1

For set A and group G, $A \subset G$; prove: $A = \langle A \rangle$ iff $A \leq G$.

- $\Rightarrow$: Assume $A = \langle A \rangle$, then A is a subgroup of G by definition, as it is the intersection of all subgroups of G containing A. Thus, $A \leq G$.
- $\Leftarrow$: Assume A is a subgroup of G. Recall

$$\langle A \rangle = \bigcap_{\substack{A \subseteq H \\ H \leq G}} H, \tag{1}$$

so $A \subseteq H, \forall H \leq G$, so $A \subseteq \langle A \rangle$.

Assume $a \in \langle A \rangle$, then by definitiion, $a = a_1^{\varepsilon_1} a_2^{\varepsilon_2} ... a_k^{\varepsilon_k} \in A$, since A is closed under operations (a subgroud) ,and $a_i \in A$. Thus, $\langle A \rangle \subseteq A$. Therefore, $A = \langle A \rangle$.

Thus, $A = \langle A \rangle$ iff $A \leq G$. ■

2

> Prove that subgroup of S_4 generated by (1 2) and (1 3)(2 4) is isomorphic to D_8.

$$H = \langle (12), (13)(24) \rangle \leq S_4. \tag{2}$$

Consider element

$$\begin{gathered} \sigma_1 = (12)(13)(24) = (1324) \in H, \\ \sigma_2 = (12) \in H. \end{gathered} \tag{3}$$

We can check that any relation satisfied in D_8 by r, s is also satisfiend in H by σ_1, σ_2 :

$$\begin{gathered} \sigma_1^4 = (1324)^4 = e \\ \sigma_2^2 = (12)^2 = e; \\ \sigma_2\sigma_1 = (12)(1324) = (13)(24), \\ \sigma_1^{-1}\sigma_2 = (1423)(12) = (13)(24) = \sigma_2\sigma_1 \end{gathered} \tag{4}$$

we can see that relations satisfied by r is also satisfied by σ_1, and relations satisfied by s is also satisfied by σ_2.

According to D&F 3rd ed. p38, a "useful fact" that was stated (*without* a proof) is:

> If two groups $G = \langle a_1, a_2, ..., a_n \rangle$ and $H = \langle b_1, b_2, ..., b_n \rangle$ have the same relations among their generators a_i and b_i, then there is a unique homomorphism $\varphi : G \to H$ such that $\varphi(a_i) = b_i$.

Spcifically, in our case, there is a unique homomorphism $\varphi : D_8 \to H$ such that $\varphi(r) = \sigma_1, \varphi(s) = \sigma_2$.

Further, we check bijectivity by writing out all elements explicitely:

$$\begin{gathered} e \mapsto (1), s \mapsto (12), r \mapsto (1324), \\ sr \mapsto (13)(24), sr^2 \mapsto (34), sr^3 \mapsto (14)(23); \\ r^2 \mapsto (12)(34), r^3 \mapsto (1423) \end{gathered} \tag{5}$$

φ is bijective since all elements in D_8 have distinct images in H. Thus, φ is an isomorphism, and $H \cong D_8$.

3

Show a proper subgroup of $\mathbb{Q}$ that is not cyclic.

Consider the group

$$S = \left\{ \frac{a}{2^k}, a \in \mathbb{Z}, k \in \mathbb{N}_0 \right\}. \tag{6}$$

$S \leq Q$ since it is nonempty ($e \in S$), closed under addition and inverses; explicitely, for $\frac{a}{2^k}, \frac{b}{2^l} \in S$,

$$\frac{a}{2^k} + \frac{b}{2^l} = \frac{a2^l + b2^k}{2^{k+l}} \in S, \tag{7}$$

and

$$-\frac{a}{2^k} = \frac{-a}{2^k} \in S. \tag{8}$$

Now, suppose $S = \langle q \rangle, q = \frac{a}{2^b} \in S$, so that $S = \left\langle \frac{a}{2^b} \right\rangle = \left\{ n\frac{a}{b} \mid a, n \in \mathbb{Z}, b \in \mathbb{N}_0 \right\}.o'$ Then consider $s = \frac{1}{2^{b+1}} \in S$, then

