Notes on Quantum computation and quantum information by Nielsen and Chuang

Chapter 2: Linear algebra

1 vector space

- C^n: space of all n-tuple complex numbers (c numbers)

i.e. $(z_1, z_2, z_3, ..., z_n)$

- a vector space is closed under scalar multiplicationa nd addition

1.1) Dirac notation

Symbols	Meaning
$\|v\rangle$	ket, a vector in vec space
$\langle v\|$	bra, a dual vector in vec space; the complex transpose of ket $\langle v\| = (\|v\rangle^*)^\top$
$\langle v\|w\rangle$	inner product of $\|v\rangle$ and $\|w\rangle$
$\|\varphi\rangle \otimes \|\psi\rangle$	tensor product of $\|\varphi\rangle$ and $\|\psi\rangle$ abbriviates as $\|\varphi\rangle\|\psi\rangle$
$\boldsymbol{A}^*$	complex conjugate of $\boldsymbol{A}$
$\boldsymbol{A}^\top$	transpose of $\boldsymbol{A}$
$\boldsymbol{A}^\dagger$	hermitian conjugate of $\boldsymbol{A}$ i.e. $\boldsymbol{A}^\dagger = (\boldsymbol{A}^*)^\top$ $$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^\dagger = \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix} \quad (1)$$
$\langle\varphi\|\boldsymbol{A}\|\psi\rangle$	inner product betweeen $\|\varphi\rangle$ and $\boldsymbol{A}\|\psi\rangle$

1.2) Span

a set of bec $|v_1\rangle, |v_2\rangle, ..., |v_n\rangle$ spans the vector space if any vector in the space can be written as

$$|v\rangle = \sum_i a_i |v_i\rangle \quad (2)$$

for some complex numbers a_i

1.3) Linear Independence

a set of non-zero vectors $|v_1\rangle, |v_2\rangle, ..., |v_n\rangle$ are liinearly dependent if there exists a set of complex numbers $a_1, a_2, ..., a_n$, s.t.

$$a_1|v_1\rangle + a_2|v_2\rangle + ... + a_n|v_n\rangle = 0 \quad (3)$$

If the only solution to the above equation is $a_1 = a_2 = ... = a_n = 0$, then the vectors are **linearly independent**

1.4) Linear operators

A linear operator A is any linear function that

$$A\left(\sum_i a_i |v_i\rangle\right) = \sum_i a_i A(|v_i\rangle) \quad (4)$$

It is convention to write $A(|v_i\rangle) = A|v_i\rangle$

- Identity Operator $I_V : I_V|v\rangle \equiv |v\rangle$. It is convinent to write I if no confution arises.
- zero operator $0|v\rangle \equiv 0$
- composition of linear operators A and B is AB

We observe that the above is equivalent to the matrix representation of linear transformations.

In other words, for a linear operator $A : V \to W$, and suppose $|v_1\rangle, |v_2\rangle, ..., |v_m\rangle$

1.5) Hilbert Space

Given a vector basis $\{|E_i\rangle\}$, when attempting to represent a polynomial as $p = \sum_{i=0}^{\infty} a_i E_i$, the sum is in the form of, according to taylor series, an exponential function. But the exponential function is not a polynomial, i.e. outside of our vector space, so we have landed on a paradox. To avoid this, we define a **Hilbert Space** to handle infinite dimensional vector spaces.

- A Hilbert spsace is a vector space that is 1. complete and 2. has an inner product defined on it. In other words, every converging set of vectors must converge to an element **inside** the vector space.

$$|\psi\rangle \in \mathcal{H} \tag{5}$$

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2 Inner product

- Review on dot product
  - orthogonality & angle
  - norm $\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}}$

For kets $|\psi\rangle, |\varphi\rangle, |\zeta\rangle$, and scalar a, an inner product has the following rules:

- Linearity in the second argument:

$$\begin{cases} \langle\psi|\varphi + \zeta\rangle = \langle\psi|\varphi\rangle + \langle\psi|\zeta\rangle \\ \langle\psi|a\varphi\rangle = a\langle\psi|\varphi\rangle \end{cases} \tag{6}$$

