Problems and Solutions: Properties of the Riemann Integral

Problem 1. Suppose f is a continuous function on $[a,b]$, and $f(x) \geq 0$ for all $x \in [a,b]$. Prove that if $\int_a^b f = 0$, then $f(x) = 0$ for all $x \in [a,b]$.

Solution Sketch. The proof relies on Ross, Theorem 33.4(ii) [2], which states this result directly. Alternatively, prove by contradiction:

- Assume $f(x_0) > 0$ for some $x_0 \in (a,b)$.
- By continuity of f, there exists $\delta > 0$ such that for $x \in (x_0 - \delta, x_0 + \delta) \subset [a,b]$, we have $f(x) > f(x_0)/2$. Let $\alpha = f(x_0)/2 > 0$ and $[c,d] = [x_0 - \delta/2, x_0 + \delta/2]$.
- Since $f(x) \geq 0$ everywhere, and $f(x) \geq \alpha$ on $[c,d]$, we have by monotonicity of the integral [Ross, Thm. 33.4(i)]:

$$\int_a^b f(x)\,dx \geq \int_c^d f(x)\,dx \geq \int_c^d \alpha\,dx = \alpha(d-c) = \alpha\delta > 0$$

- This contradicts the given condition $\int_a^b f = 0$. Thus, the assumption must be false, and $f(x) = 0$ for all $x \in [a,b]$. (Handle endpoints a, b similarly if x_0 is an endpoint).

Problem 2. Consider the function

$$f(x) = \begin{cases} 1 & \text{if } x = 2^{-n} \text{ for } n \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

Prove that f is integrable on $[0,1]$ and that $\int_0^1 f = 0$.

Solution Sketch. Boundedness: $0 \leq f(x) \leq 1$, so f is bounded.

Lower Integral: For any partition $P = \{t_0, \ldots, t_k\}$ of $[0,1]$, any subinterval $[t_{i-1}, t_i]$ contains points where $f(x) = 0$ (e.g., irrationals). Thus, $m_i = \inf\{f(x) : x \in [t_{i-1}, t_i]\} = 0$. The lower sum is

$$L(f,P) = \sum_{i=1}^{k} m_i(t_i - t_{i-1}) = \sum_{i=1}^{k} 0 \cdot (t_i - t_{i-1}) = 0$$

The lower Darboux integral is $L(f) = \sup_P\{L(f,P)\} = 0$.

Upper Integral: We use the integrability criterion [Ross, Thm. 32.5]: f is integrable iff for any $\epsilon > 0$, there exists a partition P such that $U(f,P) - L(f,P) < \epsilon$. Since $L(f,P) = 0$, we need $U(f,P) < \epsilon$.

- Let $\epsilon > 0$. The set where $f(x) = 1$ is $S = \{2^{-n} : n \in \mathbb{N}\}$. These points accumulate only at 0.
- Choose $N \in \mathbb{N}$ large enough such that $2^{-N} < \epsilon/2$. The points $\{2^{-n} : n > N\}$ are in $[0, 2^{-N})$. The points $x_j = 2^{-j}$ for $j = 1, \ldots, N$ are in $[2^{-N}, 1]$.
- Choose $\eta > 0$ small enough such that $\eta < \epsilon/(4N)$ and the intervals $[x_j - \eta, x_j + \eta]$ are disjoint within $[2^{-N}, 1]$.
- Define a partition P using points $0, 2^{-N}$ and $x_j \pm \eta$ for $j = 1, \ldots, N$. Let $P = \{0 = t_0 < t_1 < \cdots < t_m = 1\}$.
- The upper sum is $U(f,P) = \sum_{i=1}^{m} M_i(t_i - t_{i-1})$, where $M_i = \sup_{[t_{i-1}, t_i]} f$. $M_i = 1$ only if the subinterval contains some 2^{-n}.
- Contribution from the first subinterval $[0, t_1]$ (assuming $t_1 \approx 2^{-N}$, we can ensure $t_1 \leq 2^{-N}$): Contains points 2^{-n} for $n > N$. $M_1 = 1$. We can choose $t_1 = 2^{-N}$. Contribution is $M_1(t_1 - t_0) = 1 \cdot 2^{-N} < \epsilon/2$.

