Analysis Midterm Review Notes

(Based on Sample Midterm, HW4, HW5)

These notes cover key concepts in continuity, limits, sequences, series, function sequences/series, and power series, integrating examples from the sample midterm and relevant homework problems.

1 Continuity

1.1 Definitions

Definition 1.1 (Continuity at a Point[1, Def 17.1])**.** A function $f : S \to \mathbb{R}$, where $S \subseteq \mathbb{R}$, is **continuous at** $x_0 \in S$ if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $x \in S$,

$$|x - x_0| < \delta \implies |f(x) - f(x_0)| < \varepsilon.$$

Definition 1.2 (Continuity on a Set[1, Def 17.1])**.** A function f is **continuous on** a set $S' \subseteq S$ if it is continuous at every point $x_0 \in S'$.

Definition 1.3 (Uniform Continuity[1, Def 19.1])**.** A function $f : S \to \mathbb{R}$ is **uniformly continuous on** S if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $x, y \in S$,

$$|x - y| < \delta \implies |f(x) - f(y)| < \varepsilon.$$

Remark 1.1. The key difference from pointwise continuity is that δ depends only on ε and not on the specific points $x, y \in S$. Uniform continuity is a global property on the set S.

Definition 1.4 (Lipschitz Continuity (HW5.1))**.** A function $f : I \to \mathbb{R}$ on an interval I is **Lipschitz continuous** if there exists a constant $L > 0$ (the Lipschitz constant) such that for all $x, y \in I$,

$$|f(x) - f(y)| \leq L|x - y|.$$

Definition 1.5 (Bounded Function[1, p. 123])**.** A function $f : S \to \mathbb{R}$ is **bounded** if its range $f(S)$ is a bounded subset of $\mathbb{R}$, i.e., there exists $M \geq 0$ such that $|f(x)| \leq M$ for all $x \in S$.

Definition 1.6 (Convex Function (HW4.11))**.** A function f on an interval I is **convex** if for all $x, y \in I$, and all $\lambda \in (0, 1)$,

$$f\big((1 - \lambda)x + \lambda y\big) \leq (1 - \lambda)f(x) + \lambda f(y).$$

Geometrically, the line segment connecting any two points on the graph lies above or on the graph.

1.2 Fundamental Theorems on Continuity

Theorem 1.1 (Sequential Criterion for Continuity[1, Thm 17.2])**.** *A function $f : S \to \mathbb{R}$ is continuous at $x_0 \in S$ if and only if for every sequence (x_n) in S converging to x_0, the sequence $(f(x_n))$ converges to $f(x_0)$. That is,*

$$\lim_{n\to\infty} x_n = x_0 \implies \lim_{n\to\infty} f(x_n) = f(x_0).$$

Theorem 1.2 (Algebra of Continuous Functions[1, Thm 17.4]). *If $f, g : S \to \mathbb{R}$ are functions continuous at $x_0 \in S$, then the functions $f + g$, $f - g$, $k \cdot f$ (for any constant $k \in \mathbb{R}$), and $f \cdot g$ are also continuous at x_0. Furthermore, if $g(x_0) \neq 0$, then the function f/g is continuous at x_0 (defined on $S' = \{x \in S : g(x) \neq 0\}$, assuming x_0 is in S').*

Theorem 1.3 (Composition of Continuous Functions[1, Thm 17.5]). *Let $f : S \to T$ and $g : T \to \mathbb{R}$ be functions where $S, T \subseteq \mathbb{R}$. If f is continuous at $x_0 \in S$ and g is continuous at $f(x_0) \in T$, then the composition $g \circ f : S \to \mathbb{R}$, defined by $(g \circ f)(x) = g(f(x))$, is continuous at x_0.*

Theorem 1.4 (Intermediate Value Theorem (IVT)[1, Thm 18.2]). *If $f : [a, b] \to \mathbb{R}$ is continuous on the closed interval $[a, b]$, and if y_0 is any real number between $f(a)$ and $f(b)$ (i.e., $f(a) \le y_0 \le f(b)$ or $f(b) \le y_0 \le f(a)$), then there exists at least one $c \in [a, b]$ such that $f(c) = y_0$. If y_0 is strictly between $f(a)$ and $f(b)$, then c can be chosen in the open interval (a, b).*

Corollary 1.5 (Existence of Zeros (HW4.9)). *If $f : [a, b] \to \mathbb{R}$ is continuous and $f(a)$ and $f(b)$ have opposite signs (i.e., $f(a)f(b) < 0$), then there exists $c \in (a, b)$ such that $f(c) = 0$.*

Theorem 1.6 (Properties of Continuous Functions on Compact Sets[1, Thm 18.1, 19.2]). *Let $K \subset \mathbb{R}$ be a compact set (i.e., closed and bounded) and let $f : K \to \mathbb{R}$ be continuous on K. Then:*

1. *f is bounded on K.*
2. *f attains its maximum and minimum values on K. (Extreme Value Theorem)*
3. *f is uniformly continuous on K. (Heine-Cantor Theorem)*

Theorem 1.7 (Lipschitz implies Uniform Continuity (HW5.1a)). *If $f : I \to \mathbb{R}$ is Lipschitz continuous on an interval I, then f is uniformly continuous on I.*

Proof. Let f be Lipschitz with constant $L > 0$. Let $\varepsilon > 0$ be given. We need to find $\delta > 0$ such that $|x - y| < \delta \implies |f(x) - f(y)| < \varepsilon$ for all $x, y \in I$. Choose $\delta = \varepsilon/L$. Since $\varepsilon > 0$ and $L > 0$, $\delta > 0$. Now, assume $x, y \in I$ and $|x - y| < \delta$. By the Lipschitz condition:

$$|f(x) - f(y)| \le L|x - y|.$$

Since $|x - y| < \delta = \varepsilon/L$, we have:

$$|f(x) - f(y)| < L(\varepsilon/L) = \varepsilon.$$

This holds for all $x, y \in I$. Therefore, f is uniformly continuous on I. □

Theorem 1.8 (Uniform Continuity of Sums and Compositions (HW5.2a,b)). *Let $S, T \subseteq \mathbb{R}$.*

1. *If $f, g : S \to \mathbb{R}$ are uniformly continuous on S, then $f + g$ is uniformly continuous on S.*
2. *If $f : S \to T$ is uniformly continuous on S and $g : T \to \mathbb{R}$ is uniformly continuous on T, then $g \circ f : S \to \mathbb{R}$ is uniformly continuous on S.*

Proof. (1) Let $\varepsilon > 0$. Find $\delta_f > 0$ for f w.r.t. $\varepsilon/2$ and $\delta_g > 0$ for g w.r.t. $\varepsilon/2$. Let $\delta = \min(\delta_f, \delta_g) > 0$. If $x, y \in S$ and $|x - y| < \delta$, then

$$|(f + g)(x) - (f + g)(y)| = |(f(x) - f(y)) + (g(x) - g(y))|$$

$$\le |f(x) - f(y)| + |g(x) - g(y)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

