Problems

1. **Projector and Measurement Along an Arbitrary Axis**
   A spin-1/2 system is prepared in the 'up' eigenstate of $\mathbf{S} \cdot \hat{\mathbf{n}}$.

   - Write the projector onto the 'up' outcome $(+\hbar/2)$ of a measurement of $\mathbf{S} \cdot \hat{\mathbf{b}}$.
   - Hence, compute the probability of measuring $\pm\hbar/2$ for the spin component along the $\hat{\mathbf{b}}$ axis.

2. **Quick Diagonalization of a 2×2 Hamiltonian**
   Consider the Hermitian Hamiltonian

   $$\mathbf{H} = \begin{pmatrix} E_0 + \delta & g \\ g & E_0 - \delta \end{pmatrix},$$

   where E_0, δ, and g are real constants.

   - Rewrite $\mathbf{H}$ in the Pauli basis, $\mathbf{H} = a_0\mathbf{I} + \mathbf{a} \cdot \boldsymbol{\sigma}$, and identify the scalar a_0 and the vector $\mathbf{a}$.
   - Find the eigenvalues and normalized eigenvectors of $\mathbf{H}$. Express the eigenvectors as spinors aligned with a unit vector $\hat{\mathbf{n}}$ in terms of a mixing angle β.

3. **Time Evolution under a Pauli Hamiltonian**
   Let the Hamiltonian of a system be

   $$H = E_0\mathbf{I} + \frac{\hbar\Omega}{2}\hat{\mathbf{n}} \cdot \boldsymbol{\sigma}.$$

   - Compute the unitary time-evolution operator $U(t) = e^{-iHt/\hbar}$ in a closed form. You may use the identity $(\hat{\mathbf{n}} \cdot \boldsymbol{\sigma})^2 = \mathbf{I}$.
   - If the system starts in the state $|+\rangle$ (eigenstate of S_z with eigenvalue $+\hbar/2$) at $t = 0$, find the probability of measuring $S_z = +\hbar/2$ at a later time t, given that $\hat{\mathbf{n}} = \hat{\mathbf{x}}$.

1. **Self-Adjointness Check (Sturm–Liouville Form)**
   Work on the space $L^2([-1,1])$ with the standard inner product $\langle f|g\rangle = \int_{-1}^{1} f^*(x)g(x)\,dx$.

   - For a general second-order differential operator $L = a(x)\frac{d^2}{dx^2} + b(x)\frac{d}{dx} + c(x)$, state the condition on $b(x)$ that allows L to be written in the formally self-adjoint Sturm–Liouville form.
   - For the Legendre operator, where $a(x) = -(1-x^2)$ and $c(x) = 0$, find the required $b(x)$ and state the minimal boundary condition on functions in the domain of L that ensures it is self-adjoint on $[-1,1]$.

2. **Commutators with Functions of Momentum**
   Using only the canonical commutation relation (CCR) $[x_i, p_j] = i\hbar\delta_{ij}$, evaluate the following commutators:

   (a) $[x_i, e^{\alpha p_i}]$ for a real constant α.

   (b) $[x_i, f(\mathbf{p}^2)]$ where f is an analytic function. Use this general result to find $[x_i, \mathbf{p}^2]$ and $[x_i, e^{-\lambda \mathbf{p}^2}]$.

3. **Translations and Expectation Values**
   The finite translation operator is given by $T(\boldsymbol{\ell}) = \exp\left(-\frac{i}{\hbar}\boldsymbol{\ell}\cdot\mathbf{p}\right)$.

   (a) Compute the commutator $[x_j, T(\boldsymbol{\ell})]$.

   (b) Show that $T^\dagger(\boldsymbol{\ell})\,\mathbf{x}\,T(\boldsymbol{\ell}) = \mathbf{x} + \boldsymbol{\ell}$. Use this to show that for any normalized state $|\alpha\rangle$, the expectation value of position in the translated state $T(\boldsymbol{\ell})|\alpha\rangle$ is shifted by $\boldsymbol{\ell}$.

1. **Finite Square Well — Quantization and Counting States**
   Consider the symmetric finite square well $V(x) = 0$ for $|x| < a$ and $V(x) = V_0$ for $|x| > a$, with $V_0 > 0$. For bound states ($E < V_0$), define the dimensionless quantities $\xi = a\sqrt{2mE}/\hbar$ and $R = a\sqrt{2mV_0}/\hbar$.

