Physics 415 - Lecture 33: Degenerate Fermi Gas

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Summary

- Fermi Gas (FG): $\overline{n}(\epsilon) = \frac{1}{e^{\beta(\epsilon-\mu)}+1}$ (Fermi function).
- Density of States (3D free particles, spin J, degeneracy $g = 2J+1$):

$$\rho(\epsilon) = \frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\sqrt{\epsilon}$$

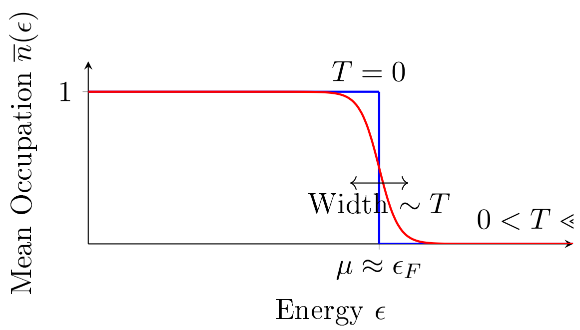
  Total number of particles $N = g\int_0^\infty d\epsilon \rho(\epsilon)\overline{n}(\epsilon)$. Grand Potential $\Phi = -gT\int_0^\infty d\epsilon \rho(\epsilon)\ln(1+e^{-\beta(\epsilon-\mu)})$.

- T=0: $\overline{n}(\epsilon) = \Theta(\epsilon_F - \epsilon)$ (step function). Fermi energy $\epsilon_F = \mu(T=0)$.

$$\epsilon_F = \frac{\hbar^2}{2m}\left(\frac{6\pi^2 n}{g}\right)^{2/3} \quad (n = N/V)$$

  Ground state energy $E_0 = \frac{3}{5}N\epsilon_F$. Ground state pressure $p_0 = \frac{2}{5}n\epsilon_F$.

- T>0: Define Fermi Temperature $T_F = \epsilon_F$ (using $k_B = 1$). Regime $0 < T \ll T_F$ is the "degenerate" Fermi gas. Only particles within $\sim T$ of ϵ_F participate in thermal excitation. Effective number $N_{eff} \sim N(T/T_F)$. Qualitative estimate for heat capacity: $E \approx E_0 + N_{eff}T \sim E_0 + NT^2/T_F \implies C_V = (\partial E/\partial T)_V \sim NT/T_F \propto T$.



Comment on Specific Heat of Metals: Total $C_V = C_V^{(el)} + C_V^{(latt)}$. Conduction electrons (FG): $C_V^{(el)} = \gamma_{el}T$. Lattice vibrations (phonons): $C_V^{(latt)} = AT^3$ (Debye T^3 law, derived later). $C_V = \gamma_{el}T + AT^3$. Experimentally verified by plotting C_V/T vs T^2: $C_V/T = \gamma_{el} + AT^2$. Should be linear. Holds well for many metals (see Reif Fig. 9.16.4).

Quantitative Analysis for $0 < T \ll T_F$

We need to evaluate integrals of the form $I = \int_0^\infty f(\epsilon)\overline{n}(\epsilon)d\epsilon$ where $f(\epsilon) \sim \rho(\epsilon)$ for N and $f(\epsilon) \sim \epsilon\rho(\epsilon)$ for E.

$$I = \int_0^\infty d\epsilon \frac{f(\epsilon)}{e^{\beta(\epsilon-\mu)} + 1}$$

We need to evaluate this integral in the regime $T \ll \epsilon_F$ (or $\beta\mu \gg 1$), where the Fermi function changes rapidly near $\epsilon = \mu \approx \epsilon_F$.

Sommerfeld Expansion

Introduce $\phi(\epsilon) = \int_0^\epsilon f(\epsilon')d\epsilon'$, so $f(\epsilon) = \phi'(\epsilon)$. Integrate I by parts: $u = \overline{n}(\epsilon)$, $dv = f(\epsilon)d\epsilon$. $du = (\partial\overline{n}/\partial\epsilon)d\epsilon$, $v = \phi(\epsilon)$.

$$I = [\overline{n}(\epsilon)\phi(\epsilon)]_0^\infty - \int_0^\infty \phi(\epsilon)\left(\frac{\partial\overline{n}}{\partial\epsilon}\right)d\epsilon$$

Assume $\phi(0) = 0$. $\overline{n}(\infty) = 0$. Boundary terms vanish.

$$I = -\int_0^\infty \phi(\epsilon)\left(\frac{\partial\overline{n}}{\partial\epsilon}\right)d\epsilon$$

The derivative $-\partial\overline{n}/\partial\epsilon = -\frac{\partial}{\partial\epsilon}[e^{\beta(\epsilon-\mu)} + 1]^{-1} = -(-1)[e^{\beta(\dots)} + 1]^{-2}(e^{\beta(\dots)})(\beta) = \beta\frac{e^{\beta(\epsilon-\mu)}}{(e^{\beta(\epsilon-\mu)}+1)^2}$. This function $(-\partial\overline{n}/\partial\epsilon)$ is sharply peaked around $\epsilon = \mu$ with width $\sim T$, and looks like a broadened negative delta function as $T \to 0$.

