[S1r 3.2, S2 3.3, S3 3.4] Consider the 2×2 matrix defined by

$$U = \frac{a_0 + i\boldsymbol{\sigma} \cdot \mathbf{a}}{a_0 - i\boldsymbol{\sigma} \cdot \mathbf{a}} = (a_0 + i\boldsymbol{\sigma} \cdot \mathbf{a})(a_0 - i\boldsymbol{\sigma} \cdot \mathbf{a})^{-1},$$

where a_0 is a real number and $\mathbf{a}$ is a three-dimensional vector with real components.
(a) Prove that U is unitary and unimodular.
(b) In general, a 2×2 unitary unimodular matrix represents a rotation in three dimensions. Find the axis and angle of rotation appropriate for U in terms of a_0, a_1, a_2, and a_3.

Unitarity ($U^\dagger U = 1$):
Let

$$\begin{gathered} N = a_0 + i\boldsymbol{\sigma} \cdot \boldsymbol{a}, \\ \Rightarrow a_0 - i\boldsymbol{\sigma} \cdot \boldsymbol{a} = N^\dagger, \left[N, N^\dagger\right] = 0. \end{gathered} \tag{1}$$

Then

$$U^\dagger = \left(N\left(N^\dagger\right)^{-1}\right)^\dagger = \left(N^\dagger\right)^{-1\dagger} N^\dagger = N^{-1} N^\dagger \tag{2}$$

$$UU^\dagger = N\left(N^\dagger\right)^{-1} N^{-1} N^\dagger = N N^{-1} \left(N^\dagger\right)^{-1} N^\dagger = 1 \cdot 1 = 1 \tag{3}$$

and is thus unitary.

Unimodularity ($\det U = 1$):
Using the identity $\det(cI + \boldsymbol{\sigma} \cdot \boldsymbol{v}) = c^2 + |\boldsymbol{v}|^2$ (where $|\boldsymbol{v}|^2 = \boldsymbol{v} \cdot \boldsymbol{v}$ for complex vectors, here just a^2):

$$\det U = \frac{\det(a_0 + i\boldsymbol{\sigma} \cdot \boldsymbol{a})}{\det(a_0 - i\boldsymbol{\sigma} \cdot \boldsymbol{a})} = \frac{a_0^2 + |\boldsymbol{a}|^2}{a_0^2 + |\boldsymbol{a}|^2} = 1. \tag{4}$$

b. Axis $\hat{n}$ and Angle θ

We seek U in standard form $\cos\left(\frac{\theta}{2}\right) - i(\boldsymbol{\sigma} \cdot \hat{n}) \sin\left(\frac{\theta}{2}\right)$.

Multiply numerator and denominator by $N = a_0 + i\boldsymbol{\sigma} \cdot \boldsymbol{a}$:

$$U = \frac{(a_0 + i\boldsymbol{\sigma} \cdot \boldsymbol{a})^2}{a_0^2 + \boldsymbol{a}^2} = \frac{a_0^2 - \boldsymbol{a}^2 + 2ia_0(\boldsymbol{\sigma} \cdot \boldsymbol{a})}{a_0^2 + \boldsymbol{a}^2} \tag{5}$$

Separating real and imaginary parts:

$$U = \underbrace{\frac{a_0^2 - \boldsymbol{a}^2}{a_0^2 + \boldsymbol{a}^2}}_{\cos(\theta/2)} + i(\boldsymbol{\sigma} \cdot \boldsymbol{a}) \frac{2a_0}{a_0^2 + \boldsymbol{a}^2} \tag{6}$$

Matching terms with starndard form:

$$\begin{gathered} \hat{n} = -\frac{\boldsymbol{a}}{|\boldsymbol{a}|}, \\ \cos\left(\frac{\theta}{2}\right) = \frac{a_0^2 - \boldsymbol{a}^2}{a_0^2 + \boldsymbol{a}^2} \Rightarrow \quad \theta = 2 \arccos\left(\frac{a_0^2 - \boldsymbol{a}^2}{a_0^2 + \boldsymbol{a}^2}\right). \end{gathered} \tag{7}$$

