Complex Power

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Contents

1 Review of Complex Numbers **1**
1.1 Traditional Representations 1
1.2 Alternative Representations Used in Power Calculations 1

2 Real Average Power **2**

3 Complex Power **3**
3.1 Ohm's Law 4
3.1.1 Complex Power is Additive for Circuit Elements in Series *and* in Parallel 5
3.2 Current Drawn by a Load in Parallel with a Capacitor 5
3.3 Power-Factor Correction 6

1. Review of Complex Numbers

1.1. Traditional Representations

Given real numbers P and Q, the complex number $\mathbb{S} = P + jQ$ is often represented in polar form as

$$\mathbb{S} = |\mathbb{S}|\angle\theta = |\mathbb{S}|e^{j\theta} = |\mathbb{S}|\big(\cos\theta + j\sin\theta\big) = |\mathbb{S}|\cos\theta + j|\mathbb{S}|\sin\theta.$$

In other words, the rectangular coordinates P and Q are obtained from polar form by

$$P = |\mathbb{S}|\cos\theta \quad \text{and} \quad Q = |\mathbb{S}|\sin\theta.$$

1.2. Alternative Representations Used in Power Calculations

The complex number $\mathbb{S}$ can also be specified by the triple $(P, \cos\theta, s)$, where $s = 1$ if $Q \geq 0$ and $s = -1$ if $Q < 0$. To justify this claim, it is enough to show that we can find Q, since the triple already gives us P. To begin, we use the equation $P = |\mathbb{S}|\cos\theta$ and the fact that the triple tells us both P and $\cos\theta$. Thus, we can compute

$$|\mathbb{S}| = \frac{P}{\cos\theta}.$$

Since $|\mathbb{S}|^2 = P^2 + Q^2$, and since the triple gives us the sign of Q, we can compute

$$Q = s\sqrt{|\mathbb{S}|^2 - P^2}$$

as claimed. Note that we can compute $\mathbb{S}$ from pairs such as (P,θ) and (Q,θ) as well.

2. Real Average Power

Figure 1 shows a circuit element with impedance Z (a complex number), across which we have the sinusoidal voltage $v(t) = V\cos(\omega t + \theta_v)$, and through which flows the current $i(t) = I\cos(\omega t + \theta_i)$, following the passive-sign convention. Keep in mind that V and I are positive numbers. Then as shown in Section 11.2 of the textbook, the ***average* power** absorbed is

$$P = \tfrac{1}{2} VI\cos(\theta_v - \theta_i),$$

and

$$\text{pf} := \cos(\theta_v - \theta_i)$$

is called the **power factor**. Not surprisingly, pfa $:= \theta_v - \theta_i$ is called the **power-factor *angle***. Since the cosine is an even function, we cannot recover the power-factor angle from knowledge of the power factor alone. The power factor is said to be *lagging* if the pfa is greater than or equal to zero, and is said to be *leading* if the pfa is negative.

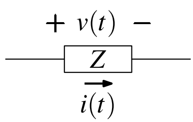

Figure 1. Voltage $v(t)$ applied to an impedance Z with current $i(t)$ flowing through it.

As shown in Section 11.4 of the textbook, the **rms** values of $v(t)$ and $i(t)$ are

$$V_{\text{rms}} := \frac{V}{\sqrt{2}} \quad \text{and} \quad I_{\text{rms}} := \frac{I}{\sqrt{2}},$$

respectively. Putting this all together, we can write

$$P = V_{\text{rms}} I_{\text{rms}} \cos(\theta_v - \theta_i). \tag{1}$$

3. Complex Power

The rms voltage and current phasors corresponding to $v(t)$ and $i(t)$ are the complex numbers

$$\mathbf{V}_{\text{rms}} := V_{\text{rms}}\angle\theta_v \quad \text{and} \quad \mathbf{I}_{\text{rms}} := I_{\text{rms}}\angle\theta_i.$$

Notice that we have used boldface letters for phasors, but italic letters for their magnitudes. Thus, $|\mathbf{V}_{\text{rms}}| = V_{\text{rms}}$ and $|\mathbf{I}_{\text{rms}}| = I_{\text{rms}}$. With this notation, we rewrite (1) as

$$P = |\mathbf{V}_{\text{rms}}|\,|\mathbf{I}_{\text{rms}}|\cos(\theta_v - \theta_i).$$

