Metric spaces

Many results from the previous sections can be framed in a more general way, using another level of abstraction.

Consider the proof that the sum of two Cauchy sequences is also Cauchy:

$$(s_n) \text{ Cauchy} \Rightarrow \forall \varepsilon > 0 \quad \exists N_1 \text{ s.t. } m,n > N_1 \Rightarrow |s_m - s_n| < \varepsilon$$
$$(t_n) \text{ Cauchy} \Rightarrow \forall \varepsilon > 0 \quad \exists N_2 \text{ s.t } m,n > N_2 \Rightarrow |t_m - t_n| < \varepsilon$$

Take $m,n > \max\{N_1, N_2\}$. A critical component of the proof is the triangle inequality.

$$|s_m + t_m - (s_n + t_n)| \le |s_m - s_n| + |t_m - t_n| = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \ \ldots$$

The proof could still apply in a different situation other than for $\mathbb{R}$, so long as the triangle inequality holds. This motivates our next definition.

<u>Definition</u> Let S be a set, and let d be a function defined for all pairs (x,y) of elements in S, such that:

(1) $d(x,x)=0 \quad \forall x \in S$ and $d(x,y)>0 \quad \forall x,y \in S$ where $x \neq y$.

(2) $d(x,y) = d(y,x) \quad \forall x,y \in S$.

(3) $d(x,z) \leq d(x,y) + d(y,z) \quad \forall x,y,z \in S$.

Then d is a distance function or metric, and the pair (S,d) is a metric space.

<u>Example</u> Define $\mathbb{R}^k$ to be k-dimensional space, given by the set of all k-tuples

$$\underline{x} = (x_1, x_2, \ldots, x_k)$$

Then the Euclidean metric is

$$d(\underline{x},\underline{y}) = \sqrt{\sum_{i=1}^{k} (x_i - y_i)^2}$$

Alternative metrics are given by

$$d_p(\underline{x},\underline{y}) = \sqrt[p]{\sum_{i=1}^{k} |x_i - y_i|^p}$$

Matches the Euclidean metric when $p=2$. It is a valid metric for all $p \geq 1$. As p gets large, the largest term $|x_i - y_i|$ will dominate the sum.

This motivates the special definition for $p=\infty$:

$$d_\infty(\underline{x}, \underline{y}) = \max_{i \in \{1,\dots,k\}} |x_i - y_i|.$$

Definition A sequence (s_n) converges to s if $\lim_{n\to\infty} d(s_n, s) = 0$. A sequence (s_n) is a Cauchy sequence if for each $\varepsilon > 0$, $\exists N$ s.t. $m, n > N$ implies $d(s_m, s_n) < \varepsilon$.

The metric space is complete if every Cauchy sequence in S converges to some element in S. In $\mathbb{R}$, this notion of completeness is interchangeable with the completeness axiom.

Topological concepts.

Consider a metric space (S, d):

(a) A neighborhood of $p \in S$ is a set

$$N_r(p) = \{ q \in S \mid d(p,q) < r \}. \quad \begin{pmatrix} r>0 \\ r \in \mathbb{R} \end{pmatrix}$$

(b) A point p is a limit point of a set $E \subseteq S$ if every neighborhood of p contains a $q \neq p$ such that $q \in E$.

(c) If $p \in E$ and it is not a limit point, it is called an isolated point.

(d) E is closed if every limit point of E is a point of E.

(e) A point p is an interior point of E if there is a neighborhood N of p such that $N \subseteq E$.

(f) E is open if every point of E is an interior point of E.

(g) The complement of E (denoted by E^c) is the set of all points $p \in \mathcal{S}$ such that $p \notin E$.

(h) E is bounded if there is a real number M and a point $q \in \mathcal{S}$ such that $d(p,q) < M$ for all $p \in E$.

