

ECE 601/901 Fall 2024
Homework #4
Due 11/12 at 11:59 pm on Canvas, in pdf format

Guidelines:

- Typed solutions or handwritten solutions (as long as they are legible) are accepted.
- Please show as much work as possible and compile your responses into a single pdf document.
- You are welcome to work together on the problems, but please write your solutions in your own words.

1. (a) (15 points) Calculate the expectation values $\langle r \rangle$, in terms of a_0 , for the 1s, 2s, and 3p states in the hydrogen atom and compare these results with expectation from the Bohr model. Please show your work, but you may use a software to solve this problem.

$$\begin{aligned}\psi_{1s}(r, \theta, \phi) &= \left(\frac{1}{\sqrt{\pi a_0^3}}\right) e^{-\frac{r}{a_0}} \rightarrow |\psi_{1s}|^2 = \left(\frac{1}{\pi a_0^3}\right) e^{-\frac{2r}{a_0}} \\ \psi_{2s}(r, \theta, \phi) &= \left(\frac{1}{4\sqrt{2\pi a_0^3}}\right) \left(2 - \frac{r}{a_0}\right) e^{-r/2a_0} \rightarrow |\psi_{2s}|^2 = \left(\frac{1}{32\pi a_0^3}\right) \left(2 - \frac{r}{a_0}\right)^2 e^{-\frac{r}{a_0}} \\ \psi_{3p,m=0}(r, \theta, \phi) &= \left(\frac{\sqrt{2}}{81\sqrt{\pi a_0^3}}\right) \left(6 - \frac{r}{a_0}\right) \left(\frac{r}{a_0}\right) e^{-\frac{r}{3a_0}} \cos \theta \\ &\rightarrow |\psi_{3p,m=0}|^2 = \left(\frac{2}{6561\pi a_0^3}\right) \left(6 - \frac{r}{a_0}\right)^2 \left(\frac{r}{a_0}\right)^2 e^{-\frac{2r}{3a_0}} \cos^2 \theta\end{aligned}$$

To calculate the expectation value $\langle r \rangle$, we perform the calculation:

$$\begin{aligned}\langle r \rangle &= \int_0^{2\pi} \int_0^\pi \int_0^\infty \psi^* r \psi r^2 \sin \theta dr d\theta d\phi \\ \langle r \rangle &= \int_0^{2\pi} \int_0^\pi \int_0^\infty r^3 |\psi|^2 \sin \theta dr d\theta d\phi\end{aligned}$$

We recognize that for ψ_{1s} and ψ_{2s} , integration over the polar and azimuthal components gives 4π . For $\psi_{3p,m=0}$, integration over θ and ϕ gives $4\pi/3$.

We can use a software, such as WolframAlpha (<https://www.wolframalpha.com/>) to solve the integral. Below are also some example MATLAB codes that can be used for the symbolic integration.

```
syms r a_0 positive;

% Wave functions for the 1s, 2s, and 3p states in a hydrogen atom

% 1s state: n=1, l=0
R_1s = (1 / sqrt(pi * a_0^3)) * exp(-r / a_0);

% 2s state: n=2, l=0
R_2s = (1 / (4 * sqrt(2 * pi * a_0^3))) * (2 - r / a_0) * exp(-r / (2 * a_0));

% 3p state: n=3, l=1
R_3p = (sqrt(2) / (81 * sqrt(pi * a_0^3))) * (6 - r/a_0) * (r / a_0) * exp(-r / (3 * a_0));
```

```
% Calculate the expectation values of r for each state
expectation_1s = int(r^3 *conj(R_1s)*R_1s, r, 0, inf)
expectation_2s = int(r^3 * conj(R_2s)*R_2s, r, 0, inf)
expectation_3p = int(r^3 * conj(R_3p)*R_3p, r, 0, inf)
```

$$\begin{aligned}\langle r \rangle_{1s} &= \frac{3a_0}{8\pi}(4\pi) = \frac{3}{2}a_0 \\ \langle r \rangle_{2s} &= \frac{3a_0}{2\pi}(4\pi) = 6a_0 \\ \langle r \rangle_{3p} &= \frac{75a_0}{8\pi}\left(\frac{4\pi}{3}\right) = \frac{25}{2}a_0\end{aligned}$$

According to the Bohr model, the orbital radius for the electron scales with n and a_0 according to $r_n = n^2 a_0$:

$$\begin{aligned}r_1 &= a_0 \\ r_2 &= 4a_0 \\ r_3 &= 9a_0\end{aligned}$$

In general, the Bohr model underestimates the mean orbital radius.

