Chapter X: Dirac Fermions in Condensed Matter

A Concise Introduction for Harry

April 23, 2025

1 Introduction: Dirac Physics in Solids

The Dirac equation, fundamental to relativistic quantum mechanics, describes spin-1/2 particles and predicts the energy-momentum relation:

$$E^2 = (|\mathbf{p}|c)^2 + (m_0 c^2)^2. \tag{1}$$

Remarkably, its mathematical structure also governs the behavior of electron quasiparticles in certain solids, known as **Dirac materials**. In these systems, the speed of light c is replaced by a Fermi velocity $v \ll c$, and the rest mass m_0 by an effective mass m. These quasiparticles often possess a *pseudospin* degree of freedom alongside their momentum. This chapter introduces the Hamiltonians describing these quasiparticles and explores their key properties relevant to your project. We set $\hbar = 1$.

2 Dirac Hamiltonians in Low Dimensions

Dirac Hamiltonians are characterized by being linear in momentum. They act on multi-component spinors.

2.1 One Dimension (1D)

A typical 1D Dirac Hamiltonian is:

$$\mathcal{H}_{1D} = v\hat{p}_x\sigma_x + m\sigma_z. \tag{2}$$

Here, $\hat{p}_x = -i\partial/\partial x$, and $\sigma_{x,z}$ are Pauli matrices acting on a 2-component spinor. In momentum space ($\hat{p}_x \to p$), the Hamiltonian becomes $\mathcal{H}_{1D}(p) = \begin{pmatrix} m & vp \\ vp & -m \end{pmatrix}$. Its eigenvalues yield the dispersion relation:

$$E_\pm(p) = \pm\sqrt{m^2 + (vp)^2}. \tag{3}$$

This shows two energy bands separated by a gap $2|m|$ at $p = 0$. If $m = 0$, the gap closes, forming a **Dirac point**. (*Project: Find eigenstates and plot dispersion*).

2.2 Two Dimensions (2D)

In 2D, relevant for graphene or topological insulator surfaces, a common form is:

$$\mathcal{H}_{2D} = v(\hat{p}_x\sigma_x + \hat{p}_y\sigma_y) + m\sigma_z. \tag{4}$$

In momentum space ($\hat{\mathbf{p}} \to \mathbf{p} = (p_x, p_y)$), the Hamiltonian matrix is $\mathcal{H}_{2D}(\mathbf{p}) = \begin{pmatrix} m & v(p_x - ip_y) \\ v(p_x + ip_y) & -m \end{pmatrix}$. The eigenvalues give a similar dispersion:

$$E_\pm(\mathbf{p}) = \pm\sqrt{m^2 + v^2|\mathbf{p}|^2}. \tag{5}$$

Again, a gap $2|m|$ exists at $\mathbf{p} = 0$ unless $m = 0$, which leads to **Dirac cones**. (*Project: Find eigenstates and plot dispersion*).

2.3 Three Dimensions (3D)

In 3D, the Hamiltonian typically involves 4×4 matrices (α_i, β) satisfying the Clifford algebra:

$$\mathcal{H}_{3D} = v \sum_{i=x,y,z} \hat{p}_i\alpha_i + m\beta. \tag{6}$$

This yields the dispersion $E_\pm(\mathbf{p}) = \pm\sqrt{m^2 + v^2|\mathbf{p}|^2}$. (*Project: Clarify the specific 3D Hamiltonian form given and find its eigenstates/values*).

3 Momentum Space Geometry: Berry Phase Concepts

Eigenstates $|\psi(\mathbf{p})\rangle$ contain geometric information revealed as $\mathbf{p}$ varies. The **Berry connection** measures the infinitesimal phase shift:

$$\mathcal{A}_i(\mathbf{p}) = i \langle\psi(\mathbf{p})| \frac{\partial}{\partial p_i} |\psi(\mathbf{p})\rangle. \tag{7}$$

It acts like a vector potential in momentum space. Its curl gives the **Berry curvature**. In 2D:

$$\Omega(\mathbf{p}) = \frac{\partial \mathcal{A}_y}{\partial p_x} - \frac{\partial \mathcal{A}_x}{\partial p_y}. \tag{8}$$

$\Omega(\mathbf{p})$ acts like a momentum-space magnetic field. Its integral over the 2D Brillouin zone, for a gapped band, gives a topological invariant, the integer **Chern number** C:

$$C = \frac{1}{2\pi} \int d^2p\, \Omega(\mathbf{p}). \tag{9}$$

A non-zero Chern number signals non-trivial topology. (*Project: Calculate $\mathcal{A}_i$ and Ω for $\mathcal{H}_{2D}$*).

