1

Consider permutation bringing BINGE to BEGIN, call it σ, How many inversion it has? Let σ' and σ'' be the permutations bringing BEING to BINGE and to BEGIN. Explain why $\text{inv}(\sigma')$ and $\text{inv}(\sigma'')$ must be of opposite parity. Verify this by calculating these inversion numbers.

We lable BINGE as (12345) and BEGIN as (15423). We see that the permulation that sends BINGE to BEGIN is

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 5 & 4 & 2 & 3 \end{pmatrix} = (2\,5\,3\,4) \tag{1}$$

We count the inversions: $(4,5), (2,5), (3,5), (2,4), (3,4)$. So $\boxed{\text{inv}(\sigma) = 5.}$

Now, we take σ' as BEING to BINGE, and σ'' as BEING to BEGIN. We notice that

$$\begin{aligned} \sigma'' &= \sigma \circ \sigma' \\ \Rightarrow \text{inv}(\sigma'') &= \text{inv}(\sigma) + \text{inv}(\sigma') = 5 + \text{inv}(\sigma') = \text{inv}(\sigma') + 1 \bmod 2 \end{aligned} \tag{2}$$

So it's obvious that σ' and σ'' have different parities.

To verify, consider

$$\sigma' = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 4 & 5 & 2 \end{pmatrix} \tag{3}$$

inversions of σ' are $(2,3), (2,4), (2,5)$. So $\text{inv}(\sigma') = 3$, which is odd.

Consider

$$\sigma'' = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 5 & 3 & 4 \end{pmatrix} \tag{4}$$

The inversions are $(3,5), (4,5)$. So $\text{inv}(\sigma'') = 2$, which is even – different parity from σ'.

2

[based on D&F 3.5.3-4]

a. Prove that S_n $(n \geq 2)$ is generated by the $n-1$ transpositions $(1\,i)$, $(2 \leq i \leq n)$.

b. Prove that S_n $(n \geq 2)$ is generated by the $n-1$ transpositions $(i\,i+1)$, $(1 \leq i \leq n-1)$.

c. Prove that S_n $(n \geq 2)$ is generated by the transposition $(1\,2)$ and the long cycle $(1\,2\,3\,\ldots\,n)$.

d. Deduce that A_n $(n \geq 3)$ is generated by the 3-cycles $(1\,i\,j)$ $(2 \leq i < j \leq n)$.

e. Conclude that if $h : A_n \to G$ is a homomorphism from A_n to an abelian group G then all $g \in h(A_n)$ must satisfy $g^3 = 1$.

a.

Recall that $S_n = \langle (ij) \rangle$, where $1 \leq i \leq j \leq n$. For each such pair of i, j transposition, we can write

$$(i\,j) = (1\,i)(1\,j)(1\,i). \tag{5}$$

In other words, each of the generator of S_n can be written in terms of generators $\{(1\,k)|\ 2 \leq k \leq n\}$ where the size of this new generating set is $n-1$. Therefore , S_n can be generated by $n-1$ transpositions $(1\,i), (2 \leq i \leq n)$.

b.

Define an adjacent transposition $t_i := (i\,i+1)$. We show that for $1 \leq i \leq n-1, j = i+k$:

$$\begin{aligned} (i\,j) &= (i\,i+1)(i+1\,i+2)\ldots(j-1\,j)(j-2\,j-1)\ldots(i\,i+1) \\ &= t_i\,t_{i+1}\ldots\,t_{j-1}\,t_{j-2}\ldots\,t_i \end{aligned} \tag{6}$$

so that each $(i\,j)$ can be decomposed into a product of adjacent transpositions. Since the set $\{(i\,j) \mid 1 \leq i < j \leq n\}$ generates S_n, the set of adjacent transpositions $\{(i\,i+1) \mid 1 \leq i \leq n-1\}$ also generates S_n.

c

Consider the transposition of two adjacent elements $(k\,k+1)$ (where k is arbitrary). This can be expressed in terms of the transposition $(1\,2)$ and the long cycle $\rho := (1\,2\,\ldots n)$ as follows:

$$(k\,k+1) = \rho^{k-1}\,(1\,2)\,\rho^{-(k-1)} \tag{7}$$

Concretely, we can follow the mapping: $\rho^{-(k-1)} : k \to 1, \quad (1\,2) : 1 \to 2, \quad \rho^{k-1} : 2 \to k+1.$

By varing k, we see that $\langle (1\,2), \rho \rangle$ generates all adjacent transpositions, which in turn generates S_n.

d.

