f(x)-0| = |x| \cdot |\sin \frac{1}{x}| \leq |x| < \delta = \varepsilon$$



\square

CONTINUITY (Ch. 6)

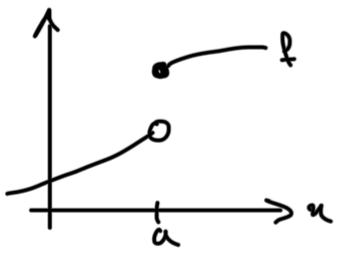
Def Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a function and $a \in \mathbb{R}$.

We say f is continuous at a if $\lim_{x \rightarrow a} f(x) = f(a)$. In other words: $\forall \varepsilon > 0, \exists \delta > 0$ s.t.

$$|x-a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon.$$

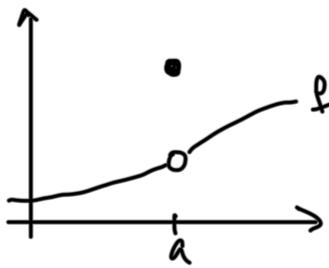
No "0<". Limit is $f(a)$.

We allow $x=a$ now.



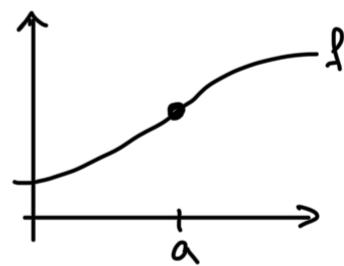
Not continuous:

$\lim_{x \rightarrow a} f(x)$ does
not exist



Not continuous:

$\lim_{x \rightarrow a} f(x)$ exists,
not equal to $f(a)$



Continuous:

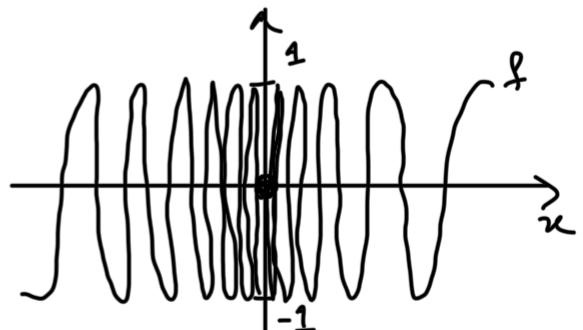
$\lim_{x \rightarrow a} f(x) = f(a)$

- In groups:

Ex Are the following functions continuous at $x=0$?

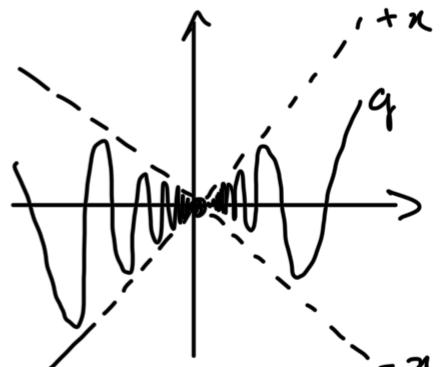
$$\textcircled{1} \quad f(x) = \begin{cases} \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

No, f is not continuous at $x=0$, since $\lim_{x \rightarrow 0} f(x)$ does not exist (by the second example today).



$$\textcircled{2} \quad g(x) = \begin{cases} x \cdot \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

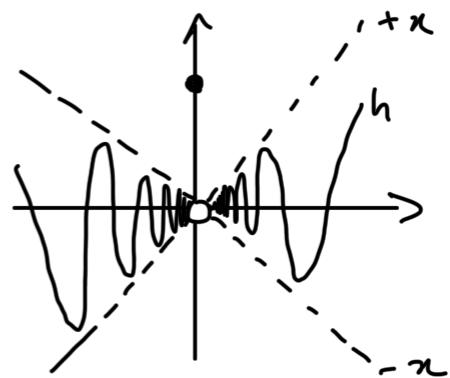
Yes, g is continuous at $x=0$, since we know $\lim_{x \rightarrow 0} g(x) = 0 = g(0)$ (by the third example today).



$$\textcircled{3} \quad h(x) = \begin{cases} x \cdot \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

No, h is not continuous
at $x=0$, since

$$\lim_{x \rightarrow 0} h(x) = 0 \neq h(0).$$



- Remember: $\lim_{x \rightarrow 0} h(x)$ does not depend on $h(0)$.
As $h(x) = g(x) \quad \forall x \neq 0$, then $\lim_{x \rightarrow 0} h(x) = \lim_{x \rightarrow 0} g(x)$.

Last time: f is continuous at $a \iff \lim_{x \rightarrow a} f(x) = f(a)$

- We can combine continuous functions with addition/multiplication:

Thm Let $f, g: \mathbb{R} \rightarrow \mathbb{R}$ be functions and $a \in \mathbb{R}$.

If f and g are continuous at a , then:

- ① cf is continuous at $a, \forall c \in \mathbb{R}$.
- ② $f+g$ is continuous at a .
- ③ $f \cdot g$ is continuous at a .

Pf: As f and g are continuous at a , we know

$$\lim_{x \rightarrow a} f(x) = f(a) \quad \text{and} \quad \lim_{x \rightarrow a} g(x) = g(a).$$

So, by the limit laws for addition/multiplication (lecture 11), we have:

- ① $\lim_{x \rightarrow a} (cf)(x) = c \cdot f(a) \quad \forall c \in \mathbb{R}$.
- ② $\lim_{x \rightarrow a} (f+g)(x) = f(a) + g(a)$.
- ③ $\lim_{x \rightarrow a} (f \cdot g)(x) = f(a) \cdot g(a)$. □

- We can also divide, if the denominator is nonzero:

Thm If f and g are continuous at a and $g(a) \neq 0$, then $\frac{f}{g}$ is continuous at a .

Pf: As f and g are continuous at a , we know

$$\lim_{x \rightarrow a} f(x) = f(a) \text{ and } \lim_{x \rightarrow a} g(x) = g(a).$$

Together with $g(a) \neq 0$, we conclude

$$\lim_{x \rightarrow a} \left(\frac{f}{g}\right)(x) = \frac{f(a)}{g(a)}$$

by the limit law for division (lecture 12). \square

• Lastly, we can also compose continuous functions:

Thm If f is continuous at a and g is continuous at $f(a)$, then $g \circ f$ is continuous at a .

Pf: We know

$$\lim_{x \rightarrow a} f(x) = f(a) \text{ and } \lim_{y \rightarrow f(a)} g(y) = g(f(a)).$$

So, by the limit law for composition (lecture 12) we have

$$\lim_{x \rightarrow a} (g \circ f)(x) = g(f(a)). \quad \square$$

Ex $f(x) = 4x^3 + 21x$ is continuous at a , $\forall a \in \mathbb{R}$.

In lecture 11, we proved $\lim_{x \rightarrow a} (4x^3 + 21x) = 4a^3 + 21a$ $\forall a \in \mathbb{R}$.

