ECE 535: Introduction to Quantum Sensing

Wave-particle nature of light and matter

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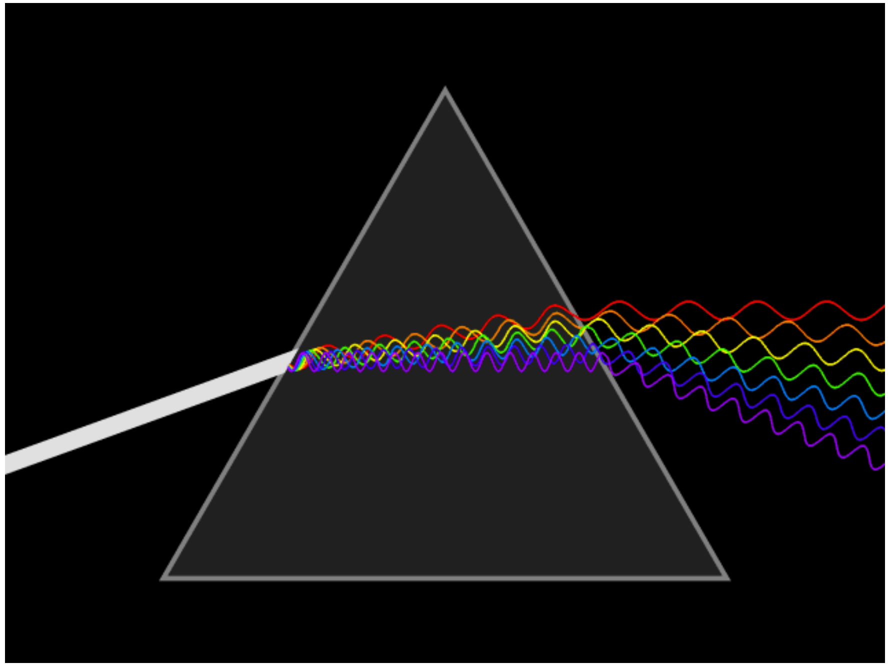
Topics

- Light as electromagnetic waves
- Challenges to wave theory of light
  - Blackbody radiation
  - Photoelectric effect
  - Atomic spectra
- Properties of photons and their interactions with charged matter
  - Changing energy
  - Changing linear momentum
  - Changing angular momentum
- De Broglie’s hypothesis

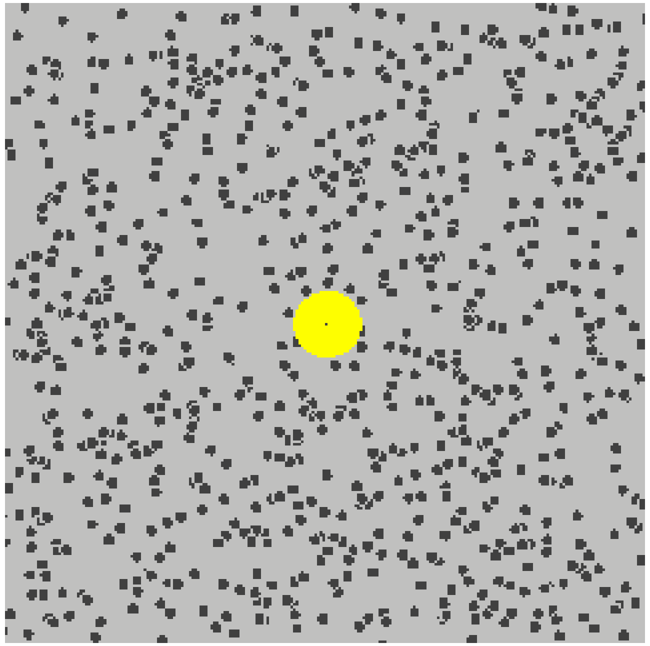
Prevailing thoughts on the nature of electromagnetism and matter before the 20th century

- Wave theory of light
  - Phenomena of diffraction and interference
  - Light behaves as an electromagnetic wave
  - Maxwell's equations



Credit: Lucas Veira, Public Domain

- Atomic theory of matter
  - Dalton and Avogadro's laws
  - Brownian motion
  - Discovery of subatomic particles

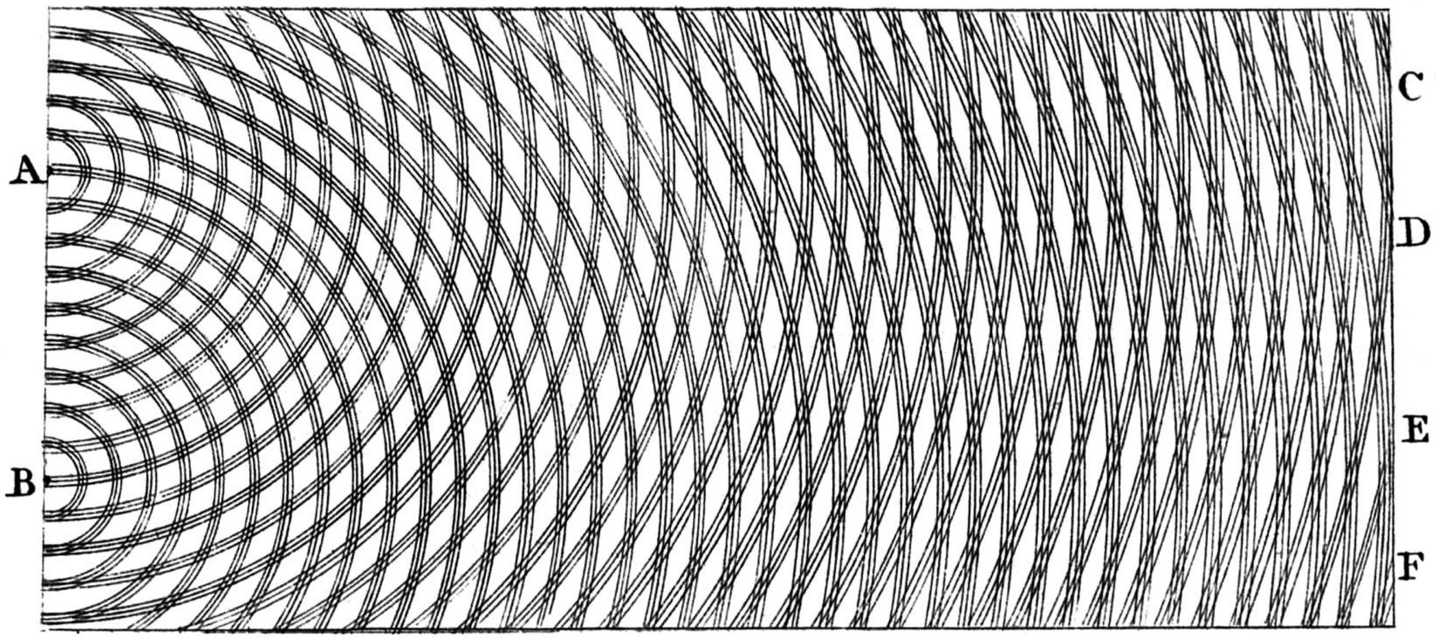


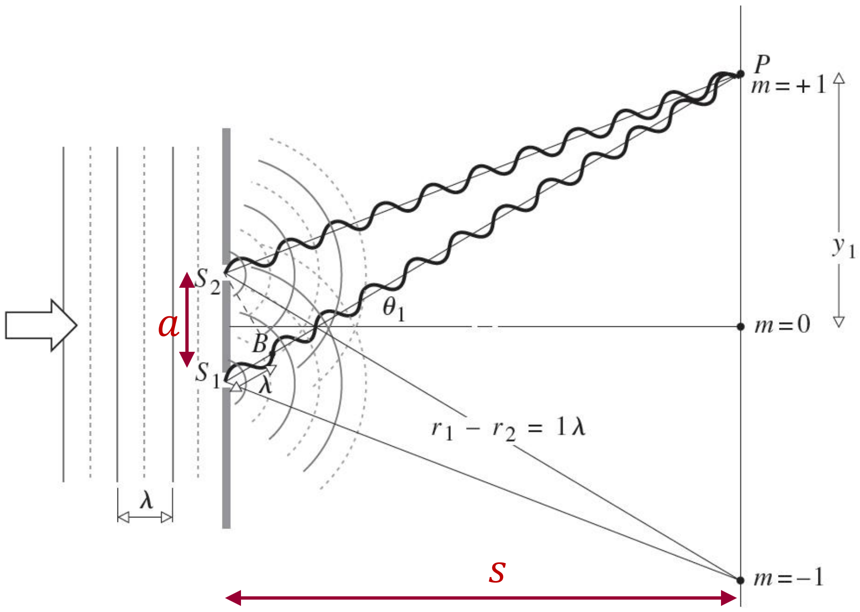
Credit: Lookang (Wikipedia), distributed under CC BY-SA 3.0

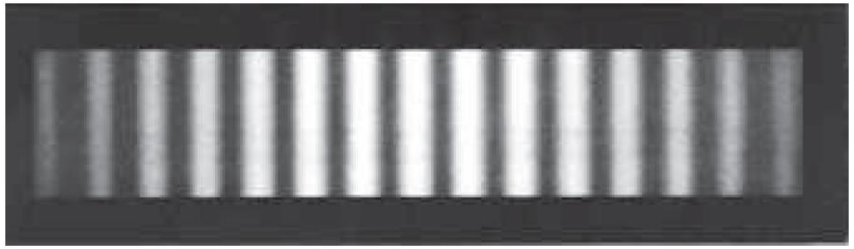
Wave theory of light

- Huygens' wave theory of light explained reflection and refraction
- Young performed double slit interference experiments and calculated wavelengths of light of different colors
- Fresnel combined Huygens' principle with the concept of wave superposition to explain diffraction and interference

Young's sketch of two-slit diffraction of water waves (1803)







Source: E. Hecht, Optics, 5th Edition

Maxwell framed light as electromagnetic wave (~1861)

$\boldsymbol{E}$ and $\boldsymbol{B}$ are vector electric and magnetic fields

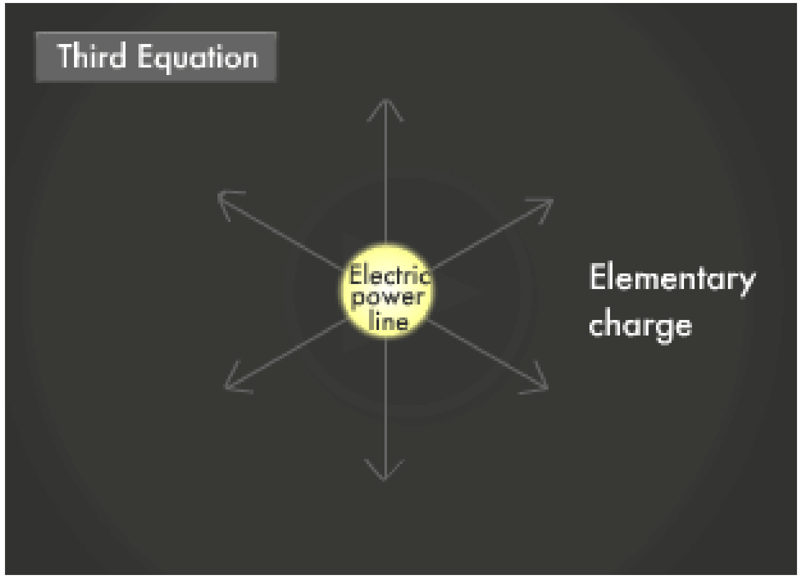
<u>Gauss's law</u>

divergence

$$\vec{\nabla} \cdot \vec{\mathcal{E}} = \rho / \epsilon_0$$

$$\int \vec{\nabla} \cdot \vec{\mathcal{E}} \, dV = \frac{Q}{\epsilon_0}$$

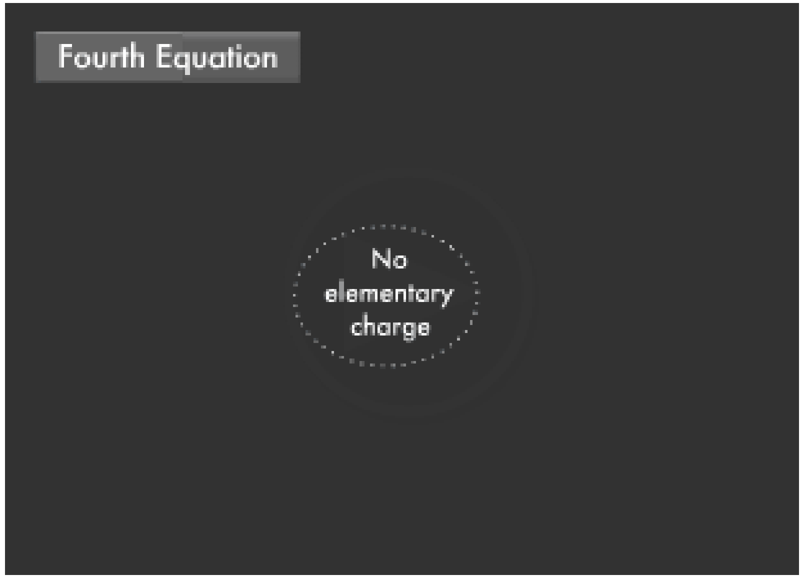
ρ is the total electric charge density
ϵ_0 is the permittivity of free space



<u>Gauss's law for magnetism</u>

$$\vec{\nabla} \cdot \vec{B} = 0$$

No magnetic monopoles

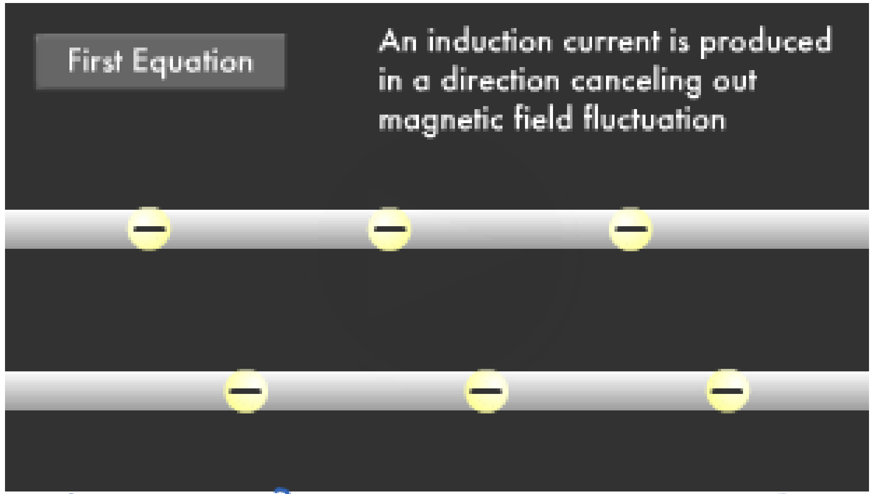


Animations from Canon Technology (https://global.canon/en/technology/s_labo/light/001/11.html)

Maxwell framed light as electromagnetic wave (~1861)

Faraday's law of induction

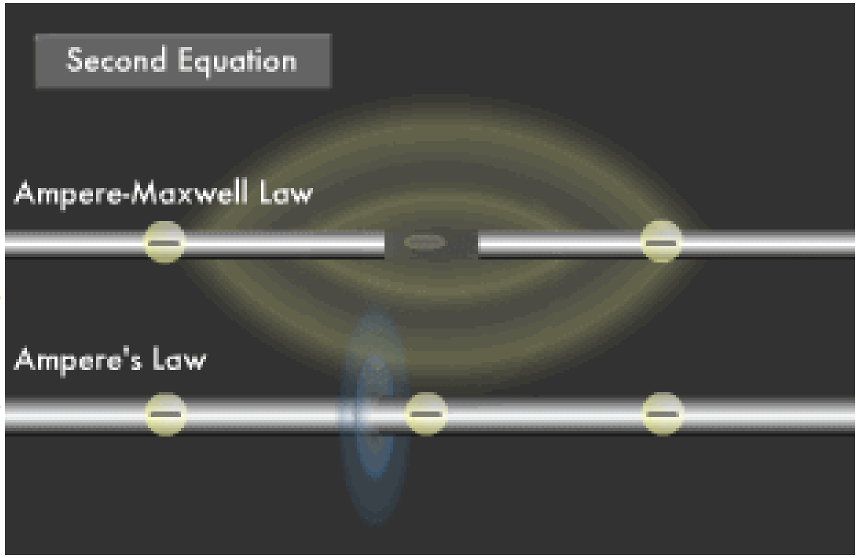
$$\vec{\nabla}\times\vec{\mathcal{E}} = -\frac{\partial\vec{B}}{\partial t}$$



Ampere's law

$$\vec{\nabla}\times\vec{B} = \mu_0\left(\vec{J} + \epsilon_0\frac{\partial\vec{\mathcal{E}}}{\partial t}\right)$$

$\vec{J}$ is the total electric current density
ϵ_0 is the permittivity of free space
μ_0 is the permeability of free space
Speed of light $c = \frac{1}{\sqrt{\epsilon_0\mu_0}} = 3\times10^8\ m/s$



$\vec{\nabla}\times(\vec{\nabla}\times\vec{\mathcal{E}} = -\frac{\partial\vec{B}}{\partial t})$

RHS: $\vec{\nabla}\times\left(-\frac{\partial\vec{B}}{\partial t}\right) = -\frac{\partial}{\partial t}(\underbrace{\vec{\nabla}\times\vec{B}}_{\mu_0\vec{J}+\mu_0\epsilon_0\frac{\partial\vec{\mathcal{E}}}{\partial t}})$

$= -\mu_0\cancel{\frac{\partial\vec{J}}{\partial t}} - \mu_0\epsilon_0\frac{\partial^2\vec{\mathcal{E}}}{\partial t^2}$

LHS: $\vec{\nabla}\times\vec{\nabla}\times\vec{\mathcal{E}} = \vec{\nabla}\cdot\underbrace{(\vec{\nabla}\cdot\vec{\mathcal{E}})}_{\text{const}}^{0} - \nabla^2\vec{\mathcal{E}}$

$\nabla^2\vec{\mathcal{E}} = \underbrace{\mu_0\epsilon_0}_{1/c^2}\frac{\partial^2\mathcal{E}}{\partial t^2}$ wave eqn in vacuum

In medium: $\mu_0\epsilon_0\epsilon_r = \frac{1}{v^2}$

$\mu_0\epsilon_0 \underset{\text{refractive index}}{n^2} = \frac{1}{\underset{\text{phase velocity}}{v^2}}$

Animations from Canon Technology (https://global.canon/en/technology/s_labo/light/001/11.html)

Light is an electromagnetic wave

$\vec{\mathcal{E}}$ and $\vec{B}$ vector fields obey the wave equation (shown here for $\vec{\mathcal{E}}$ only):

$$\nabla^2 \vec{\mathcal{E}} - \frac{1}{c^2}\frac{\partial^2 \vec{\mathcal{E}}}{\partial t^2} = 0$$

with sinusoidal solution $\vec{\mathcal{E}}(\vec{r},t) = Re\left\{\vec{A}\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right\}$.

polarization unit vector

Amplitude $\vec{A} = |\vec{A}|\, \hat{e}_p$

frequency

Angular frequency $\omega = 2\pi\nu$

Wave vector $|\vec{k}| = \frac{2\pi}{\lambda}$

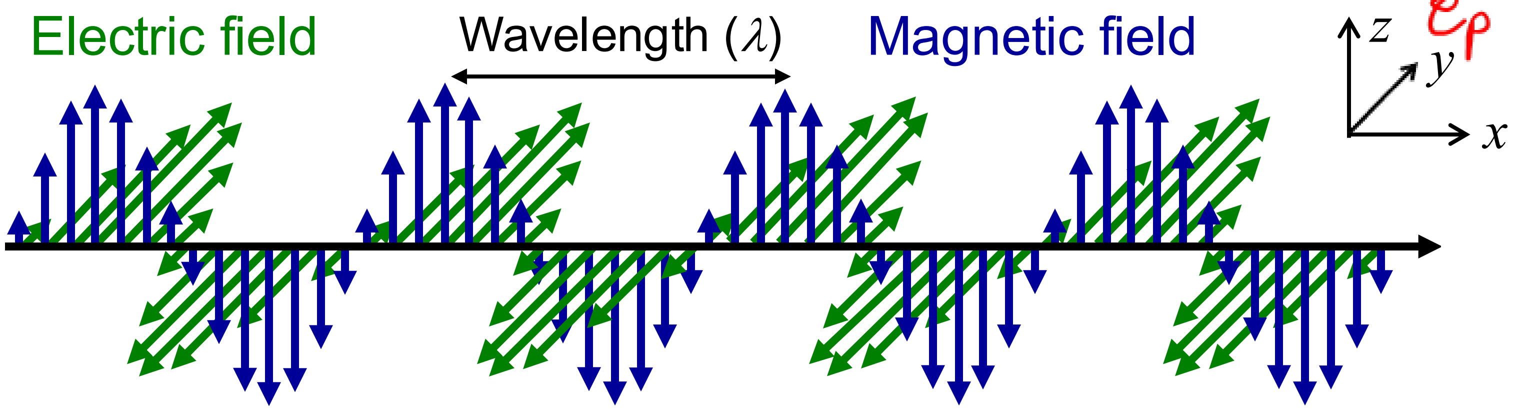
$\vec{k} = (k_0, 0, 0)$

$\hat{e}_p \quad \hat{y}$

$\vec{k} \perp \vec{\mathcal{E}}$

$\vec{k} \perp \vec{B}$

$\vec{\mathcal{E}} \perp \vec{B}$



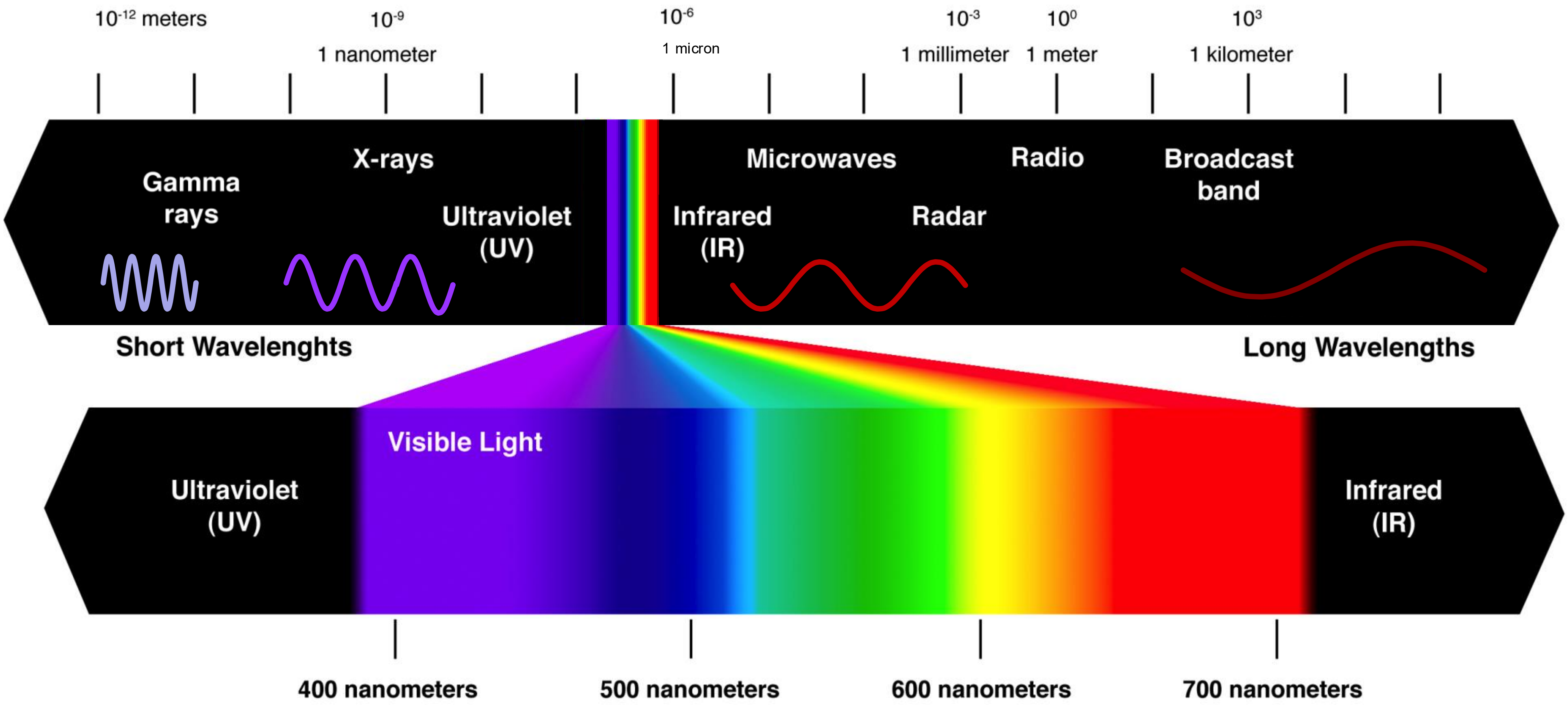
Different wavelengths (distances between the peaks) or frequencies (the rate at which the peaks pass by) correspond to different colors.

