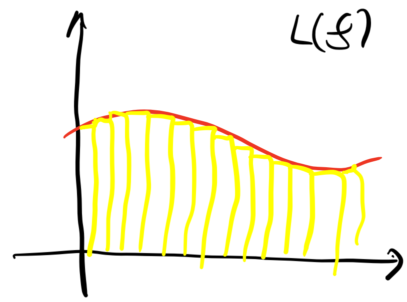
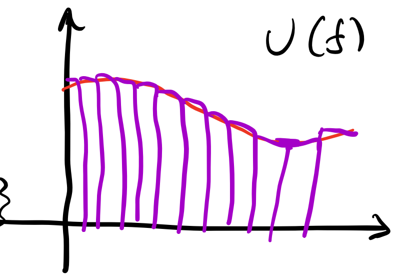
Riemann integration (continued)

The bounds imply that $U(f)$ and $L(f)$ are real numbers. Can be proved that $L(f) \leq U(f)$.
We say f is integrable on $[a,b]$ if $L(f) = U(f)$.



Then
$$\int_a^b f = \int_a^b f(x)\,dx = L(f) = U(f).$$
Called the Darboux integral.

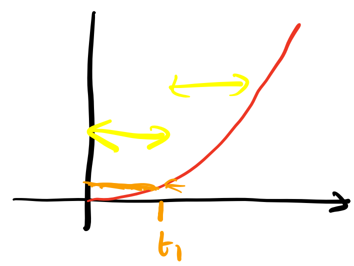


Integration example

Consider $f(x) = x^3$ and $\int_0^b f(x)\,dx$.

Consider a partition $P = \{0 = t_0 < t_1 < \dots < t_n = b\}$

Define $t_n = \frac{kb}{n}$.



Then
$$U(f,P) = \sum_{k=1}^{n} t_k^3 (t_k - t_{k-1})$$
$$= \sum_{k=1}^{n} \frac{k^3 b^3}{n^3}\left(\frac{b}{n}\right)$$
$$= \frac{b^4}{n^4} \sum_{k=1}^{n} k^3 = \frac{b^4}{n^4}\left[\frac{n(n+1)}{2}\right]^2$$ ← can prove by induction
$$= \frac{b^4}{n^4}\left(\frac{n^2+n}{2}\right)^2 = \frac{b^4}{n^4}\left(\frac{n^4+2n^3+n^2}{4}\right)$$
$$= \frac{b^4}{4}\left(1 + \frac{2}{n} + \frac{1}{n^2}\right)$$

$$L(f,P) = \sum^{n} t^3 \;(t_k - t_{k-1})$$

$$L(f,P) = \sum_{k=1}^{n} t_{k-1}^3 (t_k - t_{k-1})$$

$$= \sum_{k=1}^{n} (k-1)^3 \frac{b^3}{n^3} \frac{b}{n} = \frac{b^4}{n^4} \sum_{k=1}^{n} (k-1)^3$$

$$= \frac{b^4}{n^4} \sum_{l=1}^{n-1} l^3 = \frac{b^4}{n^4} \left[\frac{(n-1)n}{2} \right]^3$$

$$= \frac{b^4}{4} \left\{ 1 - \frac{2}{n} + \frac{1}{n^2} \right\}$$

As $n \to \infty$ $\quad U(f,P) \to \frac{b^4}{4}$

$$\Rightarrow U(f) = \inf_P \{U(f,P)\} \leq \frac{b^4}{4}$$

As $n \to \infty$ $\quad L(f,P) \to \frac{b^4}{4}$

$$\Rightarrow L(f) = \sup_P \{L(f,P)\} \geq \frac{b^4}{4}$$

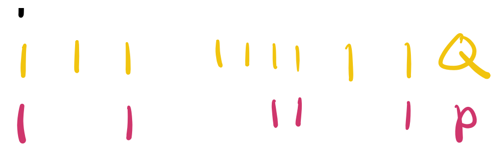
$L(f) \leq U(f)$

Hence $L(f) = U(f)$ and f is integrable on $[0,b]$ with value $b^4/4$.

Lemma

Suppose P and Q are partitions of $[a,b]$ and $P \subseteq Q$. Then

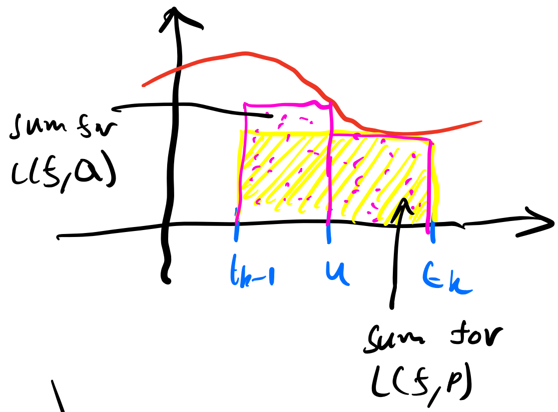
$$L(f,P) \le L(f,Q) \le U(f,Q) \le U(f,P).$$



Proof Suppose $P = \{a = t_0 < t_1 < \dots < t_n = b\}$

$Q = \{a = t_0 < t_1 < \dots < t_{k-1} < u < t_k < \dots < t_n = b\}$

Then



$$\begin{aligned} &L(f,Q) - L(f,P) \\ &= m(f,[t_{k-1},u])(u - t_{k-1}) \\ &\quad + m(f,[u,t_k])(t_k - u) \\ &\quad - m(f,[t_{k-1},t_k])(t_k - t_{k-1}) \end{aligned}$$

Since

$$\begin{aligned} m(f,[t_{k-1},t_k])(t_k - t_{k-1}) &= m(f,[t_{k-1},t_k])(t_k - u + u - t_{k-1}) \\ &= m(f,[t_{k-1},t_k])(t_k - u) \\ &\quad + m(f,[t_{k-1},t_k])(u - t_{k-1}) \\ &\le m(f,[t_{k-1},u])(u - t_{k-1}) \\ &\quad + m(f,[u,t_k])(t_k - u) \end{aligned}$$

Hence $L(f,Q) - L(f,P) \ge 0$.

$$L(f,Q) \ge L(f,P).$$

Lemma Let f be a bounded function on $[a,b]$ and P and Q be partitions of $[a,b]$. Then $L(f,P) \leq U(f,Q)$.

