Sequences

A **sequence** is a function from $\{n \in \mathbb{Z} \mid n \geq m\}$ to $\mathbb{R}$ where m is usually 0 or 1. It can be written as $s(1), s(2), s(3), \ldots$, or $(s_1, s_2, s_3, \ldots)$, or $(s_n)_{n \in \mathbb{N}}$

Examples $\quad s_n = n \qquad (1, 2, 3, 4, \ldots)$

$s_n = \frac{1}{n} \qquad (1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots)$

$s_1 = 1,\ s_{n+1} = \frac{s_n}{2} \qquad (1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots)$

$s_n = (-1)^n \qquad (-1, 1, -1, 1, -1, 1, \ldots)$

(*) $s_1 = 1,\ s_{n+1} = \begin{cases} 2 & \text{if } s_n = 1 \\ 3 & \text{if } s_n = 2 \\ 1 & \text{if } s_n = 3 \end{cases} \qquad (1, 2, 3, 1, 2, 3, 1, 2, 3, \ldots)$

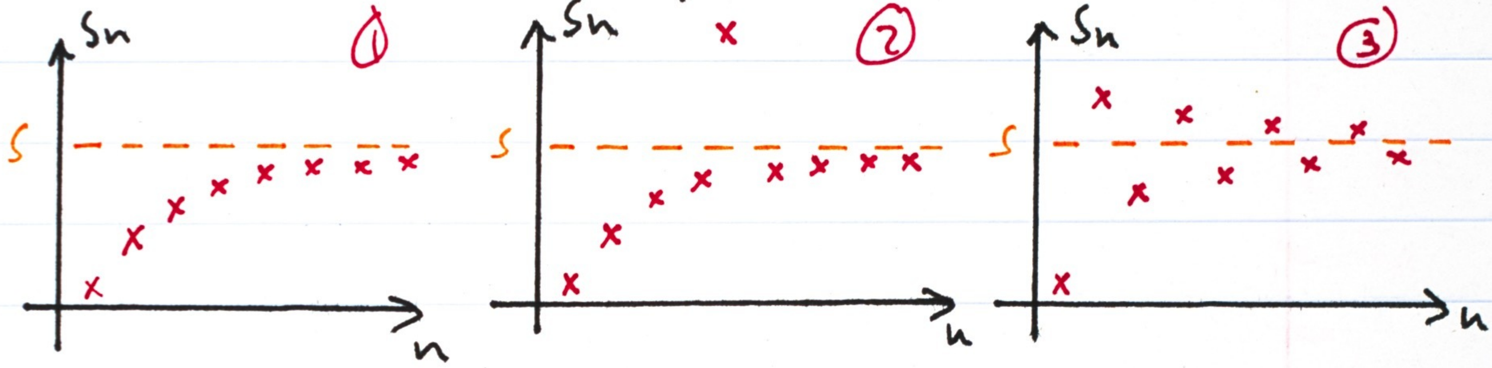
$s_n = \begin{cases} 1 & \text{if } n = 2^m \text{ for some } m \\ 0 & \text{otherwise} \end{cases} \qquad (1, 1, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, \ldots)$

It is important to distinguish between a sequence (written with round brackets) from its set of values (written with curly brackets). For example the set of values for (*) is $\{1, 2, 3\}$ or equivalently $\{2, 3, 1\}$

since it is unordered.

Convergence

Consider the three sequences shown graphically.



How do we define whether these sequences converge to a limit s? Example 1 looks like it is converging to s, and getting closer on each step. Example 2 has a blip at $n=5$, but after that its behavior is the same as 1. Convergence should not matter about transitory behavior – the properties for large n matter. Example 3 also looks convergent even though it both overshoots and undershoots the limit.

For any distance, we want s_n to ultimately always be closer than that distance.

