- Welcome to Math 421!
- Textbook: "Calculus" by Spivak, ch. 1–14
- Syllabus is on Canvas
- 2 in-class midterms and a final exam — take note of the dates!
- Weekly homework due on Wednesdays at the beginning of class on Gradescope
- Homework 1 is posted today
- I highly recommend that you type your homework using LaTeX — see Canvas for some resources

Office hours: In the MLC
M 1–2pm (Edited: Sep. 4)
Tu 10:30–11:30 am

Email: laurens@wisc.edu

- Please ask (and answer) any question concerning the course (e.g. material, logistics) on Piazza

INTRO TO LOGIC

Def A statement is a sentence that is either true or false, but not both.

Ex ① "6 is an even integer" is a (true) statement

② "4 is an odd integer" is a (false) statement

③ "Today is Tuesday" is a statement

Let A and B be statements. We write:

A	if A is true
not A	if A is false
A and B	if both A and B are true
A or B	if A is true, or B is true, or both A and B are true

- Inclusive; <u>not</u> "either A or B"

$A \Rightarrow B$ if (A and B) or (not A)

- We say "A implies B" or "If A then B"
- In this case, B is at least as true as A

<u>Ex</u> "If $\underbrace{\text{a number is less than 10}}_{A}$, then $\underbrace{\text{it's less than 20}}_{B}$"

Taking number =		we get
	5	$T \Rightarrow T$
	15	$F \Rightarrow T$
	25	$F \Rightarrow F$

We write

$A \Leftrightarrow B$ if A and B are true together or false together

- We say "A is equivalent to B" or "A if and only if B"

Truth table:

A	B	not A	A and B	A or B	$A \Rightarrow B$	$A \Leftrightarrow B$
T	T	F	T	T	T	T
T	F	F	F	T	F	F
F	T	T	F	T	T	F
F	F	T	F	F	T	T

- Recall: "$A \Rightarrow B$" means "(A and B) or (not A)"

Ex Using a truth table, show that
$A \Rightarrow B$ is equivalent to (not A) or B.

Solution:

A	B	$A \Rightarrow B$	not A	(not A) or B
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

These two statements are true together or false together. □

Hw Using a truth table, prove De Morgan's laws:

① not (A and B) = (not A) or (not B)

② not (A or B) = (not A) and (not B)

Ex Negate the following sentence:

If $\underbrace{\text{I speak in front of the class}}_{A}$, then $\underbrace{\text{I am nervous}}_{B}$.

Solution: not ($A \Rightarrow B$) = not ((not A) or B)

by previous ex

$= (\text{not } (\text{not } A)) \text{ and } (\text{not } B)$

De Morgan's law

$= A \text{ and } (\text{not } B)$

The negation is:
"I speak in front of the class and I am not nervous". $\square$

Quantifiers:

$\forall$ reads "for all" or "for every"
$\exists$ reads "there exists" or "there is"

Let X be a set, and let $P(x)$ be a statement about elements $x \in X$.
The negation of "$\forall x \in X$, $P(x)$ is true"
is "$\exists x \in X$ s.t. $P(x)$ is false"
The negation of "$\exists x \in X$ s.t. $P(x)$ is true"
is "$\forall x \in X$, $P(x)$ is false"

Ex Negate the following:
There is a rectangle that is not a square.

Solution: x = a rectangle $\quad$ $P(x)$ = "x is not a square"
X = all rectangles

The negation is: "Every rectangle is a square". $\square$

Ex Negate the following:

Every $\underbrace{\text{student}}_{x \in X}$ $\underbrace{\text{had coffee}}_{P(x)}$ or $\underbrace{\text{is late for class}}_{Q(x)}$.

Solution:

$$\text{not}\,(\forall x \in X,\ P(x) \text{ or } Q(x))$$
$$= \exists\, x \in X \text{ s.t. not } (P(x) \text{ or } Q(x))$$
$$= \exists\, x \in X \text{ s.t. } (\text{not } P(x)) \text{ or } (\text{not } Q(x))$$

The negation is:

"There is a student that did not have coffee and is not late for class." $\square$

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Tu 10:30 – 11:30 am

Ex Suppose A and B are statements, and we know (A or B) is true. Can we conclude (not A) $\Rightarrow$ B ?

Solution:

A	B	A or B	not A	(not A) $\Rightarrow$ B
T	T	T	F	T
T	F	T	F	T
F	T	T	T	T
~~F~~	~~F~~	~~F~~		

• Recall: For "P $\Rightarrow$ Q", the possibilities
$\left\{\begin{array}{l} T \Rightarrow T \\ F \Rightarrow T \\ F \Rightarrow F \end{array}\right.$
are all allowed

Yes, the statement (not A) $\Rightarrow$ B is true. $\square$

Sets

A *set* is a collection of objects.

Let A and B be sets. We write:

$x \in A$ if x is an element of A

$x \notin A$ if x is not an element of A

$A \cap B = \{ x : x \in A \text{ and } x \in B \}$

"such that"

$A \cup B \quad = \{x : x \in A \text{ or } x \in B\}$

$A \subseteq B \quad \text{if } \forall x \in A,\ x \in B$

$A = B \quad \text{if } A \subseteq B \text{ and } B \subseteq A$

- One way to describe a set is by listing all of the objects the set contains:

Ex $S = \{1, 2, 3\}$ is a set. We have $3 \in S$, but $4 \notin S$.

Some important sets:

$\emptyset = \{\}$ "Empty set"

$\mathbb{N} = \{1, 2, 3, \ldots\}$ "Natural numbers"

$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ "Integers"

$\mathbb{Q} = \{\frac{m}{n} : m \in \mathbb{Z},\ n \in \mathbb{N}\}$ "Rational numbers"

$\mathbb{R}$ "Real numbers"

$\mathbb{C}$ "Complex numbers"

- We'll have more to say about these sets later

Ex
- $S = \{x \in \mathbb{N} : x > 0 \text{ and } x < 4\}$
- $\emptyset = \{x \in \mathbb{Z} : x^2 = 2\}$
- $\mathbb{N} = \{x \in \mathbb{Z} : x > 0\}$
- $\mathbb{Z} = \mathbb{N} \cup \{0\} \cup \{-n : n \in \mathbb{N}\}$

- $\mathbb{Q} = \{ \frac{m}{n} : m, n \in \mathbb{Z} \text{ with } n \neq 0 \}$
- $\mathbb{Q} \subseteq \mathbb{R}$, but $\mathbb{R} \not\subseteq \mathbb{Q}$. E.g., $\pi \in \mathbb{R}$ and $\pi \notin \mathbb{Q}$.
- $\mathbb{C} = \{ a + ib : a, b \in \mathbb{R} \}$

INTRO TO PROOFS

"if and only if"

<u>Def</u> ① $x \in \mathbb{N}$ is <u>even</u> iff $\exists y \in \mathbb{N}$ s.t. $x = 2y$.

② $x \in \mathbb{N}$ is <u>odd</u> iff $\exists y \in \mathbb{N} \cup \{0\}$ s.t. $x = 2y + 1$.

<u>Ex</u> Prove that for any $x \in \mathbb{N}$, if x is odd then $x+1$ is even.

<u>Rmk</u> • $5 = 2 \cdot 2 + 1$ is odd, $6 = 2 \cdot 3$ is even
$421 = 2 \cdot 210 + 1$ is odd, $422 = 2 \cdot 211$ is even

- A proof is an explanation which convinces the reader why a statement is <u>always</u> true

<u>Solution</u>: Let $x \in \mathbb{N}$.

- This is a "for all" statement, so we start with any natural number. We want to show that the statement "x odd $\Rightarrow$ $x+1$ even" is true regardless of which $x \in \mathbb{N}$ we start with.

Suppose x is odd.

- If x isn't odd, then the statement is automatically true

Then $\exists y \in \mathbb{N} \cup \{0\}$ s.t. $x = 2y + 1$.

- This is the definition of x being odd.
- We want to show $x+1$ is even, i.e. $\exists z \in \mathbb{N}$ s.t. $x+1 = 2z$.

Adding 1 to both sides of this equation, we get $x+1 = 2y+2 = 2\cdot(y+1)$.

- So $z = y+1$ works, and $z \in \mathbb{N}$.

Set $z = y+1$. Then $z \in \mathbb{N}$ and $x+1 = 2z$. Therefore $x+1$ is even. $\square$

<u>Ex</u> Prove that $\forall x, y \in \mathbb{N}$, if x is odd and y is odd then xy is odd.

<u>Solution:</u> Let $x, y \in \mathbb{N}$. Suppose x and y are odd. Then $\exists\, m \in \mathbb{N} \cup \{0\}$ s.t. $x = 2m+1$ and $\exists\, n \in \mathbb{N} \cup \{0\}$ s.t. $y = 2n+1$. Therefore

$$\begin{aligned} xy &= (2m+1)(2n+1) \\ &= 4mn + 2m + 2n + 1 \\ &= 2(2mn+m+n) + 1 \end{aligned}$$

Set $z = 2mn+m+n$. Then $z \in \mathbb{N} \cup \{0\}$ and $xy = 2z+1$, and so xy is odd. $\square$

<u>Ex</u> Prove that $\forall x \in \mathbb{N}$, x is even if and only if $x+1$ is odd.

<u>Rmk</u> "$A \Leftrightarrow B$" is equivalent to "$A \Rightarrow B$ and $B \Rightarrow A$".

<u>Solution</u>: Let $x \in \mathbb{N}$.

$\boxed{\Rightarrow}$: Suppose x is even. Then $\exists y \in \mathbb{N}$ s.t. $x = 2y$. Therefore $x+1 = 2y+1$ with $y \in \mathbb{N} \cup \{0\}$, so $x+1$ is odd.

$\boxed{\Leftarrow}$: Suppose $x+1$ is odd. Then $\exists y \in \mathbb{N} \cup \{0\}$ s.t. $x+1 = 2y+1$. Therefore $x = 2y$ with $y \in \mathbb{N}$. (Note that $y \neq 0$, since otherwise $y = 0 \Rightarrow x = 0 \Rightarrow x \notin \mathbb{N}$.) So x is even. $\square$

- Last time: "Prove that $\forall n \in \mathbb{N}$, $P(n)$ is true"
- Those were direct proofs: we fixed an arbitrary $n \in \mathbb{N}$, and were able to prove $P(n)$.
- Sometimes that's not enough! Here's another useful technique for proving statements about natural numbers:

<u>Mathematical induction</u>:

Suppose we want to prove that a statement $P(n)$ about $n \in \mathbb{N}$ is true $\forall n \in \mathbb{N}$. Then it suffices to verify two steps:

1. Base case: $P(1)$ is true
2. Inductive step: If $P(n)$ is true for some $n \in \mathbb{N}$, then $P(n+1)$ is true. (In other words, "$\forall n \in \mathbb{N}$, $P(n) \Rightarrow P(n+1)$".)

- $$\left.\begin{matrix} P(1) \\ P(1) \Rightarrow P(2) \end{matrix}\right\} \Rightarrow \left.\begin{matrix} P(2) \\ P(2) \Rightarrow P(3) \end{matrix}\right\} \Rightarrow \left.\begin{matrix} P(3) \\ P(3) \Rightarrow P(4) \end{matrix}\right\} \Rightarrow \ldots$$

  In this way, $P(n)$ is true $\forall n \in \mathbb{N}$
- Induction is really a theorem. It's true, but we won't prove it in this class (because we would have to construct $\mathbb{N}$ from scratch)

<u>Ex</u> Prove that $\forall n \in \mathbb{N}$,

$$1+2+\dots+n = \frac{n(n+1)}{2}.$$

<u>Solution</u>: We will argue by induction.

- Here, $P(n)$ is the statement $1+2+\dots+n = \frac{n(n+1)}{2}$.

Base case: For $n=1$, the statement $1 = \frac{1\cdot 2}{2}$ is true.

Inductive step: Assume $1+2+\dots+n = \frac{n(n+1)}{2}$ is true for some $n \in \mathbb{N}$.

- We want to show $P(n+1)$, i.e. $1+2+\dots+(n+1) = \frac{(n+1)(n+2)}{2}$
- How do we go from $1+2+\dots+n$ to $1+2+\dots+(n+1)$?

Adding $n+1$ to both sides, we get

$$\begin{aligned} 1+2+\dots+n+(n+1) &= \frac{n(n+1)}{2} + (n+1) \\ &= \frac{n(n+1)+2(n+1)}{2} \\ &= \frac{(n+1)(n+2)}{2} \end{aligned}$$

So the formula is true for $n+1$.

By induction, we conclude that the formula holds $\forall n \in \mathbb{N}$. $\square$

<u>Ex</u> Prove that $\forall n \in \mathbb{N}$,

$$1^2+2^2+\dots+n^2 = \frac{n(n+1)(2n+1)}{6}.$$

<u>Solution</u>: We'll use induction.

$1\cdot 2\cdot 3$

Base case: For $n=1$, $1^2 = \frac{\quad}{6}$ is true.

Inductive step: Assume

$$1^2 + 2^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

for some $n \in \mathbb{N}$. Adding $(n+1)^2$ to both sides, we get

$$\begin{aligned} 1^2 + 2^2 + \dots + (n+1)^2 &= \frac{n(n+1)(2n+1)}{6} + (n+1)^2 \\ &= \frac{n+1}{6}\left[n(2n+1) + 6(n+1)\right] \\ &= \frac{n+1}{6}(2n^2 + 7n + 6) \\ &= \frac{n+1}{6}(n+2)(2n+3) \\ &= \frac{(n+1)(n+2)(2n+3)}{6} \end{aligned}$$

So the formula is true for $n+1$.

By induction, we conclude that the formula holds $\forall n \in \mathbb{N}$. $\square$

- In groups:

Ex Prove that $\forall n \in \mathbb{N}$,

$$1 + 4 + 7 + \dots + (3n+1) = \frac{(n+1)(3n+2)}{2}.$$

Ex Compute the following sum for $n \in \mathbb{N}$, and prove your answer:

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \dots + \frac{1}{n(n+1)}$$

Scratch work: $\frac{1}{n} - \frac{1}{n+1} = \frac{(\cancel{n}+1) - \cancel{n}}{n(n+1)} = \frac{1}{n(n+1)}$

$$\Rightarrow \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \dots + \frac{1}{n(n+1)}$$

$$= \frac{1}{1} - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} + \cancel{\dots} + \cancel{\frac{1}{n}} - \frac{1}{n+1}$$

$$= 1 - \frac{1}{n+1} = \frac{n}{n+1}$$

- Now that we know the answer, we can prove it

Solution: We will prove that $\forall n \in \mathbb{N}$,

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \dots + \frac{1}{n(n+1)} = \frac{n}{n+1}$$

by induction.

Base case: For $n=1$, $\frac{1}{1\cdot 2} = \frac{1}{1+1}$ is true.

Inductive step: Assume

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \dots + \frac{1}{n(n+1)} = \frac{n}{n+1}$$

is true for some $n \in \mathbb{N}$. Adding $\frac{1}{(n+1)(n+2)}$ to both sides, we get

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \dots + \frac{1}{n(n+1)} + \frac{1}{(n+1)(n+2)} = \frac{n}{n+1} + \frac{1}{(n+1)(n+2)}$$

$$= \frac{n(n+2)+1}{(n+1)(n+2)}$$

$$= \frac{n^2+2n+1}{(n+1)(n+2)}$$

$$= \frac{(n+1)^{\cancel{2}}}{\cancel{(n+1)}(n+2)}$$

$$= \frac{n+1}{\ldots}$$

$- \quad n+2$

So the formula is true for $n+1$.

By induction, we conclude that the formula holds $\forall n \in \mathbb{N}$. $\square$

Properties of Numbers (Ch. 1)

The real numbers $\mathbb{R}$ satisfy:

(A1) Closure: if $a, b \in \mathbb{R}$ then $a+b \in \mathbb{R}$

(A2) Commutativity: $\forall a, b \in \mathbb{R}$, $a+b = b+a$

(A3) Associativity: $\forall a, b, c \in \mathbb{R}$, $(a+b)+c = a+(b+c)$

(A4) Identity: $\exists\, 0 \in \mathbb{R}$ s.t. $a+0 = 0+a = a$ $\forall a \in \mathbb{R}$

(A5) Inverse: $\forall a \in \mathbb{R}$, $\exists -a \in \mathbb{R}$ s.t.

$$a+(-a) = (-a)+a = 0$$

(M1) Closure: if $a, b \in \mathbb{R}$ then $a \cdot b \in \mathbb{R}$

(M2) Commutativity: $\forall a, b \in \mathbb{R}$, $a \cdot b = b \cdot a$

(M3) Associativity: $\forall a, b, c \in \mathbb{R}$, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$

(M4) Identity: $\exists\, 1 \in \mathbb{R}$ s.t. $a \cdot 1 = 1 \cdot a = a$ $\forall a \in \mathbb{R}$

(M5) Inverse: $\forall a \in \mathbb{R}$ with $a \neq 0$, $\exists\, a^{-1} \in \mathbb{R}$

s.t. $a \cdot a^{-1} = a^{-1} \cdot a = 1$

(D) Distributivity: $\forall a, b, c \in \mathbb{R}$, $(a+b) \cdot c = a \cdot c + b \cdot c$

<u>Rmk</u> A set F with addition and multiplication operations satisfying these 11 properties is called a <u>field</u>. E.g.:

- $\mathbb{N}$ is not a field, (A4) fails
- $\mathbb{Z}$ is not a field, (M5) fails

- $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ are fields
- These properties are all we need in order to use $+$ and $\cdot$

<u>Proposition 1</u> If $a, b, c \in \mathbb{R}$ s.t. $a+b = a+c$, then $b = c$.

<u>Rmk</u> In particular, taking $c=0$ we get: $a+b = a \Rightarrow b = 0$.

<u>Proof</u>: Assume $a, b, c \in \mathbb{R}$ s.t. $a+b = a+c$. By (A5), $\exists\, -a \in \mathbb{R}$. Adding this to the left, we get:

$$-a + (a+b) = -a + (a+c)$$

By (A3), this implies

$$(-a+a) + b = (-a+a) + c$$

By (A5),

$$0 + b = 0 + c$$

By (A4),

$$b = c. \qquad \square$$

<u>Prop 2</u> If $a, b, c \in \mathbb{R}$ s.t. $a \cdot b = a \cdot c$ and $a \neq 0$, then $b = c$.

<u>Rmk</u> In particular: $a \cdot b = a$ and $a \neq 0 \Rightarrow b = 1$.

<u>Pf</u>: Suppose $a, b, c \in \mathbb{R}$ s.t. $a \cdot b = a \cdot c$ and $a \neq 0$.

As $a \neq 0$, $\exists a^{-1} \in \mathbb{R}$ by (M5). Together,

$$a^{-1} \cdot (a \cdot b) = a^{-1} \cdot (a \cdot c)$$

By (M3),

$$(a^{-1} \cdot a) \cdot b = (a^{-1} \cdot a) \cdot c$$

By (M5),

$$1 \cdot b = 1 \cdot c$$

By (M4),

$$b = c. \qquad \square$$

<u>Prop 3</u> $\forall a \in \mathbb{R}$, $a \cdot 0 = 0 = 0 \cdot a$.

<u>Pf</u>: Let $a \in \mathbb{R}$. Then

$$a \cdot 0 \underset{\substack{\uparrow \\ (A4)}}{=} a \cdot (0+0) \underset{\substack{\uparrow \\ (D)}}{=} a \cdot 0 + a \cdot 0$$

By Prop 1, we may cancel $a \cdot 0$ from both sides and conclude $a \cdot 0 = 0$.

By (M2), $0 \cdot a = a \cdot 0 = 0$. $\square$

<u>Prop 4</u> If $a, b \in \mathbb{R}$ and $a \cdot b = 0$, then $a = 0$ or $b = 0$.

<u>Pf</u>: Suppose $a \cdot b = 0$. If $a = 0$ then we're done, so assume $a \neq 0$. We want to show $b = 0$.

As $a \neq 0$, then $\exists a^{-1} \in \mathbb{R}$ by (M5).

Multiplying $a \cdot b = 0$ by a^{-1} on the left we get

Multiplying $a \cdot b = 0$ by a^{-1} on the left, we get

$$a^{-1} \cdot (a \cdot b) = a^{-1} \cdot 0 \underset{\text{Prop 3}}{=} 0$$

On the other hand,

$$a^{-1} \cdot (a \cdot b) \underset{(M3)}{=} (a^{-1} \cdot a) \cdot b \underset{(M5)}{=} 1 \cdot b \underset{(M4)}{=} b$$

Altogether, we get $b = a^{-1} \cdot (a \cdot b) = 0$, as desired. $\square$

<u>Prop 5</u> $\forall a, b \in \mathbb{R}$, $(-a) \cdot b = -(a \cdot b) = a \cdot (-b)$.

