For the finite square well

$$V = \begin{cases} 0 & |x| < a \\ V_0 & |x| > a \end{cases}, \tag{1}$$

1. Determine the odd parity eigenfunctions and their associated energy eigenvalues for this potential, and discuss limiting behaviour as $V_0 \to 0, V_0 \to \infty$.
2. FInd accurate numerical values for the boundstate energy eigenvalues of a particle in the aboe finite square well potential, in which the parameter

$$R \equiv \sqrt{\frac{2mV_0a^2}{\hbar^2}} = 4. \tag{2}$$

Find solutions graphically and numerically.

1. Odd parity

Lable region I : $x \leftarrow a$, II : $-a < x < a$, III : $x > a$. From lecture: $\psi_{\mathrm{II}} = a\cos kx + B\sin kx$, $\psi_{\mathrm{I}} = De^{\kappa x}$, $\psi_{\mathrm{III}} = Fe^{\kappa x}$, with $k = \frac{\sqrt{2mE}}{\hbar}$, $\kappa = \frac{\sqrt{2m(V_0-E)}}{\hbar}$.

Imposing odd parity $\psi(-x) = -\psi(x)$, we have $A = 0, D = -F$. Imposing boundary condition of continuity and smoothness at $x = a$, we have

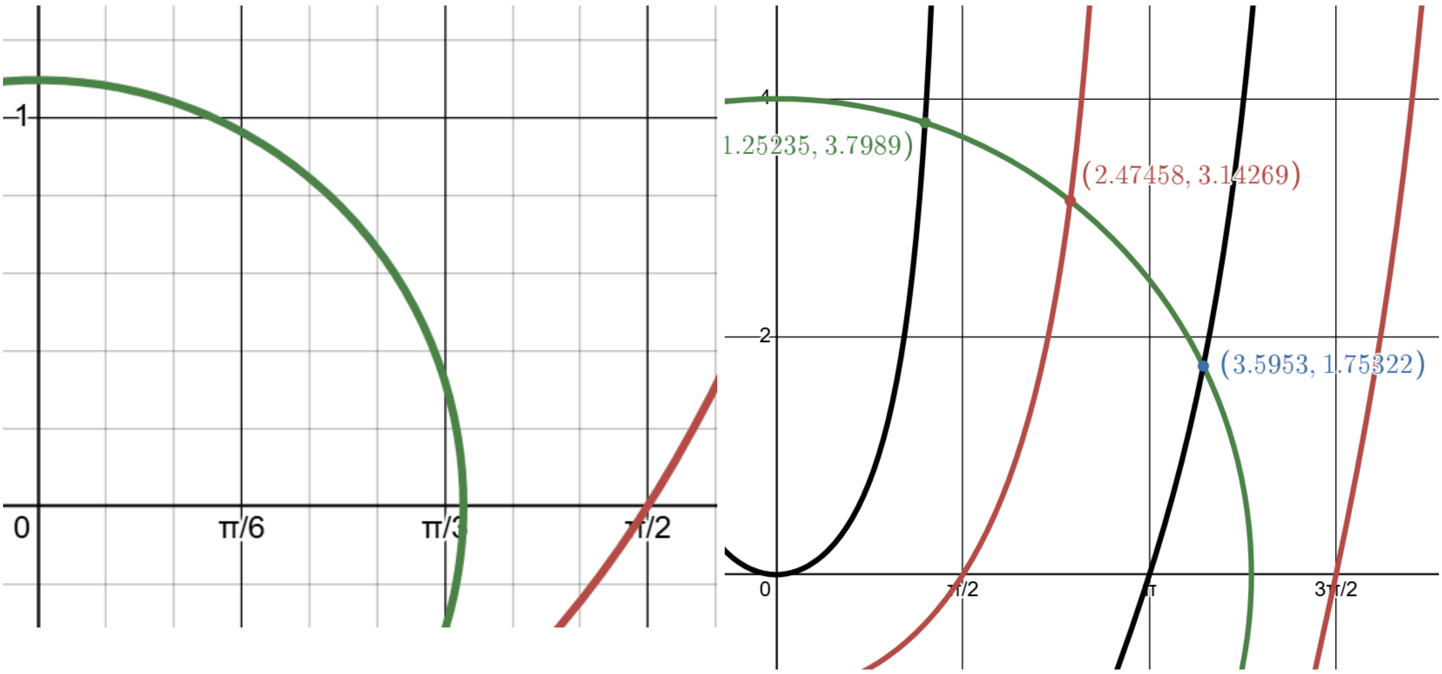
$$B\sin ka = Fe^{-\kappa a}, \quad B\cos ka = -\kappa Fe^{-\kappa a}. \tag{3}$$

