Continuity

Functions f

Domain: set on which f is defined written as $\mathrm{dom}(f)$.

A rule/formula specifying $f(x)$ at x in $\mathrm{dom}(f)$.

Real-valued function: $\mathrm{dom}(f) \subseteq \mathbb{R}$ and $f(x) \in \mathbb{R}$ for all $x \in \mathrm{dom}(f)$.

f represents the function
$f(x)$ represents the value at x.

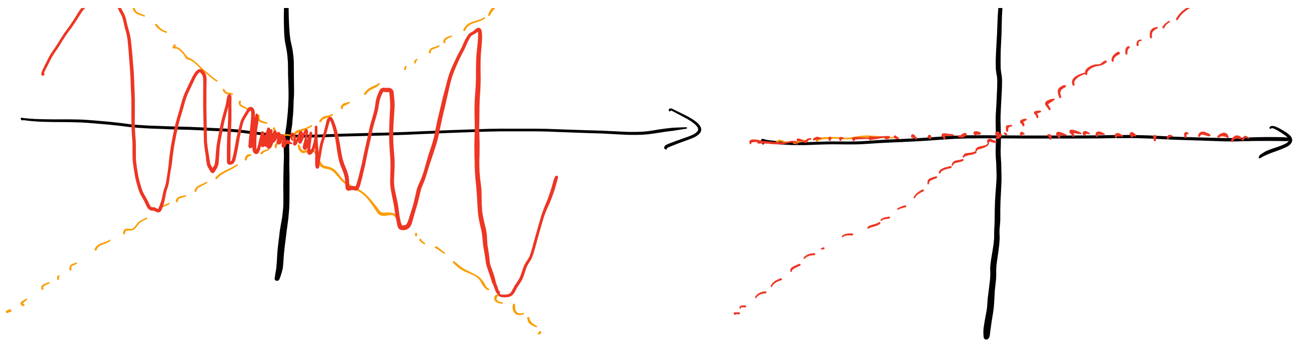
Sometimes the domain is omitted. Assume it's valid on a "natural domain": the largest subset of $\mathbb{R}$ on which the function is well-defined.

e.g. $f(x) = \frac{1}{x}$ $\quad \{x \in \mathbb{R} \mid x \neq 0\}$

$g(x) = \sqrt{4-x^2}$ $\quad [-2,2]$.

Continuity

$$f_1(x) = \begin{cases} x \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases} \qquad f_2(x) = \begin{cases} 0 & x \notin \mathbb{Q} \\ x & x \in \mathbb{Q} \end{cases}$$



Definition

Let f be a real-valued function whose domain is a subset of $\mathbb{R}$. The function f is **continuous at x_0** in $\mathrm{dom}(f)$ if for every sequence (x_n) in $\mathrm{dom}(f)$ that converges to x_0, $\lim_{n\to\infty} f(x_n) = f(x_0)$. If f is continuous at each point of a set $S \subseteq \mathrm{dom}(f)$, then f is said to be **continuous on S**.

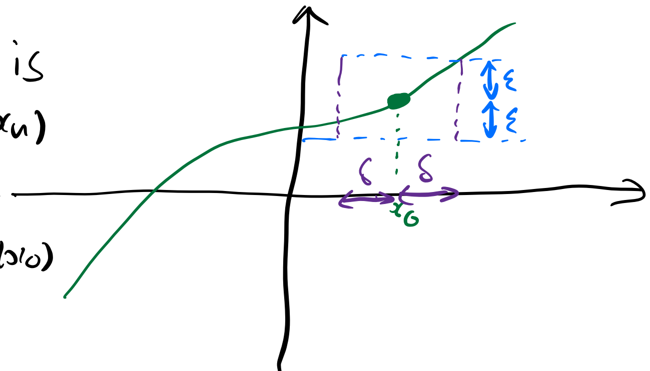
Example Polynomial $p(x) = \sum_{k=0}^{N} a_k x^k$

$\lim_{n\to\infty} x_n = x_0$ $\qquad$ $\lim_{n\to\infty} p(x_n) = p(x_0)$.

$\Rightarrow$ all polynomials are continuous for $\mathbb{R}$

Theorem Let f be a real-valued function, whose domain is a subset of $\mathbb{R}$. Then f is continuous at $x_0 \in dom(f)$ if and only if

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t } x \in dom(f) \text{ and } |x - x_0| < \delta \text{ then } |f(x) - f(x_0)| < \varepsilon. \quad (*)$$



Proof Suppose that $(*)$ is true. Take a sequence (x_n) in $dom(f)$ s.t $\lim_{n\to\infty} x_n = x_0$.
Aim to prove $\lim_{n\to\infty} f(x_n) = f(x_0)$

Let $\varepsilon > 0$

$\exists \delta > 0$ s.t $x \in dom(f)$ and $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon.$

$\exists N$ s.t $n > N \Rightarrow |x - x_0| < \delta.$

Hence $n > N$ implies $|f(x_n) - f(x_0)| < \varepsilon.$

$$\Rightarrow \lim_{n\to\infty} f(x_n) = f(x_0).$$

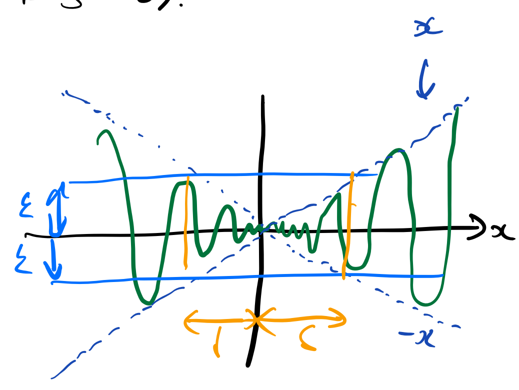
Assume f in continuous but $(*)$ fails. So for each $n \in \mathbb{N}$ for $|x_n - x_0| < \frac{1}{n}$.

$\exists x_n \in dom(f)$ s.t $|f(x_n) - f(x_0)| \geq \varepsilon.$

$\lim x_n = x_0$ $\quad \lim f(x_n) \neq f(x_0).$

Example

$$f_1(x) = \begin{cases} x \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0. \end{cases}$$



Is f_1 continuous at 0?

$x \neq 0$

$|f_1(x) - f_1(0)| = |x \sin \frac{1}{x}| < |x|.$

$|f_1(x) - f_1(0)| = 0$ $\quad x = 0.$

Pick $\delta = \varepsilon$ in our definition $(*)$ then the function will be continuous at $x = 0$.

Theorem Let f be a real-valued function with $\mathrm{dom}(f) \subseteq \mathbb{R}$. If f is continuous at x_0 in $\mathrm{dom}(f)$, then $|f|$ and kf are too.

Proof Suppose $\lim_{n\to\infty} x_n = x_0$ then $\lim_{n\to\infty} f(x_n) = f(x_0)$.

By sequence convergence theorems, $\lim_{n\to\infty} kf(x_n) = kf(x_0)$

$\Rightarrow kf$ is continuous at x_0.

For $|f|$: $\lim_{n\to\infty} |f(x_n)| = |f(x_0)|$ $\quad |f|$ is continuous at x_0.

Theorem Let f and g be real-valued functions that are continuous at $x_0 \in \mathbb{R}$. Then

$f+g$ is continuous at x_0

fg is continuous at x_0.

f/g is continuous at x_0 if $g(x_0) \neq 0$.

Proof First two follow from sequence convergence theorems. For the third, examine (x_n) in $\mathrm{dom}(f) \cap \{x \in \mathrm{dom}(g) \mid g(x) \neq 0\}$, s.t $x_n \to x_0$.

$$\lim_{n\to\infty} \left(\frac{f}{g}\right)(x_n) = \lim_{n\to\infty} \frac{f(x_n)}{g(x_n)} = \frac{f(x_0)}{g(x_0)}$$

Theorem If f is continuous at x_0 and g is continuous at $f(x_0)$, the $g \circ f$ is continuous at x_0.