- Contribution from intervals covering $x_1, \ldots, x_N$: Each x_j is contained in one or two subintervals of total length at most $2\eta = \epsilon/(2N)$. $M_i = 1$ for these. Total contribution from these N points is bounded by $N \cdot (2\eta) = N \cdot (\epsilon/(2N)) = \epsilon/2$.
- Contribution from other intervals: These contain no points 2^{-n}, so $M_i = 0$. Contribution is 0.
- Total upper sum:
$$U(f,P) < \frac{\epsilon}{2} + \frac{\epsilon}{2} + 0 = \epsilon$$

Since $U(f,P)$ can be made arbitrarily small, the upper integral $U(f) = \inf_P\{U(f,P)\} = 0$.

Conclusion: Since $L(f) = U(f) = 0$, f is integrable on $[0,1]$ and $\int_0^1 f = 0$.

Problem 3. Construct an example of a function where $f(x)^2$ is integrable on $[0,1]$ but $f(x)$ is not.

Solution Sketch. Define $f : [0,1] \to \mathbb{R}$ by
$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \cap [0,1] \\ -1 & \text{if } x \in (\mathbb{R} \setminus \mathbb{Q}) \cap [0,1] \end{cases}$$

Integrability of f: For any partition P, any subinterval $[t_{k-1}, t_k]$ contains both rational and irrational numbers.
$$m_k = \inf_{x \in [t_{k-1}, t_k]} f(x) = -1, \quad M_k = \sup_{x \in [t_{k-1}, t_k]} f(x) = 1$$

The Darboux sums are:
$$L(f,P) = \sum_{k=1}^{n} (-1)(t_k - t_{k-1}) = -1, \quad U(f,P) = \sum_{k=1}^{n} (1)(t_k - t_{k-1}) = 1$$

The lower and upper integrals are $L(f) = -1$ and $U(f) = 1$. Since $L(f) \neq U(f)$, f is not integrable [Ross, Def. 32.3].

Integrability of f^2: Consider $g(x) = f(x)^2$. If x is rational, $g(x) = (1)^2 = 1$. If x is irrational, $g(x) = (-1)^2 = 1$. So, $g(x) = f(x)^2 = 1$ for all $x \in [0,1]$. This is a constant function. Constant functions are continuous, and continuous functions on $[a,b]$ are integrable [Ross, Thm. 33.2]. Therefore, f^2 is integrable on $[0,1]$.

Problem 4. Construct an example of a sequence of functions (f_n) on $[0,1]$ such that $f_n \to 0$ pointwise, but the sequence $s_n = \int_0^1 f_n$ diverges.

Solution Sketch. Consider the sequence $f_n(x) = n^2 x(1-x^2)^n$ for $x \in [0,1]$ [4].

Pointwise Convergence:

- If $x = 0$, $f_n(0) = 0$ for all n, so $\lim_{n\to\infty} f_n(0) = 0$.
- If $0 < x \leq 1$, let $r = 1 - x^2$. Then $0 \leq r < 1$. We need $\lim_{n\to\infty} n^2 x r^n$. Since $0 \leq r < 1$, $\lim_{n\to\infty} n^p r^n = 0$ for any p [Rudin, Thm. 3.20(d)]. Thus,
$$\lim_{n\to\infty} f_n(x) = x \lim_{n\to\infty} (n^2 r^n) = x \cdot 0 = 0$$

So, $f_n \to 0$ pointwise on $[0,1]$.

Sequence of Integrals: Calculate $s_n = \int_0^1 f_n(x)\,dx$.
$$s_n = \int_0^1 n^2 x(1-x^2)^n\,dx$$

Use substitution $u = 1 - x^2$, $du = -2x\,dx$. When $x = 0, u = 1$. When $x = 1, u = 0$.
$$s_n = n^2 \int_1^0 u^n \left(-\frac{1}{2}\,du\right) = \frac{n^2}{2} \int_0^1 u^n\,du = \frac{n^2}{2} \left[\frac{u^{n+1}}{n+1}\right]_0^1 = \frac{n^2}{2(n+1)}$$

Divergence: Evaluate the limit of s_n:
$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{n^2}{2n+2} = \lim_{n\to\infty} \frac{n}{2 + 2/n} = +\infty$$

The sequence of integrals s_n diverges.

