

# Homework Guide: Assignment 6

## Problem 1

(a) Let  $f$  be a uniformly continuous function on  $\mathbb{R}$ , and define the sequence of functions  $f_n(x) = f(x - \frac{1}{n})$ . Prove that  $f_n \rightarrow f$  uniformly.

(b) Suppose that  $f$  is a continuous function on  $\mathbb{R}$ , and again define  $f_n(x) = f(x - \frac{1}{n})$ . Find an example where  $f_n$  does not uniformly converge to  $f$ .

## Relevant Definitions and Theorems

**Definition** (Uniform Continuity <sup>1</sup>). A function  $f : S \rightarrow \mathbb{R}$  is uniformly continuous on  $S$  if for every  $\epsilon > 0$ , there exists a  $\delta > 0$  such that for all  $x, y \in S$ , if  $|x - y| < \delta$ , then  $|f(x) - f(y)| < \epsilon$ .

**Definition** (Uniform Convergence <sup>2</sup>). A sequence of functions  $(f_n)$  on a set  $S$  converges uniformly to a function  $f$  on  $S$  if for every  $\epsilon > 0$ , there exists an  $N \in \mathbb{N}$  such that for all  $n > N$  and all  $x \in S$ ,  $|f_n(x) - f(x)| < \epsilon$ .

**Definition** (Continuity <sup>3</sup>). A function  $f : S \rightarrow \mathbb{R}$  is continuous on  $S$  if it is continuous at every point  $x_0 \in S$ .

## Solution Outline

(a)

1. Let  $\epsilon > 0$ .
2. Since  $f$  is uniformly continuous on  $\mathbb{R}$ , there exists  $\delta > 0$  such that  $|y - z| < \delta \implies |f(y) - f(z)| < \epsilon$ .
3. We need to show  $\forall \epsilon > 0, \exists N \in \mathbb{N}$  such that  $n > N \implies |f_n(x) - f(x)| < \epsilon$  for all  $x \in \mathbb{R}$ .
4. Consider  $|f_n(x) - f(x)| = |f(x - \frac{1}{n}) - f(x)|$ .
5. Let  $y = x - \frac{1}{n}$  and  $z = x$ . Then  $|y - z| = |-\frac{1}{n}| = \frac{1}{n}$ .
6. We need  $|y - z| < \delta$ , which means  $\frac{1}{n} < \delta$ .
7. Choose  $N \in \mathbb{N}$  such that  $N > 1/\delta$ .
8. If  $n > N$ , then  $\frac{1}{n} < \frac{1}{N} < \delta$ .
9. Thus, for  $n > N$ ,  $|y - z| = \frac{1}{n} < \delta$ , which implies  $|f(y) - f(z)| < \epsilon$ .
10. This means  $|f(x - \frac{1}{n}) - f(x)| < \epsilon$  for all  $x \in \mathbb{R}$  when  $n > N$ .
11. By definition,  $f_n \rightarrow f$  uniformly on  $\mathbb{R}$ .

(b)

1. We need a continuous function on  $\mathbb{R}$  that is not uniformly continuous. Consider  $f(x) = x^2$ .
2.  $f(x) = x^2$  is continuous on  $\mathbb{R}$ .
3. Define  $f_n(x) = f(x - \frac{1}{n}) = (x - \frac{1}{n})^2$ .
4. Consider the difference:

$$|f_n(x) - f(x)| = |(x - \frac{1}{n})^2 - x^2| = |x^2 - \frac{2x}{n} + \frac{1}{n^2} - x^2| = |-\frac{2x}{n} + \frac{1}{n^2}|.$$

5. For uniform convergence, for a given  $\epsilon > 0$ , we need  $\exists N$  such that for  $n > N$ ,  $|-\frac{2x}{n} + \frac{1}{n^2}| < \epsilon$  for all  $x \in \mathbb{R}$ .
6. Let  $\epsilon = 1$ . Suppose such an  $N$  exists.
7. Choose  $n > N$ . Let  $x = n$ .
8. Then  $|f_n(n) - f(n)| = |-\frac{2n}{n} + \frac{1}{n^2}| = |-2 + \frac{1}{n^2}|$ .

<sup>1</sup>See Definition 19.1 in Ross, K. A. \*Elementary Analysis: The Theory of Calculus\*. Springer, 2013.

<sup>2</sup>See Definition 24.1 in Ross, K. A. \*Elementary Analysis: The Theory of Calculus\*. Springer, 2013.

<sup>3</sup>See Definition 17.1 in Ross, K. A. \*Elementary Analysis: The Theory of Calculus\*. Springer, 2013.

