ECE 535: Introduction to Quantum Sensing

Uncertainties in measurements

Jennifer Choy

Fall 2025

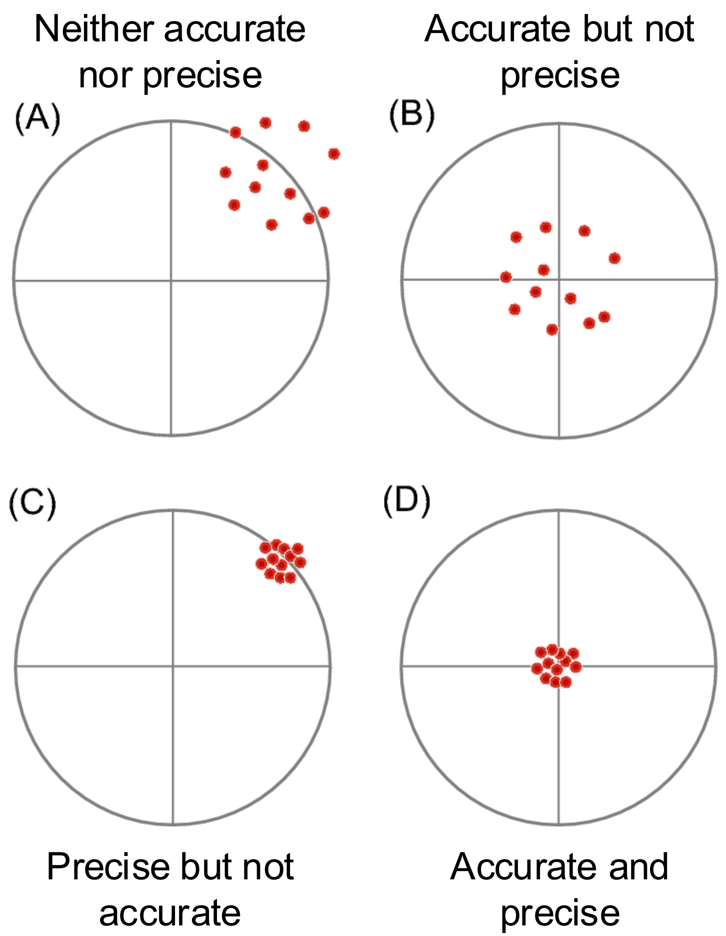
WISCONSIN
UNIVERSITY OF WISCONSIN–MADISON

Topics

- Error quantification in measurements
- Limits due to the uncertainty principle
  - Concept of squeezing in measurements
- Limits in measurements due to statistics (projection noise)
- Limits in measurements from discrete nature of light and matter (shot noise)
- Vacuum fluctuations

Error analysis in experiments

<u>Recall: Accuracy vs precision</u>



Natl Sci Rev, Volume 3, Issue 2, June 2016, Pages 189–200

- Uncertainties in measurements can be categorized into errors of precision and accuracy
- Precision is related to the distribution of random errors, as a result of experimental conditions and/or the physical process being measured
  - Error of precision can be estimated by repeating measurements and decreases with increasing number of experiments
- Accuracy is related to systematic errors, which could be due to instrumentation
  - Error of accuracy can be inferred by referencing an experiment against a known standard or a measurement with better accuracy
- Usually one of these error sources will dominate

References:

https://reference.wolfram.com/applications/eda/ExperimentalErrorsAndErrorAnalysis.html
https://faraday.physics.utoronto.ca/PVB/Harrison/ErrorAnalysis/

Is there such a thing called *over precision*?

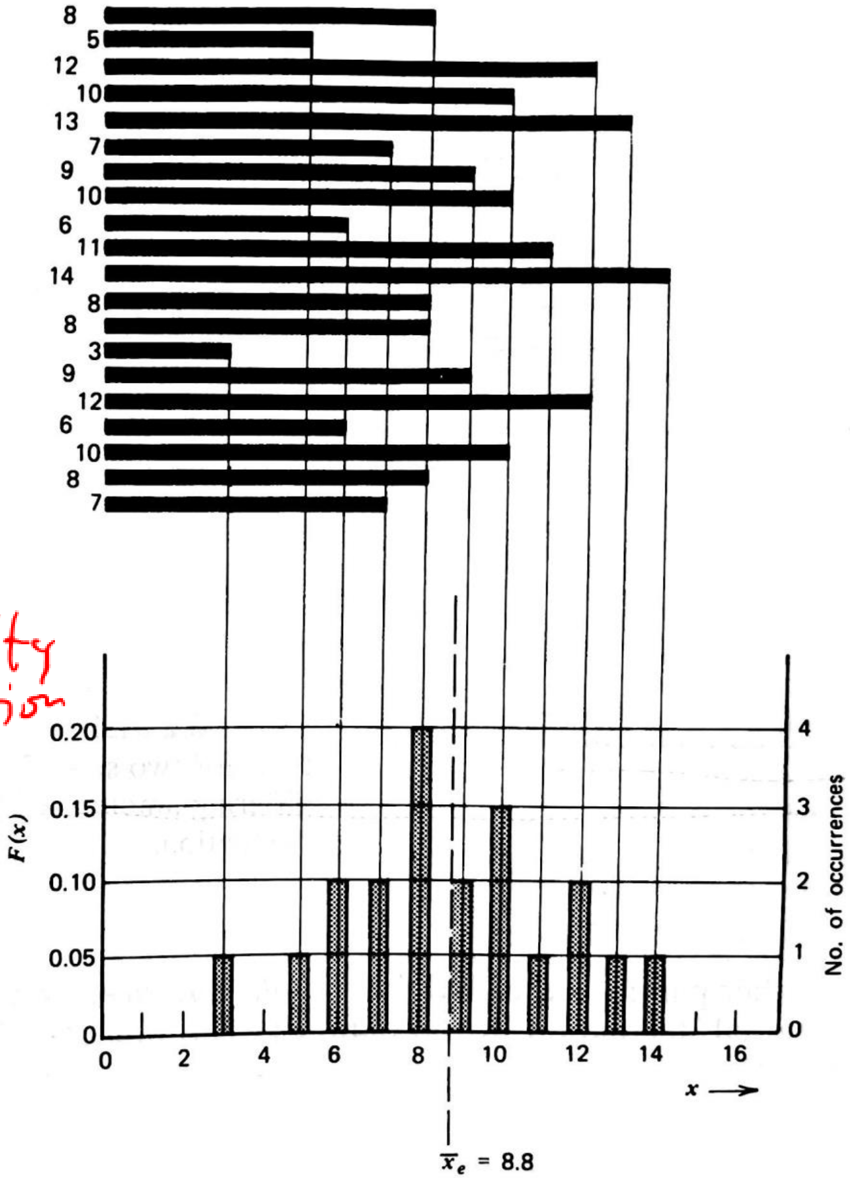
Jen went to the store to buy a crinoid fossil. She was told by the owner that the fossil is 350 million years old. She took the fossil home and displayed it on a shelf.

6 months later, some of her friends came over and asked her how old is the fossil.

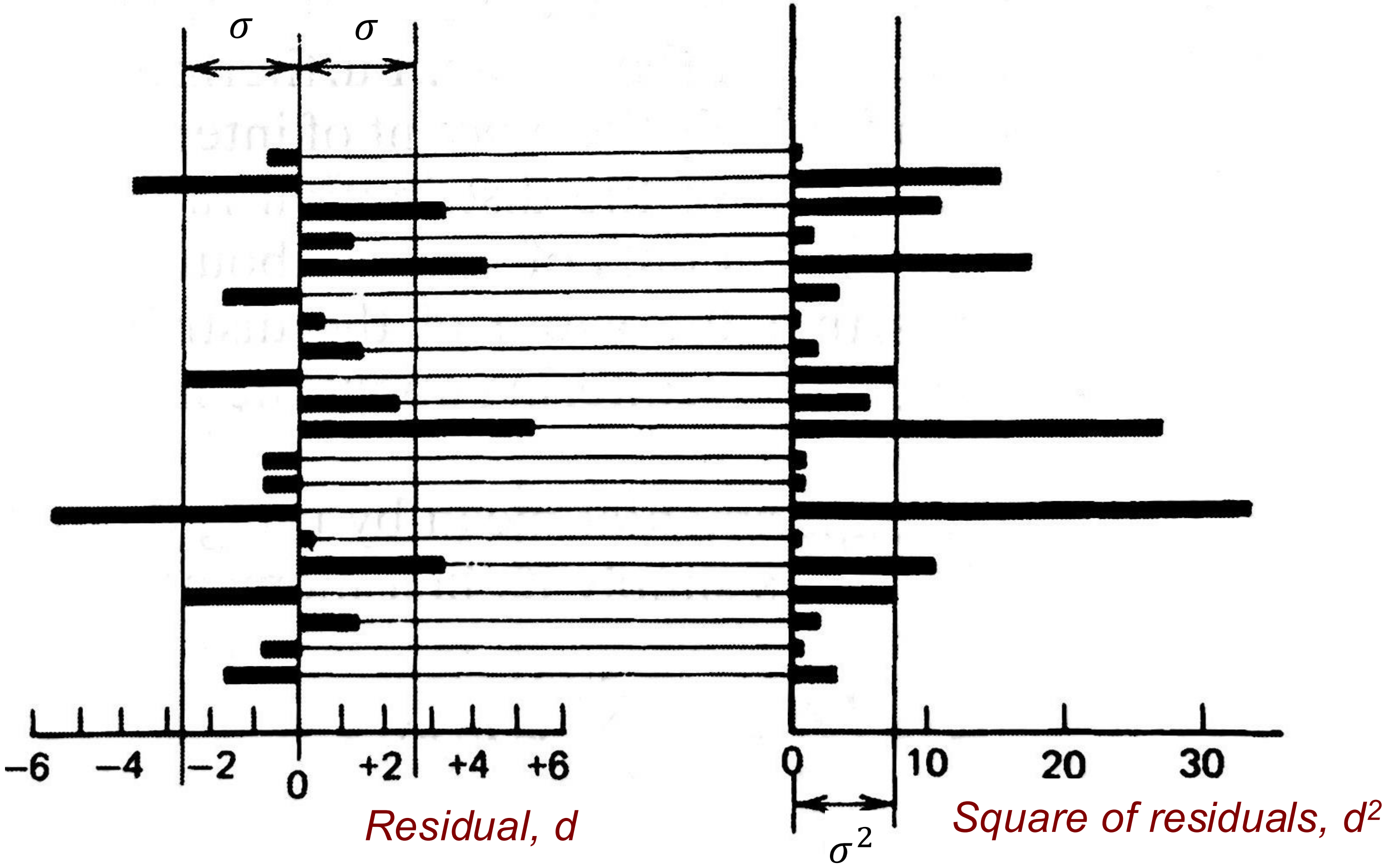
She proudly proclaimed “350 million years and 6 months!”

Determining the precision

Dataset



probability distribution



Scatter of data around mean is a measure for *randomness*

Residual $d_i \equiv x_i - \bar{x}_e$

$$\bar{x}_e = \frac{\sum_i^N x_i}{N}$$

Variance $\sigma^2 \equiv \frac{1}{N-1}\sum_{i=1}^{N}(x_i - \bar{x}_e)^2$

σ is the **standard deviation** and quantifies the amount of fluctuation in the data

Statistical models that describe the distribution of errors

- Can we predict statistical behavior under certain assumptions?

  Consider a *binary process* with only a *true* or *false* result

Table 3.2 Examples of Binary Processes

Trial	Definition of Success	Probability of Success $\equiv p$
Tossing a coin	Heads	$1/2$
Rolling a die	A six	$1/6$
Observing a given radioactive nucleus for a time t	The nucleus decays during the observation	$1 - e^{-\lambda t}$

$$P(x) = \frac{\text{no. occurrences of } x}{\text{no. measurements } (= N)}$$

- Common statistical models

Binomial Distribution

General model for *binary processes* with constant probability p for a certain outcome

Cumbersome to use with a large sample size (n)

Probability of getting x number of a certain outcome

$$P(x) = \boxed{\frac{n!}{(n-x)!\,x!}}\boxed{p^x(1-p)^{n-x}}$$

\# of permutations

Joint probability of having success x times in a sequence

Mean $$\bar{x} = \sum_{x=0}^{n} xP(x) = pn$$

Variance $$\sigma^2 \equiv \sum_{x=0}^{n} (x-\bar{x})^2 P(x) = np(1-p) = \bar{x}(1-p)$$

Statistical models that describe the distribution of errors

- Can we predict statistical behavior under certain assumptions?

