# 1



Unitary Unimodular Matrices

**Given:**

$$U = \frac{a_0 + i\boldsymbol{\sigma}\cdot\boldsymbol{a}}{a_0 - i\boldsymbol{\sigma}\cdot\boldsymbol{a}} \tag{1}$$

where $a_0$ is real and $\boldsymbol{a}$ is a real vector.

## a. Unitarity and Unimodularity

**Unitarity ($U^\dagger U = 1$):**

Let $N = a_0 + i\boldsymbol{\sigma}\cdot\boldsymbol{a}$. The denominator is $a_0 - i\boldsymbol{\sigma}\cdot\boldsymbol{a} = N^\dagger$ (since $\boldsymbol{\sigma}$ is Hermitian and scalars are real). Because $N$ and $N^\dagger$ commute, we write $U = N\left(N^\dagger\right)^{-1}$.

$$U^\dagger = \left(N\left(N^\dagger\right)^{-1}\right)^\dagger = \left(N^\dagger\right)^{-1\dagger} N^\dagger = N^{-1}N^\dagger \tag{2}$$

$$UU^\dagger = N\left(N^\dagger\right)^{-1} N^{-1}N^\dagger = NN^{-1}\left(N^\dagger\right)^{-1}N^\dagger = 1\cdot 1 = 1. \tag{3}$$

**Unimodularity ($\det \boldsymbol{U} = 1$):**

Using the identity $\det(cI + \boldsymbol{\sigma}\cdot\boldsymbol{v}) = c^2 + |\boldsymbol{v}|^2$ (where $|\boldsymbol{v}|^2 = \boldsymbol{v}\cdot\boldsymbol{v}$ for complex vectors, here just $a^2$):

$$\det U = \frac{\det(a_0 + i\boldsymbol{\sigma}\cdot\boldsymbol{a})}{\det(a_0 - i\boldsymbol{\sigma}\cdot\boldsymbol{a})} = \frac{a_0^2 + |\boldsymbol{a}|^2}{a_0^2 + |\boldsymbol{a}|^2} = 1. \tag{4}$$

## b. Axis $\hat{n}$ and Angle $\theta$

We convert $U$ to standard form $\cos\left(\frac{\theta}{2}\right) - i(\boldsymbol{\sigma}\cdot\hat{n})\sin\left(\frac{\theta}{2}\right)$. Multiply numerator and denominator by $N = a_0 + i\boldsymbol{\sigma}\cdot\boldsymbol{a}$:

$$U = \frac{(a_0 + i\boldsymbol{\sigma}\cdot\boldsymbol{a})^2}{a_0^2 + \boldsymbol{a}^2} = \frac{a_0^2 - \boldsymbol{a}^2 + 2ia_0(\boldsymbol{\sigma}\cdot\boldsymbol{a})}{a_0^2 + \boldsymbol{a}^2} \tag{5}$$

Separating real and imaginary parts:

$$U = \underbrace{\frac{a_0^2 - \boldsymbol{a}^2}{a_0^2 + \boldsymbol{a}^2}}_{\cos(\theta/2)} + i(\boldsymbol{\sigma}\cdot\boldsymbol{a})\frac{2a_0}{a_0^2 + \boldsymbol{a}^2} \tag{6}$$

Matching the imaginary part $-i(\boldsymbol{\sigma}\cdot\hat{n})\sin\left(\frac{\theta}{2}\right)$:

$$\hat{n} = -\frac{\boldsymbol{a}}{|\boldsymbol{a}|}, \quad \sin\left(\frac{\theta}{2}\right) = \frac{2a_0|\boldsymbol{a}|}{a_0^2 + \boldsymbol{a}^2}. \tag{7}$$

Thus, $\theta = 2\arccos\left(\frac{a_0^2 - \boldsymbol{a}^2}{a_0^2 + \boldsymbol{a}^2}\right)$.

# 2

Euler Rotations

Given the matrix $D^{\left(\frac{1}{2}\right)}(\alpha, \beta, \gamma)$, we find the equivalent single rotation angle $\theta$. The trace (character) of a rotation matrix is basis independent:

$$\chi(\theta) = \mathrm{Tr}(D) = 2\cos\left(\frac{\theta}{2}\right) \tag{8}$$

Computing the trace of the given matrix:

$$
\begin{aligned}
\mathrm{Tr}(D) &= e^{-\frac{i(\alpha+\gamma)}{2}} \cos\left(\frac{\beta}{2}\right) + e^{\frac{i(\alpha+\gamma)}{2}} \cos\left(\frac{\beta}{2}\right) \\
&= \cos\left(\frac{\beta}{2}\right)\left(e^{-\frac{i(\alpha+\gamma)}{2}} + e^{\frac{i(\alpha+\gamma)}{2}}\right) \\
&= 2\cos\left(\frac{\beta}{2}\right)\cos\left(\frac{\alpha+\gamma}{2}\right)
\end{aligned} \tag{9}
$$

Equating the traces yields:

$$
\theta = 2\arccos\left(\cos\left(\frac{\beta}{2}\right)\cos\left(\frac{\alpha+\gamma}{2}\right)\right) \tag{10}
$$

# 3

Operator Eigenfunctions

**Objective:** Show $|\psi\rangle = e^{-iJ_z\varphi/\hbar}e^{-iJ_y\theta/\hbar}|j,m\rangle$ is an eigenstate of $\hat{O} = J_x\cos\varphi\sin\theta + J_y\sin\varphi\sin\theta + J_z\cos\theta$ with eigenvalue $m\hbar$.

