1 Quantum Fisher information

eq.5 on page 5 of [1] shows

$$\begin{aligned}|\psi\rangle_{\text{in}} &= \text{QWP}_2\left(\frac{\pi}{4}\right) \text{HWP}(p)\, \text{QWP}_1(q)\, |H\rangle \\ &= e^{i\left(-2p+q+\frac{\pi}{4}\right)} \begin{pmatrix} \cos\left(\frac{\pi}{4}-q\right) \\ e^{i\left(4p-2q-\frac{\pi}{2}\right)} \sin\left(\frac{\pi}{4}-q\right) \end{pmatrix}\end{aligned} \quad \{1\}$$

and by adjusting p, q s.t. $\theta_0 = \frac{\pi}{2} - 2q, \varphi = 4p - 2q - \frac{\pi}{2}$, the parameterized state is given as

$$|\psi\rangle_{\text{in}} = \cos\left(\frac{\theta_0}{2}\right)|H\rangle + e^{i\varphi}\sin\left(\frac{\theta_0}{2}\right)|V\rangle \quad \{2\}$$

When passed thorugh the sucrose solution, gaining a phase of $\theta = \alpha l c$, the state becomes

$$|\psi\rangle_{\text{out}} = \cos\left(\frac{\theta+\theta_0}{2}\right)|H\rangle + e^{i\varphi}\sin\left(\frac{\theta+\theta_0}{2}\right)|V\rangle \quad \{3\}$$

This is a pure state, since $\cos^2\left(\frac{\theta}{2}\right) + \sin^2\left(\frac{\theta}{2}\right) = 1$. For a pure state, its quantum Fisher Information can be found by the following equation [2].

$$F_\theta = 4\left[\langle\partial_\theta\psi|\partial_\theta\psi\rangle + \left(\langle\partial_\theta\psi|\psi\rangle\right)^2\right] \quad \{4\}$$

Where,

$$|\partial_\theta\psi\rangle = -\frac{1}{2}\sin\left(\frac{\theta+\theta_0}{2}\right)|H\rangle + \frac{1}{2}e^{i\varphi}\cos\left(\frac{\theta+\theta_0}{2}\right)|V\rangle \quad \{5\}$$

$$\langle\partial_\theta\psi| = -\frac{1}{2}\sin\left(\frac{\theta+\theta_0}{2}\right)\langle H| + \frac{1}{2}e^{-i\varphi}\cos\left(\frac{\theta+\theta_0}{2}\right)\langle V| \quad \{6\}$$

$$\Rightarrow \langle\partial_\theta\psi|\partial_\theta\psi\rangle = \left(\frac{1}{4}\sin^2\left(\frac{\theta+\theta_0}{2}\right)\right) + \left(\frac{1}{4}\cos^2\left(\frac{\theta+\theta_0}{2}\right)\right) = \frac{1}{4} \quad \{7\}$$

$$\begin{aligned}\left(\langle\partial_\theta\psi|\psi\rangle\right)^2 &= \frac{1}{4}\sin^2\left(\frac{\theta+\theta_0}{2}\right)\cos^2\left(\frac{\theta+\theta_0}{2}\right) + \frac{1}{4}\sin^2\left(\frac{\theta+\theta_0}{2}\right)\cos^2\left(\frac{\theta+\theta_0}{2}\right) \\ &-\frac{1}{2}\sin^2\left(\frac{\theta+\theta_0}{2}\right)\cos^2\left(\frac{\theta+\theta_0}{2}\right) \\ &= 0\end{aligned} \quad \{8\}$$

collecting Equation 7, Equation 8,

$$\boxed{F_Q = 4\cdot\left(\frac{1}{4}+0\right) = 1} \quad \{9\}$$

2 Contextual Quantum Fisher Information via quasiprobability

2.1 Necessary Ingredients

2.1.1 Born's Rule

For quantum state $|\psi\rangle$ with eigenstates $|H\rangle, |V\rangle$ etc. , the probability of the meauserment outcome falls on eigensate $|H\rangle$is :

$$P(H) = |\langle H|\psi\rangle|^2. \tag{10}$$

2.1.2 Projective Measurements

The experiment involves two types of measurements:

- $\mathcal{A}$: measurement in H, V basis , with $\Pi_H = |H\rangle\langle H|, \Pi_V = |V\rangle\langle V|$
- $\mathcal{B}$: measurement in D, A basis, with $\Pi_D = |D\rangle\langle D|, \Pi_A = |A\rangle\langle A|$, where

$$|D\rangle = \frac{1}{\sqrt{2}}(|H\rangle + |V\rangle); \quad |A\rangle = \frac{1}{\sqrt{2}}(|H\rangle - |V\rangle). \tag{11}$$