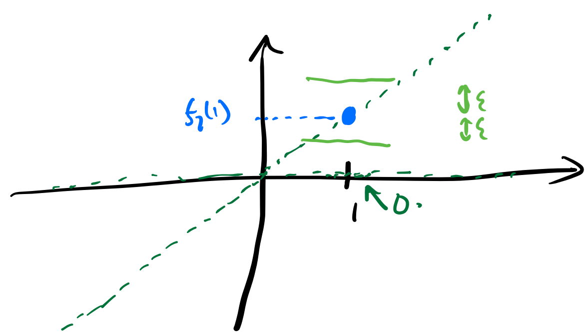
Example $\max(f,g) = \frac{1}{2}(f+g) + \frac{1}{2}|f-g|$

$$= \begin{cases} \frac{1}{2}(f+g) + \frac{1}{2}(f-g) & f \geq g \\ \frac{1}{2}(f+g) + \frac{1}{2}(g-f) & f \leq g \end{cases}$$

$$= \begin{cases} f & f \geq g \\ g & f \leq g \end{cases}$$

$\Rightarrow$ $\max(f,g)$ is continuous at x_0 if f and g are continuous there.

$$f_2(x) = \begin{cases} 0 & x \notin \mathbb{Q} \\ x & x \in \mathbb{Q} \end{cases}$$



$f_2(1) = 1$

$x_n = 1 + \frac{\sqrt{2}}{n} \quad \notin \mathbb{Q}$

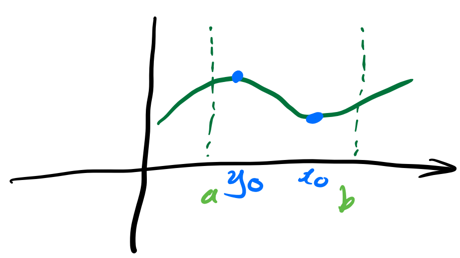
$f_2(x_n) = 0 \qquad \lim_{n\to\infty} f_2(x_n) = 0 \neq f_2(1).$

Theorem Let f be a continuous real-valued function on a closed interval $[a,b]$. Then f is a bounded function. Moreover, f assumes its maximum and minimum values on $[a,b]$. $\exists\, x_0, y_0 \in [a,b]$ s.t

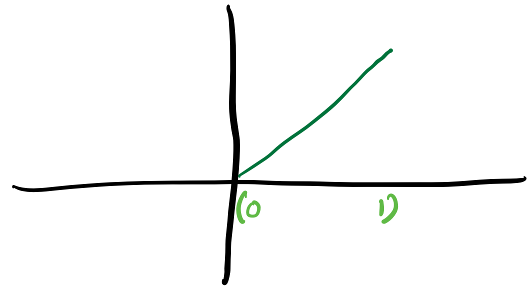
$$f(x_0) \leq f(x) \leq f(y_0) \qquad \forall x \in [a,b].$$

[Compared to open intervals. $(0,1)$. $f(x) = 1/x$

continuous but not bounded



$(0,1)$ $f(x) = x$ maximum value is not attained.



Theorem Suppose that f is not bounded. Then for each $n \in \mathbb{N}$, $\exists\, x_n$ s.t $|f(x_n)| > n$. By the Bolzano-Weierstraß theorem, $\exists$ a subsequence (x_{n_k}) that converges to x_0. x_0 must be within the closed interval.

Since f is continuous at x_0, $\lim\limits_{k \to \infty} f(x_{n_k}) = f(x_0)$.

but $\lim\limits_{k \to \infty} |f(x_{n_k})| = \infty$ $\divideontimes$ $\Rightarrow f$ is bounded.

Let $M = \sup \{ f(x) \mid x \in [a,b] \}$. M is finite.

For each $n \in \mathbb{N}$, $\exists\, y_n \in [a,b]$ such that $M - \frac{1}{n} < f(y_n) < M$.

Then $\lim\limits_{n \to \infty} f(y_n) = M$.

By the Bolzano-Weierstraß theorem, $\exists$ a convergent subsequence y_{n_k} converging to a limit y_0 in $[a,b]$.

Since f is continuous at y_0, then $f(y_0) = \lim\limits_{k \to \infty} f(y_{n_k})$

$$\lim_{k \to \infty} f(y_{n_k}) = \lim_{n \to \infty} f(y_n) = M.$$

$$f(y_0) = M.$$

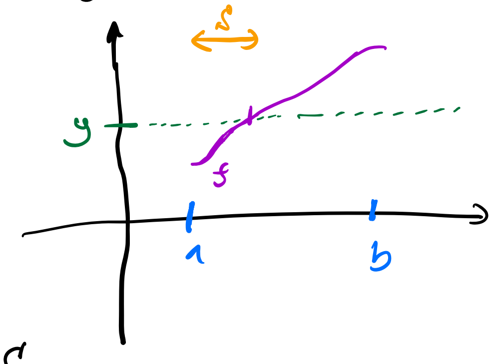
For f achieves its minimum apply the same procedure to $-f$.

Intermediate value theorem

Let f be a continuous real-valued function on an interval I. Then whenever $a, b \in I$, $a < b$ and $f(a) < y < f(b)$* or $f(b) < y < f(a)$, then $\exists$ at

$f(a) = \ldots$

least one $x \in (a,b)$ s.t. $f(x) = y$.



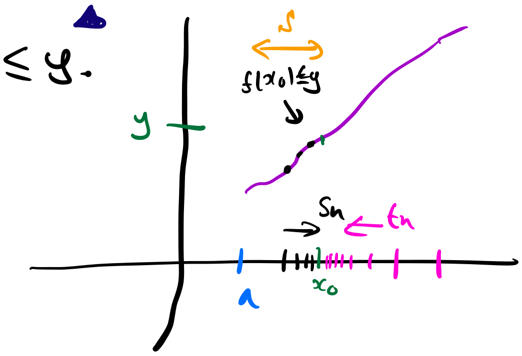
Proof Let $S = \{x \in [a,b] \mid f(x) < y\}$

$a \in S$ so S is non-empty, so $\sup S = x_0$ where $x_0 \in [a,b]$.

For all $n \in \mathbb{N}$, $x_0 - \frac{1}{n}$ is not an upper bound for S, so $\exists s_n \in S$ s.t $x_0 - \frac{1}{n} < s_n \leq x_0$. Hence $\lim_{n\to\infty} s_n = x_0$, and

since $f(s_n) < y$ $\forall n$,

$$f(x_0) = \lim_{n\to\infty} f(s_n) \leq y. \quad \blacktriangle$$



Let $t_n = \min\{b, x_0 + \frac{1}{n}\}$

Since $x_0 \leq t_n \leq x_0 + \frac{1}{n}$, then

$$\lim_{n\to\infty} t_n = x_0.$$

$t_n \in [a,b]$ but $t_n \notin S$ so $f(t_n) \geq y$.

$$f(x_0) = \lim_{n\to\infty} t_n \geq y \quad \heartsuit$$

$\heartsuit + \blacktriangle$ shows that $f(x_0) = y$.

Corollary

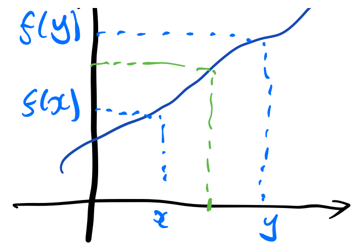
If f is a continuous real-valued function on an interval I, then $f(I)$ is also an interval or a single point.

Proof $y_0, y_1 \in f(I) \Rightarrow y_0 < y < y_1$ $y \in f(I)$.

Definition

Consider a function $f(x)$ on an interval I. Then $f(x)$ is strictly increasing if for all $x, y \in I$ where $x < y$, then

$$f(x) < f(y).$$



In cases like this, an inverse function can be unambiguously defined, so that $f^{-1} \circ f(x) = x$.

$$y = f(x)$$
$$f^{-1}(y) = x.$$

<u>Theorem</u> Let f be a continuous strictly increasing function on an interval I. Then $f(I)$ is an interval J by the above corollary, and f^{-1} represents a function on J. f^{-1} is continuous and strictly increasing.

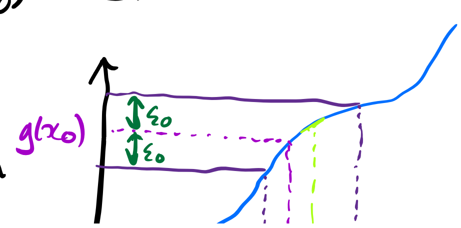
<u>Proof</u> Suppose $a, b \in J$ s.t $a < b$. Then $\exists c, d \in I$ s.t $f(c) = a$ and $f(d) = b$. Since $a \neq b$, $c \neq d$. We must have $c < d$ because if $c > d$ then $f(c) > f(d)$. Hence defining $c = f^{-1}(a)$ and $d = f^{-1}(b)$ we see
$$f^{-1}(a) < f^{-1}(b).$$
$\Rightarrow f^{-1}$ is strictly increasing.

