P1

Evaluate the Wallis Integral (for $n \to \infty$)

$$I = \int_0^{\frac{\pi}{2}} \sin^n(t)\,\mathrm{d}t. \tag{1}$$

Write $\sin^n t = \exp(n \ln(\sin t))$, then the integral is

$$\int_0^{\frac{\pi}{2}} e^{nf(t)}\,\mathrm{d}t, \quad f(t) = \ln(\sin t). \tag{2}$$

Taking $f(t) = \cot(t) = 0 \Rightarrow t_0 = \frac{\pi}{2}$ at boundary, with $f''\left(\frac{\pi}{2}\right) = -1 < 0$, so consider subsitution $u = \frac{\pi}{2} - t$. This shifts the maximum and gives

$$\sin t = \cos u \approx 1 - \frac{u^2}{2}, \quad (u \sim 0) \tag{3}$$

We see that $f(t) = \ln\left(1 - \frac{u^2}{2}\right) \approx -\frac{u^2}{2}$, and thus

$$I = \int_0^{\frac{\pi}{2}} e^{-\frac{nu^2}{2}}\,\mathrm{d}u \approx \int_0^{\infty} e^{-\frac{nu^2}{2}}\,\mathrm{d}u. \tag{4}$$

Setting $x = \frac{\sqrt{n}u}{\sqrt{2}}$, $\mathrm{d}u = \sqrt{\frac{2}{n}}\,\mathrm{d}x$, and via half a Gaussian integral, we have

$$I \approx \sqrt{\frac{2}{n}} \int_0^{\infty} e^{-x^2}\,\mathrm{d}x = \boxed{\sqrt{\frac{\pi}{2n}}.} \tag{5}$$

P2

Find the asymptotic form of the Bessel function in $x \to \infty$ limit

$$I_n(x) = \frac{1}{\pi} \int_0^\pi e^{x \cos\theta} \cos(n\theta) \, \mathrm{d}\theta. \tag{6}$$

We read off the Laplace integral form

$$I = \frac{1}{\pi} \int_0^\pi e^{x f(\theta)} g(\theta) \, \mathrm{d}\theta \tag{7}$$

with $f(\theta) = \cos\theta$, $g(\theta) = \cos n\theta$.

Find maximum with $f'(\theta) = -\sin\theta = 0 \Rightarrow \theta_0 = 0$ or π, $f''(0) = -1 < 0$. The stationary point $\theta_0 = 0$ is at boundary, so we consider expansion to second order around $\theta_0 = 0$, with

$$\cos\theta \approx 1 - \frac{\theta^2}{2}, \quad \cos(n\theta) \approx 1. \tag{8}$$

and so

$$I(x) \approx \frac{1}{\pi} \int_0^\pi e^{x(1-\theta^2)} \, \mathrm{d}\theta = \frac{e^x}{\pi} \int_0^\pi e^{-(x/2)\theta^2} \, \mathrm{d}\theta \tag{9}$$

Gaussian $e^{-(\frac{x}{2})\theta^2}$ decays rapidly (width $\sim z^{-\frac{1}{2}} \ll \pi$), so extend to ∞ with exponentially small error:

$$I(x) \approx \frac{e^x}{\pi} \int_0^\infty e^{-(x/2)\theta^2} \, \mathrm{d}\theta \underset{u=\sqrt{x/2}\,\theta}{=} \frac{e^x}{\pi} \sqrt{\frac{2}{x}} \int_0^\infty e^{-u^2} \, \mathrm{d}u = \boxed{\frac{e^x}{\sqrt{2\pi x}}.} \tag{10}$$

P3

In the limit of $n \to \infty, x > 1$, find the asymptotic form of the Legendre polynomial

$$P_n(x) = \frac{1}{\pi} \int_0^\pi \left(x + \sqrt{x^2 - 1} \cos\theta\right)^n \mathrm{d}\theta. \tag{11}$$

We read off the Laplace integral form with $f(\theta) = \ln\left(x + \sqrt{x^2 - 1} \cos\theta\right)$:

$$P_n(x) = \frac{1}{\pi} \int_0^\pi e^{n f(\theta)} \mathrm{d}\theta. \tag{12}$$

Examine $f(\theta)$, with $\omega\sqrt{x^2 - 1} > 0, \beta = x + \omega > 1 :$

$$f'(\theta) = -\frac{\omega \sin(\theta)}{\omega \cos\theta + x} < 0, \quad (\theta \in (0, \pi)) \tag{13}$$

so $f(\theta)$ is monotonically decreasing in $(0, \pi)$ with maximum at boundary $\theta_0 = 0$. Further,

$$f''(\theta) = -\frac{\omega \cos\theta(\omega \cos\theta + x) + \omega^2 \sin^2\theta}{(\omega \cos\theta + x)^2}, \quad f''(0) = -\frac{\omega^2 + \omega x}{(\omega + x)^2} = -\frac{\omega}{\beta} < 0. \tag{14}$$

