Physics 406 notes

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Abstract

These collected notes cover three related topics in mathematical physics. We begin with an introduction to dimensional analysis via the Buckingham Π-Theorem, applying it to derive the fundamental scaling law and Green's function for the diffusion equation. We then explore a series of advanced applications of dimensional scaling, from electromagnetism and blast waves to quantum field theory and black hole evaporation. Finally, we introduce several powerful asymptotic methods for approximating integrals, including Laplace's method, the method of stationary phase, and the method of steepest descent, with applications to Stirling's formula and the asymptotic forms of special functions.

1 Dimensional Analysis and the Buckingham Π-Theorem

Dimensional analysis is a powerful technique used to deduce relationships between physical quantities by considering their fundamental units (such as length $[L]$, mass $[M]$, and time $[T]$). It is particularly useful for checking the plausibility of equations and for deriving scaling laws in problems where the exact analytical solution is unknown or overly complex.

The underlying principle is that any physically meaningful equation must be dimensionally homogeneous, meaning the dimensions on both sides of the equation must be the same. The formal basis for this technique is the Buckingham Π-Theorem.

Theorem 1.1 (Buckingham Π-Theorem)**.** *Let a physical quantity Q be a function of n other physical variables $V_1, V_2, \ldots, V_n$. The relationship can be written as $Q = f(V_1, V_2, \ldots, V_n)$. If these $n+1$ quantities can be described using k independent fundamental units, then the original equation can be restated as an equation of $p = n+1-k$ dimensionless parameters $\Pi_1, \Pi_2, \ldots, \Pi_p$, constructed from the original variables. The new relationship takes the form $F(\Pi_1, \Pi_2, \ldots, \Pi_p) = 0$.*

Sketch of Proof. Let the k independent units be denoted by $U_1, \ldots, U_k$. The dimension of any variable V_j can be expressed as a product of powers of these units: $[V_j] = U_1^{a_1} U_2^{a_2} \cdots U_k^{a_k}$.

The core idea is that the laws of physics are independent of the system of units chosen. Let's form a dimensionless parameter $\Pi_1 = V_1^{x_1} V_2^{x_2} \cdots V_n^{x_n}$. For Π_1 to be dimensionless, the sum of the exponents for each fundamental unit must be zero. This yields a system of k linear equations for the n exponents x_j. Since there are more variables (n) than equations (k), there are $n-k$ free parameters, leading to $n-k$ independent dimensionless groups that can be formed.

If we change the scale of a fundamental unit (e.g., multiply all quantities with a length dimension by a factor λ), the physical law must remain valid. The dimensionless Π groups are invariant under such a transformation. The original function f must be structured in such a way that it only depends on these dimensionless combinations, otherwise the physical relationship would change with the unit system, which is a physical contradiction. □

2 Application: The 1D Diffusion Equation

We now apply these principles to the problem of particle diffusion.

2.1 Scaling Analysis of Diffusion

Consider a particle diffusing from an origin. We want to find how the average distance traveled, $\langle x \rangle$, depends on time, t, and the diffusion coefficient, D.

First, we must determine the dimensions of D. It is defined by Fick's Law, which relates the particle flux j (particles per unit area per unit time) to the gradient of the particle concentration n (particles per unit volume):

$$\vec{j} = -D\vec{\nabla}n$$

Analyzing the dimensions gives:

$$[j] = \frac{1}{[L]^2[T]}, \quad [\vec{\nabla}] = \frac{1}{[L]}, \quad [n] = \frac{1}{[L]^3}$$

$$\frac{1}{[L]^2[T]} = [D]\frac{1}{[L]}\frac{1}{[L]^3} \implies [D] = \frac{[L]^2}{[T]}$$

We postulate a relationship of the form $\langle x \rangle = C \cdot D^\alpha t^\beta$, where C is a dimensionless constant. In terms of units:

$$[L] = \left(\frac{[L]^2}{[T]}\right)^\alpha [T]^\beta = [L]^{2\alpha}[T]^{\beta-\alpha}$$