(2) Let $\varepsilon > 0$. Find $\delta_g > 0$ for g w.r.t. ε. Find $\delta_f > 0$ for f w.r.t. δ_g. If $x, y \in S$ and $|x - y| < \delta_f$, then $|f(x) - f(y)| < \delta_g$. Let $u = f(x), v = f(y)$. Then $u, v \in T$ and $|u - v| < \delta_g$, so by uniform continuity of g, $|g(u) - g(v)| < \varepsilon$. That is, $|(g \circ f)(x) - (g \circ f)(y)| < \varepsilon$. The required δ for $g \circ f$ is δ_f. □

Theorem 1.9 (Density of $\mathbb{Q}$ and Continuous Functions (HW4.7))**.** *Let $f, g : I \to \mathbb{R}$ be continuous functions on an interval I.*

1. *If $f(q) = 0$ for all rational numbers $q \in I \cap \mathbb{Q}$, then $f(x) = 0$ for all $x \in I$.*
2. *If $f(q) = g(q)$ for all rational numbers $q \in I \cap \mathbb{Q}$, then $f(x) = g(x)$ for all $x \in I$.*

Proof. (1) Let $x \in I$. Since $I \cap \mathbb{Q}$ is dense in I (as $\mathbb{Q}$ is dense in $\mathbb{R}$), there exists a sequence (q_n) in $I \cap \mathbb{Q}$ such that $q_n \to x$. By assumption, $f(q_n) = 0$ for all n. Since f is continuous at x, by the sequential criterion:

$$f(x) = \lim_{n\to\infty} f(q_n) = \lim_{n\to\infty} 0 = 0.$$

(2) Apply part (1) to the function $h(x) = f(x) - g(x)$. h is continuous on I, and $h(q) = f(q) - g(q) = 0$ for all $q \in I \cap \mathbb{Q}$. Therefore, $h(x) = 0$ for all $x \in I$, which means $f(x) = g(x)$ for all $x \in I$. $\square$

Theorem 1.10 (Growth of Uniformly Continuous Functions (HW5.3))**.** *If $f : \mathbb{R} \to \mathbb{R}$ is uniformly continuous, then there exist constants $A, B > 0$ such that $|f(x)| \le A + B|x|$ for all $x \in \mathbb{R}$. (Linear growth bound).*

Proof. Since f is uniformly continuous, for $\varepsilon = 1$, there exists $\delta > 0$ such that $|u-v| < \delta \implies |f(u)-f(v)| < 1$. Let $x \in \mathbb{R}$. Case 1: $x \ge 0$. Let $n = \lfloor x/\delta \rfloor$. So $n \le x/\delta < n+1$. Consider points $0, \delta, 2\delta, \ldots, n\delta, x$. The distance between consecutive points is $\le \delta$. Using the triangle inequality:

$$|f(x) - f(0)| = \left|(f(x) - f(n\delta)) + \sum_{k=1}^{n}(f(k\delta) - f((k-1)\delta))\right|$$

$$\le |f(x) - f(n\delta)| + \sum_{k=1}^{n} |f(k\delta) - f((k-1)\delta)|.$$

Since $|x - n\delta| < \delta$ and $|k\delta - (k-1)\delta| = \delta$, each term in the sum and the first term are less than 1 (or ≤ 1 if δ was chosen s.t. $|u - v| \le \delta \implies |f(u) - f(v)| \le 1$). There are $n + 1$ terms total.

$$|f(x) - f(0)| < (n+1) \cdot 1 = n + 1.$$

Since $n \le x/\delta$, we have $n + 1 \le x/\delta + 1$.

$$|f(x) - f(0)| < \frac{x}{\delta} + 1.$$

Case 2: $x < 0$. Let $n = \lfloor |x|/\delta \rfloor$. Consider points $x, x + \delta, \ldots, x + n\delta, 0$. A similar argument gives

$$|f(0) - f(x)| < (n+1) \cdot 1 \le \frac{|x|}{\delta} + 1.$$

Combining cases: For all $x \in \mathbb{R}$, $|f(x) - f(0)| < \frac{|x|}{\delta} + 1$. Using $|f(x)| \le |f(0)| + |f(x) - f(0)|$, we get

$$|f(x)| < |f(0)| + \frac{|x|}{\delta} + 1 = (|f(0)| + 1) + \frac{1}{\delta}|x|.$$

Choose $A = |f(0)| + 1$ and $B = 1/\delta$. Both are positive constants. Thus, $|f(x)| \le A + B|x|$. $\square$

Theorem 1.11 (Convexity and Continuity (HW4.11))**.** *If f is convex on an open interval (a, b), then f is continuous on (a, b).*

1.3 Examples and Counterexamples

Example 1.1 (Uniform vs. Lipschitz (HW5.1b)). **Problem Statement:** Find an example of a function g defined on an interval I that is uniformly continuous but not Lipschitz continuous.

Solution: Consider $g(x) = \sqrt{x}$ on $I = [0, 1]$.

- **Uniform Continuity:** g is continuous on the compact interval $[0, 1]$. By Theorem 1.6, g is uniformly continuous on $[0, 1]$.
- **Not Lipschitz:** Assume g is Lipschitz with constant $L > 0$. Then for all $x \in (0, 1]$ and $y = 0$, we need $|\sqrt{x} - \sqrt{0}| \leq L|x - 0|$, which means $\sqrt{x} \leq Lx$. Dividing by $\sqrt{x}$ gives $1 \leq L\sqrt{x}$, or $\sqrt{x} \geq 1/L$. This inequality cannot hold for all $x \in (0, 1]$, because we can choose x such that $0 < x < (1/L)^2$. For instance, if $x = 1/(4L^2)$ (assuming $L \geq 1/2$ so $x \leq 1$), then $\sqrt{x} = 1/(2L)$, and the inequality becomes $1/(2L) \geq 1/L$, which implies $1/2 \geq 1$, a contradiction. Alternatively, the difference quotient $\frac{g(x)-g(0)}{x-0} = \frac{\sqrt{x}}{x} = \frac{1}{\sqrt{x}}$ is unbounded as $x \to 0^+$. A Lipschitz function must have bounded difference quotients.

Thus, $g(x) = \sqrt{x}$ on $[0, 1]$ is uniformly continuous but not Lipschitz.

Example 1.2 (Product not Uniformly Continuous (HW5.2c)). **Problem Statement:** Show that there exist uniformly continuous functions f, g from S to $\mathbb{R}$ such that $f \cdot g$ is not uniformly continuous.

Solution: Let $S = \mathbb{R}$. Let $f(x) = x$ and $g(x) = x$.

- f and g are uniformly continuous on $\mathbb{R}$: They are Lipschitz with $L = 1$, since $|f(x) - f(y)| = |x - y| = 1 \cdot |x - y|$. By HW5.1a, they are uniformly continuous.
- The product is $h(x) = f(x)g(x) = x^2$. We show h is not uniformly continuous on $\mathbb{R}$. By Theorem 1.10, if h were uniformly continuous, it would satisfy $|h(x)| \leq A + B|x|$ for some constants $A, B > 0$. However, $|x^2|$ grows quadratically, faster than any linear function $A + B|x|$ for large $|x|$. For example, $\lim_{x\to\infty} \frac{x^2}{A+Bx} = \infty$. Thus, $h(x) = x^2$ cannot be uniformly continuous on $\mathbb{R}$.

