Recall Differentiation laws

① $(cf)'(a) = cf'(a) \quad \forall c \in \mathbb{R}$

② $(f+g)'(a) = f'(a) + g'(a)$

③ $(fg)'(a) = f'(a)g(a) + f(a)g'(a)$

④ $\left(\frac{f}{g}\right)'(a) = \dfrac{f'(a)g(a) - f(a)g'(a)}{g(a)^2}$

⑤ $(g \circ f)'(a) = g'(f(a)) \cdot f'(a)$

Thm (Chain rule) Let $I, J \subseteq \mathbb{R}$ be open intervals, $f: I \to J$ and $g: J \to \mathbb{R}$ be functions, and $a \in I$. If f is differentiable at a and g is differentiable at $f(a)$, then $g \circ f$ is differentiable at a and

$$(g \circ f)'(a) = g'(f(a)) \cdot f'(a).$$

Idea: For $h \neq 0$,

$$\frac{g(f(a+h)) - g(f(a))}{h} = \underbrace{\frac{g(f(a+h)) - g(f(a))}{f(a+h) - f(a)}} \cdot \underbrace{\frac{f(a+h) - f(a)}{h}}$$

$= \dfrac{g(f(a)+k) - g(f(a))}{k}$ $\quad \to f'(a)$ as $h \to 0$

with $k(h) = f(a+h) - f(a) \to 0$ as $h \to 0$

$\longrightarrow g'(f(a))$ as $h \to 0$

Problem What if $k = 0$? In fact, we could have $k(h) = 0$ infinitely often near $h = 0$! E.g.

- $f(x) = \text{constant} \Rightarrow k(h) = f(a+h) - f(a) = 0 \quad \forall h$

  (Well, maybe this isn't really a problem, since the other factor $f'(0)$ is 0...)
- $f(x) = x^2 \sin\frac{1}{x}$, $a=0 \Rightarrow k(h) = h^2 \sin\frac{1}{h}$

Pf: For $h \neq 0$,

$$\frac{g(f(a+h)) - g(f(a))}{h} = \phi(h) \cdot \frac{f(a+h) - f(a)}{h} \qquad (\star)$$

where

$$\phi(h) = \begin{cases} \dfrac{g(f(a+h)) - g(f(a))}{f(a+h) - f(a)} & \text{if } f(a+h) \neq f(a) \\ g'(f(a)) & \text{if } f(a+h) = f(a) \end{cases}$$

Note that in both cases, the equation $(\star)$ holds:

- If $f(a+h) \neq f(a)$: $f(a+h) - f(a)$ cancels out, as before
- If $f(a+h) = f(a)$: $\text{RHS}(\star) = g'(f(a)) \cdot \dfrac{f(a+h) - f(a)}{h} = 0$

  $\text{LHS}(\star) = \dfrac{g(f(a+h)) - g(f(a))}{h} = 0$

Claim: $\phi(h)$ is continuous at $h=0$, i.e.

$\lim_{h \to 0} \phi(h) = g'(f(a))$. Fix $\varepsilon > 0$. Want: $\exists \delta > 0$ s.t.

$$|h| < \delta \Rightarrow \underbrace{|\phi(h) - g'(f(a))|}_{} < \varepsilon$$

$$= \begin{cases} \dfrac{g(f(a+h)) - g(f(a))}{f(a+h) - f(a)} - g'(f(a)) \\ g'(f(a)) - g'(f(a)) = 0 \quad \text{already } < \varepsilon! \end{cases}$$

So it suffices to show: $\exists\, \delta > 0$ s.t.

$$|h| < \delta,\ f(a+h) \neq f(a) \implies \left| \frac{g(f(a+h)) - g(f(a))}{f(a+h) - f(a)} - g'(f(a)) \right| < \varepsilon$$

We know:

① g is differentiable at $f(a)$

$$\implies \lim_{k \to 0} \frac{g(f(a)+k) - g(f(a))}{k} = g'(f(a))$$

$\implies \exists\, \delta_1 > 0$ s.t. $|k| < \delta_1$ implies

$$\left| \frac{g(f(a)+k) - g(f(a))}{k} - g'(f(a)) \right| < \varepsilon$$

② f is differentiable at a

$\implies f$ is continuous at a $\implies \lim_{h \to 0} f(a+h) = f(a)$

$\implies \exists\, \delta_2 > 0$ s.t. $|h| < \delta_2$ implies $|f(a+h) - f(a)| < \delta_1$

Altogether,

$$|h| < \delta_2 \implies |\underbrace{f(a+h) - f(a)}_{\text{Take } = k \text{ in } ①}| < \delta_1$$

$$\implies \left| \frac{g(f(a+h)) - g(f(a))}{f(a+h) - f(a)} - g'(f(a)) \right| < \varepsilon$$

as desired. This finishes the claim.

