# Homework sheet 2 – Due 02/16/2025

**Comment:** *Part of this exercise includes functions of matrices. These are defined by the Taylor series of the corresponding function, e.g.* $e^M = \sum_{k=0} \frac{M^k}{k!}$ *for any matrix* $M$*.*

**Problem 1: Matrix Operations** [1 + 2 + 1 + 2 + 2 + 1 +1 = 10 points]

In this exercise we prove some useful matrix identities.

a) For matrices $A, B, C$, prove

$$[A, BC] = B[A, C] + [A, B]C. \tag{1}$$

b) Prove the Bianchi identity for matrices $A, B, C$

$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0. \tag{2}$$

c) Prove that

$$[A, B]^\dagger = -[A^\dagger, B^\dagger] \text{ and } [A, B]^T = -[A^T, B^T]. \tag{3}$$

d) Consider two matrices $A, B$ such that $[A, B] = C$ where $[A, C] = 0 = [B, C]$

Prove

$$e^{\alpha A} B e^{-\alpha A} = B + \alpha C \tag{4}$$

for arbitrary $\alpha \in \mathbb{C}$.

e) Campbell-Baker-Hausdorff formula. Consider matrices $A, B, C$ with the same properties as in the previous problem. Show

$$e^A e^B = e^{A+B+C/2}. \tag{5}$$

**Hint:** *Define a function* $T(\alpha) = e^{\alpha A} e^{\alpha B}$ *and first study its* $\alpha$*-derivative. Use the result from part d) .*

f) For an involutory matrix $A$ (i.e. $A^2 = \mathbf{1}$), prove

$$e^{i\alpha A} = \cos(\alpha)\mathbf{1} + i \sin(\alpha) A \tag{6}$$

g) Prove that, for any diagonalizable matrix $M$

$$\ln(\det(M)) = \mathrm{tr}\,(\ln(M)). \tag{7}$$

**Problem 2: Single-qubit gates** [1 +1 + 1 +2 + 2 + 2 + 1= 10 points]

In quantum information theory it is common practice to denote the Pauli gates as

$$X = \sigma_x, Y = \sigma_y, Z = \sigma_z. \tag{8}$$

Rotations about the x-axis are denoted $R_x(\theta_x) = e^{-i\theta_x X/2}$ (and analogously for y and z).

a) Show that, up to a phase, the $\pi/8$ gate $T = R_z(\pi/4)$.

b) Show that, up to a phase, the Hadamard gate is a concatenation of $R_y(\theta_y)$ and $R_z(\theta_z)$. Determine the angles $\theta_y, \theta_z$.

c) Show that, since $YXY = -X$,

$$YR_X(\theta_x)Y = R_X(-\theta_x),$$

(i.e. the direction of rotation is reversed by $Y$).

d) Show the following identities for single-qubit gates

$$HXH = Z, HYH = -Y, HZH = X. \tag{9}$$

e) Show that (up to a phase)

$$HTH = R_x(\pi/4). \tag{10}$$

f) Calculate eigenstates of $X, Y$ along with corresponding eigenvalues.

g) Show that the eigenstate of $X$ with eigenvalue +1 can be obtained by applying $R_y(\pi/2)$ on $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Illustrate this statement on the Bloch sphere.

**Problem 3: Higher spin systems. Spin-1 systems** [2 + 2 + 1 + 3 +2= 10 points]

a) In the lectures we found out that $\hat{J}_\pm |j, m\rangle = \hbar c_\pm |j, m \pm 1\rangle$, but did not determine the constant $c_\pm$. Calculate $c_\pm$ (we assume $c_\pm > 0$) when all $|j, m\rangle$ are orthonormalized.

b) For spin-$j$ systems consider the normalized magnetizations $\hat{m}_i = \hat{J}_i/[\hbar j]$ and calculate the Heisenberg-bound on the combined uncertainty of $\hat{m}_x$ and $\hat{m}_y$. Why is $j \to \infty$ sometimes called the semiclassical limit?

c) Explicitly present $\hat{J}_x, \hat{J}_y, \hat{J}_z$ for spin-1 systems (in the basis where $\hat{J}_z$ is diagonal).