  Consider a *binary process* with only a *true* or *false* result

Table 3.2 Examples of Binary Processes

Trial	Definition of Success	Probability of Success $= p$
Tossing a coin	Heads	$1/2$
Rolling a die	A six	$1/6$
Observing a given radioactive nucleus for a time t	The nucleus decays during the observation	$1 - e^{-\lambda t}$

$$P(x) = \frac{\text{no. occurrences of } x}{\text{no. measurements } (= N)}$$

- Common statistical models

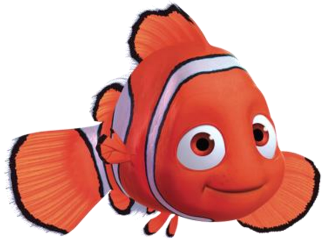
Binomial Distribution

General model for *binary processes* with constant probability p for a certain outcome

Cumbersome to use with a large sample size (n)

Poisson Distribution

Simplification for large n (less computationally intensive); assumes $p \ll 1$



$$P(x) = \frac{(pn)^x e^{-pn}}{x!} = \frac{(\bar{x})^x e^{-\bar{x}}}{x!}$$

$$\bar{x} = pn = \sigma^2$$

mean = variance

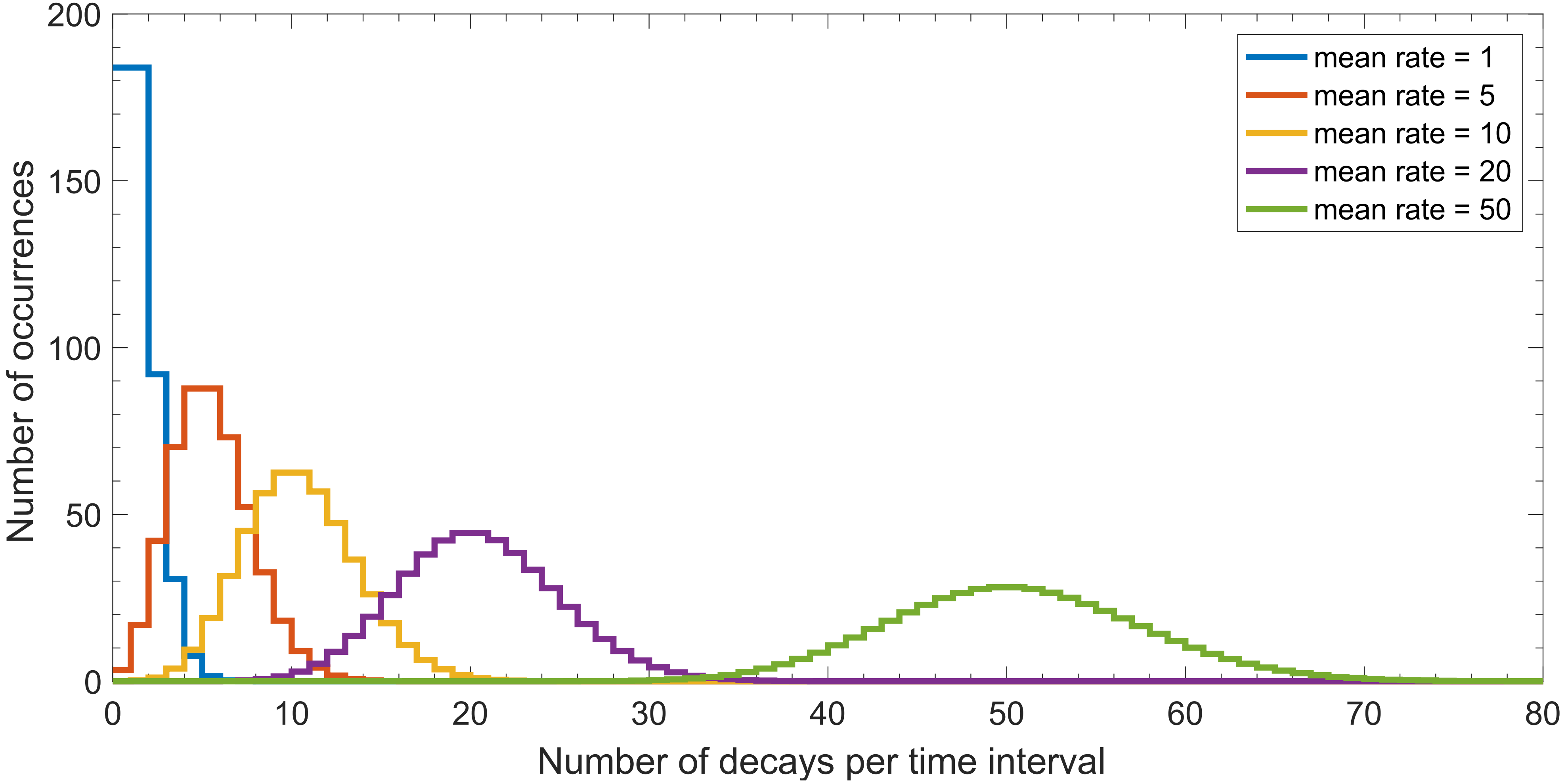
Example of Poisson distribution: radioactive decay

also fluorescence decay for many independent emitters

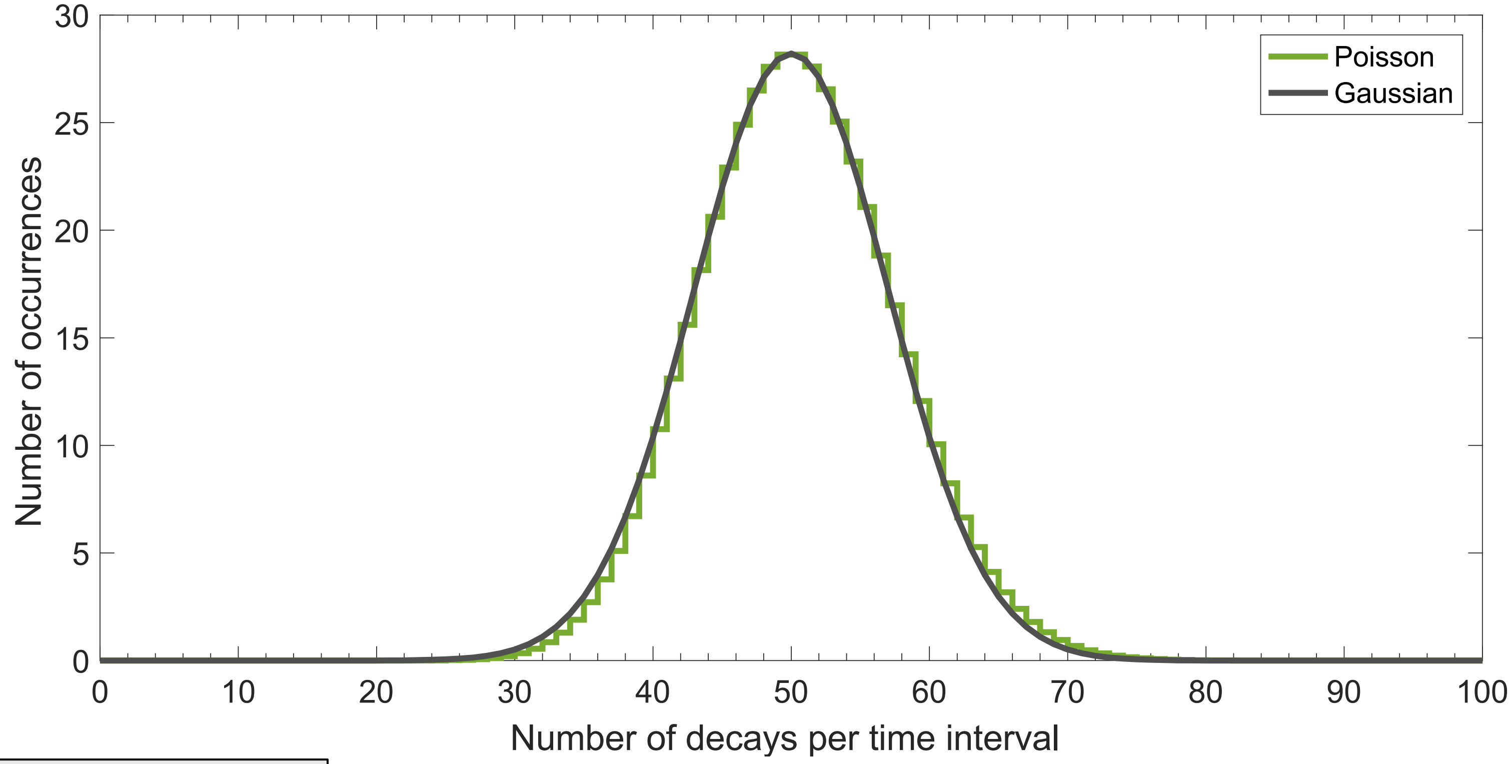
Probability of getting n decays in time t:

$$P(n,t) = \frac{(\lambda t)^n e^{-\lambda t}}{n!}$$

λ is the decay constant



Large mean → distribution looks like Gaussian



Binomial Distribution

General model for *binary processes* with constant probability p

Cumbersome to use with a large sample size (n)

⇨

Poisson Distribution

Simplification for large n (less computationally intensive); assumes $p \ll 1$

⇨

Normal Distribution

Valid when $np \gg 1$

Also known as *Gaussian Distribution*

Gaussian or normal distribution

$\int P(x)\, dx = 1$

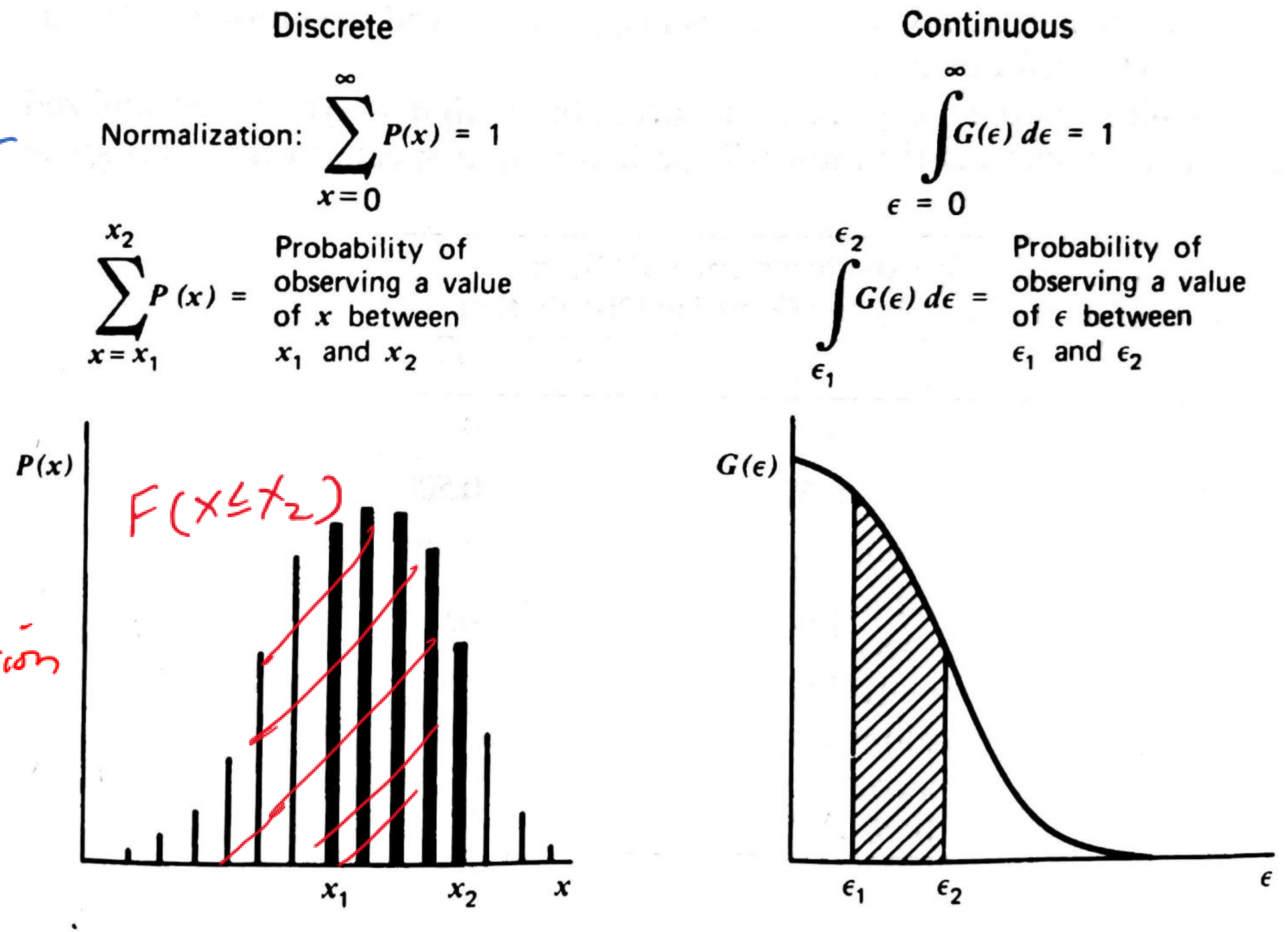
- **Probability function:** $P(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\bar{x})^2}{2\sigma^2}}$
- **Mean**: $\bar{x} = pn$

Statistics for photons

- **Variance:** $\sigma^2 = \bar{x}$ for *Poisson statistics*
- We can use discrete or continuous versions

Cumulative distribution function

$F(x \leq x_2)$



Why are Gaussian distributions important in experiments?

