

Problem 1

Prove the following statements:

1. there exists a number $1 < x < 2$ that solves the equation $x^2 - x - 1 = 0$.
 2. There exists a number $x \in \mathbb{R}$ that solves the equation $x^5 - x + 1 = 0$.
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Solution: Recall Intermediate Value theorem : if $f : [a, b] \rightarrow \mathbb{R}$ is continuous, and $f(a) < 0 < f(b)$, then $\exists c \in [a, b]$ s.t. $f(c) = 0$

1. From lecture, we know that since $f(x) = x^2 - x - 1$ is a polynomial function, it's continuous on \mathbb{R} .
Noticing that $f(1) = -1$, $f(2) = 1$, by IVT, there exists $x \in [1, 2]$ such that $f(x) = 0$. Specifically, since $f(1) \neq 0$, $f(2) \neq 0$, $x \in (1, 2)$. ■
2. Similarly, we know that since $f(x) = x^5 - x + 1$ is a polynomial function, it's continuous on \mathbb{R} .
Noticing that $f(-2) = -29 < 0$, $f(-1) = 1 > 0$, by IVT, there exists $x \in [-2, -1]$ such that $f(x) = 0$. Specifically, since $f(-2) \neq 0$, $f(-1) \neq 0$, $x \in (-2, -1) \in \mathbb{R}$. ■

Problem 2

Let $a < b$ be numbers and $f : [a, b] \rightarrow \mathbb{R}$ be a function. We say that $x \in [a, b]$ is a fixed point for f if $f(x) = x$. Prove that if f is continuous and $f(x) \in [a, b]$ for all $x \in [a, b]$, then f has a fixed point.

Solution:

Consider an auxiliary function $g : [a, b] \rightarrow \mathbb{R}$, $g(x) = f(x) - x$. Since f and x are both continuous functions, g is also continuous.

Thus the existence of a fixed point for f is equivalent to the existence of a root for g .

Noticing:

$$g(a) = f(a) - a; \quad g(b) = f(b) - b, \quad (1)$$

and that

$$f(x) \in [a, b] \forall x \in [a, b] \Leftrightarrow f(a) \geq a; f(b) \leq b, \quad (2)$$

we have:

$$g(a) \geq 0; \quad g(b) \leq 0. \quad (3)$$

In the cases where $g(a) = 0$ or $g(b) = 0$, we have $f(a) = a$ or $f(b) = b$, respectively.

In the rest of the cases, by IVT, $\exists c \in [a, b]$, s.t. $g(c) = 0$. i.e. $f(c) = c$.

Therefore, $\exists x \in [a, b]$, s.t. $f(x) = x$, and thus proving the existence of a fixed point for f . ■

Problem 3

Let $f : [0, 1] \rightarrow \mathbb{R}$ be a continuous function such that $f(0) = f(1)$. Prove that there exists $x \in [0, \frac{1}{2}]$ such that $f(x) = f(x + \frac{1}{2})$. Hint: consider the function $g(x) = f(x) - f(x + \frac{1}{2})$. Is it possible for $g(0)$ and $g(\frac{1}{2})$ to both be positive?

Solution:

Consider the suggested function $g(x) = f(x) - f(x + \frac{1}{2})$. Since f is continuous, $g : [0, 1] \rightarrow \mathbb{R}$ is also continuous. Noticing

$$\begin{aligned} g(0) &= f(0) - f\left(\frac{1}{2}\right) = f(1) - f\left(\frac{1}{2}\right), \\ g\left(\frac{1}{2}\right) &= f\left(\frac{1}{2}\right) - f(1) = f\left(\frac{1}{2}\right) - f(0), \end{aligned} \tag{4}$$

and that

$$\begin{aligned} g(0) + g\left(\frac{1}{2}\right) &= f(1) - f\left(\frac{1}{2}\right) + f\left(\frac{1}{2}\right) - f(1) = 0 \\ \Rightarrow g(0) &= -g\left(\frac{1}{2}\right) \end{aligned} \tag{5}$$

So $g(0)$ and $g(\frac{1}{2})$ cannot both be positive. i.e.

$$\begin{aligned} g(0) \geq 0, g\left(\frac{1}{2}\right) &\leq 0, \\ \text{or, } g(0) \leq 0, g\left(\frac{1}{2}\right) &\geq 0. \end{aligned} \tag{6}$$

In the special cases where $g(0) = 0$ or $g(\frac{1}{2}) = 0$, we have $f(0) = f(\frac{1}{2})$ or $f(\frac{1}{2}) = f(1)$, respectively.

In the rest of the cases, IVT implies that there exists $x \in [0, \frac{1}{2}]$ such that $g(x) = 0$, i.e. $f(x) = f(x + \frac{1}{2})$. ■

Problem 4

For each of the following functions $f : [-1, 1] \rightarrow \mathbb{R}$, find all global extrema and find the points $x \in [-1, 1]$ at which f attains these extrema.

1.
$$f(x) = \begin{cases} 1 - x & \text{if } x \geq 0 \\ 1 + x & \text{if } x < 0. \end{cases} \quad (7)$$

2.
$$f(x) = \begin{cases} 1 - x & \text{if } x \geq 0 \\ -1 - x & \text{if } x < 0. \end{cases} \quad (8)$$

3.
$$f(x) = \begin{cases} 1 - x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases} \quad (9)$$

Problem 5

Let $h > 0$. Prove that there is a point on the parabola

$$\{(x, x^2) \in \mathbb{R}^2 : -10 \leq x \leq 10\}, \quad (10)$$

that is closest to the point $(0, h)$.

Solution:

Problem 6

Let $a < b$ be numbers and $f, g, h : [a, b] \rightarrow \mathbb{R}$ be functions.

1. Prove that if f is continuous, then $|f|$ has a global maximum. Given a continuous function f we define $\|f\|$ to be equal to this value. (i.e. the global maximum of $|f|$).
 2. Prove that if g is continuous, then $\|cg\| = |c| \cdot \|g\|$ for any $c \in \mathbb{R}$.
 3. Prove that if g and h are continuous, then $\|g + h\| \leq \|g\| + \|h\|$.
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