Solutions to sample midterm questions

1. Let $x = 1 + \sqrt{1+\sqrt{2}}$. Then

$$\begin{aligned} x - 1 &= \sqrt{1+\sqrt{2}}, \\ (x-1)^2 &= 1 + \sqrt{2}, \\ x^2 - 2x + 1 &= 1 + \sqrt{2}, \\ x^2 - 2x &= \sqrt{2}, \\ x^4 - 4x^3 + 4x^2 &= 2, \\ x^4 - 4x^3 + 4x^2 - 2 &= 0. \end{aligned}$$

   Hence if $x = p/q$, then p divides 2 and q divides 1. The only possibilities are ± 1, and ± 2. But $\sqrt{1+\sqrt{2}} > 1$, and thus $x > 2$. Thus x must be irrational.

2. Define $a_n = 8^n/(n!)^2$. Then

$$\begin{aligned} \frac{a_{n+1}}{a_n} &= \frac{8^{n+1}}{((n+1)!)^2}\frac{(n!)^2}{8^n} \\ &= \frac{8}{(n+1)^2} \to 0 \end{aligned}$$

   and therefore by the ratio test, $\sum 8^n/(n!)^2$ converges.

   Now consider $\sum(-1)^n b_n$ where $b_n = 1/\sqrt{n^2+n}$. Since n and n^2 are both increasing functions, $n^2 + n$ is an increasing function also, and hence $1/\sqrt{n^2+n}$ is a decreasing function. In addition, $b_n < 1/n$ for all n, so $b_n \to 0$ as $n \to \infty$. Therefore, the series $\sum(-1)^n b_n$ satisfies the conditions for the alternating series theorem, and hence it converges.

   For the third series, make use of the root test where $a_n = 6^n/n^n$. Then

$$(a_n)^{1/n} = \frac{6}{n}, \tag{1}$$

   which converges to zero as $n \to \infty$. Hence $\sum 6^n/n^n$ converges. For the fourth series, since $n + 1/2 \leq 2n$ for all $n \in \mathbb{N}$, then

$$\frac{1}{n + 1/2} \geq \frac{1}{2n} \tag{2}$$

   for all $n \in \mathbb{N}$. Since $\sum \frac{1}{n}$ diverges, so does $\sum \frac{1}{2n}$, and hence by the comparison test, $\sum 1/(n + 1/2)$ does also.

3. (a) Let $x \in S \cup T$. Then either $x \in S$ so $x \leq \sup S$, or $x \in T$ so $x \leq \sup T$. Hence, $x \leq \max\{\sup S, \sup T\}$. Thus $\max\{\sup S, \sup T\}$ is an upper bound for $\sup S \cup T$.

   Now suppose that m is an upper bound for $S \cup T$. Hence $m \geq x$ for all $x \in S \cup T$. Thus $m \geq s$ for all $s \in S$, so $m \geq \sup S$ as $\sup S$ is the least upper bound for S. Similarly $m \geq t$ for all $t \in T$. Hence $m \geq \sup T$ as $\sup T$ is the least upper bound of T. Therefore $m \geq \max\{\sup S, \sup T\}$. Hence $\max\{\sup S, \sup T\}$ is an upper bound, and it is the least upper bound, so it must equal $\sup S \cup T$.

   Now consider $x \in S \cap T$. Hence $x \in S$ and $x \in T$. Then $x \leq \sup S$ and $x \leq \sup T$, so $x \leq \min\{\sup S, \sup T\}$, and therefore $\sup S \cap T \leq \min\{\sup S, \sup T\}$.

   (b) For a non-empty set A, $\sup A \neq -\infty$, so it suffices to consider when the suprema become positive infinity. Suppose $\sup S = \infty$. Then S is not bounded above. Hence $S \cup T$ is not bounded above. Therefore $\sup S \cup T = \infty$ and the identity still holds.

   For the second identity, if $\sup S = \infty$, then $\min\{\sup S, \sup T\} = \sup T$. Since $\sup T$ is an upper bound for T, it is also an upper bound for $S \cap T$, and hence the identity still holds.

   The same arguments can be applied if $\sup T = \infty$.

