$$f(x) = \frac{1}{i\pi} P\int_{-\infty}^{+\infty} \frac{f(x')dx'}{x'-x}$$

$$f(x) = \mathrm{Re}\, f(x) + i\, \mathrm{Im}\, f(x)$$

$$\mathrm{Re} f(x) + i\, \mathrm{Im}\, f(x) = -\frac{i}{\pi} P\int_{-\infty}^{+\infty} \frac{[\mathrm{Re}\, f(x') + i\, \mathrm{Im}\, f(x')]\,dx'}{x'-x}$$

By comparing now real and imaginary parts:

$$\boxed{\begin{aligned} \mathrm{Re}\, f(x) &= \frac{1}{\pi} P\int_{-\infty}^{+\infty} \frac{\mathrm{Im}\, f(x')\,dx'}{x'-x} \\ \mathrm{Im}\, f(x) &= -\frac{1}{\pi} P\int_{-\infty}^{+\infty} \frac{\mathrm{Re}\, f(x')\,dx'}{x'-x} \end{aligned}}$$

Asymptotic Methods – Saddle point Approximation

$$F(\lambda) = \int_a^b \varphi(t)\, e^{\lambda f(t)}\, dt \qquad \lambda \gg 1$$

Assume that $f(t)$ is such that it has sharp maximum at the point t_0 inside the range of integration $t_0 \in [a, b]$. It is then expected that dominant region of integration comes from the vicinity of t_0.

Lemma: Consider $F(\lambda) = \int_0^a \varphi(t)\, e^{-\lambda t^\alpha}\, dt$ where $\alpha > 0$ and $0 < a \leq \infty$, and the function

$$\varphi(t) = t^\beta (c_0 + c_1 t + \ldots + c_n t^n + \ldots) \qquad \beta > -1$$

Then: $$\boxed{F(\lambda) \simeq \sum_{n=0}^{\infty} \frac{c_n}{\alpha} \Gamma\left(\frac{\beta+n+1}{\alpha}\right) \lambda^{-\frac{\beta+n+1}{\alpha}}}$$

here $\Gamma(x)$ is Euler Gamma function.

$$F(\lambda) = \int_0^a \varphi(t) e^{-\lambda t^\alpha} dt = \left\{ \begin{aligned} &\tau = \lambda t^\alpha \to t = \left(\frac{\tau}{\lambda}\right)^{\frac{1}{\alpha}} \\ &d\tau = \lambda \alpha t^{\alpha-1} dt \end{aligned} \right.$$

$$\left. dt = \frac{d\tau}{\lambda\alpha} \frac{1}{t^{\alpha-1}} = \frac{d\tau}{\lambda\alpha} \left(\frac{\lambda}{\tau}\right)^{\frac{\alpha-1}{\alpha}} = \frac{d\tau}{\lambda\alpha} \left(\frac{\lambda}{\tau}\right)^{1-\frac{1}{\alpha}} \right\}$$

$$= \frac{1}{\alpha} \int_0^{\lambda a^\alpha} \varphi\left[\left(\frac{\tau}{\lambda}\right)^{\frac{1}{\alpha}}\right] e^{-\tau} \tau^{\frac{1}{\alpha}-1} \lambda^{-\frac{1}{\alpha}} d\tau =$$

$$= \frac{1}{\alpha} \sum_{K=0} \left(\frac{\tau}{\lambda}\right)^{\beta/\alpha} C_K \left(\frac{\tau}{\lambda}\right)^{\frac{K}{\alpha}} e^{-\tau} \tau^{\frac{1}{\alpha}-1} \lambda^{-\frac{1}{\alpha}} d\tau$$

$$F(\lambda) = \frac{1}{\alpha} \sum_{K=0}^{\infty} \frac{C_K}{\lambda^{\frac{\beta+K+1}{\alpha}}} \int_0^{\lambda a^\alpha} \tau^{\frac{\beta+K+1}{\alpha}-1} e^{-\tau} d\tau$$

$$\int_0^{\lambda a^\alpha} (\dots) d\tau = \int_0^{\infty} (\dots) d\tau - \int_{\lambda a^\alpha}^{\infty} (\dots) d\tau$$

$$F(\lambda) = \frac{1}{\alpha} \sum_{K=0}^{\infty} \frac{C_K}{\lambda^{\frac{\beta+K+1}{\alpha}}} \left[\underbrace{\int_0^{\infty} \tau^{\frac{\beta+K+1}{\alpha}-1} e^{-\tau} d\tau}_{\Gamma\left(\frac{\beta+K+1}{\alpha}\right)} - \underbrace{\int_{\lambda a^\alpha}^{\infty} \tau^{\frac{\beta+K+1}{\alpha}-1} e^{-\tau} d\tau}_{\text{exponentially small contribution } O(e^{-\lambda a^\alpha})} \right]$$

