P1

Consider a particle subject to a one-dimensional constant force F.
(a) Show that the momentum space propagator is

$$\langle p|U(t,0)|p'\rangle = \delta(p-p'-Ft)e^{i(p'^3-p^3)/(6m\hbar F)}.$$

(b) Then, show that

$$K(x,t;x',0)=\langle x|U(t,0)|x'\rangle = \left(\frac{m}{2\pi\hbar it}\right)^{1/2}\exp\left[\frac{i}{\hbar}\left(\frac{m(x-x')^2}{2t}+\frac{1}{2}Ft(x+x')-\frac{F^2t^3}{24m}\right)\right].$$

a.

In momentum space, $\varphi_E(p)$ satisfy the schrodinger equation

$$\begin{aligned}\left(p^2+i\hbar F\frac{\mathrm{d}}{\mathrm{d}p}\right)\varphi_E(p) &= E\varphi_E(p)\\ \Rightarrow i\hbar F\frac{\mathrm{d}\varphi_E}{\mathrm{d}p} &= \left(E-\frac{p^2}{2m}\right)\varphi_E\\ \Rightarrow \varphi_E(p) &= N\exp\left(\frac{i}{\hbar F}\left(Ep-\frac{p^3}{6m}\right)\right).\end{aligned} \tag{1}$$

and the time evolution operator is

$$U(t,0)=e^{-iHt/\hbar}=\int e^{-iEt/\hbar}|E\rangle\langle E|\,\mathrm{d}E \tag{2}$$

$$\begin{aligned}\langle p|U(t,0)|p'\rangle &= \int \mathrm{d}E e^{-iEt/\hbar}\langle p|E\rangle\langle E|p'\rangle\\ &= \int e^{-iEt/\hbar}\varphi_E(p)\varphi_E^*(p)\,\mathrm{d}E\\ &= N^2\int e^{-iEt/\hbar}\exp\left[\frac{i}{\hbar F}\left(Ep-\frac{p^3}{6m}\right)\right]\exp\left[-\frac{i}{\hbar F}\left(Ep'-\frac{p'^3}{6m}\right)\right]\mathrm{d}E\\ &= N^2\exp\left[\frac{i}{6m\hbar F}(p'^3-p^3)\right]\int_{-\infty}^{\infty}\mathrm{d}E\exp\left[\frac{iE}{\hbar F}(p-p'-Ft)\right]\end{aligned} \tag{3}$$

Using the fourier transform identity $\int_{-\infty}^{\infty} e^{ikx}\,\mathrm{d}x = 2\pi\delta(x)$, we have

$$\langle p|U(t,0)|p'\rangle = N^2\exp\left[\frac{i(p'^3-p^3)}{6\hbar mF}\right]2\pi\hbar F\delta(p-p'-Ft). \tag{4}$$

Normalization reqires $N^2=\frac{1}{2\pi\hbar F}$, so we have

$$\boxed{\langle p|U(t,0)|p'\rangle = \exp\left[\frac{i(p'^3-p^3)}{6\hbar mF}\right]\delta(p-p'-Ft).} \tag{5}$$

b.

The propagator

$$K(x,t;x',0)=\langle x|U(t,0)|x'\rangle = \int\langle x|p\rangle\langle p|U|p'\rangle\langle p'|x'\rangle\,\mathrm{d}p\,\mathrm{d}p' \tag{6}$$

Using $\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}}e^{ipx/\hbar}$, we have

$$\begin{aligned}
K &= \int \frac{1}{2\pi\hbar}e^{ipx/\hbar}\exp\Big[\frac{i}{6m\hbar F}(p'^3 - p^3)\Big]\delta(p - p' - Ft)e^{-ip'\frac{x'}{\hbar}}\,\mathrm{d}p\,\mathrm{d}p' \\
&= \frac{1}{2\pi\hbar}\int \mathrm{d}p\exp\Big[\frac{i}{\hbar}(px - (p - Ft)x')\Big]\exp\Big[\frac{i}{6m\hbar F}((p - Ft)^3 - p^3)\Big] \\
&= \frac{1}{2\pi\hbar}\exp\left[\frac{i}{\hbar}\left(Ftx' - \frac{F^2t^3}{6m}\right)\right]\int_{-\infty}^{\infty}\mathrm{d}p\exp\left[\frac{i}{\hbar}\left(-\frac{t}{2m}p^2 + \left(x - x' + \frac{Ft^2}{2m}\right)p\right)\right]
\end{aligned} \tag{7}$$

