Notes on Contextual Quantum Metrology

Quantum Fisher information

eq.5 on page 5 of [1] shows

$$\begin{aligned} |\psi\rangle_{\text{in}} &= \mathrm{QWP}_2\left(\frac{\pi}{4}\right) \mathrm{HWP}(p)\, \mathrm{QWP}_1(q)\, |H\rangle \\ &= e^{i\left(-2p+q+\frac{\pi}{4}\right)} \begin{pmatrix} \cos\left(\frac{\pi}{4}-q\right) \\ e^{i\left(4p-2q-\frac{\pi}{2}\right)} \sin\left(\frac{\pi}{4}-q\right) \end{pmatrix} \end{aligned} \tag{1}$$

and by adjusting p, q s.t. $\theta_0 = \frac{\pi}{2} - 2q$, $\varphi = 4p - 2q - \frac{\pi}{2}$, the parameterized state is given as

$$|\psi\rangle_{\text{in}} = \cos\left(\frac{\theta_0}{2}\right)|H\rangle + e^{i\varphi}\sin\left(\frac{\theta_0}{2}\right)|V\rangle \tag{2}$$

When passed thorugh the sucrose solution, gaining a phase of $\theta = \alpha l c$, the state becomes

$$|\psi\rangle_{\text{out}} = \cos\left(\frac{\theta+\theta_0}{2}\right)|H\rangle + e^{i\varphi}\sin\left(\frac{\theta+\theta_0}{2}\right)|V\rangle \tag{3}$$

This is a pure state, since $\cos^2\left(\frac{\theta}{2}\right) + \sin^2\left(\frac{\theta}{2}\right) = 1$. For a pure state, its quantum Fisher Information can be found by the following equation [2].

$$F_\theta = 4\left[\langle\partial_\theta\psi|\partial_\theta\psi\rangle + \left(\langle\partial_\theta\psi|\psi\rangle\right)^2\right] \tag{4}$$

Where,

$$|\partial_\theta\psi\rangle = -\frac{1}{2}\sin\left(\frac{\theta+\theta_0}{2}\right)|H\rangle + \frac{1}{2}e^{i\varphi}\cos\left(\frac{\theta+\theta_0}{2}\right)|V\rangle \tag{5}$$

$$\langle\partial_\theta\psi| = -\frac{1}{2}\sin\left(\frac{\theta+\theta_0}{2}\right)\langle H| + \frac{1}{2}e^{-i\varphi}\cos\left(\frac{\theta+\theta_0}{2}\right)\langle V| \tag{6}$$

$$\Rightarrow \langle\partial_\theta\psi|\partial_\theta\psi\rangle = \left(\frac{1}{4}\sin^2\left(\frac{\theta+\theta_0}{2}\right)\right) + \left(\frac{1}{4}\cos^2\left(\frac{\theta+\theta_0}{2}\right)\right) = \frac{1}{4} \tag{7}$$

$$\begin{aligned} \left(\langle\partial_\theta\psi|\psi\rangle\right)^2 &= \frac{1}{4}\sin^2\left(\frac{\theta+\theta_0}{2}\right)\cos^2\left(\frac{\theta+\theta_0}{2}\right) + \frac{1}{4}\sin^2\left(\frac{\theta+\theta_0}{2}\right)\cos^2\left(\frac{\theta+\theta_0}{2}\right) \\ &\quad -\frac{1}{2}\sin^2\left(\frac{\theta+\theta_0}{2}\right)\cos^2\left(\frac{\theta+\theta_0}{2}\right) \\ &= 0 \end{aligned} \tag{8}$$

collecting Equation 7, Equation 8,

$$\boxed{F_Q = 4 \cdot \left(\frac{1}{4} + 0\right) = 1} \tag{9}$$

Contextual Quantum Fisher Information via quasiprobability

Necessary Ingredients from Measurement Theroy

- Born's Rule For quantum state $|\psi\rangle$ with eigenstates $|H\rangle, |V\rangle$:

$$P(H) = |\langle H|\psi\rangle|^2 \tag{10}$$

- Projection Measurements The experiment involves tro types of measurements:
  - A: H/V measurement, with $\Pi_H =$

Bibliography

[1] J. Jae, J. Lee, M. S. Kim, K.-G. Lee, and J. Lee, “Contextual quantum metrology,” *npj Quantum Information*, vol. 10, no. 1, Jul. 2024, doi: 10.1038/s41534-024-00862-5.

[2] M. Barbieri, “Optical Quantum Metrology,” *PRX Quantum*, vol. 3, no. 1, p. 10202–10203, Jan. 2022, doi: 10.1103/PRXQuantum.3.010202.