P1

Extramize the functional that describes defects in crystals

$$\mathcal{F}[\theta(x)] = \int_{-\infty}^{\infty} \left[\frac{\kappa}{2} \left(\frac{\mathrm{d}\theta}{\mathrm{d}x} \right)^2 + V(1 - \cos\theta(x)) \right] \mathrm{d}x \tag{1}$$

subject to boudnary conditions $\theta(-\infty) = 0, \theta(\infty) = 2\pi$. Further, find the energy cost $E = \mathcal{F}[\Theta]$ for Θ that extramizes F.

We read off the functional

$$F(\theta, \theta') = \frac{\kappa}{2}\theta'^2 + V(1 - \cos\theta). \tag{2}$$

To use Euler-Lagrange equation, we first prepare

$$\frac{\partial F}{\partial \theta} = v \sin\theta; \quad \frac{\partial F}{\partial \theta'} = \kappa\theta'. \tag{3}$$

Then E-L gives

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}x}(\kappa\theta') - V\sin\theta = 0 \\ \Rightarrow\ &\theta'' = \frac{1}{\xi^2}\sin\theta, \quad \xi = \sqrt{\frac{\kappa}{V}}. \end{aligned} \tag{4}$$

This non-linear ODE can be solved indirectly had we noticed that F is independent of x, and so Beltrami identity applies:

$$\begin{aligned} &F - \theta' \frac{\mathrm{d}F}{\mathrm{d}\theta'} = C \\ \Rightarrow\ &\frac{\kappa}{2}\theta'^2 - V(1 - \cos\theta) = C \end{aligned} \tag{5}$$

Boundary conditions give $x \to \pm\infty \Rightarrow \theta' = 0, \cos\theta = 1 \Longrightarrow C = 0$. Thus we solve

$$\frac{\kappa}{2}\theta'^2 = V(1 - \cos\theta) \Rightarrow \theta'^2 = \frac{4}{\xi^2}\sin^2\left(\frac{\theta}{2}\right) \tag{6}$$

to get

$$\begin{aligned} &\frac{\mathrm{d}\theta}{2\sin\left(\frac{\theta}{2}\right)} = \frac{\mathrm{d}x}{\xi} \\ \Rightarrow\ &\frac{1}{2}\csc\left(\frac{\theta}{2}\right)\mathrm{d}\theta = \frac{1}{\xi}\mathrm{d}x. \end{aligned} \tag{7}$$

Since $\int \csc u \, \mathrm{d}u = \ln\left(\tan\left(\frac{u}{2}\right)\right)$, we take integration on both sides over an arbidrary range $[x_0, x]$. this yields

$$\boxed{\theta = \arctan\left[\frac{\exp(x - x_0)}{\xi}\right] \equiv \Theta.} \tag{8}$$

The corresponding functional is thus (with $\frac{\kappa}{2}\Theta'^2 = V(1 - \cos\Theta)$)

$$F(\Theta, \Theta') = 4V\sin^2\left(\frac{\theta}{2}\right) = 4V\,\mathrm{sech}^2\left(\frac{x - x_0}{\xi}\right). \tag{9}$$

The energy cost is therefore

$$E = \int_{-\infty}^{\infty} F\,\mathrm{d}x = 4V\int_{-\infty}^{\infty}\mathrm{sech}^2\left(\frac{x - x_0}{\xi}\right)\mathrm{d}x = 8V\xi = \boxed{8\sqrt{\kappa V}.} \tag{10}$$

P2

Apply calculus of variations to find coupled equations for ψ and $\boldsymbol{A}$ that extramize $\mathcal{F}$, where $\mathcal{F}$ is the free energy functional of a superconductor:

$$\mathcal{F}[\psi(\boldsymbol{r}), \boldsymbol{A}(\boldsymbol{r})] = \int_V \left(-\,|\psi|^2 + \frac{1}{2}\,|\psi|^4 + \left| \left(-\frac{i}{\kappa}\nabla - \boldsymbol{A} \right) \psi \right|^2 + \boldsymbol{H}^2 \right) \mathrm{d}V \tag{11}$$

