

For the finite square well

$$V = \begin{cases} 0 & |x| < a \\ V_0 & |x| > a \end{cases} \quad (1)$$

1. Determine the odd parity eigenfunctions and their associated energy eigenvalues for this potential, and discuss limiting behaviour as $V_0 \rightarrow 0$, $V_0 \rightarrow \infty$.
2. Find accurate numerical values for the boundstate energy eigenvalues of a particle in the above finite square well potential, in which the parameter

$$R \equiv \sqrt{\frac{2mV_0a^2}{\hbar^2}} = 4. \quad (2)$$

Find solutions graphically and numerically.

1. Odd parity

Label region I : $x < -a$, II : $-a < x < a$, III : $x > a$. From lecture: $\psi_{\text{II}} = A \cos kx + B \sin kx$, $\psi_{\text{I}} = De^{\kappa x}$, $\psi_{\text{III}} = Fe^{\kappa x}$, with $k = \frac{\sqrt{2mE}}{\hbar}$, $\kappa = \frac{\sqrt{2m(V_0-E)}}{\hbar}$.

Imposing odd parity $\psi(-x) = -\psi(x)$, we have $A = 0$, $D = -F$. Imposing boundary condition of continuity and smoothness at $x = a$, we have

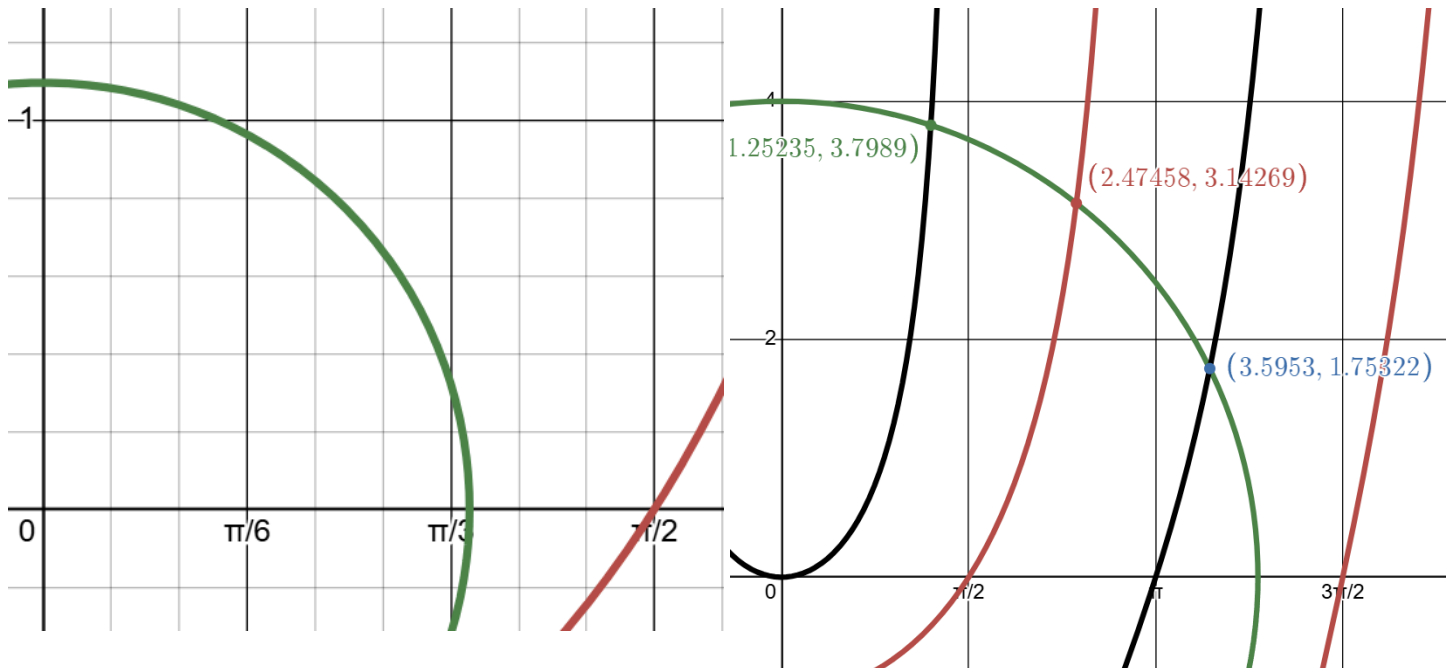
$$B \sin ka = Fe^{-\kappa a}, \quad B \cos ka = -\kappa Fe^{-\kappa a}. \quad (3)$$

