Physics 731 Lecture Notes 2

Summary: Postulates of Quantum Mechanics

These notes summarize the postulates of quantum mechanics and explore several basic quantum mechanical concepts and systems. These include spin $1/2$ systems, sets of commuting observables, uncertainty relations, unitary transformations and change of basis, function/Hilbert spaces, proper vs. improper bases, position eigenstates and momentum eigenstates, translations, and the (minimum-uncertainty) Gaussian wavepacket. Helpful references include **S1r**, **S2**, **S3**, Chapter 1, Shankar 4, 9, and Byron/Fuller, among many others.

Postulates of Quantum Mechanics

In classical mechanics, particle motion can be described precisely by $x(t)$, $p(t)$, with time evolution given by Hamilton's equations:

$$\dot{x} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial x},$$

where H is the Hamiltonian of the system.

In quantum mechanics, the state of the system is described by a vector in a Hilbert space, which we will denote here as $|\alpha\rangle$. In the Schrödinger picture, the state vector $|\alpha\rangle$ evolves in time, with its time evolution given by the Schrödinger equation:

$$i\hbar \frac{d}{dt}|\alpha(t)\rangle = H|\alpha(t)\rangle, \tag{1}$$

where H is again the Hamiltonian of the system. Physical observables correspond to self-adjoint (Hermitian) operators acting on the space.

The *normalized* state vector $|\alpha\rangle$ has the following probabilistic interpretation. Consider a physical observable described by the Hermitian operator A, and let $A|a_i\rangle = a_i|a_i\rangle$ represent the eigenvalues and *normalized* eigenvectors of A. The $\{|a_i\rangle\}$ form a complete orthonormal set of basis vectors spanning the space. Upon a measurement of A, the following postulates hold:

- The outcome of a measurement of A is one of the a_i, with probability $|\langle a_i|\alpha\rangle|^2$.
- If the measurement of A yields the value a_i, the system is suddenly changed by the measurement process to the new state $|a_i\rangle$, *i.e.*

  $$|\alpha\rangle = \sum_i |a_i\rangle\langle a_i|\alpha\rangle \to |a_i\rangle. \tag{2}$$

  This is often called the "reduction of the wave packet" or the "collapse of the wavefunction." This is the Copenhagen interpretation of QM; it is the only interpretation we will consider in this course.
- A note about degeneracy: if the outcome of the measurement of A is a degenerate eigenvalue a_i, the system is suddenly reduced to the subspace of the degenerate eigenvectors that correspond to a_i. (Please recall that it is important to choose an orthonormal basis of these degenerate eigenvectors when describing the system.)

The expectation value of A with respect to the *normalized* state $|\alpha\rangle$ is defined as $\langle A\rangle = \langle\alpha|A|\alpha\rangle$. The interpretation of $\langle A\rangle$ is that it is a weighted average of the possible outcomes of the measurement of A:

$$\langle A\rangle = \sum_{i,j}\langle\alpha|a_j\rangle\langle a_j|A|a_i\rangle\langle a_i|\alpha\rangle = \sum_i a_i|\langle a_i|\alpha\rangle|^2. \tag{3}$$

Important note: if the state vector $|\alpha\rangle$ and/or the eigenvectors $|\alpha\rangle$ are not normalized, the above expressions must be modified accordingly. The best practice is to be sure to work with normalized states. From now on, all vectors are taken to be normalized in the expressions given in these and future notes unless it is explicitly stated otherwise.

