ECE 535 Fall 2025
Homework #1 (60 points)
Due 9/17 at 11:59 pm on Canvas, in pdf format

Guidelines:

- Please submit a pdf document to Canvas with handwritten solutions, with your approach to each problem and the steps taken clearly laid out and written legibly. In cases where the solution requires plotting, the computer-generated plot should be accompanied by a brief handwritten explanation of your approach. This formatting requirement is worth **5 points** of the point total for each homework.
- Undergraduate students who wish to attempt the extra problem will receive up to 5 additional points for that homework.

1. (10 points) **Defining the second and the meter** The SI defines the meter in terms of the speed of light and the second. Today, the second is realized using the cesium (Cs)-133 hyperfine transition, but in the future optical transitions may be used instead. In this problem you will compare the cesium microwave frequency to an optical frequency (strontium (Sr) around 698 nm) for measurements of length and time.

(a) (3 points) Look up the value of the speed of light in vacuum, using a resource like the NIST database of constants (CODATA; https://physics.nist.gov/cuu/Constants/index.html). Using this SI defining constant, calculate the frequency of the optical transition of the Sr-87 optical clock, at 698.445 nm. Write your answer in THz.

The SI-defined speed of light in vacuum is c = 299 792 458 m/s.

$$\nu = \frac{c}{\lambda} = 4.2923 \times 10^{14}\ \text{Hz} = 429\ \text{THz}$$

(b) (3 points) Compute the ratio between the Sr clock frequency and the Cs clock frequency (also found using CODATA) that is used to define the second. Approximately how many ticks from the Cs clock occur during one oscillation of the Sr optical clock?

The SI-defined Cs hyperfine (clock) frequency is ν_{Cs} = 9 192 631 770 Hz. The ratio between the Cs clock frequency and Sr clock frequency is 2.14×10^{-5}.

(c) (2 points) Suppose you want to measure the time it takes light tuned to the strontium (Sr) optical clock to travel 1 meter. Compare the number of cycles (of Sr vs Cs clock) within this time.

Light needs around 3.33 ns (from $\frac{1\,m}{c}$) to travel a meter in vacuum. This duration corresponds to $3.33 \times 10^{-9}\ s\left(\frac{1}{\nu_{Sr}}\right) = 1.43 \times 10^{6}$ oscillations in a Sr clock, and $3.33 \times 10^{-9}\ s\left(\frac{1}{\nu_{Cs}}\right) =$ 30.6 oscillations in a Cs clock.

(d) (2 points) Explain why it may be advantageous to use an optical standard for length and time measurements.

With an optical standard, there are a factor of $\sim 10^5$ more oscillations or "tick marks" for the same time window, in comparison to a GHz-microwave standard, making the optical standard more precise.

2. (15 points) **Planck's radiation law**

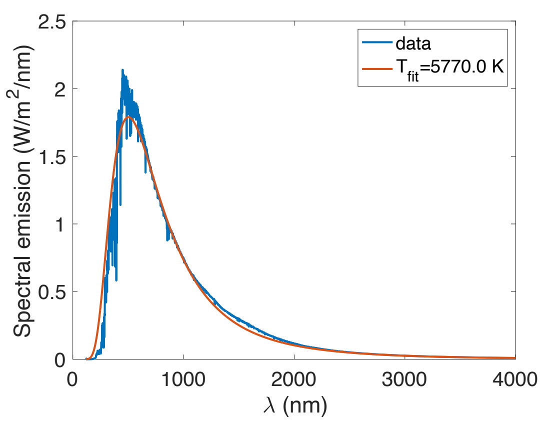
(a) (8 points) Using data provided in [this spreadsheet](), fit the measured spectral emission from the sun to the Planck's radiation law and estimate the temperature of the sun T_{sun}. **Note that the sun's radiation only reaches earth over a small solid angle and so you may have to modify the expression for $\frac{dI}{d\lambda}$.** Plot the data and fitted curve on the same figure and compare your estimate of the sun's temperature with reported values on the web.

