Series

Write $\sum_{k=m}^{n} a_k = a_m + a_{m+1} + \ldots + a_n$.

When we write $\sum_{n=m}^{\infty} a_n$, we consider the sequence of partial sums

$$S_n = a_m + a_{m+1} + \ldots + a_n = \sum_{k=m}^{n} a_k$$

The infinite series $\sum_{n=m}^{\infty} a_n$ is said to converge if the sequence of partial sums (S_n) converges to a real number S, in which case $\sum_{n=m}^{\infty} a_n = S$. Equivalent to

$$\lim S_n = S \quad \text{or} \quad \lim_{n \to \infty} \left(\sum_{k=m}^{n} a_k \right) = S.$$

A series that does not converge is said to diverge. Write that $\sum_{n=m}^{\infty} a_n$ diverges to $\pm\infty$ and $\sum_{n=m}^{\infty} a_n = \pm\infty$ provided that $\lim S_n = \pm\infty$.

Many properties do not depend on the exact values or precisely where the series begins. Write $\sum a_n$ to refer to the series in general.

Example Geometric series

$$\sum_{k=0}^{n} (a)^n = 1 + a + a^2 + \ldots = \frac{1 - a^{n+1}}{1-a} \qquad (a \neq 1)$$

Previously showed that $\lim_{n\to\infty} a^n = 0$ if $|a|<1$, in which case

$$\sum_{k=0}^{\infty} a^n = \frac{1}{1-a}.$$

Example $\sum_{k=1}^{n} \frac{1}{k^p}$ converges only if $p>1$

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6} \qquad \sum_{k=1}^{\infty} \frac{1}{k^4} = \frac{\pi^4}{90}$$

$\sum_{k=1}^{\infty} \frac{1}{k^3} = 1.2020569\ldots$ Apéry's constant - no known closed-form expression.

Riemann zeta function

$\zeta(p) = \sum_{k=1}^{n} \frac{1}{k^p}$ can be extended to an analytic function in the complex plane.

Zeros at $-2, -4, -6, \ldots$

Also zeros along the line $\operatorname{Re} p = \frac{1}{2}$, e.g. $\zeta(\frac{1}{2} + i14.1347\ldots) = 0$. Currently unknown whether all zeros are along this line (apart from $-2, -4, \ldots$). (Millennium Prize Problem).

Cauchy criterion

(s_n) is a Cauchy sequence if

$$\forall \varepsilon > 0 \quad \exists N \text{ s.t. } m, n > N \Rightarrow |s_m - s_n| < \varepsilon.$$

Equivalent to saying that $\sum a_k$ satisfies

$$\forall \varepsilon > 0 \quad \exists N \text{ s.t. } m \geq n > N \Rightarrow |s_{m-1} - s_n| < \varepsilon$$

$$\Leftrightarrow \left| \sum_{k=m}^{n} a_k \right| < \varepsilon$$

Corollary If $\sum a_n$ converges, then $\lim a_n = 0$

Proof Using the Cauchy criterion, we know that $\forall \varepsilon > 0 \quad \exists N$ s.t. $n > N \Rightarrow |a_n| < \varepsilon$, by putting $m = n$. Hence $a_n \to 0$.

Comparison test Let $\sum a_n$ be a series where $a_n \geq 0 \;\; \forall n$. Then

(i) If $\sum a_n$ converges and $|b_n| \leq a_n \;\; \forall n$, then $\sum b_n$ converges.

(ii) If $\sum a_n = \infty$ and $b_n \geq a_n \ \forall n$, then $\sum b_n = \infty$.

<u>Proof</u> (i) Consider the Cauchy criterion

$$\left| \sum_{k=m}^{n} b_k \right| \leq \sum_{k=m}^{n} |b_k| \leq \sum_{k=m}^{n} a_k$$

For all $\varepsilon > 0$, $\exists N$ s.t. $m \geq n > N \Rightarrow \left| \sum_{k=m}^{n} a_k \right| < \varepsilon$ and hence the same is true for $\sum b_n$.

(ii) $\sum_{k=1}^{n} a_k \leq \sum_{k=1}^{n} b_k$

$$\forall M > 0 \quad \exists N \text{ s.t. } \quad n > N \Rightarrow \sum_{k=1}^{n} a_k > M$$

$$\Rightarrow \sum_{k=1}^{n} b_k > M \quad \boxed{\Rightarrow \sum b_n = \infty}$$

<u>Definition</u> $\sum a_k$ is absolutely convergent if $\sum |a_k|$ converges. Note that $\sum |a_k|$ is monotonic and always has a meaningful limit. Absolutely convergent series are convergent.