<u>Proof</u> $P \cup Q$ is also a partition of $[a,b]$

$$L(f,P) \leq L(f,P\cup Q) \leq U(f,P\cup Q) \leq U(f,Q).$$

<u>Theorem</u> If f is a bounded function on $[a,b]$ then $L(f) \leq U(f)$.

<u>Proof</u> $L(f,P)$ is a lower bound for the set $\{U(f,Q) \mid Q \text{ is a partition of } [a,b]\}$

Hence $L(f,P)$ must be less than or equal to the infimum of this set.

$$L(f,P) \leq U(f)$$

$U(f)$ is an upper bound for all $L(f,P)$

Apply same argument to show $L(f) \leq U(f)$.

<u>Theorem</u> A bounded function f on $[a,b]$ is integrable if and only if $\forall \varepsilon > 0, \exists$ a partition P of $[a,b]$ such that

$$U(f,P) - L(f,P) < \varepsilon. \quad (*)$$

Proof Suppose f is integrable $\exists$ partitions

P_1 and P_2 of $[a,b]$ such that

$$L(f,P_1) > L(f) - \frac{\varepsilon}{2}$$

$$U(f,P_2) < U(f) + \frac{\varepsilon}{2}.$$

For $P = P_1 \cup P_2$

$$\begin{aligned} U(f,P) - L(f,P) &\leq U(f,P_1) - L(f,P_1) \\ &< U(f) + \frac{\varepsilon}{2} - \left(L(f) - \frac{\varepsilon}{2}\right) \\ &= U(f) - L(f) + \varepsilon. \end{aligned}$$

Since f is integrable, then $U(f) = L(f)$.
So $(*)$ holds.

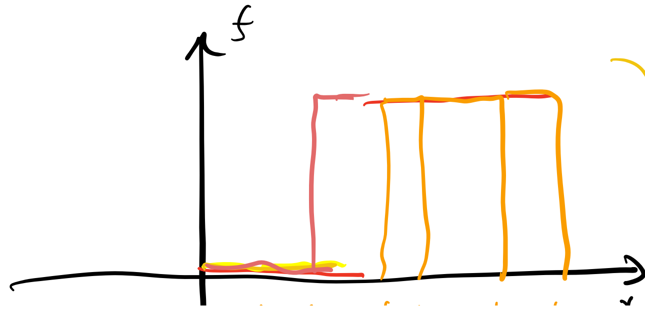
Suppose $\forall \varepsilon > 0$ $(*)$ holds. Then

$$\begin{aligned} U(f) \leq U(f,P) &= U(f,P) - L(f,P) + L(f,P) \\ &< \varepsilon + L(f,P) \leq \varepsilon + L(f). \end{aligned}$$

Since ε is arbitrary, $U(f) \leq L(f)$. Hence $L(f) = U(f)$.

The function

$$f(x) = \begin{cases} 0 & x \in \mathbb{Q} \\ 1 & x \notin \mathbb{Q} \end{cases}$$



is not integrable.
Continuity is not required for integrability e.g.
$f(x) = \begin{cases} 0 & x < 1 \\ 1 & x \geq 1 \end{cases}$, then $\int_0^2 f = 1$.

<u>**Definition**</u> The mesh of a partition P is the maximum length of subintervals comprising P. If

$$P = \{a = t_0 < t_1 < \ldots < t_n = b\}$$

then $\text{mesh}(P) = \max\{t_k - t_{k-1} \mid k = 1, 2, \ldots, n\}$.

<u>**Theorem**</u> A bounded function f on $[a,b]$ is integrable if and only if for each $\varepsilon > 0$, $\exists \delta > 0$ such that

$$\text{mesh}(P) < \delta \Rightarrow U(f,P) - L(f,P) < \varepsilon.$$

for all partitions P of $[a,b]$.

<u>**Proof**</u> Suppose f is integrable on $[a,b]$. Let $\varepsilon > 0$ and select $P_0 = \{a = u_0 < u_1 < \ldots < u_m = b\}$ a partition of $[a,b]$ such that

$$U(f, P_0) - L(f, P_0) < \frac{\varepsilon}{2}.$$

Since f is bounded, there exists $B > 0$ such that $|f(x)| \leq B$ $\forall x \in [a,b]$. Let $\delta = \frac{\varepsilon}{8mB}$, where m is the number of intervals in P_0.

Consider $P = \{a = t_0 < t_1 < \dots < t_n = b\}$ with $\text{mesh}(P) < \delta$. Define $Q = P \cup P_0$. If Q has one more element than P, then

$$L(f,Q) - L(f,P) \leq B\,\text{mesh}(P) - (-B)\,\text{mesh}(P)$$
$$= 2B\,\text{mesh}(P).$$

Q has at most m elements not in P so

$$L(f,Q) - L(f,P) \leq 2mB\,\text{mesh}(P).$$
$$< 2mB\delta = \frac{\varepsilon}{4}.$$

Hence $L(f,P_0) - L(f,P) < \frac{\varepsilon}{4}$.

Similarly $U(f,P) - U(f,P_0) < \frac{\varepsilon}{4}$.

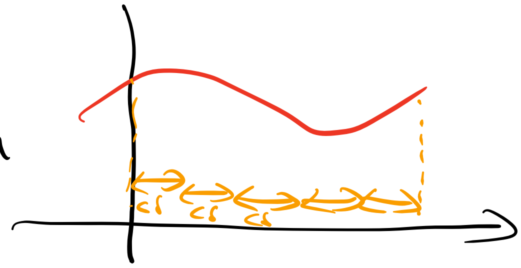
$$U(f,P) - L(f,P) < U(f,P_0) - L(f,P_0) + \frac{\varepsilon}{2}$$
$$< \varepsilon.$$

Theorem Every continuous function on $[a,b]$ is integrable.

Proof Consider $\varepsilon > 0$. Since f is uniformly continuous on $[a,b]$ $\exists\, \delta > 0$ s.t $\forall x, y \in [a,b]$ and $|x-y| < \delta$. $\Rightarrow |f(x) - f(y)| < \frac{\varepsilon}{b-a}$.

consider a partition $P = \{a = t_0 < t_1 < \dots < t_n = b\}$, such $\max\{t_k - t_{k-1} \mid k = 1, \dots, n\} < \delta$.