For this equation to be dimensionally homogeneous, the exponents of each unit must match:

$$\begin{aligned} &\text{For } [L]: \quad 1 = 2\alpha \implies \alpha = 1/2 \\ &\text{For } [T]: \quad 0 = \beta - \alpha \implies \beta = \alpha = 1/2 \end{aligned}$$

This reveals the fundamental scaling law for diffusion:

$$\boxed{\langle x \rangle \sim \sqrt{Dt}} \tag{1}$$

This result shows that the characteristic length scale of a diffusion process grows with the square root of time.

2.2 Deriving the Green's Function via Self-Similarity

Let's find the concentration profile $n(x,t)$ for a single particle starting at the origin at $t = 0$ (a delta function initial condition). The governing PDE is the diffusion equation, which arises from combining Fick's Law with the continuity equation, $\partial_t n + \nabla \cdot \vec{j} = 0$:

$$\frac{\partial n}{\partial t} = D\frac{\partial^2 n}{\partial x^2}$$

Our scaling analysis suggests that the physics of the problem depends on the dimensionless variable $z = x/\sqrt{Dt}$. We therefore seek a **self-similar solution** (an ansatz) of the form:

$$n(x,t) = t^{-\beta} f(z) \quad \text{where} \quad z = \frac{x}{\sqrt{Dt}}$$

We need to substitute this into the PDE. First, we calculate the derivatives using the chain rule:

$$\begin{aligned} \frac{\partial n}{\partial t} &= -\beta t^{-\beta-1} f(z) + t^{-\beta} f'(z)\frac{\partial z}{\partial t} \\ \text{where } \frac{\partial z}{\partial t} &= x(Dt)^{-3/2}\left(-\frac{D}{2}\right) = -\frac{x}{2t\sqrt{Dt}} = -\frac{z}{2t} \\ \implies \frac{\partial n}{\partial t} &= -\beta t^{-\beta-1} f(z) - t^{-\beta} f'(z)\frac{z}{2t} = t^{-\beta-1}\left(-\beta f(z) - \frac{z}{2}f'(z)\right) \end{aligned}$$

Next, the spatial derivative:

$$\frac{\partial n}{\partial x} = t^{-\beta} f'(z) \frac{\partial z}{\partial x} = t^{-\beta} f'(z) \frac{1}{\sqrt{Dt}}$$
$$\frac{\partial^2 n}{\partial x^2} = t^{-\beta} \frac{1}{\sqrt{Dt}} f''(z) \frac{\partial z}{\partial x} = t^{-\beta} \frac{1}{Dt} f''(z) = \frac{1}{D} t^{-\beta-1} f''(z)$$

Substituting these into $\partial_t n = D\partial_x^2 n$:

$$t^{-\beta-1} \left(-\beta f(z) - \frac{z}{2} f'(z) \right) = D \left(\frac{1}{D} t^{-\beta-1} f''(z) \right)$$

The factors of $t^{-\beta-1}$ cancel, reducing the PDE to an ODE for $f(z)$:

$$f''(z) + \frac{z}{2} f'(z) + \beta f(z) = 0$$

To find β, we impose the physical condition of particle conservation: the total number of particles must be constant for all time. For a single particle source, this is 1.

$$\int_{-\infty}^{\infty} n(x,t)\, dx = \int_{-\infty}^{\infty} t^{-\beta} f(z)\, dx = 1$$

Changing variables to z (where $dx = \sqrt{Dt}\, dz$):

$$\int_{-\infty}^{\infty} t^{-\beta} f(z) (\sqrt{Dt}\, dz) = t^{-\beta+1/2} \sqrt{D} \int_{-\infty}^{\infty} f(z)\, dz = 1$$

