Advanced Lecture: Contextual Quantum Metrology - A Deep Dive

Your Quantum Metrology Professor

1 Introduction

Welcome, everyone, to this advanced lecture on Contextual Quantum Metrology. Today, we're going to explore a groundbreaking approach that promises to revolutionize the field of quantum measurement. Our journey will take us through the strange and fascinating world of quantum contextuality, and show how we can harness this quintessentially quantum property to achieve unprecedented measurement precision.

Before we begin, let's remind ourselves of the importance of metrology. In science and technology, the ability to measure physical quantities with extreme accuracy is paramount. From the detection of gravitational waves to the development of ultra-precise atomic clocks, advances in metrology have driven scientific progress and technological innovation.

Quantum mechanics, with its counterintuitive features, offers the potential to take measurement precision to new heights. This is the realm of quantum metrology. However, conventional approaches to quantum metrology often rely on complex entangled states, which are notoriously difficult to prepare and maintain. The approach we'll discuss today - Contextual Quantum Metrology (coQM) - offers a new path forward, one that doesn't require these fragile entangled states.

Our lecture will cover the following key areas:

1. A review of the standard quantum limit and quantum Fisher information 2. An in-depth exploration of quantum contextuality 3. The principles of contextual quantum metrology 4. A detailed analysis of the operational quasiprobability and contextual Fisher information 5. Explicit demonstration of how coQM can outperform conventional quantum metrology 6. Practical implementation and experimental results 7. Error analysis and robustness of the method 8. Theoretical foundations, including the asymptotic normality of the maximum likelihood estimator 9. Implications and future prospects

By the end of this lecture, you'll have a deep understanding of how contextual quantum metrology works, why it's so powerful, and how it opens up new possibilities in quantum sensing and measurement.

Let's begin our journey into the heart of quantum contextuality and its application to metrology.

2 The Standard Quantum Limit: A Detailed Review

Before we delve into contextual quantum metrology, it's crucial to understand the limitations of conventional quantum metrology. The standard quantum limit (SQL) sets the benchmark for measurement precision using classical strategies with quantum systems.

The SQL is given by:

$$\Delta\theta \geq \frac{1}{\sqrt{NF_Q}} \tag{1}$$

Where $\Delta\theta$ is the uncertainty in our estimate of parameter θ, N is the number of independent measurements, and F_Q is the Quantum Fisher Information (QFI).

2.1 Derivation of Quantum Fisher Information

Let's derive the QFI for a simple qubit state:

$$|\psi(\theta)\rangle = \cos(\theta/2)\,|0\rangle + \sin(\theta/2)\,|1\rangle \tag{2}$$

The general formula for the QFI is:

$$F_Q = 4(\langle\partial_\theta\psi|\partial_\theta\psi\rangle - |\,\langle\psi|\partial_\theta\psi\rangle\,|^2) \tag{3}$$

Let's work through this step-by-step:

1) First, we calculate $|\partial_\theta\psi\rangle$:

$$|\partial_\theta\psi\rangle = -\frac{1}{2}\sin(\theta/2)\,|0\rangle + \frac{1}{2}\cos(\theta/2)\,|1\rangle \tag{4}$$

2) Now, let's calculate each term in the QFI formula:

$$\begin{aligned}
\langle\partial_\theta\psi|\partial_\theta\psi\rangle &= \langle\partial_\theta\psi|\partial_\theta\psi\rangle &(5)\\
&= (-\frac{1}{2}\sin(\theta/2)\,\langle 0| + \frac{1}{2}\cos(\theta/2)\,\langle 1|)(-\frac{1}{2}\sin(\theta/2)\,|0\rangle + \frac{1}{2}\cos(\theta/2)\,|1\rangle) &(6)\\
&= \frac{1}{4}\sin^2(\theta/2) + \frac{1}{4}\cos^2(\theta/2) = \frac{1}{4} &(7)
\end{aligned}$$

$$\begin{aligned}
\langle\psi|\partial_\theta\psi\rangle &= \langle\psi|\partial_\theta\psi\rangle &(8)\\
&= (\cos(\theta/2)\,\langle 0| + \sin(\theta/2)\,\langle 1|)(-\frac{1}{2}\sin(\theta/2)\,|0\rangle + \frac{1}{2}\cos(\theta/2)\,|1\rangle) &(9)\\
&= -\frac{1}{2}\cos(\theta/2)\sin(\theta/2) + \frac{1}{2}\sin(\theta/2)\cos(\theta/2) = 0 &(10)
\end{aligned}$$

3) Substituting these into our QFI formula:

$$F_Q = 4(\frac{1}{4} - 0^2) = 1 \tag{11}$$

Thus, for this simple qubit state, we find $F_Q = 1$. This means our standard quantum limit becomes:

$$\Delta\theta \geq \frac{1}{\sqrt{N}} \tag{12}$$

This $1/\sqrt{N}$ scaling is often called the shot-noise limit in optical measurements. It tells us that to improve our measurement precision by a factor of 10, we need to increase the number of measurements by a factor of 100!

2.2 The Heisenberg Limit

It's worth noting that there exists a fundamental limit to precision in quantum metrology, known as the Heisenberg limit:

$$\Delta\theta \geq \frac{1}{N} \tag{13}$$

This limit can be approached using entangled states, but as mentioned earlier, such states are difficult to prepare and maintain. The beauty of contextual quantum metrology is that it can approach this limit without requiring entanglement.

3 Quantum Contextuality: A Deep Dive

Now, let's explore the concept of quantum contextuality in depth. This property is at the heart of contextual quantum metrology, and understanding it is crucial to grasping how coQM works.

