Statistical mechanics II

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Physics 716.
Lecture course taught in Fall 2025 @ U Wisconsin-Madison

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Contents

I Theory of equilibrium (quantum) phase transitions **7**

1 Generalities about spontaneous symmetry breaking **7**

1.1 Recap: Basics of phase transitions 7

1.1.1 Definitions 7

1.1.2 The Ginzburg-Landau free energy 8

1.1.3 Critical behavior at mean field level. 10

1.2 Beyond Ginzburg-Landau theory. The role of fluctuations 12

1.2.1 Heuristics of the field integral in statistical mechanics 12

1.2.2 Upper critical dimension and limits of mean-field theory. 13

1.2.3 Fluctuation correction and lower critical dimension for systems with continuous symmetries 13

1.2.4 Lower critical dimension for systems with discrete symmetries . . . 15

1.3 Quantum to classical mapping and quantum phase transitions. Rotor model 16

1.3.1 Mean-field free energy 17

1.3.2 Effective action 18

1.3.3 Calculation of derivative terms 20

1.4 Feynman diagrammatics 20

1.5 Introduction to the renormalization group 22

1.5.1 Philosophy and basic recipe 22

1.5.2 Perturbing about the upper critical dimension: Wilson Fisher fixed point and RG at 4-ϵ dimensions 24

1.5.3 Perturbing about the lower critical dimension: non-linear-sigma model RG at 2 + ϵ dimensions 30

1.5.4 Berezinskii Kosterlitz Thouless transition in 2D 34

2 The Ising model and its connections to quantum information science 38

2.1 Transfer matrix solution in 2+0 D. Kramers-Wannier duality. Relationship to topological 1D superconductor 39

2.1.1 Transfermatrix representation 39

2.1.2 1D transfer matrix solution 41

2.1.3 2D: Kramers Wannier duality 42

2.1.4 2D Classical to Quantum Mapping: 1D transverse field Ising model 43

2.1.5 Fermionic representation and topology 46

2.1.6 Exact solution of critical exponents of the 2D Ising model 48

2.2 Optional sections on (1+1)D symmetry protected phases. 51

2.2.1 Optional section (to be prepared): The tetron – a topological qubit 51

2.2.2 Optional section: Symmetries, Symmetry fractionalization and ground state degeneracy. 51

2.2.3 Optional Section: Interacting Kitaev like chains: $\mathbb{Z}_8$ classification . 53

2.2.4 Optional section: Jordan Wigner transformation for periodic boundary conditions, Non-invertible symmetries 54

2.3 2D Transverse field Ising model, $\mathbb{Z}_2$ gauge theory and toric code: Introduction to topological order 55

2.3.1 $\mathbb{Z}_2$ lattice gauge theory, Kramers-Wannier duality in 2D and the toric code 55

2.3.2 Solution of the Toric Code 57

2.3.3 Perturbations: $g, J \neq 0$ 62

II Non-equilibrium theory: 63

3 Classical theory: Langevin and Fokker-Planck **63**

4 (Quantum) Master equation, Lindbladians and noise **63**

5 Keldysh formalism **63**

5.1 Coherent states 63

6 Boltzmann kinetic equation and H theorem **65**

Literature, scope, grades

Scope of this class:

This graduate level class is a natural continuation of Stat Mech I, Physics 715, taught in Spring 2025 by Alex Levchenko and – naturally – avoids all topics covered there. The syllabus reads:
"Statistical foundations, Liouville's theorem, Gibbs ensembles, classical and quantum distribution functions, entropy and temperature, connection with thermodynamics, partition functions, quantum gases, non-ideal gases, phase transitions and critical phenomena, non-equilibrium problems, Boltzmann equation and the H-theorem, transport properties, applications of statistical mechanics to selected problems."
To the best of my knowledge, the final non-equilibrium part was not covered or covered only partially in Stat Mech I.
As promised in the advertisements, this class...
... covers classical and quantum phase transitions, models for topological quantum computation, topological field theories, non-equilibrium physics, quantum noise.
...teaches advanced theoretical techniques: path integral methods, renormalization group, instanton calculus, lattice gauge theories, Keldysh technique.
...prepares for independent research in quantum sciences, solid state physics, plasma physics, general theoretical physics.

To achieve all of this, several topics are only touched upon, leaving more detailed reading to the students.

Literature

- Altland, A. and Simons, B. "Condensed Matter Field Theory", Cambridge University Press, 2010. Most of chapter 1
- Auerbach, A. "Interacting electrons and quantum magnetism", Springer, 1994. Chapter 1.5.3.
- Itzykson, C., Drouffe, J. " Statistical Field Theory", Cambridge University Press (1989). Chapter 2.1.
- Sachdev, S. "Quantum phase transitions", Cambridge University Press, 2011. Chapter 1.3

Topics not covered include

- Spin-coherent state path integral

- Quantum criticality in the presence of fermions in dimensions larger than 1 (Gross-Neveu problems, Landau damping in the presence of a Fermi surface)
- U(1) gauge theories and ...
- ... topological order in fractional quantum Hall systems
- bosonization
- conformal field theory
- field theory of disordered systems

Part I

Theory of equilibrium (quantum) phase transitions

1 Generalities about spontaneous symmetry breaking

1.1 Recap: Basics of phase transitions

Summary of this section:

- We refresh our knowledge about spontaneous symmetry breaking and
- ... basics about critical exponents at the mean-field level.

1.1.1 Definitions

- Definition: **Spontaneous symmetry breaking** (SSB).
  Let a given Hamiltonian or action (classical or quantum) display certain global symmetries.
  The symmetry is said to be "broken" spontaneously if the ground state transforms non-trivially under this symmetry.
  Examples:
  - The Hamiltonian describing a collection of water molecules is translationally and rotationally symmetric. An ice crystal spontaneously breaks these symmetries by locking the molecules.
  - The partition sum, e.g. Eq. (109) below, describing an easy axis magnet is $\mathbb{Z}_2$ symmetric, but the magnet on a compass needle breaks that symmetry (it can distinguish north and south pole).
- Definition: **Phase transitions and order of the phase transition**
  A phase transition is a non-analyticity in the free energy.
  In an nth order phase transition the free energy and all of its derivitives up to $(n-1)$th order are continuous.

- Definition: **Order parameter.**
  Consider an explicitly symmetry breaking source field $\vec{h}(\mathbf{x})$ in the partition sum (i.e. the generating functional) $\mathcal{Z}[\vec{h}]$. The expectation of the order parameter field

$$\langle\vec{\phi}(\mathbf{x})\rangle = \lim_{\vec{h}\to 0}\lim_{\mathrm{Vol}\to\infty}\frac{1}{\mathcal{Z}[\vec{h}]}\frac{\delta\mathcal{Z}}{\delta\vec{h}(\mathbf{x})} \tag{1}$$

  is non-vanishing in the phase with SSB, only.
  Comments:

  - For simplicity we here assume that $\vec{h},\vec{\phi}$ are vectors.
  - For the example of magnets, $\vec{h}$ is an external magentic field and $\vec{\phi}$ is the magnetization and their coupling in the Hamiltonian is $\delta H = -\int d^D x\vec{h}(\mathbf{x})\cdot\vec{\phi}(\mathbf{x})$.
  - It is crucial that the thermodynamic limit Vol $\to\infty$ is taken before the source field is switched off.
  - Equivalently, the phase with SSB is the one with long-range-order (even in the absence of a source field), i.e.

$$\langle\vec{\phi}_i(\mathbf{x})\vec{\phi}_j(\mathbf{x}')\rangle_{\vec{h}=0} \stackrel{|\mathbf{x}-\mathbf{x}'|\to\infty}{\longrightarrow} \text{const.} \neq 0. \tag{2}$$

  - From the definition of an nth order of the phase transition, we see that the order parameter develops continuously for $n \geq 2$.

- Definition: **Universality and universal critical exponents**.
  Continuous phase transitions (i.e. 2nd order and higher) are characterized by power-law behavior in thermodynamics and response functions. The exponents of these power laws are the same for a multitude of rather distinct physical systems and only depend on the symmetry and spatiotemporal dimension of the system.

1.1.2 The Ginzburg-Landau free energy

The Ginzburg-Landau free energy is a useful concept in the general theory of spontaneous symmetry breaking.

Consider a real N vector $\vec{\phi}$ ("order parameter") and

$$F_{\mathrm{GL}}[\vec{\phi}] = \int d^D x \alpha \sum_i [\partial_{x_i}\vec{\phi}^T][\partial_{x_i}\vec{\phi}] + \beta\vec{\phi}^T\vec{\phi} + \frac{\lambda}{2}(\vec{\phi}^T\vec{\phi})^2 \tag{3}$$

Comments

- Note that, in this section β is not the inverse temperature.
- Examples for this kind of effective action are
  - for $\vec{\phi} = \phi \in \mathbb{R}$: Easy-axis ("Ising"-) magnets. This theory is also called ϕ^4 theory. An explicit microscopic derivation is relegated to the exercises.
  - $\vec{\phi} \in \mathbb{R}^2$: Easy-plane ("XY"-)magnets. The superfluid transition is equivalent to this problem under the identification of $\vec{\phi} \propto (\mathrm{Re}(\Delta), \mathrm{Im}(\Delta))$
  - $\vec{\phi} \in \mathbb{R}^3$: fully spin-rotational ("Heisenberg"-) magnets.
  - $\vec{\phi} \in \mathbb{R}^4$: magnets breaking SU(2) completely (non-coplanar order, e.g. spiral or tetrahedral).
- The parameter $\beta = T - T_c$ changes sign at the transition, leading to a Mexican hat potential with minimum at $|\vec{\phi}| = \phi_0 = \sqrt{-\beta/\lambda}$
- The free energy is invariant under transformations

$$\vec{\phi} \to R\vec{\phi} \tag{4}$$

  where $R \in O(N)$.
- Below T_c, ...
  - we find a finite expectation value of $|\vec{\phi}| = \phi_0 > 0$ thus $\vec{\phi}$ has to point into a specific selection breaking the O(N) symmetry. This is the defining characteristic of *spontaneous symmetry breaking:* The ground state of a system has less symmetries than the underlying Hamiltonian (or action).
  - we can parametrize solutions with the same energy as $\vec{\phi} = \phi_0(\pi_1, \dots, \pi_{N-1}, \sqrt{1 - \sum_i \pi_i^2})$, assuming $\sum_i \pi_i^2 \leq 1$ and π_i $(i = 1, \dots N-1)$ real fields.
    * Similarly to the phase mode in superfluids, all π_i are gapless modes (typically, their dispersion relation is linear).
    * They are called "Goldstone bosons" in view of Goldstone's theorem:
      For each generator of the O(N) that is spontaneously broken, there is a scalar boson.
    * Here, the symmetries that are spontaneously broken are associated to the $N-1$ ways we can rotate $\vec{\phi} = \phi_0(1, 0, \dots 0)$ away from pointing to the "north-pole" of $\mathbb{S}^N$.
  - The first instance of such Goldstone bosons were pions in nuclear physics, hence the notation (there, the gaplessness is approximate).

1.1.3 Critical behavior at mean field level.

The Ginzburg-Landau functional allows to capture a variety of features at a second order phase transition. As the order parameter field $\vec{\phi}$ becomes massless at T_c. This leads to a variety of universal power-laws

- order parameter $|\vec{\phi}| \sim |T_c - T|^\beta$ at $\vec{h} = 0$ and $T < T_c$
- order parameter $|\vec{\phi}| \sim |\vec{h}|^{1/\delta}$ at $T = T_c$
- specific heat $C_V \sim |T - T_c|^{-\alpha}$
- susceptibility $\chi \sim |T - T_c|^{-\gamma}$.
- correlation length $\xi \sim |T - T_c|^{-\nu}$

Here, we also added an external source field

$$\delta F_{GL}[\vec{\phi}, \vec{h}] = -\int d^D x \vec{h}^T \vec{\phi}, \tag{5}$$

to the Ginzburg-Landau action (it is taken to zero at the end of any calculation). This field explicitly breaks the symmetry of the system.

Note:A first order transition is captured, e.g., for $T < T_c$, $N = 1$ and tuning h across zero: Here, the relative energy of local minima of the free energy changes.

Now we evaluate these universal power-laws at mean-field level. Below T_c the mean-field equation in the presence of the field is

$$-\vec{h} + 2\lambda(\beta/\lambda + |\vec{\phi}|^2)\vec{\phi} = 0. \tag{6}$$

Away from criticality and at zero field $|\vec{\phi}| = \phi_0 \theta(T_c - T)$ and $\phi_0 = \sqrt{-\beta/\lambda}$. Taking into account $\vec{h}$ induced corrections away from criticality

$$\vec{\phi} = (\phi_0 + \frac{h}{4\phi_0^2 \lambda})\hat{h}\theta(T_c - T) + \frac{\vec{h}}{2\beta}\theta(T - T_c) \tag{7}$$

Right at $T = T_c$ but non-zero h

$$\vec{\phi} = \hat{h}(h/2\lambda)^{1/3}. \tag{8}$$

- **Order parameter exponent at $\vec{h} = 0$.** As discussed $|\vec{\phi}| \sim \sqrt{-\beta} \sim \sqrt{T_c - T}$ and thus $\beta = 1/2$

- **Order parameter exponent at finite field but $T = T_c$:** From the solution above we read off $\delta = 3$

- **Specific heat exponent:** Inserting the mean-field solution $|\vec{\phi}| = \phi_0$ we find a free energy
$$f \equiv \frac{F}{L^D} = -\frac{\beta^2}{2\lambda} = -\frac{[T-T_c]^2}{2\lambda}\theta(T_c - T). \tag{9}$$
From this we find a contribution to the specific heat which characteristically jumps at T_c
$$C_V = -T\frac{\partial^2 f}{\partial T^2} = \frac{T}{\lambda}\theta(T_c - T). \tag{10}$$
While there is a characteristic jump, the specific heat does not diverge at criticality and $\alpha = 0$ at mean-field level. *(Keep in mind that there are additional contributions to $C(T)$ which are only weakly T dependent near T_c.*

- **Susceptibility exponent:** Taking the definition of susceptibility
$$\chi = -\frac{d^2F/L^D}{dh^2} = \frac{d(\hat{h}\cdot\vec{\phi})}{dh} \propto \frac{1}{|T_c - T|}, \tag{11}$$
i.e. $\gamma = 1$.

- **Correlation length exponent:** We observe that the $\alpha\nabla^2 \sim \alpha q^2$ enters with the same power as β in the free energy. By power counting we thus see that the typical lenght scale of the system behaves as
$$\xi \sim \sqrt{1/\beta} \Rightarrow \nu = 1/2. \tag{12}$$
A more mathematical argument follows: At $\lambda = 0$ and $T > T_c$, the $\vec{\phi}$ field correlator is (see next chapter)
$$D_\phi(\mathbf{q}) = [\alpha\mathbf{q}^2 + \beta]^{-1}. \tag{13}$$
Thus, we see that the correlation length ($\equiv$ typical length scale at which order parameter fluctuations die off) is $\xi = \sqrt{\alpha/\beta} \sim |T - T_c|^{-1/2}$ and $\nu = 1/2$. *(A similar argument can be made below T_c, expanding near the minimum of the Mexican hat. The mass of the longitudinal model also leads to $\nu = 1/2$.)*

1.2 Beyond Ginzburg-Landau theory. The role of fluctuations

Summary of this section:

- We here include fluctuations on top of the mean-field solution...
- ... which leads to the notion of upper critical dimension (at and above which the transition is mean-field like) and lower critical dimension (at and below which there is no symmetry breaking).

1.2.1 Heuristics of the field integral in statistical mechanics

- As per the standard definition of the partition sum, one ought to sum over all field configurations $\vec{\phi}(\mathbf{x})$ and weigh their contribution with the factors $e^{-F[\vec{\phi}(\mathbf{x})]/T}$
- However, as $\vec{\phi}(\mathbf{x})$ are continuum variables instead of a discrete partition sum we have a functional integral

$$\mathcal{Z} = \int \prod_{i=1}^{N} \mathcal{D}\phi_i e^{-S[\vec{\phi}]}, \quad S[\vec{\phi}] = F[\vec{\phi}(\mathbf{x})]/T. \tag{14}$$

The measure $\mathcal{D}\phi_i \sim \prod_{\mathbf{x}} d\phi_i(\mathbf{x})$ follows from the continuum limit (see Exercises for an explicit derivation in the context of the Ising model and Sec. 1.3, below).

- Physically, this mathematical procedure encodes the role of thermal fluctuations.
- Correlators are defined

$$\langle \phi_{i_1}(\mathbf{x}_1)\phi_{i_2}(\mathbf{x}_2)\dots\phi_{i_n}(\mathbf{x}_n)\rangle = \frac{1}{\mathcal{Z}}\int \prod_{i=1}^{N} \mathcal{D}\phi_i \phi_{i_1}(\mathbf{x}_1)\phi_{i_2}(\mathbf{x}_2)\dots\phi_{i_n}(\mathbf{x}_n) e^{-S[\vec{\phi}]}. \tag{15}$$

- In the limit $\lambda = 0$, the problem becomes exactly soluble by Fourier transform

$$\mathcal{Z} = \int \prod_{i=1}^{N} \prod_{\mathbf{q}}{}' d\phi'_{i,\mathbf{q}} d\phi''_{i,\mathbf{q}} e^{-\sum'_{\mathbf{q}} |\phi_{i,\mathbf{q}}|^2 [\alpha \mathbf{q}^2 + \beta]/T} = \text{const.} \times \prod_{i=1}^{N} \prod_{\mathbf{q}}{}' \frac{1}{\alpha \mathbf{q}^2 + \beta} \tag{16}$$

Here, the prime on the sum and product imply that only half of the momenta should be taken into account, because $\vec{\phi}_{\mathbf{q}} = \vec{\phi}^*_{-\mathbf{q}}$ relates complex conjugates of $\phi_{i\mathbf{q}} = \phi'_{i\mathbf{q}} + i\phi''_{i\mathbf{q}}$ and we introduced a β-independent constant.