   (a) Write down the transcendental equations that determine the energies of the even and odd parity bound states.

   (b) Show that an odd-parity bound state can only exist if the "strength" of the well satisfies $R > \pi/2$.

   (c) For a well with strength $R = 4$, determine the total number of bound states and specify their parities.

2. **SHO Matrix Elements and Ground-State Variance**
   For the 1D simple harmonic oscillator, the position and momentum operators can be written as:

$$x = \sqrt{\frac{\hbar}{2m\omega}}(a + a^\dagger), \qquad p = i\sqrt{\frac{m\omega\hbar}{2}}(a^\dagger - a).$$

   (a) Using the properties of the creation and annihilation operators, derive the matrix elements $\langle n'|x|n\rangle$ and $\langle n'|p|n\rangle$ in the energy eigenbasis. State the selection rules.

   (b) Use this formalism to evaluate the ground-state variance of position, $\langle 0|x^2|0\rangle$.

3. **Attractive Delta Potential — Bound State**
   Consider a particle in a one-dimensional attractive delta-function potential, $V(x) = -\lambda\delta(x)$ with $\lambda > 0$.

   (a) Solve for the energy E of the bound state and find the corresponding normalized wavefunction $\psi(x)$.

   (b) Compute the expectation values $\langle x\rangle$ and $\langle p\rangle$ for this bound state.

Solutions

Solution 1: The spin operator is $\mathbf{S} = (\hbar/2)\boldsymbol{\sigma}$. The eigenvalues of $\mathbf{S}\cdot\hat{\mathbf{b}}$ are $\pm\hbar/2$, which correspond to eigenvalues of ± 1 for the operator $\boldsymbol{\sigma}\cdot\hat{\mathbf{b}}$.

- Using the spectral representation $A = \sum_i \omega_i \Lambda_i$, the projector onto an eigenstate is $\Lambda_i = |\omega_i\rangle\langle\omega_i|$. For a two-level system like spin-1/2, the projectors for $\boldsymbol{\sigma}\cdot\hat{\mathbf{b}}$ are $P_\pm = \frac{1}{2}(\mathbf{I}\pm\boldsymbol{\sigma}\cdot\hat{\mathbf{b}})$. The projector onto the $+\hbar/2$ outcome is therefore:

$$\boxed{P_{+\hbar/2} = \frac{1}{2}(\mathbf{I} + \boldsymbol{\sigma}\cdot\hat{\mathbf{b}})}$$

- The initial state is the 'up' eigenstate of $\mathbf{S}\cdot\hat{\mathbf{n}}$, which means the expectation value of the spin vector in this state is $\langle\mathbf{S}\rangle = (\hbar/2)\hat{\mathbf{n}}$, or $\langle\boldsymbol{\sigma}\rangle = \hat{\mathbf{n}}$. The probability of an outcome is the expectation value of its projector.

$$P(\pm\hbar/2 \text{ along } \hat{\mathbf{b}}) = \langle P_{\pm\hbar/2}\rangle = \left\langle \frac{1}{2}(\mathbf{I}\pm\boldsymbol{\sigma}\cdot\hat{\mathbf{b}})\right\rangle = \frac{1}{2}(1\pm\langle\boldsymbol{\sigma}\rangle\cdot\hat{\mathbf{b}})$$

Substituting $\langle\boldsymbol{\sigma}\rangle = \hat{\mathbf{n}}$, we get:

$$\boxed{P(\pm\hbar/2 \text{ along } \hat{\mathbf{b}}) = \frac{1}{2}(1\pm\hat{\mathbf{n}}\cdot\hat{\mathbf{b}})}$$

Solution 2:

- Any 2×2 matrix can be expanded as $\mathbf{M} = a_0\mathbf{I} + \mathbf{a}\cdot\boldsymbol{\sigma}$. We find $a_0 = \frac{1}{2}\mathrm{Tr}(\mathbf{M})$ and $a_k = \frac{1}{2}\mathrm{Tr}(\sigma_k\mathbf{M})$. For the given Hamiltonian $\mathbf{H}$:

$$a_0 = \frac{1}{2}\mathrm{Tr}\begin{pmatrix} E_0+\delta & g \\ g & E_0-\delta \end{pmatrix} = \frac{1}{2}(E_0+\delta+E_0-\delta) = E_0.$$