<u>Pf</u>: Let $a, b \in \mathbb{R}$. Then

$$(-a) \cdot b + a \cdot b \underset{(D)}{=} (-a + a) \cdot b \underset{(A5)}{=} 0 \cdot b \underset{\text{Prop 3}}{=} 0$$

Adding $-(a \cdot b)$ to both sides, we conclude

$$(-a) \cdot b = -(a \cdot b).$$

Similarly,

$$a \cdot (-b) + a \cdot b \underset{(M2)}{=} (-b) \cdot a + b \cdot a \underset{(D)}{=} (-b + b) \cdot a$$

$$\underset{(A5)}{=} 0 \cdot a \underset{\text{Prop 3}}{=} 0$$

Adding $-(a \cdot b)$ to both sides, we conclude

$$a \cdot (-b) = -(a \cdot b).$$ $\square$

$$a \cdot (-b) = -(a \cdot b). \qquad \square$$

Last time: $\forall a, b \in \mathbb{R}, \quad (-a)\cdot b = -(a\cdot b) = a\cdot(-b).$

Idea: ① $(-a)\cdot b + a\cdot b = \ldots = 0$

② $(-a)\cdot b = -(a\cdot b)$

• Using a similar idea, we can also prove:

Prop 6 $\forall a, b \in \mathbb{R}, \quad (-a)\cdot(-b) = a\cdot b.$

Pf: Let $a, b \in \mathbb{R}$. Then

$$
\begin{aligned}
(-a)\cdot(-b) + (-(a\cdot b)) &\overset{\text{Prop 5}}{=} (-a)\cdot(-b) + a\cdot(-b) \\
&\overset{(D)}{=} (-a+a)\cdot(-b) \\
&\underset{(A5)}{=} 0\cdot(-b) \underset{\text{Prop 3}}{=} 0
\end{aligned}
$$

Adding $a\cdot b$ to both sides, we conclude $(-a)\cdot(-b) = a\cdot b.$ $\square$

• What else do we know about $\mathbb{R}$?

Inequalities "$<$" on $\mathbb{R}$ satisfy:

- trichotomy: For each $a, b \in \mathbb{R}$, one and only one of the following statements holds:

$$a<b, \quad a=b, \quad \text{or} \quad b<a$$

- transitivity: $\forall a, b, c \in \mathbb{R}$, if $a<b$ and

Transitivity: $\forall a,b,c \in \mathbb{R}$, if $a<b$ and $b<c$ then $a<c$.

Addition and multiplication satisfy:

(O1) $\forall a,b,c \in \mathbb{R}$, if $a<b$ then $a+c<b+c$

(O2) $\forall a,b,c \in \mathbb{R}$, if $a<b$ and $0<c$ then $a \cdot c < b \cdot c$

Rmk ① A relation "$<$" on a set S satisfying trichotomy and transitivity is called an order relation. E.g., inequalities "$<$" on $\mathbb{Z}$ is an order relation.

② A field F with an order relation "$<$" that also satisfies (O1) and (O2) is called an ordered field. E.g.,

- $\mathbb{Q}$ and $\mathbb{R}$ are ordered fields
- $\mathbb{C}$ is not an ordered field

Notation We write

$$
\begin{aligned}
a > b &\quad \text{if } b<a \\
a \leq b &\quad \text{if } a<b \text{ or } a=b \\
a \geq b &\quad \text{if } b \leq a
\end{aligned}
$$

- I.e.: Once you have "$<$", you also know what "$>$", "$\leq$", and "$\geq$" mean.

Prop 7 $\forall a \in \mathbb{R}$, $a > 0 \iff -a < 0$.

Pf: Let $a \in \mathbb{R}$.

$\Rightarrow$: Suppose $a > 0$. Then:

$$a > 0 \overset{(O1)}{\Longrightarrow} \underbrace{a + (-a)}_{\substack{= \\ 0 \\ \text{by (A5)}}} > \underbrace{0 + (-a)}_{\substack{= \\ -a \\ \text{by (A4)}}}$$

$$\Rightarrow \quad 0 < -a.$$

$\Leftarrow$: Suppose $-a < 0$. Then:

$$-a < 0 \overset{(O1)}{\Longrightarrow} \underbrace{-a + a}_{\substack{= \\ 0 \\ \text{by (A5)}}} < \underbrace{0 + a}_{\substack{= \\ a \\ \text{by (A4)}}}$$

$$\Rightarrow \quad 0 < a. \qquad \square$$

Prop 8 $\forall a, b, c \in \mathbb{R}$, if $a < b$ and $c < 0$ then $a \cdot c > b \cdot c$.

Pf: Suppose $a < b$ and $c < 0$. By Prop 7 we know $-c > 0$, and so (O2) implies

$$\underbrace{a \cdot (-c)}_{\substack{= \\ -(a \cdot c)}} < \underbrace{b \cdot (-c)}_{\substack{= \\ -(b \cdot c)}} \quad \text{by Prop 5.}$$

Using (O1), we add $ac+bc$ to both sides:

$$-ac + (ac+bc) < -bc + (ac+bc)$$

Now,

$$-ac + (ac+bc) \overset{(A3)}{=} (-ac+ac) + bc \overset{(A5)}{=} 0 + bc \overset{(A4)}{=} bc$$

and

$$-bc + (ac+bc) \overset{(A2)}{=} -bc + (bc+ac) \overset{(A3)}{=} (-bc+bc) + ac$$
$$\overset{(A5)}{=} 0 + ac \overset{(A4)}{=} ac$$

Altogether, we conclude $bc < ac$. $\square$

<u>Prop 9</u> $\forall a, b \in \mathbb{R}$,

① if $a > 0$ and $b > 0$ then $a \cdot b > 0$.

② if $a < 0$ and $b < 0$ then $a \cdot b > 0$.

③ if $a \neq 0$ then $a^2 > 0$.

<u>Rmk</u> In particular, $1 = 1^2 > 0$.

<u>Pf</u>: ① As $b > 0$,

$$a > 0 \overset{(O2)}{\Longrightarrow} a \cdot b > \underbrace{0 \cdot b}_{\substack{\parallel \\ 0 \text{ by Prop 3}}}$$

② Suppose $a < 0$ and $b < 0$. Then $-a > 0$

② Suppose $a<0$ and $b<0$. Then $-a>0$ and $-b>0$ by Prop 7. Applying ① to $-a$ and $-b$, we get

$$(-a)\cdot(-b) > 0$$
$$\overset{\shortparallel}{ab} \text{ by Prop 6.}$$

③ Suppose $a \neq 0$. By trichotomy, either $a>0$ or $a<0$.

- Case $a>0$: By ①, $a \cdot a > 0$.
- Case $a<0$: By ②, $a \cdot a > 0$. $\square$

<u>Prop 10</u>

① $\forall a \in \mathbb{R}$, if $a>0$ then $a^{-1}>0$.

② $\forall a,b \in \mathbb{R}$, if $0<a<b$ then $0<b^{-1}<a^{-1}$.

<u>Pf</u>: ① Suppose this statement were false. Then $\exists\, a \in \mathbb{R}$ s.t. $a>0$ but $a^{-1} \leq 0$. Note that $a^{-1} \neq 0$, since otherwise

$$a^{-1}=0 \implies a \cdot a^{-1} = 0$$

but we already know $a \cdot a^{-1} = 1 \neq 0$. So $a^{-1}<0$. Then:

$$a>0 \overset{\text{Prop 8}}{\implies} a\cdot a^{-1} < 0 \overset{\text{(M5)}}{\implies} 1<0$$

This contradicts that $1>0$. Therefore the statement ① was actually true.

(2) Suppose $0<a<b$. Then $a^{-1}>0$ and $b^{-1}>0$ by (1), and so $a^{-1}\cdot b^{-1}>0$ by Prop 9.

Using (O2), we multiply $0<a<b$ by $a^{-1}\cdot b^{-1}$:

$$0\cdot(a^{-1}\cdot b^{-1}) < a\cdot(a^{-1}\cdot b^{-1}) < b\cdot(a^{-1}\cdot b^{-1})$$

Now we simplify each of these terms:

$$0\cdot(a^{-1}\cdot b^{-1}) \underset{\uparrow \atop \text{Prop } 3}{=} 0$$

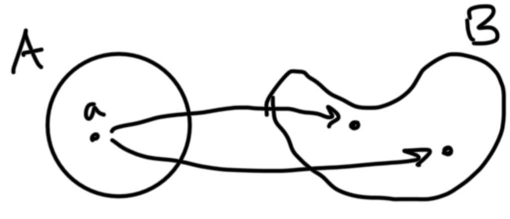
$$a\cdot(a^{-1}\cdot b^{-1}) \underset{\uparrow \atop (M3)}{=} (a\cdot a^{-1})\cdot b^{-1} \underset{\uparrow \atop (M5)}{=} 1\cdot b^{-1} \underset{\uparrow \atop (M4)}{=} b^{-1}$$

$$b\cdot(a^{-1}\cdot b^{-1}) \underset{\uparrow \atop (M2)}{=} b\cdot(b^{-1}\cdot a^{-1}) \underset{\uparrow \atop (M3)}{=} (b\cdot b^{-1})\cdot a^{-1} \underset{\uparrow \atop (M5)}{=} 1\cdot a^{-1} \underset{\uparrow \atop (M4)}{=} a^{-1}$$

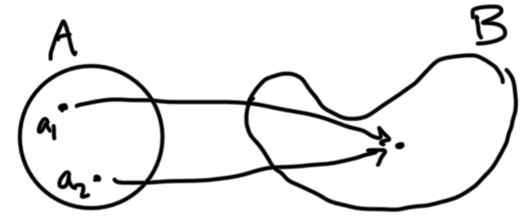
Altogether, we conclude $0<b^{-1}<a^{-1}$. $\square$

FUNCTIONS (Ch. 3)

<u>Def</u> Let A, B be nonempty sets. A <u>function</u> $f : A \to B$ is a way of associating to each element $a \in A$ exactly one element in B, denoted $f(a)$.



Not a function



OK

A is called the <u>domain</u> of f.
B is called the <u>range</u> of f.

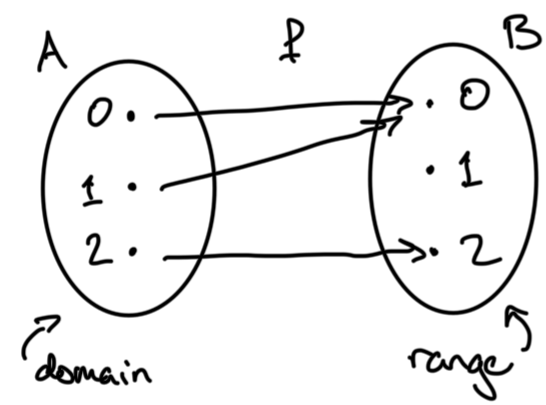
<u>Ex</u> Let $A = \{0, 1, 2\} = B$. Consider $f(x) = x^2 - x$.
(or "$x \mapsto x^2 - x$")

Is $f : A \to B$ a function? Yes.

$0 \mapsto 0$

$1 \mapsto 0$

$2 \mapsto 2$



<u>Ex</u> For the same sets A, B, are the following functions from A to B?

① 

② 

No.

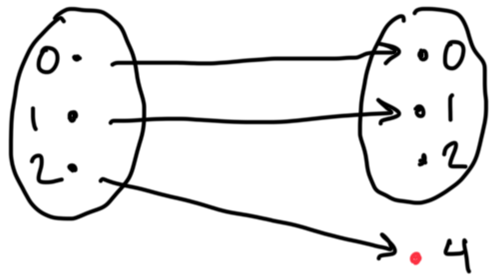
No.

③ $x \mapsto x^2$

$0 \mapsto 0$

$1 \mapsto 1$

$2 \mapsto 4$



No.

<u>Def</u> Let $f: A \to B$ be a function.

① If $X \subseteq A$, the <u>image</u> of X under f is

$$f(X) = \{ f(a) : a \in X \}.$$

② The <u>image</u> of f is $f(A)$.

③ If $Y \subseteq B$, the <u>preimage</u> of Y under f is

$$f^{-1}(Y) = \{ a \in A : f(a) \in Y \}.$$

<u>Rmk</u> Warning: Here, "f^{-1}" is not the inverse function of f! In particular:

- $f^{-1}(Y)$ always makes sense, even when f does not have an inverse.
- f and f^{-1} do not cancel in general.

<u>Ex</u> $A = \{0, 1, 2\} = B$, $f: A \to B$

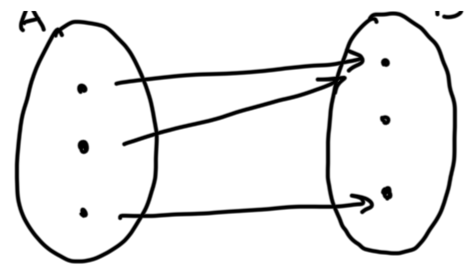
$0 \mapsto 0$

$1 \mapsto 0$

$2 \mapsto 2$

- $f(\{0,1\}) = \{ f(x) : x \in \{0,1\} \}$

$\quad = \{0\}$



- image of f
  $= f(A) = \{0, 2\}$
- $f^{-1}(\{1,2\}) = \{x \in A : f(x) \in \{1,2\}\}$
  $= \{2\}$
- $f(\{1,2\}) = \{0,2\}$
- $f(f^{-1}(\{1,2\})) = f(\{2\}) = \{2\} \neq \{1,2\}$
- $f^{-1}(f(\{1,2\})) = f^{-1}(\{0,2\}) = \{0,1,2\} \neq \{1,2\}$
- Notice how f and f^{-1} don't cancel!

Prop Let $f : A \to B$ be a function.

① $\forall Y \subseteq B, \quad f(f^{-1}(Y)) \subseteq Y$.

② $\forall X \subseteq A, \quad f^{-1}(f(X)) \supseteq X$.

Pf: ① Let $Y \subseteq B$. Want to show: $\forall b \in f(f^{-1}(Y))$, we have $b \in Y$. Let $b \in f(f^{-1}(Y))$. By definition of image,

$$b \in f(f^{-1}(Y)) = \{ f(x) : x \in f^{-1}(Y) \},$$

so $\exists\, a \in f^{-1}(Y)$ s.t. $f(a) = b$. By definition of preimage,

$$a \in f^{-1}(Y) = \{ x \in A : f(x) \in Y \},$$

so $f(a) \in Y$. Altogether, we conclude $b = f(a) \in Y$, as desired.

(2) Let $X \subseteq A$. Want: $\forall a \in X$, $a \in f^{-1}(f(X))$.

Let $a \in X$. Then

$$f(a) \in f(X) = \{ f(x) : x \in X \}$$

by definition of image. So

$$a \in f^{-1}(f(X)) = \{ x \in A : f(x) \in f(X) \}$$

by definition of preimage. $\square$

<u>Def</u> Let $f : A \to B$ be a function.

(1) f is <u>surjective</u> (or <u>onto</u>) iff $f(A) = B$.

In other words: $\forall b \in B$, $\exists a \in A$ s.t. $f(a) = b$.

(2) f is <u>injective</u> (or <u>one-to-one</u>) iff it satisfies: if $a_1, a_2 \in A$ s.t. $f(a_1) = f(a_2)$ then $a_1 = a_2$.

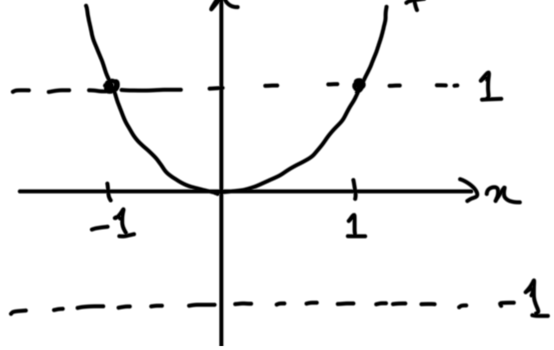
(3) f is <u>bijective</u> iff it is injective and surjective.

<u>Rmk</u> These properties depend not only on the function f, but also on the domain and range.

<u>Ex</u> $f(x) = x^2$

(1) $f : \mathbb{R} \to \mathbb{R}$ is...

- not injective: $-1, 1 \in \mathbb{R}$, $f(-1) = f(1)$, and $-1 \neq 1$
- not surjective: $-1 \in \mathbb{R}$, but $f(x) \neq -1$ $\forall x \in \mathbb{R}$



(2) $A = \{x \in \mathbb{R} : x \geq 0\}$.

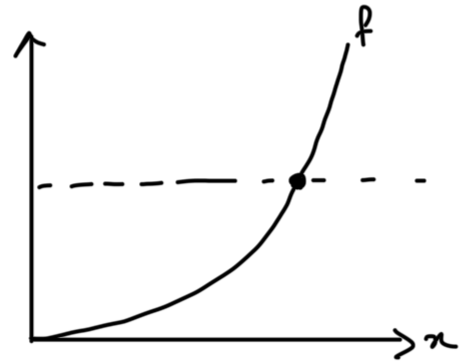
$f : A \to A$ is...

- injective:

  $a_1, a_2 \geq 0$, $a_1^2 = a_2^2 \Rightarrow a_1 = a_2$

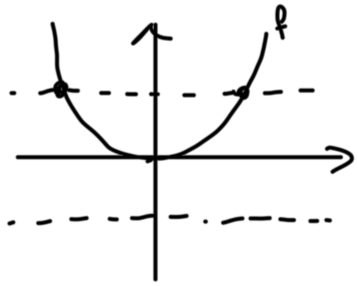
- surjective: $\forall\, b \geq 0$,

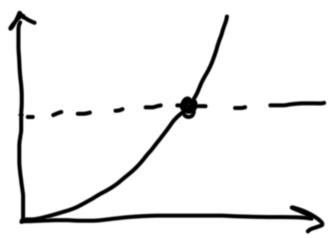
  $\exists\, a \geq 0$ s.t. $a^2 = b$



<u>Last time</u> $f: A \to B$

- f injective: $a_1, a_2 \in A$ s.t. $f(a_1) = f(a_2) \Rightarrow a_1 = a_2$
- f surjective: $\forall b \in B$, $\exists a \in A$ s.t. $f(a) = b$





bijective

<u>Lemma</u> If $f: A \to B$ is bijective, then: $\forall b \in B$, $\exists$ exactly one $a \in A$ s.t. $f(a) = b$.

<u>Pf</u>: Let $b \in B$. By definition of surjective, $\exists$ at least one $a \in A$ s.t. $f(a) = b$.

Claim: There is at most one $a \in A$ s.t. $f(a) = b$. Suppose $a_1, a_2 \in A$ s.t. $f(a_1) = b = f(a_2)$. By definition of injective, $a_1 = a_2$. $\square$

- So, for a bijective function, $f^{-1}: B \to A$ makes sense as a function:

<u>Def</u> Let $f: A \to B$ be bijective. The <u>inverse</u> of f is the function $f^{-1}: B \to A$, $f^{-1}(b) = a$, where $a \in A$ is the unique element in A s.t. $f(a) = b$.

Rmk If f is bijective then $f^{-1}(f(a)) = a$

$\forall a \in A$ and $f(f^{-1}(b)) = b \quad \forall b \in B$.

<u>Def</u> Let A, B, C be nonempty sets and let $f: A \to B$, $g: B \to C$ be functions. The <u>composition</u> of g with f is the function $g \circ f : A \to C$, $(g \circ f)(a) = g(f(a))$.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend right, "g \circ f"'] & B \arrow[r, "g"] & C \end{tikzcd}$$

<u>Ex</u> Let $f, g: \mathbb{R} \to \mathbb{R}$, $f(x) = 2x$, $g(x) = x+1$.

Then

$$(g \circ f)(x) = g(f(x)) = g(2x) = 2x+1$$
$$(f \circ g)(x) = f(g(x)) = f(x+1) = 2(x+1)$$

Not equal!

<u>Rmk</u> Composition of functions is not commutative.

<u>Prop</u> If $f: A \to B$ and $g: B \to C$ are bijective, then $g \circ f : A \to C$ is bijective and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[rr, bend left, "g \circ f"] & B \arrow[r, shift left, "g"] \arrow[l, shift left, "f^{-1}"] & C \arrow[l, shift left, "g^{-1}"] \arrow[ll, bend left] \end{tikzcd}$$

$(g\circ f)^{-1} = f^{-1}\circ g^{-1}$

Pf: Injective: Let $a_1, a_2 \in A$ s.t. $(g\circ f)(a_1) = (g\circ f)(a_2)$.
Want: $a_1 = a_2$. We know

$$g(f(a_1)) = (g\circ f)(a_1) = (g\circ f)(a_2) = g(f(a_2))$$

As g is injective,

$$f(a_1),\ f(a_2) \in B,\quad g(f(a_1)) = g(f(a_2)) \Rightarrow f(a_1) = f(a_2).$$

Likewise, as f is injective,

$$a_1, a_2 \in A,\quad f(a_1) = f(a_2) \Rightarrow a_1 = a_2.$$

So $a_1 = a_2$, as desired.

Surjective: Let $c \in C$. Want: $\exists\ a \in A$ s.t. $(g\circ f)(a) = c$. As $c \in C$ and g is surjective, then $\exists\ b \in B$ s.t. $g(b) = c$. Similarly, as $b \in B$ and f is surjective, then $\exists\ a \in A$ s.t. $f(a) = b$. Altogether,

$$(g\circ f)(a) = g(f(a)) = g(b) = c,$$

as desired.

Claim: $(g\circ f)^{-1}(c) = (f^{-1}\circ g^{-1})(c)\ \ \forall c \in C$.

Let $c \in C$. By definition, $(g\circ f)^{-1}(c)$ is equal to the unique $a_1 \in A$ s.t. $(g\circ f)(a_1) = c$. We will show that $a_2 = (f^{-1}\circ g^{-1})(c)$ satisfies this property:

$$(g\circ f)(a_2) = g(f(f^{-1}(g^{-1}(c))))$$

$\underbrace{\qquad\qquad}_{=g^{-1}(c)}$, since f^{-1} is the inverse function of f

$$= g(g^{-1}(c)) = c$$

since g^{-1} is the inverse function of g.