- The 2-point correlator at $\lambda = 0$ is easily evaluated to be

$$D_{ij}(\mathbf{x}_i - \mathbf{x}_j) = \langle \phi_i(\mathbf{x}_i)\phi_j(\mathbf{x}_j)\rangle \Leftrightarrow \langle \phi_i(\mathbf{q})\phi_j(\mathbf{p})\rangle = \underbrace{\frac{T}{2(\alpha \mathbf{q}^2 + \beta)}}_{D(\mathbf{q})} \delta_{ij}\delta_{\mathbf{q}+\mathbf{p},0} \tag{17}$$

1.2.2 Upper critical dimension and limits of mean-field theory.

We assume $T > T_c$ and calculate the correction to the specific heat due to thermal fluctuations. We can safely drop the quartic interaction term in Eq. (3) and use Eq. (16) above. From this we obtain

$$
\begin{aligned}
C_V{}^{(\text{fluct})} &= -T\frac{d^2F/L^D}{dT^2} = T^2\frac{d^2\ln(\mathcal{Z})/L^D}{dT^2} \qquad (18)\\
&\simeq \frac{T_c^2}{L^D}\frac{d\ln\mathcal{Z}}{d\beta^2}\\
&= -T_c^2\frac{d}{d\beta^2}\frac{1}{L^D}\sum_{i,\mathbf{q}}{}^{'}\ln(\alpha\mathbf{q}^2+\beta)\\
&= \frac{N}{2}T_c^2\int\frac{d^Dq}{(2\pi)^D}\frac{1}{(\beta+\alpha\mathbf{q}^2)^2}\\
&= \frac{N}{2}\left(\frac{T_c}{T-T_c}\right)^{2-D/2}(T_c/\alpha)^{D/2}I_D \qquad (19)
\end{aligned}
$$

Here, $I_D = \int d^Dx/[(2\pi)^D(x^2+1)^2]$ is a dimensionless integral, and at $\simeq$ we only kept the most singular contribution near T_c.

Comments

- Just above T_c, the order parameter fluctuations lead to an increase in specific heat.
- This fluctuation correction diverge at T_c as long as $D < 4$.
  - Thus, the mean field behavior (=neglecting fluctuation corrections near T_c) is only accurate if $D > 4$
  - For $D < 4$ there is a critical regime around T_c, where non-Gaussian fluctuations become important.
  - The limits of this temperature window is defined by the Ginzburg-Levanyuk criterion: Mean field behavior is a good description as long as $C_V{}^{(\text{fluct})} < \Delta C = T_c/\lambda$

1.2.3 Fluctuation correction and lower critical dimension for systems with continuous symmetries

We calculate the fluctuation corrections to the expectation value of the order parameter (e.g. the magnetization) below T_c.

- Effective action of Goldstone modes: Use the
  - Ginzburg Landau functional, Eq. (3)
  - and the parametrization of $\vec{\phi} = \phi_0 \left(\pi_1, \ldots, \pi_{N-1}, \sqrt{1 - \sum_i^{N-1} \pi_i^2} \right)$.

  to obtain

$$F_{\text{Goldstone}} = \sum_{i=1}^{N-1} \int' \frac{d^D q}{(2\pi)^D} \alpha \phi_0^2 \mathbf{q}^2 |\pi_i(\mathbf{q})|^2. \tag{20}$$

- Using this, we calculate the correction to the vacuum expectation value of the order parameter

$$\begin{aligned}
\langle \vec{\phi}(\mathbf{x}) \rangle &= \phi_0 \left\langle \begin{pmatrix} \pi_1(\mathbf{x}) \\ \vdots \\ \pi_{N-1}(\mathbf{x}) \\ \sqrt{1 - \sum_i \pi_i^2(\mathbf{x})} \end{pmatrix} \right\rangle \\
&\simeq \phi_0 \hat{e}_N \left(1 - \sum_i \langle \pi_i^2(\mathbf{x}) \rangle / 2 \right) \\
&= \phi_0 \hat{e}_N \left[1 - \frac{N-1}{4} \underbrace{\int' \frac{d^D q}{(2\pi)^D} \frac{T}{\alpha \phi_0^2 \mathbf{q}^2}}_{\frac{\text{Vol}(\mathbb{S}^D)}{2(2\pi)^D \alpha \phi_0^2} \int_{L^{-1}} dq q^{D-3}} \right].
\end{aligned} \tag{21}$$

Here, $\text{Vol}(\mathbb{S}^D)$ is the volume of the $D-$sphere and we introduced system size L as an IR cut-off.

- The Goldstone-mode induced corrections to the expectation value of the order parameter are negative (= fluctuations tend to suppress the ordered state)
- As the integral $\int_{L^{-1}} dq q^{D-3}$ diverges for $D \leq 2$ in the thermodynamic limit, the suppression of the the mean-field expectation value of $\vec{\phi}$ diverges for $D \leq 2$.
- Analyzing these fluctuation corrections more carefully, leads to the *Mermin-Wagner-Hohenberg theorem:*
  "Continuous symmetry cannot be spontaneously broken at finite temperature in $D \leq 2$."
  Comments
  * $D = 2$ is called the lower critical dimension.

* Note that at zero temperature, even systems with two spatial dimensions may display spontaneously continuous symmetries (essentially because the imaginary time direction in the field integral adds an extra dimension to the problem, see Sec. 1.3).
* Further note several "quasi-" loop holes to the Mermin-Wagner theorem:
  · For $N = 2$ and $D = 2$, fluctuation corrections are less effective because the order parameter manifold is flat. As a consequence, there is a phase transition between a phase with algebraic order and a disordered phase with short-range correlations, see Sec. 1.5.4.
  · For $D = 1+1$ quantum theories, algebraic (i.e. quasi-long-range ordered phases) can also exist when the Berry phase associated to topological field configurations can also prevent the disordered phase, see Sec. 1.5.3 for an example.

1.2.4 Lower critical dimension for systems with discrete symmetries

- For discrete symmetries there are no gapless Goldstone bosons. As a consequence, the SSB is more stable at low D.

- The exact solution of the classical Ising model in $D = 2$ spatial dimensions, Sec. 2.1 below, demonstrates that discrete symmetries can be broken in 2D. In fact, the lower critical dimension is $D = 1$.

- To see this consider the difference in the free energy between an ordered state $\phi(\mathbf{x}) = \phi_0$ and the state with a domain wall separating different spatial regions with different local ground state $\phi(\mathbf{x}) \simeq \pm\phi_0$. We find

$$\Delta F = E_{\text{kink}} L^{D-1} - TD\ln(L). \tag{22}$$

  (L in units of lattice spacing) The first term is the energetic cost of a domain wall. The second one the entropic gain due to the kink. Clearly, for $D > 1$, the energetic cost overpowers the entropic gain and the ordered state is stable. For $D = 1$, the entropic gain favors the spontaneous formation and proliferation of domain walls (kinks) and thus long-range order is lost.

1.3 Quantum to classical mapping and quantum phase transitions. Rotor model

In this section we introduce the **Quantum to classical mapping**:
"Quantum many-body problems in d dimensions map to classical models in $D = d+1$ dimensions (the extra dimension being time)."
To be concrete...

- ... we consider the path integral description of a Josephson-junction...
- ... which we explicitly demonstrate to be determined by an action of the Ginzburg-Landau form.

Note that the action of the quantum problem may however have extra imaginary contributions from the quantum mechanical (Berry) phases. Such contributions are the origin of topological terms, Sec. 1.5.3

We consider the rotor model on a $d-$dimensional hypercube ($\phi \in [0, 2\pi)$) in the limit of small temperatures.

$$\hat{H} = \sum_i -\frac{\partial^2_{\phi_i}}{2M} - \frac{1}{2}\sum_{ij}[J_{ij}e^{i(\phi_i - \phi_j)} + H.c.]. \tag{23}$$

Physically, this model describes

- superconducting islands on vertices (superconducting phase ϕ_i, number operator $\hat{N} = -i\partial_{\phi_i}$)
- Josephson couplings J_{ij} between them (on physical grounds, those are usually nearest neighbor $J_{ij} = 0$ if $|\mathbf{x}_i - \mathbf{x}_j| > 1$ and $J_{ij} = J$ else). Technically we have to assume J_{ij} positive definite *(This may be achieved by adding a term $\propto \delta_{ij}$ to it.)*
- In the context of Josephson junction arrays $1/M$ is the charging energy (often denoted E_C).
- When JM is large, the ϕ all want to order (this is the superfluid, broken U(1)) symmetry). When JM is small, the charging energy term wins and occupation number is a good quantum number (the system becomes an insulator, each rotor has angular momentum zero/i.e. each superconducting island has excess charge zero in the ground state).
- In the exercises you consider a an analogous Hamiltonian with $(-i\partial_\phi - A)$. Then, A the background charge (given by the gate voltage V_g) on each island. To keep things

simple, we drop A here, but it is the source of interesting imaginary terms in the action, see Sec. 1.5.3. Such imaginary terms stem from quantum mechanical phases and thus transcend the quantum to classical correspondence.

We interpret the phase on each vertex as a first quantized coordinate and use the quantum mechanical path integral representation of the partition sum $\mathcal{Z} = \text{tr}\,[e^{-\hat{H}/T}]$ as in Homework sheet 1

$$\mathcal{Z} = \int \prod_i d\phi_i e^{-\int_0^{1/T} d\tau \sum_i \frac{M}{2}\dot{\phi}_i^2 + \frac{1}{2}\int_0^\beta d\tau \sum_{ij}[J_{ij}e^{i(\phi_i-\phi_j)}+H.c.]} \tag{24}$$

$$= \int \prod_i (dz_i' dz_i'') \prod_i d\phi_i e^{-\int_0^{1/T} d\tau \sum_{ij} z_i^* \frac{[J^{-1}]_{ij}}{2} z_j + \sum_i [\; e^{-i\phi_i} z_i + z_i^* e^{i\phi_i}]} e^{-\int_0^{1/T} d\tau \sum_i \frac{M}{2}\dot{\phi}_i^2} \tag{25}$$

$$= \int \prod_i (dz_i' dz_i'') e^{-S_{\text{eff}}[z]} \tag{26}$$

- We dropped unimportant constants at equality signs.
- On each site we introduced complex integration variables ("Hubbard-Stratonovich" fields) $z_i = z_i' + iz_i''$ to decouple the Josephson interactions. (It's easier to understand the second equality sign starting in the second line and going back to the first by completing the square).
- Recall from the exercises that $e^{i\phi(0)} = e^{i\phi(\tau=1/T)}$, z fields inherit this property.
- In the third equality we formally "integrated out" the ϕ fields which defines the effective action of z.

1.3.1 Mean-field free energy

Before calculating the action for any field configuration, we firs consider the static, homogeneous case $z_i(\tau) = z$. This yields the mean-field free energy.

- From the kinetic term we obtain the free energy per site

$$f_{\text{kin}} = \frac{|z^2|}{2J(0)}. \tag{27}$$

where we used the Fourier transform $\sum_i e^{-i\mathbf{p}(\mathbf{x}_i-\mathbf{x}_j)}[J^{-1}]_{ij} = 1/J(\mathbf{p})$

- The term from the rotor is best calculated in operator formalism (rather than path integral formalism), i.e. we use

$$\begin{aligned}\hat{H}_{\mathrm{MF}}(z) &= -\frac{\partial_\phi^2}{2M} - [\, e^{-i\phi} z + z^* e^{i\phi}] \\ &= \sum_l \left[|l\rangle \frac{l^2}{2M} \langle l| - \underbrace{(|l\rangle\, z\, \langle l+1| + |l+1\rangle\, z^*\, \langle l|)}_{\hat{V}} \right]\end{aligned} \tag{28}$$

  to calculate the partition sum per site. Here we switched to angular momentum basis which is more useful.

$$\mathcal{Z}_{\mathrm{MF}}(z) = \mathrm{tr}\,[e^{-\hat{H}_{\mathrm{MF}}(z)}/T] = \sum_{l=-\infty}^{\infty} e^{-E_l(z)/T} \tag{29}$$

  The eigenenergies can be calculated using perturbation theory and are

$$E_l(z) = E_l^{(0)}(z) + E_l^{(2)}(z) + E_l^{(4)}(z). \tag{30}$$

  At small temperatures $\beta \gg M$ only the ground state contributes, it has energy corrections

$$\begin{aligned}E_0^{(2)}(z) &= \sum_{\pm} \frac{|\,\langle 0|\hat{V}| \pm 1\rangle\,|^2}{-E_{\pm 1}^{(0)}} \\ &= -4M|z|^2\end{aligned} \tag{31}$$

$$\begin{aligned}E_0^{(4)}(z) &= \underbrace{\sum_{\pm} \frac{|\,\langle 0|\hat{V}| \pm 1\rangle \langle \pm 1|\hat{V}| \pm 2\rangle\,|^2}{-[E_{\pm 1}^{(0)}]^2 E_{\pm 2}^{(0)}}}_{=-2|z|^4 \times (2M)^2 \times (2M/2^2)} - \underbrace{E_0^{(2)}}_{=-4M|z|^2} \underbrace{\sum_{\pm} \frac{|\,\langle 0|\hat{V}| \pm 1\rangle\,|^2}{[E_{\pm 1}]^2}}_{=2|z|^2 (2M)^2} \\ &= M^3 |z|^4 [-8 + 32] = 24|z|^4 M^3\end{aligned} \tag{32}$$

  with this we find the total free energy near zero temperature

$$f = \frac{|z|^2}{2}[\frac{1}{J(0)} - 8M] + 24M^3|z|^4 \tag{33}$$

- Note the mean-field transition at $1 = 8MJ(0)$ (large $J(0)M$ correspond to broken U(1) symmetry) and the standard Ginzburg Landau form with positive quartic term.

1.3.2 Effective action

We want to determine the effective action up to quartic order in z fields. By Taylor expansion of the exponential and reexponentiation it has the form

$$
\begin{aligned}
S_{\text{eff}}[z] = &-\frac{1}{2}\int_{\tau_1,\tau_2}\sum_{i_1,i_2}\langle[\, e^{-i\phi_{i_1}} z_{i_1} + z^*_{i_1} e^{i\phi_{i_1}}]_{\tau_1}[\, e^{-i\phi_{i_2}} z_{i_2} + z^*_{i_2} e^{i\phi_{i_2}}]_{\tau_2}\rangle \\
&-\frac{1}{24}\int_{\tau_{1,2,3,4}}\sum_{i_{1,2,3,4}}\langle\langle\prod_{\alpha=1}^{4}[\, e^{-i\phi_{i_\alpha}} z_{i_\alpha} + z^*_{i_\alpha} e^{i\phi_{i_\alpha}}]_{\tau_\alpha}\rangle\rangle
\end{aligned}
\tag{34}
$$

All angular brackets represent expectation values with respect to the free rotor action $\int_0^{1/T} d\tau \sum_i \frac{M}{2}\dot{\phi}_i^2$. Given the periodicity of fields this integral can be extended to $\int_{-\infty}^{\infty} d\tau \sum_i \frac{M}{2}\dot{\phi}_i^2$ at zero temperature.

However, we are only interested in slow fields z and for those we only keep derivatives for the quadratic term. With this procedure we find a quantum action of the form of the Ginzburg-Landau free energy Eq. (3)

$$
\boxed{S_{\text{eff}}[\psi] = \int d\tau \int d^d x \frac{\alpha}{v^2}|\partial_\tau \psi|^2 + \alpha \sum_i |\partial_{x_i}\psi|^2 + \beta|\psi|^2 + \frac{\lambda}{2}|\psi|^4.}
\tag{35}
$$

where $\psi_i = [J^{-1}]_{ij} z_j/\sqrt{2}$ and all coefficients can be determined microscopically

$$
\begin{aligned}
\beta &= J(0)[1 - 8MJ(0)]\,\textit{(as above)} && \text{(36a)}\\
\alpha &= [J''(0) - 16MJ(0)J''(0)/2] \simeq -J''(0)/2 \ \textit{(for unit lattice constant)} && \text{(36b)}\\
\alpha/v^2 &= M\frac{(8MJ(0))^2}{2} \simeq M/2 && \text{(36c)}\\
\Rightarrow v &= \sqrt{-J''(0)/M}, && \text{(36d)}\\
\lambda &= 4J(0)^4 \times 48M^3 \simeq 3J(0)/8. && \text{(36e)}
\end{aligned}
$$

The $\simeq$ denotes values near criticality. By comparison to the case of constant ψ, we have determined β, λ in the previous section. The other constants are derived in the next section.

Before that we comment that

- The d dimensional quantum problem near criticality resembles the $d+1$ dimensional $N = 2$ Ginzburg Landau theory
- We have seen the first instance of a qauntum phase transition (at $T = 0$ there is a transition between a "Mott" insulator at small $J(0)M$ and a superfluid at large $J(0)M$)

- We here also see the first microscopic derivation of an effective field theory.
- Note that the phase transition we found has dynamical critical exponents z (i.e. correlation length and correlation time diverge with the same power of β, as can be guessed based on the analogous appearance of temporal and spatial derivatives in the action).

1.3.3 Calculation of derivative terms

To determine α, v we study the quadratic terms in more detail

- The Hubbard-Stratonovich decoupling term yields

$$\sum_{ij} z_i^* \frac{J_{ij}^{-1}}{2} z_j = \int \frac{d^d p}{(2\pi)^d} \frac{|z(\mathbf{p})|^2}{2J(\mathbf{p})} = \int \frac{d^d p}{(2\pi)^d} J(\mathbf{p}) |\psi(\mathbf{p})|^2 \tag{37}$$

- The second order contribution of Eq. (34) is

$$\begin{aligned} - \int_{\tau_1,\tau_2} \sum_{i_1,i_2} \langle [\, e^{-i\phi_{i_1}} z_{i_1}]_{\tau_1} [z_{i_2}^* e^{i\phi_{i_2}}]_{\tau_2} \rangle &= - \sum_i \int_{\tau,\Delta\tau} e^{-\frac{|\Delta\tau|}{2M}} z_i(\tau + \Delta\tau) z_i^*(\tau) \\ &= - \sum_i \int d\tau [4M |z_i(\tau)|^2 - 16 M^3 |\dot{z}_i(\tau)|^2] \qquad (38) \\ &= - \int d\tau \int \frac{d^d p}{(2\pi)^d} [8 M J(\mathbf{p})^2 |\psi(\mathbf{p},\tau)|^2 \\ &\quad - 32 M^3 J(\mathbf{p})^2 |\dot{\psi}(\mathbf{p},\tau)|^2]. \qquad (39) \end{aligned}$$

- We assume $J(\mathbf{p}) = J(0) + \mathbf{p}^2 J''(0)/2$ (usually $J''(0) < 0$) for isotropic Josephson couplings and sum up the two contributions. From this Eqs. (36) follows immediately. Note that β has been derived consistently using two different methods in the present and previous subsection.

