1

Consider free oscillations of a circular membrane of radius R which is fixed at its perimeter. The initial displacement of the membrane is given by $u_{t=0} = AJ_0(\mu_k r/R)$, where μ_k is a particular positive root of the equation $J_0(\mu) = 0$, where J_0 is the zeroth order Bessel function. The initial velocity of the membrane is zero, namely $\partial_t u|_{t=0} = 0$. Find the displacement $u(r, \phi, t)$ of the membrane at all times.

Consider azimuthal symmetry, the membrane is described by

$$\frac{1}{r}\partial_r(r\partial_r u) = \frac{1}{v^2}\partial_{tt}u \tag{1}$$

Taking separation $u = F(r)T(t)$,

$$\frac{1}{F(r)}\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}F}{\mathrm{d}r}\right) = \frac{1}{v^2 T(t)}\frac{\mathrm{d}^2T}{\mathrm{d}t^2} \equiv -\lambda^2 \tag{2}$$

we get two ODEs:

$$T'' + (\lambda v^2)T = 0; \quad r^2F'' + rF' + (\lambda r)^2 F = 0 \tag{3}$$

which carry general solutions

$$\begin{aligned} T(t) &= c_1\cos(\lambda vt) + c_2\sin(\lambda vt); \\ F(r) &= c_3 J_0(\lambda r) + c_4 Y_0(\lambda r) \end{aligned} \tag{4}$$

Desplacement finiteness at center requires $c_4 = 0$. The boundary condition at edge $r = R$ requires $F(R) = 0$. So

$$c_3 J_0(\lambda R) = 0 \quad \Rightarrow J_0(\lambda R) = 0 \tag{5}$$

This is an eigenvalue equation. Let μ_n be the n-th root of $J_0(x) = 0$, then

$$\lambda_n R = \mu_n \quad \Rightarrow \lambda_n = \frac{\mu_n}{R}, n = 1, 2... \tag{6}$$

The total solution is the superposition of all eigenmodes:

$$u(r,t) = \sum_{n=1}^{\infty} F_n(r)T_n(t) = \sum_{n=1}^{\infty} J_0\left(\frac{\mu_0 r}{R}\right)\left[A_n\cos\left(\frac{\mu_n vt}{R}\right) + B_n\sin\left(\frac{\mu_n vt}{R}\right)\right] \tag{7}$$

Initial condition reqires:
1. $\partial_t u(r,t)|_{t=0} = 0$, so

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = \sum_n B_n \mu_n \frac{v}{R} J_0\left(\mu_n \frac{r}{R}\right) = 0 \tag{8}$$

This implies all $B_n = 0$.

2. $u_{t=0} = AJ_0(\frac{\mu_k r}{R})$ so

$$\mu(r,0) = \sum_n A_n J_0\left(\frac{\mu_n r}{R}\right) = AJ_0\left(\frac{\mu_k r}{R}\right) \quad \Rightarrow A_n = A\delta_{nk} \tag{9}$$

Collecting results, we have

$$\mu(r,\varphi,t) = AJ_0\left(\frac{\mu_k r}{R}\right)\cos\left(\frac{v\mu_k t}{R}\right) \tag{10}$$

2

> Consider infinitely long cylinder of radius R. Its surface is kept at a constant temperature T_0. The initial temperature everywhere inside the cylinder is equation to zero. Find temperature inside the cylinder as a function of time.

Let the thermal diffusivity be denoted by α. With axial and azimuthal symmetry the heat equation reduces to

$$\partial_t u = \alpha \frac{1}{r}\partial_r(r\,\partial_r u), \qquad 0 \le r < R,\, t > 0, \tag{11}$$

with boundary: $u(R,t) = T_0$, and u finite at $r = 0$, initial data: $u(r,0) = 0$.

