Use Frobenius ansatz to the Hermite equation

$$y'' - 2xy' + 2ny = 0 \tag{1}$$

Derive indicial equation, derive recursion relation between the expansion coefficient, and construct several polynomials

Let $y = \sum_{k=0}^{\infty} a_k x^{k+r}$ with $a_0 \neq 0$. We have the derivatives:

$$y' = \sum_{k=0}^{\infty}(k+r)a_k x^{k+r-1}; \quad y'' = \sum_{k=0}^{\infty}(k+r)(k+r-1)a_k x^{k+r-2}. \tag{2}$$

Plugging into the Hermite equation:

$$0 = \sum_{k=0}^{\infty}(k+r)(k+r-1)a_k x^{k+r-2} - 2x\sum_{k=0}^{\infty}(k+r)a_k x^{k+r-1} + 2n\sum_{k=0}^{\infty} a_k x^{k+r}. \tag{3}$$

To find the indicial equation and recursion relation, we require the sum of the coefficients of each power of x to be zero. Let's isolate the first few terms after re-indexing the first sum:

$$r(r-1)a_0 x^{r-2} + (r+1)ra_1 x^{r-1} + \sum_{k=0}^{\infty}[(k+r+2)(k+r+1)a_{k+2} - 2(k+r)a_k + 2na_k]x^{k+r} = 0; \tag{4}$$

The coefficient of the lowest power of x (i.e., x^{r-2}) gives the indicial equation:

$$r(r-1) = 0 \Longrightarrow r = 0 \quad \text{or} \quad 1. \tag{5}$$

Imposing the coefficient of x^{r-1} to be zero gives $(r+1)ra_1 = 0$. This implies $a_1 = 0$ when $r = 1$. The rest of the terms give the recursion relation:

$$\boxed{a_{k+2} = \frac{2(k+r-n)}{(k+r+2)(k+r+1)}a_k} \quad (r = 0 \quad \text{or} \quad r = 1, k = 0, 1, 2, ...) \tag{6}$$

For $r = 0$, the recursion relation becomes

$$a_{k+2} = \frac{2(k-n)}{(k+2)(k+1)}a_k. \tag{7}$$

Starting with a_0, we have

$$a_2 = -na_0, \quad a_4 = -\frac{n(2-n)}{6}a_0, \quad a_6 = \frac{n(n-2)(n-4)}{90}a_0, ... \tag{8}$$

we can write this into

$$y_{\text{even}} = a_0\left(1 - nx^2 + \frac{n(n-2)}{6}x^4 - ...\right) \tag{9}$$

Similarly, starting with a_1, we can write

$$y_{\text{odd}} = a_1\left(x + \frac{1-n}{3}x^3 + \frac{(1-n)(3-n)}{30}x^5 + ...\right) \tag{10}$$

and a general solution for $r = 0$ is $y = y_{\text{even}} + y_{\text{odd}}$. We notice that $r = 1$ gives the same series solution with leading constant a_0 instead of a_1. It is therefore sufficient to consider only $r = 0$.

We notice that the recursion terminates on $k - n = 0 \Rightarrow k = n$.

Now consider $H_1(x)$ with $n = 1$. Recursion :

$$a_{k+2} = \frac{2(k-1)}{(k+2)(k+1)}a_k. \tag{11}$$

Odd n gaurantees finite y_{odd} polynomial, so to get finite polynomial solution, we kill y_{even} by setting $a_0 = 0$. We thus have

$$y = a_1 x + \frac{1-1}{3} a_1 x^3 = a_1 x. \tag{12}$$

Convintionally, we set leading coefficient $a_1 = 2^n = 2$, and the Hermite polynomial to the first order is thus

$$\boxed{H_1(x) = 2x.} \tag{13}$$

Similarly, consider $H_2(x)$ with $n = 2$. Recursion :

$$a_{k+2} = \frac{2(k-2)}{(k+2)(k+1)} a_k. \tag{14}$$

Even n gaurantees finite y_{even} polynomial, so to get finite polynomial solution, we kill y_{od} by setting $a_1 = 0$. We thus have

$$y = a_0 - 2a_0 x^2. \tag{15}$$

Convintionally, we set leading coefficient $-2a_0 = 2^n = 4$, and the Hermite polynomial to the second order is thus

$$\boxed{H_2(x) = 4x^2 - 2.} \tag{16}$$

We can continue to read off several more: $H_3(x) = 8x^3 - 12x, H_4(x) = 16x^4 - 48x^2 + 12$.

