Problem 1: Lipschitz and Uniform Continuity

Review Notes

- **Definition 19.1 (Uniform Continuity):** Let f be a real-valued function defined on a set $S \subseteq \mathbb{R}$. Then f is **uniformly continuous** on S if for every $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x, y \in S$, if $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$.
  - The key difference from pointwise continuity is that δ depends only on ϵ, not on the specific points x or y in S.
- **Definition (Lipschitz Continuity):** A real-valued function f on an interval I is said to be **Lipschitz continuous** if there exists a constant $L > 0$ such that for all $x, y \in I$,

  $$|f(x) - f(y)| \leq L|x - y|.$$

  The constant L is called a Lipschitz constant for f.
- **Relationship:** Lipschitz continuity is a stronger condition than uniform continuity.

Solution

(a) **Show that if a function is Lipschitz continuous, then it is uniformly continuous.**

Let $f : I \to \mathbb{R}$ be a Lipschitz continuous function on an interval I. By definition, there exists a constant $L > 0$ such that for all $x, y \in I$,

$$|f(x) - f(y)| \leq L|x - y|.$$

We want to show that f is uniformly continuous on I. Let $\epsilon > 0$ be given. We need to find a $\delta > 0$ such that for all $x, y \in I$, if $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$.

Choose $\delta = \frac{\epsilon}{L}$. Since $L > 0$ and $\epsilon > 0$, we have $\delta > 0$. Now, let $x, y \in I$ such that $|x - y| < \delta$. Using the Lipschitz condition, we have

$$|f(x) - f(y)| \leq L|x - y|.$$

Since $|x - y| < \delta = \frac{\epsilon}{L}$, we can substitute this into the inequality:

$$|f(x) - f(y)| \leq L|x - y| < L\left(\frac{\epsilon}{L}\right) = \epsilon.$$

Thus, for any $\epsilon > 0$, we found a $\delta = \epsilon/L > 0$ such that for all $x, y \in I$, if $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$. Therefore, f is uniformly continuous on I.

(b) **Find an example of a function g defined on an interval I that is uniformly continuous but not Lipschitz continuous.**

Consider the function $g(x) = \sqrt{x}$ defined on the interval $I = [0, 1]$.

- **Uniform Continuity:** The function $g(x) = \sqrt{x}$ is continuous on the closed and bounded interval $[0, 1]$. By Theorem 19.5 (Heine-Cantor Theorem), a continuous function on a compact set (like $[0, 1]$) is uniformly continuous. Thus, $g(x) = \sqrt{x}$ is uniformly continuous on $[0, 1]$.
- **Not Lipschitz Continuous:** Assume, for the sake of contradiction, that $g(x) = \sqrt{x}$ is Lipschitz continuous on $[0, 1]$. Then there exists a constant $L > 0$ such that for all $x, y \in [0, 1]$,

  $$|\sqrt{x} - \sqrt{y}| \leq L|x - y|.$$

  Let's choose $y = 0$. Then for all $x \in (0, 1]$, we must have

  $$|\sqrt{x} - \sqrt{0}| \leq L|x - 0|$$

  $$\sqrt{x} \leq Lx.$$

Dividing by $\sqrt{x}$ (since $x > 0$), we get

$$1 \leq L\sqrt{x}$$

which implies

$$\frac{1}{L} \leq \sqrt{x}.$$

This inequality must hold for all $x \in (0, 1]$. However, as $x \to 0^+$, $\sqrt{x} \to 0$. We can choose x small enough such that $\sqrt{x} < \frac{1}{L}$. For example, choose $x = \frac{1}{4L^2}$. If $L \geq 1/2$, then $x = 1/(4L^2) \leq 1/(4(1/4)) = 1$, so $x \in (0, 1]$. Then $\sqrt{x} = \frac{1}{2L}$, and the inequality becomes

$$\frac{1}{L} \leq \frac{1}{2L},$$

which simplifies to $1 \leq \frac{1}{2}$, a contradiction. Alternatively, consider the ratio for $y = 0$:

$$\frac{|g(x) - g(0)|}{|x - 0|} = \frac{\sqrt{x}}{x} = \frac{1}{\sqrt{x}}.$$

As $x \to 0^+$, this ratio $\frac{1}{\sqrt{x}} \to \infty$. If g were Lipschitz, this ratio would be bounded by L. Since the ratio is unbounded, g cannot be Lipschitz continuous on $[0, 1]$.

Therefore, $g(x) = \sqrt{x}$ on $I = [0, 1]$ is uniformly continuous but not Lipschitz continuous.

Problem 2: Uniform Continuity and Operations

Review Notes

- **Definition 19.1 (Uniform Continuity):** As defined in Problem 1.
- **Theorem 19.4:** If $f : S \to \mathbb{R}$ is uniformly continuous on S and $g : T \to \mathbb{R}$ is uniformly continuous on T, where $f(S) \subseteq T$, then the composition $g \circ f : S \to \mathbb{R}$ is uniformly continuous on S.
- **Theorem 19.6 (Algebraic Properties for Uniform Continuity):** If f and g are uniformly continuous functions from a set $S \subseteq \mathbb{R}$ to $\mathbb{R}$, then:
  - $f + g$ is uniformly continuous on S.
  - kf is uniformly continuous on S for any constant k.
  - The product $f \cdot g$ is not necessarily uniformly continuous on S. However, if both f and g are bounded on S, then $f \cdot g$ is uniformly continuous on S.

Solution

(a) **Let $S \subseteq \mathbb{R}$, $f : S \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$ be uniformly continuous functions. Prove $g \circ f : S \to \mathbb{R}$ is uniformly continuous.** *(Note: The problem statement specifies $g : \mathbb{R} \to \mathbb{R}$, so $f(S) \subseteq \mathbb{R}$ is trivially satisfied).*

Let $\epsilon > 0$ be given. Since g is uniformly continuous on $\mathbb{R}$, there exists a $\delta_g > 0$ such that for all $u, v \in \mathbb{R}$, if $|u - v| < \delta_g$, then $|g(u) - g(v)| < \epsilon$.

Since f is uniformly continuous on S, for this $\delta_g > 0$ (treating δ_g as an ϵ for f), there exists a $\delta_f > 0$ such that for all $x, y \in S$, if $|x - y| < \delta_f$, then $|f(x) - f(y)| < \delta_g$.

Now, let $x, y \in S$ such that $|x - y| < \delta_f$. By the uniform continuity of f, we have $|f(x) - f(y)| < \delta_g$. Let $u = f(x)$ and $v = f(y)$. Then $u, v \in f(S) \subseteq \mathbb{R}$, and we have $|u - v| < \delta_g$. By the uniform continuity of g, since $|u - v| < \delta_g$, we have $|g(u) - g(v)| < \epsilon$. Substituting back $u = f(x)$ and $v = f(y)$, we get

$$|g(f(x)) - g(f(y))| < \epsilon.$$

This means $|(g \circ f)(x) - (g \circ f)(y)| < \epsilon$.