Dividing these and setting $\eta = \kappa a, \xi = ka$, we have

$$\xi\cot\xi = -\eta, \quad \xi^2 + \eta^2 = R^2. \tag{4}$$

Limiting behaviour:

- as $V_0 \to \infty, \eta \to \infty$ and $\cot\xi \to -\infty$ satisfied at $\boldsymbol{\eta = n\pi}, \quad \boldsymbol{n \in \mathbb{Z}}$**.**
- as $V_0 \to 0, R \to 0$, but as seen below (LEFT) that $\xi\cot\xi = -\eta(\xi, \eta > 0)$ has solution only for $R \geq \frac{\pi}{2}$, the limit of $V_0 \to 0$ yields **no odd parity bound states.**)



2. Numerical and graphical

1. Graphically (ABOVE, RIGHT), The equations to solve are:

- Even States: $\xi\tan(\xi) = \eta$ (black)
- Odd States: $-\xi\cot(\xi) = \eta$ (red)

- Constraint: $\xi^2 + \eta^2 = 16$ (Green)

From a graphical analysis, we find three bound states:
$(\xi, \eta) = (1.25235, 3.7989)$,
$(\xi, \eta) = (2.47458, 3.14269)$,
$(\xi, \eta) = (3.5953, 1.75322)$

2. Numerically, we solve

$$\begin{cases} \xi \tan \xi = \sqrt{16 - \xi^2} \\ -\xi \cot \xi = \sqrt{16 - \xi^2} \end{cases} \tag{5}$$

and the solutions are identical as above:

$$\xi_1 = 1.25235, \xi_2 = 2.47458, \xi_3 = 3.5953. \tag{6}$$

and the energies are

$$\begin{gathered} E = V_0 \left(\frac{\xi^2}{16} \right) \\ \Rightarrow E_1 = 0.09797, E_2 = 0.3829, E_3 = 0.8077 \end{gathered} \tag{7}$$

2

Show that for spinless particles moving in 1D, the energy spectrum of bound states is always non-degenrate.

Assume not: exists $\psi_i(x)$, $(i = 1, 2)$ s.t.

$$-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2\psi_i}{\mathrm{d}x^2} + V(x)\psi_i = E\psi_i, \quad E = E_1 = E_2. \tag{8}$$

In particular, $\psi_1(x), \psi_2(x)$ are linear independent. This would imply that their wronskian

$$W(x) = \psi_1{\psi_2}' - \psi_2{\psi_1}' = 0 \tag{9}$$

However, we notice that

$$W'(x) = 2\frac{m}{\hbar^2}(\psi_1(V - E))\psi_2 - \psi_2(V - E)\psi_1) = 0 \tag{10}$$

so $W(x) = \text{const}$. Further, since ψ_1, ψ_2 are bound states,

$$\lim_{x\to\pm\infty} \psi_i(x) = 0 \Rightarrow W(\pm\infty) = 0 \Rightarrow W(x) = 0 \quad \forall x. \tag{11}$$

This contradicts the linear independence of ψ_1, ψ_2. Thus, the energy spectrum of bound states is always non-degenerate.