2

Apply Frobenius method around $x = 0$ to the Legendre equation

$$(1 - x^2)y'' - 2xy' + n(n+1)y = 0. \tag{17}$$

Let $y = \sum_{k=0}^{\infty} a_k x^{k+r}$. Plugging into the Legendre equation, we have

$$0 = (1 - x^2)\sum_{k=0}^{\infty}(k+r)(k+r-1)a_k x^{k+r-2} - 2x\sum_{k=0}^{\infty}(k+r)a_k x^{k+r-1} + n(n+1)\sum_{k=0}^{\infty} a_k x^{k+r}. \tag{18}$$

We re-index the first and second term, and isolate the first few terms to match the summation bounds:

$$\begin{aligned} 0 = r(r-1)a_0 x^{r-2} + r(r+1)a_1 x^{r-1} + \sum_{0}^{\infty} x^{k+r}\big[&(k+r+2)(k+r+1)a_{k+2} \\ &-(k+r)(k+r-1)a_k - 2(k+r)a_k + n(n+1)a_k\big] \end{aligned} \tag{19}$$

Imposing the coefficient of the lowest power of x (i.e., x^{r-2}) to be zero gives the indicial equation:

$$r(r-1) = 0 \Longrightarrow r = 0 \quad \text{or} \quad 1. \tag{20}$$

Imposing the coefficient of x^{r-1} to be zero gives $a_1 = 0$ when $r = 1$. The rest of the terms give the recursion relation:

$$a_{k+2} = \frac{(k+r)(k+r+1) - n(n+1)}{(k+r+2)(k+r+1)} a_k, \quad (r = 0 \quad \text{or} \quad r = 1; k = 0, 1, 2, \ldots) \tag{21}$$

We consider the case $r = 0$. (the case $r = 1$ gives redundant solutions of odd series only.) The recursion relation becomes

$$\boxed{a_{k+2} = \frac{k(k+1) - n(n+1)}{(k+2)(k+1)} a_k} \tag{22}$$

and the termination condition is

$$k(k+1) - n(n+1) = 0 \Longrightarrow k = n. \tag{23}$$

We consider $P_1(x)$ with $n = 1$. Odd n gaurantees the odd series terminate, so kill even series with $a_0 \equiv 0$. We then have $y = a_1 x$. Convintionally, normalization is set so that $P_n(1) = 1 \Rightarrow a_1 = 1$. Thus

$$\boxed{P_1(x) = x.} \tag{24}$$

Similarly, for $P_2(x)$ with $n = 2$, even series terminate, so kill odd series with $a_1 \equiv 0$. We then have $a_2 = -3a_0$, and $y = a_0(1 - 3x^2)$. Convintionally, normalization is set so that $P_n(1) = 1 \Rightarrow a_0 = -\frac{1}{2}$. Thus

$$\boxed{P_2(x) = -\frac{1}{2}(3x^2 - 1).} \tag{25}$$

3

Apply Frobenius method around $x = 0$ to the Lauguerre equation

$$xy'' + (1 - x)y' + ny = 0, \tag{26}$$

where n is a non-negative integer. Find Lauguerre polynomials $L_n(x)$ for several low orders n

Let $y = \sum_{k=0}^{\infty} a_k x^{k+r}$. Plugging in, we have

$$\sum_{k=0}^{\infty}(k+r)(k+r-1)a_k x^{k+r-1} + \sum_{0}^{\infty}(k+r)a_k x^{k+r-1} - \sum_{k=0}^{\infty}(k+r)a_k x^{k+r} + n\sum_{k=0}^{\infty} a_k x^{k+r} = 0 \tag{27}$$

Re-indexing the first two sums and isolating the first few terms, we have

$$\begin{aligned} r(r-1)a_0 x^{r-1} + \sum_{k=0}^{\infty}(k+r+1)(k+r)a_{k+1}x^{k+r} + ra_0 x^{r-1} + \sum_{k=0}^{\infty}(k+r+1)a_{k+a}x^{k+r} \\ - \sum_{k=0}^{\infty}(k+r)a_k x^{k+r} + n\sum_{k=0}^{\infty} a_k x^{k+r} = 0 \end{aligned} \tag{28}$$