taking subsitution $u = x - x' + \frac{Ft^2}{2m}$:

$$\begin{aligned}
K &= \frac{m}{2\pi i\hbar t}^{\frac{1}{2}}\exp\left[\frac{i}{\hbar}\left(Ftx' - \frac{F^2t^3}{6m}\right)\right]\exp\left[\frac{i}{\hbar}\frac{mu^2}{2t}\right] \\
&\boxed{= \left(\frac{m}{2\pi i\hbar t}\right)^{\frac{1}{2}}\exp\left[\frac{i}{\hbar}\left(\frac{m}{2t}(x - x')^2 + \frac{1}{2}Ft(x + x') - \frac{F^2t^3}{24m}\right)\right]}
\end{aligned} \tag{8}$$

as desired.

P2

For a one-dimensional simple harmonic oscillator, given an initial localized state

$$\psi(x,0) = \frac{1}{\pi^{1/4}\sqrt{d}} e^{-(x-a)^2/(2d^2)},$$

and the closed-form solution of the propagator, explicitly do the appropriate Gaussian integral to show that the state $\psi(x,t)$ is

$$\psi(x,t) = \frac{1}{\pi^{1/4}\sqrt{d}} \frac{1}{\left(\cos\omega t + \frac{i\hbar}{m\omega d^2}\sin\omega t\right)^{1/2}} e^{\frac{im\omega x^2}{2\hbar\tan\omega t} - \frac{im\omega x a}{\hbar\sin\omega t} + \frac{im\omega a^2}{2\hbar\tan\omega t}} e^{-\frac{(m\omega x/(\hbar\sin\omega t) - am\omega/(\hbar\tan\omega t))^2}{2(1/d^2 - im\omega/(\hbar\tan\omega t))}},$$

and that the probability density $|\psi(x,t)|^2$ is

$$|\psi(x,t)|^2 = \frac{1}{\sqrt{\pi}d} \frac{1}{(\cos^2\omega t + (\hbar/(m\omega d^2))^2 \sin^2\omega t)^{1/2}} e^{-[(x - a\cos\omega t)^2/[d^2(\cos^2\omega t + (\hbar/(m\omega d^2))^2\sin^2\omega t)]]}.$$

The propagated state is given as

$$\psi(x,t) = \int K(x,t;x',0)\psi(x,0)\,\mathrm{d}x', \tag{9}$$

where

$$\psi(x,0) = \frac{1}{\pi^{1/4}\sqrt{d}} \exp[-(x-a)^2/(2d^2)] \tag{10}$$

and

$$K(x,t;x',0) = \left[\frac{m\omega}{2\pi i\hbar\sin\omega t}\right]^{1/2} \exp\left[\frac{im\omega[(x^2+x'^2)]\cos\omega t - 2xx'}{2\hbar\sin\omega t}\right]. \tag{11}$$

All we need to do is to *simply* carry out the gaussian integral:

$$\psi(x,t) = \frac{1}{\pi^{\frac{1}{4}}\sqrt{d}} \left(\frac{m\omega}{2\pi i\hbar\sin\omega t}\right)^{1/2} \int_{-\infty}^{\infty} e^I\,\mathrm{d}x' \tag{12}$$

where

$$I \equiv \frac{im\omega[(x^2+x'^2)\cos\omega t - 2xx']}{2\hbar\sin\omega t} - \frac{(x'-a)^2}{2d^2}. \tag{13}$$

