

Use Frobenius ansatz to the Hermite equation

$$y'' - 2xy' + 2ny = 0 \quad (1)$$

Derive indicial equation, derive recursion relation between the expansion coefficient, and construct several polynomials

Let  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$  with  $a_0 \neq 0$ . We have the derivatives:

$$y' = \sum_{k=0}^{\infty} (k+r) a_k x^{k+r-1}; \quad y'' = \sum_{k=0}^{\infty} (k+r)(k+r-1) a_k x^{k+r-2}. \quad (2)$$

Plugging into the Hermite equation:

$$0 = \sum_{k=0}^{\infty} (k+r)(k+r-1) a_k x^{k+r-2} - 2x \sum_{k=0}^{\infty} (k+r) a_k x^{k+r-1} + 2n \sum_{k=0}^{\infty} a_k x^{k+r}. \quad (3)$$

To find the indicial equation and recursion relation, we require the sum of the coefficients of each power of  $x$  to be zero. Let's isolate the first few terms after re-indexing the first sum:

$$r(r-1)a_0 x^{r-2} + (r+1)ra_1 x^{r-1} + \sum_{k=0}^{\infty} [(k+r+2)(k+r+1)a_{k+2} - 2(k+r)a_k + 2na_k] x^{k+r} = 0; \quad (4)$$

The coefficient of the lowest power of  $x$  (i.e.,  $x^{r-2}$ ) gives the indicial equation:

$$r(r-1) = 0 \Rightarrow r = 0 \text{ or } 1. \quad (5)$$

Imposing the coefficient of  $x^{r-1}$  to be zero gives  $(r+1)ra_1 = 0$ . This implies  $a_1 = 0$  when  $r = 1$ . The rest of the terms give the recursion relation:

$$a_{k+2} = \frac{2(k+r-n)}{(k+r+2)(k+r+1)} a_k \quad (r = 0 \text{ or } r = 1, k = 0, 1, 2, \dots) \quad (6)$$

For  $r = 0$ , the recursion relation becomes

$$a_{k+2} = \frac{2(k-n)}{(k+2)(k+1)} a_k. \quad (7)$$

Starting with  $a_0$ , we have

$$a_2 = -na_0, \quad a_4 = -\frac{n(2-n)}{6}a_0, \quad a_6 = \frac{n(n-2)(n-4)}{90}a_0, \dots \quad (8)$$

we can write this into

$$y_{\text{even}} = a_0 \left( 1 - nx^2 + \frac{n(n-2)}{6}x^4 - \dots \right) \quad (9)$$

Similarly, starting with  $a_1$ , we can write

$$y_{\text{odd}} = a_1 \left( x + \frac{1-n}{3}x^3 + \frac{(1-n)(3-n)}{30}x^5 + \dots \right) \quad (10)$$

and a general solution for  $r = 0$  is  $y = y_{\text{even}} + y_{\text{odd}}$ . We notice that  $r = 1$  gives the same series solution with leading constant  $a_0$  instead of  $a_1$ . It is therefore sufficient to consider only  $r = 0$ .

We notice that the recursion terminates on  $k - n = 0 \Rightarrow k = n$ .

Now consider  $H_1(x)$  with  $n = 1$ . Recursion :

$$a_{k+2} = \frac{2(k-1)}{(k+2)(k+1)} a_k. \quad (11)$$

Odd  $n$  guarantees finite  $y_{\text{odd}}$  polynomial, so to get finite polynomial solution, we kill  $y_{\text{even}}$  by setting  $a_0 = 0$ . We thus have

$$y = a_1 x + \frac{1-1}{3} a_1 x^3 = a_1 x. \quad (12)$$

Conventionally, we set leading coefficient  $a_1 = 2^n = 2$ , and the Hermite polynomial to the first order is thus

$$\boxed{H_1(x) = 2x.} \quad (13)$$

Similarly, consider  $H_2(x)$  with  $n = 2$ . Recursion :

$$a_{k+2} = \frac{2(k-2)}{(k+2)(k+1)} a_k. \quad (14)$$

Even  $n$  guarantees finite  $y_{\text{even}}$  polynomial, so to get finite polynomial solution, we kill  $y_{\text{od}}$  by setting  $a_1 = 0$ . We thus have

$$y = a_0 - 2a_0 x^2. \quad (15)$$

Conventionally, we set leading coefficient  $-2a_0 = 2^n = 4$ , and the Hermite polynomial to the second order is thus

$$\boxed{H_2(x) = 4x^2 - 2.} \quad (16)$$

We can continue to read off several more:  $H_3(x) = 8x^3 - 12x$ ,  $H_4(x) = 16x^4 - 48x^2 + 12$ .

