Midterm Review Notes: Key Definitions and Theorems

These notes summarize the key definitions, theorems, and concepts relevant to the sample midterm problems, primarily based on Ross's *Elementary Analysis*. Examples are drawn directly from the sample problems and solutions provided, rephrased for clarity and rigor.

1 Continuity

1.1 Definitions

Definition 1.1 (Continuity at a Point [1])**.** Let f be a real-valued function whose domain is a subset S of $\mathbb{R}$. The function f is **continuous at** $x_0 \in S$ if, for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $x \in S$ and $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \epsilon$.

Definition 1.2 (Continuity on a Set [1])**.** If f is continuous at every point of a set $S' \subseteq S$, then f is said to be **continuous on** S'.

Definition 1.3 (Uniform Continuity [2])**.** Let f be a real-valued function defined on a set $S \subseteq \mathbb{R}$. Then f is **uniformly continuous on** S if for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $x, y \in S$ and $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$.

Remark 1.1. The key difference from pointwise continuity is that δ depends only on ϵ and not on the specific points $x, y \in S$. Uniform continuity is a global property on the set S.

Definition 1.4 (Bounded Function [4])**.** A function f defined on a set S is **bounded** if its range $f(S) = \{f(x) : x \in S\}$ is a bounded subset of $\mathbb{R}$. Equivalently, there exists a constant $M \geq 0$ such that $|f(x)| \leq M$ for all $x \in S$.

1.2 Theorems

Theorem 1.1 (Intermediate Value Theorem (IVT) [3])**.** *If f is a continuous function on a closed interval $[a, b]$, and if y_0 is any value between $f(a)$ and $f(b)$ (i.e., $f(a) < y_0 < f(b)$ or $f(b) < y_0 < f(a)$), then there exists at least one $c \in (a, b)$ such that $f(c) = y_0$.*

1.3 Examples

Example 1.1 (Application of IVT - Sample Problem 5). **Problem Statement:** Suppose f is continuous on $[0, 2]$ and $f(0) = f(2)$. Prove that there exist $x, y \in [0, 2]$ where $|x - y| = 1$ and $f(x) = f(y)$.

Solution: We seek points x and y such that $y = x + 1$ (or $x = y + 1$) and $f(x) = f(y)$. This suggests considering the difference $f(x) - f(x + 1)$.

Define an auxiliary function $g : [0, 1] \to \mathbb{R}$ by $g(x) = f(x) - f(x + 1)$. Since f is continuous on $[0, 2]$, and $x \mapsto x + 1$ is continuous, the composition $x \mapsto f(x + 1)$ is continuous on $[0, 1]$. Therefore, g is continuous on $[0, 1]$ as the difference of continuous functions.

Evaluate g at the endpoints of its domain $[0, 1]$:

$$g(0) = f(0) - f(1)$$

$$g(1) = f(1) - f(2)$$

Using the given condition $f(0) = f(2)$, we can rewrite $g(1)$:

$$g(1) = f(1) - f(0) = -(f(0) - f(1)) = -g(0)$$

Now consider two cases for the value of $g(0)$:

1. **Case 1: $g(0) = 0$.** If $g(0) = 0$, then $f(0) - f(1) = 0$, which means $f(0) = f(1)$. We can choose $x = 0$ and $y = 1$. Then $x, y \in [0, 2]$, $|x-y| = |0-1| = 1$, and $f(x) = f(y)$. The condition is satisfied.

2. **Case 2: $g(0) \neq 0$.** If $g(0) \neq 0$, then $g(1) = -g(0)$ implies that $g(0)$ and $g(1)$ have opposite signs. Since g is continuous on the closed interval $[0, 1]$, and 0 is a value between $g(0)$ and $g(1)$, the Intermediate Value Theorem guarantees the existence of some $c \in (0, 1)$ such that $g(c) = 0$. By definition of g, $g(c) = f(c) - f(c + 1) = 0$, which means $f(c) = f(c + 1)$. Let $x = c$ and $y = c + 1$. Since $c \in (0, 1)$, we have $x \in (0, 1)$ and $y \in (1, 2)$, so both $x, y \in [0, 2]$. Also, $|x-y| = |c-(c+1)| = |-1| = 1$, and $f(x) = f(y)$. The condition is satisfied.

