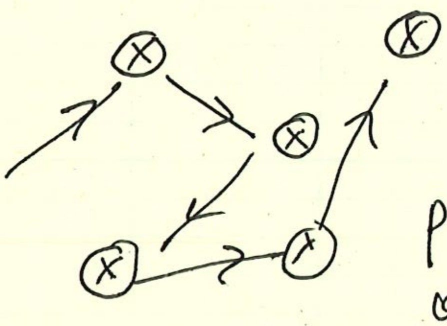
Dimensional Analysis

- Dimensional analysis (DA) can help to obtain order-of-magnitude estimates. DA is especially useful if the number of variables in the problem is equal to the number of independent units.

  In more complicated cases, the number of variables exceeds the number of independent units DA will give an answer up to a scaling function of several variables, the number of which being equal to the difference between the number of dimensional quantities and independent units [see Buckingham Π-theorem]

- Consider the problem of diffusion as a starting example. In this problem we would like to relate the average distance traveled by particle $\langle r \rangle$ from origin over time t to diffusion coefficient.



The latter is defined as a proportionality coefficient relating the particle flux (the number of particles crossing unit area per unit time) to a gradient of the number density

$$\vec{j} = -D\vec{\nabla}n \quad (\text{Fick's Law})$$

$$[j] = \frac{1}{\underbrace{[S]}_{[L]^2}[T]} = [D]\frac{1}{[L]}\cdot\frac{1}{[L]^3} \quad \text{In } d=3$$

$$\frac{1}{[L]^2[T]} = [D]\frac{1}{[L]^4} \Rightarrow \boxed{[D] = \frac{[L]^2}{[T]}}$$

We suppose that $\langle r\rangle = D^\alpha \cdot t^\beta$

Converting this to units we get:

$$[L] = \left(\frac{[L]^2}{[T]}\right)^\alpha [T]^\beta \quad \boxed{\begin{aligned}1&=2\alpha\\ \beta-\alpha&=0\end{aligned}}$$

Therefore $\langle r\rangle \sim \sqrt{D\cdot t}$

DA <u>does</u> <u>not</u> <u>fix</u> the numerical factor

⊙ Suppose now we want to determine the number density of particles at a point r and time t given that

We started with one particle at origin. Notice that the number density has units $1/[L]^3$ so to construct the natural scale for n we can use either r itself or $\langle r\rangle = \sqrt{Dt}$, namely $n \sim 1/r^3$ or $n \sim (Dt)^{-3/2}$. However, this does not determine n uniquely as it can depend on the dimensionless variable $r/\langle r\rangle = r/\sqrt{Dt}$. We can only say that $n(r,t)$ depends on $r/\sqrt{Dt}$ via some dimensionless function:

$$n(r,t) = \frac{1}{r^3} F_1\left(\frac{r}{\sqrt{Dt}}\right) \text{ or}$$

$$n(r,t) = \frac{1}{(Dt)^{3/2}} F_2\left(\frac{r}{\sqrt{Dt}}\right)$$

In fact the difference between these two forms is completely superficial

$$n(r,t) = \frac{1}{r^3} F_1\left(\frac{r}{\sqrt{Dt}}\right) = \frac{1}{(Dt)^{3/2}} \frac{(Dt)^{3/2}}{r^3} F_1\left(\frac{r}{\sqrt{Dt}}\right)$$

$$= \frac{1}{(Dt)^{3/2}} F_2\left(\frac{r}{(Dt)^{1/2}}\right) \Rightarrow \boxed{F_2(x) = \frac{1}{x^3} F_1(x)}$$

The function $F_1(x)$ is to be determined by actually solving the problem. It consists of combining Fick's law with the continuity equation for the particle density:

$$\underbrace{\partial_t n + \mathrm{div}\,\vec{j} = 0 \qquad \vec{j} = -D\vec{\nabla} n}$$

$$\partial_t n - \mathrm{div}(D\vec{\nabla} n) = 0$$

If D is independent of $(r, n, ..)$ then

$$\boxed{\partial_t n - D\nabla^2 n = 0}$$ Diffusion equation (PDE)

⊙ For this particular problem (DA) can lead to a complete solutions for 1d diffusion problem. Indeed, for the initial condition $n(t=0, x) = n_0(x)$ and bounded solution $n(x \to \pm\infty) \to 0$.

