Last time: Differentiation laws

① $(cf)'(a) = c \cdot f'(a) \quad \forall c \in \mathbb{R}$

② $(f+g)'(a) = f'(a) + g'(a)$

Thm (Product rule) Let $I \subseteq \mathbb{R}$ be an open interval, $f, g : I \to \mathbb{R}$, and $a \in I$. If f and g are differentiable at a, then fg is differentiable at a and

$$(fg)'(a) = f'(a)g(a) + f(a)g'(a).$$

Pf: For $h \neq 0$,

$$\frac{f(a+h)g(a+h) - f(a)g(a)}{h}$$

$$= \frac{f(a+h)g(a+h) - f(a)g(a+h) + f(a)g(a+h) - f(a)g(a)}{h}$$

$$= \frac{f(a+h) - f(a)}{h} \cdot g(a+h) + f(a) \cdot \frac{g(a+h) - g(a)}{h}$$

As f and g are differentiable at a, we know:

① $\lim\limits_{h \to 0} \frac{f(a+h) - f(a)}{h} = f'(a)$

② $\lim\limits_{h \to 0} \frac{g(a+h) - g(a)}{h} = g'(a)$

③ g continuous at a (by lecture 22)

$\Rightarrow \lim\limits_{x \to a} g(x) = g(a)$

$$\Rightarrow \lim_{h\to 0} g(a+h) = g(a)$$

Therefore, by the limit laws,

$$\lim_{h\to 0} \frac{f(a+h)g(a+h) - f(a)g(a)}{h}$$

$$= \lim_{h\to 0}\left[\frac{f(a+h)-f(a)}{h}\cdot g(a+h) + f(a)\cdot \frac{g(a+h)-g(a)}{h}\right]$$

$$= f'(a)\cdot g(a) + f(a)\cdot g'(a)$$

So $(fg)'(a) = f'(a)g(a) + f(a)g'(a)$. $\square$

<u>Thm</u> (Quotient rule) Let $I \subseteq \mathbb{R}$ be an open interval, $f, g: I \to \mathbb{R}$, and $a \in I$. If f and g are differentiable at a and $g(a) \neq 0$, then $\frac{f}{g}$ is differentiable at a and

$$\left(\frac{f}{g}\right)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{g(a)^2}.$$

<u>Pf</u>: It suffices to show

$$\left(\frac{1}{g}\right)'(a) = -\frac{g'(a)}{g(a)^2}.$$

Indeed, once we prove this, then by the product rule we have

$$\left(f\cdot\frac{1}{g}\right)'(a) = f'(a)\cdot\frac{1}{g(a)} + f(a)\cdot\left(\frac{1}{g}\right)'(a)$$

$$= \frac{f'(a)}{g(a)} - \frac{f(a)g'(a)}{g(a)^2}$$

$$= \frac{f'(a)g(a) - f(a)g'(a)}{g(a)^2}.$$

Want: $\lim_{h \to 0} \frac{\frac{1}{g(a+h)} - \frac{1}{g(a)}}{h} = -\frac{g'(a)}{g(a)^2}$. As g is continuous at a (by Lecture 22) and $g(a) \neq 0$, then $\exists \delta > 0$ s.t. $g(x) \neq 0 \ \forall x \in (a-\delta, a+\delta)$ (by Lecture 15). So for $h \in (-\delta, \delta)$, $\frac{1}{g(a+h)}$ makes sense and

$$\frac{\frac{1}{g(a+h)} - \frac{1}{g(a)}}{h} = \frac{g(a) - g(a+h)}{h \cdot g(a) \cdot g(a+h)}$$

$$= -\frac{g(a+h) - g(a)}{h} \cdot \frac{1}{g(a)g(a+h)}$$

As g is differentiable at a, we know:

① $\lim_{h \to 0} \frac{g(a+h) - g(a)}{h} = g'(a)$

② g continuous at a (by Lecture 22)

$$\Rightarrow \lim_{x \to a} g(x) = g(a)$$

$$\Rightarrow \lim_{h \to 0} g(a+h) = g(a) \quad \text{and } g(a) \neq 0$$

$$\Rightarrow \lim_{h \to 0} \frac{1}{g(a)g(a+h)} = \frac{1}{g(a)^2}$$

Together, by the limit laws,

$$\lim_{h \to 0} \frac{\frac{1}{g(a+h)} - \frac{1}{g(a)}}{h} = \lim_{h \to 0} \left[-\frac{g(a+h) - g(a)}{h} \cdot \frac{1}{g(a)g(a+h)} \right]$$

$$= -g'(a) \cdot \frac{1}{g(a)^2}$$

So $\left(\frac{1}{g}\right)'(a) = -\frac{g'(a)}{g(a)^2}$. $\square$

<u>Ex</u> $\forall n \in \mathbb{Z}$, $\frac{d}{dx}(x^n) = nx^{n-1}$.

<u>Pf</u>: Case: $n \geq 0$. Already know (Lecture 24).

Case: $n < 0$. Then $x^n = \frac{1}{x^{-n}}$ with $-n > 0$.

Fix $a \neq 0$. We know $g(x) = x^{-n}$ is differentiable with $g'(a) = -na^{-n-1}$ and $g(a) = a^{-n} \neq 0$. So, by the quotient rule,

$$(x^n)'(a) = \left(\frac{1}{g}\right)'(a) = -\frac{g'(a)}{g(a)^2}$$

$$= -\frac{(-n)a^{-n-1}}{(a^{-n})^2} = na^{-n-1+2n} = na^{n-1}. \quad \square$$

<u>Ex</u> Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \frac{x}{x^2+1}$. Prove f is differentiable, and find its derivative.

<u>Pf</u>: Fix $a \in \mathbb{R}$. Then

$$(x)'(a) = 1, \qquad (x^2+1)'(a) = 2a$$

(These are polynomials, which we already know are differentiable on $\mathbb{R}$.) As $a^2+1 \neq 0$ (since $a^2 \geq 0$), then by the quotient rule we have

$$f'(a) = \frac{1 \cdot (a^2+1) - a \cdot 2a}{(a^2+1)^2} = \frac{-a^2+1}{(a^2+1)^2}. \qquad \square$$

<u>Thm</u> (Chain rule) Let $I, J \subseteq \mathbb{R}$ be open intervals, $f : I \to J$ and $g : J \to \mathbb{R}$ be functions, and $a \in I$. If f is differentiable at a and g is differentiable at $f(a)$, then $g \circ f$ is differentiable at a and

$$(g \circ f)'(a) = g'(f(a)) \cdot f'(a).$$