Central limit theorem

Let $X_1, X_2, \ldots, X_n$ be a set of independent variables with the same distribution, with mean μ and standard deviation σ. For a large number of n, the mean of X $(\equiv \bar{X}_n)$ is approximately a normal distribution with the same mean as X but variance σ^2/n:

$$\bar{X}_n \approx \mathcal{N}\left(\mu, \frac{\sigma^2}{n}\right)$$

$$P(x) = \frac{1}{\sqrt{2\pi}}\frac{1}{\sigma}\exp\left(-\frac{(X-\mu)^2}{2\sigma^2}\right)$$

$$\sigma = \sqrt{\frac{p(1-p)}{n}}$$ projection noise

https://mathworld.wolfram.com/CentralLimitTheorem.html

Example from political polling:

A two-candidate race has true support for Candidate A of $p = 0.48$ (and thus 0.52 for B). A polling firm takes a simple random sample of likely voters and reports the sample proportion $\hat{p}$ favoring A. With $n = 500$, approximate the probability that the poll shows A *leading by more than 2 percentage points* (i.e., $\hat{p} > 0.51$).

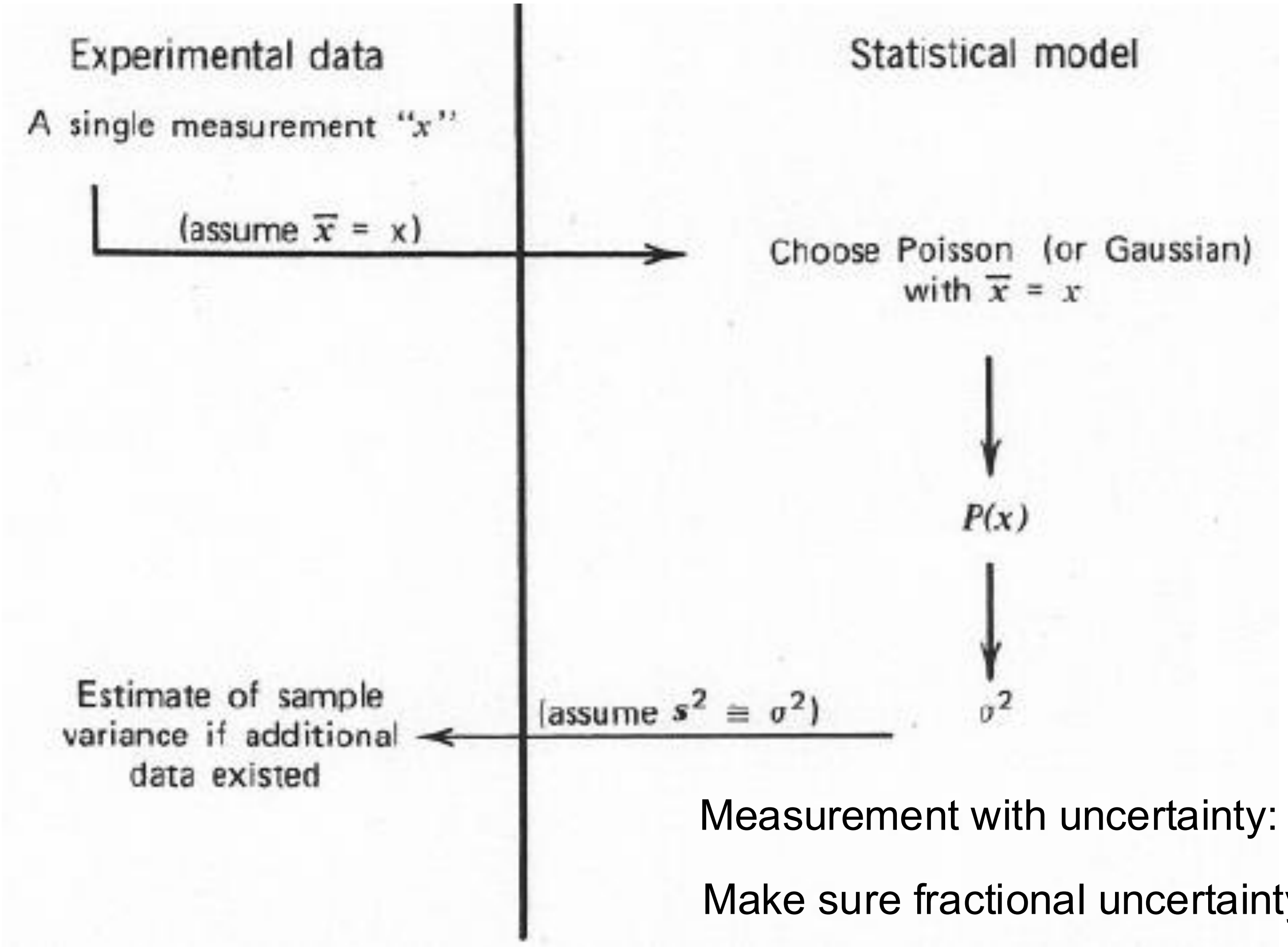
$\hat{p}$ will have a normal distribution with mean $p = 0.48$ and variance

$$\frac{p(1-p)}{n} \rightarrow \sigma = \sqrt{\frac{0.48(1-0.52)}{500}} = 0.022$$

$$P(\hat{p} > 0.51) = 1 - P(\hat{p} \leq 0.51) = 8.9\%$$

See sim_binomial_CLT.m

Single measurement of a random event (*e.g.,* photon count rate on a detector)



$\mathcal{E}(t)$ → Detector

$I(t) \propto \dot{n}$ ← photon flux

Measurement with uncertainty: $\boxed{x \pm \sqrt{x}}$

Make sure fractional uncertainty $\frac{1}{\sqrt{x}}$ is sufficiently small

Error propagation: how to estimate errors when we combine two or more measured quantities?

Gaussian error propagation law

Let *u(x,y,z)* be a statistical quantity with *x*, *y*, and *z* as statistically independent variables. Each variable has the measurement uncertainty σ_i.

Then the combined uncertainty of σ_u of *u(x,y,z)* is

$$\sigma_u^2 = \left(\frac{\partial u}{\partial x}\right)^2 \sigma_x^2 + \left(\frac{\partial u}{\partial y}\right)^2 \sigma_y^2 + \left(\frac{\partial u}{\partial z}\right)^2 \sigma_z^2 + \ldots$$

$$u = u(x,y,z,\ldots)$$

Error propagation: sums and differences

Simple sum or difference with a constant:

$$u = x \pm a$$

$$\frac{\partial u}{\partial x} = 1$$

$$\boxed{\sigma_u = \sigma_x}$$

Involving two or more independent variables:

$$u = x \pm y$$

$$\frac{\partial u}{\partial x} = 1, \frac{\partial u}{\partial y} = \pm 1$$

$$\sigma_u^2 = (1)^2 \sigma_x^2 + (\pm 1)^2 \sigma_y^2$$

$$\boxed{\sigma_u = \sqrt{\sigma_x^2 + \sigma_y^2}}$$

Weighted:

$$u = ax \pm by$$

$$\frac{\partial u}{\partial x} = a, \frac{\partial u}{\partial y} = \pm b$$

$$\boxed{\sigma_u^2 = a^2 \sigma_x^2 + b^2 \sigma_y^2 + 2ab\sigma_{xy}^2}$$

0 if x and y are uncorrelated

Error propagation rules for other operations

Error propagation equation:

For $u = u(x, y, z, \ldots)$, where all variables are uncorrelated and errors are small

$$\sigma_u^2 \approx \left(\frac{\partial u}{\partial x}\right)^2 \sigma_x^2 + \left(\frac{\partial u}{\partial y}\right)^2 \sigma_y^2 + \left(\frac{\partial u}{\partial z}\right)^2 \sigma_z^2 + \ldots$$

Sums and Differences:

$$u = ax \pm by$$

$$\sigma_u = \sqrt{a^2\sigma_x^2 + b^2\sigma_y^2}$$

Multiplication or division:

$$u = axy \text{ or } u = \frac{ax}{y}$$

Fractional uncertainty

$$\frac{\sigma_u}{u} = \sqrt{\left(\frac{\sigma_x}{x}\right)^2 + \left(\frac{\sigma_y}{y}\right)^2}$$

Powers:

$$u = ax^b$$

$$\frac{\sigma_u}{u} = b\frac{\sigma_x}{x}$$

Exponentials:

$$u = ae^{bx}$$

$$\frac{\sigma_u}{u} = b\sigma_x$$

(a, b are constants)

Mean of measurements

- To calculate the estimated mean in a measurement:

$$\bar{x}_{est} = \frac{\sum_{i=1}^{N} x_i}{N}$$

Each measurement has an uncertainty of σ_x.

- What is the error with N measurements?

$$x_{sum} = x_1 + x_2 + \dots$$

$$\sigma_{x_{sum}} = \sqrt{\underbrace{\sigma_x + \dots}_{N}} = \sqrt{N \sigma_x^2}$$

$$\sigma_{\bar{x}_{est}} = \frac{\sqrt{N}\,\sigma_x}{N} = \frac{\sigma_x}{\sqrt{N}}$$

- How many times (in terms of N) do we need to do the measurement to reduce the error by half?

Do measurements $4 \times N$

Another approach: Monte Carlo error analysis

repeated random sampling

- Concept: For $u = u(x, y, z, \ldots)$, randomly sample independent variables based on your knowledge of their mean, standard deviation, and distribution (*e.g.*, Gaussian) and calculate u with each set of random values. The standard deviation in the computed u is then the estimated uncertainty.
- Need random number generator and a large number of samples. To be computationally efficient, should avoid using for loops.

