SI and CGS Unit Systems

⊙ In the SI system (International System of Units), there are seven fundamental (base) units from which all other units are derived:

1. Length → meter (m)
2. Mass → kilogram (kg)
3. Time → seconds (s)
4. Electric current → ampere (A)
5. Temperature → kelvin (K)
6. Amount of substance → mole (mol)
7. Luminous intensity → candela (cd)

All other units (Newton, Joule, Volt, etc.) are derived units, expressed in terms of these seven units.

⊙ In contrast, in the CGS system (centimeter - gram - second) the fundamental units are

1. Length → centimeter (cm)
2. Mass → gram (g)
3. Time → second (s)

However, depending on the field, extended versions of CGS introduce additional base units

4. Electrostatic (esu): introduces the statcoulomb for electric charge
5. Electromagnetic (emu): introduces the abampere for current
6. Gaussian: mixes the two for E&M.

⊙ Since Ampere is treated on the same footing with the mechanical units its units ~~are~~ is independent of the other three. Therefore, one needs to introduce additional dimensional quantities: the vacuum permittivity $\varepsilon_0 = 8.854 \times 10^{-12}\ C^2/N\cdot m^2$ and vacuum permeability
$\mu_0 = 1.257 \times 10^{-6}\ N/A^2$

Coulomb's Law is SI : $$\boxed{F = \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_2}{r^2}}$$

In contrast, in CGS system of units

Coulomb's Law in CGS $$\boxed{F = \frac{q_1 q_2}{r^2}}$$

This relation defines the unit of charge

$$[F] = \frac{[Q]^2}{[L]^2} \qquad \text{Recall: } [F] = \frac{[M][L]}{[T]^2}$$

$$[Q] = \frac{[M]^{1/2}[L]^{3/2}}{[T]}$$

This is a statcoulomb, which amounts to ($1/10c$) of the Coulomb (C) (c is the speed of light $\mu_0 \varepsilon_0 c^2 = 1$).

SI system	CGS system
$\operatorname{div} \vec{D} = \frac{\rho}{\varepsilon_0}$	$\operatorname{div} \vec{D} = 4\pi\rho$
$\operatorname{div} \vec{B} = 0$	$\operatorname{div} \vec{B} = 0$
$\operatorname{curl} \vec{E} = -\frac{\partial \vec{B}}{\partial t}$	$\operatorname{curl} \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$
$\operatorname{curl} \vec{H} = \vec{j} + \frac{\partial \vec{D}}{\partial t}$	$\operatorname{curl} \vec{H} = \frac{4\pi}{c}\vec{j} + \frac{1}{c}\frac{\partial \vec{D}}{\partial t}$

We also need constitutive relations, connecting $\vec{E}$ and $\vec{D}$ to polarization $\vec{P}$, and $\vec{B}$ and $\vec{H}$ to magnetization $\vec{M}$.

$\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$	$\vec{D} = \vec{E} + 4\pi\vec{P}$
$\vec{H} = \frac{1}{\mu_0}\vec{B} - \vec{M}$	$\vec{H} = \vec{B} - 4\pi\vec{M}$

Notice immediately that in CGS D, E, P, H, B, M have the same units. This is quite convenient, as we can compare the fields directly, rather than evaluating the forces they produce

$$\vec{F} = q\vec{E} + \frac{q}{c}[\vec{v} \times \vec{B}]$$

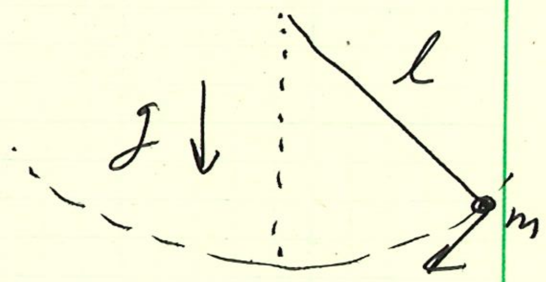
$$[E] = [D] = [P] = [H] = [B] = [M] = \frac{[Q]}{[L]^2} = \frac{[M]^{1/2}}{[T][L]^{1/2}}$$