Consider a state in H / V basis $|\psi\rangle = \alpha|H\rangle + \beta|V\rangle$, it can be expressed in D / A basis as

$$|\psi\rangle = \frac{1}{2}\sqrt{2}(\alpha + \beta)|D\rangle + \frac{1}{\sqrt{2}}(\alpha - \beta)|A\rangle. \tag{12}$$

2.1.3 Operational Quasiprobability

$$w(a, b|\theta) = p(a, b|\mathcal{A}, \mathcal{B}, \theta) + \frac{1}{2}(\ p(b|\mathcal{B}, \theta) - p(b|\mathcal{A}, \mathcal{B}, \theta)\ ) \tag{13}$$

- In the sucrose measurement experiment, $a \in \{H, V\}$, $b \in \{D, A\}$, and θ is the phase shift due to the sucrose solution.

2.1.4 Contexual Fisher information

$$F_{\text{co}} = \sum_{a,b} w(a, b|\theta)\ [\partial_\theta \ln w(a, b|\theta)]^2 \tag{14}$$

2.2 Calculating Operational QUasiprobability

There are three major terms in Equation 13, so we will calculate them seperatedly as follows.

2.2.1 Joint probability $p(a, b|\mathcal{A}, \mathcal{B}, \theta)$

Let a and b represent the specific measurement outcomes corresponding to the projection operators $\Pi_{\mathcal{A}} = |A\rangle\langle A|$ and $\Pi_{\mathcal{B}} = |B\rangle\langle B|$. i.e. $a = |A\rangle, b = |B\rangle$.

The joint probability $p(a, b|\mathcal{A}, \mathcal{B}, \theta)$ can be interpreted as the probability of sequentially measuring $|A\rangle$ and then $|B\rangle$ when performing the measurements associated with $\Pi_{\mathcal{A}}$ followed by $\Pi_{\mathcal{B}}$.

According to the postulate of quantum measurement, this cascaded measurement can be represented as a single measurement with the combined operator $\Pi_{\{\mathcal{B}\mathcal{A}\}} = \Pi_{\mathcal{B}}\Pi_{\mathcal{A}}$. Thus, the joint probability can be written as:

$$\begin{aligned} p(a, b|\mathcal{A}, \mathcal{B}, \theta) &= \langle\psi(\theta)|\Pi^\dagger_{\{\mathcal{B}\mathcal{A}\}}\Pi_{\{\mathcal{B}\mathcal{A}\}}|\psi(\theta)\rangle \\ &= \langle\psi(\theta)|(\Pi_{\mathcal{B}}\Pi_{\mathcal{A}})^\dagger(\Pi_{\mathcal{B}}\Pi_{\mathcal{A}})|\psi(\theta)\rangle \\ &= \langle\psi(\theta)|\Pi^\dagger_{\mathcal{A}}\Pi^\dagger_{\mathcal{B}}\Pi_{\mathcal{B}}\Pi_{\mathcal{A}}|\psi(\theta)\rangle \end{aligned} \tag{15}$$

Since $\Pi_{\mathcal{A}}$ and $\Pi_{\mathcal{B}}$ are Hermitian ($\Pi_{\mathcal{A}}^{\dagger} = \Pi_{\mathcal{A}}$ and $\Pi_{\mathcal{B}}^{\dagger} = \Pi_{\mathcal{B}}$) and idempotent ($\Pi_{\mathcal{A}}^2 = \Pi_{\mathcal{A}}$ and $\Pi_{\mathcal{B}}^2 = \Pi_{\mathcal{B}}$), this simplifies to:

$$p(a, b | \mathcal{A}, \mathcal{B}, \theta) = \langle \psi(\theta) | \Pi_{\mathcal{A}} \Pi_{\mathcal{B}} \Pi_{\mathcal{A}} | \psi(\theta) \rangle \tag{16}$$

Noticing $\Pi_{\mathcal{A}} = |A\rangle\langle A|, \Pi_{\mathcal{B}} = |B\rangle\langle B|$, we can simplify this further to:

$$\begin{aligned} p(a, b | \mathcal{A}, \mathcal{B}, \theta) &= \langle \psi || A\rangle \langle A || B\rangle \langle B || A\rangle \langle A || \psi\rangle \\ &\boxed{= |\langle A|\psi\rangle|^2 \, |\langle B|A\rangle|^2} \end{aligned} \tag{17}$$

Using Equation 17, we can calculate $p(a, b| \; \mathcal{A}, \mathcal{B}, \theta)$ for $a \in \{H, V\}$ and $b \in \{D, A\}$.

