**ECE 601/901 Fall 2024**
**Homework #6**
**Due 12/16 at 11:59 pm on Canvas, in pdf format**

*Guidelines*:

- Typed solutions or handwritten solutions (as long as they are legible) are accepted.
- Please show as much work as possible and compile your responses into a single pdf document.
- You are welcome to work together on the problems, but please write your solutions in your own words.

1. (15 points) Consider an atom interferometer using cesium atoms (mass = $2.2\times10^{-25}$ kg). The atoms are launched vertically with an initial velocity of 2 m/s.
   (a) (3 points) Calculate the maximum height reached by the atoms as well as the total time of flight between when the atoms are launched and when they return to the initial position (hint: this is a kinematic problem).

Max height can be calculated by $h_{max} = \frac{v_{init}^2}{2g} = 0.2\ m$.

We'll also need the total time of flight: $T_{flight} = \frac{2v_{init}}{g} = 0.41\ s$

   (b) (6 points) If the interferometer uses counter-propagating Raman beams with wavelength near $6s - 6p_{3/2}$ transitions, *approximate* the effective momentum transfer to the atoms with each Raman interaction. Please use cesium atomic data here: https://steck.us/alkalidata/cesiumnumbers.pdf

Transition wavelength is approximately $\lambda = 852\ nm$

Since Raman is a 2-photon process and the beams will be counter-propagating, the effective momentum change will be $\Delta p_{eff} = 2\hbar k = 2\hbar\left(\frac{2\pi}{\lambda}\right) = 1.55 \times 10^{-27}\ kg\ m/s$

   (c) (6 points) Assuming that the time-of-flight is twice the interferometer dwell time for a Mach-Zehnder-type interferometer, calculate the total phase shift experienced by the atom interferometer due to earth's gravity.

The interferometer dwell time $T = \frac{T_{flight}}{2} = 0.205\ s$

The gravitational phase shift is $\phi_{grav} = k_{eff} g T^2 = \left(2 \times \frac{2\pi}{\lambda}\right) g T^2 = 6.1 \times 10^6\ rad$

2. (10 points) Consider the Ramsey interferometry sequence for an atomic clock based on the ground hyperfine states:
   (a) (8 points) What is the probability of finding the atom in the **ground** state after the second $\pi/2$ pulse if: (i) The microwave frequency exactly matches the atomic transition. (ii) The microwave frequency is detuned by $\delta = \pi/T$, where $T$ is the free evolution time.

The probability of the atom being in the excited state after the interferometry sequence (series of $\pi/2$ pulses) is:

$$|c_2|^2 = \left\{\frac{\sin \delta\tau_p/2}{\delta\tau_p/2}\right\}^2 \cos^2\frac{\delta T}{2};\ \delta = \omega - \omega_0$$

For the ground state:

$$|c_1|^2 = 1 - \left\{\frac{\sin \delta\tau_p/2}{\delta\tau_p/2}\right\}^2 \cos^2 \frac{\delta T}{2}$$

For small detuning, $\left\{\frac{\sin \delta\tau_p/2}{\delta\tau_p/2}\right\}^2 \to 1$.

If the detuning is zero, $\cos^2 \frac{\delta T}{2} = 1$. Therefore $|c_1|^2 = 0$.

If the detuning is $\frac{\pi}{T}$, $\cos^2 \frac{\delta T}{2} = 0$. Therefore $|c_1|^2 = 1$.

(b) (2 points) Explain qualitatively why the sensitivity improves with longer free evolution times.

In Ramsey interferometry, the *width* of the interference fringes in frequency space scales as $\sim 1/T$. A longer free-evolution time means a narrower resonance in frequency, thereby improving the clock (or interferometer) sensitivity to small frequency shifts.

3. (5 points) A single NV center in diamond shows Zeeman splitting of its $m_s = \pm 1$ states with frequency separation $\Delta f = 56.5\ MHz$.
   (a) (3 points) Given that the gyromagnetic ratio for NV centers is $\gamma = 28.024\ GHz/T$, calculate the magnetic field magnitude.