<u>Ratio test</u> A series $\sum a_n$ of non-zero terms

(i) Converges absolutely if $\limsup \left| \frac{a_{n+1}}{a_n} \right| < 1$

(ii) Diverges if $\liminf \left| \frac{a_{n+1}}{a_n} \right| > 1$

(iii) Otherwise $\liminf \left| \frac{a_{n+1}}{a_n} \right| \leq 1 \leq \limsup \left| \frac{a_{n+1}}{a_n} \right|$ and the test provides no information

Root test let $\sum a_n$ be a series and let $\alpha = \limsup |a_n|^{1/n}$. The series $\sum a_n$

(i) Converges absolutely if $\alpha < 1$
(ii) Diverges if $\alpha > 1$
(iii) otherwise $\alpha = 1$ and the test provides no information.

Example $\sum \frac{n^2}{2^n}$

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^2 2^n}{n^2 2^{n+1}} = \frac{(1+1/n)^2}{2} \to 1/2$$

$\Rightarrow$ absolute convergence

Proof of the root test

(i) Suppose $\alpha < 1$. Choose $\varepsilon > 0$ s.t $\alpha + \varepsilon < 1$. Then $\exists N$ s.t

$$\alpha - \varepsilon < \sup\{|a_n|^{1/n} \mid n > N\} < \alpha + \varepsilon$$

Hence $|a_n|^{1/n} < \alpha + \varepsilon \quad \forall n > N$.

$$|a_n| < (\alpha+\varepsilon)^n < 1$$

Then $\displaystyle\sum_{n=N+1}^{k} |a_n| < \sum_{n=N+1}^{k} (\alpha+\varepsilon)^n \leftarrow$ convergent

$$\Rightarrow \sum_{n=1}^{\infty} a_n \text{ converges.}$$

(ii) If $\alpha > 1$, then there exists a subsequence $|a_n|^{1/n}$ that has limit >1. Hence $|a_n| > 1$ for infinitely many terms and the series diverges.

<u>Proof of the ratio test</u>

By previous theorem

$$\liminf \left|\frac{a_{n+1}}{a_n}\right| \leq \liminf |a_n|^{1/n} \leq \limsup |a_n|^{1/n} \leq \limsup \left|\frac{a_{n+1}}{a_n}\right|$$

and hence

$$\liminf \left|\frac{a_{n+1}}{a_n}\right| \leq \alpha \leq \limsup \left|\frac{a_{n+1}}{a_n}\right|.$$

Thus the results follow from the root test.

<u>Examples</u> $\quad \sum \frac{1}{n!} \quad \left|\left[\frac{(n+1)!}{n!}\right]^{-1}\right| = \frac{1}{n}$ absolutely convergent

$$\sum_{n=0}^{\infty} 2^{(-1)^n - n} = 2 + \frac{1/2}{2} + \frac{2}{4} + \frac{1/2}{8} + \frac{2}{16} + \frac{1/2}{32} + \frac{2}{64} + \dots$$

$$= 2 + 1/4 + 1/2 + 1/16 + \frac{1}{8} + \frac{1}{64} + \frac{1}{32} + \dots$$

$$\frac{1}{8} = \liminf \left|\frac{a_{n+1}}{a_n}\right| < 1 < \limsup \left|\frac{a_{n+1}}{a_n}\right| = 2.$$

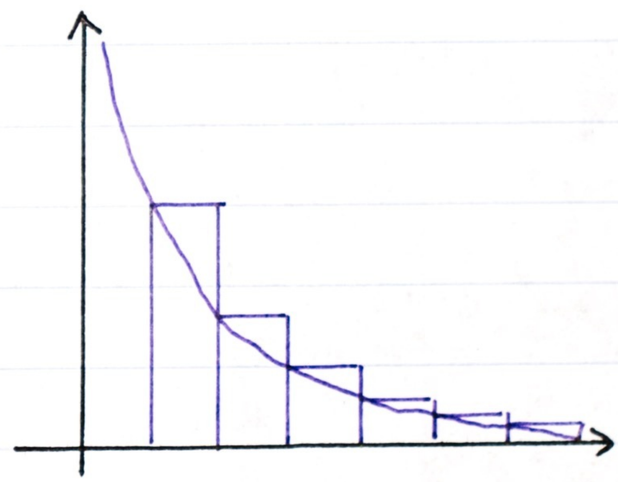
Thus in this case the ratio test provides no information. But

$$(a_n)^{1/n} = 2^{1/n - 1} \Rightarrow \lim (a_n)^{1/n} = 2^{-1} = 1/2$$

and the series converges via the root test.