$$\frac{1}{2^{b+1}} = n\frac{a}{2^b} \Rightarrow na = \frac{1}{2} \tag{9}$$

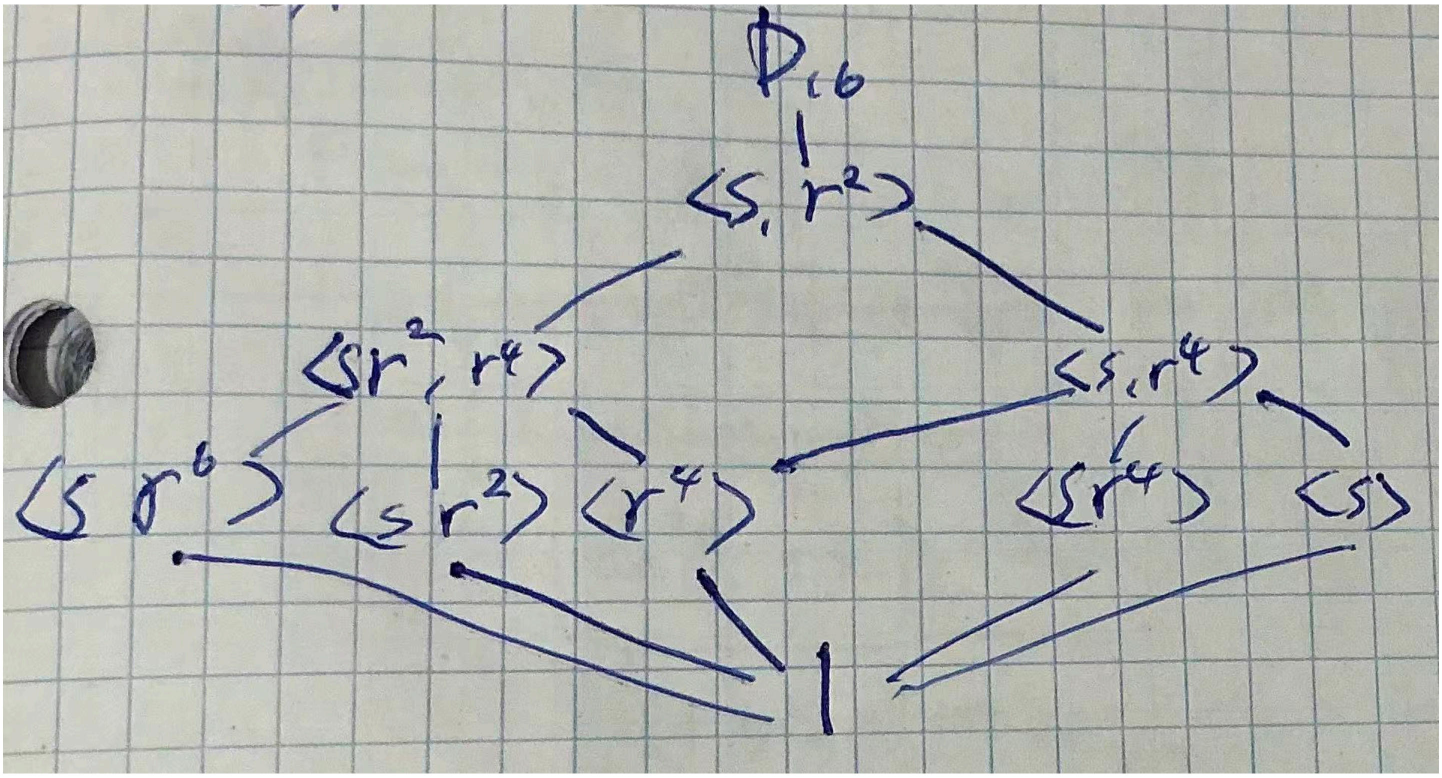
but this is impossible since $na \in \mathbb{Z}$. Thus, S is not cyclic.

Also, $S \neq \mathbb{Q}$ since $\frac{1}{3} \notin S$. Thus, S is a proper subgroup of $\mathbb{Q}$ that is not cyclic.

4

[D&F 2.5.2ad] List all subgroups of D_{16} that satisfy the given conditions:
a. Subgroups that are contained in $\langle sr^2, r^4 \rangle$,
b. Subgroups that contain $\langle s \rangle$.

We consider the sublattice of D_{16} :



We can thus read off for the following parts:

a.

$$\langle sr^2, r^4 \rangle, \langle sr^6 \rangle, \langle sr^2 \rangle, \langle r^4 \rangle. \tag{10}$$

b.

$$D_{16}, \langle s \rangle, \langle s, r^2 \rangle, \langle s, r^4 \rangle \tag{11}$$

5.

[based on D&F 3.1.6] Define $\varphi : \mathbb{R}^\times \to \{\pm 1\}$ by letting $\varphi(x)$ be x divided by the absolute value of x. Describe the fibers of φ and prove that φ is a homomorphism.

