1 Notes on Quantum computation and quantum information by Nielsen and Chuang

2 Chapter 2: Linear algebra

2.I vector space

- C^n: space of all n-tuple complex numbers (c numbers)

i.e. $(z_1, z_2, z_3, ..., z_n)$

- a vector space is closed under scalar multiplicationa nd addition

2.II Dirac notation

Symbols	Meaning
$\|v\rangle$	ket, a vector in vec space
$\langle v\|$	bra, a dual vector in vec space; the complex transpose of ket $\langle v\| = (\|v\rangle^*)^T$
$\langle v\|w\rangle$	inner product of $\|v\rangle$ and $\|w\rangle$
$\|\varphi\rangle \otimes \|\psi\rangle$	tensor product of $\|\varphi\rangle$ and $\|\psi\rangle$ abbriviates as $\|\varphi\rangle\|\psi\rangle$
$\boldsymbol{A}^*$	complex conjugate of $\boldsymbol{A}$
$\boldsymbol{A}^T$	transpose of $\boldsymbol{A}$
$\boldsymbol{A}^\dagger$	hermitian conjugate of $\boldsymbol{A}$ i.e. $\boldsymbol{A}^\dagger = (\boldsymbol{A}^*)^T$ $$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^\dagger = \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix}$$
$\langle\varphi\|\boldsymbol{A}\|\psi\rangle$	inner product betweeen $\|\varphi\rangle$ and $\boldsymbol{A}\|\psi\rangle$

2.III Span

a set of bec $|v_1\rangle, |v_2\rangle, ..., |v_n\rangle$ spans the vector space if any vector in the space can be written as

$$|v\rangle = \sum_i a_i |v_i\rangle$$

for some complex numbers a_i

2.IV Linear Independence

a set of non-zero vectors $|v_1\rangle, |v_2\rangle, ..., |v_n\rangle$ are liinearly dependent if there exists a set of complex numbers $a_1, a_2, ..., a_n$, s.t.

$$a_1|v_1\rangle + a_2|v_2\rangle + ... + a_n|v_n\rangle = 0$$

If the only solution to the above equation is $a_1 = a_2 = ... = a_n = 0$, then the vectors are **linearly independent**

2.V Linear operators

A linear operator A is any linear function that

$$A\left(\sum_i a_i|v_i\rangle\right) = \sum_i a_i A(|v_i\rangle)$$

It is convention to write $A(|v_i\rangle) = A|v_i\rangle$

- Identity Operator $I_V : I_V|v\rangle \equiv |v\rangle$. It is convinent to write I if no confution arises.
- zero operator $0|v\rangle \equiv 0$
- composition of linear operators A and B is AB

We observe that the above is equivalent to the matrix representation of linear transformations.

In other words, for a linear operator $A : V \to W$, and suppose $|v_1\rangle, |v_2\rangle, ..., |v_m\rangle$

2.VI Hilbert Space

Given a vector basis $\{|E_i\rangle\}$, when attempting to represent a polynomial as $p = \sum_{i=0}^{\infty} a_i E_i$, the sum is in the form of, according to taylor series, an exponential function. But the exponential function is not a polynomial, i.e. outside of our vector space, so we have landed on a paradox. To avoid this, we define a **Hilbert Space** to handle infinite dimensional vector spaces.

- A Hilbert spsace is a vector space that is 1. complete and 2. has an inner product defined on it. In other words, every converging set of vectors must converge to an element **inside** the vector space.

$$|\psi\rangle \in \mathcal{H}$$

2.VII Inner product

- Review on dot product
  - orthogonality & angle
  - norm $\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}}$

For kets $|\psi\rangle, |\varphi\rangle, |\zeta\rangle$, and scalar a, an inner product has the following rules:

- Linearity in the second argument:

$$\begin{cases} \langle\psi|\varphi + \zeta\rangle = \langle\psi|\varphi\rangle + \langle\psi|\zeta\rangle \\ \langle\psi|a\varphi\rangle = a\langle\psi|\varphi\rangle \end{cases}$$

- COmplex commutation:

$$\langle\psi|\varphi\rangle = \langle\varphi|\psi\rangle^*$$

- Positive definiteness (think of norm):

$$|\psi\rangle \neq 0 \Rightarrow \langle\psi|\psi\rangle > 0$$

- Magnitude of a vector:

$$\||\psi\rangle\| = \sqrt{\langle\psi|\psi\rangle}$$

- Orthogonality:

$$\langle\psi|\varphi\rangle = 0 \Rightarrow |\psi\rangle \text{ and } |\varphi\rangle \text{ are orthogonal}$$

- antilinearity in the first argument:

$$\langle a\psi + b\zeta|\varphi\rangle = a^*\langle\psi|\varphi\rangle + b^*\langle\zeta|\varphi\rangle$$

2.VIII Orthonormal basis

$\{|E_i\rangle\}$ s.t. $\langle E_i | E_j \rangle = \delta_{ij}$ is an orthonormal basis, with kroneker delta $\delta_{ij} = \begin{cases} 1 & i{=}j \\ 0 & i{\neq}j \end{cases}$

In English, the inner product of two vectors is 1 if they are the same (norm = 1), and 0 if they are different (orthogonal).

Using the orthonormal basis, we can write any vector as a linear combination of the basis vectors:

$$|\psi\rangle = \sum_i c_i |E_i\rangle$$

Notice that

$$\begin{aligned}\langle E_i|\psi\rangle &= \left\langle E_i \middle| \sum_j c_j E_j \right\rangle \\ &= \sum_j c_j \langle E_i|E_j\rangle \\ &= c_i\end{aligned}$$

And we use the above to calculate the coefficients c_i.

2.VIII.I Inner product between two vectors

$$\begin{aligned}\langle\psi|\varphi\rangle &= \left\langle \sum_i c_i E_i \middle| \sum_j d_j E_j \right\rangle \\ &= \sum_i \sum_j c_i^* d_j \langle E_i|E_j\rangle \\ &= \sum_i \sum_j c_i^* d_j \delta_{ij} \\ &= \sum_i c_i^* d_i\end{aligned}$$

When $c, d \in \mathbb{N}$, $\langle\varphi|\psi\rangle = \sum_i c_i d_i$ is simply the dot product.