Math 521: Assignment 2 (due 5 PM, February 21)

1. Find an example of two sequences (a_n) and (b_n) such that (a_n) and $(a_n b_n)$ both converge, but (b_n) does not.

2. For each of the following statements about a sequence (a_n) and a real number a, either prove the result or find a counterexample:

   (a) If $\lim a_n = a$ then $\lim |a_n| = |a|$.

   (b) If $\lim |a_n| = 0$ then $\lim a_n = 0$.

   (c) If $\lim |a_n| = |a|$ then $\lim a_n = a$.

3. Prove that if $\lim a_n = 0$ and $|b_n - b| \le a_n$ for all $n \in \mathbb{N}$, then $\lim b_n = b$.

4. Prove that if $\lim a_n = \infty$ and (b_n) is bounded below, then $\lim(a_n + b_n) = \infty$.

5. Let $x_1 = 2$ and define
$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right). \tag{1}$$

   (a) Show that x_n^2 is always greater than or equal to 2, and use this to show that x_n is a non-increasing sequence. Use this to show that (x_n) converges, and deduce that $\lim x_n = \sqrt{2}$.

   (b) Each x_n is a rational number, and thus Eq. (1) provides a method to generate accurate rational approximations to $\sqrt{2}$, similar to question 2 about the Pell numbers P_n on Assignment 1. Show that $x_5 = (P_{15} + P_{16})/P_{16}$.

   (c) **Optional.** Is there a general relationship between x_n and the Pell numbers?

6. Consider the sequence (x_n) where $x_1 = \frac{1}{4}$ and
$$x_{n+1} = r x_n (1 - x_n) \tag{2}$$
for $n \in \mathbb{N}$, where $r > 0$.

   (a) For the case when $r = \frac{3}{4}$, prove that $\lim x_n = 0$.

   (b) Consider the case when $r = \frac{5}{2}$, and introduce the sequence (z_n) such that $x_n = \frac{3}{5} + z_n$. Derive a recurrence relation giving z_{n+1} in terms of z_n. Prove that if $z_n \in (-\frac{1}{10}, \frac{1}{10})$ then
$$|z_{n+1}| \le \tfrac{3}{4}|z_n|. \tag{3}$$

   (c) Using the result from part (b), or otherwise, prove that (x_n) converges to $\frac{3}{5}$.

   (d) **Optional.** Does (x_n) converge when $r = 3.82$?

   (e) **Optional.** Does (x_n) converge when $r = 3.83$? Suppose the sequence (y_n) satisfies $y_n = x_{3n}$. Does (y_n) converge?

7. (a) Suppose that (x_n) is a convergent sequence. Prove that the sequence (y_n) given by the averages,

$$y_n = \frac{1}{n}\sum_{i=1}^{n} x_i \tag{4}$$

also converges to the same limit.

(b) Find an example where (y_n) converges but (x_n) does not.

(c) **Optional.** Suppose (z_n) diverges to ∞ and define $x_n = (-1)^n z_n$ for all $n \in \mathbb{N}$. Is it possible to find an example where (y_n) converges?

8. Let (a_n) be a sequence such that for $n \in \mathbb{N}$,

$$a_n = \begin{cases} 1 & \text{for } n \text{ odd,} \\ -1 - 2^{-n} & \text{for } n \text{ even.} \end{cases} \tag{5}$$

Calculate the monotonic sequences

$$u_N = \inf\{a_n : n > N\}, \qquad v_N = \sup\{a_n : n > N\} \tag{6}$$

for each $N \in \mathbb{N}$. Determine $\liminf a_n$ and $\limsup a_n$.

9. Define the function $f : \mathbb{N} \to \mathbb{N}$ so that $f(n)$ equals the largest power of 2 that divides n. Table 1 shows examples of evaluating f. Let (s_n) be a sequence.

(a) Find the set of subsequential limits when $s_n = \frac{1}{f(n)}$.

(b) Find the set of subsequential limits when $s_n = f(3n)$.

(c) **Optional.** Find the set of subsequential limits when $s_n = f(n+1) + \frac{1}{f(n)}$.

10. Let (a_n) be a sequence such that $\liminf |a_n| = 0$. Prove that there is a subsequence (a_{n_k}) such that $\sum_{k=1}^{\infty} a_{n_k}$ converges.

11. Determine the convergence or divergence of each of the following series defined for $n \in \mathbb{N}$:

$$\text{(a)} \sum_n \frac{1}{2^n + n^2}, \quad \text{(b)} \sum_n \frac{\cos n}{n^2}, \quad \text{(c)} \sum_n \frac{1}{\sqrt{n!}},$$

$$\text{(d)} \sum_n \exp\left(-n + 2\cos\tfrac{\pi n}{2}\right), \quad \text{(e)} \sum_n \frac{n!}{n^n}.$$

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$f(n)$	1	2	1	4	1	2	1	8	1	2	1	4	1	2	1	16

Table 1: The first sixteen values of the function f defined in question 9.

12. **Optional – iterated limits.**[1] Let a_{mn} be a doubly-indexed array where $m, n \in \mathbb{N}$. What should $\lim_{m,n\to\infty} a_{mn}$ represent?

(a) Let $a_{mn} = m/(m+n)$ and compute the iterated limits

$$\lim_{n\to\infty}\left(\lim_{m\to\infty} a_{mn}\right), \qquad \lim_{m\to\infty}\left(\lim_{n\to\infty} a_{mn}\right). \tag{7}$$

Define $\lim_{m,n\to\infty} a_{mn} = a$ to mean that for all $\epsilon > 0$, there exists N such that for all $m, n > N$, $|a_{mn} - a| < \epsilon$.

(b) Consider the two different definitions,

$$a_{mn} = \frac{1}{m+n}, \qquad a_{mn} = \frac{mn}{m^2+n^2}. \tag{8}$$

For each definition, determine whether $\lim_{m,n} a_{mn}$ exists, and whether the iterated limits in Eq. (7) exist. How do these three values compare?

(c) Find an example where $\lim_{m,n} a_{mn}$ exists but neither iterated limit can be computed.

(d) Assume $\lim_{m,n} a_{mn} = a$, and assume that for each fixed $m \in \mathbb{N}$, $\lim_{n\to\infty}(a_{mn}) \to b_m$. Show $\lim_{m\to\infty} b_m = a$.

(e) Prove that if $\lim_{m,n} a_{mn}$ exists and the iterated limits both exist, then all three limits must be equal.

[1] This is adapted from *Understanding Analysis* by S. Abbot, exercise 2.3.13.