# Physics 731

Assignment #1, Solutions

1. (a) $\text{Tr}(\mathbf{1}_{2\times 2}) = 2$. We also have Tr $\sigma_\text{i} = 0$ and Tr $\sigma_i\sigma_j = 2\delta_{ij}$. Therefore,

$$\text{Tr}X = 2a_0 = \sum_{i=1}^{2} X_{ii}, \quad \text{Tr}(\sigma_j X) = 2a_j = \sum_{i,k=1}^{2} (\sigma_j)_{ik} X_{kj}.$$

(b) Using the above, we see that $a_0 = (X_{11} + X_{22})/2$, $a_1 = (X_{12} + X_{21})/2$, $a_2 = i(X_{12} - X_{21})/2$, and $a_3 = (X_{11} - X_{22})/2$. Note that any $2 \times 2$ matrix can be written in this way. (We will make use of this result on multiple occasions.)

2. (a) Given that $A|a'\rangle = a'|a'\rangle$ (and there is no degeneracy),

$$\Pi_{a'}(A - a')|a''\rangle = \Pi_{a'}(a'' - a')|a''\rangle = 0,$$

since the product includes $a' = a''$. Therefore,

$$\Pi_{a'}(A - a')|\alpha\rangle = 0$$

for any $|\alpha\rangle$ (*i.e.*, it is the null operator).
(b) Act on $|a'\rangle$:

$$\Pi_{a''\neq a'} \frac{(A - a'')}{(a' - a'')}|a'\rangle = \Pi_{a''\neq a'} \frac{(a' - a'')}{(a' - a'')}|a'\rangle = |a'\rangle.$$

Act on $|a'''\rangle \neq |a'\rangle$:

$$\Pi_{a''\neq a'} \frac{(A - a'')}{(a' - a'')}|a'''\rangle = \Pi_{a''\neq a'} \frac{(a''' - a'')}{(a' - a'')}|a'''\rangle = 0.$$

Hence, $\Pi_{a''\neq a'}(A - a'')/(a' - a'')$ is a projection operator onto the state $|a'\rangle$.
(c) For a spin $1/2$ system, we can write an arbitrary ket as $|\alpha\rangle = c_1|+\rangle + c_2|-\rangle$, with $|c_1|^2 + |c_2|^2 = 1$. The operator of (a) acting on $|\alpha\rangle$ is

$$(S_z - \hbar/2)(S_z + \hbar/2)|\alpha\rangle = (S_z - \hbar/2)(S_z + \hbar/2)(c_1|+\rangle + c_2|-\rangle).$$

Now we note that $(S_z + \hbar/2)|-\rangle = 0$, and $(S_z - \hbar/2)|+\rangle = 0$. Hence, $(S_z - \hbar/2)(S_z + \hbar/2)|\alpha\rangle = 0$. For (b), we have for example, taking $a' = -\hbar/2$,

$$\frac{S_z - \hbar/2}{(-\hbar/2 - \hbar/2)}|\alpha\rangle = -\frac{1}{\hbar}(S_z - \hbar/2)(c_1|+\rangle + c_2|-\rangle) = -\frac{1}{\hbar}(-\hbar)|-\rangle = |-\rangle,$$

which shows this is the projection operator onto the state $|-\rangle$. Similarly, if we take $a' = +\hbar/2$, we obtain the projection operator onto the state $|+\rangle$.

3. $\hat{n} = \cos\beta\hat{z} + (\cos\alpha\hat{x} + \sin\alpha\hat{y})\sin\beta$, and thus

$$\mathbf{S}\cdot\hat{n} = S_z\cos\beta + S_x\sin\beta\cos\alpha + S_y\sin\beta\sin\alpha = \frac{\hbar}{2}\begin{pmatrix} \cos\beta & \sin\beta e^{-i\alpha} \\ \sin\beta e^{i\alpha} & -\cos\beta \end{pmatrix}.$$

Since $\det(\boldsymbol{\sigma}\cdot\hat{\mathrm{n}}-\lambda\mathbb{1}) = (\lambda-\cos\beta)(\lambda+\cos\beta)-\sin\beta^2 = \lambda^2-1$, the eigenvalues of $\mathbf{S}\cdot\hat{n}$ are $\pm\hbar/2$. The eigenvector $|n,+\rangle = a_+|+\rangle + b_+|-\rangle$ for the $+\hbar/2$ eigenvalue is determined from

$$\frac{\hbar}{2}\begin{pmatrix} \cos\beta-1 & \sin\beta e^{-i\alpha} \\ \sin\beta e^{i\alpha} & -\cos\beta-1 \end{pmatrix}\begin{pmatrix} a_+ \\ b_+ \end{pmatrix} = 0,$$

which has the solution

$$\frac{b_+}{a_+} = \frac{\sin\beta e^{i\alpha}}{1+\cos\beta} = \frac{2\sin\frac{\beta}{2}\cos\frac{\beta}{2}e^{i\alpha}}{2\cos^2\frac{\beta}{2}} = \frac{\sin\frac{\beta}{2}e^{i\alpha}}{\cos\frac{\beta}{2}}.$$

Hence, $|n,+\rangle = \cos\frac{\beta}{2}|+\rangle + \sin\frac{\beta}{2}e^{i\alpha}|-\rangle$.

