P1

They are all governed by the diffusion equation:

$$\begin{aligned} \partial_t \varphi(\boldsymbol{r}, t) &= D \nabla^2 \varphi(\boldsymbol{r}, t) \\ \Rightarrow [D] &= \frac{[L]^2}{[T]}. \end{aligned} \tag{1}$$

1. Penetration of viscous flow into medium

For a viscous fluid, we consider the following properties and dimensions:

- Dynamic viscosity μ, which relates shear stress to the rate of strain. :

$$[\mu] = \frac{[\text{force}]}{[\text{area}]}[\text{time}] = \frac{[M][L][T]^{-2}}{[L]^2}[T] = [M][L]^{-1}[T]^{-1}. \tag{2}$$

- Density ρ :

$$[\rho] = [M][L]^{-3} \tag{3}$$

We then relate the diffusion constant D (in this context, the kinematic viscosity) to these properties:

$$\begin{aligned} D &\propto \mu^{\gamma} \rho^{\beta} \\ \Rightarrow \frac{[L]^2}{[T]} &= \left([M][L]^{-1}[T]^{-1}\right)^{\gamma} \left([M][L]^{-3}\right)^{\beta} \\ &\Rightarrow \gamma = 1, \beta = -1, \end{aligned} \tag{4}$$

and from which we have

$$\boxed{D \propto \mu \rho^{-1}.} \tag{5}$$

Now, considering a characteristic length scale l, it must be related to both D and a time scale τ as

$$\begin{aligned} [l] &= [D]^a [\tau]^b = \left([L]^2 [T]^{-1}\right)^a ([T])^b \\ &\Rightarrow a = \frac{1}{2}, b = \frac{1}{2}, \end{aligned} \tag{6}$$

and thus

$$\boxed{l \propto \sqrt{D\tau} \propto \sqrt{\frac{\mu\tau}{\rho}}}. \tag{7}$$

2. Thermal propagation into medium

For a thermal conductive medium, we consider the following properties and dimensions:

- Thermal conductivity k, which relates heat flux to the temperature gradient. :

$$[k] = \frac{[\text{power}]}{[\text{length}][\text{temp}]} = \frac{[L]^2[M][T]^{-3}}{[L][\Theta]} = [M][L][T]^{-3}[\Theta]^{-1}. \tag{8}$$

- Density ρ :

$$[\rho] = [M][L]^{-3} \tag{9}$$

- Spcific heat capacity c_p: The amount of heat energy required to raise the temperature of a unit mass of a substance by one degree.

$$\left[c_p\right] = \frac{[\text{energy}]}{[\text{mass}][\text{temp}]} = \frac{[M][L]^2[T]^{-2}}{[M][\Theta]} = [L]^2[T]^{-2}[\Theta]^{-1}. \tag{10}$$

We then relate diffusion constant D to these properties:

$$
\begin{aligned}
&D \propto k^{\gamma}\rho^{\beta}c_p^{\alpha} \\
\Rightarrow \frac{[L]^2}{[T]} &= \left([M][L][T]^{-3}[\Theta]^{-1}\right)^{\gamma}\left([M][L]^{-3}\right)^{\beta}\left([M][\Theta])[L]^2[T]^{-2}[\Theta]^{-1}\right)^{\alpha} \\
&\Rightarrow \alpha = \beta = -1, \gamma = -\beta = 1,
\end{aligned}
\tag{11}
$$

and from which we have

$$
\boxed{D \propto k\rho^{-1}c_p^{-1}.}
\tag{12}
$$

Now, considering a characteristic length scale l, it must be related to both D and a time scale τ as

$$
\begin{aligned}
[l] = [D]^a[\tau]^b &= \left([L]^2[T]^{-1}\right)^a([T])^b \\
&\Rightarrow a = \frac{1}{2}, b = \frac{1}{2},
\end{aligned}
\tag{13}
$$

and thus

$$
\boxed{l \propto \sqrt{D\tau} \propto \sqrt{\frac{k\tau}{\rho c_p}}}.
\tag{14}
$$

3. Penetration of EM wave in a conducting medium

For an electrically conducting medium, we consider the following properties and dimensions:

- Magnetic permeability μ, which describes the material's response to a magnetic field. :

$$
[\mu] = \frac{[\text{force}]}{[\text{current}]^2} = \frac{[M][L][T]^{-2}}{[I]^2} = [M][L][T]^{-2}[I]^{-2}.
\tag{15}
$$

- Electrical conductivity σ, which relates current density to the electric field. :

$$
[\sigma] = \frac{[\text{current}]^2[\text{time}]^3}{[\text{mass}][\text{length}]^3} = [M]^{-1}[L]^{-3}[T]^3[I]^2.
\tag{16}
$$

