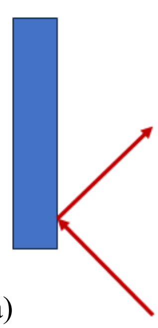
ECE 535 Fall 2025
Homework #2
Due Thursday 10/2 at 11:59 pm on Canvas, in pdf format
Note: Solutions will be posted by Friday night ahead of Exam #1 on 10/6 so please submit on time.

Guidelines:

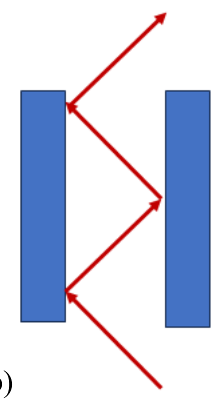
- Please submit a pdf document to Canvas with handwritten solutions, with your approach to each problem and the steps taken clearly laid out and written legibly. In cases where the solution requires plotting, the computer-generated plot should be accompanied by a brief handwritten explanation of your approach. This formatting requirement is worth **5 points** of the point total for each homework.
- Undergraduate students who wish to attempt the extra problem will receive up to 5 additional points for that homework.

1. (15 points) **Photon reflection off a mirror**
   (a) (5 points) A single photon at a wavelength of 1000 nm is incident on a perfect mirror at an angle of 45 degrees, as in the figure below. The mirror starts out at rest and has mass of 1 microgram. After the photon is reflected, what is the direction and speed of the mirror?
   (b) (10 points) Now instead of a single mirror, you have two identical mirrors, each with mass of 1 microgram. The same photon is incident on one of the mirrors at the same angle, but the orientation is such that the photon bounces of the left mirror, then off the right one, and then off the left one again, as in the figure below. After the photon makes all of its reflections, what is the direction and speed of each of the two mirrors?

   (Hint: For both parts, you may assume that the mirror's motion can be described as non-relativistic. This will simplify your conservation formulas for momentum and energy, but *please state any simplifications and justify them*).



(a)



(b)

Since the mirror is relatively massive, we consider its recoil motion to be very small. Therefore, the reflected photon's wavelength is assumed to be identical to be the incident wavelength.

For (a), we start with the conservation of momentum along the horizontal direction:
$-\frac{h}{\lambda}\cos 45^\circ = +\frac{h}{\lambda}\cos 45^\circ + p_x$, where p_x is the momentum transferred to the mirror.

$$p_x = -\frac{2h}{\lambda}\cos 45^\circ = mv_x \rightarrow v_x = -\frac{2h}{m\lambda}\cos 45^\circ$$

In the vertical direction, momenta for the photon and the mirror are:

$$+\frac{h}{\lambda}\sin 45^\circ = +\frac{h}{\lambda}\sin 45^\circ + p_y$$
$$p_y = 0$$

Therefore, the mirror moves in the $-x$ direction at a speed of $9.3706 \times 10^{-19}\ m/s$.

For (b), we apply the conservation of momentum at every reflection. Note that there will be no net motion of the mirrors in the y direction as in the case for (a). The first reflection is solved by (a) and we will relabel the velocity acquired by mirror 1 as $v_{x,1}$. At the second reflection, the momentum for the second mirror is

$$\frac{h}{\lambda}\cos 45^{\circ} = -\frac{h}{\lambda}\cos 45^{\circ} + p_{2,x}$$
$$p_{2,x} = \frac{2h}{\lambda}\cos 45^{\circ} = mv_{2,x}$$

The second mirror will be moving in the $+x$ direction at a speed of $9.3706 \times 10^{-19}\ m/s$.