Thus $\theta = 0$ is a stationary point at the boudnary. We expand around $\theta_0 = 0 :$

$$f(\theta) \approx \ln\left(\beta\left(1 - \frac{\omega\theta^2}{2\beta}\right)\right) = \ln\beta - \frac{\omega\theta^2}{2\beta}. \tag{15}$$

So

$$P_n(x) \approx \frac{1}{\pi} \int_0^\pi e^{n\left(\ln\beta - \frac{\omega\theta^2}{2\beta}\right)} \mathrm{d}\theta = \frac{\beta^n}{\pi} \int_0^\pi e^{-\frac{n\omega}{2\beta}\theta^2} \mathrm{d}\theta. \tag{16}$$

Since $e^{-\frac{n\omega}{2\beta}\theta^2}$ decays rapidly with $n \gg 0$ (width $\sim \sqrt{\frac{\beta}{n\omega}} \ll \pi$), we extend the upper limit to infinity with exponentially small error, and use half Gaussian to estimate:

$$P_n(x) \approx \frac{\beta^n}{\pi} \int_0^\infty e^{-\frac{n\omega}{2\beta}\theta^2} \mathrm{d}\theta = \frac{\beta^n}{\pi} \sqrt{\frac{2\beta}{n\omega}} \int_0^\infty e^{-u^2} \mathrm{d}u = \boxed{\frac{\beta^{n+\frac{1}{2}}}{\sqrt{2\pi n\omega}}}. \tag{17}$$

P4

In the limit of $\nu \to \infty, x > 0$, find the asymptotic form of the McDonald function

$$K_\nu(x) = \frac{1}{2} \int_{-\infty}^{\infty} \exp(\nu t - x \cosh t)\, \mathrm{d}t \tag{18}$$

We read off the Laplace integral form with $\Phi(t) = \nu t - x \cosh t$:

$$K_\nu(x) = \frac{1}{2} \int_{-\infty}^{\infty} e^{\Phi(t)}\, \mathrm{d}t. \tag{19}$$

Examine $\Phi(t)$, find stationary point with

$$\Phi'(t) = \nu - x \sinh t = 0 \quad \Rightarrow t_0 = \operatorname{arcsinh}\left(\frac{\nu}{x}\right), \tag{20}$$

which is an interior point given the infinite bound. Further,

$$\Phi''(t) = -x \cosh t, \quad \Phi''(t_0) = -x \cosh\left(\operatorname{arcsinh}\left(\frac{\nu}{x}\right)\right) = -\sqrt{\nu^2 + x^2} < 0. \tag{21}$$

Thus t_0 is a maximum in the interior. We expand around t_0:

$$\Phi(t) \approx \Phi(t_0) + \left(\frac{1}{2}\right)\Phi''(t_0)(t - t_0)^2, \quad \Phi(t_0) = \nu \operatorname{arcsinh}\left(\frac{\nu}{x}\right) - \sqrt{\nu^2 + x^2}. \tag{22}$$

So

$$\begin{aligned} K_\nu(x) &\approx \frac{1}{2} \int_{-\infty}^{\infty} \exp\left(\Phi(t_0) + \left(\frac{1}{2}\right)\Phi''(t_0)(t - t_0)^2\right) \mathrm{d}t \\ &= \left(\frac{1}{2}\right) e^{\Phi(t_0)} \int_{-\infty}^{\infty} \exp\left(-\left(\frac{\sqrt{\nu^2 + x^2}}{2}\right)(t - t_0)^2\right) \mathrm{d}t. \end{aligned} \tag{23}$$

Since this is an interior maximum, we use the full Gaussian integral. Setting

$$u = \frac{(\nu^2 + x^2)^{1/4}}{\sqrt{2}} \cdot (t - t_0) \quad \Rightarrow \mathrm{d}t = \frac{\sqrt{2}}{(\nu^2 + x^2)^{1/4}}\, \mathrm{d}u, \tag{24}$$

we have:

$$K_\nu(x) \approx \left(\frac{1}{2}\right) e^{\Phi(t_0)} \frac{\sqrt{2}}{(\nu^2 + x^2)^{\frac{1}{4}}} \int_{-\infty}^{\infty} e^{-u^2}\, \mathrm{d}u = \boxed{\sqrt{\frac{\pi}{2}} \frac{\exp\left(\nu \operatorname{arcsinh}\left(\frac{\nu}{x}\right) - \sqrt{\nu^2 + x^2}\right)}{(\nu^2 + x^2)^{\frac{1}{4}}}}. \tag{25}$$