Thus, for any $\epsilon > 0$, we found a $\delta = \delta_f > 0$ such that for all $x, y \in S$, if $|x - y| < \delta$, then $|(g \circ f)(x) - (g \circ f)(y)| < \epsilon$. Therefore, $g \circ f$ is uniformly continuous on S.

(b) **Let f and g be two uniformly continuous functions from S to $\mathbb{R}$. Prove that $f+g$ is uniformly continuous.**

Let $\epsilon > 0$ be given. Since f is uniformly continuous on S, there exists a $\delta_f > 0$ such that for all $x, y \in S$, if $|x-y| < \delta_f$, then $|f(x)-f(y)| < \epsilon/2$. Since g is uniformly continuous on S, there exists a $\delta_g > 0$ such that for all $x, y \in S$, if $|x-y| < \delta_g$, then $|g(x)-g(y)| < \epsilon/2$.

Choose $\delta = \min(\delta_f, \delta_g)$. Since $\delta_f > 0$ and $\delta_g > 0$, we have $\delta > 0$. Now, let $x, y \in S$ such that $|x-y| < \delta$. Since $|x-y| < \delta \leq \delta_f$, we have $|f(x)-f(y)| < \epsilon/2$. Since $|x-y| < \delta \leq \delta_g$, we have $|g(x)-g(y)| < \epsilon/2$.

Consider the function $h(x) = f(x)+g(x)$. We want to show $|h(x)-h(y)| < \epsilon$. Using the triangle inequality:

$$\begin{aligned}|h(x)-h(y)| &= |(f(x)+g(x))-(f(y)+g(y))| \\ &= |(f(x)-f(y))+(g(x)-g(y))| \\ &\leq |f(x)-f(y)|+|g(x)-g(y)|.\end{aligned}$$

Substituting the bounds we found:

$$|h(x)-h(y)| < \frac{\epsilon}{2}+\frac{\epsilon}{2} = \epsilon.$$

Thus, for any $\epsilon > 0$, we found a $\delta > 0$ such that for all $x, y \in S$, if $|x-y| < \delta$, then $|(f+g)(x)-(f+g)(y)| < \epsilon$. Therefore, $f+g$ is uniformly continuous on S.

(c) **Show that there exist uniformly continuous functions f and g from S to $\mathbb{R}$ such that the multiplication $f \cdot g$ is not uniformly continuous.**

Let $S = \mathbb{R}$. Consider the functions $f(x) = x$ and $g(x) = x$.

- **Uniform Continuity of f and g:** Let $\epsilon > 0$ be given. Choose $\delta = \epsilon$. Then for any $x, y \in \mathbb{R}$, if $|x-y| < \delta$, then
$$|f(x)-f(y)| = |x-y| < \delta = \epsilon.$$
So $f(x) = x$ is uniformly continuous on $\mathbb{R}$. Similarly, $g(x) = x$ is uniformly continuous on $\mathbb{R}$.
- **Product $f \cdot g$:** The product is $h(x) = f(x)g(x) = x^2$. We need to show that $h(x) = x^2$ is not uniformly continuous on $\mathbb{R}$. We can show this by negating the definition of uniform continuity. We need to find an $\epsilon > 0$ such that for every $\delta > 0$, there exist $x, y \in \mathbb{R}$ with $|x-y| < \delta$ but $|h(x)-h(y)| \geq \epsilon$.
Let $\epsilon = 1$. Let $\delta > 0$ be any positive number. We need to find $x, y \in \mathbb{R}$ such that $|x-y| < \delta$ and $|x^2-y^2| \geq 1$. Choose $x = \frac{1}{\delta}+\frac{\delta}{2}$ and $y = \frac{1}{\delta}$. Then $|x-y| = |\frac{\delta}{2}| = \frac{\delta}{2} < \delta$. Now consider the difference in the function values:
$$\begin{aligned}|h(x)-h(y)| &= |x^2-y^2| = |(x-y)(x+y)| \\ &= \left|\frac{\delta}{2}\left(\left(\frac{1}{\delta}+\frac{\delta}{2}\right)+\frac{1}{\delta}\right)\right| \\ &= \left|\frac{\delta}{2}\left(\frac{2}{\delta}+\frac{\delta}{2}\right)\right| \\ &= \left|1+\frac{\delta^2}{4}\right| = 1+\frac{\delta^2}{4}.\end{aligned}$$
Since $\delta > 0$, $\frac{\delta^2}{4} > 0$, so $1+\frac{\delta^2}{4} > 1$. Thus, we have found x, y such that $|x-y| < \delta$ but $|h(x)-h(y)| = 1+\frac{\delta^2}{4} \geq 1 = \epsilon$.
Since this holds for any $\delta > 0$, the function $h(x) = x^2$ is not uniformly continuous on $\mathbb{R}$.

Therefore, the product of two uniformly continuous functions is not necessarily uniformly continuous.

Problem 3: Growth of Uniformly Continuous Functions on $\mathbb{R}$

Review Notes

- **Definition 19.1 (Uniform Continuity):** As defined previously.
- **Triangle Inequality:** For any real numbers a, b, $|a+b| \le |a| + |b|$. Also, $|a-b| \ge ||a| - |b||$.
- **Idea:** Uniform continuity limits how fast the function can grow. If a function grows too quickly (e.g., quadratically like x^2), it cannot be uniformly continuous on $\mathbb{R}$.

Solution

Let $f : \mathbb{R} \to \mathbb{R}$ be a uniformly continuous function. We want to prove that there exist constants $A > 0$ and $B > 0$ such that $|f(x)| \le A + B|x|$ for all $x \in \mathbb{R}$.

Since f is uniformly continuous on $\mathbb{R}$, for $\epsilon = 1$, there exists a $\delta_0 > 0$ such that for all $u, v \in \mathbb{R}$, if $|u - v| < \delta_0$, then $|f(u) - f(v)| < 1$. Let's choose $\delta = \delta_0/2$. Then $\delta > 0$. If $|u - v| \le \delta$, then $|u - v| < \delta_0$, which implies $|f(u) - f(v)| < 1$.

Consider any $x \in \mathbb{R}$. If $x = 0$, then $|f(0)| \le A + B|0|$ requires $|f(0)| \le A$. So we will need to choose A large enough to accommodate this.

Let $x > 0$. Choose an integer n such that $n\delta \le x < (n+1)\delta$. Note $n = \lfloor x/\delta \rfloor \ge 0$. Consider the points $0, \delta, 2\delta, \ldots, n\delta, x$. The distance between consecutive points in the sequence $0, \delta, 2\delta, \ldots, n\delta$ is δ. The distance between $n\delta$ and x is $x - n\delta < (n+1)\delta - n\delta = \delta$. So, the distance between any two consecutive points in $0, \delta, 2\delta, \ldots, n\delta, x$ is less than or equal to δ.