At the third reflection,

$$-\frac{h}{\lambda}\cos 45^{\circ} - p_{x,1} = \frac{h}{\lambda}\cos 45^{\circ} + p_{x,1}^{final}$$
$$p_{x,1}^{final} = -\frac{2h}{\lambda}\cos 45^{\circ} - \frac{2h}{\lambda}\cos 45^{\circ} = -\frac{4h}{\lambda}\cos 45^{\circ} = mv_{x,1}^{final}$$
$$v_{x,1}^{final} = -1.8741 \times 10^{-18}\ m/s.$$

2. (15 points) **Angular momentum transfer by photons**
   (a) (5 points) Left-handed circularly polarized (σ^{+}) light ~~polarized along the $\hat{y}$ direction~~ propagates along the $\hat{z}$ direction and reflects off a perfect mirror at $z = 0$. What is the electric field and polarization of the reflected light?

A left-handed circularly polarized light traveling along $+\hat{z}$ will have the following form:

$$\vec{\mathcal{E}}_{inc} = \mathcal{E}_0 e^{i(kz-\omega t)}\left(\hat{x} + e^{\frac{i\pi}{2}}\hat{y}\right) = \mathcal{E}_0 e^{i(kz-\omega t)}(\hat{x} + i\hat{y})$$

With the reflection, the wavevector direction changes from $+\hat{z}$ to $-\hat{z}$ and each component picks up a π phase shift, therefore:

$$\vec{\mathcal{E}}_{refl} = \mathcal{E}_0 e^{i(-kz-\omega t)} e^{i\pi}(\hat{x} + i\hat{y})$$

which corresponds to a right-handed circularly polarized electric field for the reflected wave (visualize using the MATLAB code provided in class if need to). Note that the significance of this result is that the handedness of the polarization changes with reflection off a mirror (at normal incidence).

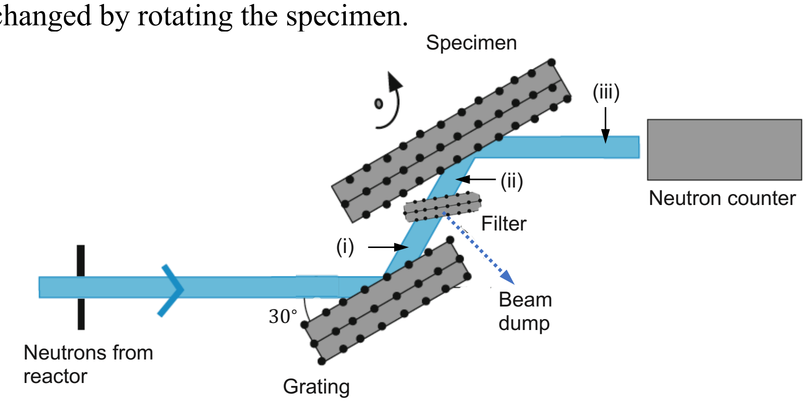
   (b) (10 points) A right-handed circularly polarized photon at $\lambda = 632\ nm$ is normally incident on a perfect mirror which is free to rotate. Calculate the change in angular momentum for both the photon and mirror upon reflection. Justify your answer.

The angular momentum of a σ^{+} (left-handed circular polarized) photon is $\hbar$. Upon reflection, the photon still has an angular momentum of $(-\hbar)(-\hat{z}) = \hbar$, and therefore the change in angular momentum for the photon is 0. For the mirror, $L_{mirror} = 0$ both before and after the reflection due to conservation of angular momentum.

3. (15 points) **Neutron spectrometer**
   Neutrons from a nuclear reactor are collimated through a slit before entering the spectrometer setup shown below, which involves neutron diffraction events at three diffracting elements (grating, filter, and the specimen). In this question, you will consider the effect of each

diffracting element, which can be assumed to consist of a different crystal with only one diffracting plane whose spacing is specified by the lattice spacing of the material:

- The grating crystal has a lattice spacing of 5.4 angstrom and is used to select neutron velocities based on the incident angle (30°). The grating-diffracted beam is indicated by (i) in the diagram.
- The filter crystal lets the first-order neutron velocity component pass through and diffracts off higher-order components, resulting in a monochromatic beam in (ii).
- The monochromatic neutrons in (ii) are used to diffract off of the specimen, with the diffracted beam at (iii) measured by a neutron counter. The angle of incidence can be changed by rotating the specimen.



