

1 Quantum Fisher information

The experimental setup designed by J.Jae et. al generates the following quantum state after going through the cascaded QWP-HWP-QWP plates:

$$\begin{aligned} |\psi\rangle_{\text{in}} &= \text{QWP}_2\left(\frac{\pi}{4}\right) \text{HWP}(p) \text{QWP}_1(q) |H\rangle \\ &= e^{i(-2p+q+\frac{\pi}{4})} \begin{pmatrix} \cos\left(\frac{\pi}{4}-q\right) \\ e^{i(4p-2q-\frac{\pi}{2})} \sin\left(\frac{\pi}{4}-q\right) \end{pmatrix} \end{aligned} \quad \{1\}$$

and by adjusting p, q s.t. $\theta_0 = \frac{\pi}{2} - 2q, \varphi = 4p - 2q - \frac{\pi}{2}$, the parameterized state is given as

$$|\psi\rangle_{\text{in}} = \cos\left(\frac{\theta_0}{2}\right) |H\rangle + e^{i\varphi} \sin\left(\frac{\theta_0}{2}\right) |V\rangle \quad \{2\}$$

When passed through the sucrose solution, gaining a phase of $\theta = \alpha lc$, the state becomes

$$|\psi\rangle_{\text{out}} = \cos\left(\frac{\theta + \theta_0}{2}\right) |H\rangle + e^{i\varphi} \sin\left(\frac{\theta + \theta_0}{2}\right) |V\rangle \quad \{3\}$$

This is a pure state, since $\cos^2\left(\frac{\theta}{2}\right) + \sin^2\left(\frac{\theta}{2}\right) = 1$. For a pure state, its quantum Fisher Information can be found by the following equation [1].

$$F_\theta = 4 \left[\langle \partial_\theta \psi | \partial_\theta \psi \rangle + (\langle \partial_\theta \psi | \psi \rangle)^2 \right] \quad \{4\}$$

Where,

$$|\partial_\theta \psi\rangle = -\frac{1}{2} \sin\left(\frac{\theta + \theta_0}{2}\right) |H\rangle + \frac{1}{2} e^{i\varphi} \cos\left(\frac{\theta + \theta_0}{2}\right) |V\rangle \quad \{5\}$$

$$\langle \partial_\theta \psi | = -\frac{1}{2} \sin\left(\frac{\theta + \theta_0}{2}\right) \langle H | + \frac{1}{2} e^{-i\varphi} \cos\left(\frac{\theta + \theta_0}{2}\right) \langle V | \quad \{6\}$$

$$\Rightarrow \langle \partial_\theta \psi | \partial_\theta \psi \rangle = \left(\frac{1}{4} \sin^2\left(\frac{\theta + \theta_0}{2}\right) \right) + \left(\frac{1}{4} \cos^2\left(\frac{\theta + \theta_0}{2}\right) \right) = \frac{1}{4} \quad \{7\}$$

$$\begin{aligned} (\langle \partial_\theta \psi | \psi \rangle)^2 &= \frac{1}{4} \sin^2\left(\frac{\theta + \theta_0}{2}\right) \cos^2\left(\frac{\theta + \theta_0}{2}\right) + \frac{1}{4} \sin^2\left(\frac{\theta + \theta_0}{2}\right) \cos^2\left(\frac{\theta + \theta_0}{2}\right) \\ &\quad - \frac{1}{2} \sin^2\left(\frac{\theta + \theta_0}{2}\right) \cos^2\left(\frac{\theta + \theta_0}{2}\right) \\ &= 0 \end{aligned} \quad \{8\}$$

collecting Equation 7, Equation 8,

$$\boxed{F_Q = 4 \cdot \left(\frac{1}{4} + 0 \right) = 1,} \quad \{9\}$$

which agrees with what was claimed on [2], page 3.