Let $D_i = -\frac{i}{\kappa}\partial_i - A_i$, $H_i = \varepsilon_{ijk}\partial_j A_k$, and define

$$F(\psi, \psi^*, \boldsymbol{A}, \partial_i\psi, \partial_i\psi^*, \partial_i A_j) = -\,|\psi|^2 + \frac{1}{2}\,|\psi|^4 + |D_i\,\psi|^2 + H_i^2. \tag{12}$$

Here we use a genralized Euler Lagrange equation for $F(\varphi_j, \partial_i\varphi_j)$:

$$\frac{\partial F}{\partial \varphi_j} - \partial_i \left(\frac{\partial F}{\partial(\partial_i \varphi_j)} \right) = 0. \tag{13}$$

We will deal with $\varphi_j = \psi^*$ and $\varphi_j = A_j$ separately. For $\varphi_j = \psi^*$, we have

$$\begin{aligned} \frac{\partial F}{\partial \psi^*} &= -\psi + |\psi|^2\psi - A_i D_i \psi, \\ \frac{\partial F}{\partial(\partial_i\psi^*)} = \frac{\partial}{\partial(\partial_i\psi^*)} \left(-\frac{i}{\kappa}\partial_i\psi - A_i\psi \right) &\left(\frac{i}{\kappa}\partial_i\psi^* - A_i\psi^* \right) = \frac{i}{\kappa} D_i \psi. \end{aligned} \tag{14}$$

Then EL reads

$$\begin{aligned} &\frac{\partial F}{\partial \psi^*} - \partial_i \left(\frac{\partial F}{\partial(\partial_i\psi^*)} \right) = 0 \\ \Rightarrow\ &-\psi + |\psi|^2\psi - A_i D_i\psi - \frac{i}{\kappa}\partial_i D_i \psi = 0 \\ \Rightarrow\ &\boxed{-\psi + |\psi|^2\psi + D_i D_i \psi = 0.} \end{aligned} \tag{15}$$

On the other hand, for $\varphi_j = A_j$, defining $D_i^\dagger = \left(\frac{i}{\kappa}\partial_i - A_i \right)$ we have

$$\begin{aligned} \frac{\partial F}{\partial A_j} &= -\left[\psi D_j^\dagger \psi^* + \psi^* D_j \psi \right], \\ \frac{\partial F}{\partial(\partial_i A_j)} &= 2H_k \frac{\partial H_k}{\partial(\partial_i A_j)} = 2H_k \varepsilon_{kij} \\ \Rightarrow \partial_i \left(\frac{\partial F}{\partial(\partial_i A_j)} \right) &= 2\varepsilon_{kij}\partial_i H_k = -2(\nabla \times \boldsymbol{H})_j. \end{aligned} \tag{16}$$

The EL equation thus reads

$$\begin{aligned} -\left[\psi D_j^\dagger \psi^* + \psi^* D_j \psi \right] + 2(\nabla \times \boldsymbol{H})_j = 0 \Longrightarrow &\boxed{\nabla \times \boldsymbol{H} = \boldsymbol{J},} \\ \text{where } J_i = \frac{1}{2}\left[\psi D_i^\dagger \psi^* + \psi^* D_i \psi \right] &= \frac{1}{2}[\frac{i}{\kappa}(\psi\partial_i\psi^* - \psi^*\partial_i\psi) - (\psi A_i \psi^* + \psi^* A_i \psi) \\ = \frac{1}{2}\left[\frac{i}{\kappa}(-2i\,\mathrm{Im}(\psi^*\partial_i\psi)) - 2A_i|\psi|^2 \right] &= \frac{1}{\kappa}\,\mathrm{Im}(\psi^*\partial_i\psi) - A_i|\psi|^2. \end{aligned} \tag{17}$$