**Proof:** Let $U = e^{-iJ_z\varphi/\hbar}e^{-iJ_y\theta/\hbar}$. We check if $U^\dagger\hat{O}U = J_z$.

$$\hat{O}_{\text{rot}} = U^\dagger\hat{O}U = e^{iJ_y\theta/\hbar}\left(e^{iJ_z\varphi/\hbar}\hat{O}e^{-iJ_z\varphi/\hbar}\right)e^{-iJ_y\theta/\hbar} \tag{11}$$

1. **Rotation by $-\varphi$ about z:** $J_x \to J_x\cos\varphi - J_y\sin\varphi$, $J_y \to J_x\sin\varphi + J_y\cos\varphi$. Substituting into $\hat{O}$, the $\sin\theta$ terms simplify:

$$\sin\theta\left(J_x\left(c_\varphi^2 + s_\varphi^2\right)\right) + J_z\cos\theta = J_x\sin\theta + J_z\cos\theta \tag{12}$$

2. **Rotation by $-\theta$ about y:** $J_x \to J_x\cos\theta - J_z\sin\theta$, $J_z \to J_z\cos\theta + J_x\sin\theta$.

$$\begin{aligned}\hat{O}_{\text{rot}} &= (J_x\cos\theta - J_z\sin\theta)\sin\theta + (J_z\cos\theta + J_x\sin\theta)\cos\theta \\ &= J_x(\sin\theta\cos\theta - \sin\theta\cos\theta) + J_z(\cos^2\theta + \sin^2\theta) = J_z\end{aligned} \tag{13}$$

Since $U^\dagger\hat{O}U = J_z$, then $\hat{O}U = UJ_z$. Applying to eigenstate $|j,m\rangle$:

$$\hat{O}|\psi\rangle = \hat{O}U|j,m\rangle = UJ_z|j,m\rangle = m\hbar\, U|j,m\rangle = m\hbar\,|\psi\rangle. \tag{14}$$

# 4

Angular Momentum Matrices (j=1)

Let $\mathcal{J} = \boldsymbol{J}^{(1)}\cdot\hat{n}/\hbar$. The eigenvalues are $1, 0, -1$.

## a. Characteristic Equation

Since the eigenvalues are $\lambda \in \{1, 0, -1\}$, the operator satisfies:

$$(\mathcal{J}-1)(\mathcal{J}-0)(\mathcal{J}+1) = \mathcal{J}(\mathcal{J}^2-1) = 0 \Rightarrow \mathcal{J}^3 = \mathcal{J}. \tag{15}$$

## b. Rotation Formula

The Taylor expansion of $D = e^{-i\mathcal{J}\varphi}$:

$$D = 1 - i\mathcal{J}\varphi - \frac{\mathcal{J}^2\varphi^2}{2!} + \frac{i\mathcal{J}^3\varphi^3}{3!} + \ldots \tag{16}$$

Using $\mathcal{J}^3 = \mathcal{J}$ (and $\mathcal{J}^4 = \mathcal{J}^2$), we group terms:

$$\begin{gathered} D = 1 - i\mathcal{J}\left(\varphi - \frac{\varphi^3}{3!} + \ldots\right) - \mathcal{J}^2\left(\frac{\varphi^2}{2!} - \frac{\varphi^4}{4!} + \ldots\right) \\ D = 1 - i\mathcal{J}\sin\varphi - \mathcal{J}^2(1-\cos\varphi) \end{gathered} \tag{17}$$

## c. Matrix for $\beta$ rotation ($\hat{n} = \hat{y}$)

We use $\mathcal{J}_y = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$.

$$\mathcal{J}_y^2 = \frac{1}{2}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix} \tag{18}$$

Substituting into the formula from (b) with $\varphi = \beta$:

$$d^{(1)}(\beta) = I - i\mathcal{J}_y\sin\beta - \mathcal{J}_y^2(1-\cos\beta) \tag{19}$$

Calculating term by term yields:

$$d^{(1)}(\beta) = \begin{pmatrix} \frac{1+\cos\beta}{2} & -\frac{\sin\beta}{\sqrt{2}} & \frac{1-\cos\beta}{2} \\ \frac{\sin\beta}{\sqrt{2}} & \cos\beta & -\frac{\sin\beta}{\sqrt{2}} \\ \frac{1-\cos\beta}{2} & \frac{\sin\beta}{\sqrt{2}} & \frac{1+\cos\beta}{2} \end{pmatrix} \tag{20}$$