Equivalent metrics Two metrics d_1 and d_2 on $\mathcal{S}$ are equivalent if $\forall x \in \mathcal{S}, \forall \varepsilon > 0, \exists \delta > 0$ such that

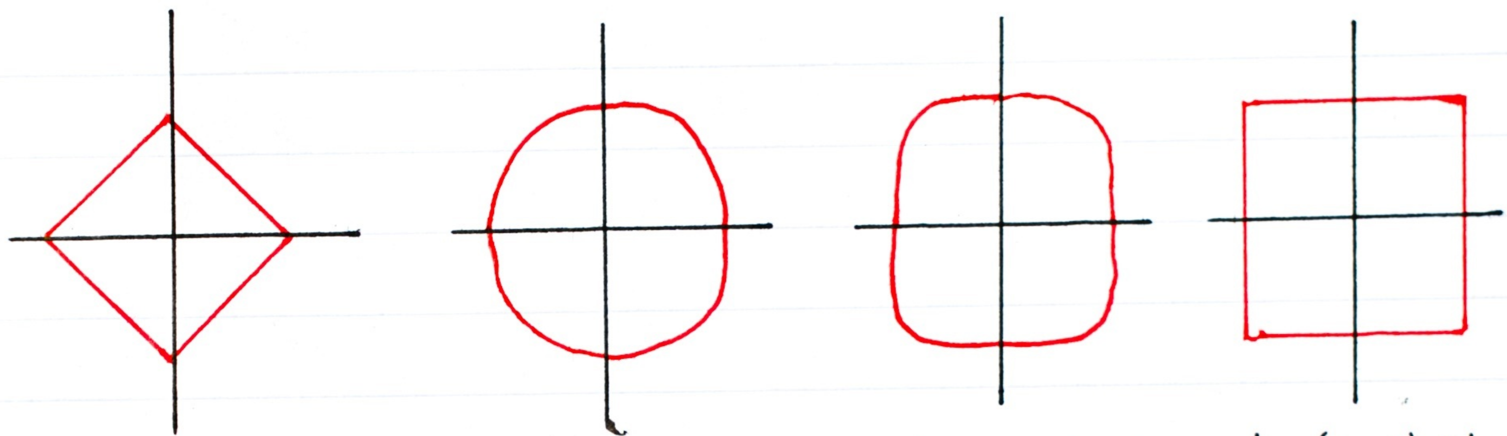
Neighborhood with respect to d_1 $\rightarrow$

$$N_\delta^{(d_1)}(x) \subseteq N_\varepsilon^{(d_2)}(x)$$

and vice versa.

Examples of neighborhoods in $\mathbb{R}^2$

$\underset{\sim}{0} = (0,0)$



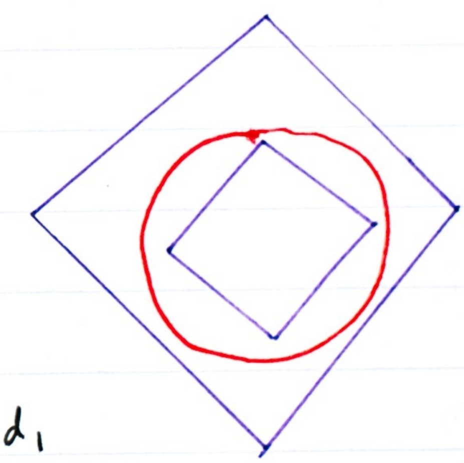
$d_1(\underset{\sim}{x}, \underset{\sim}{0}) < 1$ diamond

$d_2(\underset{\sim}{x}, \underset{\sim}{0}) < 1$ circle

$d_4(\underset{\sim}{x}, \underset{\sim}{0}) < 1$ superellipse

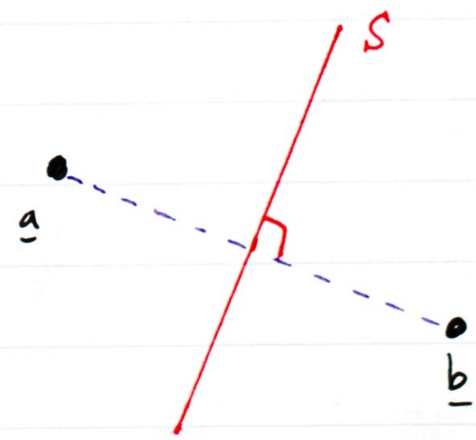
$d_\infty(\underset{\sim}{x}, \underset{\sim}{0}) < 1$ square

Graphically, the equivalency of neighborhood can be shown by nesting them inside each other. Fore example diamonds fit inside and outside a circle and hence d_1 and d_2 are equivalent.