Collecting above, we arrive at

$$\begin{cases} -\psi + |\psi|^2\psi + D_i D_i \psi = 0 \\ \nabla \times \boldsymbol{H} = \boldsymbol{J} \\ \boldsymbol{J} = \frac{1}{\kappa}\,\mathrm{Im}(\psi^*\nabla\psi) - \boldsymbol{A}|\psi|^2 \end{cases}. \tag{18}$$

P3

Find the kink configuration $\varphi(x)$ and its energy cost for the functional

$$\mathcal{F}[\varphi] = \int_{-\infty}^{\infty} \left[\frac{\kappa}{2} \left(\frac{\mathrm{d}\varphi}{\mathrm{d}x} \right)^2 + \frac{V}{4} \left(\varphi^2 - \varphi_0^2 \right)^2 \right] \mathrm{d}x \tag{19}$$

subject to boundary conditions $\varphi(-\infty) = -\varphi_0, \varphi(\infty) = \varphi_0$.

We read off the functional

$$F(\varphi, \varphi') = \frac{\kappa}{2} \varphi'^2 + \frac{V}{4} \left(\varphi^2 - \varphi_0^2 \right)^2. \tag{20}$$

Similar to P1, we notice that F is independent of x, and so Beltrami identity applies:

$$\begin{aligned} F - \varphi' \frac{\mathrm{d}F}{\mathrm{d}\varphi'} &= C \\ \Rightarrow -\frac{\kappa}{2} \varphi'^2 + \frac{V}{4} \left(\varphi^2 - \varphi_0^2 \right)^2 &= C \end{aligned} \tag{21}$$

Boundary conditions give $x \to \pm\infty \Rightarrow \varphi' = 0, \varphi^2 = \varphi_0^2 \Longrightarrow C = 0$. Thus we solve

$$\frac{\kappa}{2} \varphi'^2 = \frac{V}{4} \left(\varphi^2 - \varphi_0^2 \right)^2 \Rightarrow \varphi' = \sqrt{\frac{V}{2\kappa}} \left(\varphi_0^2 - \varphi^2 \right) \tag{22}$$

to get

$$\frac{\mathrm{d}\varphi}{\varphi_0^2 - \varphi^2} = \sqrt{\frac{V}{2\kappa}}\, \mathrm{d}x. \tag{23}$$

Since $\int \frac{\mathrm{d}u}{a^2 - u^2} = \frac{1}{a} \operatorname{artanh}\left(\frac{u}{a}\right)$, we take integration on both sides over an arbitrary range $[x_0, x]$. this yields

$$\frac{1}{\varphi_0} \operatorname{artanh}\left(\frac{\varphi}{\varphi_0} \right) = \sqrt{\frac{V}{2\kappa}} (x - x_0). \tag{24}$$

Defining a characteristic length $\xi = \sqrt{\frac{\kappa}{V\varphi_0^2}}$, the equation simplifies to $\operatorname{artanh}\left(\frac{\varphi}{\varphi_0}\right) = \frac{x - x_0}{\sqrt{2}\xi}$.

$$\boxed{\varphi = \varphi_0 \tanh\left[\frac{x - x_0}{\sqrt{2}\xi} \right] \equiv \Phi.} \tag{25}$$