See code simpleMC.m on Canvas

Example

We want to determine the resistance R of a cylindrical conductor based on the resistivity of the material ρ, the length of the cylinder L, and the radius r of the cylinder cross-section. The relationship between the resistance and cylinder properties is:

$$R = \frac{\rho L}{\pi r^2}$$

A cylinder made out of an alloy material has the following properties:

$$\rho = (4.41 \pm 0.03) \times 10^{-5}\ \Omega \cdot m$$
$$L = 0.1 \pm 0.0005\ m$$
$$r = (20 \pm 0.01) \times 10^{-6} m$$

- Calculate R and its uncertainty using the standard error propagation approach.

$$R = \frac{\rho L}{y} \qquad y = \pi r^2 \qquad \frac{\sigma_y}{y} = \frac{2\pi r \sigma_r}{\pi r^2} = \frac{2\sigma_r}{r}$$

$$\left(\frac{\sigma_R}{R}\right)^2 = \left(\frac{\sigma_\rho}{\rho}\right)^2 + \left(\frac{\sigma_L}{L}\right)^2 + \underbrace{\left(\frac{\sigma_y}{y}\right)^2}_{4\left(\frac{\sigma_r}{r}\right)^2} = 29.8\,\Omega$$

$$\boxed{R = 3509 \pm 29.8\,\Omega}$$

Example

We want to determine the resistance R of a cylindrical conductor based on the resistivity of the material ρ, the length of the cylinder L, and the radius r of the cylinder cross-section. The relationship between the resistance and cylinder properties is:

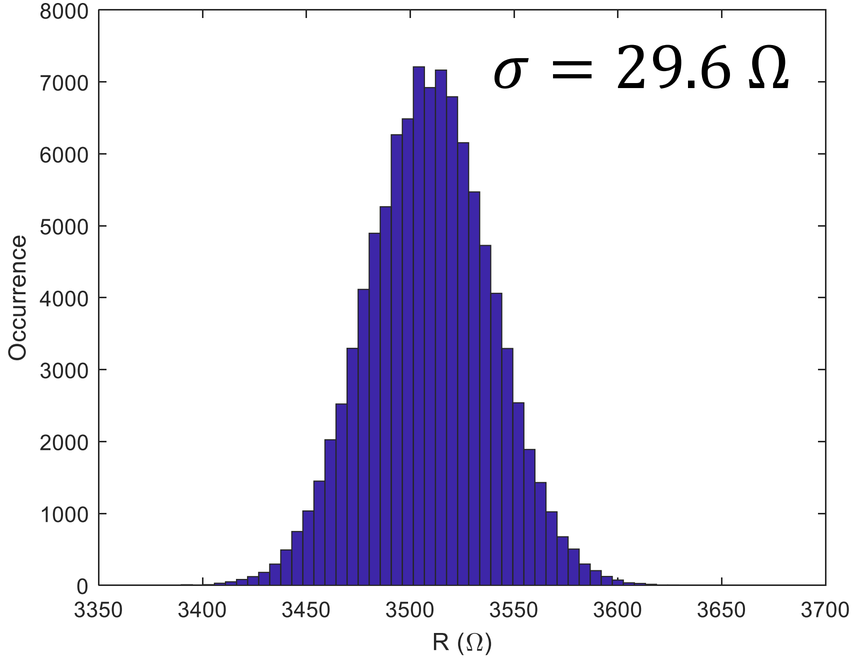
$$R = \frac{\rho L}{\pi r^2}$$

A cylinder made out of an alloy material has the following properties:

$$\rho = (4.41 \pm 0.03) \times 10^{-5}\ \Omega \cdot m$$
$$L = 0.1 \pm 0.0005\, m$$
$$r = (20 \pm 0.01) \times 10^{-6} m$$

- Calculate the uncertainty in R using the Monte Carlo approach.

See code errPropExample_mc.m on Canvas



Example

We want to determine the resistance R of a cylindrical conductor based on the resistivity of the material ρ, the length of the cylinder L, and the radius r of the cylinder cross-section. The relationship between the resistance and cylinder properties is:

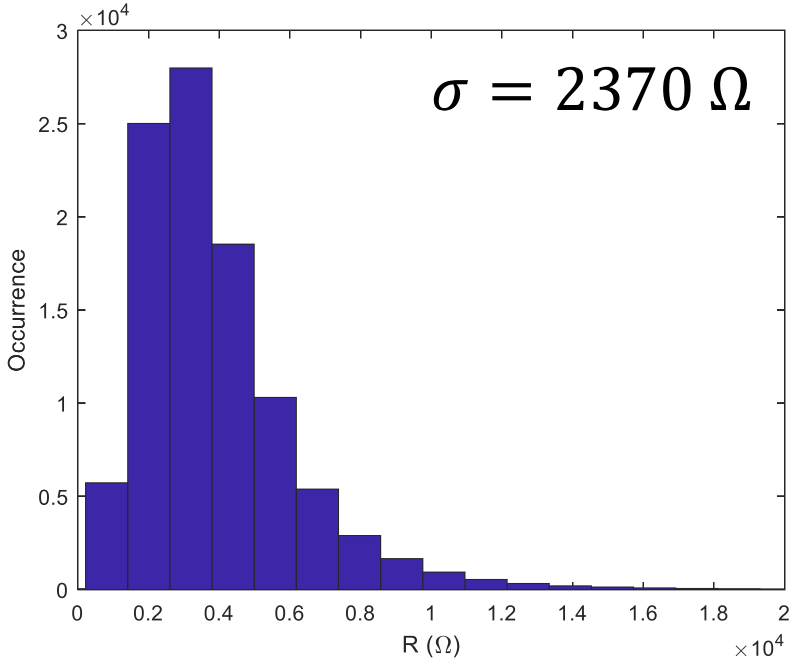
$$R = \frac{\rho L}{\pi r^2}$$

Now consider the case of much larger errors for the measurements:

$$\rho = (4.41 \pm 1.2) \times 10^{-5}\ \Omega \cdot m$$
$$L = 0.1 \pm 0.025\ m$$
$$r = (20 \pm 3.5) \times 10^{-6} m$$

Gaussian error propagation:

$$\sigma = 1786\ \Omega$$

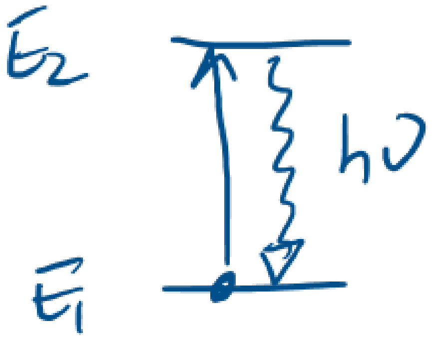


What are the fundamental limits to uncertainties in measurements?

- Uncertainty principle
- Statistical fluctuations due to probabilistic nature of quantum states (projection noise)
- Fluctuations from discreteness of light and matter (shot noise)
- Vacuum fluctuations

Bandwidth of a single photon

A photon is emitted by an atom at a decay rate of 1 ns. What is the bandwidth (frequency uncertainty) of the photon?



$$\underbrace{\Delta E}_{h\Delta\nu}\ \underbrace{\Delta t}_{1\,\mathrm{ns}} \geq \frac{\hbar}{2} = \frac{h}{4\pi}$$

$$\Delta\nu\,\Delta t \geq \frac{1}{4\pi}$$

$$\Delta\nu_{min} \sim 8\times10^{7}\ \mathrm{Hz} \sim 80\ \mathrm{MHz}$$

Trading off position-momentum uncertainties to improve a measurement of electron energy (simple example of squeezing)

An electron is prepared such that its position is known to $\Delta x \approx 10^{-12}$ m.

- Estimate the uncertainty in the kinetic energy of the electron (ΔE_{kin}). Assume that the momentum uncertainty is on the order of the mean momentum of the electron.
- Say the electron position uncertainty can be changed by a factor of 100 to improve the energy uncertainty. Which way would you change the position uncertainty? What is the improved energy uncertainty?

$$\Delta x \Delta p \geq \frac{\hbar}{2} \rightarrow \Delta p \geq \frac{\hbar}{2\Delta x} \sim 5.3 \times 10^{-23} \text{ kg m/s}$$

$$\Delta E_{kin} = \Delta\left(\frac{p^2}{2m_e}\right) \approx \frac{\Delta p \cdot p}{m_e} \sim 1.5 \times 10^{-15} \text{ J}$$

Increase uncertainty in x: $\Delta x' \approx 10^{-10}$ m

Squeezing factor in x:

$$S_x = \frac{\Delta x'}{\Delta x} = 100$$

Squeezing factor in p

$$S_y = \frac{\Delta p'}{p} = \frac{1}{100}$$

ΔE_{kin} decreases by factor of $10^2 - 10^4$

Quantum projection noise (QPN): measurement uncertainty from statistics

An electron has equal probability of being found in the left (L) or right (R) side of a region. What is the variance in the measurement of the electron position?

Electron wavefunction in which there is equal likelihood ($P = 1/2$) for the electron to be on the left and right region:

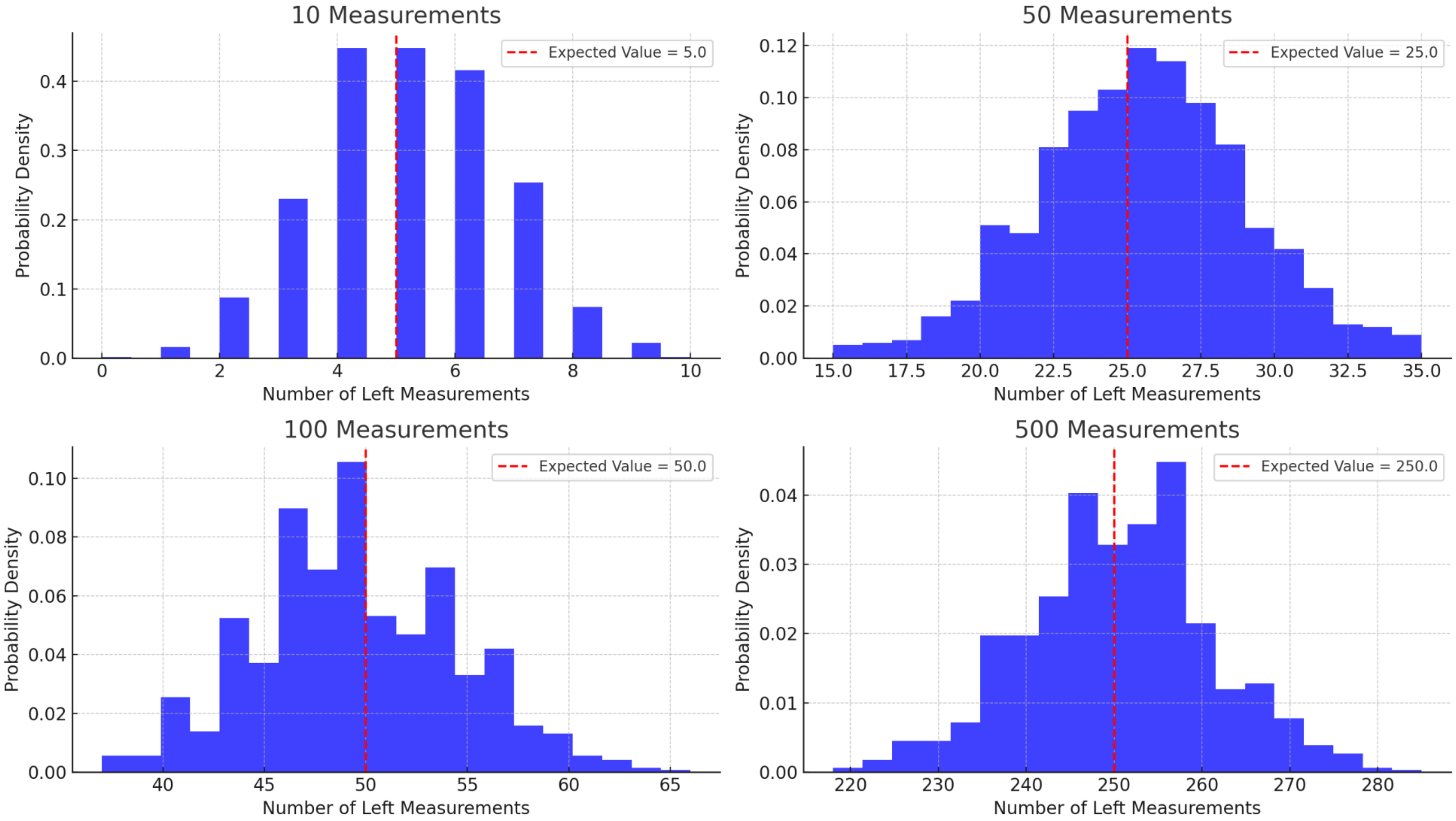
$$\psi(x) = \frac{1}{\sqrt{2}} \left(\psi_L(x) + \psi_R(x)\right)$$

Variance of finding the electron on the left side after N measurements:

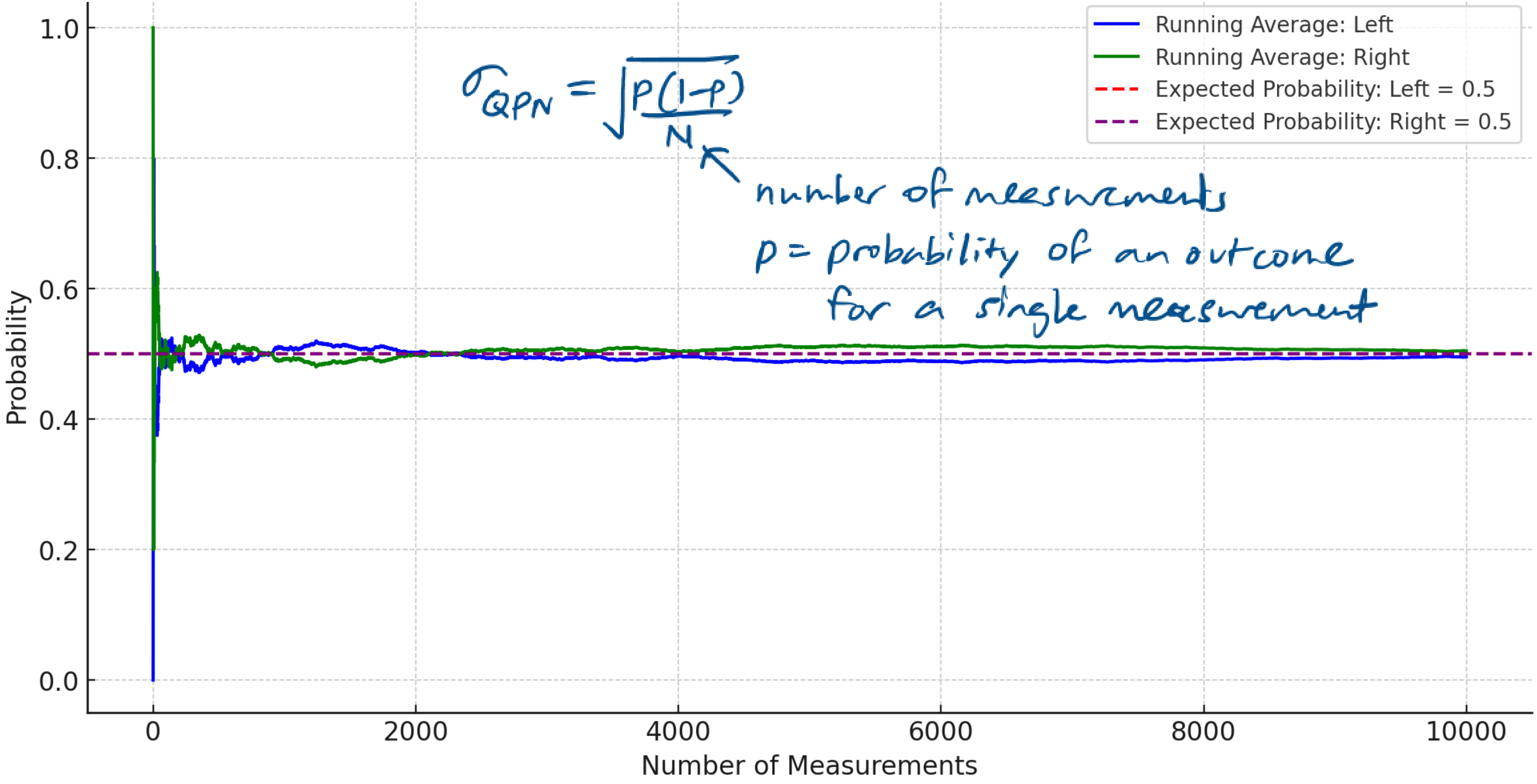
$$Var_L = \sigma_L^2 = \frac{1}{4N}$$

Same for the measuring the electron on the right:

$$Var_R = \sigma_R^2 = \frac{1}{4N}$$



Quantum projection noise (QPN): measurement uncertainty from statistics



$$\sigma_{QPN} = \sqrt{\frac{p(1-p)}{N}}$$

N = number of measurements

p = probability of an outcome for a single measurement

Another example of QPN: observing electron diffraction

- Probability distribution of the electron's position on the screen

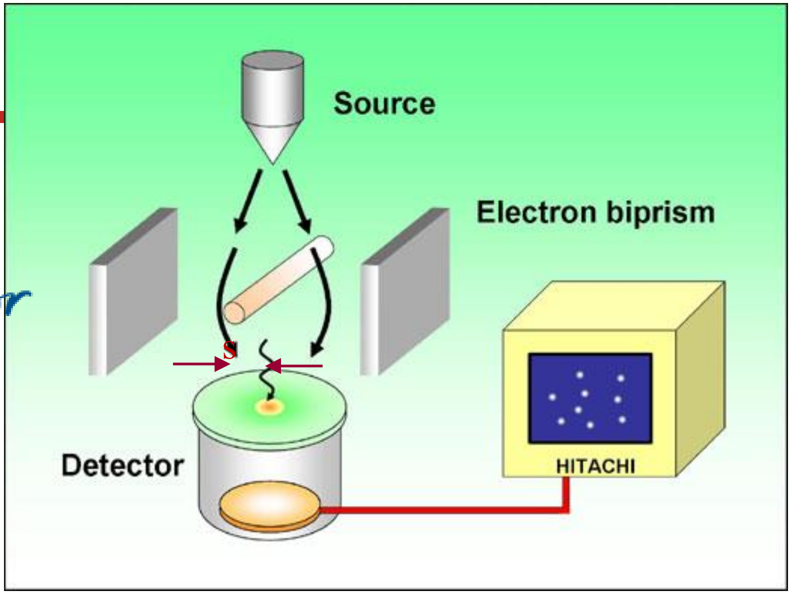
$$|\psi_s(x)|^2 \propto \frac{1}{2}\left(1+\cos\left(\frac{2\pi s x}{\lambda z_0}\right)\right)$$

$s \to$ slit width

$z_0 \to$ source-detector distance

$\lambda \to$ de Broglie λ

- What are relevant length scales here?
  - $\lambda \sim 0.005 - 1\ nm$
  - $s \sim 1\ micron$
  - $z_0 \sim 10\ cm$



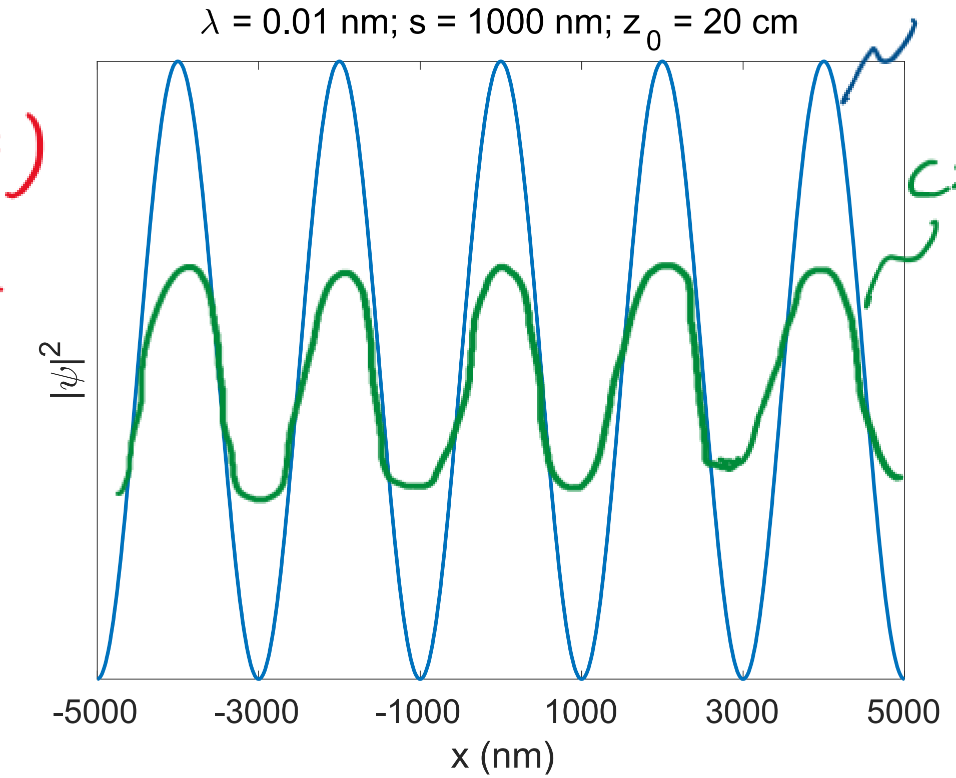
https://www.hitachi.com/rd/research/materials/quantum/doubleslit/index.html