Ex Let $p: \mathbb{R} \rightarrow \mathbb{R}$ be a polynomial: $\exists N \in \mathbb{N} \cup \{0\}$ and $c_0, c_1, \dots, c_N \in \mathbb{R}$ s.t.

$$p(x) = c_0 + c_1 x + c_2 x^2 + \dots + c_N x^N.$$

Then p is continuous at $a \in \mathbb{R}$.

Pf: Let $a \in \mathbb{R}$.

Claim: The function $f(n)=1$ is continuous at a .

Given $\varepsilon > 0$, we can take $S=10$ and

$$0 < |x-a| < S \Rightarrow |f(x)-1| = 0 < \varepsilon.$$

Claim: The function x^n is continuous at $a \in \mathbb{R}$. We will argue by induction:

① Base case: $\lim_{x \rightarrow a} x = a$. We did this on Lecture 11. (Just take $S=\varepsilon$.)

② Inductive step: Assume $\lim_{x \rightarrow a} x^n = a^n$ for some $n \in \mathbb{N}$. By the limit law for multiplication,

$$\lim_{x \rightarrow a} x^n = a^n, \quad \lim_{x \rightarrow a} x = a \Rightarrow \lim_{x \rightarrow a} x^{n+1} = a^{n+1}.$$

So x^{n+1} is continuous at a .

By induction, we conclude that x^n is continuous at $a \in \mathbb{R}$.

Let $p(x) = c_0 + c_1 x + \dots + c_N x^N$. By the claims, we know x^n is continuous at a for $n=0$ and for $n=1, 2, \dots, N$. Then $c_n x^n$ is continuous at a for each $n=0, 1, \dots, N$ (by the first theorem today). Therefore $p(x) = c_0 + c_1 x + \dots + c_N x^N$ is continuous at a (by the same theorem). \square

$$\text{Ex } f(x) = \begin{cases} \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

① f is continuous at $a \neq 0$: By the previous example, $g(x) = x$ is continuous at $a \in \mathbb{R}$. Given $a \neq 0$, we have $g(a) = a \neq 0$, and so $f(x) = \frac{1}{g(x)}$ is also continuous at a .

② f is not continuous at $x=0$: Want $\lim_{x \rightarrow 0} f(x) \neq 0$.

In fact, we'll prove $\lim_{x \rightarrow 0} f(x)$ does not exist.

Suppose towards a contradiction $\lim_{x \rightarrow 0} f(x) = l$ for some $l \in \mathbb{R}$. Consider $\varepsilon = 1$.

Let $\delta > 0$. Then $x = \min\left\{\frac{\delta}{2}, \frac{1}{|l|+10}\right\}$

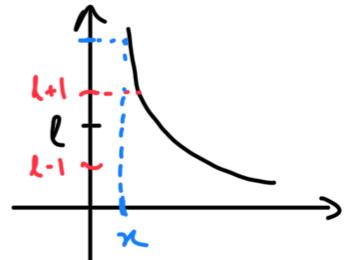
satisfies

$$0 < x \leq \frac{\delta}{2} \Rightarrow 0 < |x| < \delta$$

$$0 < x \leq \frac{1}{|l|+10} \Rightarrow f(x) = \frac{1}{x} \geq |l| + 10$$

$$\Rightarrow |f(x) - l| \geq ||l| + 10 - |l|| = 10 \geq \varepsilon$$

So $\lim_{x \rightarrow 0} f(x) \neq l$, a contradiction. \square



Recall: f continuous at $a \iff \lim_{x \rightarrow a} f(x) = f(a)$
 $\iff \forall \varepsilon > 0, \exists \delta > 0$ s.t. $|x - a| < \delta$ implies $|f(x) - f(a)| < \varepsilon$.

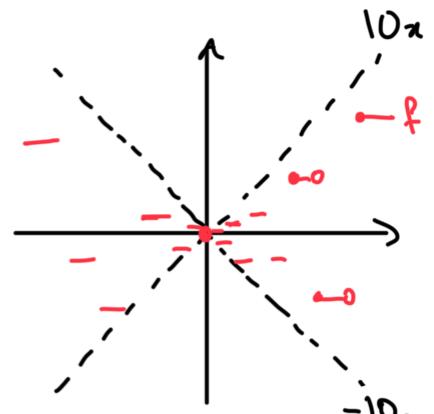
Ex Suppose $f: \mathbb{R} \rightarrow \mathbb{R}$ satisfies $|f(x)| \leq 10|x| \quad \forall x \in \mathbb{R}$.

Prove that f is continuous at 0.

Pf: Note that $|f(0)| \leq 0$, and so $f(0) = 0$. Want: $\lim_{x \rightarrow 0} f(x) = 0$.

Let $\varepsilon > 0$. Set $\delta = \frac{\varepsilon}{10}$. Then

$$\begin{aligned} |x| < \delta &\Rightarrow |f(x) - 0| = |f(x)| \\ &\leq 10|x| \\ &< 10\delta = \varepsilon. \end{aligned}$$



□

Prop let $f: \mathbb{R} \rightarrow \mathbb{R}$ and $a \in \mathbb{R}$. If f is continuous at a and $f(a) > 0$, then $\exists \delta > 0$ s.t. $f(x) > 0 \quad \forall x \in (a-\delta, a+\delta)$.

Pf: Consider $\varepsilon = \frac{f(a)}{2} > 0$.

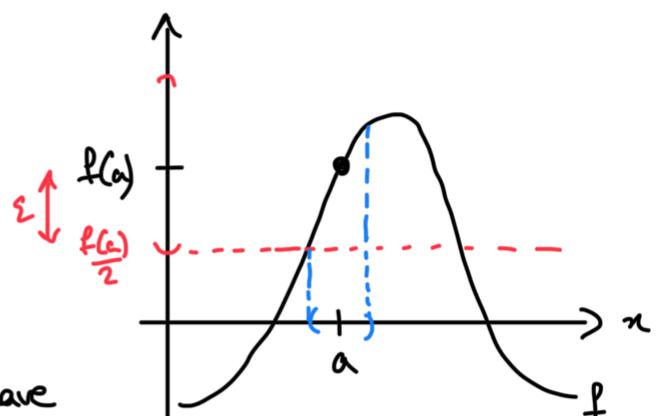
As f is continuous at a ,

$\exists \delta > 0$ s.t.