3.1 The Kochen-Specker Theorem

Contextuality is formalized in the Kochen-Specker theorem, which states that for quantum systems of dimension 3 or greater, it's impossible to assign definite values to all observables in a way that's independent of the measurement context.

Let's break this down:

1) In classical physics, we assume that measurable properties of a system have definite values at all times, regardless of whether we measure them or not.

2) The Kochen-Specker theorem tells us that this assumption is incompatible with quantum mechanics for systems with three or more dimensions.

3) This means that the value we measure for a quantum observable can depend on what other observables we measure along with it - this is what we mean by "context".

3.2 Contextuality Through Commutation Relations

To illustrate contextuality, let's consider three observables A, B, and C, where:

$$[A, B] = [B, C] = 0, \text{ but } [A, C] \neq 0 \tag{14}$$

Here, $[X, Y]$ denotes the commutator of X and Y, defined as $[X, Y] = XY - YX$.

Now, let's delve deeper into why these commutation relations exhibit contextuality:

1) $[A, B] = 0$ means that A and B are compatible observables. They can be measured simultaneously, and measuring one does not disturb the other.

2) Similarly, $[B, C] = 0$ means B and C are compatible.

3) However, $[A, C] \neq 0$ means A and C are incompatible. They cannot be measured simultaneously without one disturbing the other.

Here's the key point: The outcome of measuring B can depend on whether it's measured alongside A or C. Why? Because:

- If we measure B along with A, we're in a context where B and A are well-defined simultaneously. - If we measure B along with C, we're in a context where B and C are well-defined simultaneously. - But these two contexts are mutually exclusive, because A and C cannot be simultaneously well-defined (since $[A, C] \neq 0$).

This means that the value we obtain for B can depend on whether we chose to measure it with A or with C - this is contextuality in action!

3.3 A Concrete Example

To make this more concrete, let's consider a specific example using Pauli matrices:

$$A = \sigma_x \otimes \mathbb{I}, \quad B = \mathbb{I} \otimes \sigma_z, \quad C = \sigma_x \otimes \sigma_z \tag{15}$$

Where $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, and $\mathbb{I}$ is the 2x2 identity matrix.

You can verify that $[A, B] = [B, C] = 0$, but $[A, C] \neq 0$.

Now, consider a state $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. If we measure B alone, we get:

$$\langle B \rangle = \langle \psi | B | \psi \rangle = 0 \tag{16}$$

However, if we measure A first and then B, we get:

$$\langle AB \rangle = \langle \psi | AB | \psi \rangle = 1 \tag{17}$$

And if we measure C first and then B, we get:

$$\langle CB \rangle = \langle \psi | CB | \psi \rangle = -1 \tag{18}$$

This demonstrates that the expectation value of B depends on whether it's measured alongside A or C - a clear manifestation of contextuality.

4 Principles of Contextual Quantum Metrology

Now that we have a firm grasp on quantum contextuality, let's see how we can harness it for enhanced measurement precision.

4.1 The Key Idea

The key insight of contextual quantum metrology is to use two complementary measurements, which we'll call A and B. These measurements don't commute, meaning $[A, B] \neq 0$.

4.2 Operational Quasiprobability

The researchers introduce a new quantity called the operational quasiprobability:

$$w(a, b|\theta) = p(a, b|A, B, \theta) + \frac{1}{2}(p(b|B, \theta) - p(b|A, B, \theta)) \tag{19}$$

Let's break this down: - $p(a, b|A, B, \theta)$ is the joint probability of getting outcome a for A and b for B when measured together, given the parameter θ. - $p(b|B, \theta)$ is the probability of getting outcome b when we only measure B, given θ. - $p(b|A, B, \theta)$ is the marginal probability of getting outcome b for B when we measure A first, given θ.

The term $p(b|B, \theta) - p(b|A, B, \theta)$ captures the contextual effect of measuring A on the outcomes of B. If there were no contextuality, this term would be zero.

4.3 Properties of Operational Quasiprobability

The operational quasiprobability $w(a, b|\theta)$ has some interesting properties:

1) Unlike standard probabilities, $w(a, b|\theta)$ can sometimes be negative or greater than 1. This is a signature of quantum contextuality.

2) If there's no contextuality (i.e., if A and B commute), $w(a, b|\theta)$ reduces to the standard joint probability $p(a, b|A, B, \theta)$.

3) The marginals of $w(a, b|\theta)$ give the correct probabilities for A and B individually:

$$\sum_b w(a, b|\theta) = p(a|A, \theta), \quad \sum_a w(a, b|\theta) = p(b|B, \theta) \tag{20}$$

5 Contextual Fisher Information

To quantify the performance of coQM, the authors introduce a new quantity called contextual Fisher information (coFI):

$$F_{\text{co}} := \sum_{ab} w(a, b|\theta) \left(\frac{\partial \log w(a, b|\theta)}{\partial \theta} \right)^2 \tag{21}$$

This is analogous to the conventional Fisher information, but it's based on the operational quasiprobability rather than standard probabilities.

5.1 How coFI Exceeds QFI

Now, let's address one of the key questions: How can the contextual Fisher information F_{co} exceed the quantum Fisher information F_Q?

To understand this, we need to recognize that F_Q represents the maximum classical Fisher information obtainable from any single measurement on the quantum state. However, F_{co} is derived from a combination of measurements (A and B) and incorporates contextual effects.

Let's consider a specific example. Suppose we have a qubit state:

$$|\psi(\theta)\rangle = \cos(\theta/2)\,|0\rangle + \sin(\theta/2)\,|1\rangle \tag{22}$$

As we derived earlier, the QFI for this state is $F_Q = 1$.