   (c) Consider $S = \{1, 3\}$ and $T = \{1, 2\}$. Then $\sup S = 3$ and $\sup T = 2$, so $\min\{\sup S, \sup T\} = 2$. However, $S \cap T = \{1\}$ and so $\sup S \cap T = 1 < 2$.

4. Let $\lim s_n = s$. Since s_n converges, there exists an N_1 such that $n > N_1$ implies that $|s_n - s| < 1$. Hence $-1 < s_n - s$ and $s_n > s - 1$.

   Now pick $M > 0$. Since t_n diverges, there exists an N_2 such that

$$t_n > 1 - s + M \tag{3}$$

   for all $n > N_2$. Hence for $n > \max\{N_1, N_2\}$,

$$s_n + t_n > (s - 1) + 1 - s + M = M \tag{4}$$

   and thus $s_n + t_n$ diverges to infinity.

5. Since the lower limit of A is an open interval, it does not have a minimum, however $\inf A = 0$. Since A is not bounded above, it does not have a maximum. $\sup A = \infty$ for sets not bounded above.

   Since B has no smallest element, the minimum does not exist. However, since the fractions become arbitrarily close to 0, $\inf B = 0$. The maximum is given by $\max B = 1/2$, attained for the case when $n = 1$, and hence $\sup B = \max B = 1/2$.

6. For three values 0, 1, and 2,

$$d_1(0, 1) + d_1(1, 2) = 1^4 + 1^4 = 2 \tag{5}$$

but

$$d_1(0,2) = 2^4 = 16 \tag{6}$$

and hence the triangle inequality is violated, so d_1 is not a metric.

Since $d_2(0,0) = 1$, it does not satisfy the property that $d(x,x) = 0$ for all $x \in \mathbb{R}$, and hence d_2 is not a metric. Since $d_3(0,1) = 2$, and $d_3(1,0) = 1$ it is not symmetric, and hence it is not a metric.

7. (a) If $r^{1/4}$ is rational, then it can be written as p/q for $p \in \mathbb{Z}$ and $q \in \mathbb{N}$. But then $r = p^4/q^4$, which would be rational also. Hence if r is irrational, then $r^{1/4}$ is irrational.

   Similarly, if $r+1$ is rational, it can be written as p/q for $p \in \mathbb{Z}$ and $q \in \mathbb{N}$. But then $r = (r+1) - 1 = (p/q) - 1 = (p-q)/q$ is rational. Therefore if r is irrational, then $r+1$ is irrational.

   (b) Write $x = \sqrt{2} + \sqrt{3}$. Then

$$\begin{aligned} x - \sqrt{2} &= \sqrt{3}, \\ x^2 - 2\sqrt{2}x + 2 &= 3, \\ x^2 - 1 &= 2\sqrt{2}x, \\ x^4 - 2x^2 + 1 &= 8x^2, \\ x^4 - 10x^2 + 1 &= 0. \end{aligned}$$

   By the rational zeroes theorem, any rational solution should have the form $x = p/q$ where $p = \pm 1$ and $q = \pm 1$. Therefore $x = \pm 1$. But

$$(1)^4 - 10(1)^2 + 1 = -8, \qquad (-1)^4 - 10(-1)^2 + 1 = -8 \tag{7}$$

   and neither possibility satisfies the equation. Hence $\sqrt{2} + \sqrt{3}$ is rational. By part (a), $(\sqrt{2} + \sqrt{3})^{1/4}$ is rational, and therefore $(\sqrt{2} + \sqrt{3})^{1/4} + 1$ is rational.

8. Choose an element $t \in T$. Then either

   - $t \in S$. Hence $t \leq \sup S$.
   - There exists $s \in S$ such that $s = -t$. Hence $s \geq \inf S$, and therefore $t \leq -\inf S$.

   Thus either $t \leq \sup S$ or $t \leq -\inf S$ so $t \leq \max\{\sup S, -\inf S\}$. Hence it is an upper bound.