4 Bulk-Boundary Correspondence

A profound principle connects the bulk topology (like C) to the existence of protected boundary states.

- **1D:** A domain wall where the mass $m(x)$ changes sign in $\mathcal{H}_{1D}$ binds a zero-energy state (Jackiw-Rebbi mechanism), potentially carrying fractional fermion number. (*Project: Yufei's task*).
- **2D:** A non-zero Chern number C implies $|C|$ gapless, chiral **edge states** propagating along the boundary. These are robust against disorder and underpin the Quantum Hall Effect (QHE). (*Project: Yufei's task*).

5 Landau Levels for 2D Dirac Fermions

Applying a perpendicular magnetic field $\mathbf{B} = B\hat{\mathbf{z}}$ to the 2D system (4) leads to quantization into **Landau Levels (LLs)**. We replace $\hat{\mathbf{p}}$ with the canonical momentum $\hat{\mathbf{\Pi}} = \hat{\mathbf{p}} - q\mathbf{A}$, where q is the charge (e.g., $q = -e$ for electrons) and $\mathbf{B} = \nabla \times \mathbf{A}$. The components satisfy $[\hat{\Pi}_x, \hat{\Pi}_y] = iqB$.

Following your professor's notes (Section A.3), ladder operators $b, b^\dagger$ are defined based on $\hat{\Pi}_\pm = \hat{\Pi}_x \pm i\hat{\Pi}_y$, satisfying $[b, b^\dagger] = 1$. The definitions depend on the sign of qB. Let $M = mv_0^2$ (using v_0 for velocity as in notes) be the mass term and $\Omega_c = \sqrt{2|qB|}v_0$. The notes provide the Hamiltonian in terms of these operators.

Case 1: $qB < 0$ (e.g., electrons, $B > 0$)

$$\mathcal{H} = \begin{pmatrix} M & -\Omega_c b \\ -\Omega_c b^\dagger & -M \end{pmatrix}. \tag{10}$$

Case 2: $qB > 0$ (e.g., electrons, $B < 0$)

$$\mathcal{H} = \begin{pmatrix} M & -\Omega_c b^\dagger \\ -\Omega_c b & -M \end{pmatrix}. \tag{11}$$

Note: These specific forms might differ slightly from standard textbook derivations; we proceed assuming these are correct for the intended system.

The resulting energy spectrum is indexed by $n \in \mathbb{Z}$:

$$E_n = \mathrm{sgn}(n)\sqrt{M^2 + \Omega_c^2|n|} \quad \text{for } n \neq 0, \tag{12}$$

and a unique zeroth level:

$$E_0 = \mathrm{sgn}(qB)M. \tag{13}$$

This spectrum features $\pm E$ symmetry and a characteristic $\sqrt{|n|B}$ dependence. The $n = 0$ level's energy depends crucially on the signs of qB and M.

Each level n has a degeneracy $N_{deg} = |qB|A/(2\pi) = BA/\Phi_0$, where $\Phi_0 = 2\pi/|q|$ is the flux quantum (with $\hbar = 1$). The eigenstates $|n,k\rangle_D$ (given in the notes) involve specific combinations of the standard oscillator states $||n|\rangle$ and $||n|-1\rangle$. The $n = 0$ eigenstate is particularly simple: it's fully polarized in the pseudospin basis, aligned with σ_z if $qB > 0$ ($E_0 = M$) and anti-aligned if $qB < 0$ ($E_0 = -M$).

The notes also provide the average particle density at zero chemical potential (half-filling), obtained by summing over negative energy states:

$$\bar{n}_0 = -\mathrm{sgn}(qBm)\frac{1}{2}\frac{|qB|}{2\pi} = -\mathrm{sgn}(qBm)\frac{1}{2}\frac{B}{\Phi_0}. \tag{14}$$

The factor of 1/2 and the sign dependence arise from the unique nature of the $n = 0$ LL relative to the Fermi level. (*Project: Solve for the spectrum (confirming* (12), (13)*), calculate degeneracy, discuss spectrum vs M, calculate $\bar{n}_0$ and $d\bar{n}_0/dB$*).

6 Outlook

The concepts covered – Dirac Hamiltonians, Berry curvature, bulk-boundary correspondence, and Landau levels – are essential tools for understanding modern condensed matter phenomena like the QHE in graphene, topological insulators, Chern insulators, and Weyl semimetals. This project provides practical experience with these foundational calculations.