The following code can be used to fit and plot the solar spectral data. The key is to make sure that the unit for wavelength is consistent between your model and the data.

```
syms x

load solar_spectrum.mat
bckbody = fittype('a/(x^5)/(exp(6.626e-34*3e8/x/1.38e-23/b)-1)');
lambda=solar(:,1)*1e-9;Int=solar(:,2);[fitobj,gof,output] =
fit(lambda,Int,bckbody,'startpoint',[1e-19 5780]);

fitobj
figure; plot(lambda*1e9,Int,'linewidth',2), hold on;
plot(lambda*1e9,bckbody(8.185e-30,5770,lambda),'linewidth',2)
 xlabel('\lambda (nm)')
 ylabel('Spectral emission (W/m^2/nm)')
 legend('data',sprintf('T_{fit}=%.1f K',fitobj.b))
 xlim([0 4000])
 set(gca,'FontSize',20)
```



The fitted temperature is a reasonably consistent with the average temperature of the surface of the sun, which is 5778 K.

(b) (2 points) Explain the deviations between the spectral emission data from the sun and the Planck's radiation law.

The deviations between the spectral emission data and Planck's radiation law can be due to (1) absorption, refraction, and reflection of solar radiation by Earth's atmosphere and (2) absorption and emission of light by atomic and molecular species in the sun. This is evidenced by the sharp absorption lines observed in the visible-light region (~400 – 800 nm) of the data.

(c) (5 points) The sun has a radius $R(= 7 \times 10^8\ m)$ while the radius of the earth is $r(= 6.37 \times 10^6\ m)$. The mean distance between the sun and the earth is $L(= 1.5 \times 10^{11}\ m)$. One can assume that both the sun and the earth absorb all electromagnetic radiation and that earth has reached a steady-state temperature T_{earth} that does not change with time. Find an approximate expression for the temperature T_{earth} of the earth in terms of r, R, L, and T_{sun}. How does it compare with your answer in (a).

We will use the Stefan-Boltzmann law for this problem, which describes the power P radiated by an object based on its temperature T and surface area A:

$$P = \sigma A T^4$$

where $\sigma = 5.67 \times 10^{-8}\ W \cdot m^{-2} \cdot K^{-4}$. The power radiated by the sun is:

$$P_{sun} = \sigma(4\pi R^2)T_{sun}^4$$

A fraction of P_{sun} will reach the earth, based on the fraction of the area subtended by the earth from the sun's perspective. Assuming that all the power reaching the earth is being absorbed, the power absorbed by the earth is:

$$P_{earth,abs} = \frac{\pi r^2}{4\pi L^2} P_{sun} = \frac{\pi \sigma r^2 R^2 T_{sun}^4}{L^2}$$

In steady state, the earth emits the same power it absorbs, and therefore

$$P_{earth,em} = \sigma(4\pi r^2)T_{earth}^4 = \frac{\pi \sigma r^2 R^2 T_{sun}^4}{L^2}$$

Solving for T_{earth}:

$$T_{earth}^4 = \frac{R^2 T_{sun}^4}{4L^2}$$

3. (15 points) **Johnson noise thermometry** You are to determine the temperature of a wire by taking measurements of the thermal noise voltage V_{rms} across this wire with known resistance $R = 150\ \Omega$ over a bandwidth $\Delta\nu = 10^6\ Hz$. If the average RMS voltage was $V_{rms} = 2.1 \times 10^{-6}\ V$.

(a) (2 points) What is the noise power associated with the measured V_{rms} and resistance R?

$$P = \frac{V_{rms}^2}{R} = 2.9 \times 10^{-14}\ W$$

(b) (3 points) What is the temperature of the wire based on the measured V_{rms}?

$$T = \frac{V_{rms}^2}{4k_B R \Delta\nu} = 532\ \mathrm{K}$$

(c) (6 points) Explain why the Johnson noise in a wire, $P = k_B T(\Delta\nu)$, scales linearly with the temperature T and measurement bandwidth $\Delta\nu$, whereas Stefan–Boltzmann law, $\frac{P}{A} = \sigma T^4$, scales as T^4 and independent of any measurement bandwidth.

The Stefan-Boltzmann law describes the total power per area from thermal emission of an object. It comes from integrating the energy density per unit frequency of the thermal emission over all frequencies:

$$\frac{P}{A} = \frac{c}{4}\int_0^\infty \frac{8\pi h\nu^3}{c^3}\frac{1}{e^{h\nu/(k_B T)} - 1}\, d\nu$$

The integral yields a T^4 scaling of P/A and is independent of bandwidth since we are considering contributions over all frequencies to the thermal power. Note that the expression for the integrand (energy density per unit frequency) comes from multiplying the mode density for electromagnetic (EM) fields in three dimensions ($8\pi\nu^2/c^3$) and the mean energy per mode ($\langle E\rangle = \frac{h\nu}{e^{h\nu/(k_B T)} - 1}$) [see notes from Topic 1b and supplementary notes].

