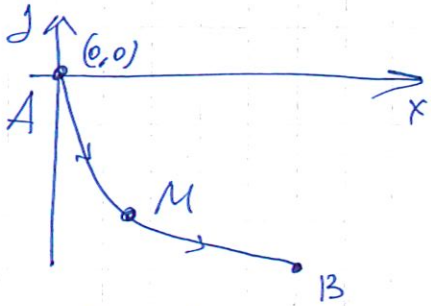
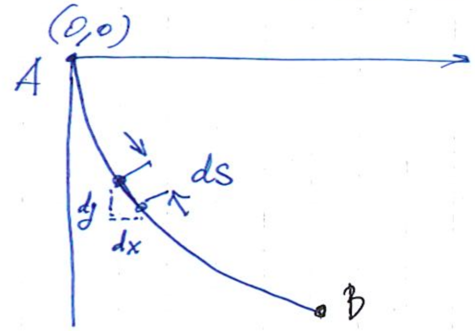
Calculus of Variations

History: In 1696 Johann Bernoulli of the University of Basel posed the following problem: given two points A and B in a vertical plane, find the path $A \to B$ which the moveable particle M will traverse in the shortest time, assuming that its acceleration is due to gravity only and ther is no friction. This is famous $\underbrace{\text{BRACHISTO}}_{\text{shortest}}\underbrace{\text{CHRONE}}_{\text{Time}}$ problem from the Greek.





$dt = \frac{ds}{v}$ instantenious velocity

$ds^2 = dx^2 + dy^2 \Rightarrow ds = \sqrt{1+(y')^2}\,dx$

$\frac{mv^2}{2} = mgy \Rightarrow \quad v = \sqrt{2gy}$

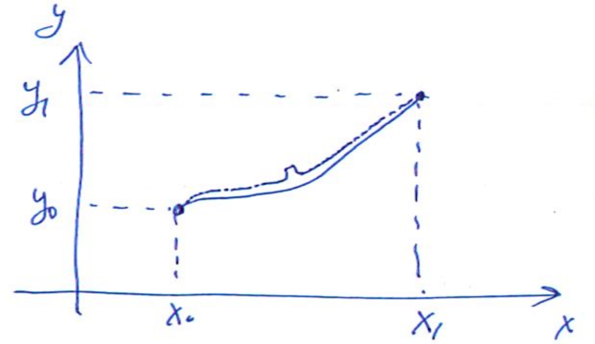
$$\boxed{t[y(x)] = \frac{1}{\sqrt{2g}} \int_{x_A}^{x_B} \frac{\sqrt{1+(y')^2}}{\sqrt{y}}\,dx}$$

$t[y(x)]$ is called a functional

Minimization (maximization) of a functional is the subject of calculus of variations.

→ Euler equation for the functionals of the type:

$$\boxed{v[y(x)] = \int_{x_0}^{x_1} F(x, y(x), y'(x))\, dx}$$



$$y(x,\varepsilon) = y(x) + \varepsilon\, \delta y(x).$$

$\varepsilon \to 0$

$\delta y(x)$ is called variation

$v[y(x,\varepsilon)] = \varphi(\varepsilon)$ so that in the sense of parameter ε extremum of the functional corresponds to the extremum of the function $\varphi(\varepsilon)$, namely: $\varphi'(0) = 0$

$$\varphi(\varepsilon) = \int_{x_0}^{x_1} F(x, y(x,\varepsilon), y'(x,\varepsilon))\, dx$$

$$\varphi'(\varepsilon) = \int_{x_0}^{x_1} \left[\frac{\partial F}{\partial y} \frac{\partial}{\partial \varepsilon} y(x,\varepsilon) + \frac{\partial F}{\partial y'} \frac{\partial}{\partial \varepsilon} y'(x,\varepsilon)\right] dx$$

$$F_y = \frac{\partial}{\partial y} F(x, y(x,\varepsilon), y'(x,\varepsilon))$$

$$F_{y'} = \frac{\partial}{\partial y'} F(x, y(x,\varepsilon), y'(x,\varepsilon))$$

$$\frac{\partial}{\partial \varepsilon} y(x,\varepsilon) = \delta y \qquad \frac{\partial}{\partial \varepsilon} y'(x,\varepsilon) = \delta y'$$

