1

[D&F 4.2.2] List the elements of S_3 as 1, $(1\,2)$, $(2\,3)$, $(1\,3)$, $(1\,2\,3)$, $(1\,3\,2)$ and label these with the integers 1, 2, 3, 4, 5, 6 respectively. Exhibit the image of each element of S_3 under the left regular representation of S_3 into S_6.

From HW 7 we know that S_n $(n \geq 2)$ is generated by the transposition (12) and the n-cycle $(12...n)$. Further, noticing $\sigma : S_3 \to S_6$ is a homomorphism, we can find the image of these two generators to find the image of the whole group.

First, find the image $\sigma(12)$ by considering its action on the set $\{1,2,3,4,5,6\}$:

$$\begin{aligned}
(12)(1) &= (12) \quad 1 \to 2 \\
(12)(12) &= 1 \quad 2 \to 1 \\
(12)(23) &= (231) \quad 3 \to 5 \\
(12)(13) &= (132) \quad 4 \to 6 \\
(12)(21) &= (213) \quad 5 \to 4 \\
(12)(3) &= 3 \quad 6 \to 3
\end{aligned} \tag{1}$$

Thus, we have $\sigma(12) = (12)(35)(46)$.

Similarly, find the image $\sigma(123)$ by considering its action on the set $\{1,2,3,4,5,6\}$:

$$\begin{aligned}
(123)(1) &= (123) \quad 1 \to 5 \\
(123)(12) &= (13) \quad 2 \to 4 \\
(123)(23) &= (12) \quad 3 \to 2 \\
(123)(13) &= (23) \quad 4 \to 3 \\
(123)(123) &= (132) \quad 5 \to 6 \\
(123)(132) &= 1 \quad 6 \to 1
\end{aligned} \tag{2}$$

Thus we have $\sigma(123) = (156)(243)$.

Then by homomorphism,

$$\begin{gathered}
\sigma(23) = \sigma((12)(123)) = \sigma(12) \circ \sigma(13) = (12)(35)(46)(156)(243) = (13)(26)(45) \\
\sigma(13) = \sigma((12)(123)^2) = \sigma(12) \circ \sigma(123)^2 = (12)(35)(46)(165)(234) = (14)(25)(36) \\
\sigma(132) = \sigma((123)^2) = (165)(234)
\end{gathered} \tag{3}$$

2

[D&F 4.2.11] Let G be a finite group and let $\pi : G \to S_G$ be the left regular representation.

a. Prove that if x is an element of G of order n and $|G| = mn$, then $\pi(x)$ is a product of m n-cycles.

b. Deduce that $\pi(x)$ is an odd permutation if and only if $|x|$ is even and $|G|/|x|$ is odd.

a.

For each $g \in G$, consider its orbit under the action $\pi(x)$:

$$\mathcal{O}_g = \{g, xg, x^2 g, ..., x^{n-1} g\}. \tag{4}$$

It's clear that $|\mathcal{O}| = n$, since $x^n g = eg = g$. We see that the action $\pi(x)$ partitions G into disjoint orbits of size n. Thus, $|G|$ is a multiple of n. Particularly, since $|G| = mn$, there are m such orbits.

The permutation $\pi(x)$ on each orbit acts as a cyclic shift. This means that the restriction of $\pi(x)$ on $\mathcal{O}_g$ is an n-cycle. Since there are m disjoint orbits, the permutation representation of $\pi(x)$ is a product of m disjoint n-cycles.

b

Recall from HW7 that an n-cycle can be written as n-1 transpositions. $\pi(x)$ is therefore written as $m \cdot (n-1)$ transpositions. Thus, in order for $\pi(x)$ to be an odd permutation, m and $n-1$ must be odd. This means that n is even and m is odd. In other words,

$$n = |x| \text{ is odd}, \quad m = |G|/\ |x| \text{ is even} \ . \tag{5}$$

3

> [D&G 4.3.2-3] Find all conjugacy classes and their sizes in the following groups:
> a. D_8,
> b. $Z_2 \times S_3$.

a

Let

$$D_8 = \langle r, s \mid r^4 = s^2 = 1, srs = r^{-1} \rangle, \quad D_8 = \{1, r, r^2, r^3, s, sr, sr^2, sr^3\}. \tag{6}$$