At steady state the solution is constant, $u_\infty(r) \equiv T_0$. Homogenize the boundary by

$$w(r,t) := T_0 - u(r,t). \tag{12}$$

Then

$$\partial_t w = \alpha \frac{1}{r}\partial_r(r\,\partial_r w), \quad w(R,t) = 0, \quad w(r,0) = T_0, \quad w \text{ finite at } r = 0. \tag{13}$$

Seek $w(r,t) = F(r)T(t)$:

$$\frac{1}{F}\frac{1}{r}\frac{d}{dr}\left(r\,\frac{dF}{dr}\right) = \frac{1}{\alpha T}\frac{dT}{dt} = -\lambda^2. \tag{14}$$

Hence

$$T'(t) = -\alpha\lambda^2 T(t) \Rightarrow T(t) = C\,e^{-\alpha\lambda^2 t}, \tag{15}$$

$$\frac{1}{r}\frac{d}{dr}\left(r\,\frac{dF}{dr}\right) + \lambda^2 F = 0 \Leftrightarrow F'' + \frac{1}{r}F' + \lambda^2 F = 0. \tag{16}$$

Let $x = \lambda r$ and $F(r) = y(x)$. Then y solves the Bessel equation of order 0:

$$y'' + \frac{1}{x}y' + y = 0 \quad \Rightarrow \quad y = AJ_0(x) + BY_0(x). \tag{17}$$

Finiteness at $r = 0$ forces $B = 0$. The boundary condition $w(R,t) = 0$ requires

$$F(R) = J_0(\lambda R) = 0. \tag{18}$$

Thus $\lambda = \lambda_n := \mu_n/R$, where μ_n is the n-th positive root of J_0. The eigenfunctions are

$$F_n(r) = J_0\!\left(\frac{\mu_n r}{R}\right), \qquad T_n(t) = e^{-\alpha(\mu_n/R)^2 t}. \tag{19}$$

Expand w:

$$w(r,t) = \sum_{n=1}^{\infty} A_n\, J_0\!\left(\frac{\mu_n r}{R}\right) e^{-\alpha\mu_n^2 t/R^2}. \tag{20}$$

Determine A_n from $w(r,0) = T_0$. Use orthogonality (weight r):

$$\int_0^R r\, J_0\!\left(\frac{\mu_m r}{R}\right) J_0\!\left(\frac{\mu_n r}{R}\right) dr = \frac{R^2}{2}\,[J_1(\mu_n)]^2\,\delta_{mn}. \tag{21}$$

Hence

$$A_n = \frac{2}{R^2[J_1(\mu_n)]^2}\int_0^R r\,T_0\,J_0\!\left(\frac{\mu_n r}{R}\right) dr. \tag{22}$$

Evaluate the integral using $d[xJ_1(x)]/dx = xJ_0(x)$ and $x = \mu_n r/R$:

$$\int_0^R r\, J_0\!\left(\frac{\mu_n r}{R}\right) dr = \frac{R^2}{\mu_n} J_1(\mu_n). \tag{23}$$

Therefore

$$A_n = \frac{2T_0}{\mu_n J_1(\mu_n)}. \tag{24}$$

Since $u = T_0 - w$,

$$\boxed{u(r,t) = T_0 \left[1 - \sum_{n=1}^{\infty} \frac{2}{\mu_n J_1(\mu_n)} J_0\!\left(\frac{\mu_n r}{R}\right) \exp\!\left(-\alpha \frac{\mu_n^2}{R^2}\, t \right) \right].} \tag{25}$$

3

> Solve Poisson equation $\Delta u = 0$ for the spherical shell with the inner radius $r = 1$ and outer radius $r = 2$ subject to the boundary conditions such that $u_{r=1} = 3\sin^2\theta\sin(2\phi)$ and $u_{r=2} = 3\cos\theta$.

