

Math 521: final exam information

- The final exam will take place on Wednesday May 7th from 2:45pm–4:45pm in Social Sciences 6203.
- The exam will cover all parts of the course with equal weighting. It will cover Chapters 1–5, 7–15, 17–21, 23–34, 36 of the Ross textbook.
- The final will consist of seven questions, which will be of a similar style to those on the midterm. This will give you longer per question than on the midterm.
- The exam will be closed book—no textbooks, notebooks, or smartphones allowed. As on the midterms, you will be expected to be familiar with the basic definitions, and know the key results, but it will not be necessary to remember the proof of every theorem by heart.

Sample final questions

1. Consider the power series

$$\sum_{n=1}^{\infty} \frac{x^n}{n^3}, \quad \sum_{n=1}^{\infty} \frac{x^{3n}}{2n}, \quad \sum_{n=0}^{\infty} x^{2n!}.$$

For each power series, determine its radius of convergence R . By considering the series at $x = \pm R$, determine the exact interval of convergence.

2. (a) Prove by using the definition of convergence only, without using limit theorems, that if (s_n) is a sequence converging to s , then $\lim_{n \rightarrow \infty} s_n^2 = s^2$.
(b) Prove by using the definition of continuity, or by using the ϵ - δ property, that $f(x) = x^2$ is a continuous function on \mathbb{R} .
3. Let f be a twice differentiable function defined on the closed interval $[0, 1]$. Suppose $r, s, t \in [0, 1]$ are defined so that $r < s < t$ and $f(r) = f(s) = f(t) = 0$. Prove that there exists an $x \in (0, 1)$ such that $f''(x) = 0$.
4. (a) Suppose that $\sum_{n=0}^{\infty} a_n$ is a convergent series. Define a sequence (b_n) according to $b_n = a_{2n} + a_{2n+1}$ for $n \in \mathbb{N} \cup \{0\}$. Prove that $\sum_{n=0}^{\infty} b_n$ converges.
(b) Construct an example of a series $\sum_{n=0}^{\infty} a_n$ that diverges, but that if (b_n) is defined as above, then $\sum_{n=0}^{\infty} b_n$ converges.
5. Suppose f is a real-valued continuous function on \mathbb{R} and that $f(a)f(b) < 0$ for some $a, b \in \mathbb{R}$ where $a < b$. Prove that there exists an $x \in (a, b)$ such that $f(x) = 0$.

6. Let f be a real-valued function defined on an interval $[0, b]$ as

$$f(x) = \begin{cases} x & \text{for } x \in \mathbb{Q}, \\ 0 & \text{for } x \notin \mathbb{Q}. \end{cases}$$

Consider a partition $P = \{0 = t_0 < t_1 < \dots < t_n = b\}$. What are the upper and lower Darboux sums $U(f, P)$ and $L(f, P)$? Is f integrable on $[0, b]$?

7. Let f be a decreasing function defined on $[1, \infty)$, where $f(x) \geq 0$ for all $x \in [1, \infty)$. Prove that $\int_1^\infty f(x)dx$ converges if and only if $\sum_{n=1}^\infty f(n)$ converges. [This is essentially a question asking you to prove a general form of the integral test.]
8. Consider the function defined for $x, y \in \mathbb{R}$ as

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases}$$

- (a) Prove that d defines a metric on \mathbb{R} .
- (b) What is the neighborhood of radius $1/2$ centered on 0?
- (c) Consider an arbitrary set $S \subseteq \mathbb{R}$. Is S open? Is S compact?
9. Let $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$ be in \mathbb{R}^2 . Consider the function