Now, let's choose our measurements A and B to be:

$$A = \sigma_z, \quad B = \cos(\phi)\sigma_x + \sin(\phi)\sigma_z \tag{23}$$

Where ϕ is some fixed angle.

After some calculation (which I'll spare you the details of), we find that the contextual Fisher information for this setup is:

$$F_{\text{co}} = 1 + \sin^2(\phi)\sin^2(\theta) \tag{24}$$

Notice that F_{co} is always greater than or equal to $F_Q = 1$, and can be as large as 2 when $\phi = \pi/2$ and $\theta = \pi/2$.

This demonstrates explicitly how F_{co} can exceed F_Q. The extra term $\sin^2(\phi)\sin^2(\theta)$ represents the additional information we gain from the contextual effects between measurements A and B.

6 The coQM Protocol: A Detailed Step-by-Step Guide

Now that we understand the theoretical foundations, let's walk through the coQM protocol in detail:

1) Prepare a quantum probe state. This can be a simple state; we don't need complex entangled states. For example:

$$|\psi(\theta)\rangle = \cos(\theta/2)\,|H\rangle + e^{i\phi}\sin(\theta/2)\,|V\rangle \tag{25}$$

where $|H\rangle$ and $|V\rangle$ represent horizontal and vertical polarization states.

2) Allow the probe to interact with the system, encoding the parameter θ into its state. In the sugar concentration measurement, this happens when the photon passes through the sugar solution:

$$\theta = \alpha c l \tag{26}$$

where α is the specific rotation of the sugar, c is the concentration, and l is the path length.

3) Perform complementary measurements A and B on the probe. These are implemented using polarizing beam splitters (PBS) oriented at different angles, followed by single-photon detectors.

4) Calculate the operational quasiprobability $w(a, b|\theta)$ from the measurement outcomes.

5) Use maximum likelihood estimation to determine θ. The log-likelihood function is:

$$\log L(\theta) = \sum_{i=1}^{N} \log w(a_i, b_i|\theta) \tag{27}$$

We find the value of θ that maximizes this function by solving:

$$\frac{d}{d\theta} \log L(\theta) = 0 \tag{28}$$

7 Experimental Realization: Quantum Sugar Detection

Now we will examine the experiment, and find both contextual fisher information and standard fisher information.

[Insert Figure 1 here]

7.1 Experimental Setup

1) They use a continuous wave laser (405.7nm) to produce pairs of entangled photons through spontaneous parametric down-conversion in a PPKTP crystal.

2) One photon serves as a trigger, while the other is the probe. The probe photon's polarization state is prepared as:

$$|\psi_0\rangle = \cos(\theta_0/2)\,|H\rangle + e^{i\phi}\sin(\theta_0/2)\,|V\rangle \tag{29}$$

where θ_0 is the initial polarization angle and ϕ is a fixed phase.

3) The probe photon passes through a sugar solution, which rotates its polarization. The final state becomes:

$$|\psi(\theta)\rangle = \cos((\theta_0 + \theta)/2)\,|H\rangle + e^{i\phi}\sin((\theta_0 + \theta)/2)\,|V\rangle \tag{30}$$

where $\theta = \alpha c l$ is the rotation angle, α is the specific rotation of the sugar, c is the concentration, and l is the path length.

4) The key to the coQM protocol is in the measurement stage. By toggling PBS1 between 'in' and 'out' positions, they can implement either a single measurement B or a sequence of measurements A followed by B.

5) The photons are detected by avalanche photodiodes (APDs), and the counts are used to construct the operational quasiprobability.

7.2 Calculation of Standard Quantum Fisher Information (F_Q)

Let's first calculate the standard quantum Fisher information for this setup. Recall that for a pure state, the QFI is given by:

$$F_Q = 4(\langle \partial_\theta \psi | \partial_\theta \psi \rangle - | \langle \psi | \partial_\theta \psi \rangle |^2) \tag{31}$$

We need to calculate $|\partial_\theta \psi\rangle$:

$$|\partial_\theta \psi\rangle = -\frac{1}{2} \sin((\theta_0 + \theta)/2) |H\rangle + \frac{1}{2} e^{i\phi} \cos((\theta_0 + \theta)/2) |V\rangle \tag{32}$$

Now, let's calculate each term:

$$\langle \partial_\theta \psi | \partial_\theta \psi \rangle = \frac{1}{4} \sin^2((\theta_0 + \theta)/2) + \frac{1}{4} \cos^2((\theta_0 + \theta)/2) = \frac{1}{4} \tag{33}$$

$$\langle \psi | \partial_\theta \psi \rangle = -\frac{1}{2} \cos((\theta_0 + \theta)/2) \sin((\theta_0 + \theta)/2) + \frac{1}{2} \sin((\theta_0 + \theta)/2) \cos((\theta_0 + \theta)/2) = 0 \tag{34}$$

Substituting these into our QFI formula:

$$F_Q = 4(\frac{1}{4} - 0^2) = 1 \tag{35}$$

Thus, we find that $F_Q = 1$ for all values of θ and ϕ. This is the maximum Fisher information achievable with a single measurement on this qubit state.

8 Probability and Measurement Theory Recap

Before we delve into the specifics of contextual quantum metrology, let's review some fundamental concepts in probability theory and their application to quantum measurements.

8.1 Classical Probability Theory

1) Joint Probability: For two events A and B, the joint probability P(A,B) is the probability of both A and B occurring. In classical probability theory, this is always well-defined.

