

Extramize the functional that describes defects in crystals

$$\mathcal{F}[\theta(x)] = \int_{-\infty}^{\infty} \left[ \frac{\kappa}{2} \left( \frac{d\theta}{dx} \right)^2 + V(1 - \cos \theta(x)) \right] dx \quad (1)$$

subject to boundary conditions  $\theta(-\infty) = 0, \theta(\infty) = 2\pi$ . Further, find the energy cost  $E = \mathcal{F}[\Theta]$  for  $\Theta$  that extramizes  $F$ .

We read off the functional

$$F(\theta, \theta') = \frac{\kappa}{2} \theta'^2 + V(1 - \cos \theta). \quad (2)$$

To use Euler-Lagrange equation, we first prepare

$$\frac{\partial F}{\partial \theta} = V \sin \theta; \quad \frac{\partial F}{\partial \theta'} = \kappa \theta'. \quad (3)$$

Then E-L gives

$$\begin{aligned} \frac{d}{dx}(\kappa \theta') - V \sin \theta &= 0 \\ \Rightarrow \theta'' &= \frac{1}{\xi^2} \sin \theta, \quad \xi = \sqrt{\frac{\kappa}{V}}. \end{aligned} \quad (4)$$

This non-linear ODE can be solved indirectly had we noticed that  $F$  is independent of  $x$ , and so Beltrami identity applies:

$$\begin{aligned} F - \theta' \frac{dF}{d\theta'} &= C \\ \Rightarrow \frac{\kappa}{2} \theta'^2 - V(1 - \cos \theta) &= C \end{aligned} \quad (5)$$

Boundary conditions give  $x \rightarrow \pm\infty \Rightarrow \theta' = 0, \cos \theta = 1 \Rightarrow C = 0$ . Thus we solve

$$\frac{\kappa}{2} \theta'^2 = V(1 - \cos \theta) \Rightarrow \theta'^2 = \frac{4}{\xi^2} \sin^2 \left( \frac{\theta}{2} \right) \quad (6)$$

to get

$$\begin{aligned} \frac{d\theta}{2 \sin(\frac{\theta}{2})} &= \frac{dx}{\xi} \\ \Rightarrow \frac{1}{2} \csc \left( \frac{\theta}{2} \right) d\theta &= \frac{1}{\xi} dx. \end{aligned} \quad (7)$$

Since  $\int \csc u \, du = \ln(\tan(\frac{u}{2}))$ , we take integration on both sides over an arbitrary range  $[x_0, x]$ . this yields

$$\theta = \arctan \left[ \frac{\exp(x - x_0)}{\xi} \right] \equiv \Theta. \quad (8)$$

The corresponding functional is thus (with  $\frac{\kappa}{2} \Theta'^2 = V(1 - \cos \Theta)$ )

$$F(\Theta, \Theta') = 4V \sin^2 \left( \frac{\Theta}{2} \right) = 4V \operatorname{sech}^2 \left( \frac{x - x_0}{\xi} \right). \quad (9)$$

The energy cost is therefore

$$E = \int_{-\infty}^{\infty} F \, dx = 4V \int_{-\infty}^{\infty} \operatorname{sech}^2 \left( \frac{x - x_0}{\xi} \right) dx = 8V\xi = \boxed{8\sqrt{\kappa V}}. \quad (10)$$

## P2

Apply calculus of variations to find coupled equations for  $\psi$  and  $\mathbf{A}$  that extremize  $\mathcal{F}$ , where  $\mathcal{F}$  is the free energy functional of a superconductor:

$$\mathcal{F}[\psi(\mathbf{r}), \mathbf{A}(\mathbf{r})] = \int_V \left( -|\psi|^2 + \frac{1}{2} |\psi|^4 + \left| \left( -\frac{i}{\kappa} \nabla - \mathbf{A} \right) \psi \right|^2 + \mathbf{H}^2 \right) dV \quad (11)$$

Let  $D_i = -\frac{i}{\kappa} \partial_i - A_i$ ,  $H_i = \varepsilon_{ijk} \partial_j A_k$ , and define

$$F(\psi, \psi^*, \mathbf{A}, \partial_i \psi, \partial_i \psi^*, \partial_i A_j) = -|\psi|^2 + \frac{1}{2} |\psi|^4 + |D_i \psi|^2 + H_i^2. \quad (12)$$

