Applications of the Residue Theorem

$$\boxed{\text{Type-I Integrals: } I = \int_0^{2\pi} R(\cos\varphi, \sin\varphi)\, d\varphi}$$

If $R(u,v)$ is rational function of u and v then by change of variables $z = e^{i\varphi}$ the above integral can be converted into the loop integral in the complex plane. Since $\varphi \in [0, 2\pi]$ then variable z makes a circle of radius $|z| = 1$ in the complex plane in the positive direction (counter-clockwise).

$$I = \oint_{|z|=1} R_1(z)\, dz \qquad R_1(z) = -\frac{i}{z} R\left[\frac{1}{2}\left(z+\frac{1}{z}\right), \frac{1}{2i}\left(z-\frac{1}{z}\right)\right]$$

$$I = 2\pi i \sum_{k=1}^{N} \operatorname{Res}\left[R_1(z)\right]_{z=z_k}$$

where z_k are poles of $R_1(z)$ that lie inside $|z|=1$.

$\boxed{\text{Exp. 1}}$

$$I = \int_0^{2\pi} \frac{d\varphi}{1 - 2a\cos\varphi + a^2} \qquad |a| < 1$$

$$z = e^{i\varphi} \Rightarrow dz = i e^{i\varphi} d\varphi = i z\, d\varphi$$

$$I = \oint_{|z|=1} \frac{dz}{iz} \frac{1}{1 - 2a \cdot \frac{1}{2} \cdot \left(z + \frac{1}{z}\right) + a^2} = \oint_{|z|=1} \frac{i\, dz}{a z^2 - (a^2+1) z + a}$$

$$a z^2 - (a^2+1) z + a = 0 \quad \begin{array}{l} \nearrow\ z_1 = a \in |z| = 1 \\ \searrow\ z_2 = 1/a \notin |z| = 1 \text{ (out side)} \end{array}$$

$$\operatorname*{Res}_{z=a}\left[\frac{i}{az^2-(a^2+1)z+a}\right]=\left[\frac{i}{2az-(a^2+1)}\right]_{z=a}=\frac{i}{a^2-1}$$

$$\boxed{\int_0^{2\pi}\frac{d\varphi}{1-2a\cos\varphi+a^2}=\frac{2\pi}{1-a^2}}$$

$\boxed{\text{Exp. 2}}$

$$I=\int_0^{2\pi}\frac{dx}{(p+q\cos x)^2}\qquad p>q>0$$

$$z=e^{i\varphi}\Rightarrow\quad dz=iz\,d\varphi$$

$$I=\oint_{|z|=1}\frac{dz}{iz\left[p+\frac{q}{2}\left(z+\frac{1}{z}\right)\right]^2}=-i\oint_{|z|=1}\frac{z\,dz}{\left(\frac{q}{2}z^2+pz+\frac{q}{2}\right)^2}$$

$$\frac{q}{2}z^2+pz+\frac{q}{2}=0\ \begin{matrix}\nearrow & z_1=\frac{1}{q}\left(-p+\sqrt{p^2-q^2}\right)\ \in\ |z|=1\\ \searrow & z_2=\frac{1}{q}\left(-p-\sqrt{p^2-q^2}\right)\ \notin\ |z|=1\end{matrix}$$

$$\frac{q}{2}z^2+pz+\frac{q}{2}=\frac{q}{2}(z-z_1)(z-z_2)$$

$$I=-\frac{4i}{q^2}\oint_{|z|=1}\frac{z\,dz}{(z-z_1)^2(z-z_2)^2}=\frac{8\pi}{q^2}\operatorname{Res}\left[\frac{z}{(z-z_1)^2(z-z_2)^2}\right]_{z=z_1}$$

$$=\frac{8\pi}{q^2}\frac{d}{dz}\left[\frac{z}{(z-z_2)^2}\right]_{z=z_1}=\frac{8\pi}{q^2}\left[\frac{1}{(z_1-z_2)^2}-\frac{2z_1}{(z_1-z_2)^3}\right]=$$

$$=-\frac{8\pi}{q^2}\frac{z_1+z_2}{(z_1-z_2)^3}=-\frac{8\pi}{q^2}\frac{-2p/q}{\left(\left(2/q\right)\sqrt{p^2-q^2}\right)^3}=\frac{2\pi p}{(p^2-q^2)^{3/2}}$$

$$\boxed{\int_0^{2\pi}\frac{d\varphi}{(p+q\cos\varphi)^2}=\frac{2\pi p}{(p^2-q^2)^{3/2}}}$$

$\boxed{\text{Exp. 3}}$ $\quad I = \int_0^{\pi} \cot(\varphi - a)\, d\varphi \qquad a = \alpha + i\beta \qquad \beta \neq 0$

$$z = e^{2i(\varphi - a)} \Rightarrow dz = 2i\, e^{2i(\varphi-a)} d\varphi = 2iz\, d\varphi$$

$$\cot(\varphi - a) = \frac{\cos(\varphi-a)}{\sin(\varphi-a)} = \frac{\frac{1}{2}\left[e^{i(\varphi-a)} + e^{-i(\varphi-a)}\right]}{\frac{1}{2i}\left[e^{i(\varphi-a)} - e^{-i(\varphi-a)}\right]} = i\,\frac{z+1}{z-1}.$$

For $\varphi \in [0, 2\pi]$ new variable z changes in the circle:

$$|z| = \left|e^{2i\varphi}\, e^{-2i(\alpha+i\beta)}\right| = e^{2\beta}\left|e^{2i\varphi - 2i\alpha}\right| = e^{2\beta}$$

$$I = \oint_{|z|=e^{2\beta}} i\,\frac{z+1}{z-1}\,\frac{dz}{2iz} = \frac{1}{2}\oint_{|z|=e^{2\beta}} \frac{z+1}{z(z-1)}\, dz$$

→ For $\beta > 0$ the radius of the circle $e^{2\beta} > 1$ so that both $z=0$ and $z=1$ (poles) are within the integration region.

$$\operatorname*{Res}_{z=0}\left[\frac{z+1}{z(z-1)}\right] = \left(\frac{z+1}{z-1}\right)_{z=0} = -1$$

$$\operatorname*{Res}_{z=1}\left[\frac{z+1}{z(z-1)}\right] = \left(\frac{z+1}{z}\right)_{z=1} = 2$$

For $\beta < 0$ the radius of the circle $e^{2\beta} < 1$ and only the pole $z=0$ is inside. As a result:

$\beta > 0$: $\quad I = \frac{2\pi i}{2}(-1+2) = i\pi$

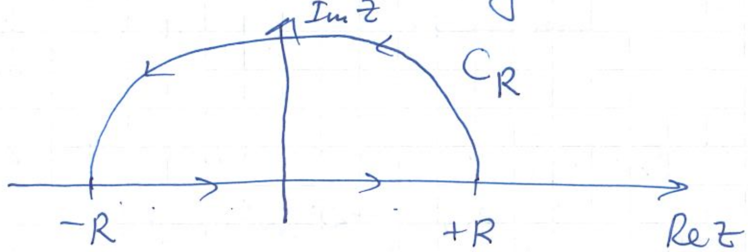
$\beta < 0$: $\quad I = \frac{2\pi i}{2}(-1) = -i\pi$

$$\boxed{\int_0^{\pi} \cot(\varphi - a)\, d\varphi = i\pi\, \operatorname{sign}(\operatorname{Im} a)}$$

$$\boxed{\text{Type-II Integrals:} \quad I = \int_{-\infty}^{+\infty} R(x)\,dx}$$

We can not apply Residue theorem to this integral directly. However, consider auxiliary contour:

$$\oint_{\Gamma_R} R(z)\,dz$$



$$\oint_{\Gamma_R} = \int_{-R}^{+R} dx + \int_{C_R} dz$$

If $\lim_{R\to\infty} \int_{C_R} R(z)\,dz \to 0$ then

$$\boxed{\int_{-\infty}^{+\infty} R(x)\,dx = 2\pi i \operatorname*{Res}_{\mathrm{Im}(z_n)>0} \left[R(z)\right]_{z=z_n}}$$

This requires that $R(z)$ at $z\to\infty$ decays at least as $|R(z)| \le C|z|^{-2}$. Indeed, then

$$\left|\int_{C_R} R(z)\,dz\right| \le \frac{C}{R^2}\cdot\pi\cdot R \to 0 \text{ as } R\to\infty$$

In complete analogy one may choose to close contour in the lower half-plane, then:

$$\boxed{\int_{-\infty}^{+\infty} R(x)\,dx = -2\pi i \operatorname*{Res}_{\mathrm{Im}(z_n)<0} \left[R(z)\right]_{z=z_n}}$$

$\boxed{\text{Exp. 1}}$

$$I = \int_{-\infty}^{+\infty} \frac{dx}{(x^2+a^2)^3} \qquad R(z) = \frac{1}{(z^2+a^2)^3}$$