Expand $\phi(\epsilon)$ in a Taylor series around $\epsilon = \mu$ (since the peak is narrow):

$$\phi(\epsilon) \approx \phi(\mu) + (\epsilon-\mu)\phi'(\mu) + \frac{1}{2}(\epsilon-\mu)^2\phi''(\mu) + \dots$$

Substitute into the integral for I:

$$I \approx \int_0^\infty [\phi(\mu) + (\epsilon-\mu)\phi'(\mu) + \frac{1}{2}(\epsilon-\mu)^2\phi''(\mu) + \dots]\left(-\frac{\partial\overline{n}}{\partial\epsilon}\right)d\epsilon$$

Since $(-\partial\overline{n}/\partial\epsilon)$ is sharply peaked near $\mu \gg T$, we can extend the lower limit to $-\infty$ with negligible error.

$$I \approx \phi(\mu)\underbrace{\int_{-\infty}^\infty \left(-\frac{\partial\overline{n}}{\partial\epsilon}\right)d\epsilon}_{=1} + \phi'(\mu)\underbrace{\int_{-\infty}^\infty (\epsilon-\mu)\left(-\frac{\partial\overline{n}}{\partial\epsilon}\right)d\epsilon}_{=0 \text{ (odd integrand)}} + \frac{1}{2}\phi''(\mu)\underbrace{\int_{-\infty}^\infty (\epsilon-\mu)^2\left(-\frac{\partial\overline{n}}{\partial\epsilon}\right)d\epsilon}_{=\pi^2T^2/3} + \dots$$

The integrals can be evaluated. $\int_{-\infty}^\infty(-\partial\overline{n}/\partial\epsilon)d\epsilon = [-\overline{n}]_{-\infty}^\infty = -(0-1) = 1$. The second integral vanishes because the integrand is odd around $\epsilon = \mu$. The third integral can be shown to be $\int_{-\infty}^\infty(\epsilon-\mu)^2(-\partial\overline{n}/\partial\epsilon)d\epsilon = \frac{\pi^2}{3}T^2$. Thus,

$$I = \int_0^\infty f(\epsilon)\overline{n}(\epsilon)d\epsilon \approx \phi(\mu) + \frac{\pi^2}{6}T^2\phi''(\mu) + O(T^4)$$

Since $\phi(\mu) = \int_0^\mu f(\epsilon)d\epsilon$ and $\phi''(\mu) = f'(\mu)$:

$$I \approx \int_0^\mu f(\epsilon)d\epsilon + \frac{\pi^2}{6}T^2 f'(\mu)$$

This is the Sommerfeld expansion (lowest order correction in T^2).

Chemical Potential $\mu(T)$

Apply Sommerfeld expansion to the integral for N: $f(\epsilon) = g\rho(\epsilon)$.

$$N = \int_0^\infty g\rho(\epsilon)\overline{n}(\epsilon)d\epsilon \approx g\int_0^\mu \rho(\epsilon)d\epsilon + \frac{\pi^2}{6}T^2 g\rho'(\mu)$$

At $T = 0$, $\mu = \epsilon_F$ and $N = g\int_0^{\epsilon_F} \rho(\epsilon)d\epsilon$. For $T > 0$, expand the first term around ϵ_F:

$$g\int_0^\mu \rho(\epsilon)d\epsilon = g\int_0^{\epsilon_F} \rho(\epsilon)d\epsilon + g\int_{\epsilon_F}^\mu \rho(\epsilon)d\epsilon \approx N + g(\mu - \epsilon_F)\rho(\epsilon_F)$$

Substitute this into the expression for N:

$$N \approx N + g\rho(\epsilon_F)(\mu - \epsilon_F) + \frac{\pi^2}{6}T^2 g\rho'(\mu)$$

(We approximate $\rho'(\mu) \approx \rho'(\epsilon_F)$ in the small T^2 term).

$$0 \approx g\rho(\epsilon_F)(\mu - \epsilon_F) + \frac{\pi^2}{6}T^2 g\rho'(\epsilon_F)$$