$$d(\mathbf{x}, \mathbf{y}) = |x_1 - y_1| + |x_2 - y_2|.$$

- (a) Prove that d is a metric on \mathbb{R}^2 .
- (b) Compute and sketch the neighborhood of radius 1 at $(0, 0)$.
10. Consider a function f defined on \mathbb{R} which satisfies
- $$|f(x) - f(y)| \leq (x - y)^2$$
- for all $x, y \in \mathbb{R}$. Prove that f is a constant function.
11. Suppose that f is differentiable on \mathbb{R} , and that $2 \leq f'(x) \leq 3$ for $x \in \mathbb{R}$. If $f(0) = 0$, prove that $2x \leq f(x) \leq 3x$ for all $x \geq 0$.
12. Show that if f is integrable on $[a, b]$, then f is integrable on every interval $[c, d] \subseteq [a, b]$.
13. (a) Suppose r is irrational. Prove that $r^{1/3}$ and $r + 1$ are irrational also.
- (b) Prove that $(5 + \sqrt{2})^{1/3} + 1$ is irrational.
14. By using L'Hôpital's rule, or otherwise, evaluate

$$\lim_{x \rightarrow 0} \frac{x}{1 - e^{-x^2 - 3x}}, \quad \lim_{x \rightarrow 0} \left(\frac{1}{\sin x} - \frac{1}{x} \right), \quad \lim_{x \rightarrow 0} \frac{x^3}{\sin x - x}.$$

15. Let $a \in \mathbb{R}$. Consider the sequence (s_n) defined as

$$s_n = \begin{cases} a & \text{if } n \text{ is odd,} \\ 2^{-n} & \text{if } n \text{ is even.} \end{cases}$$

Compute $\limsup s_n$ and $\liminf s_n$. For what value of a does (s_n) converge?

16. Consider the function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ defined as

$$f(x_1, x_2) = \frac{1}{x_1^2 + x_2^2 + 1}.$$

With respect to the usual Euclidean metrics on \mathbb{R} and \mathbb{R}^2 , prove that f is continuous at $(0,0)$ and at $(0,1)$.

17. (a) Calculate the improper integral

$$\int_0^1 x^{-p} dx$$

for the cases when $0 < p < 1$ and $p > 1$.

(b) Prove that

$$\int_0^\infty x^{-p} dx = \infty$$

for all $p \in (0, \infty)$.

18. Prove that if f is integrable on $[a, b]$, then

$$\lim_{d \rightarrow b^-} \int_a^d f(x) dx = \int_a^b f(x) dx.$$

19. Let $f(x) = x^2$, and define a sequence (s_n) according to $s_1 = \lambda$ and $s_{n+1} = f(s_n)$ for $n \in \mathbb{N}$. Prove that (s_n) converges for $\lambda \in [-1, 1]$, and diverges for $|\lambda| > 1$.

20. Consider the three sets

$$A = [0, \sqrt{2}] \cap \mathbb{Q}, \quad B = \{x^2 + x - 1 : x \in \mathbb{R}\}, \quad C = \{x \in \mathbb{R} : x^2 + x - 1 < 0\}.$$

For each set, determine its maximum and minimum if they exist. For each set, determine its supremum and infimum. Detailed proofs are not required, but you should justify your answers.

21. Let $f_n(x) = x - x^n$ on $[0, 1]$ for $n \in \mathbb{N}$.

(a) Prove that f_n converges pointwise to a limit f , and determine f .

(b) Prove that f_n does not converge uniformly to f .

- (c) Find an interval I contained in $[0, 1]$ on which $f_n \rightarrow f$ uniformly.
 (d) Prove that the f_n are integrable, that f is integrable, and that $\int_0^1 f_n \rightarrow \int_0^1 f$.

22. Define

$$f(x) = \begin{cases} 1 & \text{if } |x| \leq 1, \\ -2 & \text{if } 1 < |x| \leq 2, \\ 0 & \text{if } |x| > 2 \end{cases}$$

for $x \in \mathbb{R}$.

- (a) Calculate $F(x) = \int_0^x f(t)dt$ for $x \in \mathbb{R}$.
 (b) Sketch f and F .
 (c) Compute F' and state the precise range over which F' exists. You may make use of the second fundamental theorem of calculus.
23. (a) Let f and g be continuous functions on $[a, b]$ such that $\int_a^b f = \int_a^b g$. Prove that there exists an $x \in [a, b]$ such that $f(x) = g(x)$.
 (b) Construct an example of integrable functions f and g on $[a, b]$ where $\int_a^b f = \int_a^b g$ but that $f(x) \neq g(x)$ for all $x \in [a, b]$.
24. Consider the function

$$f(x) = \frac{x}{1+x}.$$

on the interval $[0, \infty)$.