Using the triangle inequality repeatedly:

$$f(x) - f(0) = (f(x) - f(n\delta)) + (f(n\delta) - f((n-1)\delta)) + \cdots + (f(\delta) - f(0)).$$

Taking absolute values:

$$|f(x) - f(0)| = \left|(f(x) - f(n\delta)) + \sum_{k=1}^{n} (f(k\delta) - f((k-1)\delta))\right|$$

$$\le |f(x) - f(n\delta)| + \sum_{k=1}^{n} |f(k\delta) - f((k-1)\delta)|.$$

Since the distance between consecutive points is $\le \delta$:

- $|x - n\delta| < \delta$, so $|f(x) - f(n\delta)| < 1$.
- For $k = 1, \ldots, n$, $|k\delta - (k-1)\delta| = |\delta| = \delta$. So, $|f(k\delta) - f((k-1)\delta)| < 1$.

So,

$$|f(x) - f(0)| \le 1 + \sum_{k=1}^{n} 1 = 1 + n.$$

We know $n = \lfloor x/\delta \rfloor$, so $n \le x/\delta$. Substituting this, we get:

$$|f(x) - f(0)| \le 1 + \frac{x}{\delta}.$$

Using the triangle inequality $|f(x)| - |f(0)| \le |f(x) - f(0)|$, we have:

$$|f(x)| \le |f(0)| + |f(x) - f(0)| \le |f(0)| + 1 + \frac{x}{\delta}.$$

Since $x > 0$, $|x| = x$. So for $x > 0$:

$$|f(x)| \le (|f(0)| + 1) + \frac{1}{\delta}|x|.$$

Now, let $x < 0$. Let $y = -x > 0$. Choose an integer m such that $m\delta \le y < (m+1)\delta$. Note $m = \lfloor y/\delta \rfloor = \lfloor -x/\delta \rfloor \ge 0$. Consider the points $x, x+\delta, x+2\delta, \dots, x+m\delta, 0$. Let $x_k = x + k\delta$. The points are $x_0, x_1, \dots, x_m, 0$. The distance between consecutive points x_{k-1} and x_k is δ. The distance between $x_m = x + m\delta$ and 0 is $|x + m\delta| = |-y + m\delta| = |y - m\delta|$. Since $m\delta \le y < (m+1)\delta$, we have $0 \le y - m\delta < \delta$. So $|x_m - 0| < \delta$. Using the triangle inequality:

$$f(0) - f(x) = (f(0) - f(x_m)) + (f(x_m) - f(x_{m-1})) + \cdots + (f(x_1) - f(x_0)).$$

$$|f(0) - f(x)| \le |f(0) - f(x_m)| + \sum_{k=1}^{m} |f(x_k) - f(x_{k-1})|.$$

Since the distance between consecutive points is $\le \delta$, each term is less than 1.

$$|f(0) - f(x)| \le 1 + \sum_{k=1}^{m} 1 = 1 + m.$$

We know $m = \lfloor -x/\delta \rfloor$, so $m \le -x/\delta$.

$$|f(x) - f(0)| = |f(0) - f(x)| \le 1 + m \le 1 - \frac{x}{\delta}.$$

Using the triangle inequality $|f(x)| - |f(0)| \le |f(x) - f(0)|$:

$$|f(x)| \le |f(0)| + |f(x) - f(0)| \le |f(0)| + 1 - \frac{x}{\delta}.$$

Since $x < 0$, $|x| = -x$. So for $x < 0$:

$$|f(x)| \le (|f(0)| + 1) + \frac{1}{\delta}|x|.$$

Combining the cases $x > 0$, $x < 0$, and $x = 0$: We can choose $A = |f(0)| + 1$ and $B = \frac{1}{\delta}$. Both A and B are positive constants (since $\delta > 0$). Then, for all $x \in \mathbb{R}$, we have:

$$|f(x)| \le A + B|x|.$$

This completes the proof.

Problem 4: Limits of a Rational Function

Review Notes

- **Definition (Limit of a Function):** Let f be a function defined on $S \subseteq \mathbb{R}$, let a be a limit point of S. We say $\lim_{x\to a} f(x) = L$ if for every $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x \in S$, if $0 < |x - a| < \delta$, then $|f(x) - L| < \epsilon$.
- **Definition (One-Sided Limits):**
  - **Right-hand limit:** $\lim_{x\to a^+} f(x) = L$ if for every $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x \in S$, if $a < x < a + \delta$, then $|f(x) - L| < \epsilon$.
  - **Left-hand limit:** $\lim_{x\to a^-} f(x) = L$ if for every $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x \in S$, if $a - \delta < x < a$, then $|f(x) - L| < \epsilon$.
- **Theorem 20.10:** $\lim_{x\to a} f(x) = L$ if and only if $\lim_{x\to a^+} f(x) = L$ and $\lim_{x\to a^-} f(x) = L$.
- **Infinite Limits:** We say $\lim_{x\to a^+} f(x) = +\infty$ if for every $M > 0$, there exists a $\delta > 0$ such that for all x, if $a < x < a + \delta$, then $f(x) > M$. Similar definitions hold for $-\infty$ and for $x \to a^-$ or $x \to a$.
- **Rational Functions:** A function $f(x) = P(x)/Q(x)$, where P, Q are polynomials. Limits can often be evaluated by substitution, unless $Q(a) = 0$. If $Q(a) = 0$ and $P(a) \ne 0$, there is a vertical asymptote at $x = a$. If $Q(a) = 0$ and $P(a) = 0$, there might be a hole or a vertical asymptote.

Solution

The function is $f(x) = \frac{1}{(x+1)^2(x-2)}$. The domain is $\mathbb{R} \setminus \{-1, 2\}$.

(a) **Sketch the function $f(x) = (x+1)^{-2}(x-2)^{-1}$.**

(a) **Vertical Asymptotes:** The denominator is zero when $x = -1$ or $x = 2$. The numerator is 1 (never zero). So, we have vertical asymptotes at $x = -1$ and $x = 2$.

(b) **Horizontal Asymptotes:** As $x \to \pm\infty$, the denominator $(x+1)^2(x-2) \approx x^2 \cdot x = x^3$ behaves like x^3. So, $f(x) \approx \frac{1}{x^3}$.

$$\lim_{x\to\infty} \frac{1}{(x+1)^2(x-2)} = 0$$

$$\lim_{x\to-\infty} \frac{1}{(x+1)^2(x-2)} = 0$$

There is a horizontal asymptote at $y = 0$.

