Solve the non-linear KdV equation

$$u_t + 6uu_x + u_{xxx} = 0 \tag{1}$$

using d'Alembert's principle substitution ; find a particular single-parametric solution.

Consider a moving frame ansatz of $u(x,t) = U(\xi) = U(x-ct)$, where $\xi := x - ct$. The derivatives become

$$\begin{cases} u_t = -cU'(\xi) \\ u_x = U'(\xi) \\ u_{xxx} = U'''(\xi) \end{cases} \tag{2}$$

The PDE then reads

$$-cU' + 3UU' + U''' = 0. \tag{3}$$

Integrating once with respect to ξ gives

$$-cU' + 6UU' + U'' = A \tag{4}$$

for some integration constant A. Localized wave solution require that U, U', U'' vanish as $|\xi| = \infty$, so we set $A = 0$. Rearranging gives

$$U'' = cU - 3U^2 \tag{5}$$

which can be muultipled by U' and integrated again with inverse product law to give

$$\frac{1}{2}(U')^2 = \frac{c}{2}U^2 - U^3 + B. \tag{6}$$

Again, localized wave solutions require $B = 0$. Thus, we have

$$U'^2 = cU^2 - 2U^3 = U^2(c - 2U). \tag{7}$$

Separating variables and integrating gives

$$U(\xi) = \frac{c}{2}\operatorname{sech}^2\left(\frac{\sqrt{c}}{2}(\xi - \xi_0)\right) \tag{8}$$

which in original variables is

$$u(x,t) = \frac{c}{2}\operatorname{sech}^2\left(\frac{\sqrt{c}}{2}[x - ct - x_0]\right), \quad c > 0. \tag{9}$$

This is a solitary wave solution of the KdV equation traveling at parametrized speed c.

2

Consider the non linear Liouville equation

$$u_{tt} - u_{xx} = g\exp(u), \quad g > 0. \tag{10}$$

Using the d'Alambert principle substitution find a particular single-parametric solution of this equation

Consider a moving frame ansatz of $u(x,t) = U(\xi) = U(x - ct)$, where $\xi := x - ct$. The derivatives become

$$\begin{cases} u_{tt} = c^2 U''(\xi) \\ u_{xx} = U''(\xi) \end{cases} \tag{11}$$

The PDE then reads

$$c^2 U'' - U'' = g\exp(U) \Rightarrow (c^2 - 1)U'' = g\exp(U). \tag{12}$$

Multiplying by U' and integrating once with respect to ξ gives

$$\frac{c^2 - 1}{2}(U')^2 = g\exp(U) + C \tag{13}$$

for some integration constant C. For a particular explicit solution, set $C = 0$. Thus,

$$(U')^2 = \frac{2g}{c^2 - 1}\exp(U). \tag{14}$$

Assuming $\frac{g}{c^2-1} > 0$ for real solutions, we have

$$U' = \pm\sqrt{\frac{2g}{c^2 - 1}}\exp\left(\frac{U}{2}\right). \tag{15}$$

Separating variables (taking the negative branch for illustration) and integrating gives

$$-2\exp\left(-\frac{U}{2}\right) = -\sqrt{\frac{2g}{c^2 - 1}}(\xi - \xi_0), \tag{16}$$

so

$$\exp\left(-\frac{U}{2}\right) = \frac{1}{2}\sqrt{\frac{2g}{c^2 - 1}}(\xi - \xi_0). \tag{17}$$

Solving for U yields

$$U(\xi) = -2\ln\left[\frac{1}{2}\sqrt{\frac{2g}{c^2 - 1}}(\xi - \xi_0)\right] \tag{18}$$

which in original variables is

$$u(x,t) = -2\ln[\alpha(x - ct - x_0)], \quad \alpha = \frac{1}{2}\sqrt{\frac{2g}{c^2 - 1}}, \tag{19}$$

with $c^2 > 1$ if $g > 0$. This is a singular (logarithmic blow-up) traveling wave solution of the equation traveling at parametrized speed c.

3

Solve the non-linear Schrodinger equation

$$i\psi_t\psi_{xx} - \omega_0\psi + g\psi|\psi|^2 = 0, \quad g > 0 \tag{20}$$

Consider a separable ansatz of $\psi(x,t) = A(x)\exp(i\Psi(t))$. The derivatives become

$$\begin{cases} \psi_t = i\dot{\Psi}A\exp(i\Psi) \\ \psi_{xx} = A''\exp(i\Psi) \end{cases} \tag{21}$$

Substitute into the equation

$$i\Big(i\dot{\Psi}A\exp(i\Psi)\Big) - A''\exp(i\Psi) - \omega_0 A\exp(i\Psi) + gA^3\exp(i\Psi) = 0 \tag{22}$$