Main Theorem: Assume that integral is convergent

$$\int_a^b |\varphi(t)| e^{\lambda f(t)} dt \leq M$$ and $f(t)$ takes its

maximal value at point $t_0 \in [a,b]$ where $f(t)$ can be approximated by the series ($|t-t_0| < \delta$):

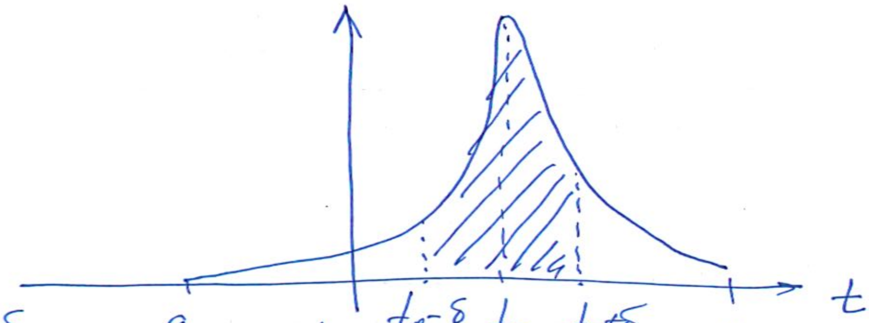
$$f(t) \simeq f(t_0) + a_2 (t-t_0)^2 + \dots + a_n (t-t_0)^n + \dots$$

Assume also that $t = \psi(\tau)$ near $\tau = 0$ is found from the equation $f(t_0) - f(t) = \tau^2$ such that

$$\varphi[\psi(\tau)]\, \psi'(\tau) = \sum_{n=0}^{\infty} c_n \tau^n$$

Then initial integral has following asymptotic expansion:

$$\boxed{F(\lambda) = \int_a^b \varphi(t) e^{\lambda f(t)} dt \simeq e^{\lambda f(t_0)} \sqrt{\frac{\pi}{\lambda}} \sum_{n=0}^{\infty} \frac{c_{2n}}{\lambda^n} \frac{(2n)!}{4^n n!}}$$



$$F(\lambda) = \int_{t_0-\delta}^{t_0+\delta} + \int_a^{t_0-\delta} + \int_{t_0+\delta}^{b}$$

$$\int_{t_0-\delta}^{t_0+\delta} \varphi(t) e^{\lambda f(t)} dt = e^{\lambda f(t_0)} \int_{t_0-\delta}^{t_0+\delta} \varphi(t) e^{\lambda [f(t) - f(t_0)]} dt \Rightarrow$$

$$f(t) - f(t_0) = \tau^2 \Rightarrow \tau^2 = -a_2 (t-t_0)^2 - a_3 (t-t_0)^3 - \dots$$

$$\tau = (t-t_0)\sqrt{-a_2 - a_3 (t-t_0) \dots} = \sum_{n=1}^{\infty} \tilde{a}_n (t-t_0)^n$$

Now we want to invert this series and get

$$t = \psi(\tau) = t_0 + \sum_{n=1}^{\infty} c_n \tau^n$$

Note: $c_1 = \frac{1}{\sqrt{-a_2}} = \sqrt{-\frac{2}{f''(t_0)}}$

$$\int_{t_0-\delta}^{t_0+\delta} = e^{\lambda f(t_0)} \int_{-\delta}^{\delta} \underbrace{\varphi[\psi(\tau)]\psi'(\tau)}_{\tilde{\varphi}(\tau)} e^{-\lambda \tau^2} d\tau =$$

$$e^{\lambda f(t_0)} \int_{-\delta}^{0} \tilde{\varphi}(\tau) e^{-\lambda\tau^2} d\tau + \int_{0}^{\delta} \tilde{\varphi}(\tau) e^{-\lambda \tau^2} d\tau =$$

$$= e^{\lambda f(t_0)} \int_0^{\delta} [\tilde{\varphi}(\tau) + \tilde{\varphi}(-\tau)] e^{-\lambda \tau^2} d\tau$$

$$\tilde{\varphi}(\tau) + \tilde{\varphi}(-\tau) = 2\sum_{n=0}^{\infty} c_{2n} \tau^{2n}$$

$$2\int_0^{\delta} \sum_{n=0}^{\infty} c_{2n} \tau^{2n} e^{-\lambda \tau^2} d\tau \cdot e^{\lambda f(t_0)} = \sum_{n=0}^{\infty} \frac{c_{2n} \Gamma(n+1/2)}{\lambda^{n+1/2}} e^{\lambda f(t_0)}$$