In both cases, we have found $x, y \in [0, 2]$ such that $|x - y| = 1$ and $f(x) = f(y)$.

2 Convergence of Numerical Series

2.1 Definitions

Definition 2.1 (Convergence of a Series)**.** A series $\sum_{n=1}^{\infty} a_n$ **converges** to a real number S if the sequence of partial sums (s_k), where $s_k = \sum_{n=1}^{k} a_n$, converges to S. If the sequence of partial sums diverges, the series **diverges**.

2.2 Convergence Tests

Theorem 2.1 (Comparison Test [5])**.** *Let $\sum a_n$ and $\sum b_n$ be series such that $0 \leq a_n \leq b_n$ for all n sufficiently large.*

1. *If $\sum b_n$ converges, then $\sum a_n$ converges.*
2. *If $\sum a_n$ diverges, then $\sum b_n$ diverges.*

Theorem 2.2 (Limit Comparison Test [6])**.** *Let $\sum a_n$ and $\sum b_n$ be series with positive terms ($a_n > 0, b_n > 0$ for n sufficiently large).*

1. *If $\lim_{n\to\infty}(a_n/b_n) = L$ where $0 < L < \infty$, then $\sum a_n$ and $\sum b_n$ either both converge or both diverge.*
2. *If $\lim_{n\to\infty}(a_n/b_n) = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.*
3. *If $\lim_{n\to\infty}(a_n/b_n) = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.*

Theorem 2.3 (Alternating Series Test [7])**.** *Let (a_n) be a sequence such that*

1. *$a_n \geq 0$ for all n (sufficiently large),*

2. $a_{n+1} \le a_n$ *for all* n *(sufficiently large) (i.e.,* (a_n) *is eventually non-increasing),*

3. $\lim_{n\to\infty} a_n = 0$.

Then the alternating series $\sum(-1)^n a_n$ *and* $\sum(-1)^{n+1} a_n$ *converge.*

Remark 2.1 (Important Series).

- The **harmonic series** $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges [8].
- The **p-series** $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \le 1$ [9].

2.3 Examples

Example 2.1 (Applying Convergence Tests - Sample Problem 4a). **Problem Statement:** Determine whether the following series converge or diverge:

$$S_1 = \sum_{n=2}^{\infty} \frac{1}{\sqrt{n^2-1}}, \qquad S_2 = \sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n^2-1}}$$

Solution:

1. **Analysis of S_1:** Let $a_n = \frac{1}{\sqrt{n^2-1}}$. The terms a_n are positive for $n \ge 2$. We compare a_n with $b_n = \frac{1}{n}$. The harmonic series $\sum b_n = \sum \frac{1}{n}$ diverges. Let's use the Limit Comparison Test:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1/\sqrt{n^2-1}}{1/n} = \lim_{n\to\infty} \frac{n}{\sqrt{n^2-1}}$$

$$= \lim_{n\to\infty} \frac{n}{\sqrt{n^2(1-1/n^2)}} = \lim_{n\to\infty} \frac{n}{n\sqrt{1-1/n^2}} = \lim_{n\to\infty} \frac{1}{\sqrt{1-1/n^2}} = \frac{1}{\sqrt{1-0}} = 1$$

   Since the limit is $L = 1$, and $0 < L < \infty$, and $\sum b_n$ diverges, the series $\sum a_n = S_1$ also **diverges** by the Limit Comparison Test.