   Now suppose that l is an upper bound for T. Then $l \geq t$ for all elements $t \in T$. Hence $l \geq |s|$ for all elements $s \in S$, and thus

$$-l \leq s \leq l \tag{8}$$

   for all elements in s, from which the following two deductions can be made:

- Since $s \leq l$ for all s, then $l \geq \sup S$ since $\sup S$ is the least upper bound for S.
- Since $-l \leq s$ for all s, then $-l \leq \inf S$ since $\inf S$ is the greatest lower bound for S. Hence $l \geq -\inf S$.

These two results show that $l \geq \max\{\sup S, -\inf S\}$. Hence $\max\{\sup S, -\inf S\}$ is an upper bound for T and it is the least upper bound, so it must be $\sup T$.

9. (a) Write $x = \sqrt{3} - \sqrt{2}$. Then $(x + \sqrt{2})^2 = 3$ and

$$
\begin{aligned}
x^2 + 2\sqrt{2}x + 2 &= 3, \\
x^2 - 1 &= -2\sqrt{2}x, \\
x^4 - 2x^2 + 1 &= 8x^2, \\
x^4 - 10x^2 + 1 &= 0.
\end{aligned}
$$

If x is rational, so that $x = p/q$ with $p \in \mathbb{Z}$ and $q \in \mathbb{N}$, then p divides 1 and q divides 1, so the only possibilities are $x = \pm 1$. But

$$1^4 - 10(1)^2 + 1 = 8, \qquad (-1)^4 - 10(-1)^2 + 1 = 8$$

and so neither possibility satisfies the equation. Hence x is irrational.

(b) For the first set

$$\min A = 0, \qquad \inf A = 0, \qquad \sup A = \sqrt{3} - \sqrt{2}.$$

$\max A$ is undefined because $\sqrt{3} - \sqrt{2}$ is irrational from part (a), but by the denseness of $\mathbb{Q}$ there are arbitrarily close rational numbers to it. The second set contains arbitrarily large positive numbers (*e.g.* n for $n \in \mathbb{N}$) and arbitrarily large negative numbers (*e.g.* $-n$ for $n \in \mathbb{N}$) and hence

$$\inf B = -\infty, \qquad \sup B = \infty.$$

$\min B$ and $\max B$ do not exist.

10. Choose $\epsilon = 1$. There exists an N such that $n > N$ implies that

$$|n^2 a_n - c| < \epsilon = 1.$$

Using the reverse triangle inequality $|n^2 a_n| < |c| + 1$, and hence

$$|a_n| < \frac{|c| + 1}{n^2}.$$

The finite number of terms for $n \leq N$ have no effect on the convergence properties, so consider the terms for $n > N$. Since $\sum 1/n^2$ converges, it follows that $\sum |a_n|$ converges by the comparison test, and therefore $\sum a_n$ converges absolutely.

