

# Introduction

This section, 1.1, is the foundation for that new way of thinking. Before we even touch it, let's review in detail the concepts from your undergraduate course that these notes assume you have mastered.

## Part 1: The Foundation - What You Should Remember

Think of this as our toolkit. Everything in Section 1.1 is built on these ideas.

### 1. Thermodynamics and Free Energy

The central principle of thermodynamics for a system at constant volume and temperature is that it will arrange itself to **minimize its Helmholtz Free Energy,  $F$** .

- **Definition:**  $F = U - TS$ 
  - $U$  is the internal energy. Systems like to be in low-energy states.
  - $S$  is the entropy. Systems like to be in high-entropy (highly disordered) states.
  - $T$  is the temperature, which controls the balance.
- **The Competition:** Free energy is all about the competition between energy and entropy.
  - At **low temperatures**, the  $TS$  term is small, so minimizing  $F$  is mostly about minimizing energy  $U$ . This leads to **ordered states** (like a solid crystal or an aligned magnet).
  - At **high temperatures**, the  $TS$  term dominates. Minimizing  $F$  is mostly about maximizing entropy  $S$ . This leads to **disordered states** (like a gas or a non-magnetic metal).
- **A phase transition is the point where the balance tips.** It's the temperature at which the system decides it's better to switch from being ordered to disordered (or vice-versa).

### 2. The Partition Function ( $Z$ )

The partition function is the single most important object in statistical mechanics. It is the bridge connecting the microscopic details of a system to its macroscopic thermodynamic properties.

- **Definition (Canonical Ensemble):** For a system at temperature  $T$ , the partition function is the sum over all possible microscopic states  $i$  of the system:

$$Z = \sum_{\text{states } i} e^{-E_i/k_B T} = \sum_{\text{states } i} e^{-\beta E_i}$$

where  $E_i$  is the energy of state  $i$ , and  $\beta = 1/(k_B T)$ .

- **The Bridge to Thermodynamics:** The magic of the partition function is that once you calculate it, you can find the free energy with a simple formula:

$$F = -k_B T \ln Z$$

And from the free energy, you can derive everything else (entropy, pressure, specific heat, etc.) by taking derivatives.

### 3. Phase Transitions (The Undergraduate View)

You likely learned about the Ehrenfest classification:

- **First-Order Transitions:** The first derivatives of the free energy (like entropy  $S = -(\partial F/\partial T)_V$ ) are discontinuous. This means there is a **latent heat**. Examples: boiling water, melting ice.
- **Second-Order (or Continuous) Transitions:** The first derivatives of  $F$  are continuous, but the *second* derivatives (like specific heat  $C_V = -T(\partial^2 F/\partial T^2)_V$ ) are discontinuous or divergent. There is **no latent heat**. Examples: the ferromagnetic-paramagnetic transition, the superfluid transition.
- **The focus of these notes is on continuous transitions.**

## 4. The Ising Model: Our Favorite Example

This is the quintessential model for a phase transition.

- **The Setup:** Imagine a grid (a lattice). On each site  $i$ , there is a "spin"  $\sigma_i$  that can only be  $+1$  (up) or  $-1$  (down).
- **The Physics (Hamiltonian):** The energy of a configuration of spins is given by:

$$H = -J \sum_{\langle i,j \rangle} \sigma_i \sigma_j - h \sum_i \sigma_i$$

- The first term describes the interaction between **nearest neighbors**  $\langle i,j \rangle$ . If  $J > 0$ , the energy is lowest when neighboring spins are the same (ferromagnetism).
- The second term describes the interaction with an **external magnetic field**  $h$ .
- **The Symmetry:** This is the most crucial part for Section 1.1. If the external field  $h = 0$ , the Hamiltonian has a  **$\mathbf{Z}_2$  symmetry**. This means if you take any configuration of spins and flip every single one ( $\sigma_i \rightarrow -\sigma_i$  for all  $i$ ), the energy *does not change*. The laws of physics for the magnet don't have a preference for "up" or "down".

## 5. Mean-Field Theory (The Undergraduate View)

Solving the Ising model exactly is very hard. The simplest approximation is mean-field theory.

- **The Idea:** Instead of considering the interaction of a spin  $\sigma_i$  with its specific neighbors  $\sigma_j$ , we replace the neighbors with their *average* value, the magnetization  $M = \langle \sigma \rangle$ .
- **The Result:** This turns a complex many-body problem into a simple single-body problem, where each spin just feels an "effective" magnetic field from the average magnetization of all other spins.
- **The Limitation:** This approximation **ignores fluctuations**. It assumes the environment of every spin is perfectly average, but in reality, a spin's neighbor might be pointing the "wrong" way. This becomes very important near the phase transition.

## Part 2: A Slow Walk Through Section 1.1

With our toolkit ready, let's tackle the text. The goal of this section is to create a theory that describes the universal behavior of all continuous phase transitions *near the critical point*. We want to zoom in on what's happening right at  $T_c$  and ignore the messy details far away from it.