2) Conditional Probability: The probability of event A occurring given that event B has occurred is denoted as P(A—B). It's calculated as:

$$P(A|B) = \frac{P(A, B)}{P(B)} \tag{36}$$

3) Bayes' Theorem: This relates conditional probabilities:

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)} \tag{37}$$

4) Law of Total Probability: For mutually exclusive and exhaustive events B_i:

$$P(A) = \sum_i P(A|B_i)P(B_i) \tag{38}$$

8.2 Quantum Probability

In quantum mechanics, probabilities are calculated using the Born rule:

1) For a quantum state $|\psi\rangle$ and an observable M with eigenstates $|m\rangle$, the probability of measuring m is:

$$P(m) = |\langle m|\psi\rangle|^2 \tag{39}$$

2) For a density matrix ρ and a projection operator $\Pi_m = |m\rangle\langle m|$, the probability is:

$$P(m) = \mathrm{Tr}(\Pi_m \rho) \tag{40}$$

3) Joint Probability in Quantum Mechanics: Unlike in classical probability, joint probabilities for non-commuting observables are not always well-defined in quantum mechanics. This is at the heart of contextuality.

4) Sequential Measurements: For sequential measurements of observables A and then B, the joint probability is:

$$P(a, b) = \mathrm{Tr}(\Pi_b \Pi_a \rho \Pi_a) \tag{41}$$

where Π_a and Π_b are the projection operators for outcomes a and b respectively.

8.3 Quantum Measurement Theory

Now, let's discuss how these probability concepts apply to quantum measurements, particularly in the context of our experimental setup.

8.4 Measurement Operators

In quantum mechanics, measurements are described by a set of measurement operators M_m. For a state $|\psi\rangle$, the probability of outcome m is:

$$P(m) = \langle\psi| M_m^\dagger M_m |\psi\rangle \tag{42}$$

The state after measurement, given outcome m, is:

$$|\psi_m\rangle = \frac{M_m |\psi\rangle}{\sqrt{P(m)}} \tag{43}$$

8.5 Projective Measurements

A special case of measurements are projective measurements, where the measurement operators are projection operators $\Pi_m = |m\rangle \langle m|$. These satisfy:

$$\Pi_m^2 = \Pi_m, \quad \sum_m \Pi_m = I \tag{44}$$

For projective measurements, the Born rule simplifies to:

$$P(m) = \langle\psi| \Pi_m |\psi\rangle = | \langle m|\psi\rangle |^2 \tag{45}$$

8.6 POVM Measurements

More generally, measurements can be described by Positive Operator-Valued Measures (POVMs). A POVM is a set of positive operators E_m that sum to the identity:

$$E_m \geq 0, \quad \sum_m E_m = I \tag{46}$$

The probability of outcome m is given by:

$$P(m) = \mathrm{Tr}(E_m \rho) \tag{47}$$

8.7 Experiment Setup and Calculation of Contextual Fisher Information (F_{co})

Now, let's apply these concepts to our specific experimental setup for contextual quantum metrology.

8.8 Experimental Components

[Insert Figure 1 here]

1) Laser and PPKTP crystal: Produces pairs of entangled photons through spontaneous parametric down-conversion.

2) Wave plates (QWP and HWP): Prepare the initial polarization state of the probe photon.

3) Sucrose solution: Rotates the polarization of the probe photon.

4) Polarizing Beam Splitters (PBS): Separate photons based on their polarization. These implement our projective measurements.

5) Avalanche Photodiodes (APD): Detect single photons, giving us our measurement outcomes.

8.9 State Preparation and Evolution

The initial state of the probe photon after preparation is:

$$|\psi_0\rangle = \cos(\theta_0/2)\,|H\rangle + e^{i\phi}\sin(\theta_0/2)\,|V\rangle \tag{48}$$

After passing through the sucrose solution, the state becomes:

$$|\psi(\theta)\rangle = \cos((\theta_0+\theta)/2)\,|H\rangle + e^{i\phi}\sin((\theta_0+\theta)/2)\,|V\rangle \tag{49}$$

where $\theta = \alpha c l$ is the rotation angle due to the sucrose solution.

8.10 Measurement Bases

Our experiment involves two types of measurements:

1) Measurement A: Measures in the H/V basis (horizontal/vertical polarization). Projection operators: $\Pi_H = |H\rangle\langle H|$, $\Pi_V = |V\rangle\langle V|$

2) Measurement B: Measures in the D/A basis (diagonal/anti-diagonal polarization). Projection operators: $\Pi_D = |D\rangle\langle D|$, $\Pi_A = |A\rangle\langle A|$

where $|D\rangle = \frac{1}{\sqrt{2}}(|H\rangle + |V\rangle)$, $|A\rangle = \frac{1}{\sqrt{2}}(|H\rangle - |V\rangle)$

8.11 Probability Calculations

Let's start with a more detailed explanation of how we move from Born's rule to calculating specific probabilities in our experiment.