Then within any interval $[t_{k-1}, t_k]$, $|f(x) - f(y)| < \frac{\varepsilon}{b-a}$ for all $x, y \in [t_{k-1}, t_k]$.

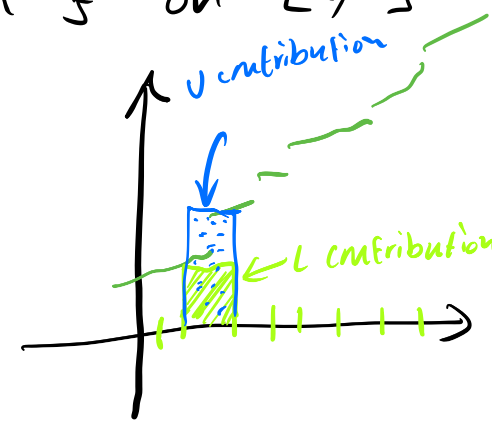


$$M(f, [t_{k-1}, t_k]) - m(f, [t_{k-1}, t_k]) < \frac{\varepsilon}{b-a}.$$

$$U(f,P) - L(f,P) < \sum_{k=1}^{n} \frac{\varepsilon}{b-a}(t_k - t_{k-1}) = \varepsilon.$$

Theorem

Every monotonic function f on $[a,b]$ is integrable



Proof Assume that f is increasing and $f(a) < f(b)$ since otherwise f is a constant function.

Since $f(a) \le f(x) \le f(b)$ for any $x \in [a,b]$, so f is bounded on $[a,b]$. Choose $\varepsilon > 0$.

Select a partition $P = \{a = t_0 < t_1 < t_2 < \dots < t_n = b\}$ with $\text{mesh}(P) < \frac{\varepsilon}{f(b) - f(a)}$.

$$U(f,P) - L(f,P) = \sum_{k=1}^{n} \left(M(f, [t_{k-1}, t_k]) - m(f, [t_{k-1}, t_k])\right) \cdot (t_k - t_{k-1})$$

$$= \sum_{k=1}^{n} \left(f(t_k) - f(t_{k-1}) \right)(t_k - t_{k-1})$$

$$< \sum_{k=1}^{n} \left(f(t_k) - f(t_{k-1}) \right) \frac{\varepsilon}{f(b) - f(a)}$$

$$= \frac{\varepsilon}{f(b) - f(a)} \sum_{k=1}^{n} \left(f(t_k) - f(t_{k-1}) \right).$$

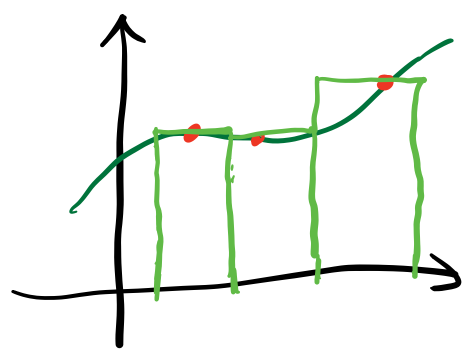
$$= \frac{\varepsilon}{f(b) - f(a)} \left[f(b) - f(a) \right] = \varepsilon.$$

$\Rightarrow f$ is integrable.

Definition Let f be a bounded function on $[a,b]$, and let $P = \{a = t_0 < t_1 < \ldots < t_n = b\}$ be a partition on $[a,b]$. A Riemann sum of f associated with the partition P is a sum of the form

$$\sum_{k=1}^{n} f(x_k)(t_k - t_{k-1})$$

where $x_k \in [t_{k-1}, t_k]$



A function is Riemann integrable on $[a,b]$ if $\exists r$ so that $\forall \varepsilon > 0, \exists \delta > 0$ s.t

$$|S - r| < \varepsilon$$

for every Riemann sum S of f associated with a partition P having $\mathrm{mesh}(P) < \delta$.
r is the Riemann integral of f on $[a,b]$.

$\mathcal{R}\int_a^b$.

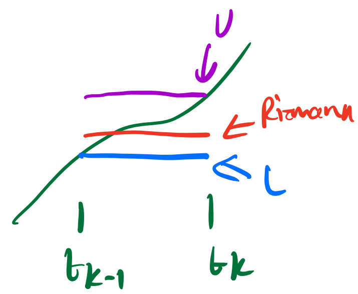
Theorem A bounded function f on $[a,b]$ is Riemann integrable if and only if it is Darboux integrable.

Proof Suppose f is Darboux integrable. Let $\varepsilon>0$, and choose $\delta>0$ such that $\operatorname{mesh}(P)<\delta$ then

$U(f,P)-L(f,P)<\varepsilon$.

Consider showing $\left|S-\int_a^b f\right|<\varepsilon$.

We have $L(f,P)\le S\le U(f,P)$.



$$U(f,P)<L(f,P)+\varepsilon\le L(f)+\varepsilon=\int_a^b f+\varepsilon$$

$$L(f,P)>U(f)-\varepsilon=\int_a^b f-\varepsilon$$

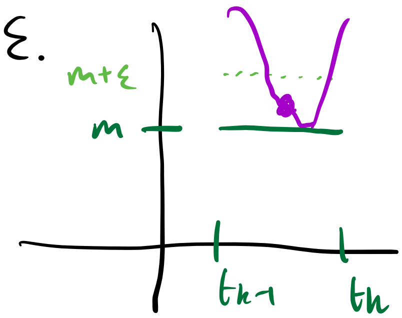
$$\int_a^b f-\varepsilon<S<\int_a^b f+\varepsilon$$

$$\Rightarrow \left|S-\int_a^b f\right|<\varepsilon.$$

Suppose f is Riemann integrable. Consider $\varepsilon>0$. $\exists\,\delta>0$ and r such for all partitions P with $\operatorname{mesh}(P)<\delta$, any Riemann sum S satisfies $|S-r|<\varepsilon$.

Choose $x_k \in [t_{k-1}, t_k]$ so that

$$f(x_k) < m(f, [t_{k-1}, t_k]) + \varepsilon.$$



Do this for all intervals.

$$S < L(f, P) + \varepsilon(b-a).$$

$$L(f) \geq L(f,P) \geq S - \varepsilon(b-a)$$
$$> r - \varepsilon - \varepsilon(b-a).$$

Since ε is arbitrary $L(f) \geq r$.

Similarly $U(f) \leq r$. Hence $L(f) = U(f)$ and f is Darboux integrable.