Applications of Dimensional Analysis

[#1] Mathematical and Physical Pendula

A point mass (m) is connected by a weightless cord of length (l) to a frictionless pivot. Find the period of oscillation

$$t \propto m^{\alpha} g^{\beta} l^{\gamma}$$



$$[T] = [M]^{\alpha} \frac{[L]^{\beta}}{[T]^{2\beta}} [L]^{\gamma}$$

$$\left.\begin{aligned} [T]&: \quad 1 = -2\beta \\ [M]&: \quad \alpha = 0 \\ [L]&: \quad 0 = \beta + \gamma \end{aligned}\right\} \quad \begin{aligned} \alpha &= 0 \\ \beta &= -1/2 \\ \gamma &= 1/2 \end{aligned}$$

$$\boxed{t \propto \sqrt{\frac{l}{g}}}$$

(!) **Note:** mass could not have entered the result because it cancels out from the equation of motion $m\ddot{\vec{r}} = m\vec{g} \Rightarrow \ddot{\vec{r}} = \vec{g}$

Consider now a physical pendulum, i.e. a solid body oscillating about an axis with pivot at a distance l from the center of mass. Now the relevant variables

$$\{l, m, g, I\}$$

The units of moment of inertia $[I]=[M][L^2]$
so we have four variables and
three independent units in a set M, L, T.
Thus according to Π-theorem the
period can be determined only up to
a scaling function of a single dimensionless
variable. One possible chooce is:

$$\boxed{t=\sqrt{\frac{l}{g}}\,F\left(\frac{I}{ml^2}\right)}$$

This is as far as DA alone can take us.
To make further progress, we need
some additional inside of function F.
Let's have a look at the equation of motion

$$\frac{d\vec{L}}{dt}=\vec{T} \qquad \begin{array}{ll}\vec{L} & \text{angular momentum}\\ \vec{T} & \text{torque}\end{array}$$

$$\vec{L}=I\vec{\Omega}$$

$$\frac{dL}{dt}=\frac{d}{dt}(I\Omega)=\frac{d}{dt}\left(I\frac{d\theta}{dt}\right)=I\frac{d^2\theta}{dt^2}$$

For an oscillator $\theta(t)=\theta_0 e^{-i\omega t}$

$$\frac{dL}{dt}=-I\omega^2\theta_0 e^{-i\omega t}$$

The key point to observe is that the

right-hand-side (torque) is independent of the moment of inertia: it's just force times distance. Therefore we see that the frequency of oscillation and moment of inertia enter only in a particular combination $I\omega^2$! Since $\omega \sim 1/t \Rightarrow t \sim 1/\omega \sim \sqrt{I}$ as $\omega \sim 1/\sqrt{I}$. This is only possible if $F(x) = C\sqrt{x}$, so finally:

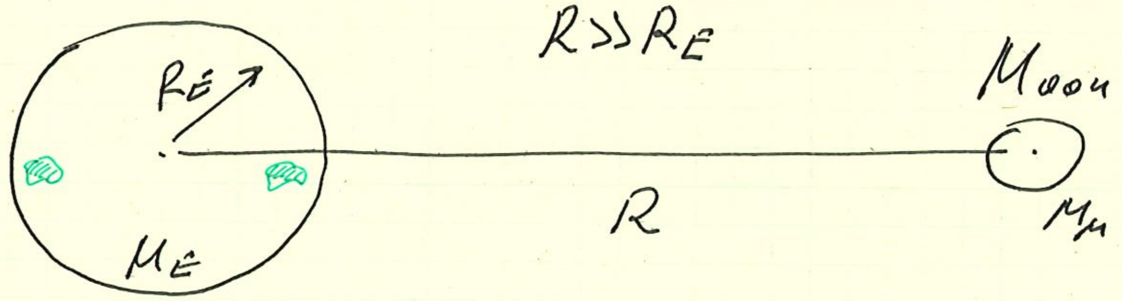
$$\boxed{t \sim \sqrt{\frac{l}{g}}\sqrt{\frac{I}{ml^2}} \sim \sqrt{\frac{I}{mgl}}}$$

(!) Note on "Fusion of Variables"

How come we got so lucky that the scaling function was determined just up to a pure number? In fact, we used two arguments: (i) the frequency of oscillation and the moment of inertia form a particular combination $I\omega^2$; (ii) the mass of the pendulum and free-fall acceleration can only enter in a product mg

[#2] Tidal Effect

Can we estimate the magnitude of ocean tides?