Important example: Spin 1/2 particle. The vector space is $V^2(\mathbb{C})$, and the state $|\alpha\rangle$ is a two-component spinor. The spin operators are

$$S_i = \frac{\hbar}{2}\sigma_i, \tag{4}$$

where $\sigma_{i=x,y,z}$ (sometimes written as $\sigma_{i=1,2,3}$) are the Pauli matrices. The standard representation of the Pauli matrices is

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{5}$$

Note that it is common to denote the three Pauli matrices using vector notation as $\boldsymbol{\sigma}$, where the σ_i are the components of $\boldsymbol{\sigma}$, in which case we can write

$$\mathbf{S} = \frac{\hbar}{2}\boldsymbol{\sigma}.$$

One can easily solve the eigenvalue problem for each of the S_i operators (usually we work in basis where S_z is diagonal, as above). The eigenvalues are $\pm\hbar/2$, with the following normalized eigenstates:

$$|S_z;+\rangle = |+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |S_z;-\rangle = |-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{6}$$

$$|S_x;\pm\rangle = |\pm\rangle_x = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \pm 1 \end{pmatrix}, \quad |S_y;\pm\rangle = |\pm\rangle_y = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \pm i \end{pmatrix}. \tag{7}$$

The spin operators obey the following relations:

$$[S_i, S_j] = i\hbar\epsilon_{ijk}S_k, \quad \{S_i, S_j\} = \frac{\hbar^2}{2}\delta_{ij}\mathbb{1}_{2\times 2}. \tag{8}$$

These can be obtained from the relations $[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k$, and $\{\sigma_i, \sigma_j\} = 2\delta_{ij}\mathbb{1}_{2\times 2}$. Note also that $S^2 = \sum_i S_i^2 = \frac{3}{4}\hbar^2\mathbb{1}_{2\times 2}$, and that $[S_i, S^2] = 0$. The spin $1/2$ particle is a prototypical simple QM example: be sure to be familiar with it!

Sets of Commuting Observables. If two operators A and B commute, they are said to be *compatible*; otherwise, they are *incompatible*. For compatible observables, it is possible to find a set of mutual eigenstates of both operators A and B.

While true in all cases, this is of particular significance if any one of the operators (say A) has degenerate eigenvalues. In this case, the eigenvalues of A are not enough to label the states. If diagonalizing B in the basis of eigenvectors of A then breaks the degeneracy, the states can be fully labeled by their eigenvalues with respect to both A and B: $|a_i, b_j\rangle$. An ON basis of eigenstates of A could also be obtained via the Gram-Schmidt procedure, but a more physically relevant thing to do is to find enough compatible observables to break the degeneracy completely. Such a set is called a *maximal* or *complete* set of commuting observables.

For compatible observables A and B, there is no interference between measurements of A and measurements of B, while for incompatible observables, the measurements generally interfere. This can easily be seen by construction; examples and an extended discussion can be found in the text.

Generalized Uncertainty Relation. Again, we begin with an observable A (with $A|a_i\rangle = a_i|a_i\rangle$) acting on the state $|\alpha\rangle$. Recalling the definition of $\langle A\rangle$, the expectation value of A, the operator ΔA is defined as

$$\Delta A = A - \langle A\rangle \mathbb{1}. \tag{9}$$

The *dispersion* or *variance* of A is

$$\langle(\Delta A)^2\rangle = \langle A^2\rangle - \langle A\rangle^2 = \sum_i a_i^2 |\langle a_i|\alpha\rangle|^2 - \left(\sum_j a_j |\langle a_j|\alpha\rangle|^2\right)^2. \tag{10}$$

Clearly, the dispersion of an operator with respect to one of its eigenstates is zero. The dispersion thus measures the "fuzziness" of the state. The generalized uncertainty relation is (proof: **S1r, S2, S3** 1.4)

$$\langle(\Delta A)^2\rangle\langle(\Delta B)^2\rangle \geq \frac{1}{4}|\langle[A,B]\rangle|^2. \tag{11}$$

Hence, for two compatible operators, it is always possible to find a state in which both $\langle(\Delta A)^2\rangle$ and $\langle(\Delta B)^2\rangle$ are zero or arbitrarily small, while for incompatible observables, this is not possible because there is a lower bound on their product.