Dividing these and setting $\eta = \kappa a$, $\xi = ka$, we have

$$\xi \cot \xi = -\eta, \quad \xi^2 + \eta^2 = R^2. \quad (4)$$

Limiting behaviour:

- as $V_0 \rightarrow \infty$, $\eta \rightarrow \infty$ and $\cot \xi \rightarrow -\infty$ satisfied at $\eta = n\pi$, $n \in \mathbb{Z}$.
- as $V_0 \rightarrow 0$, $R \rightarrow 0$, but as seen below (LEFT) that $\xi \cot \xi = -\eta$ ($\xi, \eta > 0$) has solution only for $R \geq \frac{\pi}{2}$, the limit of $V_0 \rightarrow 0$ yields **no odd parity bound states.**)



2. Numerical and graphical

1. Graphically (ABOVE, RIGHT), The equations to solve are:

- Even States: $\xi \tan(\xi) = \eta$ (black)

- Odd States: $-\xi \cot(\xi) = \eta$ (red)
- Constraint: $\xi^2 + \eta^2 = 16$ (Green)

From a graphical analysis, we find three bound states:

$$(\xi, \eta) = (1.25235, 3.7989),$$

$$(\xi, \eta) = (2.47458, 3.14269),$$

$$(\xi, \eta) = (3.5953, 1.75322)$$

2. Numerically, we solve

$$\begin{cases} \xi \tan \xi = \sqrt{16 - \xi^2} \\ -\xi \cot \xi = \sqrt{16 - \xi^2} \end{cases} \quad (5)$$

and the solutions are identical as above:

$$\xi_1 = 1.25235, \xi_2 = 2.47458, \xi_3 = 3.5953. \quad (6)$$

and the energies are

$$E = V_0 \left(\frac{\xi^2}{16} \right) \quad (7)$$

$$\Rightarrow E_1 = 0.09797, E_2 = 0.3829, E_3 = 0.8077$$

Show that for spinless particles moving in 1D, the energy spectrum of bound states is always non-degenerate.

Assume not: exists $\psi_i(x)$, ($i = 1, 2$) s.t.

$$-\frac{\hbar^2}{2m} \frac{d^2 \psi_i}{dx^2} + V(x) \psi_i = E \psi_i, \quad E = E_1 = E_2. \quad (8)$$

In particular, $\psi_1(x), \psi_2(x)$ are linear independent. This would imply that their wronskian

$$W(x) = \psi_1 \psi_2' - \psi_2 \psi_1' = 0 \quad (9)$$

However, we notice that

$$W'(x) = 2 \frac{m}{\hbar^2} (\psi_1 (V - E) \psi_2 - \psi_2 (V - E) \psi_1) = 0 \quad (10)$$

so $W(x) = \text{const.}$ Further, since ψ_1, ψ_2 are bound states,

$$\lim_{x \rightarrow \pm\infty} \psi_i(x) = 0 \Rightarrow W(\pm\infty) = 0 \Rightarrow W(x) = 0 \quad \forall x. \quad (11)$$

This contradicts the linear independence of ψ_1, ψ_2 . Thus, the energy spectrum of bound states is always non-degenerate.

Use the Hermite generating function

$$g(y, t) = e^{-t^2+2ty} = \sum_{n=0}^{\infty} H_n(y) \frac{t^n}{n!} \quad (12)$$

1. To prove the following properties

$$H_n(y) = e^{\frac{y^2}{2}} \left(y - \frac{d}{d} y \right)^n e^{-\frac{y^2}{2}}, \quad (13)$$

$$H'_n(y) = 2nH_{n-1}(y)$$

$$H_{n+1}(y) = 2yH_n(y) - 2nH_{n-1}(y).$$

2. Then evaluate

$$\int_{-\infty}^{\infty} dy e^{-y^2} H_n(y) H_{n'}(y) \quad (14)$$

1

a

Recall that $H_n(y) = \left(\frac{\partial}{\partial t} \right)^n \Big|_{t=0} g(y, t)$. Let $u = t - y$, $\partial u = \partial t$. Then $g = e^{-u^2} e^{y^2}$. Then from definition,

$$H_n(y) = \left(\frac{\partial}{\partial u} \right)^n \left(e^{-u^2} e^{y^2} \right) \Big|_{u=-y} = e^{y^2} \left(\frac{\partial}{\partial u} \right)^n e^{-u^2} \Big|_{u=-y} = (-1)^n e^{y^2} \left(\frac{\partial}{\partial y} \right)^n e^{-y^2} \quad (15)$$