Key insight: the tidal effect arises from the differential gravitational pull generated by Moon across the Earth i.e. from the Moon's potential (it is therefore a dipolar type interaction

$$F_{Tide} \simeq G\frac{M_M \delta M}{R^2}\frac{R_E}{R} \Rightarrow U_{Tide}(R_E) \simeq G\frac{M_M \delta M R_E^2}{R^3}$$

Now if surface is lifted by a height h a fluid parcel at the top of the bulge is sitting higher in Earth's gravity field. The potential energy change $U \sim \delta M g h$

$$\delta M g h \simeq G\frac{M_M \delta M R_E^2}{R^3} \Rightarrow \frac{GM_E}{R_E^2}\cdot h \simeq G\frac{M_M R_E^2}{R^3}$$

$$\boxed{h \simeq \frac{M_M}{M_E}\cdot\frac{R_E^4}{R^3}}$$

$R_E \simeq 6.4\times10^6\,m$ $R \simeq 3.8\times10^8\,m$

$M_M \simeq 7.34\times10^{22}\,kg$ $M_E \simeq 5.9\times10^{24}\,kg$

$$h \simeq \left(\frac{7.34\cdot10^{22}}{5.9\cdot10^{24}}\right)\left(\frac{6.4\cdot10^6}{3.8\cdot10^8}\right)^3 6.4\cdot10^6 \simeq 10^{-2}(8\cdot10^{-6})(6.4\times10^6)$$

$$h \simeq (1/2)\,m$$

[#3] Nuclear bomb explosion

The explosion "mushroom", which contains hot gas, is separated from cold air outside by a well-defined surface front. Find dependence of the radius $R(t)$ of the front on time.

Key insights: assume that the relevant quantity is the energy released in the explosion (E). One also must realize that gases must do work on expanding into the cold atmosphere outside. Therefore the ambient pressure and temperature outside the front are also relevant variables; however, one can "fuse" these parameters into the mass density of air (ρ), which by the Boyle-Mariotte law, depends on both the pressure (P) and temperature (T).

$$R(t) \propto E^{\alpha} \rho^{\beta} t^{\gamma}$$

$$[L] = \left[\frac{M L^2}{T^2}\right]^{\alpha} \left[\frac{M}{L^3}\right]^{\beta} [T]^{\gamma}$$

$$\boxed{R \propto t^{2/5}}$$

$$[L]: \quad 1 = 2\alpha - 3\beta \qquad \alpha = 1/5$$

$$[M]: \quad 0 = \alpha + \beta \qquad \beta = -1/5$$

$$[T]: \quad 0 = -2\alpha + \gamma \qquad \gamma = 2/5$$

$$E \propto \frac{\rho R^5}{t^2}$$

[#4] Schwinger's pair production

In quantum electrodynamics the vacuum is unstable towards production of electron-positron pairs in the strong enough electric field. Estimate that field strength E_c?

We try (DA) is (SI)-system

$$E_c \propto m_e^{\alpha} c^{\beta} \hbar^{\gamma} e^{\delta}$$

$[E] = [MLT^{-3}I^{-1}]$ from (Force / charge)

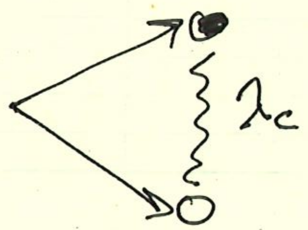
$[m_e] = [M]$ $\quad$ $[c] = [LT^{-1}]$

$\hbar = [ML^2T^{-1}]$ from (Action) $\quad$ $[e] = [I \cdot T]$

$[M]$:	$1 = \alpha + \gamma$	$\alpha = 2$
$[L]$:	$1 = \beta + 2\gamma$	$\beta = 3$
$[T]$:	$-3 = -\beta - \gamma + \delta$	$\gamma = -1$
$[I]$	$-1 = \delta$	$\delta = -1$

$$\boxed{E_c \simeq \frac{m^2 c^3}{e\hbar}}$$

Physical interpretation



$e E_c l \sim m_e c^2$

$l \sim \lambda_c \sim \hbar / m_e c$

(!) Note: what if we decide to use CGS? Then we will run into a trouble as we have four variables $m, e, c, \hbar$ but only three fundamental units. The way out is through Π-theorem ~~thus~~ since this problem has a natural dimensionless parameter

$$\alpha = \frac{e^2}{\hbar c} \Rightarrow E_c = f(\alpha)\, m_e^a c^b \hbar^c$$

(!) Note: The full formula for the rate of pair production (Schwinger 1951) gives the number of e-h pairs created per unit volume and per unit time

$$\boxed{W = \frac{(eE)^2}{4\pi^3 \hbar^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \exp\left(-\frac{\pi n m^2 c^3}{eE\hbar}\right)}$$

$e^{-\pi n E_c / E}$

⊙ The pre-exponential factor $\propto E^2$ represents the density of states in the phase space for the produced particles. This is a familiar result from classical E&M $\Rightarrow$ sum over n accounts for the fact that particles can be created in different quantum states.