Noticing that $\Gamma(n+1/2) = \frac{(2n)!\sqrt{\pi}}{4^n n!}$ we get

main formula of a theorem.

<u>Note</u>: If $f(t)$ has maximum at $t=a$ and $f'(a) \neq 0$

$$\boxed{F(\lambda) = \int_a^b \varphi(t) e^{\lambda f(t)} dt \simeq \frac{e^{\lambda f(a)}}{\lambda} \sum_{n=0}^{\infty} \frac{n! a_n}{\lambda^n}}$$

$\boxed{\text{Exp. 1}}$ $\quad \Gamma(\lambda+1) = \int_0^\infty x^\lambda e^{-x} dx \qquad \Gamma(\lambda+1)\underset{\lambda \gg 1}{-} ?$

$x = \lambda t \qquad \Gamma(\lambda+1) = \lambda^{\lambda+1} \int_0^\infty t^\lambda e^{-\lambda t} dt \Rightarrow$

$$\Gamma(\lambda+1) = \lambda^{\lambda+1} e^{-\lambda} \int_0^\infty e^{-\lambda(t-1-\ln t)} dt \approx$$

$$\begin{cases} \varphi(t) = 1 \quad f(t) = -(t-1-\ln t) \\ f'(t)\Big|_{t=t_0} = -\left(1 - \frac{1}{t_0}\right) = 0 \Rightarrow \boxed{t_0 = 1} \\ f''(t) = -\frac{1}{t^2}\Big|_{t=t_0} = -1 \end{cases}$$

$$\approx \lambda^{\lambda+1} e^{-\lambda} \int_0^\infty e^{-\frac{\lambda}{2}(t-t_0)^2} dt = \lambda^{\lambda+1} e^{-\lambda} \int_{-\infty}^{+\infty} e^{-\frac{\lambda}{2}t^2} dt$$

$$\left\{ \frac{\lambda}{2} t^2 = x^2 \Rightarrow t\sqrt{\frac{\lambda}{2}} = x \right\}$$

$$= \lambda^{\lambda+1} e^{-\lambda} \sqrt{\frac{2}{\lambda}} \int_{-\infty}^{+\infty} e^{-x^2} dx = \sqrt{\frac{2\pi}{\lambda}} \lambda^{\lambda+1} e^{-\lambda}$$

$$\boxed{\Gamma(\lambda+1) \simeq \sqrt{2\pi\lambda} \left(\frac{\lambda}{e}\right)^\lambda + \dots}$$

Exp. 2 Modified Bessel function of the 2^{nd} kind has the following integral representation:

$$K_\nu(\rho) = \frac{1}{2}\int_0^\infty e^{-\frac{\rho}{2}\left(x+\frac{1}{x}\right)} \frac{dx}{x^{1-\nu}}$$

We apply saddle point approximation to calculate $K_\nu(\rho)$ in the limit $\rho \to \infty$.

$$K_\nu(\rho) = \int_0^\infty e^{\rho f(x)} g(x)\, dx \qquad \begin{cases} f(x) = -\frac{1}{2}\left(x+\frac{1}{x}\right) \\ f'(x) = -\frac{1}{2}\left(1-\frac{1}{x^2}\right) \\ f''(x) = -\frac{1}{x^3} \end{cases}$$

$f'(x) = 0 \Rightarrow x^2 - 1 = 0 \Rightarrow x = \pm 1$

Only point $x = 1$ is within integration region!