In a wire, the EM modes can be considered to be in 1D. Here, the mode density is $4/c$ while the mean energy per mode remains the same regardless of the dimensionality. The mean energy density per frequency is thus

$$\frac{4}{c}\frac{1}{e^{h\nu/(k_B T)} - 1}$$

If we were to integrate all frequencies for the total power we would get:

$$P \propto 4\int_0^\infty \frac{h\nu}{e^{\frac{h\nu}{k_B T}} - 1}\, d\nu = \frac{2\pi^2 k_B^2}{3h} T^2$$

We get a T^2 scaling from contributions of all frequencies.

At a low-frequency regime (e.g., RF or microwaves) where $\frac{h\nu}{k_B T} \ll 1$, $e^{\frac{h\nu}{k_B T}} \approx 1 + \frac{h\nu}{k_B T} + \cdots$. $\frac{h\nu}{e^{h\nu/(k_B T)} - 1} = h\nu/(h\nu/k_B T) \approx k_B T$. Additionally if we were to consider contributions of only a narrow frequency band $\Delta\nu$, then the noise power is:

$$P = 4\, k_B T\, \Delta\nu$$

(d) (4 points) Does the model of the Johnson noise work at optical frequencies (say 600 nm)? Justify your answer.

No.

Note $k_B T = 1.38 \times 10^{-23}\, J$ at 1 K, so $h\nu = 6.6 \times 10^{-25}$ J for a 1-GHz signal is much smaller. Meanwhile, $h\nu = 3.3 \times 10^{-19}$ J for 600-nm light, so the assumption that $\frac{h\nu}{e^{h\nu/(k_B T)} - 1} \approx k_B T$ is no longer valid.

At optical frequencies and generally, the noise power within a frequency band is

$$P = \frac{4h\nu}{e^{h\nu/(k_B T)} - 1}\Delta\nu$$

This result leads to a roll off of noise power at high frequencies.

4. (5 points) Calculate the de Broglie wavelength for the following:
   (a) An electron that has been accelerated through a potential difference of 54 V.
   (b) A 10-MeV electron.

The de Broglie wavelength of an electron can be related to its kinetic energy E_{kin} and rest mass m_0:

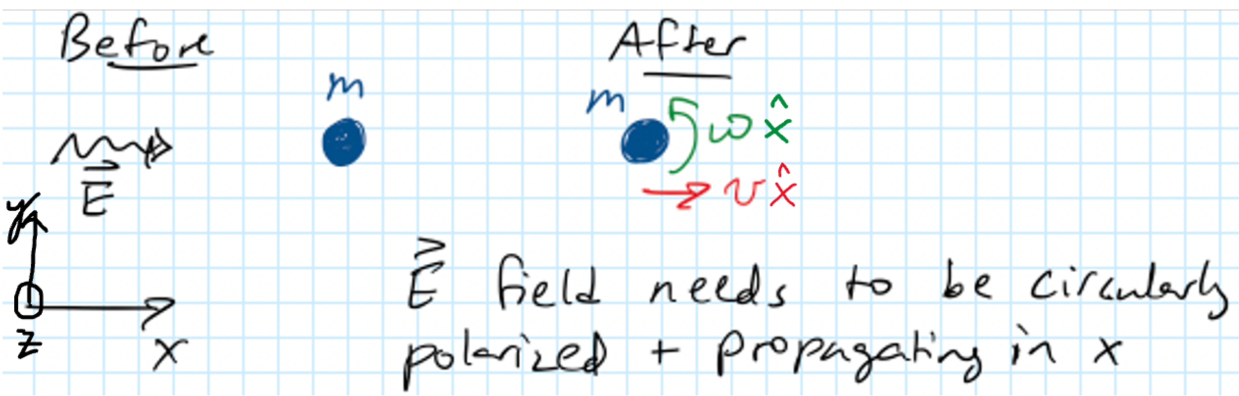
$$\lambda = \frac{hc}{\sqrt{E_{kin}^2 + 2E_{kin}m_o c^2}}$$

where E_{kin} is in joules; $h = 6.626 \times 10^{-34} J \cdot s$; $m_0 = 9.1095 \times 10^{-31}\ kg$; $c = 3 \times 10^8 \frac{m}{s}$.