9. If  $n \geq 2$ , then  $0 < \frac{1}{n^2} \leq \frac{1}{4}$ , so  $|-2 + \frac{1}{n^2}| = 2 - \frac{1}{n^2} \geq 2 - \frac{1}{4} = \frac{7}{4} > 1$ .
  10. This contradicts the assumption that  $|f_n(x) - f(x)| < 1$  for all  $x$  when  $n > N$ .
  11. Therefore, the convergence is not uniform for  $f(x) = x^2$ .
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## Problem 2

Let  $f$  be a bounded function on  $[0, 1]$  so that  $|f(x)| \leq M$  for all  $x \in [0, 1]$ . Show that the Bernstein polynomials  $B_n f$  are all bounded by  $M$ .

### Relevant Definitions and Theorems

**Definition** (Bounded Function <sup>4</sup>). A function  $f : S \rightarrow \mathbb{R}$  is bounded if there exists  $M \in \mathbb{R}$  such that  $|f(x)| \leq M$  for all  $x \in S$ .

**Definition** (Bernstein Polynomials <sup>5</sup>). For a function  $f$  defined on  $[0, 1]$ , the  $n$ -th Bernstein polynomial for  $f$  is

$$(B_n f)(x) = \sum_{k=0}^n \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right).$$

**Theorem** (Binomial Theorem). For any real numbers  $a, b$  and any integer  $n \geq 0$ ,

$$(a + b)^n = \sum_{k=0}^n \binom{n}{k} a^k b^{n-k}.$$

### Solution Outline

1. Let  $x \in [0, 1]$ . The  $n$ -th Bernstein polynomial is:

$$(B_n f)(x) = \sum_{k=0}^n \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right).$$

2. Take the absolute value:

$$|(B_n f)(x)| = \left| \sum_{k=0}^n \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right) \right|.$$

3. Apply the triangle inequality:

$$|(B_n f)(x)| \leq \sum_{k=0}^n \left| \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right) \right|.$$

4. Since  $x \in [0, 1]$ , the terms  $\binom{n}{k}$ ,  $x^k$ , and  $(1-x)^{n-k}$  are non-negative. Thus,

$$|(B_n f)(x)| \leq \sum_{k=0}^n \binom{n}{k} x^k (1-x)^{n-k} \left| f\left(\frac{k}{n}\right) \right|.$$

5. We are given  $|f(y)| \leq M$  for all  $y \in [0, 1]$ . Since  $\frac{k}{n} \in [0, 1]$ , we have  $|f(\frac{k}{n})| \leq M$ .

6. Substitute this bound:

$$|(B_n f)(x)| \leq \sum_{k=0}^n \binom{n}{k} x^k (1-x)^{n-k} M.$$

7. Factor out  $M$ :

$$|(B_n f)(x)| \leq M \left( \sum_{k=0}^n \binom{n}{k} x^k (1-x)^{n-k} \right).$$

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<sup>4</sup>See Definition 13.1 in Ross, K. A. \*Elementary Analysis: The Theory of Calculus\*. Springer, 2013.

<sup>5</sup>See Section 26 in Ross, K. A. \*Elementary Analysis: The Theory of Calculus\*. Springer, 2013.

8. By the Binomial Theorem with  $a = x$  and  $b = 1 - x$ :

$$\sum_{k=0}^n \binom{n}{k} x^k (1-x)^{n-k} = (x + (1-x))^n = 1^n = 1.$$

9. Substitute this back:

$$|(B_n f)(x)| \leq M \cdot 1 = M.$$

10. This holds for all  $x \in [0, 1]$  and all  $n \in \mathbb{N}$ . Thus,  $B_n f$  are uniformly bounded by  $M$ .

### Problem 3

Let  $f$  and  $g$  be differentiable on an open interval  $I$  and let  $a \in I$ . Define

$$h(x) = \begin{cases} f(x) & \text{for } x < a, \\ g(x) & \text{for } x \geq a. \end{cases}$$

Prove that  $h$  is differentiable at  $a$  if and only if both  $f(a) = g(a)$  and  $f'(a) = g'(a)$ .

#### Relevant Definitions and Theorems

**Definition** (Differentiability at a Point <sup>6</sup>). A function  $h$  defined on an open interval containing  $a$  is differentiable at  $a$  if the limit  $h'(a) = \lim_{x \rightarrow a} \frac{h(x) - h(a)}{x - a}$  exists.

**Theorem** (Differentiability implies Continuity <sup>7</sup>). If  $h$  is differentiable at  $a$ , then  $h$  is continuous at  $a$ .

**Definition** (Continuity at a Point <sup>8</sup>). A function  $h$  is continuous at  $a$  if  $\lim_{x \rightarrow a} h(x) = h(a)$ . This requires  $\lim_{x \rightarrow a^-} h(x) = \lim_{x \rightarrow a^+} h(x) = h(a)$ .