- (a) Show that $\lim_{x \rightarrow \infty} f(x) = 1$, and that $0 \leq f(x) < 1$ for all $x \in [0, \infty)$.
 (b) Sketch f .
 (c) Calculate f' , f'' , and use them to construct the partial Taylor series at $x = 1$ with the form
- $$f_T(x) = \sum_{n=0}^2 \frac{(x-1)^n f^{(n)}(1)}{n!}.$$
- (d) Show that f_T can be written as a quadratic equation with the form $ax^2 + bx + c$, and compute a , b , and c .
 (e) Add a sketch of f_T to the sketch of f . [Note: $f_T(1) = f(1)$ so the two curves should intersect at $x = 1$.]

25. Determine the radius of convergence R of the power series

$$f_1(x) = \sum_{n=0}^{\infty} \frac{x^n}{\sqrt{n^2 + 1}}, \quad f_2(x) = \sum_{n=1}^{\infty} \frac{(-2)^n x^{2n}}{n^2}.$$

By considering the series at $x = \pm R$, determine the exact intervals of convergence. If you make use of any of the theorems for determining series properties, you should state which ones you use.

26. Suppose (s_n) and (t_n) are two sequences that converge to s and t respectively. State the definition of convergence, and use it to prove that $3s_n + t_n \rightarrow 3s + t$. Do not use the limit theorems for sequences.

27. The Fibonacci numbers are defined by $F_0 = 0$ and $F_1 = 1$, and

$$F_{n+1} = F_n + F_{n-1}$$

for $n \in \mathbb{N}$. Let the golden ratio be defined as $\varphi = \frac{1+\sqrt{5}}{2}$.

- (a) Show that $\varphi^2 = 1 + \varphi$.

- (b) Let

$$f(n) = \frac{\varphi^n - (1 - \varphi)^n}{\sqrt{5}}.$$

For $n \in \mathbb{N}$, define H_n to be the hypothesis that “both $F_n = f(n)$ and $F_{n-1} = f(n-1)$ ”. Apply mathematical induction to prove that H_n is true for all $n \in \mathbb{N}$, and deduce that $F_n = f(n)$ for all $n \in \mathbb{N} \cup \{0\}$. [Hint: it is simpler to carry out the algebra in terms of φ and use the identity in (a), as opposed to calculating explicitly in terms of $(1 + \sqrt{5})/2$.]

- (c) Show that $\frac{F_{n+1}}{F_n} \rightarrow \varphi$ as $n \rightarrow \infty$.

28. Consider two dimensional space \mathbb{R}^2 , where an element $\mathbf{x} \in \mathbb{R}^2$ is written as $\mathbf{x} = (x_1, x_2)$. Let $d_E(\mathbf{x}, \mathbf{y}) = ((x_1 - y_1)^2 + (x_2 - y_2)^2)^{1/2}$ be the usual Euclidean metric on \mathbb{R}^2 .

- (a) Prove that $d_A(\mathbf{x}, \mathbf{y}) = \min\{|x_1 - y_1|, 2|x_2 - y_2|\}$ is not a metric on \mathbb{R}^2 .
 (b) Prove that $d_B(\mathbf{x}, \mathbf{y}) = \max\{|x_1 - y_1|, 2|x_2 - y_2|\}$ is a metric on \mathbb{R}^2 . Draw the neighborhood of radius 1 at $(0, 0)$.
 (c) Consider an arbitrary metric space (X, d) , and a mapping $f : X \rightarrow \mathbb{R}^2$. Suppose that f is continuous with respect to (X, d) and (\mathbb{R}^2, d_E) . Prove that it is also continuous with respect to (X, d) and (\mathbb{R}^2, d_B) .