2 Contextual Quantum Fisher Information via quasiprobability

2.1 Necessary Ingredients

2.1.1 Born's Rule

For quantum state $|\psi\rangle$ with eigenstates $|H\rangle, |V\rangle$ etc., the probability of the measurement outcome falls on eigensate $|H\rangle$ is :

$$P(H) = |\langle H|\psi\rangle|^2. \quad \{10\}$$

2.1.2 Projective Measurements

The experiment design in the paper involves two types of measurements [2]:

- \mathcal{A} : measurement in H, V basis, with $\Pi_H = |H\rangle\langle H|, \Pi_V = |V\rangle\langle V|$
- \mathcal{B} : measurement in D, A basis, with $\Pi_D = |D\rangle\langle D|, \Pi_A = |A\rangle\langle A|$, where

$$|D\rangle = \frac{1}{\sqrt{2}}(|H\rangle + |V\rangle); \quad |A\rangle = \frac{1}{\sqrt{2}}(|H\rangle - |V\rangle). \quad \{11\}$$

Consider a state in H / V basis $|\psi\rangle = \alpha|H\rangle + \beta|V\rangle$, it can be expressed in D / A basis as

$$|\psi\rangle = \frac{1}{2}\sqrt{2}(\alpha + \beta)|D\rangle + \frac{1}{\sqrt{2}}(\alpha - \beta)|A\rangle. \quad \{12\}$$

2.1.3 Operational Quasiprobability

$$w(a, b|\theta) = p(a, b|\mathcal{A}, \mathcal{B}, \theta) + \frac{1}{2}(p(b|\mathcal{B}, \theta) - p(b|\mathcal{A}, \mathcal{B}, \theta)) \quad \{13\}$$

- In the sucrose measurement experiment, $a \in \{H, V\}$, $b \in \{D, A\}$, and θ is the phase shift due to the sucrose solution.

2.1.4 Contextual Fisher information

$$F_{\text{co}} = \sum_{a,b} w(a, b|\theta) [\partial_\theta \ln w(a, b|\theta)]^2 \quad \{14\}$$

2.2 Calculating Operational Quasiprobability

There are three major terms in Equation 13, namely the joint probability $p(a, b|\mathcal{A}, \mathcal{B}, \theta)$, conditional probability $p(b|\mathcal{B}, \theta)$, and marginal probability $p(b|\mathcal{A}, \mathcal{B}, \theta)$. We will calculate them separately in the succeeding subsections.

2.2.1 Joint probability $p(a, b|\mathcal{A}, \mathcal{B}, \theta)$

Let a and b represent the specific measurement outcomes corresponding to the projection operators $\Pi_{\mathcal{A}} = |A\rangle\langle A|$ and $\Pi_{\mathcal{B}} = |B\rangle\langle B|$. i.e. $a = |A\rangle, b = |B\rangle$.

The joint probability $p(a, b|\mathcal{A}, \mathcal{B}, \theta)$ can be interpreted as the probability of sequentially measuring $|A\rangle$ and then $|B\rangle$ when performing the measurements associated with $\Pi_{\mathcal{A}}$ followed by $\Pi_{\mathcal{B}}$.

According to the postulate of quantum measurement, this cascaded measurement can be represented as a single measurement with the combined operator

$\Pi_{\{\mathcal{B}, \mathcal{A}\}} = \Pi_{\mathcal{B}}\Pi_{\mathcal{A}}$. Thus, the joint probability can be written as:

$$\begin{aligned} p(a, b|\mathcal{A}, \mathcal{B}, \theta) &= \langle \psi(\theta) | \Pi_{\{\mathcal{B}, \mathcal{A}\}}^\dagger \Pi_{\{\mathcal{B}, \mathcal{A}\}} | \psi(\theta) \rangle \\ &= \langle \psi(\theta) | (\Pi_{\mathcal{B}}\Pi_{\mathcal{A}})^\dagger (\Pi_{\mathcal{B}}\Pi_{\mathcal{A}}) | \psi(\theta) \rangle \\ &= \langle \psi(\theta) | \Pi_{\mathcal{A}}^\dagger \Pi_{\mathcal{B}}^\dagger \Pi_{\mathcal{B}} \Pi_{\mathcal{A}} | \psi(\theta) \rangle \end{aligned} \quad \{15\}$$