Solve Laplace's equation in the shell $1 < r < 2$:

$$\nabla^2 u = 0, \qquad u(1,\theta,\phi) = 3\sin^2\theta\sin(2\phi), \quad u(2,\theta,\phi) = 3\cos\theta. \tag{26}$$

In spherical coordinates,

$$\nabla^2 u = \frac{1}{r^2}\partial_r(r^2 u_r) + \frac{1}{r^2\sin\theta}\partial_\theta(\sin\theta\, u_\theta) + \frac{1}{r^2\sin^2\theta}u_{\phi\phi}. \tag{27}$$

Let $u = R(r)\Theta(\theta)\Phi(\phi)$. Then

$$\frac{1}{R}\frac{d}{dr}(r^2R') + \frac{1}{\Theta}\frac{1}{\sin\theta}\frac{d}{d\theta}(\sin\theta\,\Theta') + \frac{1}{\Phi}\frac{1}{\sin^2\theta}\Phi'' = 0. \tag{28}$$

Set $\Phi'' + m^2\Phi = 0 \Rightarrow \Phi = \cos m\phi$ or $\Phi = \sin m\phi$ with $m \in \mathbb{Z}_{\geq 0}$. The polar equation becomes the associated Legendre ODE

$$\frac{1}{\sin\theta}\frac{d}{d\theta}(\sin\theta\,\Theta') - \frac{m^2}{\sin^2\theta}\Theta + \ell(\ell+1)\Theta = 0, \tag{29}$$

whose regular solutions are $\Theta(\theta) = P_\ell^m(\cos\theta)$ with $\ell \geq m$. The radial equation is

$$r^2R'' + 2rR' - \ell(\ell+1)R = 0 \quad \Rightarrow \quad R_\ell(r) = A_{\ell m}r^\ell + B_{\ell m}r^{-(\ell+1)}. \tag{30}$$

Thus the relevant real expansion is

$$u(r,\theta,\phi) = \sum_{\ell,m}\left(A_{\ell m}r^\ell + B_{\ell m}r^{-(\ell+1)}\right)P_\ell^m(\cos\theta)\begin{cases}\cos m\phi\\ \sin m\phi\end{cases}. \tag{31}$$

Use the Legendre polynomial expressions:

$$P_1(\cos\theta) = \cos\theta, \qquad P_2^2(\cos\theta) = 3(1-\cos^2\theta) = 3\sin^2\theta. \tag{32}$$

Hence $3\cos\theta \implies (\ell, m) = (1,0)$, and $3\sin^2\theta\sin(2\phi) = P_2^2(\cos\theta)\sin(2\phi) \implies (\ell,m) = (2,2)$.

Therefore

$$u = (Ar + Br^{-2})P_1(\cos\theta) + \left(Cr^2 + Dr^{-3}\right)P_2^2(\cos\theta)\sin(2\phi). \tag{33}$$

Matching the boundary conditions:

At $r = 1$:

$$(A+B)P_1 + (C+D)P_2^2\sin(2\phi) = P_2^2\sin(2\phi) \Rightarrow A + B = 0, \quad C + D = 1. \tag{34}$$

At $r = 2$:

$$\left(2A + \tfrac{1}{4}B\right)P_1 + \left(4C + \tfrac{1}{8}D\right)P_2^2\sin(2\phi) = 3P_1 \Rightarrow 2A + \tfrac{1}{4}B = 3, \quad 4C + \tfrac{1}{8}D = 0. \tag{35}$$

Solving these linear equations gives

$$A = \frac{12}{7}, \quad B = -\frac{12}{7}, \qquad C = -\frac{1}{31}, \quad D = \frac{32}{31}. \tag{36}$$

We thus arrive at

$$\boxed{u(r,\theta,\phi) = \frac{12}{7}\left(r - \frac{1}{r^2}\right)\cos\theta + \frac{1}{31}\left(\frac{96}{r^3} - 3r^2\right)\sin^2\theta\sin(2\phi), \quad 1 < r < 2.} \tag{37}$$

4

Consider oscillations of a one-dimensional string with fixed ends driven by a distributed time dependent force $f(x,t)$. The displacement of the string is described by the wave equation $\partial_t^2 u = s^2 \partial_x^2 u + f(x,t)$. The initial displacements and velocities are given by respectively $u_{t=0} = u_0(x)$ and $\partial_t u|_{t=0} = u_1(x)$. The boundary conditions are $u_{x=0} = u_{x=l} = 0$. Find $u(x,t)$ and express expansion coefficients as Fourier integrals.