(a) (5 points) Which de Broglie wavelengths can be found in the beam at (i)? What are the corresponding neutron velocities?

When the neutron beam impinges on the crystal at an angle of θ, partial waves will be reflected by the parallel crystal planes (spaced apart by d which is the lattice constant). These partial waves interfere constructively if the path difference between them is an integer multiple of the de Broglie wavelength of the neutrons. This Bragg diffraction condition can be written as

$$n\lambda_{dB} = 2d\sin\theta$$

$$\lambda_{dB} = \frac{2(5.4\ A)\sin 30^\circ}{n} = \frac{5.4\ A}{n} \text{ where } n = 1,2,3$$

The lowest order wavelengths correspond to slow and non-relativistic neutrons and thus the neutron velocity is related to the de Broglie wavelength by:

$$\lambda_{dB} = \frac{h}{p} = \frac{h}{m_n v}$$

$$v = \frac{h}{m_n\lambda_{dB}} = n\frac{6.626\times 10^{-34}}{(1.674927\times 10^{-27})(5.4\times 10^{-10})}\frac{m}{s} = 732.59\,n\,m/s$$

(b) (4 points) Assume that there are only two velocity components in (i). Now, you are asked to design a neutron filter which eliminates the higher-order velocity component. You have limited material choices to construct this filter: pyrolitic graphite (lattice constant = 2.46 angstroms) and copper (lattice constant = 3.61 angstroms). Which material would you pick to construct the filter? Justify your response. Please assume here that as long as the diffraction condition is satisfied, 100% of the velocity component will be diffracted. Additionally, any un-diffracted components will pass through the filter without being scattered or absorbed.

In order to diffract the higher-order wavelengths $n \geq 2$, the lattice constant of the filter material needs to satisfy the following condition:

$$\lambda_{dB} > 2d\max(\sin\theta)$$

$$\frac{\lambda_{dB}}{2d} > 1$$

$$d < 2.7\ A$$

Therefore, pyrolitic graphite is the appropriate material to use for the filter.

(c) (4 points) At what angle should the filter, made of the material you selected in (b), be oriented relative to the incident beam (i)?

The filter needs to be oriented at an angle α that satisfies the Bragg condition for the second-order wavelength:

$$2.7 = 2(2.46)\sin\alpha$$
$$\alpha = \sin^{-1} 0.5488$$
$$\alpha = 33.28^{\circ}$$

(d) (2 points) Describe a measurement procedure to determine the lattice spacing of the specimen. Include a discussion on any limitations of the technique.

We can determine the lattice spacing of the specimen by putting it on a rotating mount, and rotating it around the vertical axis relative the to the direction of the monochromatic neutrons. We will count the number of neutrons at the detector. The count rate measured by the detector will be at a maximum whenever the Bragg condition is satisfied. That angle θ_2 can be used to determine the lattice spacing d_2 using:

$$d_2 = 2.7\,A\sin\theta$$

The range of lattice constants that can be measured will be limited by the wavelength of the incoming beam, so that d_2 needs to be greater than 2.7 A in order for an intensity maximum to be observed.

4. (15 points) **Determination of the fine structure constant**
   (a) (6 points) Suppose we are able to experimentally determine the following in a gamma-emitting atom:
      i. The mass difference ΔM of two nuclear energy levels in terms of atomic mass units (amu), which is related to the mass in kilograms kg by the Avogadro's number N_A: $mass\ [amu] = mass\ [kg]\ 1000N_A$
      ii. The wavelength λ (in meters) of the gamma ray emitted when transitioning between the nuclear energy levels.

   By relating the energy of the gamma photon with the mass difference of the atom in grams, show that measurements of $\Delta M\ [amu]$ and $\lambda\ [m]$ can give us the product of two fundamental constants $N_A\hbar$.

