Notes on Physics 531:
Intro Quantum Mechanics

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1 Algebra on Hilbert space $\mathcal{H}$

1.1 Braket Notation and properties

1.1.1 Kets and Bras

- **Ket $|\psi\rangle$:** represents a quantum state vector in a Hilbert space $\mathcal{H}$. It's a column vetor in Dirac notation:

$$|\psi\rangle = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_N \end{pmatrix} = c_1|\varphi_1\rangle + c_2|\varphi_2\rangle + \ldots + c_N|\varphi_N\rangle, \tag{1}$$

with $|\varphi_i\rangle$ are basis vectors in the Hilbert space.

- **Bra $\langle\varphi|$**: represents a linear functional that maps kets to complex numbers. It's a row vector, the conjugate transpose of the corresponding ket:

$$\langle\varphi| = |\varphi\rangle^\dagger = \begin{pmatrix} c_1^* & c_2^* & \ldots & c_N^* \end{pmatrix}. \tag{2}$$

1.1.2 Inner Product

- **Def:** The inner product of a bra $\langle\varphi|$, ket $|\psi\rangle$ is denoted as $\langle\varphi|\psi\rangle$. This results in a complex number. It represents the projection of state $|\psi\rangle$ onto state φ .

$$\langle\varphi|\psi\rangle = \begin{pmatrix} \varphi_1^* & \varphi_2^* & \ldots & \varphi_N^* \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_N \end{pmatrix} = \sum_{i=1}^{N} \varphi_i^* \psi_i = c_1\varphi_1^* + c_2\varphi_2^* + \ldots + c_N\varphi_N^*. \tag{3}$$

1.1.2.1 Properties of the inner product

- Conjugate symmetry (Hermitian property)

$$\langle\varphi|\psi\rangle = \langle\psi|\varphi\rangle^*. \tag{4}$$

- Linearity in the second argument:

$$\langle\varphi|a\psi_1 + b\psi_2\rangle = a\langle\varphi|\psi_1\rangle + b\langle\varphi|\psi_2\rangle. \tag{5}$$

- Anti-linearity in the first argument:

$$\langle a\varphi_1 + b\varphi_2|\psi\rangle = a^*\langle\varphi_1|\psi\rangle + b^*\langle\varphi_2|\psi\rangle. \tag{6}$$

- Positive-definiteness: The inner product of a state with itself is a non-negative real number, and it is zero if and only if the state is the zero vector.

$$\langle\psi|\psi\rangle \geq 0, \qquad \langle\psi|\psi\rangle = 0 \Leftrightarrow |\psi\rangle = 0. \tag{7}$$

- For orthonormal basis states $|i\rangle, |j\rangle$,

$$\langle i|j\rangle = \delta_{ij}. \tag{8}$$

1.2 Operators and Operations

1.2.1 Outer product

- Def: The outer product of a ket $|\psi\rangle$ and a bra $\langle\varphi|$ is denoted as $|\psi\rangle\langle\varphi|$. This results in a **linear operator**.

$$|\psi\rangle\langle\varphi| = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_N \end{pmatrix} \begin{pmatrix} \varphi_1^* & \varphi_2^* & \dots & \varphi_N^* \end{pmatrix} = \begin{pmatrix} \psi_1\varphi_1^* & \psi_1\varphi_2^* & \dots & \psi_1\varphi_N^* \\ \psi_2\varphi_1^* & \psi_2\varphi_2^* & \dots & \psi_2\varphi_N^* \\ \vdots & & & \\ \psi_N\varphi_1^* & \psi_N\varphi_2^* & \dots & \psi_N\varphi_N^* \end{pmatrix}. \tag{9}$$

- The outer product operator $\langle\psi|\varphi\rangle$ acts on a ket $|\chi\rangle$ to give another ket:

$$(\langle\psi|\varphi\rangle)\,|\chi\rangle = |\psi\rangle\,(\langle\varphi|\chi\rangle) = \langle\varphi|\chi\rangle\,|\psi\rangle, \tag{10}$$

  where $\langle\varphi|\chi\rangle$ is a complex number, and it scales the state $|\psi\rangle$.