2

[S1r 3.8, S2 3.9, S3 3.10] Consider a sequence of Euler rotations represented by

$$\begin{aligned} D^{(1/2)}(\alpha,\beta,\gamma) &= \exp\left(\frac{-i\sigma_3\alpha}{2}\right)\left(\frac{-i\sigma_2\beta}{2}\right)\left(\frac{-i\sigma_3\gamma}{2}\right) \\ &= \begin{pmatrix} e^{-i(\alpha+\gamma)/2}\cos(\beta/2) & -e^{-i(\alpha-\gamma)/2}\sin(\beta/2) \\ e^{i(\alpha-\gamma)/2}\sin(\beta/2) & e^{i(\alpha+\gamma)/2}\cos(\beta/2) \end{pmatrix}. \end{aligned}$$

Because of the group properties of rotations, we expect that this sequence of operations is equivalent to a single rotation about some axis by an angle θ. Find θ.

Given the matrix $D^{(\frac{1}{2})}(\alpha,\beta,\gamma)$, we find the equivalent single rotation angle θ. The trace (character) of a rotation matrix is basis independent:

$$\chi(\theta) = \mathrm{Tr}(D) = 2\cos\left(\frac{\theta}{2}\right) \tag{8}$$

Computing the trace of the given matrix:

$$\begin{aligned} \mathrm{Tr}(D) &= e^{-\frac{i(\alpha+\gamma)}{2}}\cos\left(\frac{\beta}{2}\right) + e^{\frac{i(\alpha+\gamma)}{2}}\cos\left(\frac{\beta}{2}\right) \\ &= \cos\left(\frac{\beta}{2}\right)\left(e^{-\frac{i(\alpha+\gamma)}{2}} + e^{\frac{i(\alpha+\gamma)}{2}}\right) \\ &= 2\cos\left(\frac{\beta}{2}\right)\cos\left(\frac{\alpha+\gamma}{2}\right) \end{aligned} \tag{9}$$

Equating the traces yields:

$$\theta = 2\arccos\left(\cos\left(\frac{\beta}{2}\right)\cos\left(\frac{\alpha+\gamma}{2}\right)\right). \tag{10}$$

3

Show by explicit calculation that if $|j\,m\rangle$ is an eigenfunction of J_z with eigenvalue $m\hbar$, then

$$e^{-iJ_z\varphi/\hbar}e^{-iJ_y\theta/\hbar}|j\,m\rangle$$

is an eigenfunction of the operator

$$J_x\cos\varphi\sin\theta + J_y\sin\varphi\sin\theta + J_z\cos\theta$$

with eigenvalue $m\hbar$.

Proof: Let $U = e^{-iJ_z\varphi/\hbar}e^{-iJ_y\theta/\hbar}$. We check if $U^\dagger\hat{O}U = J_z$.

$$\hat{O}_{\text{rot}} = U^\dagger\hat{O}U = e^{iJ_y\theta/\hbar}\left(e^{iJ_z\varphi/\hbar}\hat{O}e^{-iJ_z\varphi/\hbar}\right)e^{-iJ_y\theta/\hbar} \tag{11}$$

1. **Rotation by $-\varphi$ about z:** $J_x \to J_x\cos\varphi - J_y\sin\varphi$, $J_y \to J_x\sin\varphi + J_y\cos\varphi$. Substituting into $\hat{O}$, the $\sin\theta$ terms simplify:

$$\sin\theta\left(J_x\left(c_\varphi^2 + s_\varphi^2\right)\right) + J_z\cos\theta = J_x\sin\theta + J_z\cos\theta \tag{12}$$

2. **Rotation by $-\theta$ about y:** $J_x \to J_x\cos\theta - J_z\sin\theta$, $J_z \to J_z\cos\theta + J_x\sin\theta$.

$$\begin{aligned}\hat{O}_{\text{rot}} &= (J_x\cos\theta - J_z\sin\theta)\sin\theta + (J_z\cos\theta + J_x\sin\theta)\cos\theta \\ &= J_x(\sin\theta\cos\theta - \sin\theta\cos\theta) + J_z(\cos^2\theta + \sin^2\theta) = J_z\end{aligned} \tag{13}$$