Notice the identity

$$\frac{d}{dy} (e^{-y^2/2} g) = -e^{-y^2/2} \left(y - \frac{d}{dy} \right) g, \quad (16)$$

we have

$$\left(\frac{d}{dy} \right)^n (e^{-y^2}) = \left(\frac{d}{dy} \right)^n (e^{-y^2/2} e^{-y^2/2}) = -e^{-y^2/2} \left(y - \frac{d}{dy} \right)^n e^{-y^2/2}. \quad (17)$$

And so

$$H_n(y) = (-1)^n e^{y^2} \left(\frac{d}{dy} \right)^n e^{-y^2} = (-1)^{2n} e^{y^2/2} \left(y - \frac{d}{dy} \right)^n e^{-y^2/2} = \boxed{e^{y^2/2} \left(y - \frac{d}{dy} \right)^n e^{-y^2/2}}, \quad (18)$$

as wanted.

b

Notice that

$$\frac{\partial g}{\partial t} = 2tg(y, t) = \sum_{n=0}^{\infty} \frac{2t^{n+1}}{n!} H_n(y) = \sum_{n=0}^{\infty} \underbrace{2(n+1) \frac{t^{n+1}}{(n+1)!}}_{*} H_n(y) = \sum_{n=0}^{\infty} H'_n(y) \frac{t^n}{n!}. \quad (19)$$

But (*) is also

$$2nH_{n-1}(y) \frac{t^n}{n!}. \quad (20)$$

Therefore

$$H'_n(y) = 2nH_{n-1}(y). \quad (21)$$

as wanted.

c

$$\begin{aligned}\partial_t g &= (2y - 2t)g = \sum_{n=0}^{\infty} \left(2yH_n(y) \frac{t^n}{n!} - 2H_n(y) \frac{t^{n+1}}{n!} \right) \\ &\stackrel{n:=n-1}{=} \sum_{n=0}^{\infty} \left(2yH_n(y) \frac{t^n}{n!} - 2nH_{n-1}(y) \frac{t^n}{n!} \right)\end{aligned}\quad (22)$$

But $\partial_t g$ is also

$$\sum_{n=0}^{\infty} \frac{t^{n-1}}{(n-1)!} H_n(y) = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_{n+1}(y), \quad (23)$$

and so

$$H_{n+1}(y) = 2yH_n(y) - 2nH_{n-1}(y). \quad (24)$$

2.

Consider

$$\int_{-\infty}^{\infty} e^{-y^2} g(y, t) g(y, s) dy = e^{-(t^2+s^2)} \int_{-\infty}^{\infty} e^{-y^2+2(t+s)y} dy. \quad (25)$$

Complete the square in the exponent of the integrand: $-y^2 + 2(t+s)y = -(y - (t+s))^2 + (t+s)^2$. Thus,

$$\int_{-\infty}^{\infty} e^{-(y-(t+s))^2+(t+s)^2} dy = e^{(t+s)^2} \int_{-\infty}^{\infty} e^{-u^2} du = e^{(t+s)^2} \sqrt{\pi}, \quad (26)$$

where $u = y - (t+s)$. Substituting back,

$$\int_{-\infty}^{\infty} e^{-y^2} g(y, t) g(y, s) dy = e^{-(t^2+s^2)} e^{(t+s)^2} \sqrt{\pi} = e^{-(t^2+s^2)+t^2+s^2+2ts} \sqrt{\pi} = e^{2ts} \sqrt{\pi}. \quad (27)$$

On the other hand, expanding the generating functions gives

$$g(y, t)g(y, s) = \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} \frac{t^m s^l}{m! l!} H_m(y) H_l(y), \quad (28)$$

so

$$\int_{-\infty}^{\infty} e^{-y^2} g(y, t) g(y, s) dy = \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} \frac{t^m s^l}{m! l!} \int_{-\infty}^{\infty} e^{-y^2} H_m(y) H_l(y) dy. \quad (29)$$