Problem 5. Suppose that f is continuous on (a, b), where a may be $-\infty$ and b may be ∞. If $\int_a^b |f(x)|dx < \infty$, show that the integral $\int_a^b f(x)dx$ exists and is finite.

Solution Sketch. Define the positive and negative parts of f:

$$f^+(x) = \max\{f(x), 0\}, \quad f^-(x) = \max\{-f(x), 0\}$$

These are continuous and non-negative. We have the relations:

$$f(x) = f^+(x) - f^-(x), \quad |f(x)| = f^+(x) + f^-(x)$$

From these, we get the inequalities:

$$0 \le f^+(x) \le |f(x)|, \quad 0 \le f^-(x) \le |f(x)|$$

We are given that $\int_a^b |f(x)|\, dx$ converges. By the comparison test for improper integrals [Ross, Exer. 36.6], since f^+, f^- are non-negative and bounded above by the integrable function $|f|$, their improper integrals must also converge:

$$\int_a^b f^+(x)\, dx < \infty, \quad \int_a^b f^-(x)\, dx < \infty$$

The improper integral $\int_a^b f(x)\, dx$ exists if the limits defining it exist and are finite. For $c \in (a, b)$:

$$\int_a^b f\, dx = \lim_{y \to a^+} \int_y^c f\, dx + \lim_{z \to b^-} \int_c^z f\, dx$$

Since $f = f^+ - f^-$, and the integrals of f^+ and f^- converge, the limits for f must exist:

$$\lim_{y \to a^+} \int_y^c f\, dx = \lim_{y \to a^+} \int_y^c f^+\, dx - \lim_{y \to a^+} \int_y^c f^-\, dx \quad \text{(exists and is finite)}$$

$$\lim_{z \to b^-} \int_c^z f\, dx = \lim_{z \to b^-} \int_c^z f^+\, dx - \lim_{z \to b^-} \int_c^z f^-\, dx \quad \text{(exists and is finite)}$$

Therefore, $\int_a^b f(x)\, dx$ exists and is finite, equaling $\int_a^b f^+\, dx - \int_a^b f^-\, dx$.

Problem 6. (a) Suppose $|f(x)| \le M$ for $x \in [a, b]$. Show $|f(x)^2 - f(y)^2| \le 2M|f(x) - f(y)|$.

(b) Prove that if f is integrable on $[a, b]$, then so is f^2.

(c) Suppose f, g are integrable on $[a, b]$. Show fg is integrable.

Solution Sketch. (a) Use the difference of squares:

$$|f(x)^2 - f(y)^2| = |(f(x) - f(y))(f(x) + f(y))| = |f(x) - f(y)||f(x) + f(y)|$$

By the triangle inequality and the bound $|f(z)| \le M$:

$$|f(x) + f(y)| \le |f(x)| + |f(y)| \le M + M = 2M$$

Substituting gives the result:

$$|f(x)^2 - f(y)^2| \le |f(x) - f(y)|(2M)$$

(b) If f is integrable, it is bounded, say $|f(x)| \le M$. Then $|f(x)^2| \le M^2$, so f^2 is bounded. Use the integrability criterion $U(h, P) - L(h, P) < \epsilon$ [Ross, Thm. 32.5]. Let $\epsilon > 0$. For any subinterval $I_k = [t_{k-1}, t_k]$ of a partition P, let $M_k(h) = \sup_{I_k} h$ and $m_k(h) = \inf_{I_k} h$. From (a), for $x, y \in I_k$: $|f(x)^2 - f(y)^2| \le 2M|f(x) - f(y)|$. This implies:

$$M_k(f^2) - m_k(f^2) \le 2M(M_k(f) - m_k(f))$$

Summing over the partition:

$$U(f^2, P) - L(f^2, P) = \sum_k (M_k(f^2) - m_k(f^2))\Delta t_k \le \sum_k 2M(M_k(f) - m_k(f))\Delta t_k$$

$$U(f^2, P) - L(f^2, P) \le 2M(U(f, P) - L(f, P))$$

Since f is integrable, choose P such that $U(f, P) - L(f, P) < \epsilon/(2M)$ (assume $M > 0$; if $M = 0$, $f = 0$, $f^2 = 0$ is integrable). Then $U(f^2, P) - L(f^2, P) < 2M(\epsilon/(2M)) = \epsilon$. Thus f^2 is integrable.