$$|x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon.$$

So, for $x \in (a-\delta, a+\delta)$ we have

$$\begin{aligned} -\varepsilon < f(x) - f(a) < \varepsilon &\Rightarrow f(x) > f(a) - \varepsilon = f(a) - \frac{f(a)}{2} \\ &= \frac{f(a)}{2} > 0 \end{aligned}$$



as desired. \square

Rmk What if $f(a) < 0$? Applying the Proposition to $g(x) = -f(x)$, we get

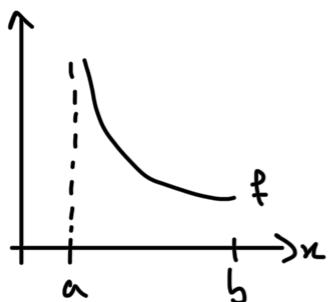
$$\left. \begin{array}{l} g \text{ continuous at } a \\ g(a) = -f(a) > 0 \end{array} \right\} \Rightarrow \exists \delta > 0 \text{ s.t. } g(x) > 0 \quad \forall x \in (a-\delta, a+\delta)$$

$$\Rightarrow f(x) = -g(x) < 0 \quad \forall x \in (a-\delta, a+\delta)$$

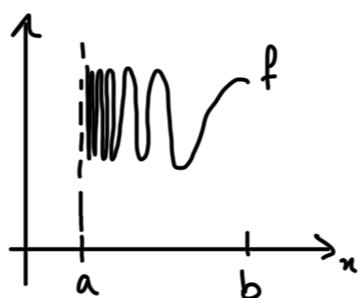
Def Let $a < b$. We say:

- ① $f: \mathbb{R} \rightarrow \mathbb{R}$ is continuous if f is continuous at $x \in \mathbb{R}$.
- ② $f: (a, b) \rightarrow \mathbb{R}$ is continuous if f is continuous at $x \in (a, b)$.
- ③ $f: [a, b] \rightarrow \mathbb{R}$ is continuous if f is continuous at $x \in (a, b)$, $\lim_{x \rightarrow a^+} f(x) = f(a)$, and $\lim_{x \rightarrow b^-} f(x) = f(b)$.

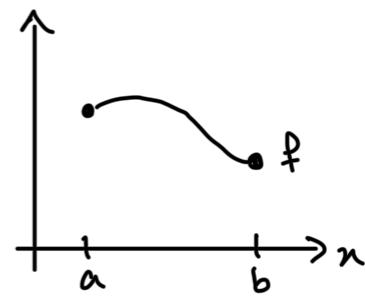
• It depends not just on f , but also the domain!



Continuous on (a, b) .
Not continuous
on $[a, b]$.



Continuous on (a, b) .
Not continuous
on $[a, b]$.



Continuous on $[a, b]$.

- Similarly, the domain can be any interval I .
- If I includes an endpoint, we require that the one-sided limit of f exists at that point.

Ex ① Polynomials $p(n)$ are continuous on \mathbb{R} .

② $\frac{1}{n}$ is continuous on $(-\infty, 0)$ and $(0, \infty)$.

③ Rational functions $\frac{p(x)}{q(x)}$, where p, q are polynomials, are continuous where $q(x) \neq 0$.

④ $|x|$ is continuous on \mathbb{R} .

⑤ \sqrt{x} is continuous on $[0, \infty)$.

⑥ $e^x, \sin x, \cos x$ are continuous on \mathbb{R} .

⑦ Any sum/product/composition of these are continuous, on the domain where they are defined. E.g.
 $\sin \frac{1}{x}$ is continuous on $(0, \infty)$ by ② and ⑥.

- We've already proved ① and ②. ③ is similar.
- ④ is on HW5.
- We won't prove all of these, but here is ⑤:

Ex Prove $f: [0, \infty) \rightarrow \mathbb{R}$, $f(x) = \sqrt{x}$ is continuous.

Pf: There are 2 conditions to check.

① f is continuous at a $\forall a \in (0, \infty)$: Recall from lecture 12 that $\lim_{n \rightarrow \infty} \sqrt{n} = \sqrt{a} \quad \forall a > 0$.

② $\lim_{x \rightarrow 0^+} f(x) = 0$: let $\varepsilon > 0$. Set $S = \varepsilon^2$. Then
 $\delta > 0$ and

$$0 < x < \delta \Rightarrow |f(x) - 0| = \sqrt{x} < \sqrt{\delta} = \varepsilon.$$

□

Ex Prove $g(x) = 4x^3 + 2\ln x$ is continuous on $[1, 2]$.

Pf: As g is a polynomial, we know $\lim_{x \rightarrow a} g(x) = g(a)$

$\forall a \in \mathbb{R}$ (by lecture 14). In particular,

$$\textcircled{1} \quad \lim_{x \rightarrow a} g(x) = g(a) \quad \forall a \in (1, 2)$$

$$\textcircled{2} \quad \lim_{x \rightarrow 1^+} g(x) = g(1)$$

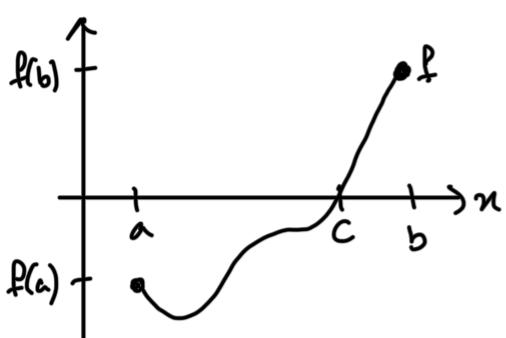
$$\textcircled{3} \quad \lim_{x \rightarrow 2^-} g(x) = g(2)$$

(Remember, $\lim_{x \rightarrow a} g(x) = l \Rightarrow \lim_{x \rightarrow a^+} g(x) = l = \lim_{x \rightarrow a^-} g(x)$.)

Therefore g is continuous on $[1, 2]$. \square

3 IMPORTANT THEOREMS (Ch. 7)

Thm 1 If $f: [a, b] \rightarrow \mathbb{R}$ is continuous and $f(a) < 0 < f(b)$, then $\exists c \in [a, b]$ s.t. $f(c) = 0$.



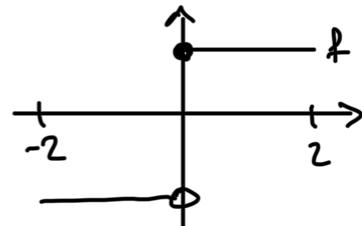
- In order to go from below to above the x-axis, a continuous function must cross the x-axis at some point

- We'll prove this later (Ch. 8)
- First, let's digest the statement of the theorem

Q: Why do we need "continuous"?

A: E.g. $f: [-2, 2] \rightarrow \mathbb{R}$, $f(x) = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases}$

Then $f(-2) < 0 < f(2)$,
but $\nexists c \in [-2, 2]$ s.t. $f(c) = 0$.



Intermediate value theorem (IVT) Suppose $f: [a, b] \rightarrow \mathbb{R}$ is continuous. If $y \in \mathbb{R}$ satisfies $f(a) < y < f(b)$ or $f(a) > y > f(b)$, then $\exists c \in (a, b)$ s.t. $f(c) = y$.

- Continuous functions can't skip a value.