$\lambda \nu = c$

$[m]\,[1/s] \quad [m/s]$

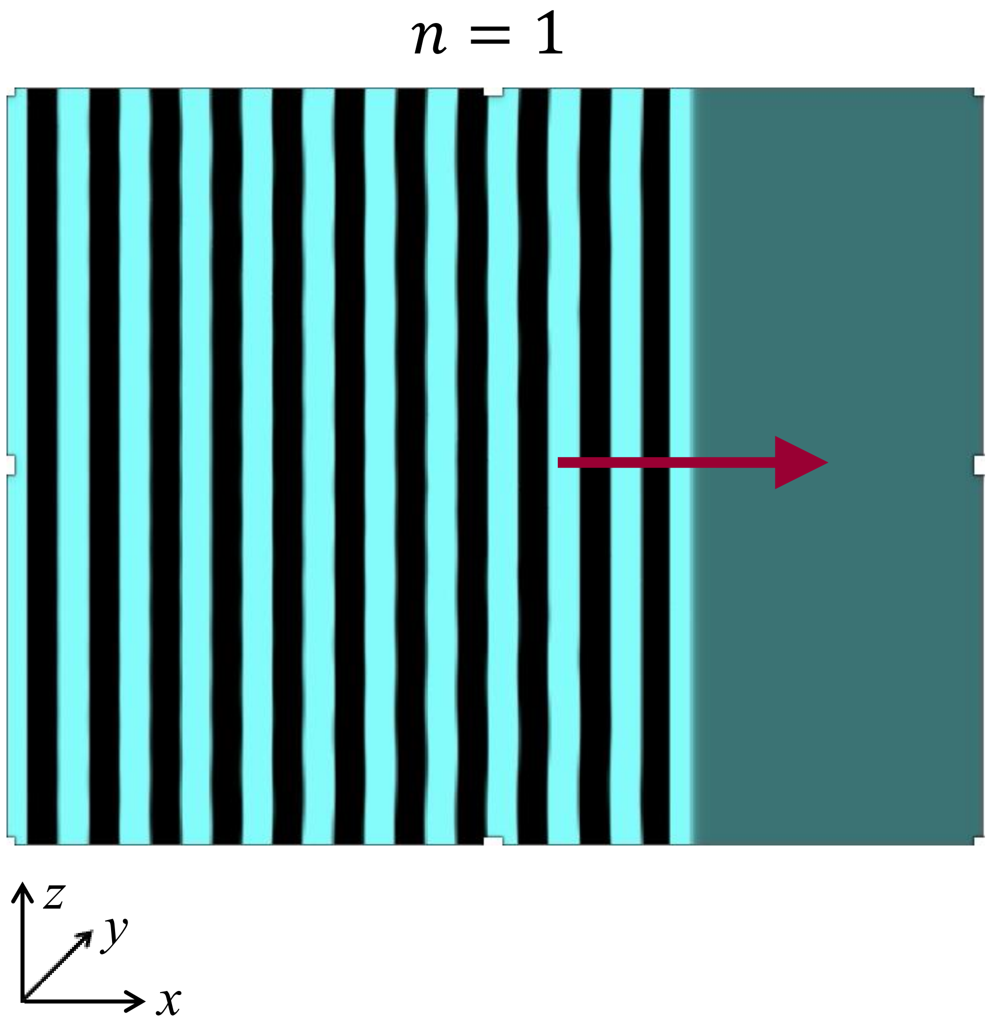
Animation from R. Trebino

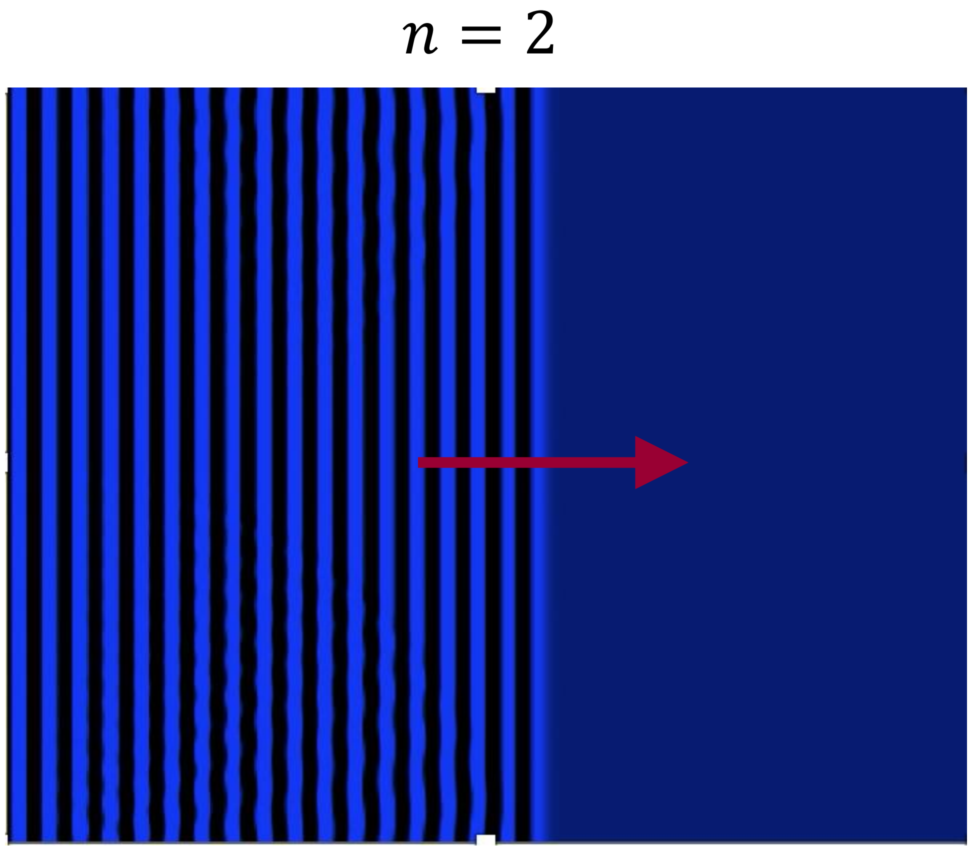
Electromagnetic spectrum



Credit: R. Trebino's lecture notes on Optics

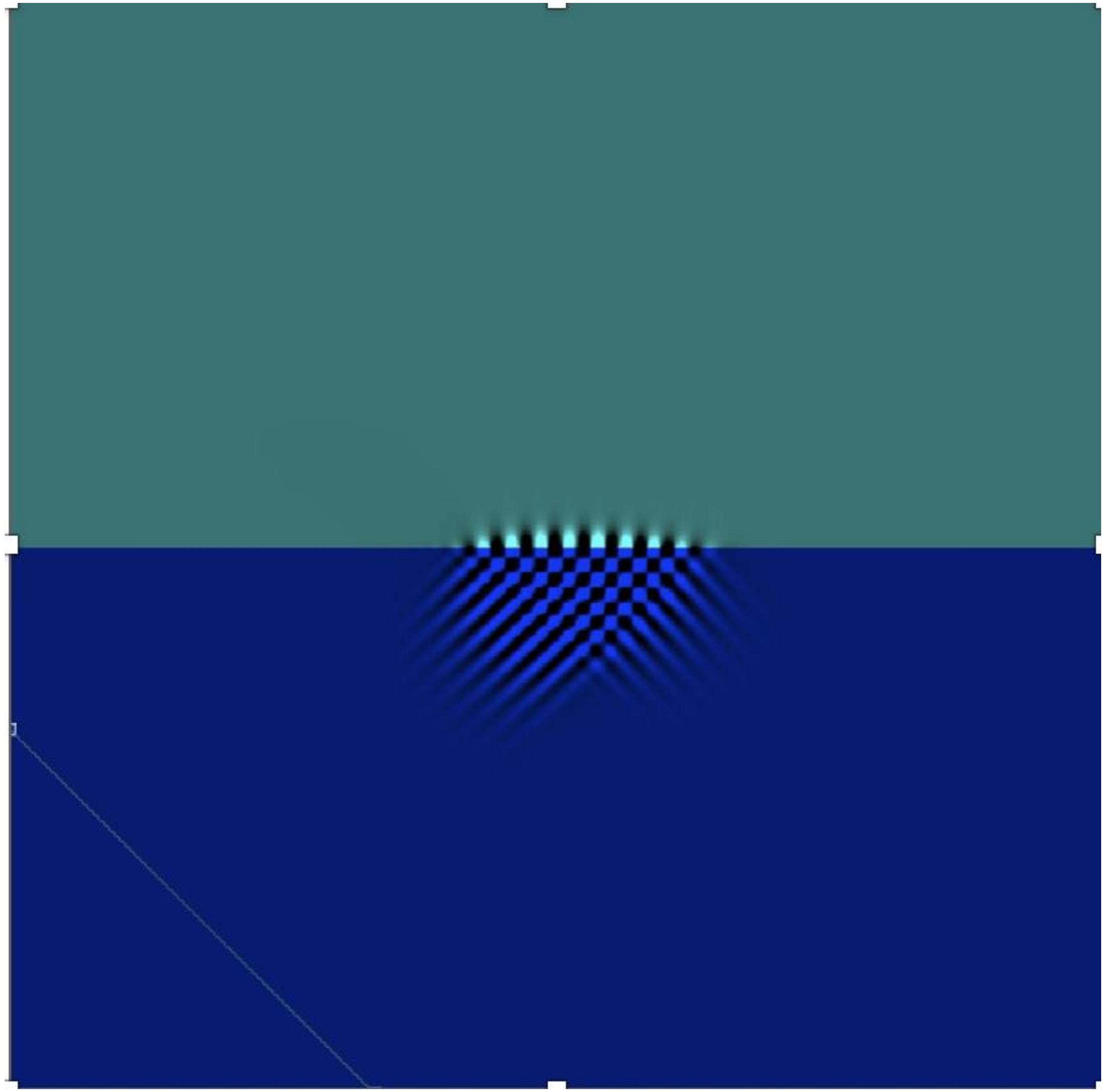
Let's try to visualize the waves

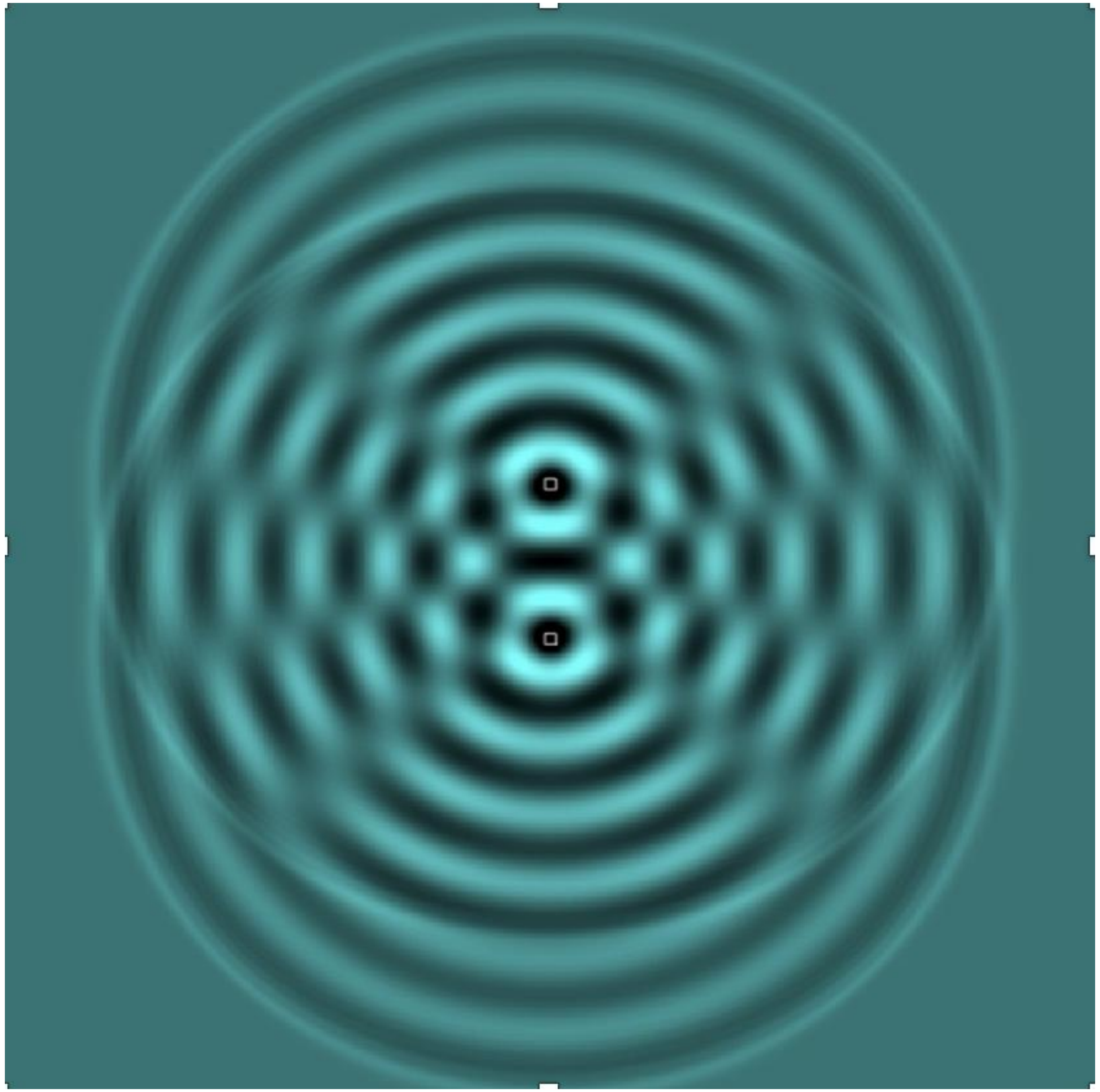




Animations generated using https://www.falstad.com/ripple/

What phenomena do you see in the animations? Choices: reflection, refraction, diffraction, interference





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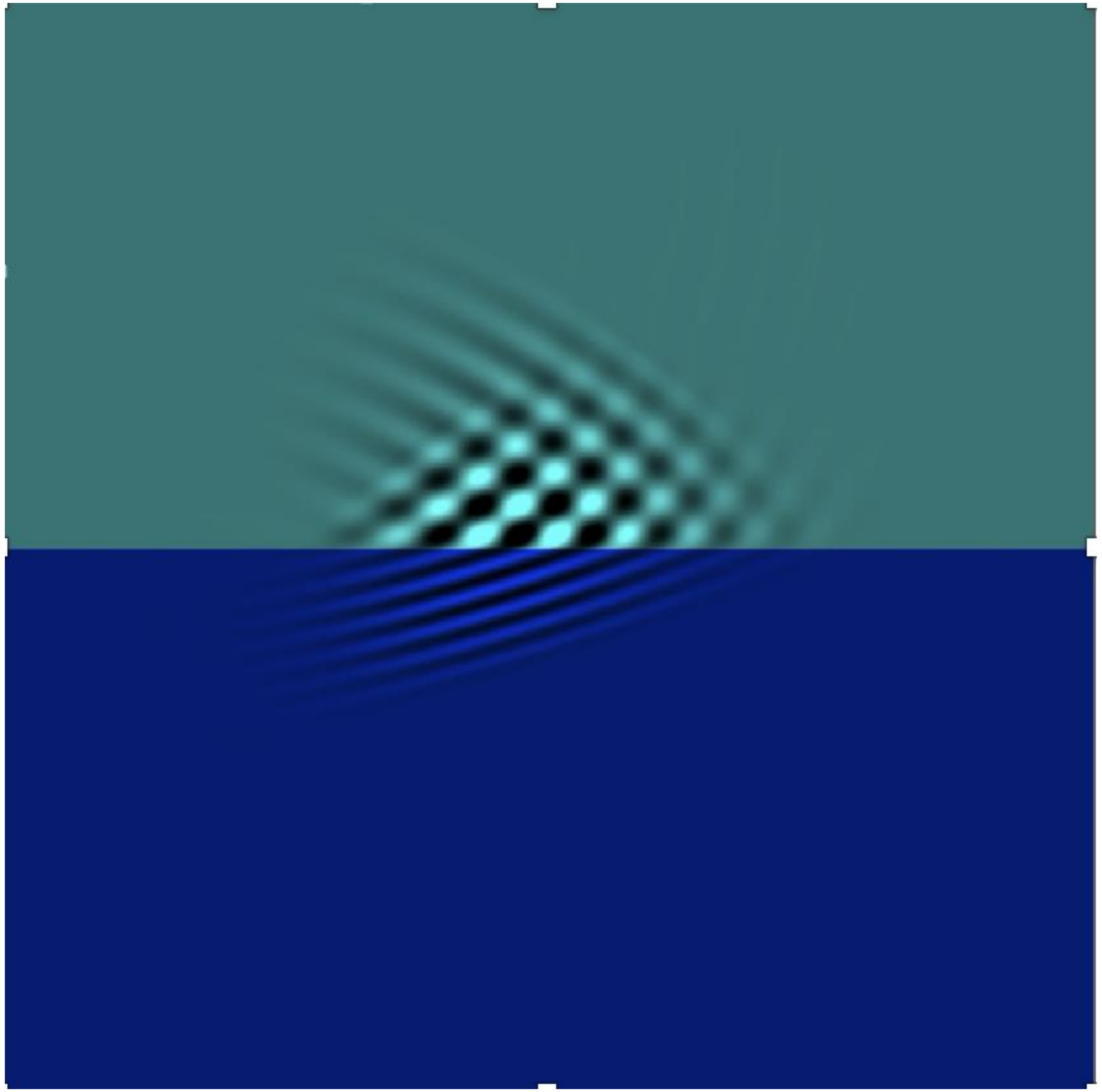