(c) **Behavior near asymptotes:**

- Near $x = 2$:
  - As $x \to 2^+$, $x > 2$, so $x - 2 > 0$. Also $(x+1)^2$ is positive (approx $3^2 = 9$). So the denominator is small and positive. $f(x) \to +\infty$.
  - As $x \to 2^-$, $x < 2$, so $x - 2 < 0$. Also $(x+1)^2$ is positive. So the denominator is small and negative. $f(x) \to -\infty$.
- Near $x = -1$:
  - As $x \to -1^+$, $x > -1$, so $x + 1$ is small and positive. $(x+1)^2$ is small and positive. $x - 2$ is negative (approx -3). So the denominator is small and negative $(+ \times - = -)$. $f(x) \to -\infty$.
  - As $x \to -1^-$, $x < -1$, so $x + 1$ is small and negative. $(x+1)^2$ is small and positive. $x - 2$ is negative (approx -3). So the denominator is small and negative $(+ \times - = -)$. $f(x) \to -\infty$.

(d) **Intercepts:**

- y-intercept: Set $x = 0$. $f(0) = \frac{1}{(0+1)^2(0-2)} = \frac{1}{1\cdot(-2)} = -\frac{1}{2}$. The y-intercept is $(0, -1/2)$.
- x-intercept: Set $f(x) = 0$. $\frac{1}{(x+1)^2(x-2)} = 0$. This equation has no solution as the numerator is never zero. There are no x-intercepts.

(e) **Sign analysis:**

- For $x > 2$: $(x+1)^2 > 0$, $x - 2 > 0$. $f(x) > 0$.
- For $-1 < x < 2$: $(x+1)^2 > 0$, $x - 2 < 0$. $f(x) < 0$.
- For $x < -1$: $(x+1)^2 > 0$, $x - 2 < 0$. $f(x) < 0$.

Sketch description: Draw axes, mark points -1 and 2 on x-axis, mark $-1/2$ on y-axis. Draw vertical lines at $x = -1$ and $x = 2$. Draw horizontal line at $y = 0$. Plot the y-intercept $(0, -1/2)$. Region $x > 2$: Starts from $+\infty$ near $x = 2$, decreases towards the horizontal asymptote $y = 0$. Region $-1 < x < 2$: Starts from $-\infty$ near $x = -1$, passes through $(0, -1/2)$, goes down to $-\infty$ near $x = 2$. Region $x < -1$: Approaches the horizontal asymptote $y = 0$ from below as $x \to -\infty$. Decreases towards $-\infty$ as $x \to -1^-$.

(b) **Determine the limits.**

- $\lim_{x\to 2^+} f(x)$: As $x \to 2^+$, $x > 2$. Then $x - 2 \to 0^+$ and $(x+1)^2 \to (2+1)^2 = 9$. The denominator $(x+1)^2(x-2)$ approaches $9 \times 0^+ = 0^+$. The numerator is 1.

$$\lim_{x\to 2^+} \frac{1}{(x+1)^2(x-2)} = +\infty.$$

- $\lim_{x\to 2^-} f(x)$: As $x \to 2^-$, $x < 2$. Then $x - 2 \to 0^-$ and $(x+1)^2 \to 9$. The denominator $(x+1)^2(x-2)$ approaches $9 \times 0^- = 0^-$. The numerator is 1.

$$\lim_{x\to 2^-} \frac{1}{(x+1)^2(x-2)} = -\infty.$$

- $\lim_{x\to -1^+} f(x)$: As $x \to -1^+$, $x > -1$. Then $x + 1 \to 0^+$, so $(x+1)^2 \to 0^+$. Also $x - 2 \to -1 - 2 = -3$. The denominator $(x+1)^2(x-2)$ approaches $0^+ \times (-3) = 0^-$. The numerator is 1.

$$\lim_{x\to -1^+} \frac{1}{(x+1)^2(x-2)} = -\infty.$$

- $\lim_{x\to -1^-} f(x)$: As $x \to -1^-$, $x < -1$. Then $x + 1 \to 0^-$, so $(x+1)^2 \to 0^+$. Also $x - 2 \to -3$. The denominator $(x+1)^2(x-2)$ approaches $0^+ \times (-3) = 0^-$. The numerator is 1.

$$\lim_{x\to -1^-} \frac{1}{(x+1)^2(x-2)} = -\infty.$$

(c) **Determine $\lim_{x\to 2} f(x)$ and $\lim_{x\to -1} f(x)$ if they exist.**

- For the limit $\lim_{x\to 2} f(x)$ to exist, the left-hand and right-hand limits must exist and be equal (Theorem 20.10). From part (b), $\lim_{x\to 2^+} f(x) = +\infty$ and $\lim_{x\to 2^-} f(x) = -\infty$. Since these are not equal (and not finite), the two-sided limit $\lim_{x\to 2} f(x)$ does not exist.
- For the limit $\lim_{x\to -1} f(x)$ to exist, the left-hand and right-hand limits must exist and be equal. From part (b), $\lim_{x\to -1^+} f(x) = -\infty$ and $\lim_{x\to -1^-} f(x) = -\infty$. Since both one-sided limits tend to $-\infty$, we can say that the limit exists in the extended sense:

$$\lim_{x\to -1} f(x) = -\infty.$$

If "exist" means "exist as a finite real number", then the limit does not exist. The limit $\lim_{x\to -1} f(x)$ does not exist as a finite real number. However, it is often written as $\lim_{x\to -1} f(x) = -\infty$.

Problem 5: Limits and Inequalities

Review Notes

- **Definition (One-Sided Limits):** As defined in Problem 4.
- **Limit Laws**:** If $\lim_{x\to a^+} f(x) = L$ and $\lim_{x\to a^+} g(x) = M$, then $\lim_{x\to a^+}(f(x) + g(x)) = L + M$, $\lim_{x\to a^+}(f(x)g(x)) = LM$, etc.
- **Theorem 20.5 (Order Properties of Limits):** Assume $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$.
  - If $f(x) \le g(x)$ for all x in some interval $(a - \delta_0, a + \delta_0) \setminus \{a\}$, then $L \le M$.
  - This theorem also holds for one-sided limits.

Solution

Suppose $L_1 = \lim_{x\to a^+} f_1(x)$ and $L_2 = \lim_{x\to a^+} f_2(x)$ exist.

(a) **Prove that if $f_1(x) \le f_2(x)$ for some interval (a, b), then $L_1 \le L_2$.**

Assume, for the sake of contradiction, that $L_1 > L_2$. Let $\epsilon = \frac{L_1 - L_2}{2}$. Since $L_1 > L_2$, we have $\epsilon > 0$.

By the definition of the right-hand limit $L_1 = \lim_{x\to a^+} f_1(x)$, there exists a $\delta_1 > 0$ such that for all x, if $a < x < a + \delta_1$, then $|f_1(x) - L_1| < \epsilon$. This implies $L_1 - \epsilon < f_1(x) < L_1 + \epsilon$. In particular, $f_1(x) > L_1 - \epsilon$.

