1

Evaluate the integral over circular contour $C : x^2 + y^2 = 2x$:

$$\oint_C \frac{1}{z^2+1}\,\mathrm{d}z \tag{1}$$

Find the pole by factoring the denominator

$$z^2 + 1 = (z+i)(z-i) = 0 \Rightarrow \quad z_1 = i, z_2 = -i \tag{2}$$

However, notice that the contour C can be written as

$$x^2 + y^2 = 2x \quad \Rightarrow (x-1)^2 + y^2 = 1 \tag{3}$$

i.e. a circle centered at $(1, 0)$ with radius 1.

From this, we see that

$$|i - 1| = \sqrt{1+1} = \sqrt{2} > 1 \tag{4}$$

and

$$|-i - 1| = \sqrt{1+1} = \sqrt{2} > 1 \tag{5}$$

so that both poles lie outside the contour and the integral is analytic within and on C. By Cauchy's Integral Theorem, the value of the integral is therefore 0.

2

Evaluate the integral with contour $C : |z-2| = \frac{1}{2}$.

$$\oint_C \frac{z}{(z-1)(z-2)^2}\,\mathrm{d}z \tag{6}$$

The contour is a circle centered at $x = 2$ with radius $\frac{1}{2}$. We find the following poles by setting the denominator equal to zero:

$$\begin{aligned} z_1 &= 1, \quad \text{order} = 1, \text{outside contour;} \\ z_2 &= 2, \quad \text{order} = 2, \text{in contour.} \end{aligned} \tag{7}$$

We thus only focus on residue at $z = 2$.

$$\mathrm{Res}(f, z_2 = 2) = \lim_{z\to 2} \frac{\mathrm{d}}{\mathrm{d}z}\left((z-2)^2 \frac{1}{(z-1)(z-2)^2}\right) = -1 \tag{8}$$

Then by the Residue Theorem, we have

$$\oint_C f(z)\,\mathrm{d}z = 2\pi i(-1) = -2\pi i. \tag{9}$$

3

For the contour C: $|z| = 2$, find

$$\oint_C \frac{1}{(z-3)(z^5-1)}\,\mathrm{d}z. \tag{10}$$

We first realize that the contour is a circle centered at the origin with radius 2.

We find the poles:

- $z_1 = 3$, outside contour.
- $z^5 = 1 \Rightarrow z = \exp[2\pi i k/5], k = 0, 1, ..., 4$. All five roots lie within the contour since $|\exp[2\pi i k/5]| = 1 < 2$. We denote these roots as z_2, z_3, z_4, z_5, z_6.

We use the fact that sum of the residues of $f(z)$ at all its finite singularities, plus the residue of $f(z)$ at infinity, is zero. Here,

$$\sum_{z=2}^{6} \mathrm{Res}(f, z_i) + \mathrm{Res}(f, z_1) = 0 \quad \Rightarrow \quad \sum_{z=2}^{6} \mathrm{Res}(f, z_i) = -\mathrm{Res}(f, z_1), \tag{11}$$

where

$$\mathrm{Res}(f, z_1) = \lim_{z \to 3} (z-3)\frac{1}{(z-3)(z^5-1)} = \frac{1}{242}. \tag{12}$$

So by the Residue Theorem, we have

$$\oint_C f(z)\,\mathrm{d}z = 2\pi i \left(-\frac{1}{242}\right) = -\frac{1}{121}\pi i. \tag{13}$$

4

Evaluate the integral with contour C: $|z| = 1$.

$$\oint_C \frac{z^3}{2z^4 + 1} \, \mathrm{d}z \tag{14}$$

The contour is a circle centered at the origin with radius 1.

We find the poles by solving $2z^4 + 1 = 0$, which gives $z^4 = -\frac{1}{2}$. The four poles are:

$$z_k = \left(\frac{1}{2}\right)^{1/4} \exp\left(\frac{i(\pi + 2\pi k)}{4}\right) \tag{15}$$

for $k = 0, 1, 2, 3$.

Since $|\left(\frac{1}{2}\right)^{\frac{1}{4}}| < 1$, all four poles lie inside the contour C.

For each simple pole z_k, we calculate the residue using

$$\operatorname{Res}(f, z_k) = \frac{P(z_k)}{Q'}(z_k) \tag{16}$$

where $P(z) = z^3$ and $Q(z) = 2z^4 + 1$:

$$Q'(z) = 8z^3, \tag{17}$$

so

$$\operatorname{Res}(f, z_k) = \frac{z_k^3}{8z_k^3} = \frac{1}{8}. \tag{18}$$

The sum of residues is $4 \cdot \left(\frac{1}{8}\right) = \frac{1}{2}$.