Let $\delta\mu = \mu - \epsilon_F$.

$$\delta\mu \approx -\frac{\pi^2}{6}T^2 \frac{\rho'(\epsilon_F)}{\rho(\epsilon_F)}$$

Since $\rho(\epsilon) = AV\sqrt{\epsilon}$, $\rho'(\epsilon) = \frac{1}{2}AV\epsilon^{-1/2} = \rho(\epsilon)/(2\epsilon)$. $\rho'(\epsilon_F)/\rho(\epsilon_F) = 1/(2\epsilon_F)$.

$$\delta\mu \approx -\frac{\pi^2}{6}T^2 \frac{1}{2\epsilon_F} = -\frac{\pi^2}{12}\frac{T^2}{\epsilon_F}$$

So the chemical potential decreases slightly from ϵ_F as T increases:

$$\mu(T) \approx \epsilon_F \left[1 - \frac{\pi^2}{12}\left(\frac{T}{\epsilon_F}\right)^2\right]$$

Since $T \ll \epsilon_F$, the correction is small, $\delta\mu \ll \epsilon_F$.

Internal Energy $E(T)$

Apply Sommerfeld expansion to $E = \int_0^\infty g\epsilon\rho(\epsilon)\overline{n}(\epsilon)d\epsilon$. Here $f(\epsilon) = g\epsilon\rho(\epsilon)$.

$$E \approx \int_0^\mu g\epsilon\rho(\epsilon)d\epsilon + \frac{\pi^2}{6}T^2 \left.\left[\frac{\mathrm{d}}{\mathrm{d}\epsilon}(g\epsilon\rho(\epsilon))\right]\right|_{\epsilon=\mu}$$

Expand the first term around ϵ_F:

$$\int_0^\mu g\epsilon\rho(\epsilon)d\epsilon \approx \int_0^{\epsilon_F} g\epsilon\rho(\epsilon)d\epsilon + (\mu - \epsilon_F)[g\epsilon_F\rho(\epsilon_F)] = E_0 + g\delta\mu\epsilon_F\rho(\epsilon_F)$$

Evaluate the derivative in the second term at $\mu \approx \epsilon_F$: $f'(\epsilon) = \frac{\mathrm{d}}{\mathrm{d}\epsilon}(g\epsilon\rho(\epsilon)) = g(\rho(\epsilon) + \epsilon\rho'(\epsilon))$. $f'(\epsilon_F) = g(\rho(\epsilon_F) + \epsilon_F\rho'(\epsilon_F))$.

$$E \approx E_0 + g\delta\mu\epsilon_F\rho(\epsilon_F) + \frac{\pi^2}{6}T^2 g[\rho(\epsilon_F) + \epsilon_F\rho'(\epsilon_F)]$$

Substitute $\delta\mu \approx -\frac{\pi^2}{6}T^2\frac{\rho'(\epsilon_F)}{\rho(\epsilon_F)}$:

$$E \approx E_0 + g\left(-\frac{\pi^2}{6}T^2\frac{\rho'(\epsilon_F)}{\rho(\epsilon_F)}\right)\epsilon_F\rho(\epsilon_F) + \frac{\pi^2}{6}T^2 g\rho(\epsilon_F) + \frac{\pi^2}{6}T^2 g\epsilon_F\rho'(\epsilon_F)$$

$$E \approx E_0 - \frac{\pi^2}{6}T^2 g\epsilon_F \rho'(\epsilon_F) + \frac{\pi^2}{6}T^2 g\rho(\epsilon_F) + \frac{\pi^2}{6}T^2 g\epsilon_F \rho'(\epsilon_F)$$

The terms with $\rho'(\epsilon_F)$ cancel.

$$E(T) \approx E_0 + \frac{\pi^2}{6} g\rho(\epsilon_F) T^2$$

Recall $E_0 = \frac{3}{5}N\epsilon_F$ and $\rho(\epsilon_F) = \frac{3N}{2g\epsilon_F}$.

$$E(T) \approx \frac{3}{5}N\epsilon_F + \frac{\pi^2}{6} g \left(\frac{3N}{2g\epsilon_F}\right) T^2 = \frac{3}{5}N\epsilon_F + \frac{\pi^2}{4} N \frac{T^2}{\epsilon_F}$$

$$E(T) \approx \frac{3}{5}N\epsilon_F \left[1 + \frac{5\pi^2}{12}\left(\frac{T}{\epsilon_F}\right)^2\right]$$

(Note: Source has $5\pi^2/12$ factor).