Definition s_n converges to s if

$$\forall \varepsilon > 0 \quad \exists N \text{ such that } n > N \Rightarrow |s_n - s| < \varepsilon.$$

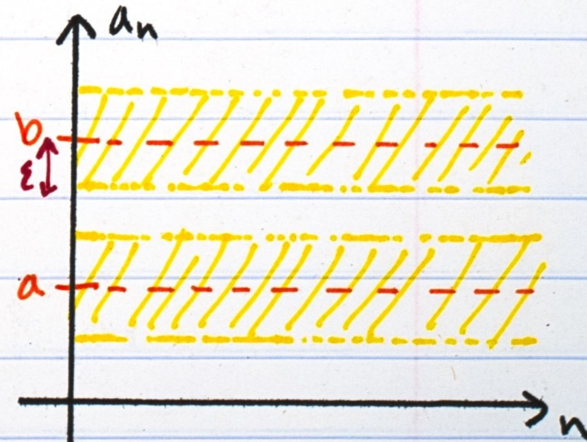
If so, we write $\lim_{n\to\infty} s_n = s$ as the limit of (s_n). Also $s_n \to s$.

If (s_n) has no limit then it is said to diverge.

Notes
- N can be forced to be an integer
- ε represents any positive number, but it is typical in analysis to use ε (and δ) in cases where the interesting/challenging values are small.

<u>Theorem</u> Limits are unique

<u>Proof</u> Suppose $a_n \to a$ and $a_n \to b$ but $b \neq a$. Take $a < b$: this can always be achieved, by switching a and b if necessary. Hence $d = b - a > 0$.



Consider the diagram, and suppose that $\varepsilon = d/3$. Then if $a_n \to a$ there must be a point where a_n is within the yellow region of distance ε from a.
But similarly if $a_n \to b$ there must be a point where a_n is within the yellow region of distance ε

from b. But since these regions are distinct, this is impossible.

Mathematically: • $\exists N_1$ s.t. $\forall n > N_1$ $|a_n - a| < \frac{d}{3}$.

• $\exists N_2$ s.t. $\forall n > N_2$ $|a_n - b| < \frac{d}{3}$.

Choose n s.t $n > N_1$ and $n > N_2$. Then, using the triangle inequality,

$$|a-b| = |(a-a_n) - (b-a_n)|$$

$$\leq |a-a_n| + |b-a_n| \leq \frac{d}{3} + \frac{d}{3} = \frac{2d}{3},$$

which is a contradiction since $|a-b| = d$. Hence limits are unique.

$s_n = (-1)^n$ diverges

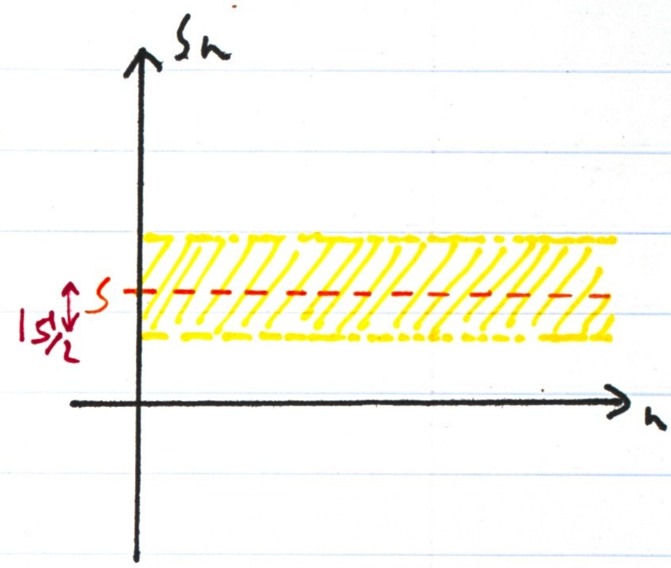
Proof Suppose that $s_n \to a$. Then $\exists N$ s.t $\forall n > N$ $|s_n - a| < 1/2$. Now choose n_1 even and n_2 odd such that $n_1, n_2 > N$. Then

$$2 = |1-(-1)| = |s_{n_1} - s_{n_2}| = |(s_{n_1} - a) - (s_{n_2} - a)|$$

$$\leq |s_{n_1} - a| + |s_{n_2} - a| \leq \frac{1}{2} + \frac{1}{2}. \quad ⋇$$

Example Let (s_n) be a convergent sequence such that $s_n \neq 0 \ \forall n \in \mathbb{N}$ and $\lim s_n = s \neq 0$. Then $\inf\{|s_n| \mid n \in \mathbb{N}\} > 0$.

