ECE 535 Fall 2025
Homework #3
Due Thursday 10/23 at 11:59 pm on Canvas, in pdf format

Guidelines:

- Please submit a pdf document to Canvas with handwritten solutions, with your approach to each problem and the steps taken clearly laid out and written legibly. In cases where the solution requires plotting, the computer-generated plot should be accompanied by a brief handwritten explanation of your approach. This formatting requirement is worth **5 points** of the point total for each homework.
- Undergraduate students who wish to attempt the extra problem will receive up to 5 additional points for that homework.

1. (10 points) A nanomechanical resonator is a tiny vibrating beam or membrane that can move back and forth like a spring. These devices can detect extremely small forces or masses, and they are often modeled as quantum harmonic oscillators where the energy levels are $E_n = \left(n + \frac{1}{2}\right)\hbar\omega$, where n is an integer and ω is the angular oscillation frequency. Consider such a resonator with:

- Effective mass $m = 10^{-15}$ kg
- Linear oscillation frequency $\nu = 1$ MHz

(a) (4 points) Calculate the numerical value of the zero-point energy of the resonator.

(b) (3 points) The zero-point fluctuation amplitude is

$$x_{\text{zpf}} = \sqrt{\frac{\hbar}{2m\omega}}.$$

Compute x_{zpf}. What is the physical meaning of this fluctuation and what limitation does it impose on the ability of this sensor to measure small displacements?

(c) (3 points) At temperature $T = 10$ mK, compute the thermal energy $k_B T$.
Compare $k_B T$ with $\hbar\omega$ and determine if the resonator is likely to be in its ground state.

2. (10 points) Two perfectly conducting parallel plates are used inside a nanoscale sensor. Each plate has an area of $A = (10\ \mu\text{m}) \times (10\ \mu\text{m}) = 100\ \mu\text{m}^2$. They are separated by a distance $d = 100$ nm.

(a) (4 points) Compute the magnitude of the Casimir force and describe if the force is attractive or repulsive.

(b) (3 points) Compare this force to the weight of a water droplet of mass 1 μg.

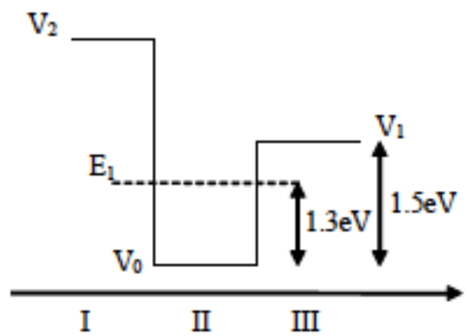
(c) (3 points) Explain why Casimir effects are important in micro- and nano-scale devices.

3. (10 points) **One-dimensional barrier well**
A one-dimensional potential well has a barrier of height 1.5 eV on the right-hand side and a barrier higher than this on the left-hand side. This potential well has an energy eigenstate for an electron at 1.3 eV. All energy values are relative to the bottom of the well.

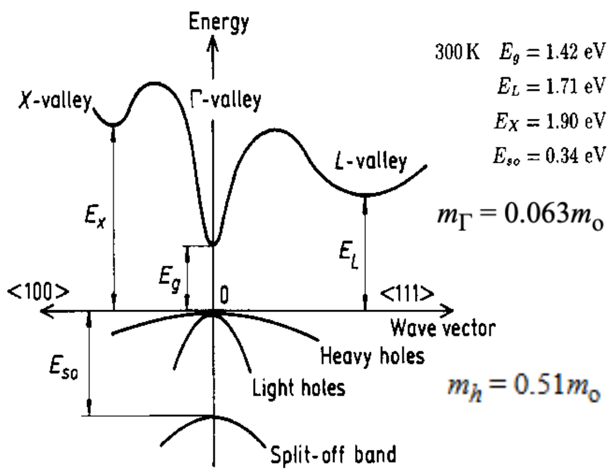
State the general form of the wavefunction solution (without solving for the expression of the normalizing constant) in the following two cases and give the values for any wavevector magnitude k and/or decay constant κ in these wavefunctions:

