Notes on Contextual Quantum Metrology

1 Quantum Fisher information

eq.5 on page 5 of [1] shows

$$
\begin{aligned}
|\psi\rangle_{\text{in}} &= \mathrm{QWP}_2\left(\frac{\pi}{4}\right)\mathrm{HWP}(p)\,\mathrm{QWP}_1(q)\,|H\rangle \\
&= e^{i\left(-2p+q+\frac{\pi}{4}\right)}\begin{pmatrix} \cos\left(\frac{\pi}{4}-q\right) \\ e^{i\left(4p-2q-\frac{\pi}{2}\right)}\sin\left(\frac{\pi}{4}-q\right)\end{pmatrix}
\end{aligned} \tag{1}
$$

and by adjusting p, q s.t. $\theta_0 = \frac{\pi}{2} - 2q, \varphi = 4p - 2q - \frac{\pi}{2}$, the parameterized state is given as

$$
|\psi\rangle_{\text{in}} = \cos\left(\frac{\theta_0}{2}\right)|H\rangle + e^{i\varphi}\sin\left(\frac{\theta_0}{2}\right)|V\rangle \tag{2}
$$

When passed thorugh the sucrose solution, gaining a phase of $\theta = \alpha l c$, the state becomes

$$
|\psi\rangle_{\text{out}} = \cos\left(\frac{\theta+\theta_0}{2}\right)|H\rangle + e^{i\varphi}\sin\left(\frac{\theta+\theta_0}{2}\right)|V\rangle \tag{3}
$$

This is a pure state, since $\cos^2\left(\frac{\theta}{2}\right) + \sin^2\left(\frac{\theta}{2}\right) = 1$. For a pure state, its quantum Fisher Information can be found by the following equation [2].

$$
F_\theta = 4\left[\langle \partial_\theta \psi | \partial_\theta \psi\rangle + \left(\langle \partial_\theta \psi | \psi\rangle\right)^2\right] \tag{4}
$$

Where,

$$
|\partial_\theta \psi\rangle = -\frac{1}{2}\sin\left(\frac{\theta+\theta_0}{2}\right)|H\rangle + \frac{1}{2}e^{i\varphi}\cos\left(\frac{\theta+\theta_0}{2}\right)|V\rangle \tag{5}
$$

$$
\langle \partial_\theta \psi| = -\frac{1}{2}\sin\left(\frac{\theta+\theta_0}{2}\right)\langle H| + \frac{1}{2}e^{-i\varphi}\cos\left(\frac{\theta+\theta_0}{2}\right)\langle V| \tag{6}
$$

$$
\Rightarrow \langle \partial_\theta \psi | \partial_\theta \psi\rangle = \left(\frac{1}{4}\sin^2\left(\frac{\theta+\theta_0}{2}\right)\right) + \left(\frac{1}{4}\cos^2\left(\frac{\theta+\theta_0}{2}\right)\right) = \frac{1}{4} \tag{7}
$$

$$
\begin{aligned}
\left(\langle \partial_\theta \psi | \psi\rangle\right)^2 &= \frac{1}{4}\sin^2\left(\frac{\theta+\theta_0}{2}\right)\cos^2\left(\frac{\theta+\theta_0}{2}\right) + \frac{1}{4}\sin^2\left(\frac{\theta+\theta_0}{2}\right)\cos^2\left(\frac{\theta+\theta_0}{2}\right) \\
&\quad -\frac{1}{2}\sin^2\left(\frac{\theta+\theta_0}{2}\right)\cos^2\left(\frac{\theta+\theta_0}{2}\right) \\
&= 0
\end{aligned} \tag{8}
$$

collecting Equation 7, Equation 8,

$$
\boxed{F_Q = 4\cdot\left(\frac{1}{4}+0\right) = 1} \tag{9}
$$

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2 Contextual Quantum Fisher Information via quasiprobability

2.1 Necessary Ingredients

2.1.1 Born's Rule

For quantum state $|\psi\rangle$ with eigenstates $|H\rangle, |V\rangle$:

$$P(H) = |\langle H|\psi\rangle|^2 \tag{10}$$

2.1.2 Projective Measurements

The experiment involves two types of measurements:

- A: measurement in H / V basis , with $\Pi_H = |H\rangle\langle H|, \Pi_V = |V\rangle\langle V|$
- B: measurement in D / A basis, with $\Pi_D = |D\rangle\langle D|, \Pi_A = |A\rangle\langle A|$, where

$$|D\rangle = \frac{1}{\sqrt{2}}(|H\rangle + |V\rangle), |A\rangle = \frac{1}{\sqrt{2}}(|H\rangle - |V\rangle) \tag{11}$$

  Consider a state in H / V basis $|\psi\rangle = \alpha|H\rangle + \beta|V\rangle$, it can be expressed in D / A basis as

$$|\psi\rangle = \frac{1}{2}\sqrt{2}(\alpha + \beta)|D\rangle + \frac{1}{\sqrt{2}}(\alpha - \beta)|A\rangle \tag{12}$$

2.1.3 Operational Quasiprobability

$$w(a, b|\theta) = p(a, b|A, B, \theta) + \frac{1}{2}p(b|B, \theta) - p(b|A, B, \theta) \tag{13}$$

- In the sucrose measurement experiment, (a, b) can take (H, D), (H, A),(V, D), (V, A)

2.1.4 Joint probability $p(a, b|A, B, \theta)$

Let a and b represent the specific measurement outcomes corresponding to the projection operators $\Pi_A = |A\rangle\langle A|$ and $\Pi_B = |B\rangle\langle B|$.

The joint probability $p(a, b|A, B, \theta)$ can be interpreted as the probability of sequentially measuring $|A\rangle$ and then $|B\rangle$ when performing the measurements associated with Π_A followed by Π_B.

According to the postulate of quantum measurement, this cascaded measurement can be represented as a single measurement with the combined operator $\Pi_{\{BA\}} = \Pi_B \Pi_A$. Thus, the joint probability can be written as:

$$\begin{aligned} p(a, b|A, B, \theta) &= \langle\psi(\theta)|\Pi_{\{BA\}}^\dagger \Pi_{\{BA\}}|\psi(\theta)\rangle \\ &= \langle\psi(\theta)|(\Pi_B \Pi_A)^\dagger (\Pi_B \Pi_A)|\psi(\theta)\rangle \\ &= \langle\psi(\theta)|\Pi_A^\dagger \Pi_B^\dagger \Pi_B \Pi_A|\psi(\theta)\rangle \end{aligned} \tag{14}$$

Since Π_A and Π_B are Hermitian ($\Pi_A^\dagger = \Pi_A$ and $\Pi_B^\dagger = \Pi_B$) and idempotent ($\Pi_A^2 = \Pi_A$ and $\Pi_B^2 = \Pi_B$), this simplifies to:

$$p(a, b|A, B, \theta) = \langle\psi(\theta)|\Pi_A \Pi_B \Pi_A|\psi(\theta)\rangle \tag{15}$$

Noticing $\Pi_A = |A\rangle\langle A|, \Pi_B = |B\rangle\langle B|$, we can simplify this further to:

$$\begin{aligned} p(a, b|A, B, \theta) &= \langle\psi||A\rangle\langle A||B\rangle\langle B||A\rangle\langle A||\psi\rangle \\ &= |\langle A|\psi\rangle|^2 \, |\langle B|A\rangle|^2 \end{aligned} \tag{16}$$

2.1.5 Contexual Fisher information

$$F_{\text{co}} = \sum_{a,b} w(a,b|\theta)\,[\partial_\theta \ln w(a,b|\theta)]^2 \tag{17}$$

Bibliography

[1] J. Jae, J. Lee, M. S. Kim, K.-G. Lee, and J. Lee, "Contextual quantum metrology," *npj Quantum Information*, vol. 10, no. 1, Jul. 2024, doi: 10.1038/s41534-024-00862-5.

[2] M. Barbieri, "Optical Quantum Metrology," *PRX Quantum*, vol. 3, no. 1, p. 10202–10203, Jan. 2022, doi: 10.1103/PRXQuantum.3.010202.