The kinetic energy of the electron in (a) is $E_{kin} = eV = 54\ eV = 8.64 \times 10^{-18}\ J$ and therefore $\lambda = 1.67 \times 10^{-10}\ m = 0.167\ nm$.

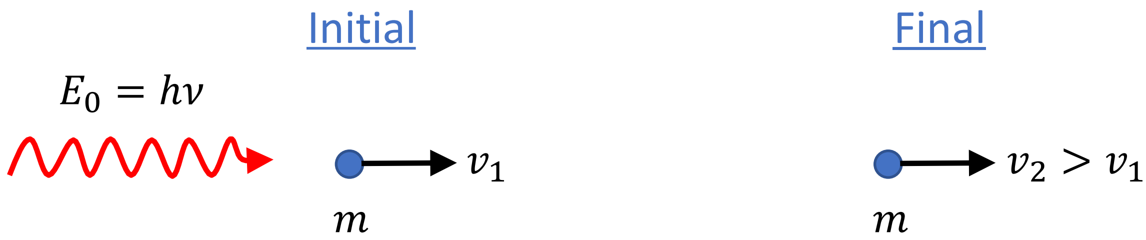
For (b), $E_{kin} = 1.6 \times 10^{-12}\ J$, and $\lambda = 1.183 \times 10^{-13}\ m$.

5. (10 points) A laser pointer is normally incident on a small spherical particle (where the particle size is much smaller than the beam size) and is fully absorbed without loss by the particle, causing it to rotate. Provide a schematic drawing of the interaction, including the coordinate system, and write a valid expression for the electric field of the laser based on your drawing.



A valid electric field would have wavevector $\vec{k} \parallel \hat{x}$ and be circularly polarized with field components along $\hat{y}$ and $\hat{z}$ having a phase difference of $\pm\frac{\pi}{2}$. Since $e^{\pm\frac{i\pi}{2}} = \pm i$, we can write the expression as: $\vec{\mathcal{E}}(x,t) = \mathcal{E}_0 e^{i(k_0 x - \omega_0 t)}(\hat{y} \pm i\hat{z})$. Note that the angular frequency of the light, ω_0, is not the same as the rotational frequency of the particle shown above.

6. (10 points) **Extra problem for graduate students: can a free electron absorb a photon?**
   (a) (8 points) Consider a scenario in which a free electron with velocity v_1 absorbs a photon with energy $h\nu$ and increases its velocity to v_2. Evaluate whether it is possible for both energy and momentum to be conserved in this scenario.



Energy and momentum conservation yield:

$$\gamma_1 mc^2 + h\nu = \gamma_2 mc^2$$
$$\frac{h\nu}{c} + \gamma_1 mc\beta_1 = \gamma_2 mc\beta_2$$

where $\beta_1 = \frac{v_1}{c}$; $\beta_2 = \frac{v_2}{c}$; $\gamma_1 = \frac{1}{\sqrt{1-\beta_1^2}}$; $\gamma_2 = \frac{1}{\sqrt{1-\beta_2^2}}$

Subtracting the two equations, we get:

$$\gamma_1 mc(1-\beta_1) = \gamma_2 mc^2(1-\beta_2)$$
$$\gamma_1(1-\beta_1) = \gamma_2(1-\beta_2)$$

This is satisfied when $\gamma_1 = \gamma_2$ and $\beta_1 = \beta_2$, which leads to the conclusion that $v_1 = v_2 = c$ (not possible because the object has mass m and violates our initial assumptions). Therefore, photo-absorption by a free electron is not allowed.

(b) (2 points) In a sentence or two, describe the condition under which the absorption of a photon by an electron is allowed.

In Compton scattering the scattered photon has momentum $\frac{h\nu'}{c} \neq \frac{h\nu}{c}$ and energy $h\nu' < h\nu$, thus satisfying the conservation of energy and momentum. For full absorption of the photon by an electron, we need interactions with the nucleus and other electrons to compensate for the photon recoil.