- The exponential factor arises from quantum tunneling

$$\exp(iS/\hbar) \longrightarrow \exp(-|S|/\hbar)$$

tunneling under the barrier

$W = eEL$ ← tunneling distance

$W \sim m_e c^2$ for produced pair

Therefore $L \sim m_e c^2 / eE$

From action $S = \oint |p|\,dr \sim (m_e c) \cdot L$

$$S \simeq (m_e c)\left(\frac{m_e c^2}{eE}\right) \simeq \frac{m^2 c^3}{eE}$$

$$\exp(-|S|/\hbar) \simeq \exp\left(-\frac{m^2 c^3}{\hbar e E}\right)$$

Finally:

$$\boxed{W(E) \propto \underbrace{E^2}_{\text{"classical" phase space}} \underbrace{\exp\left(-\#\, E_c/E\right)}_{\text{quantum} \Rightarrow \text{ instantons}}}$$

[#5] Red shift

Suppose that mathematical pendulum is taken to the top of a high-rise at height h above the surface of Earth. How would its period change because the red shift?

By definition, the relative change in the period $\Delta t/t$ is a dimensionless quantity, hence we need to construct a dimensionless combination of the three relevant parameters g, c, m, h

$$\frac{\Delta t}{t} = F\left(\frac{g}{m^{\alpha} h^{\beta} c^{\gamma}}\right)$$

The argument must be dimensionless

$$[g] = \frac{[L]}{[T]^2} = [M]^{\alpha}[L]^{\beta}\frac{[L]^{\gamma}}{[T]^{\gamma}}$$

$$\alpha = 0 \quad \beta = -1 \quad \gamma = -2$$

$$\boxed{\frac{\Delta t}{t} = F\left(\frac{gh}{c^2}\right) \simeq \frac{gh}{c^2}}$$

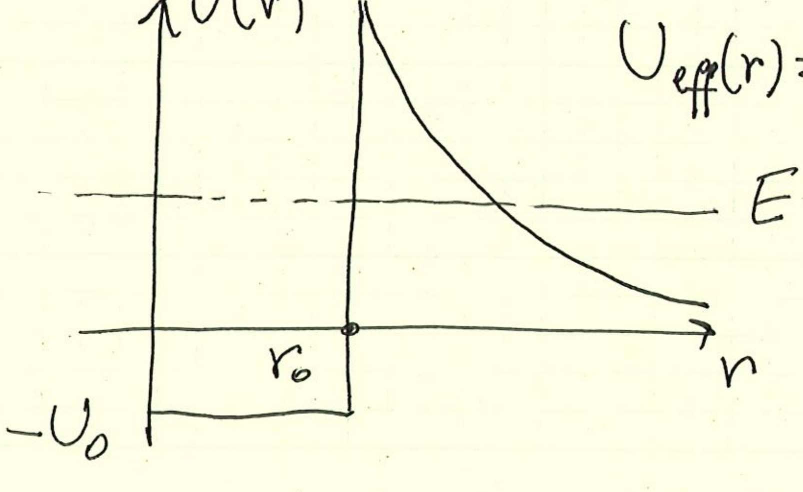
Note that $gh/c^2 \ll 1$. What can we say about $F(x)$ in the limit $x \ll 1$. If this is a regular function $F(x) \simeq a \cdot x$ (a could be zero for symmetry reasons)

[#6] Thermonuclear fusion

TF in stars is a process where light nuclei fuse to create heavier ones releasing extra energy. This process is governed by two competing factors: (i) the need for high-energy particles to overcome Coulomb repulsion, and (ii) the fact most particles in a star's core have relatively low energies.

Landau-Gamow [1933]: estimated the rate of fusion reaction.