**Theorem** (Existence of Limit <sup>9</sup>). The limit  $\lim_{x \rightarrow a} G(x)$  exists and equals  $L$  if and only if the left-hand limit  $\lim_{x \rightarrow a^-} G(x)$  and the right-hand limit  $\lim_{x \rightarrow a^+} G(x)$  both exist and equal  $L$ .

#### Solution Outline

( $\Rightarrow$ ) Assume  $h$  is differentiable at  $a$ .

1. Since  $h$  is differentiable at  $a$ , it is continuous at  $a$  (Thm 28.2).
2. Continuity at  $a$  implies  $\lim_{x \rightarrow a^-} h(x) = \lim_{x \rightarrow a^+} h(x) = h(a)$ .
3. Evaluate these using the definition of  $h$ :
  - $h(a) = g(a)$ .
  - $\lim_{x \rightarrow a^-} h(x) = \lim_{x \rightarrow a^-} f(x) = f(a)$  (since  $f$  is continuous at  $a$ ).
  - $\lim_{x \rightarrow a^+} h(x) = \lim_{x \rightarrow a^+} g(x) = g(a)$  (since  $g$  is continuous at  $a$ ).
4. Thus,  $f(a) = g(a)$ . This is the first condition.
5. Since  $h$  is differentiable at  $a$ , the limit  $h'(a) = \lim_{x \rightarrow a} \frac{h(x) - h(a)}{x - a}$  exists.
6. The left-hand and right-hand limits of the difference quotient must exist and be equal.
7. Left-hand derivative:

$$\lim_{x \rightarrow a^-} \frac{h(x) - h(a)}{x - a} = \lim_{x \rightarrow a^-} \frac{f(x) - g(a)}{x - a}.$$

Using  $f(a) = g(a)$ :

$$\lim_{x \rightarrow a^-} \frac{f(x) - f(a)}{x - a} = f'(a)$$

(since  $f$  is differentiable at  $a$ ).

<sup>6</sup>See Definition 28.1 in Ross, K. A. \*Elementary Analysis: The Theory of Calculus\*. Springer, 2013.

<sup>7</sup>See Theorem 28.2 in Ross, K. A. \*Elementary Analysis: The Theory of Calculus\*. Springer, 2013.

<sup>8</sup>See Definition 17.1 in Ross, K. A. \*Elementary Analysis: The Theory of Calculus\*. Springer, 2013.

<sup>9</sup>See Definition 20.1 in Ross, K. A. \*Elementary Analysis: The Theory of Calculus\*. Springer, 2013.

8. Right-hand derivative:

$$\lim_{x \rightarrow a^+} \frac{h(x) - h(a)}{x - a} = \lim_{x \rightarrow a^+} \frac{g(x) - g(a)}{x - a} = g'(a)$$

(since  $g$  is differentiable at  $a$ ).

9. For  $h'(a)$  to exist, the one-sided derivatives must be equal:  $f'(a) = g'(a)$ . This is the second condition.

( $\Leftarrow$ ) **Assume**  $f(a) = g(a)$  **and**  $f'(a) = g'(a)$ .

1. We need to show that  $h'(a) = \lim_{x \rightarrow a} \frac{h(x) - h(a)}{x - a}$  exists.

2. Examine the left-hand and right-hand limits of the difference quotient.

3. Left-hand derivative:

$$\lim_{x \rightarrow a^-} \frac{h(x) - h(a)}{x - a} = \lim_{x \rightarrow a^-} \frac{f(x) - g(a)}{x - a}.$$

Using  $f(a) = g(a)$ :

$$\lim_{x \rightarrow a^-} \frac{f(x) - f(a)}{x - a} = f'(a).$$

4. Right-hand derivative:

$$\lim_{x \rightarrow a^+} \frac{h(x) - h(a)}{x - a} = \lim_{x \rightarrow a^+} \frac{g(x) - g(a)}{x - a} = g'(a).$$

5. We are given  $f'(a) = g'(a)$ . Let  $L = f'(a) = g'(a)$ .

6. The left-hand derivative is  $L$  and the right-hand derivative is  $L$ .

7. Since the left-hand and right-hand limits exist and are equal, the limit exists:

$$\lim_{x \rightarrow a} \frac{h(x) - h(a)}{x - a} = L.$$

8. Therefore,  $h$  is differentiable at  $a$  and  $h'(a) = L$ .

## Problem 4

Suppose  $f$  is differentiable at  $a$ . Define

$$L_1(a, h) = \frac{f(a + h) - f(a - h)}{2h}, \quad L_2(a, h) = \frac{-f(a + 2h) + 8f(a + h) - 8f(a - h) + f(a - 2h)}{12h}.$$

(a) Prove that  $\lim_{h \rightarrow 0} L_i(a, h) = f'(a)$  for  $i = 1, 2$ . (b) Consider  $f(x) = x^5$ . How does  $|L_i(a, h) - f'(a)|$  behave as  $h \rightarrow 0$ ?