The ***complex*** **power** absorbed by a circuit element is defined as

$$\begin{aligned}\mathbb{S} := \mathbf{V}_{\text{rms}}\mathbf{I}_{\text{rms}}^* &= |\mathbf{V}_{\text{rms}}|\,|\mathbf{I}_{\text{rms}}|\angle(\theta_v - \theta_i)\\ &= |\mathbf{V}_{\text{rms}}|\,|\mathbf{I}_{\text{rms}}|\cos(\theta_v - \theta_i) + j|\mathbf{V}_{\text{rms}}|\,|\mathbf{I}_{\text{rms}}|\sin(\theta_v - \theta_i).\end{aligned}$$

We see that

$$\mathbb{S} = P + jQ, \tag{2}$$

where $P = \operatorname{Re}\mathbb{S}$ is the average power given above (also called the ***real*** **power**), and

$$Q := \operatorname{Im}\mathbb{S} = |\mathbf{V}_{\text{rms}}|\,|\mathbf{I}_{\text{rms}}|\sin(\theta_v - \theta_i)$$

is called the ***reactive*** **power**. Since $|\mathbb{S}| = |\mathbf{V}_{\text{rms}}|\,|\mathbf{I}_{\text{rms}}|$, we can also write

$$P = |\mathbb{S}|\cdot\text{pf} \quad \text{and} \quad Q = |\mathbb{S}|\sin(\theta_v - \theta_i),$$

where pf is the power factor, and $\text{pfa} = \theta_v - \theta_i$ is the power-factor angle, both discussed in the preceding section. Notice that since the sine function is odd, the sign of Q is determined by the sign of the pfa. Thus, a lagging power factor corresponds to $Q \geq 0$, and a leading power factor corresponds to $Q < 0$.

The quantity $|\mathbb{S}|$ is called the ***apparent*** **power**, and it follows that

$$|\mathbb{S}| = \frac{P}{\text{pf}} \quad \text{and} \quad |\mathbb{S}| = \frac{Q}{\sin(\theta_v - \theta_i)}. \tag{3}$$

For example, if we know the average power P and the power factor pf, then we can compute the apparent power $|\mathbb{S}|$ using left-hand equation in (3), and it follows from (2) that

$$Q = \pm\sqrt{|\mathbb{S}|^2 - P^2}, \tag{4}$$

where we take the plus sign if the power factor is lagging, and we take the minus sign if the power factor is leading. This is illustrated in Figure 2. By considering (4) and the left-hand equation in (3), it can be seen that $|Q|$ decreases as the power factor increases to one.

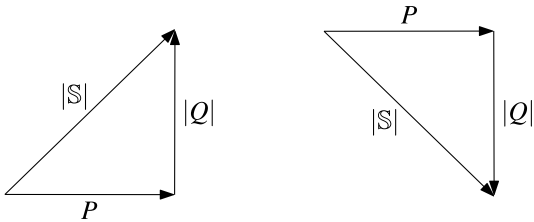

Figure 2. Power triangles illustrating the components of $\mathbb{S} = P + jQ$. The triangle on the left corresponds to a lagging power factor ($Q > 0$). The triangle on the right corresponds to a leading power factor ($Q < 0$).

3.1. Ohm's Law

From $\mathbf{V}_{\text{rms}} = \mathbf{I}_{\text{rms}} Z$, we see that

$$Z = \frac{\mathbf{V}_{\text{rms}}}{\mathbf{I}_{\text{rms}}} = \frac{|\mathbf{V}_{\text{rms}}|}{|\mathbf{I}_{\text{rms}}|} \angle(\theta_v - \theta_i). \tag{5}$$

Example 1 (ELI the ICE man). For an inductor, $Z = j\omega L$ is a complex number with angle 90°, and $\theta_v = \theta_i + 90^\circ$ says that voltage leads current by 90°. In contrast, for a capacitor, $Z = 1/(j\omega C) = -j/(\omega C)$ is a complex number with angle -90°, and $\theta_v = \theta_i - 90^\circ$ says that voltage lags current; equivalently, $\theta_i = \theta_v + 90^\circ$, which says that current leads voltage. These relationships can be remembered with the phrase "ELI the ICE man," where "E" stands for **electromotive force**, which is another name for voltage. Thus, ELI means that voltage leads current in an inductor, while ICE means current leads voltage in a capacitor.