2. **Analysis of S_2:** This is an alternating series $\sum(-1)^n a_n$ with $a_n = \frac{1}{\sqrt{n^2-1}}$. We check the conditions of the Alternating Series Test:

   - $a_n = \frac{1}{\sqrt{n^2-1}} > 0$ for $n \ge 2$. (Condition 1 satisfied)
   - Is (a_n) non-increasing? Consider $f(x) = \sqrt{x^2-1}$ for $x \ge 2$. Since $x^2 - 1$ is increasing for $x \ge 2$, $\sqrt{x^2-1}$ is increasing. Therefore, $a_n = 1/f(n)$ is decreasing for $n \ge 2$. (Condition 2 satisfied)
   - Does $a_n \to 0$?

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{\sqrt{n^2-1}} = 0$$

     (Condition 3 satisfied)

   Since all three conditions are met, the series S_2 **converges** by the Alternating Series Test.

3 Sequences and Series of Functions

3.1 Definitions

Definition 3.1 (Pointwise Convergence [10]). A sequence of functions (f_n) defined on $S \subseteq \mathbb{R}$ **converges pointwise** to f on S if, for each $x \in S$, $\lim_{n\to\infty} f_n(x) = f(x)$.

Definition 3.2 (Uniform Convergence [10]). A sequence of functions (f_n) defined on $S \subseteq \mathbb{R}$ **converges uniformly** to f on S if, for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ (depending only on ϵ) such that for all $n > N$ and for all $x \in S$, we have $|f_n(x) - f(x)| < \epsilon$.

Definition 3.3 (Uniformly Cauchy [11]). A sequence of functions (f_n) on S is **uniformly Cauchy** if, for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $m, n > N$ and for all $x \in S$, we have $|f_n(x) - f_m(x)| < \epsilon$.

3.2 Key Theorems

Theorem 3.1 (Cauchy Criterion for Uniform Convergence [12]). *A sequence of functions (f_n) converges uniformly on S if and only if it is uniformly Cauchy on S.*

Theorem 3.2 (Continuity of the Limit Function [13]). *If (f_n) is a sequence of continuous functions on S and $f_n \to f$ uniformly on S, then f is continuous on S.*

Theorem 3.3 (Uniform Continuity of the Limit Function (cf. Sample Problem 1)). *If (f_n) is a sequence of uniformly continuous functions on an interval I, and $f_n \to f$ uniformly on I, then f is uniformly continuous on I.*

Theorem 3.4 (Interchange of Limits [14]). *Let (f_n) be a sequence of continuous functions on $[a, b]$ converging uniformly to f on $[a, b]$. If (x_n) is a sequence in $[a, b]$ with $x_n \to x \in [a, b]$, then $\lim_{n\to\infty} f_n(x_n) = f(x)$.*

Theorem 3.5 (Boundedness of the Limit Function [15]). *If (f_n) is a sequence of bounded functions on S and $f_n \to f$ uniformly on S, then f is bounded on S.*

Theorem 3.6 (Weierstrass M-Test [16]). *Let (f_n) be functions on S. If there exist constants $M_n \geq 0$ such that $|f_n(x)| \leq M_n$ for all $x \in S$ and $\sum M_n$ converges, then $\sum f_n$ converges uniformly on S.*

3.3 Examples

Example 3.1 (Uniform Convergence $\implies$ Uniformly Cauchy - Sample Problem 6). **Problem Statement:** Let $f_n \to f$ uniformly on S. Prove (f_n) is uniformly Cauchy on S.

