Notes on Physics 415:
Statistical and Thermal Physics

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Contents

1 Basic Statistical Methods

1.1 Random walk: Binomial distribution and the Emergence of Gaussian

- ***Example:*** We introduce important ideas from Probability via an example of **1D random walk:**

> Consider a drunkard walking along a straight line, starting from Origin $x = 0$, and taking randoms steps of length l at regular intervals. Each step is independent of the last. He takes a probability p of steping to the left, and $1 - p$ to step to the right. After taking N steps, what is the probability that the walker is at position $x = ml$?

Let $P_N(m)$ be the position $x = ml$ of the drunkard after N steps; denote n_1 = number of steps to the left, $n_2 = N - n_1$ number of steps to the right. Notice that $-N \leq m \leq N, N = n_1 + n_2, m = n_1 - n_2$.

Then the number of walking combinations, indexed with either direction left (n_1) or right (n_2), is given by the binomial coefficient:

$$\binom{N}{n_1} = \frac{N!}{n_1!(N - n_1)!} = \frac{N!}{n_1!\, n_2!} = \binom{N}{n_2}. \tag{1}$$

Then the probability of the walker taking n_1 steps to the left and n_2 steps to the right is given by the **binomial distribution:**

$$P_N(n_1) = \frac{N!}{n_1! n_2!}\, p^{n_1} q^{n_2}. \tag{2}$$

Noticing $n_1 = \left(\frac{N+m}{2}\right)$ and $n_2 = \left(\frac{N-m}{2}\right)$:

$$P_N(m) = \frac{N!}{\left[\frac{N+m}{2}\right]!\, \left[\frac{N-m}{2}\right]!}\, p^{\frac{N+m}{2}} (1-p)^{\frac{N-m}{2}}. \tag{3}$$

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1.1.1 General Notions from probability:

Let X be a random variable, taking N possible values $x_1, x_2, ..., x_N$ with associated probabilities $P(x_1), P(x_2), ..., P(x_N)$. Note that $0 \leq P(x_i) \leq 1, \sum_{i=1}^{N} P(x_i) = 1$

- Mean: $\overline{x} := \sum_{i=1}^{N} P(x_i) x_i$ Var: $\text{var}(x) := \overline{(x - \overline{x}^2)} = \sum_{i=1}^{N} P(x_i)\, (x_i - \overline{x})^2 = \overline{x^2} - \overline{x}^2$
- RMS: $\Delta x_{\text{rms}} = \sqrt{\overline{x^2} - \overline{x}^2}$
- For Binobial Distribution: $\overline{x} = Np, \quad$ Dispersion : $\text{var}(x) = Npq, \quad \Delta x_{\text{rms}} = \sqrt{Npq}$

  Relative Width: $\dfrac{\Delta n_{1,\text{rms}}}{\overline{n_1}} = \dfrac{q}{p} \dfrac{1}{\sqrt{N}} \to 0 \quad (n \gg 1)$

1.1.2 Central Limit Theorem: Appox. of Binom.

Recall Equation 2, taking logrithm on both sides:

$$\ln(P_N(m)) = \ln(N!) - \ln(n!) - \ln(N - n)! + n \ln(p) + (N - n) \ln(q). \tag{4}$$

For $N \gg 1$, we can approximate using Stirling's formula:

$$N! \approx \sqrt{2\pi N} N^N e^{-N}, \tag{5}$$

and further algebra gives

$$P_N(m) \approx \sqrt{\frac{N}{2\pi n(N-n)}}\,\exp\left[-N\,f\left(\frac{n}{N}\right)\right], \qquad (N \gg 1) \tag{6}$$

where

$$f(x) = [x\ln x + (1-x)\ln(1-x)] - [x\ln p + (1-x)\ln q]. \tag{7}$$

For N large, P_N peakes sharply near max $\tilde{n} = Np$, which is found by maximizing $f(x)$. Expanding $f(x)$ about $\tilde{n}$, and taking $n \approx \tilde{n}$ in P_N we have :

$$\boxed{P_N(m) \approx \frac{1}{\sqrt{2\pi Npq}}\exp\left[-\frac{(n-Np)^2}{2Npq}\right],} \tag{8}$$

which is a Gaussian distribution with mean $\mu = \overline{x} = Np, \quad \sigma^2 = Npq, \quad \Delta x_{\text{rms}} = \sqrt{Npq}$.

1.2 Probability Distribution with Multivariables

Consider two r.v. u, v , which can assume possible values u_i, v_i for $i = 1, 2, ..., M; j = 1, 2, ..., N$.

- Normalization conditoin

$$\sum_{i=1}^{M}\sum_{j=1}^{N} P(u_i, v_j) = 1. \tag{9}$$

- Unconditioned prob. distribution:

$$P(u_i) = \sum_{j=1}^{N} P(u_i, v_j), \quad P(v_j) = \sum_{i=1}^{M} P(u_i, v_j). \tag{10}$$

- Statistical independence:

$$P(u_i, v_j) = P(u_i)P(v_j), \tag{11}$$

  in which case the mean of the product is the product of the means:

$$\overline{uv} = \overline{u}\,\overline{v}. \tag{12}$$

1.3 Continuous probability distribution

For continuous r.v. $x \in (a_1, a_2)$, assign value of r.v. to $f(x)$.

The probability density function $p(x)$ is normalized:

$$\int_{a_1}^{a_2} p(x)\,\mathrm{d}x = 1. \tag{13}$$

The mean and variance are defined as:

$$\overline{x} = \int_{a_1}^{a_2} f(x)p(x)\,\mathrm{d}x, \quad \operatorname{var}(x) = \int_{a_1}^{a_2} (x-\overline{x})^2 p(x)\,\mathrm{d}x. \tag{14}$$

- Especially, $p(x)\,\mathrm{d}x$ represents prob. to find x in $[x, x+\mathrm{d}x]$.
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