4. (a) Solving the eigenvalue problem explicitly, we have

$$\det\begin{pmatrix} H_{11}-\lambda & H_{12} \\ H_{12} & H_{22}-\lambda \end{pmatrix} = \lambda^2-(H_{11}+H_{22})\lambda + H_{11}H_{22}-H_{12}^2 = 0,$$

and hence

$$\lambda_\pm = \frac{1}{2}(H_{11}+H_{22}) \pm \frac{1}{2}\sqrt{(H_{11}-H_{22})^2+4H_{12}^2} \equiv \frac{1}{2}(H_{11}+H_{22}\pm\sqrt{\Lambda}).$$

The eigenvectors $|\alpha_\pm\rangle = \begin{pmatrix} a_\pm \\ b_\pm \end{pmatrix}$ are obtained from the relations (you can use either one):

$$(H_{11}-\lambda_\pm)a_\pm + H_{12}b_\pm = 0, \quad H_{12}a_\pm + (H_{22}-\lambda_\pm)b_\pm = 0,$$

such that

$$\begin{aligned} |\alpha_+\rangle &= \frac{1}{\sqrt{N}}\begin{pmatrix} \lambda_+-H_{22} \\ H_{12} \end{pmatrix} = \frac{1}{\sqrt{N}}\begin{pmatrix} \frac{1}{2}(H_{11}-H_{22})+\frac{1}{2}\sqrt{\Lambda} \\ H_{12} \end{pmatrix} \\ |\alpha_-\rangle &= \frac{1}{\sqrt{N}}\begin{pmatrix} -H_{12} \\ H_{11}-\lambda_- \end{pmatrix} = \frac{1}{\sqrt{N}}\begin{pmatrix} -H_{12} \\ \frac{1}{2}(H_{11}-H_{22})+\frac{1}{2}\sqrt{\Lambda} \end{pmatrix}, \end{aligned}$$

with $N = \frac{1}{4}((H_{11}-H_{22})+\sqrt{\Lambda})^2 + H_{12}^2$. For $H_{12}\to 0$, take $H_{11}\geq H_{22}$, then $\sqrt{\Lambda}\to H_{11}-H_{22}$. In this limit, $\lambda_+\to H_{11}$, $|\alpha_+\rangle \to \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and $\lambda_-\to H_{22}$, $|\alpha_-\rangle \to \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

(b) Use

$$H = h_0 + \boldsymbol{\sigma}\cdot\mathbf{h},$$

in which

$$h_0 = \frac{1}{2}\mathrm{Tr}H = \frac{1}{2}(H_{11}+H_{22}), \;\; \mathbf{h} = \frac{1}{2}\mathrm{Tr}(\boldsymbol{\sigma}H);\; h_1 = H_{12},\; h_2 = 0,\; h_3 = \frac{1}{2}(H_{11}-H_{22}).$$

Recall $\boldsymbol{\sigma}\cdot\hat{n}|\hat{n};\pm\rangle = \pm|\hat{n};\pm\rangle$, where

$$\begin{aligned} |\hat{n};+\rangle &= \cos\frac{\beta}{2}|+\rangle + e^{i\alpha}\sin\frac{\beta}{2}|-\rangle \\ |\hat{n};-\rangle &= -e^{-i\alpha}\sin\frac{\beta}{2}|+\rangle + \cos\frac{\beta}{2}|-\rangle. \end{aligned}$$

Clearly $\alpha = 0$. The eigenvalues of $H$ are then $\lambda_\pm = h_0 \pm |\mathbf{h}|$, with eigenvectors $|\hat{n};\pm\rangle$, in which $\cot\frac{\beta}{2} = (1+\cos\beta)/\sin\beta = (|\mathbf{h}|+h_3)/h_1$. Using the expressions for $h_0$ and $\mathbf{h}$, the expressions from part (a) are recovered.

5. (a) The state of the system is given by

$$|\hat{n};+\rangle = \cos\frac{\gamma}{2}|+\rangle + \sin\frac{\gamma}{2}|-\rangle.$$

The possible outcomes of the measurement of $S_x$ are $\pm\hbar/2$. The probability of measuring $\pm\hbar/2$ for $S_x$ is equal to $|_x\langle\pm|\hat{n};+\rangle|^2 = (1\pm\sin\gamma)/2$. Note that the probabilities sum to 1, as they should.
(b) The dispersion in $S_x$ is given by

$$\langle(\Delta S_x)^2\rangle = \langle S_x^2\rangle - \langle S_x\rangle^2.$$

Since $S_x^2 = (\hbar^2/4)\mathbb{1}$, $\langle S_x^2\rangle = \hbar^2/4$. Furthermore,

$$\langle S_x\rangle = \frac{\hbar}{2}\langle\hat{n};+|\sigma_x|\hat{n};+\rangle = \frac{\hbar}{2}\begin{pmatrix}\cos\frac{\gamma}{2} & \sin\frac{\gamma}{2}\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\cos\frac{\gamma}{2}\\ \sin\frac{\gamma}{2}\end{pmatrix} = \frac{\hbar}{2}\sin\gamma,.$$