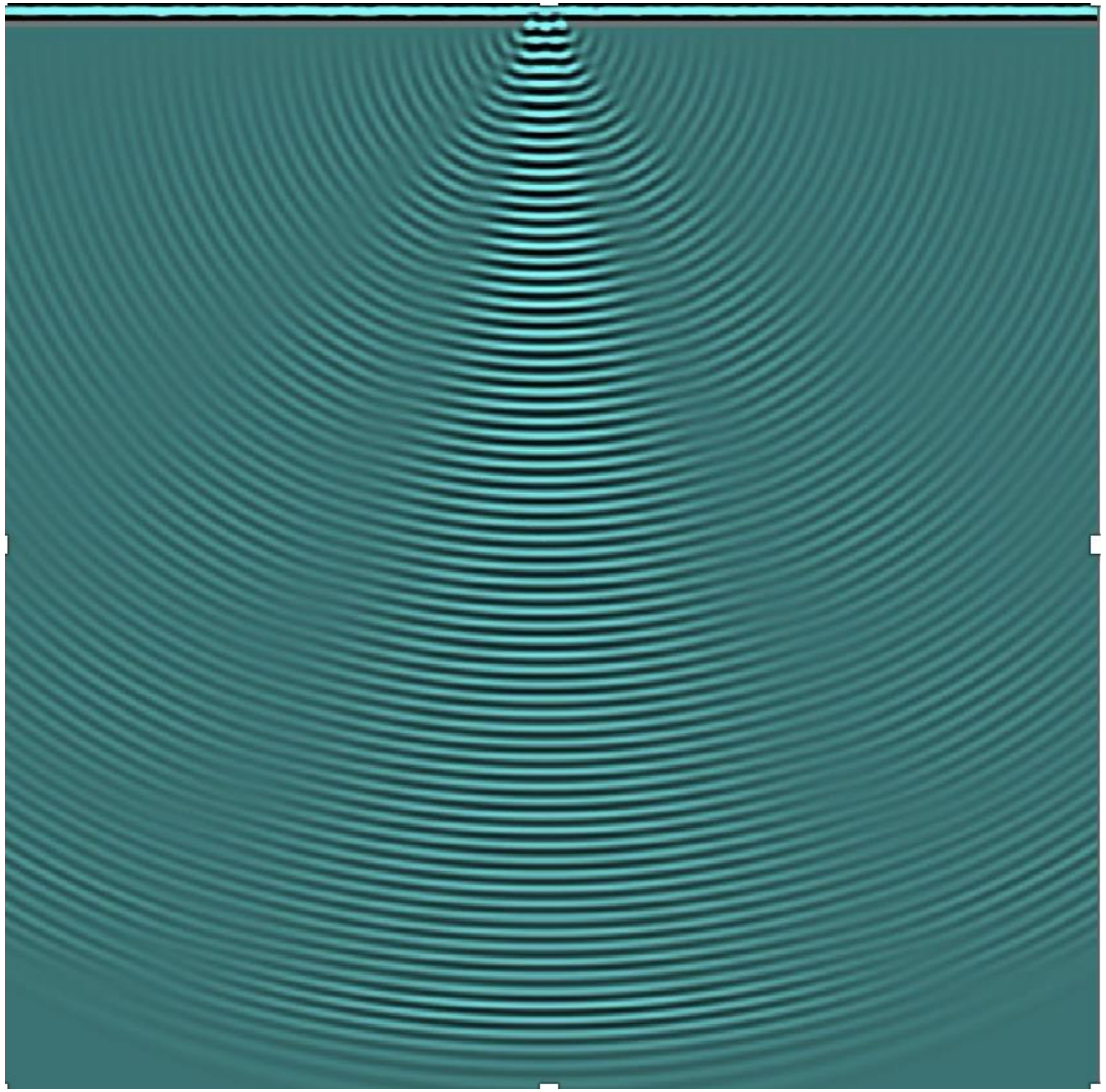

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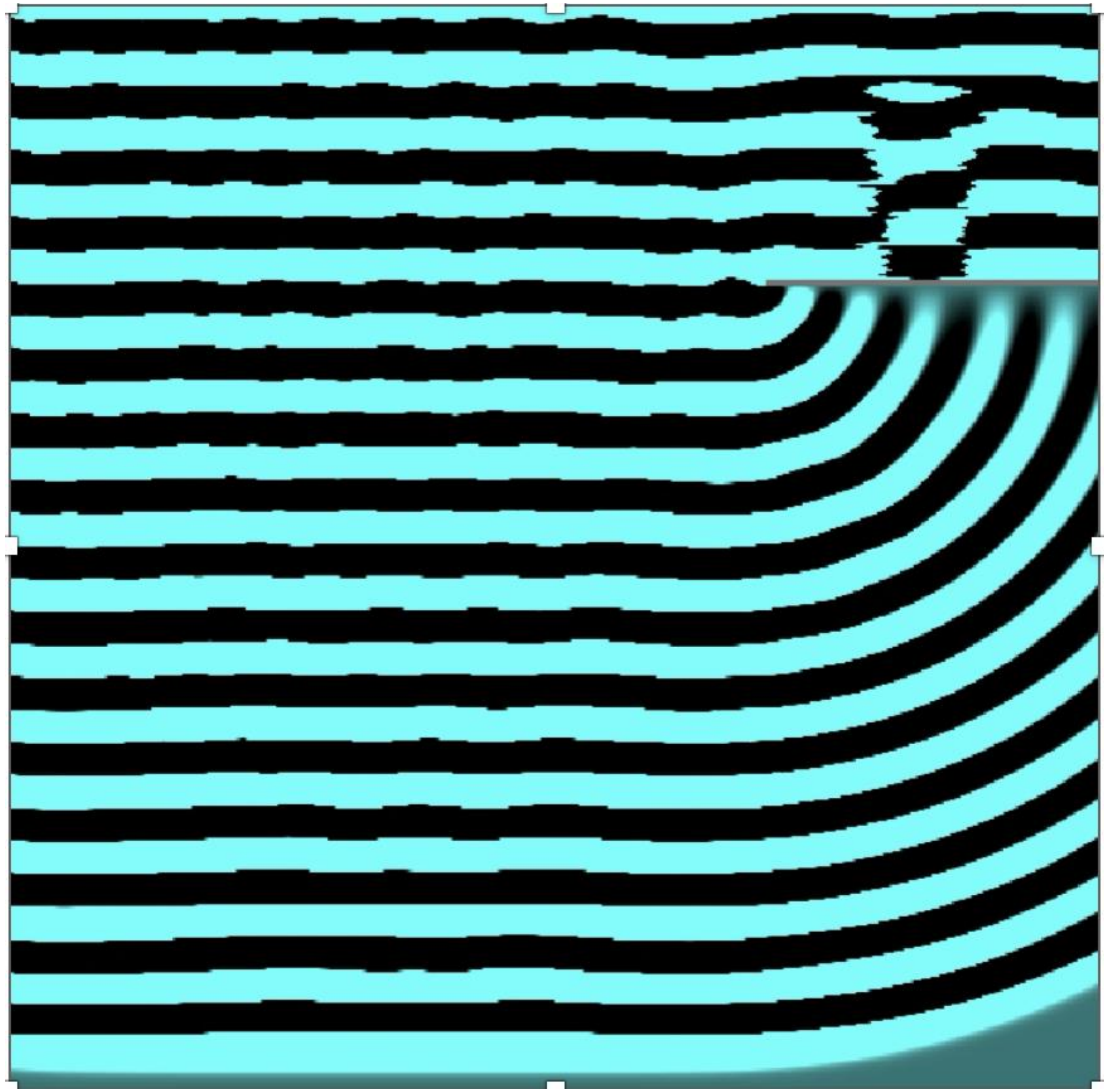
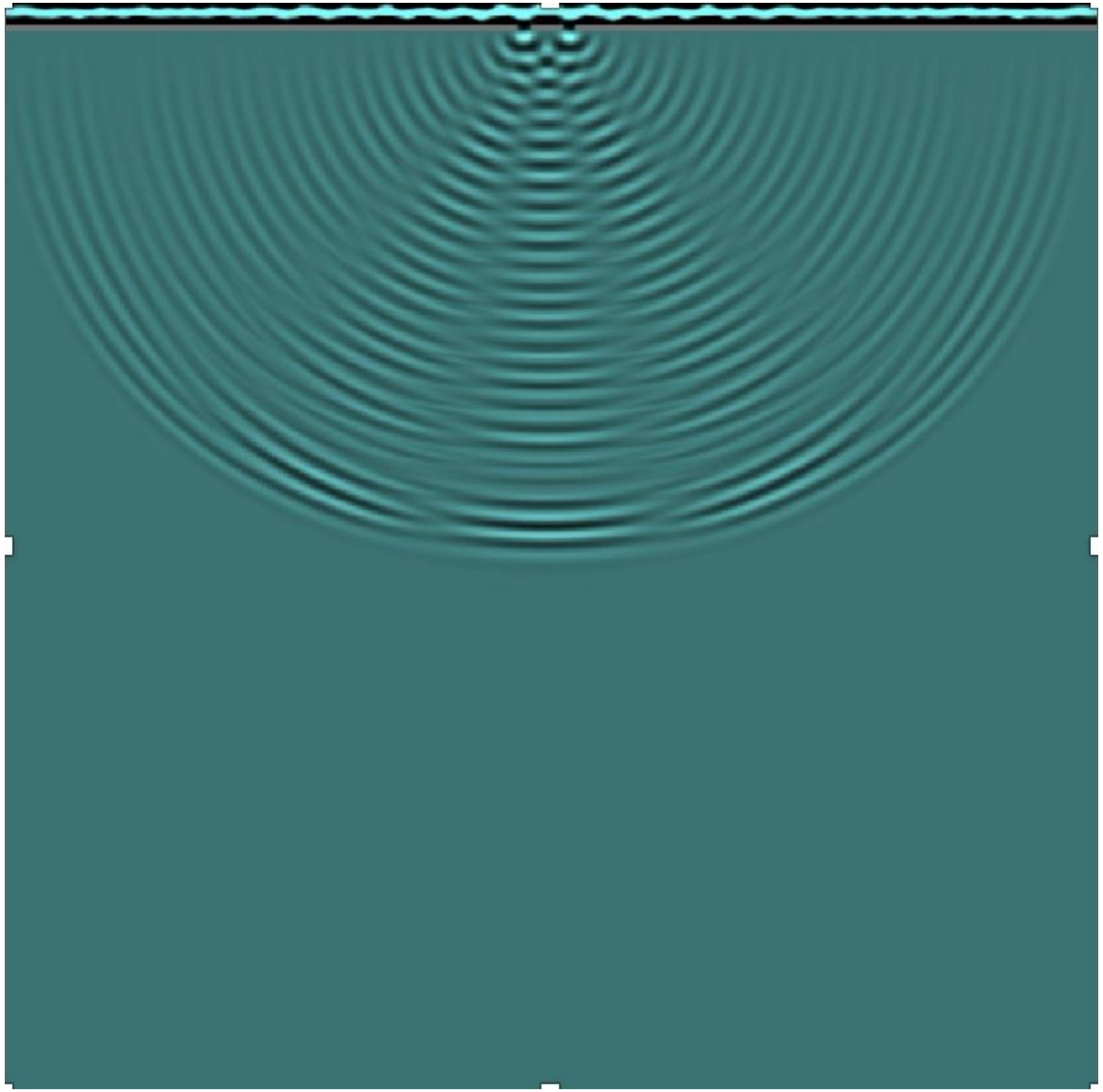
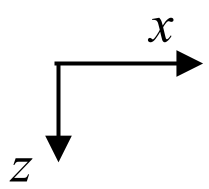
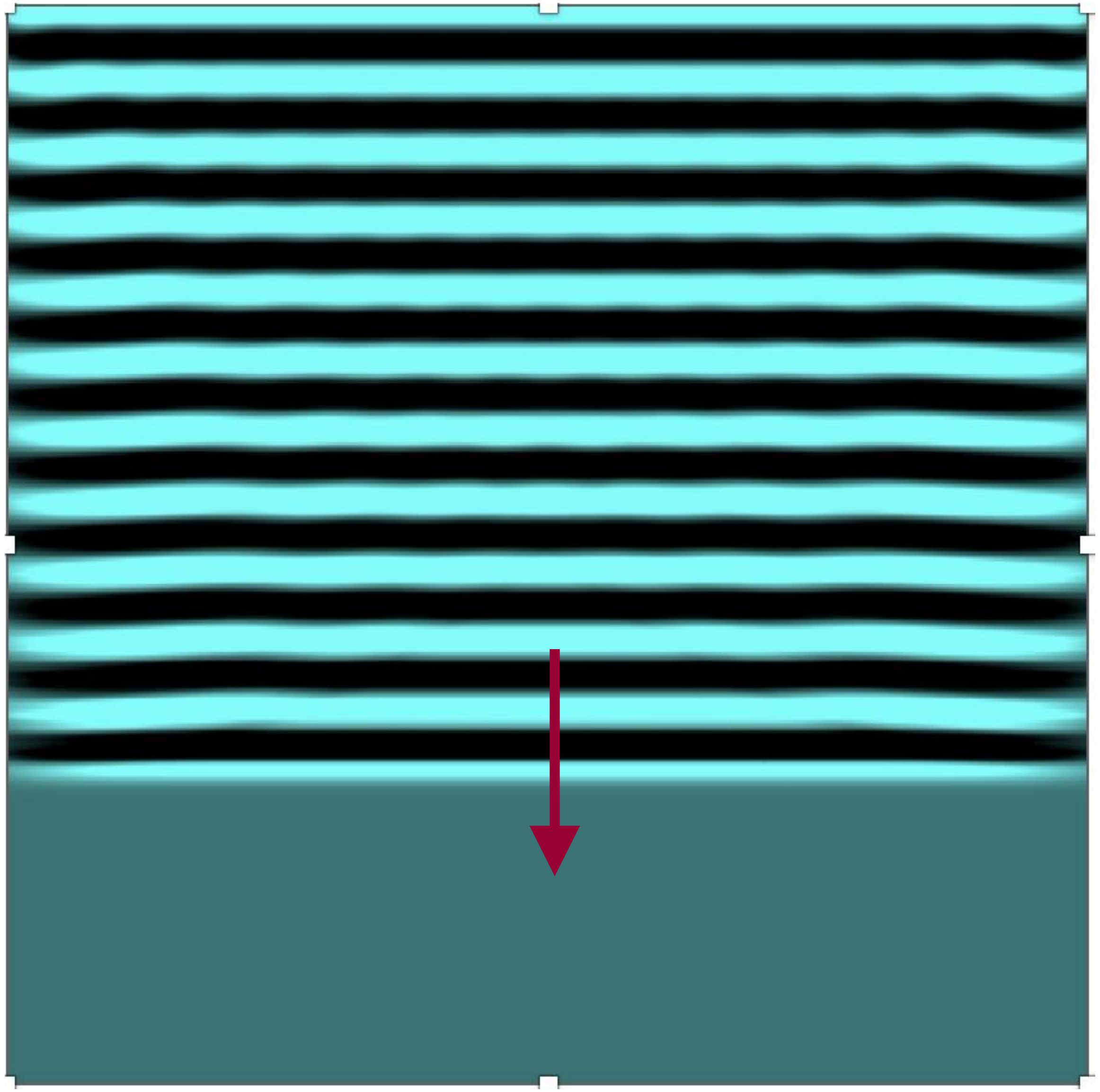




Animations generated using https://www.falstad.com/ripple/

Write a valid expression for the electric field of this wave





$$\vec{k} = (0, 0, k)$$

$$\vec{k} \perp \vec{\mathcal{E}}$$

$$\vec{\mathcal{E}}(z,t) \propto e^{i(kz-\omega t)}\,\hat{y}$$

Animations generated using https://www.falstad.com/ripple/

In-class exercise: use code to animate the electric field and discuss the following

MATLAB code: E_field_t_3d.m

- Work with a partner if you don't have a laptop with you. Provide your answers on Top Hat.
- Explore different values of $\mathcal{E}_{0,x}, \mathcal{E}_{0,y}$, and $\delta = \delta_x - \delta_y$
- What are the conditions for
  - Linear polarization?

    Either $\mathcal{E}_{0,x}$ or $\mathcal{E}_{0,y}$ is zero

    Generally: $\delta = 0$ or π ; $\mathcal{E}_{0,x}$ and $\mathcal{E}_{0,y}$ can be arbitrary
  - Circular polarization?

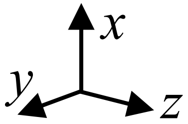
    $\delta = \pi/2$ or $-\pi/2$

    <u>AND</u> $\mathcal{E}_{0,x} = \mathcal{E}_{0,y}$
- What do you call waves that that neither linear nor circular? What do they look like?

  Elliptical

Linear polarization

- Wave propagating along $\hat{z}$:



$$\vec{\mathcal{E}}(z,t) = \vec{\mathcal{E}}_0 e^{i(kz-\omega t)} = \mathcal{E}_x \hat{x} + \mathcal{E}_y \hat{y}$$

where $\mathcal{E}_x = \mathcal{E}_{0,x} e^{i(kz-\omega t+\delta_x)}$

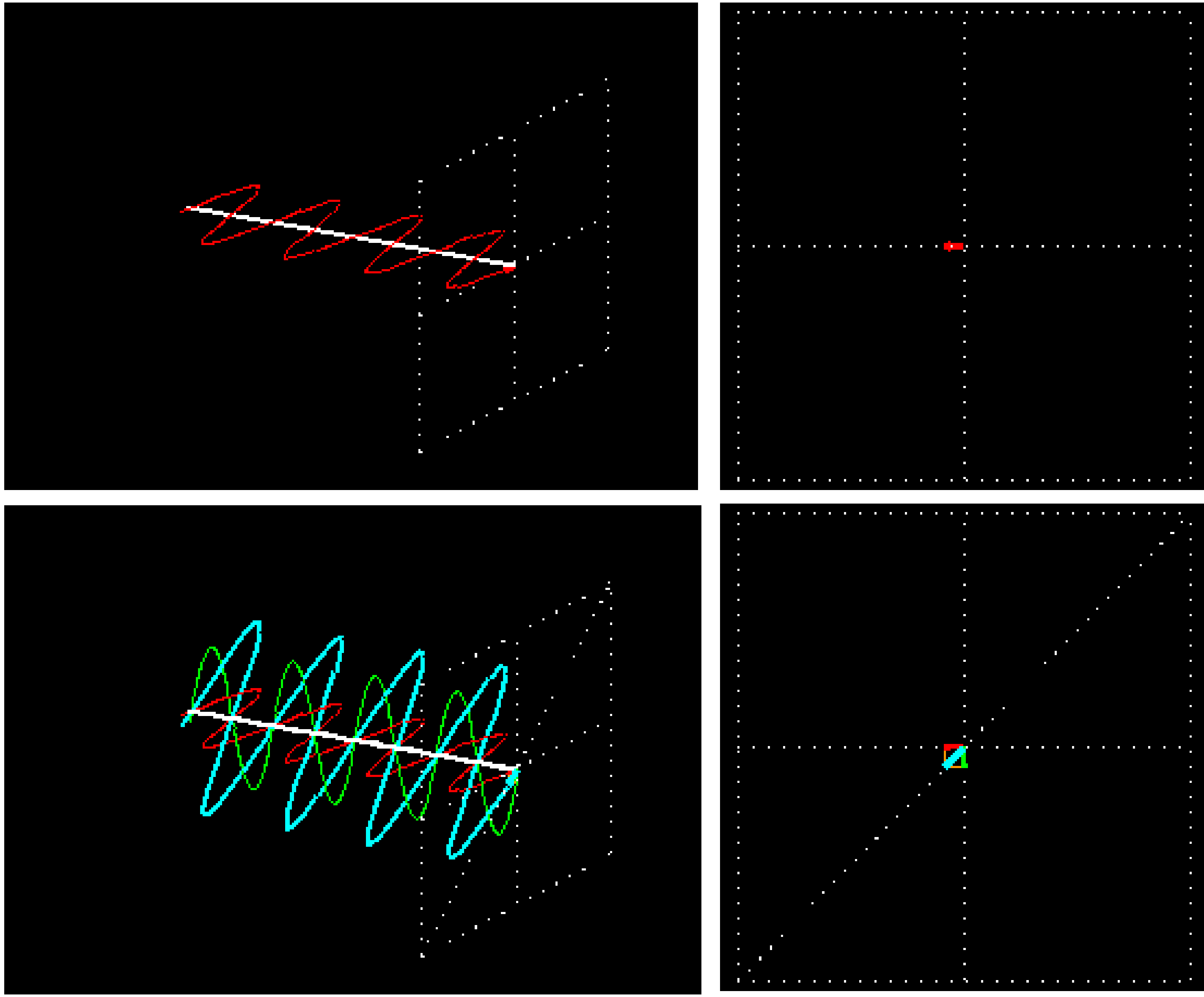
$\mathcal{E}_y = \mathcal{E}_{0,y} e^{i(kz-\omega t+\delta_y)}$

- $\mathcal{E}_x$ and $\mathcal{E}_y$ form a **polarization basis**: you can form any light polarization from the superposition of these two
  - Other polarization bases possible as well

- **Linear polarization** is when
  - $\mathcal{E}_y = 0$ or $\mathcal{E}_x = 0$
  - If $\mathcal{E}_{0,x} \neq 0$ and $\mathcal{E}_{0,y} \neq 0$

    $\delta = \delta_x - \delta_y = 0$ OR $\delta = \delta_x - \delta_y = \pi$

    - ➢ $\mathcal{E}_x$ and $\mathcal{E}_y$ oscillate in phase or exactly out of phase
    - ➢ Precise orientation depends on $\mathcal{E}_{0,x}/\mathcal{E}_{0,y}$



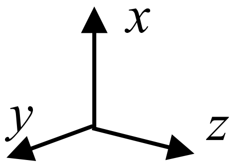
Circular polarization

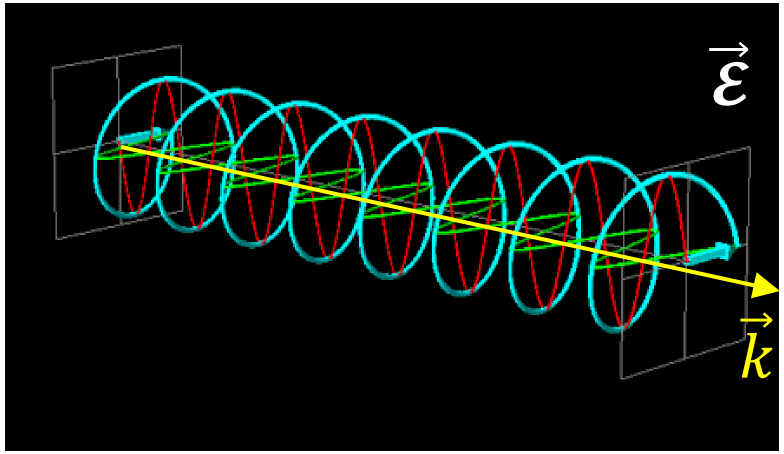
- Wave propagating along $\hat{z}$:

$$\vec{\mathcal{E}}(z,t) = \vec{\mathcal{E}}_0 e^{i(kz-\omega t)} = \mathcal{E}_x \hat{x} + \mathcal{E}_y \hat{y}$$

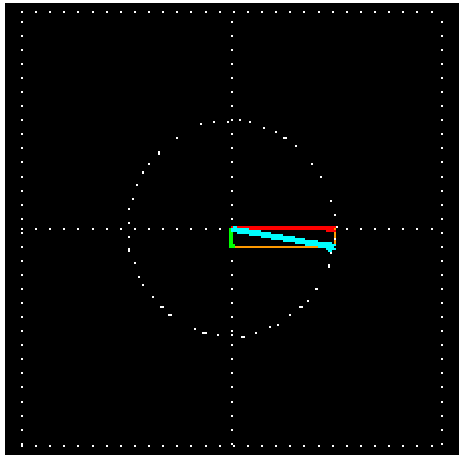
where $\mathcal{E}_x = \mathcal{E}_{0,x} e^{i(kz-\omega t+\delta_x)}$

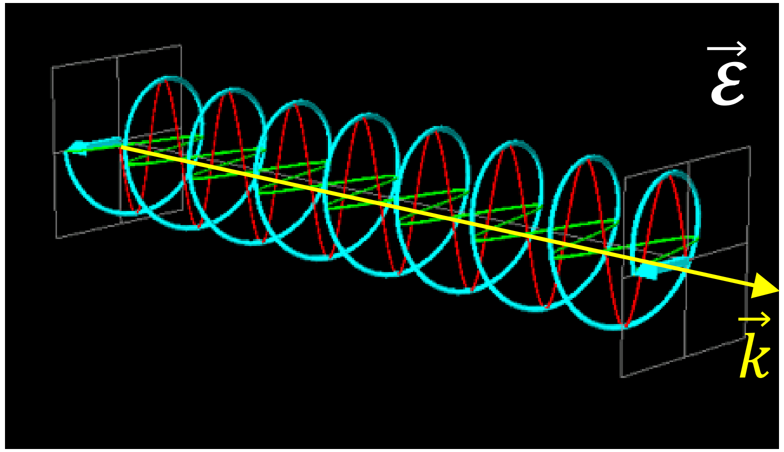
$\mathcal{E}_y = \mathcal{E}_{0,y} e^{i(kz-\omega t+\delta_y)}$





- **Circular polarization** is when
  - $\delta = \delta_x - \delta_y = \pm\pi/2$ AND $\mathcal{E}_{0,x} = \mathcal{E}_{0,y}$





Animations from R. Trebino's lecture notes on Optics

Most general polarization type: elliptical polarization

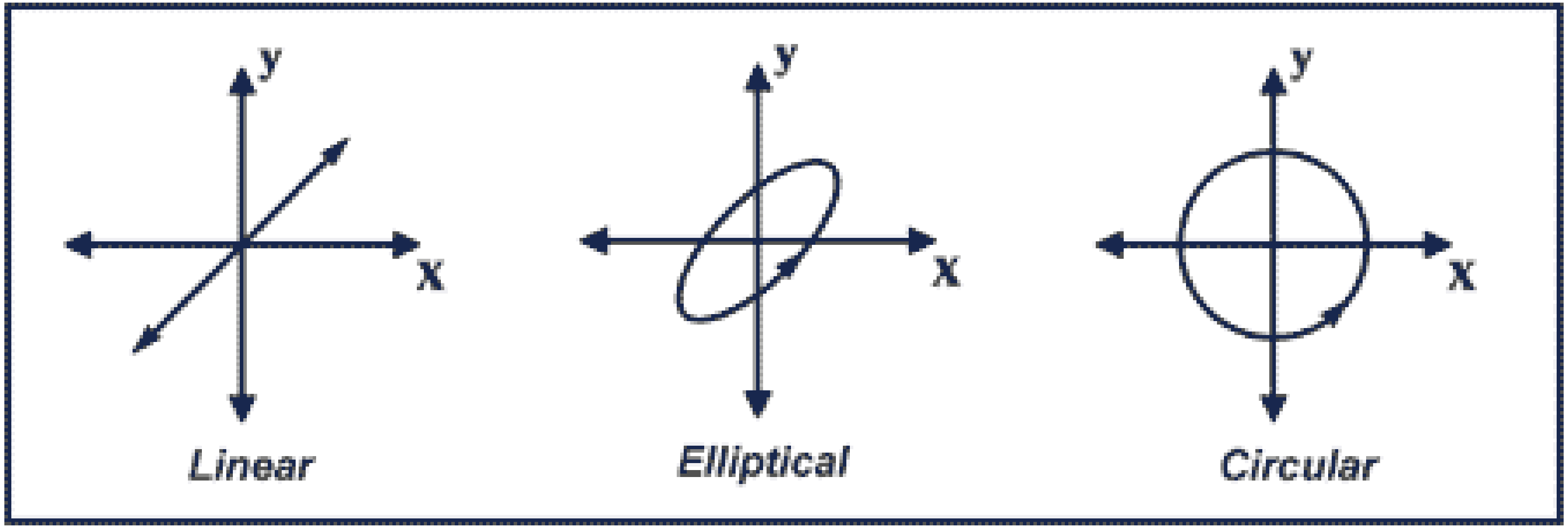
- Wave propagating along $\hat{z}$:

$$\vec{\mathcal{E}}(z,t) = \vec{\mathcal{E}}_0 e^{i(kz-\omega t)} = \mathcal{E}_x \hat{x} + \mathcal{E}_y \hat{y}$$

where $\mathcal{E}_x = \mathcal{E}_{0,x} \mathrm{e}^{i(kz-\omega t+\delta_x)}$

$\mathcal{E}_y = \mathcal{E}_{0,y} \mathrm{e}^{i(kz-\omega t+\delta_y)}$

- Any other combination of $\delta = \delta_x - \delta_y$ and $\mathcal{E}_{0,x}/\mathcal{E}_{0,y}$ give elliptical polarization