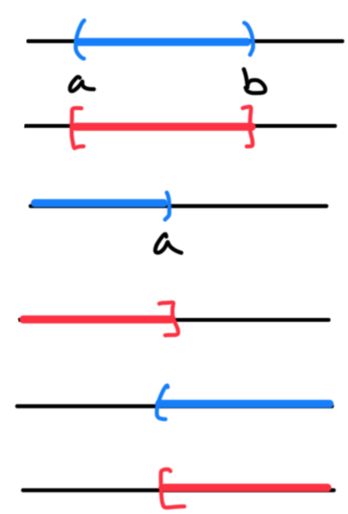
So $a_2 = a_1$, and thus $(g\circ f)^{-1}(c) = (f^{-1}\circ g^{-1})(c)$. $\square$

Subsets of $\mathbb{R}$

<u>Def</u> A set $I \subseteq \mathbb{R}$ is an <u>interval</u> iff $\forall x,y,z \in \mathbb{R}$, $x,z \in I$ and $x<y<z \Rightarrow y \in I$.

<u>Ex</u> Given $a,b\in\mathbb{R}$ with $a<b$,

- $(a,b) = \{x\in\mathbb{R} : a<x<b\}$
- $[a,b] = \{x\in\mathbb{R} : a\le x\le b\}$
- $(-\infty,a) = \{x\in\mathbb{R} : x<a\}$
- $(-\infty,a] = \dots$
- (a,∞)
- $[a,\infty)$



- These are all examples of intervals
- let's prove this in one case:

<u>Lem</u> $\forall a,b\in\mathbb{R}$ with $a<b$, (a,b) is an interval.

Pf: Let $a<b$. Suppose $x, z \in (a,b)$ and $x<y<z$.
We want to show $y \in (a,b)$, i.e. $a<y<b$:

$$x \in (a,b) \Rightarrow \left.\begin{array}{r} a<x<b \\ x<y \end{array}\right\} \Rightarrow a<y$$

$$z \in (a,b) \Rightarrow \left.\begin{array}{r} a<z<b \\ y<z \end{array}\right\} \Rightarrow y<b$$

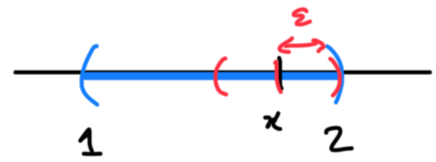
So $a<y<b$, as desired. $\square$

<u>Def</u> A set $\mathcal{U} \subseteq \mathbb{R}$ is <u>open</u> iff $\forall x \in \mathcal{U}$, $\exists \varepsilon > 0$ s.t. $(x-\varepsilon, x+\varepsilon) \subseteq \mathcal{U}$.

<u>Ex</u> ① $(1,2)$ is open:

Let $x \in (1,2)$. Then $1 < x < 2$.

Set $\varepsilon = \min\{\underbrace{2-x}_{>0}, \underbrace{x-1}_{>0}\}$.



Then $\varepsilon > 0$. Moreover, we have $(x-\varepsilon, x+\varepsilon) \subseteq (1,2)$:

$$y \in (x-\varepsilon, x+\varepsilon) \Rightarrow \left\{ \begin{array}{l} y < x+\varepsilon \overset{\varepsilon \le 2-x}{\le} x + (2-x) = 2 \\ y > x - \varepsilon \underset{\varepsilon \le x-1}{\ge} x - (x-1) = 1 \end{array} \right\} \Rightarrow y \in (1,2)$$

- Note that ε is equal to whichever number is smaller, and so ε is $\le$ both numbers.

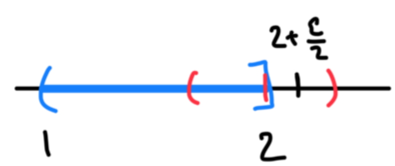
② $(1,2]$ is not open:

- not (open) = not ($\forall x \in \mathcal{U}$, $\exists \varepsilon > 0$ s.t. $(x-\varepsilon, x+\varepsilon) \subseteq \mathcal{U}$)

  $= \exists x \in \mathcal{U}$ s.t. $\forall \varepsilon > 0$, $(x-\varepsilon, x+\varepsilon) \not\subseteq \mathcal{U}$.

Consider $2 \in (1,2]$. Let $\varepsilon > 0$.

Then $(2-\varepsilon, 2+\varepsilon) \not\subseteq (1,2]$, since $2 + \frac{\varepsilon}{2} \in (2-\varepsilon, 2+\varepsilon)$ but $2 + \frac{\varepsilon}{2} \notin (1,2]$.



③ $\mathbb{R}$ is open: Let $x \in \mathbb{R}$. Set $\varepsilon = 10$.

Then $(x-\varepsilon, x+\varepsilon) \subseteq \mathbb{R}$.

④ $\emptyset$ is open: The definition "$\underbrace{\forall x \in \emptyset}$, ..." is vacuously true.

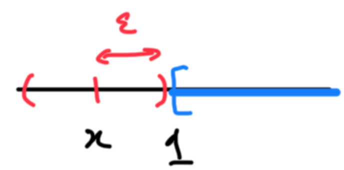
- There are no such x, so there is nothing to check!

<u>Def</u> A set $F \subseteq \mathbb{R}$ is <u>closed</u> iff $\underbrace{F^c}_{\text{"F complement"}} = \{x \in \mathbb{R} : x \notin F\}$ is open.

<u>Ex</u> ① $[1,2]$ is closed: Fix $x \in [1,2]^c$. Then $x \notin [1,2]$, so $x < 1$ or $x > 2$.

- Note: not $(x \geq 1$ and $x \leq 2) = x < 1$ or $x > 2$.

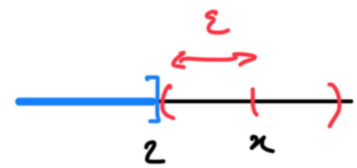
Case: $x < 1$. Set $\varepsilon = 1 - x$.



Then $\varepsilon > 0$ and $(x-\varepsilon, x+\varepsilon) \subseteq [1,2]^c$:

$$y \in (x-\varepsilon, x+\varepsilon) \Rightarrow y < x+\varepsilon = x + (1-x) = 1$$
$$\Rightarrow y \notin [1,2].$$

Case: $x > 2$. Set $\varepsilon = x - 2$.

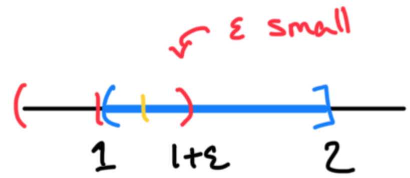


Then $\varepsilon > 0$ and $(x-\varepsilon, x+\varepsilon) \subseteq [1,2]^c$:

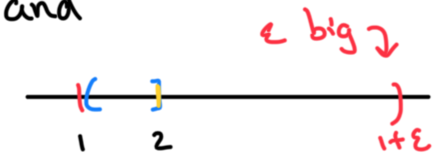
$$y \in (x-\varepsilon, x+\varepsilon) \Rightarrow y > x-\varepsilon = x-(x-2) = 2$$
$$\Rightarrow y \notin [1,2].$$

Altogether, $[1,2]^c$ is open, and so $[1,2]$ is closed.

② $(1,2]$ is not closed: I.e., $(1,2]^c$ is not open. Consider $1 \in (1,2]^c$.



Let $\varepsilon > 0$. Then $(1-\varepsilon, 1+\varepsilon) \not\subseteq (1,2]^c$, since $1+\min\{\frac{\varepsilon}{2}, 1\} \in (1-\varepsilon, 1+\varepsilon)$ and $1+\min\{\frac{\varepsilon}{2}, 1\} \in (1,2]$.



③ $\mathbb{R}$ is closed: $\mathbb{R}^c = \emptyset$ is open.

④ $\emptyset$ is closed: $\emptyset^c = \mathbb{R}$ is open.

<u>Rmk</u> Warning: "not open" $\neq$ "closed".

A set can be...

- open: e.g. $(1,2)$
- closed: e.g. $[1,2]$
- not open and not closed: e.g. $(1,2]$
- both open and closed: e.g. $\mathbb{R}, \emptyset$.

<u>Q:</u> A, B open $\Longrightarrow$ $A \cap B$ open? $A \cup B$ open?

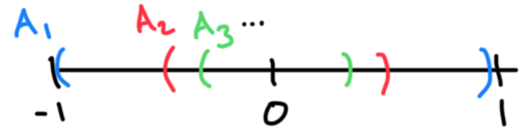
- Yes, and you can use more than 2 sets:

<u>Prop 1</u> $\forall n \in \mathbb{N}$, if $A_1, \ldots, A_n \subseteq \mathbb{R}$ are open

then $A_1 \cap \ldots \cap A_n$ is open.

- Any intersection of finitely many open sets is open.

<u>Ex</u> $A_n = \left(-\frac{1}{n}, \frac{1}{n}\right)$



- $A_1 \cap A_2 \cap \ldots \cap A_n = \left(-\frac{1}{n}, \frac{1}{n}\right)$ is open
- $A_1 \cap A_2 \cap \ldots = \{x \in \mathbb{R} : x \in A_n \ \forall n \in \mathbb{N}\}$

  $= \{0\}$ is not open

- "Finitely many" is the best we can do.

<u>Last time</u>: A open $\overset{\text{def}}{\iff}$ $\forall x \in A$, $\exists\, \varepsilon > 0$ s.t. $(x-\varepsilon, x+\varepsilon) \subseteq A$.

E.g. $(1,2)$.

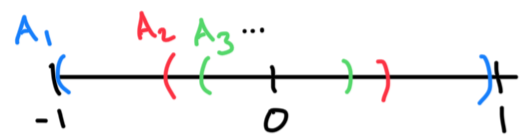
<u>Q</u>: A, B open $\Longrightarrow$ $A \cap B$ open? $A \cup B$ open?

- Yes, and you can use more than 2 sets:

<u>Prop 1</u> $\forall n \in \mathbb{N}$, if $A_1, \ldots, A_n \subseteq \mathbb{R}$ are open then $A_1 \cap \ldots \cap A_n$ is open.

- Any intersection of finitely many open sets is open.

<u>Ex</u> $A_n = \left(-\frac{1}{n}, \frac{1}{n}\right)$



- $A_1 \cap A_2 \cap \ldots \cap A_n = \left(-\frac{1}{n}, \frac{1}{n}\right)$ is open
- $A_1 \cap A_2 \cap \ldots = \{x \in \mathbb{R} : x \in A_n \ \forall n \in \mathbb{N}\}$

$= \{0\}$ is not open

- "Finitely many" is the best we can do.

<u>Pf</u>: Suppose $A_1, \ldots, A_n \subseteq \mathbb{R}$ are open. Let $x \in A_1 \cap \ldots \cap A_n$. Want: $\exists\, \varepsilon > 0$ s.t. $(x-\varepsilon, x+\varepsilon) \subseteq A_1 \cap \ldots \cap A_n$.

As $x \in A_1$ and A_1 is open, $\exists\, \varepsilon_1 > 0$ s.t. $(x-\varepsilon_1, x+\varepsilon_1) \subseteq A_1$. Likewise, as $x \in A_2$ and A_2 is open, $\exists \varepsilon_2 > 0$ s.t. $(x-\varepsilon_2, x+\varepsilon_2) \subseteq A_2$. Similarly, for each $i = 1, 2, \ldots, n$, we see $\exists \varepsilon_i > 0$ s.t. $(x-\varepsilon_i, x+\varepsilon_i) \subseteq A_i$.

Set $\varepsilon = \min\{\varepsilon_1, \ldots, \varepsilon_n\}$. Then $\varepsilon > 0$ and $\varepsilon \leq \varepsilon_i$ $\forall i = 1, \ldots, n$. So

$$(x-\varepsilon, x+\varepsilon) \subseteq (x-\varepsilon_i, x+\varepsilon_i) \subseteq A_i \quad \text{for } i = 1, \ldots, n$$

$$\Rightarrow \quad (x-\varepsilon, x+\varepsilon) \subseteq A_1 \cap A_2 \cap \ldots \cap A_n$$

as desired. $\square$

<u>Prop 2</u> $\forall$ set I, if $A_i \subseteq \mathbb{R}$ is open $\forall i \in I$, then $\bigcup_{i \in I} A_i = \{x \in \mathbb{R} : \exists i \in I \text{ s.t. } x \in A_i\}$ is open

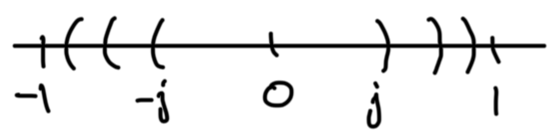
- Any union of open sets is open, even if there are infinitely many

<u>Ex</u> In particular, taking $I = \{1, 2, \ldots, n\}$ we get: $A_1, \ldots, A_n$ open $\Rightarrow$ $A_1 \cup \ldots \cup A_n$ open.

<u>Ex</u> $I = (0,1)$, $A_j = (-j, j)$ for $j \in I$.

$$\bigcup_{j \in I} A_j = \{x \in \mathbb{R} : \exists j \in (0,1) \text{ s.t. } -j < x < j\}$$

$$= (-1, 1)$$



is open.

<u>Pf</u>: Suppose $A_i \subseteq \mathbb{R}$ is open $\forall i \in I$. Let $x \in \bigcup_{i\in I} A_i$. Then $\exists j \in I$ s.t. $x \in A_j$. As A_j is open, $\exists \varepsilon > 0$ s.t. $(x-\varepsilon, x+\varepsilon) \subseteq A_j$. Then $(x-\varepsilon, x+\varepsilon) \subseteq A_j$, since:

$$y \in (x-\varepsilon, x+\varepsilon) \Rightarrow y \in A_j \Rightarrow y \in \bigcup_{i\in I} A_i .$$

Therefore $\bigcup_{i \in I} A_i$ is open. $\square$

<u>Rmk</u> A set X with a collection of subsets satisfying Props 1 and 2 (and containing $\emptyset$ and X) is called a <u>topological space</u>.

LIMITS (Ch. 5)

Recall that for $a \in \mathbb{R}$, we define

$$|a| = \begin{cases} a & \text{if} \quad a \geq 0 \\ -a & \text{if} \quad a < 0 \end{cases}$$

<u>Lem</u> (Triangle inequality) $\forall a, b \in \mathbb{R}$, $|a+b| \leq |a| + |b|$.

<u>Idea</u>: 4 cases, based on the signs of a, b. See Ch. 1 for a proof.

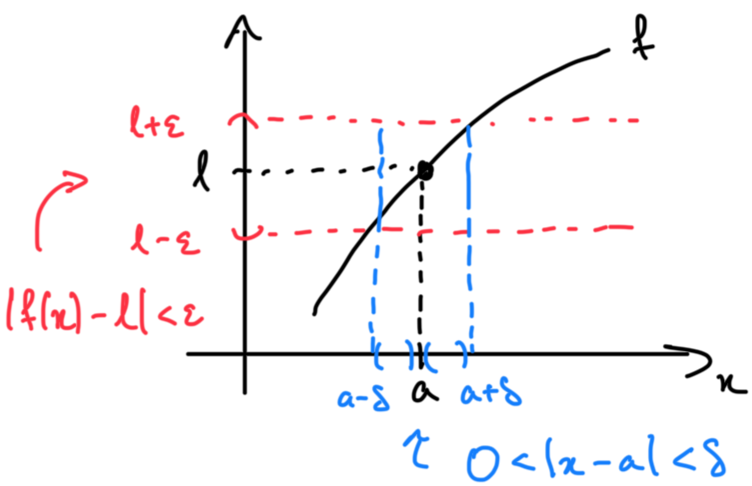
<u>Cor</u> (Reverse triangle inequality) $\forall a, b \in \mathbb{R}$, $|a+b| \geq |a| - |b|$.

Pf: $|a| = |a+b+(-b)| \underset{\text{lem}}{\leq} |a+b| + |b|$. $\square$

Def Let $f:\mathbb{R}\to\mathbb{R}$ be a function and $a, \ell \in \mathbb{R}$. We say that the <u>limit</u> of $f(x)$ as x approaches a is ℓ if: $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t.

$$x \in \mathbb{R} \text{ and } 0 < |x-a| < \delta \implies |f(x) - \ell| < \varepsilon.$$

When this is true, we write "$\lim_{x\to a} f(x) = \ell$" or "$f(x) \to \ell$ as $x \to a$".



- Heuristically: Can we make $f(x)$ as close to ℓ as we like by requiring that x is sufficiently close (but not equal!) to a

① Midterm 1 is next Friday (Oct 4), in class

② It will be based on Homeworks 1-4

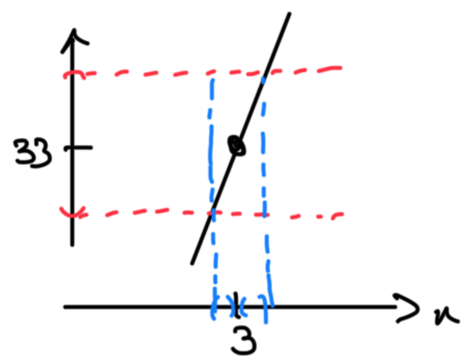
③ I will post a practice midterm later this week.

Last time: $\lim_{x \to a} f(x) = \ell \overset{\text{def}}{\iff} \forall \varepsilon > 0,\ \exists \delta > 0$ s.t. $0 < |x-a| < \delta$ implies $|f(x) - \ell| < \varepsilon$.

Ex Do the following limits exist?

① $f(x) = 4x + 21$ as $x \to 3$

Claim: $\lim_{x \to 3} f(x)$ exists and is equal to 33. Let $\varepsilon > 0$.



Scratch work: Want $\delta > 0$ s.t.

$$0 < |x-3| < \delta \implies |4x + 21 - 33| < \varepsilon$$

$$|4x - 12| = 4|x-3|$$

$$\implies |x-3| < \frac{\varepsilon}{4} \quad \leftarrow \text{Set } = \delta$$

Set $\delta = \frac{\varepsilon}{4}$. Then $\delta > 0$ and

$$0 < |x-3| < \delta \implies |f(x) - 33| = |4x + 21 - 33|$$

$$= 4|x-3| < 4\delta = \varepsilon.$$

So $\lim_{x\to 3} f(x) = 33$. $\square$

(2) $f(x) = x^2$ as $x \to 2$.

Claim: $\lim_{x\to 2} f(x) = 4$. Let $\varepsilon > 0$.

Scratch work:

$0 < |x-2| < \delta \implies |x^2-4| < \varepsilon$

$|x^2-4| = |x+2|\cdot|x-2| < \varepsilon$

$\implies |x-2| < \dfrac{\varepsilon}{|x+2|}$ ← Not good enough...

- $\delta > 0$ needs to be a constant, independent of x

As long as we choose $\underline{\delta \le 1}$, then

$|x-2| < \delta \le 1 \implies -1 < x-2 < 1$

$\implies 3 < x+2 < 5 \implies |x+2| < 5$

So

$|x^2-4| = |x+2|\cdot|x-2| < 5|x-2| < \varepsilon$

$\implies |x-2| < \frac{\varepsilon}{5}$ ← Need $\delta \le \frac{\varepsilon}{5}$

Set $\delta = \min\{1, \frac{\varepsilon}{5}\}$. Then $\delta > 0$ and

$$0 < |x-2| < \delta \implies \begin{cases} |x+2| \le |x-2| + |4| < \delta + 4 \underset{\delta\le 1}{\le} 5 \\ |x^2-4| = |x+2|\cdot|x-2| < 5\delta \underset{\delta \le \frac{\varepsilon}{5}}{\le} \varepsilon. \end{cases}$$

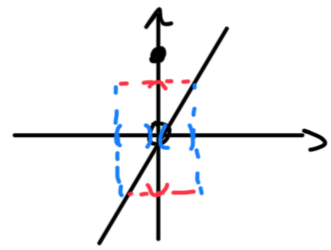
So $\lim x^2 = 4$.

$x \to 2$

- In groups:

③ $f(x) = \begin{cases} 2x & x \neq 0 \\ 1 & x = 0 \end{cases}$ as $x \to 0$.

- The value of $\lim_{x \to a} f(x)$ does not depend on $f(a)$!



Claim: $\lim_{x \to 0} f(x) = 0$.

Let $\varepsilon > 0$. Set $\delta = \frac{\varepsilon}{2}$.
Then $\delta > 0$ and

$$\underbrace{0 < |x - 0| < \delta}_{\hookrightarrow\, x \neq 0} \implies |\underbrace{f(x)}_{= 2x} - 0| = 2|x| < 2\delta = \varepsilon.$$

$\square$

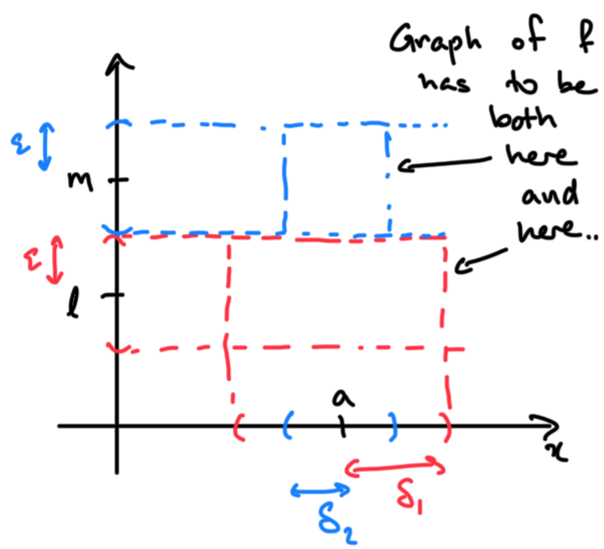
- We say "the" limit... why can't there be 2?

<u>Prop</u> (Uniqueness of limits) Let $f: \mathbb{R} \to \mathbb{R}$ be a function and $a, \ell, m \in \mathbb{R}$. If $\lim_{x \to a} f(x) = \ell$ and $\lim_{x \to a} f(x) = m$, then $\ell = m$.