Equating the two expressions yields

$$\sum_{m=0}^{\infty} \sum_{l=0}^{\infty} \frac{t^m s^l}{m! l!} \int_{-\infty}^{\infty} e^{-y^2} H_m(y) H_l(y) dy = \sqrt{\pi} e^{2ts}. \quad (30)$$

The Taylor expansion of the right-hand side is

$$e^{2ts} = \sum_{n=0}^{\infty} \frac{(2ts)^n}{n!} = \sum_{n=0}^{\infty} \frac{2^n t^n s^n}{n!}. \quad (31)$$

For the left-hand side to match, the double sum must reproduce this only when $m = l = n$, implying that the integral vanishes unless $m = l$. Specifically, the coefficient of $\frac{t^m s^l}{m! l!}$ on the right is $\sqrt{\pi} 2^m \frac{m!}{m!} \delta_{ml}$ (zero otherwise),

so

$$\int_{-\infty}^{\infty} e^{-y^2} H_m(y) H_l(y) dy = \sqrt{\pi} 2^m m! \delta_{ml}. \quad (32)$$

Using wavefunctions, compute $\langle n' | p | n \rangle$ for the eigenstates of the 1d SHO to show that

$$\langle n' | p | n \rangle = i\sqrt{\frac{m\omega\hbar}{2}}(\sqrt{n+1}\delta_{n',n+1} - \sqrt{n}\delta_{n',n-1}) \quad (33)$$

1

In the position representation, the momentum operator is $\hat{p} = -i\hbar \frac{d}{dx}$. The matrix element is therefore given by the integral:

$$\langle n' | p | n \rangle = \int_{-\infty}^{\infty} \psi_{n'}^*(x) \left(-i\hbar \frac{d}{dx} \right) \psi_n(x) dx \quad (34)$$

The normalized energy eigenfunctions are (from lecture)

$$\psi_n(x) = C_n H_n(y) e^{-y^2/2} \quad \text{with} \quad y = \frac{x}{b} = x \sqrt{\frac{m\omega}{\hbar}} \quad (35)$$

where $C_n = (m\omega/\pi\hbar)^{1/4} (2^n n!)^{-1/2} = (b\sqrt{\pi} 2^k k!)^{-1/2}$ is the normalization constant.

First, consider:

$$\frac{d\psi_n(x)}{dx} = \frac{C_n}{b} \frac{d}{dy} (H_n(y) e^{-y^2/2}) = \frac{C_n}{b} (H_{(n)'}(y) e^{-y^2/2} - y H_n(y) e^{-y^2/2}) \quad (36)$$

Using P3:

1. $H_{(n)'}(y) = 2n H_{n-1}(y)$
2. $2y H_n(y) = H_{n+1}(y) + 2n H_{n-1}(y) \Rightarrow y H_n(y) = \frac{1}{2} H_{n+1}(y) + n H_{n-1}(y)$

Then we have

$$\frac{d\psi_n(x)}{dx} = \frac{C_n}{b} e^{-y^2/2} \left(2n H_{n-1}(y) - \left[\frac{1}{2} H_{n+1}(y) + n H_{n-1}(y) \right] \right) = \frac{C_n}{b} e^{-y^2/2} \left(n H_{n-1}(y) - \frac{1}{2} H_{n+1}(y) \right) \quad (37)$$

Now $\langle n' | p | n \rangle$, with $y (dx = b dy)$ becomes:

$$\langle n' | p | n \rangle = -i\hbar \int_{-\infty}^{\infty} (C_{n'} H_{n'}(y) e^{-y^2/2}) \left(\frac{C_n}{b} e^{-y^2/2} \left[n H_{n-1}(y) - \frac{1}{2} H_{n+1}(y) \right] \right) (b dy) \quad (38)$$

$$= -i\hbar C_{n'} C_n \int_{-\infty}^{\infty} e^{-y^2} H_{n'}(y) \left[n H_{n-1}(y) - \frac{1}{2} H_{n+1}(y) \right] dy \quad (39)$$

The integral splits into two terms. We use the orthogonality relation for Hermite polynomials,

$$\int_{-\infty}^{\infty} e^{-y^2} H_m(y) H_l(y) dy = \sqrt{\pi} 2^l l! \delta_{ml}:$$