By the Residue Theorem:

$$\oint_C f(z) \, \mathrm{d}z = 2\pi i \cdot \left(\frac{1}{2}\right) = \pi i \tag{19}$$

5

Evaluate the integral with contour C: $|z| = 1$.

$$\oint_C \frac{e^{z^2}}{z^2(z^2-9)}\,\mathrm{d}z \tag{20}$$

The contour is a circle centered at the origin with radius 1.

We find the poles:
- $z = 0$ (order 2, from z^2)
- $z = 3$ (order 1, from $z^2 - 9 = 0$)
- $z = -3$ (order 1, from $z^2 - 9 = 0$)

Since $|3| > 1$ and $|-3| > 1$, only the pole at $z = 0$ lies inside the contour.

We calculate the residue at the pole of order 2 at $z = 0$:

$$\begin{aligned}
\operatorname{Res}(f, 0) &= \lim_{z\to 0} \frac{\mathrm{d}}{\mathrm{d}z}\left[z^2 \cdot \frac{e^{z^2}}{z^2(z^2-9)}\right] \\
&= \lim_{z\to 0} \frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{e^{z^2}}{z^2-9}\right] \\
&= \lim_{z\to 0} \frac{e^{z^2}\cdot 2z \cdot \left(z^2-9\right) - e^{z^2}\cdot 2z}{\left(z^2-9\right)^2} \\
&= \lim_{z\to 0} \frac{2ze^{z^2}\left(z^2-10\right)}{\left(z^2-9\right)^2} = 0
\end{aligned} \tag{21}$$

Then by the Residue Theorem:

$$\oint_C f(z)\,\mathrm{d}z = 2\pi i \cdot 0 = 0 \tag{22}$$

6

For C a circle centered at origin, find

$$\frac{1}{2\pi i}\oint_C \sin\left(\frac{1}{z}\right)\mathrm{d}z. \tag{23}$$

Expand $\sin\left(\frac{1}{z}\right)$ in its Laurent series about $z = 0$:

$$\frac{\sin_1}{z} \approx \frac{1}{z} - \frac{1}{6z^3} + ... \tag{24}$$

read off the coefficient of $\frac{1}{z}$ term, which is 1. So $\mathrm{Res}(f(z), 0) = 1$ and so by the Residue Theorem, we have

$$\frac{1}{2\pi i}\oint_C f(z)\,\mathrm{d}z = \mathrm{Res}(f(z), 0) = 1 \tag{25}$$

7

For C a circle centered at the origin ,find

$$\frac{1}{2\pi i}\oint_C \sin^2\left(\frac{1}{z}\right)\mathrm{d}z. \tag{26}$$

Write

$$\sin^2\frac{1}{z} = \frac{1-\cos\left(\frac{2}{z}\right)}{2} \tag{27}$$

and expand in Laurent series about $z = 0$:

$$\sin^2\frac{1}{z} \approx \frac{1-\left(1+\frac{1}{2}\left(\frac{2}{z}\right)^2+\ldots\right)}{2} = \frac{1}{z^2} - \frac{8}{4!z^4} + \ldots \tag{28}$$

There is no $\frac{1}{z}$ term, so the residue at $z = 0$ is 0. By the Residue Theorem, we have

$$\frac{1}{2\pi i}\oint_C f(z)\,\mathrm{d}z = 0. \tag{29}$$

8

For C a circle centered at the origin and n an integer , find

$$\frac{1}{2\pi i}\oint_C z^n e^{2/z}\,\mathrm{d}z \tag{30}$$

Expand $e^{2/z}$ in its Laurent series about $z = 0$:

$$z^{2/z} \approx 1 + \frac{2}{z} + \frac{2}{z^2} + \underbrace{+\frac{1}{j!}\frac{2^j}{z^j}}_{j\text{th term}} + \ldots \tag{31}$$

Multiply by z^n:

$$z^n e^{2/z} \approx z^n + 2z^{n-1} + \ldots + \frac{2^j}{j!}z^{n-j} + \ldots \tag{32}$$

Seek the $\frac{1}{z}$ term, which occurs when $n - j = -1 \Rightarrow j = n + 1$. Thus the residue at $z = 0$ is

$$\mathrm{Res}(f(z), 0) = \frac{2^{n+1}}{(n+1)!}. \tag{33}$$

By the Residue Theorem, the integral is

$$\oint_C f(z)\,\mathrm{d}z = \frac{2^{n+1}}{(n+1)!}. \tag{34}$$

Note that for $n < -1$, we interpret factorial of negative integers as infinite, so the residue and integral is 0 in that case.