1.4 Feynman diagrammatics

Summary of this section:

- Feynman diagrams are introduced as a means to represent the perturbation theory in the quartic term of the Ginzburg Landau functional.

We introduced the field integral representation of the partition sum in Eq. (14) and derived it for a particular quantum model in the previous section.

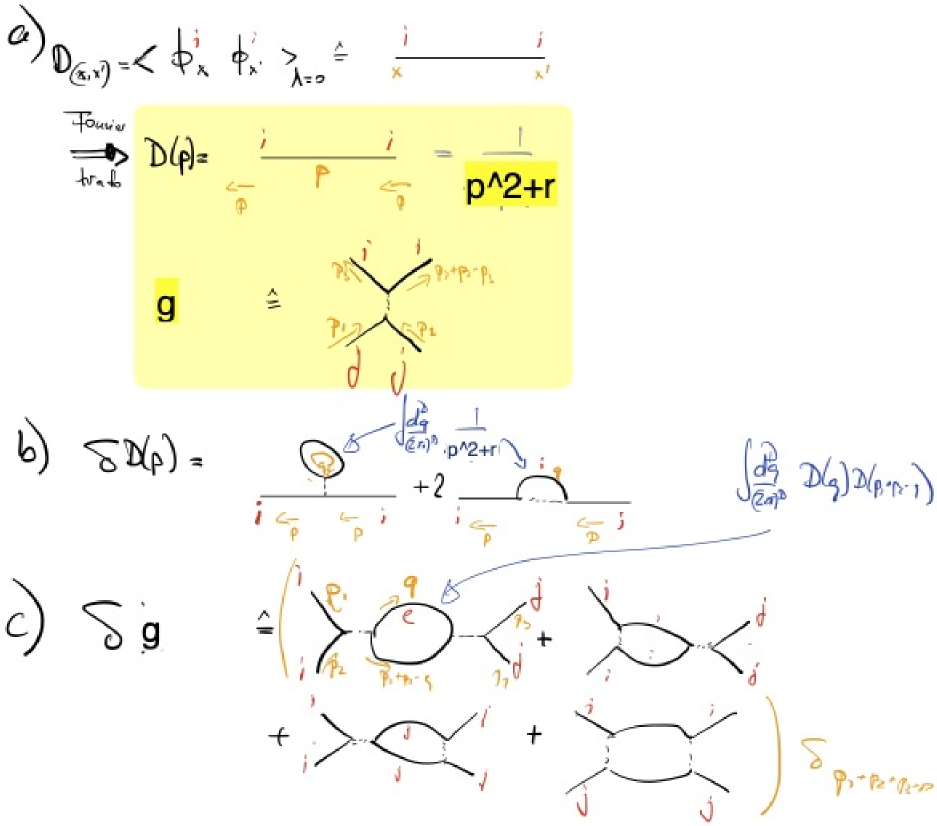


Figure 1: a) The basic elements for Feynman diagrams: The propagator (line) and the interaction vertex. b)/c) leading fluctuation corrections to the propagator and the vertex, respectively.

- Consider $\mathcal{Z} = \int \mathcal{D}\phi e^{-S[\vec{\phi}]}$ with effective action (i.e. related to the free energy as $S = F/T$)

$$S[\vec{\phi}] = \int d^D x \frac{1}{2}(\nabla\vec{\phi})^2 + \frac{r}{2}\vec{\phi}^2 + \frac{g}{4!}\vec{\phi}^4. \tag{40}$$

As compared to the previous notation, Eq. (3) we rescaled fields $\phi_i \to \sqrt{T/2\alpha}\phi_i$, and defined $r = \beta/\alpha, g = 3\lambda T/2\alpha^2$.

- We found the 2-point correlator at $g = 0$ which is

$$D_{ij}(\mathbf{x}_i - \mathbf{x}_j) = \langle \phi_i(\mathbf{x}_i)\phi_j(\mathbf{x}_j)\rangle_0 \Leftrightarrow \langle \phi_i(\mathbf{q})\phi_j(\mathbf{p})\rangle_0 = \underbrace{\frac{1}{\mathbf{q}^2 + r}}_{D(\mathbf{q})} \delta_{ij}\delta_{\mathbf{q}+\mathbf{p},0} \tag{41}$$

The index $_0$ indicates $g = 0$.

- More generally, Wick's theorem (in this formulation a simple properties of multi-dimensional Gaussian integral) allows to evaluate multi point correlators as a product of 2-point correlators

$$\langle \phi_{i_1}(\mathbf{x}_1)\phi_{i_2}(\mathbf{x}_2)\ldots\phi_{i_n}(\mathbf{x}_n)\rangle_0 \overset{n \text{ even}}{=} \sum_{\substack{\{(j_1,\mathbf{y}_1)\ldots(j_n,\mathbf{y}_n)\} \\ \text{pairings of} \\ (i_1,\mathbf{x}_1)\ldots(i_n,\mathbf{x}_n)}} D_{j_1 j_2}(\mathbf{y}_1 - \mathbf{y}_2)\ldots D_{j_{n-1}j_n}(\mathbf{y}_{n-1} - \mathbf{y}_n). \tag{42}$$

- Wick's theorem has an important practical application: When including g perturbatively, for example to the n-point correlator, it is convenient to resort to a diagrammatic representation of contributions, see Fig. 1 a) for a defnition of basic elements.

  - We first consider small g expansion of correlators to get a feeling. Note that we have to expand the exponentials both in the numerator and in the denominator (we use $1 = (i_1, \mathbf{x}_1)$ as a shorthand notation).

$$\langle \phi_1\phi_2\ldots\phi_n\rangle = \frac{\int \mathcal{D}\phi\phi_1\phi_2\ldots\phi_n e^{-S[\phi]}}{\int \mathcal{D}\phi e^{-S[\phi]}} \tag{43}$$

$$= \frac{\langle \phi_1\phi_2\ldots\phi_n e^{-\int d^D x g|\vec{\phi}(\mathbf{x})|^4/4!}\rangle_0}{\langle e^{-\int d^D x g|\vec{\phi}(\mathbf{x})|^4/4!}\rangle_0}, \tag{44}$$

    The contribution from denominator cancels the "disconnected Wick contractions" as we illustrate at 1st order and for $n = 2$

$$\langle \phi_1\phi_2\rangle|_{g^1} = \frac{g}{4!}\int d^D x[\langle \phi_1\phi_2|\vec{\phi}(\mathbf{x})|^4\rangle_0 - \langle \phi_1\phi_2\rangle_0\langle|\vec{\phi}(\mathbf{x})|^4\rangle_0] \tag{45}$$

    Clearly when ϕ_1 is contracted directly with ϕ_2 in the first term it will cancel the last term. Hence only "connected" contributions should be kept:

- Without going into details we summarize the main Feynman rules (in momentum space) which follow from the above considerations:
  * At order g^N draw all possible connected diagrams with n external legs *(= dangling lines)*, N vertices and the necessary number of lines (propagators) connecting them all.
  * label each propagator with a momentum and a flavor index i.
  * Impose momentum conservation at each vertex as well as flavor conservation along the lines.
  * Sum/Integrate over all internal momenta *(i.e. all but the n momenta at the dangling legs)*

  Feynman rules are also useful, e.g., for the corrections to the free energy. Depending on what you use them for, there are different combinatorial coefficients entering as a prefactor (not discussed here).

1.5 Introduction to the renormalization group

We have seen that, because of fluctuations, mean-field theory is not applicable to describe phase transitions below the upper critical dimension. Here we show how to incorporate those fluctuations systematically.

Summary of this section:

We explain the basics of renormalization group with 3 examples

- The Wilson-Fisher fixed point in $D = 4 - \epsilon$ (near the upper critical dimension)
- The perturbation about the lower critical dimension $D = 2 + \epsilon$
- The Berezinskii-Kosterlitz Thouless transition in $D = 2$.

For all of these examples we calculate important physical observables, e.g. the critical exponents.

1.5.1 Philosophy and basic recipe

- In Stat Mech we want to know about the low-energy/long wave length behavior of a system.
- In gapped systems, we just integrate out states above the gap (2nd order perturabtion theory).

- In gapless systems this is not possible, instead we resort to renormalization group: We iteratively integrate out short-wavelength/high energy fluctuations and step by step "renormalize" the coupling constants of the theory from there value in the UV to their value at the scale at which we probe the system.
- Technically there are several RG methods. In particular
  - In the field theory context, renormalization corresponds to iterative (in a coupling constant, $1/N$) inclusion of counter terms in the bare action in a way that observable quantities are finite. When you need only a finite number of counter terms at each perturbative step the theory is called renormalizable. The price is that, effectively, coupling constants become scale dependent.
  - In these lecture notes we consider the more condensed-matter-like approach: We integrate out high-energy degrees of freedom successively and incorporate the leading flucation corrections (quantum or thermal) into the coupling constants. This procedure can be done consistently when the leading corrections are not too strongly divergent in the infrared – this again corresponds to the renormalizability assumption.
  - Also note that finite temperature corresponds to finite size in time direction and that we impose periodic boundary conditions in time direction!

1.5.2 Perturbing about the upper critical dimension: Wilson Fisher fixed point and RG at 4-ϵ dimensions

Consider the effective action as in Eq. (40)

$$S[\vec{\phi}] = \int d^D x \frac{1}{2}(\nabla\vec{\phi})^2 + \frac{r}{2}\vec{\phi}^2 + \frac{g}{4!}\vec{\phi}^4. \tag{46}$$

Here, $D = 4-\epsilon$, $\vec{\phi}$ is a real N vector field and we assume a cut-off $\mathbf{p}^2 < \Lambda^2$ in momentum space.

We will calculate the renormalization group equations perturbatively in g, as imposed by the following condition on the dimensionless coupling constant

$$\boxed{\tilde{g} \equiv g/\Lambda^\epsilon \ll 1.} \tag{47}$$

Step 1: Splitting slow and fast modes We split our fields in momentum space ($(dp) = d^D p/(2\pi)^D$)

$$\phi(\mathbf{x}) = \int (dp) e^{i\mathbf{p}\cdot\mathbf{x}}\phi(\mathbf{p}) \tag{48}$$

into slow and fast modes

$$\phi_s(\mathbf{p}) = \phi(\mathbf{p})\theta(\tilde{\Lambda}^2 - \mathbf{p}^2) \tag{49}$$

$$\phi_f(\mathbf{p}) = \phi(\mathbf{p})\theta(\Lambda^2 - \mathbf{p}^2)\theta(\mathbf{p}^2 - \tilde{\Lambda}^2). \tag{50}$$

Here, we introduced a new, running energy scale $\tilde{\Lambda} = e^{-\ell}\Lambda$ and we will iteratively integrate out the fast modes. In order to make precise mathematical statements, we have to impose

$$\boxed{\epsilon \ll 1/\ell \ll 1}, \tag{51}$$

i.e. we stay close to the upper critical dimension (in addition to requiring Eq. (47)). Using this we find $S = S[\vec{\phi}_s] + S[\vec{\phi}_f] + S_{\text{IA}}$

$$S_{\text{IA}} = \frac{g}{12}\int d^D x \vec{\phi}_s^2\vec{\phi}_f^2 + 2(\vec{\phi}_s \cdot \vec{\phi}_f)^2. \tag{52}$$

- Note that the quadratic terms (gradient term and mass r-term) are diagonal in $\mathbf{p}$ space and hence ϕ_s, ϕ_f don't mix there
- We further used that phase space arguments to drop terms of the type $\phi_s^3\phi_f$
  - the local interaction leads to a delta function in momentum space.
  - the condition $\tilde{\Lambda} \ll \Lambda$ implies that the phase space of 3 small momenta adding up to a large momentum is negligable.
- We also droped the term $\phi_f^3\phi_s$. It leads two 2-loop corrections (e.g. sunset diagram $\mathcal{O}(g^2)$ for r) which are of subleading importance (leading correction to r is the tadpole, $O(g)$)
- We can also drop interaction terms g in $S[\vec{\phi}_f]$ as they also lead to higher order corrections terms in g.

Step 2: Integration of fast modes The integration of fast modes leads to an effective action

$$e^{-S_{\text{eff}}[\phi_s]} = \int \mathcal{D}\phi_f e^{-S[\phi_s+\phi_f]} \simeq e^{-S[\phi_s]-\langle S_{\text{IA}}\rangle_f+\frac{1}{2}\langle\langle S_{\text{IA}}^2\rangle\rangle_f} \tag{53}$$

- For calculation of the tadpole diagrams (cf. Fig. 1 b). we use ($\Omega_D = [2\pi^{D/2}/\Gamma(D/2)]/(2\pi)^D$)

$$\langle\phi_{f,i}(x)\phi_{f,j}(x)\rangle_f = \delta_{ij}[\int_{\tilde{\Lambda}}^{\Lambda} \frac{d^D p}{(2\pi)^D}\frac{1}{p^2+r} \tag{54}$$

$$\simeq \delta_{ij}\Omega_D \int_{\tilde{\Lambda}}^{\Lambda} dp[p^{D-3} - rp^{D-5}]]$$

$$= \delta_{ij}[\Omega_D\frac{\Lambda^{D-2} - \tilde{\Lambda}^{D-2}}{D-2} - r\Omega_D\frac{\Lambda^{D-4} - \tilde{\Lambda}^{D-2}}{D-4}] \tag{55}$$

leads to

$$\langle S_{\rm IA}\rangle_f = \frac{g}{12}\int d^Dx\vec{\phi}_s^2(N+2)\Omega_D[\frac{\Lambda^{D-2}-\tilde{\Lambda}^{D-2}}{D-2} - r\frac{\Lambda^{D-4}-\tilde{\Lambda}^{D-4}}{D-4}], \tag{56}$$

or effectively, in $S_{\rm eff}$

$$r\to r' \equiv r + g\Lambda^{D-2}\frac{N+2}{6}\Omega_D[\frac{1-e^{-(D-2)\ell}}{D-2} - r\Lambda^{-2}\frac{1-e^{-(D-4)\ell}}{D-4}]. \tag{57}$$

- For the calculation of the corrections to the vertex stemming from $\langle\langle S_{\rm IA}^2\rangle\rangle_f$, we keep in mind the diagrams c) in Fig. 1 and find

$$\frac{1}{2}\langle\langle S_{\rm IA}^2\rangle\rangle_f = \frac{g^2}{16}\frac{N+8}{9}\int d^Dx(\vec{\phi}_s)^4\underbrace{\int\frac{d^Dp}{(2\pi)^D(p^2+r)^2}}_{\simeq\Omega_D\frac{\Lambda^{D-4}-\tilde{\Lambda}^{D-4}}{D-4}} \tag{58}$$

or equivalently

$$\frac{g}{4!}\to\frac{g'}{4!}\equiv\frac{g}{4!}-\frac{g^2}{16}\frac{(N+8)}{9}\Omega_D\Lambda^{D-4}\frac{1-e^{-(D-4)\ell}}{D-4} \tag{59}$$

Step 3: rescaling The new action has the form

$$S_{\rm eff} = \frac{1}{2}\int^{\tilde{\Lambda}} d^Dp[\mathbf{p}^2+r']|\vec{\phi}_s(\mathbf{p})|^2 + \frac{g'}{4!}\int^{\tilde{\Lambda}}\prod_{i=1}^4 d^Dp_i[\vec{\phi}_s(\mathbf{p}_1)\cdot\vec{\phi}_s(\mathbf{p}_2)][\vec{\phi}_s(\mathbf{p}_3)\cdot\vec{\phi}_s(\mathbf{p}_4)](2\pi)^D\delta(\sum_i\mathbf{p}_i). \tag{60}$$

We want to bring it to the form of the original action (including the cut-off, which should be Λ, not $\tilde{\Lambda}$). To this end

- we write $\mathbf{p} = \frac{\tilde{\Lambda}}{\Lambda}\tilde{\mathbf{p}} = e^{-\ell}\tilde{\mathbf{p}}$ interms of $\tilde{\mathbf{p}}$ with $|\tilde{\mathbf{p}}|\in[0,\Lambda]$,

- we impose that the gradient term shall not be altered by the entire procedure. To this end we write

$$\vec{\phi}_s(\mathbf{p}) = e^{\frac{D+2}{2}\ell}\vec{\tilde{\phi}}(\tilde{\mathbf{p}}) \tag{61}$$

Using this we find

$$\begin{aligned} S_{\rm eff}[\vec{\phi}_s] &= \frac{1}{2}\int^{\Lambda} d^D\tilde{p}e^{-D\ell}[e^{-2\ell}\tilde{\mathbf{p}}^2+r']e^{(D+2)\ell}|\vec{\tilde{\phi}}(\tilde{\mathbf{p}})|^2 \\ &+ \frac{g'}{4!}\int^{\Lambda}\prod_{i=1}^4 d^D\tilde{p}_ie^{-4D\ell}e^{2(D+2)\ell}[\vec{\tilde{\phi}}(\tilde{\mathbf{p}}_1)\cdot\vec{\tilde{\phi}}(\tilde{\mathbf{p}}_2)][\vec{\tilde{\phi}}(\tilde{\mathbf{p}}_3)\cdot\vec{\tilde{\phi}}(\tilde{\mathbf{p}}_4)](2\pi)^D\delta(\sum_i\tilde{\mathbf{p}}_i)e^{D\ell} \end{aligned} \tag{62}$$

i.e. the exact original action (with the same UV cut-off Λ, and in terms of fields with a $\sim$) but with renormalized dimensionless coupling constants $\tilde{r} = r/\Lambda^2, \tilde{g} = g/\Lambda^{4-D} = g/\Lambda^{\epsilon}$

$$\tilde{r} \to \tilde{r}e^{2\ell} + e^{2\ell}\tilde{g}\frac{N+2}{6}\Omega_D\left[\frac{1-e^{-(2-\epsilon)\ell}}{2-\epsilon} + \tilde{r}\frac{1-e^{\epsilon\ell}}{\epsilon}\right] \simeq e^{2\ell}\tilde{r} + \tilde{g}\frac{N+2}{3}\frac{1}{16\pi^2}\left[\frac{e^{2\ell}-1}{2} - e^{2\ell}\tilde{r}\ell\right] \tag{63}$$

$$\tilde{g} \to \tilde{g}e^{\epsilon\ell} + e^{\epsilon\ell}\tilde{g}^2\frac{N+8}{6}\Omega_D\frac{1-e^{\epsilon\ell}}{\epsilon} \simeq e^{\epsilon\ell}\left[\tilde{g} - \tilde{g}^2\frac{N+8}{9}\frac{3}{16\pi^2}\ell\right]. \tag{64}$$

where we expanded the right hand side to leading order in $\epsilon\ell \ll 1$

Step 4: RG equations (iteration precedure) In steps 1-3 we performed a single RG step. Here we switch to repeating the procedure, each time using the result of the previous iteration step as an input. This leads to a discrete flow in $(\tilde{r}, \tilde{g})$ parameter space.