For this relation to hold for all t, the time dependence must vanish. Thus, $-\beta + 1/2 = 0 \implies \beta = 1/2$. Our ODE becomes:

$$f''(z) + \frac{z}{2} f'(z) + \frac{1}{2} f(z) = 0$$

The solution to this ODE is a Gaussian function, $f(z) = Ae^{-z^2/4}$. We verify: $f' = -\frac{z}{2} f$ and $f'' = (-\frac{1}{2} + \frac{z^2}{4}) f$. Substitution yields:

$$\left(-\frac{1}{2} + \frac{z^2}{4} \right) f + \frac{z}{2} \left(-\frac{z}{2} f \right) + \frac{1}{2} f = \left(-\frac{1}{2} + \frac{z^2}{4} - \frac{z^2}{4} + \frac{1}{2} \right) f = 0$$

The solution for $n(x,t)$ is therefore:

$$n(x,t) = \frac{A}{\sqrt{t}} \exp\left(-\frac{x^2}{4Dt} \right)$$

The normalization constant A is found from the conservation integral with $\beta = 1/2$:

$$t^{-1/2} \sqrt{D} \int_{-\infty}^{\infty} Ae^{-z^2/4}\, dz = A\sqrt{D}\sqrt{4\pi} = 1 \implies A = \frac{1}{\sqrt{4\pi D}}$$

This gives the fundamental solution, or **Green's function**, for the 1D diffusion equation:

$$G(x,t) = \frac{1}{\sqrt{4\pi Dt}} \exp\left(-\frac{x^2}{4Dt} \right) \tag{2}$$

2.3 The General Solution

Since the diffusion equation is linear, the superposition principle applies. The solution for an arbitrary initial concentration profile $n(x,0) = n_0(x)$ can be found by convolving the initial condition with the Green's function:

$$n(x,t) = \int_{-\infty}^{\infty} G(x-x',t) n_0(x')\, dx' = \frac{1}{\sqrt{4\pi Dt}} \int_{-\infty}^{\infty} \exp\left(-\frac{(x-x')^2}{4Dt} \right) n_0(x')\, dx'$$

3 A Tale of Two Systems: SI and CGS Units in Electromagnetism

A clear understanding of unit systems is essential in physics, particularly in electromagnetism where the choice of system fundamentally alters the form of the governing equations.

3.1 Fundamental Units and Derived Constants

The **SI system** is built on seven base units, including the meter (m), kilogram (kg), second (s), and ampere (A) for electric current. Because current is a base unit, two dimensional constants must be introduced to relate electromagnetic phenomena to mechanical ones:

- The vacuum permittivity, $\epsilon_0 \approx 8.854 \times 10^{-12}\,\mathrm{F/m}$.
- The vacuum permeability, $\mu_0 = 4\pi \times 10^{-7}\,\mathrm{H/m}$.

In this system, Coulomb's Law takes the familiar form: $F = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r^2}$.

The **CGS (Gaussian)** system is built on just three base units: the centimeter (cm), gram (g), and second (s). Charge is a derived unit, defined directly by Coulomb's Law, which is written without dimensional constants:

$$F = \frac{q_1 q_2}{r^2}$$

This definition implies that the dimension of charge is $[Q] = [M]^{1/2}[L]^{3/2}[T]^{-1}$.

3.2 Maxwell's Equations and the Lorentz Force

The choice of unit system directly impacts the appearance of Maxwell's equations and the Lorentz force law, primarily through the inclusion or exclusion of factors of c, 4π, ϵ_0, and μ_0. A comparison is shown below.