[See next theorem for continuity property]

<u>Theorem</u> Let g be a strictly increasing function on an interval J such that $g(J)$ is an interval I. Then g is continuous on J.

<u>Proof</u> Consider $x_0 \in J$. Restrict to x_0 not an end point. Hence $g(x_0)$ is not an endpoint, and $\exists \varepsilon_0$ s.t $(g(x_0) - \varepsilon_0, g(x_0) + \varepsilon_0) \subseteq I$.

Consider $\varepsilon > 0$. Assume $\varepsilon < \varepsilon_0$ since in the continuity definition

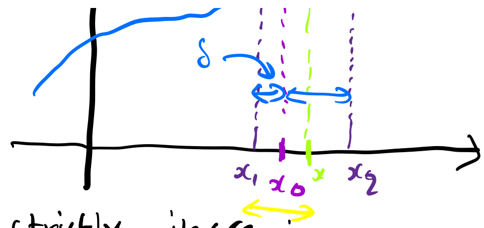


... to small ε

we can restrict to small ε.

$\exists x_1, x_2 \in J$ s.t $g(x_1) = g(x_0) - \varepsilon$

$g(x_2) = g(x_0) + \varepsilon$.



Then $x_1 < x_0 < x_2$ since it is strictly increasing.

Similarly for $x \in (x_1, x_2)$

$$g(x_1) < g(x) < g(x_2).$$

Then $|g(x_0) - g(x)| < \varepsilon$. Put $\delta = \min\{x_2 - x_0, x_0 - x_1\}$

Then $|x - x_0| < \delta$ implies that $|g(x) - g(x_0)| < \varepsilon$.

[can be thought of as a partial converse to the intermediate value theorem. A strictly increasing function with intermediate value property is continuous.]

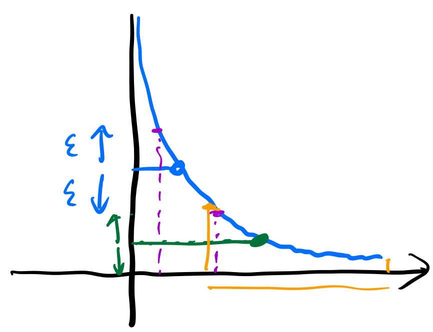
Uniform continuity

Recall what it means for a real-valued function f to be continuous on a set $S \subseteq \mathrm{dom}(f)$.

$$\forall x_0 \in S, \ \forall \varepsilon > 0, \exists \delta > 0 \text{ s.t } x \in \mathrm{dom}(f) \ \& \ |x - x_0| < \delta$$
$$\text{implies that } |f(x) - f(x_0)| < \varepsilon.$$

The choice of δ depends on the value of ε and x_0.

Example $f(x) = \frac{1}{x}$ on $(0, \infty)$.



Consider showing this function is continuous at a point $x_0 > 0$.

$$f(x) - f(x_0) = \frac{1}{x} - \frac{1}{x_0} = \frac{x_0}{x x_0} - \frac{x}{x x_0}$$

$$= \frac{x_0 - x}{x x_0}$$

Pick $\varepsilon > 0$. Suppose $|x - x_0| < \frac{x_0}{2}$, then $\frac{x_0}{2} < x < \frac{3x_0}{2}$.

$$|f(x) - f(x_0)| = \frac{|x - x_0|}{x_0 x} < \frac{|x_0 - x|}{x_0\left(\frac{x_0}{2}\right)} = \frac{2|x_0 - x|}{x_0^2}$$

Now suppose $\delta = \min\left\{\frac{x_0}{2}, \frac{\varepsilon x_0^2}{2}\right\}$. Then if $|x_0 - x| < \delta$,

$$|f(x) - f(x_0)| < \frac{2}{x_0^2} \frac{\varepsilon x_0^2}{2} = \varepsilon.$$

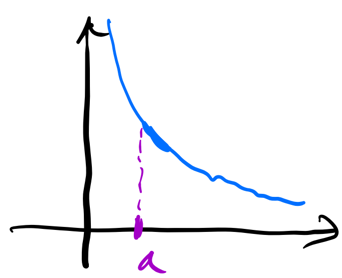
Hence f is continuous at x_0. The value of δ must become smaller as x_0 gets smaller. This is because $\frac{1}{x}$ becomes steeper as x_0 gets smaller.

<u>Definition</u> Let f be a real-valued function defined on $S \subseteq \mathbb{R}$. Then f is uniformly continuous on S if

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t } \forall x, y \in S \text{ and } |x - y| < \delta \text{ then } |f(x) - f(y)| < \varepsilon.$$

<u>Return to example</u>

$f(x) = \frac{1}{x}$ is uniformly continuous on the interval $[a, \infty)$ for $a > 0$.



Let $\varepsilon > 0$ and consider any $x, y \geq a$. Pick $\delta = \min\left\{\frac{a}{2}, \frac{\varepsilon a^2}{2}\right\}$

Then if $|x - y| < \delta$, then

$|f(x) - f(y)| < \varepsilon$ by previous result

$|f(x)-f(y)|<\varepsilon$ by previous result.

Alternatively

$$|f(x)-f(y)| = \frac{|x-y|}{xy} \le \frac{|x-y|}{a^2}$$

Hence if $\delta = a^2\varepsilon$, then $|x-y|<\delta$ implies

$$|f(x)-f(y)| < \frac{a^2\varepsilon}{a^2} = \varepsilon.$$

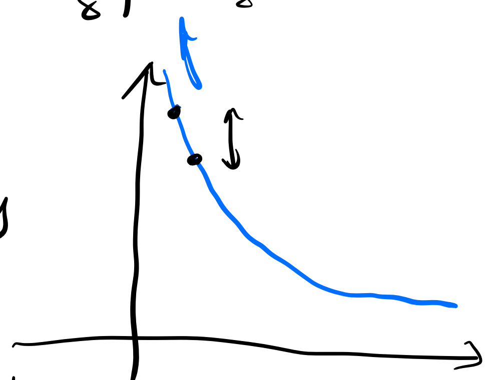
Further example

$f(x)=\frac{1}{x}$ is not uniformly continuous on $(0,\infty)$. To show this, $\forall \delta>0$ $\exists x,y\in(0,\infty)$ s.t $|x-y|<\delta$ and yet $|f(x)-f(y)|\ge 1$.

If $\delta>1$ choose $x=1, y=\frac{1}{2}$ $\left|\frac{1}{1}-\frac{1}{1/2}\right|=1.$

If $\delta\le 1$ choose $x=\delta, y=\frac{\delta}{2}$

$$\left|\frac{1}{\delta}-\frac{1}{\delta/2}\right| = \left|\frac{1}{\delta}-\frac{2}{\delta}\right| = \frac{1}{\delta}\ge 1.$$



Theorem If f is continuous on $[a,b]$ then f is uniformly continuous on $[a,b]$.

Proof Assume f is not uniformly continuous on $[a,b]$. Then $\exists\varepsilon>0$ s.t. $\forall\delta>0$, $\exists x,y\in(a,b)$ s.t $|x-y|<\delta$ but $|f(x)-f(y)|\ge\varepsilon$.

Then define $x_n, y_n \in [a,b]$ s.t $|x_n - y_n| < \frac{1}{n}$ but

$|f(x_n) - f(y_n)| \geq \varepsilon$.

By Bolzano–Weierstraß theorem $\exists$ a subsequence (x_{n_k}) of (x_n) which converges. But if $x_0 = \lim_{k\to\infty} x_{n_k}$, then $x_0 \in [a,b]$. In addition $\lim_{k\to\infty} y_{n_k} = x_0$.

But since f is continuous at x_0, then

$$f(x_0) = \lim_{k\to\infty} f(x_{n_k}) = \lim_{k\to\infty} f(y_{n_k})$$

So $\lim_{k\to\infty} [f(x_{n_k}) - f(y_{n_k})] = 0$.

