P1

Let

$$X = a_0 \mathbb{I} + \sum_{k=1}^{3} \sigma_k a_k, \tag{1}$$

where σ_k are pauli matrices and a_0, a_1 are numbers.

1. $\operatorname{Tr} X$ and $\operatorname{Tr}(\sigma_k X)$.

Noticing pauli matrices are traceless:

$$\operatorname{Tr}(X) = \operatorname{Tr}(a_0 \mathbb{I}) + \operatorname{Tr}\left(\sum_k \sigma_k a_k\right) = 2a_0. \tag{2}$$

Then consider

$$\begin{aligned} \operatorname{Tr}(\sigma_k X) &= \operatorname{Tr}(a_0 \sigma_k \mathbb{I}) + \operatorname{Tr}\left(\sum_j \sigma_k \sigma_j\right) \\ &= 0 + \sum_j a_j \operatorname{Tr}(\sigma_j \sigma_k). \end{aligned} \tag{3}$$

Using the fact that $\operatorname{Tr}(\sigma_j \sigma_k) = 2\delta_{jk}$:

$$\operatorname{Tr}(\sigma_k X) = \sum_{k=1}^{3} a_k \cdot 2\delta_{jk} = 2a_k. \tag{4}$$

2. Find a_0, a_k w.r.t. X_{ij}.

Write

$$X_{ij} = a_0 \delta_{ij} + \sum_{k=1}^{3} \left((\sigma_k)_{ij} a_k \right). \tag{5}$$

Then for diagonal elements:

$$\begin{aligned} X_{ii} &= a_0 + (\sigma_3)_{ii} a_3, \\ \Rightarrow X_{11} &= a_0 + a_3, \quad X_{22} = a_0 - a_3. \end{aligned} \tag{6}$$

Off diagonal elements: for $m \neq n$:

$$\begin{aligned} X_{mn} &= \sigma_{1_{mn}} a_1 + \sigma_{2_{mn}} a_2 \\ \Rightarrow X_{12} &= a_1 - i a_2, \quad X_{21} = a_1 + i a_2. \end{aligned} \tag{7}$$

P2.

Consider a ket space spanned by $\{|a'\rangle\}$ of Hermitian operator $\boldsymbol{A}$. No degeneracy.

1. $\Pi_{a'}(\boldsymbol{A} - a')$ is the null operator.

Proof: Since $\{|a'\rangle\}$ spans the space, it's sufficient to show that $\Pi_{a'}(\boldsymbol{A} - a')$ annilates all basisket. To show, consider arbitrary basis ket $|a''\rangle$.

$$\begin{aligned}\left[\prod_{a'}(\boldsymbol{A} - a')\right]|a''\rangle &= \dots \times (\boldsymbol{A} - a'')|a''\rangle \times \dots \\ &= \boldsymbol{A}|a''\rangle - a''|a''\rangle \\ &= a''|a''\rangle - a''|a''\rangle = 0,\end{aligned} \tag{8}$$

and thus $\Pi_{a'}(\boldsymbol{A} - a')$ annilates all basis kets, and therefore nullify all vectors in this space.

2.

Let

$$\hat{P} \equiv \prod_{a'' \neq a'} \frac{\boldsymbol{A} - a''}{a' - a''}. \tag{9}$$

Notice that $\hat{P}|a'\rangle = |a'\rangle$, $\hat{P}|a_k\rangle = 0$ (for any $a_k \neq a'$.). Explicitely:

$$\hat{P}|a'\rangle = \frac{\boldsymbol{A} - a''}{a' - a''}|a'\rangle = \frac{a' - a''}{a' - a''}|a'\rangle = |a'\rangle; \tag{10}$$

and for any $a_k \neq a'$,

$$\hat{P}|a_k\rangle = \dots \times \frac{\boldsymbol{A} - a_k}{a' - a_k} \times \dots |a_k\rangle = 0. \tag{11}$$

It's clear that $\hat{P}$ is a projection operator onto the eigenspace corresponding to the eigenvalue a'.

3. Illustrate both results using $A = S_z$ for spin 1/2 system.

Recall that the eigenkets $\{|+\rangle, |-\rangle\}$ of the Hermitian operator S_z form an orthornormal basis, just like $\{|a'\rangle\}$ in the assumption.

$$S_z|\pm\rangle = \pm\frac{\hbar}{2}|\pm\rangle, \tag{12}$$

and from which we can observe:

$$\prod_{a'}(S_z - a')|\pm\rangle = \left(S_z - \frac{\hbar}{2}\right)\left(S_z + \frac{\hbar}{2}\right)|\pm\rangle = 0. \tag{13}$$

Further,

$$\hat{P} = \prod_{a'' \neq a'} \frac{\boldsymbol{A} - a''}{a' - a''} = \begin{cases} \dfrac{S_z + \frac{\hbar}{2}}{\frac{\hbar}{2} + \frac{\hbar}{2}} & \text{for } a' = -\frac{\hbar}{2} \\ \dfrac{S_z - \frac{\hbar}{2}}{-\frac{\hbar}{2} - \frac{\hbar}{2}} & \text{for } a' = \frac{\hbar}{2} \end{cases}, \tag{14}$$

and thus for $a' = -\frac{\hbar}{2}$: $\hat{P}|+\rangle = 0$, $\hat{P}|-\rangle = 1$; for $a' = \frac{\hbar}{2}$: $\hat{P}|-\rangle = 0$, $\hat{P}|+\rangle = 1$, as expected from part 2.