1. fiber of φ.

$$\begin{aligned} \varphi^{-1}(1) &= \{x|\ x \in \mathbb{R}, x > 0\}, \\ \varphi^{-1}(-1) &= \{x \mid x \in \mathbb{R}, x < 0\}. \end{aligned} \tag{12}$$

2. Proof that φ is a homomorphism.
Consider arbitrary $x, y \in \mathbb{R}^*$, we have

$$\varphi(xy) = \frac{xy}{|xy|} = \frac{xy}{|x||y|} = \frac{x}{|x|}\frac{y}{|y|} = \varphi(x)\varphi(y). \tag{13}$$

6

[based on D&F 3.1.7] Define $\pi : \mathbb{R}^2 \to \mathbb{R}$ by $\pi((x, y)) = x + y$. Prove that π is a surjective homomorphism and describe the kernel and fibers of π geometrically.

- Homomorphism: consider arbitrary $(a, b), (c, d) \in \mathbb{R}^2$. Then

$$\begin{aligned} \pi((a,b)+(c,d)) &= \pi(a+c, b+d) \\ &= a+c+b+d \\ &= a+b+c+d = \pi(a,b)+\pi(c,d) \end{aligned} \tag{14}$$

- Surjective: for any $c \in \mathbb{R}$, we can find $(c, 0) \in \mathbb{R}^2$ such that $\pi(c, 0) = c + 0 = c$. Thus, π is surjective.

Geometrically, the kernel of π is $x + y = 0 \Rightarrow y = -x$, which is a line through the origin with slope −1 in the xy-plane.

The fibers of π are lines parallel to the kernel line, i.e. lines with slope −1. For example, the fiber $\pi^{-1}(0)$ is the kernel line $y = -x$, and the fiber $\pi^{-1}(1)$ is the line $y = -x + 1$,etc.

Let $G \leq \mathrm{GL}_2(\mathbb{R})$ be the group of matrices of the form

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}.$$

Which of the following subgroups $H_i \leq G$ is normal? If H_i is normal, describe the quotient group G/H_i.
a. H_1 consists of the matrices of the form $\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$.
b. H_2 consists of the matrices of the form $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$

a. H_1 is not normal.
Counterexample: consider

$$h_1 = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \in H, g_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in G, g_1^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \in G. \tag{15}$$

Then

$$g_1 h_1 g_1^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 0 & 1 \end{pmatrix} \notin H \tag{16}$$

There's at least one element in the conjugate $g_1 H_1 g_1^{-1}$ that is not in H_1. Thus, H_1 is not normal in G.

b. H_2 is normal.
Consider arbitrary

$$h = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, g = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}, g^{-1} = \begin{pmatrix} \frac{1}{a} & -\frac{b}{a} \\ 0 & 1 \end{pmatrix} \quad (a \neq 0). \tag{17}$$

We compute the conjugate:

$$ghg^{-1} = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{a} & -\frac{b}{a} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & ax \\ 0 & 1 \end{pmatrix} \in H_2. \tag{18}$$

Thus, for arbitrary $g \in G, h \in H_2$, we have $gH_2g^{-1} \subseteq H_2$. Therefore, H_2 is normal in G.

Now let's describe the quotient group G/H_2. We construct a surjective map $\varphi : G \to K$ such that $\ker(\varphi) = H_2$, and use the First Isomorphism Theorem to conclude $G/H_2 \cong K$. Since $a \neq 0$, consider the map $\varphi : G \to \mathbb{R}^\times$

$$\varphi\left(\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\right) = a. \tag{19}$$

We need to show that it's a surjective homomorphism with kernel H_2. Consider arbitrary

$$g_1 = \begin{pmatrix} a_1 & b_1 \\ 0 & 1 \end{pmatrix}, g_2 = \begin{pmatrix} a_2 & b_2 \\ 0 & 1 \end{pmatrix}; \quad g_1 g_2 = \begin{pmatrix} a_1 a_2 & a_1 b_2 + b_1 \\ 0 & 1 \end{pmatrix}. \tag{20}$$

We have

$$\varphi(g_1 g_2) = a_1 a_2; \quad \varphi(g_1)\varphi(g_2) = a1a_2 = \varphi(g_1 g_2). \tag{21}$$

Thus φ is a homomorphism. For arbitrary $c \in \mathbb{R}^\times$, we can find $g = \begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix} \in G$ such that $\varphi(g) = c$. Thus, φ is surjective.

The elements of the kernel of φ is described as

$$
\begin{aligned}
&\varphi\left(\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\right) = 1 \Rightarrow \quad a = 1 \\
\Rightarrow \quad &\ker(\varphi) = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \mid b \in \mathbb{R} \right\} = H_2.
\end{aligned} \tag{22}
$$

By the First Isomorphism Theorem, we have $G/H_2 \cong \mathbb{R}^{\times}$.