Proof Consider the diagram. At some point, the sequence must lie within the yellow region and all the terms will be away from zero. Mathematically, set $\varepsilon = \frac{|s|}{2}$.



Then $\exists N \in \mathbb{N}$ s.t. $n > N \Rightarrow |s_n - s| < \frac{|s|}{2}$. By the triangle inequality

$$|s| = |s_n + s - s_n| \leq |s_n| + |s - s_n|.$$

Hence $|s_n| \geq |s| - |s - s_n| \geq |s| - \frac{|s|}{2} = \frac{|s|}{2}$.

Define $m = \min\left\{\frac{|s|}{2}, |s_1|, |s_2|, \ldots, |s_N|\right\}$. Since each $|s_n| \geq m$, m is a lower bound for $\{|s_n| \mid n \in \mathbb{N}\}$ and hence

$$\inf\{|s_n| \mid n \in \mathbb{N}\} \geq m > 0.$$

Sandwich lemma

Suppose that $(a_n), (b_n), (s_n)$ are sequences such that $a_n \leq s_n \leq b_n \ \forall n$, and $s = \lim a_n = \lim b_n$. Then $\lim s_n = s$.

Proof Choose $\varepsilon > 0$. Then $\exists N_1$ s.t. $|a_n - s| < \varepsilon$ for all $n > N_1$. Therefore $a_n > s - \varepsilon \ \forall n > N_1$.

Similarly $\exists N_2$ s.t. $|b_n - s| < \varepsilon$ for all $n > N_2$. Therefore $b_n < s + \varepsilon \ \forall n > N_2$.

Choose $n > N_1$ and $n > N_2$. Then

$$s - \varepsilon < a_n \leq s_n \leq b_n < s + \varepsilon,$$

and hence $-\varepsilon < s_n - s < \varepsilon$ so $|s_n - s| < \varepsilon$.

Convergent sequences are bounded

Proof Let $(s_n)_{n \in \mathbb{N}}$ have $s_n \to s$ as $n \to \infty$. Then $\exists N \in \mathbb{N}$ s.t. $n > N \Rightarrow |s_n - s| < 1$. By the triangle inequality $|s_n| \leq |s| + |s_n - s| < |s| + 1$. Define

$$M = \max \{ |s| + 1, |s_1|, \ldots, |s_N| \}.$$

Then $|s_n| \leq M$ $\forall n \in \mathbb{N}$ and (s_n) is bounded.

If $s_n \to s$ and $k \in \mathbb{R}$, then $ks_n \to ks$.

Proof If $k=0$ then $ks_n = 0$ for all n and the result is immediately true.

Suppose $k \neq 0$, and choose $\varepsilon > 0$. Then $\exists N$ s.t. $n > N \Rightarrow |s_n - s| < \dfrac{\varepsilon}{|k|}$. Then $|ks_n - ks| < \varepsilon$.

If $s_n \to s$ and $t_n \to t$ then $s_n + t_n \to s+t$

Proof Choose $\varepsilon > 0$. Then

- $\exists N_1$ s.t. $n > N_1 \Rightarrow |s_n - s| < \frac{\varepsilon}{2}$,
- $\exists N_2$ s.t $n > N_2 \Rightarrow |t_n - t| < \frac{\varepsilon}{2}$.

Let $N = \max\{N_1, N_2\}$. Then for $n > N$,

$$|s_n + t_n - (s+t)| \leq |s_n - s| + |t_n - t| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

If $s_n \to s$ and $t_n \to t$ then $s_n t_n \to st$

Proof Since $s_n \to s$ it is bounded, so $|s_n| < M$ for all n.