Alternating series and integral tests

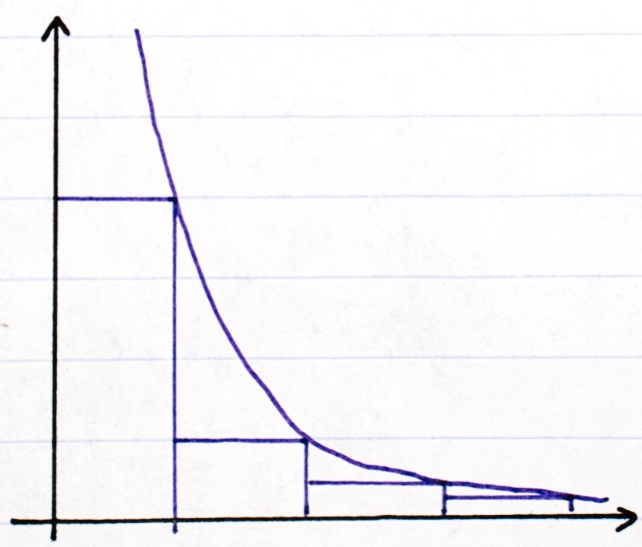
$$\sum_{k=1}^{n} \frac{1}{k} \geq \int_1^{n+1} \frac{1}{x}\, dx = \log(n+1)$$



Since $\lim\limits_{n \to \infty} \log n+1 = \infty^{(*)}$, then

$$\sum_{n=1}^{\infty} \frac{1}{n} = \infty.$$

(*) We will make integrals and function limits precise in later sections.



$$\sum_{k=1}^{n} \frac{1}{k^2} \leq \int_1^{n} \frac{1}{x^2}\, dx + 1$$

$$\leq 2 - \frac{1}{n} < 2$$

know that the series converges and its limit $\left(\frac{\pi^2}{6}\right)$ is less than 2. Same procedure shows that $\sum\limits_{k=1}^{\infty} \frac{1}{k^p}$ converges if

$p > 1$.

Alternating series theorem

Suppose that $a_1 \geq a_2 \geq a_3 \geq \ldots \geq 0$ and $\lim a_n = 0$. Then $\sum (-1)^{n+1} a_n$ converges.

$\sum (-1)^{n+1} a_n$ is called an alternating series

Proof The subsequence (s_{2n}) is increasing because

$$s_{2n+2} - s_{2n} = -a_{2n+2} + a_{2n+1} \geq 0.$$

Similarly (s_{2n-1}) is decreasing since

$$s_{2n+1} - s_{2n-1} = a_{2n+1} - a_{2n} \leq 0.$$

Now show that $s_{2m} \leq s_{2n+1}$ for all $m, n \in \mathbb{N}$. First $s_{2n} \leq s_{2n+1}$ for all n since $s_{2n+1} - s_{2n} = a_{2n+1} \geq 0$.

If $m \leq n$, $s_{2m} \leq s_{2n} \leq s_{2n+1}$. If $m \geq n$, $s_{2n+1} \geq s_{2m+1} \geq s_{2m}$.

$\Rightarrow$ • (s_{2n}) is an increasing sequence bounded above by each ~~odd~~ partial sum.

• (s_{2n+1}) is a decreasing sequence bounded below by each even partial sum.

Hence $\lim_{n\to\infty} s_{2n} = s$ and $\lim_{n\to\infty} s_{2n+1} = t$.

$$t - s = \lim_{n\to\infty} s_{2n+1} - \lim_{n\to\infty} s_{2n}$$

$$= \lim_{n\to\infty} (s_{2n+1} - s_{2n}) = \lim_{n\to\infty} a_{2n+1} = 0.$$

$$\Rightarrow s = t.$$

Hence $\lim_{n\to\infty} s_n = s$.