3

Use the Hermite generating function

$$g(y,t) = e^{-t^2+2ty} = \sum_{n=0}^{\infty} H_n(y)\frac{t^n}{n!} \tag{12}$$

1. To prove the following properties

$$\begin{gathered} H_n(y) = e^{\frac{y^2}{2}}\left(y - \frac{d}{d}y\right)^n e^{-\frac{y^2}{2}}, \\ H_n'(y) = 2nH_{n-1}(y) \\ H_{n+1}(y) = 2yH_n(y) - 2nH_{n-1}(y). \end{gathered} \tag{13}$$

2. Then evaluate

$$\int_{-\infty}^{\infty} \mathrm{d}y\, e^{-y^2} H_n(y)H_{n'}(y) \tag{14}$$

1

a

Recall that $H_n(y) = \left(\frac{\partial}{\partial t}\right)^n\Big|_{t=0} g(y,t)$. Let $u = t - y, \partial u = \partial t$. Then $g = e^{-u^2}e^{y^2}$. Then from definition,

$$H_n(y) = \left(\frac{\partial}{\partial u}\right)^n \left(e^{-u^2}e^{y^2}\right)\Bigg|_{u=-y} = e^{y^2}\left(\frac{\partial}{\partial u}\right)^n e^{-u^2}\Big|_{u=-y} = (-1)^n e^{y^2}\left(\frac{\partial}{\partial y}\right)^n e^{-y^2} \tag{15}$$

Notice the identity

$$\frac{\mathrm{d}}{\mathrm{d}y}\left(e^{-y^2/2}g\right) = -e^{-y^2/2}\left(y - \frac{\mathrm{d}}{\mathrm{d}y}\right)g, \tag{16}$$

we have

$$\left(\frac{\mathrm{d}}{\mathrm{d}y}\right)^n\left(e^{-y^2}\right) = \left(\frac{\mathrm{d}}{\mathrm{d}y}\right)^n\left(e^{-y^2/2}e^{-y^2/2}\right) = -e^{-y^2/2}\left(y - \frac{\mathrm{d}}{\mathrm{d}y}\right)^n e^{-y^2/2}. \tag{17}$$

And so

$$H_n(y) = (-1)^n e^{y^2}\left(\frac{\mathrm{d}}{\mathrm{d}y}\right)^n e^{-y^2} = (-1)^{2n} e^{y^2/2}\left(y - \frac{\mathrm{d}}{\mathrm{d}y}\right)^n e^{-y^2/2} = \boxed{e^{y^2/2}\left(y - \frac{\mathrm{d}}{\mathrm{d}y}\right)^n e^{-y^2/2}}, \tag{18}$$

as wanted.

b

Notice that

$$\frac{\partial g}{\partial t} = 2tg(y,t) = \sum_{n=0}^{\infty})\frac{2t^{n+1}}{n!}H_n(y) = \sum_{n=0}^{\infty} \underbrace{2(n+1)\frac{t^{n+1}}{(n+1)!}H_n(y)}_{*} = \sum_{n=0}^{\infty} H_n' \frac{(y)^{t^n}}{n!}. \tag{19}$$

But (*) is also

$$2nH_{n-1}(y)\frac{t^n}{n!}. \tag{20}$$

Therefore

$$H_n'(y) = 2nH_{n-1}(y). \tag{21}$$

as wanted.

c

$$\begin{aligned}\partial_t g = (2y-2t)g &= \sum_{n=0}^{\infty}\left(2yH_n(y)\frac{t^n}{n!} - 2H_n(y)\frac{t^{n+1}}{n!}\right) \\ &\underset{n:=n-1}{=} \sum_{n=0}^{\infty}\left(2yH_n(y)\frac{t^n}{n!} - 2nH_{n-1}(y)\frac{t^n}{n!}\right)\end{aligned} \tag{22}$$

But $\partial_t g$ is also

$$\sum_{n=0}^{\infty}\frac{t^{n-1}}{(n-1)!}H_n(y) = \sum_{n=0}^{\infty}\frac{t^n}{n!}H_{n+1}(y), \tag{23}$$

and so

$$H_{n+1}(y) = 2yH_n(y) - 2nH_{n-1}(y). \tag{24}$$

2.