The corresponding functional is thus (with $\frac{\kappa}{2}\Phi'^2 = \frac{V}{4}\left(\Phi^2 - \varphi_0^2\right)^2$)$F(\Phi, \Phi') = \kappa\Phi'^2 = \frac{\kappa\varphi_0^2}{2\xi^2} \operatorname{sech}^4\left(\frac{x - x_0}{\sqrt{2}\xi}\right) = \frac{V}{2}\varphi_0^4 \operatorname{sech}^4\left(\frac{x - x_0}{\sqrt{2}\xi}\right)$. The energy cost is therefore $E = \int_{-\infty}^{\infty} F\, \mathrm{d}x = \frac{V}{2}\varphi_0^4 \int_{-\infty}^{\infty} \operatorname{sech}^4\left(\frac{x - x_0}{\sqrt{2}\xi}\right) \mathrm{d}x$ Using the standard integral

$$\int_{-\infty}^{\infty} \operatorname{sech}^4(u)\, \mathrm{d}u = \frac{4}{3}, \tag{26}$$

we get

$$\boxed{E = \frac{2\sqrt{2}}{3} \varphi_0^3 \sqrt{\kappa V}.} \tag{27}$$

P4

Consider a functional of three functions

$$G[y_1, y_2, y_3] = \int_{x_0}^{x_1} F[x, y, y']\,\mathrm{d}x, \quad F[x, y, y'] = \frac{1}{2}\sum_{i,j=1}^{3} g_{ij}(y) y_{i'} y_{j'} - U(y), \tag{28}$$

where g_{ij} is a symmetric matrix that depends on y. Derive the sytem's of Euler-Lagrange equations that extramize G. Use Chistoffel symbols to simplify your result.

Using Einstein summation convention, and denote $\partial_k u = \frac{\partial u}{\partial y_k}$, we write

$$F = \frac{1}{2} g_{ij} y_{i'} y_{j'} - U(y). \tag{29}$$

And the Euler-Lagrange equation elements are found as follows

$$\begin{aligned} \frac{\partial F}{\partial y_k} &= \frac{\partial}{\partial y_k}\left[\frac{1}{2} g_{ij} y_{i'} y_{j'} - U(y)\right] = \frac{1}{2}(\partial_k g_{ij}) y_{i'} y_{j'} - \partial_k U, \\ \frac{\partial F}{\partial y'_k} &= \frac{1}{2} g_{ij} \partial_k (y_{i'} y_{j'}) = \frac{1}{2} g_{ij} (\delta_{ik} y_{j'} + y_{i'} \delta_{jk}) = g_{kj} y_{j'} \end{aligned} \tag{30}$$

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y'_k}\right) &= \frac{\mathrm{d}}{\mathrm{d}x}(g_{kj} y_{j'}) = \frac{\mathrm{d}g_{kj}}{\mathrm{d}x} y'_j + \frac{\mathrm{d}}{\mathrm{d}x} y'_j g_{kj} \\ &= \frac{\mathrm{d}y_l}{\mathrm{d}x}\frac{\partial}{\partial y_l} g_{kj} y'_j + y''_j g_{kj} = (\partial_l g_{kj}) y'_l y'_j + g_{kj} y''_j. \end{aligned} \tag{31}$$

E-L equation reads

$$\frac{1}{2}(\partial_k g_{ij}) y'_i y'_j - \partial_k U - (\partial_l g_{kj}) y'_l y'_j - g_{kj} y''_j = 0 \tag{32}$$

Rearranging

$$g_{kj} y''_j + \partial_l g_{kj} y'_l y'_j - \frac{1}{2}\partial_k g_{ij} y'_i y'_j + \partial_k U = 0. \tag{33}$$

We can use the identity $g_{mk} g_{kj} = \delta(mj)$ to simplify:

$$y''_m + \underbrace{g_{mk}\left[\partial_l g_{kj} - \frac{1}{2}\partial_k g_{ij}\right] y'_l y'_j}_{(*)} + g_{mk}\partial_k U = 0. \tag{34}$$

Since $y'_l y'_j$ is symmetric in l, j, we can symmetrize the term

$$\partial_l g_{kj} y'_l y' j = \frac{1}{2}\left[\partial_l g_{kj} + \partial_j g_{kl}\right] y'_l y'_j. \tag{35}$$