Choose $\varepsilon > 0$: • $\exists N_1$ s.t. $n > N_1 \Rightarrow |t_n - t| < \frac{\varepsilon}{2M}$

• $\exists N_2$ s.t. $n \geq N_2 \Rightarrow |s_n - s| < \frac{\varepsilon}{2(|t|+1)}$

Now look at $n > N$ where $N = \max\{N_1, N_2\}$. Then

$$
\begin{aligned}
|s_n t_n - st| &= |s_n t_n - s_n t + s_n t - st| \\
&\leq |s_n t_n - s_n t| + |s_n t - st| \\
&\leq |s_n||t_n - t| + |s_n - s||t| \\
&< M \frac{\varepsilon}{2M} + |t| \frac{\varepsilon}{2(|t|+1)} = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}
$$

If $s_n \to s$, and if $s_n \neq 0$ $\forall n$ and $s \neq 0$, then $\frac{1}{s_n} \to \frac{1}{s}$

Proof Using a previous result, $\inf\{|s_n| \mid n \in \mathbb{N}\} > 0$, and hence $\exists m > 0$ s.t. $|s_n| \geq m$ for all $n \in \mathbb{N}$.

Choose $\varepsilon > 0$. Then $\exists N \in \mathbb{N}$ such that $n > N \Rightarrow$ $|s - s_n| < \varepsilon m |s|$.

Then for $n > N$

$$\left|\frac{1}{s_n} - \frac{1}{s}\right| = \left|\frac{s}{ss_n} - \frac{s_n}{ss_n}\right| = \frac{|s - s_n|}{|ss_n|}$$

$$\leq \frac{|s-s_n|}{m|s|} < \varepsilon.$$

The previous result establish that if we have several convergent sequences, we can combine them arithmetically and still end up with a convergent sequence (with minor restrictions).

Example Suppose $a_n \to a$, $b_n \to b$, and $c_n \to c$ are convergent sequences. Then

$$\frac{3a_n}{b_n + c_n} \to \frac{3a}{b+c}$$

(so long as $b_n + c_n \neq 0 \ \forall n$ and $b + c \neq 0$)

Example $\lim_{n\to\infty} a^n = 0$ if $|a| < 1$

Proof If $a = 0$ then the result is obvious. Otherwise write $|a| = \frac{1}{1+b}$ where $b > 0$. Then

$$(1+b)^n \geq 1 + nb > nb \quad \text{for } n \in \mathbb{N}.$$

Then $|a^n - 0| = |a^n| = \frac{1}{(1+b)^n} < \frac{1}{nb}.$

Choose $\varepsilon > 0$. Then set $N = \frac{1}{\varepsilon b}$ to show that

$$|a^n - 0| < \varepsilon \qquad \forall n > N.$$

Definition Write $\lim_{n \to \infty} s_n = \infty$ if $\forall M > 0$, there exists N such that

$$n > N \Rightarrow s_n > M$$

Similarly, write $\lim_{n \to \infty} s_n = -\infty$ if $\forall M < 0$, there exists N such that

$$n > N \Rightarrow s_n < M.$$

We say that (s_n) has a limit if it converges, or diverges to $\pm\infty$

Example $\sqrt{n} + 7$ diverges to $+\infty$

Proposition Let (s_n) and (t_n) be sequences such that $\lim s_n = \infty$ and $\lim t_n > 0$ (where the limit could be finite or infinity). Then $\lim s_n t_n = \infty$

Proof There are two cases.

(a) $\lim t_n = t \in \mathbb{R}$

Then $\exists N_1$ s.t. $n > N_1 \Rightarrow |t_n - t| < t/2$

$\Rightarrow t_n > t/2$

(b) $\lim t_n = \infty$

Then $\exists N_1$ s.t. $n > N \Rightarrow t_n > 1$

Define $\lambda = \frac{t}{2}$ for case (a) and $\lambda = 1$ for case (b).
Choose $M > 0$. Since $s_n \to \infty$ $\exists N_2$ s.t. $n > N_2$ implies
$s_n > \frac{M}{\lambda}$.

Now set $N = \max\{N_1, N_2\}$. For $n > N$

$$t_n s_n > \lambda \frac{M}{\lambda} = M.$$

Hence $\lim_{n \to \infty} s_n t_n = \infty$

Monotonic sequences

Definition A sequence is called non-decreasing if $s_n \leq s_{n+1}$ $\forall n$, and non-increasing if $s_{n+1} \leq s_n$ $\forall n$. A sequence that is either non-decreasing or non-increasing is called a monotonic (or monotone) sequence.

Example • $s_n = n$ is an unbound monotic sequence.

• $s_n = 1 - \frac{1}{n}$ is a bounded monotonic sequence.

Theorem All bounded monotonic sequences converge.