(b) (5 points) A hydrogen atom is excited from the ground state with kinetic energy 12.09 eV. Using the Bohr model, what is the change in the radius of the electron? Compare this value to the change in $\langle r \rangle$ based on your result in (a).

$$\begin{aligned}-13.6 \text{ eV} \left(1 - \frac{1}{n_{exc}^2}\right) &\leq 12.09 \text{ eV} \\ n_{exc} &\leq 3\end{aligned}$$

The change in mean radius from 1s to 3p is

$$\langle r \rangle_{3p} - \langle r \rangle_{1s} = \left(\frac{25}{2} - \frac{3}{2}\right)a_0 = 11a_0$$

According to the Bohr model, this change will be $(9 - 1)a_0 = 8a_0$.

2. (15 points) A diffraction grating has 1000 lines per centimeter and an overall width of 1 cm.
 (a) (4 points) When monochromatic light of wavelength 589 nm is incident on the grating at 60° , calculate the angles at which the first and second-order maxima occur.

We use the grating equation to compute the diffracted angle θ_d :

$$\begin{aligned}m\lambda &= d(\sin \theta_{inc} - \sin \theta_d) \\ \theta_d &= \sin^{-1}(\sin \theta_{inc} - \frac{m\lambda}{d})\end{aligned}$$

where $\theta_{inc} = 60^\circ$ is the angle of incidence, m is the grating order, $\lambda = 589 \text{ nm}$, and d is the grating period.

$$d = \frac{1}{1000} 10^{-2} \text{ m} = 10^{-5} \text{ m}$$

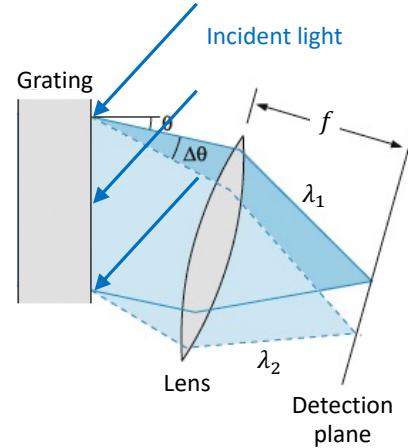
$$\begin{aligned}\sin \theta_d, \quad m=1 &= 53.82^\circ \\ \sin \theta_d, \quad m=2 &= 48.44^\circ\end{aligned}$$

- (b) (3 points) Estimate the *resolving power* of this grating for the m-th-order diffraction. Resolving power here is defined as the incident wavelength (λ) divided by the smallest wavelength separation that can be observed ($\Delta\lambda_{min}$): $\lambda/\Delta\lambda_{min}$. Discuss how increasing the number of lines would affect its resolving power.

We can consider two wavelengths to be resolved by the grating when the maximum of one wavelength coincides with the minimum of the other. For very small wavelength differences, we can find the relationship between a change in wavelength and change in diffraction angle by differentiating the grating equation:

$$m \frac{d\lambda}{d\theta_d} = -d \cos \theta_d$$

$$m \frac{\Delta\lambda}{\Delta\theta_d} = -d \cos \theta_d$$



Note that the angle of incidence is the same and therefore treated as a constant.

