Physics 415
Spring 2025
Homework 9
Due Friday, April 11, 2025

This assignment covers material in Chapter 9 of Reif. I recommend reading through the text and also Lectures Notes 27,28,30.

Problem 1: (Identical particles, adapted from Reif 9.1) Consider a system consisting of two particles, each of which can be in any one of three quantum states of respective energies 0, ϵ, and 3ϵ. The system is in contact with a heat reservoir at temperature T. This is a problem in the *canonical ensemble*, as the temperature and particle number are fixed.

(a) Write an expression for the partition function Z if the particles are *distinguishable.*

(b) What is Z if the particles obey Bose-Einstein statistics?

(c) What is Z if the particles obey Fermi-Dirac statistics?

Problem 2: (Identical particles, adapted from Reif 9.2) In class we derived the grand partition function of an ideal gas of identical particles:

$$\mathcal{Z} = \begin{cases} \prod_r 1 + e^{-\beta(\epsilon_r - \mu)}, & \text{Fermi-Dirac} \\ \prod_r \dfrac{1}{1 - e^{-\beta(\epsilon_r - \mu)}}, & \text{Bose-Einstein} \end{cases} \tag{1}$$

where $\beta = 1/T$ is the inverse temperature and μ is the chemical potential.

(a) From the expression for $\mathcal{Z}$ above, find an expression for the entropy S of an ideal Fermi-Dirac gas. Express your answer solely in terms of $\bar{n}_r$, the mean number of particles in state r (eliminate μ from your expression).

(b) Write a similar expression for the entropy S of an ideal Bose-Einstein gas.

(c) What do these expressions for S become in the classical limit when $\bar{n}_r \ll 1$?

Problem 3 (Parafermions) The Pauli exclusion principle states that at most one fermion may occupy any given single-particle state. Consider hypothetical exotic particles for which the maximum occupancy of any given single-particle state is k, where k is an integer greater than zero. (For $k = 1$, the particles are regular fermions; for $k = \infty$, the particles are regular bosons.)[1] Consider a system with one single-particle level whose energy is ϵ.

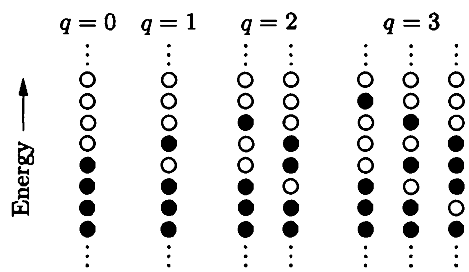
[1] Such hypothetical particles are related to so-called *parafermions*, which are "quasi-particles" that appear in condensed matter physics. These exotic quasi-particles have received significant interest owing to their potential utility for topological quantum computation.

(a) Compute the grand partition function $\mathcal{Z}$ for these particles.

(b) Compute the mean occupation $\bar{n}(T, \mu)$ of the one single-particle state. Show that $\bar{n}(T, \mu)$ reduces to the Fermi-Dirac and Bose-Einstein distributions in the appropriate limits.

(c) Sketch $\bar{n}(T, \mu)$ as a function of μ for both $T = 0$ and $T > 0$.

Problem 4 (Fermi-Dirac statistics) In this problem, you investigate how the Fermi-Dirac distribution emerges starting from a *microcanonical* approach[2]. Consider an isolated system of N identical fermions, inside a container where the allowed energy levels are nondegenerate and evenly spaced. For instance, the fermions could be trapped in a one-dimensional harmonic oscillator potential. For simplicity, neglect the fact that fermions can have multiple spin orientations (or assume that they are all forced to have the same spin orientation). Then each energy level is either occupied or unoccupied, and any allowed system state can be represented by a column of dots, with a filled dot representing an occupied level and a hollow dot representing an unoccupied level. The lowest-energy system state has all levels below a certain point occupied, and all levels above that point unoccupied. Let $\Delta\epsilon$ be the spacing between energy levels, and let q be the number of energy units (each of size $\Delta\epsilon$) in excess of the ground-state energy. Assume that $q < N$. The figure below shows all system states up to $q = 3$.



(a) Draw dot diagrams, as in the figure, for all allowed system states with $q = 4$, $q = 5$, and $q = 6$.

(b) According to the fundamental statistical postulate, all allowed system states with a given value of q are equally probable. Compute the probability of each single-particle energy level being occupied, for $q = 6$. Draw a graph of this probability as a function of the energy of the level.

(c) In the thermodynamic limit where q is large, the probability of a level being occupied should be given by the Fermi-Dirac distribution. Even though 6 is not a large number, estimate the values of μ and T that you would have to plug into the Fermi-Dirac distribution to best fit the graph you drew in part (b).

(d) Calculate the entropy of this system for each value of q from 0 to 6 (i.e., the logarithm of the number of states), and draw a graph of entropy vs. energy. Make a rough estimate of the slope of this graph near $q = 6$, to obtain another estimate of the temperature of this system at that point. Check that it is in rough agreement with your answer to part (c).

[2]This problem is based on J. Arnaud et. al., *American Journal of Physics* **67**, 215 (1999).