We try seeking solution in the form

$$\Rightarrow n(x,t) = \frac{1}{t^\beta} f(z) \qquad z = \frac{x}{\sqrt{Dt}}$$

$$\partial_t n = -\beta t^{-\beta-1} f(z) + t^{-\beta} f'(z)\frac{\partial z}{\partial t}$$

$$\frac{\partial z}{\partial t} = -\frac{1}{2}\frac{x}{\sqrt{D}\,t^{3/2}} = -\frac{z}{2t}$$

$$\partial_t n = -\beta t^{-\beta-1} f(z) - t^{-\beta} \frac{z}{2t} \cdot f'(z)$$

$$= t^{-\beta-1}\left(-\beta f - \frac{1}{2} z f'\right)$$

Next we work out spatial derivatives

$$\partial_x n = t^{-\beta} f'(z) \frac{\partial z}{\partial x} = t^{-\beta} \frac{1}{\sqrt{Dt}} f'(z)$$

$$\partial_x^2 n = t^{-\beta} \left(\frac{1}{\sqrt{Dt}}\right)\left(\frac{1}{\sqrt{Dt}}\right) f''(z)$$

$$= t^{-\beta-1} \frac{1}{D} \cdot f''(z)$$

Putting everything togethe we get:

$$\underbrace{\cancel{t^{-\beta-1}}\left(-\beta f - \frac{1}{2} z f'\right)}_{\partial_t n} = \cancel{D} \cdot \underbrace{\cancel{t^{-\beta-1}} \frac{1}{\cancel{D}} \cdot f''(z)}_{\partial_x^2 n}$$

$$\boxed{f''(z) + \frac{1}{2} z f'(z) + \beta f(z) = 0.}$$

From DA we already know that $\beta = 1/2$. But we can verify this result again based of the physical grounds. Indeed, we want our solution to preserve the number of particles. It means that $\int n(x,t)\, dx$ is independent on time. Therefore

$$\int n(x,t)\,dx = t^{-\beta}\int f(z)\,dx = \underbrace{t^{-\beta}\sqrt{Dt}\int f(z)\,dz}_{\text{must be const } t\text{-independ}}$$

$$\boxed{-\beta + \tfrac{1}{2} = 0}$$

Thus the equation we need to solve is

$$\boxed{f'' + \tfrac{1}{2} z f' + \tfrac{1}{2} f = 0}$$

This ODE is solved by $A e^{-z^2/4}$

$$f(z) = A e^{-z^2/4} \Rightarrow f' = -\frac{z}{2} f \Rightarrow f'' = \left(-\frac{1}{2} + \frac{z^2}{4}\right) f$$

$$\left(-\frac{1}{2} + \frac{z^2}{4}\right) f + \frac{1}{2} z \left(-\frac{z}{2} f\right) + \frac{1}{2} f = 0$$

As a result, the self-similar solution

$$n(x,t) = \frac{A}{\sqrt{t}} \exp\left(-\frac{x^2}{4Dt}\right)$$

The constant A can be fixed by a normalization condition

$$\int_{-\infty}^{+\infty} n(x,t)\,dx = 1 \Rightarrow \frac{A}{\sqrt{t}} \int_{-\infty}^{+\infty} e^{-\frac{x^2}{4Dt}}\,dx =$$

$$= \frac{A}{\sqrt{t}} \sqrt{4Dt} \underbrace{\int_{-\infty}^{+\infty} e^{-y^2}\,dy}_{\sqrt{\pi}} = 1 \Rightarrow A = \sqrt{\frac{1}{4\pi D}}$$

- How this result can be connected to the general solution of the initial-value-boundary problem?

