P1

Let

$$X = a_0 \mathbb{I} + \sum_{k=1}^{3} \sigma_k a_k, \tag{1}$$

where σ_k are pauli matrices and a_0, a_1 are numbers.

1. $\operatorname{Tr} X$ and $\operatorname{Tr}(\sigma_k X)$.

Noticing pauli matrices are traceless:

$$\operatorname{Tr}(X) = \operatorname{Tr}(a_0 \mathbb{I}) + \operatorname{Tr}\left(\sum_k \sigma_k a_k\right) = 2a_0. \tag{2}$$

Then consider

$$\begin{aligned} \operatorname{Tr}(\sigma_k X) &= \operatorname{Tr}(a_0 \sigma_k \mathbb{I}) + \operatorname{Tr}\left(\sum_j \sigma_k \sigma_j\right) \\ &= 0 + \sum_j a_j \operatorname{Tr}(\sigma_j \sigma_k). \end{aligned} \tag{3}$$

Using the fact that $\operatorname{Tr}(\sigma_j \sigma_k) = 2\delta_{jk}$:

$$\operatorname{Tr}(\sigma_k X) = \sum_{k=1}^{3} a_k \cdot 2\delta_{jk} = 2a_k. \tag{4}$$

2. Find a_0, a_k w.r.t. X_{ij}.

Write

$$X_{ij} = a_0 \delta_{ij} + \sum_{k=1}^{3} \left((\sigma_k)_{ij} a_k\right). \tag{5}$$

Then for diagonal elements:

$$\begin{gathered} X_{ii} = a_0 + (\sigma_3)_{ii} a_3, \\ \Rightarrow X_{11} = a_0 + a_3, \quad X_{22} = a_0 - a_3 \\ \Rightarrow \begin{cases} a_0 = \frac{1}{2}(X_{11} + X_{22}) \\ a_3 = \frac{1}{2}(X_{11} - X_{22}) \end{cases}. \end{gathered} \tag{6}$$

Off diagonal elements: for $m \neq n$:

$$\begin{gathered} X_{mn} = \sigma_{1_{mn}} a_1 + \sigma_{2_{mn}} a_2 \\ \Rightarrow X_{12} = a_1 - i a_2, \quad X_{21} = a_1 + i a_2 \\ \Rightarrow \begin{cases} a_1 = \frac{1}{2}(X_{12} + X_{21}) \\ a_2 = \frac{1}{2i}(X_{21} - X_{12}) \end{cases}. \end{gathered} \tag{7}$$

P2.

> Consider a ket space spanned by $\{|a'\rangle\}$ of Hermitian operator $\boldsymbol{A}$. No degeneracy.

1. $\Pi_{a'}(\boldsymbol{A} - a')$ is the null operator.

Proof: Since $\{|a'\rangle\}$ spans the space, it's sufficient to show that $\Pi_{a'}(\boldsymbol{A} - a')$ annilates all basisket. To show, consider arbitrary basis ket $|a''\rangle$.

$$\begin{aligned}\left[\prod_{a'}(\boldsymbol{A} - a')\right]|a''\rangle &= ... \times (\boldsymbol{A} - a'')|a''\rangle \times ... \\ &= ... \times \boldsymbol{A}|a''\rangle - a''|a''\rangle \times ... \\ &= ... \times a''|a''\rangle - a''|a''\rangle \times ... = 0,\end{aligned} \tag{8}$$

and thus $\Pi_{a'}(\boldsymbol{A} - a')$ annilates all basis kets, and therefore nullify all vectors in this space.

