

Let

$$X = a_0 \mathbb{I} + \sum_{k=1}^3 \sigma_k a_k, \quad (1)$$

where σ_k are pauli matrices and a_0, a_1 are numbers.

1. $\text{Tr } X$ and $\text{Tr}(\sigma_k X)$.

Noticing pauli matrices are traceless:

$$\text{Tr}(X) = \text{Tr}(a_0 \mathbb{I}) + \text{Tr}\left(\sum_k \sigma_k a_k\right) = 2a_0. \quad (2)$$

Then consider

$$\begin{aligned} \text{Tr}(\sigma_k X) &= \text{Tr}(a_0 \sigma_k \mathbb{I}) + \text{Tr}\left(\sum_j \sigma_k \sigma_j a_j\right) \\ &= 0 + \sum_j a_j \text{Tr}(\sigma_j \sigma_k). \end{aligned} \quad (3)$$

Using the fact that $\text{Tr}(\sigma_j \sigma_k) = 2\delta_{jk}$:

$$\text{Tr}(\sigma_k X) = \sum_{k=1}^3 a_k \cdot 2\delta_{jk} = 2a_k. \quad (4)$$

2. Find a_0, a_k w.r.t. X_{ij} .

Write

$$X_{ij} = a_0 \delta_{ij} + \sum_{k=1}^3 ((\sigma_k)_{ij} a_k). \quad (5)$$

Then for diagonal elements:

$$\begin{aligned} X_{ii} &= a_0 + (\sigma_3)_{ii} a_3, \\ \Rightarrow X_{11} &= a_0 + a_3, \quad X_{22} = a_0 - a_3 \\ \Rightarrow \begin{cases} a_0 = \frac{1}{2}(X_{11} + X_{22}) \\ a_3 = \frac{1}{2}(X_{11} - X_{22}) \end{cases} \end{aligned} \quad (6)$$

Off diagonal elements: for $m \neq n$:

$$\begin{aligned} X_{mn} &= \sigma_{1_{mn}} a_1 + \sigma_{2_{mn}} a_2 \\ \Rightarrow X_{12} &= a_1 - ia_2, \quad X_{21} = a_1 + ia_2 \\ \Rightarrow \begin{cases} a_1 = \frac{1}{2}(X_{12} + X_{21}) \\ a_2 = \frac{1}{2i}(X_{21} - X_{12}) \end{cases} \end{aligned} \quad (7)$$

P2.

Consider a ket space spanned by $\{|a'\rangle\}$ of Hermitian operator A . No degeneracy.

1. $\Pi_{a'}(A - a')$ is the null operator.

Proof: Since $\{|a'\rangle\}$ spans the space, it's sufficient to show that $\Pi_{a'}(A - a')$ annihilates all basis ket. To show, consider arbitrary basis ket $|a''\rangle$.

$$\begin{aligned} \left[\prod_{a'} (A - a') \right] |a''\rangle &= \dots \times (A - a'') |a''\rangle \times \dots \\ &= \dots \times A |a''\rangle - a'' |a''\rangle \times \dots \\ &= \dots \times a'' |a''\rangle - a'' |a''\rangle \times \dots = 0, \end{aligned} \quad (8)$$

and thus $\Pi_{a'}(A - a')$ annihilates all basis kets, and therefore nullify all vectors in this space.

2. A projection Operator

Let

$$\hat{P} \equiv \prod_{a'' \neq a'} \frac{A - a''}{a' - a''}. \quad (9)$$

Notice that $\hat{P}|a'\rangle = |a'\rangle$, $\hat{P}|a_k\rangle = 0$ (for any $a_k \neq a'$). Explicitly:

$$\hat{P}|a'\rangle = \frac{A - a''}{a' - a''} |a'\rangle = \frac{a' - a''}{a' - a''} |a'\rangle = |a'\rangle; \quad (10)$$

and for any $a_k \neq a'$,

$$\hat{P}|a_k\rangle = \dots \times \frac{A - a_k}{a' - a_k} \times \dots |a_k\rangle = 0. \quad (11)$$

It's clear that \hat{P} is a projection operator onto the eigenspace corresponding to the eigenvalue a' .