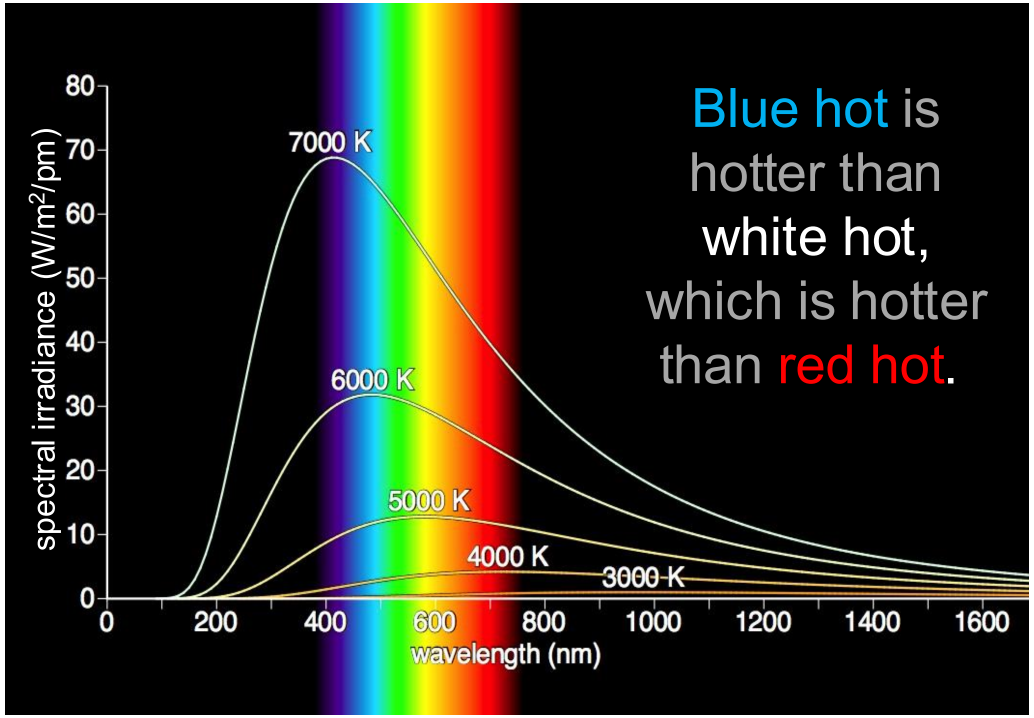
Observations that challenged classical understanding

- **Blackbody radiation**: Blackbody is expected to radiate an infinite amount of UV light, but hot objects are observed to produce a glow that peaks at a wavelength dependent on temperature and decays to zero at short wavelengths.
- **Photoelectric effect**: Light hitting metal surface should take time to build intensity to ionize electrons. Emission of electrons should then be dependent on intensity and not the wavelength of incident light. However, photoelectric effect is instantaneous and dependent only on wavelength.
- **Atomic spectra**: Light produced by atoms should be continuous in wavelength, whereas emissions from different elements are discrete and unique.

Quantization of electromagnetic energy is at the key to explaining all these phenomena.

Blackbody radiation

- When matter is heated, it both absorbs and emits light.
- A blackbody is used to model this observation: it would look black when cool since it absorbs all frequencies
  - Emission is independent of the material and depends only on their temperature
- Experimental observation: the power emitted per unit area per unit wavelength emitted by an object at a given temperature follows a distribution whose maximum shifts to shorter wavelengths with increasing temperature.

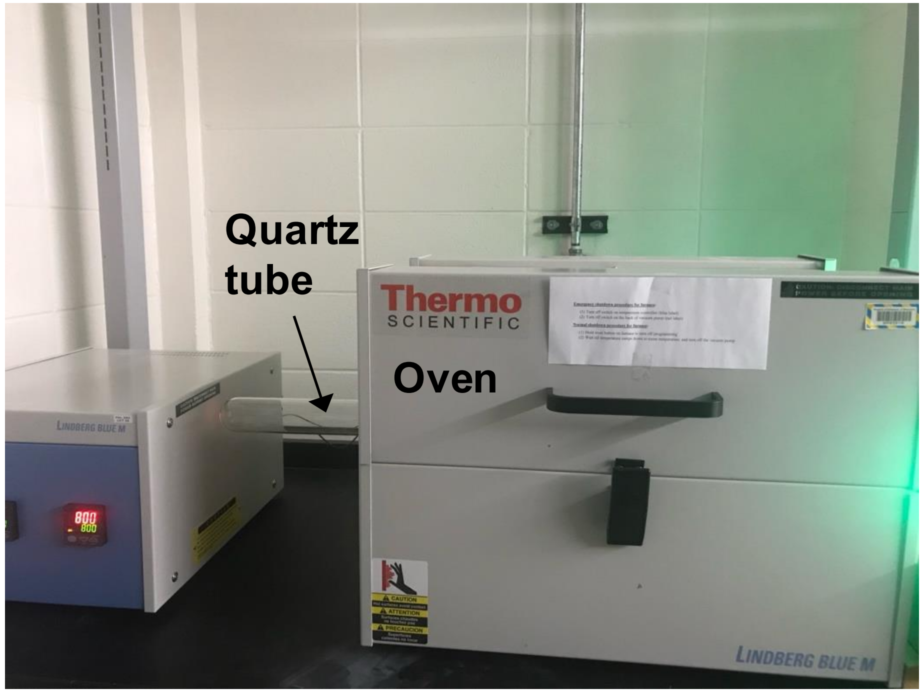


References:
Demtroder Section 3.1.1
R. Trebino's lecture notes on Modern Physics

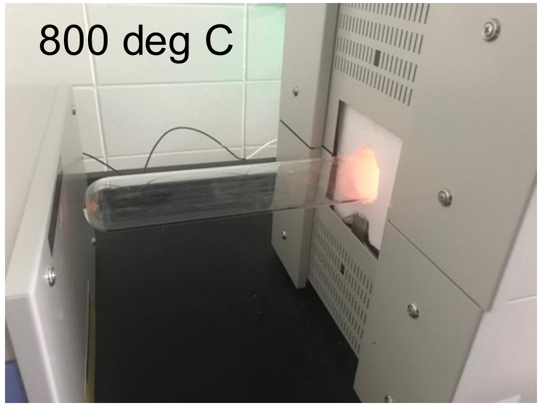
Blackbody radiation

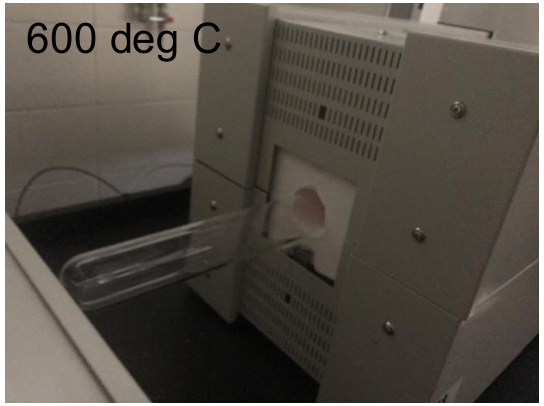
- When matter is heated, it both absorbs and emits light.
- A blackbody is used to model this observation: it would look black when cool since it absorbs all frequencies
  - Emission is independent of the material and depends only on their temperature
- The maximum of the spectrum shifts to shorter wavelengths as the temperature is increased

Example: tube furnace

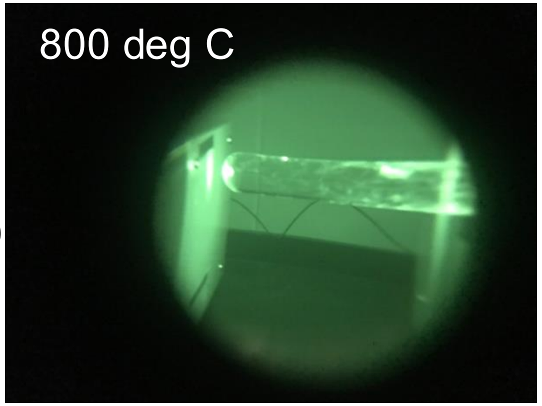


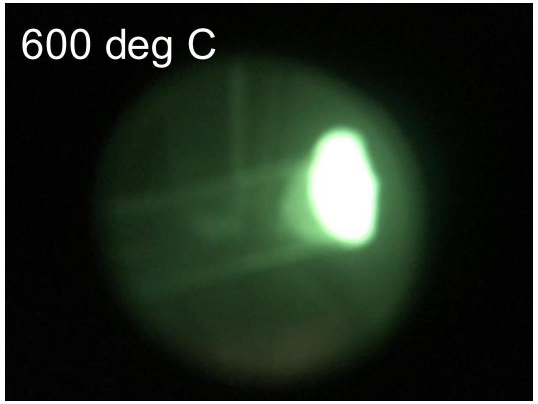
Visible
(400 – 700 nm)



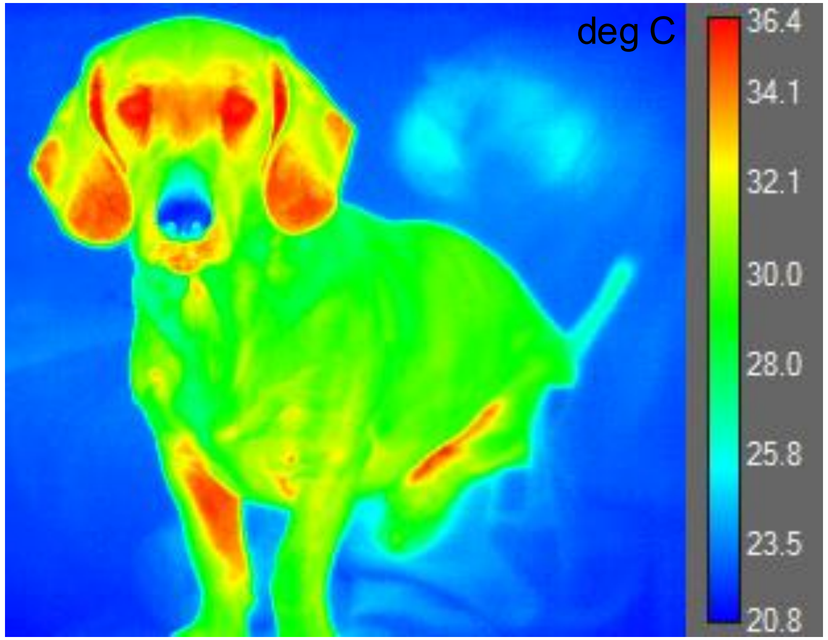


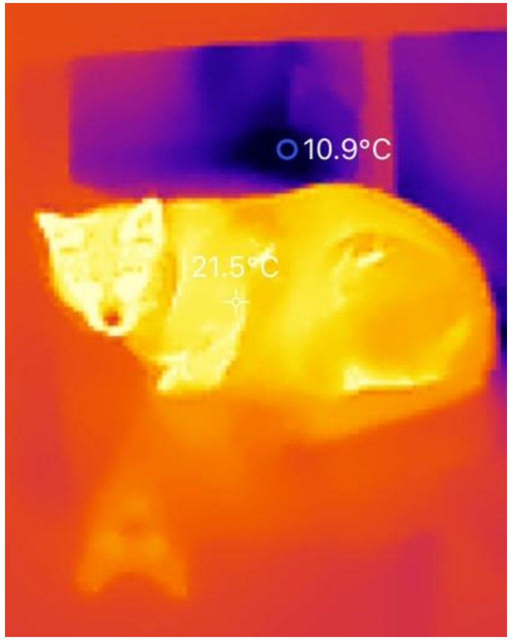
Infrared viewer
(300 – 1500 nm)

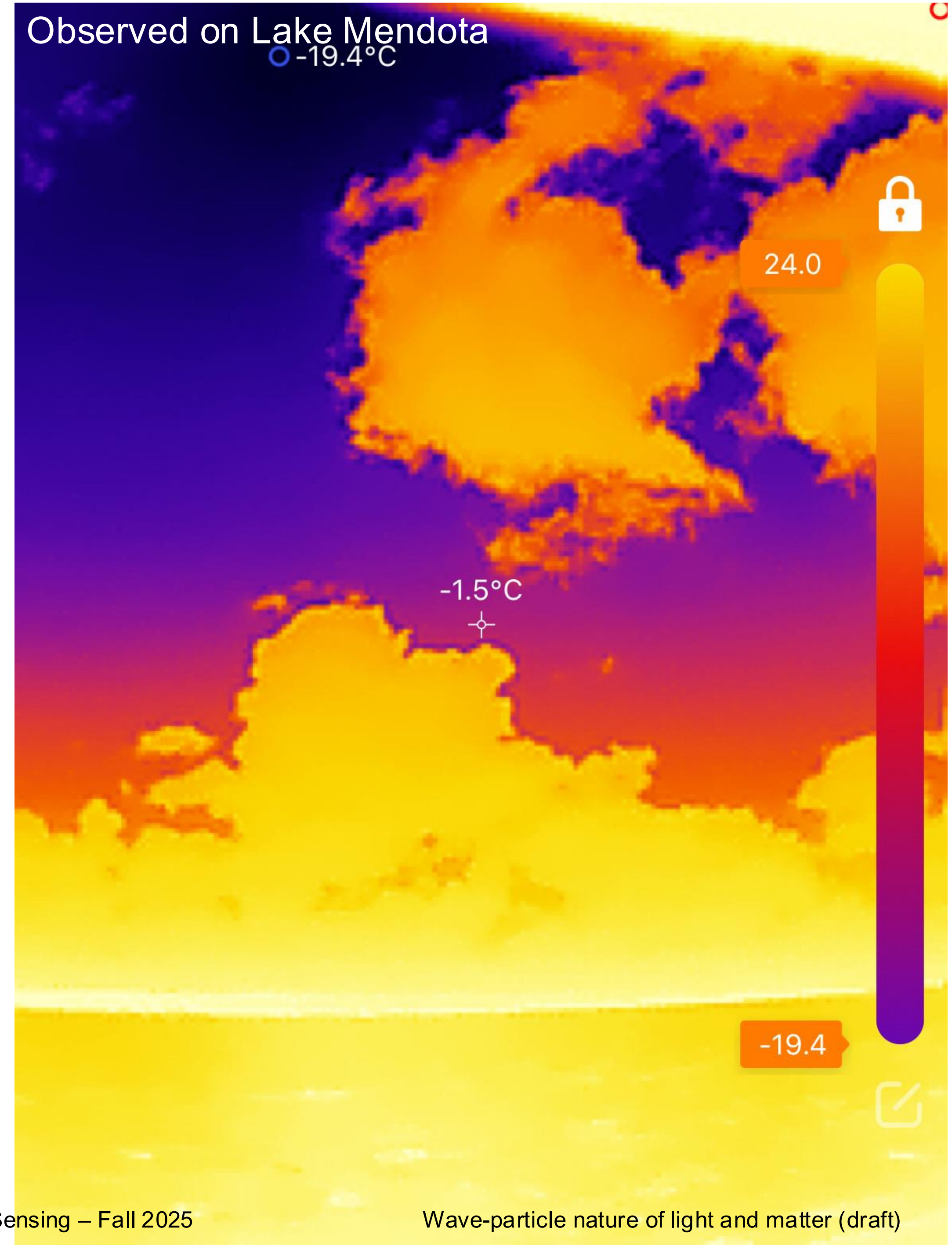




Longwave (~ 9-14 μm) thermal imaging



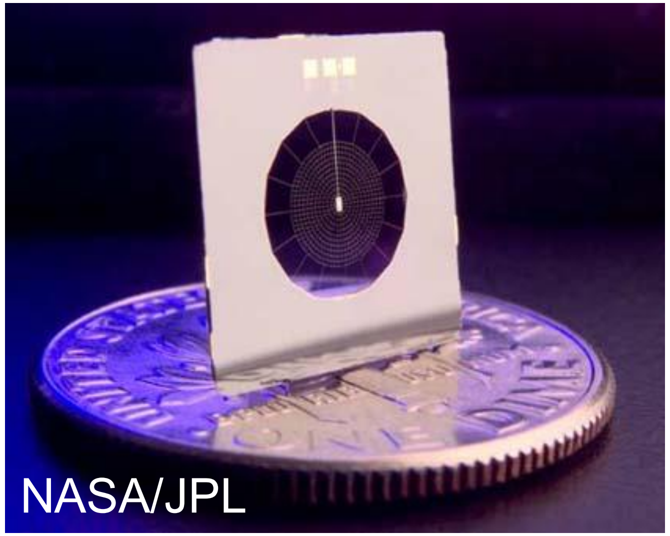


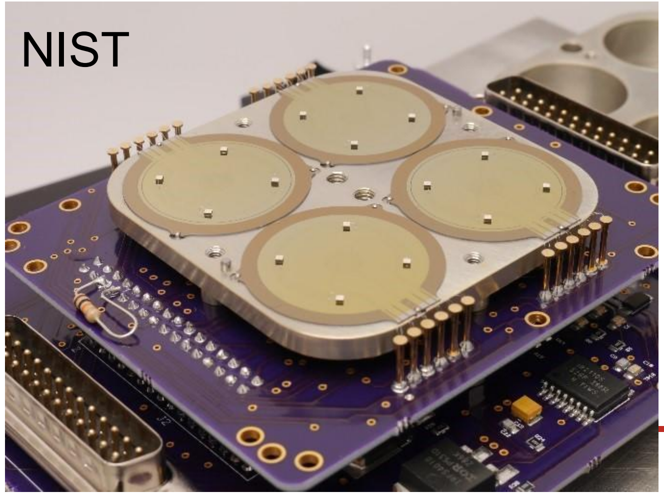


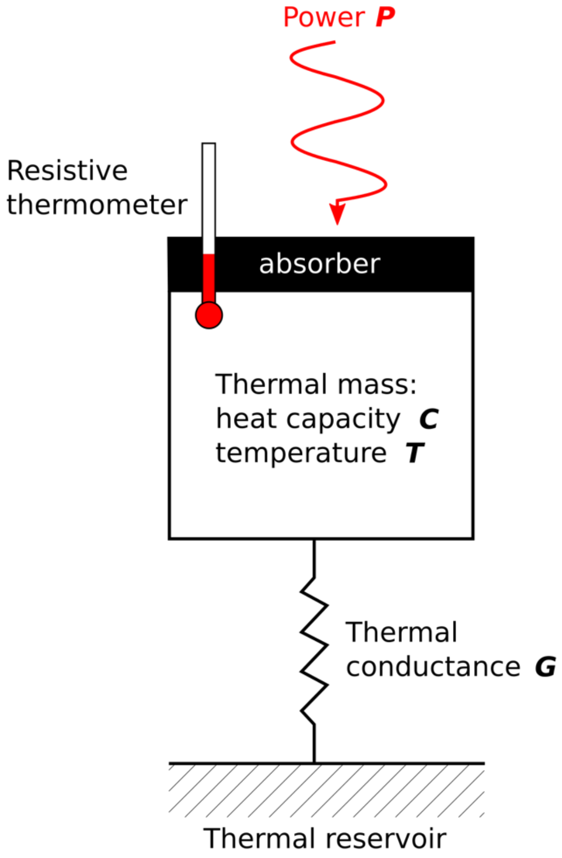
Thermal detector technologies

Classical: microbolometers

- Use materials (silicon, vanadium oxide) that change resistance when heated by infrared radiation