2. A projection Operator

Let

$$\hat{P} \equiv \prod_{a'' \neq a'} \frac{\boldsymbol{A} - a''}{a' - a''}. \tag{9}$$

Notice that $\hat{P}|a'\rangle = |a'\rangle$, $\hat{P}|a_k\rangle = 0$ (for any $a_k \neq a'$.). Explicitely:

$$\hat{P}|a'\rangle = \frac{\boldsymbol{A} - a''}{a' - a''}|a'\rangle = \frac{a' - a''}{a' - a''}|a'\rangle = |a'\rangle; \tag{10}$$

and for any $a_k \neq a'$,

$$\hat{P}|a_k\rangle = ... \times \frac{\boldsymbol{A} - a_k}{a' - a_k} \times ...|a_k\rangle = 0. \tag{11}$$

It's clear that $\hat{P}$ is a projection operator onto the eigenspace corresponding to the eigenvalue a'.

3. Illustrate both results using $\boldsymbol{A} = S_z$ for spin 1/2 system.

Recall that the eigenkets $\{|+\rangle, |-\rangle\}$ of the Hermitian operator S_z form an orthornormal basis, just like $\{|a'\rangle\}$ in the assumption.

$$S_z|\pm\rangle = \pm\frac{\hbar}{2}|\pm\rangle, \tag{12}$$

and from which we can observe:

$$\prod_{a'}(S_z - a')|\pm\rangle = \left(S_z - \frac{\hbar}{2}\right)\left(S_z + \frac{\hbar}{2}\right)|\pm\rangle = 0. \tag{13}$$

Further,

$$\hat{P} = \prod_{a'' \neq a'} \frac{\boldsymbol{A} - a''}{a' - a''} = \begin{cases} \dfrac{S_z + \frac{\hbar}{2}}{\frac{\hbar}{2} + \frac{\hbar}{2}} & \text{for } a' = -\frac{\hbar}{2} \\ \dfrac{S_z - \frac{\hbar}{2}}{-\frac{\hbar}{2} - \frac{\hbar}{2}} & \text{for } a' = \frac{\hbar}{2} \end{cases}, \tag{14}$$

and thus for $a' = -\frac{\hbar}{2}$: $\hat{P}|+\rangle = 0$, $\hat{P}|-\rangle = 1$; for $a' = \frac{\hbar}{2}$: $\hat{P}|-\rangle = 0$, $\hat{P}|+\rangle = 1$, as expected from part 2.

P3

Construct $|\boldsymbol{S}\cdot\hat{\boldsymbol{n}};+\rangle$ as a linear combination of $|+\rangle, |-\rangle$, s.t.

$$\boldsymbol{S}\cdot\hat{\boldsymbol{n}}\,|\boldsymbol{S}\cdot\hat{\boldsymbol{n}};+\rangle = \frac{\hbar}{2}|\boldsymbol{S}\cdot\hat{\boldsymbol{n}};+\rangle, \tag{15}$$

We recall that, in spherical coordinates,

$$\hat{n} = (\sin\beta\cos\alpha, \sin\beta\sin\alpha, \cos\beta); \tag{16}$$

and the definition of $\boldsymbol{S}$:

$$S_x = \frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, S_y = \frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}, S_z = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}. \tag{17}$$

Then,

$$\begin{aligned}\boldsymbol{S}\cdot\hat{\boldsymbol{n}} &= S_x n_x + S_y n_y + S_z n_z\\ &= \frac{\hbar}{2}\begin{pmatrix}\cos\beta & \sin\beta\cos\alpha - i\sin\beta\sin\alpha\\ \sin\beta\cos\alpha + i\sin\beta\sin\alpha & -\cos\beta\end{pmatrix}\\ &= \frac{\hbar}{2}\begin{pmatrix}\cos\beta & \sin\beta e^{-i\alpha}\\ \sin\beta e^{i\alpha} & -\cos\beta\end{pmatrix}.\end{aligned} \tag{18}$$