This problem involves a linear operator ∂_x^2 with homogeneous Dirichlet boundary conditions ($u(0,t) = u(l,t) = 0$). This suggests an expansion in the corresponding eigenfunctions, which are $\sin(n\pi x/l)$.

We seek a solution in the form of a sine series where the coefficients depend on time:

$$u(x,t) = \sum_{n=1}^{\infty} g_n(t) \sin\left(\frac{n\pi x}{l}\right) \tag{38}$$

We also expand the distributed force $f(x,t)$ in the same basis:

$$f(x,t) = \sum_{n=1}^{\infty} f_n(t) \sin\left(\frac{n\pi x}{l}\right) \tag{39}$$

where the Fourier coefficients $f_n(t)$ are given by the standard projection formula:

$$f_n(t) = \frac{2}{l} \int_0^l f(x,t) \sin\left(\frac{n\pi x}{l}\right) dx \tag{40}$$

Substitute the series for $u(x,t)$ and $f(x,t)$ into the governing wave equation $\partial_t^2 u = s^2 \partial_x^2 u + f(x,t)$.

$$\sum_{n=1}^{\infty} g_{(n)''}(t) \sin\left(\frac{n\pi x}{l}\right) = s^2 \sum_{n=1}^{\infty} g_n(t) \left[-\left(\frac{n\pi}{l}\right)^2 \sin\left(\frac{n\pi x}{l}\right) \right] + \sum_{n=1}^{\infty} f_n(t) \sin\left(\frac{n\pi x}{l}\right) \tag{41}$$

Combine terms under a single summation:

$$\sum_{n=1}^{\infty} \left[g_{(n)''}(t) + \left(\frac{sn\pi}{l}\right)^2 g_n(t) - f_n(t) \right] \sin\left(\frac{n\pi x}{l}\right) = 0 \tag{42}$$

Due to the orthogonality of the sine functions on $[0,l]$, this equation can only hold if the coefficient of each sine term is individually zero. This yields an independent ODE for each mode n.

Defining the natural frequencies $\omega_n = sn\pi/l$, we have:

$$g_{(n)''}(t) + \omega_n^2 g_n(t) = f_n(t) \tag{43}$$

Its general solution is the sum of the homogeneous solution and a particular solution found using Green's function :

$$g_n(t) = \underbrace{A_n \cos(\omega_n t) + B_n \sin(\omega_n t)}_{\text{Homogeneous Sol.}} + \underbrace{\int_0^t \frac{\sin[\omega_n (t-\tau)]}{\omega_n} f_n(\tau) d\tau}_{\text{Particular Sol.}} \tag{44}$$

The constants A_n and B_n are determined by the initial conditions.

We expand the initial conditions $u_0(x)$ and $u_1(x)$ in the same sine series basis:

$$u(x,0) = u_0(x) = \sum_{n=1}^{\infty} u_{0n} \sin\left(\frac{n\pi x}{l}\right) \Longrightarrow u_{0n} = \frac{2}{l} \int_0^l u_0(x) \sin\left(\frac{n\pi x}{l}\right) dx \tag{45}$$

$$\partial_t u(x,0) = u_1(x) = \sum_{n=1}^{\infty} u_{1n} \sin\left(\frac{n\pi x}{l}\right) \Longrightarrow u_{1n} = \frac{2}{l} \int_0^l u_1(x) \sin\left(\frac{n\pi x}{l}\right) dx \tag{46}$$

From our series solution for $u(x,t)$, we have $u(x,0) = \sum g_n(0)\sin(...)$ and $\partial_t u(x,0) = \sum g_{(n)'}(0)\sin(...)$. By comparing coefficients, we must have $g_n(0) = u_{0n}$ and $g_{(n)'}(0) = u_{1n}$.