We find the conjugacy classes by computing the conjugates of each element:

- $\mathcal{O}_1 = \{1\}$.
- $\mathcal{O}_{r^2} = \{r^2\}$ since r^2 commutes with r and s (hence central).
- $\mathcal{O}_r = \{r, r^3\}$ since $srs^{-1} = r^{-1} = r^3$ and $r^k r r^{-k} = r$.
- $\mathcal{O}_s = \{s, sr^2\}$ since $rsr^{-1} = sr^2$ and r^2 centralizes s.
- $\mathcal{O}_{sr} = \{sr, sr^3\}$ (e.g., $s(sr)s^{-1} = sr^{-1} = sr^3$).

So collectively, the conjugacy classes are:

$$\begin{aligned} &\mathcal{O}_1 = \{1\}, \qquad \mathcal{O}_{r^2} = \{r^2\}, \qquad \mathcal{O}_r = \{r, r^3\}, \\ &\mathcal{O}_s = \{s, sr^2\}, \mathcal{O}_{sr} = \{sr, sr^3\} \end{aligned} \tag{7}$$

Size : $1 + 1 + 2 + 2 + 2 = 8 = |D_8|$.

b

This problem is made much simpler by a general property. Let (a, σ) and (b, τ) be two elements in $\mathbb{Z}_2 \times S_3$. Their conjugate is given by:

$$(b, \tau)(a, \sigma)(b, \tau)^{-1} = (b, \tau)(a, \sigma)(b^{-1}, \tau^{-1}) = (b + a - b, \tau\sigma\tau^{-1}) = (a, \tau\sigma\tau^{-1}), \tag{8}$$

where we used additive notation for $\mathbb{Z}_2$ and the fact that it's abelian; and multiplicative notation for S_3. Therefore the conjugates of an element (a, σ) is $\{a\} \times C_{S_3}(\sigma)$.

This implies that every elemnet in $\mathbb{Z}_2$ is only conjugate to itself. Indeed, $C_{\mathbb{Z}_2}(0) = \{0\}, C_{\mathbb{Z}_2}(1) = \{1\}$

We now need to find conjugacy classes in S_3. Start by listing all elements in S_3 :

$$S_3 = \{e, (12), (13), (23), (123), (132)\}. \tag{9}$$

Recall from lecture that conjugacy classes in S_3 is grouped by cycle type. Thus we have:

- Elements of S_3: $e, (12), (13), (23), (123), (132)$.
- Class of identity (type 1^3): $\mathcal{O}_e = \{e\}$.
- Class of transpositions (type $2, 1$): $\mathcal{O}_{(12)} = \{(12), (13), (23)\}$.
- Class of 3-cycles (type 3): $\mathcal{O}_{(123)} = \{(123), (132)\}$.

Collecting these results,

$$\begin{aligned} \mathcal{O}_{(0,e)} &= \{(0, e)\}, \text{size} = 1 \\ \mathcal{O}_{(0,(12))} &= \{(0, (12)), (0, (13)), (0, (23))\}, \text{size} = 3 \\ \mathcal{O}_{(0,(123))} &= \{(0, (123)), (0, (132))\}, \text{size} = 2 \\ \mathcal{O}_{(1,e)} &= \{(1, e)\}, \text{size} = 1 \\ \mathcal{O}_{(1,(12))} &= \{(1, (12)), (1, (13)), (1, (23))\}, \text{size} = 3 \\ \mathcal{O}_{(1,(123))} &= \{(1, (123)), (1, (132))\} \text{ size} = 2 \end{aligned} \tag{10}$$

4

[D&F 4.3.13, extended]

a. Prove that if a finite group G has at most 3 conjugacy classes then the conditions that $|G|$ be a multiple of $|Z(G)|$ and of each other term $|G : C_G(g_i)|$ of the class equation of G imply that this class equation is one of

$$|G| = 1, \quad |G| = 2, \quad |G| = 3, \quad |G| = 2 + 2, \quad |G| = 1 + 2 + 3.$$

b. Show that the fourth of these is impossible, while each of the first three occurs for a unique group G.

c. In the last case, suppose orbits are e, $\{g, g'\}$, $\{h, h', h''\}$. Show that $C_G(g) = \{1, g, g^2\}$, so $|g| = 3$, and similarly $|h|$=2. Now consider hgh^{-1} and show that $G \cong S_3 \cong D_6$.