Key insights: two nuclei of charges $(Z_1 e)$ and $(Z_2 e)$ and reduced mass μ must overcome Coulomb repulsion. Quantum tunneling lets them fuse even when their kinetic energy E is smaller than the barrier.



$$U_{eff}(r) = U(r) + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}$$

$$U(r) = \frac{\alpha}{r}$$

$$\alpha = Z_1 Z_2 e^2$$

In this estimate we will consider scattering in the s-wave channel $(l=0)$. The probability for under-the-barrier tunneling can be estimated from WKB

$$W(E) \propto \exp\left[-\frac{1}{\hbar}\oint |p_r|\,dr\right]$$

$$\frac{p_r^2}{2\mu} + U(r) = E \Rightarrow |p_r| = \sqrt{2\mu\left(\frac{\alpha}{r} - E\right)}$$

$$W(E) \propto \exp\left[-\frac{2}{\hbar}\int_{r_0\to 0}^{\alpha/E} \sqrt{2\mu}\sqrt{\frac{\alpha}{r} - E}\,dr\right]$$

$$\left\{r = \frac{\alpha}{E}\cdot x\right\} \Rightarrow \exp\left[-\frac{2}{\hbar}\int_0^1 \sqrt{2\mu E}\sqrt{\frac{1}{x} - 1}\left(\frac{\alpha}{E}\right)dx\right]$$

$$W(E) \propto \exp\left[-\frac{\pi\alpha}{\hbar}\sqrt{\frac{2\mu}{E}}\right]$$

WKB does not fix pre-exponential factor. For convenience of further analysis we assume that factor is a slowly varying function with energy

$$W(E) = \frac{S(E)}{E}\, e^{-\frac{\pi\alpha}{\hbar}\sqrt{\frac{2\mu}{E}}}$$

$S(E)$ is called astrophysical factor;
$1/E$ is pulled out for convenience;

the exponential factor suggests a natural energy scale in the problem, the so-called Gamow energy

$$E_G = \left(\frac{\pi Z_1 Z_2 e^2}{\hbar}\sqrt{2\mu}\right)^2$$

In the context of atomic physics this is equivalent to a Rydberg.

The flux of particles is proportional to their velocities v, therefore the rate of fusion is governed by a thermal average of a product:

$$\langle W v\rangle_{th} = \int_0^\infty W(v)\, v\, f(v)\, dv$$

where $f(v)$ is the relative-speed Maxwell distribution function

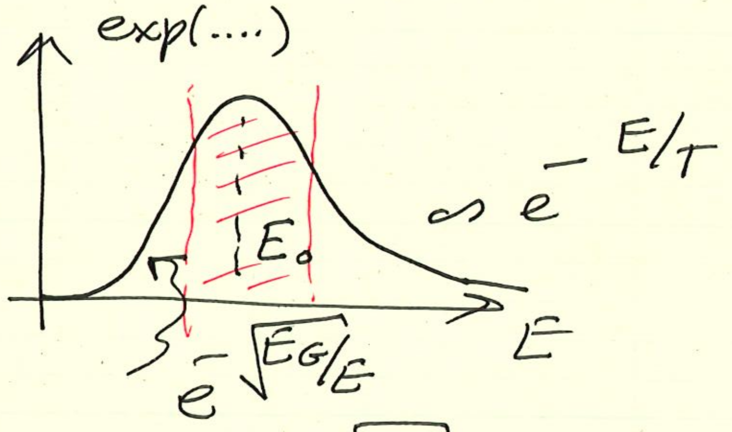
$$f(v) = (4\pi v^2)\left(\frac{\mu}{2\pi k_B T}\right)^{3/2} e^{-\frac{\mu v^2}{2k_B T}} dv$$

(in the following $k_B \equiv 1$) $\quad E = \frac{\mu v^2}{2}$

$$v = \sqrt{\frac{2E}{\mu}} \Rightarrow dv = \sqrt{\frac{1}{2\mu E}}\, dE$$

$$\langle W v\rangle = (4\pi)\left(\frac{\mu}{2\pi T}\right)^{3/2}\int_0^\infty \frac{2E}{\mu}\sqrt{\frac{2E}{\mu}}\, e^{-\frac{E}{T}}\, \frac{S(E)}{E}\, e^{-\sqrt{\frac{E_G}{E}}}\sqrt{\frac{1}{2\mu E}}\, dE$$

$$\langle W v \rangle = \frac{8\pi}{\mu^2}\left(\frac{\mu}{2\pi T}\right)^{3/2} \int_0^\infty S(E) \exp\left(-\frac{E}{T} - \sqrt{\frac{E_G}{E}}\right) dE$$



$$\frac{\partial}{\partial E}\left(\frac{E}{T} + \sqrt{\frac{E_G}{E}}\right) = \frac{1}{T} - \frac{1}{2}\frac{E_G^{1/2}}{E_0^{3/2}} = 0$$