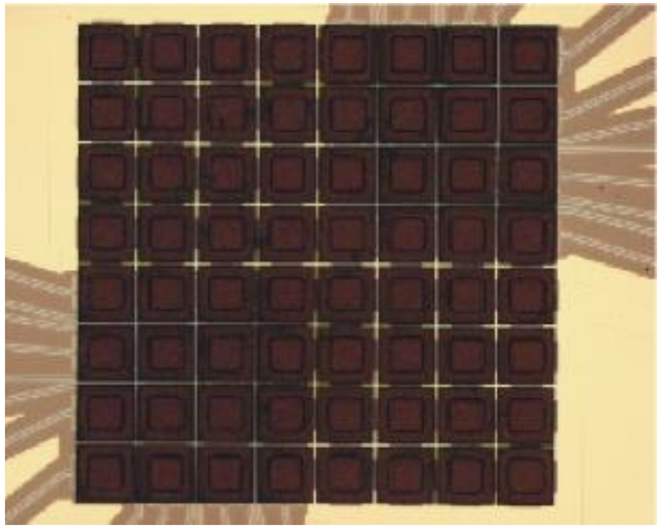


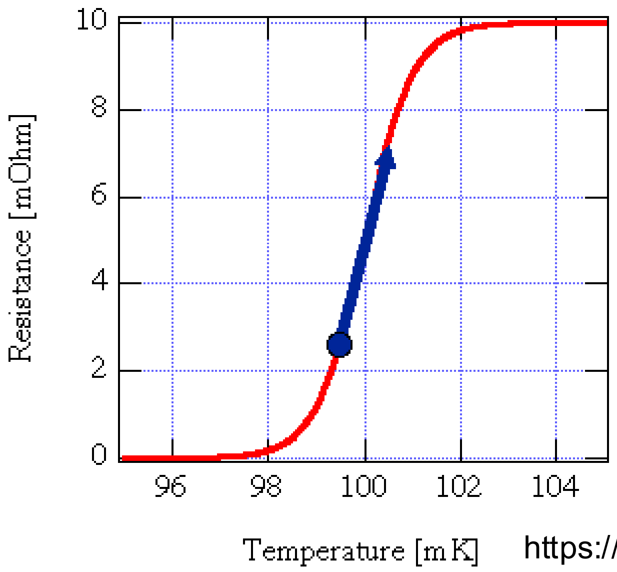




D.F. Santavicca, Wikipedia

Quantum: Transition-edge sensors





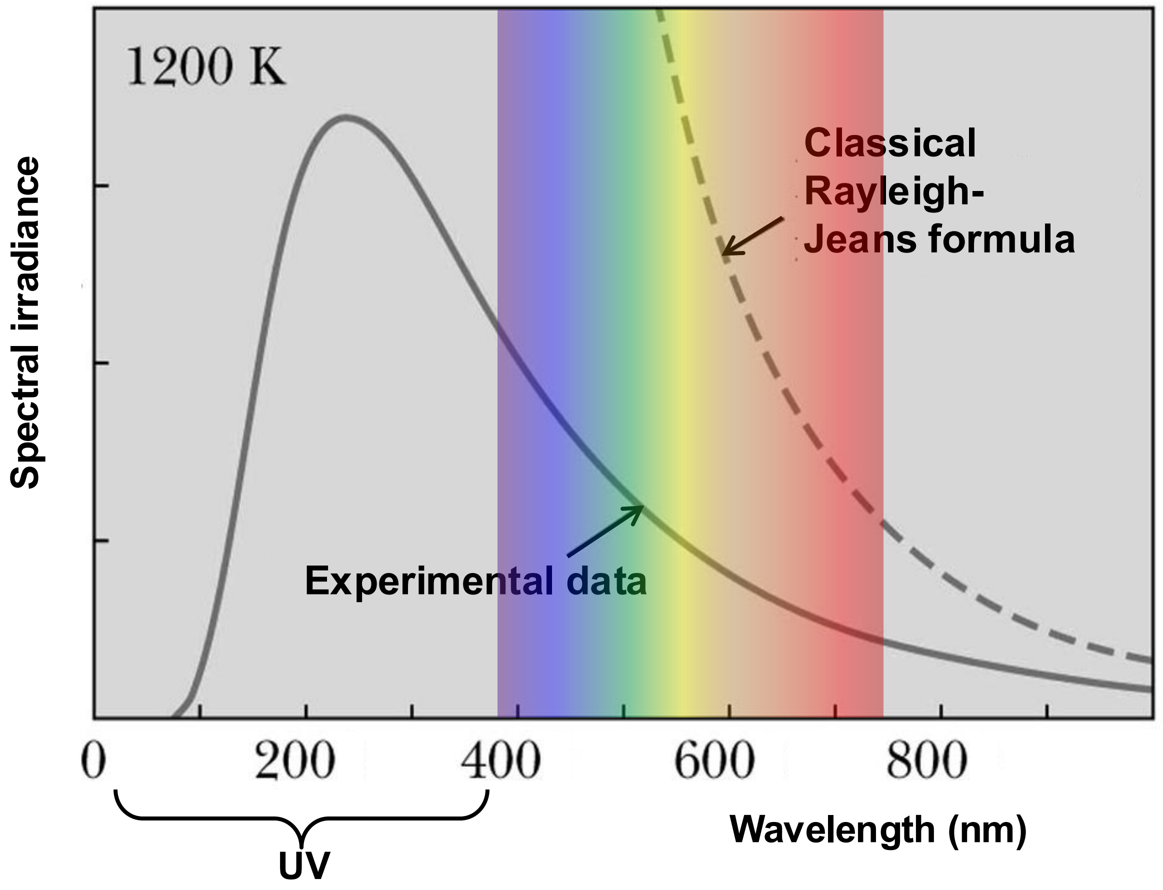
https://web.mit.edu/figueroagroup/ucal/ucal_tes/

- Use superconducting films (such as Molybdenum/Gold) operated near their transition temperatures
- Requires very low temperature
- Provides excellent thermal energy resolution (mK)

Modeling blackbody radiation

Here we will cover the core concepts; please refer to the supplementary notes posted on Canvas for more-detailed derivations

"Ultraviolet catastrophe"



Useful quantities in defining thermal radiation

Intensity (incident power per unit area) per wavelength:

$$\frac{dI}{d\lambda} = c\frac{du}{d\lambda} = \frac{8\pi c}{\lambda^4}\langle E\rangle$$

Spectral irradiance $S(\lambda, T)$ = total power radiated per area per wavelength λ per solid angle

$$S(\lambda, T) = \frac{2c}{\lambda^4}\langle E\rangle$$

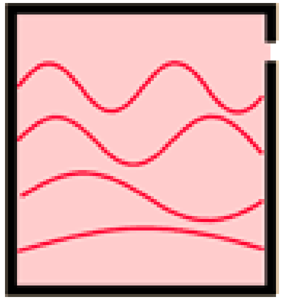
References:
Demtroder Section 3.1.1
R. Trebino's lecture notes on Modern Physics
http://hyperphysics.phy-astr.gsu.edu/hbase/mod6.html#c3

Planck's approach to modeling the blackbody

Here we will cover the core concepts; please refer to the supplementary notes posted on Canvas for more-detailed derivations

Planck's law of radiation:

- **Quantized cavity modes** $E_n = nhv$



E is the energy of the cavity mode
n is an integer
f is the frequency
h is a constant

$$\boxed{h = 6.6261 \times 10^{-34}\ \mathrm{J{\cdot}s}}$$

	#Modes per unit frequency per unit volume	Probability of occupying modes	Average energy per mode
CLASSICAL	$\frac{8\pi\nu^2}{c^3}$	Equal for all modes	kT
QUANTUM	$\frac{8\pi\nu^2}{c^3}$	Quantized modes: require hν energy to excite upper modes, less probable	$\frac{h\nu}{e^{\frac{h\nu}{kT}} - 1}$

Intensity per unit wavelength:

$$\frac{dI}{d\lambda} = \frac{8\pi hc^2}{\lambda^5}\frac{1}{e^{hc/(\lambda k_B T)}-1}$$

Spectral irradiance:

$$\frac{dI}{d\lambda} = \frac{2hc^2}{\lambda^5}\frac{1}{e^{hc/(\lambda k_B T)}-1}$$

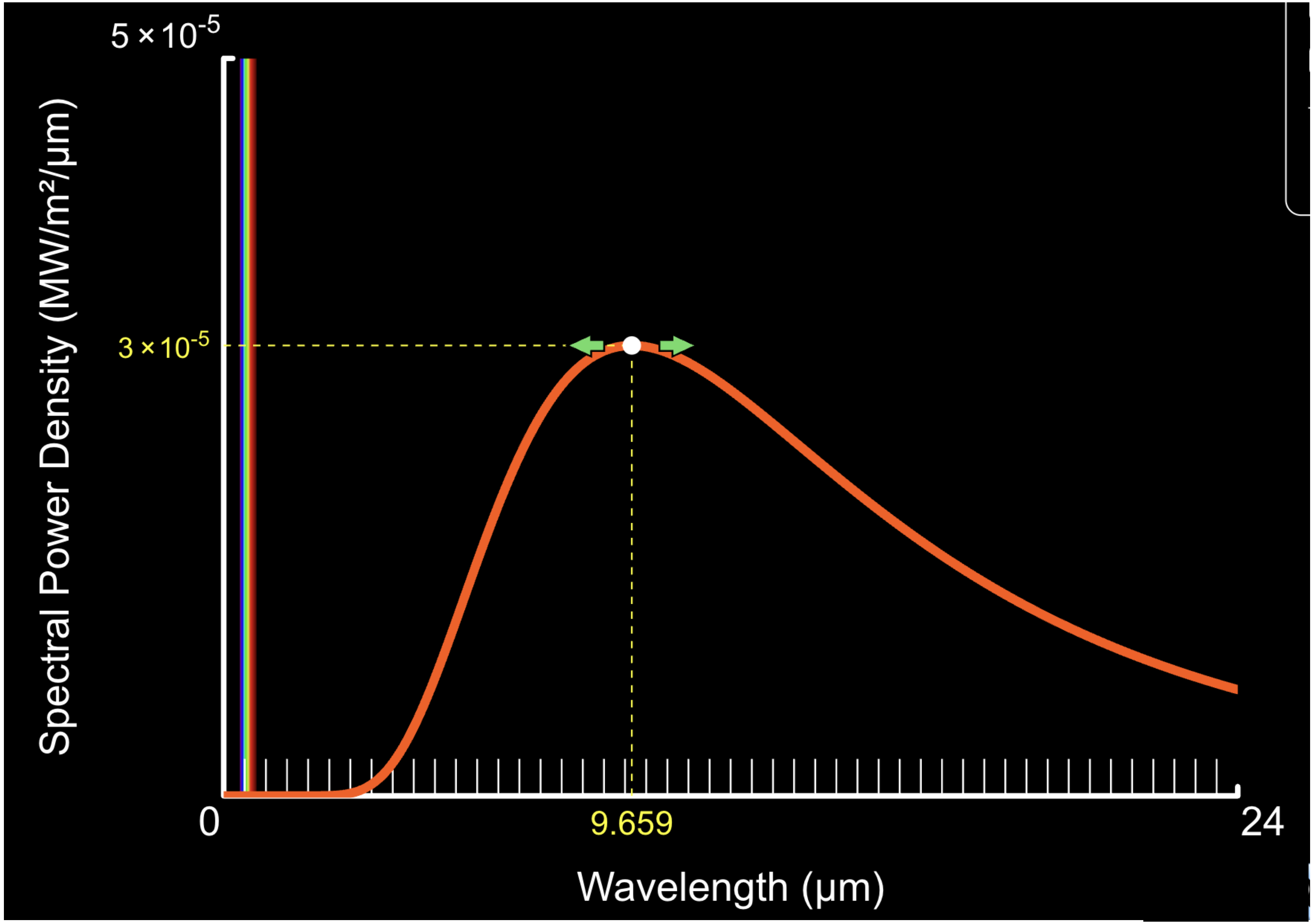
Energy density in ν:

$$u = \frac{8\pi h\nu^3}{c^3}\frac{1}{e^{h\nu/(k_B T)}-1}$$

https://phet.colorado.edu/sims/html/blackbody-spectrum/latest/blackbody-spectrum_all.html

Reference: http://hyperphysics.phy-astr.gsu.edu/hbase/mod6.html#c3

Room-temperature (300 K) blackbody spectrum



What is the emitted power per area (Stefan-Boltzmann law)?

- Energy density $u(\nu, T) = \frac{8\pi h\nu^3}{c^3}\frac{1}{e^{h\nu/(k_B T)}-1}$
- Emitted power per unit area

$$\frac{P}{A} = c \cdot \int_0^\infty u(\nu, T)\ d\nu$$

$$\frac{P}{A} = \frac{c}{2}\int_0^\infty \frac{8\pi h\nu^3}{c^3}\frac{1}{e^{h\nu/(k_B T)} - 1}\ d\nu$$

$$\frac{P}{A} = \sigma T^4$$

$$\sigma \approx 5.67 \times 10^{-8}\ Wm^{-2}K^{-4}$$

- What is the power generated by thermal noise in a wire (what is it called)? Should the results agree with Stefan-Boltzmann law?

$\langle E \rangle = \frac{h\nu}{e^{h\nu/(k_B T)}-1} \approx k_B T$ in the limit of low frequency (e.g., RF, microwaves)

R

Pos + neg freq components

Polarization

Power per frequency band ($\Delta\nu$):

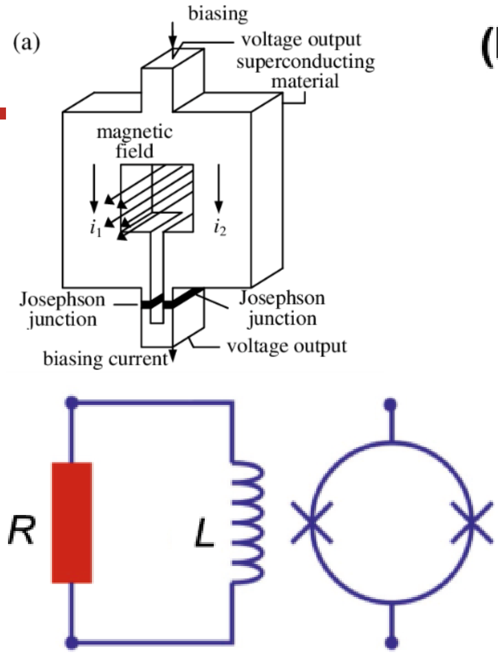
$$\frac{dP}{d(\Delta\nu)} = k_B T \longrightarrow P = k_B T(\Delta\nu) = \frac{V_{rms}^2}{R} \longrightarrow V_{rms}^2 = k_B T R(\Delta\nu) \times 2 \times 2$$

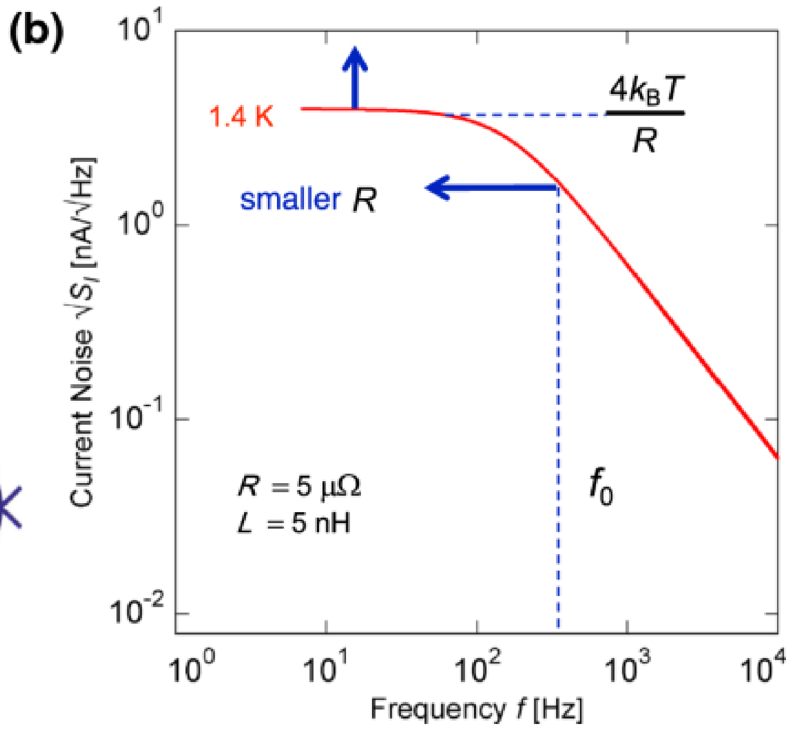
Johnson noise thermometry

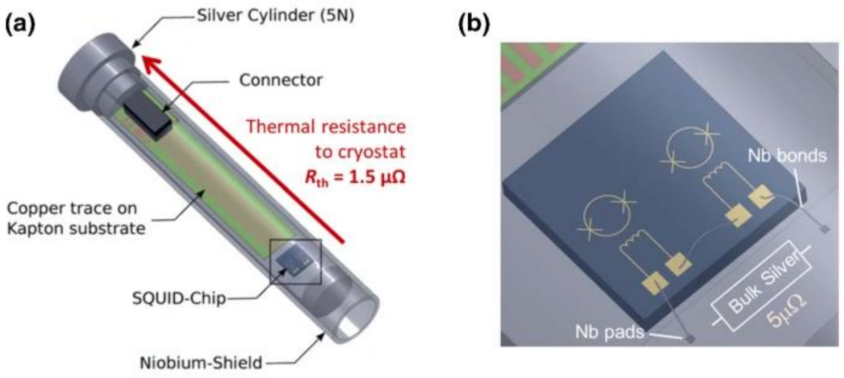
- Absolute temperature can be determined using

$$T = \frac{V_{rms}^2}{4k_B R B}$$

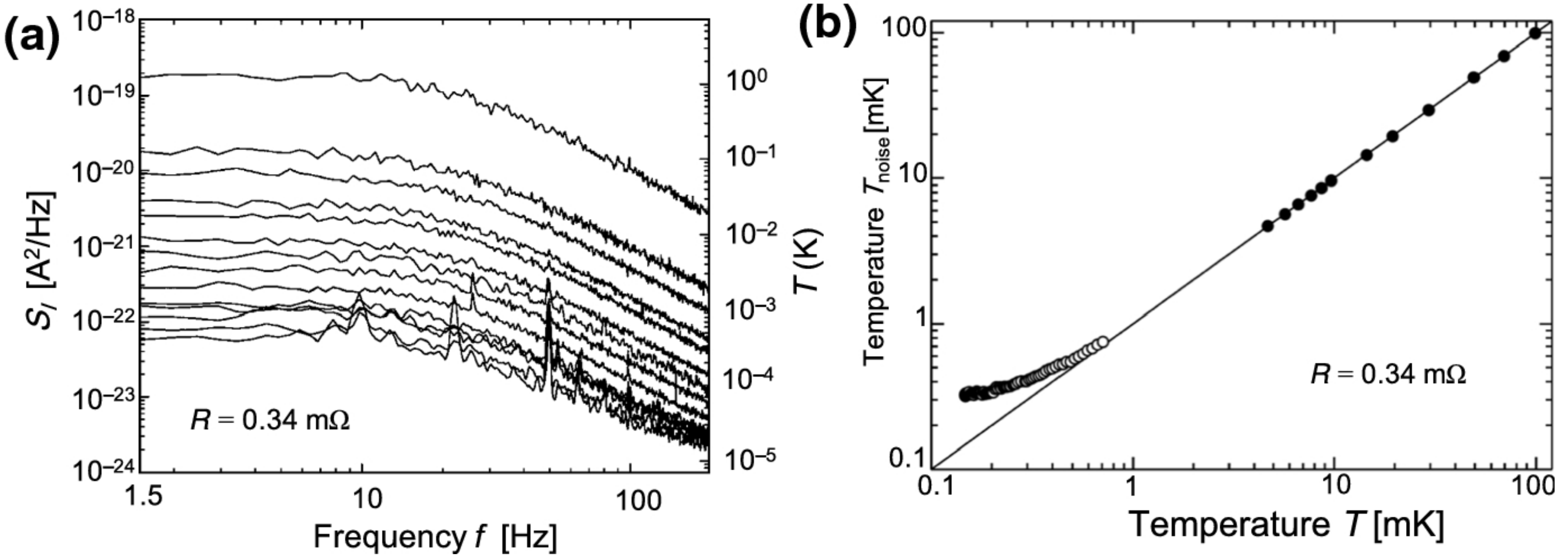
- Calibration-free: Requires knowledge of Boltzmann constant (k_B) and resistance of a conductor (R)
- Typically uses Superconducting Quantum Interference Devices (SQUIDs)







Fleischmann, Andreas, A. Reiser, and C. Enss. "Noise thermometry for ultralow temperatures." *Journal of Low Temperature Physics* 201 (2020): 803-824.

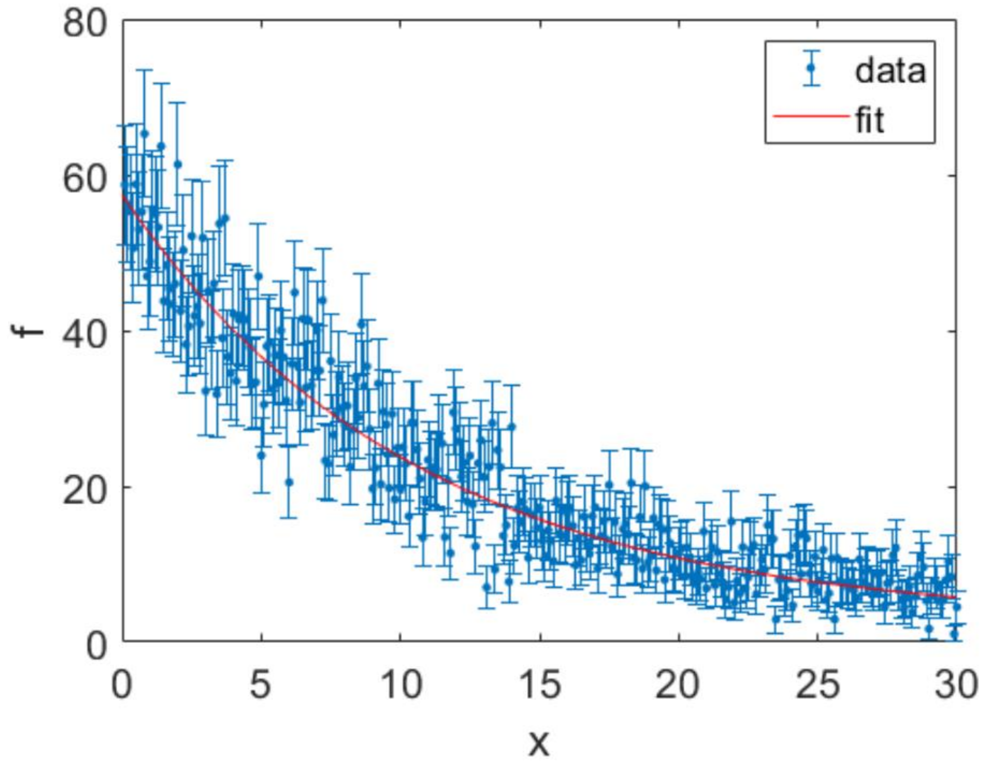


Using numerical tools for solving equations and data fitting (Homework #1)

- MATLAB examples, but you are welcome to use what you are most comfortable with.

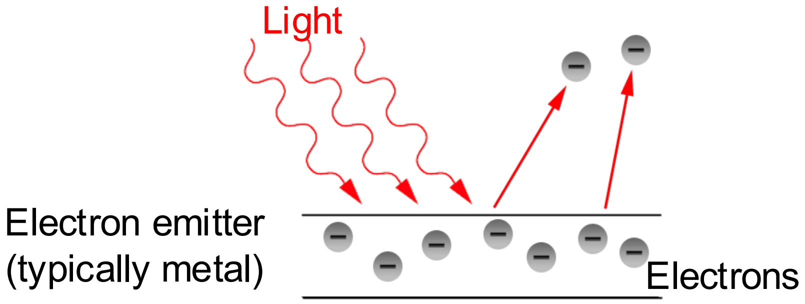
- To find analytical or numerical solutions for an equation, use the functions **syms** to construct symbolic variables; **solve** to find symbolic solutions; **vpasolve** to find the numerical solutions.