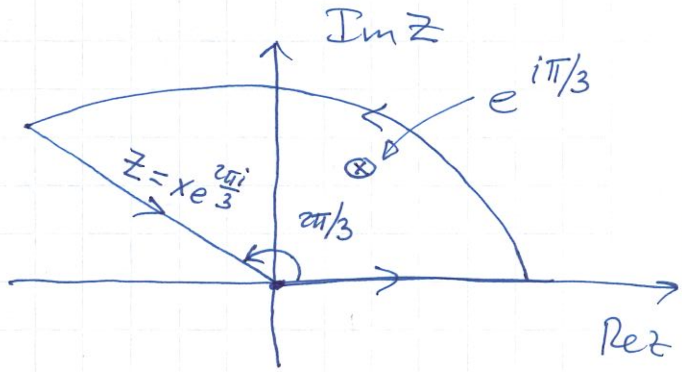
$$\oint_C \frac{dz}{(z^2+a^2)^3} = \lim_{R\to\infty} \int_{-R}^{+R} \frac{dx}{(x^2+a^2)^3} + \lim_{R\to\infty} \int_{C_R} \frac{dz}{(z^2+a^2)^3} = 2\pi i \operatorname*{Res}_{z=ia} R(z)$$

$$\operatorname*{Res}_{z=ia} \left[\frac{1}{(z^2+a^2)^3}\right] = \frac{1}{2!} \frac{d^2}{dz^2}\left[\frac{1}{(z+ia)^3}\right]_{z=ia} = \frac{1}{2}\,\frac{3\cdot 4}{(2ia)^5} = \frac{3}{16a^5 i}$$

$$\boxed{\int_{-\infty}^{+\infty} \frac{dx}{(x^2+a^2)^3} = \frac{3\pi}{8a^5}}$$

$\boxed{\text{Exp. 2}}$

$$I = \int_0^\infty \frac{dx}{x^3+1}$$



Solution Method - 1 : $\quad R(z) = \dfrac{1}{z^3+1}$

$$\oint_C \frac{dz}{z^3+1} = \int_0^\infty \frac{dx}{x^3+1} + \int_{C_R} + \int_\infty^0 \frac{e^{\frac{2\pi i}{3}}dx}{\left(e^{\frac{2\pi i}{3}}\cdot x\right)^3+1} = 2\pi i \operatorname*{Res}_{z=e^{i\pi/3}} R(z)$$

$$\left(1-e^{\frac{2\pi i}{3}}\right)\int_0^\infty \frac{dx}{x^3+1} = 2\pi i \operatorname*{Res}_{z=e^{i\pi/3}} [R(z)]$$

$$\int_0^\infty \frac{dx}{x^3+1} = \frac{2\pi i}{1-e^{2\pi i/3}} \operatorname*{Res}_{z=e^{i\pi/3}} \left[\frac{1}{(z^3+1)}\right]$$

$$z^3+1 = (z-z_1)(z-z_2)(z-z_3) \qquad z_1 = e^{i\pi/3} \qquad z_2 = e^{i\pi} \qquad z_3 = e^{-i\pi/3}$$

$$I = \frac{2\pi i}{1-e^{2\pi i/3}} \; \frac{1}{e^{i\pi/3}-e^{i\pi}} \cdot \frac{1}{e^{i\pi/3}-e^{-i\pi/3}} = $$

$$\cancel{= \frac{2\pi i}{e^{i\pi}\left(e^{+i\pi}-e^{i\pi/3}\right)} \; \frac{1}{e^{i\pi/3}-e}} \qquad e^{i\pi/3}-e^{-i\pi/3} = 2i\sin(\pi/3)$$

$$\left(1-e^{2\pi i/3}\right)\left(e^{i\pi/3}-e^{i\pi}\right) = \left(1-e^{i\pi}\cdot e^{-i\pi/3}\right)\left(e^{i\pi/3}-e^{i\pi}\right)$$

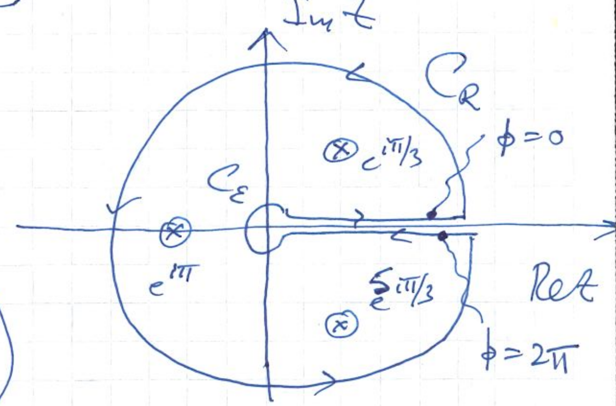
$$= \left(1+e^{-i\pi/3}\right)\left(e^{i\pi/3}+1\right) = 1+1+e^{i\pi/3}+e^{-i\pi/3} = 2\left[1+\cos(\pi/3)\right]$$

$$= \frac{2\pi i}{2\left[1+\cos(\pi/3)\right]} \; \frac{1}{2i\sin(\pi/3)} = \frac{\pi}{2\sin\pi/3\left[1+\cos(\pi/3)\right]} =$$

$$\cos\pi/3 = 1/2\,; \quad \sin\pi/3 = \frac{\sqrt{3}}{2}\,;$$

$$= \frac{\pi}{2}\;\frac{1}{\frac{\sqrt{3}}{2}\cdot\left(1+\frac{1}{2}\right)} = \frac{2\pi}{3\sqrt{3}}$$

$$\boxed{\int_0^\infty \frac{dx}{x^3+1} = \frac{2\pi}{3\sqrt{3}}}$$



Solution Method - 2: $\quad R(z) = \dfrac{\ln z}{z^3+1}$

$$\oint_c = \int_0^\infty + \int_{C_R} + \int_\infty^0 + \int_{C_\varepsilon} = 2\pi i \operatorname*{Res}_{e^{\pm i\pi/3},\, e^{i\pi}}\left(\frac{\ln z}{z^3+1}\right)$$

$$\lim_{R\to\infty}\int_{C_R}\frac{\ln z\, dz}{z^3+1} \le \lim_{R\to\infty}\frac{R\ln R}{R^3} \to 0$$

$$\lim_{\rho\to 0}\int_{C_\varepsilon}\frac{\rho\ln\rho}{\rho^3+1} \to 0$$

For $\int_0^\infty$ part $\ln z \to \ln x$

For $\int_\infty^0$ part $\ln z = \ln(xe^{2\pi i}) = \ln x + 2\pi i$

$$\int_0^\infty \frac{\ln x\, dx}{x^3+1} + \int_\infty^0 \frac{(\ln x + 2\pi i)\, dx}{x^3+1} = 2\pi i \operatorname*{Res}_{e^{\pm i\pi/3}, e^{i\pi}} \frac{\ln z}{(z^3+1)}$$

$$\boxed{\int_0^\infty \frac{dx}{x^3+1} = -\operatorname*{Res}_{e^{\pm i\pi/3}, e^{i\pi}} \left(\frac{\ln z}{z^3+1}\right)}$$

$$\frac{\ln z}{z^3+1} = \frac{\ln z}{(z-z_1)(z-z_2)(z-z_3)} \qquad \begin{cases} z_1 = e^{i\pi/3} \\ z_2 = e^{i\pi} \\ z_3 = e^{5i\pi/3} \; ! \end{cases}$$

$$\operatorname*{Res}_{z_{1,2,3}} \left[\frac{\ln z}{(z-z_1)(z-z_2)(z-z_3)}\right] = \frac{\ln(e^{i\pi/3})}{(e^{i\pi/3}-e^{i\pi})(e^{i\pi/3}-e^{5i\pi/3})}$$

$$+ \frac{\ln(e^{i\pi})}{(e^{i\pi}-e^{i\pi/3})(e^{i\pi}-e^{5i\pi/3})} + \frac{\ln(e^{5i\pi/3})}{(e^{5i\pi/3}-e^{i\pi/3})(e^{5i\pi/3}-e^{i\pi})}$$

$$= \frac{i\pi/3}{\underbrace{(e^{i\pi/3}-e^{i\pi})(e^{i\pi/3}-e^{5i\pi/3})}_{-4e^{-i\pi/3}\sin(\pi/3)\sin(2\pi/3)}} + \frac{i\pi}{\underbrace{(e^{i\pi}-e^{i\pi/3})(e^{i\pi}-e^{5i\pi/3})}_{4\sin^2(\pi/3)}} + \frac{5i\pi/3}{\underbrace{(e^{5i\pi/3}-e^{i\pi/3})(e^{5i\pi/3}-e^{i\pi})}_{-4e^{i\pi/3}\sin(\pi/3)\sin(2\pi/3)}}$$

$$= -\frac{i\pi}{12}\frac{e^{i\pi/3}}{\sin(\pi/3)\sin(2\pi/3)} + \frac{i\pi}{4\sin^2(\pi/3)} - \frac{5i\pi}{12}\frac{e^{-i\pi/3}}{\sin(\pi/3)\sin(2\pi/3)} =$$

$$= -\frac{i\pi}{12}\frac{e^{i\pi/3}+e^{-i\pi/3}}{\sin(\pi/3)\sin(2\pi/3)} + \frac{i\pi}{4\sin^2(\pi/3)} - \frac{i\pi}{3}\frac{e^{-i\pi/3}}{\sin(\pi/3)\sin(2\pi/3)} =$$

$$= -\frac{i\pi}{6}\frac{\cancel{\cos(\pi/3)}}{\sin(\pi/3)\,2\cdot\sin(\pi/3)\cancel{\cos(\pi/3)}} + \frac{i\pi}{4\sin^2\pi/3} - \frac{i\pi}{3}\frac{e^{-i\pi/3}}{\sin\pi/3\,\sin 2\pi/3}$$

$$= \frac{i\pi}{6\sin^2 \pi/3} - \frac{i\pi}{3}\,\frac{e^{-i\pi/3}}{\sin \pi/3 \,\sin 2\pi/3} = \frac{i\pi}{6\sin^2 \pi/3}\left[1 - \frac{e^{-i\pi/3}}{\cos \pi/3}\right]$$

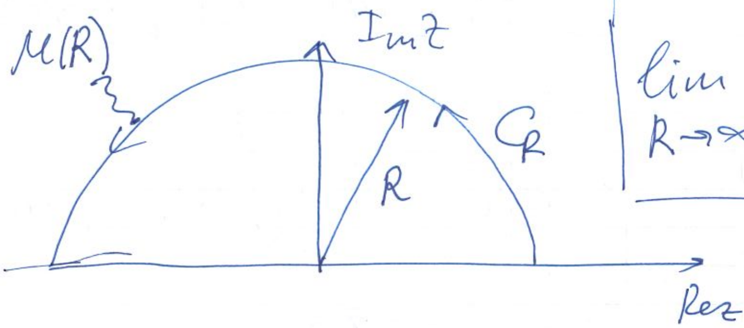
$$= \frac{i\pi}{6\sin^2(\pi/3)}\left[\frac{\cancel{\cos \pi/3} - \cancel{\cos \pi/3} + i\sin \pi/3}{\cos \pi/3}\right] = -\frac{\pi}{6\sin^2 \pi/3}\,\frac{\sin \pi/3}{\cos \pi/3}$$

$$\boxed{\operatorname*{Res}_{z_{1,2,3}}\left(\frac{\ln z}{z^3+1}\right) = -\frac{\pi}{6\sin(\pi/3)\cos(\pi/3)} = -\frac{2\pi}{3\sqrt{3}}}$$

$$\boxed{\text{Type-III Integrals}: \; I = \int_{-\infty}^{+\infty} e^{iax} R(x)\,dx}$$