<u>Pf</u>: Suppose not: $\ell \neq m$.
Set $\varepsilon = \frac{|\ell - m|}{2} > 0$. As $\lim_{x \to a} f(x) = \ell$, $\exists\, \delta_1 > 0$ s.t.

$$0 < |x - a| < \delta_1 \implies |f(x) - \ell| < \varepsilon.$$

As $\lim_{x \to a} f(x) = m$, $\exists\, \delta_2 > 0$ s.t.



$$0<|x-a|<\delta_2 \implies |f(x)-m|<\varepsilon.$$

Set $\delta=\min\{\delta_1,\delta_2\}$. Then $\delta>0$, and for any $0<|x-a|<\delta$ we have

$$\begin{aligned} 2\varepsilon = |l-m| &= |l-f(x)+f(x)-m| \\ &\leq |l-f(x)|+|f(x)-m| < \varepsilon+\varepsilon = 2\varepsilon \end{aligned}$$

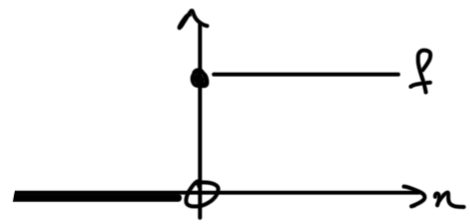
$$\implies \quad 2\varepsilon<2\varepsilon$$

which is a contradiction. $\square$

<u>Recall</u> $\lim\limits_{x \to a} f(x) = \ell \overset{\text{def}}{\iff} \forall \varepsilon > 0,\ \exists \delta > 0$ s.t.
$0 < |x-a| < \delta$ implies $|f(x) - \ell| < \varepsilon$

• How can we prove a limit <u>doesn't</u> exist?

<u>Ex</u> $f(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$ as $x \to 0$.



Claim: $\lim\limits_{x \to 0} f(x)$ does not exist. Suppose not: $\lim\limits_{x \to 0} f(x)$ exists and is equal to ℓ. Want:

$\text{not}\big(\forall \varepsilon > 0,\ \exists \delta > 0 \text{ s.t. } (x \in \mathbb{R},\ 0 < |x| < \delta \Rightarrow |f(x) - \ell| < \varepsilon)\big)$

$= \exists \varepsilon > 0$ s.t. $\forall \delta > 0,\ (x \in \mathbb{R},\ 0 < |x| < \delta,$ and $|f(x) - \ell| \geq \varepsilon)$

Set $\varepsilon = \frac{1}{2}$. Let $\delta > 0$.

• Case: $\ell \geq \frac{1}{2}$. Then $x = -\frac{\delta}{2}$ satisfies

$0 < |x| < \delta$ and $|\underbrace{f(x)}_{=0} - \ell| = \ell \geq \frac{1}{2}$.

• Case: $\ell < \frac{1}{2}$. Then $x = +\frac{\delta}{2}$ satisfies

$0 < |x| < \delta$ and $|\underbrace{f(x)}_{=1} - \ell| = 1 - \ell > \frac{1}{2}$.

A contradiction — so no such ℓ can exist. $\square$

Theorem (Limit laws) Let $f, g : \mathbb{R} \to \mathbb{R}$ be functions

Theorem (Limit laws) Let $f, g : \mathbb{R} \to \mathbb{R}$ be functions and $a \in \mathbb{R}$. If $\lim_{x \to a} f(x) = \ell$ and $\lim_{x \to a} g(x) = m$, then:

① For any $c \in \mathbb{R}$, $\lim_{x \to a} (cf)(x) = c\ell$.

② $\lim_{x \to a} (f+g)(x) = \ell + m$

③ $\lim_{x \to a} (f \cdot g)(x) = \ell \cdot m$.

Rmk If $f, g : \mathbb{R} \to \mathbb{R}$ are functions, then

- $cf : \mathbb{R} \to \mathbb{R}$, $(cf)(x) = c \cdot f(x)$
- $f+g : \mathbb{R} \to \mathbb{R}$, $(f+g)(x) = f(x) + g(x)$
- $f \cdot g : \mathbb{R} \to \mathbb{R}$, $(f \cdot g)(x) = f(x) \cdot g(x)$

are all functions.

Pf: ① Let $\varepsilon > 0$. As $\lim_{x \to a} f(x) = \ell$, we know $\exists \delta > 0$ s.t.

$$0 < |x-a| < \delta \implies |f(x) - \ell| < \frac{\varepsilon}{|c|+10}$$ ← c could be zero!

So, for $0 < |x-a| < \delta$ we have

$$|c \cdot f(x) - c \cdot \ell| = |c| \cdot |f(x) - \ell| \leq |c| \cdot \frac{\varepsilon}{|c|+10} < \varepsilon$$

Therefore $\lim_{x \to a} (c \cdot f)(x) = c \cdot \ell$.

- $cf(x) - c \cdot \ell$ is small because it is $c \cdot (\text{small})$

② Let $\varepsilon > 0$. As $\lim_{x \to a} f(x) = \ell$, $\exists\, \delta_1 > 0$ s.t.

$$0 < |x-a| < \delta_1 \quad \Rightarrow \quad |f(x) - l| < \frac{\varepsilon}{2}.$$

Likewise, as $\lim_{x \to a} g(x) = m$, $\exists\, \delta_2 > 0$ s.t.

$$0 < |x-a| < \delta_2 \quad \Rightarrow \quad |g(x) - m| < \frac{\varepsilon}{2}.$$

Set $\delta = \min\{\delta_1, \delta_2\} > 0$. Then for $0 < |x-a| < \delta$,

$$\begin{aligned} |f(x) + g(x) - (l+m)| &\leq |f(x) - l| + |g(x) - m| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

So $\lim_{x \to a} (f+g)(x) = l + m$.

- $f(x) + g(x) - (l+m)$ is small because it is equal to $(f(x) - l) + (g(x) - m)$ = small + small.

③ Let $\varepsilon > 0$. As $\lim_{x \to a} f(x) = l$, $\exists\, \delta_1 > 0$ s.t.

$$0 < |x-a| < \delta_1 \quad \Rightarrow \quad |f(x) - l| < \frac{\varepsilon}{2(|m|+10)}$$

Likewise, as $\lim_{x \to a} g(x) = m$, $\exists\, \delta_2 > 0$ s.t.

$$0 < |x-a| < \delta_2 \quad \Rightarrow \quad |g(x) - m| < \min\left\{\frac{\varepsilon}{2(|l|+10)}, 1\right\}$$

Set $\delta = \min\{\delta_1, \delta_2\} > 0$. Then, for $0 < |x-a| < \delta$,

$$\begin{aligned} |f(x) \cdot g(x) - l \cdot m| &= |f(x)g(x) - lg(x) + lg(x) - lm| \\ &\leq |f(x)g(x) - lg(x)| + |lg(x) - lm| \\ &= \underbrace{|g(x)|}_{} \cdot |f(x) - l| + |l| \cdot |g(x) - m| \end{aligned}$$

$$\leq |g(x)-m| + |m| \leq 1+|m|$$

$$< (|m|+1) \cdot \frac{\varepsilon}{2(|m|+10)} + |\ell| \cdot \frac{\varepsilon}{2(|\ell|+10)}$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

So $\lim_{x \to a} (f \cdot g)(x) = \ell \cdot m.$ $\square$

<u>Ex</u> Prove $\lim_{x \to a} (4x^3 + 21x) = 4a^3 + 21a \quad \forall a \in \mathbb{R}.$

- This would be rather cumbersome to prove using ε and δ!

<u>Pf</u>: Let $a \in \mathbb{R}$. Note that $\lim_{x \to a} x = a$. Indeed, given $\varepsilon > 0$, we can take $\delta = \varepsilon$ and

$$0 < |x-a| < \delta \implies |x-a| < \delta = \varepsilon.$$

So, by the limit laws, we have

- $\lim_{x \to a} 21x \overset{1}{=} 21a$
- $\lim_{x \to a} x^2 = \lim_{x \to a} (x \cdot x) \overset{3}{=} a \cdot a = a^2$
- $\lim_{x \to a} x^3 = \lim_{x \to a} (x \cdot x^2) \overset{3}{=} a \cdot a^2 = a^3$
- $\lim_{x \to a} 4x^3 \overset{1}{=} 4a^3$
- $\lim_{x \to a} (4x^3 + 21x) \overset{2}{=} 4a^3 + 21a.$ $\square$

Recall $\lim_{x\to a} f(x) = \ell \overset{\text{def}}{\iff} \forall \varepsilon>0,\ \exists \delta>0$ s.t.
$0<|x-a|<\delta$ implies $|f(x)-\ell|<\varepsilon$

Ex $\forall a>0$, $\lim_{x\to a} \sqrt{x} = \sqrt{a}$.

Let $a>0$ and $\varepsilon>0$.

Scratch work:

$$|\sqrt{x}-\sqrt{a}| = \frac{|x-a|}{|\sqrt{x}+\sqrt{a}|}$$

$\leftarrow$ Good, we know $|x-a|<\delta$

$\leftarrow$ Not good enough.
Want $\frac{1}{|\sqrt{x}+\sqrt{a}|} <$ constant.

As long as we pick $\delta \le a$, then

$$0<|x-a|<\delta\le a \implies -a<x-a<a$$
$$\implies 0<x \implies 0<\sqrt{x}$$
$$\implies \sqrt{a}<\sqrt{x}+\sqrt{a}$$
$$\implies 0<\frac{1}{\sqrt{x}+\sqrt{a}}<\frac{1}{\sqrt{a}}$$

So

$$|\sqrt{x}-\sqrt{a}| = \frac{|x-a|}{|\sqrt{x}+\sqrt{a}|} < \frac{\delta}{\sqrt{a}} \le \varepsilon \implies \text{Need } \delta\le\sqrt{a}\cdot\varepsilon.$$

Set $\delta = \min\{a, \sqrt{a}\cdot\varepsilon\}$. Then $\delta>0$ and

$$0<|x-a|<\delta \implies \begin{cases} x-a>-\delta \overset{\delta\le a}{\ge} -a \implies x>0 \implies \sqrt{x}>0 \\ |\sqrt{x}-\sqrt{a}| = \dfrac{|x-a|}{\sqrt{x}+\sqrt{a}} < \dfrac{\delta}{\sqrt{a}} \underset{\delta\le\sqrt{a}\cdot\varepsilon}{\le} \varepsilon \end{cases}$$

So $\lim_{x \to a} \sqrt{x} = \sqrt{a}$. $\square$

<u>Thm</u> (Limit law for division) If $\lim_{x \to a} f(x) = \ell$ and $\lim_{x \to a} g(x) = m \neq 0$, then $\lim_{x \to a} \left(\frac{f}{g}\right)(x) = \frac{\ell}{m}$.

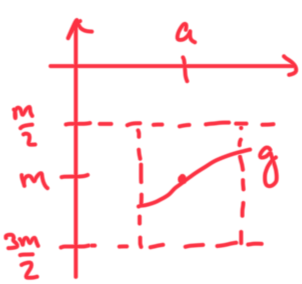
<u>Pf</u>: It suffices to show $\lim_{x \to a} \left(\frac{1}{g}\right)(x) = \frac{1}{m}$. Indeed, once we prove this, then by the limit law for multiplication (Lecture 11) we'll have $\lim_{x \to a} \left(f \cdot \frac{1}{g}\right)(x) = \ell \cdot \frac{1}{m}$.

Let $\varepsilon > 0$. As $\lim_{x \to a} g(x) = m$ and $m \neq 0$, $\exists\, \delta > 0$ s.t.

$$0 < |x-a| < \delta \implies |g(x) - m| < \min\left\{ \frac{|m|}{2}, \frac{|m|^2 \varepsilon}{2} \right\}$$

Then, for $0 < |x-a| < \delta$ we have

$$\left| \frac{1}{g(x)} - \frac{1}{m} \right| = \frac{|g(x) - m|}{|m| \cdot |g(x)|} < \frac{|g(x)-m|}{|m| \cdot \frac{|m|}{2}} < \varepsilon$$



$$|g(x) - m| < \frac{|m|}{2}$$

$$\implies -\frac{|m|}{2} < g(x) - m < \frac{|m|}{2}$$

$$\implies \begin{cases} m > 0: & g(x) > m - \frac{m}{2} = \frac{m}{2} \\ m < 0: & g(x) < m + \frac{(-m)}{2} = \frac{m}{2} \end{cases}$$

$$\implies |g(x)| > \frac{|m|}{2} \implies \frac{1}{|g(x)|} < \frac{1}{|m|/2}$$

So $\lim_{x \to a} \frac{1}{g(x)} = \frac{1}{m}$. $\square$

<u>Ex</u> Prove that $\lim_{x\to a} \frac{x^3+4x}{x^2+21} = \frac{a^3+4a}{a^2+21}$ $\forall a \in \mathbb{R}$.

<u>Pf</u>: Let $a \in \mathbb{R}$.

Claim: $\lim_{x\to a}(x^3+4x) = a^3+4a$ and $\lim_{x\to a}(x^2+21) = a^2+21$.

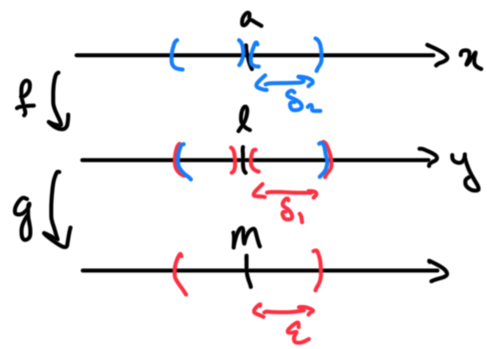
Exercise. (Just like lecture 11.)

Note that $a^2+21 \neq 0$ since $a^2 \geq 0$. So, by the limit law for division, $\lim_{x\to a} \frac{x^3+4x}{x^2+21} = \frac{a^3+4a}{a^2+21}$.

<u>Thm</u> (Limit law for composition) If $\lim_{x\to a} f(x) = \ell$, $\lim_{y\to \ell} g(y) = m$, and $g(\ell) = m$, then $\lim_{x\to a} (g\circ f)(x) = m$.

<u>Pf</u>: Fix $\varepsilon > 0$. As $\lim_{y\to\ell} g(y) = m$, $\exists\, \delta_1 > 0$ s.t.

$0 < |y-\ell| < \delta_1 \Rightarrow |g(y)-m| < \varepsilon$.



In fact, since $g(\ell) = m$, we have

$$|y-\ell| < \delta_1 \overset{①}{\Rightarrow} |g(y)-m| < \varepsilon.$$

As $\lim_{x\to a} f(x) = \ell$, $\exists\, \delta_2 > 0$ s.t.

$$0 < |x-a| < \delta_2 \overset{②}{\Rightarrow} |f(x)-\ell| < \delta_1$$

Altogether:

$$0 < |x-a| < \delta_2 \overset{②}{\Rightarrow} |f(x)-\ell| < \delta_1$$
$$\overset{①}{\Rightarrow} |g(f(x))-m| < \varepsilon.$$

So $\lim_{x\to a} (g\circ f)(x) = m.$ $\square$

Ex Prove that $\lim_{x\to a} \sqrt{x^2+21} = \sqrt{a^2+21}$ $\forall a \in \mathbb{R}$.

Pf: Note that $\sqrt{x^2+21} = (g\circ f)(x)$ for

$$f(x) = x^2+21 \quad \text{and} \quad g(y) = \sqrt{y}\,.$$

We already saw $\lim_{x\to a} f(x) = a^2+21$ (in the previous example). As $l = a^2+21 > 0$, then $\lim_{y\to l} g(y) = \sqrt{l}$ $= \sqrt{a^2+21}$ (by the first example today). So, by the limit law for composition, $\lim_{x\to a} \sqrt{x^2+21} = \sqrt{a^2+21}$.

① No office hours next week.

② Homework 5 will be due 2 weeks from today.

③ Class will be held as usual next week.

Def Let $f:\mathbb{R}\to\mathbb{R}$ be a function and $a, l \in \mathbb{R}$.
We say:

① $\lim_{x\to a^+} f(x) = l$ (or $\lim_{x\downarrow a} f(x) = l$) if:

$\forall \varepsilon > 0, \exists \delta > 0$ s.t. $0 < x-a < \delta \Rightarrow |f(x)-l| < \varepsilon$.

② $\lim_{x\to a^-} f(x) = l$ (or $\lim_{x\uparrow a} f(x) = l$) if:

$\forall \varepsilon > 0, \exists \delta > 0$ s.t. $-\delta < x-a < 0 \Rightarrow |f(x)-l| < \varepsilon$.

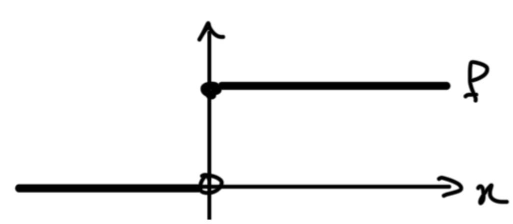
Rmk For any $f:\mathbb{R}\to\mathbb{R}$ and $a, l \in \mathbb{R}$, we have

$$\lim_{x\to a} f(x) = l \iff \lim_{x\to a^+} f(x) = l = \lim_{x\to a^-} f(x).$$

Indeed,

$$0 < |x-a| < \delta \iff 0 < x-a < \delta \text{ or } -\delta < x-a < 0.$$

Ex $f(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$



① $\lim_{x\to 0^+} f(x) = 1$: Let $\varepsilon > 0$. Set $\delta = 10$.

Then $0 < x - 0 < \delta \Rightarrow |\underbrace{f(x)}_{=1} - 1| = 0 < \varepsilon$.

② $\lim_{x \to 0^-} f(x) = 0$: Let $\varepsilon > 0$. Set $\delta = 10$.

Then $-\delta < x - 0 < 0 \implies |\underbrace{f(x)}_{=0} - 0| = 0 < \varepsilon$.

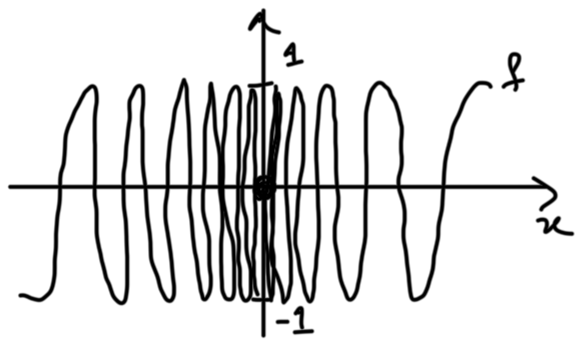
③ $\lim_{x \to 0} f(x)$ does not exist: since

$$\lim_{x \to 0^-} f(x) = 0 \neq 1 = \lim_{x \to 0^+} f(x).$$

□

- How else can $\lim_{x \to a} f(x)$ fail to exist?

<u>Ex</u> $f(x) = \begin{cases} \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$



Claim: $\lim_{x \to 0} f(x)$ does not exist. Suppose not: $\exists \ell \in \mathbb{R}$ s.t. $\lim_{x \to 0} f(x) = \ell$. Note that for any $n \in \mathbb{N}$,

$$f\left(\frac{2}{(2n+1)\pi}\right) = \sin \frac{(2n+1)\pi}{2} = \begin{cases} 1 & n \text{ even} \\ -1 & n \text{ odd} \end{cases}$$

Set $\varepsilon = 1$. Let $\delta > 0$.

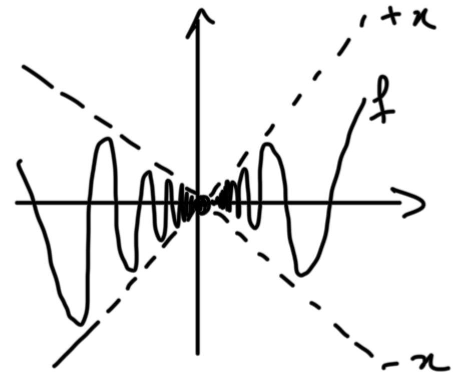
- Case: $\ell \geq 0$. Let $n \in \mathbb{N}$ be an odd number large enough so that $\frac{2}{(2n+1)\pi} < \delta$. Then $x = \frac{2}{(2n+1)\pi}$ satisfies

$$0 < |x| < \delta \quad \text{and} \quad |\underbrace{f(x)}_{=-1} - \ell| = \ell + 1 \geq 1 = \varepsilon.$$

- Case: $\ell < 0$. Let $n \in \mathbb{N}$ be an even number large enough so that $\frac{2}{(2n+1)\pi} < \delta$. Then $x = \frac{2}{(2n+1)\pi}$ satisfies

$$0 < |x| < \delta \quad \text{and} \quad |\underbrace{f(x)}_{=1} - \ell| = 1 - \ell \geq 1 = \varepsilon.$$

In both cases, we have a contradiction. $\square$



Ex $f(x) = \begin{cases} x \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$

Claim: $\lim_{x \to 0} f(x) = 0$.

Let $\varepsilon > 0$. Set $\delta = \varepsilon$. Then $\delta > 0$ and

$$0 < |x - 0| < \delta \implies |f(x) - 0| = |x| \cdot |\sin \tfrac{1}{x}| \leq |x| < \delta = \varepsilon$$

$\square$

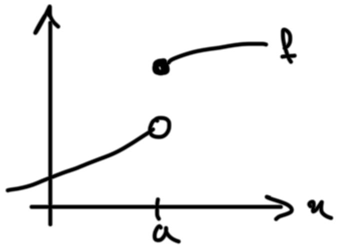
Continuity (Ch. 6)

Def Let $f : \mathbb{R} \to \mathbb{R}$ be a function and $a \in \mathbb{R}$. We say f is continuous at a if $\lim_{x \to a} f(x) = f(a)$. In other words: $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t.