By the definition of the right-hand limit $L_2 = \lim_{x\to a^+} f_2(x)$, there exists a $\delta_2 > 0$ such that for all x, if $a < x < a + \delta_2$, then $|f_2(x) - L_2| < \epsilon$. This implies $L_2 - \epsilon < f_2(x) < L_2 + \epsilon$. In particular, $f_2(x) < L_2 + \epsilon$.

We are given that $f_1(x) \le f_2(x)$ for $x \in (a, b)$. Let $b' = b$. Choose $\delta = \min(\delta_1, \delta_2, b' - a)$. Note that since $\delta_1 > 0$, $\delta_2 > 0$, and $b' > a$, we have $\delta > 0$. For any x such that $a < x < a + \delta$, we have:

(a) $a < x < a + \delta \le a + \delta_1$, so $f_1(x) > L_1 - \epsilon$.

(b) $a < x < a + \delta \le a + \delta_2$, so $f_2(x) < L_2 + \epsilon$.

(c) $a < x < a + \delta \le a + (b' - a) = b'$, so $x \in (a, b')$, which means $f_1(x) \le f_2(x)$.

Combining these inequalities for x in $(a, a + \delta)$:

$$L_1 - \epsilon < f_1(x) \le f_2(x) < L_2 + \epsilon.$$

So, $L_1 - \epsilon < L_2 + \epsilon$. Substituting $\epsilon = \frac{L_1 - L_2}{2}$:

$$L_1 - \frac{L_1 - L_2}{2} < L_2 + \frac{L_1 - L_2}{2}$$

$$\frac{2L_1 - (L_1 - L_2)}{2} < \frac{2L_2 + (L_1 - L_2)}{2}$$

$$\frac{L_1 + L_2}{2} < \frac{L_1 + L_2}{2}.$$

This is a strict inequality $\frac{L_1+L_2}{2} < \frac{L_1+L_2}{2}$, which is impossible. Therefore, our initial assumption $L_1 > L_2$ must be false. We conclude that $L_1 \le L_2$.

(b) **Suppose that $f_1(x) < f_2(x)$ for some interval (a, b). Is it always true that $L_1 < L_2$?**

No, it is not always true that $L_1 < L_2$. The limits can be equal even if the functions satisfy a strict inequality.

Counterexample: Let $a = 0$. Consider the interval $(0, 1)$ (so $b = 1$). Let $f_1(x) = 0$ for all $x \in (0, 1)$. Let $f_2(x) = x$ for all $x \in (0, 1)$.

Then for all $x \in (0, 1)$, we have $f_1(x) = 0 < x = f_2(x)$. So the condition $f_1(x) < f_2(x)$ holds on $(a, b) = (0, 1)$.

Now, let's find the limits as $x \to a^+ = 0^+$:

$$L_1 = \lim_{x\to 0^+} f_1(x) = \lim_{x\to 0^+} 0 = 0.$$

$$L_2 = \lim_{x\to 0^+} f_2(x) = \lim_{x\to 0^+} x = 0.$$

In this case, $L_1 = 0$ and $L_2 = 0$, so $L_1 = L_2$. This contradicts the claim that $L_1 < L_2$ must hold.

Therefore, $f_1(x) < f_2(x)$ on (a, b) does not imply $L_1 < L_2$.

Problem 6: Power Series Convergence

Review Notes

- **Definition (Power Series):** A power series centered at a is an infinite series of the form $\sum_{n=0}^{\infty} a_n(x-a)^n$.
- **Theorem 23.1 (Radius of Convergence):** For any power series $\sum a_n(x-a)^n$, there exists $R \in [0, \infty]$, called the **radius of convergence**, such that:
  - The series converges absolutely for $|x - a| < R$.

- The series diverges for $|x-a|>R$.
- The convergence/divergence at the endpoints $x = a \pm R$ (if $0 < R < \infty$) must be checked separately.

- **Formulas for Radius of Convergence:**
  - **Ratio Test:** If $L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ exists, then $R = 1/L$ (with $R=\infty$ if $L=0$ and $R=0$ if $L=\infty$).
  - **Root Test:** Let $\alpha = \limsup_{n\to\infty} |a_n|^{1/n}$. Then $R = 1/\alpha$ (with $R=\infty$ if $\alpha=0$ and $R=0$ if $\alpha=\infty$). The root test formula $R = 1/\limsup |a_n|^{1/n}$ always works.
- **Interval of Convergence:** The set of all x for which the power series converges. It is typically of the form $(a-R, a+R)$, $[a-R, a+R)$, $(a-R, a+R]$, or $[a-R, a+R]$ when $0<R<\infty$. If $R=0$, it's just $\{a\}$. If $R=\infty$, it's $(-\infty,\infty)$.

Solution

(a) $\sum_{n=0}^{\infty} n^2 x^n$

This is a power series centered at $a=0$ with coefficients $a_n = n^2$. We use the Ratio Test for absolute convergence. Let $b_n = n^2 x^n$.

$$\lim_{n\to\infty}\left|\frac{b_{n+1}}{b_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^2 x^{n+1}}{n^2 x^n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^2}{n^2}x\right|$$

$$= \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^2 |x| = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^2 |x| = (1)^2|x| = |x|.$$

The series converges absolutely if this limit is less than 1, i.e., $|x|<1$. The series diverges if this limit is greater than 1, i.e., $|x|>1$. The radius of convergence is $R=1$.

Now check endpoints $x=1$ and $x=-1$.

- At $x=1$: The series becomes $\sum_{n=0}^{\infty} n^2(1)^n = \sum_{n=0}^{\infty} n^2$. Since $\lim_{n\to\infty} n^2 = \infty \neq 0$, the series diverges by the Term Test (Theorem 14.6).
- At $x=-1$: The series becomes $\sum_{n=0}^{\infty} n^2(-1)^n$. Since $\lim_{n\to\infty} |n^2(-1)^n| = \lim_{n\to\infty} n^2 = \infty \neq 0$, the series diverges by the Term Test.

The interval of convergence is $(-1,1)$.

(b) $\sum_{n=1}^{\infty} \left(\frac{x}{n}\right)^n$

This is a power series $\sum_{n=1}^{\infty} \frac{1}{n^n} x^n$ centered at $a=0$ with coefficients $a_n = 1/n^n$. We use the Root Test for absolute convergence. Let $b_n = (x/n)^n$.

$$\limsup_{n\to\infty} |b_n|^{1/n} = \limsup_{n\to\infty} \left|\left(\frac{x}{n}\right)^n\right|^{1/n} = \limsup_{n\to\infty}\left|\frac{x}{n}\right|$$

$$= |x| \limsup_{n\to\infty} \frac{1}{n} = |x|\cdot 0 = 0.$$

Since the limit 0 is less than 1 for all $x\in\mathbb{R}$, the series converges absolutely for all x. The radius of convergence is $R=\infty$. The interval of convergence is $(-\infty,\infty)$.