Assume $a>0$ and that for $z \in C_R$

$\operatorname{Max}|R(z)| = M(R) \to 0$ as $R \to \infty$ then:



$$\boxed{\lim_{R\to\infty} \int_{C_R} R(z) e^{iaz}\,dz = 0}$$

In order to estimate the contribution from the integral over C_R, notice that:

$$\left|e^{iaz}\right| = \left|e^{ia(R\cos\varphi + iR\sin\varphi)}\right| = e^{-aR\sin\varphi}$$

since $a>0$ then $\left|e^{iaz}\right| \le 1$ for $z \in C_R$ and $0 \le \varphi \le \pi$. Let us use an inequality

$$\sin\varphi \geq \frac{2}{\pi}\varphi \quad \text{for} \quad 0 \le \varphi \le \pi/2$$

$$|I_{C_R}| \le \max_{z\in C_R}[R(z)] \int_0^{\pi} e^{-aR\sin\varphi} R\,d\varphi = 2RM(R)\int_0^{\pi/2} e^{-aR\sin\varphi} d\varphi$$

$$|I_{C_R}| \le 2RM(R)\int_0^{\pi/2} e^{-\frac{2aR}{\pi}\varphi} d\varphi = M(R)\,\frac{\pi}{a}\left(1-e^{-aR}\right)$$

$$|I_{C_R}| \le \frac{\pi}{a} M(R)$$

Assuming also that function $R(z)$ has no poles on the real axis we find:

$$\boxed{\int_{-\infty}^{+\infty} e^{iax} R(x)\,dx = 2\pi i \sum_{\mathrm{Im} z_k>0} \operatorname{Res}_{z_k}\left[e^{iaz} R(z)\right]}$$

Note1: If $\underline{a<0}$ then one has to close contour in the lower-half-plane, and so

$$\int_{-\infty}^{+\infty} e^{iax} R(x)\,dx = -2\pi i \sum_{\mathrm{Im} z_k<0} \operatorname{Res}_{z_k}\left[e^{iaz} R(z)\right]$$

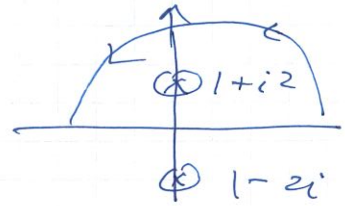
Note 2: If the function $R(z)$ is real for $z=x$ and $a>0$, then separating real and imaginary parts from above expressions:

$$\boxed{\begin{aligned} \int_{-\infty}^{+\infty} R(x)\cos(ax)\,dx &= -2\pi\,\mathrm{Im}\left\{\sum_{\mathrm{Im} z_k>0} \operatorname{Res}_{z_k}\left[e^{iaz} R(z)\right]\right\} \\ \int_{-\infty}^{+\infty} R(x)\sin(ax)\,dx &= 2\pi\,\mathrm{Re}\left\{\sum_{\mathrm{Im} z_k>0} \operatorname{Res}_{z_k}\left[e^{iaz} R(z)\right]\right\} \end{aligned}}$$

$\boxed{\text{Exp. 1}}$ $\quad I = \int_{-\infty}^{+\infty} \frac{(x-1)\cos(5x)}{x^2-2x+5}\,dx$

$$I = \oint_C \frac{(z-1)e^{i5z}\,dz}{z^2-2z+5} \qquad z_{1,2} = \frac{1}{2}\left(2 \pm i4\right) = 1 \pm i2$$

$$\operatorname{Im} z_1 > 0$$

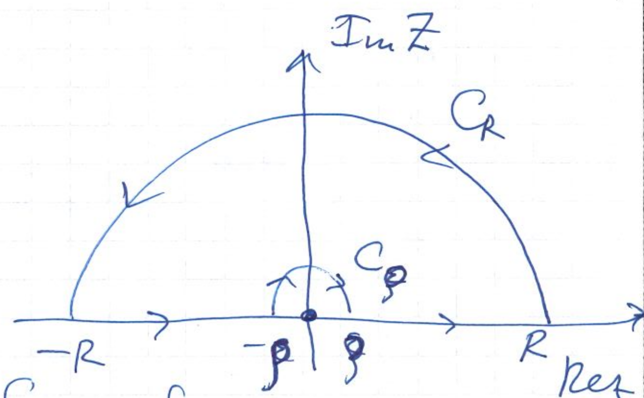


$$\operatorname*{Res}_{z=1+2i}\left[\frac{e^{5iz}(z-1)}{z^2-2z+5}\right] = \operatorname*{Res}_{z=z_1}\left[\frac{e^{5iz}(z-1)}{(z-z_1)(z-z_2)}\right] =$$

$$= \frac{(z_1-1)e^{5iz_1}}{z_1-z_2} = \frac{2i}{4i}e^{5i(1+2i)} = \frac{e^{-10}}{2}\left(\cos 5 + i\sin 5\right)$$

$$\int_{-\infty}^{+\infty} \frac{(x-1)\cos(5x)\,dx}{x^2-2x+5} = -2\pi \operatorname{Im}\{\dots\} = \boxed{-\pi e^{-10}\sin 5}$$

$\boxed{\text{Exp. 2}}$ $\quad I = \int_0^{\infty} \frac{\sin x}{x}\,dx$



$$I = \oint_C \frac{e^{iz}}{z}\,dz =$$

$$= \lim_{\substack{R\to\infty \\ \rho\to 0}} \int_{-R}^{-\rho} \frac{e^{ix}dx}{x} + \int_{\rho}^{R} \frac{e^{ix}dx}{x} + \int_{C_R} + \int_{C_\rho} = 0$$

$$\lim_{R\to\infty}\int_{C_R} \to 0 \qquad \int_{C_\rho} \frac{e^{iz}dz}{z} = \lim_{\rho\to 0}\int_{\pi}^{0} \frac{e^{i\rho e^{i\varphi}}\, i\rho e^{i\varphi}d\varphi}{\rho e^{i\varphi}} = -i\pi$$

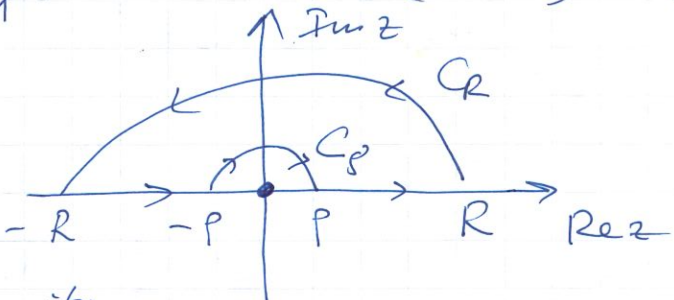
$$\lim_{\substack{R\to\infty \\ \rho\to 0}} \int_{\rho}^{R} \frac{e^{ix}-e^{-ix}}{x}\,dx = 2i \lim_{\substack{R\to\infty \\ \rho\to 0}} \int_{\rho}^{R} \frac{\sin x}{x}\,dx$$

$$2i \lim_{\substack{R\to\infty \\ \rho\to 0}} \int_\rho^R \frac{\sin x}{x}\,dx - i\pi = 0$$

$$\boxed{\int_0^\infty \frac{\sin x\,dx}{x} = \pi/2}$$

$\boxed{\text{Exp. 3}}$ $$I = \int_0^\infty \frac{\cos(ax) - \cos(bx)}{x^2}\,dx \quad (a \geq 0,\ b \geq 0)$$

$$I = \oint_c \frac{e^{iaz} - e^{ibz}}{z^2}\,dz$$



$$\int_{-R}^{-\rho} \frac{e^{iax} - e^{ibx}}{x^2}\,dx + \int_\rho^R \frac{e^{iax} - e^{ibx}}{x^2}\,dx + \int_{C_R} + \int_{C_\rho} = 0$$

$$\lim_{\rho\to 0} \int_{C_\rho} \frac{e^{iaz} - e^{ibz}}{z^2}\,dz = \lim_{\rho\to 0} \int_\pi^0 \frac{e^{ia\rho e^{i\varphi}} - e^{ib\rho e^{i\varphi}}}{\rho^2 e^{2i\varphi}}\, ie^{i\varphi}\rho\,d\varphi =$$

$$= \lim_{\rho\to 0} \int_\pi^0 \frac{(1 + ia\rho e^{i\varphi} + \dots) - (1 + ib\rho e^{i\varphi} + \dots)}{\rho^2 e^{2i\varphi}}\, ie^{i\varphi}\rho\,d\varphi = \pi(a-b)$$

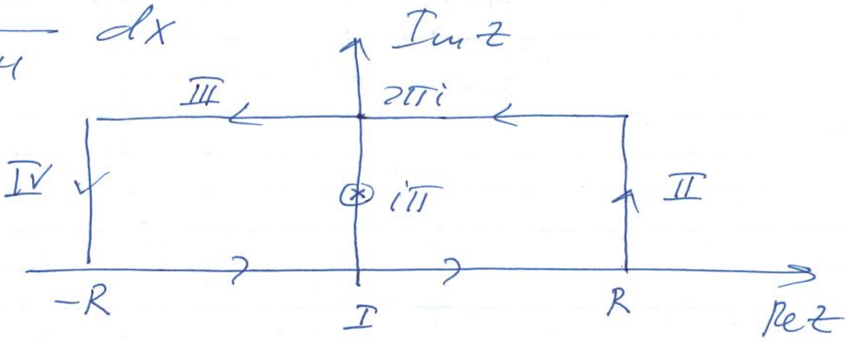
$$\lim_{\substack{R\to\infty \\ \rho\to 0}} \int_\rho^R \left[\frac{e^{iax} + e^{-iax}}{x^2} - \frac{e^{ibx} + e^{-ibx}}{x^2}\right] dx = 2 \lim_{\substack{R\to\infty \\ \rho\to 0}} \int_\rho^R \frac{\cos ax - \cos bx}{x^2}\,dx$$

$$2 \lim_{\substack{R\to\infty \\ \rho\to 0}} \int_\rho^R \frac{\cos ax - \cos bx}{x^2}\,dx + \pi(a-b) = 0$$

$$\boxed{\int_0^\infty \frac{\cos ax - \cos bx}{x^2}\,dx = \frac{\pi}{2}(b-a)}$$

$\boxed{\text{Exp. 4}}$ $\quad I = \displaystyle\int_{-\infty}^{+\infty} \frac{e^{ax}}{e^x+1}\,dx$

$$\oint_C \frac{e^{az}\,dz}{e^z+1}$$



$$\oint_C = \int_{I} + \int_{II} + \int_{III} + \int_{IV} = 2\pi i \operatorname*{Res}_{z=i\pi}\left[\frac{e^{az}}{e^z+1}\right] = 2\pi i\,\frac{e^{ia\pi}}{e^{i\pi}}$$

$$\int_{I} = \int_{-R}^{+R} \frac{e^{ax}\,dx}{e^x+1}\ ; \quad \int_{III} = \int_{R}^{-R} \frac{e^{a(x+2\pi i)}\,dx}{e^{x+2\pi i}+1} = -e^{2\pi a i}\int_{-R}^{+R}\frac{e^{ax}\,dx}{e^x+1}$$