Here we use a generalized Euler Lagrange equation for  $F(\varphi_j, \partial_i \varphi_j)$ :

$$\frac{\partial F}{\partial \varphi_j} - \partial_i \left( \frac{\partial F}{\partial (\partial_i \varphi_j)} \right) = 0. \quad (13)$$

We will deal with  $\varphi_j = \psi^*$  and  $\varphi_j = A_j$  separately. For  $\varphi_j = \psi^*$ , we have

$$\begin{aligned} \frac{\partial F}{\partial \psi^*} &= -\psi + |\psi|^2 \psi - A_i D_i \psi, \\ \frac{\partial F}{\partial (\partial_i \psi^*)} &= \frac{\partial}{\partial (\partial_i \psi^*)} \left( -\frac{i}{\kappa} \partial_i \psi - A_i \psi \right) \left( \frac{i}{\kappa} \partial_i \psi^* - A_i \psi^* \right) = \frac{i}{\kappa} D_i \psi. \end{aligned} \quad (14)$$

Then EL reads

$$\begin{aligned} \frac{\partial F}{\partial \psi^*} - \partial_i \left( \frac{\partial F}{\partial (\partial_i \psi^*)} \right) &= 0 \\ \Rightarrow -\psi + |\psi|^2 \psi - A_i D_i \psi - \frac{i}{\kappa} \partial_i D_i \psi &= 0 \\ \Rightarrow \boxed{-\psi + |\psi|^2 \psi + D_i D_i \psi = 0.} \end{aligned} \quad (15)$$

On the other hand, for  $\varphi_j = A_j$ , defining  $D_i^\dagger = (\frac{i}{\kappa} \partial_i - A_i)$  we have

$$\begin{aligned} \frac{\partial F}{\partial A_j} &= -[\psi D_j^\dagger \psi^* + \psi^* D_j \psi], \\ \frac{\partial F}{\partial (\partial_i A_j)} &= 2H_k \frac{\partial H_k}{\partial (\partial_i A_j)} = 2H_k \varepsilon_{kij} \\ \Rightarrow \partial_i \left( \frac{\partial F}{\partial (\partial_i A_j)} \right) &= 2\varepsilon_{kij} \partial_i H_k = -2(\nabla \times \mathbf{H})_j. \end{aligned} \quad (16)$$

The EL equation thus reads

$$\begin{aligned} -[\psi D_j^\dagger \psi^* + \psi^* D_j \psi] + 2(\nabla \times \mathbf{H})_j &= 0 \Rightarrow \boxed{\nabla \times \mathbf{H} = \mathbf{J}}, \\ \text{where } J_i &= \frac{1}{2} [\psi D_i^\dagger \psi^* + \psi^* D_i \psi] = \frac{1}{2} \left[ \frac{i}{\kappa} (\psi \partial_i \psi^* - \psi^* \partial_i \psi) - (\psi A_i \psi^* + \psi^* A_i \psi) \right] \\ &= \frac{1}{2} \left[ \frac{i}{\kappa} (-2i \operatorname{Im}(\psi^* \partial_i \psi)) - 2A_i |\psi|^2 \right] = \frac{1}{\kappa} \operatorname{Im}(\psi^* \partial_i \psi) - A_i |\psi|^2. \end{aligned} \quad (17)$$

Collecting above, we arrive at

$$\begin{cases} -\psi + |\psi|^2 \psi + D_i D_i \psi = 0 \\ \nabla \times \mathbf{H} = \mathbf{J} \\ \mathbf{J} = \frac{1}{\kappa} \operatorname{Im}(\psi^* \nabla \psi) - \mathbf{A} |\psi|^2 \end{cases}. \quad (18)$$

### P3

Find the kink configuration  $\varphi(x)$  and its energy cost for the functional

$$\mathcal{F}[\varphi] = \int_{-\infty}^{\infty} \left[ \frac{\kappa}{2} \left( \frac{d\varphi}{dx} \right)^2 + \frac{V}{4} (\varphi^2 - \varphi_0^2)^2 \right] dx \quad (19)$$

subject to boundary conditions  $\varphi(-\infty) = -\varphi_0, \varphi(\infty) = \varphi_0$ .