$$a_x = \frac{1}{2}\mathrm{Tr}(\sigma_x\mathbf{H}) = \frac{1}{2}\mathrm{Tr}\begin{pmatrix} g & E_0-\delta \\ E_0+\delta & g \end{pmatrix} = g.$$

$$a_y = \frac{1}{2}\mathrm{Tr}(\sigma_y\mathbf{H}) = \frac{1}{2}\mathrm{Tr}\begin{pmatrix} -ig & -i(E_0-\delta) \\ i(E_0+\delta) & ig \end{pmatrix} = 0.$$

$$a_z = \frac{1}{2}\mathrm{Tr}(\sigma_z\mathbf{H}) = \frac{1}{2}\mathrm{Tr}\begin{pmatrix} E_0+\delta & -g \\ g & -(E_0-\delta) \end{pmatrix} = \delta.$$

So, $\mathbf{H} = E_0\mathbf{I} + (g\sigma_x + \delta\sigma_z)$, with $\boxed{a_0 = E_0, \mathbf{a} = (g, 0, \delta)}$.

- The eigenvalues of $a_0\mathbf{I} + \mathbf{a}\cdot\boldsymbol{\sigma}$ are $a_0 \pm |\mathbf{a}|$.

$$\boxed{E_\pm = E_0 \pm \sqrt{g^2+\delta^2}}$$

The eigenvectors are the eigenstates of $\boldsymbol{\sigma}\cdot\hat{\mathbf{n}}$, where $\hat{\mathbf{n}} = \mathbf{a}/|\mathbf{a}|$. Let $\hat{\mathbf{n}} = (\sin\beta, 0, \cos\beta)$. Then $\cos\beta = \delta/\sqrt{g^2+\delta^2}$ and $\sin\beta = g/\sqrt{g^2+\delta^2}$. The corresponding normalized eigenvectors are:

$$\boxed{|+\rangle_{\hat{n}} = \begin{pmatrix}\cos(\beta/2)\\ \sin(\beta/2)\end{pmatrix}, \quad |-\rangle_{\hat{n}} = \begin{pmatrix}\sin(\beta/2)\\ -\cos(\beta/2)\end{pmatrix}}$$

Solution 3:

- First, factor out the identity part: $U(t) = e^{-iE_0t/\hbar}\exp\left(-i\frac{\Omega t}{2}\hat{\mathbf{n}}\cdot\boldsymbol{\sigma}\right)$. Let $\theta = \Omega t/2$. We use the Taylor series expansion and the property $(\hat{\mathbf{n}}\cdot\boldsymbol{\sigma})^{2k} = \mathbf{I}$ and $(\hat{\mathbf{n}}\cdot\boldsymbol{\sigma})^{2k+1} = \hat{\mathbf{n}}\cdot\boldsymbol{\sigma}$.

$$e^{-i\theta\hat{\mathbf{n}}\cdot\boldsymbol{\sigma}} = \sum_{k=0}^{\infty}\frac{(-i\theta)^k}{k!}(\hat{\mathbf{n}}\cdot\boldsymbol{\sigma})^k = \left(\sum_{k\text{ even}}\frac{(-i\theta)^k}{k!}\right)\mathbf{I} + \left(\sum_{k\text{ odd}}\frac{(-i\theta)^k}{k!}\right)\hat{\mathbf{n}}\cdot\boldsymbol{\sigma}$$

This simplifies to $\cos(\theta)\mathbf{I} - i\sin(\theta)\hat{\mathbf{n}}\cdot\boldsymbol{\sigma}$. Therefore:

$$\boxed{U(t) = e^{-iE_0t/\hbar}\left[\cos\left(\frac{\Omega t}{2}\right)\mathbf{I} - i\sin\left(\frac{\Omega t}{2}\right)\hat{\mathbf{n}}\cdot\boldsymbol{\sigma}\right]}$$

- The state at time t is $|\psi(t)\rangle = U(t)|+\rangle$. With $\hat{\mathbf{n}} = \hat{\mathbf{x}}$:

$$|\psi(t)\rangle = e^{-iE_0t/\hbar}\left[\cos\left(\frac{\Omega t}{2}\right)\mathbf{I} - i\sin\left(\frac{\Omega t}{2}\right)\sigma_x\right]|+\rangle$$