For the segments II and IV:

$$|f(z)|_{(II)} = \left|\frac{e^{a(R+iy)}}{e^{R+iy}+1}\right| \le \frac{e^{aR}}{e^R-1} = \frac{e^{(a-1)R}}{1-e^{-R}}$$

$$|f(z)|_{(IV)} = \left|\frac{e^{a(-R+iy)}}{e^{-R+iy}+1}\right| \le \frac{e^{-aR}}{1-e^{-R}}$$

$$\left.\begin{aligned} &\lim_{R\to\infty} \frac{e^{(a-1)R}}{1-e^{-R}} \to 0 \quad \text{if} \quad a<1 \\ &\lim_{R\to\infty} \frac{e^{-aR}}{1-e^{-R}} \to 0 \quad \text{if} \quad a>0 \end{aligned}\right\} \quad \text{We must require } \boxed{0<a<1}\,!$$

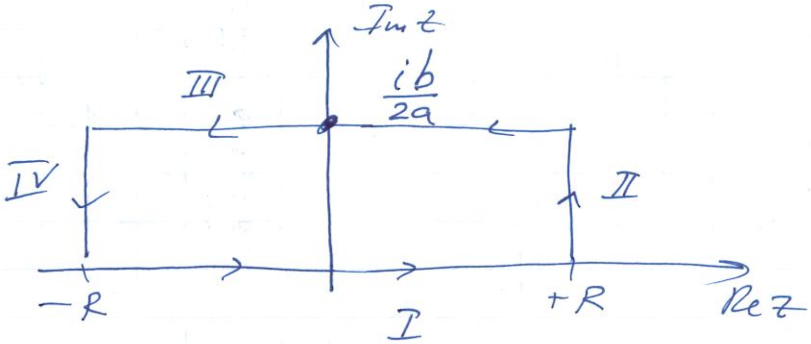
$$\left(1-e^{2\pi i a}\right)\int_{-R}^{+R}\frac{e^{ax}\,dx}{e^x+1} = -2\pi i\,e^{ia\pi}$$

$$\int_{-\infty}^{+\infty}\frac{e^{ax}\,dx}{e^x+1} = \frac{2\pi i}{e^{i\pi a}-e^{-i\pi a}} = \boxed{\frac{\pi}{\sin(\pi a)}} \quad \text{for} \quad 0<a<1$$

Exp. 5

$$I = \int_0^{\infty} e^{-ax^2} \cos(bx)\, dx$$

$$\oint_C e^{-az^2} dz$$



$$\oint_C = \int_I + \int_{II} + \int_{III} + \int_{IV} = 0 \quad \text{(No poles inside)}$$

$$\int_I = \int_{-R}^{R} e^{-ax^2} dx = \frac{2}{\sqrt{a}} \int_0^{R\sqrt{a}} e^{-t^2} dt$$

$$\int_{III} = - e^{+\frac{b^2}{4a}} \int_{-R}^{+R} e^{-ax^2} e^{-ibx} dx$$

At the segments II and IV:

$$\left| e^{-az^2} \right| = e^{-a(R^2 - y^2)} \leq e^{b^2/4a}\, e^{-aR^2} \xrightarrow{R\to\infty} 0 \quad \underline{\text{if } a>0}$$

In the limit $R \to \infty$:

$$\frac{2}{\sqrt{a}} \int_0^{\infty} e^{-t^2} dt - e^{+\frac{b^2}{4a}} \int_{-\infty}^{+\infty} e^{-ax^2} \underbrace{e^{-ibx}}_{[\cos(bx) - i\sin(bx)]} dx = 0$$

$\sin(bx) \rightarrow$ contribution drops out (odd-function)

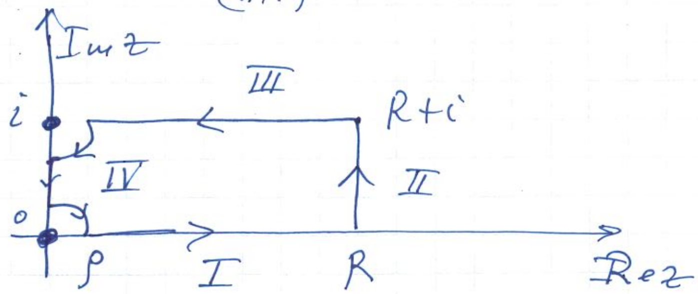
$$\sqrt{\frac{\pi}{a}} - e^{+\frac{b^2}{4a}} \int_{-\infty}^{+\infty} e^{-ax^2} \cos(bx)\, dx = 0$$

$$\boxed{I = \frac{1}{2} \sqrt{\frac{\pi}{a}}\, e^{-b^2/4a}} \qquad a>0$$

$\boxed{\text{Exp. 6}}$

$$I = \int_0^\infty e^{-\pi x}\,\frac{\sin(ax)}{\sinh(\pi x)}\,dx$$

$$\oint_C \underbrace{\frac{e^{iaz}}{e^{2\pi z}-1}}_{f(z)}\,dz$$



$$\Rightarrow f(x+i) = \frac{e^{ia(x+i)}}{e^{2\pi(x+i)}-1} = \frac{e^{-a}\cdot e^{iax}}{e^{2\pi x}-1} = e^{-a} f(x)$$

$$\int_I + \int_{III} = \int_\rho^R (1-e^{-a})\, f(x)\,dx$$

$$|f(R+iy)| = \left|\frac{e^{ia(R+iy)}}{e^{2\pi(R+iy)}-1}\right| = \frac{e^{-ay}}{(e^{2\pi R}e^{2\pi i y}-1)(e^{2\pi R}e^{-2\pi i y}-1)} \xrightarrow{R\to\infty} 0$$

$$\int_{IV} = \int_{1-\rho}^{\rho} \frac{e^{-ay}\, i\,dy}{e^{2\pi i y}-1} = -i\int_\rho^{1-\rho} \frac{e^{-ay}\,dy}{e^{2\pi i y}-1}$$

$$\int_C = (1-e^{-a})\int_\rho^R \frac{e^{iax}\,dx}{e^{2\pi x}-1} - i\int_\rho^{1-\rho}\frac{e^{-ay}\,dy}{e^{2\pi i y}-1} + \int_{C_{\rho_1}} + \int_{C_{\rho_2}} = 0$$

$\Rightarrow$ Integral near $z = i$ $\quad \int_{C_{\rho_2}}$: $\left(z - i = \rho e^{i\varphi}\right)$

$$f(z) = \frac{e^{iaz}}{e^{2\pi(z-i)}-1} = \frac{e^{ia[(z-i)+i]}}{e^{2\pi(z-i)}-1} = \frac{e^{-a} + C_1(z-i) + \cdots}{2\pi(z-i) + C_2'(z-i)^2 + \cdots}$$

$$f(z)\Big|_{\text{near } z=i} = \frac{e^{-a}}{2\pi}\,\frac{1}{z-i} + \mathcal{O}(z-i)$$

$$\int_{C_{\rho_2}} \frac{e^{iaz}\,dz}{e^{2\pi z}-1} = \frac{e^{-a}}{2\pi}\int_0^{-\pi/2} \frac{i\rho e^{i\varphi}\,d\varphi}{\rho e^{i\varphi}} = -\frac{ie^{-a}}{4} + \mathcal{O}(\rho)$$

$\Longrightarrow$ Integral near $z=0$ $\int_{C_{\rho_1}}$

$$\int_{C_{\rho_1}} = \int_{\pi/2}^{0} \frac{i\rho e^{i\varphi}\,d\varphi}{2\pi\rho e^{i\varphi}} = -\frac{i}{4} + \mathcal{O}(\rho)$$

Combining now all terms together we have:

$$(1-e^{-a})\int_\rho^R \frac{e^{iax}\,dx}{e^{2\pi x}-1} = i\int_\rho^{1-\rho} \frac{e^{-ay}\,dy}{e^{2\pi i y}-1} + \frac{i}{4}(1+e^{-a})$$

Let us take an imaginary part of this formula:

$$\operatorname{Im}\int_\rho^R \frac{e^{iax}\,dx}{(e^{2\pi x}-1)} = \int_\rho^R \frac{\sin(ax)\,dx}{e^{2\pi x}-1} = \frac{1}{2}\int_\rho^R e^{-\pi x}\frac{\sin(ax)}{\sinh(\pi x)}\,dx$$

$$\operatorname{Im} i\int_\rho^{1-\rho} \frac{e^{-ay}\,dy}{e^{2\pi i y}-1} = \operatorname{Im} i\int_\rho^{1-\rho} \frac{e^{-ay}(e^{-2\pi i y}-1)}{(e^{2\pi i y}-1)(e^{-2\pi i y}-1)}\,dy =$$

$$= \operatorname{Im} i\int_\rho^{1-\rho} \frac{e^{-ay}[\cos(2\pi y)-1-i\sin(2\pi y)]}{\underbrace{1-e^{2\pi i y}-e^{-2\pi i y}+1}} =$$

$$= \operatorname{Im} i\int_\rho^{1-\rho} \frac{e^{-ay}[\cos(2\pi y)-1-i\sin(2\pi y)]}{2[1-\cos(2\pi y)]} = -\int_\rho^{1-\rho} \frac{e^{-ay}}{2}\,dy$$

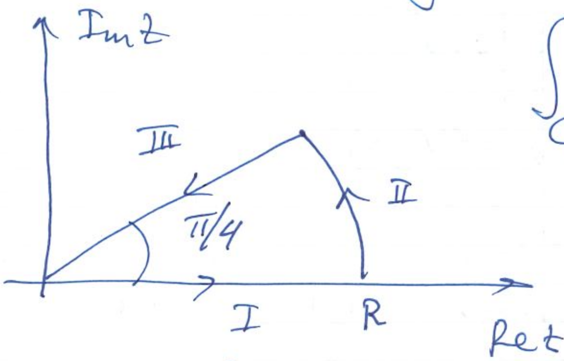
In the limit $R\to\infty$, $\rho\to 0$:

$$(1-e^{-a})\,\frac{1}{2}\int_0^\infty e^{-\pi x}\frac{\sin(ax)}{\sinh(\pi x)}\,dx = -\int_0^1 \frac{e^{-ay}\,dy}{2} + \frac{1}{4}(1+e^{-a})$$

$$\boxed{\int_0^\infty e^{-\pi x}\frac{\sin(ax)\,dx}{\sinh(\pi x)} = \frac{1}{2}\,\frac{1+e^{-a}}{1-e^{-a}} - \frac{1}{a}}$$