Example 1.3 (Piecewise Rational/Irrational (HW4.5)). **Problem Statement:** Consider $h(x) = (1 - x^2)$ if $x \in \mathbb{Q}$ and $h(x) = 0$ if $x \notin \mathbb{Q}$. Show h is continuous at ± 1 but at no other points.

Solution:

- **Continuity at $x_0 = 1$:** We have $h(1) = 1 - 1^2 = 0$. Let $\varepsilon > 0$. We need $\delta > 0$ such that $|x - 1| < \delta \implies |h(x) - h(1)| = |h(x)| < \varepsilon$. If $x \notin \mathbb{Q}$, $|h(x)| = 0 < \varepsilon$. If $x \in \mathbb{Q}$, $|h(x)| = |1 - x^2| = |1 - x||1 + x|$. Choose $\delta_1 = 1$. If $|x - 1| < \delta_1$, then $0 < x < 2$, so $|1 + x| < 3$. Then $|h(x)| < 3|x - 1|$. We want this less than ε, so we need $|x - 1| < \varepsilon/3$. Choose $\delta = \min(1, \varepsilon/3)$. If $|x - 1| < \delta$, then $|h(x)| < \varepsilon$. Thus h is continuous at $x = 1$. By symmetry ($h(x) = h(-x)$), h is also continuous at $x = -1$.
- **Discontinuity at $x_0 \neq \pm 1$:** Case 1: $x_0 \in \mathbb{Q}$. Then $h(x_0) = 1 - x_0^2 \neq 0$. Since $\mathbb{R} \setminus \mathbb{Q}$ is dense, there is a sequence of irrational numbers $y_n \to x_0$. Then $h(y_n) = 0$ for all n. So $\lim h(y_n) = 0 \neq h(x_0)$. By the sequential criterion, h is discontinuous at x_0. Case 2: $x_0 \notin \mathbb{Q}$. Then $h(x_0) = 0$. Since $\mathbb{Q}$ is dense, there is a sequence of rational numbers $q_n \to x_0$. Then $h(q_n) = 1 - q_n^2$. By continuity of polynomials, $\lim h(q_n) = 1 - x_0^2$. Since $x_0 \neq \pm 1$, $1 - x_0^2 \neq 0 = h(x_0)$. By the sequential criterion, h is discontinuous at x_0.

Example 1.4 (Oscillatory Damped (HW4.6)). **Problem Statement:** For $\alpha \in \mathbb{R}$, define $f(x) = |x|^\alpha \sin(1/x)$ for $x \neq 0$, and $f(0) = 0$. Find the exact range of α for which f is continuous at 0.

Solution: We need $\lim_{x\to 0} f(x) = f(0) = 0$. Since $-1 \leq \sin(1/x) \leq 1$, we have for $x \neq 0$:

$$-|x|^\alpha \leq f(x) \leq |x|^\alpha.$$

By the Squeeze Theorem, if $\lim_{x\to 0} |x|^\alpha = 0$, then $\lim_{x\to 0} f(x) = 0$. The limit $\lim_{x\to 0} |x|^\alpha = 0$ if and only if $\alpha > 0$. If $\alpha = 0$, $f(x) = \sin(1/x)$ for $x \neq 0$. Consider $x_n = 1/(n\pi)$. $x_n \to 0$, but $f(x_n) = \sin(n\pi) = 0$.

Consider $y_n = 1/(2n\pi + \pi/2)$. $y_n \to 0$, but $f(y_n) = \sin(2n\pi + \pi/2) = 1$. Since we get different limits (0 and 1) for sequences approaching 0, $\lim_{x\to 0} f(x)$ does not exist for $\alpha = 0$. If $\alpha < 0$, let $\beta = -\alpha > 0$. Then $f(x) = \frac{\sin(1/x)}{|x|^\beta}$. Consider $y_n = 1/(2n\pi + \pi/2) \to 0$. $f(y_n) = \frac{1}{|y_n|^\beta} = (2n\pi + \pi/2)^\beta \to \infty$. The limit is not 0. Therefore, f is continuous at 0 if and only if $\alpha > 0$.

Example 1.5 (IVT Application (Sample Midterm 5)). **Problem Statement:** Suppose f is continuous on $[0, 2]$ and $f(0) = f(2)$. Prove that there exist $x, y \in [0, 2]$ where $|x - y| = 1$ and $f(x) = f(y)$.

Solution: Define the auxiliary function $g : [0, 1] \to \mathbb{R}$ by

$$g(x) = f(x + 1) - f(x).$$

Since f is continuous on $[0, 2]$, g is continuous on $[0, 1]$ (as a difference of continuous functions). Evaluate g at the endpoints:

$$g(0) = f(1) - f(0).$$
$$g(1) = f(1 + 1) - f(1) = f(2) - f(1).$$

Using the given condition $f(0) = f(2)$, we have

$$g(1) = f(0) - f(1) = -(f(1) - f(0)) = -g(0).$$

If $g(0) = 0$, then $f(1) - f(0) = 0$, so $f(1) = f(0)$. We can choose $x = 0, y = 1$. Then $|x - y| = 1$ and $f(x) = f(y)$. If $g(0) \neq 0$, then $g(0)$ and $g(1)$ have opposite signs. Since g is continuous on $[0, 1]$, by the Intermediate Value Theorem, there must exist some $c \in (0, 1)$ such that $g(c) = 0$. This means $f(c+1) - f(c) = 0$, or $f(c+1) = f(c)$. We can choose $x = c$ and $y = c+1$. Since $c \in (0, 1)$, both $x, y \in [0, 2]$. Also $|x - y| = |c - (c + 1)| = 1$, and $f(x) = f(y)$. In either case, the desired x and y exist.

Example 1.6 (Function Defined by Supremum (HW4.10)). **Problem Statement:** Let f be continuous on $[a, b]$. Show that $f^*(x) = \sup\{f(z) : a \leq z \leq x\}$ is an increasing continuous function on $[a, b]$.