(c) Use the polarization identity [Ross, Exer. 33.8a]:

$$fg = \frac{1}{4}\left((f + g)^2 - (f - g)^2\right)$$

- If f, g are integrable, then $f+g$ and $f-g$ are integrable [Ross, Thm. 33.3].
- By part (b), $(f+g)^2$ and $(f-g)^2$ are integrable.
- The difference $(f+g)^2-(f-g)^2$ is integrable [Ross, Thm. 33.3].
- Multiplying by the constant $1/4$, fg is integrable [Ross, Thm. 33.3].

Problem 7. (a) For $u, v \in \mathbb{R}$, prove $uv \leq (u^2+v^2)/2$. If $\int_a^b f^2 = 1, \int_a^b g^2 = 1$, show $\int_a^b fg \leq 1$.

(b) Prove the Schwarz inequality: $|\int_a^b fg| \leq (\int_a^b f^2)^{1/2}(\int_a^b g^2)^{1/2}$.

(c) Let $X = C[a,b]$. Show $d(f,g) = (\int_a^b |f-g|^2)^{1/2}$ is a metric.

Solution Sketch. (a) The inequality $(u-v)^2 = u^2 - 2uv + v^2 \geq 0$ implies $u^2+v^2 \geq 2uv$, hence $uv \leq (u^2+v^2)/2$. Applying this pointwise: $f(x)g(x) \leq \frac{f(x)^2+g(x)^2}{2}$. Since f, g are integrable, f^2, g^2, fg are integrable (Problem 6). Integrate the inequality [Ross, Thm. 33.4(i)]:

$$\int_a^b f(x)g(x)\,dx \leq \int_a^b \frac{f(x)^2+g(x)^2}{2}\,dx$$

Using linearity [Ross, Thm. 33.3]:

$$\int_a^b fg \leq \frac{1}{2}\left(\int_a^b f^2 + \int_a^b g^2\right)$$

Given $\int f^2 = 1$ and $\int g^2 = 1$, we get $\int_a^b fg \leq \frac{1}{2}(1+1) = 1$.

(b) Consider the quadratic in λ:

$$Q(\lambda) = \int_a^b (f(x) + \lambda g(x))^2\,dx \geq 0$$

Expand using linearity:

$$Q(\lambda) = \int_a^b f^2\,dx + 2\lambda \int_a^b fg\,dx + \lambda^2 \int_a^b g^2\,dx$$

Let $A = \int g^2$, $B = 2\int fg$, $C = \int f^2$. Then $A\lambda^2 + B\lambda + C \geq 0$. If $A = 0$, then $g = 0$ a.e., $\int fg = 0$, inequality holds ($0 \leq 0$). If $A > 0$, the quadratic is always ≥ 0, so its discriminant must be ≤ 0:

$$B^2 - 4AC \leq 0$$

$$\left(2\int_a^b fg\right)^2 - 4\left(\int_a^b g^2\right)\left(\int_a^b f^2\right) \leq 0$$

$$4\left(\int_a^b fg\right)^2 \leq 4\left(\int_a^b f^2\right)\left(\int_a^b g^2\right)$$

$$\left(\int_a^b fg\right)^2 \leq \left(\int_a^b f^2\right)\left(\int_a^b g^2\right)$$

Taking the square root yields the Schwarz inequality:

$$\left|\int_a^b fg\right| \leq \left(\int_a^b f^2\right)^{1/2}\left(\int_a^b g^2\right)^{1/2}$$

(c) Verify metric properties for $d(f,g) = \left(\int_a^b |f-g|^2\right)^{1/2}$ on $X = C[a,b]$.