$\boxed{\text{Exp. 7}}$ $\quad I_1 = \int_0^\infty \cos(x^2)\,dx \qquad I_2 = \int_0^\infty \sin(x^2)\,dx$

Consider auxiliary function $f(z) = e^{iz^2}$



$$\int_{C_R} e^{iz^2}\,dz = 0$$

$$\int_{C_R} = \int_{I} + \int_{II} + \int_{III}$$

$$\Rightarrow \int_I = \int_0^R e^{ix^2}\,dx \,; \quad \Rightarrow \int_{III} = \int_R^0 e^{i\left(Re^{i\pi/4}\right)^2} e^{i\pi/4}\,dR =$$

$$= -\sqrt{i}\int_0^R e^{iR^2 e^{i\pi/2}}\,dR = -\sqrt{i}\int_0^R e^{-x^2}\,dx \,;$$

$$\Rightarrow \int_{II} = \int_{C_R} e^{iz^2}\,dz = \left\{ z^2 = \xi \right\} \rightarrow \frac{1}{2}\int_{C'_{R^2}} \frac{e^{i\xi}\,d\xi}{\sqrt{\xi}}$$

In the limit $R \to \infty$ this integral is zero according to the Jordan's lemma.

$$\int_0^\infty e^{ix^2}\,dx - \sqrt{i}\underbrace{\int_0^\infty e^{-x^2}\,dx}_{\frac{\sqrt{\pi}}{2}} = 0$$

$$\int_0^\infty e^{ix^2}\,dx = \sqrt{i}\,\frac{\sqrt{\pi}}{2}$$

$$\boxed{\int_0^\infty \cos(x^2)\,dx = \int_0^\infty \sin(x^2)\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}}$$

Type-IV Integrals : $I = \int_0^\infty x^{\alpha-1} R(x)\,dx$

⇒ It is assumed here that α is a <u>non-integer</u> number and $R(x)$ is rational (well-behaved) function at infinity. This integral defines so-called <u>Mellin</u> <u>transform</u>

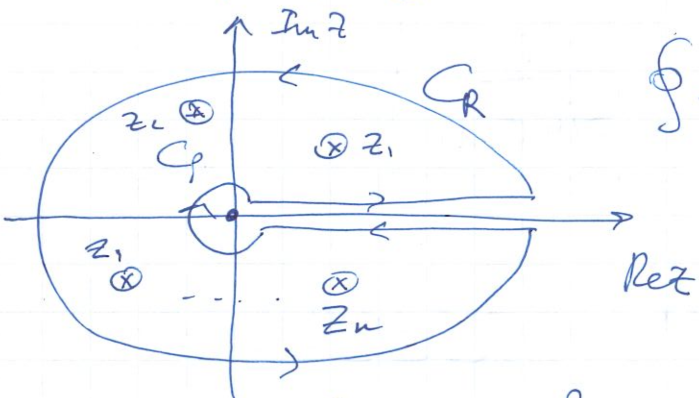
⇒ Essential assumptions about convergence

$$\lim_{z\to 0} |z|^{\alpha} R(z) \to 0 \quad \text{and} \quad \lim_{z\to\infty} |z|^{\alpha} R(z) \to 0$$

⇒ Define $h(z) = z^{\alpha-1}$; $z = re^{i\varphi} \Rightarrow h(z) = r^{\alpha-1} e^{i(\alpha-1)\varphi}$

$$\begin{cases} h(x+i0) = h(x) = x^{\alpha-1} > 0 \\ h(x-i0) = h(\tilde{x}) = x^{\alpha-1} e^{2\pi i(\alpha-1)} = e^{2\pi i\alpha} h(x) \end{cases}$$

For the complex function : $f(z) = h(z)\cdot R(z)$ we have a property : $f(\tilde{x}) = e^{2\pi i\alpha} f(x)$.



$$\oint f(z) = 2\pi i \sum_{z=z_k} \mathrm{Res}\left[z^{\alpha-1} R(z)\right]$$

$$\oint f(z)dz = \int_\rho^R f(x)dx + \int_R^\rho f(\tilde{x})dx + \int_{C_\rho} f(z)dz + \int_{C_R} f(z)dz$$

$$\lim_{\rho\to 0} \int_{C_\rho} \le \lim_{\rho\to 0} \left|\int_{C_\rho} f(z)dz\right| \le \lim_{\rho\to 0} \left\{2\pi\rho\cdot\rho^{\alpha-1}\cdot \max|R(z)|_{z=\rho}\right\} \to 0$$

provided that $\alpha > 0$ and $R(\rho)$ is not singular at $z=0$.

$$\lim_{R\to\infty} \int_{C_R} \leq \lim_{R\to\infty} \left| \int_{C_R} f(z)dz \right| \leq \lim_{R\to\infty} 2\pi R \cdot R^{\alpha-1} \max(R(z))_{C_R} \to 0$$

provided that $\alpha < 1$ if $R(z)$ decays at least as $1/z$ at infinity.

$$I - e^{2\pi i\alpha} I = 2\pi i \sum_{z=z_k} \mathrm{Res}\left[z^{\alpha-1} R(z)\right]$$

$$\boxed{I = \frac{2\pi i}{1-e^{2\pi i\alpha}} \sum_{z=z_k} \mathrm{Res}\left[z^{\alpha-1} R(z)\right]}$$

Exp. 1 $\quad I = \int_0^\infty \frac{x^{\alpha-1}}{x+1}\,dx \qquad 0<\alpha<1$

$$I = \frac{2\pi i}{1-e^{2\pi i\alpha}} \mathrm{Res}_{z=-1}\left[\frac{z^{\alpha-1}}{z+1}\right] = \frac{2\pi i}{1-e^{2\pi i\alpha}} \left(e^{i\pi}\right)^{\alpha-1} =$$

$$= -\frac{2\pi i e^{i\pi\alpha}}{e^{i\pi\alpha}\left(e^{-i\pi\alpha} - e^{i\pi\alpha}\right)} = \frac{\pi}{\sin(\pi\alpha)}$$

$$\boxed{\int_0^\infty \frac{x^{\alpha-1}dx}{x+1} = \frac{\pi}{\sin(\pi\alpha)}}$$

Exp. 2 $\quad I = \int_0^\infty \frac{x^{\alpha-1}}{(1+x^2)^2}\,dx \qquad 0<\alpha<4$

$$I = \frac{2\pi i}{1-e^{2\pi i\alpha}} \left\{ \mathrm{Res}_{z=i}\left[\frac{z^{\alpha-1}}{(z^2+1)^2}\right] + \mathrm{Res}_{z=-i}\left[\frac{z^{\alpha-1}}{(z^2+1)^2}\right] \right\}$$

$$\Rightarrow \mathrm{Res}\left[\frac{z^{\alpha-1}}{(z^2+1)^2}\right]_{z=i} = \frac{d}{dz}\left[\frac{z^{\alpha-1}}{(z+i)^2}\right]_{z=i} = \left[\frac{(\alpha-1)z^{\alpha-2}}{(z+i)^2} - \frac{2z^{\alpha-1}}{(z+i)^3}\right]_{z=i}$$

$$= \frac{(\alpha-1)\left(e^{i\pi/2}\right)^{\alpha-2}}{(2i)^2} - \frac{2\left(e^{i\pi/2}\right)^{\alpha-1}}{(2i)^3} = \frac{\alpha-1}{4} e^{\frac{i\pi\alpha}{2}} - \frac{e^{\frac{i\pi\alpha}{2}}}{4}$$

$$= \frac{\alpha-2}{4} e^{\frac{i\pi\alpha}{2}}$$

$$\Rightarrow \operatorname{Res}\left[\frac{z^{\alpha-1}}{(z^2+1)^2}\right]_{z=-i} = \frac{d}{dz}\left(\frac{z^{\alpha-1}}{(z-i)^2}\right)_{z=-i} =$$

$$= \left[\frac{(\alpha-1) z^{\alpha-2}}{(z-i)^2} - \frac{2 z^{\alpha-1}}{(z-i)^3}\right]_{z=-i} = \frac{(\alpha-1)\left(e^{3i\pi/2}\right)^{\alpha-2}}{(2i)^2} + \frac{2\left(e^{3i\pi/2}\right)^{\alpha-1}}{(2i)^3} =$$

$$= \frac{\alpha-1}{4} e^{\frac{3i\pi\alpha}{2}} - \frac{1}{4} e^{3i\pi/2} = \frac{\alpha-2}{4} e^{3i\pi\alpha/2}$$

Finally, combining all parts together:

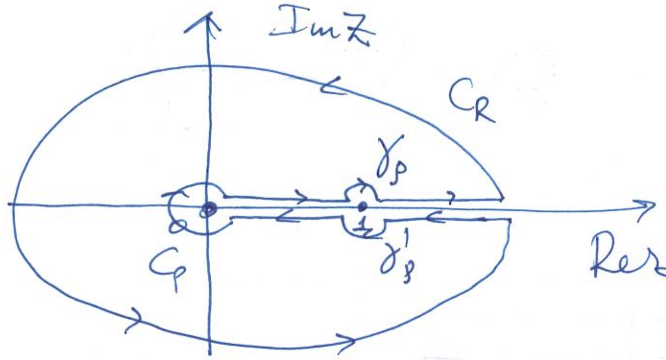
$$I = \frac{2\pi i}{1-e^{2\pi i\alpha}} \frac{\alpha-2}{4}\left(e^{\frac{i\pi\alpha}{2}} + e^{\frac{3i\pi\alpha}{2}}\right) =$$

$$= \frac{2\pi i}{\cancel{e^{i\pi\alpha}} \underbrace{\left(e^{-i\pi\alpha} - e^{i\pi\alpha}\right)}_{-2i\sin(\pi\alpha)}} \frac{\alpha-2}{4} \cancel{e^{i\pi\alpha}} \underbrace{\left(e^{-\frac{i\pi\alpha}{2}} + e^{\frac{i\pi\alpha}{2}}\right)}_{2\cos(\pi\alpha/2)} =$$

$$\boxed{I = \int_0^\infty \frac{x^{\alpha-1}\,dx}{(x^2+1)^2} = \frac{\pi(2-\alpha)}{4\sin(\pi\alpha/2)}}$$