Change of Basis. How to change from one ON basis $|a_i\rangle$ to a new basis $|b_j\rangle$: use the unitary operator

$$U = \sum_k |b_k\rangle\langle a_k|, \tag{12}$$

for which $|b_i\rangle = U|a_i\rangle$, $U^\dagger|b_i\rangle = |a_i\rangle$. Therefore, if one represents a general state $|\alpha\rangle$ as

$$|\alpha\rangle = \sum_i c_i|a_i\rangle = \sum_j d_j|b_j\rangle, \tag{13}$$

in which $c_i = \langle a_i|\alpha\rangle$ and $d_j = \langle b_j|\alpha\rangle$, it is easy to show that

$$d_j = \sum_i (U^\dagger)_{ji} c_i. \tag{14}$$

The matrix representation of an operator X in the new basis $|b_j\rangle$ is related to the matrix representation in the old basis $|a_i\rangle$ by

$$X'_{ij} = \langle b_i|X|b_j\rangle = (U^\dagger X U)_{ij}. \tag{15}$$

Function spaces/Hilbert spaces

We will now consider function spaces, which are the way in which we describe continuous systems in quantum mechanics. The function spaces we wish to consider are infinite-dimensional vector spaces for which the vectors in the space are normalizable (more precisely, such spaces are inner product spaces plus certain convergence assumptions). Note: these are the spaces traditionally referred to as *Hilbert spaces*, though as stated in previous notes, it is now commonplace to use the term Hilbert space more broadly to include relevant finite-dimensional spaces.

Consider the set of complex valued functions $f(x)$ defined with respect to a real argument $x \in [a,b]$.[1] We want to consider the space $L^2_w(a,b)$, the set of such functions defined on $[a,b]$ which are square integrable

[1]Here x is an abstract label. Later we will take x to denote position, and the space in question in position space; note that there is nothing particularly special about position space (except familiarity, perhaps). Shortly we will also consider momentum space. Both will be valuable constructs in quantum mechanics.

with weight function $w(x)$. The inner product defined on this space takes the general form

$$\langle f|g\rangle_w = \int_a^b f^*(x)g(x)w(x)dx, \tag{16}$$

in we have loosely defined $f(x) = \langle x|f\rangle$ (soon we will get to the precise definitions of $|x\rangle$ and its dual $\langle x|$). On the space, we want to introduce a complete ON basis $|i\rangle = |f_i\rangle$, for which

$$\langle i|j\rangle_w = \int_a^b f_i^*(x)f_j(x)w(x)dx, \qquad \sum_{i=1}^{\infty} |i\rangle\langle i| = \mathbb{1}. \tag{17}$$

Then, for $f \in L_w^2(a,b)$, we can write

$$|f\rangle = \sum_{i=1}^{\infty} |i\rangle\langle i|f\rangle_w = \sum_{i=1}^{\infty} |i\rangle\alpha_i, \tag{18}$$

with

$$\alpha_i = \int_a^b f_i^*(x)f(x)w(x)dx. \tag{19}$$

A linear differential operator L is a mapping from one vector in the space to another:

$$L|f\rangle = |Lf\rangle = |g\rangle. \tag{20}$$

Note that boundary conditions on the functions must also be specified.

Definition. L is *self-adjoint* if

$$\langle f|Lg\rangle_w = \langle Lf|g\rangle_w \tag{21}$$

for all $f, g \in L_w^2(a,b)$ satisfying boundary conditions.

We focus here on the case in which $w(x) = 1$; this will be the default assumption unless explicitly stated otherwise.

As an example, consider the operator defined as

$$L = -iD, \tag{22}$$

in which D is defined by

$$\langle x|D|f\rangle = \langle x|Df\rangle = \frac{d}{dx}\langle x|f\rangle = \frac{d}{dx}f(x). \tag{23}$$

Note that the factor of i is needed for L to be self-adjoint. In particular, it is straightforward to see that

$$\begin{aligned}\langle f|Lg\rangle &= \int_a^b (-if^*(x))\frac{dg(x)}{dx}dx = \int_a^b \left(-i\frac{df(x)}{dx}\right)^* g(x)dx - if^*g|_a^b \\ &= \langle Lf|g\rangle - if^*g|_a^b.\end{aligned} \tag{24}$$

Hence, $-iD$ is self-adjoint only if $f^*g|_a^b = 0$ (*i.e.*, functions vanish at endpoints, or are periodic, etc.).

