[S1r, S2, S3 2.3] An electron is subject to a uniform, time-independent magnetic field of strength $\mathbf{B}$ in the positive z-direction. At $t = 0$ the electron is known to be in an eigenstate of $\mathbf{S} \cdot \hat{\mathbf{n}}$ with eigenvalue $\hbar/2$ (*i.e.* $|\hat{n}; +\rangle$), where $\hat{\mathbf{n}}$ is a unit vector, lying in the xz-plane, that makes an angle β with the z-azis.
(a) Obtain the probability of finding the electron in the $s_x = +\hbar/2$ state as a function of time.
(b) Find the expectation value of S_x as a function of time.
(c) Show that your answers make good sense for (i) $\beta \to 0$ and (ii) $\beta \to \pi/2$.

1).
Hamiltonian under $\boldsymbol{B}$:

$$\boldsymbol{H} = -\boldsymbol{\mu} \cdot \boldsymbol{B} = \frac{egB}{2m_e} S_z \equiv \omega S_z. \tag{1}$$

Initial state

$$|\psi(0)\rangle = |\hat{n};t\rangle = \cos\left(\frac{\beta}{2}\right)|+\rangle + \sin\left(\frac{\beta}{2}\right)|-\rangle \tag{2}$$

and the time evolution is given by

$$\begin{aligned} |\psi(t)\rangle = \hat{U}(t)|\psi(0)\rangle = e^{-iHt/\hbar}|\psi(0)\rangle &= e^{-i\omega S_z t/\hbar}\left[\cos\left(\frac{\beta}{2}\right)|+\rangle + \sin\left(\frac{\beta}{2}\right)|-\rangle\right] \\ &= \cos\left(\frac{\beta}{2}\right)e^{-i\omega t/2}|+\rangle + \sin\left(\frac{\beta}{2}\right)e^{i\omega t/2}|-\rangle. \end{aligned} \tag{3}$$

So we have projection on $|+\rangle$ as

$$\langle +_x|\psi\rangle = \frac{1}{\sqrt{2}}(\langle +| + \langle -|)|\psi\rangle = \frac{1}{\sqrt{2}}\left(\cos\left(\frac{\beta}{2}\right)e^{-i\omega t/2} + \sin\left(\frac{\beta}{2}\right)e^{i\omega t/2}\right). \tag{4}$$

Denote $\Omega = \omega t/2$, we find the probability of measuring $S_x = \frac{\hbar}{2}$ is

$$|\langle +_x|\psi\rangle|^2 = \frac{1}{2}\left(\cos\left(\frac{\beta}{2}\right)e^{i\Omega} + \sin\left(\frac{\beta}{2}\right)e^{i\Omega}\right)^2 = \frac{1}{2}[1 + \sin\beta\cos(2\Omega)] = \boxed{\frac{1}{2}[1 + \sin\beta\cos(\omega t)].} \tag{5}$$

2).
To find $\langle S_x\rangle(t)$, it is eluding to work in the Hisenburg picture, since

$$\langle S_x\rangle = \langle\psi_0|S_x(t)|\psi_0\rangle, \tag{6}$$

and $S_x(t)$ can be found in the Hisenburg picture with

$$\begin{aligned} \frac{\mathrm{d}S_x}{\mathrm{d}t} &= \frac{1}{i\hbar}[S_x, H] = \frac{\omega}{i\hbar}[S_x, S_z] = -\omega S_y, \\ \frac{\mathrm{d}S_y}{\mathrm{d}t} &= \frac{\omega}{i\hbar}\left[S_y, S_z\right] = \omega S_x. \end{aligned} \tag{7}$$

Taking derivative of the first equation again, and substituting the second equation in, we have

$$\frac{\mathrm{d}^2 S_x}{\mathrm{d}t^2} = -\omega^2 S_x, \tag{8}$$

which has the solution

$$S_x = c_1\cos(\omega t) + c_2\sin(\omega t). \tag{9}$$

Initial value

$$
\begin{aligned}
S_x(0) &= c_1, \\
\frac{\mathrm{d}S_x(0)}{\mathrm{d}t} &= -c_2\omega = -\omega S_y(0)
\end{aligned}
\tag{10}
$$