We read off the functional

$$F(\varphi, \varphi') = \frac{\kappa}{2} \varphi'^2 + \frac{V}{4} (\varphi^2 - \varphi_0^2)^2. \quad (20)$$

Similar to P1, we notice that  $F$  is independent of  $x$ , and so Beltrami identity applies:

$$\begin{aligned} F - \varphi' \frac{dF}{d\varphi'} &= C \\ \Rightarrow -\frac{\kappa}{2} \varphi'^2 + \frac{V}{4} (\varphi^2 - \varphi_0^2)^2 &= C \end{aligned} \quad (21)$$

Boundary conditions give  $x \rightarrow \pm\infty \Rightarrow \varphi' = 0, \varphi^2 = \varphi_0^2 \Rightarrow C = 0$ . Thus we solve

$$\frac{\kappa}{2} \varphi'^2 = \frac{V}{4} (\varphi^2 - \varphi_0^2)^2 \Rightarrow \varphi' = \sqrt{\frac{V}{2\kappa}} (\varphi_0^2 - \varphi^2) \quad (22)$$

to get

$$\frac{d\varphi}{\varphi_0^2 - \varphi^2} = \sqrt{\frac{V}{2\kappa}} dx. \quad (23)$$

Since  $\int \frac{du}{a^2 - u^2} = \frac{1}{a} \operatorname{artanh}\left(\frac{u}{a}\right)$ , we take integration on both sides over an arbitrary range  $[x_0, x]$ . this yields

$$\frac{1}{\varphi_0} \operatorname{artanh}\left(\frac{\varphi}{\varphi_0}\right) = \sqrt{\frac{V}{2\kappa}} (x - x_0). \quad (24)$$

Defining a characteristic length  $\xi = \sqrt{\frac{\kappa}{V\varphi_0^2}}$ , the equation simplifies to  $\operatorname{artanh}\left(\frac{\varphi}{\varphi_0}\right) = \frac{x - x_0}{\sqrt{2}\xi}$ .

$$\varphi = \varphi_0 \tanh\left[\frac{x - x_0}{\sqrt{2}\xi}\right] \equiv \Phi. \quad (25)$$

The corresponding functional is thus (with  $\frac{\kappa}{2} \Phi'^2 = \frac{V}{4} (\Phi^2 - \varphi_0^2)^2$ )  $F(\Phi, \Phi') = \kappa \Phi'^2 = \frac{\kappa \varphi_0^2}{2\xi^2} \operatorname{sech}^4\left(\frac{x - x_0}{\sqrt{2}\xi}\right) = \frac{V}{2} \varphi_0^4 \operatorname{sech}^4\left(\frac{x - x_0}{\sqrt{2}\xi}\right)$ . The energy cost is therefore  $E = \int_{-\infty}^{\infty} F dx = \frac{V}{2} \varphi_0^4 \int_{-\infty}^{\infty} \operatorname{sech}^4\left(\frac{x - x_0}{\sqrt{2}\xi}\right) dx$  Using the standard integral

$$\int_{-\infty}^{\infty} \operatorname{sech}^4(u) du = \frac{4}{3}, \quad (26)$$

we get

$$E = \frac{2\sqrt{2}}{3} \varphi_0^3 \sqrt{\kappa V}. \quad (27)$$

## P4

Consider a functional of three functions

$$G[y_1, y_2, y_3] = \int_{x_0}^{x_1} F[x, y, y'] dx, \quad F[x, y, y'] = \frac{1}{2} \sum_{i,j=1}^3 g_{ij}(y) y_{i'} y_{j'} - U(y), \quad (28)$$

where  $g_{ij}$  is a symmetric matrix that depends on  $y$ . Derive the system's of Euler-Lagrange equations that extremize  $G$ . Use Christoffel symbols to simplify your result.

Using Einstein summation convention, and denote  $\partial_k u = \frac{\partial u}{\partial y_k}$ , we write

$$F = \frac{1}{2} g_{ij} y_{i'} y_{j'} - U(y). \quad (29)$$

And the Euler-Lagrange equation elements are found as follows

$$\begin{aligned} \frac{\partial F}{\partial y_k} &= \frac{\partial}{\partial y_k} \left[ \frac{1}{2} g_{ij} y_{i'} y_{j'} - U(y) \right] = \frac{1}{2} (\partial_k g_{ij}) y_{i'} y_{j'} - \partial_k U, \\ \frac{\partial F}{\partial y_{k'}} &= \frac{1}{2} g_{ij} \partial_k (y_{i'} y_{j'}) = \frac{1}{2} g_{ij} (\delta_{ik} y_{j'} + y_{i'} \delta_{jk}) = g_{kj} y_{j'} \end{aligned} \quad (30)$$