## Relevant Definitions and Theorems

**Definition** (Differentiability at a Point <sup>10</sup>).  $f$  is differentiable at  $a$  if  $f'(a) = \lim_{k \rightarrow 0} \frac{f(a+k) - f(a)}{k}$  exists. This implies  $f(a + k) = f(a) + f'(a)k + \phi(k)$  where  $\lim_{k \rightarrow 0} \frac{\phi(k)}{k} = 0$ .

**Theorem** (Taylor's Theorem <sup>11</sup>). If  $f$  has  $n + 1$  derivatives in an interval around  $a$ , then for  $x$  in that interval,

$$f(x) = \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} (x - a)^k + R_n(x),$$

where  $R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} (x - a)^{n+1}$  for some  $c$  between  $a$  and  $x$ .

<sup>10</sup>See Definition 28.1 in Ross, K. A. \*Elementary Analysis: The Theory of Calculus\*. Springer, 2013.

<sup>11</sup>See Theorem 30.2 in Ross, K. A. \*Elementary Analysis: The Theory of Calculus\*. Springer, 2013.

## Solution Outline

(a) For  $L_1(a, h)$ :

1. Rewrite  $L_1(a, h)$ :

$$L_1(a, h) = \frac{f(a+h) - f(a) - (f(a-h) - f(a))}{2h} = \frac{1}{2} \left[ \frac{f(a+h) - f(a)}{h} - \frac{f(a-h) - f(a)}{h} \right].$$

2. Let  $k = -h$ . Then  $\frac{f(a-h) - f(a)}{h} = \frac{f(a+k) - f(a)}{-k} = -\frac{f(a+k) - f(a)}{k}$ .

3. So,  $L_1(a, h) = \frac{1}{2} \left[ \frac{f(a+h) - f(a)}{h} + \frac{f(a+(-h)) - f(a)}{-h} \right]$ .

4. Take the limit as  $h \rightarrow 0$ . Since  $f$  is differentiable at  $a$ :

$$\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} = f'(a), \quad \lim_{h \rightarrow 0} \frac{f(a+(-h)) - f(a)}{-h} = f'(a).$$

5. Therefore,  $\lim_{h \rightarrow 0} L_1(a, h) = \frac{1}{2}[f'(a) + f'(a)] = f'(a)$ .

For  $L_2(a, h)$ :

1. Use the definition  $f(a+k) = f(a) + f'(a)k + \phi(k)$ , where  $\phi(k)/k \rightarrow 0$  as  $k \rightarrow 0$ .

2. Substitute into the numerator of  $L_2(a, h)$  for  $k = 2h, h, -h, -2h$ .

3. Sum the terms:

- $f(a)$  terms:  $-1 + 8 - 8 + 1 = 0$ .
- $f'(a)k$  terms:  $f'(a)[-2h + 8h - 8(-h) + (-2h)] = f'(a)[-2h + 8h + 8h - 2h] = 12f'(a)h$ .
- Remainder terms:  $R(h) = -\phi(2h) + 8\phi(h) - 8\phi(-h) + \phi(-2h)$ .

4. Numerator  $= 12f'(a)h + R(h)$ .

5.  $L_2(a, h) = \frac{12f'(a)h + R(h)}{12h} = f'(a) + \frac{R(h)}{12h}$ .

6. We need to show  $\lim_{h \rightarrow 0} \frac{R(h)}{12h} = 0$ .

$$\begin{aligned} \frac{R(h)}{12h} &= \frac{1}{12} \left[ -\frac{\phi(2h)}{h} + 8\frac{\phi(h)}{h} - 8\frac{\phi(-h)}{h} + \frac{\phi(-2h)}{h} \right] \\ &= \frac{1}{12} \left[ -2\frac{\phi(2h)}{2h} + 8\frac{\phi(h)}{h} + 8\frac{\phi(-h)}{-h} - 2\frac{\phi(-2h)}{-2h} \right]. \end{aligned}$$

7. As  $h \rightarrow 0$ , each term  $\frac{\phi(k)}{k} \rightarrow 0$ .

8. Thus,  $\lim_{h \rightarrow 0} \frac{R(h)}{12h} = \frac{1}{12}[-2(0) + 8(0) + 8(0) - 2(0)] = 0$ .

9. Therefore,  $\lim_{h \rightarrow 0} L_2(a, h) = f'(a) + 0 = f'(a)$ .

(b)

1. Let  $f(x) = x^5$ . Derivatives:  $f'(x) = 5x^4$ ,  $f''(x) = 20x^3$ ,  $f'''(x) = 60x^2$ ,  $f^{(4)}(x) = 120x$ ,  $f^{(5)}(x) = 120$ ,  $f^{(k)}(x) = 0$  for  $k \geq 6$ .