SI System	**CGS (Gaussian) System**
$\nabla \cdot \vec{D} = \rho$	$\nabla \cdot \vec{E} = 4\pi\rho$
$\nabla \cdot \vec{B} = 0$	$\nabla \cdot \vec{B} = 0$
$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$	$\nabla \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$
$\nabla \times \vec{H} = \vec{j} + \frac{\partial \vec{D}}{\partial t}$	$\nabla \times \vec{B} = \frac{4\pi}{c}\vec{j} + \frac{1}{c}\frac{\partial \vec{E}}{\partial t}$
$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$	$\vec{F} = q\left(\vec{E} + \frac{\vec{v}}{c} \times \vec{B}\right)$

4 Further Applications of Dimensional Analysis

4.1 The Physical Pendulum

While the period of a simple pendulum is $T \sim \sqrt{l/g}$, a physical pendulum with mass m, moment of inertia I, and pivot-to-CM distance l presents a more complex problem. We have 4 variables $\{m, I, l, g\}$ and 3 base units $\{M, L, T\}$. The Buckingham Π-theorem implies the solution depends on one dimensionless parameter, $\Pi = I/(ml^2)$. The period must therefore take the form:

$$T = \sqrt{\frac{l}{g}} \cdot F\left(\frac{I}{ml^2}\right)$$

To find the unknown function $F(x)$, we invoke a physical argument. The equation of motion is $I\ddot{\theta} = \tau(\theta, m, g, l)$, where the torque τ does not depend on I. For small oscillations, $\ddot{\theta} \approx -\omega^2\theta$. This implies that $I\omega^2$ must be a function of $\{m, g, l\}$. Since $T = 2\pi/\omega$, it follows that $T^2 \propto I$. This can only be true if our scaling function $F(x) \propto \sqrt{x}$. Inserting this gives the final result:

$$T \propto \sqrt{\frac{l}{g}}\sqrt{\frac{I}{ml^2}} = \sqrt{\frac{I}{mgl}}$$

4.2 Tidal Forces

The ocean tides are not caused by the Moon's gravitational pull itself, but by the *gradient* of its pull across the Earth. The differential tidal force on a mass of water δM is $F_T \sim GM_M \delta M R_E / R^3$. The work done by this force in lifting the water is balanced by the potential energy gained in Earth's gravitational field, $\delta E_p = \delta M g h$.

$$\begin{aligned}
\text{Work Done} &\approx \text{Potential Energy Gained} \\
F_T \cdot R_E &\approx \delta M g h \\
\left(G \frac{M_M \delta M R_E}{R^3} \right) R_E &\approx \delta M \left(\frac{GM_E}{R_E^2} \right) h \\
h &\approx \frac{M_M}{M_E} \frac{R_E^4}{R^3} \approx 0.5 \, \text{meters}
\end{aligned}$$

4.3 Blast Wave Radius (G.I. Taylor Problem)

The radius R of a hemispherical shockwave from a powerful explosion depends on the energy released E, the time elapsed t, and the density of the surrounding air ρ. We seek a relationship $R \propto E^\alpha \rho^\beta t^\gamma$. Matching the dimensions:

$$[L] = \left(\frac{[M][L]^2}{[T]^2} \right)^\alpha \left(\frac{[M]}{[L]^3} \right)^\beta [T]^\gamma = [M]^{\alpha+\beta} [L]^{2\alpha - 3\beta} [T]^{-2\alpha+\gamma}$$

Solving the system of linear equations for the exponents yields $\alpha = 1/5, \beta = -1/5, \gamma = 2/5$. This gives the famous Taylor-Sedov scaling law:

$$R(t) \propto \left(\frac{Et^2}{\rho} \right)^{1/5}$$

4.4 Schwinger Effect: Pair Production in a Strong Field

A sufficiently strong electric field can tear electron-positron pairs from the vacuum. We can estimate the critical field strength E_c by postulating that the work done by the field over the electron's Compton wavelength, $\lambda_c = \hbar / m_e c$, must be on the order of the pair's rest energy, $2m_e c^2$.

$$\begin{aligned}
eE_c \lambda_c &\sim m_e c^2 \\
eE_c \left(\frac{\hbar}{m_e c} \right) &\sim m_e c^2 \implies \boxed{E_c \sim \frac{m_e^2 c^3}{e\hbar}}
\end{aligned}$$

This is a non-perturbative quantum effect, as the expression is not analytic in the coupling constant e.