Eigenvalue Problem. Given a self-adjoint linear differential operator L and specified boundary conditions, we wish to find λ and f_λ such that

$$L|f_\lambda\rangle = \lambda|f_\lambda\rangle. \tag{25}$$

Usually, we will find a continuum of formal solutions until the boundary conditions are specified, which then often yield a discrete spectrum (for example, in the case of bound states).

Theorem. For self-adjoint L, then the $|f_\lambda\rangle$ form a complete ON set on the subspace of functions $\in L_w^2(a,b)$ satisfying the appropriate boundary conditions. Useful examples include:

- For the interval $[a, b] = [-1, 1]$ and the operator $L = -(1 - x^2)\frac{d^2}{dx^2} + 2x\frac{d}{dx}$, with $f(\pm 1)$ finite. Here, $(L - \lambda\mathbb{1})|f_\lambda\rangle = 0$ gives $\lambda = n(n+1)$, $|f_n\rangle = \sqrt{\frac{2n+1}{2}}P_n(x)$, where $n = 0, 1, 2, \ldots$ and $P_n(x)$ are the Legendre polynomials.

- For the interval $[a, b] = [-\pi, \pi]$, and the operator $L = \frac{d^2}{dx^2}$, with $f(\pi) = f(-\pi)$ (periodic boundary conditions). The condition $(L - \lambda\mathbb{1})|f_\lambda\rangle = 0$ gives $\lambda = n^2$ (n integer), and $f_n(x) = \frac{1}{\sqrt{2\pi}}e^{\pm inx}$.

- For the interval $[a, b] = [-\infty, \infty]$, and the $L = -\frac{d^2}{dx^2} + x^2$, with $f(\pm\infty)$ finite, we have $\lambda = 2n + 1$, and $|f_n\rangle = 2^{-n/2}\pi^{-1/4}(n!)^{-1/2}e^{-x^2/2}H_n(x)$, where $n = 0, 1, 2, \ldots \infty$, and $H_n(x)$ are the Hermite polynomials.

Proper vs. Improper bases. The above examples are all examples of "proper" bases, *i.e.*, the eigenvectors are vectors which are in the Hilbert space. However, it is also often convenient to work with "improper" bases, which are complete sets of orthogonal states which are not normalizable in the proper sense, and hence the basis vectors are not in the Hilbert space. Examples include the eigenstates of the position operator x and the eigenstates of the momentum operator p. (Here we will work in one dimension initially for simplicity, then generalize to three dimensions.)

Position operator eigenstates. For the position operator $x_{\rm op}$ (where at the moment the subscript is explicitly included for notational clarity; later it will be dropped and the context should be understood), the eigenvalue equation takes the form

$$x_{\rm op}|x'\rangle = x'|x'\rangle. \tag{26}$$

The claim is that these eigenstates are represented as delta functions, which are clearly not normalizable in the proper sense, as

$$\langle x|x'\rangle = \delta(x - x'). \tag{27}$$

However, it can be shown that these states are complete:

$$\int_{-\infty}^{\infty} dx|x\rangle\langle x| = \mathbb{1}. \tag{28}$$

We can now give meaning to the abstract statement of the inner product of two vectors in the space:

$$\langle f|g\rangle = \int dx\langle f|x\rangle\langle x|g\rangle \equiv \int f^*(x)g(x)dx, \tag{29}$$

in which we have defined $\langle x|f\rangle = f(x)$, as previously mentioned. An arbitrary function may be expanded as

$$|f\rangle = \int_{-\infty}^{\infty} |x\rangle\langle x|f\rangle dx = \int_{-\infty}^{\infty} |x\rangle f(x)dx. \tag{30}$$

