ECE 601/901 Fall 2024
Homework #5
Due 11/22 at 11:59 pm on Canvas, in pdf format
No late submission accepted

Guidelines:
- Typed solutions or handwritten solutions (as long as they are legible) are accepted.
- Please show as much work as possible and compile your responses into <u>a single pdf document</u>.
- You are welcome to work together on the problems, but please write your solutions in your own words.

1. (10 points) Consider the spin-orbit interaction for an alkali atom.
   (a) (5 points) Determine the number of degenerate states for a given n and l and their associated energies.

$$E_j = E_{nr}\left(1 - \frac{\alpha^2}{nl\left(l + \frac{1}{2}\right)(l + 1)}\left[\frac{1}{2}\left(\left(l - \frac{1}{2}\right)\left(l + \frac{1}{2}\right) - l(l + 1) - s(s + 1)\right)\right]\right.$$

$$E_{j\prime} = E_{nr}\left(1 - \frac{\alpha^2}{nl\left(l + \frac{1}{2}\right)(l + 1)}\left[\frac{1}{2}\left(\left(l + \frac{1}{2}\right)\left(l + \frac{3}{2}\right) - l(l + 1) - s(s + 1)\right)\right]\right.$$

The number of degenerate states for E_j is $(2j + 1)$, where the number of degenerate states for $E_{j\prime}$ is $(2j' + 1)$.

   (b) (4 points) Calculate the mean energy (average energy of all possible states) based on your answer in (a).

The mean energy can be calculated as: $\langle E \rangle = [(2j' + 1)E_{j\prime} + (2j + 1)E_j]/(2j' + 2j + 2)$.

$$\langle E \rangle = E_{nr} - \frac{\alpha^2}{2nl\left(l + \frac{1}{2}\right)(l + 1)(2j' + 2j + 2)}\left[(2l)\left(\left(l - \frac{1}{2}\right)\left(l + \frac{1}{2}\right) - l(l + 1) - \frac{3}{4}\right)\right.$$
$$\left. + (2l + 2)\left(\left(l + \frac{1}{2}\right)\left(l + \frac{3}{2}\right) - l(l + 1) - \frac{3}{4}\right)\right]$$

$$\langle E \rangle = E_{nr} - \frac{\alpha^2}{2nl\left(l + \frac{1}{2}\right)(l + 1)(2j' + 2j + 2)}[(2l)(-l - 1) + (2l + 2)(l)]$$

$$\langle E \rangle = E_{nr} - \frac{\alpha^2}{2nl\left(l + \frac{1}{2}\right)(l + 1)(2j' + 2j + 2)}(0) = E_{nr}$$

   (c) (1 points) Compare the mean energy to the unshifted energy E_n from the gross energy solutions.

They are the same (mean energy is not shifted).

2. (10 points) Calculate the wavelengths of emission from the decay of the 2P orbital in hydrogen for three isotopes: ^1H (nuclear spin $I = 1/2$), ^2D ($I = 1$), and ^3T ($I = 3/2$). What wavelength resolution is needed in a spectrometer to distinguish these lines? To calculate the center emission wavelength for each isotope, use the Rydberg constant and reduced mass.

For one-electron atoms, the transition wavelength between principal quantum numbers n_i and n_f is

$$\lambda^{-1} = R_\infty \frac{m_{nucleus}}{m_e + m_{nucleus}} \left(\frac{1}{n_i^2} - \frac{1}{n_f^2} \right)$$

where $R_\infty = 10973731.56816 \ m^{-1}$ and $m_e = 9.109\ 383\ 7015 \times 10^{-31} \ kg$.

The mass for the nucleus is $m_{nucleus} = \begin{cases} m_p = 1.6726 \times 10^{-27} \text{ kg for hydrogen} \\ m_p + m_n = 3.3475 \times 10^{-27} \text{ kg for deuterium} \\ m_p + 2m_n = 5.0225 \times 10^{-27} \text{ kg for tritium} \end{cases}$

The center wavelengths of the emitted photons from $n = 2 \to 1$ are then:

$$\lambda = \begin{cases} 121.56844 \ nm \text{ for hydrogen} \\ 121.53534 \ nm \text{ for deuterium} \\ 121.52431 \ nm \text{ for tritium} \end{cases}$$

The hyperfine structure for the ground state for each isotope is as follows:
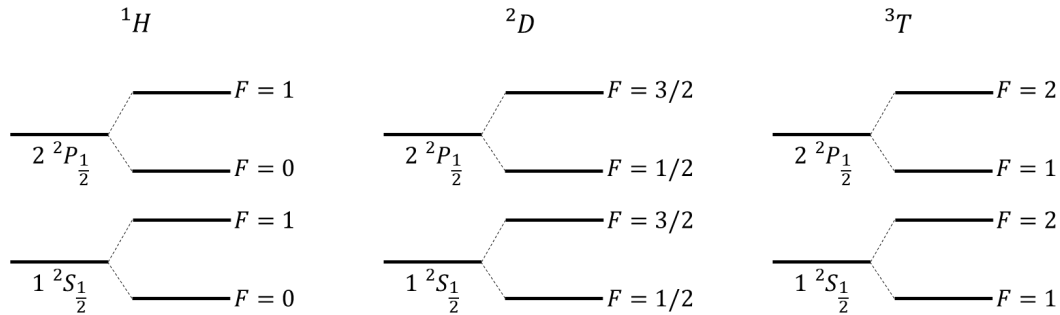
where the allowable transitions are $\Delta F = 0, \pm 1$.

