1 Problem 1: Diffusion Equation & Penetration Depths

Unified diffusion form

$$\frac{\partial X}{\partial t} = D\,\nabla^2 X, \qquad [D] = L^2T^{-1}.$$

Instances

$$\begin{array}{lll}
\text{Viscous (Stokes)}: & \rho\,\partial_t\mathbf{u} = \mu\nabla^2\mathbf{u} & \Rightarrow\ \partial_t\mathbf{u} = \nu\nabla^2\mathbf{u},\ \nu = \mu/\rho, \\
\text{Thermal}: & \rho c_p\,\partial_t T = k\nabla^2 T & \Rightarrow\ \partial_t T = \kappa\nabla^2 T,\ \kappa = \dfrac{k}{\rho c_p}, \\
\text{EM in conductor (skin effect)}: & \nabla\times\mathbf{B} = \mu_0\sigma\mathbf{E},\ \ \partial_t\mathbf{B} = -\nabla\times\mathbf{E} & \Rightarrow\ \partial_t\mathbf{B} = \eta\nabla^2\mathbf{B},\ \eta = \dfrac{1}{\mu_0\sigma}.
\end{array}$$

Penetration in time domain

$$\ell(t) \sim \sqrt{Dt} \quad (\text{since } Dt \text{ has units } L^2).$$

Oscillatory forcing (ω): take $X(z,t) = \Re\{X_0 e^{i\omega t - kz}\}$, then

$$i\omega = -Dk^2 \ \Rightarrow\ k = \frac{1+i}{\sqrt{2}}\sqrt{\frac{\omega}{D}}, \qquad \delta = \frac{1}{\Re\,k} = \sqrt{\frac{2D}{\omega}}.$$

Hence

$$\delta_{\text{visc}} = \sqrt{\frac{2\nu}{\omega}}, \qquad \delta_{\text{th}} = \sqrt{\frac{2\kappa}{\omega}}, \qquad \delta_{\text{EM}} = \sqrt{\frac{2}{\mu_0\sigma\,\omega}}.$$

2 Problem 2: Total Cross-Section for $V(r) = A/r^n$

Dimensions & classical interaction radius

$$[A] = \text{energy}\cdot L^n, \qquad \frac{A}{R^n} \sim E = \tfrac{1}{2}mv^2 \ \Rightarrow\ R \sim \left(\frac{A}{E}\right)^{1/n}.$$

Impulse (small-angle) deflection for a power law

$$F(r) = \frac{nA}{r^{n+1}}, \quad r = \sqrt{b^2+v^2t^2}, \quad \Delta p_\perp = \int_{-\infty}^{\infty} F_\perp\,dt = \frac{nA}{v}\,b\int_{-\infty}^{\infty}\frac{dt}{(b^2+v^2t^2)^{\frac{n+2}{2}}}.$$

Use $\displaystyle\int_{-\infty}^{\infty}\frac{du}{(1+u^2)^\alpha} = \sqrt{\pi}\,\frac{\Gamma(\alpha-\frac{1}{2})}{\Gamma(\alpha)}$ with $u = vt/b,\ \alpha = \frac{n+2}{2}$:

$$\Delta p_\perp = \frac{nA}{v}\,b\cdot\frac{b}{v}\cdot\frac{\sqrt{\pi}\,\Gamma(\frac{n+1}{2})}{b^{n+2}\,\Gamma(\frac{n+2}{2})} = \frac{n\sqrt{\pi}\,\Gamma\big(\frac{n+1}{2}\big)}{\Gamma\big(\frac{n+2}{2}\big)}\,\frac{A}{v}\,\frac{1}{b^n}.$$

Deflection angle $\theta(b) \simeq \Delta p_\perp/(mv)$:

$$\theta(b) \simeq C_n \frac{A}{mv^2\, b^n}, \qquad C_n := \frac{n\sqrt{\pi}\,\Gamma\big(\frac{n+1}{2}\big)}{\Gamma\big(\frac{n+2}{2}\big)} = \mathcal{O}(1).$$

Hard scattering when $\theta(b_*) \sim 1$ gives

$$b_* \sim \Big(\frac{C_n A}{mv^2}\Big)^{1/n}, \qquad \sigma_{\rm tot} \sim \pi b_*^2 \sim \pi\, C_n^{2/n}\Big(\frac{A}{E}\Big)^{2/n}.$$

Quantum regimes

$$\text{High energy } (kb_* \gg 1): \quad \sigma \sim \pi b_*^2 \quad (\text{partial-wave cutoff } \ell_{\max} \sim kb_*).$$

$$\text{Low energy } (kb_* \ll 1,\ n > 2): \quad \sigma \sim 4\pi a^2, \quad a \sim \Big(\frac{mA}{\hbar^2}\Big)^{\frac{1}{n-2}} \text{ (from } [a] = L).$$

$$\text{Long range } (n \le 2): \ \sigma_{\rm tot} \text{ diverges (e.g. Rutherford for } n = 1).$$

3 Problem 3: Quantum-Induced Fall Time of a Vertical Rod

Linearized dynamics about upright (inverted pendulum about the base)

$$I = \frac{1}{3}ML^2, \qquad \tau \simeq -Mg\frac{L}{2}\,\theta, \qquad I\ddot\theta = -\tau \ \Rightarrow\ \ddot\theta = \Omega^2\theta,\ \Omega = \sqrt{\frac{3g}{2L}}.$$

Solution: $\theta(t) = \theta_0\cosh(\Omega t) + \dfrac{\omega_0}{\Omega}\sinh(\Omega t)$, with $\omega_0 = \dot\theta(0)$.