So the solution becomes

$$
S_x(t) = S_x(0)\cos(\omega t) - S_y(0)\sin(\omega t). \tag{11}
$$

Going back to Equation 6, we have

$$
\begin{aligned}
\langle S_x\rangle(t) = \langle\psi_0|S_x(t)|\psi_0\rangle &= \left\langle\psi_0\middle|S_x(0)\cos\omega t - S_y(0)\sin\omega t\middle|\psi_0\right\rangle \\
&= \langle\psi_0|S_x(0)|\psi_0\rangle\cos\omega t - \left\langle\psi_0\middle|S_y(0)\middle|\psi_0\right\rangle\sin\omega t \\
&= \langle S_x\rangle_0\cos\omega t - \left\langle S_y\right\rangle_0\sin\omega t.
\end{aligned}
\tag{12}
$$

Since initial spin in x-z plane, $\left\langle S_y\right\rangle_0 = 0.$ and

$$
\begin{aligned}
\langle S_x\rangle_0 &= \langle\psi_0|S_x|\psi_0\rangle \\
&= \left(\cos\left(\frac{\beta}{2}\right)\langle+| + \sin\left(\frac{\beta}{2}\right)\langle-|\right)S_x\left(\cos\left(\frac{\beta}{2}\right)|+\rangle + \sin\left(\frac{\beta}{2}\right)|-\rangle\right) \\
&= \frac{\hbar}{2}\sin\beta.
\end{aligned}
\tag{13}
$$

Thus,

$$
\boxed{\langle S_x\rangle(t) = \frac{\hbar}{2}\sin\beta\cos\omega t.} \tag{14}
$$

3). Limit case of $\beta = 0, \frac{\pi}{2}$

1. $\beta = 0$:

$$
|\langle +_x|\psi\rangle|^2 = \frac{1}{2}, \quad \langle S_x\rangle(t) = 0 \tag{15}
$$

   Which means a completely mixed state in S_x measurement, and no time evolution in $\langle S_x\rangle$. This makes sense since $[S_x, S_z] \neq 0$ so definite S_z means completely uncertain S_x.

2. $\beta = \frac{\pi}{2}$:

$$
|\langle +_x|\psi\rangle|^2 = \frac{1}{2}[1+\cos(\omega t)], \quad \langle S_x\rangle(t) = \frac{\hbar}{2}\cos(\omega t). \tag{16}
$$

   This means that the initial state is fixed at $|+\rangle_x$, and the state evolution is a perfect precession around z axis.

2

[S2, S3 2.4 modified] Consider the problem of two-flavor neutrino oscillations, in which the lepton flavor eigenstates $|\nu_e\rangle$ and $|\nu_\mu\rangle$ are linear combinations of the energy eigenstates (known in this context as the "mass eigenstates") $|\nu_1\rangle$ and $|\nu_2\rangle$. The mass eigenstates have energies $E_{1,2}$, in which

$$E_i = (p^2c^2 + m_i^2c^4)^{1/2} \approx pc\left(1 + \frac{m_i^2c^2}{2p^2}\right),$$

(as neutrinos, which have very small masses compared to the typical momenta in a practical neutrino detection experiment, are highly relativistic). The flavor eigenstates can be written in terms of a flavor mixing angle θ as

$$|\nu_e\rangle = \cos\theta|\nu_1\rangle - \sin\theta|\nu_2\rangle, \qquad |\nu_\mu\rangle = \sin\theta|\nu_1\rangle + \cos\theta|\nu_2\rangle.$$

Calculate the probability of a $\nu_e \to \nu_e$ transition as a function of time, and show that it can be expressed as

$$P_{\nu_e\to\nu_e} = 1 - \sin^2 2\theta \sin^2\left(\Delta m^2c^4\frac{L}{4E\hbar c}\right),$$

where $\Delta m^2 = m_2^2 - m_1^2$, $E = pc$ is the nominal neutrino energy, and $L = ct$ is the flight distance of the neutrino.