Proof. Assume $f_n \to f$ uniformly on S. Let $\epsilon > 0$ be given. By the definition of uniform convergence, there exists $N \in \mathbb{N}$ such that for all $k > N$ and for all $x \in S$,

$$|f_k(x) - f(x)| < \frac{\epsilon}{2}$$

Now, let $m > N$ and $n > N$. For any $x \in S$, we use the triangle inequality:

$$\begin{aligned} |f_n(x) - f_m(x)| &= |(f_n(x) - f(x)) + (f(x) - f_m(x))| \\ &\leq |f_n(x) - f(x)| + |f(x) - f_m(x)| \end{aligned}$$

Since $n > N$ and $m > N$, both terms on the right are less than $\epsilon/2$.

$$|f_n(x) - f_m(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

This inequality holds for all $x \in S$ whenever $m, n > N$. Therefore, the sequence (f_n) is uniformly Cauchy on S. □

Example 3.2 (Uniform Continuity Preservation - Sample Problem 1). **Problem Statement:** Let (f_n) be uniformly continuous functions on (a, b), and $f_n \to f$ uniformly on (a, b). Prove f is uniformly continuous on (a, b).

Proof. Let $\epsilon > 0$ be given. We need to find a $\delta > 0$ such that for all $x, y \in (a, b)$, $|x-y| < \delta$ implies $|f(x) - f(y)| < \epsilon$.

1. **Use Uniform Convergence:** Since $f_n \to f$ uniformly, there exists $N \in \mathbb{N}$ such that for all $n > N$ and for all $z \in (a, b)$,

$$|f_n(z) - f(z)| < \frac{\epsilon}{3}$$

Let's fix one such index, say $n = N + 1$. So, for all $z \in (a, b)$, $|f_{N+1}(z) - f(z)| < \epsilon/3$.

2. **Use Uniform Continuity of f_{N+1}:** Since f_{N+1} is uniformly continuous on (a, b), for the value $\epsilon/3 > 0$, there exists a $\delta > 0$ such that for all $x, y \in (a, b)$,

$$|x - y| < \delta \implies |f_{N+1}(x) - f_{N+1}(y)| < \frac{\epsilon}{3}$$

3. **Combine using Triangle Inequality:** Now, let $x, y \in (a, b)$ such that $|x - y| < \delta$ (using the δ from step 2). Consider $|f(x) - f(y)|$:

$$|f(x) - f(y)| = |(f(x) - f_{N+1}(x)) + (f_{N+1}(x) - f_{N+1}(y)) + (f_{N+1}(y) - f(y))|$$

Applying the triangle inequality:

$$\leq |f(x) - f_{N+1}(x)| + |f_{N+1}(x) - f_{N+1}(y)| + |f_{N+1}(y) - f(y)|$$

Using the bounds derived in steps 1 and 2:

$$< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$$

Thus, for any $\epsilon > 0$, we found a $\delta > 0$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$. This proves f is uniformly continuous on (a, b). □

Example 3.3 (Limit Interchange - Sample Problem 3). **Problem Statement:** Let f_n be continuous on $[a, b]$, $f_n \to f$ uniformly on $[a, b]$. If $x_n \in [a, b]$ and $x_n \to x \in [a, b]$, show $\lim_{n\to\infty} f_n(x_n) = f(x)$.

Proof. Let $\epsilon > 0$ be given. We want to show there exists N such that $n > N \implies |f_n(x_n) - f(x)| < \epsilon$.

1. **Continuity of Limit Function:** Since f_n are continuous and $f_n \to f$ uniformly on $[a, b]$, the limit function f is continuous on $[a, b]$ (by Theorem 3.2).

2. **Use Continuity of f:** Since f is continuous at $x \in [a, b]$ and $x_n \to x$, there exists $N_1 \in \mathbb{N}$ such that for all $n > N_1$,

$$|f(x_n) - f(x)| < \frac{\epsilon}{2}$$

3. **Use Uniform Convergence:** Since $f_n \to f$ uniformly on $[a, b]$, there exists $N_2 \in \mathbb{N}$ such that for all $n > N_2$ and for all $y \in [a, b]$,

$$|f_n(y) - f(y)| < \frac{\epsilon}{2}$$

In particular, this holds for $y = x_n$ (since $x_n \in [a, b]$), so for $n > N_2$, $|f_n(x_n) - f(x_n)| < \epsilon/2$.