$$\varphi'(\varepsilon) = \int_{x_0}^{x_1} [F_y\, \delta y + F_{y'}\, \delta y']\, dx$$

$$\delta y' = \delta\left(\frac{dy}{dx}\right) = \frac{d}{dx}\, \delta y$$

Integrate this term by parts

$$\varphi'(\varepsilon) = \int_{x_0}^{x_1} \Big[F_y\, \delta y + \overbrace{F_{y'} \frac{d}{dx}\, \delta y}\Big]\, dx$$

$$\varphi'(\varepsilon) = \left(F_{y'}\,\delta y\right)\Big|_{x=x_0}^{x=x_1} + \int_{x_0}^{x_1} \left[F_y - \frac{d}{dx}\left(F_{y'}\right)\right] \delta y \, dx$$

Since $\delta y(x_0) = \delta y(x_1) = 0 \Rightarrow$ fixed end points

$$\varphi'(0) = \int_{x_0}^{x_1} \left(F_y - \frac{d}{dx} F_{y'}\right) \delta y(x) \, dx$$

Fundamental Theorem: If for any continuous function $\eta(x)$ defined at the interval $x \in [x_0, x_1]$

$$\int_{x_0}^{x_1} \Phi(x)\,\eta(x)\,dx = 0$$

where function $\Phi(x)$ is also continuous at that interval, then

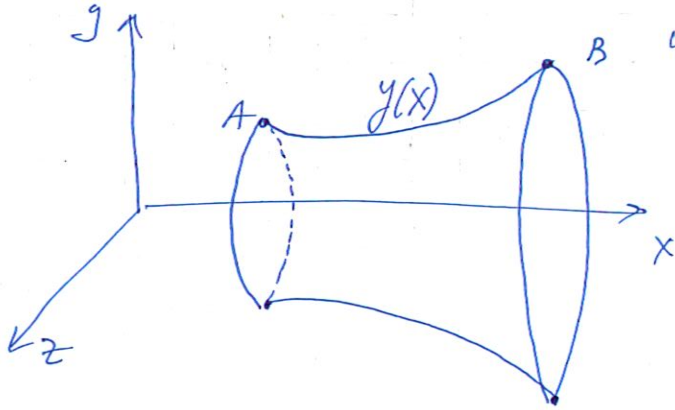
$$\boxed{\Phi(x) \equiv 0}$$

Applying (FT) to above variation and from the condition $\varphi'(0) = 0$ we find:

$$\boxed{F_y - \frac{d}{dx} F_{y'} = 0}$$ Euler (1744)

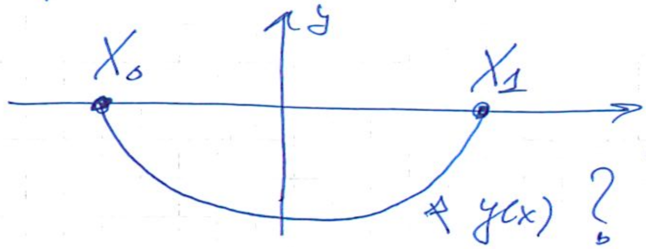
Some famous problems

(#1) The minimum surface of revolution: find curve $y(x)$ which upon revolution covers the smallest area?



(#2) Find the plane curve of given length which encloses the greatest possible area?

(#3) freely suspended chain:



$y(x)$? (under gravity only).

$\boxed{\text{Exp. 1}}$ $\quad v[y(x)] = \int_0^{\pi/2} [(y')^2 - y^2]\,dx \quad y(0)=0 \quad y(\pi/2)=1$

$$F_y = -2y \qquad F_{y'} = 2y'$$

$$F_y - \frac{d}{dx}F_{y'} = 0 \Longrightarrow y'' + y = 0$$

$$y(x) = C_1 \cos x + C_2 \sin x$$
$$y(0) = C_1 = 0$$
$$y(\pi/2) = C_2 = 1$$

$$\boxed{y(x) = \sin x}$$

$\boxed{\text{Exp. 2}}$ $\quad v[y(x)] = \int_0^1 [(y')^2 + 12xy]\,dx \quad y(0)=0 \quad y(1)=1$

$$F_y = 12x \qquad F_{y'} = 2y'$$

$$F_y - \frac{d}{dx}F_{y'} = 0 \Longrightarrow y'' - 6x = 0$$

$$y(x) = x^3 + C_1 x + C_2$$
$$y(0) = C_2 = 0$$
$$y(1) = 1 + C_1 = 1 \Rightarrow C_1 = 0$$