A $\nu_e \to \nu_e$ transition is characterized as $\langle\nu_e|\psi\rangle$ with

$$|\psi\rangle = U|\psi_0\rangle = e^{-iHt/\hbar}|\psi_0\rangle, \tag{17}$$

where $|\psi_0\rangle = |\nu_e\rangle = \cos(\theta)|\nu_1\rangle - \sin(\theta)|\nu_2\rangle$. Using $H|\nu_i\rangle = E_i|\nu_i\rangle$, we have

$$|\psi\rangle = e^{-iE_1t/\hbar}\cos\theta|\nu_1\rangle - e^{-iE_2t/\hbar}\sin\theta|\nu_2\rangle \tag{18}$$

Projecting to $|\nu_e\rangle$ we have

$$\begin{aligned}\langle\nu_e|\psi\rangle &= (\cos\theta\langle\nu_1| - \sin\theta\langle\nu_2|)\big(e^{-iE_1t/\hbar}\cos\theta|\nu_1\rangle - e^{-iE_2t/\hbar}\sin\theta|\nu_2\rangle\big) \\ &= \cos^2\theta e^{-iE_1t/\hbar} + \sin^2\theta e^{-iE_2t/\hbar}.\end{aligned} \tag{19}$$

The probability of $\nu_e \to \nu_e$ transition is thus

$$P = |\langle\nu_e|\psi\rangle|^2 = 1 - \sin^2(2\theta)\sin^2\left(\frac{\Delta Et}{2\hbar}\right), \quad (\Delta E \equiv E_2 - E_1) \tag{20}$$

Using the appriximations given in the problem,

$$\Delta E = \frac{\Delta m^2c^3}{2p} = \frac{\Delta m^2c^4}{2E}, \quad t = \frac{L}{c}. \tag{21}$$

Thesefore the probability becomes

$$P = 1 - \sin^2(2\theta)\sin^2\left(\frac{\Delta m^2c^4L}{4E\hbar c}\right), \tag{22}$$

which is exactly as wanted.

[S1r 2.9, S2 2.10, S3 2.11] A box containing a particle is divided into a right and a left compartment by a thin partition. If the particle is known to be on the right (left) side with certainty, the state is represented by the position eigenket $|R\rangle$ ($|L\rangle$), where we have neglected spatial variations within each part of the box. The most general state vector can then be written as

$$|\alpha\rangle = |R\rangle\langle R|\alpha\rangle + |L\rangle\langle L|\alpha\rangle.$$

The particle can tunnel through the partition; this tunneling effect is characterized by the Hamiltonian

$$H = \Delta(|L\rangle\langle R| + |R\rangle\langle L|),$$

where Δ is a real number with the dimension of energy.
(a) Find the normalized energy eigenkets and the corresponding energy eigenvalues.
(b) If the system at time $t = 0$ is given by the state $|\alpha\rangle$, find the state vector $|\alpha, t = t_0; t\rangle$ by applying the appropriate time-evolution operator to $|\alpha\rangle$.
(c) Suppose that at $t = 0$ the particle is on the right side with certainty. What is the probability for observing the particle on the left side as a function of time?
(d) Write down the coupled Schrödinger equations for $\langle L|\alpha, t_0 = 0; t\rangle$ and $\langle R|\alpha, t_0 = 0; t\rangle$. Show that the solutions to these equations are just what you expect from (b).
(e) Suppose in error H was written as

$$H = \Delta|L\rangle\langle R|.$$

By explicitly solving the most general time evolution problem with this Hamiltonian, show that probability conservation is violated.