Near $x=1$: $f(x) \simeq -1 - \frac{1}{2}(x-1)^2$ so that

$$K_\nu(\rho) \simeq \int_{-\infty}^{+\infty} e^{-\rho} \cdot e^{-\frac{\rho}{2}(x-1)^2} \frac{dx}{2(1)^{1-\nu}} = \frac{e^{-\rho}}{2}\int_{-\infty}^{+\infty} e^{-\frac{\rho}{2}(x-1)^2} dx$$

$$= \frac{e^{-\rho}}{2}\sqrt{\frac{2}{\rho}}\int_{-\infty}^{+\infty} e^{-y^2} dy = \sqrt{\frac{\pi}{2\rho}}\, e^{-\rho}$$

$$\boxed{K_\nu(\rho) \simeq \sqrt{\frac{\pi}{2\rho}}\, e^{-\rho} \qquad \rho \gg 1}$$

Exp. 3

$$I(\lambda) = \int_{-\infty}^{+\infty} e^{\lambda\left(\frac{x^2}{2} - \frac{x^4}{4}\right)} dx \qquad \lambda \gg 1$$

$f(x) = \frac{x^2}{2} - \frac{x^4}{4}$; $f'(x) = x - x^3$ $\quad x_0 = 0, \pm 1$

$f'' = 1 - 3x^2$

$f''(0) = 1 \to$ minimum

$f''(\pm 1) = -2 \to$ maxima $\quad f(\pm 1) = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$

$$I(\lambda) \approx 2 \cdot \int_{-\infty}^{+\infty} e^{\lambda\left[\frac{1}{4} - (x-1)^2\right]} dx =$$

for $x=+1$ and $x=-1$

$$= 2 \cdot e^{\lambda/4} \int_{-\infty}^{+\infty} e^{-\lambda x^2} dx = 2\sqrt{\frac{\pi}{\lambda}}\, e^{\lambda/4}$$

$$\boxed{I(\lambda) \approx 2\sqrt{\frac{\pi}{\lambda}}\, e^{\lambda/4}}$$

Stationary Phase Method

$$\boxed{\int_a^b \phi(x)\, e^{i\lambda f(x)} dx \approx \sqrt{\frac{\pi i}{2\lambda f''(a)}}\, \phi(a)\, e^{i\lambda f(a)}}$$

The point $x=a$ is a stationary point of $f(x)$, namely $f'(a)=0$ and $f''(a) \neq 0$

Note 1: In the case if $f(x)$ does not have stationary point the integral at $\lambda \gg 1$ can be estimated by integration by parts

$$\int_a^b \phi(x)\, e^{i\lambda f(x)} dx \approx \frac{\phi(b)\, e^{i\lambda f(b)}}{i\lambda f'(b)} - \frac{\phi(a) e^{i\lambda f(a)}}{i\lambda f'(a)} + O(1/\lambda^2)$$

Note 2: It may happen that at the Stationary point $f''(a) = 0$. In that case:

$$\int_a^b \phi(x) e^{i\lambda f(x)} dx \approx \Gamma\left(\frac{4}{3}\right) \sqrt[3]{\frac{6i}{\lambda f'''(a)}}\, \phi(a)\, e^{i\lambda f(a)}$$

Steepest Descent Method

$$I(\lambda) = \int_C \phi(z) e^{\lambda f(z)} dz \qquad \lambda \gg 1$$

where $f(z)$ and $\phi(z)$ are analytic functions in the complex plane that contains integration curve.

a) Find saddle points $f'(z) = 0$

b) Deform contour $C \to C'$ in such a way that initial and final points remain the same but when passing through saddle point along C' $e^{\lambda f(z)}$ decays in fastest possible manner.

c) Use Cauchy theorem that deformation of the contour does not change the integral

d) Apply saddle point method to estimate I.

$\boxed{\text{Exp. 1}}$ Airy function integral:

$$Ai(-x) = I(x) = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} e^{i(\omega x - \omega^3/3)} \qquad x \gg 1$$

First we make a substitution: $\omega = \sqrt{x}\,\omega'$

$$I(x) = \frac{\sqrt{x}}{2\pi} \int_{-\infty}^{+\infty} d\omega\, e^{i x^{3/2}(\omega - \omega^3/3)}$$

$$f(\omega) = i(\omega - \omega^3/3)\,; \quad f'(\omega) = i(1-\omega^2) \qquad \underline{\omega = \pm 1}$$

$$f''(\omega) = -2i\omega$$

So we have two contributions

$\omega = +1$ $\qquad f(\omega) \simeq \frac{2i}{3} - i(\omega-1)^2$

$$I_1 = \frac{\sqrt{x}}{2\pi}\int_{-\infty}^{+\infty} e^{\frac{2i}{3}x^{3/2}}\, e^{-ix^{3/2}(\omega-1)^2}\, d\omega \;\stackrel{}{=}$$