The mass difference in kg is related to the gamma energy by $\delta m = \frac{E_{photon}}{c^2} = \frac{h}{\lambda c} = \frac{2\pi\hbar}{\lambda c}$

$$\frac{\Delta M[amu]}{1000\ N_A} = \frac{2\pi\hbar}{\lambda c} \rightarrow \frac{\Delta M[amu]}{1000\ N_A} = \frac{2\pi\hbar}{\lambda c}$$
$$N_A\hbar = \frac{\Delta M[amu]\lambda c}{2000\pi}$$

   (b) (6 points) $N_A\hbar$ is known as the molar Planck constant and can be used to infer the fine constant α. Write α in terms of $N_A\hbar$, the Rydberg constant R_∞, and the electron mass in atomic mass unit M_e.

$$\alpha = \frac{e^2}{4\pi\epsilon_0\hbar c} \text{ and } R_\infty = \frac{m_e e^4}{(4\pi\hbar)^3 c\epsilon_0^2} \rightarrow \frac{\alpha^2}{R_\infty} = \frac{4\pi\hbar}{m_e c} = \frac{4\pi\hbar}{c}\frac{1000N_A}{M_e[amu]}$$

$$\alpha = \sqrt{\frac{R_\infty}{M_e}\frac{4000\pi}{c}N_A\hbar}$$

(c) (3 points) Use the data in https://physics.nist.gov/cuu/Constants/index.html to estimate the uncertainty with which we can determine α.

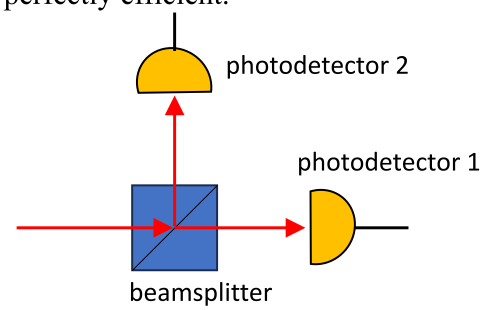
The only quantities with uncertainties are R_∞ and M_e, so the uncertainty in α becomes:

$$\frac{\delta\alpha}{\alpha} = \frac{1}{2}\sqrt{\left(\frac{\delta R_\infty}{R_\infty}\right)^2 + \left(\frac{\delta M_e}{M_e}\right)^2}$$

From the database, $\frac{\delta R_\infty}{R_\infty} = 1.9 \times 10^{-12}$ and $\frac{\delta M_e}{M_e} = 2.9 \times 10^{-11}$, so $\frac{\delta\alpha}{\alpha} \sim 1.4 \times 10^{-11}$.