- $$\begin{aligned} p(H, D \mid \mathcal{A}, \mathcal{B}, \theta) &= |\langle H|\psi\rangle|^2 |\langle D|H\rangle|^2 \\ &= \left| \begin{pmatrix} 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \cos\left(\frac{\theta+\theta_0}{2}\right) \\ e^{i\varphi} \sin\left(\frac{\theta+\theta_0}{2}\right) \end{pmatrix} \right|^2 \left| \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right|^2 \\ &= \frac{1}{2} \cos^2\left(\frac{\theta + \theta_0}{2}\right) \\ &= \frac{1}{4}(1 + \cos(\theta + \theta_0)) \end{aligned} \tag{18}$$

- Similarly,

$$\begin{aligned} p(H, A | \mathcal{A}, \mathcal{B}, \theta) &= |\langle H|\psi\rangle|^2 |\langle A|H\rangle|^2 \\ &= \frac{1}{2} \cos^2\left(\frac{\theta + \theta_0}{2}\right) \\ &= \frac{1}{4}(1 + \cos(\theta + \theta_0)) \end{aligned} \tag{19}$$

- $$\begin{aligned} p(V, D | \mathcal{A}, \mathcal{B}, \theta) &= |\langle V|\psi\rangle|^2 |\langle D|V\rangle|^2 \\ &= \frac{1}{4}(1 - \cos(\theta + \theta_0)) \end{aligned} \tag{20}$$

- $$\begin{aligned} p(V, A | \mathcal{A}, \mathcal{B}, \theta) &= |\langle V|\psi\rangle|^2 |\langle A|V\rangle|^2 \\ &= \frac{1}{4}(1 - \cos(\theta + \theta_0)) \end{aligned} \tag{21}$$

2.2.2 Conditional probability $p(b| \; \mathcal{B}, \theta)$

$p(b|\mathcal{B}, \theta)$ represents the probability of measuring outcome b when measurement $\mathcal{B}$ is performed. This can be found using the Born's rule (Equation 10). i.e.

$$p(b| \; \mathcal{B}, \theta) = |\langle B|\psi\rangle|^2 \tag{22}$$

We calculate $p(b|\mathcal{B}, \theta)$ for $b \in \{D, A\}$ as follows:

- $$\begin{aligned} p(D|\mathcal{B},\theta) &= |\langle D|\psi\rangle|^2 \\ &= \left|\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \cdot \begin{pmatrix} \cos\left(\frac{\theta+\theta_0}{2}\right) \\ e^{i\varphi}\sin\left(\frac{\theta+\theta_0}{2}\right) \end{pmatrix}\right|^2 \\ &= \left|\frac{1}{\sqrt{2}}\cos\left(\frac{\theta+\theta_0}{2}\right) + \frac{1}{\sqrt{2}}e^{i\varphi}\sin\left(\frac{\theta+\theta_0}{2}\right)\right|^2 \\ &= \left(\frac{1}{\sqrt{2}}\cos\left(\frac{\theta+\theta_0}{2}\right) + \frac{1}{\sqrt{2}}\cos\varphi\sin\left(\frac{\theta+\theta_0}{2}\right)\right)^2 \\ &\quad+\left(\frac{1}{\sqrt{2}}\sin\varphi\sin\left(\frac{\theta+\theta_0}{2}\right)\right)^2 \\ &= \frac{1}{2} + \cos\varphi\cos\left(\frac{\theta+\theta_0}{2}\right)\sin\left(\frac{\theta+\theta_0}{2}\right) \\ &\boxed{= \frac{1}{2}(1+\sin(\theta+\theta_0)\cos\varphi)} \end{aligned} \tag{23}$$

- $$\begin{aligned} p(A|\ \mathcal{B},\theta) &= |\langle A|\psi\rangle|^2 \\ &= \left|\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \cdot \begin{pmatrix} \cos\left(\frac{\theta+\theta_0}{2}\right) \\ e^{i\varphi}\sin\left(\frac{\theta+\theta_0}{2}\right) \end{pmatrix}\right|^2 \\ &= \left|\frac{1}{\sqrt{2}}\cos\left(\frac{\theta+\theta_0}{2}\right) - \frac{1}{\sqrt{2}}e^{i\varphi}\sin\left(\frac{\theta+\theta_0}{2}\right)\right|^2 \\ &= \left(\frac{1}{\sqrt{2}}\cos\left(\frac{\theta+\theta_0}{2}\right) - \frac{1}{\sqrt{2}}\cos\varphi\sin\left(\frac{\theta+\theta_0}{2}\right)\right)^2 \\ &\quad+\left(-\frac{1}{\sqrt{2}}\sin\varphi\sin\left(\frac{\theta+\theta_0}{2}\right)\right)^2 \\ &= \frac{1}{2} - \cos\varphi\cos\left(\frac{\theta+\theta_0}{2}\right)\sin\left(\frac{\theta+\theta_0}{2}\right) \\ &\boxed{= \frac{1}{2}(1-\sin(\theta+\theta_0)\cos\varphi)} \end{aligned} \tag{24}$$

