P1

Proof:

1. First, prove the relation to be equivalent:
   - Reflective: $a \sim a \Rightarrow f(a) = f(a)$, TRUE.
   - Symmetric: $a \sim b \Rightarrow f(a) = f(b) \Rightarrow f(b) = f(a) \Rightarrow b \sim a$, TRUE.
   - transitive: $a \sim b, b \sim c \Rightarrow f(a) = f(b), f(b) = f(c) \Rightarrow f(a) = f(c) \Rightarrow a \sim c$, TRUE.
2. Then, prove its equivalence classes to be the fibers of f:

   Let C be the set of equivalence classes of A under $\sim$, and let F be the set of fibers of f. We will show that $C = F$.

   Take an arbitrary element $a \in A$. The equivalence class of $a \in A$ is:

$$\begin{aligned} \{x \in A \mid x \sim a\} &= \{x \in A \mid f(x) = f(a)\} \\ &= f^{-1}\{f(a)\} \end{aligned} \tag{1}$$

   which by definition is the fiber of f.

   Since a was arbitrary, every equivalence class is a fiber of f, i.e. $C \subseteq F$.

   Conversely, let F' be an arbitrary fiber of f for some $b \in B$. Then by definition,

$$\begin{aligned} F' &= f^{-1}\{b\} \\ &= \{x \in A \mid f(x) = b\} \end{aligned} \tag{2}$$

   .

   Since f is surjective, $\exists a \in A \ s.t. \ f(a) = b$. Consider the equivalence class of a:

$$\begin{aligned} \{x \in A \mid x \sim a\} &= \{x \in A \mid f(x) = f(a)\} \\ &= \{x \in A \mid f(x) = b\} \\ &= F'. \end{aligned} \tag{3}$$

   Since F' was arbitrary, every fiber of f is an equivalence class, i.e. $F \subseteq C$. Thus, $C = F$. ■

P2

Prove by contradiction:

1. Consider an arbitrary **column** in the multiplication table of G. Suppose that the colum is *not* a permutation of G.

   Then there would be at least two identical elements in this column, which we denote as a . This implies that

   $$\exists x, y \in G, x \neq y, s.t.\ xa = ya \tag{4}$$

   Applying x^{-1} from right on both sides:

   $$\begin{aligned} x^{-1}xa &= x^{-1}ya \\ a &= x^{-1}ya \\ \Rightarrow x^{-1}y &= e. \end{aligned} \tag{5}$$

   Since inverse of an element is unique, $y = x$, which is a contradiction.

2. Similarly, consider arbitrary **row** in the multiplication table of G . Suppose that this row is *not* a permutation of G , i.e. there are at least two repeating elements, denoted as b. This implies

   $$\exists x, y \in G, x \neq y, s.t.\ xa = xb. \tag{6}$$

   Applying a^{-1} from left on both sides:

   $$\begin{aligned} xaa^{-1} &= xba^{-1} \\ x &= xba^{-1} \\ \Rightarrow ba^{-1} &= e. \end{aligned} \tag{7}$$

   Since inverse of an element is unique, $b = a$, a contradiction. ■

P3