Solution:

- **Increasing:** Let $a \leq x < y \leq b$. The set $S_x = \{f(z) : a \leq z \leq x\}$ is a subset of $S_y = \{f(z) : a \leq z \leq y\}$. Therefore, $\sup S_x \leq \sup S_y$, which means $f^*(x) \leq f^*(y)$. So f^* is increasing.
- **Continuous:** Let $x_0 \in [a, b]$ and let $\varepsilon > 0$. Since f is continuous on $[a, b]$, it is uniformly continuous (Thm 1.6). There exists $\delta > 0$ such that if $z \in [a, b]$ and $|z - x_0| < \delta$, then $|f(z) - f(x_0)| < \varepsilon$. Let $x \in [a, b]$ with $|x - x_0| < \delta$. Case 1: $x > x_0$. Then $f^*(x) = \max(f^*(x_0), \sup\{f(z) : x_0 < z \leq x\})$. For $z \in (x_0, x]$, we have $|z - x_0| \leq |x - x_0| < \delta$, so $f(z) < f(x_0) + \varepsilon \leq f^*(x_0) + \varepsilon$. Thus $\sup\{f(z) : x_0 < z \leq x\} \leq f^*(x_0) + \varepsilon$. This implies $f^*(x) \leq f^*(x_0) + \varepsilon$. Since f^* is increasing, $f^*(x) \geq f^*(x_0)$. So $|f^*(x) - f^*(x_0)| = f^*(x) - f^*(x_0) \leq \varepsilon$. Case 2: $x < x_0$. Since f^* is increasing, $f^*(x) \leq f^*(x_0)$. By the Extreme Value Theorem, f attains its max on $[a, x_0]$, say at $z_0 \in [a, x_0]$, so $f^*(x_0) = f(z_0)$. If $z_0 \leq x$, then $f^*(x_0) = f(z_0) \leq f^*(x)$, implying $f^*(x) = f^*(x_0)$. If $x < z_0 \leq x_0$, then $|z_0 - x_0| \leq |x - x_0| < \delta$. So $f(z_0) < f(x_0) + \varepsilon$. Also $f^*(x_0) = f(z_0)$. We need to show $f^*(x_0) - f^*(x) < \varepsilon$. Since $f^*(x) = \sup\{f(z) : a \leq z \leq x\}$, we have $f^*(x) \geq f(z)$ for $z \in [a, x]$. It is known that $f^*(x_0) = f(z_0)$. If we choose x close enough to x_0, specifically $|x - x_0| < \delta'$ where δ' corresponds to ε for uniform continuity of f, a more detailed argument shows $f^*(x_0) \leq f^*(x) + \varepsilon$. Combining cases, $|f^*(x) - f^*(x_0)| \leq \varepsilon$ when $|x - x_0| < \delta$. So f^* is continuous.

Example 1.7 (Convexity and Endpoints (HW4.11)). **Problem Statement:** Show a convex function on $[a, b]$ is continuous on (a, b) but need not be at a or b.

Solution: The proof for continuity on (a, b) involves showing that for any $x_0 \in (a, b)$, the function values $f(x)$ are bounded above and below by linear functions passing through $f(x_0)$ for x near x_0, leading to $|f(x) - f(x_0)| \leq K|x - x_0|$ locally (Lipschitz continuity locally implies continuity). For the endpoint counterexample, consider f on $[0, 1]$ defined by

$$f(x) = \begin{cases} 1 & \text{if } x \in (0, 1) \\ 0 & \text{if } x = 0 \text{ or } x = 1 \end{cases}$$

This function is convex (the segment between any two points lies above or on the graph), but $\lim_{x\to 0^+} f(x) = 1 \neq f(0) = 0$ and $\lim_{x\to 1^-} f(x) = 1 \neq f(1) = 0$, so it's discontinuous at the endpoints.

2 Limits of Functions

2.1 Definitions and Properties

Definition 2.1 (Limit of a Function (Neighborhood Def)[1, Def 20.1]). Let $f : S \to \mathbb{R}$, and let a be a limit point of S. We say $\lim_{x\to a} f(x) = L$ if $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $\forall x \in S$,

$$0 < |x - a| < \delta \implies |f(x) - L| < \varepsilon.$$

Definition 2.2 (One-Sided Limits[1, Def 20.8]).

- **Right-hand limit:** $\lim_{x\to a^+} f(x) = L$ if $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $\forall x \in S$, $a < x < a + \delta \implies |f(x) - L| < \varepsilon$.
- **Left-hand limit:** $\lim_{x\to a^-} f(x) = L$ if $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $\forall x \in S$, $a - \delta < x < a \implies |f(x) - L| < \varepsilon$.

Definition 2.3 (Infinite Limits[1, Sec 20]). $\lim_{x\to a} f(x) = +\infty$ if $\forall M > 0$, $\exists \delta > 0$ s.t. $\forall x \in S$, $0 < |x-a| < \delta \implies f(x) > M$. (Similar definitions for $-\infty$ and one-sided limits).

Theorem 2.1 (Two-Sided vs One-Sided Limits[1, Thm 20.10]). *Let f be defined on an interval around a, except possibly at a. Then $\lim_{x\to a} f(x) = L$ if and only if $\lim_{x\to a^+} f(x) = L$ and $\lim_{x\to a^-} f(x) = L$.*

Theorem 2.2 (Limit Laws[1, Thm 20.4]). *If $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$ (where $L, M \in \mathbb{R}$), then*

- $\lim_{x\to a}(f(x) + g(x)) = L + M$
- $\lim_{x\to a}(f(x)g(x)) = LM$
- $\lim_{x\to a}(f(x)/g(x)) = L/M$*, provided* $M \neq 0$.

These hold similarly for one-sided limits.

Theorem 2.3 (Order Properties of Limits[1, Thm 20.5] (HW5.5a)). *Assume $\lim_{x\to a^+} f(x) = L$ and $\lim_{x\to a^+} g(x) = M$ exist (as finite numbers). If there exists $\delta_0 > 0$ such that $f(x) \le g(x)$ for all $x \in (a, a+\delta_0)$, then $L \le M$.*

Proof. Assume for contradiction that $L > M$. Let $\varepsilon = (L - M)/2 > 0$. By definition of limits, there exist $\delta_1, \delta_2 > 0$ such that: If $a < x < a+\delta_1$, then $|f(x)-L| < \varepsilon \implies f(x) > L-\varepsilon = (L+M)/2$. If $a < x < a+\delta_2$, then $|g(x) - M| < \varepsilon \implies g(x) < M + \varepsilon = (L + M)/2$. Let $\delta = \min(\delta_0, \delta_1, \delta_2) > 0$. For any $x \in (a, a + \delta)$, we have: $f(x) \le g(x)$ (since $x \in (a, a + \delta_0)$). $f(x) > (L + M)/2$ (since $x \in (a, a + \delta_1)$). $g(x) < (L + M)/2$ (since $x \in (a, a + \delta_2)$). Combining these gives $(L + M)/2 < f(x) \le g(x) < (L + M)/2$, which is impossible. Therefore, the assumption $L > M$ must be false, so $L \le M$. $\square$

2.2 Examples

Example 2.1 (Limits of a Rational Function (HW5.4)). **Problem Statement:** Let $f(x) = \frac{1}{(x+1)^2(x-2)}$. Find the one-sided limits at $x = 2$ and $x = -1$, and the two-sided limits if they exist.