$$\begin{aligned} \frac{d}{dx} \left( \frac{\partial F}{\partial y_{k'}} \right) &= \frac{d}{dx} (g_{kj} y_{j'}) = \frac{dg_{kj}}{dx} y_{j'} + \frac{d}{dx} y_{j'} g_{kj} \\ &= \frac{dy_l}{dx} \frac{\partial}{\partial y_l} g_{kj} y_{j'} + y_{j'}'' g_{kj} = (\partial_l g_{kj}) y_l' y_{j'} + g_{kj} y_{j'}''. \end{aligned} \quad (31)$$

E-L equation reads

$$\frac{1}{2} (\partial_k g_{ij}) y_l' y_{j'} - \partial_k U - (\partial_l g_{kj}) y_l' y_{j'} - g_{kj} y_{j'}'' = 0 \quad (32)$$

Rearranging

$$g_{kj} y_{j'}'' + \partial_l g_{kj} y_l' y_{j'} - \frac{1}{2} \partial_k g_{ij} y_l' y_{j'} + \partial_k U = 0. \quad (33)$$

We can use the identity  $g_{mk} g_{kj} = \delta(mj)$  to simplify:

$$y_m'' + g_{mk} \underbrace{\left[ \partial_l g_{kj} - \frac{1}{2} \partial_k g_{ij} \right]}_{(*)} y_l' y_{j'} + g_{mk} \partial_k U = 0. \quad (34)$$

Since  $y_l' y_{j'}$  is symmetric in  $l, j$ , we can symmetrize the term

$$\partial_l g_{kj} y_l' y_{j'} = \frac{1}{2} [\partial_l g_{kj} + \partial_j g_{kl}] y_l' y_{j'}. \quad (35)$$

Then  $(*)$  becomes

$$\frac{1}{2} [\partial_l g_{kj} + \partial_j g_{kl} - \partial_k g_{ij}] y_l' y_{j'} = \Gamma_{lj}^k y_l' y_{j'}, \quad (36)$$

and we arrive at a simplified form:

$$y_m'' + g_{mk} \Gamma_{lj}^k y_l' y_{j'} + g_{mk} \partial_k U = 0. \quad (37)$$

## P5

Consider the Lagrangian density for a one-dimensional electron-phonon system:

$$\mathcal{L} = \frac{i}{2}(\dot{\psi}\psi^* - \psi\dot{\psi}^*) - \frac{1}{2}|\partial_x\psi|^2 + \frac{1}{2}(\partial_t u)^2 - \frac{s^2}{2}(\partial_x u)^2 + g(\partial_x u)|\psi|^2. \quad (38)$$

First, derive the equations of motion for the complex field  $\psi(x, t)$  and the real field  $u(x, t)$ . Then, find a traveling-wave solution (polaron) of the form  $\psi(x, t) = e^{i(ax-bt)}\varphi(x - Vt)$  and  $u(x, t) = U(x - Vt)$ . Finally, calculate the total energy  $E(V)$  of this solution and find the ground state energy  $E_0$ .

### 1. Equations of Motion from Variational Principle

The Lagrangian density is given by  $\mathcal{L} = \mathcal{L}_\psi + \mathcal{L}_u + \mathcal{L}_{\psi u}$ :

$$\mathcal{L} = \frac{i}{2}(\dot{\psi}\psi^* - \psi\dot{\psi}^*) - \frac{1}{2}|\partial_x\psi|^2 + \frac{1}{2}(\partial_t u)^2 - \frac{s^2}{2}(\partial_x u)^2 + g(\partial_x u)|\psi|^2 \quad (39)$$

The equations of motion are derived from the Euler-Lagrange equations.