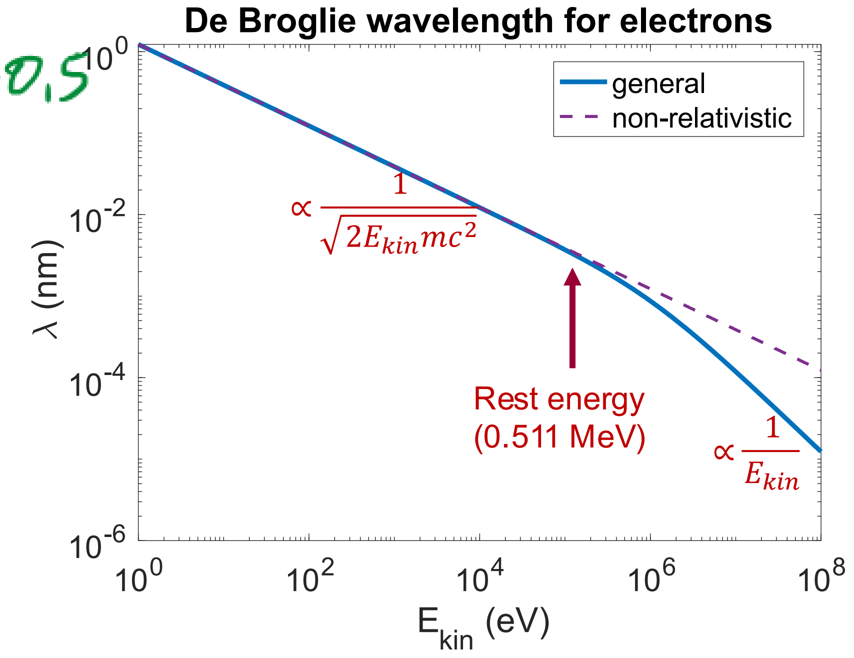
$p(\phi) = \frac{1}{2}(1 + C\cos\phi)$

C is interferometer contrast

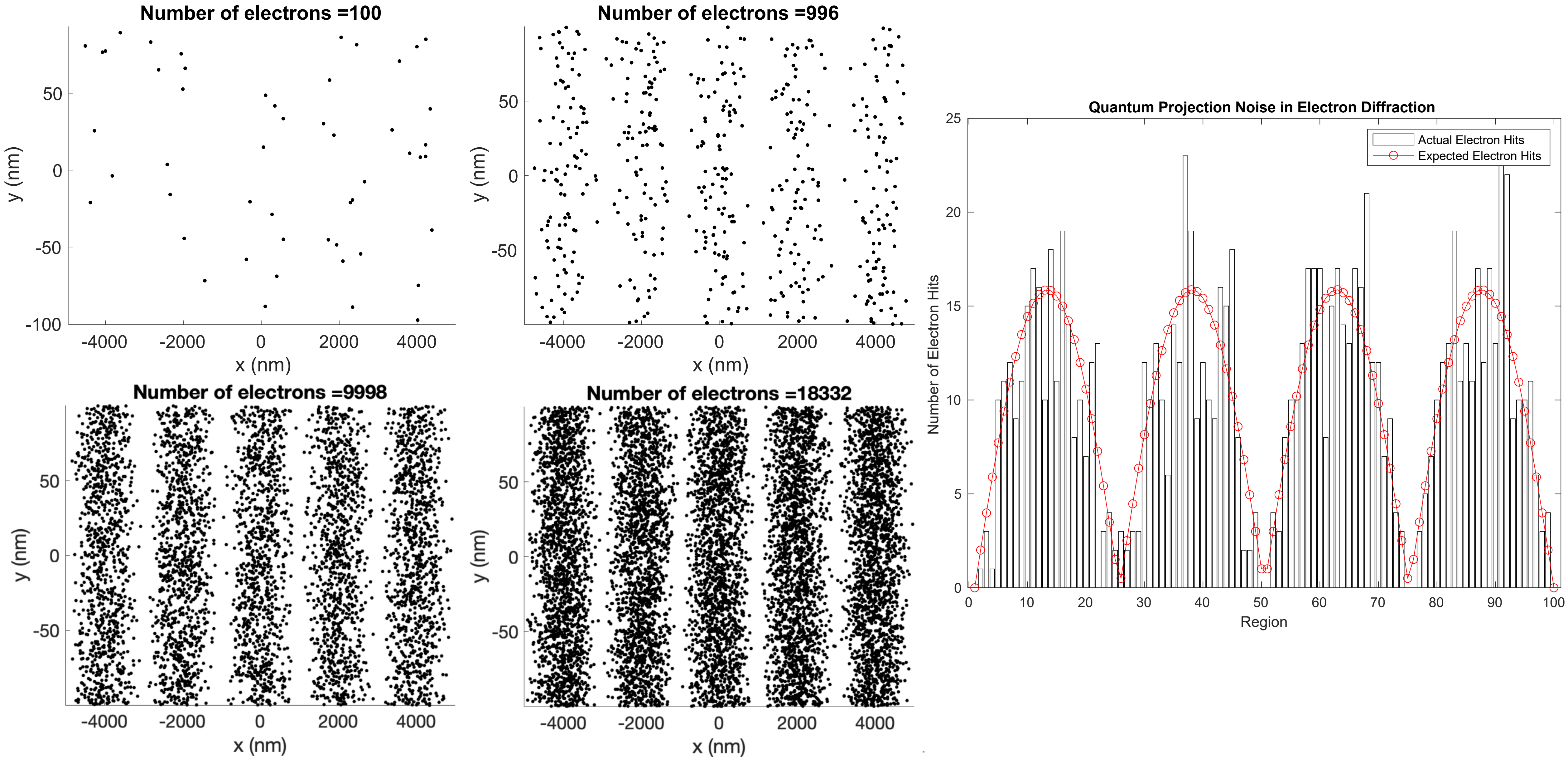
[0, 1]

$\phi = \frac{2\pi}{\lambda} \cdot \frac{s}{z_0} x$

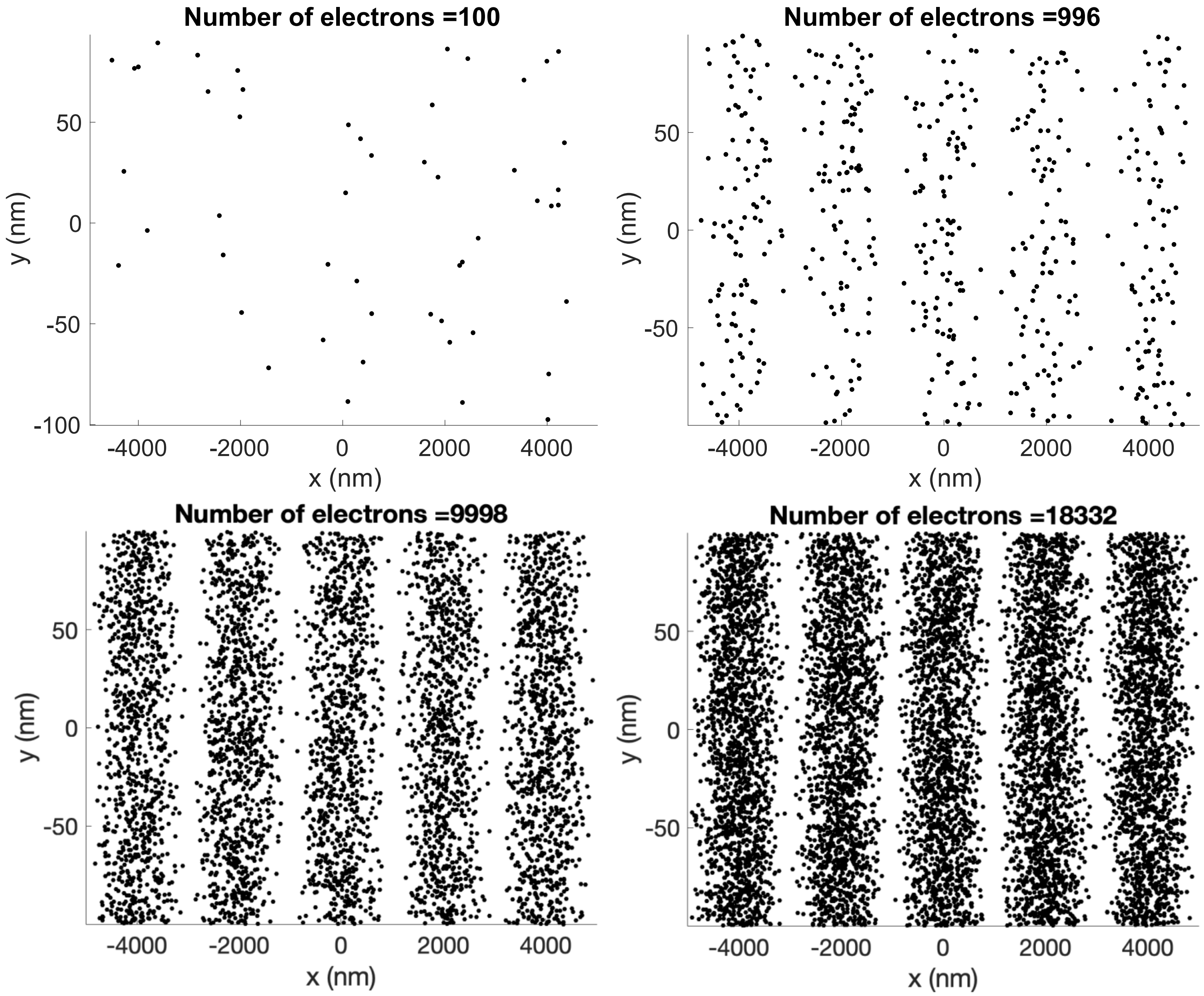




Another example of QPN: observing electron diffraction



Another example of QPN: observing electron diffraction



Electron detection at 1 pixel

Number of detected e^- : Np

total number of detected e^- $\quad$ $\frac{1}{2}(1+c\cos\phi)$

Variance in detected e^- : $\sigma_{QPN} = \sqrt{\frac{p(1-p)}{N}}$

Phase sensitivity : smallest detectable phase

$$\Delta\phi_{min} \approx \frac{\sigma_{QPN}}{\left|\frac{\partial p}{\partial \phi}\right|} \qquad \frac{\partial p}{\partial \phi} = -\frac{c}{2}\sin\phi$$

$$\boxed{\Delta\phi_{min} \approx \frac{1}{\sqrt{N}}\sqrt{\frac{1-c^2\cos^2\phi}{c^2\sin^2\phi}}}$$

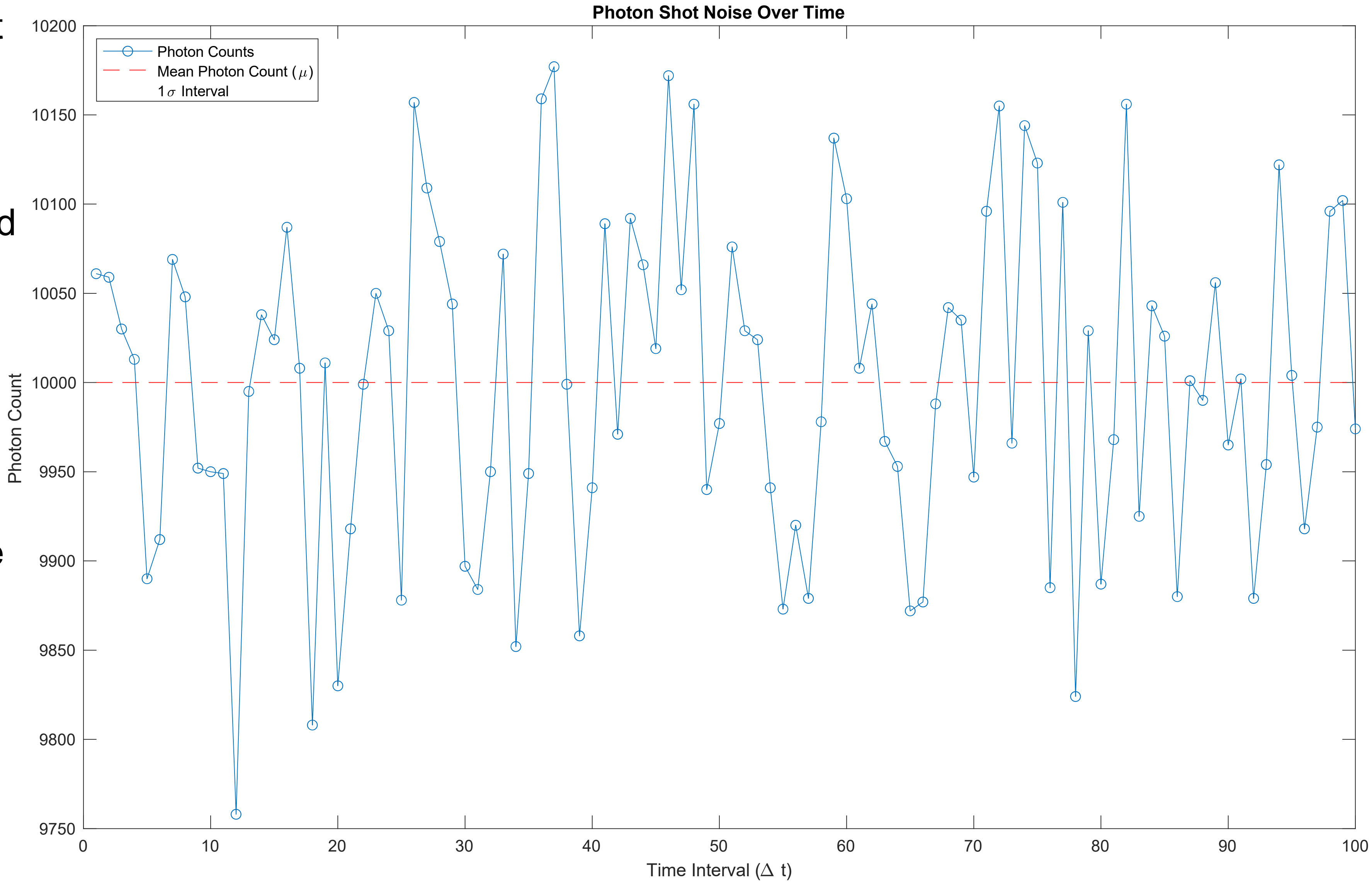
want :
c close to 1
$|\sin^2\phi| = 1$; $p = \frac{1}{2}$

Shot noise

- Coherent light (such as output of a laser) follows Poissonian photon statistics
- Discrete nature of photons and electrons leads to statistical fluctuations on short time-scales.
- Observed in the form of fluctuations in the number of particles observed over a time window

$$\sigma = \sqrt{\mu}$$

$$\frac{\sigma}{\mu} = \frac{1}{\sqrt{\mu}}$$



Shot noise numerical example

An attenuated beam from a 514-nm laser with a power of 0.1 pW is detected with a photon counter with efficiency 20%. The time interval of the counting system is set to 0.1 s. Calculate:

- Mean photon counts within a 0.1-s time window
- Standard deviation in the mean photon counts

$$1\ pW = 1 \times 10^{-12}\ W$$
$$h = 6.626 \times 10^{-34}\ J \cdot s$$

Photon generation rate: $\dot{n} = \frac{10^{-13}\ W}{hc/\lambda} = 2.6 \times 10^{5}\ s^{-1}$

Mean counts within measurement window: $\bar{N} = 0.2 \times (0.1\ s) \times (2.6 \times 10^{5}\ s^{-1}) = 5180$