Since $\Pi_{\mathcal{A}}$ and $\Pi_{\mathcal{B}}$ are Hermitian ($\Pi_{\mathcal{A}}^\dagger = \Pi_{\mathcal{A}}$ and $\Pi_{\mathcal{B}}^\dagger = \Pi_{\mathcal{B}}$) and idempotent ($\Pi_{\mathcal{A}}^2 = \Pi_{\mathcal{A}}$ and $\Pi_{\mathcal{B}}^2 = \Pi_{\mathcal{B}}$), this simplifies to:

$$p(a, b | \mathcal{A}, \mathcal{B}, \theta) = \langle \psi(\theta) | \Pi_{\mathcal{A}} \Pi_{\mathcal{B}} \Pi_{\mathcal{A}} | \psi(\theta) \rangle \quad \{16\}$$

Noticing $\Pi_{\mathcal{A}} = |A\rangle\langle A|$, $\Pi_{\mathcal{B}} = |B\rangle\langle B|$, we can simplify this further to:

$$\begin{aligned} p(a, b | \mathcal{A}, \mathcal{B}, \theta) &= \langle \psi | |A\rangle\langle A| |B\rangle\langle B| |A\rangle\langle A| | \psi \rangle \\ &= |\langle A | \psi \rangle|^2 |\langle B | A \rangle|^2 \end{aligned} \quad \{17\}$$

Using Equation 17, we can calculate $p(a, b | \mathcal{A}, \mathcal{B}, \theta)$ for $a \in \{H, V\}$ and $b \in \{D, A\}$.

- $$\begin{aligned} p(H, D | \mathcal{A}, \mathcal{B}, \theta) &= |\langle H | \psi \rangle|^2 |\langle D | H \rangle|^2 \\ &= \left| \begin{pmatrix} 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \cos\left(\frac{\theta+\theta_0}{2}\right) \\ e^{i\varphi} \sin\left(\frac{\theta+\theta_0}{2}\right) \end{pmatrix} \right|^2 \left| \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right|^2 \\ &= \frac{1}{2} \cos^2\left(\frac{\theta + \theta_0}{2}\right) \\ &= \frac{1}{4} (1 + \cos(\theta + \theta_0)) \end{aligned} \quad \{18\}$$

• Similarly,

$$\begin{aligned} p(H, A | \mathcal{A}, \mathcal{B}, \theta) &= |\langle H | \psi \rangle|^2 |\langle A | H \rangle|^2 \\ &= \frac{1}{2} \cos^2\left(\frac{\theta + \theta_0}{2}\right) \\ &= \frac{1}{4} (1 + \cos(\theta + \theta_0)) \end{aligned} \quad \{19\}$$

- $$\begin{aligned} p(V, D | \mathcal{A}, \mathcal{B}, \theta) &= |\langle V | \psi \rangle|^2 |\langle D | V \rangle|^2 \\ &= \frac{1}{4} (1 - \cos(\theta + \theta_0)) \end{aligned} \quad \{20\}$$

- $$\begin{aligned} p(V, A | \mathcal{A}, \mathcal{B}, \theta) &= |\langle V | \psi \rangle|^2 |\langle A | V \rangle|^2 \\ &= \frac{1}{4} (1 - \cos(\theta + \theta_0)) \end{aligned} \quad \{21\}$$

2.2.2 Conditional probability $p(b | \mathcal{B}, \theta)$

$p(b | \mathcal{B}, \theta)$ represents the probability of measuring outcome b when measurement \mathcal{B} is performed. This can be found using the Born's rule (Equation 10). i.e.