### 1.1.1 Definitions: The New Language

This section gives us the precise vocabulary to talk about this general theory.

#### Definition: Spontaneous Symmetry Breaking (SSB)

- **The Concept:** We just saw that the Ising model Hamiltonian (with  $h = 0$ ) is perfectly symmetric under a global spin flip. Above the critical temperature  $T_c$ , the system is a paramagnet. The average magnetization is zero. The system *looks* symmetric, just like the Hamiltonian. Below  $T_c$ , the system becomes a ferromagnet. All the spins align, but they have to *choose* a direction: either mostly up or mostly down. Let's say they choose "up". Now the system itself is no longer symmetric (it has a clear "up" direction), even though the underlying laws (the Hamiltonian) still are.
- **This is SSB:** The system's lowest energy state (the "ground state") is less symmetric than the physical laws that govern it. The symmetry is "spontaneously" broken by the system's choice.
- *The text's examples:*
  - Water molecules: The laws of physics governing them are the same in all directions (rotationally symmetric). But when they freeze, they form a crystal lattice with specific, preferred directions. The symmetry is broken.
  - Compass needle: The physics is symmetric (North and South are equivalent), but the needle *breaks* that symmetry by pointing to one of them.

### Definition: Order Parameter

- **The Concept:** How do we mathematically track SSB? We need a quantity that tells us which phase we are in.
- **The Definition:** An **order parameter** is a quantity that is **zero** in the symmetric (disordered) phase and **non-zero** in the broken-symmetry (ordered) phase.
- For our Ising magnet, the order parameter is the magnetization,  $M$ .
- **Equation (1):** This is the formal, general definition of the order parameter,  $\langle\phi(x)\rangle$ . It looks terrifying, but it's a very clever trick based on our Ising model intuition.

$$\langle\phi(x)\rangle = \lim_{h \rightarrow 0} \lim_{Vol \rightarrow \infty} \frac{1}{Z[h]} \frac{\delta Z}{\delta h(x)}$$

Let's decode it:

- $\phi(x)$  is our generalized order parameter field (think of it as magnetization at every point  $x$ ).
- $h(x)$  is a generalized external field that couples to  $\phi(x)$ .
- The fraction  $\frac{1}{Z[h]} \frac{\delta Z}{\delta h(x)}$  is just the formal way of writing the average value of  $\phi(x)$  in the presence of the field  $h(x)$ .
- **The Limits are the key:** Why this weird sequence? Imagine you want to create a ferromagnet. If you just cool it down with no external field, some parts might decide to point up, and other parts might point down, forming domains with a total magnetization of zero. To get a single magnet, you cool it in the presence of a small external field  $h$ . This field "convinces" all the spins to align in one direction.
  1.  $\lim Vol \rightarrow \infty$ : First, we make the system infinitely large. In an infinite system, it would take an infinite amount of energy to flip all the spins at once. They become "locked" in their chosen direction.
  2.  $\lim h \rightarrow 0$ : After the system is locked, we can turn off the small external field. The magnetization will remain because the system is locked. This procedure allows us to select one of the broken-symmetry ground states and calculate its non-zero order parameter.

### Definition: Universality and Universal Critical Exponents

- **The Concept:** This is one of the most profound ideas in modern physics. It states that the behavior of a system *right at the critical point* is universal—it does not depend on the microscopic details. A huge variety of systems (magnets, liquid-gas transitions, superfluids) can be described by the same mathematical framework and the same set of numbers called **critical exponents**.
- These exponents describe *how* physical quantities behave as we approach the critical temperature  $T_c$ . For example:
  - Order parameter:  $\phi \sim (T_c - T)^\beta$
  - Susceptibility (how much  $\phi$  responds to  $h$ ):  $\chi \sim |T - T_c|^{-\gamma}$
  - Specific Heat:  $C_V \sim |T - T_c|^{-\alpha}$
- Universality means that systems with the same symmetry and dimension will have the same values for  $\alpha, \beta, \gamma$ , etc.

#### 1.1.2 The Ginzburg-Landau Free Energy

This is our main tool. Instead of starting from a microscopic Hamiltonian (like for the Ising model) and trying to calculate  $F = -k_B T \ln Z$  (which is usually impossible), we take a shortcut. We are only interested in the behavior near  $T_c$ , which is governed by the order parameter  $\phi$ . So, let's just write down the free energy directly as a function of  $\phi$ .

- **The Philosophy:** We will construct the simplest possible polynomial for the free energy,  $F_{GL}[\phi]$ , that respects the symmetries of the problem.