From (a), we know that $S_n = \langle (12)\ (13) \ldots (1n) \rangle$. Since $A_n = \{\pi \in S_n, \pi \text{ is even permutation}\}$, π can be written as products even number of transpositions , i.e. $\prod_k (1k)$. This can be grouped into pairs of

$$(1i)(1j) = (1ij) \quad (2 \leq i < j \leq n) \tag{8}$$

which is exactly the set of 3-cycles that we want. Therefore, A_n is generated by the set of 3-cycles of the form

$$(1ij), \quad 2 \leq i < j \leq n. \tag{9}$$

e.

By part d, we know that for any $\sigma \in A_n$, $\sigma = \prod \tau_i$, where each τ_i is a 3-cycle $(1ij)$. We know that $\tau^3 = 1$.

Then for any $g \in h(A_n), g = h(\sigma)$. Homomorphism gives

$$g = h(\sigma) = h\Bigl(\prod \tau_i\Bigr) = \prod h(\tau_i) \tag{10}$$

and abelianity gives

$$g^3 = (h(\sigma))^3 = \prod (h(\tau_i))^3 = \prod h(\tau_i^3) = \prod h(1) = 1. \tag{11}$$

3

[D&F 3.5.12] Prove that A_n contains a subgroup isomorphic to S_{n-2} for each $n \geq 3$.

Construct a mapping (with $n \geq 3$) $\varphi : S_{n-2} \to A_n$ as

$$\varphi(\sigma) = \begin{cases} \sigma & \sigma \text{ even} \\ \sigma(n\, n-1) & \sigma \text{ odd} \end{cases} \tag{12}$$

where $\sigma \in S_{n-2}$, and it permutes the first $n-2$ elements, and leaves $n-1$ and n fixed.

We will show that φ is an injective homomorphism, and use the first isomorphism theorem to conclude that $S_{n-2} \cong \mathrm{Im}(\varphi) \subset A_n$.

First, $\mathrm{Im}(\varphi) \subset A_n$. Since for any $\sigma \in S_{n-2}$, if :

- σ even, then $\varphi(\sigma) = \sigma \in A_n$.
- σ odd, then $\varphi(\sigma) = \sigma(n\, n-1)$. Note that $(n\, n-1)$ is also odd, so the product of two odd permutations is even, and thus $\varphi(\sigma) \in A_n$.

Second, $\varphi(\sigma\tau) = \varphi(\sigma)\varphi(\tau)$ for any $\sigma, \tau \in S_{n-2}$. Indeed:

- If σ, τ have the same parity, then $\sigma\tau$ is even, and so
$$\varphi(\sigma\tau) = \sigma\tau = \varphi(\sigma)\varphi(\tau), \tag{13}$$
where the last equality holds because in the case both σ, τ are odd, $\varphi(\sigma)\varphi(\tau) = \sigma(n\, n-1)\tau(n\, n-1) = \sigma\tau$, since σ, τ commute with $(n\, n-1)$ as they only permute the first $n-2$ elements.
- If σ, τ have different parities, then $\sigma\tau$ is odd, and so
$$\varphi(\sigma\tau) = \sigma\tau(n\, n-1) = \varphi(\sigma)\varphi(\tau). \tag{14}$$

Finally, we show that $\ker(\varphi) = \{e\}$. Consider $\varphi(\sigma) = e$. If:

- σ even, then $\varphi(\sigma) = \sigma = e$, so $\sigma = e$.
- σ odd, then $\varphi(\sigma) = \sigma(n\, n-1) = e$, so $\sigma = (n\, n-1)$, which is a contradiction since $\sigma \in S_{n-2}$ only permutes the first $n-2$ elements so $\sigma \neq (n\, n-1)$.