1. **Non-negativity:** $|f-g|^2 \geq 0 \implies \int |f-g|^2 \geq 0 \implies d(f,g) \geq 0$.
2. **Identity:** $d(f,g) = 0 \iff \int |f-g|^2 = 0$. Since $|f-g|^2$ is continuous and non-negative, this holds iff $|f(x)-g(x)|^2 = 0$ for all x, which means $f(x) = g(x)$ for all x, i.e., $f = g$ [Ross, Thm. 33.4(ii)].

3. **Symmetry:** $|f-g|^2 = |g-f|^2 \implies \int |f-g|^2 = \int |g-f|^2 \implies d(f,g) = d(g,f)$.

4. **Triangle Inequality (Minkowski):** Show $d(f,h) \le d(f,g) + d(g,h)$. Let $u = f-g, v = g-h$. Need to show $(\int |u+v|^2)^{1/2} \le (\int |u|^2)^{1/2} + (\int |v|^2)^{1/2}$.

$$\int |u+v|^2 = \int (u+v)^2 = \int (u^2 + 2uv + v^2) = \int u^2 + 2\int uv + \int v^2$$

By Schwarz inequality (part b): $\int uv \le |\int uv| \le (\int u^2)^{1/2} (\int v^2)^{1/2}$.

$$\int |u+v|^2 \le \int u^2 + 2(\int u^2)^{1/2} (\int v^2)^{1/2} + \int v^2$$

The right side is $(\|u\|_2 + \|v\|_2)^2$, where $\|w\|_2 = (\int w^2)^{1/2}$.

$$\int |u+v|^2 \le \left(\left(\int u^2 \right)^{1/2} + \left(\int v^2 \right)^{1/2} \right)^2$$

Taking the square root gives the Minkowski inequality:

$$\left(\int |u+v|^2 \right)^{1/2} \le \left(\int u^2 \right)^{1/2} + \left(\int v^2 \right)^{1/2}$$

Substituting back $u = f-g, v = g-h, u+v = f-h$ gives $d(f,h) \le d(f,g) + d(g,h)$.

All properties hold, so d is a metric.

Chapter Summary: The Riemann Integral

These problems primarily cover material from Ross, Chapters 32 and 33, focusing on the definition and fundamental properties of the Riemann integral.

The core concept is the **Definition of Riemann Integrability** via Darboux sums (Ross, Def. 32.1, 32.3). A bounded function f on $[a,b]$ is integrable if its lower integral $L(f)$ equals its upper integral $U(f)$. The common value is $\int_a^b f$. The **Integrability Criterion** (Ross, Thm. 32.5), stating that f is integrable iff for every $\epsilon > 0$, there exists a partition P with $U(f,P) - L(f,P) < \epsilon$, is a key tool for proving integrability (Problems 2, 6(b)).

Essential **Properties of the Integral** include:

- **Linearity** (Ross, Thm. 33.3): $\int (cf + dg) = c\int f + d\int g$.
- **Monotonicity** (Ross, Thm. 33.4(i)): $f \le g \implies \int f \le \int g$. This leads to the property that if $f \ge 0$ is continuous and $\int f = 0$, then $f \equiv 0$ (Ross, Thm. 33.4(ii), Problem 1).
- **Absolute Value and Products**: If f,g are integrable, then $|f|$, f^2, and fg are integrable (Ross, Thm. 33.5, Exer. 33.7b, 33.8a; Problems 3, 6). The inequality $|\int f| \le \int |f|$ holds (Ross, Thm. 33.5).

Important classes of **Integrable Functions** include continuous functions (Ross, Thm. 33.2) and monotonic functions (Ross, Thm. 33.1). Functions with limited discontinuities may also be integrable (Problem 2).

The relationship between **Integration and Limits** is subtle. Pointwise convergence of f_n to f does not suffice to ensure $\int f_n \to \int f$ (Problem 4). Stronger conditions like uniform convergence (Ross, Thm. 25.2) are needed.

The **Schwarz Inequality** (Problem 7(b)) is a fundamental result relating the integral of a product to the integrals of squares. It forms the basis for the L^2 norm and metric structure on function spaces, as demonstrated by showing the triangle inequality for the L^2 metric on $C[a,b]$ (Problem 7(c)).

Finally, the concept of **Absolute Convergence** for improper integrals is introduced (Problem 5), showing that if $\int |f|$ converges, then $\int f$ converges.