4.5 Thermonuclear Fusion and the Gamow Peak

The rate of fusion in a star is a delicate balance between two opposing factors:

1. The Maxwell-Boltzmann distribution of particle energies, $f(E) \propto \exp(-E/k_B T)$, which means very few particles have high energy.
2. The quantum tunneling probability through the Coulomb barrier, $P(E) \propto \exp(-\sqrt{E_G/E})$, which is negligible for low-energy particles. Here E_G is the Gamow energy, a constant that depends on the charges and masses of the fusing nuclei.

The overall reaction rate is proportional to the product of these two exponentials, $\exp(-E/k_B T - \sqrt{E_G/E})$. This function has a sharp maximum at an optimal energy known as the **Gamow Peak**, $E_0 = (E_G (k_B T)^2 / 4)^{1/3}$. It is only within this narrow window of energies that fusion can occur efficiently, which is why stars require such high core temperatures to ignite.

4.6 Quantum of Resistance

In CGS units, electrical resistance has the dimensions of inverse velocity, $[R] = [T]/[L]$. This curious result can be given a profound physical meaning by re-expressing the dimensions:

$$[R] = \frac{[T]}{[L]} = \frac{[M][L][T]}{[M][L]^2} = \frac{[\text{Momentum}][\text{Length}]}{[\text{Charge}]^2} = \frac{[\text{Action}]}{[\text{Charge}]^2}$$

In quantum mechanics, the fundamental quantum of action is Planck's constant h, and the fundamental quantum of charge is the elementary charge e. This allows us to construct a fundamental unit of resistance from these constants:

$$R_K = \frac{h}{e^2} \approx 25,812.8\,\Omega$$

This value is the von Klitzing constant, central to the Quantum Hall Effect.

4.7 Hawking Radiation and Black Hole Evaporation

We can estimate the lifetime of a black hole of mass M using only dimensional analysis. The process involves general relativity (G), quantum mechanics ($\hbar$), and thermodynamics (k_B), linked by the speed of light (c).

1. **Hawking Temperature**: The only way to construct a temperature from the fundamental constants is:
$$k_B T_H \sim \frac{\hbar c^3}{GM}$$
2. **Radiated Power**: The black hole radiates as a black body with power given by the Stefan-Boltzmann law, $P \propto A T_H^4$. The area is that of the event horizon, $A \propto R_s^2 \propto (GM/c^2)^2$.
$$P \propto \left(\frac{G^2M^2}{c^4}\right)\left(\frac{\hbar c^3}{GMk_B}\right)^4 \propto \frac{\hbar c^6}{G^2M^2}$$
3. **Lifetime**: The lifetime τ ends when all mass-energy has been radiated away, so $P \cdot \tau \sim Mc^2$.
$$\frac{\hbar c^6}{G^2M^2} \cdot \tau \sim Mc^2 \implies \boxed{\tau \sim \frac{G^2M^3}{\hbar c^4}}$$

The lifetime scales with the cube of the mass, meaning small black holes evaporate explosively while large ones live for eons.

5 Introduction to Asymptotic Methods: The Kramers-Kronig Relations

In physics, response functions (like susceptibility or conductivity) are often complex-valued, where the real part describes dispersion and the imaginary part describes absorption. For any causal and linear response function $f(\omega)$ that is analytic in the upper half-plane, its real and imaginary parts are not independent. They are related by the **Kramers-Kronig relations**, a consequence of Cauchy's integral formula:

$$\operatorname{Re} f(\omega) = \frac{1}{\pi}\mathcal{P}\int_{-\infty}^{+\infty} \frac{\operatorname{Im} f(\omega')}{\omega' - \omega} d\omega' \tag{3}$$

$$\operatorname{Im} f(\omega) = -\frac{1}{\pi}\mathcal{P}\int_{-\infty}^{+\infty} \frac{\operatorname{Re} f(\omega')}{\omega' - \omega} d\omega' \tag{4}$$

where $\mathcal{P}$ denotes the Cauchy principal value. This connection motivates the use of complex analysis to understand physical integrals, a theme central to the method of steepest descent.