3. Illustrate both results using $A = S_z$ for spin 1/2 system.

Recall that the eigenkets $\{|+\rangle, |-\rangle\}$ of the Hermitian operator S_z form an orthonormal basis, just like $\{|a'\rangle\}$ in the assumption.

$$S_z |\pm\rangle = \pm \frac{\hbar}{2} |\pm\rangle, \quad (12)$$

and from which we can observe:

$$\prod_{a'} (S_z - a') |\pm\rangle = \left(S_z - \frac{\hbar}{2} \right) \left(S_z + \frac{\hbar}{2} \right) |\pm\rangle = 0. \quad (13)$$

Further,

$$\hat{P} = \prod_{a'' \neq a'} \frac{A - a''}{a' - a''} = \begin{cases} \frac{S_z + \frac{\hbar}{2}}{\frac{\hbar}{2} + \frac{\hbar}{2}} & \text{for } a' = -\frac{\hbar}{2} \\ \frac{S_z - \frac{\hbar}{2}}{-\frac{\hbar}{2} - \frac{\hbar}{2}} & \text{for } a' = \frac{\hbar}{2} \end{cases}, \quad (14)$$

and thus for $a' = -\frac{\hbar}{2}$: $\hat{P}|+\rangle = 0$, $\hat{P}|-\rangle = 1$; for $a' = \frac{\hbar}{2}$: $\hat{P}|-\rangle = 0$, $\hat{P}|+\rangle = 1$, as expected from part 2.

P3

Construct $|\mathbf{S} \cdot \hat{\mathbf{n}}; +\rangle$ as a linear combination of $|+\rangle, |-\rangle$, s.t.

$$\mathbf{S} \cdot \hat{\mathbf{n}} |\mathbf{S} \cdot \hat{\mathbf{n}}; +\rangle = \frac{\hbar}{2} |\mathbf{S} \cdot \hat{\mathbf{n}}; +\rangle, \quad (15)$$

We recall that, in spherical coordinates,

$$\hat{\mathbf{n}} = (\sin \beta \cos \alpha, \sin \beta \sin \alpha, \cos \beta); \quad (16)$$

and the definition of \mathbf{S} :

$$S_x = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, S_y = \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, S_z = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (17)$$

Then,

$$\begin{aligned} \mathbf{S} \cdot \hat{\mathbf{n}} &= S_x n_x + S_y n_y + S_z n_z \\ &= \frac{\hbar}{2} \begin{pmatrix} \cos \beta & \sin \beta \cos \alpha - i \sin \beta \sin \alpha \\ \sin \beta \cos \alpha + i \sin \beta \sin \alpha & -\cos \beta \end{pmatrix} \\ &= \frac{\hbar}{2} \begin{pmatrix} \cos \beta & \sin \beta e^{-i\alpha} \\ \sin \beta e^{i\alpha} & -\cos \beta \end{pmatrix}. \end{aligned} \quad (18)$$

Let eigenket $|\mathbf{S} \cdot \hat{\mathbf{n}}; +\rangle = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$. The eigenvalue equation then reads:

$$\begin{aligned} \frac{\hbar}{2} \begin{pmatrix} \cos \beta & \sin \beta e^{-i\alpha} \\ \sin \beta e^{i\alpha} & -\cos \beta \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} &= \frac{\hbar}{2} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \\ \Rightarrow \begin{pmatrix} c_1 \cos \beta + c_2 \sin \beta e^{-i\alpha} \\ c_1 \sin \beta e^{i\alpha} - c_2 \cos \beta \end{pmatrix} &= \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \end{aligned} \quad (19)$$

This yields two dependent equations

$$\begin{aligned} c_1 (\cos \beta - 1) + c_2 \sin \beta e^{-i\alpha} &= 0, \\ c_1 \sin \beta e^{i\alpha} - c_2 (\cos \beta + 1) &= 0, \end{aligned} \quad (20)$$

and from which,

$$c_1 = \frac{\sin \beta e^{-i\alpha}}{1 - \cos \beta} c_2 \Rightarrow c_1 = \frac{\cos\left(\frac{\beta}{2}\right) e^{-i\alpha}}{\sin\left(\frac{\beta}{2}\right)} c_2. \quad (21)$$