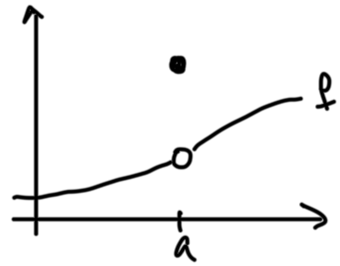
$$|x - a| < \delta \implies |f(x) - \underbrace{f(a)}_{\text{Limit is } f(a).}| < \varepsilon.$$

No "$0 <$".
We allow $x = a$ now.

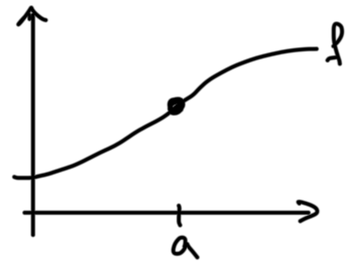
Limit is $f(a)$.



Not continuous: $\lim_{x\to a} f(x)$ does not exist



Not continuous: $\lim_{x\to a} f(x)$ exists, not equal to $f(a)$

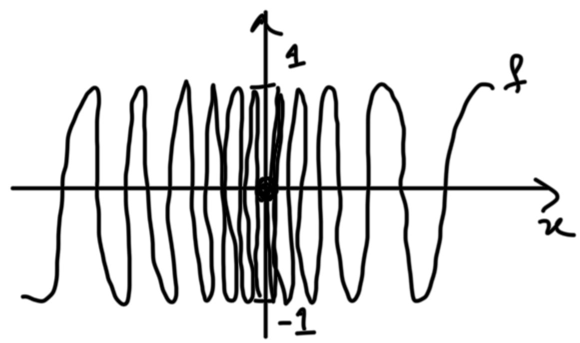


Continuous: $\lim_{x\to a} f(x) = f(a)$

- In groups:

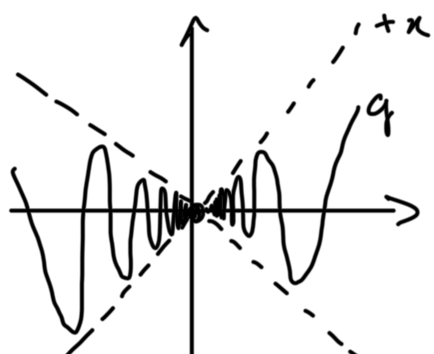
<u>Ex</u> Are the following functions continuous at $x=0$?

① $f(x) = \begin{cases} \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$



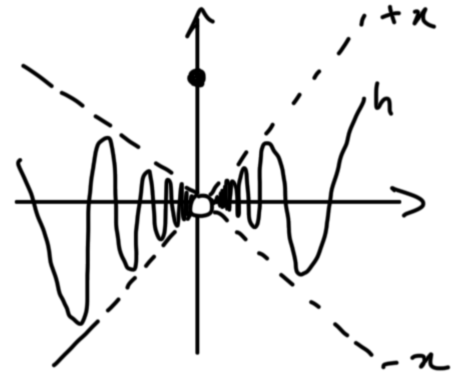
No, f is not continuous at $x=0$, since $\lim_{x\to 0} f(x)$ does not exist (by the second example today).

② $g(x) = \begin{cases} x \cdot \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$



Yes, g is continuous at $x=0$, since we know $\lim_{x\to 0} g(x) = 0 = g(0)$ (by the third example today).

(3) $h(x) = \begin{cases} x \cdot \sin \frac{1}{x} & x \neq 0 \\ 10 & x = 0 \end{cases}$



No, h is not continuous at $x = 0$, since

$$\lim_{x \to 0} h(x) = 0 \neq h(0).$$

- Remember: $\lim_{x \to 0} h(x)$ does not depend on $h(0)$.

As $h(x) = g(x)$ $\forall x \neq 0$, then $\lim_{x \to 0} h(x) = \lim g(x)$.

Last time: f is continuous at $a \overset{\text{def}}{\iff} \lim_{x\to a} f(x) = f(a)$

- We can combine continuous functions with addition/multiplication:

Thm Let $f, g : \mathbb{R} \to \mathbb{R}$ be functions and $a \in \mathbb{R}$.
If f and g are continuous at a, then:
① cf is continuous at a, $\forall c \in \mathbb{R}$.
② $f+g$ is continuous at a.
③ $f \cdot g$ is continuous at a.

Pf: As f and g are continuous at a, we know

$$\lim_{x\to a} f(x) = f(a) \quad \text{and} \quad \lim_{x\to a} g(x) = g(a).$$

So, by the limit laws for addition/multiplication (Lecture 11), we have:

① $\lim_{x\to a} (cf)(x) = c \cdot f(a) \quad \forall c \in \mathbb{R}$.

② $\lim_{x\to a} (f+g)(x) = f(a) + g(a)$.

③ $\lim_{x\to a} (f \cdot g)(x) = f(a) \cdot g(a)$. $\square$

- We can also divide, if the denominator is nonzero:

Thm If f and g are continuous at a and $g(a) \neq 0$, then $\frac{f}{g}$ is continuous at a.

Pf: As f and g are continuous at a, we know

$$\lim_{x\to a} f(x) = f(a) \quad \text{and} \quad \lim_{x\to a} g(x) = g(a).$$

Together with $g(a) \neq 0$, we conclude

$$\lim_{x\to a} \left(\frac{f}{g}\right)(x) = \frac{f(a)}{g(a)}$$

by the limit law for division (Lecture 12). $\square$

· Lastly, we can also compose continuous functions:

Thm If f is continuous at a and g is continuous at $f(a)$, then $g \circ f$ is continuous at a.

Pf: We know

$$\lim_{x\to a} f(x) = f(a) \quad \text{and} \quad \lim_{y\to f(a)} g(y) = g(f(a)).$$

So, by the limit law for composition (Lecture 12) we have

$$\lim_{x\to a} (g\circ f)(x) = g(f(a)). \qquad \square$$

Ex $f(x) = 4x^3 + 21x$ is continuous at a, $\forall a \in \mathbb{R}$.

In lecture 11, we proved $\lim_{x\to a} (4x^3 + 21x) = 4a^3 + 21a$ $\forall a \in \mathbb{R}$.

Ex Let $p : \mathbb{R} \to \mathbb{R}$ be a polynomial: $\exists\, N \in \mathbb{N} \cup \{0\}$ and $c_0, c_1, \ldots, c_N \in \mathbb{R}$ s.t.

$$p(x) = c_0 + c_1 x + c_2 x^2 + \dots + c_N x^N.$$

Then p is continuous at a $\forall a \in \mathbb{R}$.

Pf: Let $a \in \mathbb{R}$.

Claim: The function $f(x) = 1$ is continuous at a. Given $\varepsilon > 0$, we can take $\delta = 10$ and

$$0 < |x-a| < \delta \implies |f(x) - 1| = 0 < \varepsilon.$$

Claim: The function x^n is continuous at a $\forall n \in \mathbb{N}$. We will argue by induction:

1. Base case: $\lim_{x \to a} x = a$. We did this on Lecture 11. (Just take $\delta = \varepsilon$.)
2. Inductive step: Assume $\lim_{x \to a} x^n = a^n$ for some $n \in \mathbb{N}$. By the limit law for multiplication,

$$\lim_{x \to a} x^n = a^n, \quad \lim_{x \to a} x = a \implies \lim_{x \to a} x^{n+1} = a^{n+1}.$$

So x^{n+1} is continuous at a.

By induction, we conclude that x^n is continuous at a $\forall n \in \mathbb{N}$.

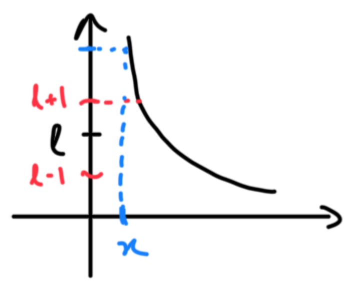
Let $p(x) = c_0 + c_1 x + \dots + c_N x^N$. By the claims, we know x^n is continuous at a for $n = 0$ and for $n = 1, 2, \dots, N$. Then $c_n x^n$ is continuous at a for each $n = 0, 1, \dots, N$ (by the first theorem today). Therefore $p(x) = c_0 + c_1 x + \dots + c_N x^N$ is continuous at a (by the same theorem). $\square$

<u>Ex</u> $f(x) = \begin{cases} \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

① f is continuous at a $\forall a \neq 0$: By the previous example, $g(x) = x$ is continuous at a $\forall a \in \mathbb{R}$. Given $a \neq 0$, we have $g(a) = a \neq 0$, and so $f(x) = \frac{1}{g(x)}$ is also continuous at a.

② f is not continuous at $x = 0$: Want $\lim_{x \to 0} f(x) \neq 0$. In fact, we'll prove $\lim_{x \to 0} f(x)$ does not exist. Suppose towards a contradiction $\lim_{x \to 0} f(x) = \ell$ for some $\ell \in \mathbb{R}$. Consider $\varepsilon = 1$.

Let $\delta > 0$. Then $x = \min\left\{ \frac{\delta}{2}, \frac{1}{|\ell|+10} \right\}$ satisfies



$$0 < x \leq \frac{\delta}{2} \quad \Rightarrow \quad 0 < |x| < \delta$$

$$0 < x \leq \frac{1}{|\ell|+10} \quad \Rightarrow \quad f(x) = \frac{1}{x} \geq |\ell| + 10$$

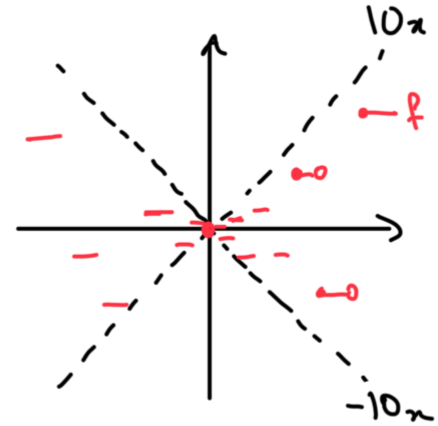
$$\Rightarrow \quad |f(x) - \ell| \geq \big||\ell|+10\big| - |\ell| = 10 \geq \varepsilon$$

So $\lim_{x \to 0} f(x) \neq \ell$, a contradiction. $\square$

Recall: f continuous at $a \iff \lim_{x\to a} f(x) = f(a)$

$\iff \forall \varepsilon>0, \exists \delta>0$ s.t. $|x-a|<\delta$ implies $|f(x)-f(a)|<\varepsilon$.

Ex Suppose $f:\mathbb{R}\to\mathbb{R}$ satisfies $|f(x)|\le 10|x|$ $\forall x\in\mathbb{R}$. Prove that f is continuous at 0.



Pf: Note that $|f(0)|\le 0$, and so $f(0)=0$. Want: $\lim_{x\to 0} f(x)=0$.

Let $\varepsilon>0$. Set $\delta=\frac{\varepsilon}{10}$. Then

$$|x|<\delta \Rightarrow |f(x)-0| = |f(x)| \le 10|x| < 10\delta = \varepsilon.$$

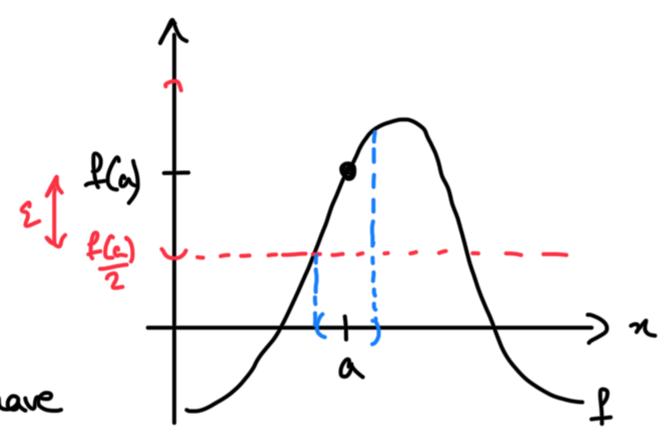
$\square$

Prop Let $f:\mathbb{R}\to\mathbb{R}$ and $a\in\mathbb{R}$. If f is continuous at a and $f(a)>0$, then $\exists\delta>0$ s.t. $f(x)>0$ $\forall x\in(a-\delta, a+\delta)$.

Pf: Consider $\varepsilon=\frac{f(a)}{2}>0$.

As f is continuous at a, $\exists\delta>0$ s.t.

$|x-a|<\delta \Rightarrow |f(x)-f(a)|<\varepsilon$.



So, for $x\in(a-\delta, a+\delta)$ we have

$$-\varepsilon < f(x)-f(a) < \varepsilon \Rightarrow f(x) > f(a)-\varepsilon = f(a)-\frac{f(a)}{2} = \frac{f(a)}{2} > 0$$

as desired. $\square$

<u>Rmk</u> What if $f(a) < 0$? Applying the Proposition to $g(x) = -f(x)$, we get

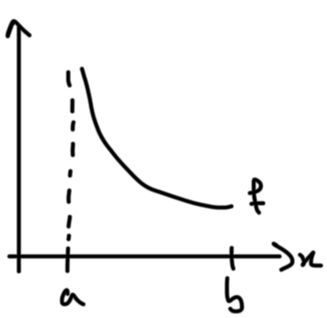
$$\left.\begin{array}{l} g \text{ continuous at } a \\ g(a) = -f(a) > 0 \end{array}\right\} \Rightarrow \exists\, \delta > 0 \text{ s.t. } g(x) > 0 \quad \forall x \in (a-\delta, a+\delta)$$

$$\Rightarrow f(x) = -g(x) < 0 \quad \forall x \in (a-\delta, a+\delta)$$

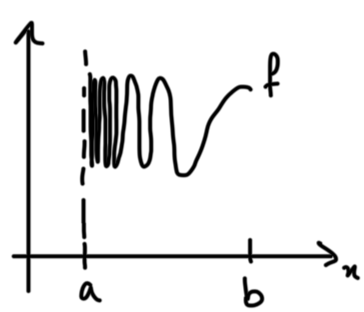
<u>Def</u> Let $a<b$. We say:

1. $f: \mathbb{R} \to \mathbb{R}$ is <u>continuous</u> if f is continuous at x $\forall x \in \mathbb{R}$.
2. $f: (a,b) \to \mathbb{R}$ is <u>continuous</u> if f is continuous at x $\forall x \in (a,b)$.
3. $f: [a,b] \to \mathbb{R}$ is <u>continuous</u> if f is continuous at x $\forall x \in (a,b)$, $\lim_{x \to a^+} f(x) = f(a)$, and $\lim_{x \to b^-} f(x) = f(b)$.

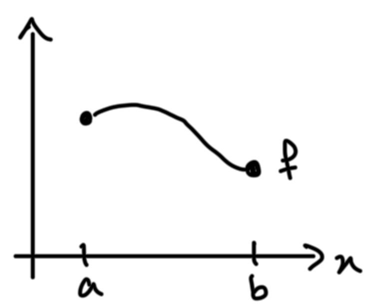
- It depends not just on f, but also the domain!



Continuous on (a,b).
Not continuous on $[a,b]$.



Continuous on (a,b).
Not continuous on $[a,b]$.



Continuous on $[a,b]$.

- Similarly, the domain can be any interval I.
- If I includes an endpoint, we require that the one-sided limit of f exists at that point.

Ex ① Polynomials $p(x)$ are continuous on $\mathbb{R}$.

② $\frac{1}{x}$ is continuous on $(-\infty, 0)$ and $(0, \infty)$.

③ Rational functions $\frac{p(x)}{q(x)}$, where p, q are polynomials, are continuous where $q(x) \neq 0$.

④ $|x|$ is continuous on $\mathbb{R}$.

⑤ $\sqrt{x}$ is continuous on $[0, \infty)$.

⑥ e^x, $\sin x$, $\cos x$ are continuous on $\mathbb{R}$.

⑦ Any sum/product/composition of these are continuous, on the domain where they are defined. E.g. $\sin \frac{1}{x}$ is continuous on $(0, \infty)$ by ② and ⑥.

- We've already proved ① and ②. ③ is similar.
- ④ is on HW5.
- We won't prove all of these, but here is ⑤:

Ex Prove $f : [0, \infty) \to \mathbb{R}$, $f(x) = \sqrt{x}$ is continuous.

Pf: There are 2 conditions to check.

① f is continuous at a $\forall a \in (0, \infty)$: Recall from lecture 12 that $\lim_{x \to a} \sqrt{x} = \sqrt{a}$ $\forall a > 0$.

② $\lim_{x \to 0^+} f(x) = 0$: Let $\varepsilon > 0$. Set $\delta = \varepsilon^2$. Then $\delta > 0$ and

$$0 < x < \delta \implies |f(x) - 0| = \sqrt{x} < \sqrt{\delta} = \varepsilon. \qquad \square$$

<u>Ex</u> Prove $g(x) = 4x^3 + 21x$ is continuous on $[1,2]$.

<u>Pf</u>: As g is a polynomial, we know $\lim_{x\to a} g(x) = g(a)$ $\forall a \in \mathbb{R}$ (by Lecture 14). In particular,

① $\lim_{x\to a} g(x) = g(a) \quad \forall a \in (1,2)$

② $\lim_{x\to 1^+} g(x) = g(1)$

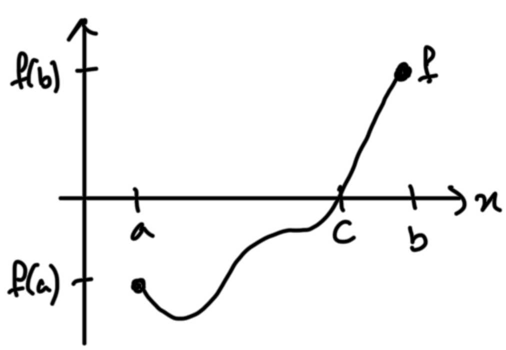
③ $\lim_{x\to 2^-} g(x) = g(2)$

(Remember, $\lim_{x\to a} g(x) = \ell \implies \lim_{x\to a^+} g(x) = \ell = \lim_{x\to a^-} g(x)$.)

Therefore g is continuous on $[1,2]$. $\square$

3 Important Theorems (Ch. 7)

<u>Thm 1</u> If $f : [a,b] \to \mathbb{R}$ is continuous and $f(a) < 0 < f(b)$, then $\exists c \in [a,b]$ s.t. $f(c) = 0$.



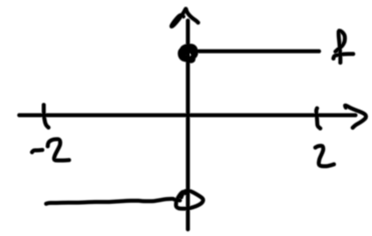
- In order to go from below to above the x-axis, a continuous function <u>must</u> cross the x-axis at some point

- We'll prove this later (Ch. 8)
- First, let's digest the statement of the theorem

<u>Q</u>: Why do we need "continuous"?

<u>A</u>: E.g. $f : [-2, 2] \to \mathbb{R}$, $\quad f(x) = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases}$

Then $f(-2) < 0 < f(2)$,
but $\nexists\, c \in [-2, 2]$ s.t. $f(c) = 0$.



<u>Intermediate value theorem</u> (IVT) Suppose $f : [a, b] \to \mathbb{R}$ is continuous. If $y \in \mathbb{R}$ satisfies $f(a) < y < f(b)$ or $f(a) > y > f(b)$, then $\exists\, c \in (a, b)$ s.t. $f(c) = y$.

- Continuous functions can't skip a value.

<u>Pf</u>: Case: $f(a) < y < f(b)$. Consider the function $g : [a, b] \to \mathbb{R}$, $g(x) = f(x) - y$. Then g is continuous, since it's the sum of the continuous functions $f(x)$ and $-y$. Moreover, $g(a) < 0 < g(b)$. So, by Theorem 1, $\exists\, c \in [a, b]$ s.t.

$$0 = g(c) = f(c) - y \quad \Rightarrow \quad f(c) = y.$$

Note $c \in (a, b)$, since $f(a), f(b) \neq y$.

Case: $f(a) > y > f(b)$. Consider $h : [a, b] \to \mathbb{R}$, $h(x) = y - f(x)$. Then h is continuous and $h(a) < 0 < h(b)$. Apply Theorem 1 as before. $\square$

<u>Cor</u> $\forall a>0$, $\exists \sqrt{a}\in\mathbb{R}$ s.t. $\sqrt{a}>0$ and $(\sqrt{a})^2=a$.

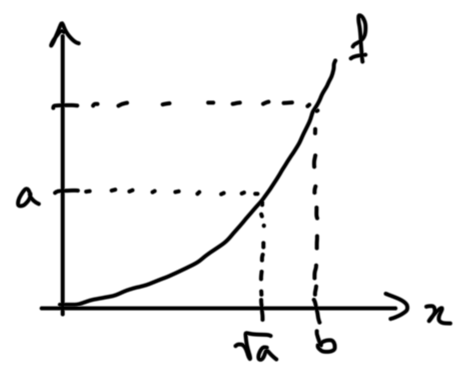
- Strictly speaking, we haven't proved this yet
- In fact, we <u>couldn't</u> have proved this with what we know so far!

<u>Rmk</u> Recall that both $\mathbb{Q}$, $\mathbb{R}$ are ordered fields. But the corollary is false for $\mathbb{Q}$: e.g. $2\in\mathbb{Q}$ but $\sqrt{2}\notin\mathbb{Q}$. So Theorem 1 can only be true for $\mathbb{R}$, not $\mathbb{Q}$, and we need another property of $\mathbb{R}$ to prove it.