The hyperfine energy shifts is: $\Delta E_{HFS} = \frac{A_j}{2}[F(F + 1) - j(j + 1) - I(I + 1)]$, where $A_{\frac{1}{2}} - 5.8 \times 10^{-6} \ eV$ for ground-state hydrogen.

For ^1H:

$$\Delta E_{HFS} = \begin{cases} \frac{A}{2}\left(-\frac{3}{2}\right) = -\frac{3A}{4} \text{ for F = 0} \\ \frac{A}{2}\left(\frac{A}{2}\right) = \frac{A}{4} \text{ for F = 1} \end{cases}$$

The separation between the ground hyperfine states is $\Delta E_{HFS} = A$. The transitions are thus separated by an energy of δE_{HFS} which needs to be converted to wavelength:

$$\left| \frac{\delta E}{\delta \lambda} \right| = \frac{hc}{\lambda^2}$$

$$\delta\lambda = \frac{\lambda^2 \Delta E_{HFS}}{hc}$$

where λ is the center wavelength.

$$\delta\lambda_H = 6.9 \times 10^{-5} \ nm$$

For ^2D:

$$\Delta E_{HFS} = \begin{cases} \frac{A}{2}(-2) = -A \text{ for F} = \frac{1}{2} \\ \frac{A}{2}(1) = \frac{A}{2} \text{ for F} = \frac{3}{2} \end{cases}$$

The separation between the ground hyperfine states is $\delta E_{HFS} = \frac{3A}{2} \rightarrow \delta\lambda_T = 10.4 \times 10^{-5} \ nm$

For ^3T:

$$\Delta E_{HFS} = \begin{cases} \frac{A}{2}\left(-\frac{5}{2}\right) = -\frac{5A}{4} \text{ for F} = 1 \\ \frac{A}{2}\left(\frac{3}{2}\right) = \frac{3A}{4} \text{ for F} = 2 \end{cases}$$

The separation between the ground hyperfine states is $\delta E_{HFS} = 2A \rightarrow \delta\lambda_{HFS} = 13.8 \times 10^{-5} \ nm$.

Need resolution of 0.01 nm to resolve isotopic shifts, and resolution of $7 \times 10^{-5} \ nm$ ($< 100 \ fm$) to resolve hyperfine splitting.

3. (7 points) The energy of the ground state in lithium is $E = -5.39 \ eV$, while that for a Rydberg state of $n = 20$ is $E = -0.034$ eV. Determine the quantum defects for these states and compare their mean Bohr radii. For the Rydberg atom, estimate the effective charge experienced by the electron and justify your answer.

The ground state of the outermost electron is in $n = 2$.

The quantum defect δ_n is related to the energy by
$$E_n = -\frac{E_0}{n^{*2}} = -\frac{-13.605693 \ eV}{(n - \delta_2)^2}$$

Therefore, $\delta_2 = 0.4112$ and $\delta_{20} = -0.004$.

The Bohr radius is $r_{n^*} = n^{*2}a_0$, so $r(n^* = 1.5888) = 2.52 \ a_0$ and $r(n^* = 20.004) = 400.17 \ a_0$.

For the Rydberg atom with high n^* number, the outermost electron is on average far away from the nucleus. Due to the screening of the nuclear charge by the two core electrons, the effective charge is $Z_{eff} = 1$.

4. (20 points) Consider a rubidium-87 atom in a weak magnetic field.

(a) (6 points) Draw the energy diagram for transitions between the $5^2S_{1/2}$ and $5^2P_{3/2}$ states. Indicate and label all energy levels and the allowed transitions.

The transitions will split according to the anomalous Zeeman effect, in which each m_j sublevel will experience a shift in energy according to $\Delta E = \mu_B g_J m_j B$, where $g_J = \frac{3}{2} + \frac{s(s+1)-l(l+1)}{2j(j+1)}$. For $5^2S_{1/2}, g_j = 2$, while for $5^2P_{3/2}, g_j = 4/3$.