- COmplex commutation:

$$\langle\psi|\varphi\rangle = \langle\varphi|\psi\rangle^* \tag{7}$$

- Positive definiteness (think of norm):

$$|\psi\rangle \neq 0 \Rightarrow \langle\psi|\psi\rangle > 0 \tag{8}$$

- Magnitude of a vector:

$$\||\psi\rangle\| = \sqrt{\langle\psi|\psi\rangle} \tag{9}$$

- Orthogonality:

$$\langle\psi|\varphi\rangle = 0 \Rightarrow |\psi\rangle \text{ and } |\varphi\rangle \text{ are orthogonal} \tag{10}$$

- antilinearity in the first argument:

$$\langle a\psi + b\zeta|\varphi\rangle = a^*\langle\psi|\varphi\rangle + b^*\langle\zeta|\varphi\rangle \tag{11}$$

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3 Orthonormal basis

$\{|E_i\rangle\}$ s.t. $\langle E_i|E_j\rangle = \delta_{ij}$ is an orthonormal basis, with kroneker delta $\delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases}$

In English, the inner product of two vectors is 1 if they are the same (norm = 1), and 0 if they are different (orthogonal).

Using the orthonormal basis, we can write any vector as a linear combination of the basis vectors:

$$|\psi\rangle = \sum_i c_i |E_i\rangle \tag{12}$$

Notice that

$$\begin{aligned} \langle E_i|\psi\rangle &= \left\langle E_i \middle| \sum_j c_j E_j \right\rangle \\ &= \sum_j c_j \langle E_i|E_j\rangle \\ &= c_i \end{aligned} \tag{13}$$

And we use the above to calculate the coefficients c_i.

3.1) Inner product between two vectors

$$\begin{aligned} \langle\psi|\varphi\rangle &= \left\langle \sum_i c_i E_i \middle| \sum_j d_j E_j \right\rangle \\ &= \sum_i \sum_j c_i^* d_j \langle E_i|E_j\rangle \\ &= \sum_i \sum_j c_i^* d_j \delta_{ij} \\ &= \sum_i c_i^* d_i \end{aligned} \tag{14}$$

When $c, d \in \mathbb{N}$, $\langle\varphi|\psi\rangle = \sum_i c_i d_i$ is simply the dot product.

4 Continuous basis

In a Hilbert space, we can represent continuous functions as a linear combination of a set of continous basis $\{|x\rangle |\ x \in R\}$. The mathematicians said so and we do not question the validity of this argument on our own.

An example would be

$$|\psi\rangle = \int \mathrm{d}x\ c(x)\ |x\rangle \tag{15}$$

Where $c(x)$ is a function that maps x to the coefficient $c(x)$.

4.1) representatio of continuous orthonormal basis

Similar to the discrete case, we can represent the continuous orthonormal basis $\{|x\rangle |\ x \in \mathbb{R}\}$ as:

$$\langle x_i|x_j\rangle = \delta(x_i - x_j) \tag{16}$$

where $\delta(x_i - x_j)$ is the Dirac delta function. We can be pragmatic and understand the Dirac Delta function as ANY function that satisfies the following properties:

$$\begin{gathered}\int \mathrm{d}x\, f(x)\delta(c-x) = f(c) \\ \delta(c-x) = \begin{cases} 1 & x = c \\ 0 & x \neq c \end{cases}\end{gathered} \tag{17}$$

4.2) Inner product between two continuous vectors

$$\begin{aligned}\langle\psi|\varphi\rangle &= \left(\int \psi(x)\langle x|\,\mathrm{d}x\right)\left(\int \varphi(y)|y\rangle\,\mathrm{d}y\right) \\ &= \iint \mathrm{d}\psi^*(x)\varphi(y)\,\mathrm{d}x\,\mathrm{d}y\,\langle x|y\rangle \\ &= \int \mathrm{d}x\psi^*(x)\underbrace{\int \mathrm{d}y\varphi(y)\delta(x-y)}_{\delta\text{ property}} \\ &= \int \mathrm{d}x\psi^*(x)\,\varphi(x)\end{aligned} \tag{18}$$

We have done a inner product of wavefunctions.