1. The first term is non-zero only if $n' = n - 1$:

$$n \int_{-\infty}^{\infty} e^{-y^2} H_{n-1}(y) H_{n-1}(y) dy = n \sqrt{\pi} 2^{n-1} (n-1)! = \frac{\sqrt{\pi}}{2} 2^n n! \quad (40)$$

2. The second term is non-zero only if $n' = n + 1$:

$$-\frac{1}{2} \int_{-\infty}^{\infty} e^{-y^2} H_{n+1}(y) H_{n+1}(y) dy = -\frac{1}{2} \sqrt{\pi} 2^{n+1} (n+1)! \quad (41)$$

The matrix element is non-zero only for $n' = n \pm 1$.

- Case 1: $n' = n - 1$

$$\langle n-1 | p | n \rangle = -i\hbar C_{n-1} C_n \left(\frac{\sqrt{\pi}}{2} 2^n n! \right) \quad (42)$$

$$= -i\hbar \frac{1}{b\sqrt{\pi}\sqrt{2^{n-1}(n-1)!2^n n!}} \left(\frac{\sqrt{\pi}}{2} 2^n n! \right) = -i\hbar \frac{\sqrt{2^n n!}}{b\sqrt{2 \cdot 2^{n-1}(n-1)!}} = -i\hbar \frac{\sqrt{n}}{b\sqrt{2}} \quad (43)$$

Using $b = \sqrt{\hbar/m\omega}$, we get $\langle n-1 | p | n \rangle = -i\sqrt{\frac{m\omega\hbar}{2}}\sqrt{n}$.

- Case 2: $n' = n+1$,

$$\langle n+1 | p | n \rangle = -i\hbar C_{n+1} C_n \left(-\frac{1}{2} \sqrt{\pi} 2^{n+1} (n+1)! \right) \quad (44)$$

$$= i\hbar \frac{1}{b\sqrt{\pi}\sqrt{2^{n+1}(n+1)!2^n n!}} \left(\frac{\sqrt{\pi}}{2} 2^{n+1} (n+1)! \right) = i\hbar \frac{\sqrt{2^{n+1}(n+1)!}}{b\sqrt{2 \cdot 2^n n!}} = i\hbar \frac{\sqrt{2(n+1)}}{b\sqrt{2}} \quad (45)$$

Using $b = \sqrt{\hbar/m\omega}$, we get $\langle n+1 | p | n \rangle = i\sqrt{\frac{m\omega\hbar}{2}}\sqrt{n+1}$.

Combining these results using the Kronecker delta gives the final expression:

$$\boxed{\langle n' | p | n \rangle = i\sqrt{\frac{m\omega\hbar}{2}} (\sqrt{n+1} \delta_{n',n+1} - \sqrt{n} \delta_{n',n-1})} \quad (46)$$

2

In the momentum representation,

$$\langle n' | p | n \rangle = \int_{-\infty}^{\infty} \phi_{n'}^*(p) p \phi_n(p) dp \quad (47)$$

The momentum-space eigenfunctions are (from lecture)

$$\phi_n(p) = (-i)^n D_n H_n(q) e^{-q^2/2} \quad \text{with} \quad q = \frac{p}{\sqrt{m\omega\hbar}} \quad (48)$$

where $D_n = (1/m\omega\pi\hbar)^{1/4} (2^n n!)^{-1/2}$ is the normalization constant.

Similar to calculation of the position matrix element $\langle n' | x | n \rangle$ in position space.

$$\langle n' | p | n \rangle = \int_{-\infty}^{\infty} \left((-i)^{n'} D_{n'} H_{n'}(q) e^{-q^2/2} \right)^* \cdot p \cdot \left((-i)^n D_n H_n(q) e^{-q^2/2} \right) dp \quad (49)$$

$$= (i)^{n'} (-i)^n D_{n'} D_n \int_{-\infty}^{\infty} e^{-q^2} H_{n'}(q) H_n(q) p dp \quad (50)$$

Changing variables $p = q\sqrt{m\omega\hbar}$ and $dp = dq\sqrt{m\omega\hbar}$:

$$= (i)^{n'} (-i)^n D_{n'} D_n (m\omega\hbar) \int_{-\infty}^{\infty} e^{-q^2} H_{n'}(q) (q H_n(q)) dq \quad (51)$$

Using $q H_n(q) = \frac{1}{2} H_{n+1}(q) + n H_{n-1}(q)$ and the orthogonality relation, we again find that the integral is non-zero only for $n' = n \pm 1$.