Uncertainty in counts: $\sigma_{\bar{N}} = \sqrt{\bar{N}} = 72$

Standard quantum limit

phase

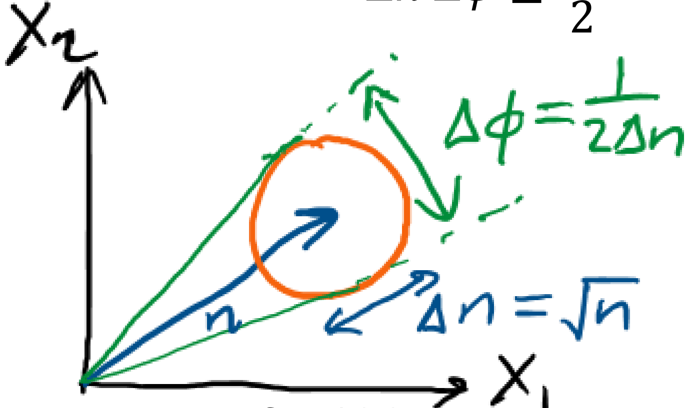
- A "coherent" electromagnetic field, say from a laser, can be represented by $\vec{\mathcal{E}} = \vec{\mathcal{E}}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)} = \vec{\mathcal{E}}_0 e^{i\phi(\vec{r},t)}$

$\sqrt{\frac{I}{c\epsilon_0}} \propto \sqrt{n}$

- The amplitude and phase of an EM field follow the uncertainty relationship:

$$\Delta n\,\Delta\phi \geq \frac{1}{2}$$

  - n is the photon number



- Example: What is the phase uncertainty of a 1064-nm laser with a power output of 300 W?

$$\dot{n} = \frac{300\,W}{hc/(1064\,nm)} = 1.6\times10^{21}\ s^{-1}$$

$$\Delta n = 4\times10^{10} = \sqrt{1.6\times10^{21}}$$ in a second

$$\Delta\phi = \frac{1}{2\Delta n} = 1.3\times10^{-12}\ rad$$

$$\Delta\phi_{SQL} = \frac{1}{2\sqrt{n}} \propto \frac{1}{\sqrt{I}}$$

increase intensity to reduce phase noise

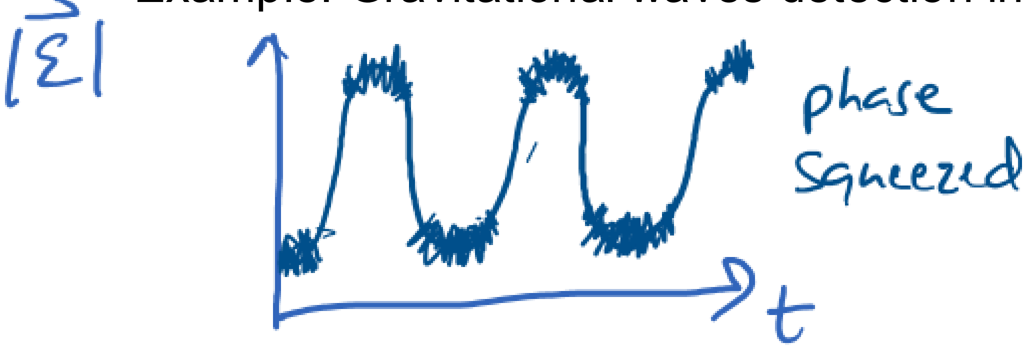
Heisenberg limit and squeezed light

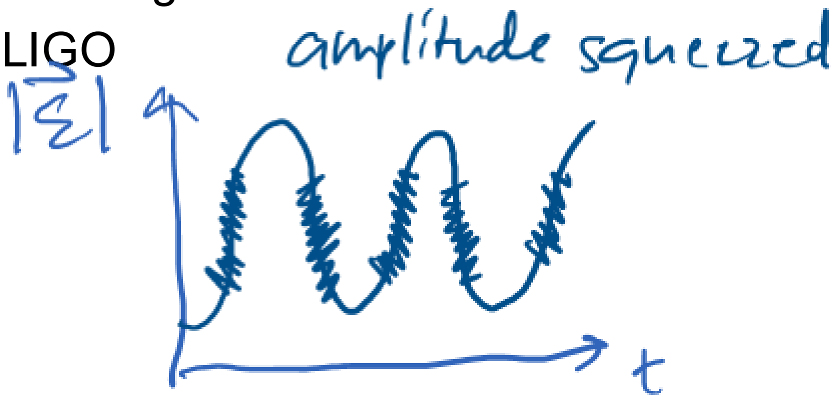
- Can we beat the standard measurement limit?
  - What is the most uncertainty we can produce in n?

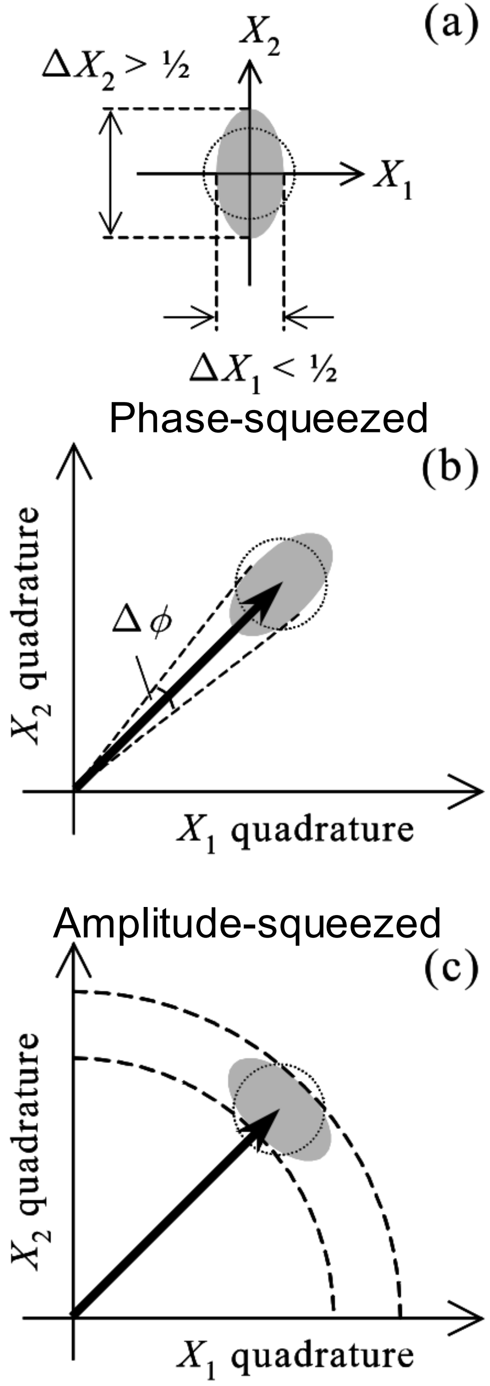
$$\Delta n \approx n$$

$$\Delta\phi_{\text{Heisenberg limit}} = \frac{1}{2n} \propto \frac{1}{I}$$

- Amplitude and phase noise can be traded off to improve precision in measurements
  - Requires nonlinear optics to generate squeezed light
  - Example: Gravitational waves detection in LIGO