Let eigenket $|\boldsymbol{S}\cdot\hat{\boldsymbol{n}};+\rangle = \begin{pmatrix}c_1\\ c_2\end{pmatrix}$. The eigenvalue equation then reads:

$$\begin{aligned}\frac{\hbar}{2}\begin{pmatrix}\cos\beta & \sin\beta e^{-i\alpha}\\ \sin\beta e^{i\alpha} & -\cos\beta\end{pmatrix}\begin{pmatrix}c_1\\ c_2\end{pmatrix} &= \frac{\hbar}{2}\begin{pmatrix}c_1\\ c_2\end{pmatrix}\\ \Rightarrow \begin{pmatrix}c_1\cos\beta + c_2\sin\beta e^{-i\alpha}\\ c_1\sin\beta e^{i\alpha} - c_2\cos\beta\end{pmatrix} &= \begin{pmatrix}c_1\\ c_2\end{pmatrix}\end{aligned} \tag{19}$$

This yields two dependent equations

$$\begin{aligned}c_1(\cos\beta - 1) + c_2\sin\beta e^{-i\alpha} &= 0,\\ c_1\sin\beta e^{i\alpha} - c_2(\cos\beta + 1) &= 0,\end{aligned} \tag{20}$$

and from which,

$$c_1 = \frac{\sin\beta e^{-i\alpha}}{1-\cos\beta}c_2 \Rightarrow \quad c_1 = \frac{\cos\left(\frac{\beta}{2}\right)e^{-i\alpha}}{\sin\left(\frac{\beta}{2}\right)}c_2. \tag{21}$$

Normalization condition for the eigen solution requires

$$\begin{aligned}|c_1|^2 + |c_2|^2 &= 1\\ \Rightarrow |c_2|^2\left(\frac{\cos^2\left(\frac{\beta}{2}\right)}{\sin^2\left(\frac{\beta}{2}\right)} + 1\right) &= 1\\ \Rightarrow |c_2|^2 &= \sin^2\left(\frac{\beta}{2}\right).\end{aligned} \tag{22}$$

Then for an arbitrary azimuthal angle α , we have

$$c_2 = \sin\left(\frac{\beta}{2}\right)e^{i\alpha}, c_1 = \cos\left(\frac{\beta}{2}\right). \tag{23}$$

Thus the eigenket is found to be

$$\boxed{\cos\left(\frac{\beta}{2}\right)|+\rangle + \sin\left(\frac{\beta}{2}\right)e^{i\alpha}|-\rangle.} \tag{24}$$

P4.

A two level system with Hamiltonian

$$H = H_{11}|1\rangle\langle 1| + H_{22}|2\rangle\langle 2| + H_{12}(|1\rangle\langle 2| + |2\rangle\langle 1|. \tag{25}$$

Find energy eigenvalues and eigenkets.

1. method 1, solving eigenvalue problem Explicitely.

Writing $\hat{H}$ in matrix form by noticing $\hat{H}_{11} = \langle 1|H|1\rangle = H_{11}, \hat{H}_{12} = \langle 1|H|2\rangle = H_{21}, \hat{H}_{22} = \langle 2|H|2\rangle = H_{22}$:

$$\hat{H} = \begin{pmatrix} H_{11} & H_{12} \\ H_{12} & H_{22} \end{pmatrix}. \tag{26}$$

Let eigenvalue be E, and the eigenvalue equation reads:

$$\begin{aligned} \det\left(\hat{H} - E\mathbb{I}\right) = 0 \quad \Rightarrow \quad \det\begin{pmatrix} H_{11} - E & H_{12} \\ H_{12} & H_{22} - E \end{pmatrix} = 0, \\ \Rightarrow (H_{11} - E)(H_{22} - E) - H_{12}^2 = 0 \\ \Rightarrow E_\pm = \frac{1}{2}\left(H_{11} + H_{22} \pm \sqrt{(H_{11} - H_{22})^2 + 4H_{12}^2}\right). \end{aligned} \tag{27}$$