$$p(b | \mathcal{B}, \theta) = |\langle B | \psi \rangle|^2 \quad \{22\}$$

We calculate $p(b | \mathcal{B}, \theta)$ for $b \in \{D, A\}$ as follows:

- $$\begin{aligned}
p(D|\mathcal{B}, \theta) &= |\langle D|\psi \rangle|^2 \\
&= \left| \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \cdot \begin{pmatrix} \cos\left(\frac{\theta+\theta_0}{2}\right) \\ e^{i\varphi} \sin\left(\frac{\theta+\theta_0}{2}\right) \end{pmatrix} \right|^2 \\
&= \left| \frac{1}{\sqrt{2}} \cos\left(\frac{\theta+\theta_0}{2}\right) + \frac{1}{\sqrt{2}} e^{i\varphi} \sin\left(\frac{\theta+\theta_0}{2}\right) \right|^2 \\
&= \left(\frac{1}{\sqrt{2}} \cos\left(\frac{\theta+\theta_0}{2}\right) + \frac{1}{\sqrt{2}} \cos \varphi \sin\left(\frac{\theta+\theta_0}{2}\right) \right)^2 \quad \{23\} \\
&\quad + \left(\frac{1}{\sqrt{2}} \sin \varphi \sin\left(\frac{\theta+\theta_0}{2}\right) \right)^2 \\
&= \frac{1}{2} + \cos \varphi \cos\left(\frac{\theta+\theta_0}{2}\right) \sin\left(\frac{\theta+\theta_0}{2}\right) \\
&\quad \boxed{= \frac{1}{2}(1 + \sin(\theta + \theta_0) \cos \varphi)}
\end{aligned}$$

- $$\begin{aligned}
p(A|\mathcal{B}, \theta) &= |\langle A|\psi \rangle|^2 \\
&= \left| \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \cdot \begin{pmatrix} \cos\left(\frac{\theta+\theta_0}{2}\right) \\ e^{i\varphi} \sin\left(\frac{\theta+\theta_0}{2}\right) \end{pmatrix} \right|^2 \\
&= \left| \frac{1}{\sqrt{2}} \cos\left(\frac{\theta+\theta_0}{2}\right) - \frac{1}{\sqrt{2}} e^{i\varphi} \sin\left(\frac{\theta+\theta_0}{2}\right) \right|^2 \\
&= \left(\frac{1}{\sqrt{2}} \cos\left(\frac{\theta+\theta_0}{2}\right) - \frac{1}{\sqrt{2}} \cos \varphi \sin\left(\frac{\theta+\theta_0}{2}\right) \right)^2 \quad \{24\} \\
&\quad + \left(-\frac{1}{\sqrt{2}} \sin \varphi \sin\left(\frac{\theta+\theta_0}{2}\right) \right)^2 \\
&= \frac{1}{2} - \cos \varphi \cos\left(\frac{\theta+\theta_0}{2}\right) \sin\left(\frac{\theta+\theta_0}{2}\right) \\
&\quad \boxed{= \frac{1}{2}(1 - \sin(\theta + \theta_0) \cos \varphi)}
\end{aligned}$$

2.2.3 Marginal probability $p(b|\mathcal{A}, \mathcal{B}, \theta)$

By definition, $p(b|\mathcal{A}, \mathcal{B}, \theta) = \sum_a p(a, b|\mathcal{A}, \mathcal{B}, \theta)$. Using results from Equation 18 to Equation 21, we can calculate $p(b|\mathcal{A}, \mathcal{B}, \theta)$ for $b \in \{D, A\}$ as follows:

- $$\begin{aligned}
p(D|\mathcal{A}, \mathcal{B}, \theta) &= p(H, D|\mathcal{A}, \mathcal{B}, \theta) + p(V, D|\mathcal{A}, \mathcal{B}, \theta) \\
&= \frac{1}{4}(1 + \cos(\theta + \theta_0)) + \frac{1}{4}(1 - \cos(\theta + \theta_0)) \quad \{25\} \\
&= \frac{1}{2}
\end{aligned}$$