__Theorem__ If f and g are integrable functions on $[a,b]$, then $c \in \mathbb{R}$

i) cf is integrable and $\int_a^b cf = c\int_a^b f$.

ii) $f+g$ is integrable and $\int_a^b (f+g) = \int_a^b f + \int_a^b g$.

__Proof__ Suppose $c > 0$. For a given partition,

$$M(cf, [t_{k-1}, t_k]) = c\,M(f, [t_{k-1}, t_k])$$

Hence $U(cf, P) = c\,U(f, P)$.

$$\left.\begin{aligned} U(cf) &= c\,U(f). \\ \text{Similarly } L(cf) &= c\,L(f). \end{aligned}\right\} \text{confirm that } cf \text{ is integrable}$$

$$\int_a^b cf = U(cf) = c\,U(f) = c\int_a^b f.$$

Choose $\varepsilon > 0$. $\exists$ partitions P_1, P_2 such that

$$U(f, P_1) - L(f, P_1) < \frac{\varepsilon}{2}$$

$$U(g, P_2) - L(g, P_2) < \frac{\varepsilon}{2}.$$

For $P = P_1 \cup P_2$

$$U(f,P) - L(f,P) < \frac{\varepsilon}{2}$$

$$U(g,P) - L(g,P) < \frac{\varepsilon}{2}.$$

$m(f+g, [t_{k-1}, t_k])$

$$\inf\{f(x)+g(x) \mid x \in S\} \geq \inf\{f(x) \mid x \in S\} + \inf\{g(x) \mid x \in S\}$$

$$m(f+g, S) \geq m(f,S) + m(g,S)$$

$$L(f+g,P) \geq L(f,P) + L(g,P).$$

$$U(f+g,P) \leq U(f,P) + U(g,P).$$

$$U(f+g,P) - L(f+g,P) < \varepsilon$$

$$\int_a^b f+g = U(f+g) \leq U(f+g,P) \leq U(f,P) + U(g,P).$$

$$< L(f,P) + L(g,P) + \varepsilon \leq L(f) + L(g) + \varepsilon$$

$$= \int_a^b f + \int_a^b g + \varepsilon.$$

Similarly

$$\int_a^b f+g \geq \int_a^b f + \int_a^b g - \varepsilon.$$

Since ε is arbitrary, $\int_a^b f+g = \int_a^b f + \int_a^b g.$

Theorem If f and g are integrable on $[a,b]$ and $f(x) \leq g(x)$ $\forall x \in [a,b]$ then $\int_a^b f \leq \int_a^b g$.

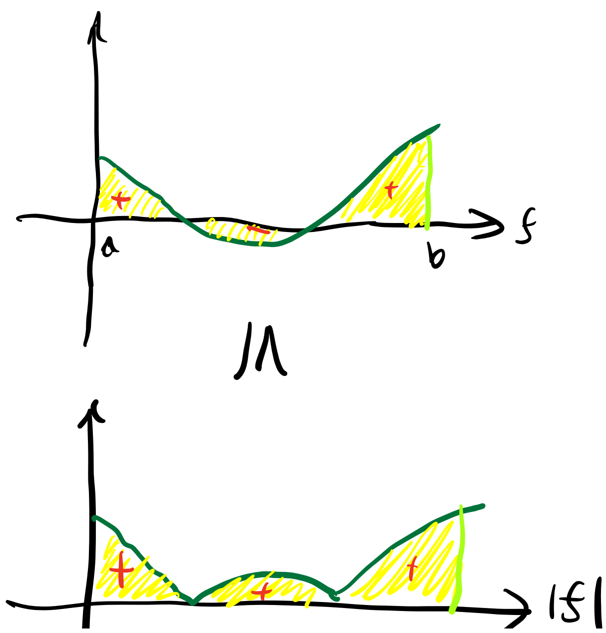
Proof The previous theorem shows that $h = g - f$ is integrable on $[a,b]$. $h(x) \geq 0$ $\forall x \in [a,b]$, so $L(h,P) \geq 0$. So $\int_a^b h = L(h) \geq 0$.

Use the previous theorem to show that

$$\int_a^b g - \int_a^b f = \int_a^b h \geq 0. \quad \boxed{\Rightarrow \int_a^b g \geq \int_a^b f}$$

Theorem If f is integrable on $[a,b]$ then $|f|$ is integrable on $[a,b]$ and $\left|\int_a^b f\right| \leq \int_a^b |f|$

$$\left|\sum_{k=1}^n f(x_k)(t_k - t_{k-1})\right| \leq \sum_{k=1}^n |f(x_k)(t_k - t_{k-1})|$$



Proof Consider any subset S of $[a,b]$.

$$M(|f|,S)-m(|f|,S)=\sup\{|f(x)|\mid x\in S\}-\inf\{|f(x)|\mid x\in S\}$$

$$=\sup\{|f(x)|\mid x\in S\}+\sup\{-|f(x)|\mid x\in S\}$$

$$=\sup\{|f(x)|-|f(y)|\mid x,y\in S\}$$

$$\leq\sup\{|f(x)-f(y)|\mid x,y\in S\}$$

$$=\sup\{f(x)-f(y)\mid x,y\in S\}.$$

$$=M(f,S)-m(f,S).$$

$$U(|f|,P)-L(|f|,P)\leq U(f,P)-L(f,P).$$

Choose $\varepsilon>0$ $\exists P$ s.t $U(f,P)-L(f,P)<\varepsilon$

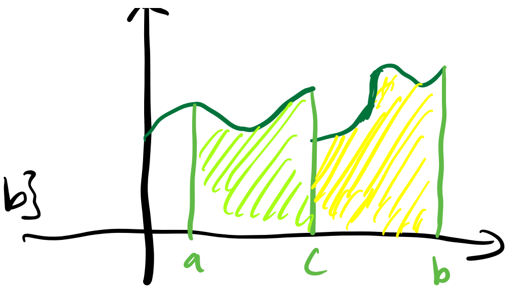
$$\Rightarrow U(|f|,P)-L(|f|,P)<\varepsilon.$$

Since $-|f|\leq f\leq |f|$ it follows that

$$-\int_a^b|f|\leq\int_a^b f\leq\int_a^b|f| \quad\Rightarrow\quad \left|\int_a^b f\right|\leq\int_a^b|f|.$$

Theorem Let f be defined on $[a,b]$. If $a<c<b$ and f is integrable on $[a,c]$ and $[c,b]$ then $\int_a^b f=\int_a^c f+\int_c^b f$ (and f is integrable on $[a,b]$).