Allowed transitions are determined by $\Delta m_j = 0, \pm 1$. For clarity, only emission processes are shown in the energy level diagram below:
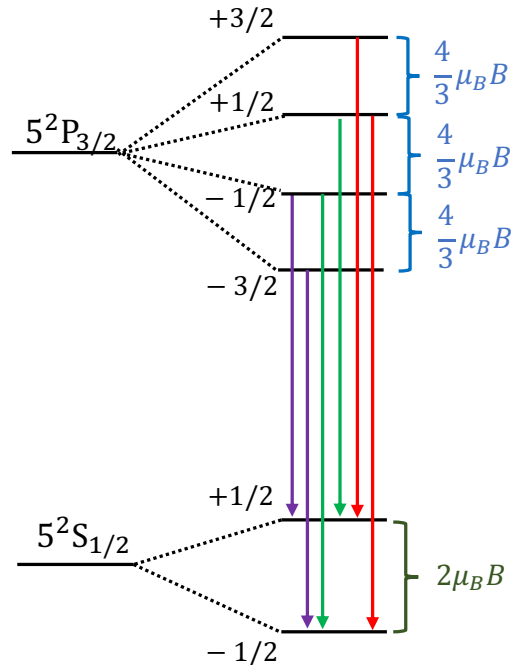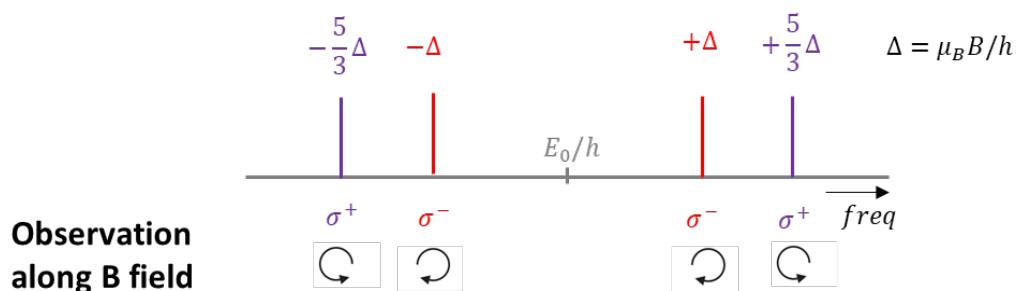

(b) (7 points) Draw the frequency spectrum (without worrying about the relative intensities of the lines) observed along the applied magnetic field and indicate the polarization of the spectral lines. Calculate the spacing between the spectral lines for an applied field of 1 G, in MHz.

When observing along the direction of the magnetic field, we can only observe the $\Delta m_j = \pm 1$ transitions, in which the emitted photon will have circular polarization around the quantization axis (direction of the field).
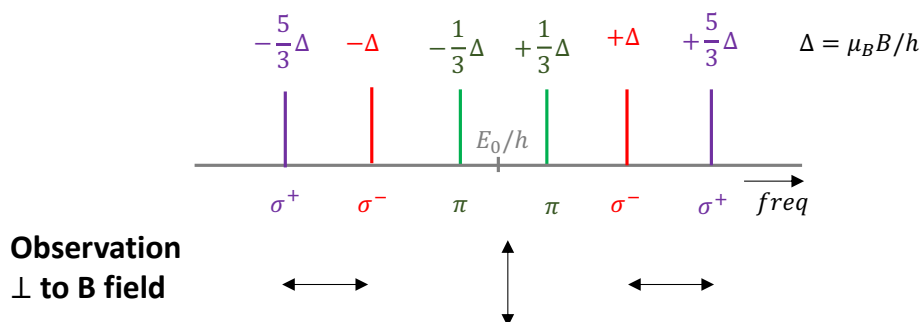The frequency splittings are calculated relative to the central frequency of the $5^2P_{3/2} \leftrightarrow 5^2S_{1/2}$ transition. For $B = 1$ G $= 10^{-4}$ T, $\Delta = \frac{\mu_B B}{h} = \frac{9.274\times10^{-24}\frac{J}{T}}{6.626\times10^{-34}J\cdot s}(10^{-4}T) = 1.3996\ MHz$.

$-\frac{5}{3}\Delta$ $-\Delta$ $+\Delta$ $+\frac{5}{3}\Delta$ $\Delta = \mu_B B/h$

E_0/h

σ^+ σ^- σ^- σ^+ $freq$

Observation along B field

(c) (7 points) Draw the frequency spectrum (without worrying about the relative intensities of the lines) observed perpendicular to the applied magnetic field and indicate the polarization of the spectral lines. Calculate the spacing between the spectral lines for an applied field of 1 G, in MHz.

When observing orthogonally to magnetic field, we can all transitions ($\Delta m_j = 0, \pm 1$) and the emitted photons will appear to have linear polarization.

$-\frac{5}{3}\Delta$ $-\Delta$ $-\frac{1}{3}\Delta$ $+\frac{1}{3}\Delta$ $+\Delta$ $+\frac{5}{3}\Delta$ $\Delta = \mu_B B/h$

E_0/h

σ^+ σ^- π π σ^- σ^+ $freq$

Observation ⊥ to B field

5. (3 points) Suppose you have an atomic clock with a fractional frequency uncertainty $\Delta v/v$ 5×10^{-12}. Calculate the maximum timing error (in seconds) that could accumulate over the period of one day. How about one month?

Since we are given the fractional frequency uncertainty, the timing error is just the fractional uncertainty times the duration:

Error in one day $= \frac{\Delta v}{v}(24 \times 60 \times 60)$ seconds $= 4.32 \times 10^{-7}\ s$

Error in one month $= \frac{\Delta v}{v}(24 \times 60 \times 60 \times 30)$ seconds $= 1.296 \times 10^{-5}\ s$

No extra problem for ECE 901 for this homework.