Since $U^\dagger\hat{O}U = J_z$, then $\hat{O}U = UJ_z$. Applying to eigenstate $|j,m\rangle$:

$$\hat{O}|\psi\rangle = \hat{O}U|j,m\rangle = UJ_z|j,m\rangle = m\hbar U|j,m\rangle = m\hbar|\psi\rangle. \tag{14}$$

4

Let $\mathcal{J} \equiv \mathbf{J}^{(1)} \cdot \hat{n}/\hbar$ for some fixed $\hat{n}$, where $\mathbf{J}^{(1)}$ are the angular momentum matrices for $j = 1$.
(a) Using the fact that the eigenvalues of $\mathcal{J}$ are $1,\ 0,\ -1$, prove that $\mathcal{J}^3 = \mathcal{J}$.
(b) Prove that $D^{(1)}(R_{\hat{n}}(\phi)) = 1 - i\mathcal{J}\sin\phi - \mathcal{J}^2(1-\cos\phi)$.
(c) Derive $d^{(1)}(\beta)$.

Let $\mathcal{J} = \boldsymbol{J}^{(1)} \cdot \hat{n}/\hbar$. The eigenvalues are $1, 0, -1$.

a. Characteristic Equation

Since the eigenvalues are $\lambda \in \{1, 0, -1\}$, the operator satisfies:

$$(\mathcal{J} - 1)(\mathcal{J} - 0)(\mathcal{J} + 1) = \mathcal{J}(\mathcal{J}^2 - 1) = 0 \Rightarrow \mathcal{J}^3 = \mathcal{J}. \tag{15}$$

b. Rotation Formula

The Taylor expansion of $D = e^{-i\mathcal{J}\varphi}$:

$$D = 1 - i\mathcal{J}\varphi - \frac{\mathcal{J}^2\varphi^2}{2!} + \frac{i\mathcal{J}^3\varphi^3}{3!} + \ldots \tag{16}$$

Using $\mathcal{J}^3 = \mathcal{J}$ (and $\mathcal{J}^4 = \mathcal{J}^2$), we group terms:

$$D = 1 - i\mathcal{J}\left(\varphi - \frac{\varphi^3}{3!} + \ldots\right) - \mathcal{J}^2\left(\frac{\varphi^2}{2!} - \frac{\varphi^4}{4!} + \ldots\right) \tag{17}$$

Recognizing the expansion of sine and cosine,

$$D = 1 - i\mathcal{J}\sin\varphi - \mathcal{J}^2(1 - \cos\varphi) \tag{18}$$

c. Matrix for β rotation ($\hat{n} = \hat{y}$)

IN the standard basis,

$$\mathcal{J}_y = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \Rightarrow \mathcal{J}_y^2 = \frac{1}{2}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix} \tag{19}$$

Substituting into the formula from (b) with $\varphi = \beta$:

$$d^{(1)}(\beta) = I - i\mathcal{J}_y\sin\beta - \mathcal{J}_y^2(1 - \cos\beta) \tag{20}$$

Calculating term by term yields:

$$d^{(1)}(\beta) = \begin{pmatrix} \frac{1+\cos\beta}{2} & -\frac{\sin\beta}{\sqrt{2}} & \frac{1-\cos\beta}{2} \\ \frac{\sin\beta}{\sqrt{2}} & \cos\beta & -\frac{\sin\beta}{\sqrt{2}} \\ \frac{1-\cos\beta}{2} & \frac{\sin\beta}{\sqrt{2}} & \frac{1+\cos\beta}{2} \end{pmatrix} \tag{21}$$

5

[S1r 3.17, S2 3.19, S3 3.25] Suppose a half-integer ℓ value, say $1/2$, were allowed for orbital angular momentum. From $L_+ Y_{\frac{1}{2}\frac{1}{2}}(\theta, \phi) = 0$, we may deduce as usual $Y_{\frac{1}{2}\frac{1}{2}}(\theta, \phi) \sim e^{i\phi/2}\sqrt{\sin\theta}$. Now try to construct $Y_{\frac{1}{2}-\frac{1}{2}}$ by (a) applying L_- to $Y_{\frac{1}{2}\frac{1}{2}}$; and (b) using $L_- Y_{\frac{1}{2}-\frac{1}{2}}(\theta, \phi) = 0$. Show that the two procedures lead to contradictory results.