Consider

$$\int_{-\infty}^{\infty} e^{-y^2} g(y,t)g(y,s)\, dy = e^{-(t^2+s^2)}\int_{-\infty}^{\infty} e^{-y^2+2(t+s)y}\, dy. \tag{25}$$

Complete the square in the exponent of the integrand: $-y^2 + 2(t+s)y = -(y-(t+s))^2 + (t+s)^2$. Thus,

$$\int_{-\infty}^{\infty} e^{-(y-(t+s))^2+(t+s)^2}\, dy = e^{(t+s)^2}\int_{-\infty}^{\infty} e^{-u^2}\, du = e^{(t+s)^2}\sqrt{\pi}, \tag{26}$$

where $u = y-(t+s)$. Substituting back,

$$\int_{-\infty}^{\infty} e^{-y^2} g(y,t)g(y,s)\, dy = e^{-(t^2+s^2)}e^{(t+s)^2}\sqrt{\pi} = e^{-(t^2+s^2)+t^2+s^2+2ts}\sqrt{\pi} = e^{2ts}\sqrt{\pi}. \tag{27}$$

On the other hand, expanding the generating functions gives

$$g(y,t)g(y,s) = \sum_{m=0}^{\infty}\sum_{l=0}^{\infty}\frac{t^m}{m!}\frac{s^l}{l!}H_m(y)H_l(y), \tag{28}$$

so

$$\int_{-\infty}^{\infty} e^{-y^2} g(y,t)g(y,s)\, dy = \sum_{m=0}^{\infty}\sum_{l=0}^{\infty}\frac{t^m s^l}{m!l!}\int_{-\infty}^{\infty} e^{-y^2}H_m(y)H_l(y)\, dy. \tag{29}$$

Equating the two expressions yields

$$\sum_{m=0}^{\infty}\sum_{l=0}^{\infty}\frac{t^m s^l}{m!l!}\int_{-\infty}^{\infty} e^{-y^2}H_m(y)H_l(y)\, dy = \sqrt{\pi}\, e^{2ts}. \tag{30}$$

The Taylor expansion of the right-hand side is

$$e^{2ts} = \sum_{n=0}^{\infty}\frac{(2ts)^n}{n!} = \sum_{n=0}^{\infty}\frac{2^n t^n s^n}{n!}. \tag{31}$$

For the left-hand side to match, the double sum must reproduce this only when $m = l = n$, implying that the integral vanishes unless $m = l$. Specifically, the coefficient of $\frac{t^m s^l}{m!l!}$ on the right is $\sqrt{\pi}\, 2^m \frac{m!}{m!}\delta_{ml}$ (zero otherwise), so

$$\int_{-\infty}^{\infty} e^{-y^2}H_m(y)H_l(y)\, dy = \sqrt{\pi}\, 2^m m!\, \delta_{ml}. \tag{32}$$

4

Using wavefunctions, compute $\langle n'|p|n\rangle$for the eigenstates of the 1d SHO to show that

$$\langle n'|p|n\rangle = i\sqrt{\frac{m\omega\hbar}{2}}\left(\sqrt{n+1}\delta_{n',n+1} - \sqrt{n}\delta_{n',n-1}\right) \tag{33}$$

1

In the position representation, the momentum operator is $\hat{p} = -i\hbar\frac{d}{dx}$. The matrix element is therefore given by the integral:

$$\langle n'|\ p\ |\ n\rangle = \int_{-\infty}^{\infty} \psi_{n'}^*(x)\left(-i\hbar\frac{d}{dx}\right)\psi_n(x)\,dx \tag{34}$$

The normalized energy eigenfunctions are (from lecture)

$$\psi_n(x) = C_n H_n(y) e^{-y^2/2} \quad \text{with} \quad y = \frac{x}{b} = x\sqrt{\frac{m\omega}{\hbar}} \tag{35}$$

where $C_n = (m\omega/\pi\hbar)^{1/4}(2^n n!)^{-1/2} = (b\sqrt{\pi}2^k k!)^{-1/2}$ is the normalization constant.