[In fact for each k there are only finitely many possible class equations for a finite group G with k conjugacy classes, and thus only finitely many possible G, though the list of all such G quickly becomes unmanageably huge.]

a.

Since G has at most 3 conjugacy classes, we write its class equation as

$$|G| = n_1 + ... + n_r, \quad r \leq 3, \tag{11}$$

where n_i is the size of the i-th class. We analyze by cases on r:

- $r = 1$. Since there's only one conjugacy class, it must be the trivial class containing only the identity element. Thus $|G| = 1$.
- $r = 2$. Write $|G| = 1 + n_2 \Rightarrow n_2 = |G| - 1$. Since n_2 divides $|G|$, we also have $|G| - 1 \mid |G| \Rightarrow |G| - 1 \mid 1 \Rightarrow |G| = 2$.
- $r = 3$. Write $|G| = 1 + n_2 + n_3$. Assume $1 \leq n_2 \leq n_3$. We know that n_2 divides $|G|$, and n_3 divides $|G|$ as well. Since $n_3 < |G|$, we have $|G| \geq 2n_3$. Thus

$$|G| = 1 + n_2 + n_3 \geq 2n_3 \quad \Rightarrow n_2 \leq n_3 \leq n_2 + 1. \tag{12}$$

  This implies that either $n_3 = n_2$ or $n_3 = n_2 + 1$.

  - If $n_3 = n_2$, then $|G| = 1 + 2n_2$. Since $n_2 \mid |G|$, we have $2n_2 \mid 1 + 2n_2 \Rightarrow 2n_2 \mid 1 \Rightarrow n_2 = 1$. Thus $|G| = 3$.
  - If $n_3 = n_2 + 1$, then $|G| = 1 + n_2 + (n_2 + 1) = 2n_2 + 2$. Since $n_2 \mid |G|$, we have $n_2 \mid 2(n_2 + 1) \Rightarrow n = 1$ or 2.

    If $n_2 = 1$, then $|G| = 4$. However this implies that G is abelian and thus has 4 conjugacy classes, contradicting our assumption. So $n_2 = 2$. Thus $|G| = 1 + 2 + 3 = 6$.

Collecively, if G has at most 3 conjugacy classes, then $|G| = 1, 2, 3,$ or $1 + 2 + 3$,

b.

We have shown in part a that, the forth option $|G| = 2 + 2 = 4$, is impossible.

For the first three options:

- $|G| = 1$, the trivial group, is unique.
- $|G| = 2$, whose only possibility is the cyclic group $\mathbb{Z}_2$, is unique.
- $|G| = 3$, with the only possiblity being the cyclic group $\mathbb{Z}_3$, is unique.

c.

Let the classes be $\mathcal{O}_e = \{e\}$, $\mathcal{O}_g = \{g, g'\}$, and $\mathcal{O}_h = \{h, h', h''\}$.

Analyze the element g:

The size of its class is $|\mathcal{O}_g| = 2$. The size of its centralizer is $|C_G(g)| = |G|\mathcal{O}_g| = \frac{6}{2} = 3$. Since $\langle g \rangle \subseteq C_G(g)$, the order of g must divide 3. The order cannot be 1 (as $g \neq e$). Thus, the order of g is 3. This implies $C_G(g) = \langle g \rangle = \{1, g, g^2\}$. The other element in the class, g', must be $g^2 = g^{-1}$.

Analyze the element h:

The size of its class is $|\mathcal{O}_h| = 3$. The size of its centralizer is $|C_G(h)| = \frac{|G|}{|\mathcal{O}_h|} = \frac{6}{3} = 2$. This implies the order of h is 2, and $C_G(h) = \langle h \rangle = \{1, h\}$.

Consider the conjugation hgh^{-1}:

This is the conjugate of g by h. It must be an element in the class $\mathcal{O}_g$. Could $hgh^{-1} = g$? This would mean h is in the centralizer of g, $C_G(g)$. But we found $C_G(g) = \{1, g, g^2\}$ contains only elements of order 1 or 3, while h has order 2. So $hgh^{-1} \neq g$. The only other element in $\mathcal{O}_g$ is $g' = g^{-1}$. Therefore, we must have $hgh^{-1} = g^{-1}$.