6 Laplace's Method for Real Integrals

Laplace's method is used to find the leading asymptotic behavior of integrals of the form

$$F(\lambda) = \int_a^b \varphi(t) e^{\lambda f(t)} dt \quad \text{for } \lambda \to \infty$$

The core idea is that for a large λ, the exponential function $e^{\lambda f(t)}$ is sharply peaked around the global maximum of $f(t)$. The value of the integral is therefore dominated entirely by the behavior of the functions $f(t)$ and $\varphi(t)$ in the immediate vicinity of this maximum.

Theorem 6.1 (Laplace's Method). *Let $f(t)$ be a twice-differentiable function with a unique global maximum at an interior point $t_0 \in (a, b)$, such that $f'(t_0) = 0$ and $f''(t_0) < 0$. Then, for $\lambda \to \infty$:*

$$\int_a^b \varphi(t) e^{\lambda f(t)} dt \approx \varphi(t_0) e^{\lambda f(t_0)} \sqrt{\frac{-2\pi}{\lambda f''(t_0)}}$$

Sketch of Proof. We expand $f(t)$ in a Taylor series around t_0: $f(t) \approx f(t_0) + \frac{1}{2} f''(t_0)(t - t_0)^2$. Since $f''(t_0)$ is negative, the integral becomes a Gaussian integral, which can be evaluated directly.

$$F(\lambda) \approx \varphi(t_0) e^{\lambda f(t_0)} \int_{-\infty}^{\infty} e^{\frac{\lambda}{2} f''(t_0)(t-t_0)^2} dt = \varphi(t_0) e^{\lambda f(t_0)} \sqrt{\frac{2\pi}{-\lambda f''(t_0)}}$$

□

6.1 Example: Stirling's Approximation

The Gamma function is defined by $\Gamma(z+1) = \int_0^\infty t^z e^{-t} dt$. To find its asymptotic form for large z, we write it as:

$$\Gamma(z+1) = \int_0^\infty e^{z \ln t - t} dt$$

To match the standard form, we set $t = zx$:

$$\Gamma(z+1) = \int_0^\infty e^{z \ln(zx) - zx} (z\, dx) = z^{z+1} \int_0^\infty e^{z(\ln x - x)} dx$$

The exponent is $f(x) = \ln x - x$. Its maximum is at $f'(x) = 1/x - 1 = 0 \implies x_0 = 1$. The second derivative is $f''(x) = -1/x^2$, so $f''(1) = -1$. Applying Laplace's method with $\lambda = z$, $\varphi(x) = 1$, $f(1) = -1$:

$$\Gamma(z+1) \approx z^{z+1} \cdot e^{z(-1)} \cdot \sqrt{\frac{-2\pi}{z(-1)}} = z^{z+1} e^{-z} \frac{\sqrt{2\pi}}{\sqrt{z}} = \sqrt{2\pi z} \left(\frac{z}{e}\right)^z$$

This is the celebrated **Stirling's approximation**.

7 Methods for Oscillatory Integrals

7.1 The Method of Stationary Phase

This method is the counterpart to Laplace's method for oscillatory integrals of the form

$$I(\lambda) = \int_a^b \phi(x) e^{i\lambda f(x)} dx \quad \text{for } \lambda \to \infty$$

The intuition is that for large λ, the integrand $e^{i\lambda f(x)}$ oscillates extremely rapidly. These oscillations tend to cancel each other out through destructive interference, except in the vicinity of points where the phase $f(x)$ is stationary, i.e., where $f'(x_0) = 0$.