Heat Capacity C_V

$$C_V = \left(\frac{\partial E}{\partial T}\right)_V \approx \frac{\partial}{\partial T}\left(E_0 + \frac{\pi^2}{6} g\rho(\epsilon_F) T^2\right)$$

$$C_V = \frac{\pi^2}{6} g\rho(\epsilon_F)(2T) = \frac{\pi^2}{3} g\rho(\epsilon_F) T$$

Substitute $\rho(\epsilon_F) = 3N/(2g\epsilon_F)$:

$$C_V = \frac{\pi^2}{3} g \left(\frac{3N}{2g\epsilon_F}\right) T = \frac{\pi^2}{2} N \left(\frac{T}{\epsilon_F}\right)$$

This confirms the specific heat is indeed linear in T for $T \ll \epsilon_F$, as argued qualitatively before.

Appendix: Proof of Integral Result

Prove that $I_1 = \int_0^\infty dx \frac{x}{e^x+1} = \frac{\pi^2}{12}$. Use geometric series for $1/(1+e^{-x}) = \sum_{n=0}^\infty (-1)^n e^{-nx}$.

$$\frac{1}{e^x+1} = \frac{e^{-x}}{1+e^{-x}} = e^{-x}\sum_{n=0}^\infty (-1)^n e^{-nx} = \sum_{n=0}^\infty (-1)^n e^{-(n+1)x}$$

$$I_1 = \int_0^\infty dx\, x \sum_{n=0}^\infty (-1)^n e^{-(n+1)x}$$

Swap sum and integral (assume convergence):

$$I_1 = \sum_{n=0}^\infty (-1)^n \int_0^\infty dx\, x e^{-(n+1)x}$$

Let $y = (n+1)x$, $x = y/(n+1)$, $dx = dy/(n+1)$.

$$\int_0^\infty x e^{-(n+1)x} dx = \int_0^\infty \frac{y}{n+1} e^{-y} \frac{dy}{n+1} = \frac{1}{(n+1)^2}\int_0^\infty y e^{-y} dy$$

The integral is $\Gamma(2) = 1! = 1$.

$$I_1 = \sum_{n=0}^\infty \frac{(-1)^n}{(n+1)^2}$$

Let $k = n + 1$. k runs from 1 to ∞. $n = k - 1$, $(-1)^n = (-1)^{k-1}$.

$$I_1 = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^2} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots$$

This is the alternating sum $\eta(2)$. We know the Riemann zeta function $\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$. $\eta(s) = (1-2^{1-s})\zeta(s)$. For $s = 2$: $\eta(2) = (1-2^{-1})\zeta(2) = (1-1/2)(\pi^2/6) = (1/2)(\pi^2/6) = \pi^2/12$. ✓

Euler's Proof for $\zeta(2) = \pi^2/6$: Consider the Taylor expansion of $\sin x$: $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \implies \frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \dots \quad (*)$ Alternatively, view $\sin x$ as an infinite polynomial with roots at $x = n\pi$ ($n = 0, \pm 1, \pm 2, \dots$). We can write $\sin x$ as a product over its roots (like $P(x) = c(x - r_1)(x - r_2)\dots$). For $\sin x$, normalization requires $\sin x/x \to 1$ as $x \to 0$.

$$\sin x = Cx \prod_{n=1}^{\infty}(x - n\pi) \prod_{n=1}^{\infty}(x + n\pi) = Cx \prod_{n=1}^{\infty}(x^2 - n^2\pi^2)$$

$$\frac{\sin x}{x} = C \prod_{n=1}^{\infty}(x^2 - n^2\pi^2)$$

To match $\sin x/x \to 1$ as $x \to 0$, we need $C \prod(-\pi^2 n^2) = 1$? No. Factor out constants:

$$\frac{\sin x}{x} = C' \prod_{n=1}^{\infty}(1 - \frac{x^2}{n^2\pi^2})$$

As $x \to 0$, the product goes to 1. So $C' = 1$.

$$\frac{\sin x}{x} = \left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{4\pi^2}\right)\left(1 - \frac{x^2}{9\pi^2}\right)\dots$$

Expand this product and compare the coefficient of x^2 with the Taylor series $(*)$. Coefficient of x^2 in product: $(-\frac{1}{\pi^2}) + (-\frac{1}{4\pi^2}) + (-\frac{1}{9\pi^2}) + \dots = -\frac{1}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}$. Coefficient of x^2 in Taylor series: $-1/3! = -1/6$. Equating coefficients:

$$-\frac{1}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2} = -\frac{1}{6}$$

$$\implies \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$$