Proof Let s_n be a non-decreasing sequence. Let $S = \{s_n \mid n \in \mathbb{N}\}$ and let $u = \sup S$. $u \in \mathbb{R}$ by the completeness axiom.

Now choose $\varepsilon > 0$. $u - \varepsilon$ is not an upper bound for S, so $\exists N$ s.t. $s_N > u - \varepsilon$.

Since s_n is non-decreasing $s_n > u - \varepsilon \ \forall n > N$. Also $s_n \leq u \ \forall n > N$, so $s_n < u + \varepsilon \ \forall n > N$. Hence

$$|s_n - u| < \varepsilon$$

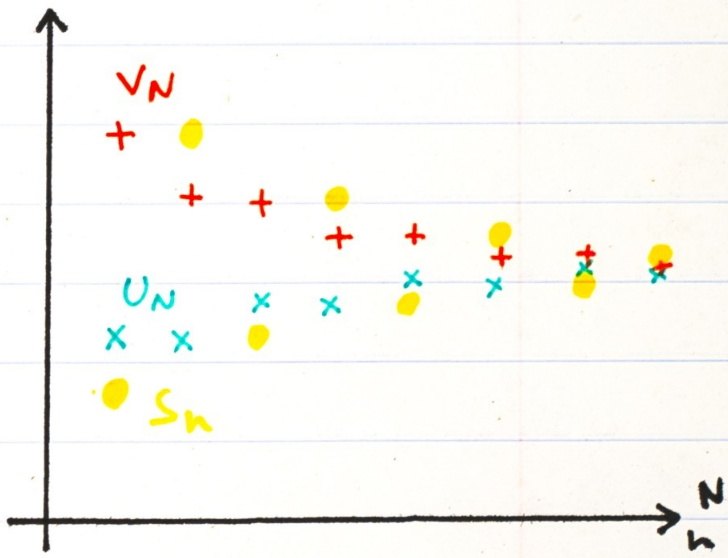
and s_n converges to u.

Theorem If (s_n) is an unbounded non-decreasing sequence, then $\lim_{n \to \infty} s_n = \infty$.

Proof Since $\{s_n \mid n \in \mathbb{N}\}$ is bounded below by s_1 and is unbounded, it must be unbounded above. Hence for any $M > 0$, $\exists N \in \mathbb{N}$ s.t $s_N > M$. Since

(s_n) is non-decreasing, $s_n \geq s_N > M$ for all $n > N$ and hence $\lim_{n\to\infty} s_n = \infty$.

Corollary If (s_n) is monotonic, it either converges, or diverges to $\pm\infty$. Thus $\lim s_n$ is always meaningful.



For a given sequence (s_n) define associated sequences (u_N) and (v_N) as

$$u_N = \inf \{ s_n \mid n > N \}$$

$$v_N = \sup \{ s_n \mid n > N \}$$

Hence $u_1 \leq u_2 \leq u_3 \leq \ldots$ and it is a non-dereasing sequence. Also, $v_1 \geq v_2 \geq v_3 \geq \ldots$ and it is a non-increasing sequence. Define

$$\limsup_{n\to\infty} s_n = \lim_{N\to\infty} v_N = \lim_{N\to\infty} \sup \{ s_n \mid n > N \}$$

$$\liminf_{n\to\infty} s_n = \lim_{N\to\infty} u_N = \lim_{N\to\infty} \inf \{ s_n \mid n > N \}.$$

Theorem (a) $\lim s_n$ exists $\Rightarrow$ $\liminf s_n = \lim s_n = \limsup s_n$

(b) $\liminf s_n = \limsup s_n$ $\Rightarrow$ $\lim s_n$ is defined and $\lim s_n = \liminf s_n = \limsup s_n$.

These results also hold if the limits are $\pm\infty$

Definition A sequence (s_n) is called a Cauchy sequence if for all $\varepsilon > 0$, $\exists N$ such that if $m, n > N$ then $|s_n - s_m| < \varepsilon$.