Definition A function is **piecewise monotonic** if $\exists$ a partition $P=\{a=t_0<t_1<\ldots<t_n=b\}$ such that f is monotonic on each interval (t_{k-1},t_k). The function is **piecewise continuous** if $\exists$ a partition P of $[a,b]$ such that f is uniformly continuous on (t_{k-1},t_k).

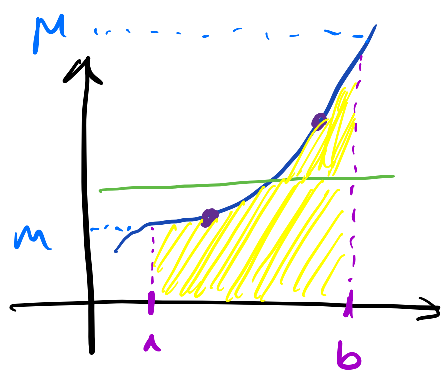


Piecewise continuous and monotonic functions are integrable.

Theorem Intermediate value theorem for integrals

If f is a continuous function on $[a,b]$, then for at least one $x\in(a,b)$,

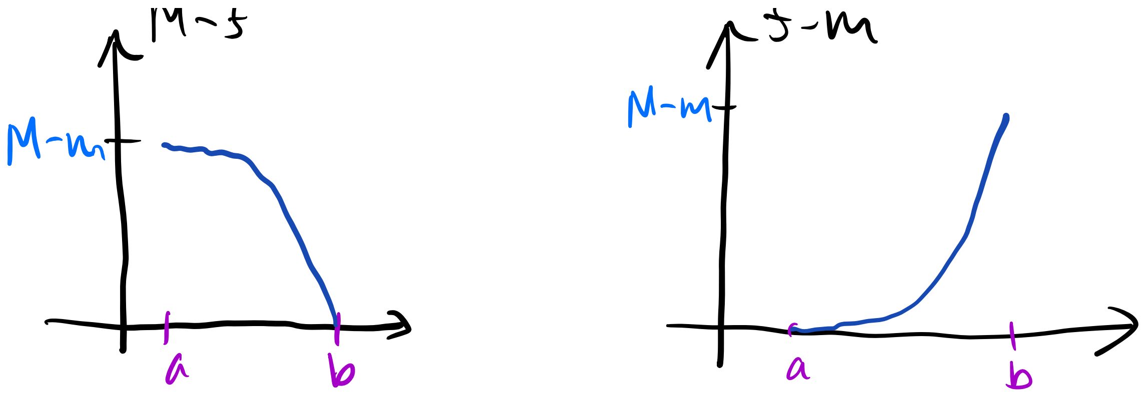
$$f(x)=\frac{1}{b-a}\int_a^b f$$



Proof Let M and m be the maximum and minimum values of f on $[a,b]$. If $m=M$ then f is a constant function and $f(x)=\frac{1}{b-a}\int_a^b f$ for all $x\in\{a,b\}$.

Otherwise $m<M$, and there exist distinct x_0,y_0 in $[a,b]$ such that $f(x_0)=m$, and $f(y_0)=M$.

The functions $M-f$ and $f-m$ are non-negative and not identically zero.



$M-f \geq 0$

$f-m \geq 0$

Previous results $\int_a^b M-f \geq 0$ $\int_a^b f-m \geq 0.$

$$\int_a^b M-f > 0 \qquad \int_a^b f-m > 0$$

HW8 Q1.

$$\int_a^b m < \int_a^b f < \int_a^b M.$$

$$(b-a)m < \int_a^b f < (b-a)M.$$

$$m < \frac{1}{b-a}\int_a^b f < M.$$

Apply the intermediate value theorem for continuous functions between x_0 and y_0.

$\exists x$ between x_0 and y_0 such that

$$f(x) = \frac{1}{b-a}\int_a^b f.$$

<u>Fundamental theorem of calculus</u>

If g is continuous function on $[a,b]$ that is differentiable on (a,b) and g' is integrable on $[a,b]$, then

$$\int_a^b g' = g(b) - g(a).$$

<u>Proof</u> Choose $\varepsilon > 0$. There exists a partition $P = \{a = t_0 < t_1 < \ldots < t_n = b\}$ of $[a,b]$ such that

$$U(g', P) - L(g', P) < \varepsilon.$$

Apply the Mean value theorem to each interval $[t_{k-1}, t_k]$. $\exists x_k \in (t_{k-1}, t_k)$ such that

$$(t_k - t_{k-1})\, g'(x_k) = g(t_k) - g(t_{k-1}).$$

Hence
$$g(b) - g(a) = \sum_{k=1}^{n} [g(t_k) - g(t_{k-1})].$$

$\leq M(g', [t_{k-1}, t_k])$

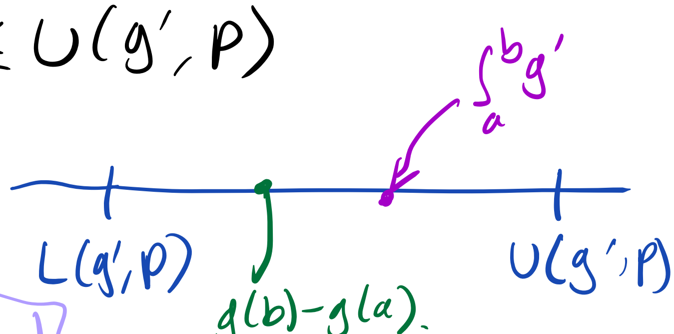
$$= \sum_{k=1}^{n} g'(x_k)(t_k - t_{k-1}).$$

$\geq m(g', [t_{k-1}, t_k])$

$$L(g', P) \leq g(b) - g(a) \leq U(g', P)$$

$$L(g', P) \leq \int_a^b g' \leq U(g', P)$$

$$\boxed{\begin{aligned}&\sum_{k=1}^{n} g'(x_k)(t_k - t_{k-1}) \\ &\leq \sum_{k=1}^{n} M(g', [t_{k-1}, t_k])(t_k - t_{k-1})\end{aligned}}$$



$$\boxed{\begin{array}{l} k=1 \\ = U(g',P) \end{array}}$$

$\longleftrightarrow$
$< \varepsilon$

Hence $\left| \int_a^b g' - (g(b) - g(a)) \right| < \varepsilon$

ε is arbitrary, so $\int_a^b g' = g(b) - g(a)$.

