P1

Proof:

1. First, prove the relation to be equivalent:
   - Reflective: $a \sim a \Rightarrow f(a) = f(a)$, TRUE.
   - Symmetric: $a \sim b \Rightarrow f(a) = f(b) \Rightarrow f(b) = f(a) \Rightarrow b \sim a$, TRUE.
   - transitive: $a \sim b, b \sim c \Rightarrow f(a) = f(b), f(b) = f(c) \Rightarrow f(a) = f(c) \Rightarrow a \sim c$, TRUE.
2. Then, prove its equivalence classes to be the fibers of f:

   Let C be the set of equivalence classes of A under $\sim$, and let F be the set of fibers of f. We will show that $C = F$.

   Take an arbitrary element $a \in A$. The equivalence class of $a \in A$ is:

$$\begin{aligned} \{x \in A \mid x \sim a\} &= \{x \in A \mid f(x) = f(a)\} \\ &= f^{-1}\{f(a)\} \end{aligned} \tag{1}$$

   which by definition is the fiber of f.

   Since a was arbitrary, every equivalence class is a fiber of f, i.e. $C \subseteq F$.

   Conversely, let F' be an arbitrary fiber of f for some $b \in B$. Then by definition,

$$\begin{aligned} F' &= f^{-1}\{b\} \\ &= \{x \in A \mid f(x) = b\} \end{aligned} \tag{2}$$

   .

   Since f is surjective, $\exists a \in A \ s.t. \ f(a) = b$. Consider the equivalence class of a:

$$\begin{aligned} \{x \in A \mid x \sim a\} &= \{x \in A \mid f(x) = f(a)\} \\ &= \{x \in A \mid f(x) = b\} \\ &= F'. \end{aligned} \tag{3}$$

   Since F' was arbitrary, every fiber of f is an equivalence class, i.e. $F \subseteq C$. Thus, $C = F$. ■

P2

Prove by contradiction:

1. Consider an arbitrary **column** in the multiplication table of G. Suppose that the colum is *not* a permutation of G.

   Then there would be at least two identical elements in this column, which we denote as a . This implies that

   $$\exists x, y \in G, x \neq y, s.t.\ xa = ya \tag{4}$$

   Applying x^{-1} from right on both sides:

   $$\begin{aligned} x^{-1}xa &= x^{-1}ya \\ a &= x^{-1}ya \\ \Rightarrow x^{-1}y &= e. \end{aligned} \tag{5}$$

   Since inverse of an element is unique, $y = x$, which is a contradiction.

2. Similarly, consider arbitrary **row** in the multiplication table of G . Suppose that this row is *not* a permutation of G , i.e. there are at least two repeating elements, denoted as b. This implies

   $$\exists x, y \in G, x \neq y, s.t.\ xa = xb. \tag{6}$$

   Applying a^{-1} from left on both sides:

   $$\begin{aligned} xaa^{-1} &= xba^{-1} \\ x &= xba^{-1} \\ \Rightarrow ba^{-1} &= e. \end{aligned} \tag{7}$$

   Since inverse of an element is unique, $b = a$, a contradiction. ∎

3. Multiplication tables are special cases of Latin squares, in particular, they hold hold the property of associativity. This restricts the set of possible Latin squares, because:

   The group operation must be associative, menaing for every single combiniation of three elements, $a, b, c \in G, (ab)c = a(bc)$.

   In a table, this means:

   - let entry $(a, b) := d$ and entry $(d, c) := e$, then we must have entry (d, c) equal to entry (a, e).

   This is a strong restriction on the possible arrangements of elements in a Latin square, and thus only a small subset of Latin squares can be multiplication tables of groups.

P3

We check each axiom one by one:

Closure: Satisfied.

For any $a, b \in \mathbb{R}, a + b \in \mathbb{R}_{\text{ext}}$.
If at least one of the numbers is ∞, the sum is $\infty \in \mathbb{R}_{\text{ext}}$.

associativity: Satisfied.

We want to show that for any $a, b, c \in \mathbb{R}_{\text{ext}}, (a + b) + c = a + (b + c)$. We have two cases:

- If all elements are real, then the sum is trivially associative.
- If at least one element is ∞, then both sides equal ∞.

Identity: Satisfied.

The identity element is $0 \in \mathbb{R}_{\text{ext}}$. For any $a \in \mathbb{R}_{\text{ext}}, a + 0 = 0 + a = a$.

Inverse: NOT satisfied.

Assume not, then for $\infty \in \mathbb{R}_{\text{ext}}, \exists a \in \mathbb{R}_{\text{ext}} s.t. a + \infty = 0$. This is a contradiction, since $a + \infty = \infty$ for any $a \in \mathbb{R}_{\text{ext}}$.

Therefore, $(\mathbb{R}_{\text{ext}}, +)$ is not a group. ■

P4

$$G = \{z \in \mathbb{C} \mid z^n = 1 \text{ for some } n \in \mathbb{Z}^+\} \tag{8}$$

a. Prove that G is a group under multiplication.

We check for each axiom:

Closure:

let $a, b \in G$, then $a^{n_1} = 1, b^{n_2} = 1$, for some $n_1, n_2 \in \mathbb{Z}^+$. Need to show that $ab \in G \Leftrightarrow (ab)^k = 1$ for some $k \in \mathbb{Z}^+$.

Take $k = n_1 n_2$, then

$$(ab)^k = a^{n_1 n_2} b^{n_1 n_2} = 1^{n_2} 1^{n_1} = 1. \tag{9}$$

Exists such k, and so $ab \in G$, i.e. closure is satisfied.

Assoc.

Taivially satisfied, as $G \subset \mathbb{C}$, each element is a complex number, and multiplication of complex numbers is associative.

Identity.

Trivially satisfied, as $1 \in G$ (take $n = 1$), and for any $a \in G, a1 = 1a = a$.

Inverse.

Consider arbitrary $a \in G$. Exists $n \in \mathbb{Z}^+ \; s.t. \; a^n = 1$. Rewriting,

$$a^{n-1} a = 1 \Rightarrow a^{n-1} = a^{-1}. \tag{10}$$

Since $\left(z^{n-1}\right)^n = (z^n)^{n-1} = 1$, $z^{n-1} \in G$.

Therefore, $(G, \times)$ is a group. ■

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b. $(G, +)$ is not a group.

Assume identity exists, then for any $a \in G$,

$$e + a = a + e = a. \tag{11}$$

Since $a, e \in \mathbb{C}$, the identity must be 0. However, $0 \notin G$, since $0^n = 0$ for any $n \in \mathbb{Z}^+$, a contradiction. ■

P5