Objective: Demonstrate that assuming half-integer orbital angular momentum leads to a contradiction in position space wavefunctions.

Given:
- Ladder operators: $L_\pm = \hbar e^{\pm i\varphi}\left(\pm\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\varphi}\right)$.
- Top state assumption: $L_+ Y_{\frac{1}{2},\frac{1}{2}}(\theta, \varphi) = 0 \Rightarrow Y_{\frac{1}{2},\frac{1}{2}} \sim e^{i\frac{\varphi}{2}}\sqrt{\sin\theta}$.

a. Construct $Y_{\frac{1}{2},-\frac{1}{2}}$ by applying L_-

We apply the lowering operator to the top state $f(\theta, \varphi) = e^{i\frac{\varphi}{2}}\sqrt{\sin\theta}$.

$$L_- f = \hbar e^{-i\varphi}\left(-\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\varphi}\right)\left(e^{i\frac{\varphi}{2}}\sqrt{\sin\theta}\right) \tag{22}$$

Calculate the derivatives:

1. $$\frac{\partial}{\partial\theta}\sqrt{\sin\theta} = \frac{1}{2}(\sin\theta)^{-\frac{1}{2}}\cos\theta \tag{23}$$

2. $$\frac{\partial}{\partial\varphi}e^{i\frac{\varphi}{2}} = \frac{i}{2}e^{i\frac{\varphi}{2}} \tag{24}$$

Substitute back:

$$\begin{aligned} L_- f &\propto e^{-i\varphi}\left[-\frac{\cos\theta}{2\sqrt{\sin\theta}} + i\cot\theta\left(\frac{i}{2}\right)\right]e^{i\frac{\varphi}{2}}\sqrt{\sin\theta} \\ &= e^{-i\frac{\varphi}{2}}\left[-\frac{\cos\theta}{2\sqrt{\sin\theta}} - \frac{1}{2}\frac{\cos\theta}{\sin\theta}\sqrt{\sin\theta}\right] \\ &= e^{-i\frac{\varphi}{2}}\left[-\frac{\cos\theta}{2\sqrt{\sin\theta}} - \frac{\cos\theta}{2\sqrt{\sin\theta}}\right] = -e^{-i\frac{\varphi}{2}}\frac{\cos\theta}{\sqrt{\sin\theta}} \end{aligned} \tag{25}$$

Thus, via lowering:

$$Y^{(a)}_{\frac{1}{2},-\frac{1}{2}} \propto e^{-i\frac{\varphi}{2}}\frac{\cos\theta}{\sqrt{\sin\theta}} \tag{26}$$

b. Construct $Y_{\frac{1}{2},-\frac{1}{2}}$ by using the annihilation condition

Since $m = -\frac{1}{2}$ is the bottom of the ladder for $j = \frac{1}{2}$, we must have $L_- Y_{\frac{1}{2},-\frac{1}{2}} = 0$. Let $g(\theta, \varphi) = \Theta(\theta)e^{-i\frac{\varphi}{2}}$.

$$L_- g = \hbar e^{-i\varphi}\left(-\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\varphi}\right)\Theta(\theta)e^{-i\frac{\varphi}{2}} = 0 \tag{27}$$

$$\left(-\theta'(\theta) + i\cot\theta\left(-\frac{i}{2}\right)\Theta\right) = 0 \Rightarrow \theta' + \frac{1}{2}\cot\theta\Theta = 0 \tag{28}$$

Rearranging $\frac{d\Theta}{\Theta} = \frac{1}{2}\frac{\cos\theta}{\sin\theta}d\theta$, integrating gives $\ln\Theta = \frac{1}{2}\ln(\sin\theta)$, so $\Theta \propto \sqrt{\sin\theta}$. Thus, via the bottom-rung condition:

$$Y^{(b)}_{\frac{1}{2},-\frac{1}{2}} \propto e^{-i\frac{\varphi}{2}}\sqrt{\sin\theta} \tag{29}$$

Conclusion: The function from (a) diverges at poles $\theta = 0, \pi$, while (b) vanishes. Since $Y^{(a)} \neq Y^{(b)}$, half-integer OAM is inconsistent with single-valued position-space wavefunctions.