2. Use Taylor expansion around  $a$ :

$$f(a+h) = f(a) + f'(a)h + \frac{f''(a)}{2}h^2 + \frac{f'''(a)}{6}h^3 + \frac{f^{(4)}(a)}{24}h^4 + \frac{f^{(5)}(a)}{120}h^5.$$

For  $L_1(a, h)$ :

3. Calculate  $f(a+h) - f(a-h)$  using the expansion. Odd power terms add, even power terms cancel.

$$f(a+h) - f(a-h) = 2f'(a)h + 2\frac{f'''(a)}{6}h^3 + 2\frac{f^{(5)}(a)}{120}h^5 = 2f'(a)h + \frac{f'''(a)}{3}h^3 + \frac{f^{(5)}(a)}{60}h^5.$$

$$4. L_1(a, h) = \frac{f(a+h) - f(a-h)}{2h} = f'(a) + \frac{f'''(a)}{6}h^2 + \frac{f^{(5)}(a)}{120}h^4.$$

$$5. L_1(a, h) - f'(a) = \frac{f'''(a)}{6}h^2 + \frac{f^{(5)}(a)}{120}h^4.$$

$$6. \text{Substitute derivatives of } x^5: f'''(a) = 60a^2, f^{(5)}(a) = 120.$$

$$L_1(a, h) - f'(a) = \frac{60a^2}{6}h^2 + \frac{120}{120}h^4 = 10a^2h^2 + h^4.$$

$$7. |L_1(a, h) - f'(a)| = |10a^2h^2 + h^4| = O(h^2) \text{ as } h \rightarrow 0 \text{ (unless } a = 0, \text{ then } O(h^4)). \text{ For } L_2(a, h):$$

$$8. \text{Calculate } -f(a+2h) + 8f(a+h) - 8f(a-h) + f(a-2h) \text{ using Taylor expansions.}$$

$$9. \text{Coefficients of } h^j/j! \text{ for } j = 0, 2, 3, 4 \text{ cancel out.}$$

$$10. \text{Coefficient of } h^1/1!: f'(a)[-2 + 8 - 8(-1) + (-2)] = 12f'(a).$$

$$11. \text{Coefficient of } h^5/5!: f^{(5)}(a)[-(2)^5 + 8(1)^5 - 8(-1)^5 + (-2)^5] = f^{(5)}(a)[-32 + 8 + 8 - 32] = -48f^{(5)}(a).$$

$$12. \text{Numerator} = 12f'(a)h + \frac{-48f^{(5)}(a)}{120}h^5 = 12f'(a)h - \frac{2}{5}f^{(5)}(a)h^5.$$

$$13. L_2(a, h) = \frac{12f'(a)h - \frac{2}{5}f^{(5)}(a)h^5}{12h} = f'(a) - \frac{1}{30}f^{(5)}(a)h^4.$$

$$14. L_2(a, h) - f'(a) = -\frac{1}{30}f^{(5)}(a)h^4.$$

$$15. \text{Substitute } f^{(5)}(a) = 120:$$

$$L_2(a, h) - f'(a) = -\frac{1}{30}(120)h^4 = -4h^4.$$

$$16. |L_2(a, h) - f'(a)| = |-4h^4| = 4h^4 = O(h^4) \text{ as } h \rightarrow 0. \text{ Comparison:}$$

$$17. \text{The error for } L_1 \text{ is } O(h^2), \text{ while the error for } L_2 \text{ is } O(h^4).$$

$$18. L_2 \text{ converges to } f'(a) \text{ faster than } L_1 \text{ as } h \rightarrow 0.$$

## Problem 5

For a real-valued function  $f$ ,  $x$  is a fixed point if  $f(x) = x$ . Show that if  $f$  is differentiable on an interval with  $f'(x) \neq 1$ , then  $f$  can have at most one fixed point.

### Relevant Definitions and Theorems

**Definition** (Fixed Point). A point  $x$  such that  $f(x) = x$ .

**Theorem** (Mean Value Theorem (MVT) <sup>12</sup>). If  $g$  is continuous on  $[a, b]$  and differentiable on  $(a, b)$ , then there exists  $c \in (a, b)$  such that  $g(b) - g(a) = g'(c)(b - a)$ .