Optimal energy: $E_0 = \left(T^2 E_G/4\right)^{1/3}$

At the optimal energy $\exp(\cdots) = \exp\left(-\frac{3E_0}{T}\right)$

Putting everything together one finds

$$\boxed{\langle W v \rangle = \sqrt{\frac{8}{\pi\mu}} \frac{S(E_0)}{T^{3/2}} e^{-3E_0/T} \underbrace{\sqrt{\frac{4E_0 T}{3}}}}$$

Additional factor coming from Gaussian integral near Gamov's peak

- $e^{-\frac{3E_0}{T}} = \exp\left[-\#\left(\frac{E_G}{T}\right)^{1/3}\right]$ $\quad E_G \propto Z^4$
- $S(E)$ encodes nuclear physics
- Star's core must reach $T \sim 10^6 K$ (for Sun) to ignite and sustain thermonuclear synthesis.

Consider proton-proton fusion as example
For $T \simeq 10^7 K \rightarrow 1.3\,keV$

$p+p \rightarrow (Z_1 = Z_2 = 1)$ $\mu = m_p/2$ the Gamow scal

$E_G \approx 493\,keV$ then $E_0 \simeq 5.9\,keV$

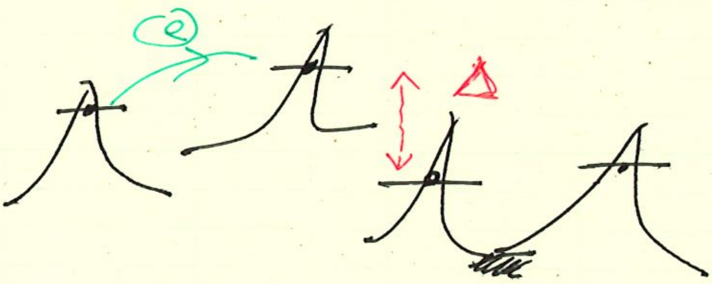
The exponential suppression factor from the presented analysis is

$$\exp(-3E_0/T) \approx \exp(-14) \sim 10^{-6}$$

So even before including the smallness of weak-interaction $(p+p \rightarrow d + e^+ + \nu_e)$ S-factor, the Gamow tunneling + Maxwell tail gives about a million-fold suppression. The extra small S-factor (and weak interaction matrix element) makes the real $p+p$ rate in the Sun very small, this explains why stars burn slowly on gigayear timescales.

[#7] Hopping conductivity

In disordered systems electronic states are localized. Electron transport in this regime is dominated by a mechanism called <u>variable</u>-<u>range</u>-<u>hopping</u>



$$|\psi|^2 \propto \exp(-2r/\xi)$$

Key insights:

(i) localized states with length ξ

(ii) tunneling between sites at distance r produces a factor $e^{-2r/\xi}$ for the hopping matrix element squared

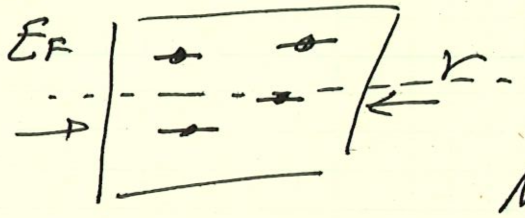
(iii) hoping betwee cites with energy difference Δ carries Boltzmann suppression factor $\exp(-\Delta/T)$

(iv) the total hopping conductivity consists of two factors

$$\exp[-S(r,\Delta)] = \exp\left[-\left(\frac{2r}{\xi} + \frac{\Delta}{T}\right)\right]$$

(v) hopping conductivity is dominated by hops that minimize S-function.

Case-I: Mott limit of VRH



Volume $\sim r^d$

Density of States $g(E_F)$

Number of levels $\propto r^d g(E_F)$

Level spacing $\propto \dfrac{1}{g r^d}$

$$S(r) = \frac{2r}{\xi} + \frac{1}{g r^d T}$$

Minimize $S \Rightarrow \dfrac{\partial S}{\partial r} = 0 \Rightarrow r_{opt}^{d+1} \simeq \dfrac{d\xi}{2gT}$

Evaluate $S(r_{opt}) \Rightarrow S_{min} \propto r_{opt}/\xi \sim (T_M/T)^{\frac{1}{d+1}}$

where $T_M \sim \dfrac{1}{g\xi^d}$

$$\boxed{\sigma_{Mott} = \sigma_0 \exp\left[-\left(\frac{T_M}{T}\right)^{\frac{1}{d+1}}\right]}$$