1. At $t = 0$, the integral in the solution for $g_n(t)$ vanishes, giving:

$$g_n(0) = A_n \cos(0) + B_n \sin(0) = A_n. \tag{47}$$

Thus, $A_n = u_{0n}$.

2. Differentiating $g_n(t)$ (using the Leibniz rule for the integral) gives:

$$g_{(n)'}(t) = -\omega_n A_n \sin(\omega_n t) + \omega_n B_n \cos(\omega_n t) + \int_0^t \cos[\omega_n(t-\tau)] f_n(\tau) d\tau. \tag{48}$$

At $t = 0$, this simplifies to $g_{(n)'}(0) = \omega_n B_n$. Thus, $\omega_n B_n = u_{1n}$, which means $B_n = u_{1n}/\omega_n$.

Assembling the pieces gives the complete solution

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{l}\right)\left[A_n \cos(\omega_n t) + \frac{B_n}{\omega_n}\sin(\omega_n t) + \int_0^t \frac{\sin[\omega_n(t-\tau)]}{\omega_n} f_n(\tau) d\tau\right]} \tag{49}$$

where $\omega_n = \frac{sn\pi}{l}$ and the coefficients are the following Fourier integrals:

$$A_n = \frac{2}{l}\int_0^l u_0(x') \sin\left(\frac{n\pi x'}{l}\right) dx' \tag{50}$$

$$B_n = \frac{2}{l}\int_0^l u_1(x') \sin\left(\frac{n\pi x'}{l}\right) dx' \tag{51}$$

$$f_n(\tau) = \frac{2}{l}\int_0^l f(x',\tau) \sin\left(\frac{n\pi x'}{l}\right) dx' \tag{52}$$

5

In some cases wave equation acquires higher order gradient terms such as in the case of bending oscillations of a solid rode with the fixed endpoint. Consider such a PDE in the form $\partial_t^2 y + a^2 \partial_x^4 y = 0$. The boundary conditions are as follows, on one end $y|_{x=0} = \partial_x y|_{x=0} = 0$ and on the other end $\partial_x^2 y|_{x=l} = \partial_x^3 y|_{x=l} = 0$ for $0 < x < l$. The initial conditions are as follows $y|_{t=0} = f(x)$ and $\partial_t y|_{t=0} = g(x)$. By using the method of separation of variables, reduced this PDE to the corresponding Sturm-Liouville problem and find an equation for the corresponding eigenvalues and eigenfunction. Find the generic solution for this boundary problem.

We solve the PDE for bending oscillations of a solid rod, given by

$$\frac{\partial^2 y}{\partial t^2} + a^2 \frac{\partial^4 y}{\partial x^4} = 0. \tag{53}$$

We assume a separable solution of the form $y(x,t) = X(x)T(t)$. Substituting this into the PDE yields

$$X(x)T''(t) + a^2 X^{(4)}(x)T(t) = 0. \tag{54}$$

Dividing by $a^2 X(x)T(t)$ allows us to separate the variables. We introduce a separation constant $-\lambda^4$ for convenience, ensuring oscillatory solutions in time.

$$\frac{T''(t)}{a^2 T(t)} = -\frac{X^{(4)}(x)}{X(x)} = -\lambda^4. \tag{55}$$

This results in two ordinary differential equations:

$$\begin{aligned} T''(t) + (a\lambda^2)^2 T(t) &= 0 \\ X^{(4)}(x) - \lambda^4 X(x) &= 0 \end{aligned} \tag{56}$$

The general solution to the spatial ODE is

$$X(x) = C_1 \cos(\lambda x) + C_2 \sin(\lambda x) + C_3 \cosh(\lambda x) + C_4 \sinh(\lambda x). \tag{57}$$