Pf: Case: $f(a) < y < f(b)$. Consider the function $g: [a, b] \rightarrow \mathbb{R}$, $g(x) = f(x) - y$. Then g is continuous, since it's the sum of the continuous functions $f(x)$ and $-y$. Moreover, $g(a) < 0 < g(b)$. So, by Theorem 1, $\exists c \in [a, b]$ s.t.

$$0 = g(c) = f(c) - y \Rightarrow f(c) = y.$$

Note $c \in (a, b)$, since $f(a), f(b) \neq y$.

Case: $f(a) > y > f(b)$. Consider $h: [a, b] \rightarrow \mathbb{R}$, $h(x) = y - f(x)$. Then h is continuous and $h(a) < 0 < h(b)$. Apply Theorem 1 as before. \square

Cor $\forall a > 0$, $\exists \sqrt{a} \in \mathbb{R}$ s.t. $\sqrt{a} > 0$ and $(\sqrt{a})^2 = a$.

- Strictly speaking, we haven't proved this yet
- In fact, we couldn't have proved this with what we know so far!

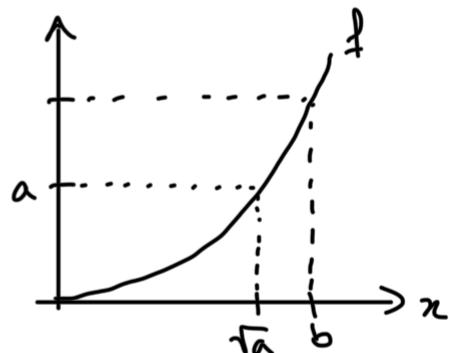
Rmk Recall that both \mathbb{Q}, \mathbb{R} are ordered fields. But the corollary is false for \mathbb{Q} : e.g. $2 \in \mathbb{Q}$ but $\sqrt{2} \notin \mathbb{Q}$. So Theorem 1 can only be true for \mathbb{R} , not \mathbb{Q} , and we need another property of \mathbb{R} to prove it.

- We'll introduce this property next week

Pf (of Cor): Let $a > 0$.

Consider $f(x) = x^2$ and

$$b = \begin{cases} a & \text{if } a > 1 \\ 2 & \text{if } a \leq 1. \end{cases}$$



Then:

① $f: [0, b] \rightarrow \mathbb{R}$ is continuous: since it's a polynomial.

② $f(0) < a < f(b)$

$$\overbrace{0}^{\sim} \quad \overbrace{= \begin{cases} a^2 & \text{if } a > 1 \\ 4 & \text{if } a \leq 1 \end{cases}}^{\sim}$$

So, by the I.V.T., $\exists c \in (0, b)$ s.t. $f(c) = a$. Set

$$\sqrt{a} = c.$$

$$\overbrace{\sqrt{a} > 0}^{\sim}$$

$$\overbrace{(\sqrt{a})^2 = a}^{\sim}$$

□

- Next: 2nd important theorem

Def Let $A \subseteq \mathbb{R}$ and $f: A \rightarrow \mathbb{R}$. We say:

- ① f is bounded above if $\exists M \in \mathbb{R}$ s.t.

$$f(x) \leq M \quad \forall x \in A.$$

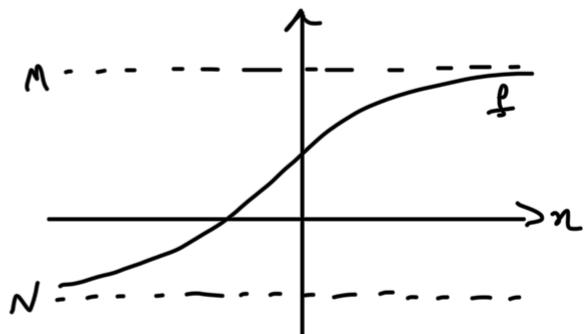
- ② f is bounded below if $\exists N \in \mathbb{R}$ s.t.

$$f(x) \geq N \quad \forall x \in A.$$

- ③ f is bounded if

both ① and ②

are true.



Thm 2 If $f: [a, b] \rightarrow \mathbb{R}$ is continuous, then f is bounded above.

- We'll also prove this later (Ch. 8)

Cor If $g: [a, b] \rightarrow \mathbb{R}$ is continuous, then g is bounded.

Pf: Bounded above: by Theorem 2.

Bounded below: As $-g$ is also continuous

on $[a, b]$, then by Theorem 2, $\exists M \in \mathbb{R}$ s.t.

$$-g(x) \leq M \quad \forall x \in [a, b] \Rightarrow g(x) \geq -M \quad \forall x \in [a, b]$$

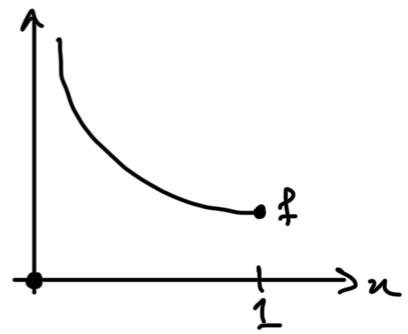
So g is bounded below by $-M$. □

Q: Why do we need "continuous"?

A: E.g. $f: [0, 1] \rightarrow \mathbb{R}$,

$$f(x) = \begin{cases} \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Then f is not bounded above.



Last time: 3 important theorems

① IVT (special case):

$$\left. \begin{array}{l} f: [a,b] \rightarrow \mathbb{R} \text{ continuous} \\ f(a) < 0 < f(b) \end{array} \right\} \Rightarrow \exists c \in [a,b] \text{ s.t. } f(c) = 0$$

② $f: [a,b] \rightarrow \mathbb{R}$ continuous $\Rightarrow f$ is bounded above

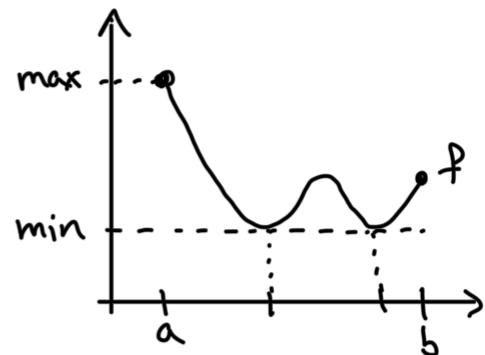
• Next: 3rd important theorem

Def Let $A \subseteq \mathbb{R}$, $f: A \rightarrow \mathbb{R}$, and $y \in \mathbb{R}$. We say:

① y is a global maximum of f if $\exists a \in A$ s.t.
 $f(a) = y$ and $f(x) \leq y \quad \forall x \in A$.

② y is a global minimum of f if $\exists a \in A$ s.t.
 $f(a) = y$ and $f(x) \geq y \quad \forall x \in A$.

③ y is a global extremum of f if it is a global
maximum or minimum.