#### Equation for $\psi(x, t)$

We vary with respect to the independent field  $\psi^*$ . The Euler-Lagrange equation is:

$$\frac{\partial \mathcal{L}}{\partial \psi^*} - \partial_t \left( \frac{\partial \mathcal{L}}{\partial \dot{\psi}^*} \right) - \partial_x \left( \frac{\partial \mathcal{L}}{\partial (\partial_x \psi^*)} \right) = 0 \quad (40)$$

The required partial derivatives are:

- $$\frac{\partial \mathcal{L}}{\partial \psi^*} = \frac{i}{2}\dot{\psi} + g(\partial_x u)\psi \quad (41)$$

- $$\frac{\partial \mathcal{L}}{\partial \dot{\psi}^*} = -\frac{i}{2}\psi \quad (42)$$

- $$\frac{\partial \mathcal{L}}{\partial (\partial_x \psi^*)} = -\frac{1}{2}(\partial_x \psi) \quad (43)$$

Substituting these into the equation gives:

$$\left( \frac{i}{2}\dot{\psi} + g(\partial_x u)\psi \right) - \partial_t \left( -\frac{i}{2}\psi \right) - \partial_x \left( -\frac{1}{2}\partial_x \psi \right) = 0 \quad (44)$$

$$\Rightarrow \frac{i}{2}\dot{\psi} + g(\partial_x u)\psi + \frac{i}{2}\dot{\psi} + \frac{1}{2}\partial_x^2 \psi = 0 \quad (45)$$

This simplifies to the first equation of motion:

$$\boxed{i\partial_t \psi = -\frac{1}{2}\partial_x^2 \psi - g(\partial_x u)\psi} \quad (46)$$

#### Equation for $u(x, t)$

We vary with respect to the real field  $u$ :

$$\frac{\partial \mathcal{L}}{\partial u} - \partial_t \left( \frac{\partial \mathcal{L}}{\partial (\partial_t u)} \right) - \partial_x \left( \frac{\partial \mathcal{L}}{\partial (\partial_x u)} \right) = 0 \quad (47)$$

The partial derivatives are:

- $$\frac{\partial \mathcal{L}}{\partial u} = 0 \quad (48)$$

- $$\frac{\partial \mathcal{L}}{\partial (\partial_t u)} = \partial_t u \quad (49)$$

- $$\frac{\partial \mathcal{L}}{\partial (\partial_x u)} = -s^2(\partial_x u) + g|\psi|^2 \quad (50)$$

Substituting these yields:

$$0 - \partial_t(\partial_t u) - \partial_x(-s^2 \partial_x u + g|\psi|^2) = 0 \quad (51)$$

This simplifies to the second equation of motion:

$$\partial_t^2 u - s^2 \partial_x^2 u + g \partial_x(|\psi|^2) = 0 \quad (52)$$

## 2. Traveling-Wave Solution

We seek a solution where the electron and the string deformation propagate together at a velocity  $V$ . We introduce the co-moving coordinate  $\xi = x - Vt$  and use the ansatz:

$$\psi(x, t) = e^{i(ax-bt)} \varphi(\xi), \quad u(x, t) = U(\xi) \quad (53)$$

where  $\varphi(\xi)$  is a real-valued profile function, and  $a, b$  are constants. The derivatives transform as  $\partial_t = -V \frac{d}{d\xi}$  and  $\partial_x = \frac{d}{d\xi}$ .

Applying these transformations to the ansatz (denoting  $\frac{d}{d\xi}$  with a prime):

$$\bullet \quad \partial_t \psi = e^{i(ax-bt)} (-ib\varphi - V\varphi') \quad (54)$$

$$\bullet \quad \partial_x \psi = e^{i(ax-bt)} (ia\varphi + \varphi') \quad (55)$$

$$\bullet \quad \partial_x^2 \psi = e^{i(ax-bt)} (-a^2\varphi + 2ia\varphi' + \varphi'') \quad (56)$$

$$\bullet \quad \partial_t u = -VU', \quad \partial_t^2 u = V^2U'' \quad (57)$$

$$\bullet \quad \partial_x u = U', \quad \partial_x^2 u = U'' \quad (58)$$

Substituting these into the first equation of motion and canceling the phase factor  $e^{i(ax-bt)}$  gives:

$$i(-ib\varphi - V\varphi') = -\frac{1}{2}(-a^2\varphi + 2ia\varphi' + \varphi'') - gU'\varphi \quad (59)$$

$$\Rightarrow b\varphi - iV\varphi' = \frac{a^2}{2}\varphi - ia\varphi' - \frac{1}{2}\varphi'' - gU'\varphi \quad (60)$$

Separating the real and imaginary parts:

• **Imaginary Part:**

$$-V\varphi' = -a\varphi' \Rightarrow a = V. \quad (61)$$

This aligns the phase velocity with the group velocity, a feature of Galilean invariance.