- We'll introduce this property next week

<u>Pf</u> (of Cor): Let $a>0$.
Consider $f(x)=x^2$ and

$$b=\begin{cases} a & \text{if } a>1 \\ 2 & \text{if } a\le 1.\end{cases}$$



Then:

① $f:[0,b]\to\mathbb{R}$ is continuous: since it's a polynomial.

② $\underbrace{f(0)}_{0}<a<\underbrace{f(b)}_{=\begin{cases} a^2 & \text{if } a>1 \\ 4 & \text{if } a\le 1\end{cases}}$

So, by the IVT, $\exists \underbrace{c\in(0,b)}_{\sqrt{a}>0}$ s.t. $\underbrace{f(c)=a}_{(\sqrt{a})^2=a}$. Set $\sqrt{a}=c$. $\square$

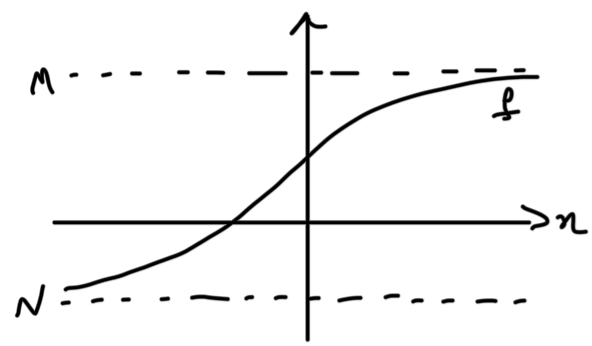
- Next: 2^{nd} important theorem

<u>Def</u> Let $A \subseteq \mathbb{R}$ and $f: A \to \mathbb{R}$. We say:

① f is <u>bounded above</u> if $\exists M \in \mathbb{R}$ s.t.
$f(x) \le M \quad \forall x \in A.$

② f is <u>bounded below</u> if $\exists N \in \mathbb{R}$ s.t.
$f(x) \ge N \quad \forall x \in A.$

③ f is <u>bounded</u> if both ① and ② are true.



<u>Thm 2</u> If $f: [a,b] \to \mathbb{R}$ is continuous, then f is bounded above.

- We'll also prove this later (Ch. 8)

<u>Cor</u> If $g: [a,b] \to \mathbb{R}$ is continuous, then g is bounded.

<u>Pf</u>: Bounded above: by Theorem 2.

Bounded below: As $-g$ is also continuous on $[a,b]$, then by Theorem 2, $\exists M \in \mathbb{R}$ s.t.

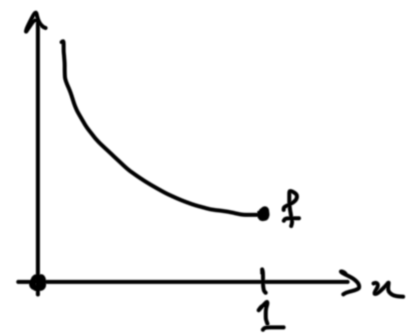
$$-g(x) \le M \quad \forall x \in [a,b] \implies g(x) \ge -M \quad \forall x \in [a,b]$$

So g is bounded below by $-M$. $\square$

<u>Q</u>: Why do we need "continuous"?

<u>A</u>: E.g. $f : [0,1] \to \mathbb{R}$,

$$f(x) = \begin{cases} \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$



Then f is not bounded above.

Last time: 3 important theorems

① IVT (special case):

$$\left.\begin{array}{l} f:[a,b]\to\mathbb{R} \text{ continuous} \\ f(a)<0<f(b) \end{array}\right\} \Rightarrow \begin{array}{l}\exists c\in[a,b] \text{ s.t.} \\ f(c)=0\end{array}$$

② $f:[a,b]\to\mathbb{R}$ continuous $\Rightarrow$ f is bounded above

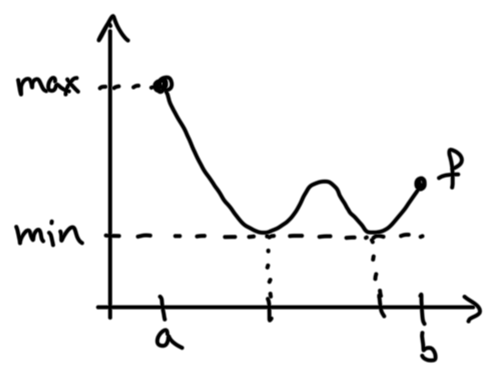
- Next: 3rd important theorem

Def Let $A\subseteq\mathbb{R}$, $f:A\to\mathbb{R}$, and $y\in\mathbb{R}$. We say:

① y is a global maximum of f if $\exists a\in A$ s.t. $f(a)=y$ and $f(x)\leq y$ $\forall x\in A$.

② y is a global minimum of f if $\exists a\in A$ s.t. $f(a)=y$ and $f(x)\geq y$ $\forall x\in A$.

③ y is a global extremum of f if it is a global maximum or minimum.

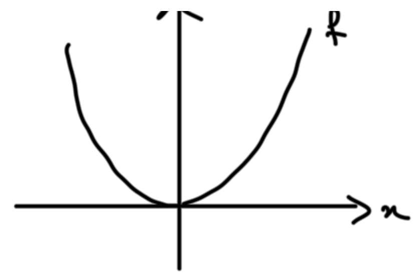


Rmk y is a global max/min of f $\Rightarrow$ f is bounded above/below by y.

- But "$\Leftarrow$" is false in general.

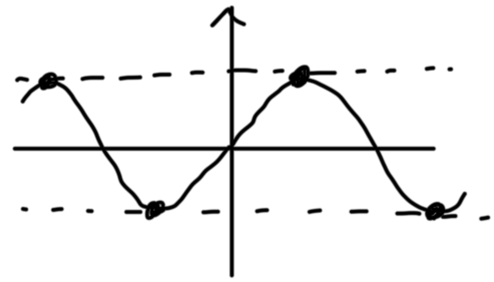
Ex ① $f:\mathbb{R}\to\mathbb{R}$, $f(x)=x^2$

- Global min $f(0) = 0$
- Bounded below by 0
- Not bounded above
- No global max



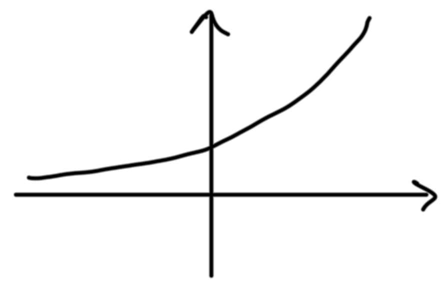
② $g : \mathbb{R} \to \mathbb{R}$, $g(x) = \sin x$

- Global min $-1 = g(-\frac{\pi}{2}) = g(\frac{3\pi}{2}) = \ldots$
- Bounded below by -1
- Global max $1 = g(\frac{\pi}{2}) = g(-\frac{3\pi}{2}) = \ldots$
- Bounded above by 1



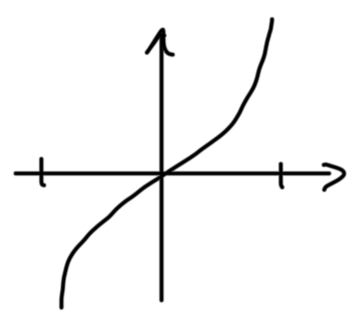
③ $h : \mathbb{R} \to \mathbb{R}$, $h(x) = e^x$

- Not bounded above
- Bounded below by 0
- No global extrema



④ $t : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$, $t(x) = \tan x$

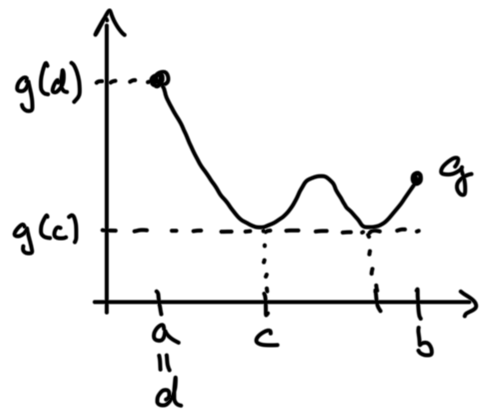
- Not bounded above
- Not bounded below
- No global extrema



<u>Thm 3</u> If $f : [a,b] \to \mathbb{R}$ is continuous, then f has a global maximum.

<u>Extreme value theorem</u> (EVT) If $g : [a,b] \to \mathbb{R}$ is continuous, then g has a global maximum and

minimum, i.e. $\exists\, c, d \in [a,b]$ s.t. $g(c) \le g(x) \le g(d)$ $\forall x \in [a,b]$.



Pf: Global max: by Thm 3.

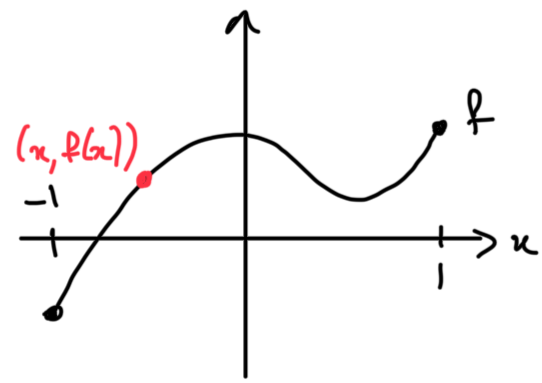
Global min: As $-g$ is also continuous on $[a,b]$, then by Theorem 3 $\exists\, c \in [a,b]$ s.t.

$$-g(x) \le -g(c) \quad \forall x \in [a,b]$$

$$\Rightarrow \quad g(x) \ge g(c) \quad \forall x \in [a,b].$$

So $g(c)$ is a global min of g. $\square$

Ex Let $f : [-1,1] \to \mathbb{R}$ be continuous. Prove that $\exists$ a point on the graph of f that is closest to the origin.



Rmk The plane is the set

$$\mathbb{R}^2 = \{ (x,y) : x, y \in \mathbb{R} \}.$$

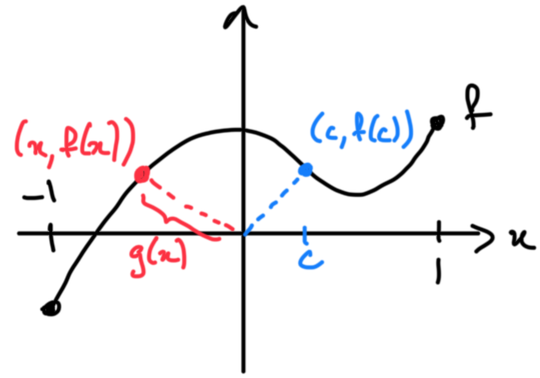
The graph of f is the subset

$$\{ (x, f(x)) : x \in [-1,1] \}.$$

Solution: The distance from a point $(x,y) \in \mathbb{R}^2$ to the origin is $\sqrt{x^2+y^2}$. So, the distance from a point $(x, f(x))$ on the graph of f to the origin is $\sqrt{x^2 + f(x)^2}$.

Let $g:[-1,1]\to\mathbb{R}$, $\quad g(x)=\sqrt{x^2+f(x)^2}$.

Want: $\exists\, c\in[-1,1]$ s.t. $g(c)$ is a global min of g. Then, $(c, f(c))$ is the desired point on the graph of f.



It suffices to prove that g is continuous. Once we know this, then g must have a global min by the EVT.

Claim: $g(x)=\sqrt{x^2+f(x)^2}$ is continuous on $[-1,1]$.

As f is continuous on $[-1,1]$, then so is f^2 (by lecture 14). We also know x^2 is continuous on $[-1,1]$, since it's a polynomial.

- Recall: A polynomial is continuous at any $a\in\mathbb{R}$, by lecture 14
- This implies continuous on $[-1,1]$, as in Lecture 16

Together, the sum $x^2+f(x)^2$ is continuous (by lecture 14). Therefore $g(x)=\sqrt{x^2+f(x)^2}$ is continuous, as its the composition of $x^2+f(x)^2\geq 0$ and $y\mapsto\sqrt{y}$, which is continuous on $[0,\infty)$ by lecture 15.

$\square$

Last time: 3 important theorems

① IVT (special case):

$$\left.\begin{array}{l} f:[a,b]\to\mathbb{R} \text{ continuous} \\ f(a)<0<f(b) \end{array}\right\} \Rightarrow \exists\, c\in[a,b] \text{ s.t. } f(c)=0$$

② $f:[a,b]\to\mathbb{R}$ continuous $\Rightarrow$ f bounded above

③ EVT (special case):

$f:[a,b]\to\mathbb{R}$ continuous $\Rightarrow$ f has a global max

- Goal: Prove these theorems
- First: A new tool

LEAST UPPER BOUNDS (Ch. 8)

Def Let $A\subseteq\mathbb{R}$ and $M,N\in\mathbb{R}$. We say:

① A is bounded above by M if $a\le M$ $\forall a\in A$.
If also $M\in A$, then M is the maximum of A.

② A is bounded below by N if $a\ge N$ $\forall a\in A$.
If also $N\in A$, then N is the minimum of A.

③ A is bounded if it is bounded above and bounded below.

Ex $A=\{\frac{1}{2},1\}$

- maximum is 1: $a\le 1$ $\forall a\in A$ and $1\in A$
- bounded above by 1

- minimum is $\frac{1}{2}$: $a \geq \frac{1}{2}$ $\forall a \in A$ and $\frac{1}{2} \in A$
- bounded below by $\frac{1}{2}$
- bounded

<u>Ex</u> $\{\frac{1}{n} : n \in \mathbb{N}\}$

- maximum is 1: $\frac{1}{n} \leq 1$ $\forall n \in \mathbb{N}$
  and $1 = \frac{1}{n}$ for $n = 1$
- bounded above by 1
- bounded below by 0: $\frac{1}{n} \geq 0$ $\forall n \in \mathbb{N}$
- no minimum: Suppose $\exists n_0 \in \mathbb{N}$ s.t. $\frac{1}{n} \geq \frac{1}{n_0}$ $\forall n \in \mathbb{N}$. But $n_0 + 1 \in \mathbb{N}$ and $\frac{1}{n_0+1} < \frac{1}{n_0}$, a contradiction.

<u>Ex</u> $[0,1)$

- minimum is 0:
  $x \geq 0$ $\forall x \in [0,1)$ and $0 \in [0,1)$
- bounded below by 0
- no maximum: Suppose $\exists m \in [0,1)$ s.t. $x \leq m$ $\forall x \in [0,1)$. Then $\frac{m+1}{2} \in [0,1)$ but $\frac{m+1}{2} > m$, a contradiction.
- bounded above by 1: $x \leq 1$ $\forall x \in [0,1)$
- bounded above by 10: $x \leq 10$ $\forall x \in [0,1)$.

- This set has many upper bounds: any $M \geq 1$
- But $M = 1$ is the best we can do...

<u>Def</u> Let $A \subseteq \mathbb{R}$ and $L \in \mathbb{R}$. We say:

① L is the <u>least upper bound</u> (or <u>supremum</u>) of A if:

- L is an upper bound of A, and
- if M is an upper bound of A then $L \leq M$.

When this is true, we write $L = \sup A$.

② L is the <u>greatest lower bound</u> (or <u>infimum</u>) of A if:

- L is a lower bound of A, and
- if M is a lower bound of A then $L \geq M$.

When this is true, we write $L = \inf A$.

<u>Ex</u> $[0,1)$

① $\sup [0,1) = 1$:

- 1 is an upper bound: previous example.
- Suppose M is another upper bound. Want: $M \geq 1$. Suppose not: $M < 1$. Set $x = \max\{\frac{1+M}{2}, 0\}$. Then $x \in [0,1)$ but $x > M$, a contradiction.

② $\inf [0,1) = 0$.

- 0 is a lower bound: previous example.
- Suppose M is another lower bound. Then $M \leq x \;\forall\, x \in [0,1)$. In particular, for $x = 0$ we have $M \leq 0$.

<u>Lem</u> Let $A \subseteq \mathbb{R}$. If $\sup A = L$ and $\sup A = M$, then $L = M$.

<u>Pf</u>: Suppose $\sup A = L$ and $\sup A = M$. Then

$$\left.\begin{array}{l} \sup A = L \Rightarrow L \text{ is an upper bound of } A \\ \qquad\qquad\quad\; M \text{ is a least upper bound of } A \end{array}\right\} \Rightarrow M \leq L$$

Similarly,

$$\left.\begin{array}{l} \sup A = M \Rightarrow M \text{ is an upper bound of } A \\ \qquad\qquad\quad\; L \text{ is a least upper bound of } A \end{array}\right\} \Rightarrow L \leq M$$

Together, we conclude $L = M$. $\square$

- Not every set has a least upper bound:

<u>Ex</u> ① $\sup [0, \infty)$ does not exist: Suppose $\sup [0, \infty) = L \in \mathbb{R}$. Then L is an upper bound for $[0, \infty)$. But $L+1 \in [0, \infty)$ and $L+1 > L$, a contradiction.

② $\sup \emptyset$ does not exist: Any $M \in \mathbb{R}$ is an upper bound, since "$x \leq M \;\forall x \in \emptyset$" is vacuously true. So $\nexists$ a least upper bound.

The real numbers $\mathbb{R}$ satisfy the <u>least upper bound property</u>: $\forall A \subseteq \mathbb{R}$ that is nonempty and bounded above, $\exists$ a least upper bound $\sup A \in \mathbb{R}$.

Rmk ① $\mathbb{Q}$ and $\mathbb{R}$ are both ordered fields, but $\mathbb{Q}$ does not satisfy the least upper bound property. E.g. $A = \{x \in \mathbb{Q} : x^2 \leq 2\}$ is:

- nonempty: $0^2 \leq 2 \Rightarrow 0 \in A$
- bounded above: by $\frac{3}{2}$, since $x > \frac{3}{2} \Rightarrow x^2 > \frac{9}{4} > 2$.
  So $x \in A \Rightarrow x^2 \leq 2 \Rightarrow x \leq \frac{3}{2}$.

But $\sup A = \sqrt{2} \notin \mathbb{Q}$.

② It turns out that $\mathbb{R}$ is the unique ordered field satisfying the least upper bound property.

- We won't prove this in this course. But this is what makes $\mathbb{R}$ special.

- Midterm 1 distribution and solutions are on Canvas
- There are letter grade cut-offs in the syllabus
- I will replace your lowest midterm score with your final exam score, if it benefits your grade

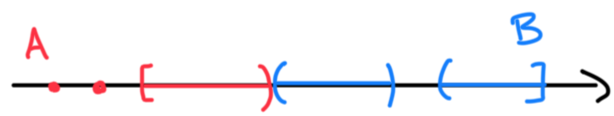
Last time:

① $\sup A = L \iff \begin{cases} \cdot\ a \le L \quad \forall a \in A \\ \cdot\ M \text{ upper bound} \Rightarrow L \le M \end{cases}$

② least upper bound property: $\forall A \subseteq \mathbb{R}$ nonempty and bounded above, $\exists \sup A \in \mathbb{R}$.

Rmk $\mathbb{R}$ also satisfies the greatest lower bound property: $\forall B \subseteq \mathbb{R}$ nonempty and bounded below, $\exists \inf B \in \mathbb{R}$. HW: least upper bound property $\Rightarrow$ greatest lower bound property.

Ex Let $A, B \subseteq \mathbb{R}$ be nonempty and bounded. Suppose $a \le b$ $\forall a \in A$ and $\forall b \in B$.



Prove that $\sup A \le \inf B$.

Pf: As $A, B \subseteq \mathbb{R}$ are nonempty and bounded, we know $\sup A$ and $\inf B$ exist.

Note that $\forall b \in B$, b is an upper bound for A, since $a \le b$ $\forall a \in A$. As $\sup A$ is the least upper bound, we have $\sup A \le b$.

This shows that $\sup A \le b$, $\forall b \in B$. So $\sup A$ is a lower bound for B. As $\inf B$ is the greatest lower bound, we must have $\sup A \le \inf B$. $\square$

<u>Prop</u> (Archimedean property) $\forall x \in \mathbb{R}$, $\exists\, n \in \mathbb{N}$ s.t. $n > x$.

<u>Pf</u>: Suppose not: $\exists\, x_0 \in \mathbb{R}$ s.t. $\forall n \in \mathbb{N}$, $n \le x_0$. Then $\mathbb{N} \subseteq \mathbb{R}$ would satisfy:

- $\mathbb{N} \ne \emptyset$: since $1 \in \mathbb{N}$
- $\mathbb{N}$ is bounded above: by x_0

So, by the least upper bound property, $\exists\, L \in \mathbb{R}$ s.t. $L = \sup \mathbb{N}$. As $L - 1 < L$, then $L-1$ cannot be an upper bound for $\mathbb{N}$. So $\exists\, n_0 \in \mathbb{N}$ s.t. $n_0 > L-1$.

(• Recall: $L-1$ upper bound $\Leftrightarrow$ $n \le L-1$ $\forall n \in \mathbb{N}$)

But then $n_0 + 1 \in \mathbb{N}$ and $n_0 + 1 > L$, so L is not an upper bound — a contradiction. $\square$

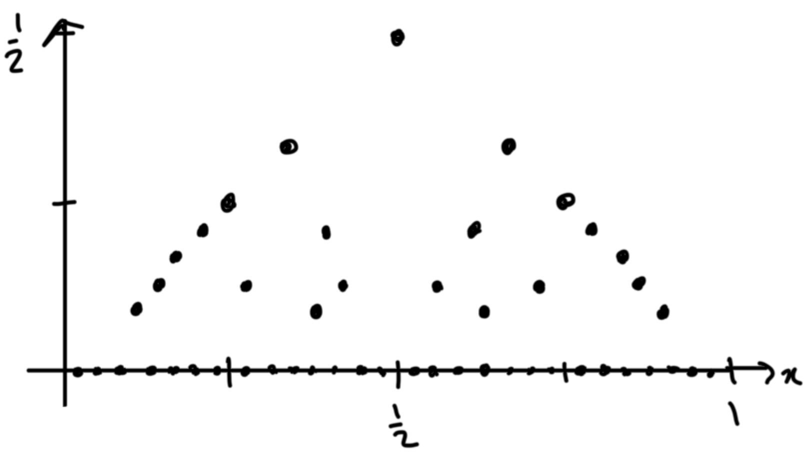
<u>Cor</u> $\forall \varepsilon > 0$, $\exists\, n \in \mathbb{N}$ s.t. $\frac{1}{n} < \varepsilon$.