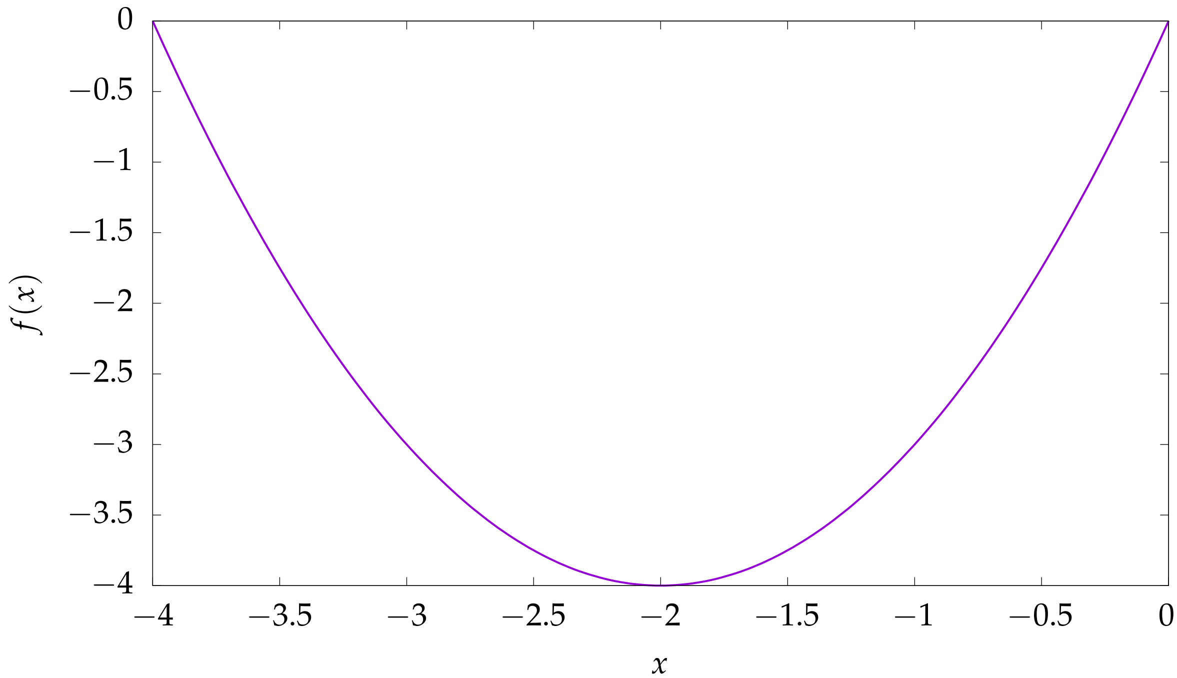

Figure 1: Graph of the quadratic function considered in question 11.

11. (a) From the initial condition, $s_0 = 1$. Consider $n \in \mathbb{N} \cup \{0\}$ and suppose that $s_n \geq n+1$. Then

$$s_{n+1} = 4s_n + s_n^2 \geq 4s_n \geq 4n+4 \geq n+2.$$

Hence by induction $s_n \geq n+1$ for all $n \in \mathbb{N} \cup \{0\}$. Since $\lim_{n\to\infty}(n+1) = \infty$ it follows that $\lim_{n\to\infty} s_n = \infty$ also.

(b) The quadratic can be rewritten as $f(x) = (x+2)^2 - 4$. Hence its minimum is -4 at $x = -2$.

(c) The quadratic is plotted in Fig. 1.

(d) Part (b) and the sketch show that if $s_n \in [-4, 0]$ then $(s_n + 2)^2 \in [0, 4]$ and hence $s_{n+1} = f(s_n) \in [-4, 0]$. Hence if $s_0 \in [-4, 0]$, then $s_n \in [-4, 0]$ for all $n \in \mathbb{N} \cup \{0\}$ by induction. Since (s_n) is bounded, it has a convergent subsequence by the Bolzano–Weierstraß theorem.

12. (a) From the definition $d(x, y) = 0$ if and only if $x = y$, and $d(x, y) > 0$ for all $x \neq y$. If $x = y$ then $d(x, y) = d(y, x)$ and if $x \neq y$ then $d(x, y) = 1 - |x - y| = 1 - |y - x| = d(y, x)$ so the function is symmetric. To check the triangle inequality, note that if $y = z$ then $d(x, z) = d(x, y) + 0 = d(x, y) + d(y, z)$ and the triangle inequality holds. The same argument holds for $x = y$. If both $x \neq y$ and $y \neq z$

then

$$d(x,y) + d(y,z) = (1 + |x-y|) + (1 + |y-z|) \geq 2 + |x-z| > d(x,z).$$

Therefore the triangle inequality holds, and hence d is a metric.

(b) For any $x \in \mathbb{R}$, the neighborhood of radius $\frac{1}{2}$ is

$$N_{1/2}(x) = \{y \in \mathbb{R} : d(x,y) < \tfrac{1}{2}\} = \{x\}$$

since $d(x,y) > 1$ when $x \neq y$. Hence each set $\{x\}$ for $x \in \mathbb{R}$ is open, since it is a neighborhood. Define $G_0 = \{0\}$ and $G_n = \{\frac{1}{n}\}$ for $n \in \mathbb{N}$. Then

$$\bigcup_{k=0}^{\infty} G_k = A$$

and hence the collection $\{G_k\}_{k=0}^{\infty}$ form an open cover of A. For any $n \in \mathbb{N}$, each set $\{G_n\}$ must be part of any subcover of A since it is the only set that contains $\frac{1}{n}$. Hence $\{G_k\}_{k=0}^{\infty}$ has no finite subcover of A. Therefore A is not compact.