Math 421, Section 1
Homework 2
(Name)

Problem 1. Prove that for any $x, y \in \mathbb{N}$, if x is odd and y is odd then $x + y$ is even.

Solution: Suppose $x, y \in \mathbb{N}$ are odd, then $\exists n, m \in \mathbb{N} \cup \{0\}\ \ s.t.\ x = 2n + 1, y = 2m + 1$.

$$x + y = 2n + 1 + 2m + 1 = 2(n + m + 1). \tag{1}$$

It is clear that since $(n + m + 1) \in \mathbb{N}, x + y$ is even. $\square$

$$\iiint_{\text{Cube}} \left(-e^{-x} - e^{-y} - e^{-z}\right), dV = l^2 \Big[\Big(e^{-\left(x_0+\frac{l}{2}\right)} - e^{-\left(x_0-\frac{l}{2}\right)}\Big) + \Big(e^{-\left(y_0+\frac{l}{2}\right)} - e^{-\left(y_0-\frac{l}{2}\right)}\Big) + \Big(e^{-\left(z_0+\frac{l}{2}\right)} - e^{-(} \tag{2}$$

Problem 2. Prove that for any $x \in \mathbb{N}$, if x is odd then x^3 is odd.

Solution: Suppose x is odd, i.e. $\exists n \in \mathbb{N} \cup \{0\}\; s.t.\, x = 2n + 1$

$$x^3 = (2n+1)^3 = 8n^3 + 12n^2 + 6n + 1 = 2(4n^3 + 6n^2 + 3n) + 1. \tag{3}$$

It is clear that since $(4n^3 + 6n^2 + 3n) \in \mathbb{N} \cup \{0\}, x^3$ is odd.

$\square$

Problem 3. Using induction, prove that for all $n \in \mathbb{N}$ we have

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2.$$

Solution: [Base case]: For $n = 1$, we have $1 = 1$, which is true.

[Inductive step]: Suppose the statement is true for $\exists n \in \mathbb{N}$, i.e.

$$1 + 3 + \cdots + 2n - 1 = n^2 \tag{4}$$

Then for $n = n + 1$ we have:

$$\begin{align} 1 + 3 + \cdots + 2n - 1 + 2(n + 1) - 1 &= n^2 + 2n + 1 \tag{5} \\ &= (n + 1)^2 \tag{6} \end{align}$$

So the formula is true for $n + 1$. Thus, by induction, the statement is true for all $n \in \mathbb{N}$.

$\square$

Problem 4. Compute the following sum:

$$\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\cdots+\frac{1}{(2n-1)(2n+1)}.$$

Prove that your answer is true for all $n\in\mathbb{N}$ using induction.

Solution: By noticing $\frac{1}{(2n-1)(2n+1)}=\frac{1}{2}(\frac{1}{2n-1}-\frac{1}{2n+1})$, a rough calculation suggests that the sum should be $\frac{1}{2}-\frac{1}{4n+2}$. It is proved by induction as follows:

[base case]: For $n=1$, we have

$$\frac{1}{1\cdot 3}=\frac{1}{2}-\frac{1}{6}=\frac{1}{3},$$

which is true.

[Inductive case]: Suppose the statement is true for $\exists n\in\mathbb{N}$, i.e.

$$\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\cdots+\frac{1}{(2n-1)(2n+1)}=\frac{1}{2}-\frac{1}{4n+2} \tag{7}$$

Then for $n=n+1$,

$$\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\cdots+\frac{1}{(2n-1)(2n+1)}+\frac{1}{(2n+1)(2n+3)}=\frac{1}{2}-\frac{1}{4n+2}+\frac{1}{(2n+1)(2n+3)} \tag{8}$$

$$=\frac{1}{2}-\frac{1}{4n+2}+\frac{1}{2}(\frac{1}{2n+1}-\frac{1}{2n+3}) \tag{9}$$

$$=\frac{1}{2}-\frac{1}{4n+6} \tag{10}$$

$$=\frac{1}{2}-\frac{1}{4(n+1)+2} \tag{11}$$

So the formula is true for $n+1$. Thus, by induction, the statement is true for all $n\in\mathbb{N}$.

□

Problem 5. Prove the following statements for all $a, b \in \mathbb{R}$:

(a) $-a + (-b) = -(a + b)$.
(b) If $a, b \neq 0$ then $a^{-1} \cdot b^{-1} = (ab)^{-1}$.

Carefully justify every step using properties of $\mathbb{R}$ stated in lecture.

Solution: [a]: Consider the original equation,

$$-a + (-b) = -(a + b) \tag{12}$$

Adding (a+b) to both sides, we can find that it is equivalent to

$$-a + (-b) + (a + b) = -(a + b) + (a + b) \tag{13}$$

Applying inverse addition to the right side, and apply associativity to the left, this is equivalent to

$$-a + a + (-b) + b = 0 \tag{14}$$

Therefore, to prove the original statement is equivalent to prove Equation 14. By the inverse addition property, we have

$$-a + a = 0, \ -b + b = 0 \tag{15}$$

$$\Rightarrow -a + a + (-b) + b = 0 \tag{16}$$

The statement is thus proved.

[b]: Suppose $a, b \neq 0$, we have

$$a^{-1} \cdot b^{-1} \cdot ab \overset{\text{commutivity}}{=} a^{-1} \cdot a \cdot b^{-1} \cdot b \overset{\text{inverse}}{=} 1 \tag{17}$$

$$\text{also, } (ab)^{-1} \cdot ab \overset{inverse}{=} 1 \tag{18}$$

By transivity, we have

$$a^{-1} \cdot b^{-1} \cdot ab = (ab)^{-1} \cdot ab \tag{19}$$

$$\overset{\text{prop.1}}{\Rightarrow} a^{-1} \cdot b^{-1} = (ab)^{-1} \tag{20}$$

The statement is thus proved. □

Problem 6. Prove the following statements for all $a, b, c, d \in \mathbb{R}$:

(a) If $a < b$ and $c < d$ then $a + c < b + d$.
(b) If $0 < a < b$ and $0 < c < d$ then $ac < bd$.

Solution: [a]: Suppose $a < b, c < d$, then by O1,

$$a + c < b + c \tag{21}$$
$$b + c < d + b. \tag{22}$$

By Transitivity,

$$a + c < d + b \tag{23}$$

By commutivity,

$$a + c < b + d \tag{24}$$

Thus proved the inequality.

[b]: Suppose $0 < a < b$, $0 < c < d$. Then by O2,

$$a \cdot c < b \cdot c \tag{25}$$
$$b \cdot c < b \cdot d \tag{26}$$

By transitivity,

$$ac < bd \tag{27}$$

Thus proves the inequality. $\square$