Dividing by $\exp(i\Psi)$ gives

$$-\dot{\Psi}A - A'' - \omega_0 A + gA^3 = 0 \tag{23}$$

Rearranging yields

$$A'' = gA^3 - \Big(\omega_0 + \dot{\Psi}\Big)A \tag{24}$$

For separation of variables, set $\omega_0 + \dot{\Psi} = \mu$ (constant), so $\dot{\Psi} = \mu - \omega_0$ and

$$A'' = gA^3 - \mu A \tag{25}$$

Multiply by A' and integrate once

$$\frac{1}{2}(A')^2 = \frac{g}{4}A^4 - \frac{\mu}{2}A^2 + C \tag{26}$$

For a dark soliton solution with $A(x) \to \pm\sqrt{\frac{\mu}{g}}$ as $|x| \to \infty$, set $C = \frac{\mu^2}{4g}$ so that $A' = 0$ at the asymptotic value. This gives

$$(A')^2 = \frac{g}{2}A^4 - \mu A^2 + \frac{\mu^2}{2g} = \left(\frac{g}{2}\right)\left(A^2 - \frac{\mu}{g}\right)^2 \tag{27}$$

Taking square roots and separating variables

$$\int d\frac{A}{A^2 - \frac{\mu}{g}} = \pm\sqrt{\frac{g}{2}}\int dx \tag{28}$$

Integrating yields

$$\frac{1}{2\sqrt{\frac{\mu}{g}}}\ln\left|\frac{A - \sqrt{\frac{\mu}{g}}}{A + \sqrt{\frac{\mu}{g}}}\right| = \pm\sqrt{\frac{g}{2}}(x - x_0) \tag{29}$$

Solving for $A(x)$ with appropriate boundary conditions gives

$$A(x) = \sqrt{\frac{\mu}{g}}\tanh\left(\sqrt{\frac{\mu}{2}}(x - x_0)\right) \tag{30}$$

Thus the particular solution is

$$\psi(x,t) = \sqrt{\frac{\mu}{g}}\tanh\left(\sqrt{\frac{\mu}{2}}(x - x_0)\right)\exp(i(\mu - \omega_0)t), \quad \mu > 0 \tag{31}$$

This is a single-parametric dark soliton solution of the defocusing NLS equation with parameter μ.

4

Solve the Klein Gorden equation

$$u_{tt} - u_{xx} - u + u^3 = 0, \tag{32}$$

using d'Alembert's principle substitution ; find a particular single-parametric solution.

Consider a moving frame ansatz of $u(x,t) = U(\xi) = U(x - ct)$, where $\xi := x - ct$. The derivatives become

$$\begin{cases} u_{tt} = c^2 U''(\xi) \\ u_{xx} = U''(\xi) \end{cases} \tag{33}$$

The PDE then reads

$$c^2 U'' - U'' - U + U^3 = 0 \Rightarrow \quad (c^2 - 1)U'' = U - U^3. \tag{34}$$

Multiplying by U' and integrating once with respect to ξ gives

$$\frac{c^2 - 1}{2}(U')^2 - \frac{1}{2}U^2 + \frac{1}{4}U^4 = C \tag{35}$$

for some integration constant C. For a particular heteroclinic (kink) solution connecting $U(\pm\infty) = \pm 1$ with $U'(\pm\infty) = 0$, we impose such boundary conditiion to find C:

$$C = -\frac{1}{2} \cdot 1 + \frac{1}{4} \cdot 1 = -\frac{1}{4}. \tag{36}$$

Thus,

$$(U')^2 = \frac{(1 - U^2)^2}{2(1 - c^2)}, \tag{37}$$

requiring $|c| < 1$ for real solutions. Separating variables gives

$$\frac{dU}{1 - U^2} = \pm \frac{d\xi}{\sqrt{2(1 - c^2)}}. \tag{38}$$

Integrating yields

$$\frac{1}{2} \ln| \frac{1 + U}{1 - U} | = \pm \frac{\xi - \xi_0}{\sqrt{2(1 - c^2)}}, \tag{39}$$

so

$$U(\xi) = \tanh\left(\frac{\xi - \xi_0}{\sqrt{2(1 - c^2)}} \right). \tag{40}$$

In original variables,

$$u(x,t) = \tanh\left(\frac{x - ct - x_0}{\sqrt{2(1 - c^2)}} \right), \quad |c| < 1, \tag{41}$$

a single-parameter kink soliton traveling at speed c with asymptotics ± 1.