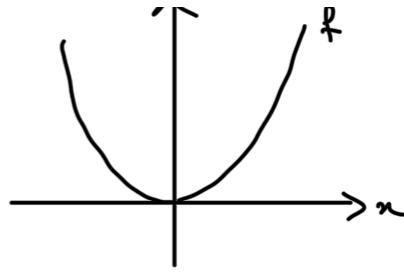


Rmk y is a global max/min of $f \Rightarrow f$ is
bounded above/below by y .

• But " \leq " is false in general.

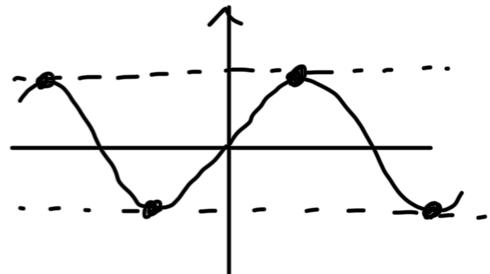
Ex ① $f: \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = x^2$

- Global min $f(0) = 0$
- Bounded below by 0
- Not bounded above
- No global max



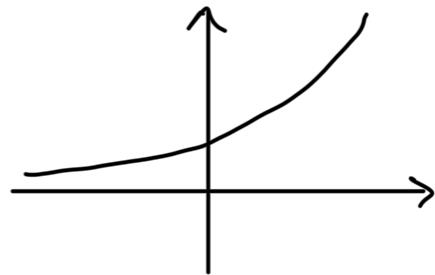
② $g: \mathbb{R} \rightarrow \mathbb{R}$, $g(x) = \sin x$

- Global min $-1 = g(-\frac{\pi}{2}) = g(\frac{3\pi}{2}) = \dots$
- Bounded below by -1
- Global max $1 = g(\frac{\pi}{2}) = g(-\frac{3\pi}{2}) = \dots$
- Bounded above by 1



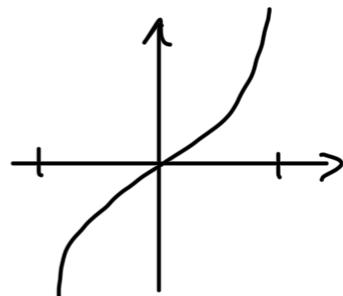
③ $h: \mathbb{R} \rightarrow \mathbb{R}$, $h(x) = e^x$

- Not bounded above
- Bounded below by 0
- No global extrema



④ $t: (-\frac{\pi}{2}, \frac{\pi}{2}) \rightarrow \mathbb{R}$, $t(x) = \tan x$

- Not bounded above
- Not bounded below
- No global extrema



Thm 3 If $f: [a, b] \rightarrow \mathbb{R}$ is continuous, then f has a global maximum.

Extreme value theorem (EVT) If $g: [a, b] \rightarrow \mathbb{R}$ is continuous, then g has a global maximum and

minimum, i.e. $\exists c, d \in [a, b]$ s.t. $g(c) \leq g(x) \leq g(d)$ $\forall x \in [a, b]$.

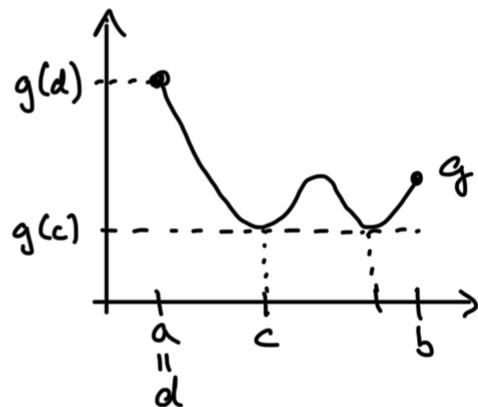
Pf: Global max: by Thm 3.

Global min: As $-g$ is also continuous on $[a, b]$, then by Theorem 3 $\exists c \in [a, b]$ s.t.

$$-g(x) \leq -g(c) \quad \forall x \in [a, b]$$

$$\Rightarrow g(x) \geq g(c) \quad \forall x \in [a, b].$$

So $g(c)$ is a global min of g . \square

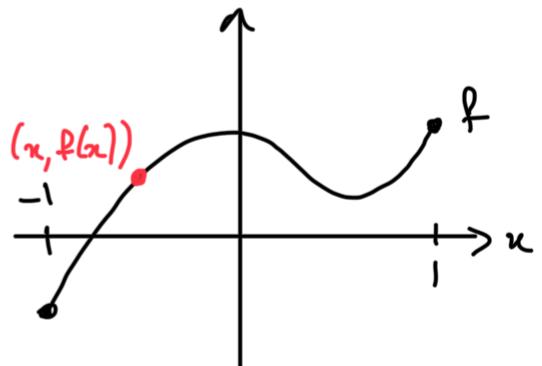


Ex let $f: [-1, 1] \rightarrow \mathbb{R}$ be continuous. Prove that \exists a point on the graph of f that is closest to the origin.

Rmk The plane is the set

$$\mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}.$$

The graph of f is the subset $\{(x, f(x)) : x \in [-1, 1]\}$.



Solution: The distance from a point $(x, y) \in \mathbb{R}^2$ to the origin is $\sqrt{x^2 + y^2}$. So, the distance from a point $(x, f(x))$ on the graph of f to the origin is $\sqrt{x^2 + f(x)^2}$.

let $g: [-1, 1] \rightarrow \mathbb{R}$, $g(x) = \sqrt{x^2 + f(x)^2}$.

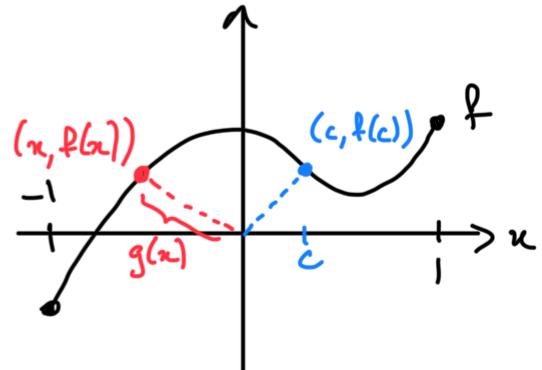
Want: $\exists c \in [-1, 1]$ s.t. $g(c)$ is a global min of g . Then, $(c, f(c))$ is the desired point on the graph of f .

It suffices to prove that g is continuous. Once we know this, then g must have a global min by the EVT.

Claim: $g(x) = \sqrt{x^2 + f(x)^2}$ is continuous on $[-1, 1]$. As f is continuous on $[-1, 1]$, then so is f^2 (by lecture 14). We also know x^2 is continuous on $[-1, 1]$, since it's a polynomial.

- Recall: A polynomial is continuous at any $a \in \mathbb{R}$, by lecture 14
- This implies continuous on $[-1, 1]$, as in lecture 16

Together, the sum $x^2 + f(x)^2$ is continuous (by lecture 14). Therefore $g(x) = \sqrt{x^2 + f(x)^2}$ is continuous, as its the composition of $x^2 + f(x)^2 \geq 0$ and $y \mapsto \sqrt{y}$, which is continuous on $[0, \infty)$ by lecture 15.



