Math 521: Midterm 1 information

- The course midterm will take place in class on Wednesday March 12, from 9:57am–10:42am.
- The exam will cover everything in class up to the end of the lecture on February 28.
- The exam is closed book—no textbooks, notebooks, calculators, or smartphones allowed. You will not be expected to know every theorem by heart, but you will be expected to remember the basic definitions, such as suprema/infima, convergence/divergence, subsequential limits, interiors and closures. You should also be familiar with the limit theorems for determining convergence of sequences, and the tests for determining series convergence and divergence.
- There will be three questions in total. At least one question will involve constructing a mathematical proof.
- If a question uses the word "prove", then you will be expected to write down a mathematical proof similar to those in the textbook or given in the homework solutions. If a question uses weaker langauge, such as "determine", "justify", or "compute", a less rigorous argument will still receive full credit.
- You can assume a number of basic results, such as
  - ◇ $\sum n^{-p}$ converges if and only if $p > 1$,
  - ◇ the triangle inequality: $|a| + |b| \geq |a + b|$ for all $a, b \in \mathbb{R}$,
  - ◇ $|b| < a$ if and only if $-a < b < a$,
  - ◇ $\lim_{n\to\infty} a^n = 0$ for $|a| < 1$,
  - ◇ $\lim_{n\to\infty} n^{1/n} = 1$.

Sample midterm questions

The following questions are of a similar style to the ones that will be on the midterm. They are designed to test familiarity with basic concepts, and will generally be more straightforward than some of the questions on the homework.

1. Prove that $1 + \sqrt{1 + \sqrt{2}}$ is irrational.
2. Consider the following series defined for $n \in \mathbb{N}$:

$$\sum_n \frac{8^n}{(n!)^2}, \qquad \sum_n \frac{(-1)^n}{\sqrt{n^2 + n}}, \qquad \sum_n \frac{6^n}{n^n}, \qquad \sum_n \frac{1}{n + 1/2}.$$

For each series, determine whether they converge or diverge. If you make use of any of the theorems for determining series properties, you should state which one you use.

3. (a) Let S and T be non-empty bounded subsets of $\mathbb{R}$. Prove that $\sup S \cup T = \max\{\sup S, \sup T\}$ and $\sup S \cap T \leq \min\{\sup S, \sup T\}$.
   (b) Extend part (a) to the cases where S and T are not bounded.
   (c) Give an example where $\sup S \cap T < \min\{\sup S, \sup T\}$.

4. Suppose that (s_n) is a convergent sequence and (t_n) is a sequence that diverges to ∞. Prove that
$$\lim_{n\to\infty} s_n + t_n = \infty.$$

5. Consider the two sets
$$A = (0, 1] \cup [4, \infty), \qquad B = \left\{ \frac{1}{2n} \ : \ n \in \mathbb{N} \right\}.$$
For each set, determine its maximum and minimum if they exist. For each set, determine its supremum and infimum. Detailed proofs are not required, but you should justify your answers.

6. Prove that the functions
$$d_1(x, y) = (x - y)^4, \qquad d_2(x, y) = 1 + |x - y|,$$
$$d_3(x, y) = \begin{cases} x - y & \text{if } x > y, \\ 2(y - x) & \text{if } x \leq y, \end{cases}$$
are not metrics on $\mathbb{R}$.

7. (a) Prove that if r is irrational, then $r^{1/4}$ and $r + 1$ are also irrational.
   (b) Prove that $(\sqrt{2} + \sqrt{3})^{1/4} + 1$ is irrational.

8. Let S be a non-empty bounded subset of $\mathbb{R}$. Define $T = \{|x| \ : \ x \in S\}$ to be the set of all absolute values of elements in S. Prove that $\sup T = \max\{\sup S, -\inf S\}$.

9. (a) Prove that $\sqrt{3} - \sqrt{2}$ is irrational.
   (b) Consider the sets
$$A = [0, \sqrt{3} - \sqrt{2}] \cap \mathbb{Q}, \qquad B = [0, \sqrt{3} - \sqrt{2}] \cup \mathbb{Q}.$$
   For each set, determine its maximum, minimum, supremum, and infimum, or state that these values do not exist. Detailed proofs are not required, but you should justify your answers.

10. Let (a_n) be a sequence and suppose that $\lim_{n\to\infty} n^2 a_n = c \in \mathbb{R}$. Prove that $\sum a_n$ converges absolutely.

11. Consider the function $f(x) = 4x + x^2$, and the sequence (s_n) defined recursively as

$$s_{n+1} = f(s_n).$$

(a) If $s_0 = 1$, prove that $\lim_{n\to\infty} s_n = \infty$.

(b) What is the minimum of f on the interval $[-4, 0]$?

(c) Sketch the function f on the interval $[-4, 0]$.

(d) If $s_0 \in [-4, 0]$, prove that (s_n) has a convergent subsequence.

12. (a) Consider the function $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined as

$$d(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 1 + |x - y| & \text{if } x \neq y. \end{cases}$$

Prove that d is a metric on $\mathbb{R}$.

(b) For the metric space $(\mathbb{R}, d)$, is the set

$$A = \{\tfrac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$$

compact? Prove your assertion.