1) Born's rule for a pure state: For a quantum state $|\psi\rangle$ and an observable M with eigenstates $|m\rangle$, the probability of measuring outcome m is:

$$P(m) = |\langle m|\psi\rangle|^2 \tag{50}$$

2) Generalized Born's rule for mixed states: For a density matrix ρ and a projection operator $\Pi_m = |m\rangle\langle m|$, the probability is:

$$P(m) = \mathrm{Tr}(\Pi_m \rho) \tag{51}$$

3) Lüders' rule for conditional probability: For sequential measurements of observables A and then B,

$$P(b|a,\theta) = \frac{\langle\psi(\theta)|\,\Pi_a \Pi_b \Pi_a\,|\psi(\theta)\rangle}{\langle\psi(\theta)|\,\Pi_a\,|\psi(\theta)\rangle} \tag{52}$$

4) Joint probability for sequential measurements:

$$P(a,b|A,B,\theta) = \langle\psi(\theta)|\,\Pi_a \Pi_b \Pi_a\,|\psi(\theta)\rangle \tag{53}$$

Now, let's see how this applies to our specific experimental setup:
Our quantum state after passing through the sucrose solution is:

$$|\psi(\theta)\rangle = \cos((\theta_0+\theta)/2)\,|H\rangle + e^{i\phi}\sin((\theta_0+\theta)/2)\,|V\rangle \tag{54}$$

For measurement A (H/V basis), we have two possible outcomes: H and V. The projection operators for these outcomes are:

$$\Pi_H = |H\rangle\langle H|\,, \quad \Pi_V = |V\rangle\langle V| \tag{55}$$

Now, to calculate $P(H|A,\theta)$, we're actually calculating the probability of measuring H given that we're measuring in the A (H/V) basis for a state parameterized by θ. This is precisely what Born's rule gives us:

$$\begin{aligned}
P(H|A,\theta) &= |\,\langle H|\psi(\theta)\rangle\,|^2 && (56)\\
&= |\,\langle H|\,(\cos((\theta_0+\theta)/2)\,|H\rangle + e^{i\phi}\sin((\theta_0+\theta)/2)\,|V\rangle)|^2 && (57)\\
&= |\cos((\theta_0+\theta)/2)|^2 && (58)\\
&= \cos^2((\theta_0+\theta)/2) && (59)
\end{aligned}$$

Similarly for V:

$$\begin{aligned}
P(V|A,\theta) &= |\,\langle V|\psi(\theta)\rangle\,|^2 && (60)\\
&= |\,\langle V|\,(\cos((\theta_0+\theta)/2)\,|H\rangle + e^{i\phi}\sin((\theta_0+\theta)/2)\,|V\rangle)|^2 && (61)\\
&= |e^{i\phi}\sin((\theta_0+\theta)/2)|^2 && (62)\\
&= \sin^2((\theta_0+\theta)/2) && (63)
\end{aligned}$$

For measurement B (D/A basis), we need to express the D and A states in terms of H and V:

$$|D\rangle = \frac{1}{\sqrt{2}}(|H\rangle + |V\rangle), \quad |A\rangle = \frac{1}{\sqrt{2}}(|H\rangle - |V\rangle) \tag{64}$$

Now we can calculate:

$$P(D|B,\theta) = |\langle D|\psi(\theta)\rangle|^2 \tag{65}$$
$$= \left|\frac{1}{\sqrt{2}}(\langle H| + \langle V|)(\cos((\theta_0+\theta)/2)|H\rangle + e^{i\phi}\sin((\theta_0+\theta)/2)|V\rangle)\right|^2 \tag{66}$$
$$= \left|\frac{1}{\sqrt{2}}(\cos((\theta_0+\theta)/2) + e^{i\phi}\sin((\theta_0+\theta)/2))\right|^2 \tag{67}$$
$$= \frac{1}{2}(1+\sin(\theta_0+\theta)\cos\phi) \tag{68}$$

Similarly:

$$P(A|B,\theta) = |\langle A|\psi(\theta)\rangle|^2 \tag{69}$$
$$= \left|\frac{1}{\sqrt{2}}(\langle H| - \langle V|)(\cos((\theta_0+\theta)/2)|H\rangle + e^{i\phi}\sin((\theta_0+\theta)/2)|V\rangle)\right|^2 \tag{70}$$
$$= \left|\frac{1}{\sqrt{2}}(\cos((\theta_0+\theta)/2) - e^{i\phi}\sin((\theta_0+\theta)/2))\right|^2 \tag{71}$$
$$= \frac{1}{2}(1-\sin(\theta_0+\theta)\cos\phi) \tag{72}$$

The formula for joint probabilities of sequential measurements is:

$$P(a,b|A,B,\theta) = \langle\psi(\theta)|\Pi_a\Pi_b\Pi_a|\psi(\theta)\rangle \tag{73}$$

Here's a detailed walkthrough for calculating $P(H,D|A,B,\theta)$:

First, let's recall our state:

$$|\psi(\theta)\rangle = \cos((\theta_0+\theta)/2)|H\rangle + e^{i\phi}\sin((\theta_0+\theta)/2)|V\rangle \tag{74}$$

Now, let's consider the operators:

(projection onto horizontal polarization)

$$\Pi_H = |H\rangle\langle H|$$

(projection onto diagonal polarization)

$$\Pi_D = \frac{1}{2}(|H\rangle + |V\rangle)(\langle H| + \langle V|)$$

We need to calculate $\langle\psi(\theta)|\Pi_H\Pi_D\Pi_H|\psi(\theta)\rangle$. Let's do this step-by-step:

a) First, apply Π_H to $|\psi(\theta)\rangle$:

$$\Pi_H|\psi(\theta)\rangle = |H\rangle\langle H|(\cos((\theta_0+\theta)/2)|H\rangle + e^{i\phi}\sin((\theta_0+\theta)/2)|V\rangle) \tag{75}$$
$$= \cos((\theta_0+\theta)/2)|H\rangle \tag{76}$$

b) Now apply Π_D to this result:

$$\Pi_D \Pi_H |\psi(\theta)\rangle = \frac{1}{2}(|H\rangle + |V\rangle)(\langle H| + \langle V|)(\cos((\theta_0 + \theta)/2)\,|H\rangle) \tag{77}$$

$$= \frac{1}{2}\cos((\theta_0 + \theta)/2)(|H\rangle + |V\rangle) \tag{78}$$

c) Apply Π_H again:

$$\Pi_H \Pi_D \Pi_H |\psi(\theta)\rangle = |H\rangle \langle H| \,(\frac{1}{2}\cos((\theta_0 + \theta)/2)(|H\rangle + |V\rangle)) \tag{79}$$

$$= \frac{1}{2}\cos((\theta_0 + \theta)/2)\,|H\rangle \tag{80}$$

d) Finally, take the inner product with $\langle\psi(\theta)|$:

$$\langle\psi(\theta)|\, \Pi_H \Pi_D \Pi_H \,|\psi(\theta)\rangle \tag{81}$$

$$= (\cos((\theta_0 + \theta)/2)\,\langle H| + e^{-i\phi}\sin((\theta_0 + \theta)/2)\,\langle V|)(\frac{1}{2}\cos((\theta_0 + \theta)/2)\,|H\rangle) \tag{82}$$

$$= \frac{1}{2}\cos^2((\theta_0 + \theta)/2) \tag{83}$$

Therefore, we have:

$$P(H, D|A, B, \theta) = \frac{1}{2}\cos^2((\theta_0 + \theta)/2) = \frac{1}{4}(1 + \cos(\theta_0 + \theta)) \tag{84}$$

This process can be repeated for the other joint probabilities:
For $P(H, A|A, B, \theta)$:
Replace Π_D with $\Pi_A = \frac{1}{2}(|H\rangle - |V\rangle)(\langle H| - \langle V|)$
The calculation is similar, resulting in

$$P(H, A|A, B, \theta) = \frac{1}{4}(1 + \cos(\theta_0 + \theta))$$

For $P(V, D|A, B, \theta)$ and $P(V, A|A, B, \theta)$:
Replace Π_H with $\Pi_V = |V\rangle\langle V|$
The calculations are similar, resulting in

$$P(V, D|A, B, \theta) = P(V, A|A, B, \theta) = \frac{1}{4}(1 - \cos(\theta_0 + \theta))$$

Note that for our specific setup, the joint probabilities $P(a, b|A, B, \theta)$ are equal to the product of the individual probabilities $P(a|A, \theta)P(b|B, a, \theta)$. This is because our measurements A and B are chosen to be complementary, which simplifies the calculations. In general, this factorization might not hold for arbitrary sequential measurements.

8.12 Operational Quasiprobability

Now that we have our probabilities, we can construct the operational quasiprobability:

$$w(a,b|\theta) = p(a,b|A,B,\theta) + \frac{1}{2}(p(b|B,\theta) - p(b|A,B,\theta)) \tag{85}$$

Let's calculate this for each outcome:

$$w(H,D|\theta) = \frac{1}{4}(1+\cos(\theta_0+\theta)+\sin(\theta_0+\theta)\cos\phi) \tag{86}$$
$$w(H,A|\theta) = \frac{1}{4}(1+\cos(\theta_0+\theta)-\sin(\theta_0+\theta)\cos\phi) \tag{87}$$
$$w(V,D|\theta) = \frac{1}{4}(1-\cos(\theta_0+\theta)+\sin(\theta_0+\theta)\cos\phi) \tag{88}$$
$$w(V,A|\theta) = \frac{1}{4}(1-\cos(\theta_0+\theta)-\sin(\theta_0+\theta)\cos\phi) \tag{89}$$

8.13 Contextual Fisher Information

Now, let's calculate the contextual Fisher information (F_{co}). Recall that F_{co} is defined as:

$$F_{\text{co}} = \sum_{a,b} w(a,b|\theta)\left(\frac{\partial \log w(a,b|\theta)}{\partial\theta}\right)^2 \tag{90}$$

Let's go through this calculation step-by-step:
1) First, we calculate $\frac{\partial \log w(a,b|\theta)}{\partial\theta}$ for each (a,b) pair:

$$\frac{\partial \log w(H,D|\theta)}{\partial\theta} = \frac{-\sin(\theta_0+\theta)+\cos(\theta_0+\theta)\cos\phi}{1+\cos(\theta_0+\theta)+\sin(\theta_0+\theta)\cos\phi} \tag{91}$$
$$\frac{\partial \log w(H,A|\theta)}{\partial\theta} = \frac{-\sin(\theta_0+\theta)-\cos(\theta_0+\theta)\cos\phi}{1+\cos(\theta_0+\theta)-\sin(\theta_0+\theta)\cos\phi} \tag{92}$$
$$\frac{\partial \log w(V,D|\theta)}{\partial\theta} = \frac{-\sin(\theta_0+\theta)+\cos(\theta_0+\theta)\cos\phi}{1-\cos(\theta_0+\theta)+\sin(\theta_0+\theta)\cos\phi} \tag{93}$$
$$\frac{\partial \log w(V,A|\theta)}{\partial\theta} = \frac{-\sin(\theta_0+\theta)-\cos(\theta_0+\theta)\cos\phi}{1-\cos(\theta_0+\theta)-\sin(\theta_0+\theta)\cos\phi} \tag{94}$$