Solution:

- **Near** $x = 2$**:** As $x \to 2^+$, $x - 2 \to 0^+$ and $(x + 1)^2 \to 9$. Denominator $\to 9 \cdot 0^+ = 0^+$. $f(x) \to +\infty$. As $x \to 2^-$, $x - 2 \to 0^-$ and $(x + 1)^2 \to 9$. Denominator $\to 9 \cdot 0^- = 0^-$. $f(x) \to -\infty$. Since $\lim_{x\to 2^+} f(x) \neq \lim_{x\to 2^-} f(x)$, $\lim_{x\to 2} f(x)$ does not exist.
- **Near** $x = -1$**:** As $x \to -1^+$, $x+1 \to 0^+$ so $(x+1)^2 \to 0^+$. $x-2 \to -3$. Denominator $\to 0^+\cdot(-3) = 0^-$. $f(x) \to -\infty$. As $x \to -1^-$, $x+1 \to 0^-$ so $(x+1)^2 \to 0^+$. $x-2 \to -3$. Denominator $\to 0^+ \cdot (-3) = 0^-$. $f(x) \to -\infty$. Since $\lim_{x\to -1^+} f(x) = \lim_{x\to -1^-} f(x) = -\infty$, we write $\lim_{x\to -1} f(x) = -\infty$. (Note: This limit does not exist as a real number).

Example 2.2 (Limits and Strict Inequality (HW5.5b)). **Problem Statement:** Suppose $f_1(x) < f_2(x)$ for x in (a, b). Does it follow that $\lim_{x\to a^+} f_1(x) < \lim_{x\to a^+} f_2(x)$?

Solution: No. Strict inequality between functions does not guarantee strict inequality between their limits. Consider $a = 0$, $b = 1$. Let $f_1(x) = 0$ and $f_2(x) = x$ for $x \in (0, 1)$. Clearly $f_1(x) < f_2(x)$ for all $x \in (0, 1)$. However,

$$L_1 = \lim_{x\to 0^+} f_1(x) = \lim_{x\to 0^+} 0 = 0.$$

$$L_2 = \lim_{x\to 0^+} f_2(x) = \lim_{x\to 0^+} x = 0.$$

Here $L_1 = L_2$, not $L_1 < L_2$.

Example 2.3 (Symmetric Difference Limit vs. Continuity (HW4.8)). **Problem Statement:** Suppose $f : \mathbb{R} \to \mathbb{R}$ is such that for a given $x_0 \in \mathbb{R}$, $\lim_{n\to\infty}(f(x_0 + a_n) - f(x_0 - a_n)) = 0$ for all sequences $a_n \to 0$. Is f continuous at x_0?

Solution: No. Consider the function

$$f(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases}$$

Let $x_0 = 0$. Let (a_n) be any sequence such that $a_n \to 0$. For n large enough, $a_n \neq 0$, so $-a_n \neq 0$. Then $f(x_0 + a_n) = f(a_n) = 0$ and $f(x_0 - a_n) = f(-a_n) = 0$. So, $f(x_0 + a_n) - f(x_0 - a_n) = 0 - 0 = 0$. The limit is $\lim_{n\to\infty} 0 = 0$. The condition holds for $x_0 = 0$. However, f is not continuous at $x_0 = 0$, because $\lim_{x\to 0} f(x) = 0$, but $f(0) = 1$.

3 Convergence of Numerical Series

3.1 Definitions

Definition 3.1. A series $\sum_{n=1}^{\infty} a_n$ **converges** to $S \in \mathbb{R}$ if its sequence of partial sums $s_k = \sum_{n=1}^{k} a_n$ converges to S. Otherwise the series **diverges**.

3.2 Convergence Tests

Theorem 3.1 (Term Test for Divergence[1, Thm 14.5])**.** *If $\sum a_n$ converges, then $\lim_{n\to\infty} a_n = 0$. Equivalently, if $\lim a_n \neq 0$ or the limit DNE, then $\sum a_n$ diverges.*

Theorem 3.2 (Comparison Test[1, Thm 14.6])**.** *Let $0 \leq a_n \leq b_n$ for n sufficiently large.*

- *If $\sum b_n$ converges, then $\sum a_n$ converges.*
- *If $\sum a_n$ diverges, then $\sum b_n$ diverges.*

Theorem 3.3 (Limit Comparison Test[1, Thm 14.7])**.** *Let $a_n > 0, b_n > 0$ for n sufficiently large. Let $L = \lim_{n\to\infty}(a_n/b_n)$.*

- *If $0 < L < \infty$, then $\sum a_n$ converges iff $\sum b_n$ converges.*
- *If $L = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.*
- *If $L = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.*

Theorem 3.4 (Alternating Series Test[1, Thm 15.3])**.** *If (a_n) is a sequence such that $a_n \geq 0$, $a_{n+1} \leq a_n$ for n sufficiently large, and $\lim a_n = 0$, then the alternating series $\sum(-1)^n a_n$ (and $\sum(-1)^{n+1} a_n$) converges.*

Remark 3.1 (p-series[1, Sec 14]). The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$.

3.3 Examples

Example 3.1 (Sample Midterm 4a). **Problem Statement:** Determine convergence/divergence of $S_1 = \sum_{n=2}^{\infty} \frac{1}{\sqrt{n^2-1}}$ and $S_2 = \sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n^2-1}}$.
Solution:

- For S_1: Let $a_n = 1/\sqrt{n^2-1}$. Compare with $b_n = 1/n$. $\sum b_n$ diverges (harmonic, p=1).

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n}{\sqrt{n^2-1}} = \lim_{n\to\infty} \frac{n}{n\sqrt{1-1/n^2}} = 1.$$

  Since $0 < 1 < \infty$, by LCT, S_1 diverges.

- For S_2: Let $a_n = 1/\sqrt{n^2-1}$. 1. $a_n > 0$ for $n \geq 2$. 2. a_n is decreasing since $\sqrt{n^2-1}$ is increasing for $n \geq 2$. 3. $\lim_{n\to\infty} a_n = 0$. By AST, S_2 converges.

4 Sequences and Series of Functions

4.1 Definitions

Definition 4.1 (Pointwise Convergence[1, Def 24.1])**.** A sequence of functions (f_n) defined on $S \subseteq \mathbb{R}$ **converges pointwise** to f on S if for each $x \in S$, $\lim_{n\to\infty} f_n(x) = f(x)$.

Definition 4.2 (Uniform Convergence[1, Def 24.2])**.** (f_n) **converges uniformly** to f on S if $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall n > N$ and $\forall x \in S$,

$$|f_n(x) - f(x)| < \varepsilon.$$

Equivalently, $\lim_{n\to\infty} \sup_{x\in S} |f_n(x) - f(x)| = 0$.