(c) $\sum_{n=1}^{\infty} x^{n!}$

This is a power series centered at $a=0$. The coefficients a_k are: $a_k = 1$ if $k=n!$ for some $n\geq 1$ (i.e., k = 1, 2, 6, 24, 120, ...). $a_k = 0$ otherwise. The Ratio Test is difficult to apply because many coefficients are zero. We use the Root Test formula $R = 1/\alpha$ where $\alpha = \limsup_{k\to\infty} |a_k|^{1/k}$. The terms $|a_k|^{1/k}$ are either $1^{1/k}=1$ (if $k=n!$) or $0^{1/k}=0$ (if k is not a factorial). To find the limit superior, we look at the sequence $|a_k|^{1/k}$: $1,1,0,0,0,1,0,\ldots,0,1,0,\ldots$ (1 at positions 1!, 2!, 3!, ...). The supremum of the tails $\sup\{|a_m|^{1/m} : m \geq k\}$ is always 1 for any k, because there will always be a factorial $n! \geq k$ for large enough n, making $a_{n!} = 1$. So, $\alpha = \limsup_{k\to\infty} |a_k|^{1/k} = 1$. The radius of convergence is $R = 1/\alpha = 1/1 = 1$.

The series converges absolutely for $|x| < 1$ and diverges for $|x| > 1$. Check endpoints $x = 1$ and $x = -1$.

- At $x = 1$: The series becomes $\sum_{n=1}^{\infty}(1)^{n!} = \sum_{n=1}^{\infty} 1$. This series clearly diverges (terms do not go to 0).
- At $x = -1$: The series becomes $\sum_{n=1}^{\infty}(-1)^{n!}$. For $n \geq 2$, $n!$ is an even number $(2! = 2, 3! = 6, 4! = 24, \ldots)$. So $(-1)^{n!} = 1$ for $n \geq 2$. The series is $(-1)^{1!} + (-1)^{2!} + (-1)^{3!} + \cdots = -1 + 1 + 1 + 1 + \ldots$. The terms are $a_1 = -1$ and $a_n = 1$ for $n \geq 2$. Since $\lim_{n\to\infty} a_n = 1 \neq 0$, the series diverges by the Term Test.

The interval of convergence is $(-1, 1)$.

(d) $\sum_{n=0}^{\infty} 5^n x^{2n+1}$
Rewrite the series: $\sum_{n=0}^{\infty} 5^n x \cdot x^{2n} = x \sum_{n=0}^{\infty} 5^n (x^2)^n$. Let $y = x^2$. The series becomes $x \sum_{n=0}^{\infty} 5^n y^n$. This is a geometric series in y with ratio $5y$. It converges if and only if $|5y| < 1$, i.e., $|y| < 1/5$. Substituting back $y = x^2$, the series converges if $|x^2| < 1/5$, which means $x^2 < 1/5$. This inequality holds if $-\frac{1}{\sqrt{5}} < x < \frac{1}{\sqrt{5}}$.

Alternatively, use the Ratio Test on the original series $\sum b_n(x)$ where $b_n(x) = 5^n x^{2n+1}$.

$$\lim_{n\to\infty} \left| \frac{b_{n+1}(x)}{b_n(x)} \right| = \lim_{n\to\infty} \left| \frac{5^{n+1} x^{2(n+1)+1}}{5^n x^{2n+1}} \right| = \lim_{n\to\infty} \left| \frac{5^{n+1} x^{2n+3}}{5^n x^{2n+1}} \right|$$
$$= \lim_{n\to\infty} \left| 5x^2 \right| = 5|x^2| = 5x^2.$$

The series converges absolutely if $5x^2 < 1$, i.e., $x^2 < 1/5$. The series diverges if $5x^2 > 1$, i.e., $x^2 > 1/5$. The radius of convergence is $R = 1/\sqrt{5}$.

Now check endpoints $x = 1/\sqrt{5}$ and $x = -1/\sqrt{5}$.

- At $x = 1/\sqrt{5}$: The series becomes $\sum_{n=0}^{\infty} 5^n \left(\frac{1}{\sqrt{5}}\right)^{2n+1} = \sum_{n=0}^{\infty} 5^n \frac{1}{(\sqrt{5})^{2n}\sqrt{5}} = \sum_{n=0}^{\infty} 5^n \frac{1}{5^n\sqrt{5}} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{5}}$. This series diverges because the terms $\frac{1}{\sqrt{5}}$ do not approach 0.
- At $x = -1/\sqrt{5}$: The series becomes $\sum_{n=0}^{\infty} 5^n \left(-\frac{1}{\sqrt{5}}\right)^{2n+1} = \sum_{n=0}^{\infty} 5^n (-1)^{2n+1} \left(\frac{1}{\sqrt{5}}\right)^{2n+1}$. Since $2n+1$ is always odd, $(-1)^{2n+1} = -1$. The series is $\sum_{n=0}^{\infty} 5^n (-1) \frac{1}{5^n\sqrt{5}} = \sum_{n=0}^{\infty} -\frac{1}{\sqrt{5}}$. This series also diverges because the terms $-\frac{1}{\sqrt{5}}$ do not approach 0.

The interval of convergence is $(-\frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}})$.

Problem 7: Pointwise and Uniform Convergence

Review Notes

- **Definition 24.1 (Pointwise Convergence):** Let (f_n) be a sequence of functions defined on a set $S \subseteq \mathbb{R}$. The sequence (f_n) converges **pointwise** on S to a function f if for each $x \in S$, the sequence of real numbers $(f_n(x))$ converges to $f(x)$. That is, for every $x \in S$ and every $\epsilon > 0$, there exists an N such that if $n > N$, then $|f_n(x) - f(x)| < \epsilon$.
- **Definition 24.2 (Uniform Convergence):** A sequence of functions (f_n) converges **uniformly** on S to a function f if for every $\epsilon > 0$, there exists an N such that for all $n > N$ and for all $x \in S$, we have $|f_n(x) - f(x)| < \epsilon$.
  - The key difference from pointwise convergence is that N depends only on ϵ, not on x.
- **Supremum Norm Test (related to Definition 24.2):** $f_n \to f$ uniformly on S if and only if $\lim_{n\to\infty} \sup_{x\in S} |f_n(x) - f(x)| = 0$.

Solution

Let $f_n(x) = \frac{x}{n}$ for $x \in [0, \infty)$.

(a) **Find** $f(x) = \lim_{n\to\infty} f_n(x)$**.**

For any fixed $x \in [0, \infty)$, we consider the limit of the sequence $(f_n(x))_{n=1}^{\infty}$:

$$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x}{n}.$$

Since x is a fixed real number, this limit is:

$$f(x) = x \lim_{n\to\infty} \frac{1}{n} = x \cdot 0 = 0.$$

Thus, the sequence of functions (f_n) converges pointwise to the function $f(x) = 0$ on $[0, \infty)$.