• **Real Part:**

$$b\varphi = \frac{V^2}{2}\varphi - \frac{1}{2}\varphi'' - gU'\varphi. \quad (62)$$

Rearranging gives:

$$-\frac{1}{2}\varphi'' - gU'\varphi = \left(b - \frac{V^2}{2}\right)\varphi \quad (63)$$

Define  $\mu := b - \frac{V^2}{2}$ , which represents the energy of the electron in the co-moving frame. The equation becomes:

$$-\frac{1}{2}\varphi''(\xi) - gU'(\xi)\varphi(\xi) = \mu\varphi(\xi) \quad (64)$$

Next, substituting the ansatz into the second equation of motion gives:

$$V^2U'' - s^2U'' + g\frac{d}{d\xi}(\varphi^2) = 0 \Rightarrow (V^2 - s^2)U'' + g(\varphi^2)' = 0 \quad (65)$$

## Solving the Coupled ODEs

We integrate Equation 65 with respect to  $\xi$ . For a localized polaron, we assume the fields vanish at infinity ( $\varphi(\xi) \rightarrow 0$  and  $U'(\xi) \rightarrow 0$  as  $|\xi| \rightarrow \infty$ ), which sets the integration constant to zero:

$$(V^2 - s^2)U' + g\varphi^2 = 0 \implies U'(\xi) = \frac{g}{s^2 - V^2}\varphi^2(\xi) \quad (66)$$

For a localized “bright soliton” solution to exist, the induced self-interaction for the electron must be attractive. Substituting  $U'$  into Equation 63:

$$-\frac{1}{2}\varphi'' - \frac{g^2}{s^2 - V^2}\varphi^3 = \mu\varphi \quad (67)$$

The nonlinear term  $-\frac{g^2}{s^2 - V^2}\varphi^3$  is attractive if the coefficient is negative, which requires  $s^2 - V^2 > 0$ , or  $|V| < s$ . The polaron must travel slower than the speed of sound in the medium. We also need  $\mu < 0$  for a bound state. Let  $\mu = -\nu$  where  $\nu > 0$ . The equation is:

$$\varphi'' = 2\nu\varphi - \frac{2g^2}{s^2 - V^2}\varphi^3 \quad (68)$$

This is a standard nonlinear Schrödinger (NLS) equation. The solution is of the form:

$$\varphi(\xi) = A \operatorname{sech}(\alpha\xi) \quad (69)$$

Substituting this form back into the NLS equation gives the relations:  $\alpha^2 = 2\nu$  and  $A^2 = \frac{2\nu(s^2 - V^2)}{g^2} = \frac{\alpha^2(s^2 - V^2)}{g^2}$ .

We impose the normalization condition for a single electron,  $\int_{-\infty}^{\infty} |\psi|^2 dx = \int_{-\infty}^{\infty} \varphi^2 d\xi = 1$ :

$$\int_{-\infty}^{\infty} A^2 \operatorname{sech}^2(\alpha\xi) d\xi = A^2 \left[ \frac{1}{\alpha} \tanh(\alpha\xi) \right]_{-\infty}^{\infty} = \frac{2A^2}{\alpha} = 1 \quad (70)$$

Using  $A^2 = \frac{\alpha^2(s^2 - V^2)}{g^2}$ , we get  $\frac{2}{\alpha} \frac{\alpha^2(s^2 - V^2)}{g^2} = 1$ , which solves for  $\alpha$ :

$$\alpha = \frac{g^2}{2(s^2 - V^2)} \quad (71)$$

From this, we find the other parameters:

$$A^2 = \frac{\alpha}{2} = \frac{g^2}{4(s^2 - V^2)}, \quad \nu = \frac{\alpha^2}{2} = \frac{g^4}{8(s^2 - V^2)^2} \quad (72)$$

### 3. System Energy $E(V)$

The Hamiltonian (energy) density is derived from the Lagrangian density:  $\mathcal{H} = \sum \pi_i \dot{\varphi}_i - \mathcal{L}$ .