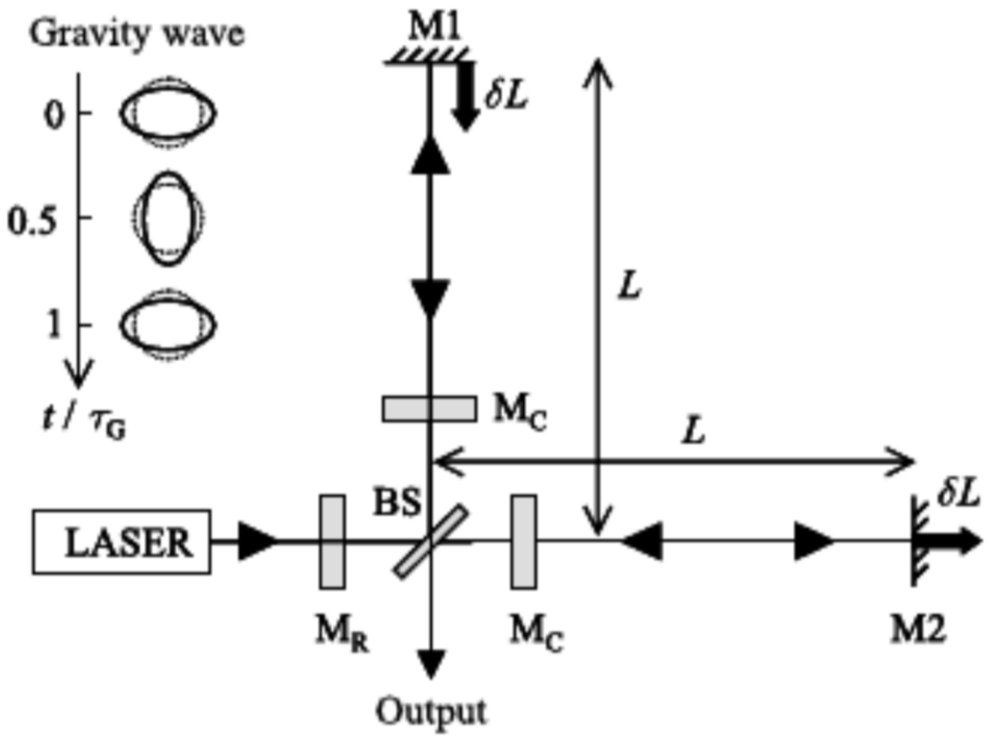


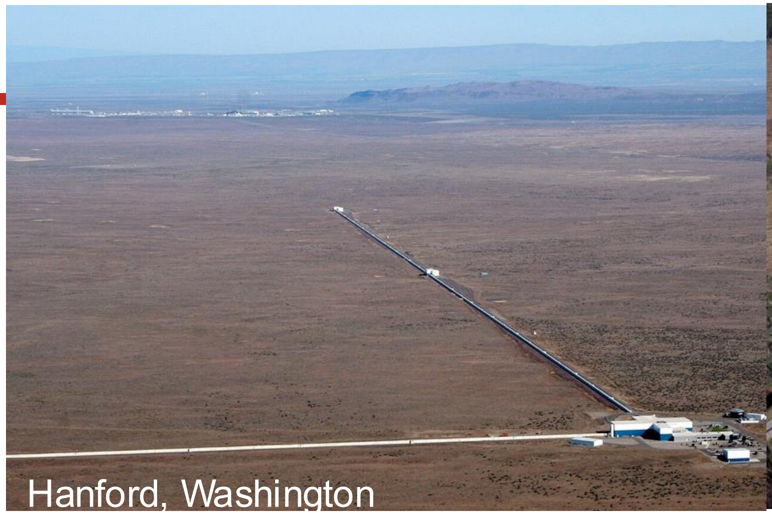


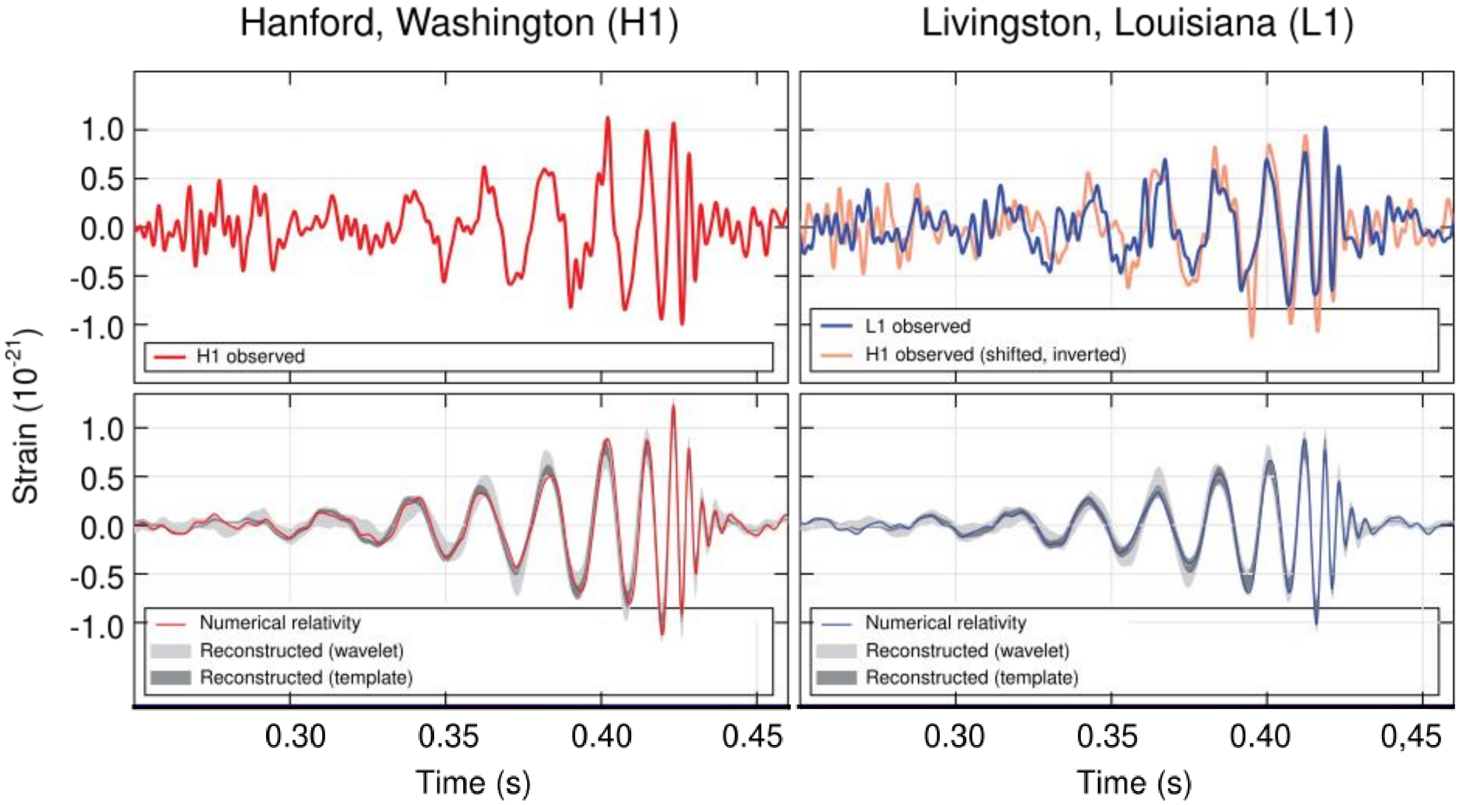
Dowling, Jonathan P. "Quantum optical metrology–the lowdown on high-N00N states." *Contemporary physics* 49.2 (2008): 125-143. https://arxiv.org/pdf/0904.0163

Exam 1 covers up to this point

Laser Interferometer Gravitational-Wave Observatory (LIGO)







Mirror displacement: $$\frac{\delta L}{\lambda} > \frac{\delta \phi}{2\pi}$$

Strain: $$h \approx \frac{\delta L}{L_{tot}}$$

Zero-point energy

- Explanation using Heisenberg uncertainty principle
  - Zero energy, corresponding to a particle sitting motionless, corresponds to precisely determined position and momentum → violates uncertainty principle
  - Distribution in momentum and position → particle energy must be greater than the minimum of the potential well

- Zero-point energy for harmonic oscillators (which approximate all potential wells near their minima):

$$E_0 = \frac{\hbar\omega}{2}\text{ , where } \omega \text{ is the angular frequency of the oscillation of the system}$$

- Experimental observations:
  - Vacuum field in quantum optics
  - ”Lamb shifts” in atomic spectra (later in the course)
  - Spontaneous emission (later in the course)

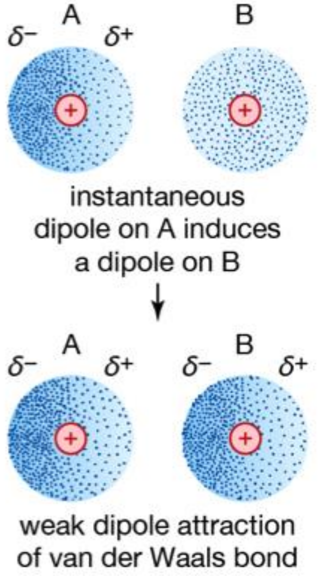
Connection of zero-point energy to vacuum field

van der Waals and Casimir forces

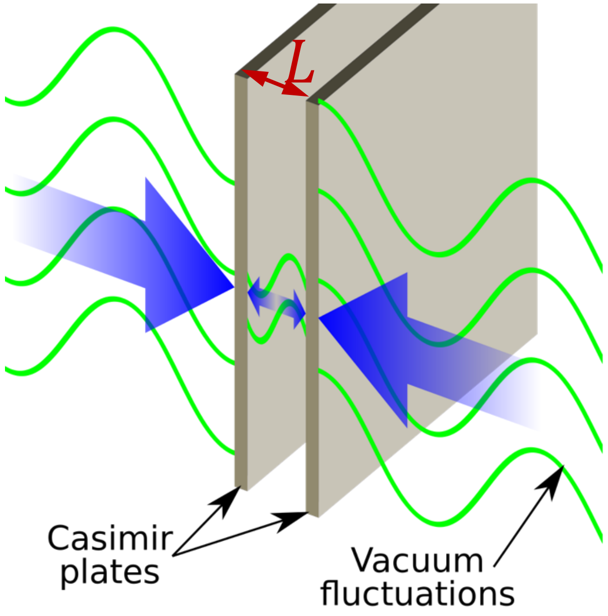
- Zero-point energies of charges in objects (molecules, small particles, etc.) induce electromagnetic fluctuations that interact at small distances (few nm) → attractive van der Waals forces
  - Instantaneous interaction
  - At larger separations, interaction is non-instantaneous due to the finite speed of light (“Casimir force”)

- Casimir force between parallel conducting mirrors in vacuum of area A

$$F_{Casimir} = -\frac{\hbar c \pi^2}{240 L^4} A$$



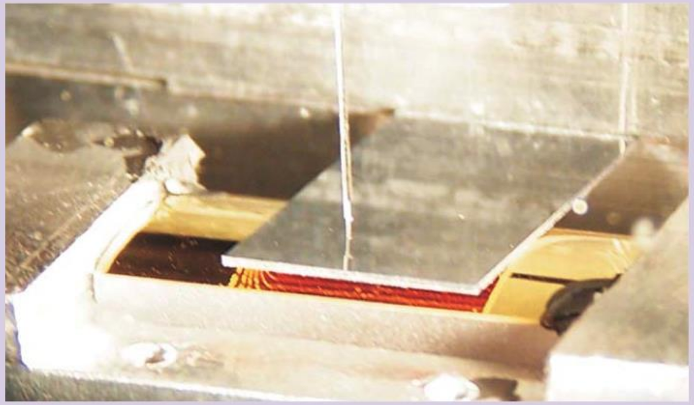
Source: Encyclopedia Britannica



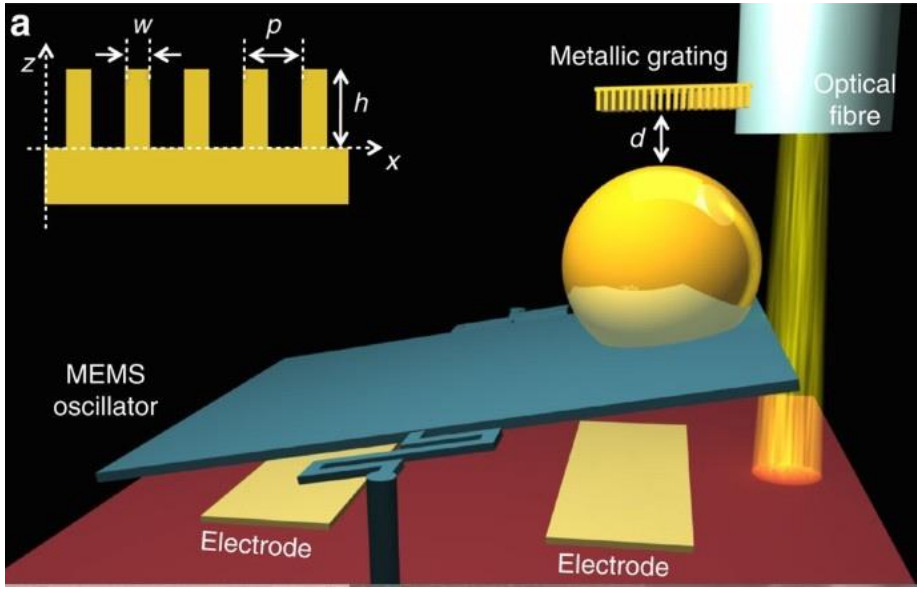
Source: Emok (Wikipedia), CC BY-SA 3.0

Measurement of Casimir forces

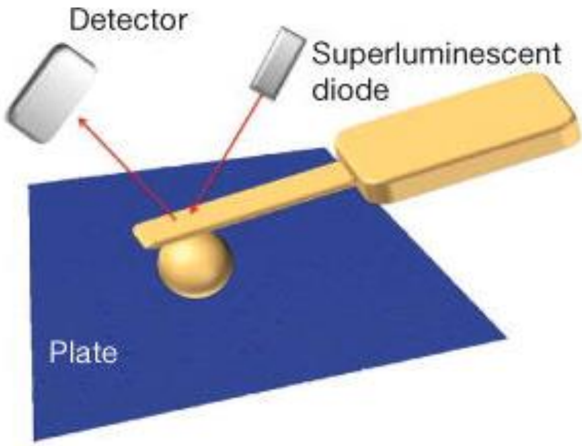
Figure 2. A plane–cylinder geometry is employed in a new Casimir-force experiment under development by Roberto Onofrio and colleagues at Dartmouth College. The plane, a gold-plated silicon wafer 1 cm wide and 200 μm thick, is clamped along one edge between two plates of aluminum and is configured as a mechanical resonator. The vertical white object is an optical fiber used for interferometric distance measurement. In the foreground is the cylinder holder, mounted on a piezoelectric transducer for fine distance control; mechanical actuators provide coarse distance and tilt control. The cylinder, a gold-coated optical-quality cylindrical lens, is 2 cm long and 1 cm wide, with a radius of curvature of 2 cm. A change in the mechanical resonance frequency of the clamped plane will provide a measure of the gradient of the Casimir force. The Dartmouth group hopes to obtain sufficient sensitivity to measure with high accuracy the thermal correction to the Casimir force. (Courtesy of Michael Brown-Hayes and Roberto Onofrio, Dartmouth College.)

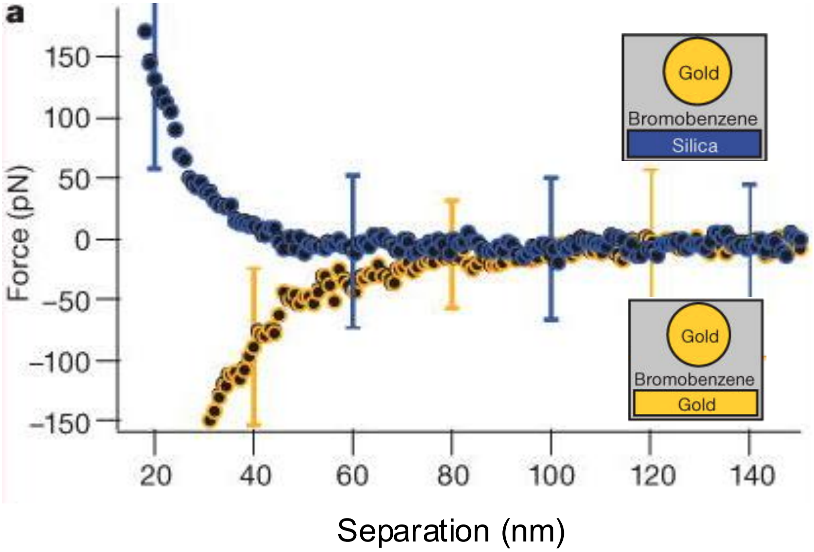


https://physicstoday.scitation.org/doi/pdf/10.1063/1.2711635



F. Intravaia et al Nature Communications 4 2515 (2013)





J. Munday et al Nature 457:170-173 (2009)