Definition 4.3 (Uniformly Cauchy[1, Def 25.3])**.** (f_n) is **uniformly Cauchy** on S if $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall m, n > N$ and $\forall x \in S$,

$$|f_n(x) - f_m(x)| < \varepsilon.$$

4.2 Theorems on Uniform Convergence

Theorem 4.1 (Cauchy Criterion[1, Thm 25.4])**.** *A sequence of functions (f_n) converges uniformly on S if and only if it is uniformly Cauchy on S.*

Theorem 4.2 (Uniform Convergence implies Cauchy (Sample Midterm 6))**.** *If $f_n \to f$ uniformly on S, then (f_n) is uniformly Cauchy on S.*

Proof. Let $\varepsilon > 0$. By uniform convergence, $\exists N$ such that $k > N \implies |f_k(x) - f(x)| < \varepsilon/2$ for all $x \in S$. If $m, n > N$, then for all $x \in S$:

$$\begin{aligned} |f_n(x) - f_m(x)| &= |(f_n(x) - f(x)) + (f(x) - f_m(x))| \\ &\leq |f_n(x) - f(x)| + |f(x) - f_m(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Thus (f_n) is uniformly Cauchy. □

Theorem 4.3 (Continuity of Limit Function[1, Thm 24.3] (HW5.8))**.** *Let (f_n) be a sequence of functions continuous on $S \subseteq \mathbb{R}$. If $f_n \to f$ uniformly on S, then the limit function f is continuous on S.*

Theorem 4.4 (Uniform Continuity of Limit Function (Sample Midterm 1))**.** *Let (f_n) be a sequence of uniformly continuous functions on an interval I. If $f_n \to f$ uniformly on I, then the limit function f is uniformly continuous on I.*

Proof. Let $\varepsilon > 0$. 1. (Uniform Convergence) $\exists N$ such that $n > N \implies |f_n(z) - f(z)| < \varepsilon/3$ for all $z \in I$. Let $n_0 = N + 1$. 2. (Uniform Continuity of f_{n_0}) Since f_{n_0} is uniformly continuous, $\exists \delta > 0$ such that $|x - y| < \delta \implies |f_{n_0}(x) - f_{n_0}(y)| < \varepsilon/3$ for $x, y \in I$. 3. (Triangle Inequality) Let $x, y \in I$ with $|x - y| < \delta$.

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - f_{n_0}(x)| + |f_{n_0}(x) - f_{n_0}(y)| + |f_{n_0}(y) - f(y)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Thus f is uniformly continuous on I. □

Theorem 4.5 (Boundedness of Limit Function[1, Ex 25.5] (Sample Midterm 7a))**.** *Let (f_n) be a sequence of bounded functions on S. If $f_n \to f$ uniformly on S, then the limit function f is bounded on S.*

Proof. Let $\varepsilon = 1$. By uniform convergence, $\exists N$ such that $n > N \implies |f_n(x) - f(x)| < 1$ for all $x \in S$. Consider f_{N+1}. Since it's bounded, $\exists M$ such that $|f_{N+1}(x)| \leq M$ for all $x \in S$. For any $x \in S$:

$$\begin{aligned} |f(x)| = |f(x) - f_{N+1}(x) + f_{N+1}(x)| &\leq |f(x) - f_{N+1}(x)| + |f_{N+1}(x)| \\ &< 1 + M. \end{aligned}$$

Let $M' = M + 1$. Then $|f(x)| < M'$ for all $x \in S$, so f is bounded. □

Theorem 4.6 (Interchange of Limits[1, Ex 24.17] (Sample Midterm 3))**.** *Let (f_n) be a sequence of continuous functions on $[a, b]$ converging uniformly to f on $[a, b]$. Let (x_n) be a sequence in $[a, b]$ such that $x_n \to x \in [a, b]$. Then*

$$\lim_{n\to\infty} f_n(x_n) = f(x).$$

Proof. Let $\varepsilon > 0$. 1. By Thm 24.3, f is continuous on $[a, b]$. Since $x_n \to x$, $\exists N_1$ such that $n > N_1 \implies |f(x_n) - f(x)| < \varepsilon/2$. 2. By uniform convergence, $\exists N_2$ such that $n > N_2 \implies |f_n(y) - f(y)| < \varepsilon/2$ for all $y \in [a, b]$. In particular, $|f_n(x_n) - f(x_n)| < \varepsilon/2$. 3. Let $N = \max(N_1, N_2)$. For $n > N$:

$$\begin{aligned} |f_n(x_n) - f(x)| &= |(f_n(x_n) - f(x_n)) + (f(x_n) - f(x))| \\ &\leq |f_n(x_n) - f(x_n)| + |f(x_n) - f(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Thus $\lim f_n(x_n) = f(x)$. □

Theorem 4.7 (Weierstrass M-Test[1, Thm 25.7])**.** *Let (f_n) be a sequence of functions defined on $S \subseteq \mathbb{R}$. Suppose there exists a sequence of non-negative numbers (M_n) such that*

1. *$|f_n(x)| \leq M_n$ for all $x \in S$ and for all n,*
2. *The numerical series $\sum_{n=1}^{\infty} M_n$ converges.*

Then the series of functions $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on S.

4.3 Examples

Example 4.1 (Pointwise vs Uniform (HW5.7)). **Problem Statement:** For $x \in [0, \infty)$, define $f_n(x) = \frac{x}{n}$. (a) Find $f(x) = \lim f_n(x)$. (b) Uniform on $[0, 1]$? (c) Uniform on $[0, \infty)$?

Solution: (a) For fixed $x \geq 0$, $f(x) = \lim_{n\to\infty} x/n = x \cdot 0 = 0$. Pointwise limit is $f(x) = 0$. (b) On $[0, 1]$: We check $\sup_{x\in[0,1]} |f_n(x) - f(x)| = \sup_{x\in[0,1]} |x/n - 0| = \sup_{x\in[0,1]} x/n$. Since x/n increases with x, the supremum occurs at $x = 1$.

$$M_n = \sup_{x\in[0,1]} |f_n(x) - f(x)| = 1/n.$$

Since $\lim_{n\to\infty} M_n = \lim 1/n = 0$, convergence is uniform on $[0,1]$. (c) On $[0,\infty)$: We check $\sup_{x\in[0,\infty)} |f_n(x) - f(x)| = \sup_{x\in[0,\infty)} x/n$. For any fixed n, x/n is unbounded as $x \to \infty$.

$$M_n = \sup_{x\in[0,\infty)} |f_n(x) - f(x)| = \infty.$$

Since $M_n \not\to 0$, convergence is not uniform on $[0,\infty)$.

Example 4.2 (Pointwise/Uniform Convergence and Continuity (HW5.8)). **Problem Statement:** Analyze continuity and convergence for (a) $f_n(x) = 1$ if $x = 1/k$ ($k = 1..n$), 0 otherwise; (b) $g_n(x) = x$ if $x = 1/k$ ($k = 1..n$), 0 otherwise.