Normalization condition for the eigen solution requires

$$\begin{aligned} |c_1|^2 + |c_2|^2 &= 1 \\ \Rightarrow |c_2|^2 \left(\frac{\cos^2\left(\frac{\beta}{2}\right)}{\sin^2\left(\frac{\beta}{2}\right)} + 1 \right) &= 1 \\ \Rightarrow |c_2|^2 &= \sin^2\left(\frac{\beta}{2}\right). \end{aligned} \quad (22)$$

Then for an arbitrary azimuthal angle α , we have

$$c_2 = \sin\left(\frac{\beta}{2}\right) e^{i\alpha}, c_1 = \cos\left(\frac{\beta}{2}\right). \quad (23)$$

Thus the eigenket is found to be

$$\cos\left(\frac{\beta}{2}\right) |+\rangle + \sin\left(\frac{\beta}{2}\right) e^{i\alpha} |-\rangle. \quad (24)$$

P4.

A two level system with Hamiltonian

$$H = H_{11}|1\rangle\langle 1| + H_{22}|2\rangle\langle 2| + H_{12}(|1\rangle\langle 2| + |2\rangle\langle 1|). \quad (25)$$

Find energy eigenvalues and eigenkets.

1. method 1, solving eigenvalue problem Explicitely.

Writing \hat{H} in matrix form by noticing $\hat{H}_{11} = \langle 1|H|1\rangle = H_{11}$, $\hat{H}_{12} = \langle 1|H|2\rangle = H_{21}$, $\hat{H}_{22} = \langle 2|H|2\rangle = H_{22}$:

$$\hat{H} = \begin{pmatrix} H_{11} & H_{12} \\ H_{12} & H_{22} \end{pmatrix}. \quad (26)$$

Let eigenvalue be E , and the eigenvalue equation reads:

$$\begin{aligned} \det(\hat{H} - E\mathbb{I}) = 0 &\Rightarrow \det \begin{pmatrix} H_{11} - E & H_{12} \\ H_{12} & H_{22} - E \end{pmatrix} = 0, \\ &\Rightarrow (H_{11} - E)(H_{22} - E) - H_{12}^2 = 0 \\ &\Rightarrow E_{\pm} = \frac{1}{2} \left(H_{11} + H_{22} \pm \sqrt{(H_{11} - H_{22})^2 + 4H_{12}^2} \right). \end{aligned} \quad (27)$$

For each E_{\pm} , we solve for the eigenvector problem $\hat{H}|E\rangle = E_{\pm}|E\rangle$, for $|E\rangle = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$

$$\begin{aligned} \begin{pmatrix} H_{11} - E_{\pm} & H_{12} \\ H_{12} & H_{22} - E_{\pm} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} &= 0 \\ \Rightarrow \begin{cases} (H_{11} - E_{\pm})c_1 + H_{12}c_2 = 0 \\ H_{12}c_1 + (H_{22} - E_{\pm})c_2 = 0 \end{cases} & \\ \Rightarrow \begin{cases} c_1 = H_{12} \\ c_2 = E_{\pm} - H_{11} \end{cases} &. \end{aligned} \quad (28)$$

Applying normalization, we have

$$|E\rangle = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \frac{1}{\sqrt{H_{12}^2 + |E_{\pm} - H_{11}|^2}} \begin{pmatrix} H_{12} \\ E_{\pm} - H_{11} \end{pmatrix}, \quad (29)$$

with E_{\pm} specified above.

2. method 2, use result from P1&P3.

Since $\{\mathbb{I}, \sigma_1, \sigma_2, \sigma_3\}$ form a basis of the 4-dimensional complex vector space of 2×2 complex matrices, any 2 -by- 2 matrix X can be expanded as

$$X = a_0\mathbb{I} + \sum_{k=1}^3 \sigma_k a_k, \quad (30)$$

and thus we can use result from P1 b to write \hat{H} :

$$\hat{H} = \begin{pmatrix} H_{11} & H_{12} \\ H_{12} & H_{22} \end{pmatrix} = a_0\mathbb{I} + \sum_{k=1}^3 \sigma_k a_k, \quad (31)$$

where

$$a_0 = \frac{1}{2}(H_{11} + H_{22}), a_1 = H_{12}, a_2 = 0, a_3 = \frac{1}{2}(H_{11} - H_{22}). \quad (32)$$