Theorem Convergent sequences are Cauchy sequences

Proof Suppose $\lim s_n = s$. Then

$$|s_n - s_m| = |s_n - s + s - s_m| \leq |s_n - s| + |s_m - s|$$

using the triangle inequality. Then $\exists N$ such that for $n > N$, $|s_n - s| < \frac{\varepsilon}{2}$. We also have $\forall m > N$ $|s_m - s| < \frac{\varepsilon}{2}$. Thus

$$|s_n - s_m| \leq |s_n - s| + |s_m - s| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Since this works for all $\varepsilon > 0$, (s_n) is a Cauchy sequence.

Theorem Cauchy sequences are bounded

Proof Use the definition with $\varepsilon = 1$ to find an $N \in \mathbb{N}$ s.t.

$$m, n > N \Rightarrow |s_n - s_m| < 1.$$

Thus $|s_n - s_{N+1}| < 1$ for all $n > N$, so $|s_n| < |s_{N+1}| + 1$, for all $n > N$. Define

$$M = \max \{ |s_{N+1}| + 1, |s_1|, |s_2|, \dots, |s_N| \}.$$

Then $|s_n| \leq M$ for all $n \in \mathbb{N}$

Theorem A sequence is a convergent sequence if and only if it is a Cauchy sequence.

Proof We only need to show that Cauchy $\Rightarrow$ convergent. Do it by showing that $\liminf s_n = \limsup s_n$ for a sequence (s_n).

Choose $\varepsilon > 0$. $\exists N$ s.t. $m, n > N \Rightarrow |s_n - s_m| < \varepsilon$.

$$\Rightarrow s_n < s_m + \varepsilon \qquad \forall m, n > N$$
$$\Rightarrow s_m + \varepsilon \text{ is an upper bound for } \{ s_n \mid n > N \}.$$

Hence $v_N = \sup \{ s_n \mid n > N \} \leq s_m + \varepsilon$ for $m > N$.

$v_N - \varepsilon$ is a lower bound for $\{s_m \mid m > N\}$

$$v_N - \varepsilon < \inf \{s_m \mid m > N\} = u_N$$

Thus $\limsup s_n \leq v_N \leq u_N + \varepsilon \leq \liminf s_n + \varepsilon$. Since this is true for arbitrary $\varepsilon > 0$, we have $\limsup s_n \leq \liminf s_n$. Since $\limsup s_n \leq \liminf s_n$, we must have $\liminf s_n = \limsup s_n$, and therefore (s_n) converges.

Subsequences

Suppose that $(s_n)_{n\in\mathbb{N}}$ is a sequence. A subsequence of this sequence has the form $(t_k)_{k\in\mathbb{N}}$ where for each k there is a positive integer n_k such that

$$n_1 < n_2 < \ldots < n_k < n_{k+1} < \ldots$$

and $t_k = s_{n_k}$.

We can also introduce a selection function $\sigma : \mathbb{N} \to \mathbb{N}$ such that $\sigma(k) = n_k$ for $k \in \mathbb{N}$. Thus

$$t_k = t(k) = s \circ \sigma(k) = s(\sigma(k)) = s(n_k) = s_{n_k}.$$

Example Suppose $s_n = n^2(-1)^n$. It has terms

$$(-1, 4, -9, 16, -25, \ldots).$$

The positive terms form a subsequence

$$(4, 16, 36, 64, 100, \ldots)$$

The subsequence is $(s_{n_k})_{k \in \mathbb{N}}$ where $n_k = 2k$. Hence $s_{n_k} = (2k)^2(-1)^{2k} = 4k^2$. The selection function is $\sigma(k) = 2k$.

Proposition Suppose that a sequence (s_n) has $s_n > 0$ for all $n \in \mathbb{N}$, and $\inf\{s_n \mid n \in \mathbb{N}\} = 0$. Then there is a subsequence (s_{n_k}) that has limit 0.

Proof Do it inductively: choose $n_1 = 1$ and define $n_1 < n_2 < n_3 < \ldots$ so that

$$s_{n_{j+1}} < \min\left\{s_{n_j}, \frac{1}{j+1}\right\} \qquad j = 1, 2, \ldots, k-1$$

we know that $\min\{s_n \mid 1 \leq n \leq n_k\} > 0$ and $\inf\{s_n \mid n > n_k\} = 0$. Hence $\exists n_{k+1} > n_k$ such that $s_{n_{k+1}} < \min\left\{s_{n_k}, \frac{1}{k+1}\right\}$.