1).
Let $|\alpha\rangle = r|R\rangle + l|L\rangle$, then the eigenequatino reads

$$\begin{aligned} H|\alpha\rangle = \Delta r|L\rangle + \Delta l|R\rangle &= Er|R\rangle + El|L\rangle \\ \Rightarrow \Delta l = Er; &\quad \Delta r = El, \\ \Rightarrow E = \pm\Delta. \end{aligned} \tag{23}$$

The corresponding eigenvectors are :

- $E = \Delta$:

$$\begin{cases} \Delta l = \Delta r \\ l^2 + r^2 = 1 \end{cases} \Rightarrow |+\rangle = \frac{\Delta}{\sqrt{2}}|R\rangle + \frac{\Delta}{\sqrt{2}}|L\rangle. \tag{24}$$

- $E = -\Delta$:

$$\begin{cases} \Delta l = -\Delta r \\ l^2 + r^2 = 1 \end{cases} \Rightarrow |-\rangle = \frac{\Delta}{\sqrt{2}}|R\rangle - \frac{\Delta}{\sqrt{2}}|L\rangle \tag{25}$$

and the statevector in the eigenbasis is

$$|\alpha\rangle = \frac{r}{\sqrt{2}}(|+\rangle + |-\rangle) + \frac{l}{\sqrt{2}}(|+\rangle - |-\rangle) = \frac{r+l}{\sqrt{2}}|+\rangle + \frac{r-l}{\sqrt{2}}|-\rangle. \tag{26}$$

2).
The time evolution in the eigenbasis is

$$|\psi\rangle \equiv |\alpha, t = t_0, t\rangle = \hat{U}|\alpha\rangle = \frac{r+l}{\sqrt{2}}e^{-i\Delta t/\hbar}|+\rangle + \frac{r-l}{\sqrt{2}}e^{i\Delta t/\hbar}|-\rangle. \tag{27}$$

3).

WIth initial condition now $|\alpha, t=0\rangle = |R\rangle$, we have $r=1, l=0$. Then the statevector becomes

$$|\psi\rangle = \frac{1}{\sqrt{2}} e^{-i\theta}|+\rangle + \frac{1}{\sqrt{2}} e^{i\theta}|-\rangle, \quad (\theta \equiv \Delta t/\hbar). \tag{28}$$

Re-expressing in the $|R\rangle, |L\rangle$ basis, we have

$$|\psi\rangle = (-i\sin\theta)|L\rangle + \cos\theta|R\rangle. \tag{29}$$

The probability of measuring $|L\rangle$ is thus

$$P = |\langle L|\psi\rangle|^2 = \sin^2\theta = \sin^2\left(\frac{\Delta t}{\hbar}\right). \tag{30}$$

4).

Using Schrodinger equation, we have

$$i\hbar\partial_t|\psi(t)\rangle = H|\psi(t)\rangle. \tag{31}$$

Apply $|L\rangle, |R\rangle$ on both sides, respectively, we have

$$\begin{aligned} \langle L|(i\hbar\partial_t|\psi(t)\rangle = \langle L|H|\psi(t)\rangle \Rightarrow \quad i\hbar\dot{C}_L = \Delta C_R, \\ \langle R|(i\hbar\partial_t|\psi(t)\rangle = \langle R|H|\psi(t)\rangle \Rightarrow \quad i\hbar\dot{C}_R = \Delta C_L. \end{aligned} \tag{32}$$

Taking derivative of the first term, and inserting the second term, we have

$$i\hbar\ddot{C}_L = -\frac{\Delta^2}{\hbar^2}C_L \tag{33}$$

with solution (letting $\theta \equiv \Delta t/\hbar$):

$$C_L(t) = A\cos\theta + iB\sin\theta. \tag{34}$$

Similarily, for C_R we have

$$C_R(t) = A\sin\theta + iB\cos\theta \tag{35}$$

Initial condition was $C_L(0) = l, C_R(0) = r$, so

$$C_L(0) = A = l, \quad C_R(0) = iB = r \tag{36}$$

Then the solution becomes

$$C_{L(t)} = l\cos\theta + r\sin\theta, \quad C_R(t) = l\sin\theta + r\cos\theta \tag{37}$$

Comparing from part B,

$$\begin{aligned} C_L = \langle L|\psi(t)\rangle = \frac{r+l}{\sqrt{2}}e^{-i\theta}\langle L|+\rangle + \frac{r-l}{\sqrt{2}}e^{i\theta}\langle L|-\rangle = \frac{1}{2}\big((r+l)e^{-i\theta} - (r-l)e^{i\theta}\big) = l\cos\theta + r\sin\theta \\ C_R = \langle R|\psi(t)\rangle = \frac{r+l}{\sqrt{2}}e^{-i\theta}\langle R|+\rangle + \frac{r-l}{\sqrt{2}}e^{i\theta}\langle R|-\rangle = \frac{1}{2}\big((r+l)e^{-i\theta} + (r-l)e^{i\theta}\big) = r\cos\theta + l\sin\theta \end{aligned} \tag{38}$$

Exactly as wanted.