Apply Frobenius method around  $x = 0$  to the Legendre equation

$$(1 - x^2)y'' - 2xy' + n(n+1)y = 0. \quad (17)$$

Let  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$ . Plugging into the Legendre equation, we have

$$0 = (1 - x^2) \sum_{k=0}^{\infty} (k+r)(k+r-1)a_k x^{k+r-2} - 2x \sum_{k=0}^{\infty} (k+r)a_k x^{k+r-1} + n(n+1) \sum_{k=0}^{\infty} a_k x^{k+r}. \quad (18)$$

We re-index the first and second term, and isolate the first few terms to match the summation bounds:

$$0 = r(r-1)a_0 x^{r-2} + r(r+1)a_1 x^{r-1} + \sum_{k=0}^{\infty} x^{k+r} [(k+r+2)(k+r+1)a_{k+2} - (k+r)(k+r-1)a_k - 2(k+r)a_k + n(n+1)a_k] \quad (19)$$

Imposing the coefficient of the lowest power of  $x$  (i.e.,  $x^{r-2}$ ) to be zero gives the indicial equation:

$$r(r-1) = 0 \implies r = 0 \text{ or } 1. \quad (20)$$

Imposing the coefficient of  $x^{r-1}$  to be zero gives  $a_1 = 0$  when  $r = 1$ . The rest of the terms give the recursion relation:

$$a_{k+2} = \frac{(k+r)(k+r+1) - n(n+1)}{(k+r+2)(k+r+1)} a_k, \quad (r = 0 \text{ or } r = 1; k = 0, 1, 2, \dots) \quad (21)$$

We consider the case  $r = 0$ . (the case  $r = 1$  gives redundant solutions of odd series only.) The recursion relation becomes

$$a_{k+2} = \frac{k(k+1) - n(n+1)}{(k+2)(k+1)} a_k \quad (22)$$

and the termination condition is

$$k(k+1) - n(n+1) = 0 \implies k = n. \quad (23)$$

We consider  $P_1(x)$  with  $n = 1$ . Odd  $n$  guarantees the odd series terminate, so kill even series with  $a_0 \equiv 0$ . We then have  $y = a_1 x$ . Conventionally, normalization is set so that  $P_n(1) = 1 \implies a_1 = 1$ . Thus

$$P_1(x) = x. \quad (24)$$

Similarly, for  $P_2(x)$  with  $n = 2$ , even series terminate, so kill odd series with  $a_1 \equiv 0$ . We then have  $a_2 = -3a_0$ , and  $y = a_0(1 - 3x^2)$ . Conventionally, normalization is set so that  $P_n(1) = 1 \implies a_0 = -\frac{1}{2}$ . Thus

$$P_2(x) = \frac{1}{2}(3x^2 - 1). \quad (25)$$

Apply Frobenius method around  $x = 0$  to the Lauguerre equation

$$xy'' + (1 - x)y' + ny = 0, \quad (26)$$

where  $n$  is a non-negative integer. Find Lauguerre polynomials  $L_n(x)$  for several low orders  $n$

Let  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$ . Plugging in, we have

$$\sum_{k=0}^{\infty} (k+r)(k+r-1)a_k x^{k+r-1} + \sum_{k=0}^{\infty} (k+r)a_k x^{k+r-1} - \sum_{k=0}^{\infty} (k+r)a_k x^{k+r} + n \sum_{k=0}^{\infty} a_k x^{k+r} = 0 \quad (27)$$

Re-indexing the first two sums and isolating the first few terms, we have

$$\begin{aligned} r(r-1)a_0 x^{r-1} + \sum_{k=0}^{\infty} (k+r+1)(k+r)a_{k+1} x^{k+r} + r a_0 x^{r-1} + \sum_{k=0}^{\infty} (k+r+1)a_{k+1} x^{k+r} \\ - \sum_{k=0}^{\infty} (k+r)a_k x^{k+r} + n \sum_{k=0}^{\infty} a_k x^{k+r} = 0 \end{aligned} \quad (28)$$