Case-II: Efros-Shklovskii VRH

Key insight: long-range Coulomb interaction between localized electrons is strong enough to produce Coulomb gap. Therefore the relevant energy cost is $\Delta(r) = \dfrac{e^2}{\varkappa r}$

$$S(r) = \frac{2r}{\xi} + \frac{e^2}{\varkappa r}$$

$$r_{opt} \simeq \sqrt{\frac{e^2\xi}{2\varkappa T}} \qquad S_{min}(r_{opt}) \simeq \sqrt{\frac{e^2}{2\varkappa\xi T}}$$

$$\boxed{\sigma_{ES}(T) = \sigma_0 \exp\left[-\sqrt{\frac{T_{ES}}{T}}\right]}$$

[#8] Quantum of Resistance

This particular application of DA played an important role in several discoveries: Anderson localization, integer and fractional Hall effects, conductance quantization in ballistic point contacts.

⊙ Let's start from the Ohm's law:

$$R = \frac{V}{I}$$

Regardless of the system of units we use $[V] = [\mathcal{E}] \cdot [L]$ and $[I] = [Q]/[T]$

$$\boxed{[R] = \frac{[\mathcal{E}][L][T]}{[Q]}}$$

We recall now from prior discussion that $[\mathcal{E}] = [Q]/[L]^2$ in the CGS so that

the units of resistance is inverse speed

$$[R] = \frac{[T]}{[L]} = \underset{\sim}{G}^{-1} \text{ conductance}$$

It is quite curious that an electrical quantity has units that allow for a mechanical interpretation. It begs a question: <u>the speed of what?</u>

To this end, notice above formula can be represented as follows:

$$[R] = \frac{[Q][E][L][T]}{[Q]^2} = \frac{[\text{Action}]}{[Q]^2}$$

The unit of action in quantum theory is h
Therefor, using the fundamental charge
we can define a fundamental unit
of resistance → quantum resistance R_Q →
also called von Klitzing constant

$$\boxed{R_Q = \frac{h}{e^2} = 25.810\ k\Omega} \quad (h = 2\pi\hbar)$$

- The same R_Q is also featured prominently in the Landauer formula, which relates conductance of a quantum conductor to its (dimensionless) transmission coefficient

$$\boxed{G = \frac{2e^2}{h} \sum_n T_n}$$

- Resistivity (or its inverse conductivity) is an intensive quantity whose units depend on the sample's dimensionality.

$\rho = R \frac{A}{L}$ L - sample length in the current direction

A - cross-section area $\perp$ to current

(3d) $[\rho] = [R][L] = [T]$ in CGS. This time has a well-defined physical meaning: an excess charge in a conductor relaxes with time exponentially with a characteristic Maxwell time $\tau = \rho$!

$$\partial_t (en) + \operatorname{div}\vec{j} = 0 \qquad \vec{j} = \sigma\vec{E} = \rho^{-1}\vec{E}$$

charge density

$$\operatorname{div}\vec{E} = 4\pi e n$$

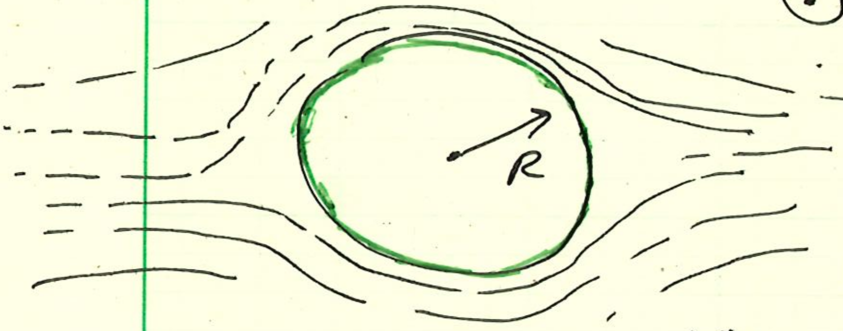
$$\partial_t (en) + \rho^{-1}\operatorname{div}\vec{E} = \partial_t (en) + 4\pi e \rho^{-1} n = 0$$

$$\partial_t n + 4\pi\rho^{-1} n = 0 \Rightarrow \boxed{n(t) = n_0 \exp\left(-{}^{4\pi t}/_{\rho}\right)}$$

⊙ Now we bring a speed of light into the game. In CGS conductance has units of speed, so what happens of the conductance becomes comparable to the speed of light? The answer to this question has to do with electromagnetic excitations in a thin metallic film that behave differently if its conductivity (σ_{2D}) is smaller or larger than c. The absorption coefficient of light polarized in the plane of incidence goes through a maximum at a certain incident angle if $\sigma_{2D} > c$. The latter condition translates into $\rho_{2D} < 188\,\Omega$.