Since $0 < s_{n_k} < \frac{1}{k}$ for all $k \in \mathbb{N}$ it converges to zero via the sandwich lemma.

Theorem If (s_n) converges, then every subsequence converges to the same limit.

Proof Let (s_{n_k}) be the subsequence. By induction we can show that $n_k \geq k$ for all k.

Now choose $\varepsilon > 0$. Then there exists an N such that $n > N$ implies $|s_n - s| < \varepsilon$ for a limit s. But if $k > N$, then $n_k \geq k > N$, and thus $|s_{n_k} - s| < \varepsilon$.

Theorem Every sequence (s_n) has a monotonic subsequence

Proof Define a term n as dominant if $s_n > s_m$ for all $m > n$.

There are two cases: (1) $\exists$ infinite dominant terms. Define (s_{n_k}) as a subsequence of them. Then $s_{n_{k+1}} < s_{n_k}$ for all k, so it is a decreasing sequence.

(2) $\exists$ only finitely many dominant terms. Choose n_1 beyond all of them. Given $N \geq n_1$, $\exists\, m > N$

such that $s_m \geq s_N$. Choose terms successively to obtain a non-decreasing sequence.

Theorem Let (s_n) be any sequence. Then there exists a monotonic sequence whose limit is $\limsup s_n$, and a monotonic sequence whose limit is $\liminf s_n$.

Theorem (Bolzano-Weierstraß) Every bounded sequence has a convergent subsequence.

Proof By the previous result, the sequence has a monotonic subsequence. Since this is bounded, it must converge.

Definition A subsequential limit is any real number or symbol that is the limit of some subsequence of (s_n).

Example (s_n) $s_n = n^{(-1)^n} = 1, 2, \frac{1}{3}, 4, \frac{1}{5}, 6, \frac{1}{7}, \ldots$

This has subsequential limits $S = \{0, \infty\}$

Theorem Let (s_n) be any sequence in $\mathbb{R}$ and let S denote the set of subsequential limits of (s_n).

Then (i) $\mathcal{S}$ is non-empty

(ii) $\sup \mathcal{S} = \lim\sup s_n$, $\inf \mathcal{S} = \lim\inf s_n$

(iii) $\lim s_n$ exists $\Leftrightarrow$ $\mathcal{S}$ has exactly one element, namely $\lim s_n$.

Proof of (ii) Consider a subsequence with limit t, given by (s_{n_k}). Then

$$t = \lim\inf s_{n_k} = \lim\sup s_{n_k}.$$

In addition, $\{s_{n_k} \mid k > N\} \subseteq \{s_n \mid n > N\}$ and hence

$$\lim\inf s_n \leq \lim\inf s_{n_k} = t = \lim\sup s_{n_k} \leq \lim\sup s_n$$

In addition we know there exist subsequences that tend to $\lim\sup s_n$ and $\lim\inf s_n$. Hence $\inf \mathcal{S} = \lim\inf s_n$ and $\sup \mathcal{S} = \lim\sup s_n$.

Theorem Let $\mathcal{S}$ denote the set of subsequential limits of a sequence (s_n). Suppose (t_n) is a sequence in $\mathcal{S} \cap \mathbb{R}$ and that $t = \lim t_n$. Then t belongs to $\mathcal{S}$

Proof Since a subsequence of (s_n) converges to t_1, $\exists\, n_1$ s.t. $|s_{n_1} - t_1| < 1$. Choose $n_1 < n_2 < \ldots < n_k$

such that

$$|s_{n_j} - t_j| < \frac{1}{j} \quad \text{for } j = 1, \dots, k.$$

Suppose $t \in \mathbb{R}$. Then

$$|s_{n_k} - t| \leq |s_{n_k} - t_k| + |t_k - t| < \frac{1}{k} + |t_k - t|.$$

Consider $\varepsilon > 0$. $\exists N_1$ s.t. $k > N_1 \Rightarrow |t_k - t| < \frac{\varepsilon}{2}$.

$\exists N_2$ s.t. $k > N_2 \Rightarrow \frac{1}{k} < \frac{\varepsilon}{2}$.