P5

In the limit of $\nu \to \infty, x > 0$, find the asymptotic form of the Weber function

$$D_{-\nu-1}(x) = \frac{e^{-x^2/4}}{\Gamma(\nu+1)} \int_0^\infty t^\nu \exp\left(-xt - \frac{t^2}{2}\right) \mathrm{d}t. \tag{26}$$

Recall the estimate for the Gamma function at large argument (from lecture):

$$\Gamma(\nu+1) \approx \sqrt{2\pi\nu}\left(\frac{\nu}{e}\right)^\nu. \tag{27}$$

We read off the Laplace integral form with $\Phi(t) = \nu \ln t - xt - \frac{t^2}{2}$:

$$J(\nu) = \int_0^\infty e^{\Phi(t)} \,\mathrm{d}t, \quad D_{-\nu-1}(x) = e^{-\frac{x^2}{4}} \frac{J(\nu)}{\Gamma(\nu+1)}. \tag{28}$$

Examine $\Phi(t)$, find stationary point with

$$\Phi'(t) = \frac{\nu}{t} - x - t = 0 \Rightarrow t^2 + xt - \nu = 0, \quad t_0 = \frac{-x + \sqrt{x^2 + 4\nu}}{2} > 0, \tag{29}$$

which is an interior point ($t_0 \sim \sqrt{\nu} \gg 0$). Further,

$$\Phi''(t) = -\frac{\nu}{t^2} - 1, \quad \Phi''(t_0) = -\left(2 + \frac{x}{t_0}\right) < 0. \tag{30}$$

Thus t_0 is a maximum in the interior. We expand around t_0:

$$\Phi(t) \approx \Phi(t_0) + \left(\frac{1}{2}\right)\Phi''(t_0)(t-t_0)^2, \quad \Phi(t_0) = \nu \ln t_0 - \frac{\nu}{2} - \frac{x\sqrt{x^2+4\nu}}{4}. \tag{31}$$

So

$$J(\nu) \approx e^{\Phi(t_0)} \int_{-\infty}^\infty \exp\left(\left(\frac{1}{2}\right)\Phi''(t_0)(t-t_0)^2\right) \mathrm{d}t. \tag{32}$$

Since this is an interior maximum far from $t = 0$ (Gaussian width $\sim \frac{1}{\sqrt{-\Phi''(t_0)}} = O(1) \ll t_0$), we use the full Gaussian integral. Setting $u = \sqrt{-\Phi''(t_0)}(t - t_0), \mathrm{d}t = \mathrm{d}\frac{u}{\sqrt{-\Phi''(t_0)}}$, we have:

$$J(\nu) \approx e^{\Phi(t_0)} \sqrt{2\frac{\pi}{-\Phi''(t_0)}} = e^{\Phi(t_0)} \sqrt{2\frac{\pi}{2 + \frac{x}{t_0}}}. \tag{33}$$

Thus

$$D_{-\nu-1}(x) \approx \frac{e^{-\frac{x^2}{4}} e^{\Phi(t_0)} \sqrt{2\frac{\pi}{2+\frac{x}{t_0}}}}{\Gamma(\nu+1)}. \tag{34}$$

Using Stirling's approximation $\frac{1}{\Gamma(\nu+1)} \approx \frac{e^\nu}{\sqrt{2\pi\nu}\nu^\nu}$, and noting $\Phi(t_0) + \frac{x^2}{4} = \nu \ln t_0 - \frac{\nu}{2} - x\frac{\sqrt{x^2+4\nu}}{4}$, we obtain the saddle-point form:

$$\boxed{\frac{\sqrt{2\pi} \exp\left(\nu \ln t_0 - \frac{\nu}{2} - \frac{x\sqrt{x^2+4\nu}}{4}\right)}{\sqrt{2 + \frac{x}{t_0}}\Gamma(\nu+1)}}. \tag{35}$$

For large ν, with $t_0 \sim \sqrt{\nu} - \frac{x}{2}$, this simplifies to the leading behavior:

$$\boxed{\frac{1}{\sqrt{2\nu}}\left(\frac{e}{\nu}\right)^{\frac{\nu}{2}} e^{-x\sqrt{\nu}}.} \tag{36}$$