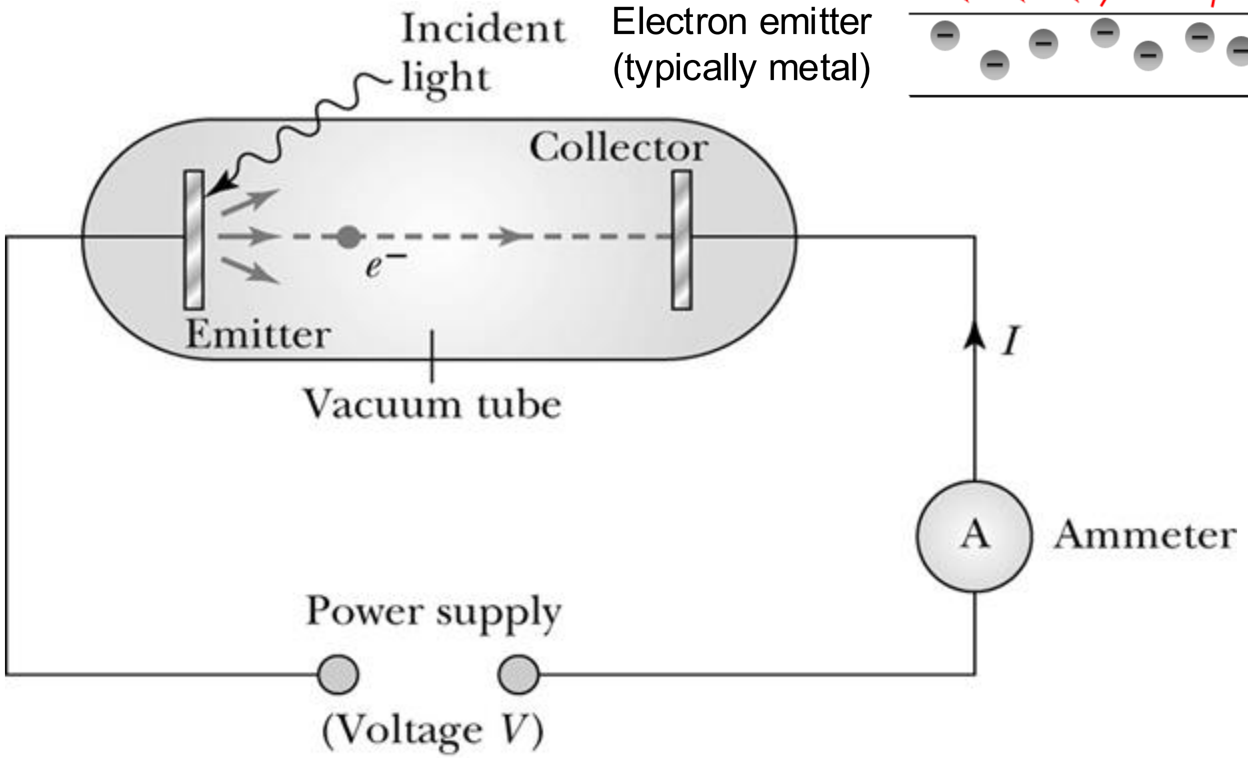
- Example script for curve fitting and error estimates
  - See script and notes posted on Canvas



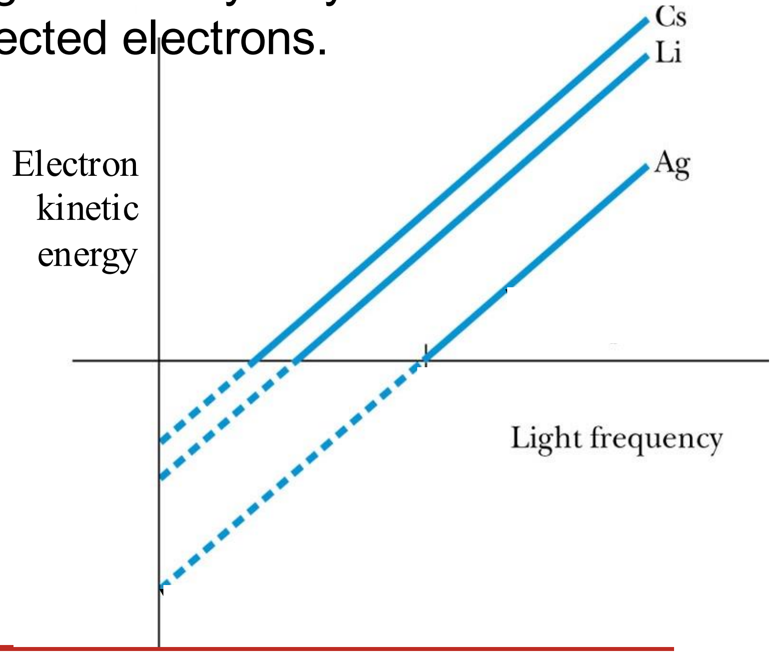
Photoelectric effect

- Incident light on the material transfer energy to the electrons, allowing them to escape.
- Experimental setup:





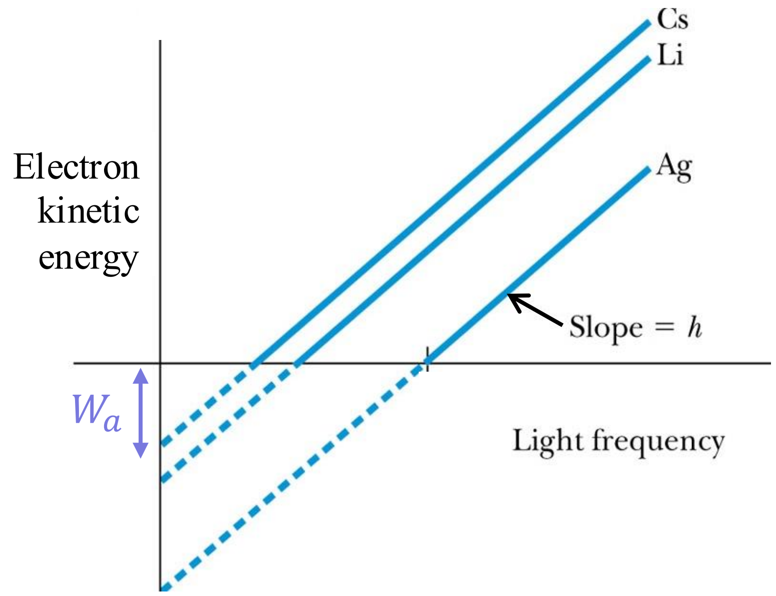
- Observations:
  - Kinetic energy of electrons depends on frequency of light.
  - For each material, there is a threshold frequency below which electrons are not emitted.
    - Electrons immediately ejected above threshold.
  - Light intensity only affects number of ejected electrons.



References:
Demtroder 3.1.6
R. Trebino's lecture notes on Modern Physics
http://hyperphysics.phy-astr.gsu.edu/hbase/mod6.html#c3

Explanation for photoelectric effect: quantization of light field (photons)

- Einstein: each absorbed photon transfers its quantized energy $h\nu$ to an electron inside the emitter, which is bound by attractive forces and requires a minimum energy W_a to leave the emitter. The maximum kinetic energy of the photo-electron:



$$E_{kin}^{max} = h\nu - W_a \qquad h = 6.6261 \times 10^{-34}\ \mathrm{J\cdot s}$$

- W_a is known as the work-function and is a function of the emitter material

Element	Work Function (eV)
Aluminum	4.08
Carbon	4.81
Copper	4.7
Gold	5.1
Platinum	6.35
Sodium	~2.3
Zinc	4.3

What is the wavelength of light needed to eject electrons from aluminum? Use: $1\ eV = 1.6 \times 10^{-19}\ J$

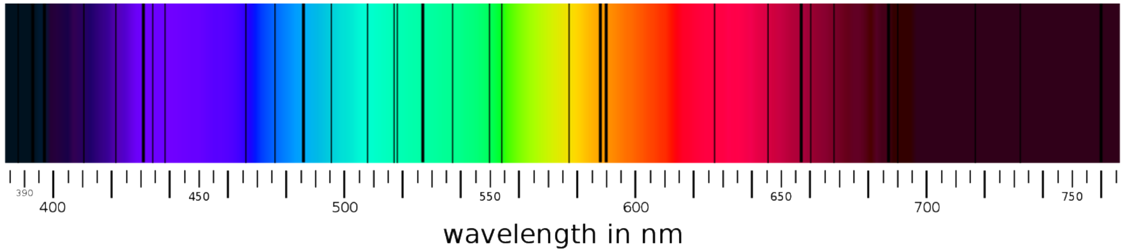
$\lambda = hc/E$

304 nm

References:
Demtroder 3.1.6
R. Trebino's lecture notes on Modern Physics
http://hyperphysics.phy-astr.gsu.edu/hbase/mod6.html#c3

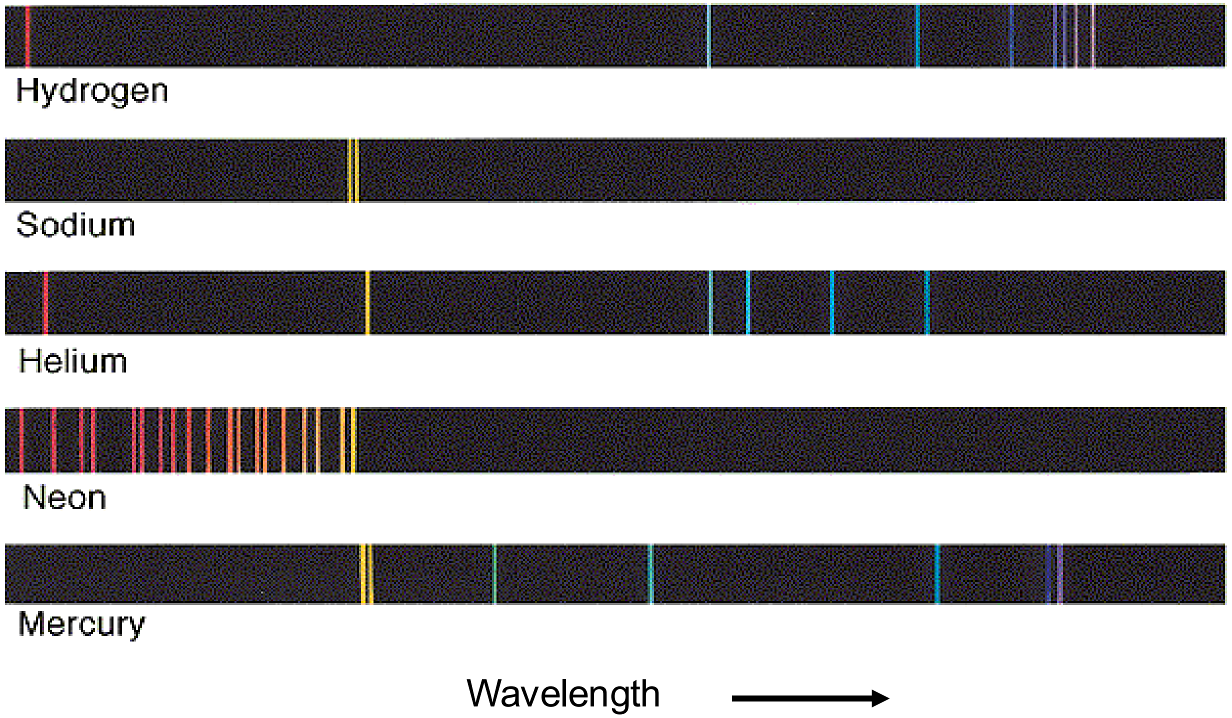
Another evidence of quantized energies of light: atomic spectra

Discrete dark (Fraunhofer) lines in the solar spectrum

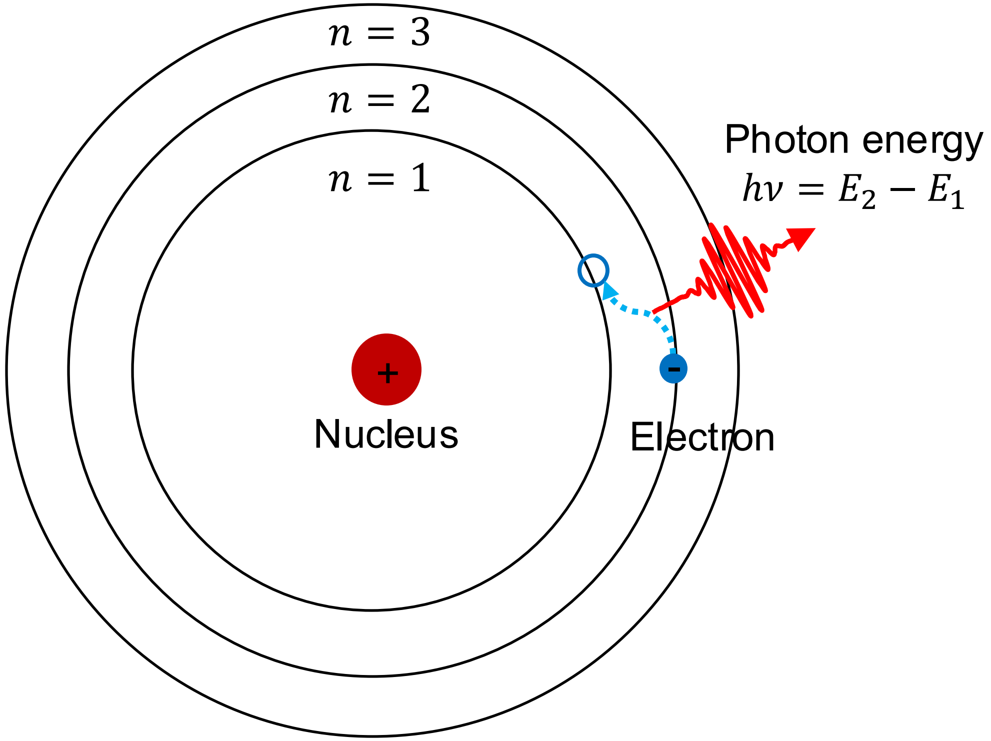


Credit: Saperaud (Wikipedia), Public Domain

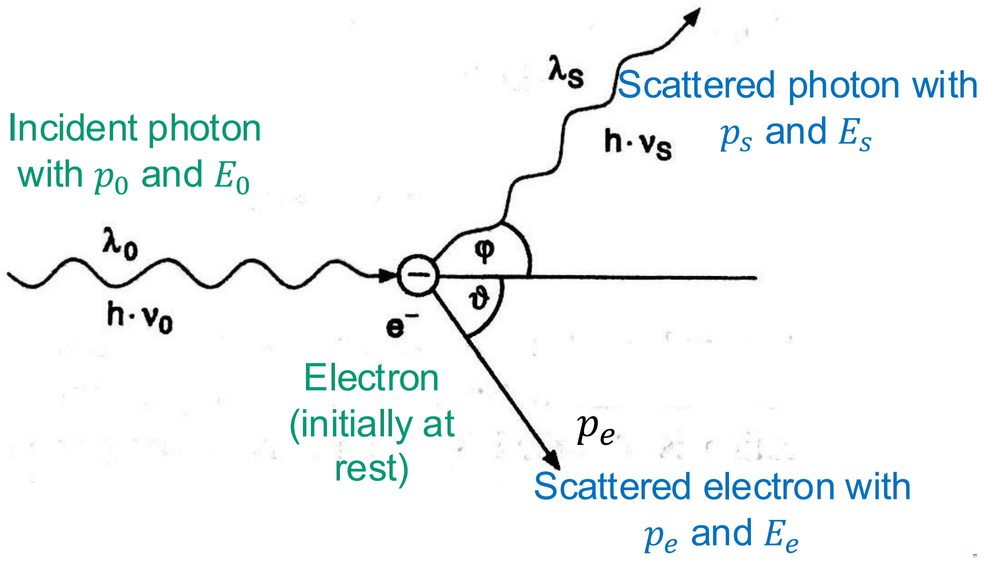
Discrete and unique spectral lines from atomic gases



Bohr model of hydrogen atom
(not a correct model!)



Compton effect illustrates that photons also carry momentum



- Elastic collision between a photon and loosely bound electron of a scattering material
- Photon has energy $\mathrm{E} = h\nu$ and momentum $p = \frac{h}{\lambda}$, where ν is the frequency of the photon and λ is the wavelength
- Electron has rest mass $m_e = 9.109 \times 10^{-31}\, kg$; its initial momentum is 0 and rest energy is $m_e c^2$
- After collision it acquires momentum p_e and energy $E_e = \sqrt{m_e^2 c^4 + p_e^2 c^2}$ (using relativistic form)

Photon energy + momentum

$$E = h\nu = \frac{hc}{\lambda}$$

$$\vec{p} = \hbar \vec{k} \quad |\vec{p}| = \hbar \cdot \frac{2\pi}{\lambda} = \frac{h}{\lambda}$$

$$E = pc$$

Electron energy + momentum

$$E_e = m c^2 = E_{kinetic} + \underbrace{E_{rest}}_{m_e c^2}$$

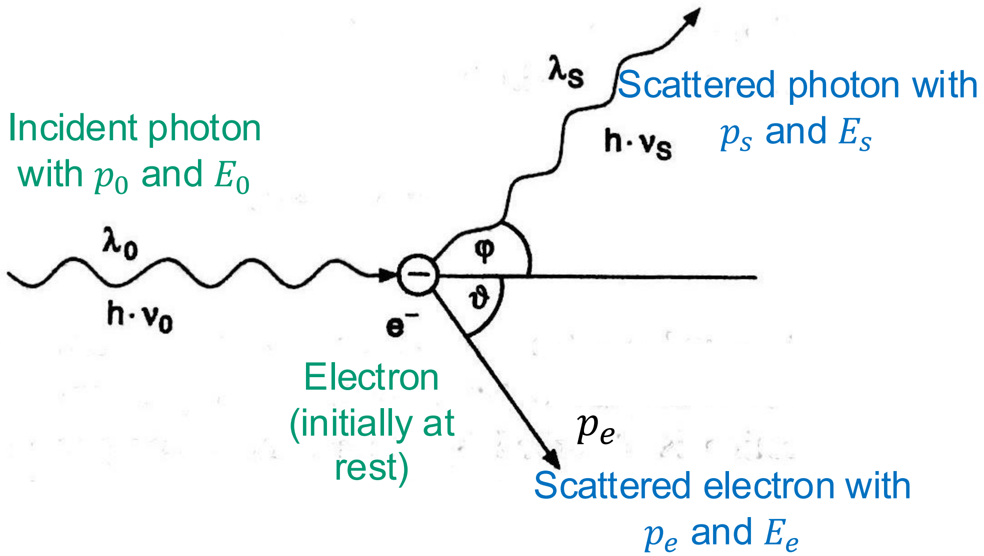
↑ relativistic mass

↑ rest mass of e^-

$$p_e = \frac{m_e v}{\sqrt{1 - v^2/c^2}}$$

Reference:
Demtroder 3.1.7

Compton effect illustrates that photons also carry momentum



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- After collision it acquires momentum p_e and energy $E_e = \sqrt{m_e^2 c^4 + p_e^2 c^2}$ (using relativistic form)

- Conservation of momentum

$$\vec{p}_0 = \vec{p}_s + \vec{p}_e$$

$$p_e^2 = p_0^2 + p_s^2 - 2\, \vec{p}_0 \cdot \vec{p}_s$$

$$\boxed{p_e^2 = p_0^2 + p_s^2 - 2 p_0 p_s \cos\phi}$$

- Conservation of energy

$$p_0 c + m_e c^2 = p_s c + \sqrt{m_e^2 c^4 + p_e^2 c^2}$$

$$\boxed{p_e^2 = p_0^2 + p_s^2 - 2 p_0 p_s + \frac{2 m_e^2 c^4 (p_0 - p_s)}{c}}$$

$$\frac{m_e^2 c^4 (p_0 - p_s)}{c} = p_0 p_s (1 - \cos\phi)$$

$$\boxed{\lambda_s - \lambda_0 = \frac{h}{m_e c}(1 - \cos\phi)}$$

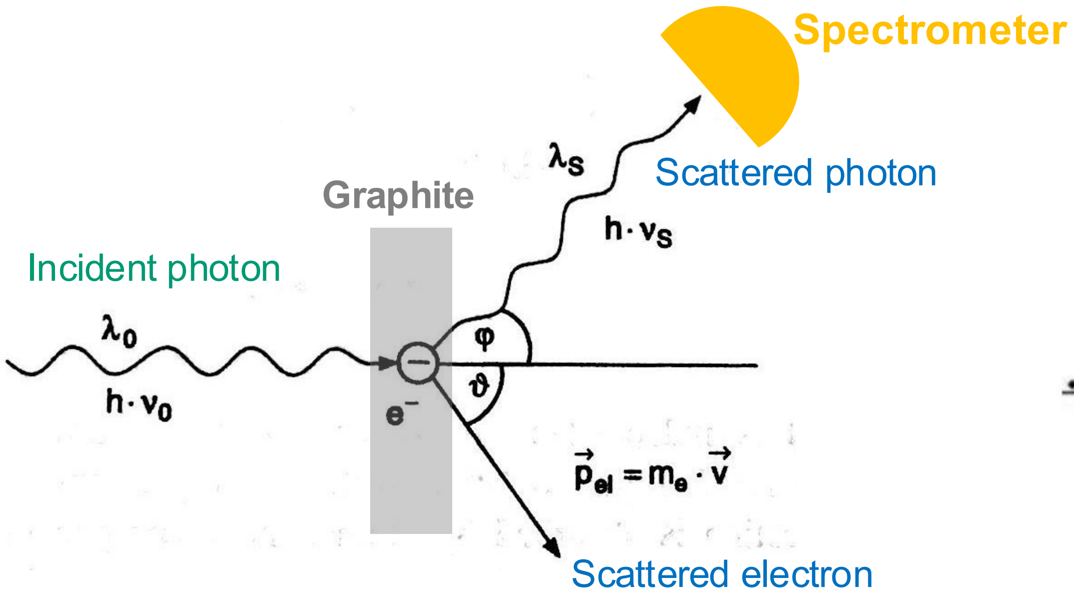
Reference:
Demtroder 3.1.7

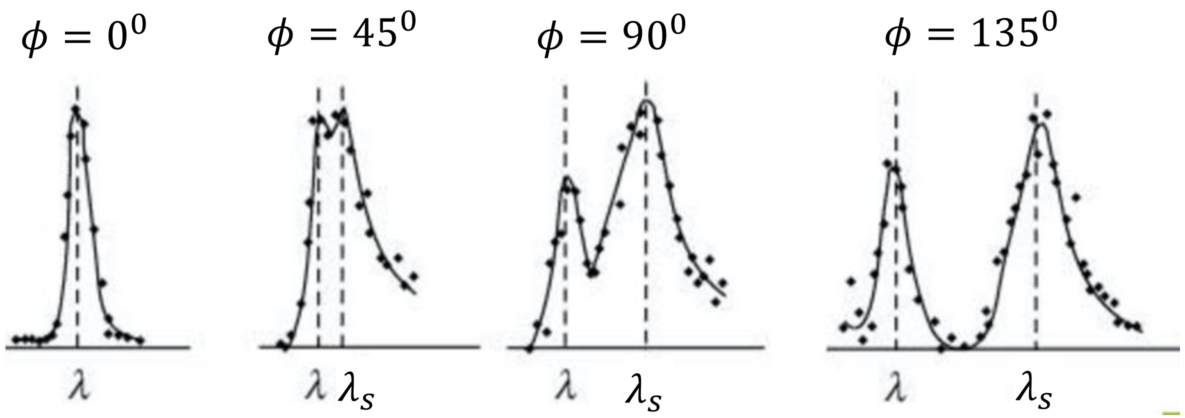
Compton effect (cont'd)

- Compton shift defines the change in wavelength:

$$\lambda_s - \lambda_0 = \frac{h}{m_e c}(1 - \cos\phi)$$

- Maximum change in wavelength $\lambda_s - \lambda_0 = \frac{h}{m_e c}(1 - \cos 180^0) = \frac{2h}{m_e c} = 4.86 \times 10^{-12}\ m$
  - Negligible in comparison to visible wavelengths ($\lambda \approx 10^{-7} m$)
  - Significant for X-rays and gammas ($\lambda \approx 10^{-10} m$)





Why do we see two peaks?