Imposing the coefficient of the lowest power of x (i.e., x^{r-1}) to be zero gives the indicial equation:

$$r^2 = 0 \Longrightarrow r = 0. \tag{29}$$

The rest of the terms give the recursion relation:

$$a_{k+1} = \frac{k+r-n}{(k+r+1)^2}a_k \overset{r=0}{=} \boxed{\frac{k-n}{(k+1)^2}a_k}, \quad (k = 0, 1, 2, ...) \tag{30}$$

From which we see that the series terminates when

$$k - n = 0 \Longrightarrow k = n. \tag{31}$$

Consider $L_1(x) : n = 1$. $a_1 = \frac{0-1}{1}a_0 = -a_0$ and so $y = a_0(1 - x)$. Conventional normalization sets $L_n(0) = 1 \Rightarrow a_0 = 1$. Thus

$$\boxed{L_1(x) = 1 - x.} \tag{32}$$

Similarly, $L_2(x)$ is given by $n = 2$. $a_1 = \frac{0-2}{1}a_0 = -2a_0$ and $a_2 = \frac{1-2}{4}a_1 = \frac{1}{2}a_0$. Thus $y = a_0\left(1 - 2x + \frac{1}{2}x^2\right)$. Conventional normalization sets $L_n(0) = 1 \Rightarrow a_0 = 1$. Thus

$$\boxed{L_2(x) = 1 - 2x + \frac{1}{2}x^2.} \tag{33}$$

Consider the Bessel equation with order $\nu = 0$

$$x^2 y'' + x y' + (x^2 - \nu^2) y = 0. \tag{34}$$

Show that the indicial equation has degenreate roots by construct explicitly the Bessel function $J_0(x)$ using Frobenius method. Then, use $y = J_0(x)\ln(x) + \sum_{k\geq 0} b_k x^k$ to find the second linearly independenly solution. Compare with the Bessel function of the second kind $Y_0(x)$.

We use the regular Frobenius ansatz $y = \sum_{k=0}^{\infty} a_k x^{k+r}$. Plugging into the Bessel equation with $\nu = 0$, we have

$$x^2 \sum_{k=0}^{\infty} (k+r)(k+r-1) a_k x^{k+r-2} + x \sum_{k=0}^{\infty} (k+r) a_k x^{k+r-1} + (x^2) \sum_{k=0}^{\infty} a_k x^{k+r} = 0. \tag{35}$$

Matching the indicies, we have

$$\begin{gathered} \sum_{k=0}^{\infty} (k+r)(k+r-1) a_k x^{k+r} + \sum_{k=0}^{\infty} (k+r) a_k x^{k+r} + \sum_{k=0}^{\infty} a_k x^{k+r+2} = 0 \\ \sum_{k=0}^{\infty} (k+r)^2 a_k x^{k+r} + \sum_{k=2}^{\infty} a_{k-2} x^{k+r} = 0 \end{gathered} \tag{36}$$

For $k = 0$, we read off the indicial equation:

$$r^2 = 0 \Longrightarrow r = 0. \tag{37}$$

For $k = 1$, we have $a_1 = 0$, gaurantees odd terms vanish. The rest of the terms give the recursion relation:

$$a_{k+2} = -\frac{1}{(k+r+2)^2} a_k \overset{r=0}{=} -\frac{1}{(k+2)^2} a_k, \quad (k = 0, 1, 2, \ldots) \tag{38}$$

Let $k = 2m$, the recursion relation gives

$$a_{2m} = a_0 (-1)^m \frac{1}{2^{2m} (m!)^2}; \quad J_0(x) = \sum_{m=0}^{\infty} a_{2m} x^{2m}. \tag{39}$$

and so the first solution is (choosing $a_0 = 1$) :

$$\boxed{J_0(x) = \sum_{m=0}^{\infty} (-1)^m \frac{x^{2m}}{2^{2m} (m!)^2}} \tag{40}$$

For the second solution, take the Frobenius-log ansatz

$$y_2(x) = J_0(x) \ln x + S(x), \quad S(x) = \sum_{k=0}^{\infty} b_k x^k. \tag{41}$$

Differentiate:

$$y_{2'} = J_{0'} \ln x + \frac{J_0}{x} + S'; \quad y_{2''} = J_{0''} \ln x + \frac{2 J_{0'}}{x} - \frac{J_0}{x^2} + S''. \tag{42}$$