4. **Combine using Triangle Inequality:** Let $N = \max\{N_1, N_2\}$. If $n > N$, then both conditions from steps 2 and 3 hold. Consider $|f_n(x_n) - f(x)|$:

$$|f_n(x_n) - f(x)| = |(f_n(x_n) - f(x_n)) + (f(x_n) - f(x))|$$

Applying the triangle inequality:

$$\leq |f_n(x_n) - f(x_n)| + |f(x_n) - f(x)|$$

Using the bounds derived in steps 3 and 2:

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Thus, for any $\epsilon > 0$, we found N such that $n > N \implies |f_n(x_n) - f(x)| < \epsilon$. This proves $\lim_{n\to\infty} f_n(x_n) = f(x)$. □

Example 3.4 (Boundedness Preservation - Sample Problem 7a). **Problem Statement:** Let (f_n) be bounded functions on S, and $f_n \to f$ uniformly on S. Prove f is bounded on S.

Proof. We need to show there exists $M' \geq 0$ such that $|f(x)| \leq M'$ for all $x \in S$.

1. **Use Uniform Convergence:** Since $f_n \to f$ uniformly, for $\epsilon = 1$, there exists $N \in \mathbb{N}$ such that for all $n > N$ and for all $x \in S$,

$$|f_n(x) - f(x)| < 1$$

Let's fix one such index, say $n = N + 1$. So, $|f_{N+1}(x) - f(x)| < 1$ for all $x \in S$.

2. **Use Boundedness of f_{N+1}:** Since f_{N+1} is a bounded function on S, there exists a constant $M \geq 0$ such that

$$|f_{N+1}(x)| \leq M \quad \text{for all } x \in S$$

3. **Combine using Triangle Inequality:** For any $x \in S$, consider $|f(x)|$:

$$|f(x)| = |f(x) - f_{N+1}(x) + f_{N+1}(x)|$$

Applying the triangle inequality:

$$\leq |f(x) - f_{N+1}(x)| + |f_{N+1}(x)|$$

Using the bounds from steps 1 and 2:

$$< 1 + M$$

Let $M' = M + 1$. We have shown that $|f(x)| < M'$ for all $x \in S$. Therefore, f is bounded on S. □

Example 3.5 (Pointwise Limit Need Not Be Bounded - Sample Problem 7b). **Problem Statement:** Give an example of a set $S \subseteq \mathbb{R}$ and a sequence of bounded functions (f_n) on S such that $f_n \to f$ pointwise on S, but f is not bounded on S.

Solution: Let $S = (0, 1]$. Consider the sequence of functions $f_n : S \to \mathbb{R}$ defined by

$$f_n(x) = \min\left\{n, \frac{1}{x}\right\} = \begin{cases} n & \text{if } 0 < x \leq 1/n \\ 1/x & \text{if } 1/n < x \leq 1 \end{cases}$$

We verify the properties:

- f_n **is Bounded:** For any fixed n, the value of $f_n(x)$ is either n or $1/x$. If $1/n < x \leq 1$, then $1 \leq 1/x < n$. So, in all cases, $0 < f_n(x) \leq n$. Thus, each f_n is bounded on S (by $M_n = n$).
- **Pointwise Convergence:** Let $x \in (0, 1]$ be fixed. Consider the limit $\lim_{n\to\infty} f_n(x)$. Choose an integer N such that $N > 1/x$. Then, for all $n > N$, we have $n > 1/x$, which implies $x > 1/n$. According to the definition of f_n, for $n > N$, $f_n(x) = 1/x$. Therefore, the sequence $(f_n(x))$ eventually becomes constant $(1/x, 1/x, \ldots)$ and converges to $1/x$. So, $f_n \to f$ pointwise on S, where the limit function is $f(x) = 1/x$.
- f **is Unbounded:** The limit function $f(x) = 1/x$ is not bounded on the interval $S = (0, 1]$. As $x \to 0^+$, $f(x) \to +\infty$. There is no constant M such that $|f(x)| \leq M$ for all $x \in (0, 1]$.