Reference:
Demtroder 3.1.7

Compton scattering exercise

- If we want to observe Compton scattering with protons, what energy source do we need? Select all that apply, and you may assume that you can at best resolve one-thousandth of the shift in wavelength for each source that you select.

$$h = 6.626 \times 10^{-34}\, J \cdot s$$
$$m_p = 1.67262 \times 10^{-27}\, kg$$

  - Deep UV light with $\lambda \sim 10^{-7}\, m$
  - X-rays with $\lambda \sim 10^{-11}\, m$
  - Gamma rays with $\lambda \sim 10^{-15}\, m$

$$\Delta\lambda \sim \frac{2h}{m_p c} \sim 2.6 \times 10^{-15}\, m$$

- What scale of energy is needed to produce such photon beams?

$$E = \frac{hc}{\lambda} \sim GeV$$

Properties of photons: energy and momentum

- Energy quanta for electromagnetic field is $E = h\nu$

$$\boxed{h = 6.6261 \times 10^{-34}\ \mathrm{J{\cdot}s}}$$

- Photon density and flux:

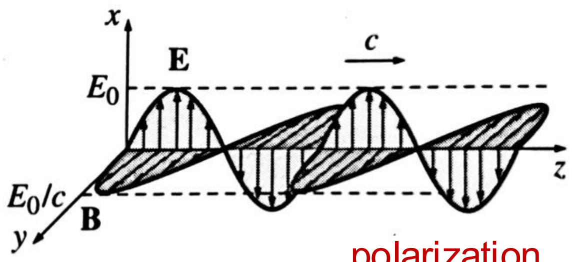
$$n = \frac{u}{h\nu}$$

$$\dot{n} = \frac{I}{h\nu} = cn$$

- Momentum: $\vec{p} = \hbar k$; $|\vec{p}| = \hbar \frac{2\pi}{\lambda} = h\nu/c$ $\quad \boxed{\hbar = \frac{h}{2\pi}}$

- Momentum density: $\vec{\mathcal{P}} = n\hbar\vec{k}$; $\left|\vec{\mathcal{P}}\right| = \frac{nh\nu}{c} = u/c$

Monochromatic plane waves from classical electromagnetism



$$\vec{\mathcal{E}}(r,t) = \mathcal{E}_0 e^{-i(\vec{k}\cdot\vec{r}-\omega t)}\,\hat{\mathrm{n}}$$

polarization vector ($\hat{\mathbf{n}} \perp \hat{\mathbf{k}}$)

$$\vec{B}(r,t) = \frac{\mathcal{E}_0}{c} e^{-i(\vec{k}\cdot\vec{r}-\omega t)}(\hat{\mathrm{k}} \times \hat{\mathrm{n}})$$

Wave vector $|\vec{\boldsymbol{k}}| = \frac{2\pi}{\lambda}$ $\quad \omega = 2\pi\nu$

$\vec{\boldsymbol{\mathcal{E}}}$ and $\vec{\boldsymbol{B}}$ are in phase and perpendicular to one another

Energy density of wave: $u = \epsilon_0 \mathcal{E}_0^2$

Energy flux density (Poynting vector): $\vec{S} = \epsilon_0\left(\vec{\mathcal{E}} \times \vec{B}\right)$

Intensity (average power/area): $I = \langle|\vec{S}|\rangle = \epsilon_0 c \mathcal{E}_0^2 = cu$

Momentum density: $\vec{\mathcal{P}} = \frac{\vec{S}}{c^2}$

Reference: D. Griffiths, Introduction to Electrodynamics

Mass of a photon

- Mass-energy relationship for a particle with rest mass m_0

$$E = mc^2$$

$$m = \frac{E}{c^2} = \frac{h\nu}{c^2}$$

$$E = \sqrt{p^2c^2 + m_0^2c^4}$$

- For a photon, $E = h\nu = pc \rightarrow m_0 = 0$
  - **Rest mass of photon is zero (but really photons at rest do not exist).**

Properties of particles with mass: De Broglie's hypothesis

- Mass particles should have wave properties similar to electromagnetic radiation (1924)
- Energy can be written as

$$h\nu = pc$$
$$h\nu = p\lambda\nu$$
$$\boxed{\lambda = \frac{h}{p}}$$

de Broglie wavelength

Definitions:

h	Planck's constant
ν	Frequency
p	Momentum
m	Mass
v	Velocity
E_{kin}	Kinetic energy

  - Another useful form: $\lambda = \frac{hc}{pc}$

- At the time of hypothesis, no wave properties of particles had been observed.

- De Broglie wavelength for photons

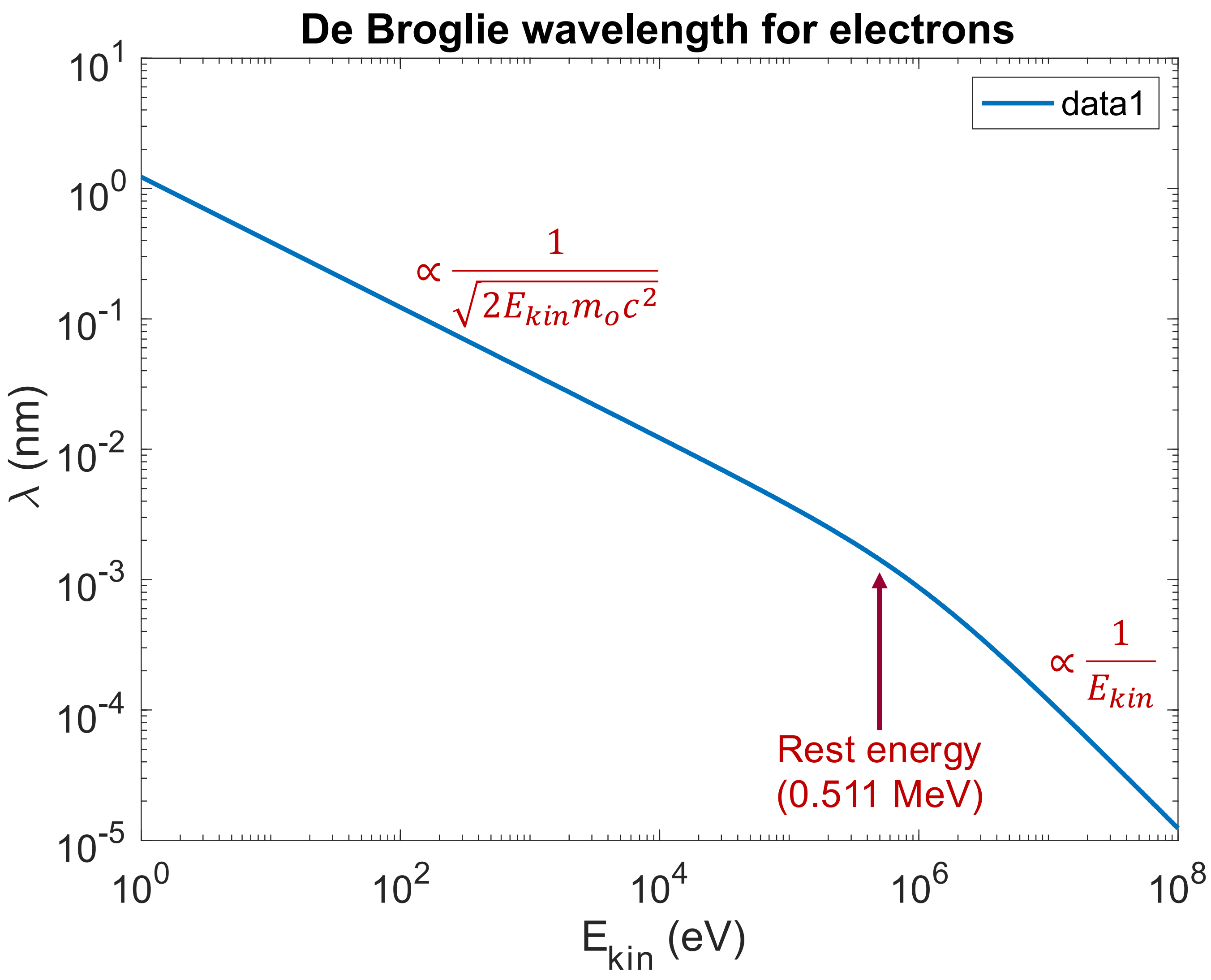
$$\lambda = \frac{hc}{pc} = \frac{hc}{E}$$

- For macroscopic objects: $\lambda = \frac{h}{mv}$
- For subatomic particles (electrons, protons, neutrons, etc.) and atoms

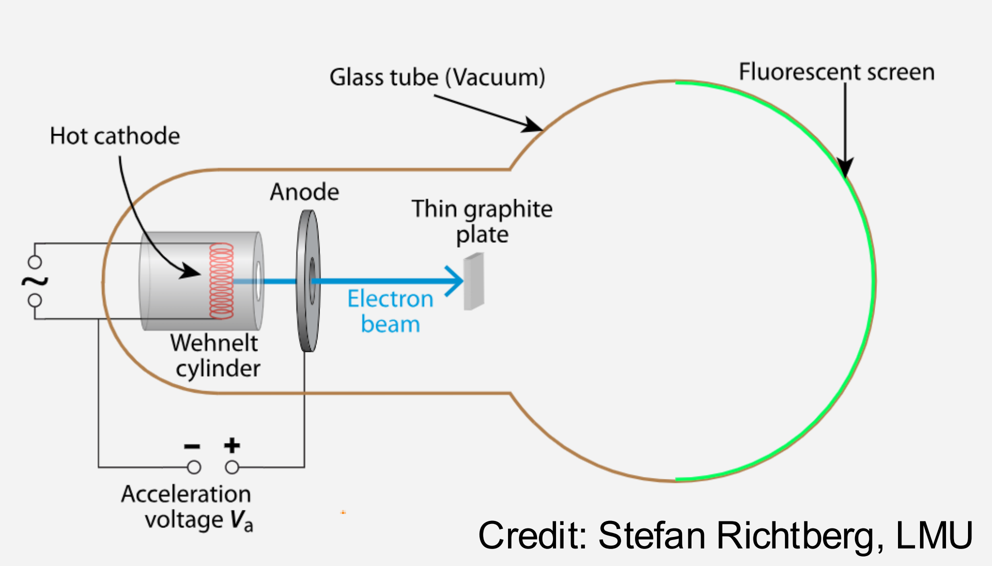
$$\lambda = \frac{hc}{\sqrt{E_{kin}^2 + 2E_{kin}m_oc^2}}$$

  - For $E_{kin} \ll m_oc^2, pc \approx \sqrt{2E_{kin}m_oc^2}$
  - For $E_{kin} \gg m_oc^2, pc \approx E_{kin}$

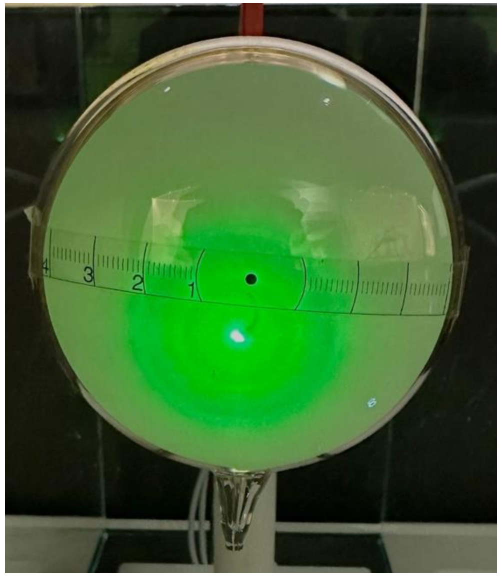
De Broglie wavelength of an electron



Example of electron diffraction
(demo at Deutsches Museum in Munich, Germany)



Credit: Stefan Richtberg, LMU



- What is causing diffraction?
- Assuming that the electrons are being accelerated by a voltage source at 10 V, what is the maximum feature size in the sample that can lead to diffraction?

$$E_{rest} = m_e c^2 = 0.511\ \text{MeV}$$

$$\lambda = \frac{hc}{\sqrt{E_{kin}^2 + 2E_{kin}\cdot E_{rest}}}$$

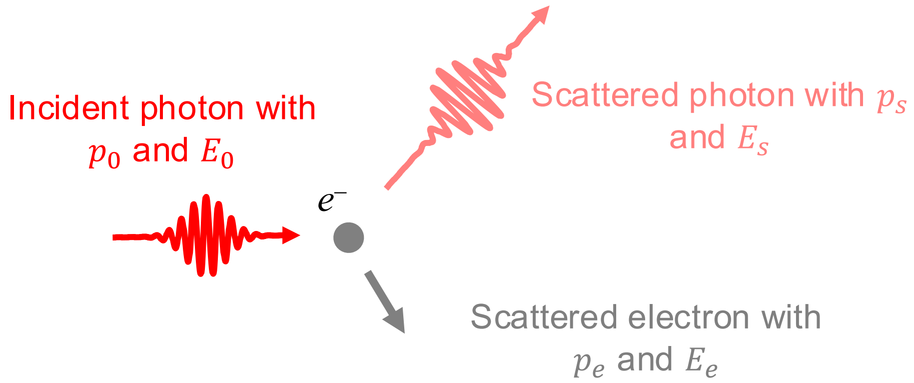
$$E_{kin} = 10\ \text{eV}$$

$$\lambda \sim hc/\sqrt{2(10\,\text{eV})(0.511\ \text{MeV})} \sim 1\ \text{nm}$$

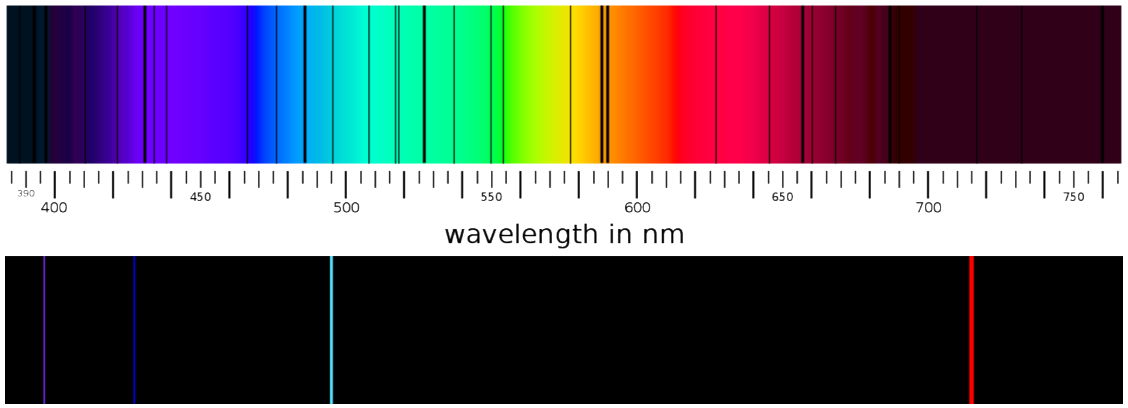
How photons interacts with matter

- Photons can only act on *charges*:
  - We describe EM radiation and their interactions with Maxwell's equations along with the Lorentz force law: $\vec{F} = q(\vec{\mathcal{E}} + \vec{v} \times \vec{B})$ where $\vec{F}$ is the force acting on a moving charge with velocity $\vec{v}$; q is the electric charge; $\vec{\mathcal{E}}$ is the electric field; $\vec{B}$ is the magnetic field.
  - When we talk about light interacting with nominally neutral objects like a wall or mirror (*e.g.,* through absorption, reflection, scattering, etc.), we are referring to its interaction with charged particles in the material.

- What are ways photons can interact with matter/charged particles?
  - Change in particle's energy
  - Change in particle's linear momentum
  - Change in particle's angular momentum



Discrete dark (Fraunhofer) lines in the solar spectrum



Hydrogen emission spectrum

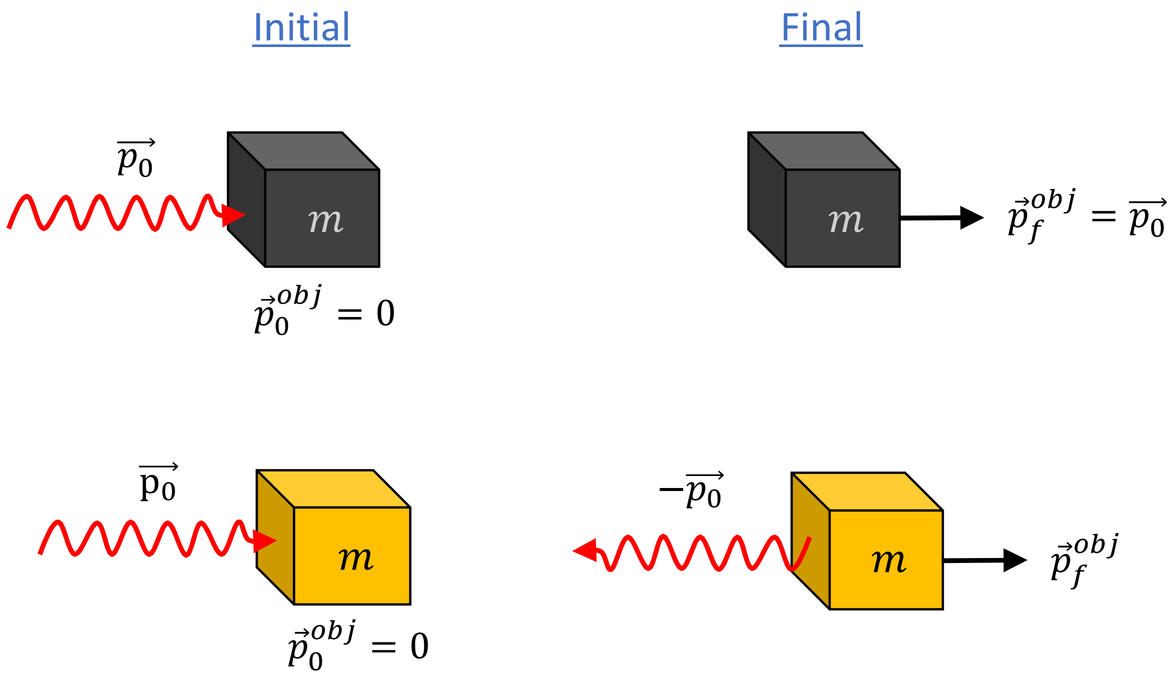
https://en.wikipedia.org/wiki/Fraunhofer_lines

Energy and momentum conversation

In space, a photon with energy $E_0 = h\nu_0$ and momentum $\vec{p}_0$ hits the surface of an object initially at rest with mass m at normal incidence.

(a) In terms of $\vec{p}_0$, what is the momentum of the object after the interaction, assuming that the surface of the object is perfectly absorptive? Round to the nearest whole number, but please include the sign.

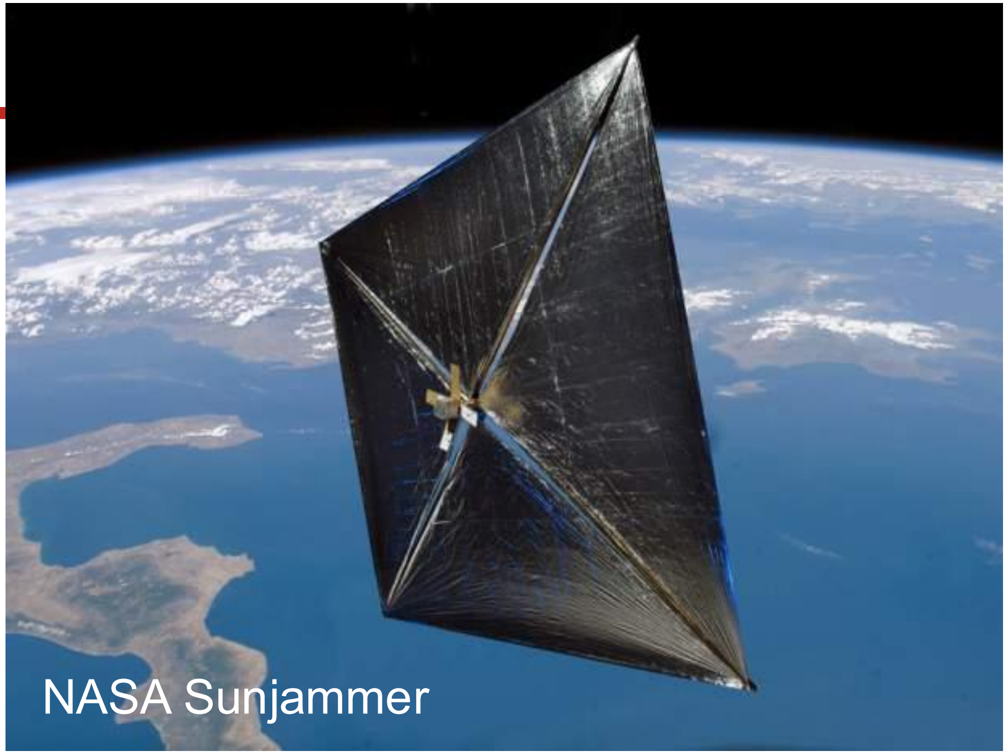
(b) In terms of $\vec{p}_0$, what is the momentum of the object after the interaction, assuming that the surface of the object perfectly reflective? Round to the nearest whole number, but please include the sign.

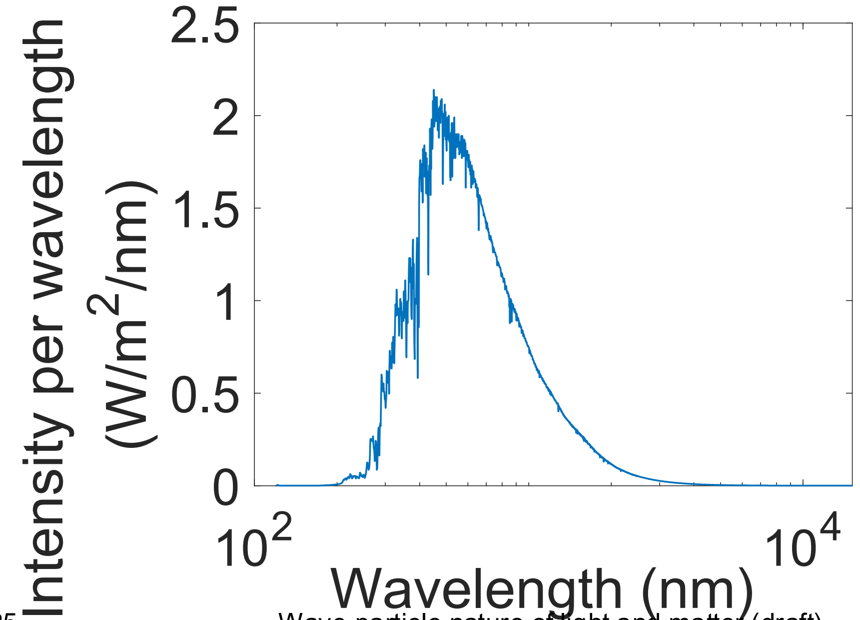


Solar sail example

A spacecraft close to Earth is equipped with a solar sail that has an area of $A = 600\ m^2$. Assuming that the solar sail is perfectly reflective:

1. How can we determine the intensity of sunlight near Earth's orbit?
2. Calculate the radiation pressure exerted on the solar sail. (Write the expression on Top Hat in terms of intensity I and speed of light c)
3. Determine the force exerted on the solar sail due to radiation pressure.
4. If the spacecraft (including the sail) has a mass of $m = 100\ kg$, calculate its acceleration due to the radiation pressure.
5. Estimate the number of photons hitting the surface of the solar sail per second.
6. Is solar sailing a practical technology?