[#5]: Landau-Lifshitz Equation [2 points]
Consider the nonlinear Landau-Lifshitz equation for the dynamics of magnetization. It describes a magnetic moment whose spatial orientation is parametrized by polar angle $\theta(x,t)$ and azimuthal angle $\phi(x,t)$. For a particular model of magnetic anisotropy these coupled equations take the form:

$$l^2\partial_x^2\theta - [1+l^2(\partial_x\phi)^2]\sin\theta\cos\theta - \omega^{-1}\partial_t\phi = 0, \quad l^2\partial_x[\sin^2\theta\partial_x\phi] - \omega^{-1}\partial_t\theta = 0, \tag{1}$$

where l is the intrinsic length scale in the problem and ω is the intrinsic frequency scale. Assuming solution in the form $\theta = \theta(x-Vt)$ and $\phi = \phi(x-Vt)$ find localized solitonic solution of these equations $\theta(\xi \to \pm\infty) \to 0$ with $\xi = x - Vt$.

This problem is *impossible* to solve in its original statement, and we will show that by leading to a contradiction.

With the subsitution $\xi = x - Vt$, write derivatives against ξ with prime, the original coupled equations become

$$l^2\theta'' - \left[1+l^2\varphi'^2\right]\sin\theta\cos\theta + \frac{V}{\omega}\varphi' = 0 \tag{42}$$

$$l^2\left(\sin^2(\theta)\varphi'\right)' + \frac{V}{\omega}\theta' = 0 \tag{43}$$

We can first integrate Equation 43 w.r.t. ξ to get $\varphi'(\xi)$.

$$\int l^2\left(\sin^2(\theta)\varphi'\right)\mathrm{d}\xi + \int \frac{V}{\omega}\,\mathrm{d}\xi = l^2\sin^2(\theta)\varphi' + \frac{V}{\omega}\theta + C = 0 \tag{44}$$

Boundary conditions at infinity is $\theta(\xi) \to 0$ as $\xi \to \infty$. This implies $C = 0$. Thus, we can separate for φ' as

$$\varphi' = -\frac{V}{\omega l^2}\frac{\theta}{\sin^2(\theta)}. \tag{45}$$

Let $a \equiv \frac{V^2}{\omega^2 l^2}$, substute above into Equation 42 gives

$$l^2\theta'' - \sin\theta\cos\theta - a\left(\frac{\theta^2\cos\theta}{\sin^3\theta} + \frac{\theta}{\sin^2\theta}\right) = 0 \tag{46}$$

Multiplying by θ' to make it into a form so that we can use the fact

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\frac{l^2}{2}\theta'^2\right) = l^2\theta''\theta, \quad \frac{\mathrm{d}}{\mathrm{d}\xi}\left(-\frac{1}{2}\sin^2\theta\right) = -\sin\theta\cos\theta\,\theta', \tag{47}$$

we find that

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\frac{l^2}{2}\theta'^2\right) + \frac{\mathrm{d}}{\mathrm{d}\xi}\left(-\frac{1}{2}\sin^2\theta\right) = a\theta'\left(\frac{\theta^2\cos\theta}{\sin^3\theta} + \frac{\theta}{\sin^2\theta}\right). \tag{48}$$

Define

$$E = \frac{l^2}{2}\theta'^2 - \frac{1}{2}\sin^2\theta, \tag{49}$$

we have then

$$E' = a\theta'\left(\frac{\theta^2\cos\theta}{\sin^3\theta} + \frac{\theta}{\sin^2\theta}\right). \tag{50}$$

We can deduce to a contradiction by studying the small angle limit $\theta \to 0$. In this limit,

$$E' \sim a\theta' \left(\frac{\theta^2}{\theta^3} + \frac{\theta}{\theta^2} \right) = 2a\frac{\theta'}{\theta}. \tag{51}$$

Without loss of generality, consider a decay rate $\theta'/\theta = \kappa < 0$ as $\xi \to \infty$. Then,

$$E' \sim 2a\kappa < 0 \quad (\xi \to \infty) \tag{52}$$

so that in this limit $E \to -\infty$.

However, the definition Equation 49 is constrained by the boundary condition

$$\theta(\xi) \to 0, \quad \theta'(\xi) \to 0 \quad \text{as } \xi \to \infty \tag{53}$$

and thus $E \to 0$ as $\xi \to \infty$. This is a *contradiction*, so no such solution exists for the original problem beyond the trivial solution $\theta(\xi) = 0, \varphi(\xi) = \text{const}$.

One can infer that a sign error in the original problem statement is likely. In fact, had we changed the sign in Equation 43 to be the following

$$l^2\left(\sin^2(\theta)\varphi'\right)' - \frac{V}{\omega}\theta' = 0, \tag{54}$$

the contradiction would be resolved and a nontrivial solution would exist. However that is beyond my scope in terms of this homework, as I have already demonstrated the impossibility of the original problem as stated.