Collectively, we have shown that φ is an injective homomorphism, so by the first isomorphism theorem,

$$S_{n-2} \cong \mathrm{Im}(\varphi) \subset A_n. \tag{15}$$

In other words, A_n contains a subgroup isomorphic to S_{n-2} for $n \geq 3$.

4

> 4. [based on D&F 4.1.1-3] Let G act on the nonempty set A.
>   a. Prove that if $a \in A$ and $g \in G$ then the stabilizer $G_{g \cdot a}$ is $gG_a g^{-1}$.
>   b. Deduce that if the action is transitive then the kernel of the action is $\bigcap_{g \in G} gG_a g^{-1}$.
>   c. In particular if G is a transitive subgroup of S_A (i.e. action of G on A is transitive) then $\bigcap_{g \in G} gG_a g^{-1} = \{1\}$.
>   d. If moreover G is abelian, show that $g(a) \neq a$ for all $g \in G - \{1\}$ and $a \in A$, and deduce $|G| = |A|$.

a

We show that $G_{ga} \subset gG_a g^{-1}$, and then $gG_a g^{-1} \subset G_{ga}$.

1. For any $h \in G_{ga}$, we have by the definition of stablizer that

$$h(ga) = ga \Rightarrow g^{-1}hga = a \Rightarrow g^{-1}hg \in G_a \tag{16}$$

   and so $h \in G_a$.

Therefore, $G_{ga} \subset gG_a g^{-1}$.

2. For any $h \in gG_a g^{-1}$, and for any $k \in G_a$, we have

$$h = gkg^{-1} \Rightarrow hga = gkg^{-1}ga = gka = ga \tag{17}$$

   and so $h \in G_{ga}$.

Therefore, $gG_a g^{-1} \subset G_{ga}$. Collectively, we have $G_{ga} = gG_a g^{-1}$.

b

We know that the kernel is the intersection of all stabilizers:

$$\ker = \{k \in G \mid kx = x \forall x \in A\} = \bigcap_{x \in A} G_x \tag{18}$$

Since the action is transitive, the orbit of any $x \in A$ is the whole set A, i.e.

$$\mathcal{O}_a = A \tag{19}$$

and so for any $x \in A$, there exists $g \in G$ such that $x = ga$. Therefore,

$$\ker = \bigcap_{x \in A} G_x = \bigcap_{g \in G} G_{ga} \tag{20}$$

by part a, we have

$$\ker = \bigcap_{g \in G} gG_a g^{-1}. \tag{21}$$

c

Given $G \leq S_A$ and G is transitive, then

$$\ker = \{g \in G | \ gs = s, \forall s \in A\} = \{e\} \tag{22}$$

since each $g \in G$ is a permutation on A, and the only permutation that fixes all elements is the identity permutation.

By part b, a transitive action has kernel equal to the intersection of all conjugates of a stabilizer. Therefore,

$$\{e\} = \ker = \bigcap_{g \in G} gG_a g^{-1}. \tag{23}$$

d

Given that G is abelian in addition to being a transitive subgroup of S_A, we have

$$gG_ag^{-1} = G_a, \tag{24}$$

since for any $h \in G_a$, $ghg^{-1} = hgg^{-1} = h \in G_a$. Therefore by part c,

$$\bigcap_{g \in G_a} gG_ag^{-1} = \bigcap_{g \in G} G_a = G_a = \{e\}. \tag{25}$$

Thus the set $G - \{e\} = G - G_a$, and so for any $g \in G - \{e\}, a \in A : g \notin G_a$, i.e.