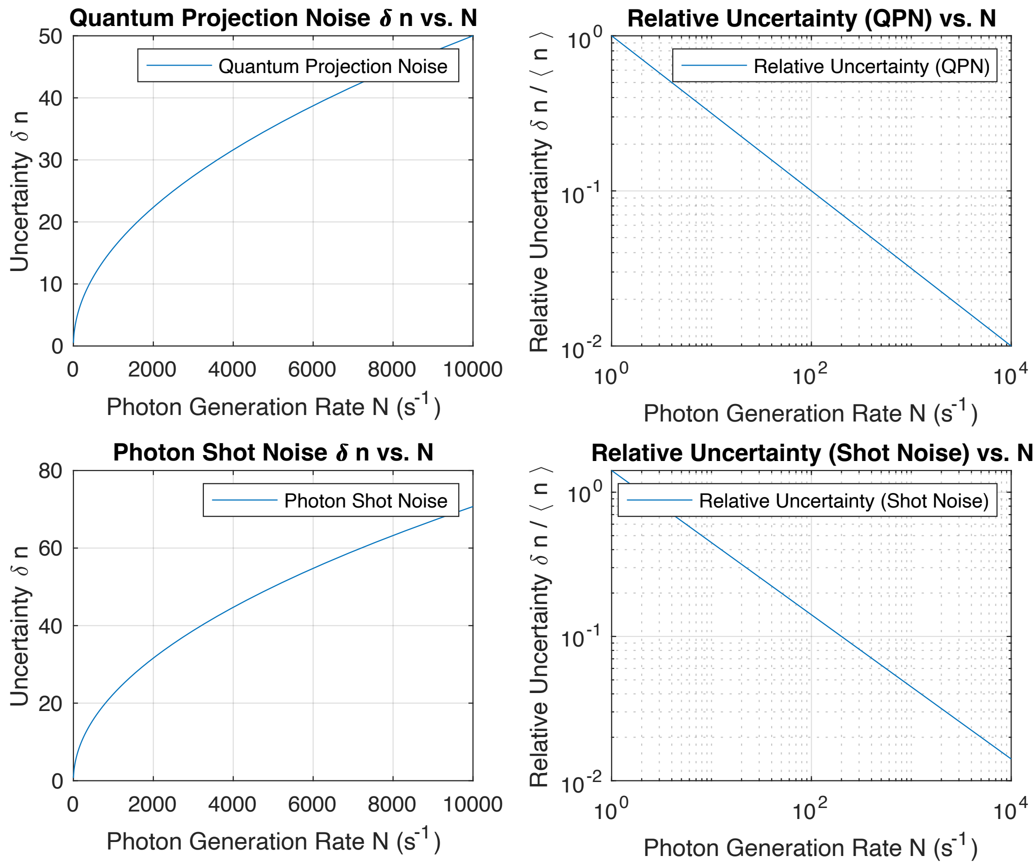
5. (10 points) **Quantum projection noise and photon shot noise**

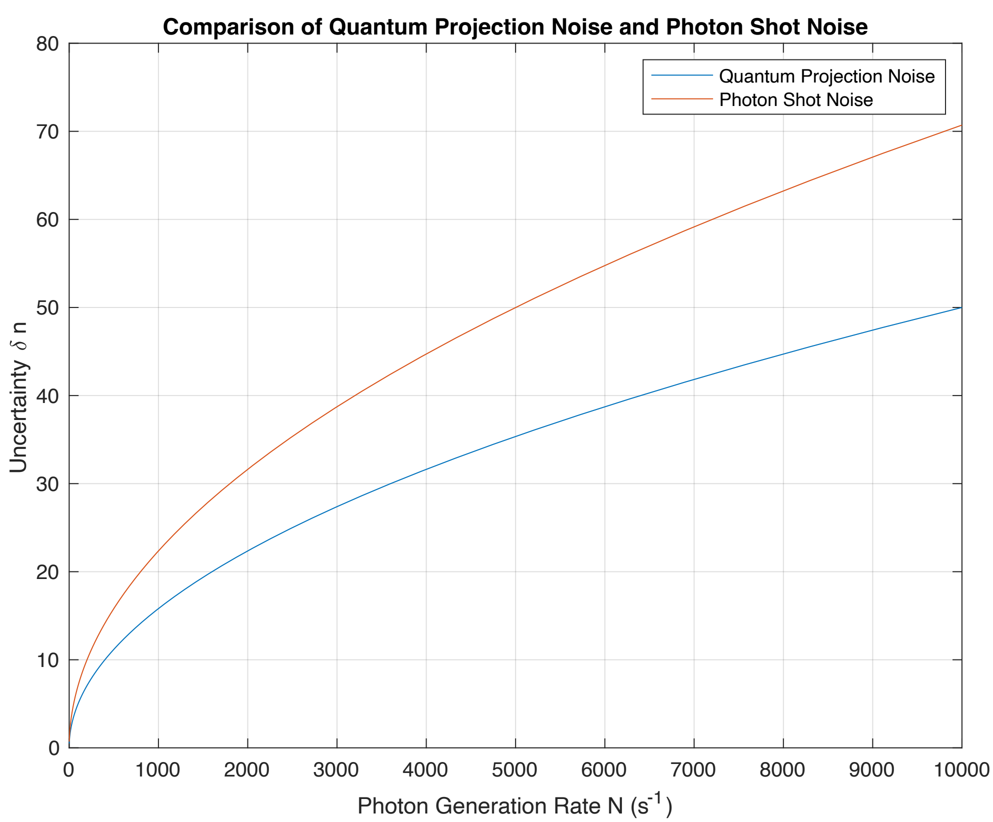
A stream of single photons is being generated at a rate of N (in units of number of photons per second) and sent through a beamsplitter. The probability of each photon taking either path after the beamsplitter is $p = 0.5$. The photons in each path are detected by a photodetector, which can be assumed to be perfectly efficient.