**Theorem** (Rolle's Theorem <sup>13</sup>). If  $g$  is continuous on  $[a, b]$ , differentiable on  $(a, b)$ , and  $g(a) = g(b)$ , then there exists  $c \in (a, b)$  such that  $g'(c) = 0$ .

### Solution Outline

#### Method 1: Using MVT directly

1. Assume for contradiction that  $f$  has two distinct fixed points,  $x_1$  and  $x_2$ , with  $x_1 < x_2$ .
2. By definition,  $f(x_1) = x_1$  and  $f(x_2) = x_2$ .
3.  $f$  is differentiable on an interval  $I$  containing  $x_1, x_2$ . Thus  $f$  is continuous on  $[x_1, x_2]$  and differentiable on  $(x_1, x_2)$ .

<sup>12</sup>See Theorem 29.3 in Ross, K. A. \*Elementary Analysis: The Theory of Calculus\*. Springer, 2013.

<sup>13</sup>See Theorem 29.2 in Ross, K. A. \*Elementary Analysis: The Theory of Calculus\*. Springer, 2013.

4. Apply MVT to  $f$  on  $[x_1, x_2]$ . There exists  $c \in (x_1, x_2)$  such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

5. Substitute  $f(x_1) = x_1$  and  $f(x_2) = x_2$ :

$$f'(c) = \frac{x_2 - x_1}{x_2 - x_1}.$$

6. Since  $x_1 \neq x_2$ ,  $x_2 - x_1 \neq 0$ . Therefore,  $f'(c) = 1$ .

7. This contradicts the hypothesis that  $f'(x) \neq 1$  for all  $x \in I$ . Since  $c \in (x_1, x_2) \subseteq I$ , we must have  $f'(c) \neq 1$ .

8. The assumption of two distinct fixed points leads to a contradiction.

9. Therefore,  $f$  can have at most one fixed point.

## Method 2: Using Rolle's Theorem

1. Define  $g(x) = f(x) - x$ . Fixed points of  $f$  are roots of  $g$ .

2.  $g$  is differentiable on  $I$  since  $f(x)$  and  $x$  are.

3.  $g'(x) = f'(x) - 1$ .

4. The hypothesis  $f'(x) \neq 1$  implies  $g'(x) = f'(x) - 1 \neq 0$  for all  $x \in I$ .

5. Assume for contradiction that  $f$  has two distinct fixed points,  $x_1$  and  $x_2$ , with  $x_1 < x_2$ .

6. Then  $g(x_1) = f(x_1) - x_1 = 0$  and  $g(x_2) = f(x_2) - x_2 = 0$ .

7.  $g$  is continuous on  $[x_1, x_2]$ , differentiable on  $(x_1, x_2)$ , and  $g(x_1) = g(x_2) = 0$ .

8. By Rolle's Theorem, there exists  $c \in (x_1, x_2)$  such that  $g'(c) = 0$ .

9. This contradicts the fact that  $g'(x) \neq 0$  for all  $x \in I$ .

10. The assumption of two distinct fixed points leads to a contradiction.

11. Therefore,  $f$  can have at most one fixed point.

## Problem 6

Find the following limits if they exist: (a)  $\lim_{x \rightarrow 0} \frac{x^3}{\sin x - x}$  (b)  $\lim_{x \rightarrow 0} \frac{\tan x - x}{x^3}$  (c)  $\lim_{x \rightarrow 0} \left( \frac{1}{\sin x} - \frac{1}{x} \right)$  (d)  $\lim_{x \rightarrow 0} \frac{1 - \cos x}{e^{3x} - 3x - 1}$

## Relevant Theorems

**Theorem** (L'Hôpital's Rule <sup>14</sup>). *If  $\lim f(x) = \lim g(x) = 0$  (or  $\pm\infty$ ) and  $\lim \frac{f'(x)}{g'(x)}$  exists, then  $\lim \frac{f(x)}{g(x)} = \lim \frac{f'(x)}{g'(x)}$ .*

## Taylor Series Expansions around 0:

- $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$
- $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$
- $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \dots$
- $e^u = 1 + u + \frac{u^2}{2!} + \frac{u^3}{3!} + \dots$

<sup>14</sup>See Theorems 30.3, 30.4 in Ross, K. A. \*Elementary Analysis: The Theory of Calculus\*. Springer, 2013.