Quantum initial uncertainties

$$\Delta\theta\,\Delta L_\theta \gtrsim \frac{\hbar}{2}, \qquad L_\theta = I\dot\theta \ \Rightarrow\ \Delta\omega_0 \sim \frac{\hbar}{I\,\Delta\theta}.$$

For large t, $\theta(t) \sim \frac{e^{\Omega t}}{2}\Big(\Delta\theta + \frac{\Delta\omega_0}{\Omega}\Big)$. Define

$$S(\Delta\theta) := \Delta\theta + \frac{\hbar}{I\Omega}\,\frac{1}{\Delta\theta} \quad \Rightarrow \quad \frac{dS}{d(\Delta\theta)} = 1 - \frac{\hbar}{I\Omega}\,\frac{1}{(\Delta\theta)^2} = 0$$

$$\Rightarrow\ \Delta\theta_* = \Big(\frac{\hbar}{I\Omega}\Big)^{1/2}, \qquad S_{\min} = 2\Big(\frac{\hbar}{I\Omega}\Big)^{1/2}.$$

Fall time from $\theta(t_f) \sim 1$:

$$\frac{e^{\Omega t_f}}{2}\,S_{\min} \sim 1 \ \Rightarrow\ t_f \sim \frac{1}{\Omega}\ln\Big(\frac{2}{S_{\min}}\Big) = \frac{1}{2\Omega}\ln\Big(\frac{I\Omega}{\hbar}\Big) \quad (\text{up to factors } \sim 1).$$

Numerics ($L = 1\,\mathrm{m},\ M = 1\,\mathrm{kg}$): $\Omega = \sqrt{3g/(2L)} \approx 3.8\,\mathrm{s}^{-1}$, $I = \frac{1}{3}ML^2 \approx 0.33\,\mathrm{kg\,m^2}$, hence

$$\frac{I\Omega}{\hbar} \sim 10^{34}, \quad t_f \sim \frac{1}{2\Omega}\ln\Big(10^{34}\Big) \approx \frac{78}{2\times 3.8}\ \mathrm{s}\ \sim\ 10\ \mathrm{s}.$$

4 Problem 4: Hovering Power Scaling for a $2\times$ Larger Helicopter

Actuator-disk (momentum) theory

$$T = Mg, \qquad T = 2\rho A v_i^2, \qquad P = T\,v_i \;\Rightarrow\; v_i = \sqrt{\frac{Mg}{2\rho A}}, \quad P = \frac{(Mg)^{3/2}}{\sqrt{2\rho A}}.$$

Geometric similarity (scale s): $M \propto s^3,\; A \propto s^2$

$$P \;\propto\; \frac{M^{3/2}}{\sqrt{A}} \;\propto\; s^{\frac{9}{2}}\, s^{-1} = s^{7/2}.$$

For $s = 2$ (all lengths doubled):

$$\boxed{\frac{P_2}{P_1} = 2^{7/2} = 8\sqrt{2} \approx 11.31} \quad \text{(independent of } \rho, g).$$

5 Problem 5: Relativistic White Dwarf Mass (Chandrasekhar Limit)

Ultra-relativistic degeneracy EOS

$$P = K_{\rm UR}\, n_e^{4/3}, \qquad K_{\rm UR} = \frac{1}{4}(3\pi^2)^{1/3}\hbar c, \qquad n_e = \frac{\rho}{\mu_e m_p}$$

$$\Rightarrow\; P = K'\,\rho^{4/3}, \quad K' = K_{\rm UR}\,(\mu_e m_p)^{-4/3}.$$

Hydrostatic scaling and mean density

$$P_c \sim \frac{GM^2}{R^4}, \qquad \rho \sim \frac{M}{R^3}.$$

Balance $P_c \sim K'\rho^{4/3}$ eliminates R:

$$K'\Big(\frac{M}{R^3}\Big)^{4/3} \sim \frac{GM^2}{R^4} \;\Rightarrow\; M \sim \Big(\frac{K'}{G}\Big)^{3/2}.$$

Hence

$$\boxed{M_{\rm Ch} \sim \frac{(K_{\rm UR})^{3/2}}{G^{3/2}}\,\frac{1}{(\mu_e m_p)^2} \;\propto\; \frac{(\hbar c)^{3/2}}{G^{3/2}}\,\frac{1}{(\mu_e m_p)^2}}.$$

With $n{=}3$ polytrope constants (Lane–Emden):

$$M = 4\pi\big[-\xi_1^2\theta'(\xi_1)\big]\left(\frac{K'}{\pi G}\right)^{3/2}, \quad \xi_1 = 6.89685, \quad -\xi_1^2\theta'(\xi_1) = 2.01824,$$

$$\Rightarrow\ M \simeq \mathcal{C}\ \frac{(\hbar c)^{3/2}}{G^{3/2}}\ \frac{1}{(\mu_e m_p)^2}, \quad \mathcal{C} = 4\pi \cdot 2.01824 \cdot \pi^{-3/2} \cdot \left(\tfrac{1}{4}(3\pi^2)^{1/3}\right)^{3/2}.$$

Numerically this yields (full structure calculation):

$$\boxed{M_{\rm Ch} \approx 1.44\, M_\odot \left(\frac{2}{\mu_e}\right)^2}.$$