First, consider:

$$\frac{d\psi_n(x)}{dx} = \frac{C_n}{b}\frac{d}{dy}\left(H_n(y)e^{-y^2/2}\right) = \frac{C_n}{b}\left(H_{(n)'}(y)e^{-y^2/2} - yH_n(y)e^{-y^2/2}\right) \tag{36}$$

Using P3:

1. $H_{(n)'}(y) = 2nH_{n-1}(y)$
2. $2yH_n(y) = H_{n+1}(y) + 2nH_{n-1}(y) \Longrightarrow yH_n(y) = \frac{1}{2}H_{n+1}(y) + nH_{n-1}(y)$

Then we have

$$\frac{d\psi_n(x)}{dx} = \frac{C_n}{b}e^{-y^2/2}\left(2nH_{n-1}(y) - \left[\frac{1}{2}H_{n+1}(y) + nH_{n-1}(y)\right]\right) = \frac{C_n}{b}e^{-y^2/2}\left(nH_{n-1}(y) - \frac{1}{2}H_{n+1}(y)\right) \tag{37}$$

Now $\langle n'|\ p\ |\ n\rangle$, with y ($dx = b\,dy$) becomes:

$$\langle n'|\ p\ |\ n\rangle = -i\hbar\int_{-\infty}^{\infty}\left(C_{n'}H_{n'}(y)e^{-y^2/2}\right)\left(\frac{C_n}{b}e^{-y^2/2}\left[nH_{n-1}(y) - \frac{1}{2}H_{n+1}(y)\right]\right)(b\,dy) \tag{38}$$

$$= -i\hbar C_{n'}C_n\int_{-\infty}^{\infty} e^{-y^2}H_{n'}(y)\left[nH_{n-1}(y) - \frac{1}{2}H_{n+1}(y)\right]dy \tag{39}$$

The integral splits into two terms. We use the orthogonality relation for Hermite polynomials, $\int_{-\infty}^{\infty} e^{-y^2}H_m(y)H_l(y)\,dy = \sqrt{\pi}\,2^l l!\,\delta_{ml}$:

1. The first term is non-zero only if $n' = n-1$:

$$n\int_{-\infty}^{\infty} e^{-y^2}H_{n-1}(y)H_{n-1}(y)\,dy = n\sqrt{\pi}2^{n-1}(n-1)! = \frac{\sqrt{\pi}}{2}2^n n! \tag{40}$$

2. The second term is non-zero only if $n' = n+1$:

$$-\frac{1}{2}\int_{-\infty}^{\infty} e^{-y^2}H_{n+1}(y)H_{n+1}(y)\,dy = -\frac{1}{2}\sqrt{\pi}2^{n+1}(n+1)! \tag{41}$$

The matrix element is non-zero only for $n' = n \pm 1$.

- Case 1: $n' = n-1$

$$\langle n-1\ |\ p\ |\ n\rangle = -i\hbar C_{n-1}C_n\left(\frac{\sqrt{\pi}}{2}2^n n!\right) \tag{42}$$

$$= -i\hbar \frac{1}{b\sqrt{\pi}\sqrt{2^{n-1}(n-1)!2^n n!}}\left(\frac{\sqrt{\pi}}{2}2^n n!\right) = -i\hbar \frac{\sqrt{2^n n!}}{b\sqrt{2\cdot 2^{n-1}(n-1)!}} = -i\hbar \frac{\sqrt{n}}{b\sqrt{2}} \tag{43}$$

Using $b = \sqrt{\hbar/m\omega}$, we get $\langle n-1 \mid p \mid n\rangle = -i\sqrt{\frac{m\omega\hbar}{2}}\sqrt{n}$.