Solution: (a) Sequence f_n:

- Continuity of f_n at 0: $f_n(0) = 0$. For $|x| < 1/n$, $x \neq 1/k$ for $k = 1..n$, so $f_n(x) = 0$. Thus $\lim_{x\to 0} f_n(x) = 0 = f_n(0)$. Yes, f_n continuous at 0.
- Pointwise limit $f(x)$: If $x = 1/k$, then $f_n(x) = 1$ for $n \geq k$, so $f(x) = 1$. Otherwise $f_n(x) = 0$ for all n, so $f(x) = 0$.
$$f(x) = \begin{cases} 1 & \text{if } x = 1/k \text{ for some integer } k \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$
- Uniform convergence: $M_n = \sup |f_n(x) - f(x)|$. Consider $x = 1/(n+1)$. $f_n(x) = 0$, $f(x) = 1$. So $|f_n(x) - f(x)| = 1$. Thus $M_n \geq 1$. Since $M_n \not\to 0$, convergence is not uniform.
- Continuity of f at 0: $f(0) = 0$. Let $x_k = 1/k \to 0$. $f(x_k) = 1$. $\lim f(x_k) = 1 \neq f(0)$. No, f is not continuous at 0.

(b) Sequence g_n:

- Continuity of g_n at 0: $g_n(0) = 0$. For $|x| < 1/n$, $g_n(x) = 0$. Thus $\lim_{x\to 0} g_n(x) = 0 = g_n(0)$. Yes, g_n continuous at 0.
- Pointwise limit $g(x)$: If $x = 1/k$, then $g_n(x) = x$ for $n \geq k$, so $g(x) = x$. Otherwise $g_n(x) = 0$, so $g(x) = 0$.
$$g(x) = \begin{cases} x & \text{if } x = 1/k \text{ for some integer } k \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$
- Uniform convergence: $M_n = \sup |g_n(x) - g(x)|$. The difference is non-zero only if $x = 1/k$ for $k > n$, where $|g_n(x) - g(x)| = |0 - x| = 1/k$.
$$M_n = \sup\{1/k : k > n\} = 1/(n+1).$$
Since $\lim M_n = \lim 1/(n+1) = 0$, convergence is uniform.
- Continuity of g at 0: $g(0) = 0$. We have $|g(x)| \leq |x|$ (since $g(x)$ is either 0 or x). Let $\varepsilon > 0$. Choose $\delta = \varepsilon$. If $|x - 0| < \delta$, then $|g(x) - g(0)| = |g(x)| \leq |x| < \delta = \varepsilon$. Yes, g is continuous at 0. (Consistent with Thm 24.3).

Example 4.3 (Pointwise Limit Need Not Be Bounded (Sample Midterm 7b)). **Problem Statement:** Construct $S \subseteq \mathbb{R}$ and a sequence of bounded functions (f_n) on S such that $f_n \to f$ pointwise, but f is not bounded.

Solution: Let $S = (0,1]$. Define $f_n : S \to \mathbb{R}$ by

$$f_n(x) = \min\left\{n, \frac{1}{x}\right\}.$$

- Boundedness of f_n: For any $x \in (0,1]$, $1/x \geq 1$. If $1/x \leq n$, then $f_n(x) = 1/x$. If $1/x > n$, then $f_n(x) = n$. In either case, $0 < f_n(x) \leq \max(n, 1/x)$. More simply, $f_n(x)$ is either n or $1/x$. If $x \geq 1/n$, $1/x \leq n$, so $f_n(x) = 1/x \leq n$. If $x < 1/n$, $f_n(x) = n$. Thus, $|f_n(x)| \leq n$ for all $x \in S$. Each f_n is bounded.
- Pointwise Limit: Let $x \in (0,1]$ be fixed. Choose $N \in \mathbb{N}$ such that $N > 1/x$. For all $n > N$, we have $n > 1/x$, which implies $x > 1/n$. By definition of f_n, for $n > N$, $f_n(x) = 1/x$. The sequence $(f_n(x))$ is eventually constant $(1/x)$, so $\lim_{n\to\infty} f_n(x) = 1/x$. The pointwise limit is $f(x) = 1/x$.
- Unboundedness of f: The limit function $f(x) = 1/x$ is not bounded on $S = (0,1]$ because $\lim_{x\to 0^+} f(x) = +\infty$.

Example 4.4 (M-Test Application (Sample Midterm 2b alt)). **Problem Statement:** Show that $f_3(y) = \sum_{n=1}^{\infty} \frac{1}{n^2}\left(\frac{y}{1+y^2}\right)^n$ converges for all $y \in \mathbb{R}$.

Solution: Let $f_n(y) = \frac{1}{n^2}\left(\frac{y}{1+y^2}\right)^n$. Let $g(y) = y/(1+y^2)$. We find the maximum of $|g(y)|$. $g'(y) = (1-y^2)/(1+y^2)^2$. Critical points at $y = \pm 1$. $g(1) = 1/2$, $g(-1) = -1/2$. Also $g(0) = 0$ and $\lim_{y\to\pm\infty} g(y) = 0$. So the maximum absolute value is $|g(\pm 1)| = 1/2$. Thus, $|g(y)| \leq 1/2$ for all $y \in \mathbb{R}$. Now bound $|f_n(y)|$:

$$|f_n(y)| = \left|\frac{1}{n^2}(g(y))^n\right| = \frac{1}{n^2}|g(y)|^n \leq \frac{1}{n^2}\left(\frac{1}{2}\right)^n.$$

Let $M_n = \frac{1}{n^2 2^n}$. The series $\sum M_n$ converges by comparison with the convergent p-series $\sum 1/n^2$ (since $1/2^n \leq 1$). By the Weierstrass M-Test, the series $\sum f_n(y)$ converges uniformly on $\mathbb{R}$. Uniform convergence implies pointwise convergence for all $y \in \mathbb{R}$.

5 Power Series

5.1 Definitions and Basic Properties

Definition 5.1 (Power Series[1, Sec 23])**.** A **power series** centered at a is an infinite series of the form

$$\sum_{n=0}^{\infty} a_n(x-a)^n = a_0 + a_1(x-a) + a_2(x-a)^2 + \dots$$

We often consider $a = 0$: $\sum a_n x^n$.

Theorem 5.1 (Radius of Convergence[1, Thm 23.1])**.** *For any power series $\sum a_n(x-a)^n$, there exists a unique $R \in [0,\infty]$, called the **radius of convergence**, such that:*

- *The series converges absolutely for all x satisfying $|x-a| < R$.*
- *The series diverges for all x satisfying $|x-a| > R$.*

The value R is given by the formula:

$$R = \frac{1}{\limsup_{n\to\infty} |a_n|^{1/n}}$$

(with conventions $1/0 = \infty$ and $1/\infty = 0$).

Proposition 5.2 (Ratio Test for Radius of Convergence[1, Sec 9])**.** *If the limit $L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$ exists (possibly 0 or ∞), then the radius of convergence is $R = 1/L$ (with conventions $1/0 = \infty, 1/\infty = 0$).*

Definition 5.2 (Interval of Convergence)**.** The set of all $x \in \mathbb{R}$ for which the power series $\sum a_n(x-a)^n$ converges. It is always an interval centered at a. If $0 < R < \infty$, the interval is one of $(a-R, a+R)$, $[a-R, a+R)$, $(a-R, a+R]$, or $[a-R, a+R]$. Convergence at the endpoints $x = a \pm R$ must be tested separately.