We now represent this discrete flow as a continuous flow encouded by differential equations (see below)

- The dominant flow is $\tilde{r}_n = e^{n2\ell}\tilde{r}_0, \tilde{g}_n = e^{n\epsilon\ell}$ for the n-th step which can be expressed as the solution of differential equations $d\tilde{r}/d\ell = 2\tilde{r}$, $d\tilde{g}/d\ell = \epsilon\tilde{g}$. The corrections to this flow are incremental, motivating to represent the entire flow as a differential equation.
- Also, the choice of ℓ is arbitrary: we need a formalism that allows to interpolate between arbitrary step sizes.

Instead of the discrete flow, we thus consider a flow as a function of infinitesimal ℓ, which we can read off from Eq. (64) by formally expanding in ℓ and differentiating

$$\frac{d\tilde{r}}{d\ell} = 2\tilde{r} + \frac{N+2}{3}\frac{1}{16\pi^2}\left(\tilde{g} - \tilde{g}\tilde{r}\right) \tag{65a}$$

$$\frac{d\tilde{g}}{d\ell} = \epsilon\tilde{g} - \frac{N+8}{9}\frac{3}{16\pi^2}\tilde{g}^2. \tag{65b}$$

Analysis of RG flow The Eqs. (65), see Fig. 2 for an illustration, describe the behavior as the system flows towards the infrared.

- Points in parameter space where the flow is stationary are called **fixed points**.

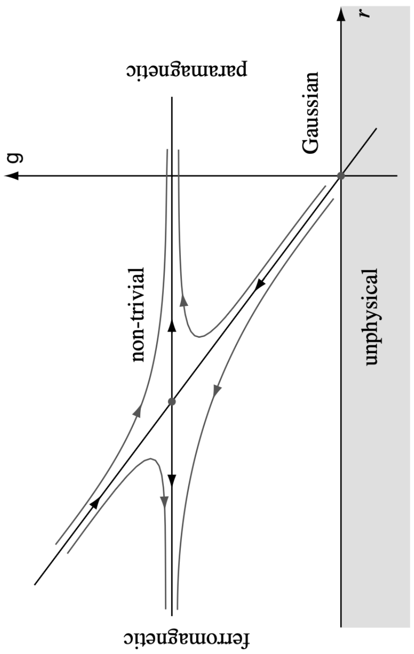


Figure 2: RG flow associated to Eqs. (65). Figure adapted from Altland & Simons, "Condensed Matter field theroy".

Exponent	**Value (to order ϵ)**
ν	$\frac{1}{2}+\frac{N+2}{4(N+8)}\epsilon$
η	$\mathcal{O}(\epsilon^2)$
γ	$1+\frac{N+2}{2(N+8)}\epsilon$
β	$\frac{1}{2}-\frac{3}{2(N+8)}\epsilon$
α	$\epsilon\cdot\frac{-N+4}{2(N+8)}$
δ	$3+\epsilon\cdot\frac{9}{N+8}$

Table 1: Critical exponents of the $O(N)$ vector model in $d=4-\epsilon$ dimensions to leading order in ϵ.

- attractive fixed points correspond to the stable phases of matter, here they are at $\tilde{r}\to\pm\infty$ corresponding to ordered and disordered phases respectively.
- repulsive fixed points (those with "relevant" perturbations driving you away from the fixed poin) represent phase transitions. Here we have two repulsive fixed points:
  - The trivial Gaussian fixed point ($\tilde{r}=0, \tilde{g}=0$) represents the physics of mean-field transition. For $\epsilon>0$ it is unstable and instead the system generically flows to the
  - ... Wilson-Fisher fixed point at

$$\tilde{r}_* \simeq -\frac{1}{2}\frac{N+2}{3}\frac{\tilde{g}_*}{16\pi^2} = -\frac{\epsilon}{6}\frac{3(N+2)}{N+8}, \tag{66}$$

$$\tilde{g}_* = \epsilon\frac{9}{N+8}\frac{16\pi^2}{3}. \tag{67}$$

- General comment on the procedure of RG: Upon integration of fast modes one typically generates higher order terms, e.g. ϕ^6 term in the present case. Such terms are dropped if they are irrelevant (it's readily seen that ϕ^6 is irrelevant near 4D). If, however, RG generates new relevant or marginal terms are generated, those should be included in the starting action and renormalized along with all other terms.
- For second order phase transitions, behavior near criticality $(\tilde{r},\tilde{g}) = (\tilde{r}_*,\tilde{g}_*) + (\tilde{\tilde{r}},\tilde{\tilde{g}})$ is characterized by linear flow equations

$$\frac{d\delta\tilde{r}}{d\ell} = \left(2-\frac{N+2}{N+8}\epsilon\right)\delta\tilde{r} + \frac{3}{16\pi^2}\frac{N+2}{N+8}\delta\tilde{g} \tag{68a}$$

$$\frac{d\delta\tilde{g}}{d\ell} = -\epsilon\delta\tilde{g} \tag{68b}$$

- Note that $\delta g = 0$ is preserved along the most relevant direction of the RG flow near the Wilson Fisher fixed point.

- Thus we find

$$\delta\tilde{r}(\ell) = e^{(2-\frac{N+2}{N+8}\epsilon)\ell}\delta\tilde{r}(0). \tag{69}$$

We define the coherence length as the scale at which $\delta\tilde{r}(\ell_\xi = \ln(\xi/a)) \sim 1$ is order unity, and we remind the reader that $\tilde{r}(0) \sim T - T_c$. Thus

$$-(2 - \frac{N+2}{N+8}\epsilon)\ell_\xi = \ln(\delta(r(0)) \Leftrightarrow \frac{\xi}{a} = \delta(r(0))^{1/(2-\frac{N+2}{N+8}\epsilon)} \tag{70}$$

or

$$\boxed{\nu = \frac{1}{2} + \frac{N+2}{N+8}\frac{\epsilon}{4}}. \tag{71}$$

Similar calculations lead to all other critical exponents, see Tab. 1

1.5.3 Perturbing about the lower critical dimension: non-linear-sigma model RG at $2 + \epsilon$ dimensions

In this section we consider an RG procedure about the lower critical dimension .

Consider the Goldstone modes in the symmetry broken phase . Their action is given by a *non-linear sigma model* in d dimensions

$$S[\hat{n}] = \frac{K}{2}\int d^Dx|\vec{\nabla}\hat{n}|^2 \tag{72}$$

where

- K is the stiffness, it has dimensions of L^{2-D}. As in the previous section it is thus useful to also consider the dimensionless stiffness $\tilde{K} = K/\Lambda^{D-2}$ (Λ is the momentum cut-off)

- $\hat{n}$ is an N-component unit vector, it is useful to parametrize $\hat{n} = O(\mathbf{x})\hat{e}_1$ (with $O(\mathbf{x})$ an orthognal matrix and $\hat{e}_N = (0, 0, ..., 0, 1)^T$.

- Note the gauge equivalence $O(\mathbf{x}) \sim O(\mathbf{x})h(\mathbf{x})$, where

$$h(\mathbf{x}) = \left(\begin{array}{c|c} h_{(N-1)\times(N-1)} & 0 \\ \hline 0 & 1 \end{array}\right) \in \mathrm{O}(N-1). \tag{73}$$

We consider dimensions $D = 2 + \epsilon$ and calculate the RG equations in the limit

$$\boxed{\tilde{K} \ll 1}. \tag{74}$$

Step 1: Splitting slow and fast modes.

We follow Polyakov's original treatment

$$\hat{n}(\mathbf{x}) = O_s(\mathbf{x})O_f(\mathbf{x})\hat{e}_1 = O_s(\mathbf{x}) \underbrace{\begin{pmatrix} \pi_1(\mathbf{x}) \\ \pi_2(\mathbf{x}) \\ \vdots \\ \pi_{N-1}(\mathbf{x}) \\ \sqrt{1 - \sum_{a=1}^{N-1} \pi_i^2(\mathbf{x})} \end{pmatrix}}_{\hat{n}_f} . \tag{75}$$

As before, in Fourier space slow (fast) fields have support for $|\mathbf{p}| \in [0, \tilde{\Lambda}]$ ($|\mathbf{p}| \in [\tilde{\Lambda}, \Lambda]$) and $\tilde{\Lambda} = e^{-\ell}\Lambda \ll \Lambda$ or

$$\boxed{\epsilon \ll 1/\ell \ll 1}. \tag{76}$$

We use

$$\vec{\nabla}[O_s\hat{n}_f] = O_s[\vec{\nabla} + \underbrace{O_s^{-1}\vec{\nabla}O_s}_{\equiv \vec{A}_s}]\hat{n}_f \tag{77}$$

Note that $A_s = -A_s^T$. We perform a gauge transformation $\hat{n}_f \to h_s(\mathbf{x})\hat{n}_f$ with $h_s(\mathbf{x})$ slow such that

$$\vec{\nabla}[O_s\hat{n}_f] \to O_s[\vec{\nabla} + \underbrace{O_s^{-1}\vec{\nabla}O_s + h_s(\mathbf{x})^{-1}\vec{\nabla}h_s(\mathbf{x})}_{\equiv \vec{A}}]\hat{n}_f \tag{78}$$

and

$$\vec{A} = \left(\begin{array}{ccc|c} \ddots & & & \vdots \\ & 0 & & \vec{A}_a \\ & & \ddots & \vdots \\ \hline \cdots & -\vec{A}_a & \cdots & 0 \end{array}\right), \tag{79}$$

so that

$$\vec{\nabla}[O_s\hat{n}_f] = O_s \left[\vec{\nabla}\hat{n}_f + \begin{pmatrix} \vdots \\ \vec{A}_a\sqrt{1-\vec{\pi}^2} \\ \vdots \\ -\sum_{a=1}^{N-1} \vec{A}_a\pi_a \end{pmatrix} \right] . \tag{80}$$

Using this the action becomes

$$S[\hat{n}] = \frac{K}{2}\int d^D x\Big[(\vec{\nabla}\sqrt{1-\vec{\pi}^2} - \sum_a \vec{A}_a\pi_a)^2 + \sum_a(\vec{\nabla}\pi_a + \vec{A}_a\sqrt{1-\vec{\pi}^2})^2\Big] \tag{81}$$

$$= S[\hat{n}_s] + \underbrace{\frac{K}{2}\int d^D x\sum_a(\vec{\nabla}\pi_a)^2}_{=S_{\vec{\pi}}} + \underbrace{\frac{K}{2}\int d^D x\sum_{ab}\pi_a\pi_b[\vec{A}_a\vec{A}_b - \delta_{ab}(\sum_c \vec{A}_c^2)]}_{S_{\mathrm{IA}}} \tag{82}$$

Here we used $\sum_{a=1}^{N-1}\vec{A}_a^2 = |\nabla\hat{n}_s|^2$ and we expanded up to second order in π (note the color coding to illustrate the orgin of terms in the second line from the first).

Step 2: Integration of fast modes We only need

$$\langle\pi_a(\mathbf{x})\pi_b(\mathbf{x})\rangle = \int_{\tilde{\Lambda}}^{\Lambda}\frac{d^D p}{(2\pi)^D}\frac{\delta_{ab}}{K\mathbf{p}^2} = \frac{\Omega_D}{K}\int_{\tilde{\Lambda}}^{\Lambda} dp p^{D-3} = \delta_{ab}\frac{\Omega_D}{K}\frac{\Lambda^\epsilon(1-e^{-\epsilon\ell})}{\epsilon}, \tag{83}$$

to see that

$$S_{\text{eff}}[\hat{n}_s] = \frac{K}{2}\left(1 + \frac{\Omega_D}{\tilde{K}}(1-(N-1))\ell\right)\int d^D x|\vec{\nabla}\hat{n}_s|^2. \tag{84}$$

Step 3: Rescaling. The effetive action which we found has UV cut-off $\tilde{\Lambda}$ (in real space, the smallest scale is $1/\tilde{\Lambda}$). We write $\mathbf{x} = (\Lambda/\tilde{\Lambda})\tilde{x}$ where the smallest real space scale of $\tilde{x}$ is $1/\Lambda$. This replaces

$$\int_{|\mathbf{x}|>1/\Lambda} d^D x|\vec{\nabla}\hat{n}_s(\mathbf{x})|^2 \to (\Lambda/\tilde{\Lambda})^{D-2}\int_{|\tilde{\mathbf{x}}|>1/\Lambda} d^D\tilde{x}|\vec{\nabla}\hat{\tilde{n}}_s(\tilde{\mathbf{x}})|^2 \tag{85}$$

All in all we found the replacement

$$\tilde{K} \to \left[\tilde{K} - \Omega_D(N-2)\ell\right]e^{\epsilon\ell} \tag{86}$$

We can use $\Omega_{2+\epsilon} \simeq \Omega_2 = \frac{1}{2\pi}$.

Step 4: RG equation Iterating the process as discussed in the previous section leads to

$$\boxed{\frac{d\tilde{K}}{d\ell} = \epsilon\tilde{K} - \frac{N-2}{2\pi} + \mathcal{O}(1/K)} \tag{87}$$

Comments for $\epsilon > 0$:

- There is a fixed point $K^* = (N-2)/(\epsilon 2\pi)$ within the perturbatively accesible large K region whcih separates the flow $K \to 0$ (disordered) from the flow to $K \to \infty$ (order).

- Near the fixed point the RG equation the RG flow is simply $d(\tilde{K}-K^*)/d\ell = \epsilon(\tilde{K}-K^*)$ so that we can extract the correlation length exponent to be

$$\nu = 1/\epsilon. \tag{88}$$

Comments for $\epsilon = 0$ (in strictly 2D):

- For any $N > 2$ the flow is towards small stiffnesses. It can be checked by expanding about the small K fixed point that this limit corresponds to a stable fixed point (= disordered phase).
- Note that the RG time to reach small $\tilde{K} \sim$ from starting from a large $\tilde{K}_0$ is exponentially large leading to a large correlation length

$$\xi \sim a e^{\frac{2\pi}{(N-2)} K_0}. \tag{89}$$

- In the case of $1+1 = 2D$ topological terms can prevent flow to zero stiffness leading to critical states (i.e. states with algebraic correlations). This is vaguely reminiscent of your observation in Exercise sheet 1, that quantum mechanical ground state degeneracies are encoded in a topological term for the path integral of the quantun particle on a ring.

  For example consider the case $N = 4$. There are instances (captured by non-Abelian bosonization) where the following extra term is added to the action

$$S_{\text{WZNW}} = i\frac{k}{12\pi}\int_0^1 ds \int d^D x \epsilon_{ijk}\epsilon_{abcd}\hat{n}_a\partial_{x_i}\hat{n}_b\partial_{x_j}\hat{n}_c\partial_{x_k}\hat{n}_d \tag{90}$$

  Here, fields $\hat{n}(\mathbf{x}, s)$ are functions of three variables, but only the surface $\hat{n}(\mathbf{x}, s = 0) = \hat{n}(\mathbf{x})$ is physical, while $\hat{n}(\mathbf{x}, s = 1)$ is fixed to an arbitrary point on the 3-sphere. This is an instance of a Wess-Zumino-term which

  - rely on a non-trivial $D+1$th homotopy group $\pi_{D+1}(\mathcal{M}) = \mathbb{Z}$.
  - The addtional term in the ction takes the form

$$S_{\text{WZW}} = i2\pi k\Gamma[Q],$$

    where $Q : \mathbb{R}^D \to \mathcal{M}$ is the quantum field (called $\hat{n}$ in the previous example) and

    * $\Gamma[Q]$ measures the normalized solid hyper-angle enclosed in a (hyper-)loop.
    * As a consequence that closing the solid angle into nord-pole or south pole is arbitrary and $\Gamma_{\text{North}}[Q] - \Gamma_{\text{South}}[Q] \in \mathbb{Z}$, it follows that the "Wess-Zumino-level" $k \in \mathbb{Z}$ has to be quantized (otherwise the partition function would be ill-defined).

- being of topological origin, the level k of the WZW term does not flow under RG flow.
- However, it impacts the flow of K which now is

$$\frac{dK}{d\ell} = -\frac{1}{\pi}\left(1 - \left(\frac{k}{2\pi K}\right)^2\right) \tag{91}$$

  The factor in bracket persists to all orders in perturbation theory and thus introduces a (potentially) strong -coupling fixed point $K = k/2\pi$.

- Note that for $N = 2$ (O(2) or U(1) model) the loop corrections vanish. This is true to all orders in perturbation theory and instead non-perturbative (exponentially small) effects dominate the RG flow (see next section).

1.5.4 Berezinskii Kosterlitz Thouless transition in 2D

- As mentioned, in 2D and for an O(2) (or U(1)) model all perturbative corrections to the stiffness vanish (this is because the order parameter manifold is flat)
- However, topological defects in the orderparameter play a crucial role: vortices
  - Inside the vortex the amplitude ϕ_0 of the order paramer field $\vec{\phi} = (\phi_1, \phi_2) = \phi_0(\cos(\phi), \sin(\phi))$ is suppressed.
  - But the phase ϕ acquires a winding around this defect.
- We will find two phases
  - a phase in which vortex-antivortex pairs are bound into small dipoles. This phase displays algebraic order parameter correlators.
  - a phase in which vortices and antivortices proliferate, forming a plasma of vortex charges. This phase displays exponentially decaying order parameter correlations (i.e. it is disordered).