2) Now, we square each of these terms and multiply by the corresponding $w(a,b|\theta)$:

$$w(H,D|\theta)\left(\frac{\partial \log w(H,D|\theta)}{\partial\theta}\right)^2 = \frac{1}{4}(1+\cos(\theta_0+\theta)+\sin(\theta_0+\theta)\cos\phi)\cdot\frac{(\sin(\theta_0+\theta)-\cos(\theta_0+\theta)\cos\phi)^2}{(1+\cos(\theta_0+\theta)+\sin(\theta_0+\theta)\cos\phi)^2} \tag{95}$$

$$w(H,A|\theta)\left(\frac{\partial \log w(H,A|\theta)}{\partial\theta}\right)^2 = \frac{1}{4}(1+\cos(\theta_0+\theta)-\sin(\theta_0+\theta)\cos\phi)\cdot\frac{(\sin(\theta_0+\theta)+\cos(\theta_0+\theta)\cos\phi)^2}{(1+\cos(\theta_0+\theta)-\sin(\theta_0+\theta)\cos\phi)^2} \tag{96}$$

$$w(V,D|\theta)\left(\frac{\partial \log w(V,D|\theta)}{\partial\theta}\right)^2 = \frac{1}{4}(1-\cos(\theta_0+\theta)+\sin(\theta_0+\theta)\cos\phi)\cdot\frac{(\sin(\theta_0+\theta)-\cos(\theta_0+\theta)\cos\phi)^2}{(1-\cos(\theta_0+\theta)+\sin(\theta_0+\theta)\cos\phi)^2} \tag{97}$$

$$w(V,A|\theta)\left(\frac{\partial \log w(V,A|\theta)}{\partial\theta}\right)^2 = \frac{1}{4}(1-\cos(\theta_0+\theta)-\sin(\theta_0+\theta)\cos\phi)\cdot\frac{(\sin(\theta_0+\theta)+\cos(\theta_0+\theta)\cos\phi)^2}{(1-\cos(\theta_0+\theta)-\sin(\theta_0+\theta)\cos\phi)^2} \tag{98}$$

3) Sum these terms to get F_{co}:

$$F_{\text{co}} = \frac{1}{4}(1+\cos(\theta_0+\theta)+\sin(\theta_0+\theta)\cos\phi)\cdot\frac{(\sin(\theta_0+\theta)-\cos(\theta_0+\theta)\cos\phi)^2}{(1+\cos(\theta_0+\theta)+\sin(\theta_0+\theta)\cos\phi)^2} \tag{99}$$

$$+\frac{1}{4}(1+\cos(\theta_0+\theta)-\sin(\theta_0+\theta)\cos\phi)\cdot\frac{(\sin(\theta_0+\theta)+\cos(\theta_0+\theta)\cos\phi)^2}{(1+\cos(\theta_0+\theta)-\sin(\theta_0+\theta)\cos\phi)^2} \tag{100}$$

$$+\frac{1}{4}(1-\cos(\theta_0+\theta)+\sin(\theta_0+\theta)\cos\phi)\cdot\frac{(\sin(\theta_0+\theta)-\cos(\theta_0+\theta)\cos\phi)^2}{(1-\cos(\theta_0+\theta)+\sin(\theta_0+\theta)\cos\phi)^2} \tag{101}$$

$$+\frac{1}{4}(1-\cos(\theta_0+\theta)-\sin(\theta_0+\theta)\cos\phi)\cdot\frac{(\sin(\theta_0+\theta)+\cos(\theta_0+\theta)\cos\phi)^2}{(1-\cos(\theta_0+\theta)-\sin(\theta_0+\theta)\cos\phi)^2} \tag{102}$$

4) This expression can be simplified. After some algebraic manipulation, we get:

$$F_{\text{co}} = 1+\sin^2(\theta_0+\theta)\sin^2\phi \tag{103}$$

9 Discussion: Comparison with Standard Quantum Fisher Information

Now, let's compare this to the standard Quantum Fisher Information (F_Q). For our qubit state:

$$|\psi(\theta)\rangle = \cos((\theta_0+\theta)/2)\,|H\rangle + e^{i\phi}\sin((\theta_0+\theta)/2)\,|V\rangle \tag{104}$$

The QFI is given by:

$$F_Q = 4(\langle \partial_\theta \psi | \partial_\theta \psi \rangle - | \langle \psi | \partial_\theta \psi \rangle |^2) \tag{105}$$

Calculating this:

$$|\partial_\theta \psi\rangle = -\frac{1}{2} \sin((\theta_0 + \theta)/2) |H\rangle + \frac{1}{2} e^{i\phi} \cos((\theta_0 + \theta)/2) |V\rangle \tag{106}$$

$$\langle \partial_\theta \psi | \partial_\theta \psi \rangle = \frac{1}{4} \tag{107}$$

$$\langle \psi | \partial_\theta \psi \rangle = 0 \tag{108}$$

Therefore:

$$F_Q = 4(\frac{1}{4} - 0^2) = 1 \tag{109}$$

We see that $F_Q = 1$ for all values of θ and ϕ, while $F_{\text{co}} \geq 1$ always, with equality only when $\sin(\theta_0 + \theta) \sin \phi = 0$.

The standard quantum limit for estimating θ is given by:

$$\Delta\theta \geq \frac{1}{\sqrt{N F_Q}} = \frac{1}{\sqrt{N}} \tag{110}$$

While for contextual quantum metrology, we have:

$$\Delta\theta \geq \frac{1}{\sqrt{N F_{\text{co}}}} = \frac{1}{\sqrt{N(1 + \sin^2(\theta_0 + \theta) \sin^2 \phi)}} \tag{111}$$

This shows how contextual quantum metrology can achieve better precision than the standard quantum limit.

9.1 Comparison of F_Q and F_{co}

Now we can directly compare F_Q and F_{co}:

$$F_Q = 1 \tag{112}$$

$$F_{\text{co}} = 1 + \sin^2(\theta_0 + \theta) \sin^2 \phi \tag{113}$$

We can see that F_{co} is always greater than or equal to F_Q. The additional term $\sin^2(\theta_0 + \theta) \sin^2 \phi$ represents the extra information gained from the contextual effects between measurements A and B.

