Physics 415
Spring 2025
Midterm 2

You have 50 minutes. There are a total of 2 problems, both of which have multiple parts. The exam is a total of 50 points. The following mathematical identities may be useful:

$$\text{Taylor series:} \quad e^x \approx 1 + x + \frac{1}{2}x^2 + \ldots, \quad x \ll 1. \tag{1}$$

$$\text{Gaussian integral:} \quad \int_{-\infty}^{\infty} \mathrm{d}x \; e^{-ax^2} = \sqrt{\frac{\pi}{a}} \tag{2}$$

Problem 1: (Spin-1) Consider a system of noninteracting atoms with spin-1 at fixed absolute temperature T and placed in an external magnetic field H. The possible energies of an individual atom are

$$\varepsilon_m = -g\mu_B H m, \tag{3}$$

where $m = -1, 0, 1$. The component of the magnetic moment along H is $\mu = g\mu_B m$.

(a) Calculate the partition function Z and Helmholtz free energy F for a single spin.

(b) Explicitly calculate the mean magnetic moment $\bar{\mu}$.

(c) The average "magnetization" of the system is $\bar{M} = n\bar{\mu}$, where n is the number of atoms per unit volume. In the limit of a weak field, $g\mu_B H \ll T$, show that the magnetization can be written $\bar{M} = \chi H$ and determine χ (the magnetic susceptibility).

(d) Determine the limiting behavior of $\bar{M}$ in the high-field limit $g\mu_B H \gg T$. Explain how you could have written this answer down without any calculation.

Problem 2: (Ideal gas with spin-1/2) Consider an ideal gas of N indistinguishable particles, each with spin-1/2, at fixed absolute temperature T. The energy of such a system in an applied magnetic field H is

$$E = \sum_{i=1}^{N} \left(\frac{\mathbf{p}_i^2}{2m} - g\mu_B H s_i \right), \tag{4}$$

where $\mathbf{p}_i$ are the momenta of the particles and $s_i = -1/2, +1/2$ are the projections of their spin along the applied field H.

(a) Treating the coordinates and momenta $(\mathbf{q}_i, \mathbf{p}_i)$ *classically*, but the spin degrees of freedom as discrete, calculate the partition function Z.

(b) Calculate the mean energy $\bar{E}$ of the gas.

(c) Calculate the heat capacity $C_V = (\partial \bar{E}/\partial T)_V$ of the gas.