We take the following subsitutions

$$A = \frac{1}{2d^2} - \frac{im\omega\cot\omega t}{2\hbar}, \quad B = \frac{q}{d^2} - \frac{im\omega x}{\hbar\sin\omega t}, \quad C = \frac{im\omega x^2\cos\omega t}{2\hbar\sin\omega t} - \frac{a^2}{2d^2}, \tag{14}$$

to write I in the form of

$$I = -Ax'^2 + Bx' + C \tag{15}$$

and so the gaussian integral term becomes

$$\int_{-\infty}^{\infty} e^I\,\mathrm{d}x' = \int \exp[-Ax'^2 + Bx' + C]\,\mathrm{d}x' = e^C \int \exp[-Ax'^2 + Bx']\,\mathrm{d}x' = \sqrt{\frac{\pi}{A}} \exp\left(\frac{B^2}{4A} + C\right). \tag{16}$$

The wavefunction is retrieved:

$$\psi(x,t) = \frac{1}{\pi^{\frac{1}{4}}\sqrt{d}}\left(\frac{m\omega}{2i\hbar A\sin\omega t}\right)^{\frac{1}{2}}\exp\left(\frac{B^2}{4A}+C\right). \tag{17}$$

Expanding out A, B, C, we have the desired result (which is algebraicly simplified via MATHEMATICA):

$$\boxed{\begin{aligned}&\psi(x,t) = \\ &\frac{1}{\pi^{1/4}\sqrt{d}}\left[\cos\omega t + i\frac{\hbar}{m\omega d^2}\sin\omega t\right]^{-1/2}\exp\left[\frac{im\omega x^2}{2\hbar\tan\omega t}-\frac{im\omega xa}{\hbar\sin\omega t}+\frac{im\omega a^2}{2\hbar\tan\omega t}\right]\exp\left[-\frac{\left(\frac{m\omega x}{\hbar\sin\omega t}-\frac{am\omega}{\hbar\tan\omega t}\right)^2}{2\left(\frac{1}{d^2}-\frac{im\omega}{\hbar\tan\omega t}\right)}\right].\end{aligned}} \tag{18}$$

The probability density is found by $|\psi(x,t)|^2 = \psi(x,t)\cdot\psi^*(x,t)$. The overall phase factors (first exp) contribute nothing to the magnitude. The prefactor's modulus is

$$1/\sqrt{\cos^2\omega t + \left(\frac{\hbar}{m\omega d^2}\right)^2\sin^2\omega t} \tag{19}$$

and the second exp's real part (after expanding the quadratic) gives the Gaussian form centered at $x = a\cos\omega t$ with effective width $d^2\left[\cos^2\omega t + \left(\frac{\hbar}{m\omega d^2}\right)^2\sin^2\omega t\right]$, yielding

$$\boxed{|\psi(x,t)|^2 = \frac{1}{\sqrt{\pi}\,d}\frac{1}{\left(\cos^2\omega t + \left(\frac{\hbar}{m\omega d^2}\right)^2\sin^2\omega t\right)^{1/2}}\exp\left[-\frac{(x-a\cos\omega t)^2}{d^2\left(\cos^2\omega t+\left(\frac{\hbar}{m\omega d^2}\right)^2\sin^2\omega t\right)}\right]} \tag{20}$$

exactly as desired.

P3

Prove that $\langle k_1|k_1'\rangle = \delta(k_1 - k_1')$ for the $E > V_0$ stationary states of the step function potential:

$$\psi_{k_1}(x) = \frac{1}{\sqrt{2\pi}}\left[\theta(-x)\left(e^{ik_1x} + \left(\frac{k_1-k_2}{k_1+k_2}\right)e^{-ik_1x}\right) + \theta(x)\left(\frac{2k_1}{k_1+k_2}\right)e^{ik_2x}\right],$$

in which $k_1 = \sqrt{2mE/\hbar^2}$ and $k_2 = \sqrt{2m(E-V_0)/\hbar^2}$.
Note that the divergent integral $\int_0^\infty e^{ikx}dx$ can be evaluated by considering the $\alpha \to 0$ limit of

$$\int_0^\infty e^{ikx-\alpha x}dx = \frac{\alpha}{\alpha^2+k^2} + \frac{ik}{\alpha^2+k^2},$$

which reduces to

$$\int_0^\infty e^{ikx}dx = \pi\delta(k) + \frac{i}{k}.$$

We rewrite $\langle k_1|k_{1'}\rangle$ by inserting the identity:

$$\langle k_1|k_{1'}\rangle = \int \langle k_1|x\rangle\langle x|k_1'\rangle\,\mathrm{d}x = \int \psi^*_{k_1}(x)\psi_{k_1'}(x)\,\mathrm{d}x \tag{21}$$