4.3) Finding coefficients of continuous basis

notice

$$\begin{aligned}\langle x_0|\psi\rangle &= \langle x_0|\left(\int \mathrm{d}x\psi(x)|x\rangle\right) \\ &= \int \mathrm{d}x\psi(x)\langle x_0|x\rangle \\ &= \int \mathrm{d}x\psi(x)\,\delta(x_0-x) \\ &= \psi(x_0)\end{aligned} \tag{19}$$

So, the coefficient function $\psi(x_0) = \langle x_0|\psi\rangle$

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5 Bra and Braket notation

5.1) Linear functionals

A linear functional is a linear function that maps a vector to a scalar:

$$L\vec{v} = c, \quad c \in \mathbb{F} \tag{20}$$

L maps from the vector space to the field of scalars, or $L : \mathbb{C}^n \to \mathbb{F}$

L is actually an $1 \times n$ matrix, or a row vector.

5.2) Dual Space $\boldsymbol{V}^*$

Set of all L, where each L is a linear funcitonal s.t. $L\boldsymbol{V} = c \in \mathbb{F}$. More rigorously we can say

$$\boldsymbol{V}^* = \{L : \boldsymbol{V} \to F \mid L \text{ is linear}\} \tag{21}$$

5.3) Bra

- $\langle\psi|$ is a shorthand of a linear functional inside a Hilbert dual space. It is an operator that when acting on a vector, will spit out a constant.

$$\langle\psi|\,|\varphi\rangle = c \in \mathbb{C} \tag{22}$$

Riesz Representation theorem

For any linear functional L_φ s.t.

$$L_\varphi \vec{v} = \mathrm{InProd}(\vec{\varphi}, \vec{v}) \tag{23}$$

where $\vec{\varphi}$ is a unique vector. In hilbert space, this unique vector is $\langle\varphi|$

So it follows that

$$\langle\varphi||\psi\rangle = \langle\varphi|\psi\rangle \tag{24}$$

5.4) Complex conjegate

Recall the fact that

$$\langle\varphi|\psi\rangle = \sum_i c_i^* d_i = \langle\varphi||\psi\rangle \tag{25}$$

where $|\varphi\rangle = \sum_i c_i |E_i\rangle$. By virtue of intuition, we can see that

$$\langle\varphi| = \left(|\varphi\rangle^*\right)^\top \tag{26}$$

5.5) Outer product: a peek

notice

$$\begin{aligned} |\psi\rangle = \sum_i c_i |A_i\rangle &= \sum_i \langle A_i|\psi\rangle\,|A_i\rangle \\ &= \sum_i |A_i\rangle\langle A_i|\psi\rangle \\ &= \sum_i |A_i\rangle\langle A_i||\psi\rangle \\ &= \left(\sum_i |A_i\rangle\langle A_i|\right)|\psi\rangle \end{aligned} \tag{27}$$

Comparing left with right, we observe that

$$\sum_i |A_i\rangle\langle A_i| = \hat{\boldsymbol{I}} \tag{28}$$

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6 Observables

We represent physical quantities or states as an linear operator. The genrealization of any physical state is an observable.

6.1) Linear operator

An Linear operator is a map on a vector space that preserves its original structure. i.e.,

$$\begin{cases} \widehat{\boldsymbol{M}}(|\psi\rangle + |\varphi\rangle) = \widehat{\boldsymbol{M}}|\psi\rangle + \widehat{\boldsymbol{M}}|\varphi\rangle \\ \widehat{\boldsymbol{M}}(a|\psi\rangle) = a\widehat{\boldsymbol{M}}|\psi\rangle \end{cases} \tag{29}$$

6.2) Definite stsates

Special states in which observables has one definite value. These states are eigenstates of the observable. They corresponds to eigenvalues, all possible values of the observable.

Therefore, we can represent physical states as linear combinations of the outcome eigenstates. It is intuitive to propose the following constrains:

1. Observables have real eigenvalues
2. Eigenstates span the entire vector space.
3. eigenstates are mutually orthogonal.