We then relate the diffusion constant D (in this context, the magnetic diffusivity) to these properties:

$$
\begin{aligned}
&D \propto \mu^{\gamma}\sigma^{\beta} \\
\Rightarrow \frac{[L]^2}{[T]} &= \left([M][L][T]^{-2}[I]^{-2}\right)^{\gamma}\left([M]^{-1}[L]^{-3}[T]^3[I]^2\right)^{\beta} \\
&\Rightarrow \gamma = -1, \beta = -1,
\end{aligned}
\tag{17}
$$

and from which we have

$$
\boxed{D \propto (\mu\sigma)^{-1}.}
\tag{18}
$$

Now, considering a characteristic length scale l, it must be related to both D and a time scale τ as

$$
\begin{aligned}
[l] = [D]^a[\tau]^b &= \left([L]^2[T]^{-1}\right)^a([T])^b \\
&\Rightarrow a = \frac{1}{2}, b = \frac{1}{2},
\end{aligned}
\tag{19}
$$

and thus

$$
\boxed{l \propto \sqrt{D\tau} \propto \sqrt{\frac{\tau}{\mu\sigma}}}.
\tag{20}
$$

P2

Cross setction has dimension of area, $[\sigma_T] = [L]^2$. We are interested in expressing σ_T in terms of the potential and parameters of collision. Note that the problem with divergent integral

$$\int_0^\infty b \, \mathrm{d}b \to \infty \tag{21}$$

is that, at large b (i.e. far away from potential source), the deflection is small, but never zero. Quantum mechanics fix this by introducing the wave nature of particles, with non-local particles negligibly affected by the potential at large distance, i.e. setting a classical limit $b_{\max}$ beyond which the particle is unaffected by the potential.

- All that word is to motivate the introduction of Planck constant h, with $[h] = [M][L]^2[T]^{-1}$.
- The potential function $V(r) = A/r^n$ gives $[A] = [E][L]^n = [M][L]^{n+2}[T]^{-2}$.
- A collision is characterized by the impact velocity v.

With $\{A, v, h\}$, we can estimate σ_T with:

$$\begin{gathered} [\sigma_T] = A^a h^b v^c \\ [L]^2 = \left([M][L]^{n+2}[T]^{-2}\right)^a \left([M][L]^2[T]^{-1}\right)^b \left([L][T]^{-1}\right)^c \\ \Rightarrow [L]^2 = [M]^{a+b} \, [L]^{a(n+2)+2b+c} \, [T]^{-2a-b-c} \end{gathered} \tag{22}$$

and from which:

$$\begin{cases} a + b = 0 \\ (2+n)a + b + c = 2 \\ -2a - b - c = 0 \end{cases} \Rightarrow \begin{cases} a = \dfrac{2}{n-1} \\ b = \dfrac{-2}{n-1} \\ c = \dfrac{-2}{n-1} \end{cases} \tag{23}$$

Thus,

$$\begin{gathered} \sigma_T \propto A^{\frac{2}{n-1}} h^{\frac{-2}{n-1}} v^{\frac{-2}{n-1}} \\ \Rightarrow \boxed{\sigma_T \propto \left(\frac{A}{hv}\right)^{\dfrac{2}{n-1}}.} \end{gathered} \tag{24}$$

P3

We treat the rod as an inverted pendulum, for which we know from classical mechanics that, the characteristic time for a pendulum scales as

$$t_c \propto \sqrt{\frac{l}{g}}. \tag{25}$$

But classical mechanics can't explain why the rod falls from unstable equilibrium. To this end we introduce quantum mechanical constant h. To this end we also introduce relevant parameters of mass m.

Buckingham theorem tells us, that if we want to express t in terms of $\{h, g, l, m\}$ having known Equation 25 a priori, we need to construct a dimensionless function f , so that

$$t \propto \sqrt{\frac{l}{g}}\, f(m, h, l, g). \tag{26}$$

Since $f(m, h, l, g)$ is dimensionless, we let

$$\begin{gathered} 1 = [h]^a [m]^b [l]^c [g]^d \\ \Rightarrow 1 = \left([M][L]^2[T]^{-1}\right)^a [M]^b [L]^c \left([L][T]^{-2}\right)^d \end{gathered} \tag{27}$$

from which,

$$\begin{cases} a + b = 0 \\ b + c + 2d = 0 \\ -2c - d = 0. \end{cases} \underset{d\equiv 1}{\Longrightarrow} \begin{cases} a = -1 \\ b = -\frac{3}{2}. \\ c = -\frac{1}{2} \end{cases} \tag{28}$$

So that

$$t \propto \sqrt{\frac{l}{g}} \cdot f\left(\frac{h}{ml^{3/2}\sqrt{g}}\right). \tag{29}$$

Further, we know for a pendulum in unstable equilibrium,

$$\begin{gathered} \ddot{\theta} \approx \omega^2 \theta \Rightarrow \theta \approx \theta_0 \exp(\omega t) \equiv \theta_0 \exp(t/t_c) \\ \Rightarrow t \sim t_c \ln\left(\frac{1}{\theta_0}\right) = \sqrt{\frac{l}{g}} \ln\left(\frac{1}{\theta_0}\right). \end{gathered} \tag{30}$$

.