  Since $\sigma_x|+\rangle = |-\rangle$, we have:

$$|\psi(t)\rangle = e^{-iE_0t/\hbar}\left(\cos\left(\frac{\Omega t}{2}\right)|+\rangle - i\sin\left(\frac{\Omega t}{2}\right)|-\rangle\right)$$

  The probability of measuring $S_z = +\hbar/2$ is $|\langle +|\psi(t)\rangle|^2$.

$$\langle +|\psi(t)\rangle = e^{-iE_0t/\hbar}\cos\left(\frac{\Omega t}{2}\right)$$

$$\boxed{P(S_z = +\hbar/2 \text{ at } t) = \cos^2\left(\frac{\Omega t}{2}\right)}$$

Solution 4:

- For L to be formally self-adjoint, the non-self-adjoint first-derivative term must be absorbable into the second-derivative term. This happens if L can be written as $L = \frac{d}{dx}\left(a(x)\frac{d}{dx}\right) + c(x)$. Expanding this gives $\frac{d}{dx}\left(a(x)\frac{d}{dx}\right) = a(x)\frac{d^2}{dx^2} + a'(x)\frac{d}{dx}$. Comparing this to the original form of L, we find the condition is $\boxed{b(x) = a'(x)}$.
- With $a(x) = -(1-x^2)$, the condition gives $b(x) = a'(x) = 2x$. The operator is $L = -(1-x^2)\frac{d^2}{dx^2} + 2x\frac{d}{dx}$. To check for self-adjointness, we examine the boundary terms from integration by parts:

$$\langle f|Lg\rangle - \langle Lf|g\rangle = [a(x)(f^*g' - (f^*)'g)]_{-1}^{1}$$

  Since $a(x) = -(1-x^2)$, we have $a(\pm 1) = 0$. Therefore, the boundary term vanishes automatically as long as the functions $f(x)$ and $g(x)$ (and their derivatives) are finite at the endpoints $x = \pm 1$. This is the minimal boundary condition.

$$\boxed{b(x) = 2x, \quad \text{Boundary Condition: functions must be finite at } x = \pm 1.}$$

Solution 5:

(a) We use the identity $[A, e^B] = [A, B]e^B$ if $[A, B]$ commutes with B. Here, $[x_i, \alpha p_i] = i\hbar\alpha$, which is a c-number and commutes with p_i. So the identity applies.

$$\boxed{[x_i, e^{\alpha p_i}] = i\hbar\alpha e^{\alpha p_i}}$$

Alternatively, expand the exponential: $[x_i, \sum_n \frac{(\alpha p_i)^n}{n!}] = \sum_n \frac{\alpha^n}{n!}[x_i, p_i^n] = \sum_n \frac{\alpha^n}{n!}(i\hbar n p_i^{n-1}) = i\hbar\alpha\sum_n \frac{(\alpha p_i)^{n-1}}{(n-1)!} = i\hbar\alpha e^{\alpha p_i}$.

(b) For an analytic function $G(\mathbf{p})$, one can prove by power series that $[x_i, G(\mathbf{p})] = i\hbar\frac{\partial G}{\partial p_i}$. Let $G(\mathbf{p}) = f(\mathbf{p}^2)$. Using the chain rule:

$$\frac{\partial}{\partial p_i}f(\mathbf{p}^2) = f'(\mathbf{p}^2)\frac{\partial(\mathbf{p}^2)}{\partial p_i} = f'(\mathbf{p}^2)(2p_i).$$

Therefore, $\boxed{[x_i, f(\mathbf{p}^2)] = 2i\hbar p_i f'(\mathbf{p}^2)}$. For $f(s) = s$, $f'(s) = 1$, so $\boxed{[x_i, \mathbf{p}^2] = 2i\hbar p_i}$. For $f(s) = e^{-\lambda s}$, $f'(s) = -\lambda e^{-\lambda s}$, so $\boxed{[x_i, e^{-\lambda\mathbf{p}^2}] = -2i\hbar\lambda p_i e^{-\lambda\mathbf{p}^2}}$.