The angular separation between the minima and maxima for N grooves can be found through the intensity of the sum of all diffracted waves from N grooves, in which each groove introduces an additional phase difference $\delta\phi = -\frac{2\pi d}{\lambda} \sin \theta_d$. Differentiating this gives:

$$\frac{\Delta(\delta\phi)}{\Delta\theta_d} = \frac{2\pi d}{\lambda} \cos \theta_d$$

Meanwhile, the total electric field is

$$\mathcal{E}_{tot} \propto e^{i\omega t} + e^{i(\omega t + \delta\phi)} + e^{i(\omega t + 2\delta\phi)} + \dots + e^{i(\omega t + (N-1)\delta\phi)}$$

which is a geometric series that becomes

$$\mathcal{E}_{tot} \propto \frac{e^{i\omega t}(1 - e^{iN\delta\phi})}{1 - e^{i\delta\phi}}$$

The intensity is

$$I_{tot} \propto |\mathcal{E}_{tot}|^2 \propto \frac{\sin^2(N\delta\phi/2)}{\sin^2(\delta\phi/2)}$$

The first minima occurs when $N\delta\phi/2 = \pi$ where the maxima is when $\delta\phi = 0$. Therefore, $\Delta(\delta\phi) = 2\pi/N$.

$$\Delta\theta_d = \frac{\Delta(\delta\phi)}{\frac{2\pi d}{\lambda} \cos \theta_d} = \frac{\left(\frac{2\pi}{N}\right)}{\frac{2\pi d}{\lambda} \cos \theta_d}$$

$$\Delta\theta_d = \frac{\lambda}{Nd \cos \theta_d}$$

Now we combine the two highlighted equations:

$$-m \Delta\lambda = \frac{\lambda}{N} \rightarrow \left| \frac{\lambda}{\Delta\lambda} \right| = m \cdot N$$

We see that $\lambda/\Delta\lambda$ scales with both the diffraction order and number of grooves.

$$\text{For this problem, } \left| \frac{\lambda}{\Delta\lambda} \right| = m \times \left(\frac{1000}{cm} \cdot 1 \text{ cm} \right) = 1000 \times m$$

Increasing the number of lines would increase the resolving power.

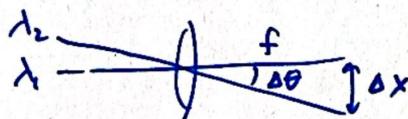
- (c) (5 points) The grating is illuminated with emission from sodium at an incident angle of 60° , after which a lens is used to collect and focus the diffracted light onto a detection plane. What is the distance between first-order diffracted lines in sodium (at 589 nm and 589.6 nm) on the detection plane, if the focal length of the lens is 1 m?

We can calculate the diffracted angle for each unique wavelength:

$$\begin{aligned}\theta_{d, \lambda_1=589 \text{ nm}} &= 53.816^\circ \\ \theta_{d, \lambda_2=589.6 \text{ nm}} &= 53.8102^\circ\end{aligned}$$

$$\Delta\theta_d = 0.0058^\circ = 1.0129 \times 10^{-4} \text{ radians}$$

On the detection plane, the lines are separated by $\Delta x \approx f\Delta\theta_d \approx 1.0129 \times 10^{-4} \text{ m} \approx 0.1 \text{ mm}$



- (d) (3 points) Is the scenario described in (c) sufficient to clearly observe the spectral lines? If not, recommend another way to resolve the lines using this grating.

Since the resolving power is 1000, the resolution $\Delta\lambda$ at 589 nm is 0.589 nm, which is on the order of the wavelength difference. Therefore it will be difficult to see the lines as distinct from each other (see illustration below of the intensity on the screen). To better resolve the lines, we can use a higher diffraction grating order.



3. (10 points) A grating spectrometer is used to study the emission spectrum of an element, which has two closely spaced emission lines at 502.1 nm and 502.3 nm. The spectrometer has a grating with 1500 grooves per mm and the grating has a width of 2 cm. The linear dispersion of the spectrometer is 4 mm/nm. The light first passes through an entrance slit before striking the grating. In this problem you will calculate and compare the limits to the spectral resolution of the spectrometer based on the finite size of the grating and the width of the entrance slit.

- (a) (3 points) The resolution based on the number of grooves illuminated (N) in a grating is determined by $\Delta\lambda_{min} = \lambda/(mN)$, where m is the diffraction order. Estimate the resolution limits due to the grating in this case as a function of m , by assuming that light is illuminating the entire width of the grating after entering the spectrometer.