Taking $N = \max\{N_1, N_2\}$, then $n > N$ implies

$$|s_{n_k} - t| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$ and hence $\lim s_{n_k} = t$ so $t \in S$.

Suppose $t = \infty$. Choose $M > 0$. $\exists N$ s.t $t_k > M+1$ $\forall k > N$. Then

$$s_{n_k} > M + 1 - \frac{1}{k} \geq M$$ so s_{n_k} diverges and $\infty \in S$

The same idea can be used for $t = -\infty$.

Theorem If (s_n) and (t_n) are sequences such that $\lim s_n = s > 0$, then $\limsup s_n t_n = s(\limsup t_n)$. Allow $s \times \infty = \infty$ and $s \times (-\infty) = -\infty$.

Proof Suppose that $\limsup t_n$ is finite, and equal to β. Then there exists a subsequence t_{n_k} such that

$$\lim_{k\to\infty} t_{n_k} = \beta$$

In addition $\lim_{k\to\infty} s_{n_k} = s$, so $\lim_{k\to\infty} s_{n_k} t_{n_k} = \beta s$.

Therefore $s \limsup t_n = s\beta \leq \limsup s_n t_n$

Note that we can ignore the first few terms of (s_n) and assume $s_n \neq 0$ for all n beyond a certain point. Then $\lim 1/s_n = \frac{1}{s}$. Now follow the same argument as above, but using $\frac{1}{s_n}$:

$$\limsup t_n = \limsup \left(\frac{1}{s_n}\right) s_n t_n$$

$$\geq \left(\frac{1}{s}\right) \limsup s_n t_n$$

And therefore $s \limsup t_n \geq \limsup s_n t_n$. Combining this with (*) shows that

$$s \limsup t_n = \limsup s_n t_n.$$

Theorem Let (s_n) be any sequence of non-zero numbers. Then $\liminf \left|\frac{s_{n+1}}{s_n}\right| \leq \liminf |s_n|^{1/n} \leq \limsup |s_n|^{1/n} \leq \limsup \left|\frac{s_{n+1}}{s_n}\right|$

Proof of third inequality

Define $\alpha = \limsup |s_n|^{1/n}$ and $L = \limsup \left|\frac{s_{n+1}}{s_n}\right|$.

Our goal is to show that $\alpha \leq L$. It suffices to show that $\alpha \leq L_1 \ \forall L_1 > L$.

Since $L = \limsup \left|\frac{s_{n+1}}{s_n}\right|$

$$= \lim_{N \to \infty} \sup \left\{ \left|\frac{s_{n+1}}{s_n}\right| : n > N \right\} < L_1$$

there exists an $N \in \mathbb{N}$ such that

$$\sup \left\{ \left|\frac{s_{n+1}}{s_n}\right| : n > N \right\} < L_1$$

and therefore

$$\left|\frac{s_{n+1}}{s_n}\right| < L_1 \quad \text{for} \quad n > N.$$

For $n > N$,

$$|s_n| = \underbrace{\left|\frac{s_n}{s_{n-1}}\right| \cdot \left|\frac{s_{n-1}}{s_{n-2}}\right| \cdots \left|\frac{s_{N+1}}{s_N}\right|}_{n-N \text{ fractions}} |s_N|$$

$$< L_1^{\,n-N} |s_N| \qquad \text{for} \quad n > N.$$

L_1 and N are fixed in this expression, so we can define $a = L_1^{-N}|s_N|$ as a positive constant. Hence

$$|s_n| < L_1^n a \qquad \text{for} \quad n > N,$$

$$|s_n|^{1/n} < L_1 a^{1/n} \qquad \text{for} \quad n > N.$$

We have $\lim_{n \to \infty} a^{1/n} = 1$. Hence

$$\kappa = \lim \sup |s_n|^{1/n} \leq L_1$$

and hence $\kappa \leq L$.

Corollary If $\lim_{n\to\infty} \left|\frac{s_{n+1}}{s_n}\right|$ exists and equals L, then $\lim_{n\to\infty} |s_n|^{1/n}$ exists and equals L.

Proof If $\lim_{n\to\infty} \left|\frac{s_{n+1}}{s_n}\right| = L$, then all four terms in the previous inequality must be equal to L.