[#9] Resistive forces in fluids

Consider a sphere of Radius R moving with velocity u through a fluid. Estimate the resistive force on a sphere.



⊙ What are the relevant parameters? It is obviously R, u, one should think of fluid mass density ρ, its viscosity η. In addition, it should matter whether the motion is subsonic or supersonic $(u>s)$ or $(u<s)$ where s is the speed of sound.

⊙ Recall that viscosity is defined as the proportionality coefficient between the velocity gradient and the ensuring stress (the latter has units of pressure = force per unit area:

$$\frac{[F]}{[L]^2} = [\eta]\cdot[\nabla u] = [\eta]\,\frac{1}{[L]}\,\frac{[L]}{[T]} = \frac{[\eta]}{[T]}$$

From here one deduces $[\eta] = \dfrac{[M]}{[L][T]}$

$$\underbrace{\{R, u, \rho, \eta, s\}}_{\text{variables}} \longleftrightarrow \underbrace{[M], [L], [T]}_{\text{units}}$$

For 5-variables and 3-units DA

can determine force up to a function of <u>two</u> dimensionless variables. Out of all the variables we have two options of constructing a quantity with unit of a force: (i) $\eta R U$ (Stoke's force) and (ii) $\rho U^2 R^2$ (drag force). In addition we can form <u>two</u> dimensionless quantities: (i) $Re = UR\rho/\eta$ (Reynolds number) and (ii) $M = U/s$ (Mach number).

$$\boxed{F = \rho U^2 R^2 \; G(Re, M)}$$

In liquids ($s \sim 10^3$ m/s) so typically $M \ll 1$ $G(Re, M \to 0) \to \tilde{G}(Re)$. If $Re \ll 1$ (viscosity dominated regime) one must have $\tilde{G}(x) \sim 1/x$ at $x \ll 1$. In contrast if $Re \gg 1$ viscosity is irrelevant and must drop out, which means that $\tilde{G}(x) \to$ const at $x \gg$

⊙ <u>Note on Stokes paradox</u>

(3d) $F = \eta U R \, G^{sphere}(Re)$ (2d) $f = \eta U \, G^{cylinder}(Re)$

$f = F/L$ force per unit of length

$Re \to 0$ means that $G^{sphere}(Re \to 0) \to$ Const

$Re \to 0$ means that if $G^{cylinder} \to$ const

then force is independent of R?!

Actually: $G^{cylinder}(x) = 4\pi \cdot \left(\frac{1}{2} - C + \ln \frac{4}{x}\right)^{-1}$

[#10] Hawking radiation

For a Black hole of mass M estimate its (evaporative) life-time via Hawking mechanism.

Hawking mechanism describes decay of a Black hole via radiation from its event horizon if it is modeled as a Black body (radiating photons, gravitons, ...). Therefore, we need first to identify its temperature (T_H) so-called Hawking temperature of a Black hole.

$$\text{From } \{\hbar, c, G, k_B\} \Rightarrow T_H \sim \frac{\hbar c^3}{G M k_B}$$

For example, for a solar-mass Black hole with $M \simeq 2\times10^{30}\,kg \Rightarrow T_H \simeq 6\times10^{-8}\,K$

The radiative power

$$P = \sum_i \int_0^\infty \frac{T_i(E)\, E\, dE}{(2\pi\hbar)} \frac{1}{e^{E/k_B T_H} - (-1)^{2S_i}} \sim \sigma_{eff} T_H^4$$

$T_i(E)$ "greybody factors" which accounts for curvature/scattering of modes at horizon (S_i) is for spin of emitted particles

$$\frac{d(Mc^2)}{dt} \sim -(\sigma_{eff} T_H^4)\cdot(4\pi R_S^2) \qquad R_S = 2GM/c^2$$

Schwarzschild radius

$$\frac{dM}{dt} \sim -\frac{\hbar c^4}{G^2 M^2} \Rightarrow \boxed{t_{evap} \sim G^2 M^3/\hbar c^4}$$