Imposing the coefficient of the lowest power of  $x$  (i.e.,  $x^{r-1}$ ) to be zero gives the indicial equation:

$$r^2 = 0 \implies r = 0. \quad (29)$$

The rest of the terms give the recursion relation:

$$a_{k+1} = \frac{k+r-n}{(k+r+1)^2} a_k \stackrel{r=0}{=} \boxed{\frac{k-n}{(k+1)^2} a_k}, \quad (k = 0, 1, 2, \dots) \quad (30)$$

From which we see that the series terminates when

$$k - n = 0 \implies k = n. \quad (31)$$

Consider  $L_1(x) : n = 1$ .  $a_1 = \frac{0-1}{1} a_0 = -a_0$  and so  $y = a_0(1 - x)$ . Conventional normalization sets  $L_n(0) = 1 \implies a_0 = 1$ . Thus

$$\boxed{L_1(x) = 1 - x.} \quad (32)$$

Similarly,  $L_2(x)$  is given by  $n = 2$ .  $a_1 = \frac{0-2}{1} a_0 = -2a_0$  and  $a_2 = \frac{1-2}{4} a_1 = \frac{1}{2} a_0$ . Thus  $y = a_0(1 - 2x + \frac{1}{2}x^2)$ . Conventional normalization sets  $L_n(0) = 1 \implies a_0 = 1$ . Thus

$$\boxed{L_2(x) = 1 - 2x + \frac{1}{2}x^2.} \quad (33)$$

Consider the Bessel equation with order  $\nu = 0$

$$x^2 y'' + xy' + (x^2 - \nu^2)y = 0. \quad (34)$$

Show that the indicial equation has degenerate roots by construct explicitly the Bessel function  $J_0(x)$  using Frobenius method. Then, use  $y = J_0(x) \ln(x) + \sum_{k \geq 0} b_k x^k$  to find the second linearly independently solution. Compare with the Bessel function of the second kind  $Y_0(x)$ .

We use the regular Frobenius ansatz  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$ . Plugging into the Bessel equation with  $\nu = 0$ , we have

$$x^2 \sum_{k=0}^{\infty} (k+r)(k+r-1) a_k x^{k+r-2} + x \sum_{k=0}^{\infty} (k+r) a_k x^{k+r-1} + (x^2) \sum_{k=0}^{\infty} a_k x^{k+r} = 0. \quad (35)$$

Matching the indicies, we have

$$\begin{aligned} \sum_{k=0}^{\infty} (k+r)(k+r-1) a_k x^{k+r} + \sum_{k=0}^{\infty} (k+r) a_k x^{k+r} + \sum_{k=0}^{\infty} a_k x^{k+r+2} &= 0 \\ \sum_{k=0}^{\infty} (k+r)^2 a_k x^{k+r} + \sum_{k=2}^{\infty} a_{k-2} x^{k+r} &= 0 \end{aligned} \quad (36)$$

For  $k = 0$ , we read off the indicial equation:

$$r^2 = 0 \implies r = 0. \quad (37)$$

For  $k = 1$ , we have  $a_1 = 0$ , guarantees odd terms vanish. The rest of the terms give the recursion relation:

$$a_{k+2} = -\frac{1}{(k+r+2)^2} a_k \stackrel{r=0}{=} -\frac{1}{(k+2)^2} a_k, \quad (k = 0, 1, 2, \dots) \quad (38)$$

Let  $k = 2m$ , the recursion relation gives

$$a_{2m} = a_0 (-1)^m \frac{1}{2^{2m} (m!)^2}; \quad J_0(x) = \sum_{m=0}^{\infty} a_{2m} x^{2m}. \quad (39)$$

and so the first solution is (choosing  $a_0 = 1$ ):

$$J_0(x) = \sum_{m=0}^{\infty} (-1)^m \frac{x^{2m}}{2^{2m} (m!)^2} \quad (40)$$

For the second solution, take the Frobenius-log ansatz

$$y_2(x) = J_0(x) \ln x + S(x), \quad S(x) = \sum_{k=0}^{\infty} b_k x^k. \quad (41)$$

Differentiate:

$$y_{2'} = J_{0'} \ln x + \frac{J_0}{x} + S'; \quad y_{2''} = J_{0''} \ln x + \frac{2J_{0'}}{x} - \frac{J_0}{x^2} + S''. \quad (42)$$