This is similar to expanding over a discrete set, except all discrete sums have now become integrals over the continuum, *i.e.* $\delta_{nm} \to \delta(x - x')$, and

$$\langle x'|x|x''\rangle = x''\delta(x' - x''), \quad \langle x'|f(x)|x''\rangle = f(x'')\delta(x' - x''). \tag{31}$$

The defining relation for the Dirac delta function is

$$\int_{-\infty}^{\infty} f(x')\delta(x - x')dx' = f(x). \tag{32}$$

The derivative of the delta function,

$$\delta'(x - x') = \frac{d}{dx}\delta(x - x') = -\frac{d}{dx'}\delta(x - x'), \tag{33}$$

satisfies

$$\int_{-\infty}^{\infty} f(x')\delta'(x - x')dx' = f'(x). \tag{34}$$

The matrix element of the self-adjoint operator $-iD$ in the improper basis of x eigenstates satisfies the required relation

$$\langle x| - iD|x'\rangle = -i\delta'(x - x') = \langle x'| - iD|x\rangle^*. \tag{35}$$

Consider a position measurement, with outcome $x'' \pm \Delta/2$ (*i.e.*, a measurement yielding x'' up to resolution Δ). The initial state

$$|\psi\rangle = \int_{-\infty}^{\infty} dx'|x'\rangle\langle x'|\psi\rangle \tag{36}$$

is then collapsed to

$$|\psi\rangle = \int_{x''-\frac{\Delta}{2}}^{x''+\frac{\Delta}{2}} dx'|x'\rangle\langle x'|\psi\rangle. \tag{37}$$

The probability of measuring $x'' \pm \Delta/2$ is given by $|\langle x''|\psi\rangle|^2\Delta$. The probability of measuring x'' in range dx'' is $|\langle x''|\psi\rangle|^2 dx$. The *probability density* is $|\langle x''|\psi\rangle|^2$.

Momentum operator eigenstates. From classical limits and relation to translations, the momentum operator can be shown to take the form

$$p = -i\hbar\frac{d}{dx} = -i\hbar D. \tag{38}$$

The eigenstates of p are given by the condition $p|p'\rangle = p'|p'\rangle$

$$\langle x|p|p'\rangle = -i\hbar\frac{d}{dx}\langle x|p'\rangle = p'\langle x|p'\rangle, \tag{39}$$

which has the (normalized) solution

$$\langle x|p'\rangle = \varphi_{p'}(x) = \frac{1}{\sqrt{2\pi\hbar}}e^{ip'x/\hbar}, \tag{40}$$

in which we have used the important relation

$$\int_{-\infty}^{\infty} dx\, e^{iqx} = 2\pi\delta(q).$$

The boundary conditions that $\varphi_{p'}(x)$ is finite at $x = \pm\infty$ dictate that p' is real. These states are orthonormal (in the Dirac delta-function sense) and complete:

$$\langle p|p'\rangle = \delta(p - p'), \quad \int_{-\infty}^{\infty} dp|p\rangle\langle p| = \mathbb{1}. \tag{41}$$

An arbitrary state $|\alpha\rangle$ can thus equally well be expanded in the $|p\rangle$ states:

$$|\alpha\rangle = \int_{-\infty}^{\infty} |p\rangle\langle p|\alpha\rangle dp = \int_{-\infty}^{\infty} |p\rangle\varphi_\alpha(p)dp, \tag{42}$$

in which $\varphi_\alpha(p)$ is the momentum space wavefunction. The momentum space wavefunction and position space wavefunction are related by Fourier transforms:

$$\psi_\alpha(x) = \langle x|\alpha\rangle = \int_{-\infty}^{\infty} dp\langle x|p\rangle\langle p|\alpha\rangle = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} e^{ipx/\hbar}\varphi_\alpha(p)dp. \tag{43}$$

$$\varphi_\alpha(p) = \langle p|\alpha\rangle = \int_{-\infty}^{\infty} dx\langle p|x\rangle\langle x|\alpha\rangle = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} e^{-ipx/\hbar}\psi_\alpha(x)dx. \tag{44}$$