The step potential kills cross terms, so

$$\langle k_1|k_1'\rangle = \frac{1}{2\pi}\left[\overbrace{\int_{-\infty}^0\left(e^{-ik_1x} + \frac{k_1-k_2}{k_1+k_2}e^{ik_1x}\right)\left(e^{ik_1'x} + \frac{k_1'-k_2'}{k_1'+k_2'}e^{-ik_1'x}\right)\mathrm{d}x}^{I_1} + \underbrace{\int_0^\infty\left(\frac{2k_1}{k_1+k_2}e^{-ik_2x}\right)\left(\frac{2k_1'}{k_1'+k_2'}e^{ik_2'x}\right)\mathrm{d}x}_{I_2}\right] \tag{22}$$

We evaluate I_1, I_2 separately. But first denote $r = \frac{k_1-k_2}{k_1+k_2}, t = \frac{2k_1}{k_1+k_2}$.

$$\begin{aligned} I_1 &= \int_{-\infty}^0 \left(e^{-ik_1x} + re^{ik_1x}\right)\left(e^{ik_1'x} + r'e^{-ik_1'x}\right)\mathrm{d}x \\ &= \int_0^\infty e^{i(k_1-k_1')x} + re^{-i(k_1+k_1')x} + r'e^{i(k_1+k_1')x} + rr'e^{-i(k_1-k_1')x}\,\mathrm{d}x \end{aligned} \tag{23}$$

Since $E > V_0$, $k_1 > 0$ and so all $\delta(k_1+k_1') = 0$. This gives

$$I_1 = \pi(1+rr')\delta(k_1-k_1') + (1-rr')\frac{i}{k_1-k_1'}. \tag{24}$$

I_2 is more direct:

$$I_2 = \int_0^\infty t\left(e^{-ik_2x}\right)\left(t'e^{ik_2'x}\right)\mathrm{d}(x) = tt'\int_0^\infty e^{-i(k_2-k_2')x}\,\mathrm{d}x = tt'(\pi\delta(k_2-k_2') - \frac{i}{k_2-k_2'} \tag{25}$$

We can combine $I_1 + I_2$, but first notice that

$$\delta(k_2-k_2') = \frac{k_2}{k_1}\delta(k_1-k_1'); \quad k_2 - k_2' = \frac{k_1}{k_2}(k_1-k_1') \tag{26}$$

then

$$I_1 + I_2 = \frac{1}{2\pi}\left[\pi\left(1 + rr' + tt'\frac{k_2}{k_1}\right)\delta(k_1-k_1') + \left(1 - rr' - tt'\frac{k_2}{k_1}\frac{i}{k_1-k_1'}\right)\right] \tag{27}$$

Further, since $t = 1_r \quad \Rightarrow rr' + tt'\frac{k_2}{k_1} = 1$, so we arrive at $\langle k_1|k_1'\rangle = \delta(k_1-k_1')$.

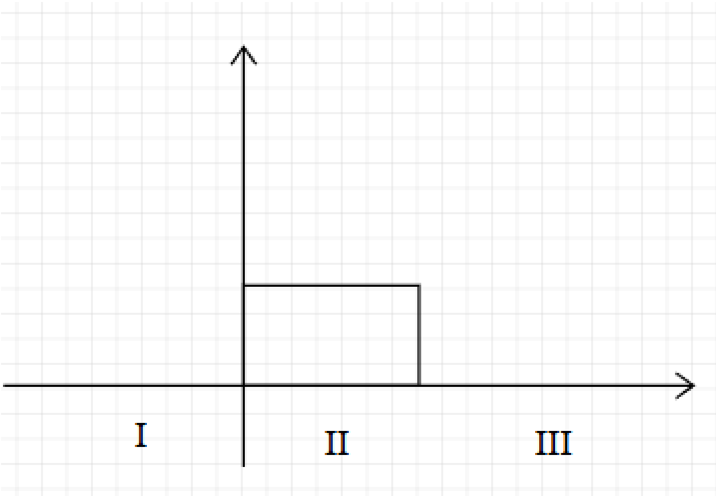
P4

A particle with mass m and energy $E > 0$ is incident from the left on a potential of the form