5)

Taking the erronous Hamiltonian

$$H' = \Delta|L\rangle\langle R|. \tag{39}$$

Then applying to Schrodinger equation, we have

$$
\begin{aligned}
\langle L|(i\hbar\partial_t|\psi\rangle) = \langle L|H'|\psi\rangle \Rightarrow i\hbar\dot{C}_L = \Delta C_R; \\
\langle R|(i\hbar\partial_t|\psi\rangle) = \langle R|H'|\psi\rangle \Rightarrow i\hbar\dot{C}_R = 0
\end{aligned}
\tag{40}
$$

With initial condition $C_L(0) = l, C_R(0) = r$, we have

$$
C_R(t) = r, \quad C_L(t) = C_L(0) + \int_0^t \Delta \frac{r}{i\hbar}\,\mathrm{d}t' = l - i\Delta rt/\hbar. \tag{41}
$$

And the total probability is

$$
P(t) = |C_L(t)|^2 + |C_R|^2 = l^2 + r^2 + \frac{\Delta^2 r^2 t^2}{\hbar^2} > 1 \tag{42}
$$

Which violates the conservation of probability, thus unphysical.

[S1r 2.21, S2 2.23, S3 2.28 modified] A particle of mass m in one dimension is trapped between two rigid walls:

$$V(x) = \begin{cases} 0, & 0 < x < L, \\ \infty, & x < 0,\ x > L. \end{cases}$$

(a) At $t = 0$ it is known to be exactly at $x = L/2$ with certainty. What are the relative probabilities for the particle to be found in various energy eigenstates? Write down the wave function for $t \geq 0$. (You need not worry about absolute normalization, convergence, and other mathematical subtleties.)
(b) As a slightly more realistic version of this problem, instead assume that the initial state is a constant for $(L/2 - \epsilon) < x < (L/2 + \epsilon)$ and zero for $x < (L/2 - \epsilon)$ and $x > (L/2 + \epsilon)$, in which ϵ is a small parameter ($\epsilon/L \ll 1$). Calculate the probabilities for the particle to be found in various energy eigenstates and determine the wave function for $t \geq 0$.

1).
We recite the time-independent solution to 1D infinite potential well:

$$u_n(x) = \langle x|n\rangle = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right), \quad E_n = \frac{\hbar^2\pi^2 n^2}{2mL^2}. \tag{43}$$

The time evolution is given by

$$\psi(x,t) = \langle x|\alpha,t\rangle = \sum_n \langle x|n\rangle\langle n|\alpha,t\rangle = \sum_n u_n(x)\langle n|e^{-iE_n t/\hbar}|\alpha,0\rangle = \sum_n c_n u_n(x) e^{-iE_n t/\hbar}, \tag{44}$$

with $c_n \equiv \langle n|\alpha,0\rangle = \int_0^L u_n^*(x)\psi(x,t)\,\mathrm{d}x$
From the problem statement, $\psi(\alpha,0) = \langle x|\alpha,0\rangle$ is known with certainty as

$$\psi(\alpha,0) = \delta\left(x - \frac{L}{2}\right) \tag{45}$$

and so

$$c_n = \int_0^L u_n^*(x)\delta\left(x - \frac{L}{2}\right)\mathrm{d}x = u_n\left(\frac{L}{2}\right) = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi}{2}\right) = \begin{cases} 0 & n \text{ even} \\ \sqrt{\frac{2}{L}}(-1)^{\frac{n-1}{2}} & n \text{ odd} \end{cases} \tag{46}$$

So for odd n, the time evolution is

$$\psi(x,t) = \sum_{n=\text{ odd}} \frac{2}{L}(-1)^{\frac{n-1}{2}}\sin\left(\frac{n\pi x}{L}\right)e^{-iE_n t/\hbar}. \tag{47}$$