For each $E_\pm$, we solve for the eigenvector problem $\hat{H}|E\rangle = E_\pm|E\rangle$, for $|E\rangle = \left(\begin{smallmatrix} c_1 \\ c_2. \end{smallmatrix}\right)$

$$\begin{aligned} \begin{pmatrix} H_{11} - E_\pm & H_{12} \\ H_{12} & H_{22} - E_\pm \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = 0 \\ \Rightarrow \begin{cases} (H_{11} - E_\pm)c_1 + H_{12}c_2 = 0 \\ H_{12}c_1 + (H_{22} - E_\pm)c_2 = 0 \end{cases} \\ \Rightarrow \begin{cases} c_1 = H_{12} \\ c_2 = E_\pm - H_{11} \end{cases}. \end{aligned} \tag{28}$$

Applying normalization, we have

$$|E\rangle = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \frac{1}{\sqrt{H_{12}^2 + |E_\pm - H_{11}|^2}}\begin{pmatrix} H_{12} \\ E_\pm - H_{11} \end{pmatrix}, \tag{29}$$

with $E_\pm$ specified above.

2. method 2, use result from P1&P3.

Since $\{\mathbb{I}, \sigma_1, \sigma_2, \sigma_3\}$ form a basis of the 4-dimensional complex vector space of 2×2 complex matrices, any 2-by-2 matrix X can be expanded as

$$X = a_0\mathbb{I} + \sum_{k=1}^{3} \sigma_k a_k, \tag{30}$$

and thus we can use result from P1 b to write $\hat{H}$:

$$\hat{H} = \begin{pmatrix} H_{11} & H_{12} \\ H_{12} & H_{22} \end{pmatrix} = a_0\mathbb{I} + \sum_{k=1}^{3} \sigma_k a_k, \tag{31}$$

where

$$a_0 = \frac{1}{2}(H_{11} + H_{22}), a_1 = H_{12}, a_2 = 0, a_3 = \frac{1}{2}(H_{11} - H_{22}). \tag{32}$$

Denote $\boldsymbol{a} = (a_1, a_2, a_3), \boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$, and then Equation 31 reads

$$
\begin{aligned}
\hat{H} &= a_0\mathbb{I} + \boldsymbol{a}\cdot\boldsymbol{\sigma} \\
&= a_0\mathbb{I} + \|\boldsymbol{a}\|(\hat{n}\cdot\boldsymbol{\sigma}) \\
&= a_0\mathbb{I} + \frac{2\|\boldsymbol{a}\|}{\hbar}(\hat{n}\cdot\boldsymbol{S}).
\end{aligned} \tag{33}
$$

The eigenvalue equation is then

$$
H|E\rangle = \left[a_0\mathbb{I} + \frac{2\|\boldsymbol{a}\|}{\hbar}(\hat{n}\cdot\boldsymbol{S})\right]|E\rangle = E|E\rangle. \tag{34}
$$

Since we knew from P3 that the eigenvalues of $\hat{n}\cdot\boldsymbol{S}$ are $\pm\frac{\hbar}{2}$, we have the energy eigenvalues:

$$
E_\pm = a_0 \pm \|\boldsymbol{a}\|, \tag{35}
$$

with $\|\boldsymbol{a}\| = \sqrt{\sum_{k=1}^{3} a_k^2}$ specified above. Further, since

$$
n_z = \cos\beta = \frac{a_3}{\|\boldsymbol{a}\|}, n_y = \frac{a_2}{\|\boldsymbol{a}\|} = 0, n_x = \sin\beta\cos\alpha = \frac{a_1}{\|\boldsymbol{a}\|}, \tag{36}
$$

we have $\alpha = 0$ or π , and

$$
\begin{aligned}
\cos\left(\frac{\beta}{2}\right) &= \sqrt{\frac{1+\cos\beta}{2}} = \sqrt{\frac{1+\frac{a_3}{\|\boldsymbol{a}\|}}{2}}, \\
\sin\left(\frac{\beta}{2}\right) &= \sqrt{\frac{1-\cos\beta}{2}} = \sqrt{\frac{1-\frac{a_3}{\|\boldsymbol{a}\|}}{2}}.
\end{aligned} \tag{37}
$$