Thus, Eigenstates are just the basis of an observable state in disguise!

$$|\psi\rangle = \sum_i c_i |E_i\rangle \tag{30}$$

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7 Probablistic Determination of observables: Born's rule

Knowing $|\psi\rangle = \sum_i c_i |E_i\rangle$, what is probability of $|\psi\rangle$ exactly measured to be $|E_i\rangle$, i.e. $P(|\psi\rangle = E_i)$

Observe a relationship between coefficient c_i to the probability of falling into the eigenstate $|E_i\rangle$, we propose the following:

$$P(E = E_i) = f(c_i) \tag{31}$$

with intuitive constrains:

$$\begin{gathered} 1. \quad P_{\text{total}} = \sum_i f(c_i) = 1 \\ 2. \quad \left| |\psi\rangle^2 \right| = \sum_i^n c_i^2 = k^2 \text{ constant length of eigenstates decomposition} \end{gathered} \tag{32}$$

from 2:

$$\begin{gathered} c_1^2 + c_2^2 + ... + c_n^2 = k^2 \\ c_n = \pm\sqrt{k^2 - a^2 - b^2 - ...} = \pm A \end{gathered} \tag{33}$$

Obviously, linear combination is invariant under positiveness, so

$$f(c) = f(-c) \tag{34}$$

Recall property one:

$$\sum_i^n f(c_i) = 1 \tag{35}$$

$$\sum_{i}^{n-1} f(c_i) + f_{c_n} = 1$$

$$\sum_{i}^{n-1} \frac{\mathrm{d}f(c_i)}{\mathrm{d}c_1} + \frac{\mathrm{d}f(c_n)}{\mathrm{d}c_1} = 0 \quad (*)$$

$$\frac{\mathrm{d}}{\mathrm{d}c_1} f(c_1) - f'\left(\sqrt{k^2 - c_1^2 - c_2^2 - \ldots}\right) \cdot \frac{c_1}{\sqrt{k^2 - c_1^2 \ldots}} = 0 \tag{36}$$

$$\frac{1}{c_1} \frac{\mathrm{d}}{\mathrm{d}c_1} f(c_1) = \frac{f'\left(\sqrt{k^2 - c_1^2 - c_2^2 - \ldots}\right)}{\sqrt{k^2 - c_1^2 - c_2^2 - \ldots}} (**)$$

Notice that if we take $\frac{\mathrm{d}}{\mathrm{d}c_2}$ on both sides at $(*)$ instead, $(**)$ becomes

$$\frac{1}{c_2} \frac{\mathrm{d}}{\mathrm{d}c_2} f(c_2) = \frac{f'\left(\sqrt{k^2 - c_1^2 - c_2^2 - \ldots}\right)}{\sqrt{k^2 - c_1^2 - c_2^2 - \ldots}} \tag{37}$$

Thus

$$\frac{1}{c_1} \frac{\mathrm{d}}{\mathrm{d}c_1} f(c_1) = \frac{1}{c_2} \frac{\mathrm{d}}{\mathrm{d}c_2} f(c_2)$$
$$\frac{\mathrm{d}}{\mathrm{d}c_1} \left(\frac{1}{c_1} \frac{\mathrm{d}}{\mathrm{d}c_1} f(c_1)\right) = \frac{\mathrm{d}}{\mathrm{d}c_1} \underbrace{\left(\frac{1}{c_2} \frac{\mathrm{d}}{\mathrm{d}c_2} f(c_2)\right)}_{\text{idpt of } c_1} = 0 \tag{38}$$

Integrating the first term as our apporach to this ODE:

$$f(c_1) = \frac{\lambda}{2} c_1^2 + \mu \tag{39}$$

It is obvious that we can generalize the above to any c_i,

$$f(c) = \frac{\lambda}{2} c^2 + \mu \tag{40}$$

We meditate on an initial condition:

$$f(0) = P(|\psi\rangle = 0 | E_i\rangle) = 0 \tag{41}$$

This is obvious as the probability of not being in any eigenstate is 0. And we can see that $f(0) = \mu = 0$

Now recall the first constrain:

$$\sum_i f(c_i) = 1 \tag{42}$$

We exploit this constrain using the following special case to find λ

$$f(k) + f(0) + f(0) + f(0) + \ldots = 1$$
$$f(k) = 1 = \frac{\lambda}{2} k^2 \tag{43}$$
$$\Rightarrow \lambda = \frac{2}{k^2}$$

Collecting the above,

$$f(c) = \frac{c^2}{k^2} \tag{44}$$

We have the freedom to choose $k = 1$

Thus, we have the Born's Rule:

$$\boxed{p(E = E_i) = f(c_i) = |c_i|^2 = |\langle E_i|\psi\rangle|^2} \tag{45}$$

When applied to a continuous basis, we have the probability density funciton

$$\begin{aligned} |\psi\rangle &= \int \mathrm{d}x \psi(x)|x\rangle \\ \Rightarrow p(x) &= |\psi(x)|^2 \end{aligned} \tag{46}$$