$\boxed{\text{Exp. 3}}$ $$I = \int_0^\infty \frac{x^{\alpha-1}}{1-x}\,dx \qquad 0 < \alpha < 1$$

This integral has to be understood as a Principal Value since integrand has singularity at the real axis $x=1$.



$$\oint f(z)\,dz = 0 = \int_{C_\rho} + \int_{C_R} + \int_{\gamma_\rho} + \int_{\gamma'_\rho} + \left[+ \int_{\rho}^{1-\rho} + \int_{1+\rho}^{R} \right] \times \left(1 - e^{2\pi i \alpha}\right)$$

$$\lim_{\rho\to 0} \int_{\gamma_\rho} f(z)\,dz = -\int_{\gamma_\rho} \frac{z^{\alpha-1}\,dz}{z-1} = -\lim_{\rho\to 0} \int_{\pi}^{0} \frac{\left(1+\rho e^{i\varphi}\right)^{\alpha-1}}{\rho e^{i\varphi}}\, i\rho e^{i\varphi}\,d\varphi = i\pi$$

$$\lim_{\rho\to 0} \int_{\gamma'_\rho} f(z)\,dz = -\int_{2\pi}^{\pi} \frac{\left(e^{2\pi i}\right)^{\alpha-1} + O(\rho)}{\rho e^{i\varphi}}\, i\rho e^{i\varphi}\,d\varphi = i\pi e^{2\pi i\alpha}$$

$$\left(1 - e^{2\pi i \alpha}\right)\cdot I + i\pi\left(1 + e^{2\pi i\alpha}\right) = 0$$

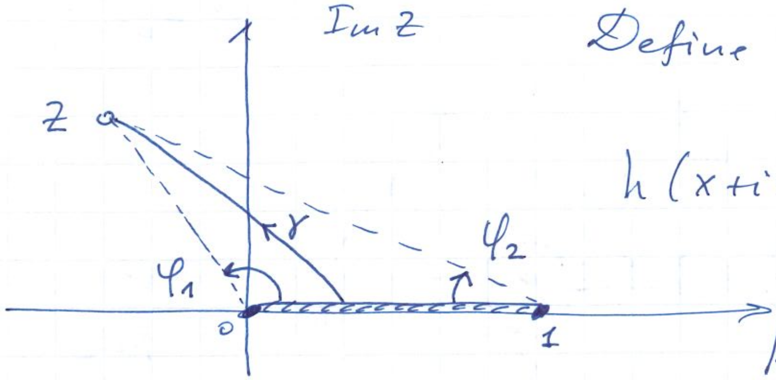
$$I = -i\pi \frac{1 + e^{2\pi i\alpha}}{1 - e^{2\pi i \alpha}} = -i\pi \frac{e^{-i\pi\alpha} + e^{i\pi\alpha}}{e^{-i\pi\alpha} - e^{i\pi\alpha}}$$

$$\boxed{\int_0^\infty \frac{x^{\alpha-1}\,dx}{1-x} = \pi \cot(\pi\alpha)}$$

$$\boxed{\text{Type V Integrals: } I = \int_0^1 \left(\frac{x}{1-x}\right)^\alpha R(x)\,dx}$$

This integral is a beta-function type.

We assume that $R(x)$ has no poles for $x \in [0,1]$ and also we require $-1 < \alpha < 1$ for convergence.



Define $h(z) = \left(\frac{z}{1-z}\right)^\alpha$

$$h(x+i0) = h(x) = \left(\frac{x}{1-x}\right)^\alpha > 0 \quad 0<x<1$$

That is to say that on the real axis (upper part of the branch cut) the function is real.

Let us find now $h(\tilde{x}) = h(x-i0)$, $0<x<1$, where $\tilde{x} = x - i0$ – is a point at the lower part of the branch cut:

$$h(z) = \left|\frac{z}{1-z}\right|^\alpha e^{i\alpha(\varphi_1 - \varphi_2)}$$

$$\varphi_1 = \Delta_\gamma \arg z \quad \text{and} \quad \varphi_2 = \Delta_\gamma \arg(1-z)$$

where γ is the curve which connects any point $x \in [0,1]$ of the upper branch and point z.

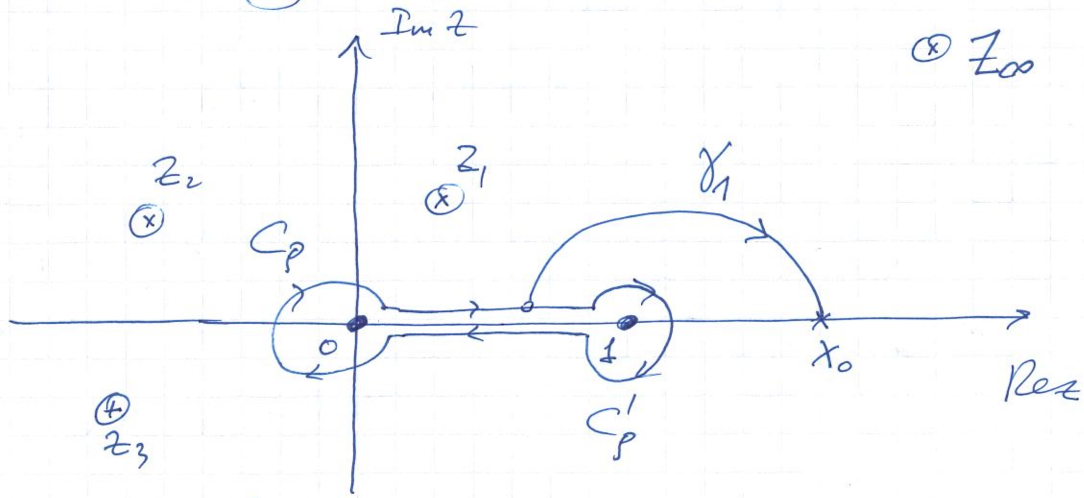
If $\tilde{x} = x - i0 = z$ then: $\boxed{\varphi_1 = 2\pi}$ $\boxed{\varphi_2 = 0}$

$$h(\tilde{x}) = e^{2\pi i\alpha}\left(\frac{x}{1-x}\right)^\alpha = e^{2\pi i \alpha} h(x)$$

Similarly:

$$f(z) = h(z)\,R(z) \rightarrow \boxed{f(\tilde{x}) = e^{2\pi i \alpha} f(x)}$$

In order to calculate the original integral consider following contour:



$$\Rightarrow \oint_C f(z)\,dz = 2\pi i \left(\sum_{n=1}^{N} \operatorname*{Res}_{z=z_n} f(z) + \operatorname*{Res}_{z=\infty} f(z) \right)$$

$$\Rightarrow \oint_C f(z)\,dz = \int_{C_\rho} f(z)\,dz + \int_\rho^{1-\rho} f(x)\,dx + \int_{1-\rho}^{\rho} f(\tilde{x})\,dx + \int_{C'_\rho} f(z)\,dz$$

$$\operatorname{Max}\left[\left(\frac{z}{1-z}\right)^\alpha R(z) \right] \leq M \cdot |z|^\alpha \rightarrow 0 \quad \text{as} \quad z = \rho e^{i\varphi} \rightarrow 0$$

$$\int_{C_\rho}, \int_{C'_\rho} \rightarrow 0 \quad \text{in the limit } \rho \rightarrow 0$$

$$\int_\rho^{1-\rho} f(x)\,dx + e^{2\pi i \alpha} \int_{1-\rho}^{\rho} f(x)\,dx = 2\pi i \left(\sum_{n=1}^{N} \operatorname*{Res}_{z=z_n} f(z) + \operatorname{Res} f(\infty) \right)$$

$$\boxed{I = \frac{2\pi i}{1 - e^{2\pi i \alpha}} \left(\sum_{n=1}^{N} \operatorname*{Res}_{z=z_n} f(z) + \operatorname{Res} f(\infty) \right)}$$

We need to look at the residue at infinity.

Assume Laurant expansion:

$$R(z) = c_0 + \frac{c_{-1}}{z} + \frac{c_{-2}}{z^2} + \dots, \quad |z| > R$$

$$h(z) = \left(\frac{z}{1-z}\right)^{\alpha} = h(\infty)\, g(z) \quad \begin{cases} g(z) = \left(1 - \frac{1}{z}\right)^{-\alpha} \\ h(\infty) = e^{i\alpha(\varphi_1 - \varphi_2)} \end{cases}$$

$\varphi_1 = \Delta_{\gamma_1} \arg z \quad \varphi_2 = \Delta_{\gamma_1} \arg(1-z)$

For the curve γ_1 on the figure: $\boxed{\varphi_1 = 0, \ \varphi_2 = -\pi}$

$$f(z) = h(z) \cdot R(z) = h(\infty)\, g(z)\, R(z) =$$

$$= e^{i\pi\alpha} \underbrace{\left(1 + \frac{\alpha}{z} + \dots\right)}_{g(z)} \underbrace{\left(c_0 + \frac{c_{-1}}{z} + \frac{c_{-2}}{z^2} + \dots\right)}_{R(z)} =$$

$$= e^{i\pi\alpha}\left(c_0 + \frac{\alpha c_0 + c_{-1}}{z} + \dots\right), \quad |z| > R$$

$$\boxed{\operatorname{Res} f(\infty) = -e^{i\pi\alpha}(\alpha c_0 + c_{-1})}$$

<u>Note</u>: The same approach without modifications can be applied to integrals of the form:

$$I = \int_a^b \left(\frac{x-a}{b-x}\right)^{\alpha} R(x)\, dx$$

$$\left(\frac{z-a}{b-z}\right)^{\alpha} = e^{i\pi\alpha}\left(1 - \frac{a}{z}\right)^{\alpha}\left(1 - \frac{b}{z}\right)^{-\alpha} = e^{i\pi\alpha}\left(1 + \frac{\alpha(b-a)}{z} + \dots\right)$$

$$\boxed{\operatorname{Res} f(\infty) = -e^{i\pi\alpha}\left[\alpha c_0 (b-a) + c_{-1}\right]}$$