As double vortices can always be considered as two single vortices on top of each other we will consider the following vortex configurations, only:

$$\boxed{\vec{\nabla} \wedge (\vec{\nabla}\phi) = 2\pi \sum_i n_i \delta(\mathbf{x} - \mathbf{R}_i)} \tag{92}$$

- Obviously the field ϕ is singular, hence curl of gradient isn't zero (note $\vec{a} \wedge \vec{b} = a_x b_y - a_y b_x$).
- $\mathbf{R}_i$ are the vortex positions
- n_i are the vortex charges
- We will only keep overall neutral vortex configurations $\sum_i n_i = 0$

Derivation of sine-Gordon theory The partition sum of the O(2) non-linear σ model in the presence of vortices is

$$\mathcal{Z} = \sum_{N=0}^{\infty} \frac{(y/2)^{2N}}{(N!)^2} \int \left(\prod_{i=1}^{2N} d^2 R_i \right) \int \mathcal{D}\phi \delta \left[\vec{\nabla} \wedge (\vec{\nabla}\phi) - 2\pi \sum_i (-1)^i \delta(\mathbf{x} - \mathbf{R}_i) \right] e^{-\frac{K}{2}\int d^2x (\vec{\nabla}\phi)^2} \tag{93}$$

Here we included

- a sum over sectors with $\sum_i |n_i| = 2N$ vortices
- included a Boltzmann weight ("fugacity") $y \sim \Lambda^2 e^{-S_{\text{core}}}$ for each vortex which is associated to the Boltzmann cost of a vortex core.
- Note that the fugacity is purpusely dimensionful so that it cancels the powers of length from the integration $d^2 R_i$ over classical vortex positions.
- devided by $1/(N!)^2$ not to overcount equivalent configurations of vortices/antivortices

Note the "functional delta function" which can be represented as

$$\delta \left[\vec{\nabla} \wedge (\vec{\nabla}\phi) - 2\pi \sum_i (-1)^i \delta(\mathbf{x} - \mathbf{R}_i) \right] = \int \mathcal{D}\theta e^{-\frac{i}{2\pi}\int d^D x \theta(\mathbf{x})[\vec{\nabla}\times(\vec{\nabla}\phi) - 2\pi\sum_i (-1)^i \delta(\mathbf{x}-\mathbf{R}_i)]} \tag{94}$$

$$= \int \mathcal{D}\theta e^{i\int d^D x \frac{\vec{\nabla}\theta \wedge \vec{\nabla}\phi}{2\pi}} e^{i\sum_i (-1)^i \theta(\mathbf{R}_i)} \tag{95}$$

Next note that

$$\sum_{N=0}^{\infty} \frac{(y/2)^{2N}}{(N!)^2} \int \left(\prod_{i=1}^{2N} d^2 R_i \right) e^{i\sum_i (-1)^i \theta(\mathbf{R}_i)}$$

$$= \sum_{N=0}^{\infty} \sum_{M=0}^{\infty} \frac{1}{N!} \left(\int d^2 R (y/2) e^{i\theta(\mathbf{R})} \right)^N \frac{1}{M!} \left(\int d^2 R (y/2) e^{-i\theta(\mathbf{R})} \right)^M \underbrace{\int \frac{d\alpha}{2\pi} e^{i\alpha(N-M)}}_{=\delta_{N,M}} \tag{96}$$

$$== \int \frac{d\alpha}{2\pi} e^{y\int d^2x \cos(\theta(\mathbf{x})+\alpha)}. \tag{97}$$

Thus

$$\mathcal{Z} = \int \mathcal{D}\theta \mathcal{D}\phi \frac{d\alpha}{2\pi} e^{-\int d^2x \left[\frac{K}{2}(\vec{\nabla}\phi)^2 - i\frac{\vec{\nabla}\theta\wedge\vec{\nabla}\phi}{2\pi} - y\cos(\theta(\mathbf{x})+\alpha)\right]} \tag{98}$$

We next

- shift the field $\theta(\mathbf{x}) \to \theta(\mathbf{x}) - \alpha$ at no cost.
- complete the square for ϕ fields

$$(\vec{\nabla}\phi - \frac{i}{2\pi K} \wedge \vec{\nabla}\theta)^2 = (\vec{\nabla}\phi)^2 + \frac{i}{\pi K}\vec{\nabla}\theta \wedge \vec{\nabla}\phi - \frac{1}{(2\pi)^2K^2}(\vec{\nabla}\theta)^2 \tag{99}$$

So that

$$\boxed{\mathcal{Z} = \int \mathcal{D}\theta e^{-S[\theta]}, \quad S[\theta] = \int d^2x \frac{1}{8\pi^2 K}(\vec{\nabla}\theta)^2 - y\cos(\theta).} \tag{100}$$

This is the "sine-Gordon model" which is dual to the BKT problem.

Renormalization Group treatment The renormalization group equations are derived in the exercises. We find

$$\frac{dK^{-1}}{d\ell} = \pi y^2/2, \tag{101}$$

$$\frac{dy}{d\ell} = (2 - \pi K)y. \tag{102}$$

Here we have absorbed a constant into y.

We plot this RG flow in Fig. 3

- For $K > 2/\pi$ the vortex fugacity flows to zero $\Rightarrow$ vortices effectively disappear from the problem.
- Thus $K > 2/\pi$, $y = 0$ is a line of fixed points (the system display algebraic correlations just like a critical point).
- For stiffnesses below the critical end point $K < 2/\pi$ vortices proliferate.
- It is one of the crucial characteristic features of the BKT transition that the stiffness (it's IR value which is probed in experiment) jumps accross the transition!
- Moreover, note that for classical 2D problems, the dimensionless stiffness $K = \mathcal{J}/T$, where $\mathcal{J}$ is the physical stiffnes (units of energy). Thus, BKT theory predicts a linear relationship between the jump in the stifness and T_{BKT}.

In the previous discussions we linearized RG equations near the critical point to extract critical exponents. We attempt the same here using $K = 2(1+x)/\pi$

$$\frac{dx}{d\ell} = -y^2 \tag{103}$$

$$\frac{dy}{d\ell} = -xy \tag{104}$$

- We see that $c = y^2 - x^2$ are preserved under RG flow and that $x = \pm y$ are the critical lines

- If we start the RG flow $1 \gg c_0 \equiv y_0^2 - x_0^2 > 0$ the flow to the disordered phase is extremely slow: We integrate

  $$\frac{dx}{d\ell} = -y^2 = -c_0 - x^2 \tag{105}$$

  for a flow from $x_0 = x(\ell = 0) \simeq 0$ to $x(\ell_*) = 0$

  $$\arctan\left(\frac{x(\ell)}{\sqrt{c_0}}\right)\Bigg|_{\ell=0}^{\ell_*} = -\sqrt{c_0}\ell_* \tag{106}$$

  $$\pi/2 = \sqrt{c_0}\ln(\xi/a) \tag{107}$$

  $$\Rightarrow \xi = ae^{\frac{\pi}{2\sqrt{y_0^2 - x^2}}} \simeq ae^{\#\sqrt{\frac{T_{\mathrm{BKT}}}{T - T_{\mathrm{BKT}}}}}. \tag{108}$$

- In the last expression we used that c_0 in the vicinity of the separatrix can be assumed to be a linear function of T_{BKT}.

- We see that RG flow at the BKT transition **cannot be linearized**. As a consequence the coherence length does not diverge as power law. In fact, there are no critical exponents to be extracted (in a way, BKT is an "infinite order" phase transition, because we have an essential singularity).

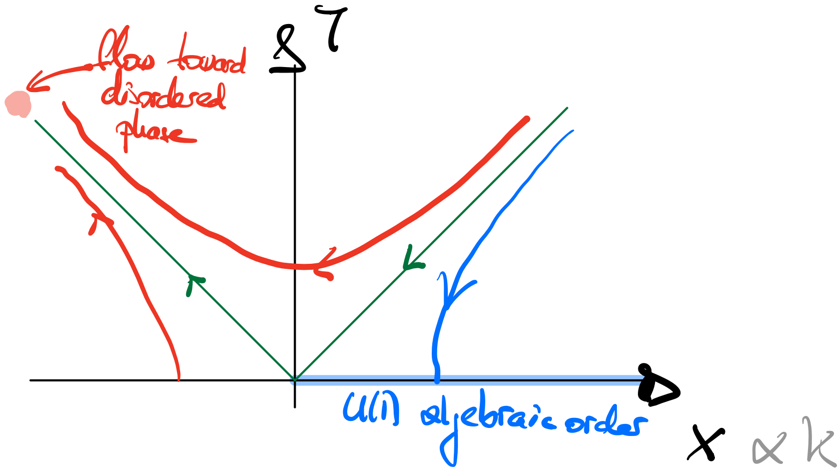


Figure 3: Flow at the Berezinskii-Kosterlitz-Thouless phase transition.

2 The Ising model and its connections to quantum information science

Summary of this section:

- We present the exact solution of the Ising model in 2D (or in 1+1 D) and make connection to the problem of the Kitaev chain.
- We show that the dual to the Ising model in 3D (or more explicitly in $2+1$D) is a $\mathbb{Z}_2$ lattice gauge theory of relevance for the theory of quantum spin liquids and topological quantum error correction codes.

The classical Ising model in d spatial dimensions is given by the following classical partition sum

$$\mathcal{Z}_{\mathcal{N}}(\beta) \quad = \tfrac{1}{2^{\mathcal{N}}} \sum_{\{\sigma_i=\pm1\}} \exp\{-\beta \sum_{ij,} J_{ij}\sigma_i\sigma_j\}. \tag{109}$$

Note that...

- ... the convention is not physically standard, but mathematically convenient and chosen such that $\mathcal{Z}(0) = 1$. It leads to an overal shift of the free energy is $F(\beta) = -\ln(\mathcal{Z}(\beta))/\beta$ as compared to the standard definition we're used to.
- ... In order to get the physical entropy in this case, use $S = -\partial_T F(1/T) + \mathcal{N}\ln(2)$.

The extra term in the definition of the entropy leads to a high temperature entropy of ln(2) per spin and a vanishing zero temperature entropy.

- ... we choose a simple lattice, e.g. (hyper) cubic and

$$J_{ij} = \begin{cases} -1/2 & i,j \text{ nearest neighbors,} \\ 0 & \text{else,} \end{cases} \tag{110}$$

where:

  - the negative sign implies ferromagnetic interactions.
  - the factor 1/2 is such that each link contributes energy $-\beta\sigma_i\sigma_j$.
  - the temperature $1/\beta$ is measured in units of the spin-spin-interaction constant

For a derivation of the ϕ^4 theory from the Ising model, see exercise sheet 1 and the solution thereof.

2.1 Transfer matrix solution in 2+0 D. Kramers-Wannier duality. Relationship to topological 1D superconductor

2.1.1 Transfermatrix representation

- Consider a square lattice with $\mathcal{N} = L_x \times L_y$ sites and periodic boundary conditions.
- We generalize Eq. (109) to the case of anisotropic interactions β_x, β_y
- Without loss of generality we express the partition sum as

$$\mathcal{Z}_{(L_x,L_y)}(\beta_x,\beta_y) = \left(\frac{1}{2}\sinh(2\beta_y)\right)^{\mathcal{N}/2} \operatorname{tr}[\hat{T}^{L_y}], \tag{111}$$

  where $\hat{T}$ is the $L_x \times L_x$ dimensional *transfer matrix* which we will derive in the following.

- The weird prefactor is just a choice, but becomes clear in Eq. (115c) below.
- It is already manifest in Eq. (111) L_y plays the role of an effective temperature of a $1D$ quantum model with $\hat{T} = e^{-H_{\mathrm{1D}}}$, with 1D quantum Hamiltonian H_{1D} to be derived below. Thus y plays the role of imaginary time. The thermodynamic limit of the 2D problem corresponds to the zero-temperature limit of the quantum problem.

- We introduce statevectors at fixed $y = 1, \ldots, L_y$ specifying the spin configuration at given fixed y. They live in a 2^{L_x} dimensional vector space we will represent it statevectors as if they where quantum mechanical states in the Hilbert space of L_x qubits.
$$|\{\sigma_y\}\rangle \equiv |\sigma_{1,y}\rangle \otimes |\sigma_{2,y}\rangle \otimes \cdots \otimes |\sigma_{L_x,y}\rangle\,, \quad \sigma_{x,y} = \pm 1. \tag{112}$$

- We will introduce Pauli matrices X_x, Y_x, Z_x acting on the 2 dimensional Hilbert space of the qubits at site x.

- We use Z as the quantization axis
$$Z_x\,|\{\sigma_y\}\rangle = \sigma_{x,y}\,|\{\sigma_y\}\rangle \tag{113}$$
  - As usual
$$Z_x \equiv \mathbf{1}_{x=1} \otimes \mathbf{1}_{x=2} \otimes \cdots \otimes Z_x \otimes \cdots \otimes \mathbf{1}_{x=L_x} \tag{114}$$
    is a shorthand notation for a single qubit operation.

- Using this we can write the transfer matrix as a product of θ (includes only operations within a given y slice) and $\tilde{\theta}$ (contains interactions between adjacent y and $y \pm 1$):
$$\boxed{\hat{T} = \theta\tilde{\theta},} \tag{115a}$$
  where
$$\theta = \exp(\beta_x \sum_x Z_x Z_{x+1}), \tag{115b}$$
$$\tilde{\theta} = \exp(\tilde{\beta}_y \sum_x X_x), \tag{115c}$$
  and
$$\tilde{\beta}_i = \operatorname{arcosh}\left(\frac{e^{\beta_i}}{\sqrt{2\sinh(2\beta_i)}}\right), \quad i = x, y. \tag{115d}$$
  Comments:
  - Eq. (115b) should be obvious in the given basis
  - For Eq. (115c)...
    * ...use $\tilde{\theta} = \prod_x \tilde{\theta}_x$ (it falls into a product of operations for each qubit).

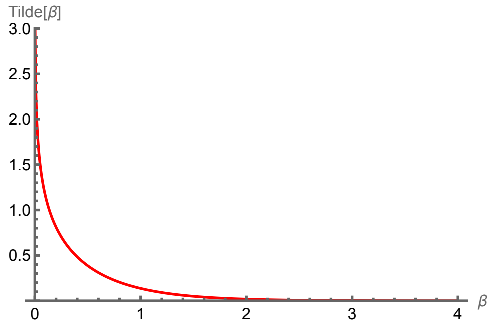


Figure 4: The Mapping $\tilde{\beta}(\beta)$ reverses high and low-temperatures

- ... it's obvious that, in the eigenbasis where Z is diagonal

$$\tilde{\theta}_x = \frac{1}{\sqrt{2\sinh(2\beta_y)}} \begin{pmatrix} e^{\beta_y} & e^{-\beta_y} \\ e^{-\beta_y} & e^{\beta_y} \end{pmatrix} \tag{116}$$

$$= \underbrace{\frac{e^{\beta}_y}{\sqrt{2\sinh(2\beta_y)}}}_{\equiv\cosh(\tilde{\beta}_y)} \left(1 + \underbrace{e^{-2\beta_y}}_{=\tanh(\tilde{\beta}_y)} X_x\right)$$

$$= e^{\tilde{\beta}_y X_x} \tag{117}$$

- Note that the prefactor is the one that cancels the prefactor in Eq. (111) up to 1/2 per site.

– Here, $x \in [1, L_x]$ and $X_{L_x+1} = X_1$ ana analogously for Y_x, Z_x. For periodic boundary conditions, both sums have $x \in [1, L_x]$, while for open boundary conditions, the first sum entering θ ends at $x = L_x - 1$.

– The mapping $\sim$: $\beta \to \tilde{\beta}(\beta)$ reverses high and low temperatures (i.e. it monotonically decreases, diverges at $\beta = 0$ and approaches zero at infinity $\beta \to \infty$), see Fig. 4. Furthermore it is involutive, i.e. $\tilde{\tilde{\beta}} = \beta$.

2.1.2 1D transfer matrix solution

For $L_x = 1$ we can drop the subscript y in β_y. We see that $\theta = 1$ and $\hat{T} = \tilde{\theta}$ so that

$$\mathcal{Z}_{(1,L_y)}(\beta) = \frac{1}{2^L_y} \mathrm{tr}\,[\begin{pmatrix} e^{\beta} & e^{-\beta} \\ e^{-\beta} & e^{\beta} \end{pmatrix}^{L_y}]$$

$$= \cosh(\beta)^{L_y} + \sinh(\beta)^{L_y}. \tag{118}$$

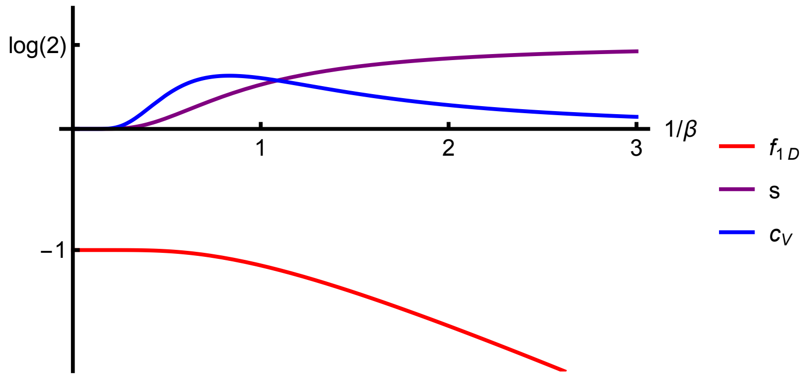


Figure 5: free energy, entropy and specific heat for the 1D Ising model

Where we used Eq. (116) and the basis independence of the basis in tr several times. Thus we find the free energy density (in the convention defined above)

$$\begin{aligned} f_{\mathrm{1D}}(\beta) &= -\frac{1}{L_y} \ln\left(\mathcal{Z}_{(1,L_y)}\right)/\beta \\ &= -\ln(\cosh(\beta))/\beta - \underbrace{\frac{1}{\beta L_y} \ln\left(1+\tanh(\beta)^{L_y}\right)}_{\overset{L_y\to\infty}{\longrightarrow} 0}. \end{aligned} \tag{119}$$

Thus, the free energy in 1D is a continuous, differentiable function fory any finite β. The entropy and specific heat densities are

$$s \equiv -\partial_T f_{1D} + \ln(2) = \ln(2\cosh(\beta)) - \beta\tanh(\beta) \tag{120}$$

$$c_v \equiv T\partial_T S = \beta^2 \mathrm{sech}^2(\beta) \tag{121}$$

There is no phase transition and no spontaneous symmetry breaking – just a crossover at the temperature associated to the interaction, see Fig. 5

2.1.3 2D: Kramers Wannier duality

Consider the dual variables ("disorder operators") on the dual lattice (i.e. the bonds of the 1D lattice), see Fig. 6

$$\tilde{Z}_b = \prod_{x<b} X_x, \tag{122}$$

$$\tilde{X}_b = Z_{b-1/2}Z_{b+1/2}. \tag{123}$$

Up to subtleties at the boundaries (indicated by $\doteq$, they are irrelevant in the thermodynamic limit but see Sec. 2.2.4 for details) we see that

$$\theta \doteq \exp(\beta_x \sum_x \tilde{X}_{x+1/2}) \tag{124}$$

$$\tilde{\theta} \doteq \exp(\tilde{\beta}_y \sum_x \tilde{Z}_{x-1/2}\tilde{Z}_{x+1/2}). \tag{125}$$

Note that $\beta \leftrightarrow \tilde{\beta}, \theta \leftrightarrow \tilde{\theta}$ have exchanged their roles.