- $$\begin{aligned}
p(A|\mathcal{A}, \mathcal{B}, \theta) &= p(H, A|\mathcal{A}, \mathcal{B}, \theta) + p(V, A|\mathcal{A}, \mathcal{B}, \theta) \\
&= \frac{1}{4}(1 + \cos(\theta + \theta_0)) + \frac{1}{4}(1 - \cos(\theta + \theta_0)) \quad \{26\} \\
&= \frac{1}{2}
\end{aligned}$$

2.2.4 Quasiprobability for $a \in \{H, V\}, b \in \{D, A\}$

Collecting the above,

- $$\begin{aligned}
w(H, D | \theta) &= p(H, D | \mathcal{A}, \mathcal{B}, \theta) + \frac{1}{2}[p(D | \mathcal{B}, \theta) - p(D | \mathcal{A}, \mathcal{B}, \theta)] \\
&= \frac{1}{4}(1 + \cos(\theta + \theta_0)) + \frac{1}{2}\left(\frac{1}{2}(1 + \sin(\theta + \theta_0) \cos \varphi) - \frac{1}{2}\right) \{27\} \\
&= \frac{1}{4}(1 + \cos(\theta + \theta_0) + \sin(\theta + \theta_0) \cos \varphi)
\end{aligned}$$
- $$\begin{aligned}
w(H, A | \theta) &= p(H, A | \mathcal{A}, \mathcal{B}, \theta) + \frac{1}{2}[p(A | \mathcal{B}, \theta) - p(A | \mathcal{A}, \mathcal{B}, \theta)] \\
&= \frac{1}{4}(1 + \cos(\theta + \theta_0) - \sin(\theta + \theta_0) \cos \varphi) \{28\}
\end{aligned}$$
- $$\begin{aligned}
w(V, D | \theta) &= p(V, D | \mathcal{A}, \mathcal{B}, \theta) + \frac{1}{2}[p(D | \mathcal{B}, \theta) - p(D | \mathcal{A}, \mathcal{B}, \theta)] \\
&= \frac{1}{4}(1 - \cos(\theta + \theta_0) + \sin(\theta + \theta_0) \cos \varphi) \{29\}
\end{aligned}$$
- $$\begin{aligned}
w(V, A | \theta) &= p(V, A | \mathcal{A}, \mathcal{B}, \theta) + \frac{1}{2}[p(A | \mathcal{B}, \theta) - p(A | \mathcal{A}, \mathcal{B}, \theta)] \\
&= \frac{1}{4}(1 - \cos(\theta + \theta_0) - \sin(\theta + \theta_0) \cos \varphi) \{30\}
\end{aligned}$$

This set of quasipaobabilities agrees with:

$$w(a, b | \theta) = (1 + (-1)^a \cos \theta + (-1)^b \sin \theta \cos \varphi) / 4, \quad \{31\}$$

as claimed in [2], page 3.