Denote $\mathbf{a} = (a_1, a_2, a_3)$, $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$, and then Equation 31 reads

$$\begin{aligned}
\hat{H} &= a_0 \mathbb{I} + \mathbf{a} \cdot \boldsymbol{\sigma} \\
&= a_0 \mathbb{I} + \|\mathbf{a}\| (\hat{n} \cdot \boldsymbol{\sigma}) \\
&= a_0 \mathbb{I} + \frac{2\|\mathbf{a}\|}{\hbar} (\hat{n} \cdot \mathbf{S}).
\end{aligned} \tag{33}$$

The eigenvalue equation is then

$$H|E\rangle = \left[a_0 \mathbb{I} + \frac{2\|\mathbf{a}\|}{\hbar} (\hat{n} \cdot \mathbf{S}) \right] |E\rangle = E|E\rangle. \tag{34}$$

Since we knew from P3 that the eigenvalues of $\hat{n} \cdot \mathbf{S}$ are $\pm \frac{\hbar}{2}$, we have the energy eigenvalues:

$$E_{\pm} = a_0 \pm \|\mathbf{a}\|, \tag{35}$$

with $\|\mathbf{a}\| = \sqrt{\sum_{k=1}^3 a_k^2}$ specified above. Further, since

$$n_z = \cos \beta = \frac{a_3}{\|\mathbf{a}\|}, n_y = \frac{a_2}{\|\mathbf{a}\|} = 0, n_x = \sin \beta \cos \alpha = \frac{a_1}{\|\mathbf{a}\|}, \tag{36}$$

we have $\alpha = 0$ or π , and

$$\begin{aligned}
\cos\left(\frac{\beta}{2}\right) &= \sqrt{\frac{1 + \cos \beta}{2}} = \sqrt{\frac{1 + \frac{a_3}{\|\mathbf{a}\|}}{2}}, \\
\sin\left(\frac{\beta}{2}\right) &= \sqrt{\frac{1 - \cos \beta}{2}} = \sqrt{\frac{1 - \frac{a_3}{\|\mathbf{a}\|}}{2}}.
\end{aligned} \tag{37}$$

Thus, the eigenkets are the same as those in P3, but with $\sin\left(\frac{\beta}{2}\right), \cos\left(\frac{\beta}{2}\right)$ specified above:

$$\begin{aligned}
|E_+\rangle &= \begin{pmatrix} \cos\left(\frac{\beta}{2}\right) \\ e^{i\alpha} \sin\left(\frac{\beta}{2}\right) \end{pmatrix}, \\
|E_-\rangle &= \begin{pmatrix} \sin\left(\frac{\beta}{2}\right) \\ -e^{i\alpha} \cos\left(\frac{\beta}{2}\right) \end{pmatrix}.
\end{aligned} \tag{38}$$

In particular, when $H_{12} < 0, n_1 < 0, \alpha = \pi$; when $H_{12} > 0, n_1 > 0, \alpha = 0$.

P5

A spin 1/2 system in an eigenstate of $\mathbf{S} \cdot \hat{\mathbf{n}}$ with eigenvalue $\frac{\hbar}{2}$. $\hat{\mathbf{n}}$ is a unit vector in xz plane with angle γ w.r.t. $+\hat{z}$.

1. The possible outcomes of a measurement of S_x and their probabilities.

From previous problems, we know that the eigenket of $\mathbf{S} \cdot \hat{\mathbf{n}}$ with eigenvalue $\frac{\hbar}{2}$, where $\alpha = 0, \beta = \gamma$, is

$$|\psi\rangle \equiv |\mathbf{S} \cdot \hat{\mathbf{n}} +\rangle = \cos\left(\frac{\gamma}{2}\right)|+\rangle + \sin\left(\frac{\gamma}{2}\right)|-\rangle. \quad (39)$$