The generalization to three dimensions is straightforward. We now have states $|\mathbf{x}\rangle$ and $|\mathbf{p}\rangle$ that satisfy

$$\langle\mathbf{x}|\mathbf{x}'\rangle = \delta^{(3)}(\mathbf{x}-\mathbf{x}'), \quad \int_{-\infty}^{\infty} d^3x|\mathbf{x}\rangle\langle\mathbf{x}| = \mathbb{1}, \tag{45}$$

$$\langle\mathbf{p}|\mathbf{p}'\rangle = \delta^{(3)}(\mathbf{p}-\mathbf{p}'), \quad \int_{-\infty}^{\infty} d^3p|\mathbf{p}\rangle\langle\mathbf{p}| = \mathbb{1}. \tag{46}$$

The bases are related by

$$\langle\mathbf{x}|\mathbf{p}\rangle = \frac{1}{(2\pi\hbar)^{3/2}}e^{i\mathbf{p}\cdot\mathbf{x}/\hbar}, \tag{47}$$

and hence

$$\psi_\alpha(\mathbf{x}) = \langle\mathbf{x}|\alpha\rangle = \int_{-\infty}^{\infty} d^3p\langle\mathbf{x}|\mathbf{p}\rangle\langle\mathbf{p}|\alpha\rangle = \frac{1}{(2\pi\hbar)^{3/2}}\int_{-\infty}^{\infty} e^{i\mathbf{p}\cdot\mathbf{x}/\hbar}\varphi_\alpha(\mathbf{p})d^3p. \tag{48}$$

$$\varphi_\alpha(\mathbf{p}) = \langle\mathbf{p}|\alpha\rangle = \int_{-\infty}^{\infty} d^3x\langle\mathbf{p}|\mathbf{x}\rangle\langle\mathbf{x}|\alpha\rangle = \frac{1}{(2\pi\hbar)^{3/2}}\int_{-\infty}^{\infty} e^{-i\mathbf{p}\cdot\mathbf{x}/\hbar}\psi_\alpha(\mathbf{x}). \tag{49}$$

It is straightforward to show that in position space, $\mathbf{x}$ acts like a multiplication by $\mathbf{x}$:

$$\begin{aligned}\langle\alpha|\mathbf{x}|\beta\rangle &= \int d^3x\, d^3x'\langle\alpha|\mathbf{x}\rangle\langle\mathbf{x}|\mathbf{x}|\mathbf{x}'\rangle\langle\mathbf{x}'|\beta\rangle = \int d^3x\, d^3x'\psi^*_\alpha(\mathbf{x})\mathbf{x}'\delta^3(\mathbf{x}-\mathbf{x}')\psi_\beta(\mathbf{x}') \\ &= \int d^3x\psi^*_\alpha(\mathbf{x})\mathbf{x}\psi_\beta(\mathbf{x}).\end{aligned} \tag{50}$$

while in momentum space, $\mathbf{x}$ is given by $i\hbar\nabla_{\mathbf{p}}$:

$$\begin{aligned}\langle\alpha|\mathbf{x}|\beta\rangle &= \int d^3p\, d^3p'\langle\alpha|\mathbf{p}\rangle\langle\mathbf{p}|\mathbf{x}|\mathbf{p}'\rangle\langle\mathbf{p}'|\beta\rangle = \int d^3p\, d^3p' d^3x d^3x'\varphi^*_\alpha(\mathbf{p})\langle\mathbf{p}|\mathbf{x}\rangle\langle\mathbf{x}|\mathbf{x}|\mathbf{x}'\rangle\langle\mathbf{x}'|\mathbf{p}'\rangle\varphi_\beta(\mathbf{p}') \\ &= \frac{1}{(2\pi\hbar)^3}\int d^3p\, d^3p' d^3x\, \varphi^*_\alpha(\mathbf{p})\mathbf{x}e^{-i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{x}/\hbar}\varphi_\beta(\mathbf{p}') \\ &= \frac{1}{(2\pi\hbar)^3}\int d^3p\, d^3p' d^3x\, \varphi^*_\alpha(\mathbf{p})i\hbar\nabla_{\mathbf{p}}\, e^{-i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{x}/\hbar}\varphi_\beta(\mathbf{p}') = \int d^3p\, \varphi^*_\alpha(\mathbf{p})i\hbar\nabla_{\mathbf{p}}(\varphi_\beta(\mathbf{p})).\end{aligned} \tag{51}$$

Similarly, in position space, $\mathbf{p} = -i\hbar\nabla_{\mathbf{x}}$, while in momentum space $\mathbf{p}$ is just a multiplication by $\mathbf{p}$. The $\mathbf{x}$ and $\mathbf{p}$ operators satisfy the following *canonical commutation relations*:

$$[x_i, x_j] = 0, \quad [p_i, p_j] = 0, \quad [x_i, p_j] = i\hbar\delta_{ij}. \tag{52}$$

Translations. The form of the momentum operator can be derived from noting that momentum is the generator of translations (for a discussion, see the text). The translation operator takes the form

$$T(\mathbf{a}) \equiv e^{-i\mathbf{p}\cdot\mathbf{a}/\hbar}, \tag{53}$$

in which $\mathbf{p}$ is the momentum operator and $\mathbf{a}$ is a Cartesian vector. Note that $T(\mathbf{a})$ is a unitary operator. The $|\mathbf{p}\rangle$ states are eigenstates of the translation operator. It is straightforward to show that the action of $T(\mathbf{a})$ on the position eigenstate $|\mathbf{x}\rangle$ is given by

$$T(\mathbf{a})|\mathbf{x}\rangle = |\mathbf{x}+\mathbf{a}\rangle. \tag{54}$$

In position space, the action of $T(\mathbf{a})$ on an arbitrary state $|\alpha\rangle$ takes the form

$$\langle\mathbf{x}|T(\mathbf{a})|\alpha\rangle = \psi_{T_a\alpha}(\mathbf{x}) = \psi_\alpha(\mathbf{x}-\mathbf{a}). \tag{55}$$

The translation group is an example of a Lie group. For infinitesimal translations, we can expand about the identity and study the resulting Lie algebra. Since the momentum operators commute with each other, the translation group is an Abelian group:

$$T(\mathbf{a})T(\mathbf{b}) = T(\mathbf{b})T(\mathbf{a}) = T(\mathbf{a}+\mathbf{b}).$$

Gaussian wave packet in 1d. An example, worth going through thoroughly, is the Gaussian wave packet, which we consider here in 1d for simplicity. In position space, it takes the form

$$\psi_\alpha(x) = \langle x|\alpha\rangle = \frac{1}{\pi^{1/4}\sqrt{d}}e^{ikx-x^2/(2d^2)}, \tag{56}$$

where d is real, and $k = p/\hbar$. Note that the probability density $|\psi_\alpha|^2$ is a Gaussian of width d. The momentum space wavefunction is also a Gaussian:

$$\varphi_\alpha(p') = \sqrt{\frac{d}{\hbar\sqrt{\pi}}}e^{-(p'-\hbar k)^2d^2/(2\hbar^2)}. \tag{57}$$

This is an example of a minimum uncertainty wavepacket:

$$\langle(\Delta x)^2\rangle\langle(\Delta p)^2\rangle = \hbar^2/4 = (1/4)|\langle[x,p]\rangle|^2. \tag{58}$$

We will return to this example often in this course, particularly when considering time evolution.