Pf: Let $\varepsilon>0$. Then $\frac{1}{\varepsilon}\in\mathbb{R}$, so by the Archimedean Property $\exists n\in\mathbb{N}$ s.t. $n>\frac{1}{\varepsilon}$. So $\frac{1}{n}<\varepsilon$. $\square$

Ex Let $f:(0,1)\to\mathbb{R}$, $f(x)=\begin{cases} 0 & \text{if } x\notin\mathbb{Q} \\ \frac{1}{q} & \text{if } x=\frac{p}{q} \text{ in lowest terms} \end{cases}$

Prove that $\lim\limits_{x\to a} f(x)=0$ for any $a\in(0,1)$.

Pf: Let $a\in(0,1)$. Fix $\varepsilon>0$. By the Corallary, $\exists n\in\mathbb{N}$ s.t. $\frac{1}{n}<\varepsilon$. The only numbers x for which $|f(x)-0|\geq\varepsilon$ are rational numbers with denominator $\leq n$:



$$B_n=\left\{\frac{1}{2},\ \frac{1}{3},\frac{2}{3},\ \frac{1}{4},\underbrace{\frac{2}{4}}_{(=\frac{1}{2})},\frac{3}{4},\ \ldots,\ \frac{1}{n},\frac{2}{n},\ldots,\frac{n-1}{n}\right\}$$

So

$$|f(x)-0|\geq\varepsilon \implies x\in B_n.$$

Note that B_n is a finite set. Therefore

$$\delta=\min\{|x-a| : x\in B_n \text{ and } x\neq a\}$$

exists and $\delta>0$. Then

$$0<|x-a|<\delta \implies x\notin B_n \implies |f(x)-0|<\varepsilon.$$

So $\lim\limits_{x\to a} f(x)=0$. $\square$

<u>Rmk</u> So, for $a \in (0,1)$ we have

- $a \notin \mathbb{Q} \Rightarrow f$ is continuous at a
- $a \in \mathbb{Q} \Rightarrow f$ is not continuous at a

<u>Thm 1</u> If $f:[a,b]\to\mathbb{R}$ is continuous and $f(a)<0<f(b)$, then $\exists\, c\in[a,b]$ s.t. $f(c)=0$.

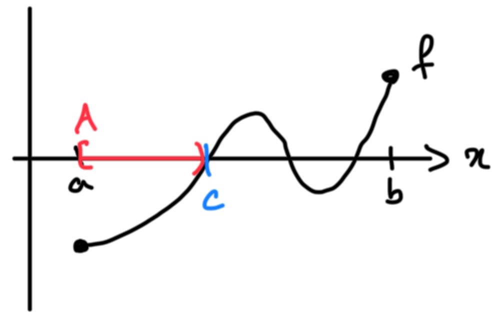
<u>Pf</u>: Define

$$A=\{x_0\in[a,b] : f(x)<0 \quad \forall x \text{ s.t. } a\le x\le x_0\}$$

Steps:

① sup A exists

② $c=\sup A$ satisfies $f(c)=0$.



Step ①: Want $\exists \sup A\in\mathbb{R}$.

- $A\neq\emptyset$: Note that $a\in A$, since $f(a)<0$, and so "$f(x)<0 \ \forall x$ s.t. $a\le x\le a$" is true.
- A is bounded above: by b, since $x_0\le b \ \forall x_0\in A$.

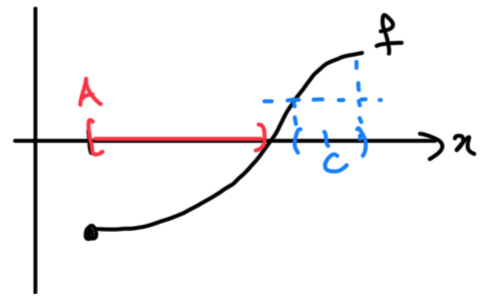
So, by the least upper bound property, $\exists \sup A$ in $\mathbb{R}$.

Step ②: Set $c=\sup A$. Note that $c\ge a$ since $a\in A$ and $c\le b$ since b is an upper bound, so $c\in[a,b]$. Want: $f(c)=0$. Suppose not: $f(c)\neq 0$. Then $f(c)>0$ or $f(c)<0$.

Case: $f(c)>0$. Recall from lecture 15:

$$\left.\begin{array}{l}\bullet\ g \text{ continuous at } c\\ \bullet\ g(c)>0\end{array}\right\} \Rightarrow \begin{array}{l}\exists\,\delta>0 \text{ s.t.}\\ g(x)>0 \quad \forall x\in(c-\delta,c+\delta)\end{array}$$

Applying this fact to $g=f$, we get: $\exists\, \delta>0$ s.t. $f(x)>0$ $\forall x\in(c-\delta, c+\delta)$. But then $c-\delta$ is an upper bound for A:



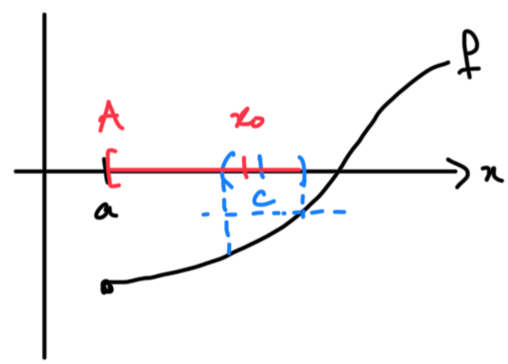
$$x_0\in A \Rightarrow \underbrace{f(x)<0}_{\Rightarrow\, x\notin(c-\delta,\, c+\delta)} \quad \forall x \text{ s.t. } a\le x\le x_0$$

$$\Rightarrow x_0\le c-\delta$$

This contradicts that $c=\sup A$ is the least upper bound.

Case: $f(c)<0$. Applying the fact from Lecture 15 to $g=-f$, we get: $\exists\,\delta>0$ s.t. $f(x)<0$ $\forall x\in(c-\delta, c+\delta)$.

On the other hand, $\exists\, x_0\in A$ s.t. $x_0\in(c-\delta, c]$; otherwise, we would have $x_0\le c-\delta$ $\forall x_0\in A$ and then c would not be the least upper bound of A.



Together, we see that $c+\frac{\delta}{2}\in A$:

$$x_0\in A \Rightarrow \left.\begin{array}{ll} f(x)<0 & \forall x\in[a, x_0] \\ f(x)<0 & \forall x\in(c-\delta, c+\delta) \end{array}\right\} \Rightarrow \begin{array}{l} f(x)<0 \\ \forall x\in[a, c+\frac{\delta}{2}] \end{array}$$

This contradicts that $c=\sup A$ is an upper bound of A.

In both cases we have a contradiction, and so we must have $f(c) = 0$. $\square$

<u>Thm 2</u> If $f : [a,b] \to \mathbb{R}$ is continuous, then f is bounded above.

- For the proof, we'll need:

<u>Lem</u> If g is continuous at c, then $\exists\, \delta > 0$ s.t. $g : (c-\delta, c+\delta) \to \mathbb{R}$ is bounded above.

<u>Pf</u>: As $\lim_{x \to c} g(x) = g(c)$, $\exists\, \delta > 0$ s.t.

$$
\begin{aligned}
|x - c| < \delta \quad &\Rightarrow \quad |g(x) - g(c)| < 1 \\
&\Rightarrow \quad g(x) - g(c) < 1 \\
&\Rightarrow \quad g(x) < g(c) + 1
\end{aligned}
$$

So $g : (c-\delta, c+\delta) \to \mathbb{R}$ is bounded above by $g(c)+1$. $\square$

<u>Exc</u> If $\lim_{x \to c^-} g(x) = g(c)$, then $\exists\, \delta > 0$ s.t. $g : (c-\delta, c] \to \mathbb{R}$ is bounded above.

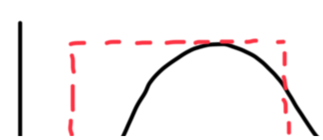
- Just replace "$|x-c| < \delta$" by "$-\delta < x - c \leq 0$"

<u>Pf</u> (of Thm 2): Define

$$A = \{x_0 \in [a,b] : f : [a, x_0] \to \mathbb{R} \text{ is bounded above}\}.$$

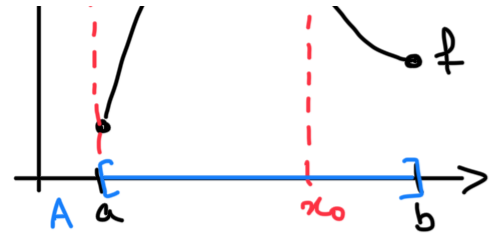
Steps:

① sup A exists



② $\sup A = b$

③ $b \in A$

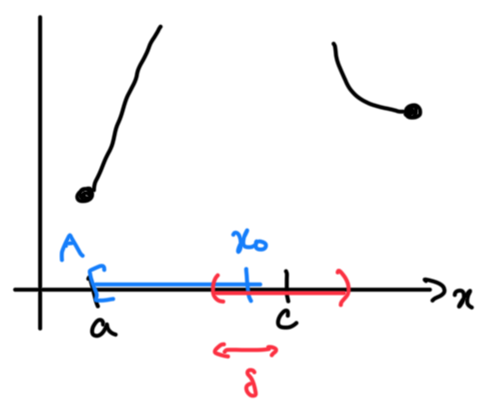


Step ①: Want $\exists \sup A \in \mathbb{R}$.

- $A \neq \emptyset$: Note $a \in A$, since $f : \{a\} \to \mathbb{R}$ is bounded above by $f(a)$.
- A is bounded above: by b, since $x_0 \leq b$ $\forall x_0 \in A$.

So, by the least upper bound property, $\exists \sup A \in \mathbb{R}$.

Step ②: Want $\sup A = b$. Suppose not: $\sup A \neq b$. As b is an upper bound for A, we already know $\sup A \leq b$. So $\sup A < b$. As f is continuous at $c = \sup A$, $\exists \delta > 0$ s.t. $f : (c-\delta, c+\delta) \to \mathbb{R}$ is bounded above by the previous lemma. As $c - \delta < c$, then $c - \delta$ is not an upper bound for A, so $\exists x_0 \in A$ s.t. $x_0 > c - \delta$. Together, we see that $c + \frac{\delta}{2} \in A$:



$$\left.\begin{array}{l} x_0 \in A \Rightarrow f(x) \leq M_1 \quad \forall x \in [a, x_0] \\ \qquad\qquad f(x) \leq M_2 \quad \forall x \in (c-\delta, c+\delta) \end{array}\right\} \Rightarrow \begin{array}{l} f(x) \leq \max\{M_1, M_2\} \\ \forall x \in [a, c+\frac{\delta}{2}] \end{array}$$

This contradicts that $c = \sup A$ is an upper bound for A. So $\sup A = b$.

Step 3 : Next time.

Thm 2 If $f:[a,b]\to\mathbb{R}$ is continuous, then f is bounded above.

Pf (of Thm 2): Define

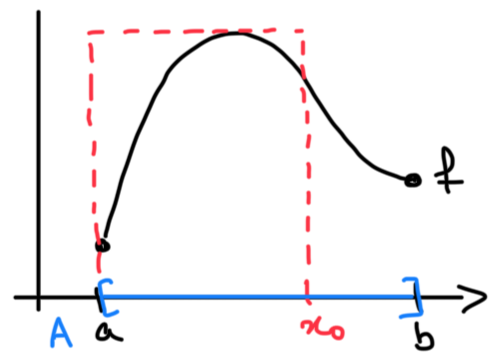
$$A=\{x_0\in[a,b] : f:[a,x_0]\to\mathbb{R} \text{ is bounded above}\}.$$

Steps:

1. sup A exists
2. sup $A = b$
3. $b\in A$

(1)–(2): Last time

(3): ← Today



Step 3: Want $b\in A$, i.e. $f:[a,b]\to\mathbb{R}$ is bounded above. Recall from Lecture 20:

$$\lim_{x\to b^-} f(x)=f(b) \implies \exists\,\delta>0 \text{ s.t. } f:(b-\delta,b]\to\mathbb{R} \text{ is bounded above}$$

As $b-\delta<b$, then $b-\delta$ cannot be an upper bound for A, so $\exists\, x_0\in A$ s.t. $x_0>b-\delta$. Together:

$$\left.\begin{array}{ll} x_0\in A \implies f(x)\le M_1 & \forall x\in[a,x_0] \\ \ f(x)\le M_2 & \forall x\in(b-\delta,b] \end{array}\right\} \implies \begin{array}{l} f(x)\le\max\{M_1,M_2\} \\ \forall x\in[a,b] \end{array}$$

So $f:[a,b]\to\mathbb{R}$ is bounded above. □

Thm 3 If $f:[a,b]\to\mathbb{R}$ is continuous, then f has a global maximum.

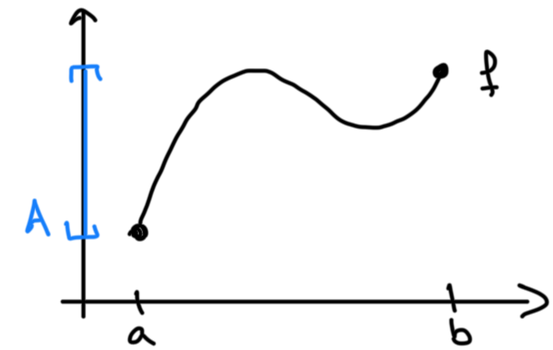
Pf: Define

$$A = f([a,b]) = \{f(x) : x \in [a,b]\}$$

Steps:

① $c = \sup A$ exists

② $c \in A$



Step ①: Want $\exists \sup A \in \mathbb{R}$.

- $A \neq \emptyset$: Note that $f(a) \in A$.
- A is bounded above: By Thm 2,

  $f : [a,b] \to \mathbb{R}$ is bounded above

  $\Rightarrow \exists M$ s.t. $f(x) \leq M \quad \forall x \in [a,b]$

  $\Rightarrow M$ is an upper bound for A.

So, by the least upper property, $\exists \sup A \in \mathbb{R}$.

Step ②: Set $c = \sup A$. Want: $\exists x \in [a,b]$ s.t. $f(x) = c$. Suppose not: $f(x) \neq c \quad \forall x \in [a,b]$.

Consider $g : [a,b] \to \mathbb{R}$, $g(x) = \dfrac{1}{c - f(x)}$.

Claim: g is continuous on $[a,b]$. Let $x_0 \in [a,b]$.

$f(x)$ is continuous at x_0

$\Rightarrow$ $c - f(x)$ is continuous at x_0 (by Lecture 14)

$\Rightarrow$ $\dfrac{1}{c - f(x)}$ is continuous at x_0, since $c - f(x_0) \neq 0$.

So, by Thm 2, $g:[a,b]\to\mathbb{R}$ is bounded above: $\exists M\in\mathbb{R}$ s.t. $g(x)=\frac{1}{c-f(x)}\le M\ \forall x\in[a,b]$. Note that $M>0$, since $f(x)<c\ \forall x\in[a,b]$. So:

$$\frac{1}{c-f(x)}\le M \implies c-f(x)\ge\frac{1}{M}\implies f(x)\le c-\frac{1}{M}$$

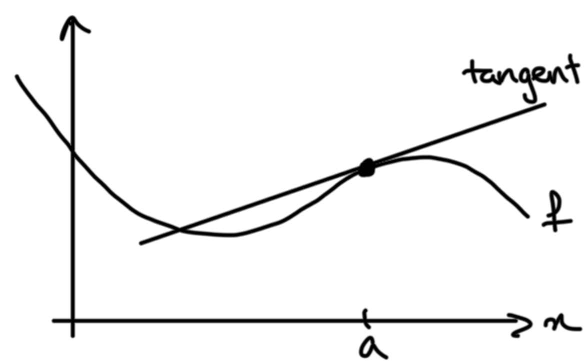
for all $x\in[a,b]$. So $c-\frac{1}{M}$ is an upper bound for A. This contradicts that $c=\sup A$ is the least upper bound. $\square$

CALCULUS

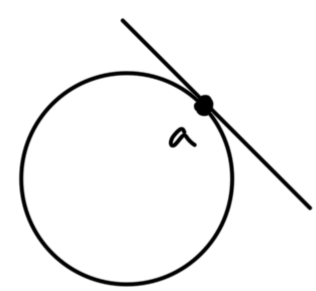
- Calculus is centered around 2 fundamental problems

<u>Q</u>: For an arbitrary function $f:\mathbb{R}\to\mathbb{R}$,

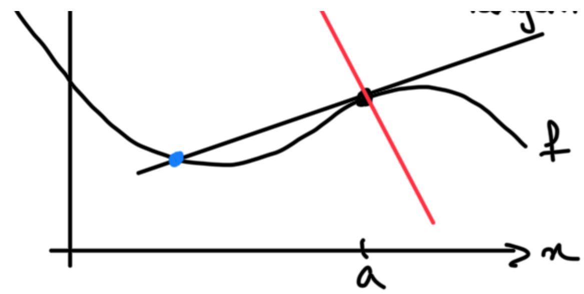
① How do we find the line tangent to the graph of f at a point $x=a$?



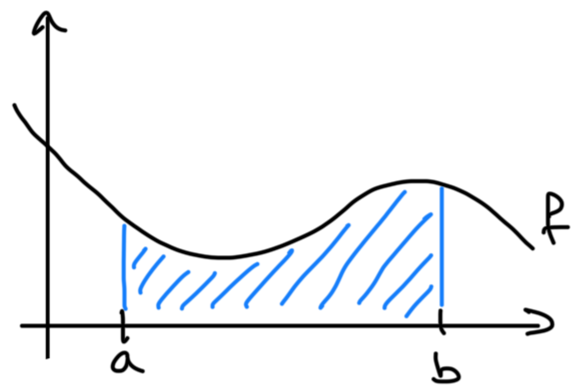
- Well, for a circle, it's the line through a that intersects the circle only once. (There's only 1 such line.)



- ... But this doesn't work for arbitrary curves:



② How do we find the graph of $f(x)$ for $a \le x \le b$?



- Well, for a circle, the area is πr^2.
- ... But this doesn't work for arbitrary curves.

- Surprisingly, these 2 seemingly independent questions are closely related

<u>A</u>: ① Derivatives (Ch. 9–12)

② Integrals (Ch. 13–14)

DERIVATIVES (Ch. 9)

<u>Def</u> Let $I \subseteq \mathbb{R}$ be an open interval, $f : I \to \mathbb{R}$ a function, and $a \in I$. We say f is <u>differentiable at a</u> if

$$\lim \frac{f(a+h)-f(a)}{\quad} \text{ exists.}$$

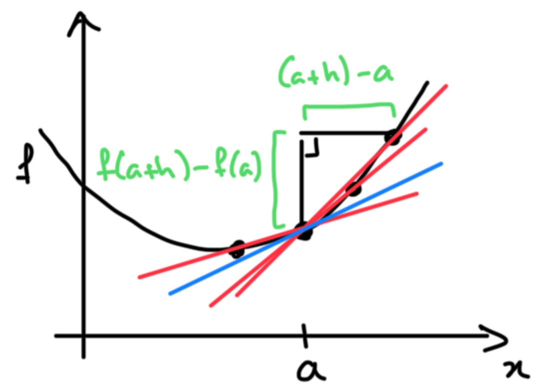
$h \to 0$ $\qquad h$

When this is true, we call the limit the <u>derivative of f at a</u> and we denote it by $f'(a)$ or $\frac{df}{dx}(a)$.

<u>Rmk</u> "I is an open interval" accounts for all of the cases $I = (b,c)$, $(-\infty, b)$, (b, ∞), $\mathbb{R}$ simultaneously. In all of these cases, $\exists\, \delta > 0$ s.t. $(a-\delta, a+\delta) \subseteq I$, so the limit makes sense.

<u>Rmk</u> $f'(a)$ is...

① The slope of the tangent line to the graph of f at $(a, f(a))$.



② The instantaneous rate of change of $f(x)$ near a:

$$f'(a) \approx \frac{f(a+h)-f(a)}{h} \quad \Rightarrow \quad f(a+h) \approx f(a) + f'(a)\cdot h$$

<u>Last time:</u>

f differentiable at a $\overset{\text{def}}{\Longleftrightarrow}$ $f'(a) = \lim\limits_{h\to 0} \dfrac{f(a+h)-f(a)}{h}$ exists

<u>Ex</u> Let $a \in \mathbb{R}$. Are the following functions $f:\mathbb{R}\to\mathbb{R}$ differentiable at a?

① $f(x) = 4x + 21$

Yes. Claim: $f'(a) = 4$. For $h \neq 0$,

$$\frac{f(a+h)-f(a)}{h} = \frac{4(\cancel{a}+h)+\cancel{21}-(4\cancel{a}+\cancel{21})}{h} = \frac{4h}{h} = 4$$

• The value of $\dfrac{f(a+h)-f(a)}{h}$ at $h=0$ does not affect the limit

So
$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0} 4 = 4.$$

② $f(x) = x^2$

Yes. Claim: $f'(a) = 2a$. For $h \neq 0$,

$$\frac{f(a+h)-f(a)}{h} = \frac{(a+h)^2 - a^2}{h} = \frac{\cancel{a^2}+2ah+h^2-\cancel{a^2}}{h} = 2a+h$$

• We know $\lim\limits_{h\to 0} p(h) = p(0)$ for any polynomial p

So
$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0} (2a+h) = 2a.$$

• In groups:

· In groups.