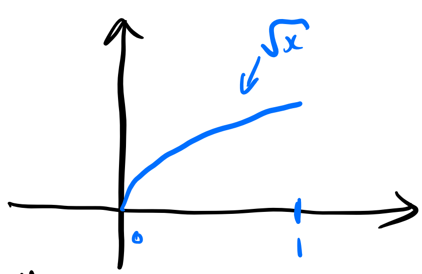
Contradiction since $|f(x_{n_k}) - f(y_{n_k})| \geq \varepsilon \ \forall k$.

Hence f is uniformly continuous on $[a,b]$.

Implies that many functions are uniformly continuous.

e.g. any polynomial on a closed interval.

$\sqrt{x}$ on $[0,1]$.



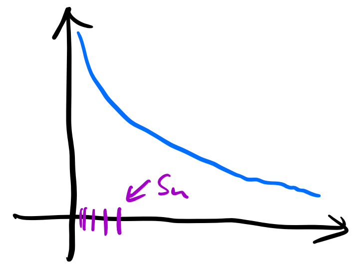
__Theorem__ If f is uniformly continuous on a set S and (s_n) is a Cauchy sequence in S, the $(f(s_n))$ is a Cauchy sequence.

__Proof__ Pick $\varepsilon > 0$ $\exists \delta$ s.t $x,y \in S$ $|x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$

Since (s_n) is Cauchy, $\exists N$ s.t. $n>N \Rightarrow |s_n - s_m| < \delta$.

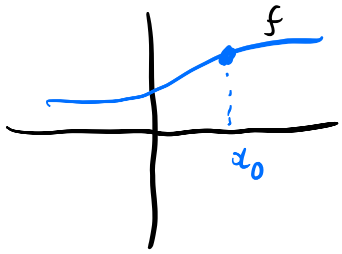
$\Rightarrow |f(s_n) - f(s_m)| < \varepsilon$.

This result requires uniform continuity.



Limits of functions

$\lim_{n\to\infty} u_n$



$\lim_{x \to x_0} f(x)$

Let S be a subset of $\mathbb{R}$, and let a be a real number or a symbol $\pm\infty$ that is the limit of some sequence in S, and let L be a real number or symbol $\pm\infty$.

We write $\lim_{x\to a^S} f(x) = L$ if

- f is a function defined on S
- for every sequence (x_n) in S with limit a, we have $\lim_{n\to\infty} f(x_n) = L$.

The limit exists if and only if f is continuous at a on S.

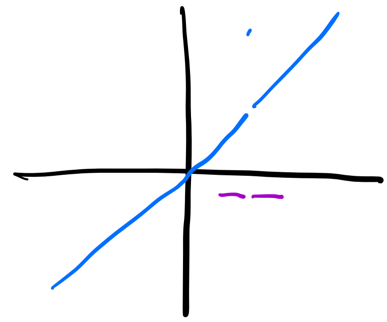
Some standard definitions

a) For $a \in \mathbb{R}$, write $\lim_{x\to a} f(x) = L$ if $\lim_{x\to a^S} f(x) = L$

for some $S = J \setminus \{a\}$ where J is an open interval

containing a

e.g $f(x) = \begin{cases} x & \text{if } x \neq 1 \\ 2 & \text{if } x = 1 \end{cases}$



$\lim_{x \to 1} f(x) = 1.$

b) Positive and negative limits

$\lim_{x \to a^+} f(x) = L$ if $\lim_{x \to a^S} f(x) = L$ for some open interval $S = (a, b)$

$\lim_{x \to a^-} f(x) = L$ if $\lim_{x \to a^S} f(x) = L$ for some open interval $S = (c, a)$

c) Infinite function limit

$\lim_{x \to \infty} f(x) = L$ if $\lim_{x \to \infty^S} f(x) = L$ where $S = (c, \infty)$

Theorem

Let f_1 and f_2 be functions for which $L_1 = \lim_{x \to a^S} f_1(x)$ and $L_2 = \lim_{x \to a^S} f_2(x)$.

i) $\lim_{x \to a^S} (f_1 + f_2)(x)$ exists and equals $L_1 + L_2$

ii) $\lim_{x \to a^S} (f_1 \cdot f_2)(x)$ exists and equals $L_1 L_2$

iii) $\lim_{x \to a^S} (f_1 / f_2)(x)$ exists and equals L_1 / L_2

provided that $L_2 \neq 0$ and $f_2(x) \neq 0$ $\forall x \in S$.

Proof (i) Consider x_n is S with limit a.

Then $L_1 = \lim_{n\to\infty} f_1(x_n)$

$L_2 = \lim_{n\to\infty} f_2(x_n)$

$$\lim_{n\to\infty} (f_1+f_2)(x_n) = \lim_{n\to\infty} f_1(x_n) + \lim_{n\to\infty} f_2(x_n)$$
$$= L_1 + L_2.$$

Theorem Let f be a function defined on $S \subseteq \mathbb{R}$ and a be a real number that is the limit of some sequence in S, and let L be a real number. Then

$\lim_{x\to a^S} f(x) = L$ if and only if

$$\forall \varepsilon>0, \exists \delta>0 \text{ s.t } x\in S \text{ and } |x-a|<\delta \text{ then } |f(x)-L|<\varepsilon. \quad (*)$$

Proof Consider a sequence (x_n) in S s.t $\lim_{n\to\infty} x_n = a$. To show $\lim_{n\to\infty} f(x_n) = L$.

Consider $\varepsilon > 0$. $\exists \delta > 0$ s.t $x \in S$ and $|x-a| < \delta$ then $|f(x) - L| < \varepsilon$.

$\exists N$ s.t. $n > N \Rightarrow |x_n - a| < \delta$

$\Rightarrow |f(x_n) - L| < \varepsilon.$

Hence $\lim_{n \to \infty} f(x_n) = L.$

Now assume $\lim_{x \to a S} f(x) = L$ but that (*) fails. Then $\exists \varepsilon > 0$ s.t. $\forall \delta > 0$, $x \in S$ and $|x - a| < \delta$ then that does not imply $|f(x) - L| < \varepsilon$.

For each $n \in \mathbb{N}$, $\exists x_n \in S$ where $|x_n - a| < \frac{1}{n}$ while $|f(x_n) - L| \geq \varepsilon$

But then $x_n \to a$ but $\lim_{n \to \infty} f(x_n) = L$ fails.

$\#$ hence property $*$ is true.

Let f be a function defined on $I \setminus \{a\}$ for an open interval I containing a.
Let $L \in \mathbb{R}$, then $\lim_{x \to a} f(x) = L$ if and only if

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } 0 < |x - a| < \delta$$
$$\text{then } |f(x) - L| < \varepsilon.$$

Theorem Let f be a function defined on $I \setminus \{a\}$. Then $\lim_{x \to a} f(x)$ exists if and only if $\lim_{x \to a^+} f(x)$ and $\lim_{x \to a^-} f(x)$ exist and are equal, in which case all three limits are equal.

Proof

If $\lim_{x \to a} f(x) = L$ then

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t } 0 < |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon.$$

Immediately follows that

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t } a < x < a + \delta \Rightarrow |f(x) - L| < \varepsilon.$$

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t } a - \delta < x < a \Rightarrow |f(x) - L| < \varepsilon.$$

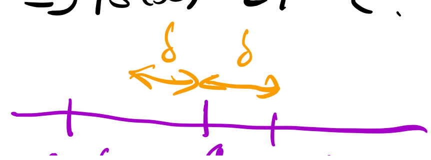
Then $\lim_{x \to a^+} f(x) = L$ and $\lim_{x \to a^-} f(x) = L$.

Suppose $\lim_{x \to a^+} f(x) = L = \lim_{x \to a^-} f(x)$.