Theorem (Integration by parts)
Suppose u and v are continuous on $[a,b]$ and are differentiable on (a,b). If u' and v' are integrable on $[a,b]$, then

$$\int_a^b u(x)v'(x)\,dx + \int_a^b u'(x)\,v(x)\,dx = u(b)v(b) - u(a)v(a).$$

Proof Let $g = uv$, Then $g' = uv' + u'v$.
Products of integrable functions are integrable. (see exercise).

$$\int_a^b g'(x)\,dx = g(b) - g(a) = u(b)\,v(b) - u(a)v(a).$$

Fundamental theorem of calculus II
Let f be an integrable function on $[a,b]$. For $x \in [a,b]$ Let $F(x) = \int_a^x f(t)\,dt$.

Then F is continuous on $[a,b]$. If f is continuous at $x_0 \in (a,b)$, then F is differentiable at x_0 and $F'(x_0) = f(x_0)$.

Proof Choose $B>0$ such that $|f(x)| \le B$, for all $x \in [a,b]$. If $x, y \in [a,b]$ where $|x-y| < \frac{\varepsilon}{B}$ and $x<y$, then

$$|F(y)-F(x)| = \left| \int_x^y f(t)\,dt \right| \le \int_x^y |f(t)|\,dt$$

$$\le \int_x^y B\,dt = B(y-x) < \varepsilon$$

$\Rightarrow F$ is uniformly continuous on $[a,b]$.

Suppose f is continuous at $x_0 \in (a,b)$.

$$\frac{F(x)-F(x_0)}{x-x_0} = \frac{1}{x-x_0} \int_{x_0}^{x} f(t)\,dt. \qquad \text{for } x \ne x_0.$$

And
$$f(x_0) = \frac{1}{x-x_0} \int_{x_0}^{x} \underbrace{f(x_0)}_{\text{constant.}}\,dt$$

Hence
$$\frac{F(x)-F(x_0)}{x-x_0} - f(x_0) = \frac{1}{x-x_0} \int_{x_0}^{x} [f(t)-f(x_0)]\,dt.$$

Let $\varepsilon > 0$. Since f is continuous, $\exists \delta > 0$ such that $t \in (a,b)$ and $|t-x_0| < \delta$ then $|f(t)-f(x_0)| < \varepsilon$.

$$\Rightarrow \left| \frac{F(x)-F(x_0)}{x-x_0} - f(x_0) \right| \le \varepsilon \qquad \text{for } x \in (a,b) \text{ and}$$

| $x - x_0$, $|x-x_0|<\delta$.

$$\Rightarrow \lim_{x \to x_0} \frac{F(x)-F(x_0)}{x-x_0} = f(x_0).$$

$F'(x_0)=f(x_0)$.

Change of variable

Let u be a differentiable function on an open interval J such that u' is continuous and let I be an open interval such that $u(x) \in I$ $\forall x \in J$. If f is continuous on I then $f \circ u$ is continuous on J and

$$\int_a^b f \circ u(x) u'(x) dx = \int_{u(a)}^{u(b)} f(u) du.$$

for $a, b \in J$.

Proof $f \circ u$ is continuous by previous result. Choose $c \in I$ and define $F(u) = \int_c^u f(t) dt$. Then $F'(u) = f(u)$ $\forall u \in I$ by FTC2. Let $g = F \circ u$. Then $g'(x) = F'(u(x))u'(x)$ $= f(u(x))u'(x)$. Hence

$$\int_a^b f \circ u(x) u'(x) dx = \int_a^b g'(x) dx = g(b) - g(a)$$

$$= F(u(b)) - F(u(a))$$

$\int^{u(b)}$ $\int^{u(a)}$

$$= \int_c f(t)\,dt - \int_c f(t)\,dt$$

$$= \int_{u(a)}^{u(b)} f(t)\,dt.$$

Improper integrals

Consider an interval $[a,b)$ where $b \in \mathbb{R}$ or $b = \infty$. Suppose f is a function that is integrable on each $[a,d]$ for $a < d < b$, and that

$$\lim_{d \to b^-} \int_a^d f(x)\,dx$$

exists as either a finite number or $\pm\infty$. Then

$$\int_a^b f(x)\,dx = \lim_{d \to b^-} \int_a^d f(x)\,dx.$$

Similarly if f is defined on $(a,b]$ where $a \in \mathbb{R}$ or $a = -\infty$, then if f is integrable on $[c,b]$ $\forall\, a < c < b$, then

$$\int_a^b f(x)\,dx = \lim_{c \to a^+} \int_c^b f(x)\,dx.$$

For f defined on (a,b) and integrable [illegible] subintervals $[c,d]$

on all closed subintervals $[c,d]$,

$$\int_a^b f(x)\,dx = \int_a^\alpha f(x)\,dx + \int_\alpha^b f(x)\,dx \qquad \alpha \in (a,b)$$

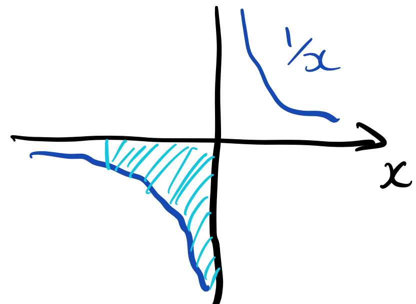
(Can be shown that the choice of α does not affect the result.)