The separation between the $0 \leftrightarrow +1$ and $0 \leftrightarrow -1$ transitions is:

$$\Delta\nu_{Zeeman} = 2\gamma B$$

The field is therefore around 1 mT.

(b) (2 points) What is the minimum detectable magnetic field if the resonance linewidth is 1 MHz?

To determine the minimum resonance linewidth, we use

$$\delta B = \frac{\Delta\nu}{2\gamma} = 17.5\ \mu T$$

4. (10 points) Using the $3^2S_{1/2}$ and $3^2P_{3/2}$ states of sodium in a magnetic field:
   (a) (5 points) List all possible hyperfine states $(F, m_F)$.

   $I = \frac{3}{2} \to F = 1,2$ and $F' = 0,1,2,3$

   Possible $3^2S_{1/2}$ states:

   $(1,-1), (1,0), (1,1), (2,-2), (2,-1), (2,0), (2,1), (2,2)$

   Possible $3^2P_{3/2}$ states:

   $(0', 0)$

   $(1', -1), (1', 0), (1', 1)$

$(2', -2), (2', -1), (2', 0), (2', 1), (2', 2)$

$(3', -3), (3', -2), (3', -1), (3', 0), (3', 1), (3', 2), (3', 3)$

(b) (5 points) Identify the light polarization required to drive $\Delta m_F = +1$ transitions and explain its orientation with respect to the magnetic field.

Want to absorb $\sigma^+$ light, which needs to propagate along the magnetic field and have a circular polarization when viewed along that direction.

5. (5 points) Describe the polarization of light beam that was initially horizontally polarized before passing through an atomic medium in a magnetic field of $10\ mT$. The atomic medium has length $l = 5\ cm$ and a Verdet constant of $10^4\ rad/T/m$.

Rotated angle is:

$$\theta_{rot} = VBl = 5\ rad$$

This is equivalent to $286°$ and thus the outgoing light will be polarized $-74°$ away from the horizontal axis.

*Extra problem for ECE 901 (15 points)*

A Mach-Zehnder-type atom interferometer is used to measure linear acceleration ($\vec{a}$) and rotation ($\vec{\Omega}$) *in space*. Rubidium 87 atoms are first launched at $1\ m/s$ and the total trajectory of the atoms is $1\ m$. For this problem, you can ignore the effects of the finite temperature of atoms.

(a) (2 points) In a rotating reference frame, the Coriolis force on the atom is given by

$$\vec{F}_{Coriolis} = -2m(\vec{\Omega} \times \vec{v})$$

Where $m$ is the mass of each atom and $\vec{v}$ is the launch velocity of the atom. Using Newton's second law, derive an expression for the total acceleration experienced by the atom, from both acceleration and rotation.

Based on Newton's second law, $\vec{F} = m\vec{a}_{non-inert}$, where $\vec{a}_{non-inert}$ is the effective acceleration experienced by the atom in the non-inertial frame. The Coriolis contribution to the acceleration is thus:

$$\vec{a}_{Coriolis} = -2(\vec{\Omega} \times \vec{v})$$

The overall effective acceleration includes contribution from the acceleration in the lab frame ($\vec{a}$):

$$\vec{a}_{non-inert} = \vec{a} - 2(\vec{\Omega} \times \vec{v})$$

(b) (5 points) Derive an expression for the phase shift experienced by the atom interferometer due to acceleration and rotation forces by integrating the total acceleration twice to obtain the effective path length.