- Case 2: $n' = n+1$,

$$\langle n+1 \mid p \mid n\rangle = -i\hbar C_{n+1}C_n\left(-\frac{1}{2}\sqrt{\pi}2^{n+1}(n+1)!\right) \tag{44}$$

$$= i\hbar \frac{1}{b\sqrt{\pi}\sqrt{2^{n+1}(n+1)!2^n n!}}\left(\frac{\sqrt{\pi}}{2}2^{n+1}(n+1)!\right) = i\hbar \frac{\sqrt{2^{n+1}(n+1)!}}{b\sqrt{2\cdot 2^n n!}} = i\hbar \frac{\sqrt{2(n+1)}}{b\sqrt{2}} \tag{45}$$

Using $b = \sqrt{\hbar/m\omega}$, we get $\langle n+1 \mid p \mid n\rangle = i\sqrt{\frac{m\omega\hbar}{2}}\sqrt{n+1}$.

Combining these results using the Kronecker delta gives the final expression:

$$\boxed{\langle n' | \ p \mid n\rangle = i\sqrt{\frac{m\omega\hbar}{2}}\left(\sqrt{n+1}\delta_{n',n+1} - \sqrt{n}\delta_{n',n-1}\right)} \tag{46}$$

2

In the momentum representation,

$$\langle n' | \ p \mid n\rangle = \int_{-\infty}^{\infty} \phi_{n'}^*(p)\, p\, \phi_n(p)\, dp \tag{47}$$

The momentum-space eigenfunctions are (from lecture)

$$\phi_n(p) = (-i)^n D_n H_n(q) e^{-q^2/2} \quad \text{with} \quad q = \frac{p}{\sqrt{m\omega\hbar}} \tag{48}$$

where $D_n = (1/m\omega\pi\hbar)^{1/4}(2^n n!)^{-1/2}$ is the normalization constant.

Similar to calculation of the position matrix element $\langle n' | \ x \mid n\rangle$ in position space.

$$\langle n' | \ p \mid n\rangle = \int_{-\infty}^{\infty} \left((-i)^{n'} D_{n'} H_{n'}(q) e^{-q^2/2}\right)^* \cdot p \cdot \left((-i)^n D_n H_n(q) e^{-q^2/2}\right) dp \tag{49}$$

$$= (i)^{n'}(-i)^n D_{n'} D_n \int_{-\infty}^{\infty} e^{-q^2} H_{n'}(q) H_n(q)\, p\, dp \tag{50}$$

Changing variables $p = q\sqrt{m\omega\hbar}$ and $dp = dq\sqrt{m\omega\hbar}$:

$$= (i)^{n'}(-i)^n D_{n'} D_n (m\omega\hbar) \int_{-\infty}^{\infty} e^{-q^2} H_{n'}(q)(qH_n(q))dq \tag{51}$$

Using $qH_n(q) = \frac{1}{2}H_{n+1}(q) + nH_{n-1}(q)$ and the orthogonality relation, we again find that the integral is non-zero only for $n' = n \pm 1$.

- Case 1: $n' = n-1$, The phase factor is $(i)^{n-1}(-i)^n = i^{-1} = -i$. The integral gives $n\sqrt{\pi}2^{n-1}(n-1)!$.

$$\langle n-1 \mid p \mid n\rangle = (-i)D_{n-1}D_n(m\omega\hbar)\left(n\sqrt{\pi}2^{n-1}(n-1)!\right) \tag{52}$$

The calculation for the constants is analogous to the position-space case, with the parameter $\sqrt{m\omega\hbar}$ replacing $1/b$.

$$\langle n-1 \mid p \mid n\rangle = (-i)\sqrt{\frac{m\omega\hbar}{2}}\sqrt{n} \tag{53}$$

- Case 2: $n' = n+1$, The phase factor is $(i)^{n+1}(-i)^n = i$. The integral gives $\frac{1}{2}\sqrt{\pi}2^{n+1}(n+1)!$.