Returning to (5), it tells us that ***the angle of complex power is equal to the angle of the impedance***. This is useful to know. For example, if we know the reactive power Q and the impedance angle $\angle Z$, then, analogous to (3), we can compute the apparent power

$$|\mathbb{S}| = \frac{Q}{\sin(\angle Z)}.$$

Recall that a passive circuit element is one that absorbs power; i.e., $P \geq 0$. This implies $\mathbb{S} = P + jQ$ lies in the right half plane, which means $-90^\circ \leq \angle\mathbb{S} \leq 90^\circ$. Since $\angle Z = \angle\mathbb{S}$, the same is true of the impedance angle. This means that the power factor, pf $= \cos(\angle Z)$ is always nonnegative. But the reactive power $Q = |\mathbb{S}|\sin(\angle Z)$ is negative if the sine is negative, which corresponds to a leading power factor.

If we substitute Ohm's law into the definition of complex power, we find that

$$\mathbb{S} = \mathbf{V}_{\text{rms}} \mathbf{I}^*_{\text{rms}} = (\mathbf{I}_{\text{rms}} Z)\mathbf{I}^*_{\text{rms}} = |\mathbf{I}_{\text{rms}}|^2 Z.$$

Taking absolute values show that we can obtain the apparent power from

$$|\mathbb{S}| = |\mathbf{I}_{\text{rms}}|^2 \, |Z|.$$

Similarly,

$$\mathbb{S} = \mathbf{V}_{\text{rms}}\mathbf{I}_{\text{rms}}^* = \mathbf{V}_{\text{rms}}(\mathbf{V}_{\text{rms}}/Z)^* = |\mathbf{V}_{\text{rms}}|^2/Z^*,$$

and

$$|\mathbb{S}| = |\mathbf{V}_{\text{rms}}|^2/|Z|.$$

3.1.1. Complex Power is Additive for Circuit Elements in Series *and* in Parallel

We show that the sum of the complex powers absorbed by circuit elements in series or in parallel is equal to the power absorbed by an equivalent impedance.

Let $\mathbf{I}_{\text{rms}}$ denote the current passing through two circuit elements connected in series. If the circuit elements have impedances Z_1 and Z_2, then the complex powers absorbed by the respective circuit elements are $|\mathbf{I}_{\text{rms}}|^2 Z_1$ and $|\mathbf{I}_{\text{rms}}|^2 Z_2$. Adding the complex powers, we get $|\mathbf{I}_{\text{rms}}|^2(Z_1 + Z_2)$, which is the complex power absorbed by an equivalent circuit element of impedance $Z_1 + Z_2$.

Now suppose the two circuit elements are connected in parallel with the voltage across them being $\mathbf{V}_{\text{rms}}$. Then the complex powers absorbed by the respective circuit elements are $|\mathbf{V}_{\text{rms}}|^2/Z_1^*$ and $|\mathbf{V}_{\text{rms}}|^2/Z_2^*$. Adding the complex powers, we get $|\mathbf{V}_{\text{rms}}|^2/Z_{\text{eq}}^*$, where $Z_{\text{eq}} = 1/[1/Z_1 + 1/Z_2]$.

3.2. Current Drawn by a Load in Parallel with a Capacitor

Consider an ideal voltage source $\mathbf{V}_{\text{rms}}$ applied to a load impedance Z. The complex power absorbed by the load is $\mathbb{S} = \mathbf{V}_{\text{rms}}\mathbf{I}_{\text{rms}}^*$. In particular, the magnitude of the current drawn by the load is $|\mathbf{I}_{\text{rms}}| = |\mathbb{S}|/|\mathbf{V}_{\text{rms}}|$. If we connect another circuit element in parallel with Z such that the apparent power of the parallel combination, call it $|\mathbb{S}_{\|}|$, is smaller than $|\mathbb{S}|$, we can reduce the current drawn from the ideal source because $\mathbf{V}_{\text{rms}}$ will not change if another element is connected in parallel with Z.

