Homework sheet 1 – Due 01/31/2025

Problem 1: Basics of linear algebra [1 + 1 + (1+1+2) + (1+1+2) = 10 points]

Consider a finite dimensional linear vector space V (its dual is denoted V^*) with inner product $\langle \cdot | \cdot \rangle : V^* \times V \to \mathbb{C}$. A simple example to be kept in mind throughout the exercise is $V = \mathbb{C}^N$ with scalar products defined as $\langle a|b\rangle = \sum_{\sigma=1}^N a_\sigma^* b_\sigma$. For our quantum mechanics applications, such a vector space is the prototype for Hilbert spaces.

a) Prove that, for a Hermitian operator $\hat{A}$ with (right) eigenstate $|a\rangle$ (i.e. $\hat{A}|a\rangle = a|a\rangle$) the dual state $\langle a|$ is a (left) eigenstate with the same eigenvalue a and that $a \in \mathbb{R}$.

b) Prove the Schwarz-inequality

$$\langle \phi|\phi\rangle \langle \psi|\psi\rangle \geq |\langle \phi|\psi\rangle|^2. \tag{1}$$

c) Consider the direct sum $V = U \oplus W$, where U, V, W are all finite dimensional linear vector spaces. *This means the following: Let $U \subseteq V, W \subseteq V$ and the space spanned by $U \cup W$ (as obtained by summing elements of U, W using the addition defined for V) is a linear subvectorspace of V. If additionally $U \cap W = \{0\}$ the sum of U and W is called direct.*

i) Show that each $|v\rangle \in V$ can be uniquely decomposed in $|v\rangle = |u\rangle + |w\rangle$ with $|u\rangle \in U, |w\rangle \in W$.

ii) Consider the map $P : V \to V$ defined by

$$P : |v\rangle = |u\rangle + |w\rangle \mapsto P|v\rangle := |w\rangle . \tag{2}$$

Show that P is a projection (i.e. it's a linear operation with $P^2 = P$). Determine the eigenspace of P.

iii) Construct an explicit eigenbasis for state vectors $|v\rangle, |u\rangle, |w\rangle$ in their respective spaces. Express the dimension of the vector space V in terms of the dimension of vector spaces U, W. Construct a matrix representation for P from subexercise ii).

d) Consider the direct product $V = U \otimes W$, where U, V, W are all finite dimensional linear vector spaces. *This means the following: Let $|u\rangle \in U, |w\rangle \in W$. The direct (outer) product is a bilinear operation $\otimes : U \times W \to V$, such that $\otimes : (|u\rangle, |w\rangle) \mapsto |u\rangle \otimes |w\rangle \equiv |u, w\rangle$.*

i) Show that the Schwarz-inequality in U, W implies the Schwarz inequality in V, i.e. that for any $|u_1, w_1\rangle, |u_2, w_2\rangle \in V$

$$\langle u_1, w_1|u_1, w_1\rangle \langle u_2, w_2|u_2, w_2\rangle \geq |\langle u_1, w_1|u_2, w_2\rangle|^2 \tag{3}$$

ii) Consider a linear operator O on U, i.e. $O : |u\rangle \mapsto O|u\rangle$. Explain why the properties of the outer product imply that this operator is represented as $O \otimes \mathbf{I}$ on elements of V, where $\mathbf{I}$ is the identity operation.

iii) Construct an explicit eigenbasis for state vectors $|v\rangle, |u\rangle, |w\rangle$ in their respective spaces. Express the dimension of the vector space V in terms of the dimension of vector spaces U, W. Construct a matrix representation for O from subexercise ii).