This example demonstrates that pointwise convergence does not preserve boundedness.

Example 3.6 (Application of M-Test - Sample Problem 2b Alternative). **Problem Statement:** Show that $f_3(y) = \sum_{n=1}^{\infty} \frac{1}{n^2} \left(\frac{y}{1+y^2}\right)^n$ converges for all $y \in \mathbb{R}$.

Solution using M-Test: Let $f_n(y) = \frac{1}{n^2} \left(\frac{y}{1+y^2}\right)^n$. We want to apply the Weierstrass M-Test.

1. **Find a Bound** M_n**:** We need to bound $|f_n(y)|$ uniformly for all $y \in \mathbb{R}$. Let $g(y) = \frac{y}{1+y^2}$. We find the maximum value of $|g(y)|$. If $y = 0$, $g(0) = 0$. If $y \neq 0$, $|g(y)| = \frac{|y|}{1+y^2}$. Consider $h(t) = \frac{t}{1+t^2}$ for $t > 0$. $h'(t) = \frac{(1+t^2)(1)-t(2t)}{(1+t^2)^2} = \frac{1-t^2}{(1+t^2)^2}$. $h'(t) = 0$ when $t = 1$. $h(1) = 1/2$. Since $\lim_{t\to 0^+} h(t) = 0$ and $\lim_{t\to\infty} h(t) = 0$, the maximum value for $t > 0$ is $1/2$. Since $g(-y) = -g(y)$, the maximum value of $|g(y)|$ for all $y \in \mathbb{R}$ is $1/2$. Therefore, for all $y \in \mathbb{R}$,

$$|f_n(y)| = \frac{1}{n^2} \left|\frac{y}{1+y^2}\right|^n \leq \frac{1}{n^2} \left(\frac{1}{2}\right)^n$$

Let $M_n = \frac{1}{n^2} \left(\frac{1}{2}\right)^n$.

2. **Check Convergence of** $\sum M_n$**:** The series is $\sum_{n=1}^{\infty} \frac{1}{n^2 2^n}$. We can use the Comparison Test. Since $0 < 1/2^n \leq 1$ for $n \geq 1$, we have $0 < M_n \leq \frac{1}{n^2}$. The series $\sum \frac{1}{n^2}$ is a convergent p-series ($p = 2 > 1$). By the Comparison Test, $\sum M_n$ converges.

3. **Conclusion:** Since $|f_n(y)| \leq M_n$ for all $y \in \mathbb{R}$ and $\sum M_n$ converges, the series $\sum f_n(y)$ converges uniformly on $\mathbb{R}$ by the Weierstrass M-Test. Uniform convergence implies pointwise convergence for all $y \in \mathbb{R}$.

4 Power Series

4.1 Definitions and Basic Properties

Definition 4.1 (Power Series [17]). A **power series** centered at 0 is a series of the form $\sum_{n=0}^{\infty} a_n x^n$.

Theorem 4.1 (Radius of Convergence [18]). *For any power series $\sum a_n x^n$, let $\beta = \limsup_{n\to\infty} |a_n|^{1/n}$. The* ***radius of convergence*** *R is defined as*

$$R = \begin{cases} 1/\beta & \text{if } 0 < \beta < \infty \\ \infty & \text{if } \beta = 0 \\ 0 & \text{if } \beta = \infty \end{cases}$$

The series converges absolutely for $|x| < R$ and diverges for $|x| > R$.

Remark 4.1 (Ratio Test for R). If $\lim_{n\to\infty} |a_{n+1}/a_n| = L$ exists, then $\beta = L$ and $R = 1/L$ (with $R = \infty$ if $L = 0$, $R = 0$ if $L = \infty$).