Plug into  $x^2 y'' + xy' + x^2 y = 0$ . The  $\ln x$ -terms cancel because  $J_0$  solves the ODE, and the non- $\ln x$  terms give

$$x^2 S'' + x S' + x^2 S + 2x J_{0'} = 0. \quad (43)$$

Now expand in series. With  $S = \sum_{k \geq 0} b_k x^k$  and

$$2x J_{0'}(x) = \sum_{m=1}^{\infty} \frac{4m(-1)^m}{2^{2m} (m!)^2} x^{2m}, \quad (44)$$

equating the coefficient of  $x^k$  yields, for  $k \geq 2$ ,

$$k^2 b_k + b_{k-2} + \begin{cases} \frac{4m(-1)^m}{2^{2m}(m!)^2} & k = 2m \\ 0 & (k \text{ odd}) \end{cases} = 0, \quad (45)$$

and from the  $x^1$  term we get  $b_1 = 0$ . By induction, all odd coefficients vanish,  $b_{2m+1} = 0$ .

Thus it suffices to work with even indices. For  $m \geq 1$ ,

$$4m^2 b_{2m} + b_{2m-2} + \frac{4m(-1)^m}{2^{2m}(m!)^2} = 0. \quad (46)$$

Introduce (ingenious insight from GPT)

$$d_m := (-1)^m 2^{2m} (m!)^2 b_{2m} \quad (m \geq 0). \quad (47)$$

Multiplying Equation 46 by  $(-1)^m 2^{2m} (m!)^2$  gives

$$4m^2 d_m - 4m^2 d_{m-1} + 4m = 0 \Rightarrow d_m - d_{m-1} = -\frac{1}{m} \quad (m \geq 1). \quad (48)$$

Hence

$$d_m = d_0 - H_m, \quad H_m := \sum_{j=1}^m \frac{1}{j}; \quad H_0 := 0. \quad (49)$$

Undoing the substitution,

$$b_{2m} = \frac{(-1)^m}{2^{2m}(m!)^2} (d_0 - H_m), \quad b_{2m+1} = 0. \quad (50)$$

Now, rewrite

$$\begin{aligned} S(x) &= \sum_{m=0}^{\infty} b_{2m} x^{2m} = \sum_{m=0}^{\infty} \frac{(-1)^m}{2^{2m}(m!)^2} (d_0 - H_m) x^{2m} \\ &= d_0 \underbrace{\sum_{m=0}^{\infty} \frac{(-1)^m}{2^{2m}(m!)^2} x^{2m}}_{J_0(x)} - \sum_{m=1}^{\infty} \frac{(-1)^m H_m}{2^{2m}(m!)^2} x^{2m}. \end{aligned} \quad (51)$$

Therefore

$$\boxed{y_2(x) = J_0(x)(\ln x + d_0) - \sum_{m=1}^{\infty} \frac{(-1)^m H_m}{2^{2m}(m!)^2} x^{2m}.} \quad (52)$$

The standard small- $x$  expansion of the Neumann function is (cite [DLMF 10.8.2](#))

$$Y_0(x) = \frac{2}{\pi} \left( \ln \frac{x}{2} + \gamma \right) J_0(x) + \frac{2}{\pi} \sum_{m=1}^{\infty} \frac{(-1)^{m+1} H_m}{2^{2m}(m!)^2} x^{2m}. \quad (53)$$

Compare this with our  $y_2$ . Choose

$$d_0 = \gamma + \ln \frac{1}{2} \quad s.t. \quad \ln x + d_0 = \ln \frac{x}{2} + \gamma, \quad (54)$$

and then define  $Y_0$  simply by scaling:

$$Y_0(x) = \frac{2}{\pi} y_2(x). \quad (55)$$

With this choice,

$$\frac{2}{\pi} y_2(x) = \frac{2}{\pi} \left( \ln \frac{x}{2} + \gamma \right) J_0(x) - \frac{2}{\pi} \sum_{m=1}^{\infty} \frac{(-1)^m H_m}{2^{2m}(m!)^2} x^{2m}, \quad (56)$$

which matches Equation 53 term-by-term because  $-(-1)^m = (-1)^{m+1}$ .