Comparing Equation 29 and Equation 30, we can make an educated estimate that the quantum effect is encapsulated in the logarithmic term, i.e. $f(\cdot) = \ln(\cdot)$ and so

$$\boxed{t \propto \sqrt{\frac{l}{g}} \ln\left(\frac{h}{ml^{3/2}\sqrt{g}}\right).} \tag{31}$$

P4

Chopers hover due to the lift caused by pressure difference created by the propeller. To this end, we consider the engine power P to be related with air density ρ, propeller diameter l, and weight of choper, G. Set

$$\begin{gathered} P \propto \rho^a l^b G^c \\ \Rightarrow [M][L]^2[T]^{-3} = \left([M][L]^{-3}\right)^a [L]^b \left([M][L][T]^{-2}\right)^c \\ \Rightarrow \begin{cases} a + c = 1 \\ -3a + b + c = 2 \\ -2c = -3 \end{cases} \Rightarrow \begin{cases} a = -\frac{1}{2} \\ b = -1 \\ c = \frac{3}{2} \end{cases}. \end{gathered} \tag{32}$$

This gives

$$P \propto \rho^{-\frac{1}{2}} l^{-1} G^{\frac{3}{2}}. \tag{33}$$

We approximate a choper as a cubic box of side length l. Then, for choper 1 with l_1, G_1, and choper 2 being twice as large in *volume* :

$$V_2 = 2V_1 \Rightarrow l_2^3 = 2l_1^3 \Rightarrow l_2 = 2^{\frac{1}{3}} l_1, \tag{34}$$

and $G_2 = 2G_1$. Then

$$P_2 = \rho^{-\frac{1}{2}} l_2^{-1} G_2^{\frac{3}{2}} = \rho^{-\frac{1}{2}} \left(2^{\frac{1}{3}} l_1\right)^{-1} (2G_1)^{\frac{3}{2}} = \rho^{\frac{1}{2}} \cdot 2^{\frac{7}{6}} \left(l_1^{-1} G_1^{\frac{3}{2}}\right) = \boxed{2^{7/6} P_1} \tag{35}$$

P5

For the White Dwarf mass limit, we derive the scaling in two steps using dimensional analysis: first, the relativistic degenerate pressure P from electron density n_e; second, balancing it against gravity for total mass M.

1. Degenerate Pressure P from n_e

Relevant parameters: $\hbar$ ($[\hbar] = [M][L]^2[T]^{-1}$), c ($[c] = [L][T]^{-1}$), n_e ($[n_e] = [L]^{-3}$).

Seek P ($[P] = [M][L]^{-1}[T]^{-2}$):

$$P \propto \hbar^a c^b n_e^d$$
$$\Rightarrow \begin{cases} a = 1 \\ 2a + b - 3d = -1 \\ -a - b = -2 \end{cases} \Rightarrow \begin{cases} a = 1 \\ b = 1 \\ d = \frac{4}{3} \end{cases}. \tag{36}$$

Thus,

$$\boxed{P \propto \hbar c n_e^{\frac{4}{3}}} \tag{37}$$

2. Gravitational Balance for M

Relevant parameters: G ($[G] = [M]^{-1}[L]^3[T]^{-2}$), $\hbar$, c, m_p ($\left[m_p\right] = [M]$).

Seek M ($[M] = [M]^1$):

$$M \propto G^a \hbar^b c^d m_p^e$$
$$\Rightarrow \begin{cases} -a + b + e = 1 \\ 3a + 2b + d = 0 \\ -2a - b - d = 0 \end{cases} \Rightarrow \begin{cases} b = -a \\ d = -a \\ e = 1 + 2a \end{cases}. \tag{38}$$

To fix the free exponent a, use the pressure-gravity balance. The electron density $n_e \propto \frac{\rho}{m_p}$, with stellar density $\rho \propto \frac{M}{R^3}$, so $n_e \propto \frac{M}{m_p R^3}$. From Step 1, $P \propto \hbar c n_e^{\frac{4}{3}} \propto \hbar c \left(\frac{M}{m_p R^3}\right)^{\frac{4}{3}}$.

Hydrostatic equilibrium scales as $P \propto G \frac{M^2}{R^4}$ (pressure gradient $\frac{P}{R}$ balances gravitational pull $G \frac{M}{R^2}$ times $\frac{M}{R^2}$).

Set them equal:

$$\hbar c \left(\frac{M}{m_p R^3}\right)^{\frac{4}{3}} \propto G \frac{M^2}{R^4}$$
$$\Rightarrow \hbar c \frac{M^{\frac{4}{3}}}{m_p^{\frac{4}{3}}} \propto G M^2 \tag{39}$$

Solve for M: $M^{2-\frac{4}{3}} \propto \left(\hbar \frac{c}{G}\right) m_p^{\frac{4}{3}}$, so $M^{\frac{2}{3}} \propto \left(\hbar \frac{c}{G}\right) m_p^{\frac{4}{3}}$.

Thus, $M \propto \left(\hbar \frac{c}{G}\right)^{\frac{3}{2}} m_p^{-2}$, implying $a = -\frac{3}{2}$, $b = \frac{3}{2}$, $d = \frac{3}{2}$, $e = -2$.

$$\boxed{M \propto \left(\frac{\hbar c}{G}\right)^{\frac{3}{2}} m_p^{-2}} \tag{40}$$