$$\boxed{y(x) = x^3}$$

If F depends only on y and y' Euler equation has first integral. Indeed:

$$F_y - \frac{d}{dx} F_{y'} = F_y - F_{xy'} - y' F_{y'y} - y'' F_{y'y'} = 0.$$

Assume $F = F(y, y')$ so $\Rightarrow$ $F_{xy'} = 0$

$$y'\left(F_y - y' F_{y'y} - y'' F_{y'y'}\right) = \frac{d}{dx}\left(F - y' F_{y'}\right) = 0$$

$$\boxed{F - y' F_{y'} = C}$$

In physical terms this integral corresponds to energy conservation.

<u>Solution to Special problem #3</u>

$$E = \rho g \int_A^B y\, ds = \rho g \int_A^B y \sqrt{1+(y')^2}\, dx$$

$$L = \int_A^B ds = \int_A^B \sqrt{1+(y')^2}\, dx$$

Lagrange multiplier

$$v[y(x)] = \int_{x_A}^{x_B} \left[\rho g\, y \sqrt{1+(y')^2} - \lambda \sqrt{1+(y')^2}\right] dx$$

It is convenient to chose $\lambda = \rho g y_0$ where y_0 is some constant, so:

$$v[y(x)] = \rho g \int_{x_A}^{x_B} (y - y_0)\sqrt{1+(y')^2}\, dx$$

Chang now variables $y - y_0 = z \Leftrightarrow y' = z'$

$$v[z(x)] = \rho g \int_A^B z \sqrt{1+(z')^2}\, dx$$

This functional is independent of x so that has first

$$\underbrace{z\sqrt{1+(z')^2} - z'\frac{z\cdot z'}{\sqrt{1+(z')^2}}}\ = C$$

$$\frac{z}{\sqrt{1+(z')^2}} = C \Rightarrow \quad z^2 = C^2\left[1+(z')^2\right] \Rightarrow$$

$$\frac{dz}{dx} = \frac{\sqrt{z^2-c^2}}{c} \Rightarrow \frac{dz}{\sqrt{z^2-c^2}} = \frac{dx}{c}$$

Change variables now $z = c\cosh t \quad dz = c\sinh t\, dt$

$$\frac{c}{\cancel{}}\cdot\frac{\sinh t}{c\sqrt{\cosh^2 t-1}}\,dt = \frac{dx}{c} \Rightarrow t = \frac{x-x_0}{c}$$

So that $z(x) = c\cosh\left(\frac{x-x_0}{c}\right)$ and finally,

$$\boxed{y(x) = y_0 + c\cosh\left(\frac{x-x_0}{c}\right)}$$

The three constants y_0, x_0 and c are chosen so that the curve passes through (x_A, y_A) and (x_B, y_B) and so that the lengt is L. This presents numerical problems, but it's always possible in principle.

Generalized functionals

$$v[y(x)] = \int_{x_0}^{x_1} F(x, y(x), y'(x), \ldots, y^{(n)}(x))\,dx$$

$$\begin{cases} y(x_0) = y_0 \quad y'(x_0) = y_0', \quad \ldots, \quad y_0^{(n-1)}(x_0) = y_0^{(n-1)} \\ y(x_1) = y_1 \quad y'(x_1) = y_1', \quad \ldots, \quad y_1^{(n-1)}(x_1) = y_1^{(n-1)} \end{cases}$$

Introduce a variation $y(\varepsilon, x) = y(x) + \varepsilon \delta y(x)$:

$$\delta v = \frac{d}{d\varepsilon}\left[\int_{x_0}^{x_1} F(x, y(\varepsilon,x), y'(\varepsilon,x), \dots, y^{(n)}(\varepsilon,x))\,dx\right]_{\varepsilon=0} =$$

$$= \int_{x_0}^{x_1}\left[F_y\,\delta y + F_{y'}\,\delta y' + F_{y''}\,\delta y'' + \dots + F_{y^{(n)}}\,\delta y^{(n)}\right] dx$$

a) $$\int_{x_0}^{x_1} F_{y'}\,\delta y'\,dx = \underbrace{F_{y'}\,\delta y(x)\Big|_{x_0}^{x_1}}_{\equiv 0} - \int_{x_0}^{x_1} \delta y(x) \frac{d}{dx}(F_{y'})\,dx$$