Exercise Consider two points $\underset{\sim}{a}, \underset{\sim}{b} \in \mathbb{R}^2$. The set

$$S = \{ \underset{\sim}{x} \in \mathbb{R}^2 \mid d(\underset{\sim}{x}, \underset{\sim}{a}) = d(\underset{\sim}{x}, \underset{\sim}{b}) \}$$



is the perpendicular bisector between $\underset{\sim}{a}$ and $\underset{\sim}{b}$. What do the sets $\{ \underset{\sim}{x} \in \mathbb{R} \mid d_p(\underset{\sim}{x}, \underset{\sim}{a}) = d_p(\underset{\sim}{x}, \underset{\sim}{b}) \}$ look like for different choices of p?

Theorem Every neighborhood is an open set

Proof Consider $E = N_r(p)$ and let q be any point of E. Then there is a positive real number h such that

$$d(p,q) = r - h.$$

Consider s such that $d(q,s) < h$. Then

$$d(p,s) \leq d(p,q) + d(q,s) < r - h + h = r,$$

so $s \in E$. Hence $N_h(q) \subseteq E$ and q is an interior point. Hence $N_r(p)$ is open.

Proposition If p is a limit point of a set E then every neighborhood of p contains infinitely many points of E.

Proof Suppose there is a neighborhood N of p that contains only finitely many points of E. Let $q_1, q_2, \ldots, q_n$ be those points of $N \cap E$ that are distinct from p. Define $r = \min\limits_{m \in \{1, \ldots, n\}} d(p, q_m)$.

Then $N_r(p) \not\ni q_i$ for all q_i, so it does not contain any point q of E such that $q \neq p$. Hence

p is not a limit point of E, which is a contradiction.

<u>Corollary</u> A finite point set has no limit points

<u>Theorem</u> $A = \left(\bigcup_{\alpha} E_\alpha \right)^c = \bigcap_{\alpha} E_\alpha^c = B$ for a collection of (possibly infinite) sets E_α.

<u>Proof</u> If $x \in A$, then $x \notin \bigcup_\alpha E_\alpha$, so $x \notin E_\alpha$ for all α. Hence $x \in E_\alpha^c$ for all α. So $x \in \bigcap_\alpha E_\alpha^c$ and $A \subseteq B$.

If $x \in B$, then $x \in E_\alpha^c$ for all α, so $x \notin E_\alpha$ for all α, so $x \notin \bigcup_\alpha E_\alpha$. Therefore $x \in \left(\bigcup_\alpha E_\alpha\right)^c$ and $B \subseteq A$. Therefore $A = B$.

<u>Example</u> $(0,1)$ is open, but not closed. To show it is open, pick $q \in (0,1)$ and define $r = \min\{q, 1-q\}$. Then $N_r(q) \subseteq (0,1)$ so q is an interior point. Hence all points in $(0,1)$ are interior points and $(0,1)$ is open.

Alternatively, note that $(0,1) = N_{1/2}(1/2)$ and neighborhoods are always open.

To show that $(0,1)$ is not closed, consider the point $0 \notin (0,1)$. Choose $\varepsilon > 0$ and define $r = \min\{\varepsilon, 1\}$. Then $r/2 \in (0,1)$ and $\frac{r}{2} \in N_\varepsilon(0)$, and $\frac{r}{2} \neq 0$.

Hence any neighborhood of 0 contains a point $q \neq 0$ such that $q \in (0,1)$, and therefore 0 is a limit point. Therefore $(0,1)$ is not closed.

Theorem A set E is open if and only if its complement is closed.

Proof Suppose E^c is closed. Choose $x \in E$. Since $x \notin E^c$, x is not a limit point of E^c. Hence there exists a neighborhood $N_r(x)$ such that $E^c \cap N_r(x) = \phi$, so $N_r(x) \subseteq E$. Hence x is an interior point. Since this is true for all $x \in E$, E is open.

