Physics 731: Assignment #1

1. **[S1r, S2 1.2, S3 1.4]** Suppose a 2×2 matrix X (not necessarily Hermitian or unitary) is written as

$$X = a_0 + \boldsymbol{\sigma} \cdot \mathbf{a} \equiv a_0 \, \mathbb{1} + \sum_{k=1}^{3} \sigma_k a_k,$$

where the σ_k are the Pauli matrices, and a_0 and a_k are numbers.
(a) How are a_0 and a_k related to $\mathrm{Tr}X$ and $\mathrm{Tr}(\sigma_k X)$?
(b) Obtain a_0 and a_k in terms of the matrix elements X_{ij}.

2. **[S1r, S2 1.7, S3 1.9]** Consider a ket space spanned by the eigenkets $|a'\rangle$ of a Hermitian operator A. There is no degeneracy.
(a) Prove that $\Pi_{a'}(A - a')$ is the null operator.
(b) What is the significance of

$$\Pi_{a'' \neq a'} \frac{(A - a'')}{(a' - a'')}?$$

(c) Illustrate (a) and (b) using A set equal to S_z of a spin $1/2$ system.

3. **[S1r, S2, 1.9, S3 1.11]** Construct $|\mathbf{S} \cdot \hat{\mathbf{n}}; +\rangle$ such that

$$\mathbf{S} \cdot \hat{\mathbf{n}} |\mathbf{S} \cdot \hat{\mathbf{n}}; +\rangle = \frac{\hbar}{2} |\mathbf{S} \cdot \hat{\mathbf{n}}; +\rangle.$$

The unit vector $\hat{\mathbf{n}}$ is characterized by the polar angle β with respect to the positive z axis and the azimuthal angle α in the $x - y$ plane, with $\alpha = 0$ corresponding to the x axis. Express your answer in terms of $|+\rangle$ and $|-\rangle$, the eigenvectors of S_z. Do this by explicitly solving the eigenvalue problem, as opposed to using rotation operators or plugging in the answer that is given in this question in the text.

4. **[S1r, S2 1.11, S3 1.12, modified]** A two-state system is characterized by the Hamiltonian

$$H = H_{11}|1\rangle\langle 1| + H_{22}|2\rangle\langle 2| + H_{12}[|1\rangle\langle 2| + |2\rangle\langle 1|],$$

where the H_{ij} are real numbers with the dimensions of energy, and $|1\rangle$ and $|2\rangle$ are the eigenkets of some observable (not H). Find the energy eigenvalues and eigenvectors. Do this in two ways: (a) solve the eigenvalue problem explicitly, and (b) use your result from the previous problem.

5. **[S1r, S2 1.12, S3 1.14, modified]** A spin $1/2$ system is known to be in an eigenstate of $\mathbf{S} \cdot \hat{\mathbf{n}}$ with eigenvalue $+\hbar/2$, where $\hat{\mathbf{n}}$ is a unit vector lying in the xz plane that makes an angle γ with the positive z axis.
(a) Suppose S_x is measured. What are the possible outcomes of the measurement and their respective probabilities?
(b) Evaluate the dispersion in S_x: $\langle (S_x - \langle S_x \rangle)^2 \rangle$. Check your answers for $\gamma = 0$, $\pi/2$, and π.
(c) How do the answers for (a) and (b) change for the case of S_y?

6. **[S1r, S2 1.20, S3 1.22]** Find the linear combination of $|+\rangle$ and $|-\rangle$ that maximizes the uncertainty product $\langle (\Delta S_x)^2 \rangle \langle (\Delta S_y)^2 \rangle$. Verify explicitly that for the linear combination you found, the uncertainty relation for S_x and S_y is not violated.