(b) **Determine whether** $f_n \to f$ **uniformly on** $[0, 1]$**.**

We need to check if $\lim_{n\to\infty} \sup_{x\in[0,1]} |f_n(x) - f(x)| = 0$. Here $f(x) = 0$, so we consider:

$$|f_n(x) - f(x)| = \left|\frac{x}{n} - 0\right| = \frac{x}{n}$$

(since $x \geq 0$ and $n \geq 1$). We need to find the supremum of this expression for $x \in [0, 1]$.

$$M_n = \sup_{x\in[0,1]} |f_n(x) - f(x)| = \sup_{x\in[0,1]} \frac{x}{n}.$$

Since $\frac{x}{n}$ is an increasing function of x (for fixed n), the supremum on $[0, 1]$ occurs at $x = 1$.

$$M_n = \frac{1}{n}.$$

Now we check if $M_n \to 0$ as $n \to \infty$:

$$\lim_{n\to\infty} M_n = \lim_{n\to\infty} \frac{1}{n} = 0.$$

Since the limit is 0, the convergence $f_n \to f$ is uniform on $[0, 1]$.

(c) **Determine whether** $f_n \to f$ **uniformly on** $[0, \infty)$**.**

We need to check if $\lim_{n\to\infty} \sup_{x\in[0,\infty)} |f_n(x) - f(x)| = 0$.

$$|f_n(x) - f(x)| = \frac{x}{n} \quad \text{for } x \in [0, \infty).$$

We need to find the supremum of this expression for $x \in [0, \infty)$.

$$M_n = \sup_{x\in[0,\infty)} |f_n(x) - f(x)| = \sup_{x\in[0,\infty)} \frac{x}{n}.$$

For any fixed n, the function $\frac{x}{n}$ is unbounded on $[0, \infty)$. For example, as $x \to \infty$, $\frac{x}{n} \to \infty$. Therefore, the supremum is infinite:

$$M_n = \sup_{x\in[0,\infty)} \frac{x}{n} = \infty$$

for every $n \geq 1$. Since M_n does not converge to 0 (it's always ∞), the convergence $f_n \to f$ is not uniform on $[0, \infty)$.

Alternatively, using the definition: Uniform convergence requires that for a given $\epsilon > 0$, there exists N such that for all $n > N$, $|f_n(x) - f(x)| < \epsilon$ for all $x \in [0, \infty)$. This means we need $\frac{x}{n} < \epsilon$ for all $x \in [0, \infty)$ when $n > N$. This is equivalent to $x < n\epsilon$ for all $x \in [0, \infty)$. But this is impossible, since x can be arbitrarily large. For any n and any ϵ, we can always find an x (e.g., $x = n\epsilon$) such that $x \not< n\epsilon$. Thus, the convergence is not uniform on $[0, \infty)$.

Problem 8: Pointwise/Uniform Convergence and Continuity

Review Notes

- **Definition (Continuity at a Point):** A function $g : S \to \mathbb{R}$ is continuous at $c \in S$ if $\lim_{x \to c, x \in S} g(x) = g(c)$. This means for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $x \in S$ and $|x - c| < \delta$, then $|g(x) - g(c)| < \epsilon$.
- **Pointwise Convergence:** As defined in Problem 7.
- **Uniform Convergence:** As defined in Problem 7.
- **Theorem 24.3 (Uniform Convergence and Continuity):** Let (f_n) be a sequence of functions on $S \subseteq \mathbb{R}$ that converges uniformly to f on S. If each f_n is continuous at a point $c \in S$, then the limit function f is also continuous at c.
  - **Corollary:** If $f_n \to f$ uniformly on S and each f_n is continuous on S, then f is continuous on S.
  - **Contrapositive:** If $f_n \to f$ pointwise on S, each f_n is continuous on S, but f is not continuous on S, then the convergence cannot be uniform.

Solution

(a) **Sequence $f_n(x)$**

$$f_n(x) = \begin{cases} 1 & \text{if } x = 1/k \text{ for } k = 1, 2, \dots, n, \\ 0 & \text{otherwise.} \end{cases}$$

Defined on $\mathbb{R}$.

- **Is each f_n continuous at 0?**
  We need to check if $\lim_{x \to 0} f_n(x) = f_n(0)$. First, $f_n(0)$. Is 0 in the set $\{1, 1/2, \dots, 1/n\}$? No, these are all positive. So, $f_n(0) = 0$ by the 'otherwise' case. Now, we need the limit $\lim_{x \to 0} f_n(x)$. Consider any $\delta > 0$. We need to evaluate $f_n(x)$ for x in $(-\delta, \delta) \setminus \{0\}$. The points where $f_n(x) = 1$ are $1, 1/2, \dots, 1/n$. Only $1/n$ is the closest to 0 among these. We can choose δ small enough such that the interval $(-\delta, \delta)$ does not contain any of the points $1, 1/2, \dots, 1/n$. Specifically, choose $\delta = 1/(n+1)$. Then $0 < |x| < \delta = 1/(n+1)$ implies x cannot be any of $1, 1/2, \dots, 1/n$ (since the smallest of these is $1/n$, and $1/(n+1) < 1/n$). So, for $0 < |x| < 1/(n+1)$, $f_n(x) = 0$. Therefore, $\lim_{x \to 0} f_n(x) = 0$. Since $\lim_{x \to 0} f_n(x) = 0$ and $f_n(0) = 0$, yes, each f_n is continuous at 0.
- **Pointwise limit $f(x)$:**
  Let $x \in \mathbb{R}$ be fixed. We want to find $f(x) = \lim_{n \to \infty} f_n(x)$. Case 1: x is of the form $1/k$ for some integer $k \geq 1$. Then for all $n \geq k$, $x = 1/k$ is in the set $\{1, 1/2, \dots, 1/n\}$, so $f_n(x) = 1$ for all $n \geq k$. The sequence $(f_n(x))$ for large n is $(1, 1, 1, \dots)$. Thus, $\lim_{n \to \infty} f_n(x) = 1$. Case 2: x is not of the form $1/k$ for any integer $k \geq 1$. Then x is never in the set $\{1, 1/2, \dots, 1/n\}$ for any n. So $f_n(x) = 0$ for all n. The sequence $(f_n(x))$ is $(0, 0, 0, \dots)$. Thus, $\lim_{n \to \infty} f_n(x) = 0$. Combining these cases, the pointwise limit function $f(x)$ is:

$$f(x) = \begin{cases} 1 & \text{if } x = 1/k \text{ for some integer } k \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

- **Does $f_n \to f$ uniformly on $\mathbb{R}$?**
  We check $\lim_{n \to \infty} \sup_{x \in \mathbb{R}} |f_n(x) - f(x)|$.

$$|f_n(x) - f(x)|$$

  If $x = 1/k$ for $k = 1, \dots, n$: $|f_n(x) - f(x)| = |1 - 1| = 0$. If $x = 1/k$ for $k > n$: $f_n(x) = 0$ (since $1/k$ is not in $\{1, \dots, 1/n\}$). $f(x) = 1$ (since x is of the form $1/k$). So $|f_n(x) - f(x)| = |0 - 1| = 1$.