- Case 1: $n' = n-1$, The phase factor is $(i)^{n-1} (-i)^n = i^{-1} = -i$. The integral gives $n\sqrt{\pi} 2^{n-1} (n-1)!$.

$$\langle n-1 | p | n \rangle = (-i) D_{n-1} D_n (m\omega\hbar) (n\sqrt{\pi} 2^{n-1} (n-1)!) \quad (52)$$

The calculation for the constants is analogous to the position-space case, with the parameter $\sqrt{m\omega\hbar}$ replacing $1/b$.

$$\langle n-1 | p | n \rangle = (-i) \sqrt{\frac{m\omega\hbar}{2}} \sqrt{n} \quad (53)$$

- Case 2: $n' = n+1$, The phase factor is $(i)^{n+1} (-i)^n = i$. The integral gives $\frac{1}{2} \sqrt{\pi} 2^{n+1} (n+1)!$.

$$\langle n+1 | p | n \rangle = (i) D_{n+1} D_n (m\omega\hbar) \left(\frac{1}{2} \sqrt{\pi} 2^{n+1} (n+1)! \right) \quad (54)$$

$$\langle n+1 | p | n \rangle = (i) \sqrt{\frac{m\omega\hbar}{2}} \sqrt{n+1} \quad (55)$$

Combining these gives the same final result

$$\boxed{\langle n' | p | n \rangle = i \sqrt{\frac{m\omega\hbar}{2}} (\sqrt{n+1} \delta_{n',n+1} - \sqrt{n} \delta_{n',n-1})} \quad (56)$$

For 1. The ground state, and 2. The first excited state, calculate the probability that a particle of mass m in the 1d SHO with freq ω is farther from the origin than the classical turning points where $E = V$.

- Let $b \equiv \sqrt{\hbar/(m\omega)}$ and $y \equiv x/b$.
- SHO energies: $E_n = (n + \frac{1}{2})\hbar\omega$.
- Classical turning points solve $E_n = \frac{1}{2}m\omega^2 x_t^2$, hence

$$x_t(n) = b\sqrt{2n+1}, \quad y_t(n) = \sqrt{2n+1}. \quad (57)$$

- Normalized SHO wavefunctions (in y):

$$\psi_n(x) = \frac{1}{\sqrt{b}} \frac{1}{\pi^{1/4} \sqrt{2^n n!}} H_n(y) e^{-y^2/2}, \quad (58)$$

so

$$|\psi_n(x)|^2 = \frac{1}{\sqrt{\pi} b} \frac{H_n(y)^2}{2^n n!} e^{-y^2}. \quad (59)$$

- “Tunneling” probability outside the classical region:

$$P_n \equiv 2 \int_{x_t(n)}^{\infty} |\psi_n(x)|^2 dx = 2 \int_{y_t(n)}^{\infty} \frac{1}{\sqrt{\pi}} \frac{H_n(y)^2}{2^n n!} e^{-y^2} dy. \quad (60)$$

- Ground state $n=0$

Here $H_0(y) = 1$, $y_t = \sqrt{1} = 1$:

$$P_0 = \frac{2}{\sqrt{\pi}} \int_1^{\infty} e^{-y^2} dy = \operatorname{erfc}(1) \approx 0.157299. \quad (61)$$

- First excited state $n=1$

- Here $H_1(y) = 2y$, $y_t = \sqrt{3}$, and

$$P_1 = \frac{2}{\sqrt{\pi}} \int_{\sqrt{3}}^{\infty} \frac{(2y)^2}{2} e^{-y^2} dy = \frac{4}{\sqrt{\pi}} \int_{\sqrt{3}}^{\infty} y^2 e^{-y^2} dy. \quad (62)$$

- Use

$$\int y^2 e^{-y^2} dy = -\frac{y}{2} e^{-y^2} + \frac{\sqrt{\pi}}{4} \operatorname{erf}(y), \quad (63)$$

to obtain for $a > 0$,

$$\int_a^{\infty} y^2 e^{-y^2} dy = \frac{\sqrt{\pi}}{4} \operatorname{erfc}(a) + \frac{a}{2} e^{-a^2}. \quad (64)$$