P3

Construct $|\boldsymbol{S}\cdot\hat{\boldsymbol{n}};+\rangle$ as a linear combination of $|+\rangle, |-\rangle$, s.t.

$$\boldsymbol{S}\cdot\hat{\boldsymbol{n}}\,|\boldsymbol{S}\cdot\hat{\boldsymbol{n}};+\rangle = \frac{\hbar}{2}|\boldsymbol{S}\cdot\hat{\boldsymbol{n}};+\rangle, \tag{15}$$

We recall that, in spherical coordinates,

$$\hat{n} = (\sin\beta\cos\alpha, \sin\beta\sin\alpha, \cos\beta); \tag{16}$$

and the definition of $\boldsymbol{S}$:

$$S_x = \frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, S_y = \frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}, S_z = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}. \tag{17}$$

Then,

$$\begin{aligned}\boldsymbol{S}\cdot\hat{\boldsymbol{n}} &= S_x n_x + S_y n_y + S_z n_z\\ &= \frac{\hbar}{2}\begin{pmatrix}\cos\beta & \sin\beta\cos\alpha - i\sin\beta\sin\alpha\\ \sin\beta\cos\alpha + i\sin\beta\sin\alpha & -\cos\beta\end{pmatrix}\\ &= \frac{\hbar}{2}\begin{pmatrix}\cos\beta & \sin\beta e^{-i\alpha}\\ \sin\beta e^{i\alpha} & -\cos\beta\end{pmatrix}.\end{aligned} \tag{18}$$

Let eigenket $|\boldsymbol{S}\cdot\hat{\boldsymbol{n}};+\rangle = \left(\begin{smallmatrix}c_1\\ c_2\end{smallmatrix}\right)$. The eigenvalue equation then reads:

$$\begin{aligned}\frac{\hbar}{2}\begin{pmatrix}\cos\beta & \sin\beta e^{-i\alpha}\\ \sin\beta e^{i\alpha} & -\cos\beta\end{pmatrix}\begin{pmatrix}c_1\\ c_2\end{pmatrix} &= \frac{\hbar}{2}\begin{pmatrix}c_1\\ c_2\end{pmatrix}\\ \Rightarrow \begin{pmatrix}c_1\cos\beta + c_2\sin\beta e^{-i\alpha}\\ c_1\sin\beta e^{i\alpha} - c_2\cos\beta\end{pmatrix} &= \begin{pmatrix}c_1\\ c_2\end{pmatrix}\end{aligned} \tag{19}$$

This yields two dependent equations

$$\begin{aligned}c_1(\cos\beta - 1) + c_2\sin\beta e^{-i\alpha} &= 0,\\ c_1\sin\beta e^{i\alpha} - c_2(\cos\beta + 1) &= 0.\end{aligned} \tag{20}$$

from which,

$$c_1 = \frac{\sin\beta e^{-i\alpha}}{1-\cos\beta}c_2 \Rightarrow \quad c_1 = \frac{\cos\left(\frac{\beta}{2}\right)e^{-i\alpha}}{\sin\left(\frac{\beta}{2}\right)}c_2. \tag{21}$$

Normalization condition for the eige solution requires

$$\begin{aligned}|c_1|^2 + |c_2|^2 &= 1\\ \Rightarrow |c_2|^2\left(\frac{\cos^2\left(\frac{\beta}{2}\right)}{\sin^2\left(\frac{\beta}{2}\right)} + 1\right) &= 1\\ \Rightarrow |c_2|^2 &= \sin^2\left(\frac{\beta}{2}\right).\end{aligned} \tag{22}$$

Then for an arbitrary azimuthal angle α , we have

$$c_2 = \sin\left(\frac{\beta}{2}\right)e^{i\alpha}, c_1 = \cos\left(\frac{\beta}{2}\right). \tag{23}$$

Thus the eigenket is found to be

$$\boxed{\cos\left(\frac{\beta}{2}\right)|+\rangle + \sin\left(\frac{\beta}{2}\right)e^{i\alpha}|-\rangle.} \tag{24}$$

P4.

A two level system with Hamiltonian

$$H = H_{11}|1\rangle\langle 1| + H_{22}|2\rangle\langle 2| + H_{12}(|1\rangle\langle 2| + |2\rangle\langle 1|. \tag{25}$$

Find energy eigenvalues and eigenkets.

1. method 1, solve eigenvalue problem Explicitely.

2. method 2, use result from P3.