Conclusion:

We have found an element g of order 3 and an element h of order 2. These two elements generate G (since no proper subgroup of a group of order 6 can contain elements of both order 2 and 3). They satisfy the relation $hgh^{-1} = g^{-1}$. This is precisely the presentation of the dihedral group D_6:

$$D_6 = \langle r, s \mid r^3 = 1, s^2 = 1, srs^{-1} = r^{-1} \rangle \tag{13}$$

The group D_6 is isomorphic to S_3 since it can be understood as permutations on the three tips of a tri-gon. Therefore, $G \cong S_3$.

5

> [D&F 4.3.27] Let g_1, g_2, ..., g_k be representatives of the conjugacy classes of the finite group G and assume these elements pairwise commute. Prove that G is abelian.
>
> [Hint: $g_1, g_2, ..., g_k \in C_G(g_i)$].

For any arbitrary $g_i \in G$, consider its centralizer

$$C_G(g_i) = \{x \in G \mid x = g_i x g_i^{-1}\}. \tag{14}$$

Since for each representative g_j we know that $g_j = g_i g_j g_i^{-1}, g_j \in C_G(g_i)$, for $1 \leq j \leq k$. THerefore the size of the centralizer is at least k, i.e., $|C_G(g_i)| \geq k$.

Recall the class equation states that the order of the group is the sum of the sizes of its distinct conjugacy classes. Without loss of generality, let $g_1 = 1$ be the representative of the identity class, so $|\mathcal{O}_{g_1}| = 1$. Thus we have

$$|G| = \sum_{i=1}^{m} |\mathcal{O}_{g_i}| = 1 + \sum_{i=2}^{m} |\mathcal{O}_{g_i}|. \tag{15}$$

Meanwhile by the orbit-stabilizer theorem, we have

$$|\mathcal{O}_g| = |G : C_G(g_i)| = \frac{|G|}{|C_G(g_i)|} \leq \frac{|G|}{k} \tag{16}$$

so that

$$\begin{gathered} |G| \leq 1 + \sum_{i=2}^{m} \frac{|G|}{k} = 1 + (m-1)\frac{|G|}{k} \\ \Rightarrow |G| \leq r. \end{gathered} \tag{17}$$

By definition, r is the number of distinct conjugacy classes that partition the group G. It is impossible for the number of subsets in a partition to be greater than the total number of elements. Thus, we must also have $r \leq |G|$. Collectively,

$$|G| = r. \tag{18}$$

A group is abelian if and only if every element commutes with every other element. This is equivalent to the center of the group being the group itself, $Z(G) = G$. An element is in the center if and only if its conjugacy class has size 1.

Therefore, since we have shown that the number of conjugacy classes is equal to the order of the group, every conjugacy class must have size 1.Thus G is abelian.

6

[D&F 4.3.30] If G is a group of odd order, prove for any nonidentity element $x \in G$ that x and x^{-1} are not conjugate in G.

Suppose for contradiction that for some non-idenetity element $x \in G$, where G is of odd order, x and x^{-1} are conjugates. That is,

$$\exists g \in G \; s.t. \; gxg^{-1} = x^{-1}. \tag{19}$$

Taking conjugate one more time, we see

$$\begin{gathered} g^2xg^{-2} = gx^{-1}g^{-1} = \left(gxg^{-1}\right)^{-1} = \left(x^{-1}\right)^{-1} = x \\ \Rightarrow g^2xg^{-2} = x. \end{gathered} \tag{20}$$

Since G is of odd order, $g^k = 1$ for some odd integer k. Further, $\gcd(2, k) = 1$ so $\langle g^2 \rangle$ generates the same subgroup as $\langle g \rangle$. There exists an integer m such that $2m = 1 \bmod k$. This implies:

$$g = g^1 = g^{2m} = \left(g^2\right)^m \tag{21}$$

So, g is a power of g^2.

From , we know that g^2 commutes with x. Since g is a power of g^2, g also commutes with x. Explicitly,

$$gxg^{-1} = x. \tag{22}$$

However this contradicts . Thus our assumption is false, and no non-identity element is conjugate to its inverse in a group of odd order. ■