Consider the attractive spherical well potential,

$$V(r) = \begin{cases} -V_0, & r < r_0, \\ 0, & r > r_0, \end{cases}$$

in which $V_0 > 0$.
(a) Show that the quantization condition for $\ell = 0$ bound states is

$$\frac{k}{\kappa} = -\tan(kr_0),$$

where $k = (2m(E+V_0)/\hbar^2)^{1/2}$ and $\kappa = (-2mE/\hbar^2)^{1/2}$.
(b) Show that there are no solutions for $V_0 < \pi^2\hbar^2/(8mr_0^2)$.
(c) Show there is one solution for $V_0 = 9\pi^2\hbar^2/(32mr_0^2)$, and find the numerical value of $(-2mEr_0^2/\hbar^2)$

Potential: $V(r) = -V_0$ for $r < r_0$ and $V(r) = 0$ for $r > r_0$.

a. Quantization Condition ($l = 0$)

Let $u(r) = rR(r)$. The radial equation is $u'' + 2\frac{m}{\hbar^2}(E - V)u = 0$.

- **Region I ($r < r_0$):** $k = \frac{\sqrt{2m(E+V_0)}}{\hbar}$. Solution $u_{\text{in}} \sim \sin(kr)$.
- **Region II ($r > r_0$):** $\kappa = \frac{\sqrt{-2mE}}{\hbar}$. Solution $u_{\text{out}} \sim e^{-\kappa r}$.

Matching logarithmic derivative $\frac{u'}{u}$ at r_0:

$$\frac{k\cos(kr_0)}{\sin(kr_0)} = \frac{-\kappa e^{-\kappa r_0}}{e^{-\kappa r_0}} \Rightarrow k\cot(kr_0) = -\kappa \tag{30}$$

Inverting the cotangent:

$$\frac{k}{\kappa} = -\tan(kr_0) \tag{31}$$

b. No solutions for shallow wells

Let $\xi = kr_0$ and $\eta = \kappa r_0$.

$$\xi^2 + \eta^2 = \frac{2m(E+V_0)r_0^2}{\hbar^2} - \frac{2mEr_0^2}{\hbar^2} = \frac{2mV_0 r_0^2}{\hbar^2} \equiv \gamma^2 \tag{32}$$

The condition is $\tan\xi = -\frac{\xi}{\eta} = -\frac{\xi}{\sqrt{\gamma^2-\xi^2}}$. For a bound state ($E < 0$), $\kappa > 0$, so RHS is negative. $\tan\xi$ is negative first in $(\frac{\pi}{2}, \pi)$. Threshold condition: The "circle" radius γ must extend to the asymptote $\frac{\pi}{2}$.

$$\gamma \geq \frac{\pi}{2} \Rightarrow \frac{2mV_0 r_0^2}{\hbar^2} \geq \frac{\pi^2}{4} \Rightarrow V_0 \geq \frac{\pi^2\hbar^2}{8mr_0^2}. \tag{33}$$

c. Numerical Value

Given $V_0 = \frac{9\pi^2\hbar^2}{32mr_0^2}$, we find $\gamma = 3\frac{\pi}{4} \approx 2.356$. We solve $\tan\xi = -\frac{\xi}{\sqrt{\gamma^2-\xi^2}}$. Using identity $\sin^2\xi = \tan^2\frac{\xi}{1+\tan^2\xi} = \frac{\xi^2}{\gamma^2}$:

$$\sin\xi = \pm\frac{\xi}{\gamma} = \frac{\xi}{3\pi/4} \tag{34}$$

Solving $x = \frac{3\pi}{4}\sin x$ numerically yields $\xi \approx 2.07$. The requested value is $\kappa r_0 = \sqrt{\gamma^2 - \xi^2} = \sqrt{(2.356)^2 - (2.07)^2} \approx 1.13$.