Observe that:

(i) $\lim_{t\to 0} \sqrt{\frac{1}{4\pi D t}} \exp\left[-\frac{(x-x')^2}{4Dt}\right] \to \delta(x-x')$;

(ii) The equation is linear thus it obeys superposition principle;

(iii) On the infinite line the equation is translationally invariant

$$\boxed{n(x,t) = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{4\pi D t}} e^{-\frac{(x-x')^2}{4Dt}} n_0(x')\, dx'}$$

$$\Downarrow$$

$$\boxed{n(x,t) = \int_{-\infty}^{+\infty} \underbrace{G(x-x',t)}_{\text{Green's function of diffusion kernel}} n_0(x')\, dx}$$

Green's function

of diffusion kernel

- The asymptotic limits of the scaling functions can be determined by invoking other considerations, such as symmetry, analyticity

Further reading:

G. I. Barenblatt

"Dimensionless Analysis" (1987)

[Gordon Publishers]

⊙ The main principle of the DA is formulated via "Buckingham Π-theorem"

Suppose that a physical quantity Q depends on $\underline{n}$ variables

$$Q = f(V_1, V_2, \dots, V_n)$$

Suppose also that only $\underline{k}$ out of $\underline{n}$ variables have independent units, whereas units of remaining $\underline{n-k}$ variables can be expressed via those of the first k:

$$[V_{k+1}] = [V_1]^{p_{k+1}} \cdots [V_k]^{r_{k+1}}$$
$$\vdots$$
$$[V_n] = [V_1]^{p_n} \cdots [V_k]^{r_n}$$

then Q can be written as follows

$$\boxed{Q = V_1^p \cdots V_k^r \, F(\Pi_1, \dots, \Pi_{n-k})}$$

where the $n-k$ dimensionless parameters $\Pi_1, \dots, \Pi_{n-k}$ are given by:

$$\left.\begin{aligned} \Pi_1 &= \frac{V_{k+1}}{V_1^{p_{k+1}} \cdots V_k^{r_{k+1}}} \\ \Pi_2 &= \frac{V_{k+2}}{V_1^{p_{k+2}} \cdots V_k^{r_{k+2}}} \\ &\vdots \\ \Pi_{n-k} &= \frac{V_n}{V_1^{p_n} \cdots V_k^{r_n}} \end{aligned}\right\}$$

and $F(\Pi_1, \dots, \Pi_{n-k})$ is the dimensionless

Sketch of the proof: First observe that the units of Q must be expressed via those of the $\underline{k}$ independent ~~units~~ variables

$$[Q] = [V_1]^p \dots [V_k]^r$$

This immediately suggests that

$$\boxed{\Pi = \frac{Q}{V_1^p \dots V_k^r}} \quad \text{is dimensionless}$$

the remaining $n-k$ combinations are given by expression defined in the theorem statement $\Pi_1, \dots, \Pi_{n-k}$.
The equation for Π can be rewritten a

$$\Pi = \frac{Q}{V_1^p \dots V_k^r} = \frac{f(V_1, \dots, V_k, V_{k+1}, \dots V_n)}{V_1^p \dots V_k^r} =$$

$$= \frac{f(V_1, \dots, V_k, \Pi_1 \times (V_1^{p_{k+1}} \dots V_k^{r_{k+1}}), \dots, \Pi_{n-k}(V_1^{p_n} \dots V_n^{r_n}))}{V_1^p \dots V_k^r}$$

$$= F(V_1, \dots, V_k, \Pi_1, \dots, \Pi_{n-k})$$

the last line here is the formal definition of function $\underline{F}$.
Now we need to prove that this new function $F(V_1, \dots, V_k, \Pi_1, \dots, \Pi_{n-k})$ **does not** depend on $V_1, \dots, V_k$.

To this end, we recall that units of $V_1, \dots, V_K$ are independent. This means that we can switch to a new unit system in such a way that any of $V_1, \dots, V_K$, e.g. V_1 is multiplied by an arbitrary factor, whereas the remaining variables are unchanged. The units of Q will change accordingly so the definition of Π remains the same. On the other hand, the first argument of function F is multiplied by an arbitrary factor, whereas all its other argument remain the same. Since the left-hand-side, $\Pi = F(V_1, \dots, V_K, \Pi_1, \dots \Pi_{n-k})$ does not change, the right-hand-side does not change either, which is only possible if F does not depend on V_1. Repeating the same argument for the remaining $K-1$ variables, we must conclude that F does not depend on $V_1, \dots, V_K$. This proves the statement of Buckingham Theorem for the variable Q

(*) E. Buckingham Phys. Rev. 4, 345 [1914]