The number of illuminated grooves is $N = 20 \text{ mm} (1500/\text{mm}) = 30000$

$$\text{Using } \lambda \sim \lambda_1 = 502.1 \text{ nm}, \Delta\lambda|_{gr} = \frac{1}{m} (0.017 \text{ nm})$$

- (b) (3 points) Now you are adjusting the entrance slit width to optimize resolution without significantly compromising the intensity of the detected light. The minimum slit width for which you are getting detectable light is 0.3 mm. What is the resolution here based on just the entrance slit width?

The entrance-slit-limited resolution, based on a slit size of s and linear dispersion $dx/d\lambda$ is

$$\begin{aligned}\Delta\lambda|_{slit} &= \frac{s}{\frac{dx}{d\lambda}} = \frac{0.3 \text{ mm}}{4 \text{ mm/nm}} \\ \Delta\lambda|_{slit} &= 0.075 \text{ nm}\end{aligned}$$

- (c) (2 points) Compare the resolutions in parts (a) and (b). Assuming first-order diffraction, which is the limiting factor in the resolution of the spectrometer?

The limiting factor is the entrance slit size: $\Delta\lambda|_{slit} > \Delta\lambda|_{gr}$

- (d) (2 points) If the incidence angle is 20° , what is the diffraction angle for the order that allows the grating spectrometer to resolve the two spectral lines?

First-order diffraction is sufficient to resolve the two lines, since the resolution limit is given by the slit entrance $\Delta\lambda|_{slit} = 0.075 \text{ nm} > (502.3 - 502.1) \text{ nm}$

We use the grating equation again to estimate the diffraction angle:

$$\begin{aligned}502.1 \text{ nm} &= \left(\frac{1500}{mm}\right)^{-1} (\sin 30^\circ - \sin \beta) \\ \beta &= -24.28^\circ\end{aligned}$$

4. (10 points) Two lasers, a bragg reflector laser and a ring cavity laser, both emit light at a central wavelength of 780 nm. The bragg reflector lasers have a linewidth of 1 GHz, while the ring cavity laser has a linewidth of 1 kHz. A Michelson interferometer is set up with one arm that can be varied in length.

- (a) (5 points) For each laser, determine the maximum path difference that can be introduced between the two arms of the interferometer before the interference pattern is lost.

The maximum path length difference before interference is lost will be very close to the coherence length of the laser.

$$L_c = \frac{c}{\Delta\nu}$$

For the bragg reflector laser, $L_c = 0.3 \text{ m}$

For the ring cavity laser, $L_c = 300,000 \text{ m}$

- (b) (5 points) Suppose you increase the path length difference in the interferometer gradually from zero: how many bright fringes will you be able to count for each source from the starting point until the visibility is significantly lost?

$$\text{Number of fringes} = \frac{kL_c}{2\pi} = \frac{L_c}{\lambda}$$

Bragg: you can count 384,615 bright fringes

Ring: 384,615,384,615 bright fringes

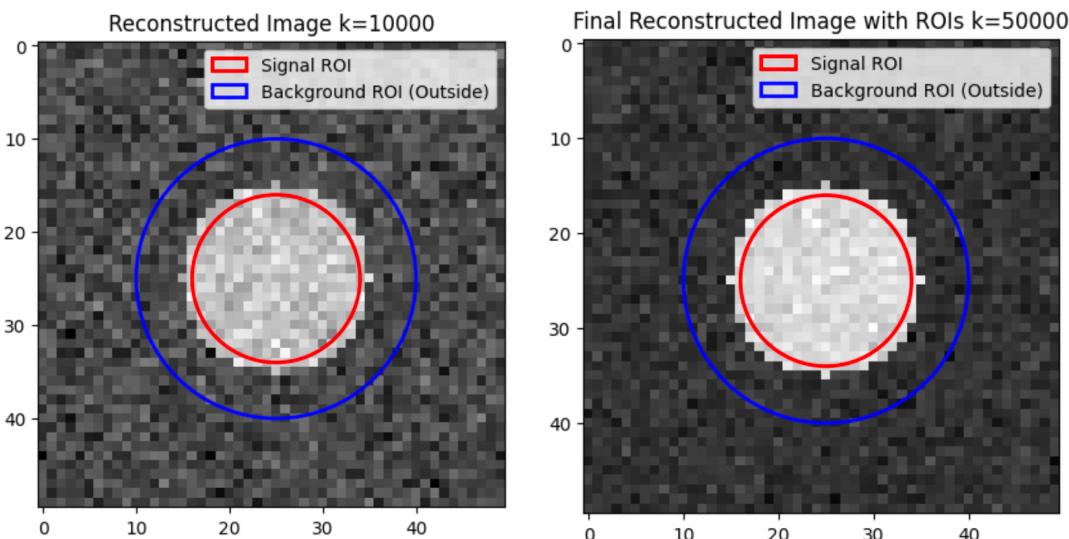
5. (10 points) In this problem, we will determine how the signal-to-noise ratio of the image is affected by the number patterns used for the image reconstruction in a classical ghost imaging problem.