$$\langle n+1 \mid p \mid n\rangle = (i)D_{n+1}D_n(m\omega\hbar)\left(\frac{1}{2}\sqrt{\pi}2^{n+1}(n+1)!\right) \tag{54}$$

$$\langle n+1 \mid p \mid n \rangle = (i)\sqrt{\frac{m\omega\hbar}{2}}\sqrt{n+1} \tag{55}$$

Combining these gives the same final result

$$\boxed{\langle n' | \ p \mid n \rangle = i\sqrt{\tfrac{m\omega\hbar}{2}}\left(\sqrt{n+1}\delta_{n\prime,n+1} - \sqrt{n}\delta_{n\prime,n-1}\right)} \tag{56}$$

5

For 1. The ground state, and 2. The first excited state, calculate the probability that a particle of mass m in the 1d SHO with freq ω is farther from the origin than the classical turning points where $E = V$.

- Let $b \equiv \sqrt{\hbar/(m\omega)}$ and $y \equiv x/b$.
- SHO energies: $E_n = (n + \frac{1}{2})\hbar\omega$.
- Classical turning points solve $E_n = \frac{1}{2}m\omega^2 x_t^2$, hence

$$x_t(n) = b\sqrt{2n+1}, \qquad y_t(n) = \sqrt{2n+1}. \tag{57}$$

- Normalized SHO wavefunctions (in y):

$$\psi_n(x) = \frac{1}{\sqrt{b}} \frac{1}{\pi^{1/4}\sqrt{2^n n!}} H_n(y)\, e^{-y^2/2}, \tag{58}$$

so

$$|\psi_n(x)|^2 = \frac{1}{\sqrt{\pi}\, b} \frac{H_n(y)^2}{2^n n!} e^{-y^2}. \tag{59}$$

- "Tunneling" probability outside the classical region:

$$P_n \equiv 2\int_{x_t(n)}^{\infty} |\psi_n(x)|^2\, dx = 2\int_{y_t(n)}^{\infty} \frac{1}{\sqrt{\pi}} \frac{H_n(y)^2}{2^n n!} e^{-y^2}\, dy. \tag{60}$$

- Ground state n=0

Here $H_0(y) = 1$, $y_t = \sqrt{1} = 1$:

$$P_0 = \frac{2}{\sqrt{\pi}} \int_1^{\infty} e^{-y^2}\, dy = \operatorname{erfc}(1) \approx 0.157299. \tag{61}$$

- First excited state n=1
- Here $H_1(y) = 2y$, $y_t = \sqrt{3}$, and

$$P_1 = \frac{2}{\sqrt{\pi}} \int_{\sqrt{3}}^{\infty} \frac{(2y)^2}{2} e^{-y^2}\, dy = \frac{4}{\sqrt{\pi}} \int_{\sqrt{3}}^{\infty} y^2 e^{-y^2}\, dy. \tag{62}$$

- Use

$$\int y^2 e^{-y^2}\, dy = -\frac{y}{2} e^{-y^2} + \frac{\sqrt{\pi}}{4} \operatorname{erf}(y), \tag{63}$$

to obtain for $a > 0$,

$$\int_a^{\infty} y^2 e^{-y^2}\, dy = \frac{\sqrt{\pi}}{4} \operatorname{erfc}(a) + \frac{a}{2} e^{-a^2}. \tag{64}$$

- With $a = \sqrt{3}$:

$$P_1 = \frac{4}{\sqrt{\pi}} \left[\frac{\sqrt{\pi}}{4} \operatorname{erfc}(\sqrt{3}) + \frac{\sqrt{3}}{2} e^{-3} \right] = \operatorname{erfc}(\sqrt{3}) + \frac{2\sqrt{3}}{\sqrt{\pi}} e^{-3}. \tag{65}$$

$$\boxed{P_1 = \operatorname{erfc}(\sqrt{3}) + \frac{2\sqrt{3}}{\sqrt{\pi}} e^{-3} \approx 0.11} \tag{66}$$