Thus, the eigenkets are the same as those in P3, but with $\sin\left(\frac{\beta}{2}\right), \cos\left(\frac{\beta}{2}\right)$ specified above:

$$
\begin{aligned}
|E_+\rangle &= \begin{pmatrix} \cos\left(\frac{\beta}{2}\right) \\ e^{i\alpha}\sin\left(\frac{\beta}{2}\right) \end{pmatrix}, \\
|E_-\rangle &= \begin{pmatrix} \sin\left(\frac{\beta}{2}\right) \\ -e^{i\alpha}\cos\left(\frac{\beta}{2}\right) \end{pmatrix}.
\end{aligned} \tag{38}
$$

In particular, when $H_{12} < 0, n_1 < 0, \alpha = \pi$; when $H_{12} > 0, n_1 > 0, \alpha = 0$.

P5

A spin 1/2 system in an eigenstate of $\boldsymbol{S} \cdot \hat{\boldsymbol{n}}$ with eigenvalue $\frac{\hbar}{2}$. $\hat{\boldsymbol{n}}$ is a unit vector in xz plane with angle γ w.r.t. $+\hat{z}$.

1. The possible outcomes of a measurement of S_x and their probabilities.

From previous problems, we know that the eigenket of $\boldsymbol{S} \cdot \hat{\boldsymbol{n}}$ with eigenvalue $\frac{\hbar}{2}$, where $\alpha = 0, \beta = \gamma$, is

$$|\psi\rangle \equiv |\boldsymbol{S} \cdot \hat{\boldsymbol{n}}\, +\rangle = \cos\left(\frac{\gamma}{2}\right)|+\rangle + \sin\left(\frac{\gamma}{2}\right)|-\rangle. \tag{39}$$

and so the measurement probabilities are

$$\begin{aligned} P\left(\frac{\hbar}{2}\right) &= |\langle +_x|\psi\rangle|^2 = \left|\frac{1}{\sqrt{2}}(\langle +| + \langle -|)|\psi\rangle\right|^2 = \left|\frac{1}{\sqrt{2}}\left(\cos\frac{\gamma}{2} + \sin\frac{\gamma}{2}\right)\right|^2 = \frac{1+\sin\gamma}{2}, \\ P\left(-\frac{\hbar}{2}\right) &= |\langle -_x|\psi\rangle|^2 = \left|\frac{1}{\sqrt{2}}(\langle +| - \langle -|)|\psi\rangle\right|^2 = \left|\frac{1}{\sqrt{2}}\left(\cos\frac{\gamma}{2} - \sin\frac{\gamma}{2}\right)\right|^2 = \frac{1-\sin\gamma}{2}. \end{aligned} \tag{40}$$

2. Find $\left\langle (\Delta S_x)^2 \right\rangle$. Check for $\gamma = 0, \frac{\pi}{2}, \pi$.

A useful identity is

$$\left\langle (\Delta S_x)^2 \right\rangle = \langle S_x^2 \rangle - (\langle S_x \rangle)^2, \tag{41}$$

where

$$\begin{aligned} S_x^2 &= \left(\frac{\hbar}{2}\right)^2 \sigma_x^2 = \left(\frac{\hbar}{2}\right)^2 \mathbb{I} \Rightarrow \langle S_x^2 \rangle = \frac{\hbar^2}{4}. \\ \langle S_x \rangle &= \langle \psi|S_x|\psi\rangle = \sum_i a_i p(a_i) = \frac{\hbar}{2} \cdot \frac{1+\sin\gamma}{2} + \left(-\frac{\hbar}{2}\right) \cdot \frac{1-\sin\gamma}{2} = \frac{\hbar}{2}\sin\gamma. \end{aligned} \tag{42}$$