$$\pi_\psi = \frac{\partial \mathcal{L}}{\partial \dot{\psi}} = \frac{i}{2}\psi^*, \quad \pi_u = \frac{\partial \mathcal{L}}{\partial \dot{u}} = \partial_t u \quad (73)$$

$$\begin{aligned} \mathcal{H} &= \left( \frac{i}{2}\psi^* \right) \dot{\psi} + \left( -\frac{i}{2}\psi \right) \dot{\psi}^* + (\partial_t u)(\partial_t u) - \mathcal{L} \\ &= \frac{1}{2}|\partial_x \psi|^2 + \frac{1}{2}(\partial_t u)^2 + \frac{s^2}{2}(\partial_x u)^2 - g(\partial_x u)|\psi|^2 \end{aligned} \quad (74)$$

The total energy is  $E(V) = \int_{-\infty}^{\infty} \mathcal{H} dx$ . We substitute the traveling wave solution into the terms of  $\mathcal{H}$ :

- $|\partial_x \psi|^2 = |e^{i(Vx - bt)}(\varphi' + iV\varphi)|^2 = (\varphi')^2 + V^2\varphi^2 \quad (75)$

- $(\partial_t u)^2 = (-VU')^2 = V^2(U')^2 \quad (76)$

- $(\partial_x u)^2 = (U')^2 \quad (77)$

• Using  $U' = \frac{g}{s^2 - V^2}\varphi^2$ , the terms involving  $u$  become:

$$\frac{1}{2}(V^2 + s^2)(U')^2 - gU'\varphi^2 = \frac{1}{2}(V^2 + s^2)\frac{g^2}{(s^2 - V^2)^2}\varphi^4 - \frac{g^2}{s^2 - V^2}\varphi^4 = -\frac{g^2(s^2 - 3V^2)}{2(s^2 - V^2)^2}\varphi^4 \quad (78)$$

The total energy integral is:

$$E(V) = \int_{-\infty}^{\infty} \left[ \frac{1}{2} ((\varphi')^2 + V^2 \varphi^2) - \frac{g^2(s^2 - 3V^2)}{2(s^2 - V^2)^2} \varphi^4 \right] d\xi \quad (79)$$

We evaluate the separate integrals using our solution for  $\varphi$ :

$$\bullet \quad \int \frac{1}{2} V^2 \varphi^2 d\xi = \frac{1}{2} V^2 \quad \int \varphi^2 d\xi = \frac{1}{2} V^2 \quad (80)$$

(from normalization).

$$\bullet \quad \int \frac{1}{2} (\varphi')^2 d\xi = \int \frac{1}{2} A^2 \alpha^2 \operatorname{sech}^2(\alpha\xi) \tanh^2(\alpha\xi) d\xi = \frac{A^2 \alpha}{3} = \frac{(\frac{\alpha}{2}) \alpha}{3} = \frac{\alpha^2}{6} = \frac{\nu}{3} = \frac{g^4}{24(s^2 - V^2)^2} \quad (81)$$

$$\bullet \quad \int \varphi^4 d\xi = \int A^4 \operatorname{sech}^4(\alpha\xi) d\xi = A^4 \frac{4}{3\alpha} = \left(\frac{\alpha}{2}\right)^2 \frac{4}{3\alpha} = \frac{\alpha}{3} = \frac{g^2}{6(s^2 - V^2)} \quad (82)$$

Combining these results:

$$\begin{aligned} E(V) &= \frac{1}{2} V^2 + \frac{g^4}{24(s^2 - V^2)^2} - \frac{g^2(s^2 - 3V^2)}{2(s^2 - V^2)^2} \left( \frac{g^2}{6(s^2 - V^2)} \right) \\ &= \frac{1}{2} V^2 + \frac{g^4}{24(s^2 - V^2)^3} [(s^2 - V^2) - 2(s^2 - 3V^2)] \\ &= \frac{1}{2} V^2 + \frac{g^4}{24(s^2 - V^2)^3} (5V^2 - s^2) \end{aligned} \quad (83)$$

This gives the final energy as a function of velocity for the subsonic polaron, valid for  $|V| < s$ :

$$\boxed{E(V) = \frac{V^2}{2} + \frac{g^4(5V^2 - s^2)}{24(s^2 - V^2)^3}} \quad (84)$$

### Ground State Energy $E_0$

The ground state energy is the energy of a static polaron, found by setting  $V = 0$ .

$$E_0 = E(0) = \frac{0}{2} + \frac{g^4(0 - s^2)}{24(s^2 - 0)^3} = \boxed{\frac{-g^4}{24s^4}} \quad (85)$$

The negative energy confirms that the formation of a static polaron is energetically favorable, representing a stable bound state.