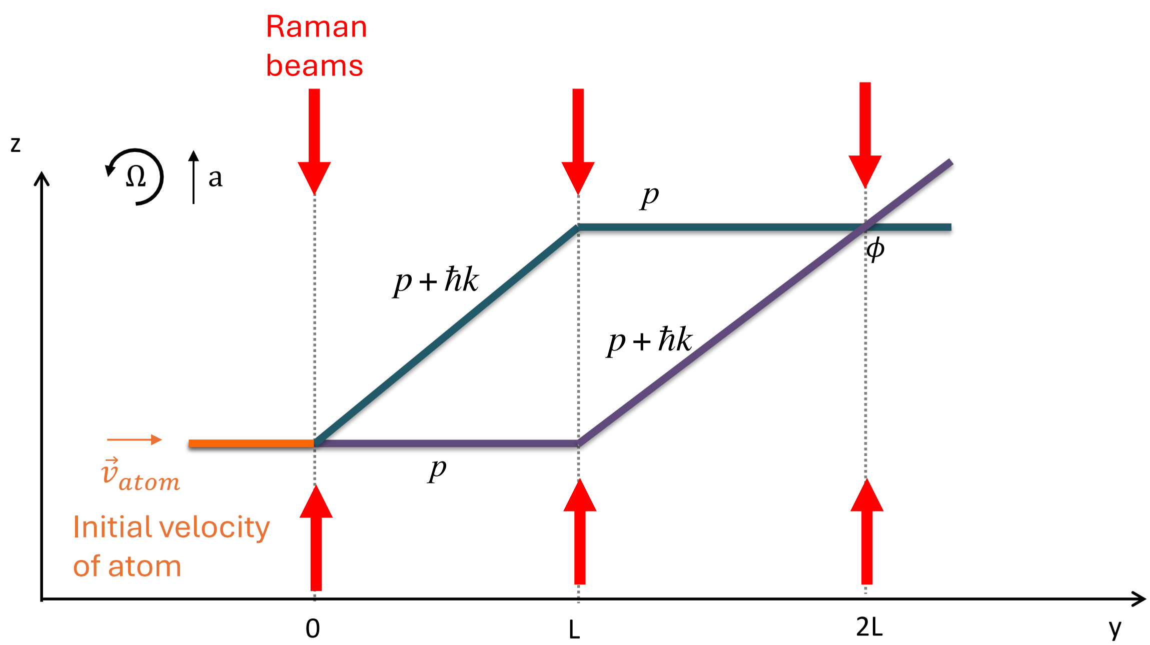
The phase picked up by the atom's matter wave scales with $\vec{k}_{eff} \cdot \vec{r}(t)$ where $\vec{k}_{eff}$ is the effective wavevector for the Raman beams and $\vec{r}(t)$ is the position of the atom. The phase difference between the two interferometer paths (1,2) at $t = T$ is:

$$\Delta\phi(t = T) = \vec{k}_{eff}\left(\vec{r}_2(t = T) - \vec{r}_1(t = T)\right)$$

The path length difference $\vec{r}_2 - \vec{r}_1|_{t=T} = \iint_0^T \vec{a}_{non-inert} dt = \frac{1}{2}\vec{a}_{non-inert}T^2$. Note that at $t = T$, a $\pi$ pulse is applied so the matter wave swaps momentum. Therefore in the second half, another $\frac{1}{2}\vec{k}_{eff}\,\vec{a}_{non-inert}T^2$ is accrued. The total phase becomes

$$\Delta\phi_{tot} = \vec{k}_{eff} \cdot \vec{a}T^2 - 2\vec{k}_{eff} \cdot (\vec{\Omega} \times \vec{v})T^2$$

(c) (3 points) Sketch a configuration for the atom interferometer, showing the directions of $\vec{v}$, $\vec{a}$, $\vec{\Omega}$, and the directions of the Raman beams.




(d) (5 points) What is the acceleration (in $m/s^2$) and rotation (in $rad/s$) contributions to an interferometer phase change of $2\pi$? You can assume the Raman wavelength to be 780 nm.

$$k_{eff} = 2 \times \frac{2\pi}{\lambda} = 1.61 \times 10^7\ m^{-1}$$

$$2T = \frac{1\ m}{1\ m/s} \rightarrow T = 0.5\ s$$

$$a = \frac{2\pi}{k_{eff}T^2} = 1.6 \times 10^{-6}\ m/s^2$$

$$\Omega = \frac{2\pi}{2k_{eff}vT^2} = 7.8 \times 10^{-7}\ rad/s$$