2)
We still have

$$\psi(x,t) = \sum_n c_n u_n(x) e^{-iE_n t/\hbar}, \quad c_n = \int_0^L u_n^*\psi(x,0)\,\mathrm{d}x. \tag{48}$$

At time 0, we now have

$$\psi(x,0) \equiv \langle x|\alpha,0\rangle = C\left[H\left(x - \frac{L}{2} + \varepsilon\right) - H\left(x - \frac{L}{2} - \varepsilon\right)\right], \tag{49}$$

where $H(x)$ is the Heaviside step function, and C is the normalization constant. Normalizing gives

$$\int_0^L |\psi(x,0)|^2 \,\mathrm{d}x = c^2(2\varepsilon) = 1 \quad \Rightarrow c = \frac{1}{\sqrt{2\varepsilon}} \tag{50}$$

and therefore the statefunction

$$\psi(x,0) = \begin{cases} \frac{1}{\sqrt{2\varepsilon}} & x \in \left[\frac{L}{2} - \varepsilon, \frac{L}{2} + \varepsilon\right] \\ 0 & \text{else} \end{cases} \tag{51}$$

and the coefficients c_n is

$$\begin{aligned} c_n &= \int_{\frac{L}{2}-\varepsilon}^{\frac{L}{2}+\varepsilon} \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)\left(\frac{1}{\sqrt{2\varepsilon}}\right) \mathrm{d}x \\ &= -\frac{\sqrt{L}}{n\pi\sqrt{\varepsilon}}\left[\cos\left(\frac{n\pi}{2} + \frac{n\pi\varepsilon}{L}\right) - \cos\left(\frac{n\pi}{2} - \frac{n\pi\varepsilon}{L}\right)\right] = \frac{2\sqrt{L}}{n\pi\sqrt{\varepsilon}} \sin\left(\frac{n\pi}{2}\right) \sin\left(\frac{n\pi\varepsilon}{L}\right) \end{aligned} \tag{52}$$

Therefore

$$\psi(x,t) = \sum_{n \text{ odd}}^{\infty} \frac{2\sqrt{2}}{n\pi\sqrt{\varepsilon}} (-1)^{\frac{n-1}{2}} \sin\left(\frac{n\pi\varepsilon}{L}\right) \sin\left(\frac{n\pi x}{L}\right) e^{-iE_n t/\hbar}. \tag{53}$$

5

[S1r 2.23, S2 2.25, S3 2.30] A particle of mass m in one dimension is bound to a fixed center by an attractive delta function potential:

$$V(x) = -\lambda\delta(x) \qquad\qquad (\lambda > 0).$$

At $t = 0$, the potential is suddenly switched off (that is, $V = 0$ for $t > 0$), leaving the wavefunction unchanged immediately after the switch $(t = 0^+)$. Find an integral expression for the wavefunction at $t > 0$. (You do not need to evaluate the integral.)

In a delta attractive potential well, the initial state is prepared as

$$\psi_0(x) = \sqrt{\kappa}e^{-\kappa|x|}, \quad \kappa = m\lambda/\hbar^2. \tag{54}$$

Immediately after $t = 0$, the potential is turned off, so the initial state is the same

$$\psi(x,0) = \psi_0(x) = \sqrt{\frac{m\lambda}{\hbar^2}}\exp\left(-\frac{m\lambda}{\hbar^2}|x|\right) \tag{55}$$

The evolution of a free particle is given by the 1D free particle propagator to be

$$\begin{aligned}\psi(x,t) &= \int_{-\infty}^{\infty} K(x,t;x',t_0)\psi(x',0)\,\mathrm{d}x \\ &= \int_{-\infty}^{\infty}\left(\left(\frac{m}{2\pi i\hbar t}\right)^{1/2}\exp\left[\frac{im(x-x')^2}{2\hbar t}\right]\right)\sqrt{\frac{m\lambda}{\hbar^2}}\exp\left[-\frac{m\lambda}{\hbar^2}|x'|\right]\mathrm{d}x'\end{aligned} \tag{56}$$