and so the measurement probabilities are

$$\begin{aligned} P\left(\frac{\hbar}{2}\right) &= |\langle +_x | \psi \rangle|^2 = \left| \frac{1}{\sqrt{2}} (\langle + | + \rangle + \langle - | - \rangle) |\psi\rangle \right|^2 = \left| \frac{1}{\sqrt{2}} \left(\cos \frac{\gamma}{2} + \sin \frac{\gamma}{2} \right) \right|^2 = \frac{1 + \sin \gamma}{2}, \\ P\left(-\frac{\hbar}{2}\right) &= |\langle -_x | \psi \rangle|^2 = \left| \frac{1}{\sqrt{2}} (\langle + | - \rangle - \langle - | - \rangle) |\psi\rangle \right|^2 = \left| \frac{1}{\sqrt{2}} \left(\cos \frac{\gamma}{2} - \sin \frac{\gamma}{2} \right) \right|^2 = \frac{1 - \sin \gamma}{2}. \end{aligned} \quad (40)$$

2. Find $\langle (\Delta S_x)^2 \rangle$. Check for $\gamma = 0, \frac{\pi}{2}, \pi$.

A useful identity is

$$\langle (\Delta S_x)^2 \rangle = \langle S_x^2 \rangle - (\langle S_x \rangle)^2, \quad (41)$$

where

$$\begin{aligned} S_x^2 &= \left(\frac{\hbar}{2}\right)^2 \sigma_x^2 = \left(\frac{\hbar}{2}\right)^2 \mathbb{I} \Rightarrow \langle S_x^2 \rangle = \frac{\hbar^2}{4}. \\ \langle S_x \rangle &= \langle \psi | S_x | \psi \rangle = \sum_i a_i p(a_i) = \frac{\hbar}{2} \cdot \frac{1 + \sin \gamma}{2} + \left(-\frac{\hbar}{2}\right) \cdot \frac{1 - \sin \gamma}{2} = \frac{\hbar}{2} \sin \gamma. \end{aligned} \quad (42)$$

From which we have

$$\langle (\Delta S_x)^2 \rangle = \frac{\hbar^2}{4} - \left(\frac{\hbar}{2} \sin \gamma\right)^2 = \boxed{\frac{\hbar^2}{4} \cos^2 \gamma} \quad (43)$$

Checking for

$$\begin{aligned} \gamma = 0 : \langle (\Delta S_x)^2 \rangle &= \frac{\hbar^2}{4}; \\ \gamma = \frac{\pi}{2} : \langle (\Delta S_x)^2 \rangle &= 0; \\ \gamma = \pi : \langle (\Delta S_x)^2 \rangle &= \frac{\hbar^2}{4} \end{aligned} \quad (44)$$

3. How do the results for 1 and 2 change for the case of S_y ?

Noticing $\hat{\mathbf{n}} = (\sin \gamma, 0, \cos \gamma)$, we can easily read off:

$$P\left(\pm \frac{\hbar}{2}\right) = \frac{1 \pm \hat{\mathbf{n}} \cdot \hat{\mathbf{y}}}{2} = \frac{1 \pm 0}{2} = \frac{1}{2}. \quad (45)$$

Then

$$\begin{aligned} \langle S_y \rangle &= \sum_i a_i p(a_i) = \frac{\hbar}{2} \cdot \frac{1}{2} + \left(-\frac{\hbar}{2}\right) \cdot \frac{1}{2} = 0, \\ \langle S_y^2 \rangle &= \frac{\hbar^2}{4}. \\ \Rightarrow \langle (\Delta S_y)^2 \rangle &= \frac{\hbar^2}{4} - 0^2 = \frac{\hbar^2}{4}. \end{aligned} \quad (46)$$

P6

Find the linear combination of $|+\rangle, |-\rangle$ that maximizes the uncertainty product

$$\langle (\Delta S_x)^2 \rangle \langle (\Delta S_y)^2 \rangle. \quad (47)$$

Verify that for the linear combination you found, the uncertainty relation for S_x, S_y is not violated.

We take the general linear combination from P3:

$$|\psi\rangle = \cos\left(\frac{\beta}{2}\right)|+\rangle + \sin\left(\frac{\beta}{2}\right)e^{i\alpha}|-\rangle, \quad (48)$$

and perform similar procedure to find $\langle S_x \rangle, \langle S_y \rangle$.

