Homework Guide: Assignment 6

Problem 1

(a) Let f be a uniformly continuous function on $\mathbb{R}$, and define the sequence of functions $f_n(x) = f(x - \frac{1}{n})$. Prove that $f_n \to f$ uniformly.

(b) Suppose that f is a continuous function on $\mathbb{R}$, and again define $f_n(x) = f(x - \frac{1}{n})$. Find an example where f_n does not uniformly converge to f.

Relevant Definitions and Theorems

Definition (Uniform Continuity [1]). *A function $f : S \to \mathbb{R}$ is uniformly continuous on S if for every $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x, y \in S$, if $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$.*

Definition (Uniform Convergence [2]). *A sequence of functions (f_n) on a set S converges uniformly to a function f on S if for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n > N$ and all $x \in S$, $|f_n(x) - f(x)| < \epsilon$.*

Definition (Continuity [3]). *A function $f : S \to \mathbb{R}$ is continuous on S if it is continuous at every point $x_0 \in S$.*

Solution Outline

(a)

1. Let $\epsilon > 0$.
2. Since f is uniformly continuous on $\mathbb{R}$, there exists $\delta > 0$ such that $|y - z| < \delta \implies |f(y) - f(z)| < \epsilon$.
3. We need to show $\forall \epsilon > 0, \exists N \in \mathbb{N}$ such that $n > N \implies |f_n(x) - f(x)| < \epsilon$ for all $x \in \mathbb{R}$.
4. Consider $|f_n(x) - f(x)| = \left|f(x - \frac{1}{n}) - f(x)\right|$.
5. Let $y = x - \frac{1}{n}$ and $z = x$. Then $|y - z| = \left|-\frac{1}{n}\right| = \frac{1}{n}$.
6. We need $|y - z| < \delta$, which means $\frac{1}{n} < \delta$.
7. Choose $N \in \mathbb{N}$ such that $N > 1/\delta$.
8. If $n > N$, then $\frac{1}{n} < \frac{1}{N} < \delta$.
9. Thus, for $n > N$, $|y - z| = \frac{1}{n} < \delta$, which implies $|f(y) - f(z)| < \epsilon$.
10. This means $\left|f(x - \frac{1}{n}) - f(x)\right| < \epsilon$ for all $x \in \mathbb{R}$ when $n > N$.
11. By definition, $f_n \to f$ uniformly on $\mathbb{R}$.

(b)

1. We need a continuous function on $\mathbb{R}$ that is not uniformly continuous. Consider $f(x) = x^2$.
2. $f(x) = x^2$ is continuous on $\mathbb{R}$.
3. Define $f_n(x) = f(x - \frac{1}{n}) = (x - \frac{1}{n})^2$.
4. Consider the difference:

$$|f_n(x) - f(x)| = \left|(x - \tfrac{1}{n})^2 - x^2\right| = \left|x^2 - \tfrac{2x}{n} + \tfrac{1}{n^2} - x^2\right| = \left|-\tfrac{2x}{n} + \tfrac{1}{n^2}\right|.$$

[1] See Definition 19.1 in Ross, K. A. *Elementary Analysis: The Theory of Calculus*. Springer, 2013.
[2] See Definition 24.1 in Ross, K. A. *Elementary Analysis: The Theory of Calculus*. Springer, 2013.
[3] See Definition 17.1 in Ross, K. A. *Elementary Analysis: The Theory of Calculus*. Springer, 2013.

5. For uniform convergence, for a given $\epsilon > 0$, we need $\exists N$ such that for $n > N$, $\left|-\frac{2x}{n} + \frac{1}{n^2}\right| < \epsilon$ for *all* $x \in \mathbb{R}$.
6. Let $\epsilon = 1$. Suppose such an N exists.
7. Choose $n > N$. Let $x = n$.
8. Then $|f_n(n) - f(n)| = \left|-\frac{2n}{n} + \frac{1}{n^2}\right| = \left|-2 + \frac{1}{n^2}\right|$.
9. If $n \geq 2$, then $0 < \frac{1}{n^2} \leq \frac{1}{4}$, so $\left|-2 + \frac{1}{n^2}\right| = 2 - \frac{1}{n^2} \geq 2 - \frac{1}{4} = \frac{7}{4} > 1$.
10. This contradicts the assumption that $|f_n(x) - f(x)| < 1$ for all x when $n > N$.
11. Therefore, the convergence is not uniform for $f(x) = x^2$.