Then $(*)$ becomes

$$\frac{1}{2}\left[\partial_l g_{kj} + \partial_j g_{kl} - \partial_k g_{ij}\right] y'_l y'_j = \Gamma^k_{lj}\, y'_l y'_j, \tag{36}$$

and we arrive at a simplified form:

$$\boxed{y''_m + g_{mk}\Gamma^k_{lj}\, y'_l y'_j + g_{mk}\partial_k U = 0.} \tag{37}$$

P5

Consider the Lagrangian density for a one-dimensional electron-phonon system:

$$\mathcal{L} = \frac{i}{2}\left(\dot{\psi}\psi^* - \psi\dot{\psi}^*\right) - \frac{1}{2}|\partial_x\psi|^2 + \frac{1}{2}(\partial_t u)^2 - \frac{s^2}{2}(\partial_x u)^2 + g(\partial_x u)|\psi|^2. \tag{38}$$

First, derive the equations of motion for the complex field $\psi(x, t)$ and the real field $u(x, t)$. Then, find a traveling-wave solution (polaron) of the form $\psi(x, t) = e^{i(ax-bt)}\varphi(x - Vt)$ and $u(x, t) = U(x - Vt)$. Finally, calculate the total energy $E(V)$ of this solution and find the ground state energy E_0.

1. Equations of Motion from Variational Principle

The Lagrangian density is given by $\mathcal{L} = \mathcal{L}_\psi + \mathcal{L}_u + \mathcal{L}_{\psi u}$:

$$\mathcal{L} = \frac{i}{2}\left(\dot{\psi}\psi^* - \psi\dot{\psi}^*\right) - \frac{1}{2}|\partial_x\psi|^2 + \frac{1}{2}(\partial_t u)^2 - \frac{s^2}{2}(\partial_x u)^2 + g(\partial_x u)|\psi|^2 \tag{39}$$

The equations of motion are derived from the Euler-Lagrange equations.

Equation for $\psi(x, t)$

We vary with respect to the independent field ψ^*. The Euler-Lagrange equation is:

$$\frac{\partial\mathcal{L}}{\partial\psi^*} - \partial_t\left(\frac{\partial\mathcal{L}}{\partial\dot{\psi}^*}\right) - \partial_x\left(\frac{\partial\mathcal{L}}{\partial(\partial_x\psi^*)}\right) = 0 \tag{40}$$

The required partial derivatives are:

- $$\frac{\partial\mathcal{L}}{\partial\psi^*} = \frac{i}{2}\dot{\psi} + g(\partial_x u)\psi \tag{41}$$
- $$\frac{\partial\mathcal{L}}{\partial\dot{\psi}^*} = -\frac{i}{2}\psi \tag{42}$$
- $$\frac{\partial\mathcal{L}}{\partial(\partial_x\psi^*)} = -\frac{1}{2}(\partial_x\psi) \tag{43}$$

Substituting these into the equation gives:

$$\left(\frac{i}{2}\dot{\psi} + g(\partial_x u)\psi\right) - \partial_t\left(-\frac{i}{2}\psi\right) - \partial_x\left(-\frac{1}{2}\partial_x\psi\right) = 0 \tag{44}$$

$$\Rightarrow \frac{i}{2}\dot{\psi} + g(\partial_x u)\psi + \frac{i}{2}\dot{\psi} + \frac{1}{2}\partial_x^2\psi = 0 \tag{45}$$

This simplifies to the first equation of motion:

$$\boxed{i\partial_t\psi = -\frac{1}{2}\partial_x^2\psi - g(\partial_x u)\psi} \tag{46}$$

Equation for $u(x, t)$

We vary with respect to the real field u:

$$\frac{\partial\mathcal{L}}{\partial u} - \partial_t\left(\frac{\partial\mathcal{L}}{\partial(\partial_t u)}\right) - \partial_x\left(\frac{\partial\mathcal{L}}{\partial(\partial_x u)}\right) = 0 \tag{47}$$

The partial derivatives are:

- $$\frac{\partial\mathcal{L}}{\partial u} = 0 \tag{48}$$
- $$\frac{\partial\mathcal{L}}{\partial(\partial_t u)} = \partial_t u \tag{49}$$
- $$\frac{\partial\mathcal{L}}{\partial(\partial_x u)} = -s^2(\partial_x u) + g|\psi|^2 \tag{50}$$

Substituting these yields:

$$0 - \partial_t(\partial_t u) - \partial_x(-s^2 \partial_x u + g|\psi|^2) = 0 \tag{51}$$

This simplifies to the second equation of motion:

$$\boxed{\partial_t^2 u - s^2 \partial_x^2 u + g\partial_x(|\psi|^2) = 0} \tag{52}$$

2. Traveling-Wave Solution

We seek a solution where the electron and the string deformation propagate together at a velocity V. We introduce the co-moving coordinate $\xi = x - Vt$ and use the ansatz:

$$\psi(x,t) = e^{i(ax-bt)}\varphi(\xi), \quad u(x,t) = U(\xi) \tag{53}$$

where $\varphi(\xi)$ is a real-valued profile function, and a, b are constants. The derivatives transform as $\partial_t = -V\frac{\mathrm{d}}{\mathrm{d}\xi}$ and $\partial_x = \frac{\mathrm{d}}{\mathrm{d}\xi}$.

Applying these transformations to the ansatz (denoting $\frac{\mathrm{d}}{\mathrm{d}\xi}$ with a prime):

- $$\partial_t \psi = e^{i(ax-bt)}(-ib\varphi - V\varphi') \tag{54}$$
- $$\partial_x \psi = e^{i(ax-bt)}(ia\varphi + \varphi') \tag{55}$$
- $$\partial_x^2 \psi = e^{i(ax-bt)}(-a^2\varphi + 2ia\varphi' + \varphi'') \tag{56}$$
- $$\partial_t u = -VU', \quad \partial_t^2 u = V^2 U'' \tag{57}$$
- $$\partial_x u = U', \quad \partial_x^2 u = U'' \tag{58}$$

Substituting these into the first equation of motion and canceling the phase factor $e^{i(ax-bt)}$ gives:

$$i(-ib\varphi - V\varphi') = -\frac{1}{2}(-a^2\varphi + 2ia\varphi' + \varphi'') - gU'\varphi \tag{59}$$

$$\Rightarrow b\varphi - iV\varphi' = \frac{a^2}{2}\varphi - ia\varphi' - \frac{1}{2}\varphi'' - gU'\varphi \tag{60}$$

Separating the real and imaginary parts:

- **Imaginary Part:**

$$-V\varphi' = -a\varphi' \Longrightarrow a = V. \tag{61}$$

  This aligns the phase velocity with the group velocity, a feature of Galilean invariance.
- **Real Part:**

$$b\varphi = \frac{V^2}{2}\varphi - \frac{1}{2}\varphi'' - gU'\varphi. \tag{62}$$

  Rearranging gives:

$$-\frac{1}{2}\varphi'' - gU'\varphi = \left(b - \frac{V^2}{2}\right)\varphi \tag{63}$$

  Define $\mu := b - \frac{V^2}{2}$, which represents the energy of the electron in the co-moving frame. The equation becomes:

$$-\frac{1}{2}\varphi''(\xi) - gU'(\xi)\varphi(\xi) = \mu\varphi(\xi) \tag{64}$$

Next, substituting the ansatz into the second equation of motion gives:

$$V^2 U'' - s^2 U'' + g\frac{\mathrm{d}}{\mathrm{d}\xi}(\varphi^2) = 0 \Longrightarrow (V^2 - s^2)U'' + g(\varphi^2)' = 0 \tag{65}$$