6

Show that for the 1d SHO (x being the position operator,)

$$\langle 0|e^{ikx}|0\rangle = \exp\left[-k^2\langle 0|x^2|0\rangle/2\right] \tag{67}$$

Start from creation and annihilation oprators:

$$x = \sqrt{\frac{\hbar}{2m\omega}}\left(a + a^\dagger\right) \Rightarrow \quad e^{ikx} = e^{i\lambda\left(a+a^\dagger\right)}, \quad \left(\lambda = k\sqrt{\frac{\hbar}{2m\omega}}\right). \tag{68}$$

Since

$$\left[i\lambda a, i\lambda a^\dagger\right] = i^2\lambda^2\left(aa^\dagger - a^\dagger a\right) = \lambda^2\left(a^\dagger a - aa^\dagger\right) = \lambda^2\left[a^\dagger, a\right] = -\lambda^2, \tag{69}$$

we can use the BCH formula to write

$$e^{i\lambda\left(a+a^\dagger\right)} = e^{i\lambda a}e^{i\lambda a^\dagger}e^{\frac{\lambda^2}{2}}. \tag{70}$$

Then

$$\langle 0|e^{ikx}|0\rangle = e^{\frac{\lambda^2}{2}}\underbrace{\left\langle 0\middle|e^{i\lambda a}e^{i\lambda a^\dagger}\middle|0\right\rangle}_{*}. \tag{71}$$

Expanding

$$e^{i\lambda a} = \sum_{m=0}^{\infty}\frac{(i\lambda a)^m}{m!}, \quad e^{i\lambda a^\dagger} = \sum_{n=0}^{\infty}\frac{\left(i\lambda a^\dagger\right)^n}{n!}, \tag{72}$$

we have (*) to be

$$\left\langle 0\middle|\sum_m\sum_n\frac{(i\lambda)^{m+n}}{m!n!}a^m\left(a^\dagger\right)^n\middle|0\right\rangle = \sum_m\sum_n\frac{(i\lambda)^{m+n}}{m!n!}\underbrace{\left\langle 0\middle|a^m\left(a^\dagger\right)^n\middle|0\right\rangle}_{**}. \tag{73}$$

Since $\left(a^\dagger\right)^n|0\rangle = \sqrt{n!}|n\rangle$, (**) becomes

$$\sqrt{n!}\langle 0|a^m|n\rangle = \sqrt{n!}\sqrt{n(n-1)...(n-m+1)}\langle 0|n-m\rangle = \sqrt{n!}\sqrt{n(n-1)...(n-m+1)}\delta_{m,n} = n!\delta_{m,n}. \tag{74}$$

So for $n = m$,

$$(*) = \sum_{n=0}^{\infty}\frac{(i\lambda)^{2n}}{n!} = \sum_n\frac{\left(-\lambda^2\right)^n}{n!} = e^{-\lambda^2}. \tag{75}$$

Then

$$\langle 0|e^{ikx}|0\rangle = e^{\frac{\lambda^2}{2}}e^{-\lambda^2} = e^{-\frac{\lambda^2}{2}}. \tag{76}$$

On the other hand, since

$$\langle 0|x^2|0\rangle = \left\langle 0\middle|\frac{\hbar}{2m\omega}\left(a^2 + aa^\dagger + a^\dagger a + a^{\dagger 2}\right)\right\rangle = \frac{\hbar}{2m\omega}\langle 0|a^\dagger a + 1|0\rangle = \frac{\hbar}{2m\omega}, \tag{77}$$

we have

$$\frac{k^2}{2}\langle 0|x^2|0\rangle = \frac{k^2}{2}\frac{\hbar}{2m\omega} = \frac{\lambda^2}{2}, \tag{78}$$

and so

$$\langle 0|e^{ikx}|0\rangle = e^{-k^2\langle 0|x^2|0\rangle/2}, \tag{79}$$

as desired.