③ $f(x) = x^3$.

Yes. Claim: $f'(a) = 3a^2$. For $h \neq 0$,

$$\frac{f(a+h)-f(a)}{h} = \frac{(a+h)^3 - a^3}{h} = \frac{\cancel{a^3} + 3a^2h + 3ah^2 + h^3 - \cancel{a^3}}{h}$$

$$= 3a^2 + 3ah + h^2$$

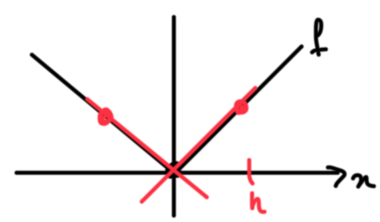
So

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0} (3a^2 + 3ah + h^2) = 3a^2.$$

④ $f(x) = |x|$ at $a = 0$.

No. Claim: $f'(0)$ does not exist. For $h \neq 0$,

$$\frac{f(0+h)-f(0)}{h} = \frac{|h|}{h} = \begin{cases} 1 & h>0 \\ -1 & h<0 \end{cases}$$



So

$$\lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h} = 1, \quad \lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h} = -1$$

$$\Rightarrow \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} \text{ does not exist}$$

<u>Prop</u> If f is differentiable at a, then f is continuous at a.

<u>Pf</u>: We know

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = f'(a), \qquad \lim_{h\to 0} h = 0$$

Therefore, by the limit law for multiplication,

$$\lim_{h\to 0}\left(f(a+h)-f(a)\right) = \lim_{h\to 0}\left(\frac{f(a+h)-f(a)}{h}\cdot h\right)$$

$$= f'(a)\cdot 0 = 0$$

This means: $\forall \varepsilon>0,\ \exists \delta>0$ s.t.

$$0<|h|<\delta \implies |f(a+h)-f(a)|<\varepsilon.$$

Taking h to be $x-a$, we get:

$$0<|x-a|<\delta \implies |f(x)-f(a)|<\varepsilon.$$

So $\lim_{x\to a} f(x) = f(a)$. $\square$

<u>Rmk</u> Continuous $\not\Rightarrow$ differentiable. E.g., $f(x)=|x|$ is continuous and not differentiable at $x=0$.

<u>Ex</u> Let $f:\mathbb{R}\to\mathbb{R}$, $$f(x)=\begin{cases} x^2 & x\geq 0 \\ -x^2 & x<0\end{cases}$$

At which points $a\in\mathbb{R}$ is f differentiable?

① Case: $a>0$. Then $f(x)=x^2\ \forall x\in(a-\delta,a+\delta)$, for $\delta=a$. So the derivative of f at a is the same as for x^2: $f'(a)=2a$.

② Case: $a<0$. Then $f(x)=-x^2\ \forall x\in(a-\delta,a+\delta)$, for $\delta=|a|$. So the derivative of f at a is the same as for $-x^2$:

$$\lim_{h\to 0}\frac{(a+h)^2-a^2}{h} = 2a$$

$$\Rightarrow \quad f'(a) = \lim_{h\to 0} \frac{-(a+h)^2 - (-a^2)}{h} = -2a.$$

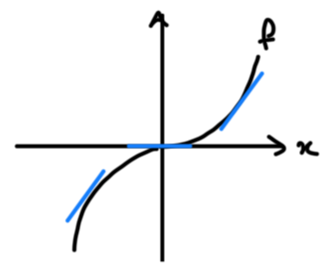
③ Case: $a=0$. For $h \neq 0$,

$$\frac{f(0+h)-f(0)}{h} = \begin{cases} \frac{h^2-0}{h} = h & \text{if } h>0 \\ \frac{-h^2-0}{h} = -h & \text{if } h<0 \end{cases} = |h|$$

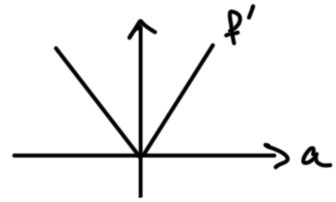
So

$$f'(0) = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = 0.$$

Altogether, f is differentiable at any $a \in \mathbb{R}$, and:



$$f'(a) = \begin{cases} 2a & a>0 \\ 0 & a=0 \\ -2a & a<0 \end{cases} = 2|a|$$

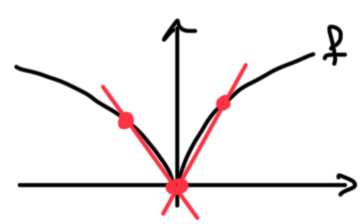


<u>Ex</u> $f(x) = \sqrt{|x|}$. Is f differentiable at $a=0$?

No. We'll prove $f'(0)$ does not exist.

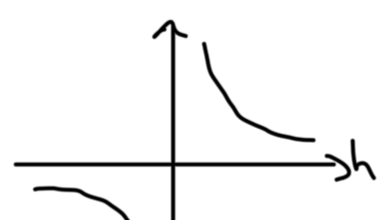
For $h \neq 0$,

$$\frac{f(0+h)-f(0)}{h} = \frac{\sqrt{|h|}}{h} = \begin{cases} \frac{1}{\sqrt{h}} & h>0 \\ -\frac{1}{\sqrt{-h}} & h<0 \end{cases}$$



Claim: $\lim_{h\to 0^+} \frac{1}{\sqrt{h}}$ does not exist.

Suppose not: $\lim_{h\to 0^+} \frac{1}{\sqrt{h}} = \ell \in \mathbb{R}$.



Consider $\varepsilon = 1$. Then $\exists\, \delta > 0$ s.t.

$$0 < |h| < \delta \implies \left| \frac{1}{\sqrt{h}} - \ell \right| < 1$$

Set $h_0 = \min\left\{ \frac{\delta}{2}, \frac{1}{(|\ell|+10)^2} \right\}$. Then

$$0 < |h_0| < \delta \quad \text{but} \quad \left| \frac{1}{\sqrt{h_0}} - \ell \right| \geq 1$$

since $h_0 > 0$ $h_0 \leq \frac{\delta}{2}$

$$0 < h_0 \leq \frac{1}{(|\ell|+10)^2}$$

$$\implies \frac{1}{\sqrt{h_0}} \geq |\ell| + 10$$

$$\implies \frac{1}{\sqrt{h_0}} - \ell \geq \underbrace{|\ell| - \ell}_{\geq 0} + 10 \geq 10$$

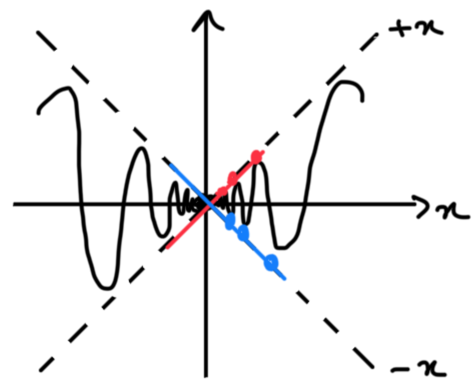
which is a contradiction.

Therefore $\lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$ does not exist. $\square$

<u>Recall</u> f differentiable at a $\overset{\text{def}}{\Longleftrightarrow}$ $f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$ exists

<u>Ex</u> Are the following functions differentiable?

① $f(x) = \begin{cases} x \cdot \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$ at $a=0$.



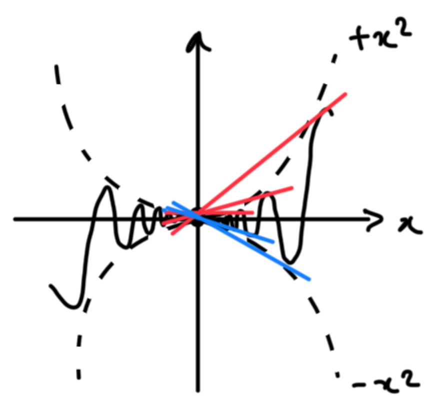
- Previously: f is continuous at 0
- Secant lines can have slope ± 1 arbitrarily close to $x=0$

No. Claim: $f'(0)$ does not exist. For $h \neq 0$,

$$\frac{f(0+h)-f(0)}{h} = \frac{h \cdot \sin \frac{1}{h} - 0}{h} = \sin \frac{1}{h}$$

By Lecture 13, we know $\lim_{h\to 0} \sin \frac{1}{h}$ does not exist.

② $g(x) = \begin{cases} x^2 \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$ at $a=0$.



- Secant lines become horizontal as $h \to 0$:

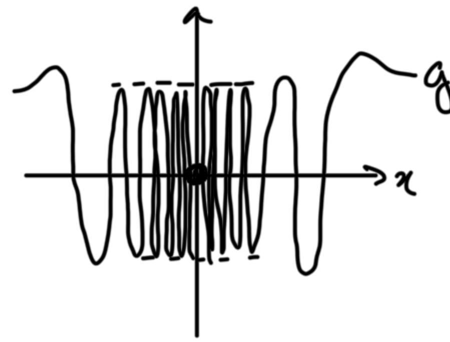
Yes. Claim: $g'(0) = 0$. For $h \neq 0$,

$$\frac{g(0+h)-g(0)}{h} = \frac{h^2 \sin\frac{1}{h} - 0}{h} = h \cdot \sin\frac{1}{h}$$

By Lecture 13, we know $\lim_{h\to 0} h\cdot\sin\frac{1}{h} = 0$.

<u>Rmk</u> This g is differentiable at any $x \in \mathbb{R}$, and

$$g'(x) = \begin{cases} 2x\sin\frac{1}{x} - \cos\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$



But g' is discontinuous at $x=0$!

<u>Def</u> Let $a<b$. We say:

① $f:\mathbb{R}\to\mathbb{R}$ is <u>differentiable</u> if f is differentiable at x $\forall x \in \mathbb{R}$. The function $f':\mathbb{R}\to\mathbb{R}$ is the <u>derivative</u> of f.

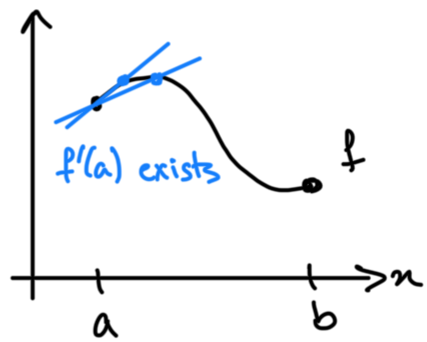
② $f:(a,b)\to\mathbb{R}$ is <u>differentiable</u> if f is differentiable at x $\forall x\in(a,b)$. The function $f':(a,b)\to\mathbb{R}$ is the <u>derivative</u> of f.

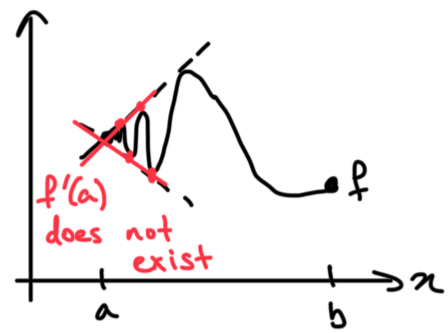
③ $f:[a,b]\to\mathbb{R}$ is <u>differentiable</u> if:

- f is differentiable at x $\forall x \in (a,b)$
- $\lim_{h\to 0^+} \frac{f(a+h)-f(a)}{h}$ exists
- $\lim_{h\to 0^-} \frac{f(b+h)-f(b)}{h}$ exists

The function $f':[a,b]\to\mathbb{R}$ is the <u>derivative</u>

of f.





- Differentiable on $[a,b]$
- $f': [a,b] \to \mathbb{R}$

- Continuous on $[a,b]$
- Not differentiable on $[a,b]$
- $f': (a,b] \to \mathbb{R}$

Ex
- Polynomials $p: \mathbb{R} \to \mathbb{R}$ are differentiable (but we haven't proved this yet!)
- $f(x) = |x|$ is differentiable on $[0,\infty)$ and $(-\infty, 0]$, but not on $\mathbb{R}$ (by lecture 22)
- $f(x) = \sqrt{x}$ is differentiable on $(0,\infty)$, but not on $[0,\infty)$
- $f(x) = \begin{cases} x \cdot \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$ is differentiable on $(0,\infty)$ and $(-\infty, 0)$, but not on $\mathbb{R}$
- $g(x) = \begin{cases} x^2 \cdot \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$ is differentiable on $\mathbb{R}$

Prop $\forall n \in \mathbb{N}$, the function $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^n$ is differentiable with derivative $f'(x) = n x^{n-1}$.

Pf: Fix $n \in \mathbb{N}$ and $x \in \mathbb{R}$. For $h \neq 0$.

$$\frac{f(a+h)-f(a)}{h} = \frac{(a+h)^n - a^n}{h}$$

$$(a+h)^n = a^n + na^{n-1}h + \frac{n(n-1)}{2}a^{n-2}h^2 + \ldots + nah^{n-1} + h^n$$

$$= \sum_{j=0}^{n} \binom{n}{j} a^{n-j} h^j, \qquad \text{where} \quad \binom{n}{j} = \frac{n!}{j!\,(n-j)!}$$

- "Binomial theorem" — see Ch. 2 problem 3 for a proof

$$\Rightarrow \quad \frac{f(a+h)-f(a)}{h} = \frac{\cancel{a^n} + na^{n-1}h + \frac{n(n-1)}{2}a^{n-2}h^2 + \ldots + h^n - \cancel{a^n}}{h}$$

$$= na^{n-1} + \frac{n(n-1)}{2}a^{n-2}h + \ldots + h^{n-1}$$

This is a polynomial in h. By Lecture 14 we know any polynomial is continuous at $h=0$, and so

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = na^{n-1} + 0 + \ldots + 0. \qquad \square$$

<u>Last time</u> $\forall n \in \mathbb{N}$, $f(x) = x^n$ is differentiable on $\mathbb{R}$, and $f'(x) = nx^{n-1}$.

<u>Def</u> Let $I \subseteq \mathbb{R}$ be an open interval, $f : I \to \mathbb{R}$, and $a \in I$. We say:

① f is <u>twice differentiable</u> at a if:

- $f : I \to \mathbb{R}$ is differentiable at x $\forall x \in I$, and
- $f' : I \to \mathbb{R}$ is differentiable at a.

When this is true, we call $(f')'(a)$ the <u>second derivative of f at a</u> and we denote it by $f''(a)$ or $\frac{d^2 f}{dx^2}(a)$.

② f is n-times differentiable at a if $f^{(n-1)} : I \to \mathbb{R}$ exists and is differentiable at a. When this is true, we call $(f^{(n-1)})'(a)$ the <u>nth derivative of f at a</u> and we denote it by $f^{(n)}(a)$ or $\frac{d^n f}{dx^n}(a)$.

<u>Ex</u> ① Polynomials $p(x)$ are k-times differentiable at any $a \in \mathbb{R}$, $\forall k \in \mathbb{N}$. E.g.,

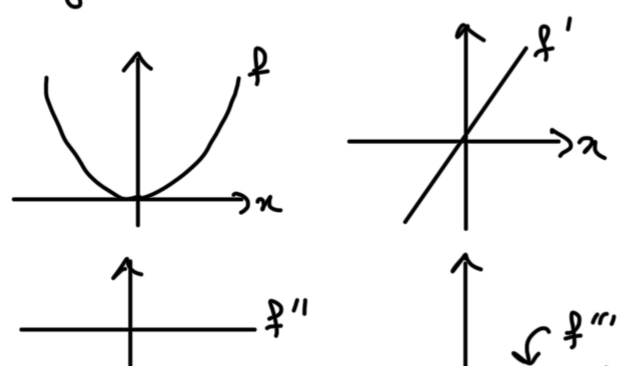
$f(x) = x^2$

$\Rightarrow f'(x) = 2x$

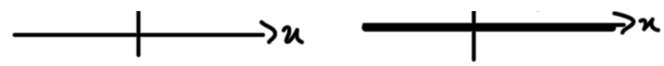
$\Rightarrow f''(x) = 2$

$\Rightarrow f'''(x) = 0$

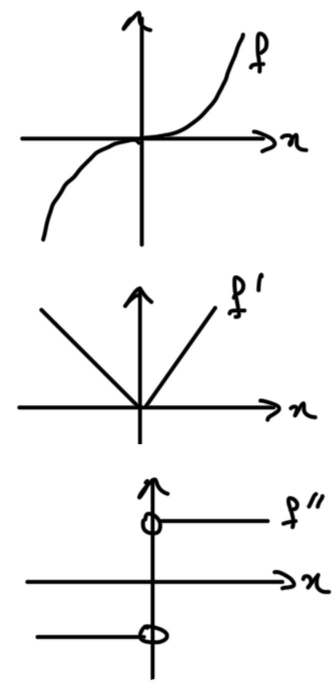
$\Rightarrow f^{(k)}(x) = 0 \quad \forall k \geq 3$



$\Rightarrow f'(x) = 0 \quad \forall x \ldots$



② $f : \mathbb{R} \to \mathbb{R}$, $\quad f(x) = \begin{cases} x^2 & x \geq 0 \\ -x^2 & x < 0 \end{cases}$



Then:

- f is differentiable on $\mathbb{R}$, $f'(x) = 2|x|$ (by lecture 22)
- f' is not differentiable at 0

So f is not twice differentiable at $x = 0$.

<u>Exc</u> Let $n \in \mathbb{N}$. The function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^n$ is k-times differentiable on $\mathbb{R}$ $\forall k \in \mathbb{N}$, and

$$f^{(k)}(a) = \begin{cases} \frac{n!}{(n-k)!} a^{n-k} & \text{if } k \leq n, \\ 0 & \text{if } k > n. \end{cases}$$

<u>Idea</u>: $f'(x) = \underbrace{n}_{\frac{n!}{(n-1)!}} x^{n-1}$

$$\Rightarrow f''(x) = n \frac{d}{dx}(x^{n-1}) = \underbrace{n(n-1)}_{\frac{n!}{(n-2)!}} x^{n-2}$$

$$\Rightarrow f'''(x) = n(n-1) \frac{d}{dx}(x^{n-2}) = \underbrace{n(n-1)(n-2)}_{\frac{n!}{(n-3)!}} x^{n-3}$$

$\vdots$

$$\Rightarrow f^{(n+1)}(x) = n(n-1) \cdots 2 \cdot 1 \frac{d}{dx}(1) = 0$$

$f^{(n+2)}(x) = 0$

$\Rightarrow f^{(n+1)}(x) = 0$

$\vdots$

Induction on $k \in \mathbb{N}$.

DIFFERENTIATION LAWS (Ch. 10)

__Thm__ Let $I \subseteq \mathbb{R}$ be an open interval, $f, g : I \to \mathbb{R}$, and $a \in I$. If f and g are differentiable at a, then:

① cf is differentiable at a $\forall c \in \mathbb{R}$, and

$$(cf)'(a) = c \cdot f'(a).$$

② $f+g$ is differentiable at a, and

$$(f+g)'(a) = f'(a) + g'(a).$$

__Pf__: ① Fix $c \in \mathbb{R}$. As f is differentiable at a,

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = f'(a).$$

So, by the limit law for multiplication by a constant,

$$\lim_{h \to 0} \left[c \cdot \frac{f(a+h) - f(a)}{h} \right] = c \cdot f'(a)$$

$$\parallel$$

$$\lim_{h \to 0} \frac{cf(a+h) - cf(a)}{h}$$

So $(cf)'(a) = c \cdot f'(a)$.

② As f and g are differentiable at a,

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = f'(a), \qquad \lim_{h\to 0} \frac{g(a+h)-g(a)}{h} = g'(a).$$

So, by the limit law for addition,

$$\lim_{h\to 0}\left[\frac{f(a+h)-f(a)}{h} + \frac{g(a+h)-g(a)}{h}\right] = f'(a)+g'(a)$$

$$\shortparallel$$

$$\lim_{h\to 0} \frac{f(a+h)+g(a+h)-(f(a)+g(a))}{h}$$

So $(f+g)'(a) = f'(a)+g'(a)$. □

<u>Cor</u> If $p:\mathbb{R}\to\mathbb{R}$ is a polynomial, then p is differentiable.

<u>Pf</u>: let

$$p(x) = c_0 + c_1 x + c_2 x^2 + \ldots + c_N x^N$$

be a polynomial, for some $N \geq 0$ and $c_0, c_1, \ldots, c_N \in \mathbb{R}$. Fix $a \in \mathbb{R}$. We will prove

$$p'(a) = c_1 + 2c_2 a + \ldots + Nc_N a^{N-1}.$$

Recall that $x \mapsto x^n$ is differentiable at a and $(x^n)'(a) = na^{n-1}$, $\forall n \in \mathbb{N}$ (by Lecture 23). So, by part ① of the previous theorem, $c_n x^n$ is differentiable at a and $(c_n x^n)'(a) = nc_n a^{n-1}$, for each $n = 1, 2, \ldots, N$. Additionally, the constant function $x \mapsto c_0$ is differentiable at a with derivative

$$(c_0)'(a) = \lim_{h\to 0} \frac{c_0 - c_0}{h} = \lim_{h\to 0} 0 = 0.$$

So, by part 2 of the previous theorem, $p(x) = c_0 + c_1 x + \ldots + c_N x^N$ is differentiable at a and

$$p'(a) = 0 + c_1 + 2c_2 a + \ldots + N c_N a^{N-1}. \qquad \square$$