Exercise 2: Pauli matrices [2 + 1+ 2 + 1+ 2+ 1 + 1 = 10 points]

The Pauli matrices are

$$\sigma_1 = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4}$$

Prove the following:

a) $\{\sigma_i, \sigma_j\} = 2\delta_{ij}$, where $\{A, B\} = AB + BA$ is the anticommutator and δ_{ij} the Kronecker delta. (This implies that the Pauli matrices form a *Clifford algebra*.)

b) $\sigma_i = \sigma_i^\dagger = -\sigma_y \sigma_i^T \sigma_y$

c) $[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k$, where $[A, B] = AB - BA$ is the commutator and ϵ_{ijk} the Levi-Civita-tensor. (The Pauli matrices fulfill the *angular momentum algebra* (up to a constant).)

d) $\mathrm{tr}\,[\sigma_i \sigma_j] = 2\delta_{ij}$

e) Any complex 2×2 matrix M can be expanded uniquely as $M = m_0 \mathbf{1} + \sum_{i=1}^3 m_i \sigma_i$, where $\mathbf{1}$ is the 2×2 identity. Determine m_0, m_i.

f) Any traceless, Hermitian 2×2 matrix H can be uniquely expanded as $H = \sum_{i=1}^3 h_i \sigma_i$, where $h_i = \mathrm{tr}\,[H\sigma_i]/2 \in \mathbb{R}$.

g) Any unitary 2×2 matrix U with unit determinant, $\det[U] = 1$, can be expanded as $U = a_0 \mathbf{1} + i \sum_i a_i \sigma_i$, where $a_{0,1,2,3}$ are all real numbers and $\sum_{i=0}^3 a_i^2 = 1$.

Exercise 3: States, Operators, Expectation values [1+2 +2 + 2+1 + 2 = 10 points]

Consider a 3-dimensional Hilbert space and the four states

$$|\psi_1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, |\psi_2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, |\psi_3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{5}$$

and

$$|\tilde{\psi}\rangle = C[|\psi_2\rangle + |\psi_3\rangle]. \tag{6}$$

Let the states $|\psi_i\rangle, i = 1, 2, 3$ be the eigenstates of an operator $\hat{A}$ with eigenvalues a_i, i.e. $\hat{A}|\psi_i\rangle = a_i|\psi_i\rangle$.

Further consider an operator $\hat{B}$, which in the basis of $|\psi_i\rangle$ is given by

$$\hat{B} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & i \\ 0 & -i & 3 \end{pmatrix}. \tag{7}$$

a) Determine the constant C such that $|\tilde{\psi}\rangle$ is normalized.

b) Determine $\hat{A}|\tilde{\psi}\rangle$ and the expectation value $\langle\tilde{\psi}|\hat{A}|\tilde{\psi}\rangle$.

c) Determine $\hat{B}|\tilde{\psi}\rangle$ and the expectation value $\langle\tilde{\psi}|\hat{B}|\tilde{\psi}\rangle$.

d) Is it possible to diagonalize $\hat{A}$ and $\hat{B}$ simultaneously? Calculate the commutator $[\hat{A}, \hat{B}]$.

e) Consider the operator $\hat{\rho}$ defined by

$$\hat{\rho} = \frac{1}{3}|\psi_1\rangle\langle\psi_1| + \frac{1}{3}|\tilde{\psi}\rangle\langle\tilde{\psi}| \tag{8}$$

and express it explicitly as a matrix in the basis of $|\psi_i\rangle, i = 1, 2, 3$.

Comment: $\hat{\rho}$ is called a density matrix. A state whose density matrix is not given by a single "ket-bra" $|.\rangle\langle.|$ is called *pure*, if instead the density matrix is a sum over $|.\rangle\langle.|$ with non-zero coefficients, the states is called *mixed*.

f) Calculate the expectation values of $\hat{A}, \hat{B}$ with respect to the mixed state $\hat{\rho}$.

Comment: For mixed states, the expectation value of an operator is $\langle\hat{O}\rangle = \text{tr}[\hat{\rho}\hat{O}]$, where tr is the trace and $\hat{O}$ and arbitrary operator. Convince yourself that for pure states, this expectation value is the same as the expectation value $\langle\hat{O}\rangle = \langle\psi|\hat{O}|\psi\rangle$ discussed in the lecture.