# Math 521: Assignment 1 (due 5 PM, February 7)

1. Prove that
$$(2n+1)+(2n+3)+(2n+5)+\ldots+(4n-1)=3n^2 \tag{1}$$
for all $n \in \mathbb{N}$.

2. For $n \in \mathbb{N} \cup \{0\}$, the Pell numbers are defined by
$$P_n = \begin{cases} 0 & \text{if } n=0, \\ 1 & \text{if } n=1, \\ 2P_{n-1}+P_{n-2} & \text{otherwise.} \end{cases} \tag{2}$$

   (a) Define
   $$f(n) = \frac{(1+\sqrt{2})^n - (1-\sqrt{2})^n}{2\sqrt{2}}. \tag{3}$$
   Let $H_n$ be the proposition that "both $P_n = f(n)$ and $P_{n-1} = f(n-1)$." Use mathematical induction to prove that $H_n$ is true for all $n \in \mathbb{N}$, and deduce that $P_n = f(n)$ for all $n \in \mathbb{N} \cup \{0\}$.

   (b) Define the rational number $\lambda = (P_8 + P_9)/P_9$. Why is $|\lambda - \sqrt{2}|$ small?

3. Prove that $\sqrt{2} + \sqrt{5}$ is irrational.

4. In the lectures, we introduced the field properties A1–A4, M1–M4, DL, and O1–O5.

   (a) Which of the field properties hold for $\mathbb{N}$?

   (b) Which of the field properties hold for $\mathbb{Z}$?

   (c) Consider the binary numbers $\mathbb{B} = \{0, 1\}$, where addition and multiplication are defined as
   $$0+0=0, \quad 0+1=1, \quad 1+1=0,$$
   $$0\times 0=0, \quad 0\times 1=0, \quad 1\times 1=1,$$
   and the ordering is defined so that $0 \le 1$. Which of the field properties hold for $\mathbb{B}$?

5. Using the field properties of $\mathbb{R}$, prove that

   (a) $0 < 1$.

   (b) If $0 < a < b$ then $0 < b^{-1} < a^{-1}$, for all $a, b \in \mathbb{R}$.

6. Consider each of the following sets:

$$A = [1,2) \cup (3,\infty), \qquad B = \{r \in \mathbb{Q} : r < 2\}, \quad C = \{r \in \mathbb{Q} : r^2 < 2\},$$
$$D = \{\tfrac{1}{m} + n : m,n \in \mathbb{N}\}, \quad E = \{\sqrt{2}, e, \pi\}, \qquad F = \{2 - x^2 : x \in \mathbb{R}\}.$$

For each set, determine its minimum and maximum if they exist. Find the infimum and supremum, writing your answers in terms of infinity for unbounded sets. Detailed proofs are not required.

7. Let $a, b, \epsilon \in \mathbb{R}$. Prove the following statements, or find a counterexample:
   (a) $a < b$ if and only if $a < b + \epsilon$ for all $\epsilon > 0$.
   (b) $a \leq b$ if and only if $a < b + \epsilon$ for all $\epsilon > 0$.

8. Let $A$ and $B$ be non-empty bounded subsets of $\mathbb{R}$ and define $A - B = \{a - b : a \in A, b \in B\}$. Prove that $\sup(A - B) = \sup A - \inf B$.

9. Let $C$ and $D$ be non-empty bounded subsets of $\mathbb{R}$. For each of the three statements, either prove the result or find a counterexample:
   (a) Define $M = \{cd : c \in C, d \in D\}$. Then $\inf M = (\inf C)(\inf D)$.
   (b) If $\sup C < \inf D$, then there exists an $r \in \mathbb{R}$ such that $c < r < d$ for all $c \in C$ and $d \in D$.
   (c) If there exists an $r \in \mathbb{R}$ such that $c < r < d$ for all $c \in C$ and $d \in D$, then $\sup C < \inf D$.

10. Define the set of irrational numbers as $\mathbb{I} = \mathbb{R} \setminus \mathbb{Q}$. Prove that if $a < b$ then there exists $x \in \mathbb{I}$ such that $a < x < b$. *(Hint: it may be useful to consider the set $\{r + \sqrt{2} : r \in \mathbb{Q}\} \subseteq \mathbb{I}$.)*

11. For each of the following sequences defined for $n \in \mathbb{N}$, calculate its limit if it exists:
    (a) $\frac{n^2+3}{n^2-3}$
    (b) $(-1)^n n$
    (c) $\frac{4n+2}{3-5n^2}$
    (d) $\sqrt{n} - \sqrt{n-1}$
    (e) $\sqrt{n^2+n} - n$
    (f) $n!/8^n$

    Detailed proofs are not required, but you should justify your answers.

12. (a) Find a sequence $(q_n)$ of rational numbers having a limit $\lim q_n$ that is an irrational number.
    (b) Find a sequence $(p_n)$ of irrational numbers having a limit $\lim p_n$ that is a rational number.

13. **Optional.** Find an infinite collection of sets $S_1, S_2, S_3, \ldots$ such that every $S_i$ has an infinite number of elements, $S_i \cap S_j = \emptyset$ for all $i \neq j$, and $\bigcup_{i=1}^{\infty} S_i = \mathbb{N}$.

14. **Optional.** By writing a computer program or otherwise, show that $\lambda$ from question 2(b) is the closest member of the set $\{p/q : p \in \mathbb{Z}, q \in \mathbb{N}, q < P_{10}\}$ to $\sqrt{2}$.