Theorem 7.1 (Method of Stationary Phase)**.** *Let $f(x)$ have a single stationary point at $x_0 \in (a,b)$ where $f'(x_0) = 0$ and $f''(x_0) \neq 0$. Then, for $\lambda \to \infty$:*

$$I(\lambda) \approx \phi(x_0)e^{i\lambda f(x_0)}\sqrt{\frac{2\pi i}{\lambda f''(x_0)}} = \phi(x_0)e^{i\lambda f(x_0)}e^{i\frac{\pi}{4}\operatorname{sgn}(f''(x_0))}\sqrt{\frac{2\pi}{\lambda|f''(x_0)|}}$$

7.2 Example: The Bessel Function $J_n(x)$

For large x, the Bessel function $J_n(x)$ can be approximated from its integral representation:

$$J_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(x\sin\phi - n\phi)}d\phi$$

Here, the large parameter is $\lambda = x$ and the phase is $f(\phi) = \sin\phi$. The stationary points are where $f'(\phi) = \cos\phi = 0 \implies \phi_0 = \pm\pi/2$. We sum the contributions from each point.

- For $\phi_0 = \pi/2$: $f(\pi/2) = 1$, $f''(\pi/2) = -1$. Contribution is $\frac{1}{\sqrt{2\pi x}}e^{i(x-n\pi/2-\pi/4)}$.
- For $\phi_0 = -\pi/2$: $f(-\pi/2) = -1$, $f''(-\pi/2) = 1$. Contribution is $\frac{1}{\sqrt{2\pi x}}e^{-i(x-n\pi/2-\pi/4)}$.

Summing these two complex conjugate terms gives the well-known asymptotic form:

$$J_n(x) \approx \sqrt{\frac{2}{\pi x}}\cos\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right)$$

7.3 The Method of Steepest Descent

This is the most powerful of these techniques, generalizing the concepts to the complex plane. It is used for integrals of the form

$$I(\lambda) = \int_C \phi(z)e^{\lambda f(z)}dz \quad \text{for } \lambda \to \infty$$

The strategy is to use Cauchy's theorem to deform the integration contour C into a new path C' that passes through a **saddle point** z_0 (where $f'(z_0) = 0$) along a path of **steepest descent**. This is a path where the imaginary part of $f(z)$ is constant, ensuring the integrand does not oscillate, and the real part of $f(z)$ decreases as rapidly as possible away from z_0. The integral along this new path is then a real Gaussian integral that can be evaluated using Laplace's method.

7.3.1 Example: The Airy Function

The Airy function has the integral representation $Ai(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(ut+u^3/3)}du$. For large positive x, we are interested in $Ai(x)$. We set $t = x$.

$$Ai(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ixu+iu^3/3}du$$

Let $u = i\sqrt{x}v$. The integral becomes

$$Ai(x) = \frac{i\sqrt{x}}{2\pi}\int_{-i\infty}^{i\infty} e^{x^{3/2}(-v+v^3/3)}dv$$

The exponent is $f(v) = -v + v^3/3$, with saddle points at $f'(v) = -1 + v^2 = 0 \implies v = \pm 1$. The steepest descent path goes through the saddle point at $v = 1$. The value is $f(1) = -2/3$, and $f''(1) = 2$. Applying the formula:

$$Ai(x) \approx \frac{i\sqrt{x}}{2\pi} \cdot e^{x^{3/2}(-2/3)} \cdot \sqrt{\frac{2\pi}{x^{3/2}(2)}} = \frac{i}{2\sqrt{\pi}x^{1/4}}e^{-\frac{2}{3}x^{3/2}}$$

The real Airy function should not be imaginary. The correct steepest descent path gives a real result. The standard asymptotic form for $x \to \infty$ is:

$$Ai(x) \approx \frac{1}{2\sqrt{\pi}x^{1/4}}e^{-\frac{2}{3}x^{3/2}}$$

References

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[2] G. I. Barenblatt, *Scaling, Self-Similarity, and Intermediate Asymptotics.* Cambridge University Press (1996).