(a) (6 points) Based on information provided through the lecture notes, calculate and plot the uncertainties in the photon detection rate δn at photodetector 1 due to the quantum projection noise and photon shot noise, as a function N (spanning 1 to 10^4 s^{-1}). For this problem, consider the two noise sources separately.

(b) (3 points) Now plot $\delta n/\langle n\rangle$ vs N, where $\langle n\rangle$ is the mean photon count rate at photodetector 1. Come up with analytical expressions for $\delta n/\langle n\rangle$ for each of the quantum projection noise and photon shot noise based on your results.


Quantum Projection Noise δ n vs. N
Quantum Projection Noise
Uncertainty δ n
Photon Generation Rate N (s^-1)
Relative Uncertainty (QPN) vs. N
Relative Uncertainty (QPN)
Relative Uncertainty δ n / ⟨ n ⟩
Photon Generation Rate N (s^-1)
Photon Shot Noise δ n vs. N
Photon Shot Noise
Uncertainty δ n
Photon Generation Rate N (s^-1)
Relative Uncertainty (Shot Noise) vs. N
Relative Uncertainty (Shot Noise)
Relative Uncertainty δ n / ⟨ n ⟩
Photon Generation Rate N (s^-1)


Comparison of Quantum Projection Noise and Photon Shot Noise
Quantum Projection Noise
Photon Shot Noise
Uncertainty δ n
Photon Generation Rate N (s^-1)

We expect the mean number of photons detected at detector 1, $\langle n \rangle$, to be $N/2$ for a large number of N.

In the scenario described, the quantum projection noise arises from the uncertainty in which each path the photon will take, while the photon shot noise is related to the inherent statistical fluctuations in the arrival time of each photon at the detector.

For the quantum projection noise, there is a probability p of 0.5 for *each photon* to reach photodetector 1. We can then model the uncertainty in the number of photons being detected at each detector as a binomial distribution, where the mean probability of detection is $\hat{p} = p = \frac{1}{2}$, and standard deviation in the mean probability is:

$$\sigma_{QPN,\hat{p}} = \sqrt{\frac{p(1-p)}{N}} = \frac{1}{2}\sqrt{\frac{1}{N}}$$

To get the fluctuations in photon number within a second, we multiply $\sigma_{QPN,\hat{p}}$ with N, the total number of photons arriving at the beamsplitter within a second

$$\sigma_{QPN,\ N} = (\delta n)_{QPN} = N\sqrt{\frac{p(1-p)}{N}} = \sqrt{Np(1-p)} = \sqrt{N/4} = \sqrt{N}/2$$

The mean number of photons detected per second is $\langle n \rangle = N\hat{p} = \frac{N}{2}$. Therefore the fractional uncertainty in photodetection at detector 1 is:

$$\left(\frac{\delta n}{\langle n \rangle}\right)_{QPN} = \frac{\sqrt{N}/2}{N/2} = 1/\sqrt{N}$$

The standard deviation in the photon shot noise is the square-root of the number of detected photons, so at each detector $\sigma_{shot,\ N} = (\delta n)_{shot} = \sqrt{N/2}$. Therefore,

$$\frac{\delta n}{\langle n \rangle} = 1/\sqrt{N/2}$$

(c) (1 point) Discuss what happens to $\delta n/\langle n \rangle$ if the photodetectors are only 75% efficient.