□

Last time: 3 important theorems

① IVT (special case):

$$\left. \begin{array}{l} f: [a, b] \rightarrow \mathbb{R} \text{ continuous} \\ f(a) < 0 < f(b) \end{array} \right\} \Rightarrow \exists c \in [a, b] \text{ s.t. } f(c) = 0$$

② $f: [a, b] \rightarrow \mathbb{R}$ continuous $\Rightarrow f$ bounded above

③ EVT (special case):

$$f: [a, b] \rightarrow \mathbb{R} \text{ continuous} \Rightarrow f \text{ has a global max}$$

- Goal: Prove these theorems
- First: A new tool

LEAST UPPER BOUNDS (Ch. 8)

Def let $A \subseteq \mathbb{R}$ and $M, N \in \mathbb{R}$. We say:

① A is bounded above by M if $a \leq M \forall a \in A$.

If also $M \in A$, then M is the maximum of A .

② A is bounded below by N if $a \geq N \forall a \in A$.

If also $N \in A$, then N is the minimum of A .

③ A is bounded if it is bounded above and bounded below.

Ex $A = \{\frac{1}{2}, 1\}$

- maximum is 1 : $a \leq 1 \forall a \in A$ and $1 \in A$
- bounded above by 1

- minimum is $\frac{1}{2}$: $a \geq \frac{1}{2} \forall a \in A$ and $\frac{1}{2} \in A$
- bounded below by $\frac{1}{2}$
- bounded

Ex $\{\frac{1}{n} : n \in \mathbb{N}\}$



- maximum is 1: $\frac{1}{n} \leq 1 \forall n \in \mathbb{N}$
and $1 = \frac{1}{1}$ for $n=1$
- bounded above by 1
- bounded below by 0: $\frac{1}{n} \geq 0 \forall n \in \mathbb{N}$
- no minimum: Suppose $\exists n_0 \in \mathbb{N}$ s.t. $\frac{1}{n} \geq \frac{1}{n_0} \forall n \in \mathbb{N}$.
But $n_0+1 \in \mathbb{N}$ and $\frac{1}{n_0+1} < \frac{1}{n_0}$, a contradiction.

Ex $[0, 1)$



- minimum is 0:
 $x \geq 0 \forall x \in [0, 1)$ and $0 \in [0, 1)$
- bounded below by 0
- no maximum: Suppose $\exists m \in [0, 1)$ s.t. $x \leq m \forall x \in [0, 1)$. Then $\frac{m+1}{2} \in [0, 1)$ but $\frac{m+1}{2} > m$, a contradiction.
- bounded above by 1: $x \leq 1 \forall x \in [0, 1)$
- bounded above by 10: $x \leq 10 \forall x \in [0, 1)$.
- This set has many upper bounds: any $M \geq 1$
- But $M=1$ is the best we can do...

Def let $A \subseteq \mathbb{R}$ and $L \in \mathbb{R}$. We say:

① L is the least upper bound (or supremum) of A if:

- L is an upper bound of A , and
- if M is an upper bound of A then $L \leq M$.

When this is true, we write $L = \sup A$.

② L is the greatest lower bound (or infimum) of A if:

- L is a lower bound of A , and
- if M is a lower bound of A then $L \geq M$.

When this is true, we write $L = \inf A$.

Ex $[0, 1)$

① $\sup [0, 1) = 1$:

- 1 is an upper bound: previous example.
- Suppose M is another upper bound. Want: $M \geq 1$.
Suppose not: $M < 1$. Set $x = \max \left\{ \frac{1+M}{2}, 0 \right\}$.
Then $x \in [0, 1)$ but $x > M$, a contradiction.

② $\inf [0, 1) = 0$.

- 0 is a lower bound: previous example.
- Suppose M is another lower bound. Then
 $M \leq x \wedge x \in [0, 1)$. In particular, for $x = 0$
we have $M \leq 0$.

Lem let $A \subseteq \mathbb{R}$. If $\sup A = L$ and $\sup A = M$, then $L = M$.

Pf: Suppose $\sup A = L$ and $\sup A = M$. Then

$$\begin{aligned} \sup A = L \Rightarrow L &\text{ is an upper bound of } A \\ M &\text{ is a least upper bound of } A \end{aligned} \quad \left. \begin{array}{l} \\ \end{array} \right\} \Rightarrow M \leq L$$

Similarly,

$$\begin{aligned} \sup A = M \Rightarrow M &\text{ is an upper bound of } A \\ L &\text{ is a least upper bound of } A \end{aligned} \quad \left. \begin{array}{l} \\ \end{array} \right\} \Rightarrow L \leq M$$

Together, we conclude $L = M$. \square

- Not every set has a least upper bound:

Ex ① $\sup [0, \infty)$ does not exist: Suppose $\sup [0, \infty) = L \in \mathbb{R}$. Then L is an upper bound for $[0, \infty)$. But $L+1 \in [0, \infty)$ and $L+1 > L$, a contradiction.

② $\sup \emptyset$ does not exist: Any $M \in \mathbb{R}$ is an upper bound, since " $x \leq M \wedge x \in \emptyset$ " is vacuously true. So \emptyset a least upper bound.

The real numbers \mathbb{R} satisfy the least upper bound property: $\forall A \subseteq \mathbb{R}$ that is nonempty

and bounded above, \exists a least upper bound $\sup A \in \mathbb{R}$.

Rmk ① \mathbb{Q} and \mathbb{R} are both ordered fields, but \mathbb{Q} does not satisfy the least upper bound property. Eg. $A = \{x \in \mathbb{Q} : x^2 \leq 2\}$ is:

- nonempty: $0^2 \leq 2 \Rightarrow 0 \in A$
- bounded above: by $\frac{3}{2}$, since $x > \frac{3}{2} \Rightarrow x^2 > \frac{9}{4} > 2$.
So $x \in A \Rightarrow x^2 \leq 2 \Rightarrow x \leq \frac{3}{2}$.

But $\sup A = \sqrt{2} \notin \mathbb{Q}$.

② It turns out that \mathbb{R} is the unique ordered field satisfying the least upper bound property.

- We won't prove this in this course. But this is what makes \mathbb{R} special.

- Midterm 1 distribution and solutions are on Canvas
 - There are letter grade cut-offs in the syllabus
 - I will replace your lowest midterm score with your final exam score, if it benefits your grade
-

Last time:

$$\textcircled{1} \quad \sup A = L \iff \begin{cases} \cdot a \leq L \quad \forall a \in A \\ \cdot M \text{ upper bound} \Rightarrow L \leq M \end{cases}$$

\textcircled{2} least upper bound property: $\forall A \subseteq \mathbb{R}$ nonempty and bounded above, $\exists \sup A \in \mathbb{R}$.