Choose $\varepsilon > 0$. Then

- $\exists \delta_1 > 0$ s.t $a < x < a + \delta_1 \Rightarrow |f(x) - L| < \varepsilon.$
- $\exists \delta_2 > 0$ s.t $a - \delta_2 < x < a \Rightarrow |f(x) - L| < \varepsilon.$

Pick $\delta = \min\{\delta_1, \delta_2\}$ then



$a-\delta_2 \quad a \quad a+\delta_1$

$$0<|x-a|<\delta \Rightarrow |f(x)-L|<\varepsilon.$$

$$\Rightarrow \lim_{x\to a} f(x) = L.$$

Power series

Suppose (a_n) is a sequence of real number. Then the series

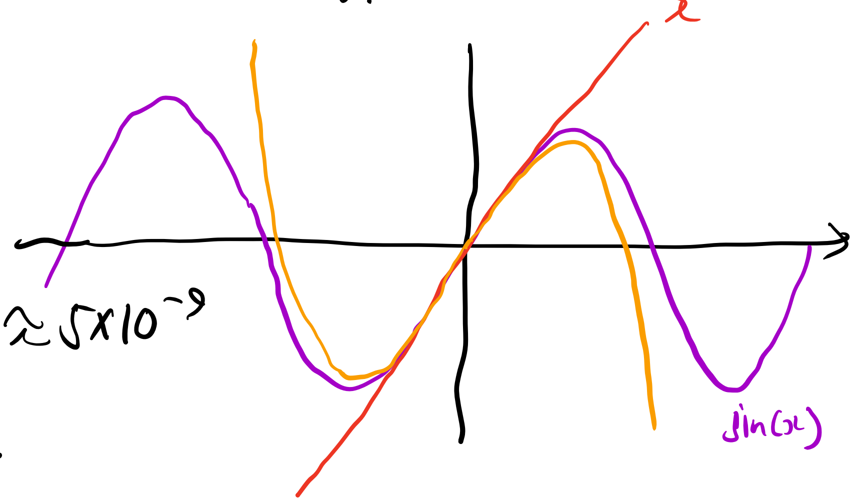
$$\sum_{n=0}^{\infty} a_n x^n$$

Use convention that $0^0=1$.

is called a power series. Main aim: to approximate other functions. e.g.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

Good for computation



$$\left|\sin x - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}\right)\right| \approx 5\times 10^{-9}$$

$x = 1/2.$

Good for computing π.

e.g. $$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$$

$$\tan^{-1} 1 = \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots$$

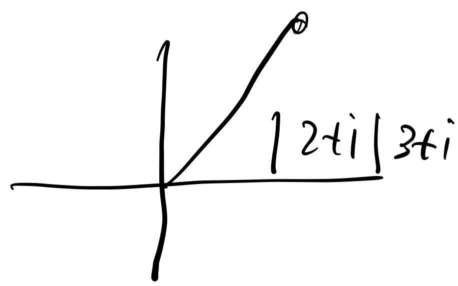
James Gregory (1638–1675).

$$\tan^{-1}\frac{1}{2} = 1 - \frac{(1/2)^3}{3} + \frac{(1/2)^5}{5} - \frac{\boxed{(1/2)^7}}{7} + \cdots$$

$\frac{1}{128}$.

$$\tan^{-1}\frac{1}{3} = 1 - \frac{(1/3)^3}{3} + \frac{(1/3)^5}{5} - \frac{(1/3)^7}{7} + \cdots$$

$$\frac{\pi}{4} = \tan^{-1} 1/2 + \tan^{-1} 1/3$$



$$\operatorname{Arg} 2+i = \tan^{-1} 1/2$$

$$\operatorname{Arg} 3+i = \tan^{-1} 1/3$$

$$(2+i)(3+i) = 6-1+5i = 5+5i$$

$$\operatorname{Arg} 5+5i = \frac{\pi}{4}.$$

Machin formula.

$$\frac{\pi}{4} = 4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239}.$$

1706 John Machin 100 digits

1844 Zacharias Dase 200 digits

[illegible] William Shanks ~~707~~ digits

1873 William Shanks ~~707~~ digits

528 onward incorrect

Seven hundred seven
Shanks did state
Digits of π
We would calculate
And none can deny
It was a good try
But he erred at 528!

Three cases: a) the power series converges $\forall x \in \mathbb{R}$

b) the power series converges at $x=0$ only

c) the power series converges for all x in a bounded interval (open / closed)

<u>Theorem</u> For the power series $\sum a_n x^n$, let $\beta = \limsup |a_n|^{1/n}$ and $R = \frac{1}{\beta}$.

Then the power series converges for $|x|<R$ and diverges for $|x|>R$. R is called the radius of convergence

(Define $R=0$ if $\beta=\infty$ and $R=\infty$ if $\beta=0$)

convergence.

Proof Use the root test. For a given x, define

$$\alpha_x = \limsup |a_n x^n|^{1/n}$$
$$= \limsup |a_n|^{1/n} |x|.$$
$$= |x| \limsup |a_n|^{1/n} = \beta |x|.$$

If $0 < \beta < \infty$ then $\alpha_x = \beta|x| = \frac{|x|}{R}$. Then if $|x| < R$ then $\alpha_x < 1$, so the series converges. If $|x| > R$ then $\alpha_x > 1$, so the series diverges.

$R = \infty$, $\beta = 0$ $\alpha_x = 0$ Series converges.

Examples $\sum_{n=0}^{\infty} x^n$ $R=1$ doesn't converge for $x = \pm 1$.

$\sum_{n=0}^{\infty} \frac{1}{n} x^n$ $R=1$ converges if $x = -1$.

$\sum_{n=0}^{\infty} \frac{1}{n^2} x^n$ $R=1$ converges at $x = \pm 1$.

Could write in a more general setting.

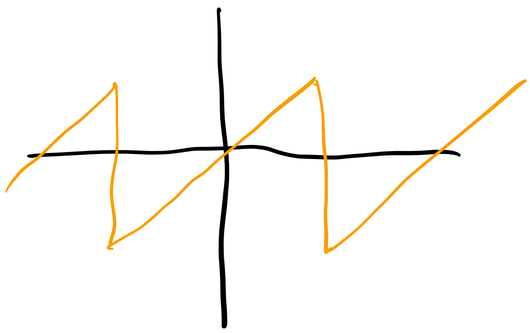
$$\sum_{n=0}^{\infty} a_n (x - x_0)^n.$$

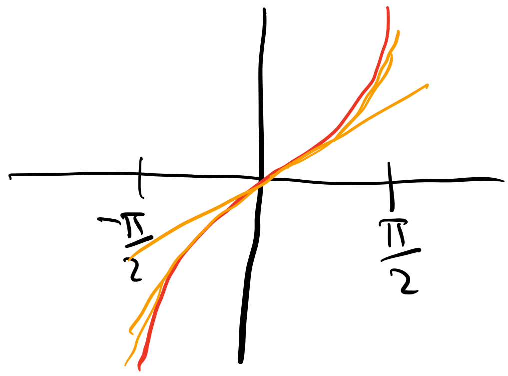
What can be said in general about this series? Any partial sum will be continuous and possible to differentiate.

Just because each partial sum is continuous does not guarantee that $\sum_{n=0}^{\infty}$ will be continuous.

$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

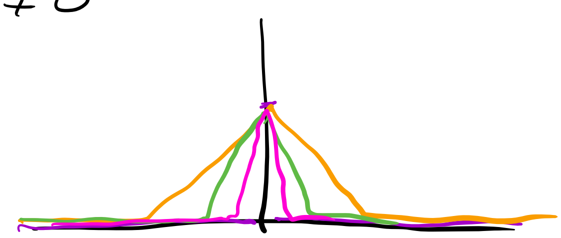
(f_n)





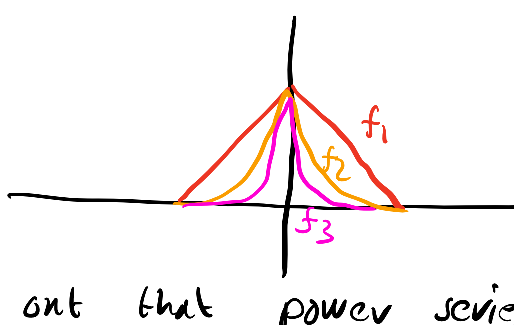
$$f(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases}$$

$$f_n(x) = \min\left(1 - n|x|, 0\right)$$



$$f_n(x) = \left(1 - |x|\right)^n$$

Motivates a stronger definition of convergence, which we call uniform convergence.



It will turn out that power series

satisfy this stronger definition.