If the detector has an efficiency of $\eta = 0.75$, then the mean photon count rate at each detector will change to:

$$\langle n \rangle = \eta\frac{N}{2}$$

The quantum projection noise is unaffected since it depends on the probabilistic behavior of the photon at the beamsplitter. However, our measurement of this noise is imperfect due to detector sampling. Therefore the projection noise will appear to look like:

$$(\delta n)_{QPN,\ \eta N} = \sqrt{N\eta p(1-\eta p)} \rightarrow \left(\frac{\delta n}{\langle n \rangle}\right)_{QPN,\ \eta N} = \frac{2\sqrt{p(1-\eta p)}}{\sqrt{\eta N}}$$

Note that full credit will be given here for recognizing that the projection noise should not intrinsically change with detection efficiency.

The photon shot noise will change to $\sigma_{shot,\ N} = \sqrt{\eta N p} = \sqrt{\eta N/2}$

The relative uncertainty for the photon shot noise becomes $\left(\frac{\delta n}{\langle n \rangle}\right)_{shot} = 1/\sqrt{\eta N/2}$

6. (10 points) **Extra question for graduate students: what is the Doppler shift of a reflected photon?**
   (a) (7 points) A photon with frequency ν_0 is normally incident on a perfectly reflective mirror at rest. The mirror, which has mass m, moves at a velocity of v, but the motion of the mirror can be described as relativistic. Using the conservation of energy and momentum, find an expression relating ν_0 to the frequency of the reflected photon (ν'). If you cannot find a closed form solution for ν', simplify the expression as much as possible.

Momentum conservation:

$$\frac{h\nu_0}{c} = -\frac{h\nu'}{c} + \frac{m_0 v}{\sqrt{1 - v^2/c^2}}$$

Energy conservation

$$h\nu_0 + m_0c^2 = h\nu' + \frac{m_0c^2}{\sqrt{1 - v^2/c^2}}$$

Let's substitute $\beta = v/c$ and rewrite the equations as

$$h\nu_0 = -h\nu' + \frac{m_0c^2\beta}{\sqrt{1-\beta^2}}$$

$$h\nu_0 = h\nu' + \left(\frac{1}{\sqrt{1-\beta^2}} - 1\right)m_0c^2$$

$$\rightarrow \frac{2h\nu_0}{m_0c^2} + 1 = \frac{\beta}{\sqrt{1-\beta^2}} + \frac{1}{\sqrt{1-\beta^2}}$$

$$\rightarrow A\sqrt{1-\beta^2} = \beta + 1, \text{ where } A = \frac{2h\nu_0}{m_0c^2} + 1$$

We can solve for β analytically using MATLAB:

```
syms A b
solve(b+1==A*sqrt(1-b^2),b)
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The valid solutions are $\beta = -1$ (not valid given the direction of the mirror's motion) and $\beta = \frac{A^2-1}{A^2+1}$
The shift in photon frequency can be found by:

$$\nu_0 - \nu' = \left(\frac{1}{\sqrt{1-\beta^2}} - 1\right)m_0c^2/h$$

   (b) (3 points) The frequency shift you solved for in (a) is the Doppler shift of a reflected photon off a moving mirror. Using relativity, the Doppler shift for a source moving away from the observer is calculated as

$$\nu' = \sqrt{\frac{c-v}{c+v}}\nu_0$$

   Compare your answer in (a) with the above formula and justify why the expressions may be different. For your comparison, feel free to use a numerical example (say for $\nu_0 =$ 500 THz, $m = 10^{-35}\,kg$).

The expressions look different but they should describe the same physical effect. For example, for $\nu_0 =$ 500 THz and $m = 10^{-35}\,kg$:

$$\beta = \frac{A^2 - 1}{A^2 + 1} = 0.5018 \rightarrow v = 0.5018\, c$$
$$\nu' = 2.8798 \times 10^{14}\ Hz$$
$$\nu' = 287.98\, THz$$

Using the Doppler shift formula,

$$\nu' = \sqrt{\frac{c - v}{c + v}}\,\nu_0 = 287.98\, THz$$

Same result!