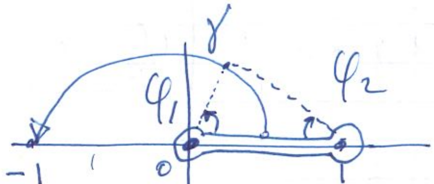
Exp. 1

$$I = \int_0^1 \left(\frac{x}{1-x}\right)^{\alpha} \frac{dx}{x+1} \qquad -1 < \alpha < 1$$

$$I = \frac{2\pi i}{1-e^{2\pi i \alpha}} \left[\operatorname*{Res}_{z=-1} f(z) + \operatorname*{Res}_{z=\infty} f(z) \right]$$

$$f(z) = h(z)\, R(z) = \left(\frac{z}{1-z}\right)^{\alpha} \cdot \frac{1}{z+1}$$

$$\Rightarrow \operatorname*{Res}_{z=-1} f(z) = h(-1) = \left.\left(\frac{z}{1-z}\right)^{\alpha}\right|_{z=-1} = e^{i\alpha(\varphi_1 - \varphi_2)}\, 2^{-\alpha}$$



$$\varphi_1 = \pi \qquad \varphi_2 = 0$$

$$\boxed{\operatorname*{Res}_{z=-1} f(z) = 2^{-\alpha} e^{i\pi\alpha}}$$

$$\Rightarrow \operatorname*{Res}_{z=\infty} f(z) = \operatorname{Res}\left[h(\infty) \left(1 - \frac{1}{z}\right)^{-\alpha} \frac{1}{z} \left(\frac{1}{1+1/z}\right) \right] = -h(\infty) =$$

$$= -e^{i\pi\alpha}$$

$$I = \frac{2\pi i}{1-e^{2\pi i\alpha}} \left(2^{-\alpha} e^{i\pi\alpha} - e^{i\pi\alpha}\right) = \boxed{\frac{\pi\,(1-2^{-\alpha})}{\sin(\pi\alpha)}}$$

Exp. 2

$$I = \int_0^1 \sqrt{\frac{1-x}{x}}\, \frac{dx}{(x+2)^2}$$

$$R(z) = \frac{1}{(z+2)^2} \qquad \text{For } z \to \infty \qquad R(z) = \frac{1}{z^2}\left(1 - \frac{4}{z} + \cdots\right)$$

$$\operatorname*{Res}_{z=\infty} \left[h(z)\, R(z)\right] = 0$$

Since $\alpha = -1/2$ for this particular example

$$I = \frac{2\pi i}{1-e^{-i\pi}} \operatorname*{Res}_{z=-2} f(z) = i\pi \frac{d}{dz}\left[h(z)\right]\Big|_{z=-2}$$

$$h(z) = \left(\frac{1}{z}-1\right)^{1/2}; \quad \frac{dh}{dz}\Big|_{z=-2} = -\frac{1}{2z^2 h(z)}\Big|_{z=-2} = -\frac{1}{8h(-2)}$$

$$h(-2) = \sqrt{\frac{3}{2}}\, e^{\overbrace{(-1/2)}^{\alpha} i\cdot(\underbrace{\pi}_{\varphi_1} - \underbrace{0}_{\varphi_2})} = \sqrt{\frac{3}{2}}\, e^{-i\pi/2} = -i\sqrt{\frac{3}{2}}$$

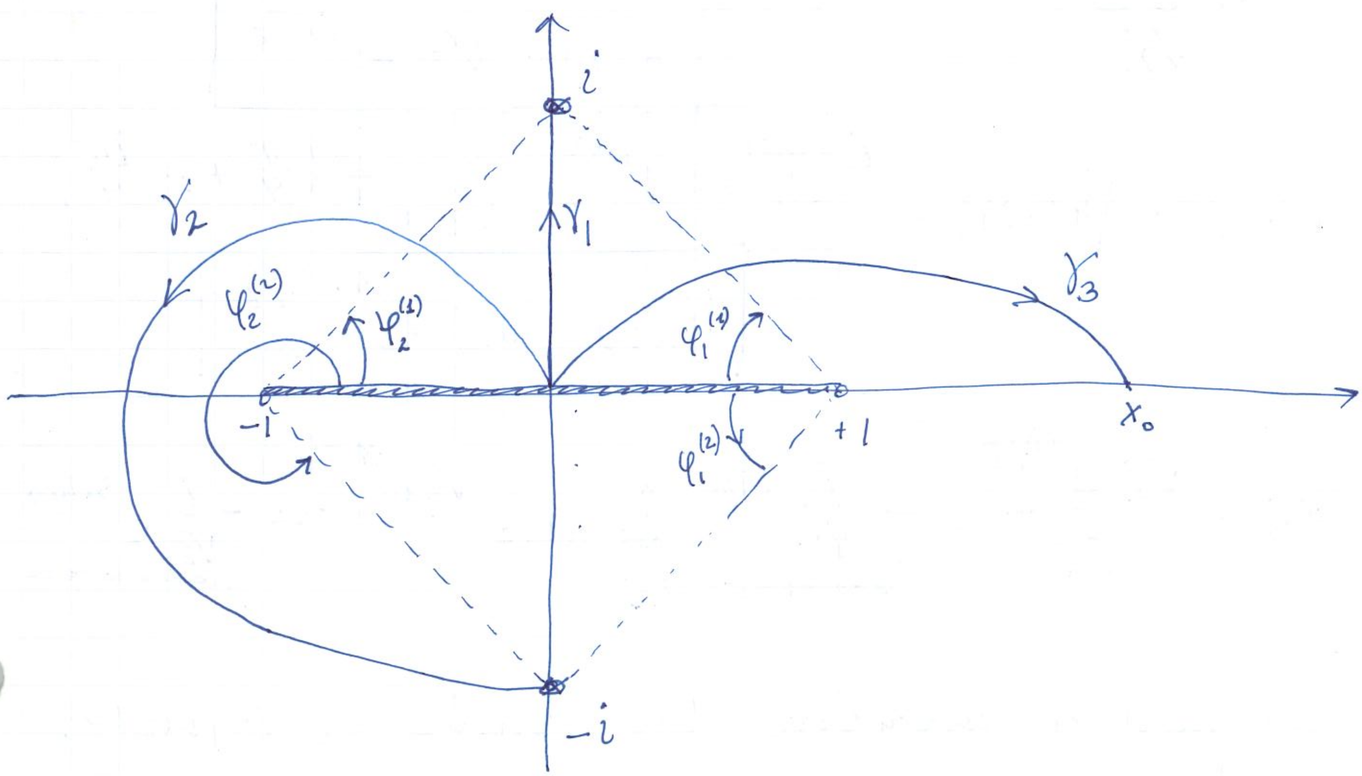
$$I = (i\pi)\left(-\frac{1}{8}\right)\cdot\frac{1}{-i\sqrt{3/2}} = \boxed{\frac{\pi}{4\sqrt{6}}}$$

$\boxed{\text{Exp. 3}}$

$$I = \int_{-1}^{+1} \frac{\sqrt[4]{(1-x)(1+x)^3}}{1+x^2}\,dx$$

$$h(z) = \sqrt[4]{(1-z)(1+z)^3}$$

for $-1<x<1$: $h(x+i0) = \sqrt[4]{(1-x)(1+x)^3}$



For $-1<x<1$ on the lower part of the branch cut

$$h(\tilde{x}) = h(x-i0) = \sqrt[4]{(1-x)(1+x)^3}\; e^{\frac{i3\varphi_1}{4}+\frac{i\varphi_2}{4}} \qquad \varphi_1 = 2\pi \quad \varphi_2 = 0$$

$$h(\tilde{x}) = e^{\frac{3i\pi}{2}} \sqrt[4]{(1-x)(1+x)^3} = -i\,h(x)$$

For the complex function $f(z) = h(z) - R(z)$ we have:

$$f(\tilde{x}) = f(x-i0) = -i f(x)$$

$$\int_{-1}^{+1} f(x)dx + \int_{1}^{-1} f(\tilde{x})dx = 2\pi i\left(\operatorname*{Res}_{z=i} f(z) + \operatorname*{Res}_{z=-i} f(z) + \operatorname*{Res}_{z=\infty} f(z)\right)$$

$$\Rightarrow \operatorname*{Res}_{z=\pm i}\left[f(z)\right] = \left(\frac{h(z)}{2z}\right)_{z=\pm i} \qquad \begin{Bmatrix} \varphi_1^{(k)} = \Delta_{\gamma_k} \arg(1-z) \\ \varphi_2^{(k)} = \Delta_{\gamma_k} \arg(1+z) \end{Bmatrix}$$

$$\rightarrow h(i) = |h(i)|\, e^{\frac{i}{4}\left(\varphi_1^{(1)} + 3\varphi_2^{(1)}\right)} =$$

$$= \underbrace{|h(i)|}_{\sqrt{2}}\, e^{\frac{i}{4}\left(-\frac{\pi}{4} + 3\cdot\frac{\pi}{4}\right)} = \sqrt{2}\, e^{i\pi/8}$$

$$\boxed{h(i) = \sqrt{2}\, e^{i\pi/8}}$$

$$\rightarrow h(-i) = |h(-i)|\, e^{\frac{i}{4}\left(\varphi_1^{(2)} + 3\varphi_2^{(2)}\right)} = \sqrt{2}\, e^{\frac{i}{4}\left(\frac{\pi}{4} + 3\cdot\frac{7\pi}{4}\right)}$$

$$\boxed{h(-i) = \sqrt{2}\, e^{11i\pi/8}}$$

$$\operatorname*{Res}_{z=i} f(z) = \frac{\sqrt{2}\, e^{i\pi/8}}{2i} = \underline{-\frac{i}{\sqrt{2}} e^{i\pi/8}}, \quad \operatorname*{Res}_{z=-i} f(z) = \frac{\sqrt{2} e^{11i\pi/8}}{-2i} = \underline{-\frac{i}{\sqrt{2}} e^{3i\pi/8}}$$

Now we need to calculate the residue at infinity.