Solutions to the solar sail problem

- To determine the total intensity of solar radiation in the vicinity of Earth, we can integrate the spectral irradiance data ($dI/d\lambda$) given in HW #1 over all wavelengths:

$$I = \int_0^\infty \frac{dI}{d\lambda}\, d\lambda$$

  - Numerical integration of the data gives us

$$I \approx 1366.5\, W/m^2$$

- The radiation pressure P (in N/m^2) from an electric field to an absorbing object is related to the intensity of the field (in W/m^2) and the speed of light c, according to $P = I/c$. Since the sail is perfectly reflective, the total radiation pressure is

$$P_{tot} = 2I/c$$

- We can get the force F exerted by the light field on the sail by

$$F = P_{tot}A = 0.0055\, N \text{ (compare this to a typical thrust from firing rocket fuel ~ 11 N/kg!)}$$

- This force is being used to accelerate the sail, so the acceleration is

$$a = \frac{F}{m} = 5.5 \times 10^{-5}\, m/s^2$$

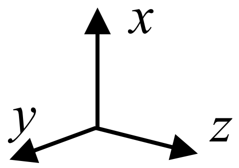
- The total number of photons hitting the sail per second can be found by dividing the total energy of the field by the energy of each individual photon ($E_{ph} = hc/\lambda$). There is a range of wavelengths for the photons from this solar radiation, so as an approximation we take the peak emission wavelength of $\lambda = 500$ nm). The number of photons on the sail per second is:

$$E_{tot} = \frac{IA}{E_{ph}} \approx 2 \times 10^{24} \text{ photons}$$

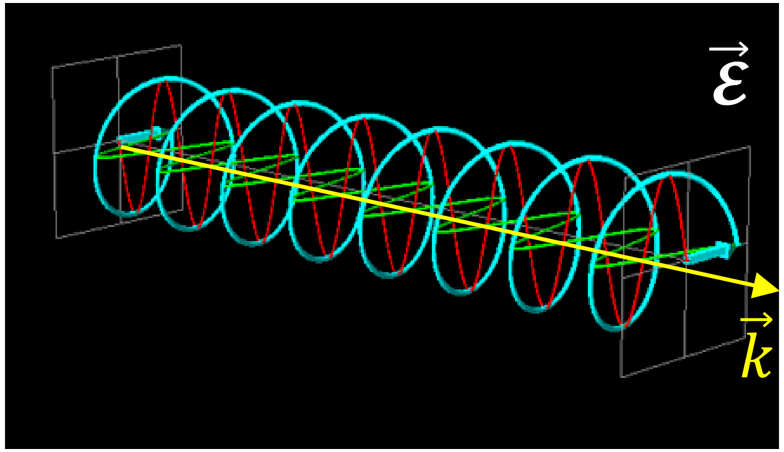
- Even though the force and acceleration values are very small, solar sailing could be practical if we consider that the radiation pressure is applied continuously and that the mass of the spacecraft could be reduced by having to carry less fuel.

Classical argument for the transfer of angular momentum between photon and electron

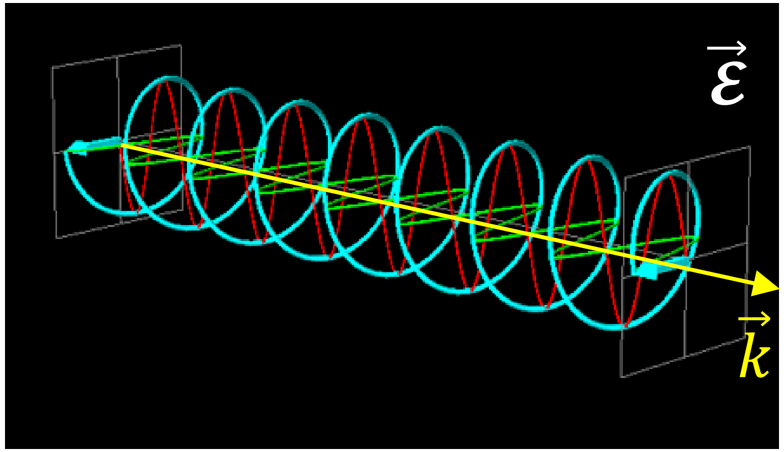
- Circularly polarized light wave
  - $\vec{\mathcal{E}} = \mathcal{E}_0 \cos(kz - \omega t)\hat{x} \mp \mathcal{E}_0 \sin(kz - \omega t)\,\hat{y}$
  - x (red) and y (green) components of $\vec{\mathcal{E}}$ are $\pm\,\pi/2$ out of phase
  - Electric field rotates around $\vec{k}$



- $\vec{\mathcal{E}}$ will drive charges in materials to undergo circular motion



Left-handed polarization



Right-handed polarization

Animations from R. Trebino's lecture notes on Optics
Reference: Hecht, Optics (Chapter 8.1)

Define $E \equiv$ energy in electric field $\vec{\mathcal{E}}$

The power P needed to be applied to an object to rotate it by angular frequency ω is

$$P = \frac{dE}{dt} = \omega|\vec{\tau}|$$

where $\vec{\tau}$ is the torque applied to the object and is $\vec{\tau} = \vec{r} \times \vec{F}$ (where $\vec{r}$ is the vector from the point where torque is measured to the point where force $\vec{F}$ is applied). This torque is also the change in the angular momentum on the object: $\vec{\tau} = \frac{d\vec{L}}{dt}$. Therefore we have

$$\frac{dE}{dt} = \omega \frac{d|\vec{L}|}{dt}$$

Integrating both sides with respect to time, we get $E = \omega\,|\vec{L}| \rightarrow |\vec{L}| = E/\omega$

Rotation of microscopic particles by light

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Optical alignment and spinning of laser-trapped microscopic particles

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Light-induced rotation of absorbing microscopic particles by transfer of angular momentum from light to the material raises the possibility of optically driven micromachines. The phenomenon has been observed using elliptically polarized laser beams[1] or beams with helical phase structure[2,3].

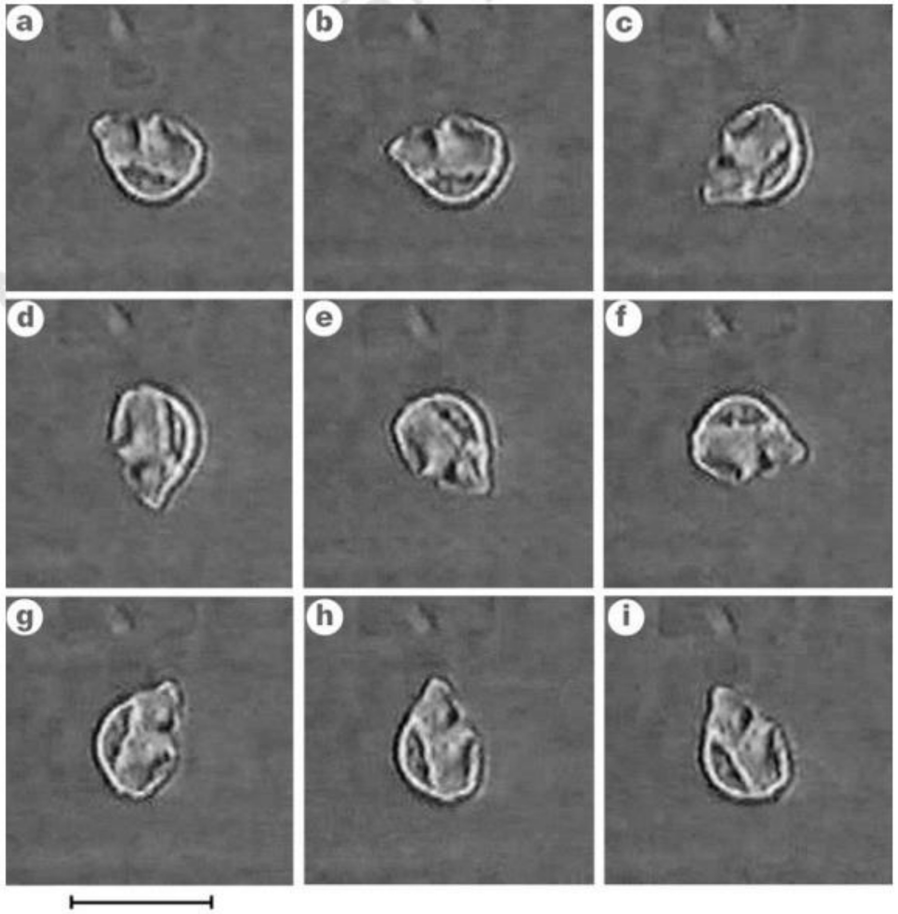
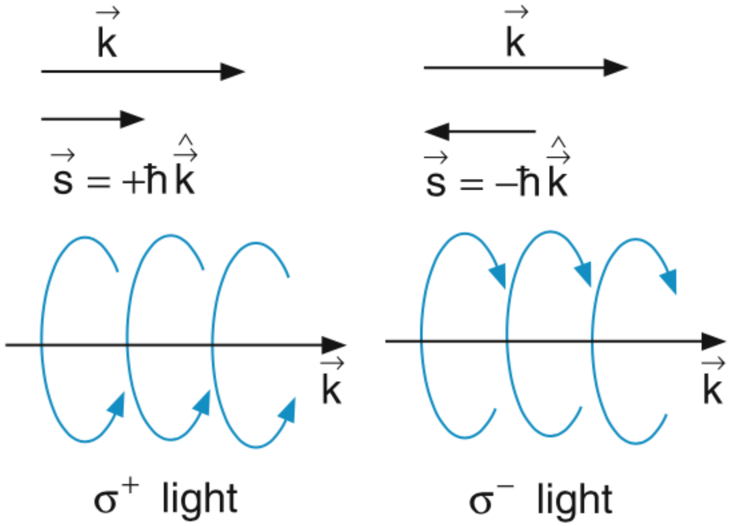


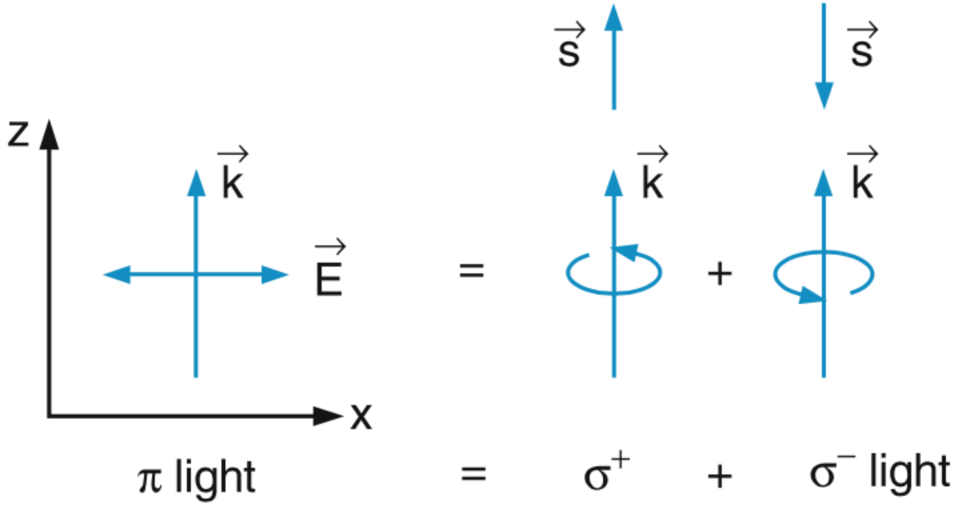
Figure 2 Nine frames of a trapped calcite crystal, showing free rotation due to an elliptically polarized trapping beam. The speed of rotation is limited by the viscous drag on the particle. As the optical torque acting on the particle depends on its orientation, the rotation speed is not constant. The frames are 40 ms apart. Scale bar, 10 μm.

What about angular momentum for a single photon

- From our classical derivation: $|\vec{L}| = E/\omega$
- We have established that photons have quantized energy $E = h\nu$

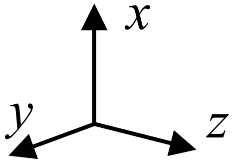


- Single photons can have spin $\vec{s}_{ph} = \pm\hbar\hat{k}$
- How about a photon in a linearly polarized field?
  - Superposition of pair of σ^+ and σ^+ photons
  - But we're still talking about a single photon, when measured the photon will either have spin $+\hbar$ or $-\hbar$ (occurring at equal likelihood)

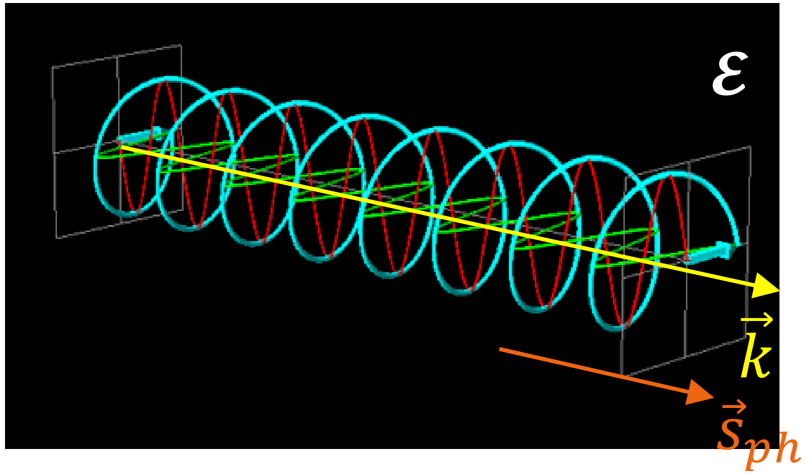


Source: W. Demtroder, "Atoms, Molecules, and Photons" (Chapter 3)

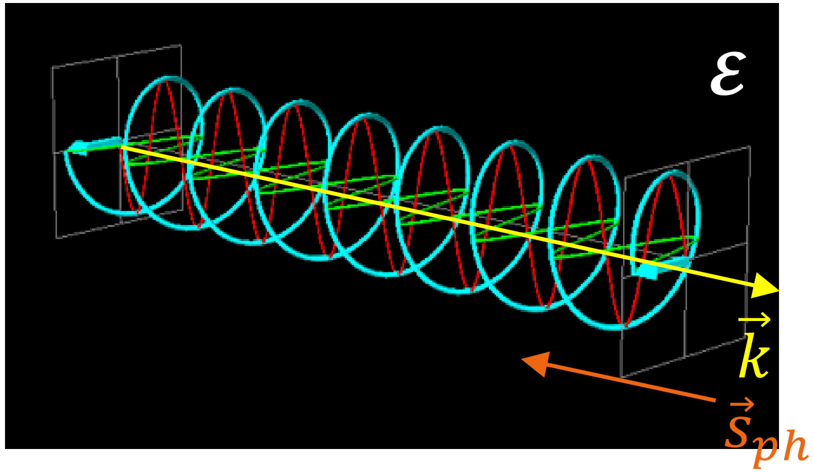
Visualizing electric field polarization and photon spin



Circular polarization

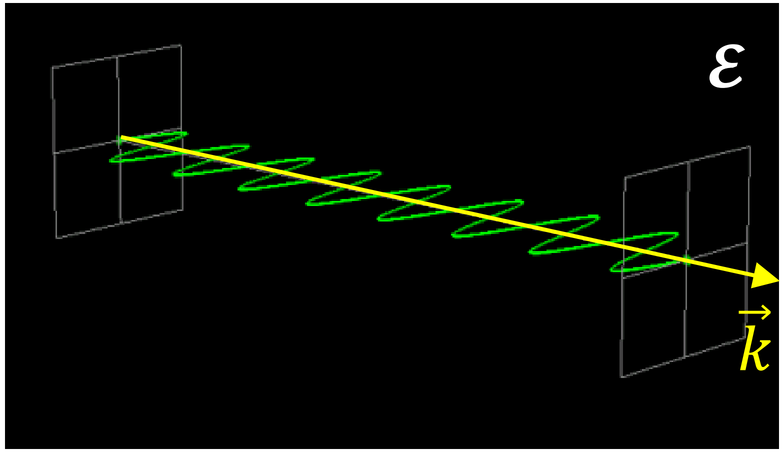


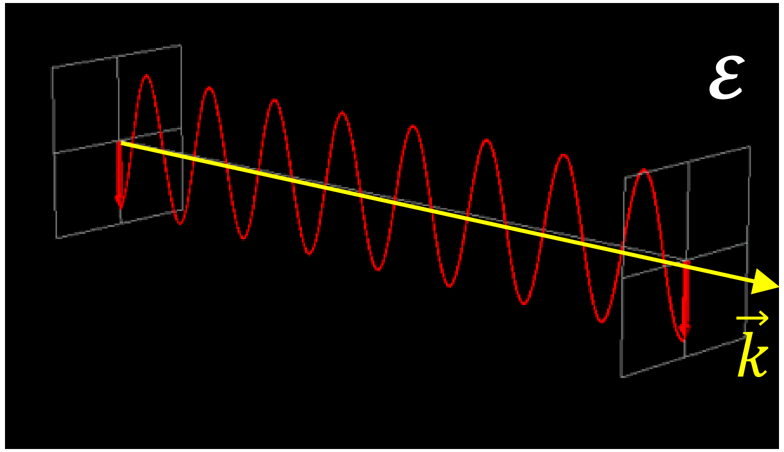
σ^+ light



σ^- light

Linear polarization (superposition of σ^+ and σ^-) → no photon spin





π light

Animations from R. Trebino's lecture notes on Optics

First measurement of angular momentum of light

JULY 15, 1936 PHYSICAL REVIEW VOLUME 50

Mechanical Detection and Measurement of the Angular Momentum of Light

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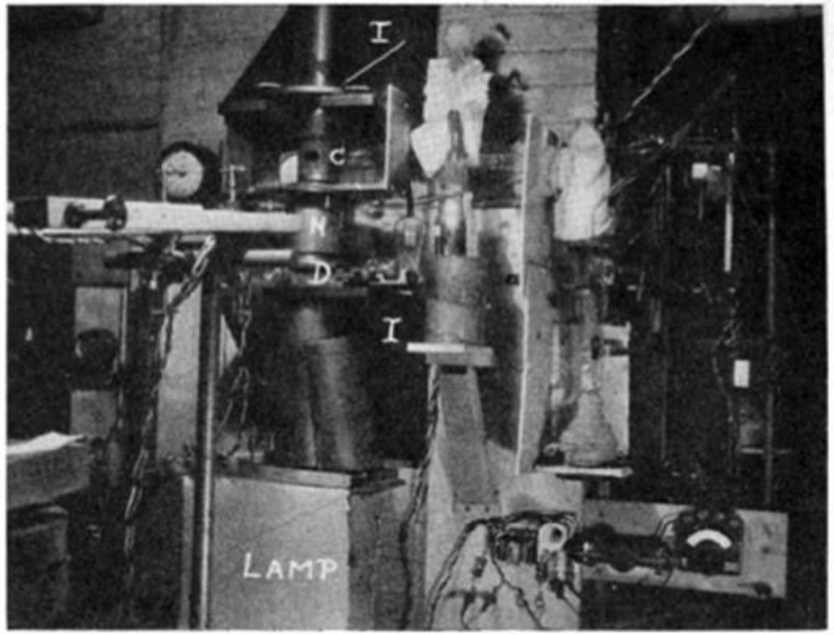
(Received May 8, 1936)

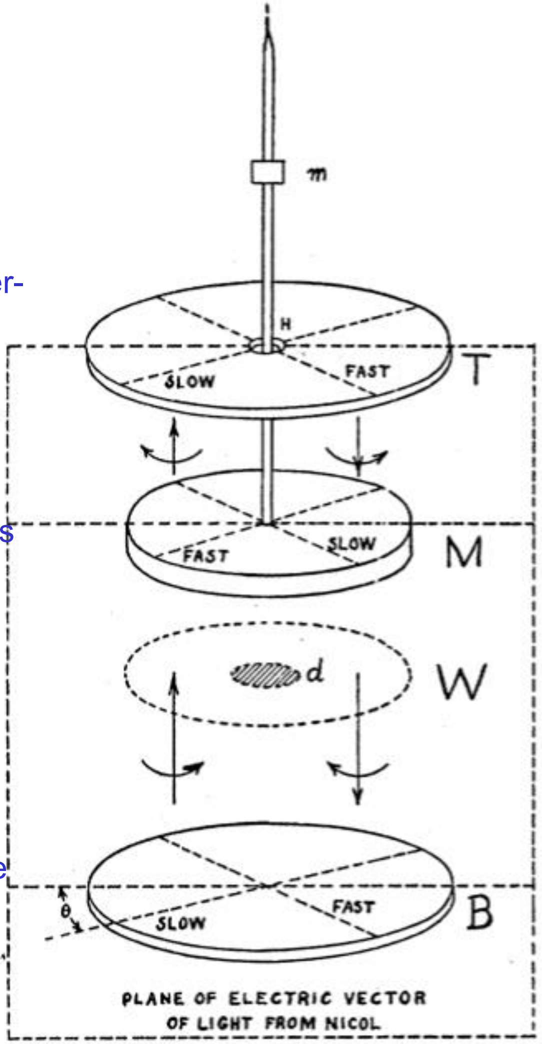
The electromagnetic theory of the torque exerted by a beam of polarized light on a doubly refracting plate which alters its state of polarization is summarized. The same quantitative result is obtained by assigning an angular momentum of $\hbar$ ($-\hbar$) to each quantum of left (right) circularly polarized light in a vacuum, and assuming the conservation of angular momentum holds at the face of the plate. The apparatus used to detect and measure this effect was designed to enhance the moment of force to be measured by an appropriate arrangement of quartz wave plates, and to reduce interferences. The results of about 120 determinations by two observers working independently show the magnitude and sign of the effect to be correct, and show that it varies as predicted by the theory with each of three experimental variables which could be independently adjusted.



Reflective quarter-waveplate

Hanging half-waveplate (changes handedness of circular pol)

Quarter-waveplate (turns linear to circular pol)



Lamp

Rotating a light sail in space

$h = 6.62607015 \times 10^{-34}$ J/s

A space probe is equipped with a circular light sail with a radius of $r = 3\ m$ and momentum of inertia $J = 10\ kg \cdot m^2$. A distant space station emits a beam of left-handed circularly polarized light toward the probe, which transfers linear and angular momentum to the sail, causing it to rotate. If the power of the beam is $P = 50\ W$ and the photons have a wavelength of $\lambda = 500\ \mathrm{nm}$. Assume that the light sail is perfectly absorptive. Calculate:

1. In terms of $\hbar$, what is the angular momentum transferred to the sail per second?
2. The torque exerted on the sail due to the angular momentum of the circularly polarized photons.
3. What angular acceleration is produced?
4. What happens if the light is changed to right-handed circularly polarized light?

Summarizing the particle and wave characteristics of electromagnetic radiation

Energy	Momentum	Spin	Mass equivalent
$E = h\nu$ $E = \hbar\omega$	$\vec{p} = \hbar\vec{k}$ $\lvert\vec{p}\rvert = \frac{h}{\lambda} = \frac{E}{c}$	$\vec{s} = \pm\hbar\hat{\mathrm{k}}$ $\lvert\vec{s}\rvert = \hbar$	$m = \frac{E}{c^2} = \frac{h}{c\lambda}$ $m_0 = 0$
Spectral energy density		**Intensity**	**Momentum density**
$u = nh\nu = \epsilon_0\mathcal{E}_0^2$		$I = nch\nu = c\epsilon_0\mathcal{E}_0^2$	$\lvert\vec{\mathcal{P}}\rvert = n\hbar k = \frac{\lvert\vec{S}\rvert}{c}$

Reference: W. Demtroder, "Atoms, Molecules, and Photons"