Definition Let (f_n) be a sequence of real-valued functions defined on $S \subseteq \mathbb{R}$. Then the sequence converges pointwise to f defined on S if $\lim_{n\to\infty} f_n(x) = f(x) \quad \forall x \in S$

$$\lim_{n\to\infty} f_n = f \text{ pointwise} \qquad f_n \to f \text{ pointwise.}$$

$$\forall x \in S \quad \forall \varepsilon > 0 \quad \exists N \text{ s.t. } |f_n(x) - f(x)| < \varepsilon \quad \forall n > N.$$

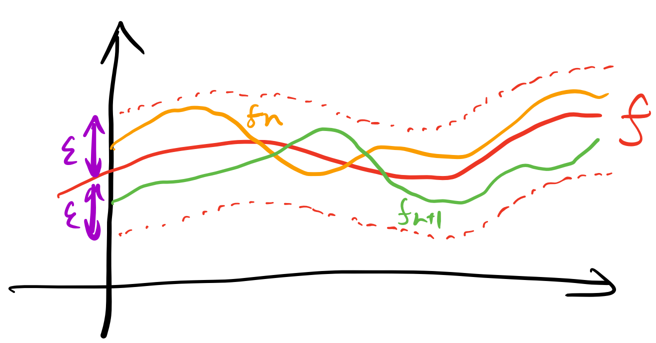
Definition Let (f_n) be a sequence of real-valued functions defined on $S \subseteq \mathbb{R}$. Then f_n converges uniformly on S to a function f if

$$\forall \varepsilon > 0 \quad \exists N \text{ s.t. } |f_n(x) - f(x)| < \varepsilon \quad \forall x \in S \quad \forall n > N.$$

In this case, write

$$\lim_{n\to\infty} f_n = f \text{ uniformly.}$$

For any $\varepsilon > 0$, the f_n have to eventually lie within a strip of width ε around f.



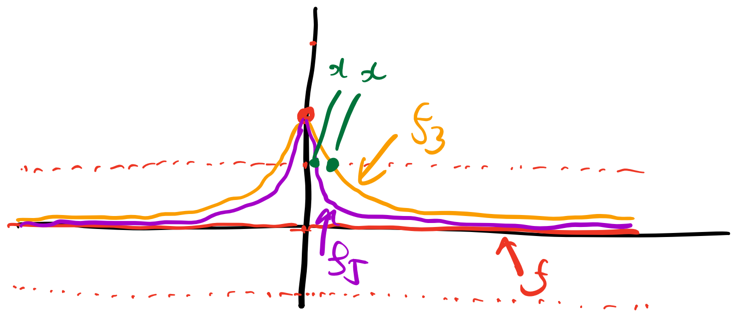
Return $f_n = (1-|x|)^n$

Then $|f_n(x) - f(x)|$ should eventually be smaller than $\varepsilon = 1/2$.

$x = 1 - 2^{-\frac{1}{N+1}}$

$1 - x = 2^{-\frac{1}{N+1}}$

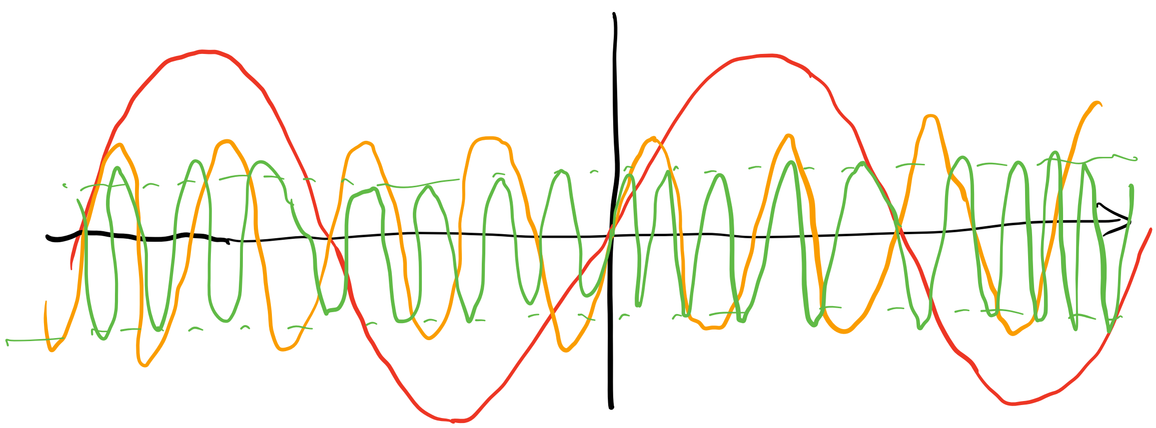
$(1-x)^{N+1} = 2^{-1} = 1/2.$



$f_{N+1}(1 - 2^{-\frac{1}{N+1}}) = 1/2 \qquad |f_{N+1}(x) - f(x)| = 1/2$

Can tolerate rapid oscillations in the f_n and still have uniform convergence.

$$f_n(x) = \frac{1}{n} \sin n^2 x$$



$|f_n(x) - 0| \leq \frac{1}{n} < \frac{1}{N} = \varepsilon.$ uniformly convergent

$n > N$

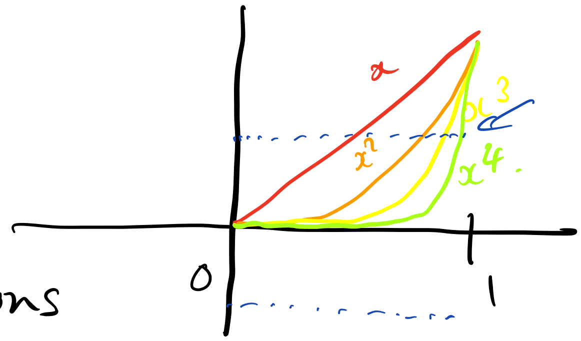
$\Downarrow$

...

$f_n(x) = x^n$ pointwise convergent.

Does not converge uniformly on $[0,1)$.

$\lim_{n\to\infty} f_n(x) = 0$ pointwise.



<u>Theorem</u> The uniform limit of continuous functions is continuous. Let (f_n) be a sequence of functions on a set $S \subseteq \mathbb{R}$ and suppose $f_n \to f$ uniformly on S and $\mathrm{dom}(f) = S$. If each f_n is continuous at x_0 in S, then f is continuous at x_0.

<u>Proof</u> Let $\varepsilon > 0$. $\exists N \in \mathbb{N}$ s.t. $n > N \Rightarrow$

$$|f_n(x) - f(x)| < \frac{\varepsilon}{3} \quad \forall x \in S$$

Hence $|f_{N+1}(x) - f(x)| < \frac{\varepsilon}{3}$ $\forall x \in S$.

f_{N+1} is continuous at x_0, so $\exists \delta > 0$ s.t.

$$x \in S \text{ and } |x - x_0| < \delta \Rightarrow |f_{N+1}(x) - f_{N+1}(x_0)| < \frac{\varepsilon}{3}$$

Then $x \in S$ and $|x - x_0| < \delta$ implies

$$|f(x) - f(x_0)| \leq |f(x) - f_{N+1}(x)| + |f_{N+1}(x) - f_{N+1}(x_0)|$$
$$+ |f_{N+1}(x_0) - f(x_0)|$$

$$|f_{N+1}(x_0) - f(x_0)|$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

$$\begin{tikzcd} f_{N+1}(x) \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] & f_{N+1}(x_0) \arrow[d, leftrightarrow] \\ f(x) \arrow[r, leftrightarrow] & f(x_0) \end{tikzcd}$$

Definition A sequence (f_n) of functions defined on $S \subseteq \mathbb{R}$ is uniformly Cauchy if

$$\forall \varepsilon > 0 \quad \exists N \quad \text{s.t.} \quad |f_n(x) - f_m(x)| < \varepsilon \quad \forall x \in S \quad \forall m, n > N.$$

Theorem Let (f_n) be a sequence of functions that are uniformly Cauchy on $S \subseteq \mathbb{R}$. Then $\exists f$ on S such that $f_n \to f$ uniformly on S.

$$\sum_{n=0}^{M} a_n x^n$$

Proof Choose $\varepsilon > 0$. Then for a fixed $x_0 \in S$,

$$|f_n(x_0) - f_m(x_0)| < \varepsilon \qquad \forall m, n > N.$$

Hence $f_n(x_0)$ is a Cauchy sequence, so it must converge to a limit $f(x_0)$.