Solving the Coupled ODEs

We integrate Equation 65 with respect to ξ. For a localized polaron, we assume the fields vanish at infinity ($\varphi(\xi) \to 0$ and $U'(\xi) \to 0$ as $|\xi| \to \infty$), which sets the integration constant to zero:

$$(V^2 - s^2)U' + g\varphi^2 = 0 \Longrightarrow U'(\xi) = \frac{g}{s^2 - V^2}\varphi^2(\xi) \tag{66}$$

For a localized "bright soliton" solution to exist, the induced self-interaction for the electron must be attractive. Substituting U' into Equation 63:

$$-\frac{1}{2}\varphi'' - \frac{g^2}{s^2 - V^2}\varphi^3 = \mu\varphi \tag{67}$$

The nonlinear term $-\frac{g^2}{s^2-V^2}\varphi^3$ is attractive if the coefficient is negative, which requires $s^2 - V^2 > 0$, or $|V| < s$. The polaron must travel slower than the speed of sound in the medium. We also need $\mu < 0$ for a bound state. Let $\mu = -\nu$ where $\nu > 0$. The equation is:

$$\varphi'' = 2\nu\varphi - \frac{2g^2}{s^2 - V^2}\varphi^3 \tag{68}$$

This is a standard nonlinear Schrödinger (NLS) equation. The solution is of the form:

$$\varphi(\xi) = A\,\mathrm{sech}(\alpha\xi) \tag{69}$$

Substituting this form back into the NLS equation gives the relations: $\alpha^2 = 2\nu$ and $A^2 = \frac{2\nu(s^2-V^2)}{g^2} = \frac{\alpha^2(s^2-V^2)}{g^2}$.

We impose the normalization condition for a single electron, $\int_{-\infty}^{\infty} |\psi|^2 \,\mathrm{d}x = \int_{-\infty}^{\infty} \varphi^2 \,\mathrm{d}\xi = 1$:

$$\int_{-\infty}^{\infty} A^2 \,\mathrm{sech}^2(\alpha\xi)\,\mathrm{d}\xi = A^2 \left[\frac{1}{\alpha}\tanh(\alpha\xi)\right]_{-\infty}^{\infty} = \frac{2A^2}{\alpha} = 1 \tag{70}$$

Using $A^2 = \frac{\alpha^2(s^2-V^2)}{g^2}$, we get $\frac{2}{\alpha}\frac{\alpha^2(s^2-V^2)}{g^2} = 1$, which solves for α:

$$\alpha = \frac{g^2}{2(s^2 - V^2)} \tag{71}$$

From this, we find the other parameters:

$$A^2 = \frac{\alpha}{2} = \frac{g^2}{4(s^2 - V^2)}, \quad \nu = \frac{\alpha^2}{2} = \frac{g^4}{8(s^2 - V^2)^2} \tag{72}$$

3. System Energy $E(V)$

The Hamiltonian (energy) density is derived from the Lagrangian density: $\mathcal{H} = \sum \pi_i \dot{\varphi}_i - \mathcal{L}$.

$$\pi_\psi = \frac{\partial\mathcal{L}}{\partial\dot{\psi}} = \frac{i}{2}\psi^*, \quad \pi_u = \frac{\partial\mathcal{L}}{\partial\dot{u}} = \partial_t u \tag{73}$$

$$\begin{aligned} \mathcal{H} &= \left(\frac{i}{2}\psi^*\right)\dot{\psi} + \left(-\frac{i}{2}\psi\right)\dot{\psi}^* + (\partial_t u)(\partial_t u) - \mathcal{L} \\ &= \frac{1}{2}|\partial_x\psi|^2 + \frac{1}{2}(\partial_t u)^2 + \frac{s^2}{2}(\partial_x u)^2 - g(\partial_x u)|\psi|^2) \end{aligned} \tag{74}$$

The total energy is $E(V) = \int_{-\infty}^{\infty} \mathcal{H}\,\mathrm{d}x$. We substitute the traveling wave solution into the terms of $\mathcal{H}$:

- $$|\partial_x\psi|^2 = \left|e^{i(Vx-bt)}(\varphi' + iV\varphi)\right|^2 = (\varphi')^2 + V^2\varphi^2 \tag{75}$$
- $$(\partial_t u)^2 = (-VU')^2 = V^2(U')^2 \tag{76}$$
- $$(\partial_x u)^2 = (U')^2 \tag{77}$$
- Using $U' = \frac{g}{s^2-V^2}\varphi^2$, the terms involving u become:

$$\frac{1}{2}(V^2 + s^2)(U')^2 - gU'\varphi^2 = \frac{1}{2}(V^2 + s^2)\frac{g^2}{(s^2 - V^2)^2}\varphi^4 - \frac{g^2}{s^2 - V^2}\varphi^4 = -\frac{g^2(s^2 - 3V^2)}{2(s^2 - V^2)^2}\varphi^4 \tag{78}$$

The total energy integral is:

$$E(V) = \int_{-\infty}^{\infty} \left[\frac{1}{2}\left((\varphi')^2 + V^2\varphi^2\right) - \frac{g^2(s^2 - 3V^2)}{2(s^2 - V^2)^2}\varphi^4 \right] \mathrm{d}\xi \tag{79}$$

We evaluate the separate integrals using our solution for φ:

- $$\int \frac{1}{2}V^2\varphi^2 \,\mathrm{d}\xi = \frac{1}{2}V^2 \quad \int \varphi^2 \,\mathrm{d}\xi = \frac{1}{2}V^2 \tag{80}$$

  (from normalization).

- $$\int \frac{1}{2}(\varphi')^2 \,\mathrm{d}\xi = \int \frac{1}{2}A^2\alpha^2 \operatorname{sech}^2(\alpha\xi)\tanh^2(\alpha\xi)\,\mathrm{d}\xi = \frac{A^2\alpha}{3} = \frac{\left(\frac{\alpha}{2}\right)\alpha}{3} = \frac{\alpha^2}{6} = \frac{\nu}{3} = \frac{g^4}{24(s^2 - V^2)^2} \tag{81}$$

  .

- $$\int \varphi^4 \,\mathrm{d}\xi = \int A^4 \operatorname{sech}^4(\alpha\xi)\,\mathrm{d}\xi = A^4\frac{4}{3\alpha} = \left(\frac{\alpha}{2}\right)^2\frac{4}{3\alpha} = \frac{\alpha}{3} = \frac{g^2}{6(s^2 - V^2)} \tag{82}$$

  .

Combining these results:

$$\begin{aligned} E(V) &= \frac{1}{2}V^2 + \frac{g^4}{24(s^2 - V^2)^2} - \frac{g^2(s^2 - 3V^2)}{2(s^2 - V^2)^2}\left(\frac{g^2}{6(s^2 - V^2)}\right) \\ &= \frac{1}{2}V^2 + \frac{g^4}{24(s^2 - V^2)^3}\left[(s^2 - V^2) - 2(s^2 - 3V^2)\right] \\ &= \frac{1}{2}V^2 + \frac{g^4}{24(s^2 - V^2)^3}(5V^2 - s^2) \end{aligned} \tag{83}$$

This gives the final energy as a function of velocity for the subsonic polaron, valid for $|V| < s$:

$$\boxed{E(V) = \frac{V^2}{2} + \frac{g^4(5V^2 - s^2)}{24(s^2 - V^2)^3}} \tag{84}$$

Ground State Energy E_0

The ground state energy is the energy of a static polaron, found by setting $V = 0$.

$$E_0 = E(0) = \frac{0}{2} + \frac{g^4(0 - s^2)}{24(s^2 - 0)^3} = \boxed{\frac{-g^4 s^2}{24 s^6}} \tag{85}$$

The negative energy confirms that the formation of a static polaron is energetically favorable, representing a stable bound state.