Thus we conclude that

$$\mathcal{Z}_{L_x,L_y}(\beta_x,\beta_y) \doteq \Big(\sinh(2\tilde{\beta}_x)\sinh(2\tilde{\beta}_y)\Big)^{-\mathcal{N}/2} \mathcal{Z}_{L_x,L_y}(\tilde{\beta}_y,\tilde{\beta}_x). \tag{126a}$$

Comments:

- We used $\sinh(2\beta_i) = \frac{1}{\sinh(2\tilde{\beta}_i)}$.
- We see that there is a duality transformation that flips low- and high temperature physics and also swaps x and y.
- The isotrpopic model is clearly self-dual when $\beta = \tilde{\beta} = \beta_c$ Defined by $\sinh(2\beta_c) = 1 \to \beta_c = \ln(\sqrt{2}+1)/2 \approx 0.44$. As we will see shortly, this is critical temperature of the 2D Ising model.
- The self-dual line of the general anisotrpic model, formally given by $\beta_x = \tilde{\beta}_y$ is given in Fig. (7).

2.1.4 2D Classical to Quantum Mapping: 1D transverse field Ising model

- We concentrate on the strongly anisotropic limit of inifinitesimal transfer matrix exponentials

$$\beta_x \to 0 \tag{127}$$

$$\tilde{\beta}_y \sim e^{-2\beta_y} \to 0. \tag{128}$$

Recall that the transition at $\beta_x = \tilde{\beta}_y$ is accessible in this limit.

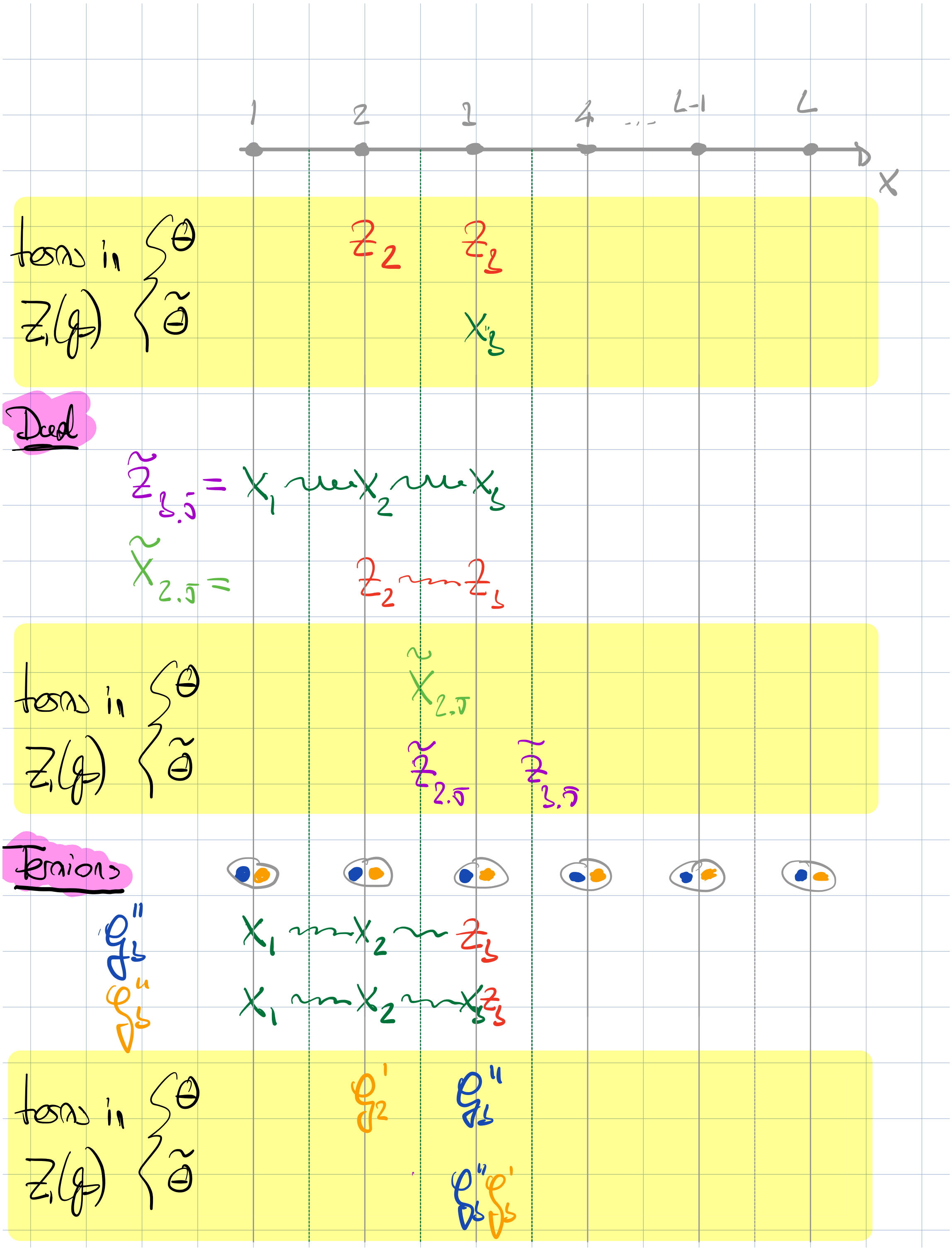


Figure 6: Kramers Wannier duality and Jordan Wigner transformation.

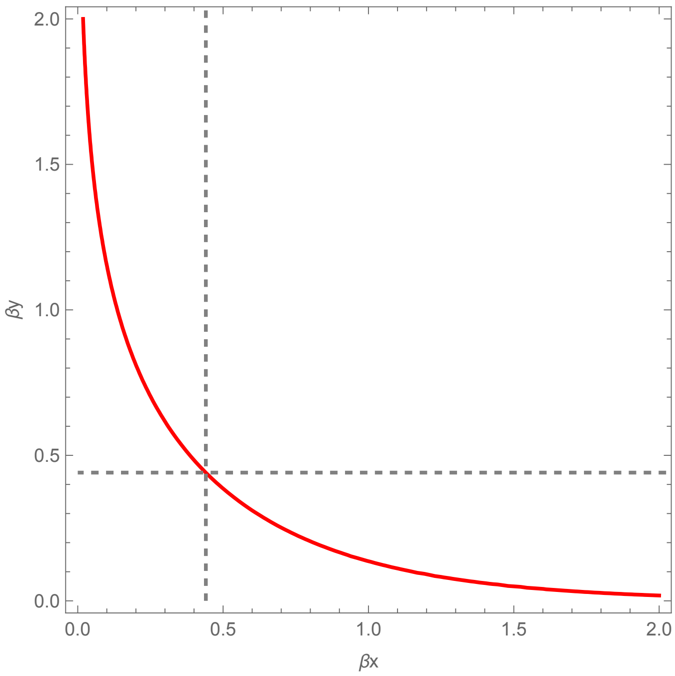


Figure 7: Self-dual line of the Kramers-Wannier transformed anisotropic 2D Ising model.

- In this limit we can expand

$$
\begin{aligned}
\theta\tilde{\theta} &\simeq \prod_x (1+\beta_x Z_x Z_{x+1}) \prod_x (1+\tilde{\beta}_y X_x) \\
&\simeq [1+\sum_x (\beta_x Z_x Z_{x+1} + \beta_y X_x)] \qquad (129) \\
&\simeq \exp[\sum_x (\beta_x Z_x Z_{x+1} + \beta_y X_x)] \qquad (130)
\end{aligned}
$$

  This leads to

$$
\mathcal{Z}_{L_x,L_y}(\beta_x,\beta_y) \simeq \mathcal{C}\mathrm{tr}\left[e^{L_y \sum_x (\beta_x Z_x Z_{x+1} + \beta_y X_x)]}\right] \equiv \mathcal{C}\mathrm{tr}\left[e^{-\frac{\hat{H}}{T_{\text{eff}}}}\right] \qquad (131)
$$

  with an unimportant $\mathcal{N}, \beta_y$ dependent constant $\mathcal{C} = (2\tilde{\beta}_y)^{-\mathcal{N}/2}$ and

$$
\hat{H} = -\sum_x (J Z_x Z_{x+1} + \lambda X_x) \qquad (132)
$$

- Thus we see that in the strongly anisotropic limit (corresponding to the continuum limit in y direction), the 2D classical model is equivalent to the quantum partition sum of the 1D transverse field Ising model with

- Effective temperature $1/(aL_y) \to T_{\text{eff}}$
- Effective Ising coupling $\beta_x/a \to J$
- Effective transverse field $\tilde{\beta}_y/a \to \lambda$.

Where a is an ultraviolet time (i.e. inverse energy) scale.

- The Kramers-Wannier-duality transformation

$$\hat{H} \to -\sum_x (J\tilde{X}_{x+1/2} + \lambda \tilde{Z}_{x-1/2}\tilde{Z}_{x+1/2}) \tag{133}$$

swaps the roles of $J \leftrightarrow \lambda$.

2.1.5 Fermionic representation and topology

For the solution of the isotropic 2D Ising model, see Itzykson and Drouffe. The solution in the isotropic point is conceptually the same but more tedious than the anisotropic limit on which we concentrate here.

We define two Majorana operators per site, see last row of Fig. 6

$$\zeta'_x = i\tilde{Z}_{x+1/2}Z_x = i\prod_{x'\leq x} X_{x'}Z_x \tag{134}$$

$$\zeta''_x = \tilde{Z}_{x-1/2}Z_x = \prod_{x'\leq x-1} X_{x'}Z_x \tag{135}$$

which all anticommute, and $\zeta'^2_x = 1 = (\zeta''_x)^2$. (Majorana fermions are "real fermions" i.e. ${\zeta'_x}^\dagger = \zeta'_x$ etc.)

It is also possible to introduce regular (complex) fermionic creation and annihilation operators by means of

$$c_x = \frac{\zeta'_x + i\zeta''_x}{2} \leftrightarrow \zeta'_x = c_x + c_x^\dagger, \quad \zeta''_x = (-i)[c_x - c_x^\dagger] \tag{136}$$

Indeed

$$\{c_x^\dagger, c_{x'}\} = \frac{1}{4}\left[\{\zeta'_x, \zeta'_{x'}\} + \{\zeta''_x, \zeta''_{x'}\}\right] = \delta_{xx'} \tag{137}$$

In terms of fermions the open chain becomes

$$H = -iJ\sum_x \zeta'_x\zeta''_{x+1} + i\lambda\sum_x \zeta'_x\zeta''_x \tag{138}$$

$$= -J\sum_x (c_x + c_x^\dagger)(c_{x+1} - c_{x+1}^\dagger) + \lambda\sum_x (c_x + c_x^\dagger)(c_x - c_x^\dagger)$$

$$= -J\sum_x (c_x^\dagger c_{x+1} + H.c.) - J\sum_x (c_x c_{x+1} + H.c.) + 2\lambda\sum_x (c_x^\dagger c_x - 1/2). \tag{139}$$

The last line demonstrates that the transverse field Ising model with open boundary conditions is equivalent to a 1D superconductor of spinless fermions with nearest neighbor pairing ("Kitaev chain") and chemical potential λ. We will now discuss its topological properties:

- For periodic boundary conditions we can use the Fourier transform

$$H = -\int \frac{dp}{2\pi} iJ\zeta'(-p)\zeta''(p)e^{ip} - i\lambda\zeta'(-p)\zeta''(p)$$

$$= \frac{1}{2}\int \frac{dp}{2\pi}(\zeta',\zeta'')_p^\dagger \underbrace{\begin{pmatrix} 0 & -iJe^{ip} + i\lambda \\ iJe^{-ip} - i\lambda & 0 \end{pmatrix}}_{\mathbf{v}(p)\cdot(\tau_x,\tau_y)} \begin{pmatrix} \zeta' \\ \zeta'' \end{pmatrix}_p . \tag{140}$$

  This Hamiltonian contains a 2 vector $\mathbf{v}(p) = (J\sin(p), J\cos(p) - \lambda)$ and two energy levels $E_\pm(p) = \pm|\mathbf{v}(p)|/2$. Away of the critical point $J = \lambda$ the single particle band structure is gapped and the mapping $p \mapsto \mathbf{v}(p)$ is a mapping $\mathbb{S}^1 \to \mathbb{R}^2\backslash\{0\}$ which allows to classify two topologically distinct gapped states (cf Fig. 8)

  - For $J > \lambda$ the curve $\mathbf{v}(p)$ in $\mathbb{R}^2\backslash\{0\}$ which is parametrized by $p \in (-\pi, \pi]$ winds around the origin (there is a nonzero **topological** winding number)
  - For $J < \lambda$, the origin is not enclosed by the curve (hence the phase is topologically trivial).

For open boundary conditions , where the sum of the J terms only run for $1 < n < L - 1$

- Topological phase ($J > \lambda$):
  - At $\lambda = 0$:
    * the operators ζ''_1 and ζ'_L don't appear. This is equivalent of saying that there are zero-energy Majorana "edge states" – a hallmark of topological states of matter.

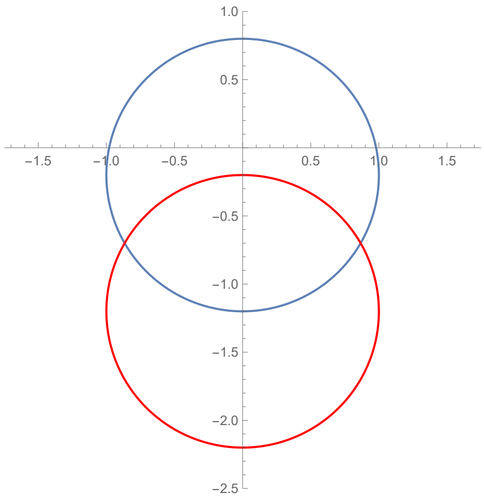


Figure 8: Plot of the vector $\mathbf{v}(p)$ for $J = 1, \lambda = 0.2$ (blue, topological phase) and $J = 1, \lambda = 1.2$ (red, trivial phase) [the winding is clockwise.]

    * The many-body ground state is given by $i\zeta'_x\zeta''_{x+1} = -1$ for all $x = 1, \dots L-1$.

- At finite $\lambda \ll J$, the edge states remain pinned at zero energy if $L \to \infty$.
- The non-local fermion $d = [\zeta''_1 + i\zeta'_L]/2$ implies a two-fold ground state degeneracy of the open chain reminiscent of the two-fold groundstate degeneracy of the different states in the transverse field Ising model
- Topologically trivial phase ($J < \lambda$): In the extreme case $J = 0$, the many-body ground state is given by $i\zeta'_x\zeta''_x = 1$, $x = 1, \dots L$.

The relationship between a bulk topological invariant (winding number) and the emergence of zero-energy boundary states in systems with open boundary conditions is a generic feature of topological states of matter, called the “bulk-boundary correspondence”.

2.1.6 Exact solution of critical exponents of the 2D Ising model

We concentrate on the regime near criticality $J = \lambda$ or $\beta_x = \tilde{\beta}_y$, where and effective field theory may be obtained from Eq. (140) by expanding about $p = 0$ (i.e. perform a gradient expantion for smoothly varying fields). Thus

$$\mathcal{H}[\zeta] = \frac{1}{2}\int dx(\zeta'(x), \zeta''(x))[-iJ\partial_x\tau_x + m\tau_y]\begin{pmatrix} \zeta'(x) \\ \zeta''(x) \end{pmatrix}. \tag{141}$$

Here, we ...

- ... absorbed J and a number into ζ fields (effectively setting the speed of fermions to unity)
- ... $m = (J - \lambda) = (\beta_x - \tilde{\beta}_y)$ is a mass that changes sign at the transition.
- ... see that the low energy theory near criticality are free massless 1D Majorana fermions.

From this we readily get

$$f_{\text{1Dquantum}} = -\underbrace{\frac{1}{2}}_{\text{take } \sqrt{\mathcal{Z}_{\text{Dirac}}}} \underbrace{2}_{\text{particle and hole Dirac bands}} T_{\text{eff}} \int_{-\infty}^{\infty} \frac{dp}{2\pi} \ln[1 + e^{-\frac{\sqrt{p^2+m^2}}{T_{\text{eff}}}}] \tag{142}$$

based on Hamiltonian formulation. This readiy leads to $f_{\text{1Dquantum}} = -T_{\text{eff}}/12\pi$ at criticality or a spcecific heat

$$C_v|_{1D} = \frac{T_{\text{eff}}}{3\pi} c, c = 1/2 \tag{143}$$

where c is called the central charge of a conformal field theory, for the Ising criticality (Majorana fermions) it is $c = 1/2$. (for complex/Dirac fermions it would be $c = 1$ instead of $c = 1/2$).

We now concentrate on a more general approach to access the 2D case and explain that from this theory one may derive the exact critical exponents of the Ising universality class in 2D, see Tab. 2. For simplicy, we concentrate on finding expressions for ν, α. One would also need to technically calculate one more exponent (e.g. β) to infer all others by scaling/hyperscaling equations, but it involves correlators of non-local functions in fermionic languagae and we refer the interested reader to Gogolin, Nersesyan, Tsvelik, "Bosonization and strongly correlated systems", in this regard.