2.3 Contextual Fisher Information (coFI)

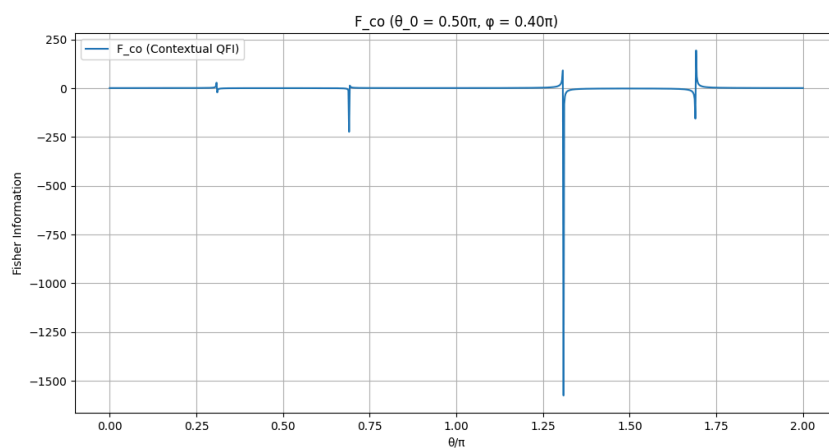
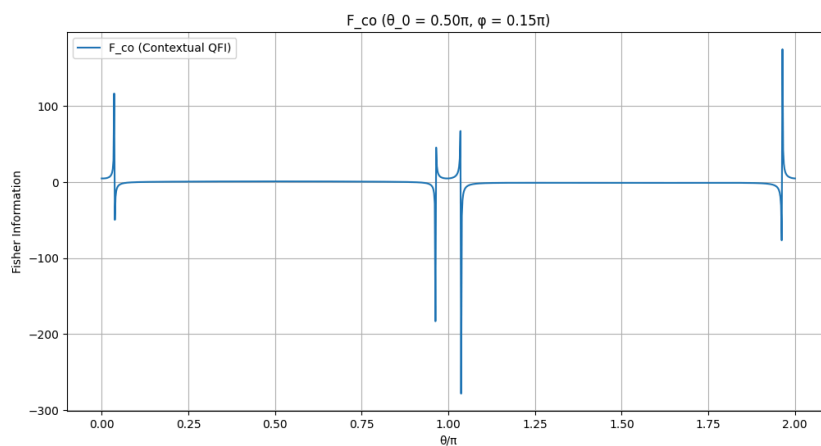
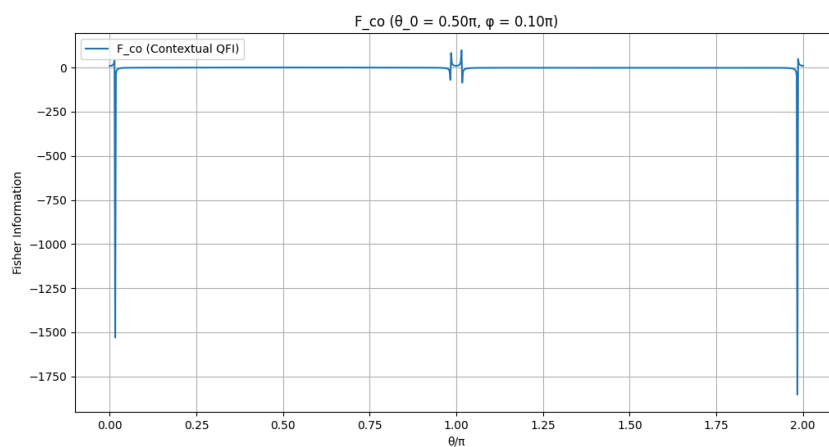
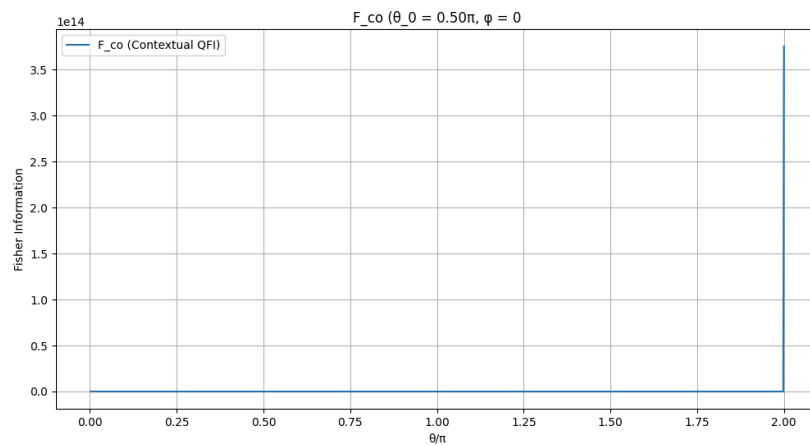
coFI is given in [2] as