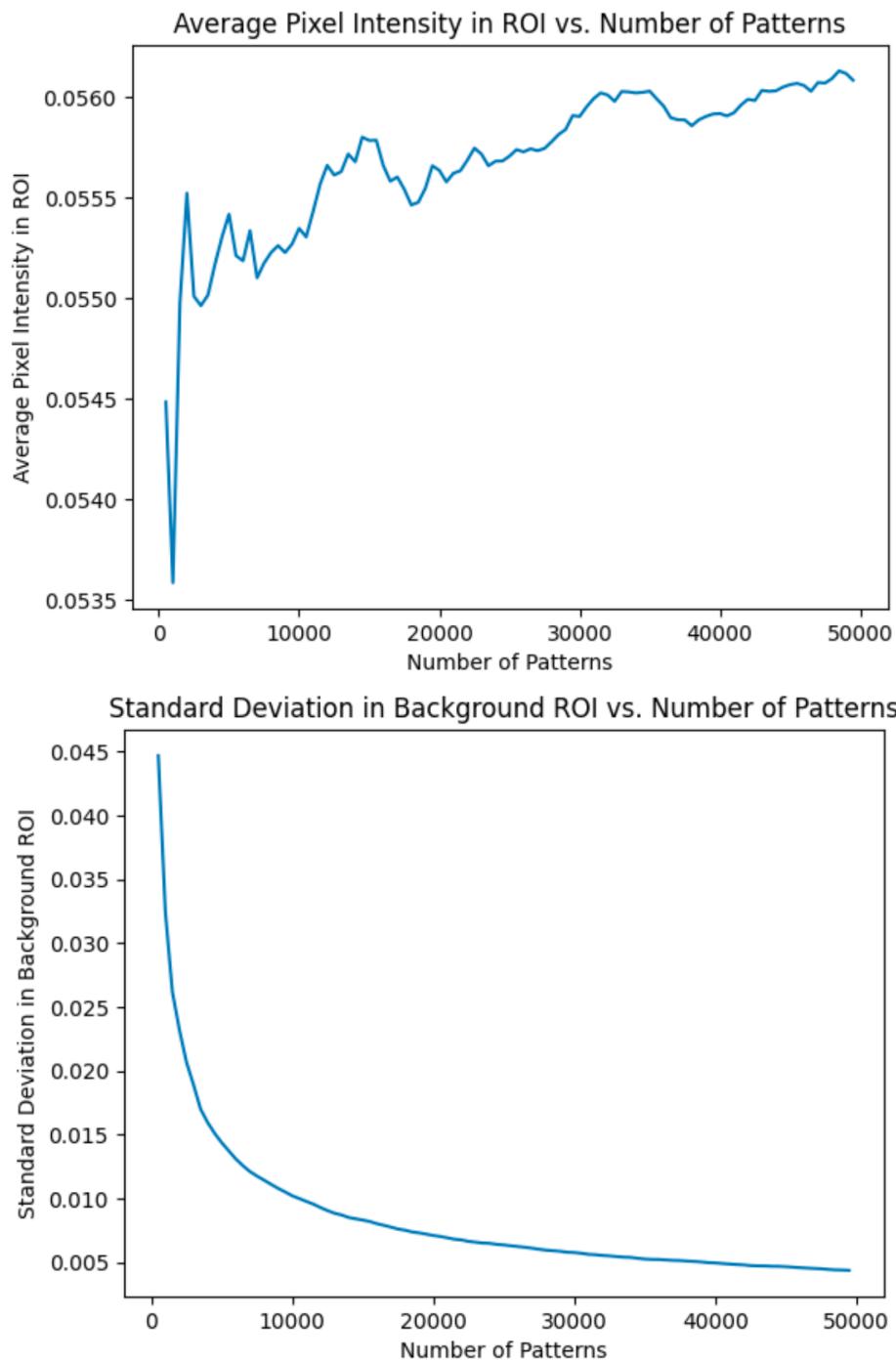
- (a) (7 points) Use the code provided in class (classical_ghost_activity.m) and make the following modifications:

- Instead of using an imaging object from an image file, generate a simple circular mask with the following commands in MATLAB:

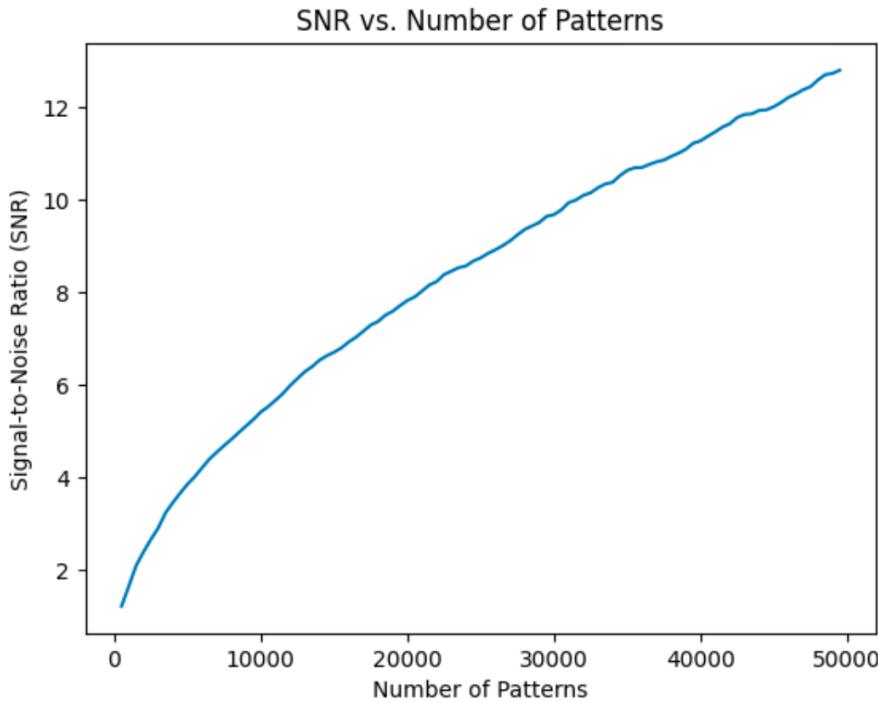
```
% Create a circular aperture as our object
object_img = zeros(width, height);
[cx, cy] = meshgrid(1:width, 1:height);
radius = 10;
mask = (cx - width/2).^2 + (cy - height/2).^2 <= radius.^2;
object_img(mask) = 1;
```

- Define region of interest (ROI) within the circular aperture (signal) and calculate the average pixel intensity within the ROI as a function of the number of patterns used for the image reconstruction. *Plot the average pixel intensity as a function of the number of patterns.*
- Define another region of interest outside of the aperture (background) and calculate the standard deviation of the intensities within the ROI as a function of the number of patterns. *Plot the standard deviation as function of the number of patterns.*
- *Include a representative reconstructed image and indicate where the ROIs are.*





- (b) (3 points) Define the signal to noise (SNR) as the average pixel intensity within the signal ROI divided by the standard deviation in the background ROI. Plot the SNR as a function of the number of patterns. Explain the trend of the data.



SNR should scale with the square root of the number of patterns, which is shown here.