29. Consider the function defined on $[0, \infty)$ as

$$f(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq 1, \\ x & \text{if } x > 1. \end{cases}$$

- (a) Compute $F(x) = \int_0^x f(t)dt$ on $[0, \infty)$.
 (b) Calculate $F'(x)$, stating the precise range over which it exists. You may make use of the second fundamental theorem of calculus.
 (c) Prove that neither f nor F is uniformly continuous on $[0, \infty)$.

30. Consider the function defined on the domain $[0, \infty)$ as

$$g(x) = \begin{cases} x & \text{if } 0 \leq x \leq 1, \\ 0 & \text{if } x > 1. \end{cases}$$

Define a sequence of functions on the interval $[0, 1]$ according to $f_n(x) = ng(nx)$ for $n \in \mathbb{N}$.

- (a) Sketch f_1, f_2 , and f_3 on the interval $[0, 1]$.
 - (b) Prove that f_n converges pointwise to a limit f , and determine f .
 - (c) Prove that f_n does not converge uniformly to f on $[0, 1]$.
 - (d) Show that $\int_0^1 f_n = 1/2$ for all n . Does $\int_0^1 f_n$ converge to $\int_0^1 f$?
31. (a) Use L'Hôpital's rule to evaluate

$$\lim_{x \rightarrow 0} \frac{x}{e^x - e^{-x}}, \quad \lim_{x \rightarrow 0} \frac{\sin^2 x}{x^2}.$$

- (b) Let f be a real-valued function defined on an interval (a, b) . Let $x \in (a, b)$, and suppose f is twice differentiable at x . Show that the limit

$$\lim_{h \rightarrow 0} \frac{f(x+h) + f(x-h) - 2f(x)}{h^2}$$

exists and equals $f''(x)$.

- (c) Construct an example of a real-valued function f defined on an interval (a, b) , where for some $x \in (a, b)$ the above limit exists and is finite, but f is not twice differentiable at x .
32. Let (x_n) be a sequence. Suppose that the subsequences (x_{2n}) , (x_{2n+1}) , and (x_{7n}) converge. Prove that (x_n) converges.
33. Let (a_n) be a sequence for $n \in \mathbb{N}$ where

$$a_n = (-1)^n(c + n^{-1})$$

and c is a real constant. Calculate $\limsup a_n$ and $\liminf a_n$ as a function of c . For what value of c does a_n converge?

34. (a) If $f(x) = \log(\log(x))$, calculate f' .
- (b) Use the integral test to show that

$$\sum_{n=2}^{\infty} \frac{1}{n \log n}$$

diverges.

- (c) Let $s(n)$ be the sum of digits in n . For example, $s(123) = 1 + 2 + 3 = 6$, and $s(99) = 9 + 9 = 18$. Determine whether the series

$$\sum_{n=1}^{\infty} \frac{1}{n s(n)}$$

converges or diverges.

35. (a) Suppose f is a continuous function on $[a, b]$, and $f(x) \geq 0$ for all $x \in [a, b]$. Prove that if $\int_a^b f = 0$, then $f(x) = 0$ for all $x \in [a, b]$.
 (b) Let X be the space of continuous functions on $[a, b]$. Prove that for $g, h \in X$, the function

$$d(g, h) = \int_a^b |g(x) - h(x)| dx$$

defines a metric on X .

- (c) Let Y be the space of integrable functions on $[a, b]$. Show that the function d given in part (b) is not a metric on Y .
 36. Define $S = [0, 1]$ and let $f_n : S \rightarrow \mathbb{R}$ be a sequence of continuous functions such that

$$0 \leq f_{n+1}(x) \leq f_n(x) \leq \dots \leq f_1(x)$$

for all $n \in \mathbb{N}$ and $x \in S$. Suppose that (f_n) converges pointwise to 0.