Next, suppose that E is open. Let x be a limit point of E^c. Then every neighborhood of x contains a point of E^c, so x is not an interior point of E. Since E is open, this means that $x \in E^c$, so E^c is closed.

Theorem (a) For any collection $\{G_\alpha\}$ of open sets, $\bigcup_\alpha G_\alpha$ is open.

(b) For any collection $\{F_\alpha\}$ of closed sets, $\bigcap_\alpha F_\alpha$ is closed.

(c) For any finite collection $G_1, G_2, \ldots, G_n$ of open sets, $\bigcap_{i=1}^{n} G_i$ is open.

(d) For any finite collection $F_1, F_2, \ldots, F_n$ of closed sets, $\bigcup_{i=1}^{n} F_i$ is closed.

Proof (a) Suppose $G = \bigcup_\alpha G_\alpha$. If $x \in G$, then $x \in G_\alpha$ for some G_α. Since x is an interior point of G_α, x is an interior point of G, and G is open.

(b) Use the identity $\bigcap_\alpha F_\alpha = \left[\bigcup_\alpha (F_\alpha^c)\right]^c$ and combine with part (a).

(c) Put $H = \bigcap_{i=1}^{n} G_i$. For any $x \in H$, $\exists$ neighborhoods N_i of x with radii r_i such that $N_i \subseteq G_i$. Define $r = \min\{r_1, r_2, \ldots, r_n\}$. Consider $N_r(x)$: Since $N_r(x) \subseteq N_i \subseteq G_i$ it follows that H is open

(d) Use the identity $\bigcup_{i=1}^{n} F_i = \left[\bigcap_{i=1}^{n} (F_i^c)\right]^c$ and combine with part (c).

Example An infinite collection of open sets $\{G_\alpha\}$ may not satisfy $\bigcap_\alpha G_\alpha$ being open.

Suppose $G_k = \left(-\frac{1}{k}, \frac{1}{k}\right)$ for $k \in \mathbb{N}$. Then $\bigcap_{k \in \mathbb{N}} G_k = \{0\}$, which is not open.

Definition If X is a metric space and $E \subseteq X$, then E' denotes the set of all limit points of E in X. The closure of E is the set $\bar{E} = E \cup E'$.

Theorem If X is a metric space and $E \subseteq X$, then

(a) $\bar{E}$ is closed

(b) $E = \bar{E}$ if and only if E is closed

(c) $\bar{E} \subseteq F$ for every closed set $F \subseteq X$ such that $E \subseteq F$.

Proof (a) If $p \in X$ and $p \notin \bar{E}$ then p is neither a point of E nor a limit point of E'. Hence p has a neighborhood that does not intersect E. The complement of $\bar{E}$ is therefore open. Hence $\bar{E}$ is closed.

(b) If $E = \bar{E}$ then E is closed by (a). If E is closed then $E' \subseteq E$ and $\bar{E} = E$.

(c) If F is closed and $F \supseteq E$, then $F \supseteq F'$. Hence $F \supseteq E'$. Therefore $F \supseteq \bar{E}$.

Theorem Let $E \subseteq \mathbb{R}$ be non-empty and bounded above. Let $y = \sup E$. Then $y \in \bar{E}$, and $y \in E$ if E is closed.

Proof If $y \in E$ then $y \in \bar{E}$. Assume $y \notin E$. For all $h > 0$ there exists $x \in E$ s.t $y - h < x < y$ because otherwise $y-h$ would be an upper bound for E. Hence y is a limit point of E. Hence $y \in \bar{E}$.

Consider $E \subseteq Y \subseteq X$. If E is an open subset of X, then for each $p \in E$ there exists $r > 0$ such that

$$N_r(p) = \{ q \in X \mid d(p,q) < r \} \subseteq E \quad (*)$$

The neighborhood definition depends on X. The pair (Y, d) is also a metric space, so the above definition $(*)$ could also be made within Y. We say that E is open relative to Y if for each $p \in E$ there exists an $r > 0$ such that

$$N_r^{(Y)}(p) = \{ q \in Y \mid d(p,q) < r \} \subseteq E.$$

Example Suppose $X = \mathbb{R}$, $Y = [-1, 1]$, and $E = (0, 1]$.