Example

$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{c\to\infty} \int_1^c \frac{1}{x^2}\,dx$$

$$= \lim_{c\to\infty} \left[-\frac{1}{x}\right]_1^c$$

$$= \lim_{c\to\infty} \left[-\frac{1}{c} + 1\right] = 1.$$



$$\int_{-1}^0 \frac{1}{x}\,dx = \lim_{c\to 0^-} \left(-\log -x\right)_{-1}^{c}$$

$$= \lim_{c\to 0^-} -\log|c| = -\infty.$$

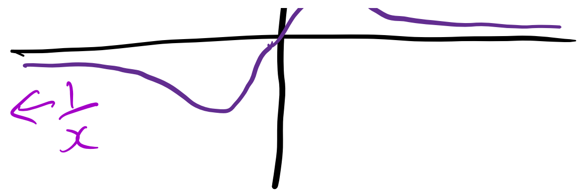
$$\int_0^d \sin x\,dx = 1 - \cos d$$

$$\int_0^\infty \sin x\,dx = \lim_{d\to\infty} 1-\cos d \quad \text{does not exist.}$$

Cauchy principal value

consider $\displaystyle\int^\infty \frac{x}{\dots}\,dx$

$_{-\infty} \qquad 1+x^2$



Regular definition

$$\int_{-\infty}^{\infty} \frac{x}{1+x^2}dx = \int_{-\infty}^{\alpha} \frac{x}{1+x^2}dx + \int_{\alpha}^{\infty} \frac{x}{1+x^2}dx \qquad \alpha = 0$$

$$= \underbrace{\lim_{a\to-\infty} \int_{a}^{\alpha} \frac{x}{1+x^2}dx}_{-\infty} + \underbrace{\lim_{b\to\infty} \int_{\alpha}^{b} \frac{x}{1+x^2}dx}_{\infty}$$

This integral value does not exist.

Can use the Cauchy principal value to obtain a solution. Take the limits concurrently to $-\infty$ and $+\infty$.

$$P\int_{-\infty}^{\infty} \frac{x}{1+x^2}dx = \lim_{a\to\infty} \int_{-a}^{a} \frac{x}{1+x^2}dx = \lim_{a\to\infty} 0 = 0.$$

$\downarrow$

$$\int_{-a}^{0} \frac{x}{1+x^2}dx + \int_{0}^{a} \frac{x}{1+x^2}dx$$

$x = -y$

$$-\int_{0}^{a} \frac{y}{1+y^2}dy + \int_{0}^{a} \frac{x}{1+x^2}dx$$

$0.$

Continuity in metric spaces

Consider two metric spaces (S, d) and (S^*, d^*).
Interested in maps $f: S \to S^*$

Definition

$f: S \to S^*$ is continuous at s_0 in S if

$\forall \varepsilon > 0, \exists \delta > 0$ such that $d(s, s_0) < \delta \Rightarrow d^*(f(s), f(s_0)) < \varepsilon$.

A function f is continuous on a subset E of S if f is continuous at each point of E.

A function is uniformly continuous on a subset E of S if

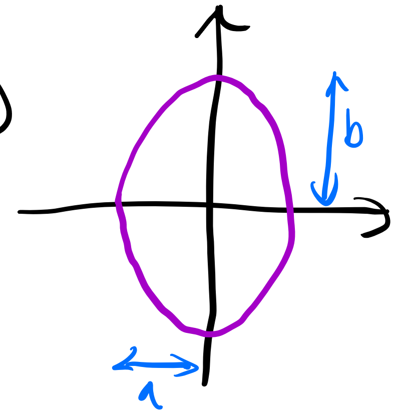
$\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. if $s, t \in E$ and $d(s,t) < \delta \Rightarrow d^*(f(s), f(t)) < \varepsilon$.

If $S = S^* = \mathbb{R}$ and $d = d^*$ is the usual Euclidean metric these match our previous definitions.

Can talk about a path: a continuous mapping $\gamma: \mathbb{R} \to \mathbb{R}^k$. The image of $\gamma(\mathbb{R})$ is called a curve.

e.g. ellipse

$\gamma(t) = (a\cos t, b\sin t)$



<u>Proposition</u> If $f_1, f_2, \ldots, f_k$ are continuous functions that are real-valued $(\mathbb{R} \to \mathbb{R})$, then

$$\gamma(t) = (f_1(t), f_2(t), \ldots, f_k(t))$$

defines a path in $\mathbb{R}^k$.

<u>Proof</u> Must show that γ is continuous.

Pick $\underline{x} = (x_1, x_2, \ldots, x_k)$, $\underline{y} = (y_1, y_2, \ldots, y_k)$.

$$d^*(\underline{x}, \underline{y}) = \left(\sum_{j=1}^{k} (x_j - y_j)^2 \right)^{1/2}$$

$$\leq \left(k \max_{j=1,\ldots,k} (x_j - y_j)^2 \right)^{1/2}$$

$$= \sqrt{k} \max_{j=1,\ldots,k} |x_j - y_j|.$$

Consider $t_0 \in \mathbb{R}$ and $\varepsilon > 0$. For $j = 1, \ldots, k$, $\exists\, \delta_j > 0$ s.t.

$$|t - t_0| < \delta_j \Rightarrow |f_j(t) - f_j(t_0)| < \frac{\varepsilon}{\sqrt{k}}$$

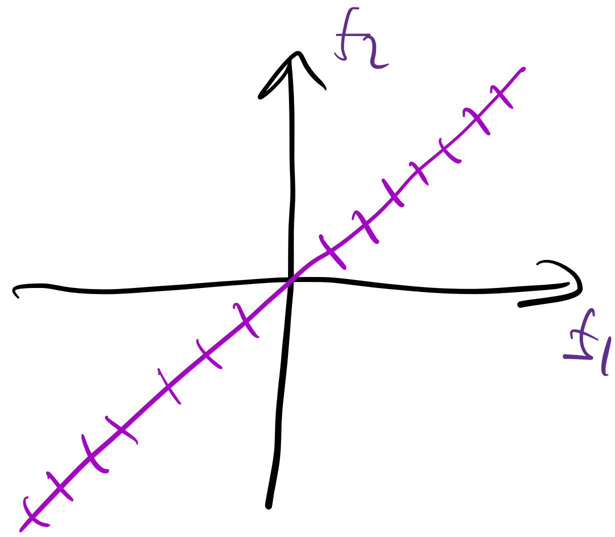
If $\delta = \min\{\delta_1, \delta_2, \ldots, \delta_k\}$ and $|t - t_0| < \delta$

$$\max\{|f_j(t) - f_j(t_0)| : j = 1, \ldots, k\} < \frac{\varepsilon}{\sqrt{k}}$$

$$d^*(\gamma(t), \gamma(t_0)) < \varepsilon \Rightarrow \gamma \text{ is continuous.}$$

