Physics 731: Assignment #2

1. (a) Consider a linear differential operator

$$L = \alpha(x)\frac{d^2}{dx^2} + \beta(x)\frac{d}{dx} + \gamma(x),$$

in which $\alpha(x)$, $\beta(x)$, and $\gamma(x)$ are all real functions. This operator acts on the functions $f(x)$ and $g(x)$ in a linear function space $\mathcal{F}$ with the inner product

$$\langle f|g\rangle = \int_{-\infty}^{\infty} f^*(x)g(x)dx.$$

Enumerate the necessary and sufficient conditions such that L is a self-adjoint operator in $\mathcal{F}$.
(b) As an example, show explicitly that the Legendre operator

$$L = -(1-x^2)\frac{d^2}{dx^2} + 2x\frac{d}{dx}$$

is self-adjoint on the interval $[-1, 1]$ under the boundary conditions that $f(\pm 1) =$ finite.

2. **[S1r, S2 1.29, S3 1.31 modified]**
(a) Show by induction that $[x_i, p_i^n] = ni\hbar p_i^{n-1}$ and $[p_i, x_i^n] = -ni\hbar x_i^{n-1}$.
(b) Gottfried (1966) states that

$$[x_i, G(\mathbf{p})] = i\hbar\frac{\partial G}{\partial p_i}, \qquad [p_i, F(\mathbf{x})] = -i\hbar\frac{\partial F}{\partial x_i}$$

can be easily derived from the fundamental commutation relations for all functions F and G that can be expressed as a power series in their arguments. Using your result from (a), verify this statement.
(c) Evaluate $[x^2, p^2]$, where $x^2 = \mathbf{x}\cdot\mathbf{x}$, $p^2 = \mathbf{p}\cdot\mathbf{p}$, and compare your result with the classical Poisson bracket $\{x^2, p^2\}_{\text{classical}}$. Recall that the Poisson bracket is given by

$$\{f, g\}_{\text{classical}} = \sum_i \left(\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial x_i}\right).$$

3. **[S1r, S2 1.30, S3 1.32]** The translation operator for a finite (spatial) displacement is given by

$$T(\mathbf{l}) = \exp\left(\frac{-i\mathbf{p}\cdot\mathbf{l}}{\hbar}\right),$$

where $\mathbf{p}$ is the momentum operator.
(a) Evaluate

$$[x_i, T(\mathbf{l})].$$

(b) Using (a) or otherwise, demonstrate how the expectation value $\langle\mathbf{x}\rangle$ changes under translation.

4. Using the relation

$$\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}}e^{ipx/\hbar},$$

and how the x and p operators act on their own eigenstates:

$$x|x'\rangle = x'|x'\rangle, \qquad p|p'\rangle = p'|p'\rangle,$$

evaluate the expression

$$\langle x|[x, p]|\alpha\rangle$$

in terms of $\psi_\alpha(x) = \langle x|\alpha\rangle$, and show that it leads to the expected result. Then show that this can be found simply by using the fact that in the $|x\rangle$ representation, p acts like $-i\hbar\, d/dx$.

5. The $1s$ (ground state) position space wave function of hydrogen is

$$\psi_{1s}(\mathbf{x}) = \langle \mathbf{x}|1s\rangle = \frac{1}{\sqrt{\pi a_0^3}} e^{-r/a_0},$$

in which r is the usual spherical radial coordinate and a_0 is the Bohr radius.
(a) Find the momentum space wave function $\varphi_{1s}(\mathbf{p})$ for this state. Note: it is useful first to prove that if $f(\mathbf{x}) = f(r)$, then

$$\int e^{-i\mathbf{q}\cdot\mathbf{x}} f(r)\, d^3x = \frac{4\pi}{q}\int_0^\infty f(r)\sin(qr) r dr.$$

(b) Starting from $\varphi_{1s}(\mathbf{p})$, determine (i) $\langle p\rangle \equiv \langle|\mathbf{p}|\rangle$ and (ii) $\langle \mathbf{p}\rangle$.

6. **[S1r, S2 1.21, S3 1.23, modified]** Evaluate the $x - p$ uncertainty product, $\langle(\Delta x)^2\rangle\langle(\Delta p)^2\rangle$, for a 1-d particle confined between two rigid walls:

$$V(x) = \begin{cases} 0, & |x| < a \\ \infty, & |x| > a \end{cases}$$

for the nth eigenstate of this potential, and evaluate the results as a numerical coefficient multiplied by $(\hbar^2/4)$ for the ground and first excited states.