Physics 415
Spring 2025
Homework 11
Due Friday, April 25, 2025

This assignment covers material in Chapter 9 of Reif. I recommend reading through the text and also Lectures Notes 34-36.

Problem 1: (BEC in a harmonic trap) In this problem, you will study the Bose-Einstein condensation (BEC) of particles in a three-dimensional harmonic trap with confining potential $U(r) = m\omega^2 r^2/2$. This is the setup used in most experiments on the BEC of dilute gases of neutral (and hence very weakly interacting) atoms. For simplicity, consider spin-0 particles, so there is no spin degeneracy.

(a) The quantum-mechanical analysis of a particle in the harmonic potential $U(r)$ given above yields the single-particle energy levels

$$\varepsilon_{n_x n_y n_z} = \hbar\omega(n_x + n_y + n_z + 3/2), \tag{1}$$

where each single-particle state is labeled by the triplet of integers n_x, n_y, n_z, each of which runs from 0 to ∞. Just as we did in class for the case of Bose-Einstein condensation of particles in a box with periodic boundary conditions (see Lecture 34), write down the expression for the particle number N as a sum over single-particle states.

(b) The energies in Eq. (1) depend only on the sum $n = n_x + n_y + n_z$, and we may therefore write $\varepsilon_n = \hbar\omega(n + 3/2)$. However, each energy level is degenerate (for example, there is a unique state for $n = 0$, there are 3 states giving $n = 1$, 6 states giving $n = 2$, etc.). Find the general formula for the degeneracy factor ρ_n for arbitrary n and use this to rewrite your result in part (a) as a sum over n.

(c) Separate out the ground state from the sum you found in part (a) (note that the ground state energy in this case is $\varepsilon_0 = (3/2)\hbar\omega$), expressing the total particle number in the form $N = N_{\varepsilon=\varepsilon_0} + N_{\varepsilon>\varepsilon_0}$. Obtain the BEC transition temperature T_c by setting $\mu = \varepsilon_0$ in $N_{\varepsilon>\varepsilon_0}$ and recall that, at $T = T_c$, $N_{\varepsilon>\varepsilon_0} = N$.

Hint: When $N \gg 1$, the sum over n may be replaced by an integration and, at the same level of approximation, you may replace ρ_n by its leading term at large n. The following integral will be useful:

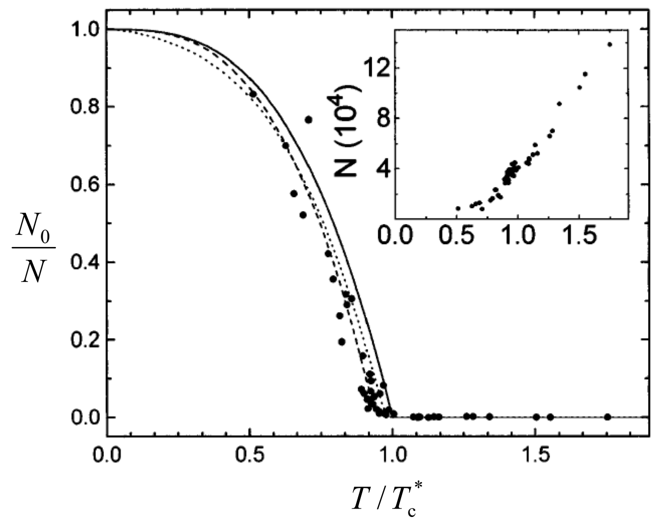
$$\int_0^\infty \frac{x^2}{e^x - 1} = 2\zeta(3) \approx 2.404. \tag{2}$$

(d) Show that number of particle in the condensate for $T \leq T_c$ obeys

$$N_{\varepsilon=\varepsilon_0} = N\left[1 - \left(\frac{T}{T_c}\right)^3\right]. \tag{3}$$

Note the different power of temperature from the result derived in lecture for the BEC of particles in a box.

Below is a plot of the condensate fraction N_0/N of a BEC of N ^{87}Rb atoms, in one of the pioneering BEC experiments – see J. Ensher *et. al.*, *Phys. Rev. Lett.* **77**, 4984 (1996). In this experiment T_c was as low as 0.28×10^{-6} K. The solid line shows the analytical result Eq. (3); other lines correspond to more detailed theories taking into account the finite number N of trapped atoms (N is shown in the inset).



Problem 2: (BEC in two dimensions) Calculate the chemical potential μ of an ideal Bose gas in two dimensions (2D), as a function of its areal density n (the number of particles per unit area). Determine whether such gas can form a BEC at low temperatures. *Hint:* Use your result for the 2D density of states from Homework 10, Problem 1a.

Problem 3: (Black-body thermodynamics, adapted from Reif 9.9) Electromagnetic radiation at temperature T_i fills a cavity of volume V. If the volume of the thermally insulated cavity is expanded quasi-statically to a volume $8V$, what the final temperature T_f?

Problem 4: (Black-body thermodynamics, Stefan-Boltzmann law) In this problem you use thermodynamic arguments to derive that the energy E of a black-body at temperature T enclosed in a volume V obeys the scaling relation $E \propto VT^4$. Take as your starting point the relation between the radiation pressure and the energy $p = E/3V$. (This can be derived in different ways, for example, from kinetic arguments.)

(a) Explain why the relation $p = E/3V$ implies that the pressure is independent of volume; i.e., $p = p(T)$.

(b) Using the fact that $p = p(T)$, obtain the Helmholtz free energy F of the black-body. Express F in terms of E.

(c) Starting from the fundamental relation $F = E - TS$, obtain a differential equation satisfied by F. Solve this differential equation and show that your solution implies $E \propto VT^4$.

Note: The proportionality factor cannot be obtained from thermodynamic arguments alone; for that we needed statistical mechanics.

Problem 5: (Black-body radiation, adapted from Reif 9.12) It has been reported that a nuclear fission explosion produces a temperature of the order of 10^{6}°K. Assuming this to be true over a sphere 10 cm in diameter, calculate approximately the following:

(a) The total rate of electromagnetic radiation from the surface of this sphere.

(b) The radiation flux (power incident per unit area) at a distance of 1 km.

(c) The wavelength corresponding to the maximum in the radiated power spectrum.

Problem 6: (Black-body radiation, adapted from Reif 9.13) The surface temperature of the sun is $T_0 = 5500$°K, its radius is $R = 7 \times 10^{10}$ cm, while the radius of the earth is $r = 6.37 \times 10^{8}$ cm. The mean distance between the sun and the earth is $L = 1.5 \times 10^{13}$ cm. In first approximation, one can assume that both the sun and the earth absorb all electromagnetic radiation incident upon them.

The earth has reached a steady state so that its mean temperature T does not change in time despite the fact that the earth constantly absorbs and emits radiation.

(a) Find an approximate expression for the temperature T of the earth in terms of the astronomical parameters mentioned above.

(b) Calculate the temperature T numerically.