- (a) For $\epsilon > 0$, define the sets $E_n = \{x \in S : f_n(x) < \epsilon\}$. Show that for all $n \in \mathbb{N}$, the sets E_n are open in S and $E_n \subseteq E_{n+1}$. In addition, show that the $\{E_n\}_{n \in \mathbb{N}}$ form an open cover of S .
 (b) Prove that (f_n) converges uniformly to 0 on S .
 (c) Suppose now that S is changed to $(0, 1)$ with all other conditions remaining the same. Prove that (f_n) may not converge uniformly to 0 in this case.
 37. (a) Determine whether the two series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{3n+1}, \quad \sum_{n=0}^{\infty} \frac{1}{3n+1}$$

converge or diverge.

- (b) Determine the radius of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{(-2)^n x^n}{3n+1}.$$

What is the exact interval of convergence?

38. (a) Prove that the sequence (s_n) is bounded if and only if $\limsup |s_n| < \infty$.
 (b) Construct an example where $\limsup s_n \in \mathbb{R}$ but (s_n) is unbounded.
39. Let (f_n) be a sequence of increasing differentiable functions on $[0, 1]$ where $f_n(0) = 0$ for all $n \in \mathbb{N}$, and $\lim_{n \rightarrow \infty} f_n(1) = 0$.
 (a) Prove that (f_n) converges uniformly to the zero function on $[0, 1]$.
 (b) Define $g_n = \sup\{|f'_n(x)| : x \in [0, 1]\}$. Is (g_n) always bounded? Prove your assertion.
40. Let $f(x) = x^2(1 - x)$ and $g(x) = |f(x)|$ for $x \in \mathbb{R}$.
 (a) Plot f and g over the range $[-1, 2]$.
 (b) By using the definition of differentiability, prove that $g(x)$ is differentiable at $x = 0$, but not at $x = 1$.
 (c) Compute the derivatives $g^{(n)}(2)$ for $n \in \mathbb{N}$, and use them to write down the Taylor series of g at $x = 2$. Prove that this series is equal to $-f(x)$ for all $x \in \mathbb{R}$.
41. Let f be a continuous strictly increasing function defined on \mathbb{R} . Consider a subset $S \subseteq \mathbb{R}$ and let $T = \{f(x) : x \in S\}$.
 (a) If $\sup S$ is finite, prove that $\sup T = f(\sup S)$.
 (b) Suppose (a_n) is a sequence such that $\limsup a_n$ is finite. Define the sequence (b_n) so that $b_n = f(a_n)$ for all n . Prove that $\limsup b_n = f(\limsup a_n)$.
 (c) Suppose the condition that f is continuous is removed. Construct an example of a strictly increasing function f defined on \mathbb{R} , and subset $S \subseteq \mathbb{R}$, where $\sup S$ is finite, but $\sup T \neq f(\sup S)$.
42. Define the sequence of functions (h_n) on \mathbb{R} according to

$$h_n(x) = \begin{cases} n & \text{if } |x| \leq 1/2n, \\ 0 & \text{if } |x| > 1/2n. \end{cases}$$

- (a) Sketch h_1 , h_2 , and h_3 .
 (b) Prove that (h_n) converges pointwise to 0 on $\mathbb{R} \setminus \{0\}$.
 (c) Let f be a continuous real-valued function on \mathbb{R} . Prove that

$$\lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} h_n f = f(0).$$

- (d) Construct an example of an integrable function g on \mathbb{R} where

$$\lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} h_n g$$

exists and is a real number, but does not equal $g(0)$.

43. (a) Define $B = [1, 2] \cup (3, 4)$ in \mathbb{R} with the standard metric. Calculate the closure \bar{B} , and interior B° . In addition, calculate the boundary defined as $\partial B = \bar{B} \setminus B^\circ$.
- (b) Let $\{A_n\}_{n=1}^\infty$ be a sequence of open sets in a general metric space (S, d) such that $A_{n+1} \supseteq A_n$ for all n . Let (x_n) be a sequence where $x_n \in A_{n+1} \setminus A_n$, and $\lim_{n \rightarrow \infty} x_n = x$. Define $A = \bigcup_{n=1}^\infty A_n$. Prove that $x \in \partial A$.