Solution 6:

(a) Using the result from the previous problem, $[x_j, G(\mathbf{p})] = i\hbar\frac{\partial G}{\partial p_j}$, with $G(\mathbf{p}) = T(\boldsymbol{\ell})$:

$$[x_j, T(\boldsymbol{\ell})] = i\hbar\frac{\partial}{\partial p_j}\exp\left(-\frac{i}{\hbar}\boldsymbol{\ell}\cdot\mathbf{p}\right) = i\hbar\left(-\frac{i}{\hbar}\ell_j\right)T(\boldsymbol{\ell}) = \ell_j T(\boldsymbol{\ell}).$$

$$\boxed{[x_j, T(\boldsymbol{\ell})] = \ell_j T(\boldsymbol{\ell})}$$

(b) We use the identity $U^\dagger AU = A + U^\dagger[A, U]$. Here $T(\boldsymbol{\ell})$ is unitary, so $T^\dagger T = \mathbf{I}$.

$$T^\dagger(\boldsymbol{\ell})x_jT(\boldsymbol{\ell}) = T^\dagger(\boldsymbol{\ell})(T(\boldsymbol{\ell})x_j - [T(\boldsymbol{\ell}), x_j]) = x_j - T^\dagger(\boldsymbol{\ell})[T(\boldsymbol{\ell}), x_j]$$

Since $[T, x_j] = -[x_j, T] = -\ell_j T$, we get:

$$T^\dagger(\boldsymbol{\ell}) x_j T(\boldsymbol{\ell}) = x_j - T^\dagger(\boldsymbol{\ell})(-\ell_j T(\boldsymbol{\ell})) = x_j + \ell_j T^\dagger(\boldsymbol{\ell}) T(\boldsymbol{\ell}) = x_j + \ell_j.$$

In vector form, $\boxed{T^\dagger(\boldsymbol{\ell})\, \mathbf{x}\, T(\boldsymbol{\ell}) = \mathbf{x} + \boldsymbol{\ell}}$. The expectation value in the translated state $|\alpha_{\boldsymbol{\ell}}\rangle = T(\boldsymbol{\ell})|\alpha\rangle$ is:

$$\langle \mathbf{x} \rangle_{\boldsymbol{\ell}} = \langle \alpha_{\boldsymbol{\ell}} | \mathbf{x} | \alpha_{\boldsymbol{\ell}} \rangle = \langle \alpha | T^\dagger(\boldsymbol{\ell}) \mathbf{x} T(\boldsymbol{\ell}) | \alpha \rangle = \langle \alpha | (\mathbf{x} + \boldsymbol{\ell}) | \alpha \rangle = \langle \mathbf{x} \rangle + \boldsymbol{\ell}.$$

Solution 7: (a) Let $\eta = a\sqrt{2m(V_0 - E)}/\hbar$. Matching the wavefunctions and their derivatives at $x = a$ for states with definite parity gives the transcendental equations:

$$\boxed{\text{Even:} \quad \xi \tan \xi = \eta} \qquad \boxed{\text{Odd:} \quad -\xi \cot \xi = \eta}$$

These must be solved simultaneously with the constraint $\xi^2 + \eta^2 = R^2$.

(b) For odd states, the equation is $\eta = -\xi \cot \xi$. Since η must be real and positive for a bound state, we must have $-\cot \xi > 0$, which means $\cot \xi < 0$. This condition holds for ξ in the intervals $(\pi/2, \pi), (3\pi/2, 2\pi), \ldots$. The lowest possible value for ξ is just above $\pi/2$. For a solution to exist, the circle $\xi^2 + \eta^2 = R^2$ must intersect the curve $\eta = -\xi \cot \xi$. This requires the radius R to be at least as large as the starting value of ξ, so we must have $\boxed{R > \pi/2}$.

(c) We count the number of intersections graphically. The circle has radius $R = 4$.

- **Even states** ($\xi \tan \xi = \eta$): Solutions exist in intervals $(0, \pi/2), (\pi, 3\pi/2), \ldots$. $R = 4$ is greater than 0 and $\pi \approx 3.14$. It is less than $2\pi \approx 6.28$. So there are intersections in the first two even-state intervals. $\Longrightarrow$ **2 even states**.
- **Odd states** ($-\xi \cot \xi = \eta$): Solutions exist in intervals $(\pi/2, \pi), (3\pi/2, 2\pi), \ldots$. $R = 4$ is greater than $\pi/2 \approx 1.57$. It is less than $3\pi/2 \approx 4.71$. So there is only an intersection in the first odd-state interval. $\Longrightarrow$ **1 odd state**.

In total, for $R = 4$, there are $\boxed{\text{3 bound states (even, odd, even)}}$.