**Comment:** *Exemplary spin-1 systems of relevance in atomic physics are* $^{87}Rb$ *and* $^{23}Na$ *(their groundstate forms a "hyperfine" triplet), which were the first cold atomic gases to display Bose-Einstein condensation (Nobel prize 2001).*

d) Convince yourself that spin nematicity operators (i.e. magnetic quadrupole operators)

$$\hat{N}_{ij} = \frac{1}{2}\{\hat{J}_i, \hat{J}_j\} - \frac{1}{3}\delta_{ij}\hat{\vec{J}}^2, \quad i, j = x, y, z. \tag{11}$$

vanish for spin-1/2 systems but they do exist for spin-1 systems. How many non-trivial $\hat{N}_{ij}$ are there for spin-1? Calculate them explicitly.

e) Which $\hat{N}_{jk}$ and $\hat{J}_i$ are compatible? Calculate $[\hat{J}_i, \hat{N}_{jk}]$ to find out.

**Problem 4: Lie Algebra for special unitary group SU(N)**[2+3 + 2 + 3 + 2 + 3= 10 points + 5 bonus points.]

For a Lie group $G$, the Lie algebra $\mathfrak{g}$ is given by the $d_G$-dimensional real vector space of generators $\lambda_a$ of the group supplemented with the Lie-bracket

$$[\lambda_a, \lambda_b] = i \sum_c f_{abc}\lambda_c. \tag{12}$$

The exponential map relates Lie algebra and Lie group

$$\exp : \mathfrak{g} \to G, \alpha \mapsto e^{i\alpha}, \tag{13}$$

In this exercise we consider the special unitary group $G = SU(N)$ of $N \times N$ unitary matrices $U$ with unit determinant $\det(U) = 1$. Elements of the Lie algebra $\alpha \in \mathfrak{su}(N)$ are $N \times N$ matrices, the Lie bracket is just the matrix commutator and the exponential map is just the matrix exponential. This is called the "fundamental representation" of the Lie algebra.

a) Prove that $\mathfrak{su}(N)$ is spanned by traceless, Hermitian matrices.

**Hint:** *Eq.* (7).

The Lie algebra can be equipped with an inner product

$$\langle \alpha, \beta \rangle = \frac{1}{2} \mathrm{tr}\,[\alpha\beta], \tag{14}$$

and we assume the $\{\lambda_a\}_{a=1}^{d_G}$ to be orthonormal with respect to this inner product, hence elements $\alpha \in \mathfrak{su}(N)$ can be expanded as $\alpha = \sum_{a=1}^{d_{SU(N)}} \alpha_a \lambda_a$.

b) Prove that $d_{SU(N)} = N^2 - 1$ and use the results from homework sheet 1 to convince yourself that the Pauli matrices form an orthonormal basis for the fundamental representation of $\mathfrak{su}(2)$. Which physical spin does the fundamental representation of SU(2) correspond to?

c) For general $N$, use the orthonormal basis of the fundamental representation to show that the "structure factors" $f_{abc}$ are real, totally antisymmetric tensors.

A "faithful representation" of a Lie algebra is an injective map $D : \alpha \mapsto D(\alpha)$, where $\alpha \in \mathfrak{g}$ and $D(\alpha)$ is a $d_{D(\mathfrak{g})} \times d_{D(\mathfrak{g})}$ dimensional matrix and the matrices $\{D(\lambda_a)\}_{a=1}^{d_G}$ fulfill the same Lie algebra as $\{\lambda_a\}_{a=1}^{d_G}$

$$[D(\lambda_a), D(\lambda_b)] = i \sum_c f_{abc} D(\lambda_c). \tag{15}$$

d) Use the Bianchi identity to show that the $d_G \times d_G$ matrices $[T_a]_{bc} = -i f_{abc}$ fulfill the Lie algebra (they form the "adjoint representation" $D(\lambda_a) = T_a$).

e) Write down the matrices of the adjoint representation of SU(2). Diagonalize one of the matrices. Which spin does this representation correspond to?

f) Write down the matrices of the orthonormal basis for the fundamental representation of SU(3) and determine a set of matrices which form an SU(2) sub-algebra.

g) In the energy window $E \lesssim 900 MeV$ only three quarks are relevant for quantum chromodynamics. They are distinguished by their flavor quantum number: up ($|u\rangle$), down ($|d\rangle$), strange ($|s\rangle$) with an approximate $SU(3)$ symmetry between them.

**Comment:** *Before this energy range was reached, aspects of particle physics could be understood by means of Heisenberg's SU(2) isospin, essentially acting in $|u\rangle$, $|d\rangle$ space. Once experiments surpassed the energy of the strange-quark rest mass $\sim 95 MeV/c^2$, SU(2) isospin flavor symmetry had to be extended to SU(3).*

i) Discuss the dimension of fundamental and adjoint SU(3) representations

ii) Based on the newly acquired knowledge on Lie algebras, explain the appearance of an octet of mesons (=quark-antiquark boundstates) for $E \lesssim 900 MeV$. Why is there only a triplet for $E \lesssim 200 MeV$?