Therefore $E \subseteq Y \subseteq X$. E is not open relative to X, since $N_r^X(1) = (1-r, 1+r)$ is not contained within E for any r.

Now look at Y. We have

$$N_r^Y(1) = \{ q \in Y \mid d(1,q) < r \}$$

$$= \begin{cases} (1-r, 1] & \text{if } r \leq 2, \\ [-1, 1] & \text{if } r > 2. \end{cases}$$

Hence $N_{1/2}^Y(1) = (1/2, 1] \subseteq E$. For $x \in (0,1)$, we have

$$N_r^Y(x) \subseteq E \qquad \text{when} \quad r = \min\{x, 1-x\}.$$

Hence E is open relative to Y.

Theorem Suppose $Y \subseteq X$. A subset E of Y is open relative to Y if and only if $E = Y \cap G$ for some open subset G of X.

Proof Suppose E is open relative to Y. For each $p \in E$, $\exists r_p > 0$ such that $N_{r_p}^Y(p) \subseteq E$. Define

$$G = \bigcup_{p \in E} N_{r_p}^X(p)$$

Using previous result that the union of open sets is open, G is an open subset of X. Since $p \in N^X_{r_p}(p)$ for all $p \in E$, it follows that $E \subseteq G \cap Y$.

Since $N^X_{r_p}(p) \cap Y \subseteq E$ for all $p \in E$, we have $G \cap Y \subseteq E$ and hence $E = G \cap Y$.

Conversely, if G is open in X and $E = G \cap Y$, every $p \in E$ has a neighborhood $V_p \subseteq G$. Then $V_p \cap Y \subseteq E$ so E is open relative to Y.

<u>Definition</u> An open cover of a set E in a metric space X is a collection $\{G_\alpha\}$ of open subsets of X such that $E \subseteq \bigcup_\alpha G_\alpha$.

<u>Definition</u> A subset K of a metric space X is compact if every open cover contains a finite subcover. If $\{G_\alpha\}$ is an open cover of K, there should be finitely many indices $\alpha_1, \ldots, \alpha_n$ such that $K \subseteq G_{\alpha_1} \cup G_{\alpha_2} \cup \ldots \cup G_{\alpha_n}$.

Every finite set is compact.

Previous results showed that the openness of a set depends on the embedding space. Compactness behaves better than this. Take the above definitions to temporarily mean that K is compact relative to X.

Theorem Suppose $K \subseteq Y \subseteq X$. Then K is compact relative to X if and only if K is compact relative to Y.

Proof Suppose K is compact relative to X. Let $\{V_\alpha\}$ be a collection of sets open relative to Y such that $K \subseteq \bigcup_\alpha V_\alpha$. By previous theorem there are sets G_α open relative to X such that $V_\alpha = Y \cap G_\alpha$ and since K is compact relative to X we have

$$K \subseteq G_{\alpha_1} \cup G_{\alpha_2} \cup \ldots \cup G_{\alpha_n} \qquad (**)$$

for some choice of finitely many indices $\alpha_1, \ldots, \alpha_n$. Since $K \subseteq Y$,

$$K \subseteq V_{\alpha_1} \cup V_{\alpha_2} \cup \ldots \cup V_{\alpha_n}. \qquad (*)$$

Hence K is compact relative to Y.

Conversely, suppose K is compact relative to Y and let $\{G_\alpha\}$ be a collection of open subsets of X that cover K and put $V_\alpha = Y \cap G_\alpha$. Then $(*)$ will hold for some choice of α_i, and since $V_\alpha \subseteq G_\alpha$, $(**)$ will be true.

This result means that we can regard compact sets as metrics in their own right. It does not make sense to talk about open spaces or closed spaces, but it does make sense to talk about compact metric spaces.

Theorem Compact subsets of metric spaces are closed.

Proof Let K be a compact subset of a metric space and consider K^c.