## Solution Outline

(a)  $\lim_{x \rightarrow 0} \frac{x^3}{\sin x - x}$

- Taylor:  $\sin x - x = (x - \frac{x^3}{6} + O(x^5)) - x = -\frac{x^3}{6} + O(x^5)$ .

$$\lim_{x \rightarrow 0} \frac{x^3}{-\frac{x^3}{6} + O(x^5)} = \lim_{x \rightarrow 0} \frac{1}{-\frac{1}{6} + O(x^2)} = -6.$$

- L'Hôpital (form  $\frac{0}{0}$ ):

$$\lim_{x \rightarrow 0} \frac{3x^2}{\cos x - 1} \stackrel{L'H}{=} \lim_{x \rightarrow 0} \frac{6x}{-\sin x} \stackrel{L'H}{=} \lim_{x \rightarrow 0} \frac{6}{-\cos x} = \frac{6}{-1} = -6.$$

(b)  $\lim_{x \rightarrow 0} \frac{\tan x - x}{x^3}$

- Taylor:  $\tan x - x = (x + \frac{x^3}{3} + O(x^5)) - x = \frac{x^3}{3} + O(x^5)$ .

$$\lim_{x \rightarrow 0} \frac{\frac{x^3}{3} + O(x^5)}{x^3} = \lim_{x \rightarrow 0} (\frac{1}{3} + O(x^2)) = \frac{1}{3}.$$

- L'Hôpital (form  $\frac{0}{0}$ ):

$$\lim_{x \rightarrow 0} \frac{\sec^2 x - 1}{3x^2} \stackrel{L'H}{=} \lim_{x \rightarrow 0} \frac{2 \sec^2 x \tan x}{6x} = \lim_{x \rightarrow 0} \frac{\sec^2 x}{3} \cdot \frac{\tan x}{x} = \frac{1}{3} \cdot 1 = \frac{1}{3}.$$

(c)  $\lim_{x \rightarrow 0} \left( \frac{1}{\sin x} - \frac{1}{x} \right)$

- Combine:  $\lim_{x \rightarrow 0} \frac{x - \sin x}{x \sin x}$ . (Form  $\frac{0}{0}$ ).
- Taylor: Numerator is  $\frac{x^3}{6} + O(x^5)$ . Denominator is  $x(x - \frac{x^3}{6} + \dots) = x^2 - \frac{x^4}{6} + \dots$

$$\lim_{x \rightarrow 0} \frac{\frac{x^3}{6} + O(x^5)}{x^2 - \frac{x^4}{6} + O(x^4)} = \lim_{x \rightarrow 0} \frac{x(\frac{1}{6} + O(x^2))}{1 + O(x^2)} = 0 \cdot \frac{1/6}{1} = 0.$$

- L'Hôpital:

$$\lim_{x \rightarrow 0} \frac{1 - \cos x}{\sin x + x \cos x} \stackrel{L'H}{=} \lim_{x \rightarrow 0} \frac{\sin x}{\cos x + \cos x - x \sin x} = \frac{0}{1 + 1 - 0} = 0.$$

(d)  $\lim_{x \rightarrow 0} \frac{1 - \cos x}{e^{3x} - 3x - 1}$

- (Form  $\frac{0}{0}$ ).
- Taylor: Numerator is  $\frac{x^2}{2} + O(x^4)$ . Denominator is  $(1 + 3x + \frac{(3x)^2}{2} + O(x^3)) - 3x - 1 = \frac{9x^2}{2} + O(x^3)$ .

$$\lim_{x \rightarrow 0} \frac{\frac{x^2}{2} + O(x^4)}{\frac{9x^2}{2} + O(x^3)} = \lim_{x \rightarrow 0} \frac{\frac{1}{2} + O(x^2)}{\frac{9}{2} + O(x)} = \frac{1/2}{9/2} = \frac{1}{9}.$$

- L'Hôpital:

$$\lim_{x \rightarrow 0} \frac{\sin x}{3e^{3x} - 3} \stackrel{L'H}{=} \lim_{x \rightarrow 0} \frac{\cos x}{9e^{3x}} = \frac{1}{9}.$$

## Problem 7

- (a) Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be twice differentiable,  $f(0) = 0$ ,  $f''(x) \geq 0$  for  $x > 0$ . Prove  $g(x) = f(x)/x$  is increasing for  $x > 0$ .
- (b) If  $f : \mathbb{R} \rightarrow \mathbb{R}$  is twice differentiable,  $f(0) = 0$  and  $g(x) = f(x)/x$  is increasing for  $x > 0$ , show  $f''(x) \geq 0$  for *some*  $x > 0$ , but not necessarily for *all*  $x > 0$ .



## Relevant Definitions and Theorems

**Definition** (Increasing Function <sup>15</sup>).  $g$  is increasing on an interval if  $x_1 < x_2$  implies  $g(x_1) \leq g(x_2)$ .

**Theorem** (Derivative Test for Increasing Function <sup>16</sup>). If  $g$  is differentiable on  $(a, b)$  and  $g'(x) \geq 0$  for all  $x \in (a, b)$ , then  $g$  is increasing on  $(a, b)$ .