Remark 4.2 (Endpoint Convergence). The convergence or divergence of the series at the endpoints $x = R$ and $x = -R$ must be checked separately using numerical series tests.

Theorem 4.2 (Uniform Convergence of Power Series [19]). *If a power series $\sum a_n x^n$ has radius of convergence $R > 0$, then for any c such that $0 < c < R$, the series converges uniformly on the interval $[-c, c]$.*

Corollary 4.1. *The function $f(x) = \sum a_n x^n$ defined by a power series is continuous on the open interval of convergence $(-R, R)$.*

4.2 Examples

Example 4.1 (Calculating Radius of Convergence - Sample Problem 2a). **Problem Statement:** Find the radius of convergence R for:

$$f_1(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}, \qquad f_2(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{2^n}$$

Solution:

1. **For $f_1(x)$:** The coefficients are $a_n = 1/n^2$ for $n \geq 1$. We use the Ratio Test limit:

$$\begin{aligned} L &= \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{1/(n+1)^2}{1/n^2} \right| = \lim_{n\to\infty} \frac{n^2}{(n+1)^2} \\ &= \lim_{n\to\infty} \left(\frac{n}{n+1} \right)^2 = \lim_{n\to\infty} \left(\frac{1}{1+1/n} \right)^2 = \left(\frac{1}{1+0} \right)^2 = 1 \end{aligned}$$

   The radius of convergence is $R_1 = 1/L = 1/1 = 1$. (Alternatively, using root test: $\beta = \limsup |1/n^2|^{1/n} = \limsup (1/n^{1/n})^2 = (1/1)^2 = 1$, so $R_1 = 1/\beta = 1$.)

2. **For $f_2(x)$:** This series involves only even powers of x. Let $y = x^2$. The series becomes $\sum_{n=0}^{\infty} \frac{y^n}{2^n} = \sum_{n=0}^{\infty} \left(\frac{1}{2^n}\right) y^n$. This is a power series in y with coefficients $b_n = 1/2^n$. Find its radius of convergence R_y. Using the Ratio Test:

$$L_y = \lim_{n\to\infty} \left| \frac{b_{n+1}}{b_n} \right| = \lim_{n\to\infty} \left| \frac{1/2^{n+1}}{1/2^n} \right| = \lim_{n\to\infty} \frac{2^n}{2^{n+1}} = \lim_{n\to\infty} \frac{1}{2} = \frac{1}{2}$$

The radius of convergence for the series in y is $R_y = 1/L_y = 2$. The series in y converges for $|y| < R_y$, i.e., $|y| < 2$. Substituting back $y = x^2$, the original series converges when $|x^2| < 2$, which means $x^2 < 2$, or $-\sqrt{2} < x < \sqrt{2}$. The radius of convergence for the series in x is $R_2 = \sqrt{2}$.

Example 4.2 (Using Endpoint Behavior for R - Sample Problem 4b). **Problem Statement:** Find R for $\sum_{n=2}^{\infty} \frac{5^n x^n}{\sqrt{n^2-1}}$, given that $\sum \frac{1}{\sqrt{n^2-1}}$ diverges and $\sum \frac{(-1)^n}{\sqrt{n^2-1}}$ converges.

Solution: Let the power series be $S(x) = \sum_{n=2}^{\infty} a_n x^n$ with $a_n = \frac{5^n}{\sqrt{n^2-1}}$.

1. **Test Endpoint $x = 1/5$:** Substitute $x = 1/5$ into the series:

$$S(1/5) = \sum_{n=2}^{\infty} \frac{5^n (1/5)^n}{\sqrt{n^2-1}} = \sum_{n=2}^{\infty} \frac{1}{\sqrt{n^2-1}}$$

We are given that this series diverges. Since the power series diverges at $x = 1/5$, the radius of convergence R must satisfy $R \leq |1/5| = 1/5$.