Exponent	**Definition**	**2D Ising Value**
α	Specific heat exponent	0
β	Order parameter exponent	$\frac{1}{8}$
γ	Susceptibility exponent	$\frac{7}{4}$
δ	Critical isotherm exponent	15
ν	Correlation length exponent	1
η	Correlation function exponent	$\frac{1}{4}$

Table 2: Critical exponents of the 2D Ising universality class

Instead of calculating the partion sum, we calculate its square.

$$\mathcal{Z}^2_{L_x,L_y} = \text{tr}\,[e^{-\mathcal{H}[\zeta]/T_{\text{eff}}}]^2 = \text{tr}\,[e^{-[\mathcal{H}[\zeta]+[\mathcal{H}[\xi]]/T_{\text{eff}}}] = \text{tr}\,[e^{-\mathcal{H}_D[\psi]/T_{\text{eff}}}] \equiv \mathcal{Z}_{\text{Dirac}}, \tag{144}$$

where $\psi'(x) = \zeta'(x) + \xi'(x)$ and analogously for $\psi''(x)$ are canonical complex fermions (we absorbed extra factors, including J into fields and coordinates to make the ψ have canonical commutation relations and set the speed of fermions to unity)

$$\mathcal{H}_D[\psi] = \int dx \begin{pmatrix} \psi'(x) \\ \psi''(x) \end{pmatrix}^\dagger [-i\partial_x \tau_x + m\tau_y] \begin{pmatrix} \psi'(x) \\ \psi''(x) \end{pmatrix}. \tag{145}$$

Thus the partition sum is the square root of the partition sum of free Dirac (complex) fermions with mass m and filled Dirac sea.

Using the coherent state path integral we find

$$\mathcal{Z}_{\text{Dirac}} = \int \mathcal{D}[\psi', \psi''] \exp\left[-\int_0^{1/T_{\text{eff}}} d\tau \int_0^{L_x} dx (\bar{\psi}'(x), \bar{\psi}''(x))[\partial_\tau - i\partial_x\tau_x + m\tau_y] \begin{pmatrix} \psi'(x) \\ \psi''(x) \end{pmatrix}\right]. \tag{146}$$

From this field theory we see that

- $-i\partial_x$ enters the with the same power as m and thus we readily find the correlation length exponent
$$\xi = m^{-1} = \frac{1}{(\beta_x - \tilde{\beta}_y)^1} \Rightarrow \nu = 1. \tag{147}$$
- time derivative and spatial derivatives appear with the same power (the dispersion is linear) leading to a *dynamical critical exponent* $z = 1$ (i.e. the correlation time τ_ξ and correlation length ξ diverge with the same power of m)
- Note that the Majorana nature of original fields does not allow for point interactions. The leading perturbations about the free-fermion fixed points are of the form $\delta\mathcal{H} = \lambda_{\text{int}} \int dx (\zeta'\partial_x\zeta')(\zeta''\partial_x\zeta'')$. Also in the squared partition sum this leads to interaction terms with 2 gradients. The scaling dimension of the field $[\psi] \sim L^{-1/2}$, from which we can immediately see that $\lambda_{\text{int}} \sim L^2$ to read off the first term in the beta function of λ_{int}
$$\frac{d\lambda_{\text{int}}}{d\ell} = -2\lambda_{\text{int}} + \dots. \tag{148}$$
so perturbations about the free Majorana fermion theory are irrelevant.

The partition sum is

$$\mathcal{Z}_{L_x,L_y} = \sqrt{\prod_\epsilon \prod_p \det[-i\epsilon + p\tau_x + m\tau_y]} = \prod_\epsilon \prod_{p>0} (\epsilon^2 + p^2 + m^2). \tag{149}$$

Or in terms of the free energy

$$f_{1D,\text{quantum}} = -\frac{1}{L_x L_y}\ln(\mathcal{Z}_{L_x,L_y}) = -\int \frac{d\epsilon}{2\pi} e^{i\epsilon 0^+} \int_0^\Lambda \frac{dp}{2\pi} \ln(\epsilon^2 + p^2 + m^2) \simeq \frac{m^2}{4\pi}\ln\left(\frac{\Lambda^2}{m^2}\right) \tag{150}$$

We note that

- ... we took the continuum limit of Riemann sums (this is the main distinction to the previous finite T_{eff} expression, Eq. (142))
- ... explicitly included the regulators $e^{i\epsilon 0}$ and Λ for otherwise divergent integrals.
- ... we only kept the most singular terms. It is more convenient to calculate the $\partial f_{\text{1Dquantum}}/\partial m^2$ and then find the antiderivative.
- ... the 2D classical free energy has extra factor of $1/\beta$

  $$f_{2D,\text{classical}} = f_{1D,\text{quantum}}/\beta \tag{151}$$

  and we use temperature in units of the Ising coupling in x-direction, i.e. $\beta_x = \beta, \beta_y = \beta\alpha$ with $\alpha \gg 1$ the anisotropy parameter.
- ... we can readily determine specific heat singularity at the critical temperature

  $$c_v|_{2D} \sim \ln(\Lambda^2/m^2) \sim -\ln|T - T_c|/T_c \tag{152}$$

  so the specific heat exponent for the 2D Ising model is $\alpha = 0$, see Fig. 9.
- ... recall that $\epsilon = p_y$ and $p = p_x$ in the 2D classical case.

2.2 Optional sections on (1+1)D symmetry protected phases.

2.2.1 Optional section (to be prepared): The tetron – a topological qubit

This section will be prepared upon demand of the students.

2.2.2 Optional section: Symmetries, Symmetry fractionalization and ground state degeneracy.

The above example of a topological phase was simple, as we could use basic topological properties of the single particle Bloch Hamiltonian. In this section, we go beyond this and investigate the symmetries underlying the gapless edge states.

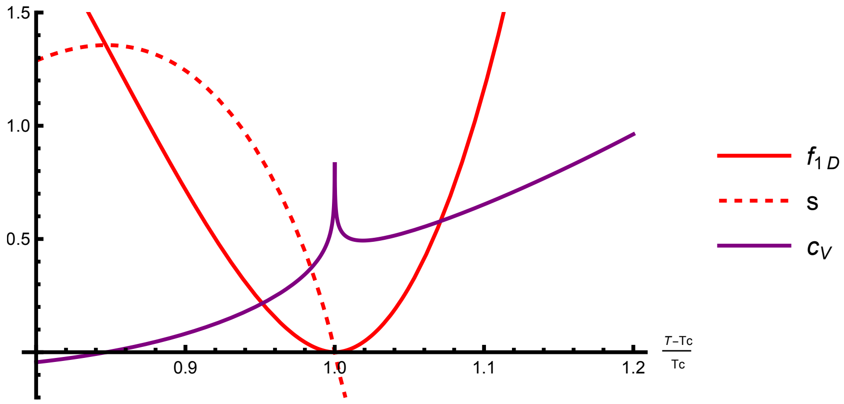


Figure 9: free energy, entropy and specific heat for the 2D Ising model near criticality as obtained from Eq. (151) and its derivatives ($\Lambda = 100$ and $\alpha = 10$ where chosen here – note that non-singular contributions were dropped, hence the non-monotonous entropy)

Consider a closed, local, fermionic quantum system with (potentially interacting) $2L$ Majorana fermions ζ'_x, ζ''_x. A physical Hamiltonian may only contain terms with an even number of fermionic operators per term, therefore the overall fermionic parity

$$P = \prod_x (i\zeta'_x\zeta''_x) \tag{153}$$

is conserved

$$[P, H] = 0. \tag{154}$$

Note that $P^2 = 1$, i.e. there are two decoupled sectors with eigenvalues $P = \pm 1$.

- In the topological trivial phase, at lowest energies we can replace $i\zeta'_x\zeta''_x \to 1$ leading to $P = 1$ (the symmetry acts trivially on the ground state)
- In the topologically non-trivial phase we can replace $i\zeta'_x\zeta''_{x+1} = -1$ so that effectively

  $$P \doteq (-1)^{L-1} i\zeta''_1\zeta'_L = P_L P_R \tag{155}$$

  The original parity operator is fractionalized in P_L, P_R which are localized at the edges, have the same algebra as P (i.e. square to 1). As a defining characteristic of the topological phase, P_L, P_R are mutually anticommuting operators.
- From the anticommuting algebra of P_L, P_R it follows that the ground state manifold is at least two fold degenerate (Clifford algebra).

This feature of the Majorana chain is a characteristic feature of SPTs called *symmetry fractionalization*.

Def: Symmetry fractionalization

Consider

- gapped 1D system of length L (i.e. unique ground state for periodic boundary conditions).
- no spontanous symmetry breaking (i.e. PBC groundstate is invariant under all symmetries)
- global symmetries $U = \otimes_{x=1}^{L} U_x$
- For open boundary conditions $U = U_L U_R$ in low-energy sector is allowed, with U_L, U_R acting non-trivially at left/right edges and the overlap of U_L, U_R vanishes for $L \to \infty$.
- The algebra of fractionalized symmetry operators defines the SPT phase
  - For bosonic systems, all U_Ls and U_Rs commute – the edges completely decouple. The $\{U_L\}$ and $\{U_R\}$ form the same algebraic group relations as the original $\{U\}$ up to possible phase factors (*projective representation, see AKLT below*)
  - For fermionic systems, additionaly U_L and U_R need not commute, as seen in the Kitaev example.

2.2.3 Optional Section: Interacting Kitaev like chains: $\mathbb{Z}_8$ classification

Above, we used Kitaev model to introduce the concept of symmetry fractionalization in 1D. Here we present some more details.

If the P symmetry was the only symmetry in the system, we found that there are two different SPTs characterized by $P_L P_R = \pm P_R P_L$ – a statement independent of interactions terms and the number of stacked Kitaev chains.

On the side: The P symmetry in fermionic systems is so fundamental (it cannot be broken by anything) that often it is not even considered a symmetry and the Kitaev chain therefore non really an SPT. Instead the word "non-invertible phase of matter" is used, see next section.

Here, we additionally impose time reversal symmetry. Assuming a stack of an arbitrary number of Kitaev chains with mutual interactions, we find a $\mathbb{Z}_8$ classification. This was anticipated by Fidkowski and Kitaev, who explicitly demonstrated that the many-body groundstate of 8 Majorana zero modes can be symmetrically gapped due to multi-particle interactions leading to 4-fermion clusters. (This can be seen as a prototypical instance of *symmetric mass generation.*)

- Time reversal symmetry is antiunitary but leaves $c_x, c_x^\dagger$ invariant

$$\begin{aligned} TiT &= -i, & T^2 &= 1 \\ Tc_xT &= c_x, & Tc_x^\dagger T &= c_x^\dagger \quad (156) \\ \Rightarrow T\zeta_x'T &= \zeta_x' & T\zeta_x''T &= -\zeta_x''. \quad (157) \end{aligned}$$

- Assuming the following fractionalized Ansatz for $T = U_L U_R \tilde{K}$, where U_L, U_R unitary. The condition $T^2 = 1$ implies $(\bar{U}_L U_L)(\bar{U}_R U_R) = 1$ if U_L, U_R commute. If they anticommute (are fermionic) $(\bar{U}_L U)(\bar{U}_R U_R) = -1$. There thus are three $\mathbb{Z}_2$ invariants a, b, c determining 8 possible different phases

$$P_L P_R = (-1)^a P_R P_L \quad (158)$$
$$\bar{U}_L U_L = (-1)^b \quad (159)$$
$$\bar{U}_R U_R = (-1)^c \quad (160)$$

For details, R. Verresen's PhD thesis contains a very pedagogical introduction.

2.2.4 Optional section: Jordan Wigner transformation for periodic boundary conditions, Non-invertible symmetries

We now consider the subtleties of the Jordan Wigner transformation in the presence of periodic boundary conditions.

- Consider the Majorana Hamiltonian, Eq. (138), for periodic boundary conditions. Clearly, for both $J = 0, \lambda > 0$ and $J > 0, \lambda = 0$ the ground state is unique and gapped. This is in obvious contrast to the transverse field Ising chain, which displays two degenerate ferromagnetic states at $\lambda = 0$. **Thus the Ising and Kitaev model are not equivalent for periodic boundary conditions!**

- Indeed, the fermionic Hamiltonian Eq. (138) with periodic boundary conditions becomes

$$H = -J \sum_{x=1}^{L-1} Z_x Z_{x+1} - JPZ_L Z_1 - \lambda \sum_{x=1}^{L} X_x, \quad (161)$$

where the fermionic parity written in terms of qubit operators is $P = \left(\prod_{x=1}^{L} X_x\right)$.

- Without going into details here, we summarize
  - All of this is related to subtleties of the Kramers-Wannier duality, Eq. (**??**) $U_{KW}^\dagger H U_{KW} = H|_{J \leftrightarrow \lambda}$

- For $J = \lambda$ the operator U_{KW} appears to be a symmetry. However, something is weird about it because the dual parity

$$P_\mu = \prod_b \tilde{X}_b \to \prod_x Z_x Z_{x+1} = 1 \tag{162}$$

seems to be mapping to exact unity always, while $P_\mu = \pm 1$.

- Thus we conclude that U_{KW} can't be bijective and can't be invertible.
- Non-invertible symmetries have attracted substantial attention in abstract field theory, as they can be seen as the origin of the gapless nature of the state at the $J = \lambda$. For details, see recent works by Seiberg and Shao, in particular the review SH Shao arXiv:2308.00747.

2.3 2D Transverse field Ising model, $\mathbb{Z}_2$ gauge theory and toric code: Introduction to topological order

In the previous sections we showed that the 1D transverse field Ising model is dual to itself. Here, we demonstrate that the 2D transverse field Ising model is dual to a lattice gauge theory.

2.3.1 $\mathbb{Z}_2$ lattice gauge theory, Kramers-Wannier duality in 2D and the toric code

Consider the following Hamiltonian ("$\mathbb{Z}_2$" lattice gauge theory)

$$H = -J \sum_{\mathbf{b}} \mu^x_{\mathbf{b}} - \lambda \sum_{\square} B_{\square}. \tag{163}$$

supplemented with the constraint

$$Q_{\mathbf{r}} = 1 \quad \forall \mathbf{r}. \tag{164}$$

Comments:

- This model is defined by spin 1/2 operators on the bonds $\mathbf{b}$ of the square lattice.

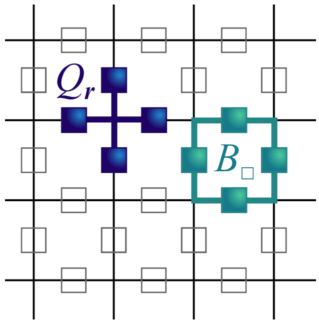


Figure 10: The terms entering the $\mathbb{Z}_2$ lattice gauge theory

- The flux term is $B_\square = \prod_{\mathbf{b}\in\partial\square} \mu^z_{\mathbf{b}}$ and the star operator is $Q_{\mathbf{r}} = \prod_{\mathbf{b} \text{ adjacent to } \mathbf{r}} \mu^x_{\mathbf{b}}$, see Fig. 10. Note that all $B_\square$ and all $Q_{\mathbf{r}}$ commute and square to 1.

- The constraint may also be imposed in a soft manner by an extra term in the Hamiltonian $(h > 0)$

$$\delta H = -h \sum_{\mathbf{r}} Q_{\mathbf{r}} \tag{165}$$

At the $J = 0$, this model is called the "toric code" and used in quantum information science. It is also a model for a quantum spin-liquid (i.e. a quantum paramagnet).

Kramers-Wannier Mapping to 2D Ising model We now show that Eq. (163) is Kramers-Wannier dual to the transverse field Ising model in 2D, very much like Eq. (**??**) is dual to the 1D transverse field Ising model.

We define original spin-variables living on the plaquettes $\mathbf{r}^*$ of the original model

$$\sigma^x_{\mathbf{r}*} = B_{\mathbf{r}*} = \prod_{\mathbf{b} \text{ adjacent to plaquette } \mathbf{r}^*} \mu^z_{\mathbf{b}} \tag{166}$$

$$\sigma^z_{\mathbf{r}*} = \prod_{\mathbf{b}\in\gamma_{\mathbf{r}^*}} \mu^x_{\mathbf{b}} \tag{167}$$

The path $\gamma_{\mathbf{r}^*}$ is a vertical line along the dual links ending at plaquette $\mathbf{r}^*$, see Fig. 11

Comments:

- Note that $\{\sigma^x_{\mathbf{r}^*}, \sigma^z_{\mathbf{r}^*}\} = 0$ and $[\sigma^x_{\mathbf{r}^*}, \sigma^z_{\mathbf{r}^*}] = 0$ for $\mathbf{r}^* \neq \mathbf{r}'^*$

- Using these variables and the gauge constraint, we find (see also Fig. 11)

$$H = -J \sum_{\langle \mathbf{r},\mathbf{r}'^* \rangle} \sigma^z_{\mathbf{r}} \sigma^z_{\mathbf{r}'^*} - \lambda \sum_{\mathbf{r}^*} \sigma^x_{\mathbf{r}^*}. \tag{168}$$

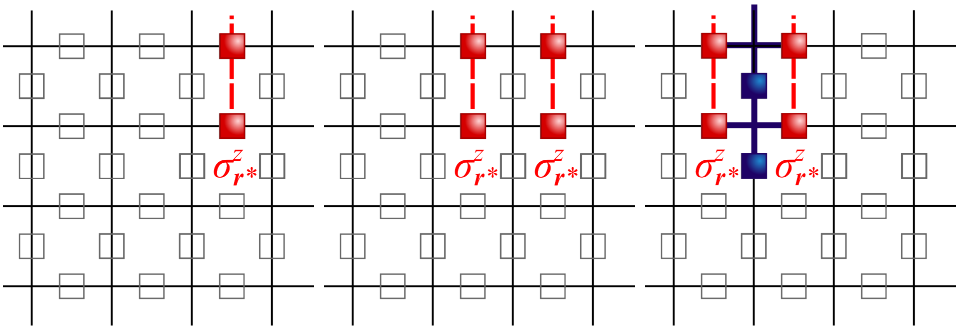


Figure 11: Left: The definition of $\sigma^z_{\mathbf{r}^*}$. Center: Two adjacent strings $\sigma^z_{\mathbf{r}^*}\sigma^z_{\mathbf{r}^*+\hat{x}}$. Right: Using the gauge constraint, we see that $\sigma^z_{\mathbf{r}^*}\sigma^z_{\mathbf{r}^*+\hat{x}} = \mu^x_{\mathbf{r}^*+\hat{x}/2}$

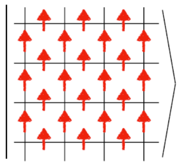
- The relationship between λ terms is obvious and follows from the definition of σ variables
- The Ising interactions $\sigma^z_{\mathbf{r}^*}\sigma^z_{\mathbf{r}^*+\hat{y}}$ are also obvious from the definition.
- The Ising interactions $\sigma^z_{\mathbf{r}^*}\sigma^z_{\mathbf{r}^*+\hat{x}}$ are less obvious, and require explicit use of the Gauss law, Eq. (164)

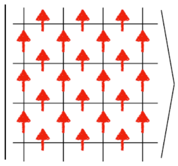
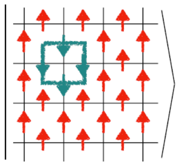
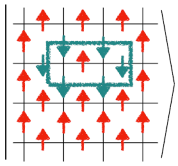
- Thus, in contrast to the 1D case, the transverse field Ising model is not self-dual, but rather maps onto a 2D $\mathbb{Z}_2$ gauge theory *Results by Franz Wegner, early 1970s).*
- As in 1D, there are two phases as a function of J/λ:
  - An Ising disordered phase $\langle\sigma^z\rangle = 0$, which corresponds to the large λ/J limit and a topologically ordered phase in the dual μ-language ("deconfining phase of the gauge theory")
  - an Ising ordered phase $\langle\sigma^z\rangle \neq 0$ which corresponds to the large J/λ limit and corresponds to the confined states ($\mu^x = 1$ throughout).