6. (10 points) **ECE 901 extra question: Scaling properties of hydrogenic atoms**

Hydrogenic ions are atoms that have one electron bound to a nucleus with atomic number $Z \geq 1$. Determine the scaling factor of Z in each of the following, relative to the values in the case of the hydrogen atom:

Hint: You should not have to use explicit wave functions for this problem. Please consider the dimensions of the quantities of interest. For example, for length, determine the scaling of the characteristic length a_c relative to Z and the Bohr radius a_0 .

We start by writing the Hamiltonian for hydrogenic atoms: $\hat{H} = -\frac{\hbar^2}{2\mu}\nabla^2 - \frac{Ze^2}{r}$. To write \hat{H} in terms of the characteristic length scale of hydrogenic atoms, we replace r with $\rho = r/Z$, such that $\nabla^2 \rightarrow Z^2 \nabla_\rho^2$. The Hamiltonian then becomes $\hat{H} = -\frac{\hbar^2 Z^2}{2\mu} \nabla_\rho^2 - \frac{Z^2 e^2}{\rho} = Z^2 \left(-\frac{\hbar^2}{2\mu} \nabla_\rho^2 - \frac{e^2}{\rho} \right)$.

The characteristic length scale of hydrogen (a_0) is thus related to the length scale of the hydrogenic atom (a_c) by a factor of Z^{-1} , such that $a_c = \frac{a_0}{Z}$.

- (a) (2 points) Expectation values of r , $1/r$, $1/r^3$, where r is the distance between the electron and the nucleus.

The characteristic length scale is $a_c = \frac{a_0}{Z}$, so $\langle r \rangle \propto Z^{-1}$; $\langle 1/r \rangle \propto Z$; $\langle 1/r^3 \rangle \propto Z^3$

- (b) (2 points) Expectation values of the potential energy (V) and total energy (E).

As shown by the Hamiltonian scaled to ρ , both the potential and kinetic energies are proportional to Z^2 . Therefore, $\langle V \rangle \propto Z^2$; $\langle E \rangle \propto Z^2$

- (c) (1 point) The probability density of finding the electron at the center of the nucleus.

From the normalization condition $\int_0^\infty \int_0^\pi \int_0^{2\pi} |\psi(r, \theta, \phi)|^2 r^2 \sin \theta \, d\phi d\theta dr = 1$, we get the scaling of the integral to (length)³. Therefore, $|\psi(r = 0)|^2 \propto Z^3$.

- (d) (2 points) Internal electric (E_c) and magnetic (B_c) fields determined using electrostatic equations.

The electrostatic electric field is $E_c = \frac{Ze}{a_c^2} \propto Z^3$.

The internal magnetic field at the electron from the relative motion of the nucleus can be obtained via the Biot-Savart law: $B_c = \frac{\mu_0}{4\pi} \oint \frac{d\vec{l} \times \hat{r}}{|r|^2} \propto \frac{Ze v_c 2\pi}{2\pi a_0 a_0} \propto \frac{Z v_c}{a_0^2} \propto Z^4$

- (e) (2 points) Electric dipole moment ($d_c = qr$, where q is the charge difference and r is the displacement between the charges) and magnetic dipole moment (μ_c , determined by the current \times area/ c).

$$d_c = (Z - -1)e \frac{a_c}{2} = \frac{(Z + 1)e a_c}{2} \propto \frac{Z + 1}{Z}$$

$$\mu_c = I \frac{2\pi a_c^2}{c} = \frac{\left(\frac{Z}{2}\right) e v_c}{2\pi a_c} \frac{2\pi a_c^2}{c} = \frac{Ze}{2c} v_c a_c \propto \frac{Z}{2}$$

- (f) (1 point) Calculate the internal magnetic field associated with a tin ion consisting of a single electron, in terms of the magnetic field for a single hydrogen atom.

Tin Sn⁺⁴⁹: $Z = 50$

$v_c = 50v_0$, where $v_0 = \alpha c$

$v_c \approx 50\alpha c = \frac{50}{137} c$ (not relativistic, but approaching relativistic limit)

$B_c \approx 6.25 \times 10^6 B_0$ (ion is highly magnetic)