Now suppose we connect a capacitor having impedance $Z_C = 1/(j\omega C)$ in parallel with Z. The complex power absorbed by the capacitor is reactive (purely imaginary) and is given by

$$\mathbb{S}_C = |\mathbf{V}_{\text{rms}}|^2/Z_C^* = -j\omega C|\mathbf{V}_{\text{rms}}|^2.$$

Thus,

$$Q_C := \operatorname{Im} \mathbb{S}_C = -\omega C|\mathbf{V}_{\text{rms}}|^2. \tag{6}$$

Keep in mind that Q_C is a negative number. If the complex power absorbed by the original load Z is $\mathbb{S} = P + jQ$, then the complex power absorbed by the parallel combination is

$$\mathbb{S}_{\|} = \mathbb{S} + \mathbb{S}_C = (P + jQ) + jQ_C = P + j(Q + Q_C). \tag{7}$$

Typically, the original load Z is inductive (Q is positive). As long as $|Q+Q_C| < Q$, we will have $|\mathbb{S}_{\|}| < |\mathbb{S}|$, which implies the current of the parallel combination, $|\mathbb{S}_{\|}|/|\mathbf{V}_{\text{rms}}|$, is less than the current of the original load, $|\mathbb{S}|/|\mathbf{V}_{\text{rms}}|$. The power triangles for $\mathbb{S} = P + jQ$ and $\mathbb{S}_{\|} = P + j(Q+Q_C)$ are illustrated in Figure 3.

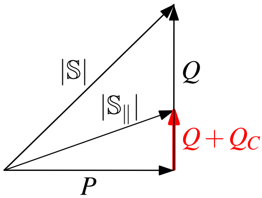

Figure 3. Power triangles illustrating the components of $\mathbb{S} = P + jQ$ and $\mathbb{S}_{\|} = P + j(Q+Q_C)$. Here Q is positive, Q_C is negative, and $Q+Q_C$ is smaller than Q.

3.3. Power-Factor Correction

Suppose the original load Z has a lagging power factor pf (i.e., $Q > 0$). Our goal is to connect an appropriate capacitor in parallel so that the power factor of the parallel combination is increased to some specified value $\widehat{\text{pf}}$, still lagging. In other words, we want to find C in (6) so that $\mathbb{S}_{\|}$ in (7) has lagging power factor $\widehat{\text{pf}}$.

Since we know P, and since $\widehat{\text{pf}}$ is given, we start by computing

$$|\mathbb{S}_{\|}| = \frac{P}{\widehat{\text{pf}}}.$$

Since the new power factor $\widehat{\text{pf}}$ is lagging, we have from (7) that

$$Q + Q_C = \sqrt{|\mathbb{S}_{\|}|^2 - P^2},$$

and we compute

$$Q_C = -Q + \sqrt{|\mathbb{S}_{\|}|^2 - P^2}. \tag{8}$$

To conclude, we use (6) to write

$$C = \frac{-Q_C}{\omega|\mathbf{V}_{\text{rms}}|^2}.$$

Example 2. Suppose a load dissipates 8 kW with 0.8 lagging power factor when connected to an ideal 240 V_{rms} source at 60 Hz. What value of C will increase the power factor to 0.95 lagging?

Solution. Since we know $P = 8$ kW and pf $= 0.8$, we compute $|\mathbb{S}| = P/\text{pf} = 8000/0.8 = 10$ kVA. Since the power factor is lagging, we can use (4) with a plus sign to find

$$Q = \sqrt{|\mathbb{S}|^2 - P^2} = 6 \text{ kVAR}.$$

Since we are told $\widehat{\text{pf}} = 0.95$, we compute $|\mathbb{S}_{\|}| = P/\widehat{\text{pf}} = 8000/.95 = 8.421$ kVA. Then we use the formula for Q_C in (8) to write

$$Q_C = -6000 + \sqrt{8421^2 - 8000^2} = -3.371 \text{ kVAR}.$$

Hence,

$$C = \frac{-(-3371)}{2\pi(60)(240^2)} = 155.2\ \mu\text{F}.$$