Since this applies to any $x_0 \in S$, $f_n \to f$ pointwise.

To show convergence is uniform, choose $\varepsilon > 0$.

Then $\exists N$ s.t $|f_n(x) - f_m(x)| < \frac{\varepsilon}{2}$ $\forall x \in S'$ and $\forall m, n > N$.

Consider a specific $m > N$. Then $\forall x \in S'$ and $\forall n > N$.

$$f_n(x) \in \left(f_m(x) - \frac{\varepsilon}{2}, f_m(x) + \frac{\varepsilon}{2}\right).$$

$$\lim_{n \to \infty} f_n(x) = f(x) \in \left[f_m(x) - \frac{\varepsilon}{2}, f_m(x) + \frac{\varepsilon}{2}\right].$$

$$|f(x) - f_m(x)| \leq \frac{\varepsilon}{2} < \varepsilon$$

$$\forall x \in S' \quad \text{and} \quad m > N.$$

Useful for series of functions, such as power series. We know each intermediate is well-defined, so we can use to reach a well-defined limit.

Theorem Consider a series of functions $\sum_{k=0}^{\infty} g_k$ defined on $S \subseteq \mathbb{R}$. Suppose that each g_k is continuous on S and that the series converges uniformly on S. Then $\sum_{k=0}^{\infty} g_k$ represents a continuous function on S.

Proof Consider sequence $f_n = \sum_{k=0}^{n} g_k(x)$. f_n is continuous since continuous functions add.

Already showed that if f_n is continuous and $f_n \to f$ uniformly, then f is continuous.

Analog of the Cauchy criterion

For a series, $\sum_{k=0}^{\infty} a_k$, the Cauchy criterion is

$$\forall \varepsilon > 0,\ \exists N \text{ s.t. } n \geq m > N \Rightarrow \left| \sum_{k=m}^{n} a_k \right| < \varepsilon.$$

For series of functions $\sum_{k=0}^{\infty} g_k(x)$

$$\forall \varepsilon > 0,\ \exists N \text{ s.t. } \left| \sum_{k=m}^{n} g_k(x) \right| < \varepsilon \quad \forall n \geq m > N,\ \forall x \in S.$$

Weierstraß M-test

Suppose (M_k) is a sequence of non-negative real numbers where $\sum_{k=0}^{\infty} M_k < \infty$. If $|g_k(x)| \leq M_k$ $\forall x \in S$ and $\forall k$, then $\sum g_k$ converges uniformly on S

Proof Since $\sum M_k$ converges, it satisfies the Cauchy criterion:

$\forall \varepsilon > 0 \quad \exists N \quad \text{s.t} \quad n \geq m > N \Rightarrow \left| \sum_{k=m}^{n} M_k \right| < \varepsilon.$

Hence

$$\left| \sum_{k=m}^{n} g_k(x) \right| \leq \sum_{k=m}^{n} |g_k(x)| \leq \sum_{k=m}^{n} M_k < \varepsilon$$

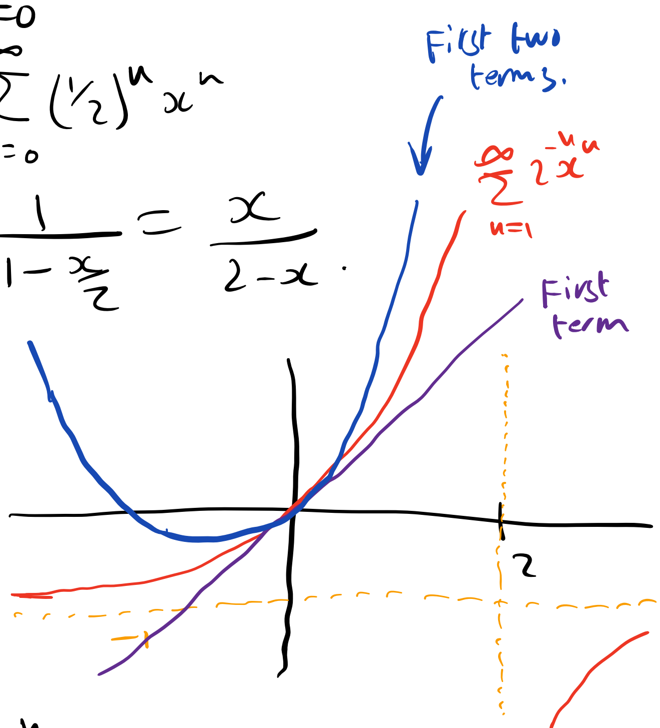
$\Rightarrow g_k$ converges uniformly on S

Example Use for power series

$$\sum_{n=1}^{\infty} 2^{-n} x^n = x \sum_{n=0}^{\infty} 2^{-n-1} x^n$$

$$= \frac{x}{2} \sum_{n=0}^{\infty} (1/2)^n x^n$$

$$= \frac{x}{2} \frac{1}{1 - \frac{x}{2}} = \frac{x}{2-x}.$$



At $x = \pm 2$ series doesn't converge.

Consider an interval $[-a, a]$ for $a < 2$.

Then $\left| 2^{-n} x^n \right| \leq \underbrace{\left(\frac{a}{2}\right)^n}_{M_n}$ $\Leftarrow$ convergent geometric series

Hence by the Weierstraß M-test, the series

converges uniformly on $[-a, a]$. The limit converges uniformly on $[-a,a]$. The limit function must be continuous. Hence continuous on $(-2,2)$.

Theorem Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence $R>0$. If $0<R_1<R$, then the power series converges uniformly on $[-R_1,R_1]$ to continuous function.

Proof Radius of convergence defined as

$$\beta = \frac{1}{R} = \limsup |a_n|^{1/n}$$

So $\sum a_n x^n$ and $\sum |a_n| x^n$ have the same radius of convergence.

$\sum |a_n| R_1^n$ converges by the root test.

$|a_n x^n| \le |a_n| R_1^n$ so the series converges uniformly by the Weierstraß M-test.

Corollary The series converges uniformly to a continuous function for the open interval $(-R,R)$.

Preview Suppose $f(x) = \sum^{\infty} a_n x^n$ for some radius of convergence R. Does $f'(x) = \sum_{n=0}^{\infty} a_n n x^{n-1}$?

First show that the two series have the same radius of convergence.

$$\sum_n n a_n x^n = x \sum_n n a_n x^{n-1}$$

so the two power series must converge for the same values of x. For $0 < R < \infty$

$$R = \frac{1}{\beta} \qquad \beta = \lim\sup |a_n|^{1/n}$$

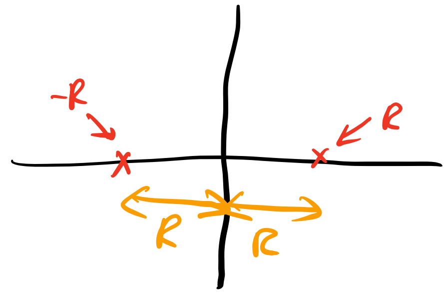
$$\lim\sup |n a_n|^{1/n} = \lim\sup |a_n|^{1/n} \cdot \underbrace{\lim |n|^{1/n}}_{1}$$
$$= \lim\sup |a_n|^{1/n}$$

Hence $\sum n a_n x^{n-1}$ has the same radius of convergence as $\sum a_n x^n$. Later we will show that f is differentiable on $(-R, R)$ and $f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$.

<u>Abel's Theorem</u> Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ be a power series with a finite positive radius of convergence R. If the series converges

at $x=R$, then f is continuous at $x=R$.

Similarly if the series converges at $x=-R$, then f is continuous at $x=-R$.