The boundary conditions at the clamped end ($x = 0$) are $X(0) = 0$ and $X'(0) = 0$.

$$\begin{aligned} X(0) = 0 &\implies C_1 + C_3 = 0 \implies C_3 = -C_1 \\ X'(0) = 0 &\implies \lambda(C_2 + C_4) = 0 \implies C_4 = -C_2 \end{aligned} \tag{58}$$

The solution simplifies to

$$X(x) = C_1(\cos(\lambda x) - \cosh(\lambda x)) + C_2(\sin(\lambda x) - \sinh(\lambda x)). \tag{59}$$

The boundary conditions at the free end ($x = l$) are $X''(l) = 0$ and $X'''(l) = 0$. This gives a homogeneous system for C_1 and C_2:

$$\begin{aligned} C_1(\cos(\lambda l) + \cosh(\lambda l)) + C_2(\sin(\lambda l) + \sinh(\lambda l)) &= 0 \\ C_1(\sin(\lambda l) - \sinh(\lambda l)) - C_2(\cos(\lambda l) + \cosh(\lambda l)) &= 0 \end{aligned} \tag{60}$$

For a non-trivial solution, the determinant of the coefficient matrix must be zero.

$$\det \begin{pmatrix} \cos(\lambda l) + \cosh(\lambda l) & \sin(\lambda l) + \sinh(\lambda l) \\ \sin(\lambda l) - \sinh(\lambda l) & -(\cos(\lambda l) + \cosh(\lambda l)) \end{pmatrix} = 0 \tag{61}$$

This yields the transcendental eigenvalue equation, whose roots define the eigenvalues λ_n:

$$\boxed{\cos(\lambda_n l)\cosh(\lambda_n l) = -1} \tag{62}$$

For each eigenvalue λ_n that satisfies this equation, the coefficients C_1 and C_2 are linearly dependent. We can find their ratio from the first homogeneous equation:

$$\frac{C_2}{C_1} = -\frac{\cos(\lambda_n l) + \cosh(\lambda_n l)}{\sin(\lambda_n l) + \sinh(\lambda_n l)}. \tag{63}$$

The corresponding eigenfunction $X_n(x)$ is unique up to a normalization constant. By setting this constant to 1, we obtain the explicit form:

$$\boxed{X_n(x) = (\cos(\lambda_n x) - \cosh(\lambda_n x)) - \left(\frac{\cos(\lambda_n l)+\cosh(\lambda_n l)}{\sin(\lambda_n l)+\sinh(\lambda_n l)}\right)(\sin(\lambda_n x) - \sinh(\lambda_n x))} \tag{64}$$

The General Solution

The solution to the temporal ODE is $T_n(t) = C_n \cos(\omega_n t) + D_n \sin(\omega_n t)$, where the angular frequencies are $\omega_n = a\lambda_n^2$. The general solution for $y(x,t)$ is a superposition of all modes:

$$y(x,t) = \sum_{n=1}^{\infty} X_n(x)[C_n \cos(\omega_n t) + D_n \sin(\omega_n t)]. \tag{65}$$

The coefficients are determined from the initial conditions, $y(x,0) = f(x)$ and $\partial_t y(x,0) = g(x)$, using the orthogonality of the eigenfunctions $X_n(x)$.

$$\begin{aligned} y(x,0) = f(x) &= \sum_{n=1}^{\infty} C_n X_n(x) \\ \partial_t y(x,0) = g(x) &= \sum_{n=1}^{\infty} \omega_n D_n X_n(x) \end{aligned} \tag{66}$$

Projecting onto the basis functions gives the coefficients:

$$C_n = \frac{\int_0^l f(x) X_n(x) dx}{\int_0^l X_n^2(x) dx} \tag{67}$$

$$D_n = \frac{1}{\omega_n} \frac{\int_0^l g(x) X_n(x) dx}{\int_0^l X_n^2(x) dx} \tag{68}$$