From which we have

$$\left\langle (\Delta S_x)^2 \right\rangle = \frac{\hbar^2}{4} - \left(\frac{\hbar}{2}\sin\gamma\right)^2 = \boxed{\frac{\hbar^2}{4}\cos^2\gamma} \tag{43}$$

Checking for

$$\begin{aligned} \gamma = 0 &: \left\langle (\Delta S_x)^2 \right\rangle = \frac{\hbar^2}{4}; \\ \gamma = \frac{\pi}{2} &: \left\langle (\Delta S_x)^2 \right\rangle = 0; \\ \gamma = \pi &: \left\langle (\Delta S_x)^2 \right\rangle = \frac{\hbar^2}{4} \end{aligned} \tag{44}$$

3. How do the results for 1 and 2 change for the case of S_y?

Noticing $\hat{n} = (\sin\gamma, 0, \cos\gamma)$, we can easily read off:

$$P\left(\pm\frac{\hbar}{2}\right) = \frac{1 \pm \hat{n} \cdot \hat{y}}{2} = \frac{1 \pm 0}{2} = \frac{1}{2}. \tag{45}$$

Then

$$\begin{aligned} \langle S_y \rangle &= \sum_i a_i p(a_i) = \frac{\hbar}{2} \cdot \frac{1}{2} + \left(-\frac{\hbar}{2}\right) \cdot \frac{1}{2} = 0, \\ \langle S_y^2 \rangle &= \frac{\hbar^2}{4}. \\ \Rightarrow \left\langle (\Delta S_y)^2 \right\rangle &= \frac{\hbar^2}{4} - 0^2 = \frac{\hbar^2}{4}. \end{aligned} \tag{46}$$

P6

Find the linear combination of $|+\rangle, |-\rangle$ that maximizes the uncertainty product

$$\left\langle (\Delta S_x)^2 \right\rangle \left\langle \left(\Delta S_y\right)^2 \right\rangle. \tag{47}$$

Verify that for the linear combination you found, the uncertainty relation for S_x, S_y is not violated.

We take the general linear combination from P3:

$$|\psi\rangle = \cos\left(\frac{\beta}{2}\right)|+\rangle + \sin\left(\frac{\beta}{2}\right)e^{i\alpha}|-\rangle, \tag{48}$$

and perform similar procedure to find $\langle S_x\rangle, \langle S_x\rangle$.

For $\langle S_x\rangle$:

$$\begin{aligned} P\left(\frac{\hbar}{2}\right) &= |\langle +_x|\psi\rangle|^2 = \left|\frac{1}{\sqrt{2}}(\langle +| + \langle -|)|\psi\rangle\right|^2 = \left|\frac{1}{\sqrt{2}}\left(\cos\left(\frac{\beta}{2}\right) + \sin\left(\frac{\beta}{2}\right)e^{i\alpha}\right)\right|^2 = \frac{1+\sin\beta\cos\alpha}{2}, \\ P\left(-\frac{\hbar}{2}\right) &= |\langle -_x|\psi\rangle|^2 = \left|\frac{1}{\sqrt{2}}(\langle +| - \langle -|)|\psi\rangle\right|^2 = \left|\frac{1}{\sqrt{2}}\left(\cos\left(\frac{\beta}{2}\right) - \sin\left(\frac{\beta}{2}\right)e^{i\alpha}\right)\right|^2 = \frac{1-\sin\beta\cos\alpha}{2}. \end{aligned} \tag{49}$$

From which we have

$$\langle S_x\rangle = \sum_i a_i p(a_i) = \frac{\hbar}{2}\cdot\frac{1+\sin\beta\cos\alpha}{2} + \left(-\frac{\hbar}{2}\right)\cdot\frac{1-\sin\beta\cos\alpha}{2} = \frac{\hbar}{2}\sin\beta\cos\alpha. \tag{50}$$