9

Find

$$\oint_{|z|=3} (1+z+z^2)\left(e^{1/z} + e^{1/(z-1)} + e^{1/(z-2)}\right) \mathrm{d}z \tag{35}$$

We recognoize pole:

$$z_1 = 0, z_2 = 1, z_3 = 2. \tag{36}$$

Since there are three terms that contribute to poles, and for each pole, only one term contribute to the residue with others being analytic, we can find the residue $\mathrm{Res}(f,0), \mathrm{Res}(f,1), \mathrm{Res}(f,2)$ separately.

- $z_1 = 0$. Only $e^{\frac{1}{z}}$ contributes to the residue. Write

$$(1+z+z^2)e^{1/z} \approx (1+z+z^2)\sum_{k=0}^{\infty} \frac{z^{-k}}{k!} = \sum_{k=0}^{\infty} \frac{1}{k!}\left(z^{-k} + z^{1-k} + z^{2-k}\right) \tag{37}$$

  We seek the $\frac{1}{z}$ term, which occurs when $k = 1, 2, 3$ respectively for the three parts. Thus

$$\mathrm{Res}(f,0) = 1 + \frac{1}{2} + \frac{1}{6} = \frac{5}{3}. \tag{38}$$

- $z_2 = 1$. Taking subsitution $\omega = z - 1$, we have

$$(1+z+z^2)e^{(1/(z-1))} = (3+3\omega+\omega^2)e^{\frac{1}{\omega}} = \sum_{k=0}^{\infty} \frac{1}{k!}\left(3\omega^{-k} + 3\omega^{1-k} + \omega^{2-k}\right). \tag{39}$$

Seek ω^{-1} terms, which occurs when $k = 1, 2, 3$ respectively for the three parts. Thus

$$\mathrm{Res}(f,1) = 3 + \frac{3}{2} + \frac{1}{6} = \frac{14}{3}. \tag{40}$$

- $z_3 = 2$. Following the same procedure, we take $\omega = z - 2$ and write

$$(1+z+z^2)e^{\frac{1}{z-2}} = (7+5\omega+\omega^2)e^{\frac{1}{\omega}} = \sum_{k=0}^{\infty} \frac{1}{k!}\left(7\omega^{-k} + 5\omega^{1-k} + \omega^{2-k}\right). \tag{41}$$

Seek ω^{-1} terms, which occurs when $k = 1, 2, 3$ respectively for the three parts. Thus

$$\mathrm{Res}(f,2) = 7 + \frac{5}{2} + \frac{1}{6} = \frac{29}{3}. \tag{42}$$

Thus by the Residue Theorem, we have

$$\oint_C f(z)\,\mathrm{d}z = 2\pi i \sum_{k=1}^{3} \mathrm{Res}(f, z_k) = 2\pi i\left(\frac{5}{3} + \frac{14}{3} + \frac{29}{3}\right) = 32\pi i. \tag{43}$$

10

Find

$$\oint_{|z|=5} \frac{z}{\sin z(1-\cos z)}\,\mathrm{d}z \tag{44}$$

We recognize poles with

$$\sin z(1-\cos z) = 0 \quad \Rightarrow z = n\pi. \tag{45}$$

Within the contour $|z| = 5$, we have poles at

$$z = 0, \pm\pi. \tag{46}$$

For $z = \pm\pi$, these are simple poles. We find using the formula

$$\operatorname{Res}\left(\frac{p(z)}{q(z)}, z_0\right) = \frac{p(z_0)}{q'(z_0)}. \tag{47}$$

So that

$$\begin{aligned} \operatorname{Res}(f,\pi) &= \frac{\pi}{\cos\pi - \cos(2\pi)} = -\frac{\pi}{2} \\ \operatorname{Res}(f,\pi) &= \frac{-\pi}{\cos(-\pi) - \cos(-2\pi)} = \frac{\pi}{2}. \end{aligned} \tag{48}$$

While for $z = 0$, it's a pole of order 2. We take the Laurent expansion around $z = 0$:

$$f(z) \approx \frac{z}{\left(z - \frac{z^3}{6}\right)\left(1 - 1\frac{x^2}{2}\right)} = \frac{z}{\frac{z^3}{2} - \frac{z^5}{12}}. \tag{49}$$

We see that there is no $\frac{1}{z}$ term, so $\operatorname{Res}(f, 0) = 0$. Thus by the Residue Theorem, we have

$$\oint_{|z|=5} f(z)\,\mathrm{d}z = 2\pi i\left(-\frac{\pi}{2} + \frac{\pi}{2}\right) = 0. \tag{50}$$