Let: $\omega = 1 + s\cdot e^{-i\pi/4}$

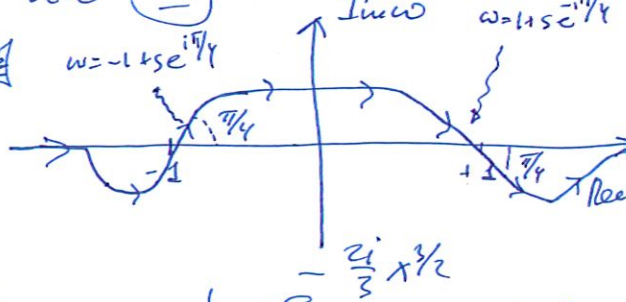
$$= \frac{\sqrt{x}}{2\pi}\, e^{-i\pi/4}\, e^{\frac{2i}{3}x^{3/2}} \int_{-\infty}^{+\infty} e^{-x^{3/2}\cdot s^2}\, ds = \left\{ x^{3/4}\cdot s = y \right\}$$

$$= \frac{\sqrt{x}}{2\pi}\, e^{-i\pi/4}\, e^{\frac{2i}{3}x^{3/2}}\, \frac{1}{x^{3/4}} \underbrace{\int_{-\infty}^{+\infty} e^{-y^2}\, dy}_{\sqrt{\pi}} \simeq \frac{1}{2\sqrt{\pi}}\, \frac{e^{\frac{2i}{3}x^{3/2}}}{x^{1/4}}\, e^{-i\pi/4}$$

$\omega = -1$ $\qquad f(\omega) = -\frac{2i}{3} + i(\omega+1)^2$

$$I_2 = \frac{\sqrt{x}}{2\pi}\int_{-\infty}^{+\infty} e^{-\frac{2i}{3}x^{3/2}}\, e^{ix^{3/2}(\omega+1)^2}\, d\omega \;\stackrel{}{=}$$

Let: $\omega = -1 + s e^{i\pi/4}$



$$= \frac{\sqrt{x}}{2\pi}\, e^{-\frac{2i}{3}x^{3/2}}\, e^{i\pi/4} \int_{-\infty}^{+\infty} e^{-x^{3/2}s^2}\, ds = \frac{1}{2\sqrt{\pi}}\, \frac{e^{-\frac{2i}{3}x^{3/2}}}{x^{1/4}}\, e^{i\pi/4}$$

$$\boxed{I = I_1 + I_2 = \frac{1}{x^{1/4}\sqrt{\pi}} \cos\left(\frac{2}{3}x^{3/2} - \frac{\pi}{4}\right)}$$

$\boxed{\text{Exp. 2}}$ Bessel function integral representation:

$$J_n(x) = \int_{-\pi}^{\pi} \frac{d\phi}{2\pi} e^{ix\sin\phi - in\phi} \quad \text{at} \quad x \gg 1$$

For large x we use stationary phase:

$$f(\phi) = \sin\phi \qquad f'(\phi) = \cos\phi = 0 \rightarrow \phi = \pm \pi/2$$

Two stationary phase points within integration path.

$$f''(\phi) = -\sin\phi$$

Near $\phi = \pi/2$: $f(\phi) \approx 1 - \frac{1}{2}(\phi - \pi/2)^2$

Near $\phi = -\pi/2$: $f(\phi) \approx -1 + \frac{1}{2}(\phi + \pi/2)^2$

$$\Rightarrow I_1(x) = \frac{1}{2\pi} \int_{\pi/2-\varepsilon}^{\pi/2+\varepsilon} e^{ix\left[1 - \frac{1}{2}(\phi - \pi/2)^2\right]} e^{in\pi/2} d\phi =$$

Deform contour so $\phi - \pi/2 = s e^{-i\pi/4}$

$$= \frac{1}{2\pi} \int_{-\varepsilon \to -\infty}^{+\varepsilon \to +\infty} e^{-i\pi/4} e^{ix} e^{\frac{in\pi}{2}} e^{-\frac{xs^2}{2}} ds \approx \sqrt{\frac{1}{2\pi x}} e^{i\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right)}$$

The contribution from near $\phi = -\pi/2$ is complex conjugate

$$\boxed{J_n = I_1 + I_2 = \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\pi n}{2} - \frac{\pi}{4}\right)}$$