- **Constructing the Formula (Equation 3):** Let's focus on a problem with the same up-down ( $Z_2$ ) symmetry as the Ising model. This means the free energy should not change if we replace  $\phi$  with  $-\phi$ . Therefore, our polynomial can only have **even powers** of  $\phi$ .

$$F_{GL}[\phi] = \int d^D x [\dots]$$

We are summing the free energy density over all of space. Inside the brackets:

1. **The Potential Terms:** We start with a Taylor series in  $\phi$ :  $A + B\phi + C\phi^2 + D\phi^3 + E\phi^4 + \dots$ 
    - The constant  $A$  doesn't matter.
    - Symmetry forbids odd powers like  $B\phi$  and  $D\phi^3$ .
    - So we are left with  $C\phi^2 + E\phi^4 + \dots$ . Let's stop at the fourth order for simplicity. We'll call the coefficients  $\beta$  and  $\lambda/2$ . So we have:  $\beta\phi^2 + \frac{\lambda}{2}(\phi^T\phi)^2$ . (The transpose notation is for vector order parameters; for a simple scalar, it's just  $\frac{\lambda}{2}\phi^4$ ). We need  $\lambda > 0$  or the energy would go to  $-\infty$  for large  $\phi$ , which is unphysical.
  2. **The Gradient Term:** What if the order parameter is not uniform in space? The system should pay an energy penalty for this. The simplest term that is always positive and respects the symmetry is  $(\partial_x\phi)^T(\partial_x\phi)$ , which is just  $(\nabla\phi)^2$ . Let's call its coefficient  $\alpha$ .
- **The Result is the Ginzburg-Landau free energy.** It's a model, a guess, but a brilliantly effective one.
  - **The Phase Transition Mechanism:** The magic is in the coefficient  $\beta$ . We make the crucial physical assumption that  $\beta$  changes with temperature and passes through zero at  $T_c$ . The simplest possible assumption is a linear one:

$$\beta \approx a(T - T_c) \quad (\text{where } a > 0)$$

Now, let's look at the shape of the potential part of the free energy,  $V(\phi) = \beta\phi^2 + \frac{\lambda}{2}\phi^4$ :

- **Above  $T_c$  ( $T > T_c$ ):**  $\beta$  is positive. The potential  $V(\phi)$  is a simple bowl shape, with its one and only minimum at  $\phi = 0$ . The system minimizes its free energy by having **zero order parameter**. This is the disordered, symmetric phase.
- **Below  $T_c$  ( $T < T_c$ ):**  $\beta$  is negative. The potential  $V(\phi)$  now looks like the bottom of a wine bottle or a "Mexican hat". The point  $\phi = 0$  is now a local *maximum* (unstable). There are two new minima located at  $\phi_0 = \pm\sqrt{-\beta/\lambda}$ . The system must **choose one of these minima**. This choice breaks the symmetry, and the system now has a **non-zero order parameter**. This is the ordered phase.

This simple, symmetry-based model has beautifully captured the essence of a continuous phase transition!

### 1.1.3 Critical Behavior at Mean Field Level

"Mean Field" in this context means we do the simplest possible thing: we find the value of  $\phi$  that minimizes the Ginzburg-Landau free energy, and we completely **ignore thermal fluctuations** that would cause  $\phi$  to jiggle around that minimum. This is equivalent to assuming  $\phi$  is uniform in space and just finding the minimum of the potential  $V(\phi)$ .

Now we can use our results from the minima to calculate the critical exponents in this approximation.

- **Order parameter exponent  $\beta$ :** This describes how the order parameter grows below  $T_c$ :  $\phi \sim (T_c - T)^\beta$ .
  - Our mean-field result for the order parameter is  $\phi_0 = \sqrt{-\beta/\lambda}$ .
  - We assumed  $\beta = a(T - T_c)$ .
  - Substituting this in:  $\phi_0 = \sqrt{-a(T - T_c)/\lambda} = \sqrt{a(T_c - T)/\lambda}$ .
  - We see that  $\phi_0$  is proportional to  $(T_c - T)^{1/2}$ .
  - Therefore, in mean-field theory, the critical exponent is  $\beta = 1/2$ .

- **Order parameter exponent  $\delta$ :** This describes the relationship between the order parameter and the field *at* the critical point:  $\phi \sim h^{1/\delta}$  at  $T = T_c$ .
  - At  $T = T_c$ , our coefficient  $\beta = 0$ .
  - The free energy, including the term for an external field, is  $F \approx \frac{\lambda}{2}\phi^4 - h\phi$ .
  - To find the minimum, we take the derivative with respect to  $\phi$  and set it to zero:  $2\lambda\phi^3 - h = 0$ .
  - Solving for  $\phi$  gives  $\phi = (h/2\lambda)^{1/3}$ .
  - We see that  $\phi$  is proportional to  $h^{1/3}$ .
  - Therefore, in mean-field theory, the critical exponent is  $\delta = 3$ .

The other exponents  $(\alpha, \gamma, \nu)$  are calculated in the text using similar logic. The key takeaway is that this simple Ginzburg-Landau model, evaluated at the mean-field level, gives a complete set of critical exponents. These values are not exactly correct for real systems (because we ignored fluctuations), but they are the essential starting point for all modern theories of critical phenomena. The rest of the course is about how to systematically improve upon this by including the effect of those fluctuations.