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8 Hermitian Operator

- New notation: for a linear operator $\widehat{M}$ and vector $|\varphi\rangle$, we write

$$\begin{aligned} \widehat{M}|\varphi\rangle &= \left|\widehat{M}\varphi\right\rangle \\ \langle\psi|\widehat{M}|\varphi\rangle &= \left\langle\psi\middle|\widehat{M}\varphi\right\rangle \end{aligned} \tag{47}$$

8.1) Hermitian adjoint

We define the hermitian adjoint of an operator $\widehat{M}$ as $\widehat{M}^\dagger$, s.t.:

$$\left\langle\widehat{M}^\dagger\psi\middle|\varphi\right\rangle = \left\langle\psi\middle|\widehat{M}\varphi\right\rangle \tag{48}$$

The hermitian adjoint yields the following properties:

$$\begin{aligned} 1.\left(\widehat{M}^\dagger\right)^\dagger &= \widehat{M} \\ 2.\left(a\widehat{M} + b\widehat{N}\right)^\dagger &= a^*\widehat{M}^\dagger + b^*\widehat{N}^\dagger \\ 3.\left(\widehat{M}\widehat{N}\right)^\dagger &= \widehat{N}^\dagger\widehat{M}^\dagger \end{aligned} \tag{49}$$

8.2) Hermitian adjoint of a scalar

for a scalar c $\in \mathbb{C}$,

$$\begin{aligned} \langle\psi|c\varphi\rangle = c\langle\psi|\varphi\rangle &= \langle c^*\psi|\varphi\rangle \\ &= \left\langle c^\dagger\psi|\varphi\right\rangle \\ \Rightarrow c^\dagger = c^* \end{aligned} \tag{50}$$

the hermitian adjoint for a scalar c is its complex conjugate.

8.3) Hermitian adjoint of a ket

for kets $|\psi\rangle, |\varphi\rangle$,

$$\begin{aligned} \langle\varphi|\psi\rangle^\dagger = \langle\varphi|\psi\rangle^* &\overset{\text{comjugate symm.}}{=} \langle\psi|\varphi\rangle \\ \langle\varphi|\psi\rangle^\dagger &= (\langle\varphi|\,|\psi\rangle)^\dagger \end{aligned} \tag{51}$$

We observe that

$$\langle\psi|\,|\varphi\rangle = |\psi\rangle^\dagger\,\langle\varphi|^\dagger \tag{52}$$

We skip the proof, but it is true that

$$\begin{aligned} |\psi\rangle^\dagger &= \langle\psi| = \left(|\psi\rangle^*\right)^\top \text{ complex transpose} \\ \langle\varphi|^\dagger &= |\varphi\rangle \end{aligned} \tag{53}$$

9 Hermitian adjoint of observables

- recap on observables $\hat{E}$
  - eigenstates $|E_i\rangle$, measured values E_i
  - rules:

$$\begin{gathered} E_i \in \mathbb{R} \\ \operatorname{span}(|E_i\rangle) = \boldsymbol{V} \\ \langle E_i | E_j \rangle = \delta_{ij} \end{gathered} \tag{54}$$

- Hermitian Operator: defined as any linear operator $\widehat{M}$ s.t. $\widehat{M}^\dagger = \widehat{M}$

We show below why any observable $\hat{E}$ is hermitian.

Consider an observable acting on a ket,

$$\begin{aligned} \hat{E}|\psi\rangle &= \hat{E}\sum_i c_i |E_i\rangle \\ &= \sum_i c_i \hat{E}|E_i\rangle \\ &= \sum_i c_i E_i |E_i\rangle \quad \text{(notice it's just } \boldsymbol{A}\vec{x} = \lambda\vec{x}\text{)} \\ &= \sum_i \langle E_i|\psi\rangle \ E_i|E_i\rangle \\ &= \sum_i E_i |E_i\rangle\langle E_i||\psi\rangle \end{aligned} \tag{55}$$

$$\Rightarrow \hat{E}|\psi\rangle = \left(\sum_i E_i |E_i\rangle\langle E_i|\right)|\psi\rangle$$

$$\boxed{\hat{E} = \sum_i E_i |E_i\rangle\langle E_i|}$$

Similarly, for a continuous basis, an observable $\vec{x} = \int \mathrm{d}x \; x \; |x\rangle \; \langle x|$