For $\langle S_x \rangle$:

$$\begin{aligned} P\left(\frac{\hbar}{2}\right) &= |\langle +_x | \psi \rangle|^2 = \left| \frac{1}{\sqrt{2}}(\langle + | + \langle - |) \psi \right|^2 = \left| \frac{1}{\sqrt{2}} \left(\cos\left(\frac{\beta}{2}\right) + \sin\left(\frac{\beta}{2}\right)e^{i\alpha} \right) \right|^2 = \frac{1 + \sin \beta \cos \alpha}{2}, \\ P\left(-\frac{\hbar}{2}\right) &= |\langle -_x | \psi \rangle|^2 = \left| \frac{1}{\sqrt{2}}(\langle + | - \langle - |) \psi \right|^2 = \left| \frac{1}{\sqrt{2}} \left(\cos\left(\frac{\beta}{2}\right) - \sin\left(\frac{\beta}{2}\right)e^{i\alpha} \right) \right|^2 = \frac{1 - \sin \beta \cos \alpha}{2}. \end{aligned} \quad (49)$$

From which we have

$$\langle S_x \rangle = \sum_i a_i p(a_i) = \frac{\hbar}{2} \cdot \frac{1 + \sin \beta \cos \alpha}{2} + \left(-\frac{\hbar}{2}\right) \cdot \frac{1 - \sin \beta \cos \alpha}{2} = \frac{\hbar}{2} \sin \beta \cos \alpha. \quad (50)$$

Similiarly, for $\langle S_y \rangle$:

$$\begin{aligned} P\left(\frac{\hbar}{2}\right) &= |\langle +_y | \psi \rangle|^2 = \left| \frac{1}{\sqrt{2}}(\langle + | + i \langle - |) \psi \right|^2 = \frac{1 + \sin \beta \sin \alpha}{2}, \\ P\left(-\frac{\hbar}{2}\right) &= |\langle -_y | \psi \rangle|^2 = \left| \frac{1}{\sqrt{2}}(\langle + | - i \langle - |) \psi \right|^2 = \frac{1 - \sin \beta \sin \alpha}{2}, \\ \Rightarrow \langle S_y \rangle &= \sum_i a_i p(a_i) = \frac{\hbar}{2} \sin \beta \sin \alpha. \end{aligned} \quad (51)$$

Also, notice

$$S_k^2 = \left(\frac{\hbar^2}{2} \mathbb{I}\right) \Rightarrow \langle S_k^2 \rangle = \frac{\hbar^2}{4}. \quad (52)$$

Collecting, we have

$$\begin{aligned} \langle (\Delta S_x)^2 \rangle &= \langle S_x^2 \rangle - (\langle S_x \rangle)^2 = \frac{\hbar^2}{4} - \left(\frac{\hbar}{2} \sin \beta \cos \alpha\right)^2 = \frac{\hbar^2}{4} (1 - \sin^2 \beta \cos^2 \alpha), \\ \langle (\Delta S_y)^2 \rangle &= \langle S_y^2 \rangle - (\langle S_y \rangle)^2 = \frac{\hbar^2}{4} - \left(\frac{\hbar}{2} \sin \beta \sin \alpha\right)^2 = \frac{\hbar^2}{4} (1 - \sin^2 \beta \sin^2 \alpha) \\ \Rightarrow \langle (\Delta S_x)^2 \rangle \langle (\Delta S_y)^2 \rangle &= \frac{\hbar^4}{16} (1 - \sin^2 \beta \cos^2 \alpha) (1 - \sin^2 \beta \sin^2 \alpha) = \frac{\hbar^4}{16} \underbrace{\left(\cos^2(\beta) + \frac{\sin^4(\beta)}{4} \sin^2(2\alpha) \right)}_{(*)}. \end{aligned} \quad (53)$$

Maximizing $(*)$ for $\alpha, \beta \in [0, 2\pi]$, we have $\max(*) = 1$ for $\beta = \pi + n\pi, n \in \mathbb{Z}$. Thus the maximum uncertainty product is

$$\max \langle (\Delta S_x)^2 \rangle \langle (\Delta S_y)^2 \rangle = \boxed{\frac{\hbar^4}{16}}. \quad (54)$$