Math 521: Assignment 3 (due 5 PM, March 7)

1. Suppose that $0 \le a_n < 1$ for all $n \in \mathbb{N}$. Prove that if $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} a_n^2$ and $\sum_{n=1}^{\infty} a_n/(1-a_n)$ also converge. Are the converse statements true?

2. Let (u_n) and (v_n) be sequences of positive real numbers for $n \in \mathbb{N}$. For each of the following statements, either prove it or provide a counterexample.

   (a) If (u_n) and (v_n) are equal except at finitely many n, then $\sum u_n$ and $\sum v_n$ either both converge or both diverge.

   (b) If (u_n) and (v_n) are equal at infinitely many n, then $\sum u_n$ and $\sum v_n$ either both converge or both diverge.

   (c) If $\sum u_n$ and $\sum v_n$ diverge, then $\sum u_n v_n$ diverges.

   (d) If $(u_n/v_n) \to 1$ as $n \to \infty$, then $\sum u_n$ and $\sum v_n$ both converge or both diverge.

   (e) If $u_n - v_n \to 0$, then $\sum u_n$ and $\sum v_n$ both converge or both diverge.

   (f) If $(u_{n+1}/u_n) > k > 1$ for infinitely many n, then $\sum u_n$ diverges.

3. Let s_n be the sum of the first n terms of the series

$$1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{5}} - \dots \tag{1}$$

   and let t_n be the sum of the first n terms of the rearranged series

$$1 + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{5}} + \frac{1}{\sqrt{7}} - \frac{1}{\sqrt{4}} + \dots, \tag{2}$$

   which is formed by taking two positive terms and then one negative term. Prove that (s_n) converges. Prove that $t_{3n} \ge s_{2n} + n/\sqrt{4n-1}$ and hence that (t_n) diverges.

4. (a) Let (a_n) be a decreasing sequence of positive numbers. Prove that the series

$$\sum_{n=1}^{\infty} a_n \qquad \text{and} \qquad \sum_{n=1}^{\infty} 2^n a_{2^n} \tag{3}$$

   both converge or both diverge.

   (b) Using part (a), prove that $\sum_{n=1}^{\infty} n^{-\alpha}$ is convergent if and only if $\alpha > 1$.

   (c) Prove that $\sum_{n=2}^{\infty} n^{-1}(\log n)^{-\alpha}$ is convergent if and only if $\alpha > 1$.

5. Using the integral test, or otherwise, determine whether the following series converge:

$$\text{(a)} \sum_n \frac{1}{\sqrt{n}\log n}, \quad \text{(b)} \sum_n \frac{1}{n(\log n)(\log(\log n))}, \quad \text{(c)} \sum_n \frac{\log n}{n}, \quad \text{(d)} \sum_n \frac{\log n}{n^2}.$$

6. **Optional.** Find a sequence (a_n) such that

$$\lim_{n\to\infty} a_n n(\log n)(\log(\log n)) = 0 \tag{4}$$

but $\sum a_n$ diverges.

7. Consider the functions defined for $x, y \in \mathbb{R}$:

$$d_1(x,y) = (x-y)^2, \qquad d_2(x,y) = \sqrt{|x-y|}, \qquad d_3(x,y) = |x^3 - y^3|,$$
$$d_4(x,y) = |x^4 - y^4|, \qquad d_5(x,y) = \min\{|x-y|, 1\}.$$

For each function, determine whether it is a metric or not. For the functions that are metrics, calculate the neighborhood $N_2(1)$.

8. Define the distance function

$$d(x,y) = \begin{cases} |x| + |y| & \text{for } x \neq y, \\ 0 & \text{for } x = y. \end{cases} \tag{5}$$

Prove that d is a metric on $\mathbb{R}$. For $0 < a < b$, show that the open interval $(a,b) = \{x \in \mathbb{R} : a < x < b\}$ is both open and closed with respect to d. Determine whether or not $(\mathbb{R}, d)$ is a complete metric space.

9. Let B be the set of all bounded sequences $\mathbf{u} = (u_1, u_2, \ldots)$. For each of the following functions, prove that it is a metric, or explain why it is not.

   (a) $d(\mathbf{u}, \mathbf{v}) = \sup\{|u_j - v_j| : j \in \mathbb{N}\}$.

   (b) $d(\mathbf{u}, \mathbf{v}) = \sum_{j=1}^{\infty} 2^{-j} |u_j - v_j|$.

   (c) $d(\mathbf{u}, \mathbf{v}) = \sum_{j=1}^{\infty} \frac{1}{j} |u_j - v_j|$.

Optional. For the three different definitions of d, determine whether or not (B, d) is a complete metric space.

10. Consider $\mathbb{R}^2$ with positions written as $\mathbf{x} = (x_1, x_2)$, and define the Euclidean norm as $\|\mathbf{x}\| = (x_1^2 + x_2^2)^{1/2}$. The Poincaré disk consists of the points $S = \{\mathbf{x} : \|\mathbf{x}\| < 1\}$ with metric

$$d(\mathbf{x}, \mathbf{y}) = \cosh^{-1}\left(1 + \frac{2\|\mathbf{x} - \mathbf{y}\|^2}{(1 - \|\mathbf{x}\|^2)(1 - \|\mathbf{y}\|^2)}\right) \tag{6}$$

for all $\mathbf{x}, \mathbf{y} \in S$. Define $r = \cosh^{-1} \frac{3}{2}$. Draw the Poincaré disk, and then calculate and draw the neighborhoods $N_r(\mathbf{x})$ for $\mathbf{x} = (0,0)$, $(\frac{1}{2}, 0)$, and $(\frac{3}{4}, 0)$.[1]

11. Let (p_n) be a Cauchy sequence in a metric space (S, d). Suppose that a subsequence (p_{n_k}) converges to a point $p \in S$. Prove that the full sequence converges to p.

[1]This can be done analytically, although you may use a computer if you wish.