Now, let’s find the hermitian adjoint of an observable

$$
\begin{aligned}
\hat{E}^\dagger &= \left(\sum_i E_i |E_i\rangle\langle E_i|\right)^\dagger \\
&= \sum_i \left(E_i |E_i\rangle\langle E_i|\right)^\dagger \\
&= \sum_i \left(|E_i\rangle\langle E_i|\right)^\dagger E_i^\dagger \\
&= \sum_i \langle E_i|^\dagger |E_i\rangle^\dagger E_i^\dagger \\
&= \sum_i \langle E_i|^\dagger |E_i\rangle^\dagger E_i^* \\
&= \sum_i E_i |E_i\rangle\langle E_i| \quad (\text{noticing } E_i^* = E_i, \text{since } E_i \in \mathbb{R}) \\
&= \hat{E}
\end{aligned}
\tag{56}
$$

$$
\boxed{\hat{E}^\dagger = \hat{E}}
\tag{57}
$$

We have proved that the observable $\hat{E}$ is an hermitian operator. ■

10 Commutator

- Two operators $\hat{A}$, $\hat{B}$ commute iff they satisfy : $\hat{A}\hat{B} = \hat{B}\hat{A}$
- Commutator: defined as $\left[\hat{A}, \hat{B}\right] = \hat{A}\hat{B} - \hat{B}\hat{A}$

  If $\hat{A}$, $\hat{B}$ are commute, then $\left\{\hat{A}, \hat{B}\right\} = 0$

Claim: Any two commutative linear operator share a simutanuous eigenbasis

Proof:

Given commutative linear operators $\hat{A}$, $\hat{B}$, and $\left[\hat{A}, \hat{B}\right] = 0$. We are to show that $\hat{A}$ and $\hat{B}$ share same eigenbasis.

Linear algebra quick fact:

Degeneracy refers to the multiplicities of eigenvalues. If a liear operator has an eigenvalue that cooresponds to N eigenvectors, we say that it is a degenerate eigenvalue with a geometric multiplicity of N. These N eigenvectors $|\varphi_i\rangle$ form an ***eigenspace*** $\left\{|\psi\rangle_i\right\}_{i=1}^N$ *A linear operator either has* ***degenerate eigenvalues*** *or not.*

- Let's consider when $\hat{A}$ does not have degenerate eigenvalues, i.e. each eigenvalue of $\hat{A}$ corresponds to unique eigenvectors of $\hat{A}$.

$$
\hat{A}|\psi_i\rangle = \lambda_i |\psi_i\rangle \quad \text{for each } i
\tag{58}
$$

  Consider $\hat{B}$ acting on $|\psi_i\rangle$:

$$
\hat{A}\underline{\hat{B}|\psi_i\rangle} = \hat{B}\hat{A}|\psi_i\rangle = \lambda_i \underline{\hat{B}|\psi_i\rangle}
\tag{59}
$$

  Notice that $\hat{B}|\psi_i\rangle$ is still an eigenvector of $\hat{A}$ with eigenvalue λ_i. Since we assumed that λ_i only corresponds to one eigenvector $|\psi_i\rangle$, $\hat{B}|\psi_i\rangle$can only be the scalar multiple of $|\psi_i\rangle$, i.e.

$$
\hat{B}|\psi_i\rangle = \mu_i |\psi_i\rangle
\tag{60}
$$

Notice that Equation 60 tells us that $|\psi_i\rangle$ is also the eigenvector of $\hat{B}$, with new eigenvalue μ_i.

We have thus shown that for any eigenvector of $\hat{A}$, it is also the eigenvector of $\hat{B}$. Putting on our mathematician hats, we say: $\{|\psi_i\rangle\}_{i=1}^{N}$ also span the eigenspace of $\hat{B}$ with eigenvalues μ_i

- Now consider when $\hat{A}$ has degenrate eigenvalues, i.e. there exsists certain eigenvalue λ s.t.

$$\hat{A}|\psi_i\rangle = \lambda \tag{61}$$