Math 415
Homework 1

Problem 1. Write each of the following as a first-order autonomous system of ODEs.

(a) $\ddot{x} + t\sqrt{1+x^2} = 0$.
(b) $\dddot{y} + y\ddot{y} = 3$.

Problem 2. Consider a particle of mass $m > 0$ and charge $q \neq 0$ traveling in 3 dimensions under the influence of a time-dependent magnetic field of magnitude $B(t)$, pointing in the direction of the z-axis. The position vector $\mathbf{r}(t)$ of the particle at time t is governed by the equation

$$m\ddot{\mathbf{r}}(t) = q\dot{\mathbf{r}}(t) \times \begin{bmatrix} 0 \\ 0 \\ B(t) \end{bmatrix}.$$

Write this as a first-order autonomous system of ODEs. (Hint: Consider the equations satisfied by the components u, v, w of the velocity vector $\dot{\mathbf{r}}$.)

Problem 3. For each of the following systems: Draw the phase portrait, classify all of the fixed points, and sketch various solutions $x(t)$.

(a) $\dot{x} = x(x-1)^2$.
(b) $\dot{x} = \frac{1}{2} - \cos x$.

Problem 4. For each of the following systems: Draw the phase portrait, classify all of the fixed points, and sketch various solutions $x(t)$.

(a) $\dot{x} = 1 - |x|$.
(b) $\dot{x} = x \ln |x|$.

Problem 5. The velocity $v(t)$ of a skydiver falling to the ground is governed by

$$m\dot{v} = mg - kv^2,$$

where $m > 0$ is the mass of the skydiver, $g > 0$ is the acceleration due to gravity, and $k > 0$ is a constant related to the amount of air resistance.

(a) Find the exact solution $v(t)$ when $v(0) = 0$. (Hint: Partial fraction decomposition.)
(b) Find the limit of $v(t)$ as $t \to \infty$. This limiting velocity is called the *terminal velocity*.
(c) Draw the phase portrait for this system, and thereby re-derive a formula for the terminal velocity. (Notice how much easier this is compared to parts (a) and (b)!)

Problem 6. Consider the system

$$\dot{x} = \pm x^k.$$

(a) For each integer $k = 1, 2, \ldots$ and each choice of $+$ or $-$, determine the stability of the fixed point $x_* = 0$.
(b) Restricting to the cases where $x_* = 0$ is stable, find the exact solution $x(t)$ when $x(0) = 1$. Does making k larger result in faster or slower convergence to the fixed point? (Hint: Check that your answer makes sense with the graph of $\dot{x}$ as a function of x.)