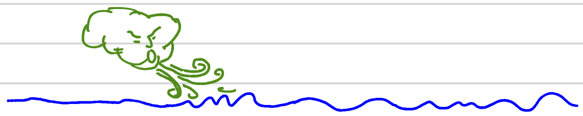
Lecture 10: Wave equation with forcing, and some Review

Reading: Stechmann 4.3.4, 5.1



$$(*) \qquad u_{tt} - c^2 u_{xx} = F(x,t) \qquad \left(\text{or } \underbrace{(\partial_t^2 - c^2\partial_x^2)}_{\Box} u = F\right)$$
$$u(x,0) = 0$$
$$u_t(x,0) = 0$$

Recall Duhamel's principle: the solution can be constructed using a family of unforced wave equations with different initial conditions.

The Family: $\{U(x,t;s)\}_{s=0}^{\infty}$

where for a given s (family member),

$$(\partial_t^2 - c^2\partial_x^2) U(x,t;s) = 0 \qquad t > s$$

$$U(x, t=s\,;\, s) = 0$$
$$U_t(x, t=s\,;\, s) = F(x,s)$$

We already know how to solve this:

$$U(x,t;s) = \frac{1}{2c}\int_{x-c(t-s)}^{x+c(t-s)} F(x',s)\,dx'$$

Finally, $u(x,t)$ in $(*)$ is given by:

$$u(x,t) = \int_{s=0}^{t} U(x,t;s)\,ds$$
$$= \frac{1}{2c}\int_{s=0}^{t}\int_{x-c(t-s)}^{x+c(t-s)} F(x',s)\,dx'\,ds$$

Let's check this:

First, recall that $\frac{d}{dt}\int_0^t g(s,t)\,ds = g(s,t)\big|_{s=t} + \int_0^t g_t(s,t)\,dt$

e.g. $\frac{d}{dt}\int_0^t (t+s)\,ds = \frac{d}{dt}\left(ts+\frac{s^2}{2}\right)\Big|_0^t = \frac{d}{dt}\left(t^2+\frac{t^2}{2}\right) = 2t+t$

while $(s+t)\big|_{s=t} + \int_0^t 1\,ds = 2t+t$

OK:

$$u_t = \underbrace{U(x,t;s)\big|_{s=t}}_{0} + \int_0^t U_t(x,t;s)\,ds$$

$$u_{tt} = \underbrace{U_t(x,t;s)\big|_{s=t}}_{F(x,t)} + \int_0^t U_{tt}(x,t;s)\,ds$$

$$F(x,t) + \int_0^t c^2 U_{xx}(x,t;s)\,ds$$

$$= F(x,t) + c^2\partial_{xx}\int_0^t U(x,t;s)\,ds$$

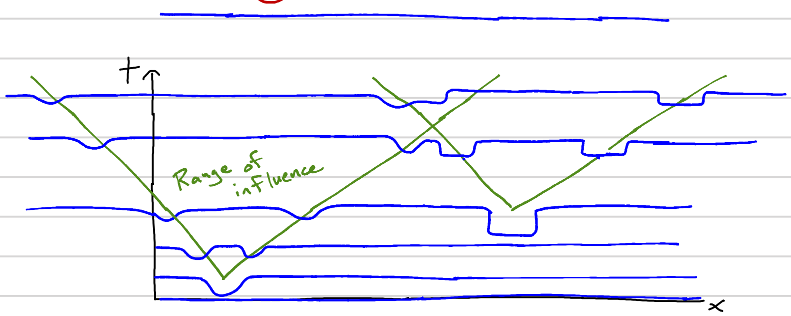
$$= F(x,t) + c^2 u_{xx}$$

Sometimes we define $\tau = t-s$, and then

$$u(x,t) = \frac{1}{2c}\int_0^t \int_{x-c\tau}^{x+c\tau} F(x', t-\tau)\,dx'\,d\tau$$

☆ Range of influence, Part II

(F(x,t)!)



Midterm Review, Part I

The midterm will be split into 3 questions, one on each of the 3 PDEs we've studied so far.

Plus some T/F.

① Heat equation / Diffusion equation

$$u_t = k\nabla^2 u$$

- Total heat is conserved

$$\frac{d}{dt}\int_D u\,dV = \int_D u_t\,dV = \int_D k\nabla^2 u\,dV = \int_D k\nabla\cdot(\nabla u)\,dV$$

$$= \int_{\partial D} k\hat{n}\cdot\nabla u\,dS = 0 \text{ as } |\partial D| \to \infty \quad (\text{if } u \searrow 0 \text{ fast enough})$$

- Fundamental solution

$$(*)\begin{cases} \Phi_t = k\nabla^2\Phi & \vec{x}\in\mathbb{R}^n,\ t>0. \\ \Phi(\vec{x},0) = \delta(\vec{x}) \\ \Phi \to 0 \text{ as } |\vec{x}| \to 0. \end{cases}$$

Solution: $\Phi(\vec{x},t) = \frac{1}{(4\pi kt)^{n/2}} e^{-|\vec{x}|^2/4kt}$

- IVP: $\begin{cases} u_t = k\nabla^2 u \\ u(\vec{x},0) = g(\vec{x}) \\ u \to 0 \text{ as } |\vec{x}| \to \infty \end{cases}$

Solution: $u(\vec{x},t) = g * \Phi = \int_{\mathbb{R}^n} g(\vec{y})\,\Phi(\vec{x}-\vec{y},t)\,dV_y$

e.g. In 1D, $u(x,t) = \int_{-\infty}^{\infty} g(y)\cdot\frac{1}{\sqrt{4\pi kt}} e^{-(x-y)^2/4kt}\,dy$

- Speed of information: Infinite!

• Forced problem:

$$(*) \quad \begin{cases} u_t - \kappa \nabla^2 u = f(\vec{x},t) & \vec{x}\in\mathbb{R}^n,\ t>0 \\ u(\vec{x},0) = 0 \\ u \to 0 \quad \text{as } |\vec{x}|\to\infty \end{cases}$$

$\longrightarrow$ Duhamel's principle.

$$u(\vec{x},t) = \int_0^t v(\vec{x},t;s)\, ds \qquad (*')$$

where
$$\begin{cases} (\partial_t - \kappa\nabla^2)\, v(\vec{x},t;s) = 0 & \vec{x}\in\mathbb{R}^n,\ t>s \\ v(\vec{x},s;s) = f(\vec{x},s) \\ v \to 0 \quad \text{as } |\vec{x}|\to\infty. \end{cases}$$

Proof? Just plug $(*')$ into $(*)$ and use the above.

Then, finally, $\quad u(\vec{x},t) = \displaystyle\int_0^t \int_{\mathbb{R}^n} f(\vec{y},s)\, \Phi(\vec{x}-\vec{y}, t-s)\, dV_{\vec{y}}\, ds$

• Forced IVP?

$$\begin{cases} u_t = \kappa\nabla^2 u + f(\vec{x},t) \\ u(\vec{x},0) = g(\vec{x}) \\ u \to 0 \quad \text{as } |\vec{x}|\to\infty \end{cases}$$

The PDE is linear, so let $u = u_1 + u_2$.

u_1 satisfies $(*)$ above, u_2 satisfies the previous IVP.