Choose $p \in K^c$. For $q \in K$, let V_q and W_q be neighborhoods of p and q, respectively of radius less than $\frac{d(p,q)}{2}$. Since K is compact, there are finitely many points $q_1, \ldots, q_n$ in K such that

$$K \subseteq W_{q_1} \cup \ldots \cup W_{q_n} = W.$$

If $V = V_{q_1} \cap \ldots \cap V_{q_n}$, then V is a neighborhood of w that does not intersect W. Hence $V \subseteq K^c$ so p is an interior point of K^c. Hence K^c is open and K is closed.

Theorem Closed subsets of compact sets are compact.

Proof Suppose $F \subseteq K \subseteq X$, F is closed (relative to X), and K is compact. Let $\{V_\alpha\}$ be an open cover of F. If F^c is adjoined to $\{V_\alpha\}$, we obtain an open cover Ω of K.

Since K is compact, there is a finite subcollection Φ of Ω that covers K and hence F. If F^c is a member of Φ we may remove it and still retain an open cover of F. Thus a finite subcollection of $\{V_\alpha\}$ covers F.

Corollary If F is closed and K is compact, then $F \cap K$ is compact.

Proof K is closed and $K \cap F$ is closed. Since $F \cap K \subseteq K$ the previous theorem can be applied.

Theorem If $\{K_\alpha\}$ is a collection of compact subsets of a metric space such that the intersection of every finite subcollection of $\{K_\alpha\}$ is non-empty, then $\bigcap_\alpha K_\alpha$ is non-empty.

Proof Fix a member K_1 of $\{K_\alpha\}$ and define $G_\alpha = K_\alpha^c$. Assume that no point of K_1 belongs to every K_α. Then the sets G_α form an open cover of K_1. Since K_1 is compact, there are finitely many indices $\alpha_1, \dots, \alpha_n$ such that $K_1 \subseteq G_{\alpha_1} \cup \dots \cup G_{\alpha_n}$. But then $K_1 \cap K_{\alpha_1} \cap \dots \cap K_{\alpha_n}$ is empty, contradicting the hypothesis.

Corollary If $\{K_n\}$ is a sequence of non-empty compact sets such that $K_n \supseteq K_{n+1}$ $\forall n$ then $\bigcap_{n=1}^{\infty} K_n$ is non-empty

Theorem If E is an infinite subset of a compact set K, then E has a limit point in K.

Proof If no point of K were a limit point of E, then each $q \in K$ would have a neighborhood V_q containing at most one point of E (i.e. q if $q \in E$).

No finite subcollection of $\{V_q\}$ can cover E, and since $E \subseteq K$ the same is true for K. This contradicts the compactness of K.

Theorem If $\{I_n\}$ is a sequence of closed intervals in $\mathbb{R}^1$ such that $I_n \supseteq I_{n+1}$ $\forall n$ then $\bigcap_{i=1}^{\infty} I_n$ is non-empty.

Proof Define $I_n = [a_n, b_n]$ and let E be the set of all a_n. Then E is non-empty and bounded above by b_1. Let $x = \sup E$. For $m, n \in \mathbb{N}$

$$a_n \leq a_{m+n} \leq b_{m+n} \leq b_m$$

so $x \leq b_m$ for each m (since it is the least upper bound for E). Since $a_m \leq x$ $\forall m \in \mathbb{N}$, $x \in I_m$ for all $m \in \mathbb{N}$ and hence $x \in \bigcap_{i=1}^{\infty} I_n$.

Definition Define numbers $a_i < b_i$ for $i = 1, \dots, k$ with $a_i, b_i \in \mathbb{R}$. The set of all points $\underline{x} = (x_1, \dots, x_k)$ in $\mathbb{R}^k$ that satisfy $a_i \leq x_i \leq b_i$ for all i is called a k-cell.

Theorem Let k be a positive integer. If $\{I_n\}$ is a sequence of k-cells such that $I_n \supseteq I_{n+1}$ for all n, then $\bigcap_{i=1}^{\infty} I_n$ is non-empty.