<u>Proof</u> Consider $f(x) = \sum_{n=0}^{\infty} a_n x^n$ with radius of convergence 1. Let series converge at 1.

write $\quad s_n = \sum_{k=0}^{n} a_k \quad f_n(x) = \sum_{k=0}^{n} a_k x^k \quad s_n = f_n(1).$

$$S = \sum_{k=0}^{\infty} a_k = f(1) \qquad \lim s_n = S.$$

$$s_k - s_{k-1} = a_k.$$

For $0<x<1$,

$$f_n(x) = s_0 + \sum_{k=1}^{n} \underbrace{(s_k - s_{k-1})}_{a_k.} x^k$$

$$= s_0 + \sum_{k=1}^{n} s_k x^k - x \sum_{k=0}^{n-1} s_k x^k$$

$$= \sum_{k=0}^{n-1} s_k (1-x) x^k + s_n x^n$$

Take limit as $n \to \infty$. Note that $x^n \to 0$.

$$f(x) = \sum_{n=0}^{\infty} s_n (1-x) x^n \qquad f(1) = S.$$

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

$$f(1) = s = s\sum_{n=0}^{\infty}(1-x)x^n$$

$$f(1) - f(x) = \sum_{n=0}^{\infty}(s-s_n)(1-x)x^n$$

Choose $\varepsilon > 0$. Since $\lim_{n\to\infty} s_n = s$, $\exists N \in \mathbb{N}$

s.t $n > N \Rightarrow |s - s_n| < \frac{\varepsilon}{2}$.

Define $g_N(x) = \sum_{n=0}^{N} |s-s_n|(1-x)x^n$.

$$|f(1)-f(x)| \leq g_N(x) + \sum_{n=N+1}^{\infty} \overbrace{|s-s_n|}^{\varepsilon/2.}(1-x)x^n.$$

$$\leq g_N(x) + \sum_{n=N+1}^{\infty}\frac{\varepsilon}{2}(1-x)x^n$$

$$< g_N(x) + \frac{\varepsilon}{2}.$$

g_N is continuous and $g_N(1) = 0$. Hence $\exists\, \delta > 0$, s.t.

$$1-\delta < x < 1 \Rightarrow g_N(x) < \frac{\varepsilon}{2}.$$

$|\cancel{g_N(1)}_{0} - g_N(x)| < \frac{\varepsilon}{2}$

$$|f(1) - f(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

If $f(x)$ has radius of convergence R, define $g(x) = f(Rx)$. g has radius of convergence 1,

so we can apply the above proof

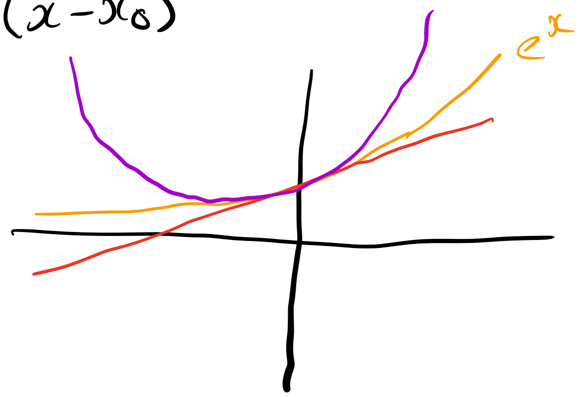
so we can apply the above proof.

To examine at $-R$, define $h(x) = f(-x)$.

Limitations of power series

Going to get to Taylor's theorem

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$$

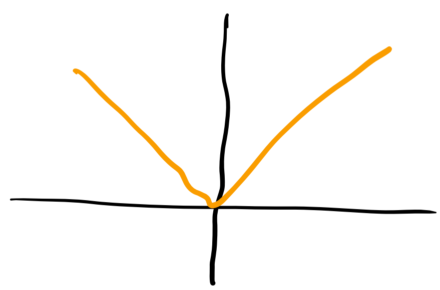


$$\frac{d}{dx} e^x = e^x.$$

$$\left.\frac{d^n}{dx^n} e^x\right|_{x=0} = 1.$$

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \qquad R = \infty$$

Going to run into problems where you can't construct derivatives. e.g $f(x) = |x|$



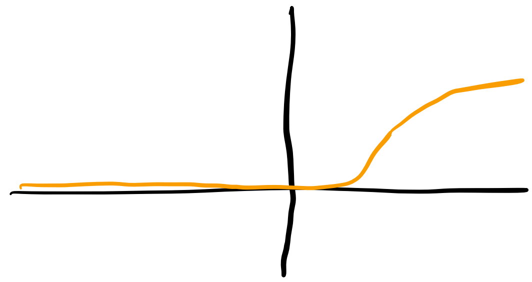
Even worse examples.

$$f(x) = \begin{cases} 0 & x \leq 0 \\ e^{-\frac{1}{x^2}} & x > 0. \end{cases}$$

$$f^{(n)}(x) = \begin{cases} 0 & x \leq 0. \\ p\left(\frac{1}{x}\right) e^{-\frac{1}{x^2}} & x > 0 \end{cases}$$

for some polynomial p.

This function is infinitely differentiable. $f^{(n)}(0)=0$ for all n.

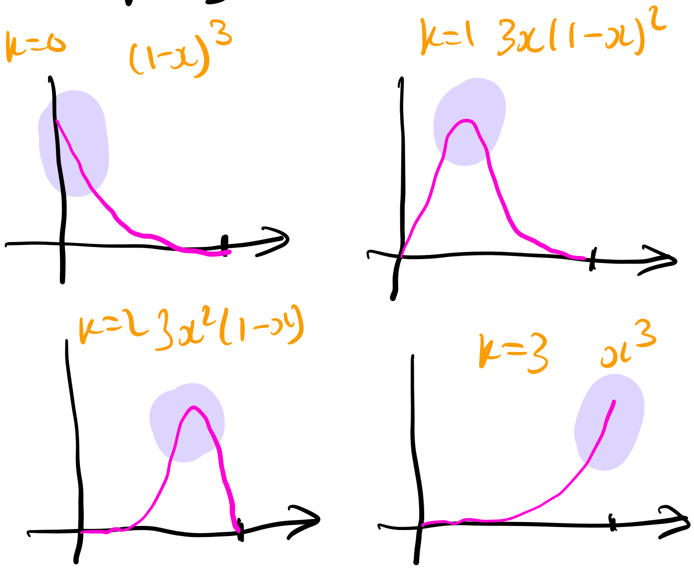


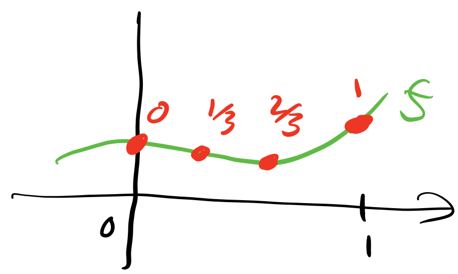
However, any continuous function on $[0,1]$ can be approximated by polynomials. (They just might not be power series.)

Can be done in terms of Bernstein polynomials. For f continuous on $[0,1]$

$$B_n f(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k (1-x)^{n-k}$$

$n=3$.





<u>Lemma</u> For every $x \in \mathbb{R}$ and $n \geq 0$,

$$\sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} = 1.$$

$\sum_{k=0} \binom{}{k}$

Proof Binomial theorem shows the LHS is equal to $(x+(1-x))^n = 1^n = 1$.

Lemma $\displaystyle\sum_{k=0}^{n} (nx-k)^2 \binom{n}{k} x^k (1-x)^{n-k} = nx(1-x) \leq \frac{n}{4}$

$$k\binom{n}{k} = \frac{kn!}{(n-k)!\,k!} = \frac{n(n-1)!}{(n-k)!\,(k-1)!} = n\binom{n-1}{k-1}.$$

$$\sum_{k=0}^{n} k\binom{n}{k} x^k (1-x)^{n-k} = n\sum_{k=1}^{n}\binom{n-1}{k-1} x^k (1-x)^{n-k}$$

0 if $k=0$

$$= nx \underbrace{\sum_{j=0}^{n-1}\binom{n-1}{j} x^j (1-x)^{n-1-j}}$$

$$= nx\,(x+(1-x))^{n-1}$$

$$= nx.$$

Similarly $\displaystyle\sum_{k=0}^{n} k(k-1)\binom{n}{k} x^k (1-x)^{n-k} = n(n-1)x^2$

$$\sum_{k=0}^{n} k^2\binom{n}{k} x^k (1-x)^{n-k} = n(n-1)x^2 + nx$$
$$= n^2x^2 + nx(1-x)$$

Since $(nx-k)^2 = n^2x^2 - 2nxk + k^2$

$$\sum_{k=0}^{n} (nx-k)^2 \binom{n}{k} x^k (1-x)^{n-k} = \cancel{n^2x^2} - 2\cancel{nxnx} + \cancel{n^2x^2}$$
$$+ nx(1-x)$$