Recall: for $h \neq 0$,

$$\frac{g(f(a+h)) - g(f(a))}{h} = \phi(h) \cdot \frac{f(a+h) - f(a)}{h}$$

By the claim,
$$\lim_{h\to 0} \phi(h) = g'(f(a)).$$
As f is differentiable at a,
$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = f'(a).$$
Therefore, by the limit law for products,
$$\begin{array}{ccc} \displaystyle\lim_{h\to 0} \frac{g(f(a+h))-g(f(a))}{h} & = & \displaystyle\lim_{h\to 0} \left[\phi(h) \cdot \frac{f(a+h)-f(a)}{h} \right] \\ \| & & \| \\ (g\circ f)'(a) & & g'(f(a)) \cdot f'(a) \end{array} \qquad \square$$

<u>Ex</u> (a.) Prove that $f:(-1,1)\to\mathbb{R}$, $f(x)=\sqrt{1-x^2}$ is differentiable, and find its derivative.

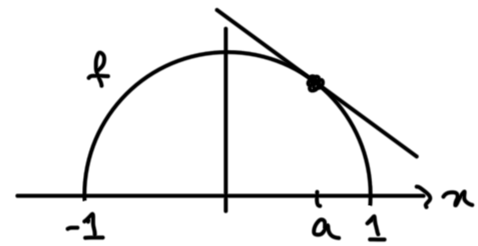
<u>Pf</u>: Write
$$f = g\circ h, \quad \text{where } g(y)=\sqrt{y}, \ h(x) = 1-x^2$$
Fix $a\in\mathbb{R}$.

- $h:(-1,1)\to(0,\infty)$ is differentiable at a since h is a polynomial, and $h'(a) = -2a$
- $g:(0,\infty)\to\mathbb{R}$ is differentiable at $y=1-a^2>0$ by Hw 8, and $g'(y) = \frac{1}{2\sqrt{y}}$

So, by the chain rule,
$$f'(a) = (g\circ h)'(a) = g'(h(a))\cdot h'(a)$$

$$= \frac{1}{2\sqrt{1-a^2}} \cdot (-2a) = \frac{-a}{\sqrt{1-a^2}}. \qquad \square$$

(b.) Find a formula for the tangent line to the graph of f at $a \in (-1,1)$.



$$L(x) = f(a) + f'(a)(x-a)$$

$$= \sqrt{1-a^2} - \frac{a}{\sqrt{1-a^2}}(x-a)$$

$$= -\frac{a}{\sqrt{1-a^2}}x + \frac{1-a^2+a^2}{\sqrt{1-a^2}} = -\frac{a}{\sqrt{1-a^2}}x + \frac{1}{\sqrt{1-a^2}}$$

(c.) Prove that this tangent line intersects the unit circle $\{(x,y) \in \mathbb{R}^2 : x^2+y^2=1\}$ exactly once.

<u>Pf</u>: Note that

$L(x)$ intersects the unit circle $\iff$ $(x, L(x))$ satisfies $x^2 + L(x)^2 = 1$.

$$x^2 + L(x)^2 = x^2 + \left(\frac{-ax+1}{\sqrt{1-a^2}}\right)^2$$

$$= x^2 + \frac{a^2x^2 - 2ax + 1}{1-a^2}$$

$$= \frac{(1-\cancel{a^2})x^2 + \cancel{a^2x^2} - 2ax + 1}{1-a^2}$$

So $\quad 1 = x^2 + L(x)^2 = \dfrac{x^2 - 2ax + 1}{1-a^2}$

$$\iff x^2 - 2ax + 1 = 1 - a^2$$

$$\iff 0 = x^2 - 2ax + a^2 = (x-a)^2$$

$$\iff \quad x = a. \qquad \square$$