Proof Let I_n consist of all points $\underline{x} = (x_1, \dots, x_n)$ such that

$$a_{n,j} \leq x_j \leq b_{n,j} \qquad (1 \leq j \leq k, n \in \mathbb{N})$$

and put $I_{n,j} = [a_{n,j}, b_{n,j}]$. Using the previous result, there are numbers x_j^* for $j = 1, \dots, k$ such that $a_{n,j} \leq x_j^* \leq b_{n,j}$ for $n \in \mathbb{N}$, $j = 1, \dots, k$.

Setting $\underline{x}^* = (x_1^*, x_2^*, \ldots, x_k^*)$, we see $\underline{x}^* \in I_n$ for $n \in \mathbb{N}$, so $\underline{x}^* \in \bigcap_{i=1}^{\infty} I_n$.

Theorem Every k-cell is compact

Proof Let I be a k-cell consisting of all points $\underline{x} = (x_1, x_2, \ldots, x_k)$ where $a_j \le x_j \le b_j$ $(1 \le j \le k)$, and define

$$\delta = \sqrt{\sum_{j=1}^{k} (b_j - a_j)^2}$$

Then $d(\underline{x}, \underline{y}) \le \delta$ if $\underline{x}, \underline{y} \in I$.

Suppose there exists an open cover $\{G_\alpha\}$ of I that has no finite subcover. Define $c_j = (a_j + b_j)/2$. The intervals $[a_j, c_j]$ and $[c_j, b_j]$ determine 2^k k-cells Q_i whose union is I. At least one of these, call it I_1, cannot be covered by any finite subcollection of $\{G_\alpha\}$. Subdivide I_1 and continue the process to obtain a sequence $\{I_n\}$ with the following properties:

(a) $I \supseteq I_1 \supseteq I_2 \supseteq I_3 \supseteq \ldots$

(b) I_n is not covered by any finite subcollection of $\{G_\alpha\}$

(c) If $\underline{x}, \underline{y} \in I_n$ then $d(\underline{x}, \underline{y}) \le 2^{-n} \delta$

Using the previous result there is a point $\underline{x}^*$ s.t. $\underline{x}^* \in I_n \ \forall n$. We must have $\underline{x}^* \in G_\alpha$ for some α. Since G_α is open, $\exists r>0$ such that $N_r(\underline{x}^*) \subseteq G_\alpha$.

Choose n large enough that $2^{-n}\delta < r$. Then $I_n \subseteq G_\alpha$, contradicting (b).

Theorem (Heine-Borel theorem) If $E \subseteq \mathbb{R}^k$ has one of the following properties, it has the other two:

(a) E is closed and bounded
(b) E is compact
(c) Every infinite subset of E has a limit point in E.

Proof If (a) holds, then $E \subseteq I$ for some k-cell I. Since closed subsets of compact sets are compact, E is compact.

Previous theorem shows that (b) implies (c). Consider showing that (c) implies (a).

If E is not bounded, then E contains points $\underline{x}_n$ with $|\underline{x}_n| > n \ \forall n \in \mathbb{N}$. The set S consisting of these points is infinite but has no limit point in $\mathbb{R}^k$. Hence (c) implies that

E is bounded.

If E is not closed, then there is a point $\underline{x}_0 \in \mathbb{R}^k$ that is a limit point of E, but $\underline{x}_0 \notin E$. $\exists\, \underline{x}_n \in E$ s.t $|\underline{x}_n - \underline{x}_0| < 1/n$ $\forall n \in \mathbb{N}$.

Define $S = \{\underline{x}_n \mid n \in \mathbb{N}\}$. S is infinite. S has $\underline{x}_0$ as a limit point, but it has no other limit point in $\mathbb{R}^k$. Choose $\underline{y} \in \mathbb{R}^k$, $\underline{y} \neq \underline{x}_0$. Then

$$d(\underline{x}_n, \underline{y}) \geq d(\underline{x}_0, \underline{y}) - d(\underline{x}_n, \underline{x}_0)$$

$$\geq d(\underline{x}_0, \underline{y}) - \frac{1}{n} \geq \frac{1}{2} d(\underline{x}_0, \underline{y})$$

for all but finitely many n. Thus $\underline{y}$ is not a limit point of S. Hence E must be closed if (c) holds.