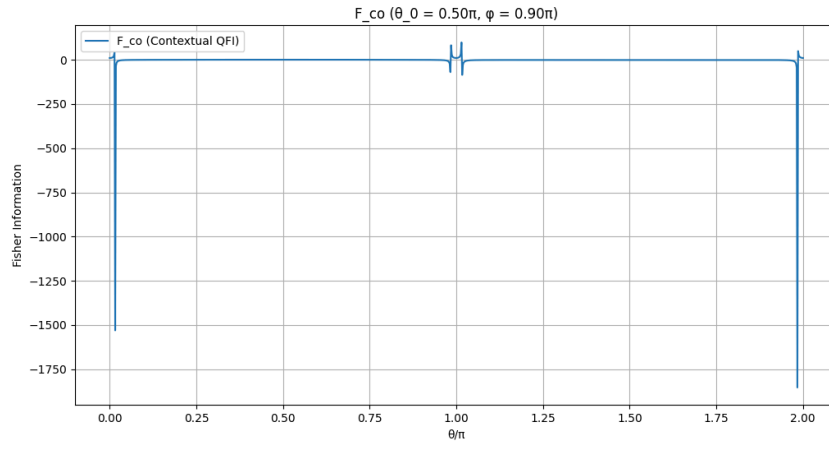
$$F_{\text{co}} = \sum_{a,b} w(a, b | \theta) [\partial_\theta \ln w(a, b | \theta)]^2 \quad \{32\}$$

Using the results from above, for $a \in \{H, V\}, b \in \{D, A\}$ we can expand it as:

$$\begin{aligned}
F_{\text{co}} &= w(H, D | \theta) [\partial_\theta \ln w(H, D | \theta)]^2 + w(H, A | \theta) [\partial_\theta \ln w(H, A | \theta)]^2 \\
&\quad + w(V, D | \theta) [\partial_\theta \ln w(V, D | \theta)]^2 + w(V, A | \theta) [\partial_\theta \ln w(V, A | \theta)]^2 \\
&= \frac{(-\cos(\varphi) \cos(\theta + \theta_0) + \sin(\theta + \theta_0))^2}{4(1 + \cos(\theta + \theta_0) + \cos(\varphi) \sin(\theta + \theta_0))} \\
&\quad + \frac{(\cos(\varphi) \cos(\theta + \theta_0) + \sin(\theta + \theta_0))^2}{4(1 + \cos(\theta + \theta_0) - \cos(\varphi) \sin(\theta + \theta_0))} \{33\} \\
&\quad + \frac{(\cos(\varphi) \cos(\theta + \theta_0) + \sin(\theta + \theta_0))^2}{4(1 - \cos(\theta + \theta_0) + \cos(\varphi) \sin(\theta + \theta_0))} \\
&\quad - \frac{(-\cos(\varphi) \cos(\theta + \theta_0) + \sin(\theta + \theta_0))^2}{4(-1 + \cos(\theta + \theta_0) + \cos(\varphi) \sin(\theta + \theta_0))}
\end{aligned}$$

We analyse numerically for the behaviour of F_{co} by taking $\theta_0 = 0.5\pi$ as suggested in the paper. [2]





Bibliography

- [1] M. Barbieri, “Optical Quantum Metrology,” *PRX Quantum*, vol. 3, no. 1, p. 10202–10203, Jan. 2022, doi: 10.1103/PRXQuantum.3.010202.
- [2] J. Jae, J. Lee, M. S. Kim, K.-G. Lee, and J. Lee, “Contextual quantum metrology,” *npj Quantum Information*, vol. 10, no. 1, Jul. 2024, doi: 10.1038/s41534-024-00862-5.