$$ga = g(a) \neq a. \tag{26}$$

Further, by Orbit-stabilizer theorem,

$$|G| = |O_a| \cdot |G_a|, \tag{27}$$

but since $G_a = \{e\}$, $|G_a| = 1$, and since the action is transitive, $|O_a| = |A|$. Therefore,

$$|G| = |A|. \tag{28}$$

5

[D&F 4.1.4] Let S_3 act on the set Ω of ordered pairs: $\{(i,j\} \mid 1 \le i,j \le 3\}$ by $a((i,j)) = (a(i), a(j))$.

a. Find the orbits of S_3 on Ω.

b. For each $\sigma \in S_3$ find the cycle decomposition of σ under this action (i.e., find its cycle decomposition when a is considered as an element of S_9 – first fix a labelling of these nine ordered pairs).

c. For each orbit $\mathcal{O}$ of S_3 acting on these nine points pick some a $a \in \mathcal{O}$ and find the stabilizer of a in S_3 .

a

It's helpful to first writeout the elements of $S_3 = \{e, (1\,2), (1\,3), (2\,3), (1\,2\,3), (1\,3\,2)\}$.

For any $\sigma \in S_3$, the action is that $\sigma(i,j) = (\sigma(i), \sigma(j))$. Notice that for $i \neq j$, $\sigma(i) \neq \sigma(j)$ and for $i = j$, $\sigma(i) = \sigma(j)$. So the action preserves the property of being a distinct or identical pair. Therefore, the action partitions Ω into two orbits

$$\begin{aligned} \mathcal{O}_1 &= \{(1,1), (2,2), (3,3)\}; \\ \mathcal{O}_2 &= \{(i,j) \mid i \neq j\}, \quad |\mathcal{O}_2| = 6 \end{aligned} \tag{29}$$

b

We first write this action into permutation representation. Labling elements of Ω as

$$\begin{aligned} &1 : (1,1), 2 : (1,2), 3 : (1,3), \\ &4 : (2,1), 5 : (2,2), 6 : (2,3), \\ &7 : (3,1), 8 : (3,2), 9 : (3,3) \end{aligned} \tag{30}$$

We can find the permutation representation of each element in S_3, and from which we can find the cycle decomposition:

$$\begin{aligned} &e : e \\ (12) : &\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 5 & 4 & 6 & 2 & 1 & 3 & 8 & 7 & 9 \end{pmatrix} = (15)(24)(36)(78)(9), \\ (13) : &\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 5 & 4 & 6 & 2 & 1 & 3 & 8 & 7 & 9 \end{pmatrix} = (19)(28)(37)(46)(5), \\ (23) : &\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 1 & 3 & 2 & 7 & 9 & 8 & 4 & 6 & 5 \end{pmatrix} = (1)(23)(47)(59)(68), \\ (123) : &\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 5 & 6 & 4 & 8 & 9 & 7 & 2 & 3 & 1 \end{pmatrix} = (159)(267)(348), \\ (132) : &\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 9 & 7 & 8 & 3 & 1 & 2 & 6 & 4 & 5 \end{pmatrix} = (195)(276)(384) \end{aligned} \tag{31}$$

c

1. For orbit $\mathcal{O}_1$, we pick $a = (1,1)$.

Stabilizer by definition is

$$G_{S_3} = \{\sigma \in S_3 \mid \sigma(1,1) = (1,1)\}. \tag{32}$$

since $\sigma(1) = 1, \sigma = e$ or (23) , and so

$$G_{S_3} = \{e, (2\,3)\}. \tag{33}$$

2. For orbit $\mathcal{O}_2$, we pick $b = (1, 2)$.

Pick $b = (1, 2)$. Stabilizer by definition is

$$G_{S_3} = \{\sigma \in S_3 \mid \sigma(1, 2) = (1, 2)\}. \tag{34}$$

Since $\sigma(1) = 1, \sigma(2) = 2, \sigma = e$, and so

$$G_{S_3} = \{e\}. \tag{35}$$