1.2.2 Norm and Normalization

- **Norm** of a state $|\varphi\rangle$ is defined as $\|\psi\| = \sqrt{\langle\psi|\psi\rangle}$. It's a real, non-negative number representing the length of the state vector.
- **Normalized State:** a state $|\psi\rangle$ is normalized if its norm is 1, i.e., $\|\psi\| = \sqrt{\langle\psi|\psi\rangle} = 1$, which means $\langle\psi|\psi\rangle = 1$. Quantum states are generally normalized to represent probabilites correctly.
- **Normalization Process**: If $|\psi\rangle$ is not normalized and $\|\psi\| \neq 0$,we can normalize it to obtain a new state $|\psi'\rangle$ by dividing by its norm:

$$|\psi'\rangle = \frac{|\psi\rangle}{\|\psi\|} = \frac{|\psi\rangle}{\sqrt{\langle\psi|\psi\rangle}}. \tag{11}$$

  It follows that $\langle\psi'|\psi'\rangle = \left\langle \frac{\psi}{\|\psi\|} \middle| \frac{\psi}{\|\psi\|} \right\rangle) = \frac{1}{(\|\psi\|)^2}\langle\psi|\psi\rangle = \frac{1}{\langle\psi|\psi\rangle}\langle\psi|\psi\rangle = 1.$

1.2.3 Linearity

- **Linearity of Ket space:** The set of all possible kets forms a vector space, meaning that if $|\psi\rangle \wedge |\varphi\rangle$ are kets, and a, b are complex numbers, then $a|\psi\rangle + b|\varphi\rangle$ is also a valid ket.
- **Linearity of a Bra space:** Similarly, bras also form a vector pace.
- **Linear Operators:** Operator A in quantum mechanics are liear operators, meaning they satisfy

$$A\,(a|\psi\rangle + b|\varphi\rangle) = a(A|\psi\rangle) + b(A|\varphi\rangle) \tag{12}$$

In braket nottaion, applying a operator A o a ket $|\psi\rangle$ results in a new ket $A|\psi\rangle$. Similiarly, applying it to a bra $\langle\varphi|$ results in a new bra $\langle\varphi|A^\dagger$, where $A^\dagger$ is the agjoint of A. .

1.2.4 Tensor Product of states and spaces

- **Composite systems**: For a system composed of two subsystems with states $|\psi\rangle_A$ in space $\mathcal{H}_A$ and $|\varphi\rangle_B$ in space $\mathcal{H}_B$, the combined sysetm's staet is described by the tensor product $|\psi\rangle_A \otimes |\varphi\rangle_B$ or simply $|\psi\rangle_A|\varphi\rangle_B$ or $|\psi, \varphi\rangle$.
- **Tensor product of kets:** If $|\psi\rangle = \sum_i a_i|i\rangle$ and $|\varphi\rangle = \sum_i b_j|j\rangle$ then :

$$|\psi\rangle \otimes |\varphi\rangle = \left(\sum_i a_i|i\rangle\right) \otimes \left(\sum_j b_j|j\rangle\right) = \sum_{i,j} a_i b_j(|i\rangle \otimes |j\rangle) \equiv \boxed{\sum_{i,j} a_i b_j|i,j\rangle}. \tag{13}$$

- **Tensor product of Bras:** Similarly, for bras $\langle\psi|_A$ and $\langle\varphi|_B$, the combined bra is $\langle\psi|_A \otimes \langle\varphi|_B = \langle\psi, \varphi|$
- **Inner product in Tensor product space**

$$(\langle\psi|_A \otimes \langle\varphi|_B)(|\chi\rangle_A \otimes |\omega\rangle_B) = \langle\psi|_A|\chi\rangle_A \cdot \langle\varphi|_B|\omega\rangle_B = \langle\psi|\chi\rangle\langle\varphi|\omega\rangle. \tag{14}$$

1.2.5 Observables and Expectation Value

- **Observable**: A physical observable is represented by a Hermitian operator O (i.e. $O^\dagger = O$).

- **Expectation Value:**The expectation value of an observable O in a state $|\psi\rangle$ is given by

$$\langle O\rangle_{\psi} = \langle\psi|O|\psi\rangle. \tag{15}$$

  This is the average value we expect to obtain if we measure the observable O when the system is in state $|\psi\rangle$.

- **Calculation of Expectation Value**: If $O = \sum_i \lambda_i |i\rangle\langle i|$ is the spectral decomposition of O , then

$$\langle\psi|O|\psi\rangle = \langle\psi|\left(\sum_i \lambda_i |i\rangle\langle i|\right)|\psi\rangle = \sum_i \lambda_i\ |\langle i|\psi\rangle|^2. \tag{16}$$

  Here, λ_i are the eigenvalues and $|i\rangle$ are the corresponding eigenvectors of O. $\left\|\langle i|\psi\rangle^2\right\|$ is the probability of obtaining the eigenvalue λ_i when measuring O in state $|\psi\rangle$.