$$V(x) = \begin{cases} V_0, & 0 < x < a \\ 0, & x < 0,\ x > a \end{cases}$$

in which $V_0 > 0$. Assuming that all conditions hold such that the plane-wave approximation can be used (*i.e.*, the wavepacket is sufficiently localized in momentum space, etc.), compute R and T and verify that $R + T = 1$ for two cases: (i) $E > V_0$, and (ii) $E < V_0$.

1. $E > V_0$



Consider$E > V_0$ first. We take the following ansatz for wavefunction in each region:

$$\psi(x) = \begin{cases} e^{ikx} + re^{-ikx} & \text{I} \\ Ce^{iqx} + De^{-iqx} & \text{II} \\ te^{ikx} & \text{III} \end{cases} \tag{28}$$

where $k = \sqrt{2mE}/\hbar$ and $q = \sqrt{2m(E - V_0)}/\hbar$

Matching the boundary conditions and cotinuity at $x = 0, a$:

$$x = 0: \quad \begin{cases} 1 + r = C + D \\ k(1 - r) = q(C - D) \end{cases} \Rightarrow \begin{pmatrix} C \\ D \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 + \frac{k}{q} & 1 - \frac{k}{q} \\ 1 - \frac{k}{q} & 1 + \frac{k}{q} \end{pmatrix}\begin{pmatrix} 1 \\ r \end{pmatrix} \tag{29}$$

$$x = a: \quad Ce^{iqa} + De^{-iqa} = te^{ika} \Rightarrow \begin{pmatrix} Ce^{iqa} \\ De^{-iqa} \end{pmatrix} = \frac{te^{ika}}{2}\begin{pmatrix} 1 + \frac{k}{q} \\ 1 - \frac{k}{q} \end{pmatrix}. \tag{30}$$

We multiply Equation 29 by $\text{diag}(e^{iqa}, De^{iqa})$ and we can equate the two boudary conditions to get

$$\frac{1}{2}\begin{pmatrix} e^{iqa}\left(1 + \frac{k}{q}\right) & e^{iqa}\left(1 - \frac{k}{q}\right) \\ e^{-iqa}\left(1 - \frac{k}{q}\right) & e^{-iqa}\left(1 + \frac{k}{q}\right) \end{pmatrix}\begin{pmatrix} 1 \\ r \end{pmatrix} = \frac{te^{ika}}{2}\begin{pmatrix} 1 + \frac{k}{q} \\ 1 - \frac{k}{q} \end{pmatrix}. \tag{31}$$

Expanding this matrix equaiton, rearranging the terms, we have

$$\begin{aligned} e^{iqa}\left(1 + \frac{k}{q}\right) + e^{iqa}\left(1 - \frac{k}{q}\right)r = te^{ika}\left(1 + \frac{k}{q}\right); \\ e^{-iqa}\left(1 - \frac{k}{q}\right) + e^{-iqa}\left(1 + \frac{k}{q}\right)r = te^{ika}\left(1 - \frac{k}{q}\right). \end{aligned} \tag{32}$$

From which we can solve for the transmission amplitude t as

$$t = \frac{\left(e^{-ika}\right)}{\left(\cos(qa) - i\dfrac{k^2 + q^2}{2kq}\sin(qa)\right)} \tag{33}$$

Similarly, we can solve for the reflection amplitude r as

$$r = \frac{\left(-i\dfrac{k^2-q^2}{2kq}\sin(qa)\right)}{\left(\cos(qa) - i\dfrac{k^2+q^2}{2kq}\sin(qa)\right)} \tag{34}$$

and the transmission and reflection coefficients T and R as

$$T = |t|^2 = \frac{1}{\cos^2(qa) + \left(\frac{k^2+q^2}{2kq}\right)^2 \sin^2(qa)} \tag{35}$$

$$R = |r|^2 = \frac{\left(\frac{k^2-q^2}{2kq}\right)^2 \sin^2(qa)}{\cos^2(qa) + \left(\frac{k^2+q^2}{2kq}\right)^2 \sin^2(qa)} \tag{36}$$

We can easily verify (using MATHEMATICA or otherwise) that $T + R = 1$ as required by probability conservation.