2.2.3 Marginal probability $p(b|\mathcal{A},\mathcal{B},\theta)$

By definition, $p(b|\mathcal{A},\mathcal{B},\theta) = \sum_a p(a,b|\mathcal{A},\mathcal{B},\theta)$. Using results from Equation 18 to Equation 21, we can calculate $p(b|\mathcal{A},\mathcal{B},\theta)$ for $b \in \{D, A\}$ as follows:

- $$\begin{aligned} p(D|\ \mathcal{A},\mathcal{B},\theta) &= p(H,D\mid\mathcal{A},\mathcal{B},\theta) + p(V,D\mid\mathcal{A},\mathcal{B},\theta) \\ &= \frac{1}{4}(1+\cos(\theta+\theta_0)) + \frac{1}{4}(1-\cos(\theta+\theta_0)) \\ &= \frac{1}{2} \end{aligned} \tag{25}$$

- $$\begin{aligned} p(A|\ \mathcal{A},\mathcal{B},\theta) &= p(H,A\mid\mathcal{A},\mathcal{B},\theta) + p(V,A\mid\mathcal{A},\mathcal{B},\theta) \\ &= \frac{1}{4}(1+\cos(\theta+\theta_0)) + \frac{1}{4}(1-\cos(\theta+\theta_0)) \\ &= \frac{1}{2} \end{aligned} \tag{26}$$

2.2.4 Quasiprobability for $a \in \{H, V\}, b \in \{D, A\}$

Collecting the above,

- $$\begin{aligned} w(H, D \mid \theta) &= p(H, D|\ \mathcal{A}, \mathcal{B}, \theta) + \frac{1}{2}[p(D|\ \mathcal{B}, \theta) - p(D|\ \mathcal{A}, \mathcal{B}, \theta)] \\ &= \frac{1}{4}(1 + \cos(\theta + \theta_0)) + \frac{1}{2}\left(\frac{1}{2}(1 + \sin(\theta + \theta_0)\cos\varphi) - \frac{1}{2}\right) \\ &= \frac{1}{4}(1 + \cos(\theta + \theta_0) + \sin(\theta + \theta_0)\cos\varphi) \end{aligned} \tag{27}$$

- $$\begin{aligned} w(H, A|\ \theta) &= p(H, A|\ \mathcal{A}, \mathcal{B}, \theta) + \frac{1}{2}[p(A|\ \mathcal{B}, \theta) - p(A|\ \mathcal{A}, \mathcal{B}, \theta)] \\ &= \frac{1}{4}(1 + \cos(\theta + \theta_0) - \sin(\theta + \theta_0)\cos\varphi) \end{aligned} \tag{28}$$

- $$\begin{aligned} w(V, D|\ \theta) &= p(V, D|\ \mathcal{A}, \mathcal{B}, \theta) + \frac{1}{2}[p(D|\ \mathcal{B}, \theta) - p(D|\ \mathcal{A}, \mathcal{B}, \theta)] \\ &= \frac{1}{4}(1 - \cos(\theta + \theta_0) + \sin(\theta + \theta_0)\cos\varphi) \end{aligned} \tag{29}$$

- $$\begin{aligned} w(V, A|\ \theta) &= p(V, A|\ \mathcal{A}, \mathcal{B}, \theta) + \frac{1}{2}[p(A|\ \mathcal{B}, \theta) - p(A|\ \mathcal{A}, \mathcal{B}, \theta)] \\ &= \frac{1}{4}(1 - \cos(\theta + \theta_0) - \sin(\theta + \theta_0)\cos\varphi)) \end{aligned} \tag{30}$$

This set of results agrees with:

$$w(a, b|\theta) = \left(1 + (-1)^a \cos\theta + (-1)^b \sin\theta\cos\varphi\right)/4, \tag{31}$$

as claimed in [1].

2.3 Contextual Fisher Information (coFI)

coFI is given in [1] as

$$F_{\text{co}} = \sum_{a,b} w(a, b|\theta)\ [\partial_\theta \ln w(a, b|\theta)]^2 \tag{32}$$

Using the results from above, we can expand it as follows:

Bibliography

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[2] M. Barbieri, “Optical Quantum Metrology,” *PRX Quantum*, vol. 3, no. 1, p. 10202–10203, Jan. 2022, doi: 10.1103/PRXQuantum.3.010202.