2. **Test Endpoint $x = -1/5$:** Substitute $x = -1/5$ into the series:

$$S(-1/5) = \sum_{n=2}^{\infty} \frac{5^n (-1/5)^n}{\sqrt{n^2-1}} = \sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n^2-1}}$$

We are given that this series converges. Since the power series converges at $x = -1/5$, the radius of convergence R must satisfy $R \geq |-1/5| = 1/5$.

3. **Conclusion:** Combining the results from both endpoints, we have $R \leq 1/5$ and $R \geq 1/5$. Therefore, the radius of convergence must be exactly $R = 1/5$.

Example 4.3 (Function Series as Power Series - Sample Problem 2b). **Problem Statement:** Show that $f_3(y) = \sum_{n=1}^{\infty} \frac{1}{n^2} \left(\frac{y}{1+y^2}\right)^n$ converges for all $y \in \mathbb{R}$.

Solution: 1. **Identify the Underlying Power Series:** Let $x = g(y) = \frac{y}{1+y^2}$. The series becomes

$$f_1(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}$$

This is a power series in x.

2. **Find Interval of Convergence for Power Series:** From Example 4.2 (or Problem 2a), the radius of convergence for $f_1(x)$ is $R = 1$. We check endpoints:

- At $x = 1$: $\sum \frac{1}{n^2}$ converges (p-series, $p = 2 > 1$).
- At $x = -1$: $\sum \frac{(-1)^n}{n^2}$ converges (by Alternating Series Test, or absolutely).

So, the power series $f_1(x)$ converges precisely for $x \in [-1, 1]$.

3. **Find the Range of the Argument Function:** Consider the argument $x = g(y) = \frac{y}{1+y^2}$. We need to determine the range of $g(y)$ for $y \in \mathbb{R}$. As shown in the M-test example (Example 3.3), the maximum value of $|g(y)|$ is $1/2$. Therefore, the range of $g(y)$ is $[-1/2, 1/2]$.

4. **Conclusion:** For any $y \in \mathbb{R}$, the value $x = g(y)$ lies in the interval $[-1/2, 1/2]$. Since $[-1/2, 1/2] \subseteq [-1, 1]$, and the power series $f_1(x)$ converges for all x in $[-1, 1]$, it follows that the series $f_3(y) = f_1(g(y))$ converges for all values of $y \in \mathbb{R}$.

References

[1] Ross, K. A. *Elementary Analysis: The Theory of Calculus*. 2nd ed., Springer, 2013, Definition 17.1.

[2] Ross, K. A. *Elementary Analysis*. Definition 19.1.

[3] Ross, K. A. *Elementary Analysis*. Theorem 18.2.

[4] Ross, K. A. *Elementary Analysis*. p. 123.

[5] Ross, K. A. *Elementary Analysis*. Theorem 14.6.

[6] Ross, K. A. *Elementary Analysis*. Theorem 14.7.

[7] Ross, K. A. *Elementary Analysis*. Theorem 15.3.

[8] Ross, K. A. *Elementary Analysis*. Example 6, Section 14.

[9] Ross, K. A. *Elementary Analysis*. Example 7, Section 14.

[10] Ross, K. A. *Elementary Analysis*. Definition 24.1.

[11] Ross, K. A. *Elementary Analysis*. Definition 25.3.

[12] Ross, K. A. *Elementary Analysis*. Theorem 25.4.

[13] Ross, K. A. *Elementary Analysis*. Theorem 24.3.

[14] Ross, K. A. *Elementary Analysis*. Exercise 24.17.

[15] Ross, K. A. *Elementary Analysis*. Exercise 25.5.

[16] Ross, K. A. *Elementary Analysis*. Theorem 25.7 (Weierstrass M-Test).

[17] Ross, K. A. *Elementary Analysis*. Section 23.

[18] Ross, K. A. *Elementary Analysis*. Theorem 23.1.

[19] Ross, K. A. *Elementary Analysis*. Theorem 26.1.