## Solution Outline

(a)

1. Define  $g(x) = f(x)/x$  for  $x > 0$ .  $g$  is differentiable for  $x > 0$ .
2. We show  $g'(x) \geq 0$  for  $x > 0$ .
3. Calculate  $g'(x)$  using the quotient rule:

$$g'(x) = \frac{xf'(x) - f(x)}{x^2}.$$

4. Let  $N(x) = xf'(x) - f(x)$ . We need to show  $N(x) \geq 0$  for  $x > 0$ .
  5. Calculate  $N'(x)$ :
- $$N'(x) = (1 \cdot f'(x) + xf''(x)) - f'(x) = xf''(x).$$
6. Given  $f''(x) \geq 0$  for  $x > 0$ . Since  $x > 0$ ,  $N'(x) = xf''(x) \geq 0$  for  $x > 0$ .
  7. Since  $N'(x) \geq 0$ ,  $N(x)$  is increasing on  $(0, \infty)$ .
  8. Consider the limit of  $N(x)$  as  $x \rightarrow 0^+$ :

$$\lim_{x \rightarrow 0^+} N(x) = \lim_{x \rightarrow 0^+} (xf'(x) - f(x)).$$

Since  $f$  is differentiable at 0 and  $f(0) = 0$ ,  $f'(0) = \lim_{x \rightarrow 0} f(x)/x$ . Since  $f''$  exists,  $f'$  is continuous.

$$\lim_{x \rightarrow 0^+} N(x) = (0 \cdot f'(0)) - f(0) = 0 - 0 = 0.$$

9. Since  $N(x)$  is increasing on  $(0, \infty)$  and  $\lim_{x \rightarrow 0^+} N(x) = 0$ , we have  $N(x) \geq 0$  for all  $x > 0$ .
10. Therefore,  $g'(x) = N(x)/x^2 \geq 0$  for  $x > 0$ .
11. By the theorem,  $g(x) = f(x)/x$  is increasing on  $(0, \infty)$ .

(b)

1. Assume  $f$  is twice differentiable,  $f(0) = 0$ , and  $g(x) = f(x)/x$  is increasing for  $x > 0$ .
2. From (a),  $g'(x) = \frac{xf'(x) - f(x)}{x^2} \geq 0$  for  $x > 0$ .
3. Let  $N(x) = xf'(x) - f(x)$ . Then  $N(x) \geq 0$  for  $x > 0$ .
4. Also  $N'(x) = xf''(x)$  and  $\lim_{x \rightarrow 0^+} N(x) = 0$ .
5. **Show  $f''(x) \geq 0$  for some  $x > 0$ :** Assume for contradiction that  $f''(x) < 0$  for all  $x > 0$ . Then  $N'(x) = xf''(x) < 0$  for all  $x > 0$ . This implies  $N(x)$  is strictly decreasing on  $(0, \infty)$ . Since  $\lim_{x \rightarrow 0^+} N(x) = 0$ , this would mean  $N(x) < 0$  for all  $x > 0$ . This contradicts  $N(x) \geq 0$ . Therefore, the assumption is false. There must exist some  $x_0 > 0$  such that  $f''(x_0) \geq 0$ .
6. **Show  $f''(x)$  is not necessarily  $\geq 0$  for all  $x > 0$ :** Consider  $f(x) = x(1 - e^{-x}) = x - xe^{-x}$ .
  - $f(0) = 0$ .  $f$  is twice differentiable.
  - $g(x) = f(x)/x = 1 - e^{-x}$  for  $x \neq 0$ .
  - $g'(x) = e^{-x} > 0$  for all  $x$ . So  $g(x)$  is increasing for  $x > 0$ .
  - $f'(x) = 1 - e^{-x} + xe^{-x}$ .
  - $f''(x) = e^{-x} + (e^{-x} - xe^{-x}) = 2e^{-x} - xe^{-x} = (2 - x)e^{-x}$ .
  - If  $x > 2$ , then  $2 - x < 0$ , so  $f''(x) < 0$ . For example,  $f''(3) = -e^{-3} < 0$ .
  - Thus,  $f''(x)$  is not non-negative for all  $x > 0$ .
7. This example satisfies the conditions but  $f''(x)$  is negative for  $x > 2$ . It does satisfy  $f''(x) \geq 0$  for some  $x > 0$  (e.g., for  $x \in (0, 2]$ ).

<sup>15</sup>See Definition 29.1 in Ross, K. A. \*Elementary Analysis: The Theory of Calculus\*. Springer, 2013.

<sup>16</sup>See Corollary 29.7 in Ross, K. A. \*Elementary Analysis: The Theory of Calculus\*, 2nd ed. Springer, 2013.