Plug into $x^2 y'' + x y' + x^2 y = 0$. The $\ln x$-terms cancel because J_0 solves the ODE, and the non-$\ln x$ terms give

$$x^2 S'' + x S' + x^2 S + 2x J_{0'} = 0. \tag{43}$$

Now expand in series. With $S = \sum_{k\geq 0} b_k x^k$ and

$$2x J_0'(x) = \sum_{m=1}^{\infty} \frac{4m(-1)^m}{2^{2m} (m!)^2} x^{2m}, \tag{44}$$

equating the coefficient of x^k yields, for $k \geq 2$,

$$k^2 b_k + b_{k-2} + \begin{cases} \frac{4m(-1)^m}{2^{2m}(m!)^2} & k = 2m \\ 0 \quad (k \text{ odd}) \end{cases} = 0, \tag{45}$$

and from the x^1 term we get $b_1 = 0$. By induction, all odd coefficients vanish, $b_{2m+1} = 0$.

Thus it suffices to work with even indices. For $m \geq 1$,

$$4m^2 b_{2m} + b_{2m-2} + \frac{4m(-1)^m}{2^{2m}(m!)^2} = 0. \tag{46}$$

Introduce (ingenious insight from GPT)

$$d_m := (-1)^m 2^{2m} (m!)^2 b_{2m} \quad (m \geq 0). \tag{47}$$

Multiplying Equation 46 by $(-1)^m 2^{2m}(m!)^2$ gives

$$4m^2 d_m - 4m^2 d_{m-1} + 4m = 0 \quad \Rightarrow \quad d_m - d_{m-1} = -\frac{1}{m} \quad (m \geq 1). \tag{48}$$

Hence

$$d_m = d_0 - H_m, \quad H_m := \sum_{j=1}^{m} \frac{1}{j}; \quad H_0 := 0. \tag{49}$$

Undoing the substitution,

$$b_{2m} = \frac{(-1)^m}{2^{2m}(m!)^2}(d_0 - H_m), \quad b_{2m+1} = 0. \tag{50}$$

Now, rewrite

$$\begin{aligned} S(x) &= \sum_{m=0}^{\infty} b_{2m} x^{2m} = \sum_{m=0}^{\infty} \frac{(-1)^m}{2^{2m}(m!)^2}(d_0 - H_m) x^{2m} \\ &= d_0 \underbrace{\sum_{m=0}^{\infty} \frac{(-1)^m}{2^{2m}(m!)^2} x^{2m}}_{J_0(x)} - \sum_{m=1}^{\infty} \frac{(-1)^m H_m}{2^{2m}(m!)^2} x^{2m}. \end{aligned} \tag{51}$$

Therefore

$$\boxed{y_2(x) = J_0(x)(\ln x + d_0) - \sum_{m=1}^{\infty} \frac{(-1)^m H_m}{2^{2m}(m!)^2} x^{2m}.} \tag{52}$$

The standard small-x expansion of the Neumann function is (cite DLMF 10.8.2)

$$Y_0(x) = \frac{2}{\pi}\left(\ln \frac{x}{2} + \gamma\right) J_0(x) + \frac{2}{\pi} \sum_{m=1}^{\infty} \frac{(-1)^{m+1} H_m}{2^{2m}(m!)^2} x^{2m}. \tag{53}$$

Compare this with our y_2. Choose

$$d_0 = \gamma + \ln \frac{1}{2} \quad s.t. \quad \ln x + d_0 = \ln \frac{x}{2} + \gamma, \tag{54}$$

and then define Y_0 simply by scaling:

$$Y_0(x) = \frac{2}{\pi} y_2(x). \tag{55}$$

With this choice,

$$\frac{2}{\pi} y_2(x) = \frac{2}{\pi}\left(\ln \frac{x}{2} + \gamma\right) J_0(x) - \frac{2}{\pi} \sum_{m=1}^{\infty} \frac{(-1)^m H_m}{2^{2m}(m!)^2} x^{2m}, \tag{56}$$

which matches Equation 53 term-by-term because $-(-1)^m = (-1)^{m+1}$.