Rmk \mathbb{R} also satisfies the greatest lower bound property: $\forall B \subseteq \mathbb{R}$ nonempty and bounded below, $\exists \inf B \in \mathbb{R}$. Hw: least upper bound property \Rightarrow greatest lower bound property.

Ex let $A, B \subseteq \mathbb{R}$ be nonempty and bounded.

Suppose $a \leq b \quad \forall a \in A$ and $\forall b \in B$.



Prove that $\sup A \leq \inf B$.

Pf: As $A, B \subseteq \mathbb{R}$ are nonempty and bounded, we know $\sup A$ and $\inf B$ exist.

Note that $\forall b \in B$, b is an upper bound

for A , since $a \leq b \forall a \in A$. As $\sup A$ is the least upper bound, we have $\sup A \leq b$.

This shows that $\sup A \leq b, \forall b \in B$. So $\sup A$ is a lower bound for B . As $\inf B$ is the greatest lower bound, we must have $\sup A \leq \inf B$. \square

Prop (Archimedean property) $\forall x \in \mathbb{R}, \exists n \in \mathbb{N}$ s.t. $n > x$.

Pf: Suppose not: $\exists x_0 \in \mathbb{R}$ st. $\forall n \in \mathbb{N}, n \leq x_0$.

Then $\mathbb{N} \subseteq \mathbb{R}$ would satisfy:

- $\mathbb{N} \neq \emptyset$: since $1 \in \mathbb{N}$
- \mathbb{N} is bounded above: by x_0

So, by the least upper bound property, $\exists L \in \mathbb{R}$ s.t. $L = \sup \mathbb{N}$. As $L-1 < L$, then $L-1$ cannot be an upper bound for \mathbb{N} . So $\exists n_0 \in \mathbb{N}$ s.t. $n_0 > L-1$.

(\because Recall: $L-1$ upper bound $\Leftrightarrow n \leq L-1 \forall n \in \mathbb{N}$)

But then $n_0 + 1 \in \mathbb{N}$ and $n_0 + 1 > L$, so L is not an upper bound — a contradiction. \square

Cor $\forall \varepsilon > 0, \exists n \in \mathbb{N}$ s.t. $\frac{1}{n} < \varepsilon$.

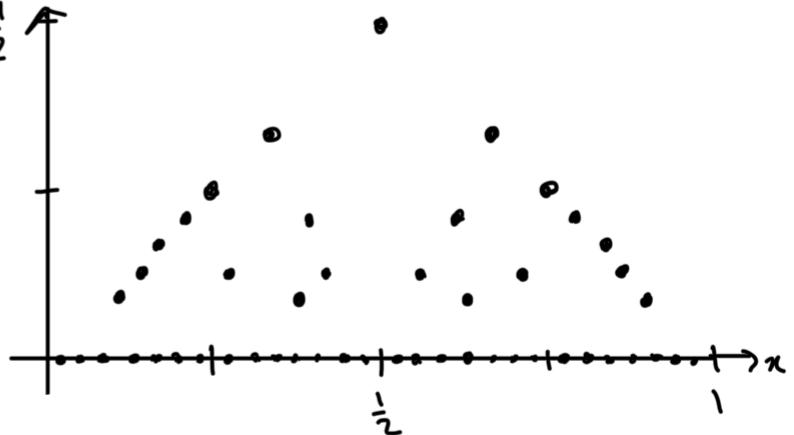
Pf: Let $\varepsilon > 0$. Then $\frac{1}{\varepsilon} \in \mathbb{R}$, so by the Archimedean Property $\exists n \in \mathbb{N}$ s.t. $n > \frac{1}{\varepsilon}$. So $\frac{1}{n} < \varepsilon$. \square

Ex let $f: (0, 1) \rightarrow \mathbb{R}$, $f(x) = \begin{cases} 0 & \text{if } x \notin \mathbb{Q} \\ \frac{1}{q} & \text{if } x = \frac{p}{q} \text{ in} \\ & \text{lowest terms} \end{cases}$

Prove that $\lim_{x \rightarrow a} f(x) = 0$ for any $a \in (0, 1)$.

Pf: let $a \in (0, 1)$.

Fix $\varepsilon > 0$. By the Corollary, $\exists n \in \mathbb{N}$ s.t. $\frac{1}{n} < \varepsilon$. The only numbers x for which $|f(x) - 0| \geq \varepsilon$ are rational numbers with denominator $\leq n$:



numbers with denominator $\leq n$:

$$B_n = \left\{ \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \dots, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n} \right\}$$

So

$$|f(x) - 0| \geq \varepsilon \Rightarrow x \in B_n.$$

Note that B_n is a finite set. Therefore

$$\delta = \min \{|x-a| : x \in B_n \text{ and } x \neq a\}$$

exists and $\delta > 0$. Then

$$0 < |x-a| < \delta \Rightarrow x \notin B_n \Rightarrow |f(x) - 0| < \varepsilon.$$

So $\lim_{x \rightarrow a} f(x) = 0$. \square

Rmk So, for $a \in (0,1)$ we have

- $a \notin \mathbb{Q} \Rightarrow f$ is continuous at a
- $a \in \mathbb{Q} \Rightarrow f$ is not continuous at a

Thm 1 If $f: [a, b] \rightarrow \mathbb{R}$ is continuous and $f(a) < 0 < f(b)$, then $\exists c \in [a, b]$ s.t. $f(c) = 0$.

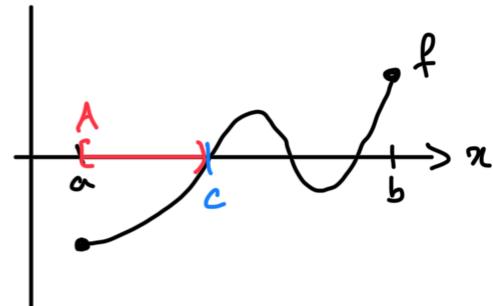
Pf: Define

$$A = \{x_0 \in [a, b] : f(x) < 0 \quad \forall x \text{ s.t. } a \leq x \leq x_0\}$$

Steps:

① $\sup A$ exists

② $c = \sup A$ satisfies $f(c) = 0$.



Step ①: Want $\exists \sup A \in \mathbb{R}$.

- $A \neq \emptyset$: Note that $a \in A$, since $f(a) < 0$, and so " $f(x) < 0 \quad \forall x \text{ s.t. } a \leq x \leq a$ " is true.
- A is bounded above: by b , since $x_0 \leq b \quad \forall x_0 \in A$.

So, by the least upper bound property, $\exists \sup A$ in \mathbb{R} .

Step ②: Set $c = \sup A$. Note that $c \geq a$ since $a \in A$ and $c \leq b$ since b is an upper bound, so $c \in [a, b]$. Want: $f(c) = 0$. Suppose not: $f(c) \neq 0$. Then $f(c) > 0$ or $f(c) < 0$.