If x is not of the form $1/k$: $f_n(x) = 0$ and $f(x) = 0$. So $|f_n(x) - f(x)| = |0 - 0| = 0$. The difference is non-zero only for $x = 1/k$ with $k > n$, where the difference is 1. So, for any n,

$$M_n = \sup_{x \in \mathbb{R}} |f_n(x) - f(x)| = \sup\{|f_n(1/k) - f(1/k)| : k > n\} = \sup\{1\} = 1.$$

Since $\lim_{n\to\infty} M_n = \lim_{n\to\infty} 1 = 1 \neq 0$, the convergence is not uniform on $\mathbb{R}$.

- **Is f continuous at 0?**
  We need to check if $\lim_{x\to 0} f(x) = f(0)$. First, what is $f(0)$? Since 0 is not of the form $1/k$ for $k \geq 1$, $f(0) = 0$. Now, consider the limit $\lim_{x\to 0} f(x)$. We need to see how $f(x)$ behaves for x near 0. In any interval $(-\delta, \delta)$ around 0 (with $\delta > 0$), no matter how small δ is, there exists an integer k such that $1/k < \delta$. For such $x = 1/k$, we have $f(x) = 1$. For example, take the sequence $x_k = 1/k$. Then $x_k \to 0$ as $k \to \infty$. But $f(x_k) = f(1/k) = 1$ for all k. So, $\lim_{k\to\infty} f(x_k) = 1$. However, we can also take a sequence like $y_k = 1/(k\sqrt{2})$. Then $y_k \to 0$. Since y_k is never of the form $1/m$, $f(y_k) = 0$ for all k. So $\lim_{k\to\infty} f(y_k) = 0$. Since we found sequences approaching 0 on which f has different limits, the limit $\lim_{x\to 0} f(x)$ does not exist. Therefore, f is not continuous at 0. (Note: This confirms the convergence is not uniform, as f_n are continuous at 0 but f is not.)

(b) **Repeat part (a) for the sequence $g_n(x)$**

$$g_n(x) = \begin{cases} x & \text{if } x = 1/k \text{ for } k = 1, 2, \ldots, n, \\ 0 & \text{otherwise.} \end{cases}$$

- **Is each g_n continuous at 0?**
  $g_n(0) = 0$ since 0 is not of the form $1/k$. Limit $\lim_{x\to 0} g_n(x)$. Choose $\delta = 1/(n+1)$. For $0 < |x| < \delta$, x is not in $\{1, \ldots, 1/n\}$, so $g_n(x) = 0$. Thus, $\lim_{x\to 0} g_n(x) = 0$. Since $\lim_{x\to 0} g_n(x) = 0$ and $g_n(0) = 0$, yes, each g_n is continuous at 0.
- **Pointwise limit $g(x)$:**
  Let $x \in \mathbb{R}$ be fixed. Find $g(x) = \lim_{n\to\infty} g_n(x)$. Case 1: $x = 1/k$ for some integer $k \geq 1$. For all $n \geq k$, $x = 1/k$ is in $\{1, \ldots, 1/n\}$, so $g_n(x) = x$. The sequence $(g_n(x))$ for large n is $(x, x, x, \ldots)$. Thus, $\lim_{n\to\infty} g_n(x) = x$. Case 2: x is not of the form $1/k$ for any integer $k \geq 1$. Then $g_n(x) = 0$ for all n. The sequence is $(0, 0, 0, \ldots)$. Thus, $\lim_{n\to\infty} g_n(x) = 0$. Combining these cases, the pointwise limit function $g(x)$ is:

$$g(x) = \begin{cases} x & \text{if } x = 1/k \text{ for some integer } k \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

  Note: $g(x)$ could also be written as $g(x) = x \cdot f(x)$ where $f(x)$ is the limit function from part (a).
- **Does $g_n \to g$ uniformly on $\mathbb{R}$?**
  We check $\lim_{n\to\infty} \sup_{x\in\mathbb{R}} |g_n(x) - g(x)|$.

$$|g_n(x) - g(x)|$$

  If $x = 1/k$ for $k = 1, \ldots, n$: $|g_n(x) - g(x)| = |x - x| = 0$. If $x = 1/k$ for $k > n$: $g_n(x) = 0$. $g(x) = x$. So $|g_n(x) - g(x)| = |0 - x| = |x| = 1/k$. If x is not of the form $1/k$: $g_n(x) = 0$ and $g(x) = 0$. So $|g_n(x) - g(x)| = 0$. The difference is non-zero only for $x = 1/k$ with $k > n$, where the difference is $1/k$. So, for any n,

$$M_n = \sup_{x\in\mathbb{R}} |g_n(x) - g(x)| = \sup\{|g_n(1/k) - g(1/k)| : k > n\} = \sup\{1/k : k > n\}.$$

  The set $\{1/k : k > n\}$ contains $1/(n+1), 1/(n+2), \ldots$. The supremum of this set is $1/(n+1)$ (achieved at $k = n+1$).

$$M_n = \frac{1}{n+1}.$$

Now we check if $M_n \to 0$ as $n \to \infty$:

$$\lim_{n\to\infty} M_n = \lim_{n\to\infty} \frac{1}{n+1} = 0.$$

Since the limit is 0, the convergence $g_n \to g$ is uniform on $\mathbb{R}$.

- **Is g continuous at 0?**
  We need to check if $\lim_{x\to 0} g(x) = g(0)$. First, $g(0) = 0$ since 0 is not $1/k$. Now, consider the limit $\lim_{x\to 0} g(x)$. We want to know if for any $\epsilon > 0$, there exists $\delta > 0$ such that if $0 < |x| < \delta$, then $|g(x) - g(0)| = |g(x)| < \epsilon$. Recall $g(x)$ is either x (if $x = 1/k$) or 0 (otherwise). So, $|g(x)|$ is either $|x|$ or 0. In both cases, $|g(x)| \leq |x|$. Let $\epsilon > 0$ be given. Choose $\delta = \epsilon$. If $x \in \mathbb{R}$ and $0 < |x| < \delta = \epsilon$, then

  $$|g(x) - g(0)| = |g(x)| \leq |x| < \delta = \epsilon.$$

  So, $|g(x)| < \epsilon$. Therefore, $\lim_{x\to 0} g(x) = 0$. Since $\lim_{x\to 0} g(x) = 0$ and $g(0) = 0$, yes, g is continuous at 0. (This is consistent with Theorem 24.3: g_n are continuous at 0, $g_n \to g$ uniformly, so g must be continuous at 0).