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2. $E < V_0$

Now take $E < V_0$. In Region II the solution is evanescent, so we use

$$\psi(x) = \begin{cases} e^{ikx} + re^{-ikx} & \text{I} \\ Ce^{\kappa x} + De^{-\kappa x} & \text{II} \\ te^{ikx} & \text{III} \end{cases} \tag{37}$$

with $k = \sqrt{2mE}/\hbar$ and $\kappa = \sqrt{2m(V_0 - E)}/\hbar$.

Matching the boundary conditions and continuity at $x = 0, a$:

$$x = 0: \quad \begin{cases} 1 + r = C + D \\ \quad ik(1-r) = \kappa(C - D) \end{cases} \Rightarrow \begin{pmatrix} C \\ D \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 - i\frac{k}{\kappa} & 1 + i\frac{k}{\kappa} \\ 1 + i\frac{k}{\kappa} & 1 - i\frac{k}{\kappa} \end{pmatrix}\begin{pmatrix} 1 \\ r \end{pmatrix} \tag{38}$$

$$x = a: \quad Ce^{\kappa a} + De^{-\kappa a} = te^{ika} \Rightarrow \begin{pmatrix} Ce^{\kappa a} \\ De^{-\kappa a} \end{pmatrix} = \frac{te^{ika}}{2}\begin{pmatrix} 1 - i\frac{k}{\kappa} \\ 1 + i\frac{k}{\kappa} \end{pmatrix}. \tag{39}$$

Multiply Equation 38 by $\mathrm{diag}(e^{\kappa a}, e^{-\kappa a})$ and equate the two boundary relations to get

$$\frac{1}{2}\begin{pmatrix} e^{\kappa a}\left(1 - i\frac{k}{\kappa}\right) & e^{\kappa a}\left(1 + i\frac{k}{\kappa}\right) \\ e^{-\kappa a}\left(1 + i\frac{k}{\kappa}\right) & e^{-\kappa a}\left(1 - i\frac{k}{\kappa}\right) \end{pmatrix}\begin{pmatrix} 1 \\ r \end{pmatrix} = \frac{te^{ika}}{2}\begin{pmatrix} 1 - i\frac{k}{\kappa} \\ 1 + i\frac{k}{\kappa} \end{pmatrix}. \tag{40}$$

Expanding the matrix equation and rearranging to isolate t and r yields

$$e^{\kappa a}\left(1 - i\frac{k}{\kappa}\right) + e^{\kappa a}\left(1 + i\frac{k}{\kappa}\right)r = te^{ika}\left(1 - i\frac{k}{\kappa}\right), \tag{41}$$

$$e^{-\kappa a}\left(1 + i\frac{k}{\kappa}\right) + e^{-\kappa a}\left(1 - i\frac{k}{\kappa}\right)r = te^{ika}\left(1 + i\frac{k}{\kappa}\right). \tag{42}$$

From these, the transmission amplitude is

$$t = \frac{\left(e^{-ika}\right)}{\left(\cosh(\kappa a) - i\dfrac{k^2-\kappa^2}{2k\kappa}\sinh(\kappa a)\right)}. \tag{43}$$

Also, the reflection amplitude is

$$r = \frac{\left(-i\dfrac{k^2+\kappa^2}{2k\kappa}\sinh(\kappa a)\right)}{\left(\cosh(\kappa a) - i\dfrac{k^2-\kappa^2}{2k\kappa}\sinh(\kappa a)\right)}. \tag{44}$$