Consider the three-dimensional isotropic simple harmonic oscillator, which has the potential

$$V(r) = \frac{1}{2}m\omega^2 r^2.$$

a) To solve for the radial eigenfunction $u(r)$ (subscripts for E and ℓ are implied), change coordinates from r to $\rho = r(\sqrt{m\omega/\hbar})$, and set

$$u(\rho) = e^{-\rho^2/2}v(\rho).$$

Starting from this form, find the differential equation for $v(\rho)$.
b) Verify that the expected energy eigenvalue condition is found by solving the differential equation found in (a). You should proceed by introducing the appropriate series solution for $v(\rho)$, keeping track of the appropriate limiting behavior as $\rho \to 0$, and determining the condition to truncate the series so as to keep the state normalizable.
c) For the lowest two levels of the principal quantum number n, write expressions for the states in both the Cartesian basis and the spherical basis. In the case of level degeneracy, provide the transformation matrix that relates the two bases.

a. Differential Equation

Let $\rho = r\sqrt{m\omega/\hbar}$. With $E = \hbar\omega\varepsilon$, substitute $u(\rho) = e^{-\frac{\rho^2}{2}}v(\rho)$:

$$-[v'' - 2\rho v' + (\rho^2 - 1)v] + \left[\frac{l(l+1)}{\rho^2} + \rho^2 - 2\varepsilon\right]e^{-\frac{\rho^2}{2}}v = 0 \tag{35}$$

Simplifying (the ρ^2 terms cancel):

$$v'' - 2\rho v' + \left[2\varepsilon - 1 - \frac{l(l+1)}{\rho^2}\right]v = 0 \tag{36}$$

b. Series Solution and Energy

Ansatz: $v(\rho) = \rho^{l+1}\sum_{k=0}^{\infty} c_k \rho^{2k}$. Recurrence relation for term ρ^j:

$$c_{j+2}[(j+2)(j+1) - l(l+1)] = c_j[2j - (2\varepsilon - 1)] \tag{37}$$

Termination at $j_{\max} = l + 1 + 2n_r$ => $2(l + 1 + 2n_r) - (2\varepsilon - 1) = 0$.

$$2\varepsilon = 2(2n_r + l) + 3 \Rightarrow E_n = \left(n + \frac{3}{2}\right)\hbar\omega, \quad (n = 2n_r + l) \tag{38}$$

c. Explicit States ($n = 0, n = 1$)

1. **Ground State ($n = 0$):** $n_r = 0, l = 0$.

$$\psi_{000}^{\text{sph}} = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{3}{4}} e^{-m\omega r^2/2\hbar} = \psi_{000}^{\text{cart}} \tag{39}$$

2. **First Excited State ($n = 1$):** Degeneracy 3.
   - **Cartesian Basis:**

$$|100\rangle \propto xe^{-\frac{r^2}{2}}, |010\rangle \propto ye^{-\frac{r^2}{2}}, |001\rangle \propto ze^{-\frac{r^2}{2}} \tag{40}$$

   - **Spherical Basis:** $Y_{10} \propto \frac{z}{r}$, $Y_{1\pm1} \propto \frac{x \pm iy}{r}$.
     - $$|110\rangle = |001\rangle_{\text{cart}} \tag{41}$$
     - $$|111\rangle = -\frac{1}{\sqrt{2}}(|100\rangle + i|010\rangle)_{\text{cart}} \tag{42}$$

- $$|11-1\rangle = \frac{1}{\sqrt{2}}(|100\rangle - i|010\rangle)_{\text{cart}} \tag{43}$$

Transformation Matrix:

$$\begin{pmatrix} |111\rangle \\ |110\rangle \\ |11-1\rangle \end{pmatrix}_{\text{sph}} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} & 0 \end{pmatrix} \begin{pmatrix} |100\rangle \\ |010\rangle \\ |001\rangle \end{pmatrix}_{\text{cart}} \tag{44}$$