5

Solve the Airy function using Frobenius method to obtain two linearly independent series solutions

$$y'' - xy = 0 \tag{57}$$

Let $y = \sum_{k=0}^{\infty} a_k x^{k+r}$, substitution gives

$$\sum_{k=0}^{\infty}(k+r)(k+r-1)a_k x^{k+r-2} - \sum_{k=0}^{\infty}(k+r)a_k x^{k+r+1} = 0 \tag{58}$$

Reindexing the second term:

$$\sum_{k=0}^{\infty}(k+r)(k+r-1)a_k x^{k+r-2} - \sum_{k=3}^{\infty}(k+r-3)a_{k-3}a_{k-3}x^{k+r-2} = 0. \tag{59}$$

Now match coefficients of each power x^{k+r-2}:

- For $k = 0$: $r(r-1)a_0 = 0 \Longrightarrow \boxed{r = 0 \text{ or } 1}$.
- For $k = 1$: $(r+1)ra_1 = 0$.
- For $k = 2$: $(r+2)(r+1)a_2 = 0 \Longrightarrow \boxed{a_2 = 0}$.
- For $k \geq 3$: $(k+r)(k+r-1)a_k - a_{k-3} = 0$, and from which we reveal the recursion relation:

$$a_k = \frac{a_{k-3}}{(k+r)(k+r-1)}, \quad k \geq 3 \tag{60}$$

We investigate the two roots of the indicial equation separately.

- **Branch $r = 0$** (Ordinary Power Series)

The head constraints for $k = 1, 2$ give no restriction on a_1 but force $a_2 = 0$. Thus two free seeds a_0 and a_1 generate two decoupled subsequences (due to the step-3 recurrence):

From a_0: $a_3 = \frac{a_0}{3 \cdot 2} = \frac{a_0}{6}$, $a_6 = \frac{a_3}{6 \cdot 5} = \frac{a_0}{180}$, etc. (indices $0, 3, 6, \ldots$).

From a_1: $a_4 = \frac{a_1}{4 \cdot 3} = \frac{a_1}{12}$, $a_7 = \frac{a_4}{7 \cdot 6} = \frac{a_1}{504}$, etc. (indices $1, 4, 7, \ldots$).

Hence

$$y(x) = a_0 \left(1 + \frac{x^3}{3!2!} + \frac{x^6}{6!5!} + \ldots\right) + a_1 \left(x + \frac{x^4}{4!3!} + \frac{x^7}{7!6!} + \ldots\right), \tag{61}$$

since $3 \cdot 2 = \frac{3!}{1!}$, $6 \cdot 5 = \frac{6!}{4!}$, etc.

- **Branch $r = 1$**: The head constraints give $a_1 = a_2 = 0$. The recurrence then produces the same "1 mod 3" subsequence as taking $r = 0$ with $a_1 \neq 0$. Thus it does not yield an independent solution beyond the two already obtained from $r = 0$.

Comparing Equation 61 to the standard Airy functions Ai(x) and Bi(x), we see that they are specific choices of these constants, selected for their distinct asymptotic behaviors.

If we choose the normalization constants

$$a_0 = \frac{1}{3^{\frac{2}{3}}\Gamma\left(\frac{2}{3}\right)} \quad \text{and} \quad a_1 = -\frac{1}{3^{\frac{1}{3}}\Gamma\left(\frac{1}{3}\right)}, \tag{62}$$

we can arrive at the specific solution:

$$y(x) = \frac{1}{3^{\frac{2}{3}}\Gamma\left(\frac{2}{3}\right)}(1 + \ldots) - \frac{1}{3^{\frac{1}{3}}\Gamma\left(\frac{1}{3}\right)}(x + \ldots). \tag{63}$$

This combination matches the definition of the **Airy function of the first kind**, $\mathrm{Ai}(x)$.

Alternatively, if we choose the constants

$$a_0 = \frac{1}{3^{\frac{2}{3}}\Gamma\left(\frac{2}{3}\right)} \quad \text{and} \quad a_1 = \frac{1}{3^{\frac{1}{3}}\Gamma\left(\frac{1}{3}\right)}, \tag{64}$$

we can arrive at the specific solution:

$$y(x) = \frac{1}{3^{\frac{2}{3}}\Gamma\left(\frac{2}{3}\right)}(1 + ...) + \frac{1}{3^{\frac{1}{3}}\Gamma\left(\frac{1}{3}\right)}(x + ...). \tag{65}$$

Which matches the definition of the **Airy function of the second kind**, $\mathrm{Bi}(x)$.