Physics 731: Assignment #4

1. **[S1r, S2, S3 2.3]** An electron is subject to a uniform, time-independent magnetic field of strength $\mathbf{B}$ in the positive z-direction. At $t = 0$ the electron is known to be in an eigenstate of $\mathbf{S} \cdot \hat{\mathbf{n}}$ with eigenvalue $\hbar/2$ (*i.e.* $|\hat{n}; +\rangle$), where $\hat{\mathbf{n}}$ is a unit vector, lying in the xz-plane, that makes an angle β with the z-azis.
(a) Obtain the probability of finding the electron in the $s_x = +\hbar/2$ state as a function of time.
(b) Find the expectation value of S_x as a function of time.
(c) Show that your answers make good sense for (i) $\beta \to 0$ and (ii) $\beta \to \pi/2$.

2. **[S2, S3 2.4 modified]** Consider the problem of two-flavor neutrino oscillations, in which the lepton flavor eigenstates $|\nu_e\rangle$ and $|\nu_\mu\rangle$ are linear combinations of the energy eigenstates (known in this context as the "mass eigenstates") $|\nu_1\rangle$ and $|\nu_2\rangle$. The mass eigenstates have energies $E_{1,2}$, in which

$$E_i = (p^2c^2 + m_i^2c^4)^{1/2} \approx pc\left(1 + \frac{m_i^2c^2}{2p^2}\right),$$

(as neutrinos, which have very small masses compared to the typical momenta in a practical neutrino detection experiment, are highly relativistic). The flavor eigenstates can be written in terms of a flavor mixing angle θ as

$$|\nu_e\rangle = \cos\theta|\nu_1\rangle - \sin\theta|\nu_2\rangle, \qquad |\nu_\mu\rangle = \sin\theta|\nu_1\rangle + \cos\theta|\nu_2\rangle.$$

Calculate the probability of a $\nu_e \to \nu_e$ transition as a function of time, and show that it can be expressed as

$$P_{\nu_e\to\nu_e} = 1 - \sin^2 2\theta \sin^2\left(\Delta m^2c^4\frac{L}{4E\hbar c}\right),$$

where $\Delta m^2 = m_2^2 - m_1^2$, $E = pc$ is the nominal neutrino energy, and $L = ct$ is the flight distance of the neutrino.

3. **[S1r 2.9, S2 2.10, S3 2.11]** A box containing a particle is divided into a right and a left compartment by a thin partition. If the particle is known to be on the right (left) side with certainty, the state is represented by the position eigenket $|R\rangle$ ($|L\rangle$), where we have neglected spatial variations within each part of the box. The most general state vector can then be written as

$$|\alpha\rangle = |R\rangle\langle R|\alpha\rangle + |L\rangle\langle L|\alpha\rangle.$$

The particle can tunnel through the partition; this tunneling effect is characterized by the Hamiltonian

$$H = \Delta(|L\rangle\langle R| + |R\rangle\langle L|),$$

where Δ is a real number with the dimension of energy.
(a) Find the normalized energy eigenkets and the corresponding energy eigenvalues.
(b) If the system at time $t = 0$ is given by the state $|\alpha\rangle$, find the state vector $|\alpha, t = t_0; t\rangle$ by applying the appropriate time-evolution operator to $|\alpha\rangle$.
(c) Suppose that at $t = 0$ the particle is on the right side with certainty. What is the probability for observing the particle on the left side as a function of time?
(d) Write down the coupled Schrödinger equations for $\langle L|\alpha, t_0 = 0; t\rangle$ and $\langle R|\alpha, t_0 = 0; t\rangle$. Show that the solutions to these equations are just what you expect from (b).
(e) Suppose in error H was written as

$$H = \Delta|L\rangle\langle R|.$$

By explicitly solving the most general time evolution problem with this Hamiltonian, show that probability conservation is violated.

4. **[S1r 2.21, S2 2.23, S3 2.28 modified]** A particle of mass m in one dimension is trapped between two rigid walls:

$$V(x) = \begin{cases} 0, & 0 < x < L, \\ \infty, & x < 0,\ x > L. \end{cases}$$

(a) At $t = 0$ it is known to be exactly at $x = L/2$ with certainty. What are the relative probabilities for the particle to be found in various energy eigenstates? Write down the wave function for $t \geq 0$. (You need not worry about absolute normalization, convergence, and other mathematical subtleties.)
(b) As a slightly more realistic version of this problem, instead assume that the initial state is a constant for $(L/2 - \epsilon) < x < (L/2 + \epsilon)$ and zero for $x < (L/2 - \epsilon)$ and $x > (L/2 + \epsilon)$, in which ϵ is a small parameter ($\epsilon/L \ll 1$). Calculate the probabilities for the particle to be found in various energy eigenstates and determine the wave function for $t \geq 0$.

5. **[S1r 2.23, S2 2.25, S3 2.30]** A particle of mass m in one dimension is bound to a fixed center by an attractive delta function potential:

$$V(x) = -\lambda\delta(x) \qquad\qquad (\lambda > 0).$$

At $t = 0$, the potential is suddenly switched off (that is, $V = 0$ for $t > 0$), leaving the wavefunction unchanged immediately after the switch ($t = 0^+$). Find an integral expression for the wavefunction at $t > 0$. (You do not need to evaluate the integral.)