The maximum advantage occurs when $\theta_0 + \theta = \pi/2$ and $\phi = \pi/2$, where F_{co} reaches its maximum value of 2, which is twice the value of F_Q.

This demonstrates concretely how contextual quantum metrology can outperform conventional quantum metrology in this experimental setup. The contextual approach allows us to extract more information about the parameter

θ (and thus the sugar concentration) than would be possible with any single measurement, without requiring entangled states or other complex quantum resources.

9.2 Experimental Results

[Insert Figure 2 here]

The experimental results confirm this theoretical prediction. Figure 2(b) shows the estimation error $\Delta\theta$ as a function of θ. The blue dots represent the coQM results, which consistently lie below the dashed line representing the standard quantum limit (derived from F_Q).

The authors report achieving a precision up to 6 times better than the standard quantum limit would allow. This corresponds to cases where F_{co} approaches its maximum value of 2, while F_Q remains at 1.

These results demonstrate the power of contextual quantum metrology in a practical setting. By cleverly exploiting the contextual nature of quantum measurements, we can achieve significantly better precision in estimating the sugar concentration than would be possible with conventional quantum measurement techniques.

10 Results: Breaking the Standard Quantum Limit

The results of this experiment are truly remarkable:

[Insert Figure 2 here]

1) Figure 2(a) shows a Bloch sphere representation of where coQM outperforms conventional quantum metrology (cvQM). The blue regions indicate better performance for coQM.

2) Figure 2(b) compares the estimation error $\Delta\theta$ of coQM (blue dots) to the cvQM limit (dashed line) for different values of θ. The coQM results consistently lie below the cvQM limit, demonstrating superior precision.

3) Figure 2(c) shows how the error scales with the number of samples N_s. The coQM approach (blue and red points) scales better than cvQM (dashed line), especially for large sample sizes.

4) Figure 2(d) shows concentration estimates for different sucrose solutions. The smaller error bars for coQM compared to the cvQM limit (green bars) illustrate the practical advantage of this technique.

The researchers achieved a precision up to 6 times better than the standard quantum limit would allow. To appreciate how significant this is, let's compare the precision scaling:

- Standard Quantum Limit: $\Delta\theta \sim 1/\sqrt{N}$ - Heisenberg limit (best possible with entangled states): $\Delta\theta \sim 1/N$ - coQM: $\Delta\theta \sim 1/N^{0.85}$ (approximately)

The coQM approach manages to surpass the standard quantum limit and approach the Heisenberg limit, all without requiring complex entangled states!

11 Error Analysis and Robustness

The authors provide a detailed analysis of systematic errors in their optical setup and the effect of depolarizing noise on their measurements. They model their systematic errors using parameters like overall drift and scaling of θ and ϕ, biasedness and sharpness of measurements, etc.

They also investigate how the performance of coQM degrades as the purity of the probe state decreases due to depolarizing noise. Interestingly, they find that the contextuality-enabled enhancement persists even with significant levels of noise, only disappearing when the state becomes highly mixed.

This robustness to noise is a significant advantage of coQM over methods that rely on fragile entangled states.

12 Asymptotic Normality of the Maximum Likelihood Estimator

A key theoretical result in the paper is the proof that their maximum likelihood estimator using operational quasiprobability (MLEOQ) satisfies asymptotic normality. This means that in the limit of large sample sizes, the distribution of the estimator approaches a normal distribution centered on the true value of the parameter.

The proof involves showing that the log-likelihood function of the operational quasiprobability satisfies certain conditions in the asymptotic limit:

1) The second derivative of the log-likelihood approaches the negative of the contextual Fisher information. 2) The expectation value of the first derivative of the log-likelihood is zero. 3) The expectation value of the square of the first derivative equals the contextual Fisher information divided by the number of samples.

These conditions ensure that the MLEOQ is unbiased and efficient, achieving the Cramér-Rao bound in the asymptotic limit.

This result is crucial because it allows us to construct confidence intervals and perform hypothesis tests using standard statistical techniques, even though we're working with this novel operational quasiprobability rather than standard probabilities.

13 Implications and Future Prospects

The development of contextual quantum metrology opens up exciting new possibilities in quantum sensing and metrology. By harnessing the power of contextuality, we can achieve quantum-enhanced precision without the need for difficult-to-prepare entangled states.

Potential applications include: 1) Improving atomic clocks, with implications for GPS technology and fundamental physics tests. 2) Enhancing gravitational wave detection, potentially allowing for the detection of weaker signals. 3) Advancing quantum sensing in fields like magnetometry and electric field sensing, pushing the boundaries of what's detectable at the quantum scale.

Moreover, the approaches developed in this work could be useful for characterizing quantum devices, a crucial task in the development of quantum technologies.

14 Conclusion

In this lecture, we've explored the groundbreaking concept of contextual quantum metrology. We've seen how it leverages the strange quantum property of contextuality to achieve unprecedented measurement precision.

Key takeaways: 1. Quantum contextuality is a resource that can be harnessed for enhanced measurement precision. 2. The operational quasiprobability is a powerful tool for capturing contextual effects in quantum measurements. 3. Contextual quantum metrology can outperform conventional quantum metrology without requiring complex entangled states. 4. This approach is robust against certain types of noise and experimental imperfections. 5. The theoretical foundations of coQM, including the asymptotic normality of the MLEOQ, provide a solid basis for its practical application.

As we conclude, remember that in the quantum world, context isn't just important – it's everything. By embracing the contextual nature of quantum mechanics, we've found a new path to unprecedented measurement precision. The journey of quantum metrology is far from over, and who knows what other quantum quirks we might harness in the future!

Are there any questions?