Similiarly, for $\left\langle S_y\right\rangle$:

$$\begin{aligned} P\left(\frac{\hbar}{2}\right) &= \left|\left\langle +_y|\psi\right\rangle\right|^2 = \left|\frac{1}{\sqrt{2}}(\langle +| + i\langle -|)|\psi\rangle\right|^2 = \frac{1+\sin\beta\sin\alpha}{2}, \\ P\left(-\frac{\hbar}{2}\right) &= \left|\left\langle -_y|\psi\right\rangle\right|^2 = \left|\frac{1}{\sqrt{2}}(\langle +| - i\langle -|)|\psi\rangle\right|^2 = \frac{1-\sin\beta\sin\alpha}{2}, \\ \Rightarrow \left\langle S_y\right\rangle &= \sum_i a_i p(a_i) = \frac{\hbar}{2}\sin\beta\sin\alpha. \end{aligned} \tag{51}$$

Also, notice

$$S_k^2 = \left(\frac{\hbar}{2}^2\mathbb{I}\right) \Rightarrow \langle S_k^2\rangle = \frac{\hbar^2}{4}. \tag{52}$$

Collecting, we have

$$\begin{aligned} \left\langle (\Delta S_x)^2\right\rangle &= \langle S_x^2\rangle - (\langle S_x\rangle)^2 = \frac{\hbar^2}{4} - \left(\frac{\hbar}{2}\sin\beta\cos\alpha\right)^2 = \frac{\hbar^2}{4}(1-\sin^2\beta\cos^2\alpha), \\ \left\langle \left(\Delta S_y\right)^2\right\rangle &= \langle S_y^2\rangle - \left(\left\langle S_y\right\rangle\right)^2 = \frac{\hbar^2}{4} - \left(\frac{\hbar}{2}\sin\beta\sin\alpha\right)^2 = \frac{\hbar^2}{4}(1-\sin^2\beta\sin^2\alpha) \\ \Rightarrow \left\langle (\Delta S_x)^2\right\rangle\left\langle \left(\Delta S_y\right)^2\right\rangle &= \frac{\hbar^4}{16}(1-\sin^2\beta\cos^2\alpha)(1-\sin^2\beta\sin^2\alpha) = \frac{\hbar^4}{16}\underbrace{\left(\cos^2(\beta) + \frac{\sin^4(\beta)}{4}\sin^2(2\alpha)\right)}_{(*)}. \end{aligned} \tag{53}$$

Maximizing $(*)$ for $\alpha, \beta \in [0, 2\pi]$, we have $\max(*) = 1$ for $\beta = \pi + n\pi, n \in \mathbb{Z}$. Thus the maximum uncertainty product is

$$\max\left\langle (\Delta S_x)^2\right\rangle\left\langle \left(\Delta S_y\right)^2\right\rangle = \boxed{\frac{\hbar^4}{16}.} \tag{54}$$

While the uncertainty relation dictates that

$$\left\langle (\Delta S_x)^2 \right\rangle \left\langle (\Delta S_y)^2 \right\rangle \geq \left(\frac{1}{4}\right) \mid \langle [S_x, S_y] \rangle |^2 = \left(\frac{1}{4}\right) \mid i\hbar S_z \mid^2 = \frac{\hbar^2}{4} \langle S_z \rangle^2, \tag{55}$$

where

$$\langle S_z \rangle = \sum_i a_i p(a_i) = \frac{\hbar}{2} \cdot \left(\cos^2\left(\frac{\beta}{2}\right)\right) + \left(-\frac{\hbar}{2}\right) \cdot \left(\sin^2\left(\frac{\beta}{2}\right)\right) = \frac{\hbar}{2} \cos\beta. \tag{56}$$

and so

$$\left\langle (\Delta S_x)^2 \right\rangle \left\langle (\Delta S_y)^2 \right\rangle \geq \frac{\hbar^4}{16}, \tag{57}$$

which is satisfied by the maximum uncertainty product we found above.