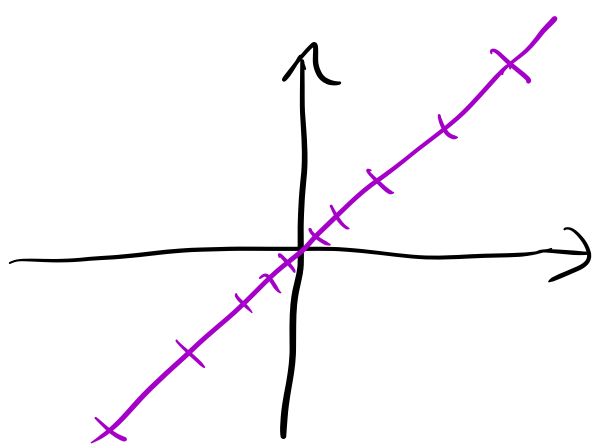
$f_1(t) = t$

$f_2(t) = t$



$f_1(t) = t^3$

$f_2(t) = t^3$



<u>Theorem</u>

Suppose (S, d) and (S^*, d^*) are two metric spaces. $f: S \to S^*$ is continuous on S iff

$f^{-1}(U)$ is an open subset of S
for every open subset U of S^*

Proof Suppose f is continuous on S. Let U be an open subset of S^*. Consider $N_\varepsilon(f(s_0))$
$s_0 \in f^{-1}(U)$. Hence $f(s_0) \in U$. Since U is open, $\exists \varepsilon > 0$ s.t $\{s^* \in S \mid d^*(s^*, f(s_0)) < \varepsilon\} \subseteq U$.

Since f is continuous at s_0, $\exists \delta > 0$ s.t

$$d(s, s_0) < \delta \Rightarrow d^*(f(s), f(s_0)) < \varepsilon$$
$$\Rightarrow f(s) \subseteq U \Rightarrow s \in f^{-1}(U).$$

$N_\delta(s_0) \subseteq U$

Hence s_0 is an interior point and $f^{-1}(U)$ is open.

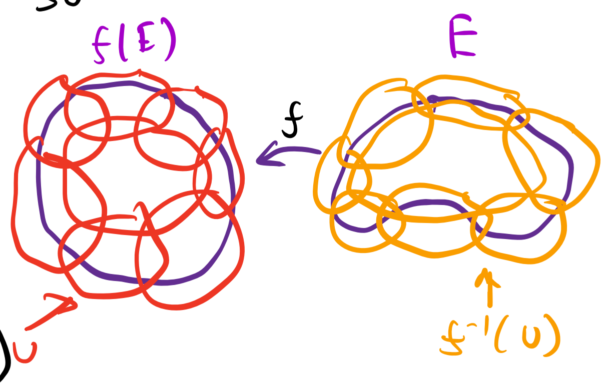
Suppose that the property holds. Consider a point $s_0 \in S$. Choose $\varepsilon > 0$, and examine neighborhood $N_\varepsilon(f(s_0))$.

Hence $F = f^{-1}(N_\varepsilon(f(s_0)))$ is open using the property. $s_0 \in F$ so $\exists \delta > 0$ s.t $N_\delta(s_0) \subseteq F$

Hence if $d(s, s_0) < \delta$ then $d^*(f(s), f(s_0)) < \varepsilon$

<u>Theorem</u> Let (S, d) and (S^*, d^*) be metric spaces, and let $f: S \to S^*$ be continuous. Suppose E is a compact subset of S. Then
i) $f(E)$ is a compact subset of S^*
ii) f is uniformly continuous on E.

<u>Proof</u> i) Let $\mathcal{U}$ be an open cover of $f(E)$.
For each $U \in \mathcal{U}$, $f^{-1}(U)$ is open in S.
Also $\{f^{-1}(U) : U \in \mathcal{U}\}$ is a cover of E, since if $x \in E$, then $f(x)$ is in $f(E)$ and $f(x) \in U'$ for some U' so $x \in f^{-1}(U')$.



Since E is compact $\exists\, U_1, U_2, \dots, U_m$ in $\mathcal{U}$ such that
$E \subseteq f^{-1}(U_1) \cup f^{-1}(U_2) \cup \dots \cup f^{-1}(U_m)$
so $\{U_1, U_2, \dots, U_m\}$ is a finite subcover of $f(E)$

$f(E)$ is compact.

$E \subseteq S \quad f: S \to S^*$

ii) Choose $\varepsilon > 0$. For each $s \in E$, $\exists\, \delta_s > 0$ s.t.

$d(s,t) < \delta_s$
$\Rightarrow d^*(f(s), f(t)) < \frac{\varepsilon}{2}$

$f^{-1}(E) = \{x \in S^* \mid f(y) = x \text{ for some } y \in E\}$

$f(x) = x^2$

$f^{-1}([1,4]) = [-2,-1] \cup [1,2]$

Define sets

$V_s = N_{\frac{\delta_s}{2}}(s)$

$= \{t \in S' \mid d(s,t) < \frac{\delta_s}{2}\}$

$\mathcal{V} = \{V_s : s \in E\}$ is an open cover of E.

By compactness, $\exists\, V_{s_1}, V_{s_2}, \dots, V_{s_n}$ s.t.

$$E \subseteq V_{s_1} \cup V_{s_2} \cup \dots \cup V_{s_n}$$

$\delta_j \sim \delta_{s_j}$

Define $\delta = \frac{1}{2} \min\{\delta_1, \delta_2, \dots, \delta_n\}$

Consider $s, t \in E$ with $d(s,t) < \delta$. Since $s \in V_{s_k}$ for some s_k, then $d(s, s_k) < \frac{\delta_{s_k}}{2}$.

$$d(t, s_k) \le d(t,s) + d(s, s_k) < \delta + \frac{\delta_{s_k}}{2} < \delta_{s_k}$$

$$d(s, s_k) < \delta_{s_k} \Rightarrow d^*(f(s), f(s_k)) < \frac{\varepsilon}{2}$$

$$d(t, s_k) < \delta_{s_k} \Rightarrow d^*(f(t), f(s_k)) < \frac{\varepsilon}{2}$$

$d^*(f(s), f(t)) < \varepsilon$. $\Rightarrow$ uniformly continuous.

Corollary

Let $f : (S, d) \to \mathbb{R}$ is continuous, and $E \subseteq S$ is compact.

i) f is bounded on E

ii) f assumes its minimum and maximum on E.

Proof

$f(E)$ is compact on $\mathbb{R}$ $\Rightarrow$ $f(E)$ is closed and bounded by the Heine-Borel theorem.