b) $$\int_{x_0}^{x_1} F_{y''}\,\delta y''\,dx = F_{y''}\,\delta y'(x)\Big|_{x_0}^{x_1} - \int_{x_0}^{x_1} \delta y'(x) \frac{d}{dx} F_{y''}\,dx =$$

$$= \underbrace{F_{y''}\,\delta y'(x)\Big|_{x_0}^{x_1}}_{\equiv 0} \underbrace{- \frac{d}{dx}F_{y''}\,\delta y(x)\Big|_{x_0}^{x_1}}_{\equiv 0} + \int_{x_0}^{x_1} \frac{d^2}{dx^2}(F_{y''})\,\delta y(x)\,dx$$

⋮ n) $$\int_{x_0}^{x_1} F_{y^{(n)}}\,\delta y^{(n)}\,dx = (-1)^n \int_{x_0}^{x_1} \frac{d^n}{dx^n} F_{y^{(n)}}\,\delta y(x)\,dx$$

$$\delta v = \int_{x_0}^{x_1} \delta y(x)\left[F_y - \frac{d}{dx}F_{y'} + \frac{d^2}{dx^2}F_{y''} + \dots + (-1)^n \frac{d^n}{dx^n} F_{y^{(n)}}\right] dx$$

$$\boxed{F_y - \frac{d}{dx}F_{y'} + \frac{d^2}{dx^2}F_{y''} + \dots + (-1)^n \frac{d^n}{dx^n} F_{y^{(n)}} = 0}$$

Euler - Poisson equation.

$\boxed{\text{Exp. 1}}$ $$v[y(x)] = \int_0^{\pi/2} \left[(y'')^2 - y^2 + x^2\right] dx \qquad \begin{matrix} y(0)=1 & y'(0)=0 \\ y(\pi/2)=0 & y'(\pi/2)=-1 \end{matrix}$$

$\{F_y = -2y;\ F_{y'} = 0;\ F_{y''} = 2y''\}$: $\quad y^{(4)} - y = 0$

$K^4 - 1 = 0 \Rightarrow K^2 = \pm 1 \Rightarrow K_{.} = \pm 1 \quad K_{..} = \pm i$

$$y(x) = C_1 e^x + C_2 e^{-x} + C_3 \cos x + C_4 \sin x$$

By using boundary conditions :

$$C_1 = 0 \quad C_2 = 0 \quad C_3 = 1 \quad C_4 = 0$$

$$\boxed{y(x) = \cos x}$$

Exp. 2

$$v[y(x)] = \int_{-l}^{l} \left[\frac{m}{2} (y'')^2 + \rho y \right] dx \qquad \begin{matrix} y(-l) = 0 & y'(-l) = 0 \\ y(l) = 0 & y'(l) = 0 \end{matrix}$$

$$F_y = \rho \quad F_{y'} = 0 \quad F_{y''} = m y''$$

$$\rho + \frac{d^2}{dx^2}(m y'') = 0 \Rightarrow y^{(4)} = -\frac{\rho}{m}$$

$$y(x) = -\frac{\rho x^4}{24m} + C_1 x^3 + C_2 x^2 + C_3 x + C_4$$

By using boundary conditions:

$$y(x) = -\frac{\rho}{24m}(x^4 - 2l^2x^2 + l^4) \Rightarrow \boxed{y(x) = -\frac{\rho}{24m}(x^2 - l^2)^2}$$

If functionals depends on many functions then one has to do variational analysis for each,

$$v[y_1, y_2, \dots, y_m] = \int_{x_0}^{x_1} F(x, y_1, y_1', \dots, y_1^{(n_1)}, y_2, y_2', \dots, y_2^{(n_2)}, \dots$$

$$\dots y_m, y_m', \dots, y_m^{(n_m)}) \, dx$$

$$\boxed{\begin{gathered} F_{y_i} - \frac{d}{dx} F_{y_i'} + \frac{d^2}{dx^2} F_{y_i''} + \dots + (-1)^{n_i} \frac{d^{n_i}}{dx^{n_i}} F_{y_i^{(n_i)}} = 0 \\ i = 1, 2, \dots, m \end{gathered}}$$

System of coupled differential equations of order $[2n_i]_{max}$