(a) (5 points) Inside the well

(b) (5 points) In the barrier on the right side



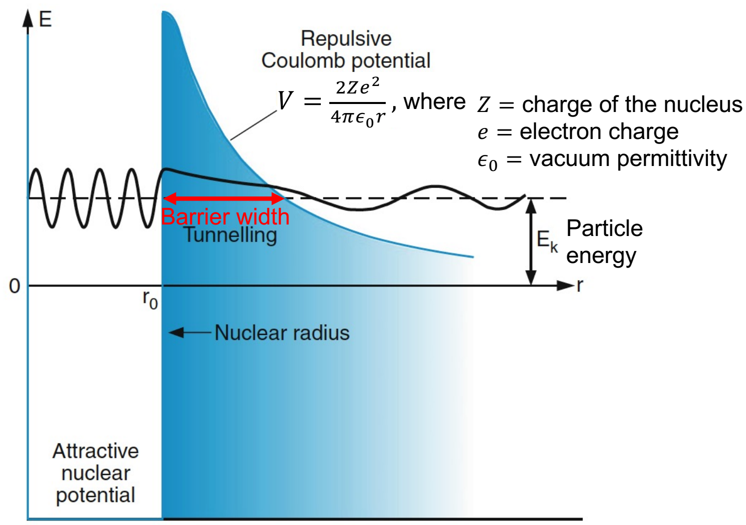
4. (10 points) Using the data for GaAs shown below, calculate the estimated emission wavelengths of a GaAs quantum well structure for a well thickness of 25 nm.



Band structure of GaAs (Ioffe)

5. (25 points) **Alpha decay of radon gas**

In this problem, we will model the alpha particle decay of radon-222 ($^{222}_{86}Rn$, where the atomic mass number is 222 and the number of protons is 86) into polonium-218 ($^{218}_{84}Po$). An alpha particle ($^{4}_{2}He$) disintegrated from the nucleus experiences the strong nuclear force within the nuclear radius and the coulomb potential outside the nucleus.



(a) (2 points) Using energy and momentum conservation, estimate the mean energy, in MeV, for alpha particles after the decay of ${}^{222}_{86}Rn$ into ${}^{218}_{84}Po$. You can use the following masses:

$$m({}^{222}_{86}Rn) = (222.017577u)\ 1.660539 \times 10^{-27}\ kg/u$$
$$m({}^{218}_{84}Po) = (218.008965u)\ 1.660539 \times 10^{-27}\ kg/u$$
$$m(\alpha) = (4.001506u)\ 1.660539 \times 10^{-27}\ kg/u$$

Compare your result to the actual alpha particle energy of 5.5904 MeV. You may use this actual value for the rest of the problem.

(b) (3 points) Calculate the Coulomb potential energy, in MeV, experienced by the alpha particle at the nuclear radius $V(r_0)$. Here r_0 is 7.2660587$\times$ 10^{-15} m.

(c) (2 points) Calculate the barrier width of the Coulomb potential.

(d) (5 points) Calculate the tunneling probability and associated decay rate (in s^{-1}) for the alpha particle, assuming a rectangular barrier with height $V(r_0)$ and the barrier width calculated in (b). Compare your result with the actual half-life of the isotope, 3.8215 days.

(e) (3 points) To better approximate the Coulomb potential, we can break the barrier into multiple segments of equal width, each consisting of a rectangular barrier with a height that is the midpoint of the segment. Construct a Coulomb potential by breaking V into <u>seven</u> segments within the barrier width. Plot this constructed potential V_{approx} along with V.

(f) (10 points) Calculate the tunneling probability and associated decay rate for the alpha particle, assuming the potential V_{approx} you constructed in (d). Compare your result with the actual half-life of the isotope, 3.8215 days.

6. **Extra problem for graduate students** (10 points): Colloidal quantum dots (for which Moungi Bawendi, Louis Brus, and Alexei Ekimov won the 2023 Nobel Prize in Chemistry) exhibit size-dependent optical emission due to quantum confinement. In this problem, we will solve the Schrodinger equation to model the quantum confinement effect in these quantum dots.

(a) (7 points) Solve the Schrodinger equation for a particle in a spherical potential where $V(r \leq a) = 0$ and $V(r > a) = \infty$. What are valid forms of the wave function and the quantized energy levels?

(b) (3 points) Cadmium Selenide (CdSe) is a common material used to synthesize quantum dots. Given the following material properties, calculate the diameter of a CdSe dot that will give an emission wavelength of 500 nm.

Effective mass of the exciton for CdSe $= 0.1009\ m_e$

Bandgap of bulk CdSe $= 1.74$ eV

Rest mass of the electron: $m_e = 9.109 \times 10^{-31}\ kg$