- With $a = \sqrt{3}$:

$$P_1 = \frac{4}{\sqrt{\pi}} \left[\frac{\sqrt{\pi}}{4} \operatorname{erfc}(\sqrt{3}) + \frac{\sqrt{3}}{2} e^{-3} \right] = \operatorname{erfc}(\sqrt{3}) + \frac{2\sqrt{3}}{\sqrt{\pi}} e^{-3}. \quad (65)$$

$$P_1 = \operatorname{erfc}(\sqrt{3}) + \frac{2\sqrt{3}}{\sqrt{\pi}} e^{-3} \approx 0.11$$

(66)

Show that for the 1d SHO (x being the position operator,)

$$\langle 0|e^{ikx}|0\rangle = \exp[-k^2\langle 0|x^2|0\rangle/2] \quad (67)$$

Start from creation and annihilation operators:

$$x = \sqrt{\frac{\hbar}{2m\omega}}(a + a^\dagger) \Rightarrow e^{ikx} = e^{i\lambda(a+a^\dagger)}, \quad \left(\lambda = k\sqrt{\frac{\hbar}{2m\omega}}\right). \quad (68)$$

Since

$$[i\lambda a, i\lambda a^\dagger] = i^2\lambda^2(aa^\dagger - a^\dagger a) = \lambda^2(a^\dagger a - aa^\dagger) = \lambda^2[a^\dagger, a] = -\lambda^2, \quad (69)$$

we can use the BCH formula to write

$$e^{i\lambda(a+a^\dagger)} = e^{i\lambda a} e^{i\lambda a^\dagger} e^{\frac{\lambda^2}{2}}. \quad (70)$$

Then

$$\langle 0|e^{ikx}|0\rangle = e^{\frac{\lambda^2}{2}} \underbrace{\langle 0|e^{i\lambda a} e^{i\lambda a^\dagger}|0\rangle}_* \quad (71)$$

Expanding

$$e^{i\lambda a} = \sum_{m=0}^{\infty} \frac{(i\lambda a)^m}{m!}, \quad e^{i\lambda a^\dagger} = \sum_{n=0}^{\infty} \frac{(i\lambda a^\dagger)^n}{n!}, \quad (72)$$

we have (*) to be

$$\left\langle 0 \left| \sum_m \sum_n \frac{(i\lambda)^{m+n}}{m!n!} a^m (a^\dagger)^n \right| 0 \right\rangle = \sum_m \sum_n \frac{(i\lambda)^{m+n}}{m!n!} \underbrace{\langle 0|a^m (a^\dagger)^n|0\rangle}_{**}. \quad (73)$$

Since $(a^\dagger)^n|0\rangle = \sqrt{n!}|n\rangle$, (**) becomes

$$\sqrt{n!}\langle 0|a^m|n\rangle = \sqrt{n!}\sqrt{n(n-1)\dots(n-m+1)}\langle 0|n-m\rangle = \sqrt{n!}\sqrt{n(n-1)\dots(n-m+1)}\delta_{m,n} = n!\delta_{m,n}. \quad (74)$$

So for $n = m$,

$$(*) = \sum_{n=0}^{\infty} \frac{(i\lambda)^{2n}}{n!} = \sum_n \frac{(-\lambda^2)^n}{n!} = e^{-\lambda^2}. \quad (75)$$

Then

$$\langle 0|e^{ikx}|0\rangle = e^{\frac{\lambda^2}{2}} e^{-\lambda^2} = e^{-\frac{\lambda^2}{2}}. \quad (76)$$

On the other hand, since

$$\langle 0|x^2|0\rangle = \left\langle 0 \left| \frac{\hbar}{2m\omega}(a^2 + aa^\dagger + a^\dagger a + a^{\dagger 2}) \right| 0 \right\rangle = \frac{\hbar}{2m\omega} \langle 0|a^\dagger a + 1|0\rangle = \frac{\hbar}{2m\omega}, \quad (77)$$

we have

$$\frac{k^2}{2} \langle 0|x^2|0\rangle = \frac{k^2}{2} \frac{\hbar}{2m\omega} = \frac{\lambda^2}{2}, \quad (78)$$

and so

$$\langle 0|e^{ikx}|0\rangle = e^{-k^2\langle 0|x^2|0\rangle/2}, \quad (79)$$

as desired.