Solve the Airy function using Frobenius method to obtain two linearly independent series solutions

$$y'' - xy = 0 \quad (57)$$

Let  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$ , substitution gives

$$\sum_{k=0}^{\infty} (k+r)(k+r-1)a_k x^{k+r-2} - \sum_{k=0}^{\infty} a_k x^{k+r+1} = 0 \quad (58)$$

Reindexing the second term:

$$\sum_{k=0}^{\infty} (k+r)(k+r-1)a_k x^{k+r-2} - \sum_{k=3}^{\infty} a_{k-3} x^{k+r-2} = 0. \quad (59)$$

Now match coefficients of each power  $x^{k+r-2}$ :

- For  $k = 0$ :  $r(r-1)a_0 = 0 \Rightarrow \boxed{r = 0 \text{ or } 1}$ .
- For  $k = 1$ :  $(r+1)ra_1 = 0$ .
- For  $k = 2$ :  $(r+2)(r+1)a_2 = 0 \Rightarrow \boxed{a_2 = 0}$ .
- For  $k \geq 3$ :  $(k+r)(k+r-1)a_k - a_{k-3} = 0$ , and from which we reveal the recursion relation:

$$a_k = \frac{a_{k-3}}{(k+r)(k+r-1)}, \quad k \geq 3 \quad (60)$$

We investigate the two roots of the indicial equation separately.

- **Branch  $r = 0$**  (Ordinary Power Series)

The head constraints for  $k = 1, 2$  give no restriction on  $a_1$  but force  $a_2 = 0$ . Thus two free seeds  $a_0$  and  $a_1$  generate two decoupled subsequences (due to the step-3 recurrence):

From  $a_0$ :  $a_3 = \frac{a_0}{3 \cdot 2} = \frac{a_0}{6}$ ,  $a_6 = \frac{a_3}{6 \cdot 5} = \frac{a_0}{180}$ , etc. (indices 0, 3, 6, ...).

From  $a_1$ :  $a_4 = \frac{a_1}{4 \cdot 3} = \frac{a_1}{12}$ ,  $a_7 = \frac{a_4}{7 \cdot 6} = \frac{a_1}{504}$ , etc. (indices 1, 4, 7, ...).

Hence

$$y(x) = a_0 \left( 1 + \frac{x^3}{3 \cdot 2} + \frac{x^6}{(3 \cdot 2)(6 \cdot 5)} + \dots \right) + a_1 \left( x + \frac{x^4}{(4 \cdot 3)} + \frac{x^7}{(4 \cdot 3)(7 \cdot 6)} + \dots \right), \quad (61)$$

- **Branch  $r = 1$** : The head constraints give  $a_1 = a_2 = 0$ . The recurrence then produces the same “1 mod 3” subsequence as taking  $r = 0$  with  $a_1 \neq 0$ . Thus it does not yield an independent solution beyond the two already obtained from  $r = 0$ .

Comparing Equation 61 to the standard Airy functions  $\text{Ai}(x)$  and  $\text{Bi}(x)$ , we see that they are specific choices of these constants, selected for their distinct asymptotic behaviors.

If we choose the normalization constants

$$a_0 = \frac{1}{3^{\frac{2}{3}} \Gamma(\frac{2}{3})} \quad \text{and} \quad a_1 = -\frac{1}{3^{\frac{1}{3}} \Gamma(\frac{1}{3})}, \quad (62)$$

we can arrive at the specific solution:

$$y(x) = \frac{1}{3^{\frac{2}{3}} \Gamma(\frac{2}{3})} (1 + \dots) - \frac{1}{3^{\frac{1}{3}} \Gamma(\frac{1}{3})} (x + \dots). \quad (63)$$

This combination matches the definition of the **Airy function of the first kind**,  $\text{Ai}(x)$ .

Alternatively, if we choose the constants

$$a_0 = \frac{3^{-\frac{1}{6}}}{\Gamma(\frac{2}{3})} \quad \text{and} \quad a_1 = \frac{3^{\frac{1}{6}}}{\Gamma(\frac{1}{3})}, \quad (64)$$

we can arrive at the specific solution:

$$y(x) = \frac{3^{-\frac{1}{6}}}{\Gamma(\frac{2}{3})}(1 + \dots) + \frac{3^{\frac{1}{6}}}{\Gamma(\frac{1}{3})}(x + \dots). \quad (65)$$

Which matches the definition of the **Airy function of the second kind**,  $\text{Bi}(x)$ .