$$g(z) = \frac{h(z)}{z} = g(\infty)\left(1-\frac{1}{z}\right)^{1/4}\left(1+\frac{1}{z}\right)^{3/4} = g(\infty)\left(1 + \frac{1}{2z} + \dots\right)$$

$$g(\infty) = e^{\frac{i}{4}\left(\varphi_1^{(3)} + 3\varphi_2^{(3)}\right)} \qquad \varphi_1^{(3)} = -\pi \qquad \varphi_2^{(3)} = 0$$

$$g(\infty) = e^{\frac{i}{4}(-\pi + 3\cdot 0)} = e^{-i\pi/4}$$

$$\begin{cases} h(z) = e^{-i\pi/4}\left(z + \frac{1}{2} + \dots\right) \\ R(z) = \frac{1}{1+z^2} = \frac{1}{z^2}\frac{1}{\left(1+\frac{1}{z^2}\right)} = \frac{1}{z^2}\left(1 - \frac{1}{z^2} + \dots\right) \end{cases}$$

$$f(z) = h(z)\cdot R(z) = e^{-i\pi/4}\left(z + \frac{1}{2} + \dots\right)\left(\frac{1}{z^2} - \frac{1}{z^4} + \dots\right) =$$

$$= e^{-i\pi/4}\left(\frac{1}{z} + \frac{1}{2z^2} + \dots\right)$$

$$\operatorname*{Res}_{z=-\infty}\left[f(z)\right] = \underline{\underline{-e^{-i\pi/4}}}$$

Finally, combining all ingredients together:

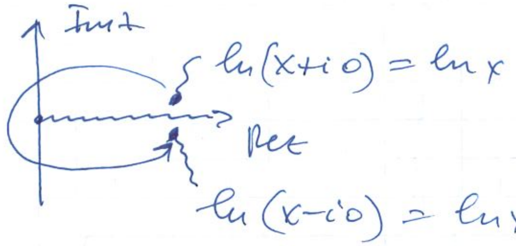
$$(1+i)\cdot I = 2\pi i\left[-\frac{i}{\sqrt{2}}e^{i\pi/8} - \frac{i}{\sqrt{2}}e^{3i\pi/8} - e^{-i\pi/4}\right]$$

$$(1+i)\, I = \sqrt{2}\pi\left[e^{i\pi/8} + e^{3i\pi/8} - \sqrt{2}e^{+i\pi/4}\right]$$

$$\downarrow$$

$$\boxed{I = \sqrt{2}\pi\left[\sqrt{2}\cos\left(\frac{\pi}{8}\right) - 1\right]}$$

Type VI Integrals: $\int_0^\infty x^{\alpha-1}(\ln x)^m R(x)\,dx$



$$f(z) = h(z)(\ln z)^m R(z) \to \quad f(x+i0) = x^{\alpha-1}(\ln x)^m R(x)$$

$$f(\tilde{x}) = f(x-i0) = e^{2\pi i\alpha} x^{\alpha-1}(\ln x + 2\pi i)^m R(x)$$

$$I_{\rho,R} = \oint_{\gamma_\rho,\Gamma_R} f(z)\,dz$$

$$\int_0^\infty [f(x) - f(\tilde{x})]\,dx = 2\pi i \sum_{z=z_k}^{n} \operatorname{Res} f(z)$$

Case 1: α is non-integer

$$\int_0^\infty [f(x) - f(\tilde{x})]\,dx = (1-e^{2\pi i\alpha})\,I + I_{m-1} + \dots + I_s$$

where $0 \le s \le m-1$. In particular for $m=1$

$$(1-e^{2\pi i\alpha})I - 2\pi i e^{2\pi i\alpha}\int_0^\infty x^{\alpha-1}R(x)\,dx = 2\pi i \sum_{z=z_k}^{n} \operatorname{Res} f(z)$$

Case 2: α is integer $\quad x^{\alpha-1}R(x) \to \tilde{R}(x)$

$$I = \int_0^\infty \ln^m x\, \tilde{R}(x)\,dx$$

Given the difference $f(x) - f(\tilde{x})$ we need to take:

$$f(z) = \ln^{m+1}(z)\,\tilde{R}(z)$$

$\boxed{\text{Exp.1}}$

$$I = \int_0^\infty \frac{\ln x \, dx}{\sqrt{x}\,(x+1)^2} \qquad (\alpha = 1/2)$$

$$\left(1-e^{2\pi i \alpha}\right) I - 2\pi i\, e^{2\pi i \alpha} \int_0^\infty \frac{dx}{\sqrt{x}\,(x+1)^2} = 2\pi i \operatorname*{Res}_{z=-1} \left[\frac{\ln z}{\sqrt{z}\,(z+1)^2}\right]$$

$\alpha = 1/2$

$$2I + 2\pi i \int_0^\infty \frac{dx}{\sqrt{x}\,(x+1)^2} = 2\pi i \operatorname*{Res}_{z=-1} \left\{\frac{\ln z}{\sqrt{z}\,(z+1)^2}\right\}$$

$$\operatorname*{res}_{z=-1} \left\{\frac{\ln z}{\sqrt{z}\,(z+1)^2}\right\} = \frac{d}{dz}\left(\frac{\ln z}{\sqrt{z}}\right) = \left[-\frac{1}{2}\,\frac{\ln z}{z^{3/2}} + \frac{1}{z \cdot z^{1/2}}\right]_{z=-1}$$

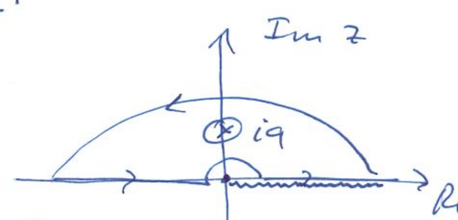
$$= \left[-\frac{1}{2}\,\frac{i\pi}{e^{+3\pi i/2}} + \frac{1}{e^{3i\pi/2}}\right] = \frac{\pi}{2} + i$$

$$\underline{\underline{2I}} + 2\pi i \int_0^\infty \frac{dx}{\sqrt{x}\,(x+1)^2} = 2\pi i \left(\pi/2 + \underline{\underline{i}}\right)$$

$$\boxed{I = -\pi}$$

$\boxed{\text{Exp.2}}$

$$I = \int_0^\infty \frac{\ln x \, dx}{x^2+a^2}$$



$$\oint_C \frac{\ln z \, dz}{z^2+a^2} = \int_{-\infty}^0 \frac{(\ln|x| + i\pi)\,dx}{x^2+a^2} + \int_0^\infty \frac{\ln x \, dx}{x^2+a^2} = 2\pi i \operatorname*{Res}_{z=ia}\left\{\frac{\ln z}{z^2+a^2}\right\}$$

$$2I + i\pi \int_0^\infty \frac{dx}{x^2+a^2} = 2\pi i \operatorname*{Res}_{z=ia}\left\{\frac{\ln z}{z^2+a^2}\right\}$$

$$\operatorname*{Res}_{ia}\left\{\frac{\ln z}{z^2+a^2}\right\} = \operatorname{Res}\left[\frac{\ln z}{(z-ia)(z+ia)}\right] = \frac{\operatorname{Ln}(ia)}{2ia} = \frac{\ln a + i\pi/2}{2ia}$$

$$2I + i\pi \int_0^\infty \frac{dx}{x^2+a^2} = \frac{\pi}{a}\left(\ln a + \frac{i\pi}{2}\right) \Rightarrow \boxed{I = \frac{\pi}{2a} \ln a}$$

Note 1: The same method can be applied to the integrals of the type:

$$\int_a^b \left[\ln\left(\frac{x-a}{b-a}\right)\right]^m R(x)\,dx$$

Note 2: $$\frac{d}{d\alpha}\int_0^\infty x^{\alpha-1}R(x)\,dx = \int_0^\infty x^{\alpha-1}\ln x\, R(x)\,dx$$

$$\boxed{\frac{d^m}{d\alpha^m}\int_0^\infty x^{\alpha-1}R(x)\,dx = \int_0^\infty x^{\alpha-1}\ln^m x\, R(x)\,dx}$$

type IV

type VI

Summation of Series

Assume $f(z)$ has simple (possibly degenerate) poles at points $a_1, a_2, \dots, a_n, \dots$ such that neither $a_n \neq 0, \pm 1, \pm 2, \dots$ In addition if $f(z)$ is such that

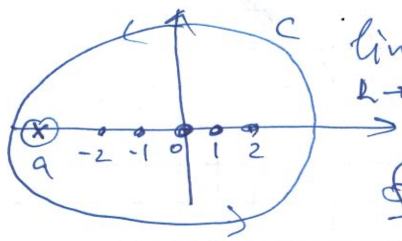
$$\lim_{n\to\infty} \int_{C_n} f(z)\cot(\pi z)\,dz = 0 \quad \text{then}$$

$$\lim_{n\to\infty} \int_{C_n} \frac{f(z)\,dz}{\sin(\pi z)}\,dz = 0$$

$$\boxed{\sum_{n=-\infty}^{+\infty} f(n) = -\pi \sum_{k=1}^{m} \operatorname{Res}\Big[f(z)\cot(\pi z)\Big]_{z=a_k}} \qquad \boxed{\sum_{n=-\infty}^{+\infty} (-1)^n f(n) = -\pi \sum_{k=1}^{m} \operatorname{Res}\left[\frac{f(z)}{\sin(\pi z)}\right]_{z=a_k}}$$

$\boxed{\text{Exp.}}$

$$\sum_{n=-\infty}^{+\infty} \frac{1}{(n+a)^2} \longrightarrow \oint_C \frac{\cot(\pi z)\,dz}{(z+a)^2}$$



$$\lim_{R\to\infty} \oint_C \frac{\cot(\pi z)\,dz}{(z+a)^2} \to 0$$

$$\oint_C \frac{\cot(\pi z)\,dz}{(z+a)^2} = 0 = 2\pi i \left\{ \sum_{z=n} \operatorname{Res} \frac{\cot(\pi z)}{(z+a)^2} + \operatorname{Res}_{z=a} \frac{\cot \pi z}{(z+a)^2} \right\}$$

$$\operatorname{Res}_{z=n}\Big[\cot(\pi z)\Big] = \frac{1}{\pi}$$

$$\frac{1}{\pi} \sum_{n=-\infty}^{+\infty} \frac{1}{(n+a)^2} + \operatorname{Res}_{z=a} \left[\frac{\cot(\pi z)}{(z+a)^2}\right] = 0$$

$$\sum_{n=-\infty}^{+\infty} \frac{1}{(n+a)^2} = -\pi \frac{d}{dz}\Big[\cot(\pi z)\Big]_{z=a} = \boxed{\frac{\pi^2}{\sin^2(\pi a)}}$$