2.3.2 Solution of the Toric Code

Groundstate We now focus on the case $J = 0$ and $\lambda > 0$ or imposing the Gauss-law, Eq. (164) by force. In the following we construct the ground state explicitly. This is best done pictorially

$$|GS_0\rangle = |\{Q_r = 1\}, \{B_\square = 1\}\rangle$$

$$= \prod_{\mathbf{r}} \frac{1 + Q_{\mathbf{r}}}{2}$$ 

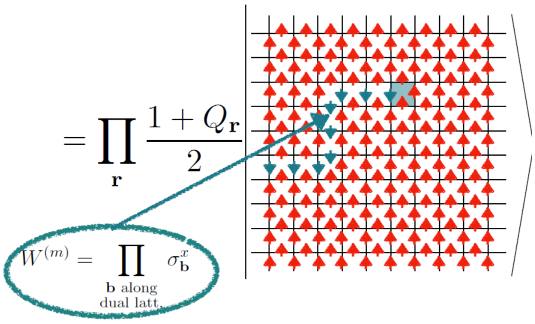
$=$  $+$  $+$  $\ldots$

Comments

- Indeed, this term has all $Q_{\mathbf{r}} = 1$ and all $B_\square = 1$:
  - The product $\prod_{\mathbf{r}} \frac{1+Q_{\mathbf{r}}}{2}$ clearly projects onto states with $Q_{\mathbf{r}} = 1$
  - Since all $B_\square$ commute with all $Q_{\mathbf{r}}$, we can commute all $B_\square$ across the projector at no cost. Then, they act on a state which clearly has zero flux through all plaquettes.
- In the third line, we expand the product over projectors
  - Note that each $Q_{\mathbf{r}}$ acting on the "all-up" state flips all four spins adjacent to a given site
  - Therefore, the expansion of the product of projectors leads to a superposition of states characterized by closed loops of flipped spins.
- Therefore, the Toric code ground state is
  - A superposition state of macroscopically many quantum states ($\Rightarrow$ highly entangled!)
  - A simple example of a "string-net-condensate"

Excitations The simplest excitation is to flip a single plaquette or star. For example, if we flip the plaquette $\square_0$, the state is pictorially represented as

$|m : \square_0\rangle =$



In any of the figures, the notation σ_b is used, but what's meant is μ_b

Comments:

- Such an excitation is called an m particle, because it corresponds to one flux. (By analogy, when we flip the star, it would be called e particle)
- Clearly, a single m has energy $2J$ above the ground state. (A single e has energy 2λ)
- We introduced the string operator $W^{(m)} = \prod_{\mathbf{b} \text{ along dual latt.}} \mu^x_{\mathbf{b}}$ along a dual lattice. By analogy there is $W^{(e)} = \prod_{\mathbf{b} \text{ along latt.}} \mu^z_{\mathbf{b}}$
- On a closed manifold (e.g. a torus) each string operator has to have zero or two ends (so an m never appears alone)
- We will see later that m and e are actually anyons, but there is one more anyon in the theory (see below).

Fusion rules:

- It's pretty obvious that $m \times m = 1$ and $e \times e = 1$, and obviously $1 \times a = a$ for any anyon a.
- However, what is $e \times m$? Clearly it's neither e, m or 1, so it's something new and we call it ϵ. Then, $\epsilon \times m = e, \epsilon \times e = m$.

Exchange statistice of excitations Let's first remember the basics about the wave functions for identical particles:

- For identical particles, an exchange of positions may at most change the wave function of the many body system by a phase $|a_1 : \mathbf{r}_1; a_2 : \mathbf{r}_2\rangle = e^{i\theta} |a_1 : \mathbf{r}_2; a_2 : \mathbf{r}_1\rangle$. *(In some more exotic cases, the phase can also be promoted to a unitary matrix)*

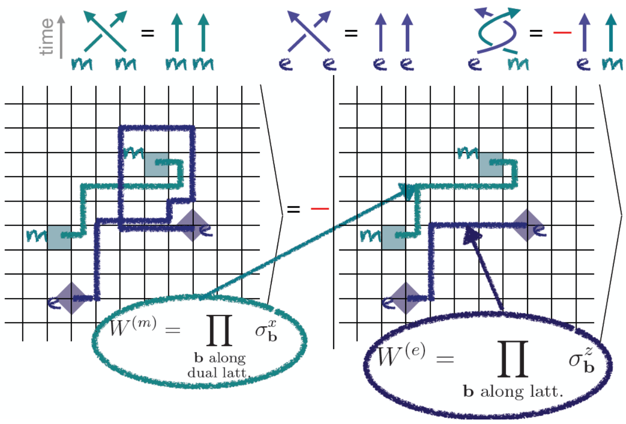


Figure 12: Pictorial representation of exchange statistics of anyons in the toric code model.

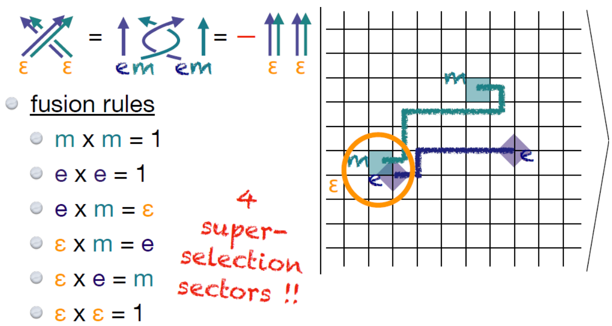


Figure 13: Fusion rules and exchange statistics of ϵ.

- Double exchange, which is equivalent to moving a_1 on a loop around a_2 , thus leads to a phase $e^{i2\theta}$.
- In 3D and for particle like excitations, any loop may be contracted to a point. Thus $e^{i2\theta} = 1$ or $\theta = 0, \pi$ ("boson" or "fermion").
- In 2D, loops can not be contracted to a single point and thus anyons with arbitrary statistical angle θ can exist.

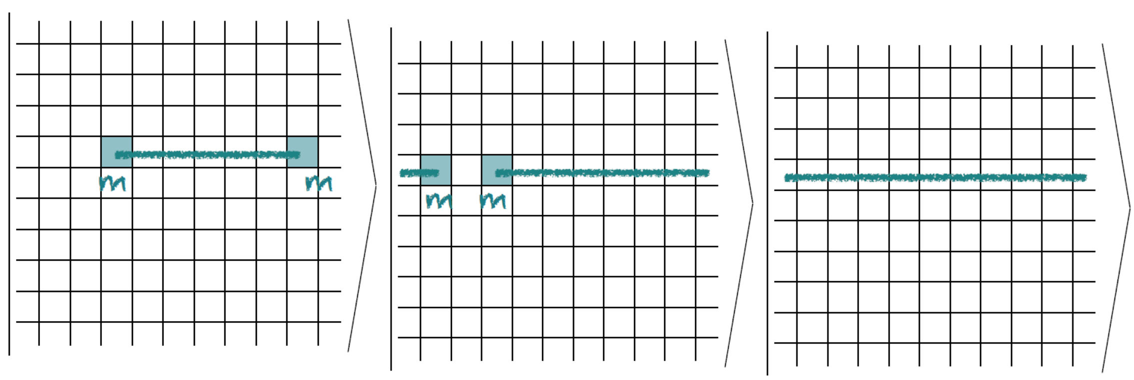
Let's now see what happens to the anyons in our system

- Clearly, moving an m around and m does not change the phase of the wave function. Thus m behaves like a boson to itself. The same is true for e.
- However, when we move an e around an m, the wave function acquires one extra $-$ sign, because the strings interact an odd number of times. Thus e and m are "mutual semions', see fig. 12
- As a corollary, ϵ particles have fermionic statistics (best seen pictorially, Fig. 13).

So in total, from a problem which only included spins ("qubits"), we were able to obtain excitations which are fermions!!!

Groundstate degeneracy on the torus The appearance of anyons also implies a non-trivial ground state degeneracy on a torus.

To see this, we successively build the magnetic holonomy $X_1 = \prod_{\mathbf{b}\in\gamma_{\text{horizontal}}} \sigma^x_{\mathbf{b}}$.



Question: Is $X_1 |GS\rangle = |GS\rangle$ (modulo a phase)?

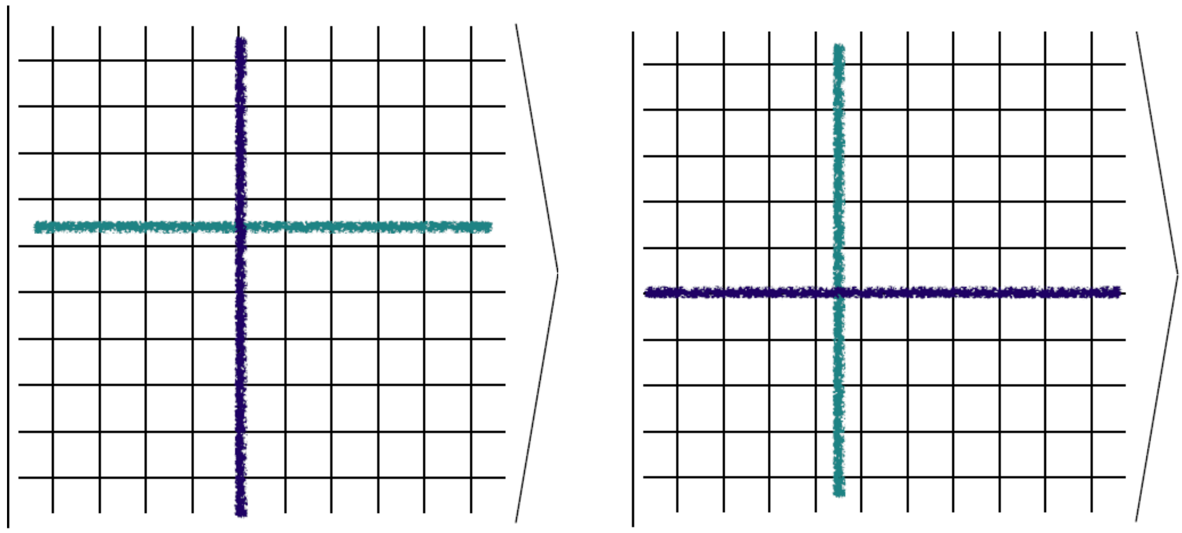


Figure 14: Left: Illustration of the two holonomies X_1, Z_1, Right: The same for X_2, Z_2.

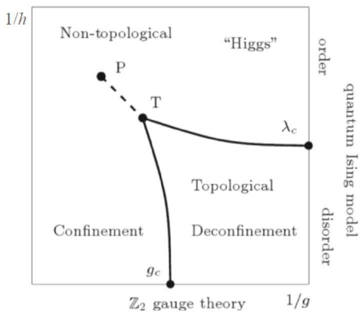
Answer: To find an answer, let's build an electric holonomy $Z_1 \prod_{\mathbf{b}\in\gamma_{\text{vertical}}}$ along vertical bonds. Clearly Z_1 anticommutes with X_1, and thus they form a 2D Hilbert space.

Similarly, one can construct yet another pair of holonomies Z_2 and X_2, which forms yet another 2D hilbert space (see Fig. 14). So in total the ground state degeneracy is 4.

2.3.3 Perturbations: $g, J \neq 0$

We briefly summarize the properties of the full Hamiltonian, Eq. (163), away from the integrable Toric Code limit $J = g = 0$. Importantly, due to the energy gap, the non-trivial topological ground state is protected towards the inclusion of finite g, J.

- Ground state splitting: $\Delta E \sim \exp(-L/\xi)$
  - robustly encode quantum information
    => "Toric Code"
  - $\langle Z_{1,2} \rangle \sim e^{-L/\xi}$: perimeter law of Wilson-loops
    <=> "Deconfinement"
- Phase Diagram
  (for J = 1/h, and K = 1/g)



[Fradkin & Shenker (1979)]

In recent years, it became apparent that the Higgs phase can be seen as an SPT phase with respect to a higher form symmetry, arXiv:2211.01376, arXiv:2303.08136. As such, the bulk phase diagram for systems with closed boundary conditions persist, yet open systems display a surface quantum phase transition separating Higgs and confining phase.

Part II

Non-equilibrium theory:

3 Classical theory: Langevin and Fokker-Planck

4 (Quantum) Master equation, Lindbladians and noise

5 Keldysh formalism

5.1 Coherent states

Problem 1: Second quantization and coherent states [2 + 4 + 4 = 10 points]

Generalizing the well known ladder operators of the quantum harmonic oscillator we define bosonic and fermionic creation/annihilation operators c_i ($i = 1, 2, \dots$ labels lattice sites, spin, or other degrees of freedom) with the property

$$[c_i, c_j^\dagger]_\mp = c_i c_j^\dagger \mp c_j^\dagger c_i = \delta_{ij}, \quad [c_i, c_j]_\mp = 0 = [c_i^\dagger, c_j^\dagger]_\mp. \tag{169}$$

Here, the upper sign refers to bosons, the lower sign to fermions. The number operators is defined as $\hat{n}_i = c_i^\dagger c_i$ and acts on a state in occupation number representation via $\hat{n}_i |n_i\rangle = n_i |n_i\rangle$, where n_i is the number of particles.

Coherent states at a given i are defined as eigenstates of the annihilation operator

$$c_i |\psi_i\rangle = \psi_i |\psi_i\rangle . \tag{170}$$

As we will see, coherent states are not orthogonal and form an overcomplete basis.

a) Operator algebra a fixed i.

   - Prove that $c_i^\dagger |n_i\rangle$ and $c_i |n_i\rangle$ are also eigenstates of the number operator.
   - Show that for the vacuum state $|0\rangle$, annihilation results in $c_i |0\rangle = 0$. *Hint: What is its norm?*
   - What are the allowed eigenvalues n_i for bosons and fermions?

b) Coherent states of bosons. For simplicity we can suppress the index i. Show that the definition implies that $\psi \in \mathbb{C}$ and that the following identities hold.

$$|\psi\rangle = e^{c^\dagger \psi} |0\rangle \tag{171a}$$
$$\langle\psi| = \langle 0| e^{\bar{\psi} c} \tag{171b}$$
$$c^\dagger |\psi\rangle = \partial_\psi |\psi\rangle \tag{171c}$$
$$\langle\psi|\psi'\rangle = e^{\bar{\psi}\psi'} \tag{171d}$$
$$\mathbf{1} = \int d(\bar{\psi}, \psi) e^{-\bar{\psi}\psi} |\psi\rangle \langle\psi| . \tag{171e}$$

Here $d(\bar{\psi}, \psi) = \frac{d\mathfrak{Re}\psi d\mathfrak{Im}\psi}{\pi}$. Finally, restoring the indices i, j, show that for hermitian positive definite matrices h_{ij}

$$\int \prod_i d(\bar{\psi}_i, \psi_i) e^{-\bar{\psi}_i h_{ij} \psi_j} = \det(h)^{-1}. \tag{172}$$

c) Coherent states for fermions.

   Info: In this case the implications of Eq. (170) *are more subtle. Consider a Fock space generated by two types of fermions and their combined coherent states* $|\psi_1, \psi_2\rangle$ *with*

$$c_i |\psi_1, \psi_2\rangle = \psi_i |\psi_1, \psi_2\rangle , i = 1, 2 \tag{173}$$
$$\Rightarrow c_i c_j |\psi_1, \psi_2\rangle = \psi_i \psi_j |\psi_1, \psi_2\rangle . \tag{174}$$

From the algebra of fermionic operators $\{c_i, c_j\} = 0$ *we conclude that also the "numbers" defining the coherent state behave the same way*

$$\{\psi_i, \psi_j\} = 0, \tag{175}$$

and we require also that they anticommute with all fermionic creation/annihilation operators. Note that this implies $\psi_i^2 = 0$*. The objects sufficing Eq.* (175) *are called "Grassmann-numbers" and they fulfill the "Grassmann-Algebra".*

Analysis with Grassmann variables:

- *We define differentiation*

$$\partial_{\psi_i}\psi_j = \delta_{ij}. \tag{176}$$

- *Let* $f(\psi_1, \psi_2 \ldots, \psi_N)$ *be a function of* N *Grassmann variables. This object is defined by Taylor expansion, which is always finite (e.g.* $f(\psi) = f(0) + f'(0)\psi$*).*
- *We define integration of Grassmann variables ("Berezin-integral") as*

$$\int d\psi\psi = 1, \int d\psi 1 = 0 \Rightarrow \int d\psi_i\psi_j = \delta_{ij} \tag{177}$$

Show the following properties of fermionic coherent states, (note the analogy to Eq. (171), except for one minus sign and the definition of the measure, which now is $d(\bar{\psi}, \psi) = d\bar{\psi}d\psi$

$$|\psi\rangle = e^{c^\dagger\psi}|0\rangle \tag{178a}$$

$$\langle\psi| = \langle 0| e^{\bar{\psi}c} \tag{178b}$$

$$c^\dagger|\psi\rangle = -\partial_\psi|\psi\rangle \tag{178c}$$

$$\langle\psi|\psi'\rangle = e^{\bar{\psi}\psi'} \tag{178d}$$

$$\mathbf{1} = \int d(\bar{\psi}, \psi)e^{-\bar{\psi}\psi}|\psi\rangle\langle\psi|. \tag{178e}$$

Finally, restoring the indices i, j, show that for any matrix h_{ij}

$$\int \prod_i d(\bar{\psi}_i, \psi_i)e^{-\bar{\psi}_i h_{ij}\psi_j} = \det(h). \tag{179}$$

Note that $\bar{\psi}$ *and* ψ *are two independent Grassmann variables.*

6 Boltzmann kinetic equation and H theorem

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