Case: $f(c) > 0$. Recall from lecture 15:

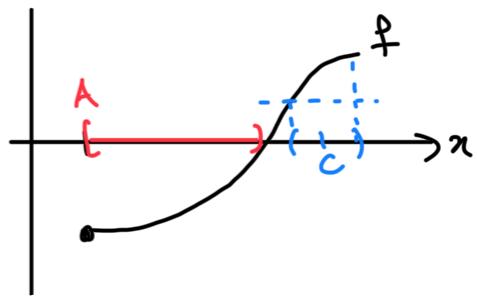
- g continuous at c
 - $g(c) > 0$
- $\Rightarrow \exists \delta > 0 \text{ s.t. } g(x) > 0 \quad \forall x \in (c - \delta, c + \delta)$

Applying this fact to $g=f$, we get: $\exists \delta > 0$ s.t. $f(x) > 0 \forall x \in (c-\delta, c+\delta)$. But then $c-\delta$ is an upper bound for A :

$$x_0 \in A \Rightarrow f(x) < 0 \quad \forall x \text{ s.t. } a \leq x \leq x_0$$

$$\Rightarrow x_0 \notin (c-\delta, c+\delta)$$

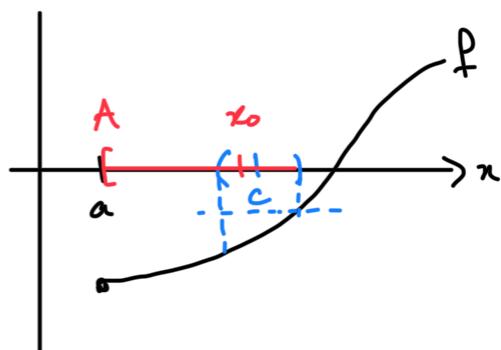
$$\Rightarrow x_0 \leq c-\delta$$



This contradicts that $c = \sup A$ is the least upper bound.

Case: $f(c) < 0$. Applying the fact from lecture 15 to $g=-f$, we get: $\exists \delta > 0$ s.t. $f(x) < 0 \forall x \in (c-\delta, c+\delta)$.

On the other hand, $\exists x_0 \in A$ s.t. $x_0 \in (c-\delta, c]$; otherwise, we would have $x_0 \leq c-\delta \forall x_0 \in A$ and then c would not be the least upper bound of A .



Together, we see that $c + \frac{\delta}{2} \in A$:

$$\left. \begin{array}{l} x_0 \in A \Rightarrow f(x) < 0 \quad \forall x \in [a, x_0] \\ f(x) < 0 \quad \forall x \in (c-\delta, c+\delta) \end{array} \right\} \Rightarrow \begin{array}{l} f(x) < 0 \\ \forall x \in [a, c + \frac{\delta}{2}] \end{array}$$

This contradicts that $c = \sup A$ is an upper bound of A .

In both cases we have a contradiction, and so we must have $f(c) = 0$. \square

Thm 2 If $f: [a, b] \rightarrow \mathbb{R}$ is continuous, then f is bounded above.

- For the proof, we'll need:

Lem If g is continuous at c , then $\exists \delta > 0$ s.t. $g: (c-\delta, c+\delta) \rightarrow \mathbb{R}$ is bounded above.

Pf: As $\lim_{n \rightarrow c} g(n) = g(c)$, $\exists \delta > 0$ s.t.

$$\begin{aligned} |x - c| < \delta &\Rightarrow |g(x) - g(c)| < 1 \\ &\Rightarrow g(x) - g(c) < 1 \\ &\Rightarrow g(x) < g(c) + 1 \end{aligned}$$

So $g: (c-\delta, c+\delta) \rightarrow \mathbb{R}$ is bounded above by $g(c) + 1$. \square

Exc If $\lim_{n \rightarrow c^-} g(n) = g(c)$, then $\exists \delta > 0$ s.t.

$g: (c-\delta, c] \rightarrow \mathbb{R}$ is bounded above.

- Just replace " $|x - c| < \delta$ " by " $-\delta < x - c \leq 0$ "

Pf (of Thm 2): Define

$$A = \{x_0 \in [a, b] : f: [a, x_0] \rightarrow \mathbb{R} \text{ is bounded above}\}.$$

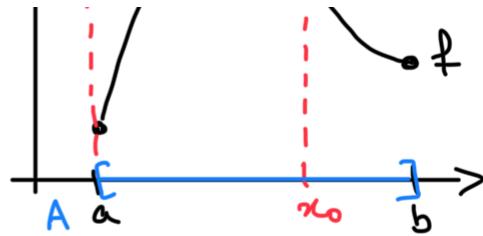
Steps:

- $\sup A$ exists



$$\textcircled{2} \quad \sup A = b$$

$$\textcircled{3} \quad b \in A$$



Step ①: Want $\exists \sup A \in \mathbb{R}$.

- $A \neq \emptyset$: Note $a \in A$, since $f: \{a\} \rightarrow \mathbb{R}$ is bounded above by $f(a)$.
- A is bounded above: by b , since $x_0 \leq b \quad \forall x_0 \in A$.

So, by the least upper bound property,
 $\exists \sup A \in \mathbb{R}$.

Step ②: Want $\sup A = b$. Suppose not:
 $\sup A \neq b$. As b is an upper bound for A , we already know $\sup A \leq b$. So $\sup A < b$. As f is continuous at $c = \sup A$, $\exists \delta > 0$ s.t.

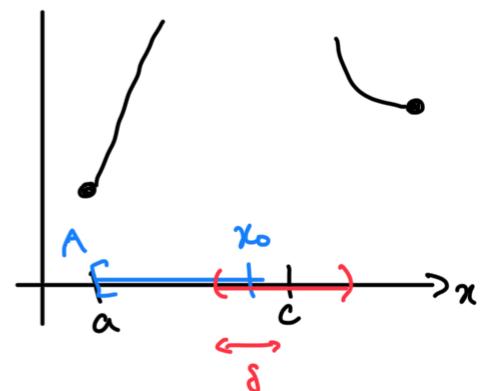
$f: (c-\delta, c+\delta) \rightarrow \mathbb{R}$ is bounded above by the previous lemma. As $c-\delta < c$, then

$c-\delta$ is not an upper bound for A , so $\exists x_0 \in A$ s.t.

$x_0 > c-\delta$. Together, we see that $c + \frac{\delta}{2} \in A$:

$$\left. \begin{array}{l} x_0 \in A \Rightarrow f(x) \leq M_1, \quad \forall x \in [a, x_0] \\ f(x) \leq M_2 \quad \forall x \in (c-\delta, c+\delta) \end{array} \right\} \Rightarrow \begin{array}{l} f(x) \leq \max\{M_1, M_2\} \\ \forall x \in [a, c + \frac{\delta}{2}] \end{array}$$

This contradicts that $c = \sup A$ is an upper bound



for A. So $\sup A = b$.

Step ③ : Next time.