Therefore the transmission and reflection coefficients are

$$T = |t|^2 = \frac{1}{\cosh^2(\kappa a) + \left(\frac{k^2-\kappa^2}{2k\kappa}\right)^2 \sinh^2(\kappa a)} = \frac{1}{1 + \left(\frac{(k^2+\kappa^2)^2}{4k^2\kappa^2}\right)\sinh^2(\kappa a)}, \tag{45}$$

$$R = |r|^2 = \frac{\left(\frac{k^2+\kappa^2}{2k\kappa}\right)^2 \sinh^2(\kappa a)}{\cosh^2(\kappa a) + \left(\frac{k^2-\kappa^2}{2k\kappa}\right)^2 \sinh^2(\kappa a)} = \frac{\left(\frac{(k^2+\kappa^2)^2}{4k^2\kappa^2}\right)\sinh^2(\kappa a)}{1 + \left(\frac{(k^2+\kappa^2)^2}{4k^2\kappa^2}\right)\sinh^2(\kappa a)}. \tag{46}$$

As a check, one verifies $T + R = 1$,as expected.

P5

Consider the spin precession problem with Hamiltonina

$$H = -\frac{eB}{mc} S_z \equiv \omega S_z. \tag{47}$$

Write Heisenberg equations of motino for the time dependent operators $S_x(t), S_y(t), S_z(t)$, and solve them to find $S_x(t), S_y(t), S_z(t)$ as functions of time.

Since the opservables has no explicit time dependence, we have, for some observable A,

$$\dot{A} = \frac{1}{i\hbar}[A, H] \tag{48}$$

So

$$\begin{aligned} \dot{S_x} &= \frac{i}{i\hbar}[S_x, \omega S_z] = \frac{\omega}{i\hbar}[S_x, S_z] = -\omega S_y, \\ \dot{S_y} &= \frac{1}{i\hbar}\left[S_y, \omega S_x\right] = \omega S_x, \\ \dot{S_z} &= \frac{\omega}{i\hbar}[S_z, S_z] = 0. \end{aligned} \tag{49}$$

We can immediately read off that $S_z(t) = S_z(0)$. To solve for $S_x(t)$ (and $S_y(t)$), we take another time derivative on $\dot{S_y}$ (or $\cdot\,(S_x)$) to get

$$\ddot{S_x} = -\omega^2 S_x; \quad \ddot{S_y} = -\omega^2 S_y \tag{50}$$

which are solved to be

$$\begin{aligned} S_x(t) &= c_1 \cos\omega t + c_2 \sin\omega t, \\ S_y(t) &= c_3 \cos\omega t + c_4 \sin\omega t. \end{aligned} \tag{51}$$

Admitting initial conditons of

$$\begin{aligned} S_x(0) = c_1, &\quad \dot{S_x}(0) = c_2\omega = -\omega S_y(0); \\ S_y(0) = c_3, &\quad \dot{S_y}(0) = c_4\omega = \omega S_x(0), \end{aligned} \tag{52}$$

we will have the solutions as

$$\boxed{\begin{aligned} S_x(t) &= S_x(0)\cos\omega t - S_y(0)\sin\omega t, \\ S_y(t) &= S_y(0)\cos\omega t + S_x(0)\sin\omega t. \end{aligned}} \tag{53}$$

P6

Let $x(t)$ be the coordinate operator for a free particle in one dimension in the Heisenberg picture. Evaluate

$$[x(t), x(0)]. \tag{54}$$

For a free particle in 1D, the Hamiltonian is $H = \frac{p^2}{2m}$. Since p commutes with H, we have

$$\dot{p} = \frac{1}{i\hbar}[p, H] = 0. \tag{55}$$

While

$$\dot{x} = \frac{1}{i\hbar}[x, H] = \frac{1}{i\hbar}\left[x, \frac{p^2}{2m}\right] = \frac{1}{i\hbar}\frac{1}{2m}i\hbar\frac{\partial p^2}{\partial p} = \frac{p}{m} = \frac{p(0)}{m}, \tag{56}$$

where we used the fact that $[x, F(\vec{p})] = i\hbar\partial_p F$.

Therefore we have the solution for $x(t)$ as

$$x(t) = x(0) + \frac{p(0)}{m}t, \tag{57}$$

which mirrors the classical trajectory of a free particle in uniform motion.

We can then evaluate the commutator as

$$[x(t), x(0)] = \left[x(0) + \frac{p(0)}{m}t, x(0)\right] = \frac{1}{m}[p(0)t, x(0)] = \boxed{-\frac{i\hbar}{m}t} \tag{58}$$