Hence, $\langle(\Delta S_x)^2\rangle = \frac{\hbar^2}{4}(1-\sin^2\gamma) = \frac{\hbar^2}{4}\cos^2\gamma$.
For $\gamma = 0$, $|\hat{n};+\rangle = |+\rangle$, which is an equal weighting of $|\pm\rangle_x$. The probability of measuring $S_x$ and obtaining $\pm\hbar/2$ is thus equal to $1/2$, and the dispersion in $S_x$ is $\hbar^2/4$, as expected. For $\gamma=\pi$, $|\hat{n};+\rangle = |-\rangle$, and similar conclusions hold. For $\gamma=\pi/2$, $|\hat{n};+\rangle = (|+\rangle+|-\rangle)/\sqrt{2} = |+\rangle_x$. The probability is then equal to 1 for measuring $S_x$ to be $\hbar/2$ and 0 for it to be $-\hbar/2$. The dispersion of $S_x$ is zero, since it is an eigenstate of $S_x$.
(c) The possible outcomes of measuring $S_y$ are $\pm\hbar/2$, with respective probabilities $|_y\langle\pm|\hat{n};+\rangle|^2 = 1/2$. This result can be understood by noting that $|\hat{n};+\rangle$ can be written in terms of the $|\pm\rangle_y$ eigenstates as follows:

$$\begin{aligned}
|\hat{n};+\rangle &= \frac{1}{\sqrt{2}}\left(\cos\frac{\gamma}{2}(|+\rangle_y+|-\rangle_y) - i\sin\frac{\gamma}{2}(|+\rangle_y-|-\rangle_y)\right)\\
&= \frac{1}{\sqrt{2}}\left(\left(\cos\frac{\gamma}{2}-i\sin\frac{\gamma}{2}\right)|+\rangle_y + \left(\cos\frac{\gamma}{2}+i\sin\frac{\gamma}{2}\right)|-\rangle_y\right)\\
&= \frac{1}{\sqrt{2}}\left(e^{-i\gamma/2}|+\rangle_y + e^{i\gamma/2}|-\rangle_y\right) = \frac{1}{\sqrt{2}}e^{-i\gamma/2}\left(|+\rangle_y + e^{i\gamma}|-\rangle_y\right).
\end{aligned}$$

Therefore, the state is an equal-weight combination of $|\pm\rangle_y$ for any value of $\gamma$. For the dispersion of $S_y$, $\langle S_y^2\rangle = \langle S_x^2\rangle = \hbar^2/4$, while $\langle S_y\rangle$ vanishes:

$$\langle S_y\rangle = \frac{\hbar}{2}\langle\hat{n};+|\sigma_y|\hat{n};+\rangle = \frac{\hbar}{2}\begin{pmatrix}\cos\frac{\gamma}{2} & \sin\frac{\gamma}{2}\end{pmatrix}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}\cos\frac{\gamma}{2}\\ \sin\frac{\gamma}{2}\end{pmatrix} = 0.$$

Hence $\langle(\Delta S_y)^2\rangle = \hbar^2/4$, and is independent of $\gamma$. This is as expected once again from the fact that the state is an equal-weight combination of $|\pm\rangle_y$ for all values of $\gamma$.

6. Let $|\alpha\rangle = a_+|+\rangle + a_-|-\rangle$, in which $|a_+|^2+|a_-|^2=1$. Then

$$\langle\alpha|S_x|\alpha\rangle = \hbar\mathrm{Re}(a_+^*a_-),\quad \langle\alpha|S_y|\alpha\rangle = \hbar\mathrm{Im}(a_+^*a_-),$$

and

$$\langle\alpha|S_x^2|\alpha\rangle = \langle\alpha|S_y^2|\alpha\rangle = \frac{\hbar^2}{4}.$$

The product of the uncertainties,

$$\langle(\Delta S_x)^2\rangle\langle(\Delta S_y)^2\rangle = \left(\frac{\hbar^2}{4} - \hbar^2(\mathrm{Re}(a_+^*a_-))^2\right)\left(\frac{\hbar^2}{4} - \hbar^2(\mathrm{Im}(a_+^*a_-))^2\right),$$

is thus maximized for $\mathrm{Re}(a_+^*a_-) = \mathrm{Im}(a_+^*a_-) = 0$, such that $\alpha = |+\rangle$ or $\alpha = |-\rangle$. Using $[S_x,S_y] = i\hbar S_z$, it is straightforward to see that the uncertainty relation is satisfied for both of these states.