Physics 731: Assignment #5

1. Consider a particle subject to a one-dimensional constant force F.
(a) Show that the momentum space propagator is

$$\langle p|U(t,0)|p'\rangle = \delta(p - p' - Ft)e^{i(p'^3 - p^3)/(6m\hbar F)}.$$

(b) Then, show that

$$K(x,t;x',0) = \langle x|U(t,0)|x'\rangle = \left(\frac{m}{2\pi\hbar it}\right)^{1/2} \exp\left[\frac{i}{\hbar}\left(\frac{m(x-x')^2}{2t} + \frac{1}{2}Ft(x+x') - \frac{F^2t^3}{24m}\right)\right].$$

2. For a one-dimensional simple harmonic oscillator, given an initial localized state

$$\psi(x,0) = \frac{1}{\pi^{1/4}\sqrt{d}}e^{-(x-a)^2/(2d^2)},$$

and the closed-form solution of the propagator, explicitly do the appropriate Gaussian integral to show that the state $\psi(x,t)$ is

$$\psi(x,t) = \frac{1}{\pi^{1/4}\sqrt{d}}\frac{1}{\left(\cos\omega t + \frac{i\hbar}{m\omega d^2}\sin\omega t\right)^{1/2}} e^{\frac{im\omega x^2}{2\hbar\tan\omega t} - \frac{im\omega xa}{\hbar\sin\omega t} + \frac{im\omega a^2}{2\hbar\tan\omega t}} e^{-\frac{(m\omega x/(\hbar\sin\omega t) - am\omega/(\hbar\tan\omega t))^2}{2(1/d^2 - im\omega/(\hbar\tan\omega t))}},$$

and that the probability density $|\psi(x,t)|^2$ is

$$|\psi(x,t)|^2 = \frac{1}{\sqrt{\pi}d}\frac{1}{(\cos^2\omega t + (\hbar/(m\omega d^2))^2\sin^2\omega t)^{1/2}} e^{-[(x-a\cos\omega t)^2/[d^2(\cos^2\omega t + (\hbar/(m\omega d^2))^2\sin^2\omega t)]]}.$$

3. Prove that $\langle k_1|k_1'\rangle = \delta(k_1 - k_1')$ for the $E > V_0$ stationary states of the step function potential:

$$\psi_{k_1}(x) = \frac{1}{\sqrt{2\pi}}\left[\theta(-x)\left(e^{ik_1x} + \left(\frac{k_1-k_2}{k_1+k_2}\right)e^{-ik_1x}\right) + \theta(x)\left(\frac{2k_1}{k_1+k_2}\right)e^{ik_2x}\right],$$

in which $k_1 = \sqrt{2mE/\hbar^2}$ and $k_2 = \sqrt{2m(E-V_0)/\hbar^2}$.
Note that the divergent integral $\int_0^\infty e^{ikx}dx$ can be evaluated by considering the $\alpha \to 0$ limit of

$$\int_0^\infty e^{ikx-\alpha x}dx = \frac{\alpha}{\alpha^2+k^2} + \frac{ik}{\alpha^2+k^2},$$

which reduces to

$$\int_0^\infty e^{ikx}dx = \pi\delta(k) + \frac{i}{k}.$$

4. A particle with mass m and energy $E > 0$ is incident from the left on a potential of the form

$$V(x) = \begin{cases} V_0, & 0 < x < a \\ 0, & x < 0,\ x > a \end{cases}$$

in which $V_0 > 0$. Assuming that all conditions hold such that the plane-wave approximation can be used (*i.e.*, the wavepacket is sufficiently localized in momentum space, etc.), compute R and T and verify that $R + T = 1$ for two cases: (i) $E > V_0$, and (ii) $E < V_0$.

5. **[S1r, S2, S3 2.1]** Consider the spin precession problem discussed in the text. It can also be solved in the Heisenberg picture. Using the Hamiltonian

$$H = -\left(\frac{eB}{mc}\right) S_z = \omega S_z,$$

write the Heisenberg equations of motion for $S_x(t)$, $S_y(t)$, $S_z(t)$. Solve them to obtain $S_{x,y,z}$ as a function of time.

6. **[S1r 2.4, S2, S3 2.5]** Let $x(t)$ be the coordinate operator for a free particle in one dimension in the Heisenberg picture. Evaluate $[x(t), x(0)]$.