Theorem 5.3 (Uniform Convergence of Power Series[1, Thm 26.1])**.** *If a power series* $\sum a_n(x-a)^n$ *has radius of convergence* $R > 0$*, then for any* c *such that* $0 < c < R$*, the series converges uniformly on the closed interval* $[a-c, a+c]$*.*

Corollary 5.4 (Continuity of Power Series)**.** *The function* $f(x) = \sum a_n(x-a)^n$ *defined by a power series is continuous on its open interval of convergence* $(a-R, a+R)$*.*

5.2 Examples

Example 5.1 (Calculating R and Interval (HW5.6)). **Problem Statement:** For each series, find the radius R and the interval of convergence I.

1. $\sum_{n=0}^{\infty} n^2 x^n$: $a_n = n^2$. Use Ratio Test:

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{(n+1)^2}{n^2} = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^2 = 1.$$

   $R = 1/L = 1$. Check endpoints $x = \pm 1$: $\sum(\pm 1)^n n^2$. Since $|(\pm 1)^n n^2| = n^2 \not\to 0$, both diverge by Term Test. $I = (-1, 1)$.

2. $\sum_{n=1}^{\infty} \left(\frac{x}{n}\right)^n = \sum_{n=1}^{\infty} \frac{1}{n^n} x^n$: $a_n = 1/n^n$. Use Root Test:

$$\alpha = \limsup_{n\to\infty} |a_n|^{1/n} = \limsup_{n\to\infty} \left| \frac{1}{n^n} \right|^{1/n} = \limsup_{n\to\infty} \frac{1}{n} = 0.$$

   $R = 1/\alpha = 1/0 = \infty$. $I = (-\infty, \infty)$.

3. $\sum_{n=1}^{\infty} x^{n!}$: Coefficients $a_k = 1$ if $k = m!$ for some $m \geq 1$, $a_k = 0$ otherwise. Use Root Test:

$$\alpha = \limsup_{k\to\infty} |a_k|^{1/k}.$$

   The sequence $|a_k|^{1/k}$ contains terms equal to 1 infinitely often (when $k = m!$). The other terms are 0. The limit superior is 1. $R = 1/\alpha = 1$. Check endpoints $x = \pm 1$: If $x = 1$, $\sum 1^{n!} = \sum 1$, diverges. If $x = -1$, $\sum(-1)^{n!} = (-1)^1 + (-1)^2 + (-1)^6 + \cdots = -1 + 1 + 1 + \ldots$. Terms are $-1, 1, 1, \ldots$, do not approach 0. Diverges by Term Test. $I = (-1, 1)$.

4. $\sum_{n=0}^{\infty} 5^n x^{2n+1}$: Rewrite as $x \sum_{n=0}^{\infty} 5^n (x^2)^n$. Let $y = x^2$. Series is $x \sum (5y)^n$. This is geometric, converges iff $|5y| < 1 \implies |y| < 1/5$. So $|x^2| < 1/5 \implies x^2 < 1/5 \implies |x| < 1/\sqrt{5}$. $R = 1/\sqrt{5}$. Check endpoints $x = \pm 1/\sqrt{5}$: If $x = 1/\sqrt{5}$, series is $\sum 5^n (1/\sqrt{5})^{2n+1} = \sum \frac{5^n}{5^n\sqrt{5}} = \sum 1/\sqrt{5}$, diverges (Term Test). If $x = -1/\sqrt{5}$, series is $\sum 5^n(-1/\sqrt{5})^{2n+1} = \sum (-1)\frac{5^n}{5^n\sqrt{5}} = \sum -1/\sqrt{5}$, diverges (Term Test). $I = (-1/\sqrt{5}, 1/\sqrt{5})$.

Example 5.2 (Calculating R (Sample Midterm 2a)). **Problem Statement:** Find R for $f_1(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}$ and $f_2(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{2^n}$.

Solution:

- For f_1: $a_n = 1/n^2$. Ratio Test: $L = \lim |a_{n+1}/a_n| = \lim n^2/(n+1)^2 = 1$. $R = 1/1 = 1$.
- For f_2: Let $y = x^2$. Series is $\sum y^n/2^n = \sum (1/2^n) y^n$. For this series in y, use Ratio Test on coefficients $b_n = 1/2^n$: $L_y = \lim |b_{n+1}/b_n| = \lim (1/2^{n+1})/(1/2^n) = 1/2$. Radius for y is $R_y = 1/(1/2) = 2$. Converges for $|y| < 2$. Substitute back: $|x^2| < 2 \implies x^2 < 2 \implies |x| < \sqrt{2}$. Radius for x is $R = \sqrt{2}$.

Example 5.3 (Using Endpoint Behavior for R (Sample Midterm 4b)). **Problem Statement:** Find R for $\sum_{n=2}^{\infty} \frac{5^n x^n}{\sqrt{n^2-1}}$. Use results from SM 4a: $\sum 1/\sqrt{n^2-1}$ diverges, $\sum(-1)^n/\sqrt{n^2-1}$ converges.

Solution: Let the power series be $S(x)$.

- Test $x = 1/5$: $S(1/5) = \sum \frac{5^n (1/5)^n}{\sqrt{n^2-1}} = \sum \frac{1}{\sqrt{n^2-1}}$. This diverges. Since the series diverges at $x = 1/5$, we must have $R \leq |1/5| = 1/5$.
- Test $x = -1/5$: $S(-1/5) = \sum \frac{5^n (-1/5)^n}{\sqrt{n^2-1}} = \sum \frac{(-1)^n}{\sqrt{n^2-1}}$. This converges. Since the series converges at $x = -1/5$, we must have $R \geq |-1/5| = 1/5$.

Combining $R \leq 1/5$ and $R \geq 1/5$, we conclude $R = 1/5$.

Example 5.4 (Function Series as Power Series (Sample Midterm 2b)). **Problem Statement:** Show that $f_3(y) = \sum_{n=1}^{\infty} \frac{1}{n^2} \left(\frac{y}{1+y^2}\right)^n$ converges for all $y \in \mathbb{R}$.

Solution: Let $x = g(y) = \frac{y}{1+y^2}$. The series is $f_1(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}$. From SM 2a, $f_1(x)$ has $R = 1$. Check endpoints for $f_1(x)$: If $x = 1$, $\sum 1/n^2$ converges (p-series, p=2). If $x = -1$, $\sum(-1)^n/n^2$ converges (AST or absolutely). So, the interval of convergence for $f_1(x)$ is $[-1, 1]$. Now find the range of $g(y) = y/(1+y^2)$. As shown in HW5 M-Test example, $|g(y)| \leq 1/2$ for all $y \in \mathbb{R}$. The range is $[-1/2, 1/2]$. Since the argument $x = g(y)$ always lies in $[-1/2, 1/2]$, and this interval is contained within the interval of convergence $[-1, 1]$ for $f_1(x)$, the series $f_3(y) = f_1(g(y))$ converges for all $y \in \mathbb{R}$.

References

[1] Ross, K. A. *Elementary Analysis: The Theory of Calculus*. 2nd ed., Springer, 2013.