1.2.6 Identity Operator

- **Completeness relation** For a complete orthonormal basis $\{|i\rangle\}$ of the Hilbert space, the identity operator can be written as

$$I = \sum_i |i\rangle\langle i|. \tag{17}$$

  This is known as the completeness relation or closure relation.

- **Action of Identity Operator**: For any state $|\psi\rangle$, $I|\psi\rangle = |\psi\rangle$.

$$I|\psi\rangle = \left(\sum_i |i\rangle\langle i|\right)|\psi\rangle = \sum_i |i\rangle\ (\langle i|\psi\rangle) = \sum_i \langle i|\psi\rangle|i\rangle = |\psi\rangle \tag{18}$$

  This shows that the identity operator leaves any state unchanged.

1.2.7 Projection Operators

- **Def**: A projection operator P_ψ onto a normalized state $|\psi\rangle$ is given by the outer product

$$P_\psi = |\psi\rangle\langle\psi|. \tag{19}$$

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- **Properties of Projection Operators**
  - Hermitian: $P_\psi^\dagger = P_\psi$
  - Idempotent: $P_\psi^2 = P_\psi$
  - Projects onto $|\psi\rangle$: for any state $|\varphi\rangle$, $P_\psi|\varphi\rangle = \langle\psi|\varphi\rangle\ |\psi\rangle$. This is the projection of $|\varphi\rangle$ onto the direction of $|\psi\rangle$.

- **Projection onto a subspace:** If we have an orthonormal basis $\{|\psi_i\rangle\}_{i=1}^k$ for a subspace, the projection operator onto this subspace is

$$P_{\text{subspace}} = \sum_{i=1}^k |\psi_i\rangle\langle\psi_i| \tag{20}$$

  In this lense we can see the completeness relation as the sum of projection operators onto each basis state $|i\rangle$ of a complete orthonormal basis $\{|i\rangle\}$. This ensures that any vector in the space can be represented as a linear combiniation of these basis vectors.

1.3 Matrix Algebra

- Important note: Multiplicative Commutation is generally not valid! Most other algebra rules hold.

1.3.1 The commutator and the anticommutator

- **Commutator**: For any two operators A and B, the commutator is defined as:

$$[A, B] = AB - BA. \tag{21}$$

  - **Useful properties:**
    - **Antisymmetry**: $[A, B] = -[B, A]$.
    - **Bilinearity**: $[aA + bB, C] = a[A, C] + b[B, C]$ and similarly $[C, aA + bB] = a[C, A] + b[C, B]$.
    - **Jacobi Identity**: $[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0$.

- **Anticommutator**: For any two operators A and B, the anticommutator is defined as:

$$\{A, B\} = AB + BA. \tag{22}$$

  - **Useful properties:**
    - **Symmetry**: $\{A, B\} = \{B, A\}$.
    - **Combination with the commutator**: In many contexts, operators are expressed in terms of both the commutator and the anticommutator to capture distinct symmetry properties.

1.3.2 Transpose, Adjoint; Hermitian and Unitary.

- **Transpose**: The transpose of an operator A, denoted as A^T, is obtained by swapping its rows and columns. For a matrix representation of A with elements A_{ij}, the transpose A^T has elements A_{ji}.

  - **Properties of Transpose**:
    - Involution: $\left(A^T\right)^T = A$
    - Linearity: $(A + B)^T = A^T + B^T$
    - Scalar Multiplication: $(cA)^T = cA^T$ for any scalar c
    - Product reversal: $(AB)^T = B^T A^T$

- **Adjoint**: The adjoint (or Hermitian conjugate) of an operator A, denoted as $A^\dagger$, is the complex conjugate transpose of A. For a matrix representation of A with elements A_{ij}, the adjoint $A^\dagger$ has elements $\left(A_{ji}\right)^*$.

  - **Properties of Adjoint**:
    - Involution: $\left(A^\dagger\right)^\dagger = A$
    - Linearity: $(A + B)^\dagger = A^\dagger + B^\dagger$
    - Scalar Multiplication: $(cA)^\dagger = c^* A^\dagger$ for any scalar c
    - Scalar Multiplication: $(AB)^\dagger = B^\dagger A^\dagger$
    - If A is **Hermitian**, then $A = A^\dagger$
    - If A is **unitary**, then $A^\dagger A = AA^\dagger = I$