Solution 8: (a) We use $a|n\rangle = \sqrt{n}|n-1\rangle$ and $a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$.

$$\langle n'|x|n\rangle = \sqrt{\frac{\hbar}{2m\omega}} \langle n'|a + a^\dagger|n\rangle = \sqrt{\frac{\hbar}{2m\omega}} (\sqrt{n}\delta_{n',n-1} + \sqrt{n+1}\delta_{n',n+1})$$

$$\langle n'|p|n\rangle = i\sqrt{\frac{m\omega\hbar}{2}} \langle n'|a^\dagger - a|n\rangle = i\sqrt{\frac{m\omega\hbar}{2}} (\sqrt{n+1}\delta_{n',n+1} - \sqrt{n}\delta_{n',n-1})$$

The selection rule for both operators is that matrix elements are non-zero only if $\boxed{\Delta n = n' - n = \pm 1}$.

(b) The variance is $\langle (\Delta x)^2 \rangle = \langle x^2 \rangle - \langle x \rangle^2$. In the ground state, $\langle x \rangle = 0$ by parity.

$$\langle 0|x^2|0\rangle = \langle 0| \left(\sqrt{\frac{\hbar}{2m\omega}} (a + a^\dagger) \right) \left(\sqrt{\frac{\hbar}{2m\omega}} (a + a^\dagger) \right) |0\rangle$$

$$= \frac{\hbar}{2m\omega} \langle 0|(a^2 + aa^\dagger + a^\dagger a + (a^\dagger)^2)|0\rangle$$

The terms $a^2|0\rangle$, $a^\dagger a|0\rangle$, and $\langle 0|(a^\dagger)^2$ are all zero. The only non-zero term is from $aa^\dagger$:

$$\langle 0|aa^\dagger|0\rangle = \langle 0|[a, a^\dagger]|0\rangle + \langle 0|a^\dagger a|0\rangle = \langle 0|1|0\rangle + 0 = 1.$$

Therefore, $\boxed{\langle 0|x^2|0\rangle = \frac{\hbar}{2m\omega}}$.

Solution 9: (a) For a bound state, $E < 0$. Let $\kappa = \sqrt{-2mE}/\hbar$. The TISE is $\psi''(x) = \kappa^2 \psi(x)$ for $x \neq 0$. The normalizable solution must have the form $\psi(x) = Ae^{-\kappa|x|}$. Integrating the TISE around $x = 0$ gives the derivative jump condition: $\psi'(0^+) - \psi'(0^-) = -\frac{2m\lambda}{\hbar^2}\psi(0)$. For our solution, $\psi'(0^+) = -A\kappa$ and $\psi'(0^-) = A\kappa$. The condition becomes $-A\kappa - A\kappa = -\frac{2m\lambda}{\hbar^2}A$, which simplifies to $2\kappa = \frac{2m\lambda}{\hbar^2}$, so $\kappa = \frac{m\lambda}{\hbar^2}$. The energy is $E = -\frac{\hbar^2\kappa^2}{2m}$, so $\boxed{E = -\frac{m\lambda^2}{2\hbar^2}}$.

To normalize, $1 = \int_{-\infty}^{\infty} |A|^2 e^{-2\kappa|x|} dx = 2|A|^2 \int_0^\infty e^{-2\kappa x} dx = |A|^2/\kappa$. So, $A = \sqrt{\kappa}$.

$$\boxed{\psi(x) = \sqrt{\frac{m\lambda}{\hbar^2}} \exp\left(-\frac{m\lambda}{\hbar^2}|x| \right)}$$

(b) The wavefunction $\psi(x)$ is a real and even function of x. The expectation value of position is $\langle x \rangle = \int_{-\infty}^{\infty} x|\psi(x)|^2 dx$. Since the integrand $x|\psi(x)|^2$ is an odd function, the integral over a symmetric domain is zero. The expectation value of momentum is $\langle p \rangle = \int_{-\infty}^{\infty} \psi^*(x)(-i\hbar \frac{d}{dx})\psi(x)dx$. Since $\psi(x)$ is real, this is $-i\hbar \int \psi(x)\psi'(x)dx$. The integrand is the product of an even function (ψ) and an odd function (ψ'), which is odd. Thus, the integral is zero.

$$\boxed{\langle x \rangle = 0, \quad \langle p \rangle = 0}$$