Problem 2

Let f be a bounded function on $[0, 1]$ so that $|f(x)| \leq M$ for all $x \in [0, 1]$. Show that the Bernstein polynomials $B_n f$ are all bounded by M.

Relevant Definitions and Theorems

Definition (Bounded Function [4]). *A function $f : S \to \mathbb{R}$ is bounded if there exists $M \in \mathbb{R}$ such that $|f(x)| \leq M$ for all $x \in S$.*

Definition (Bernstein Polynomials [5]). *For a function f defined on $[0, 1]$, the n-th Bernstein polynomial for f is*

$$(B_n f)(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right).$$

Theorem (Binomial Theorem). *For any real numbers a, b and any integer $n \geq 0$,*

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}.$$

Solution Outline

1. Let $x \in [0, 1]$. The n-th Bernstein polynomial is:
$$(B_n f)(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right).$$
2. Take the absolute value:
$$|(B_n f)(x)| = \left|\sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right)\right|.$$
3. Apply the triangle inequality:
$$|(B_n f)(x)| \leq \sum_{k=0}^{n} \left|\binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right)\right|.$$
4. Since $x \in [0, 1]$, the terms $\binom{n}{k}$, x^k, and $(1-x)^{n-k}$ are non-negative. Thus,
$$|(B_n f)(x)| \leq \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} \left|f\left(\frac{k}{n}\right)\right|.$$

[4] See Definition 13.1 in Ross, K. A. *Elementary Analysis: The Theory of Calculus*. Springer, 2013.
[5] See Section 26 in Ross, K. A. *Elementary Analysis: The Theory of Calculus*. Springer, 2013.

5. We are given $|f(y)| \le M$ for all $y \in [0,1]$. Since $\frac{k}{n} \in [0,1]$, we have $\left|f(\frac{k}{n})\right| \le M$.
6. Substitute this bound:
$$|(B_n f)(x)| \le \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} M.$$
7. Factor out M:
$$|(B_n f)(x)| \le M \left(\sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} \right).$$
8. By the Binomial Theorem with $a = x$ and $b = 1 - x$:
$$\sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} = (x + (1-x))^n = 1^n = 1.$$
9. Substitute this back:
$$|(B_n f)(x)| \le M \cdot 1 = M.$$
10. This holds for all $x \in [0,1]$ and all $n \in \mathbb{N}$. Thus, $B_n f$ are uniformly bounded by M.

Problem 3

Let f and g be differentiable on an open interval I and let $a \in I$. Define
$$h(x) = \begin{cases} f(x) & \text{for } x < a, \\ g(x) & \text{for } x \ge a. \end{cases}$$
Prove that h is differentiable at a if and only if both $f(a) = g(a)$ and $f'(a) = g'(a)$.

Relevant Definitions and Theorems

Definition (Differentiability at a Point [6]). *A function h defined on an open interval containing a is differentiable at a if the limit $h'(a) = \lim_{x\to a} \frac{h(x)-h(a)}{x-a}$ exists.*

Theorem (Differentiability implies Continuity [7]). *If h is differentiable at a, then h is continuous at a.*

Definition (Continuity at a Point [8]). *A function h is continuous at a if $\lim_{x\to a} h(x) = h(a)$. This requires $\lim_{x\to a^-} h(x) = \lim_{x\to a^+} h(x) = h(a)$.*

Theorem (Existence of Limit [9]). *The limit $\lim_{x\to a} G(x)$ exists and equals L if and only if the left-hand limit $\lim_{x\to a^-} G(x)$ and the right-hand limit $\lim_{x\to a^+} G(x)$ both exist and equal L.*

Solution Outline

(⇒) Assume h is differentiable at a.

1. Since h is differentiable at a, it is continuous at a (Thm 28.2).
2. Continuity at a implies $\lim_{x\to a^-} h(x) = \lim_{x\to a^+} h(x) = h(a)$.
3. Evaluate these using the definition of h:
   - $h(a) = g(a)$.

[6] See Definition 28.1 in Ross, K. A. *Elementary Analysis: The Theory of Calculus*. Springer, 2013.
[7] See Theorem 28.2 in Ross, K. A. *Elementary Analysis: The Theory of Calculus*. Springer, 2013.
[8] See Definition 17.1 in Ross, K. A. *Elementary Analysis: The Theory of Calculus*. Springer, 2013.
[9] See Definition 20.1 in Ross, K. A. *Elementary Analysis: The Theory of Calculus*. Springer, 2013.

- $\lim_{x\to a^-} h(x) = \lim_{x\to a^-} f(x) = f(a)$ (since f is continuous at a).
- $\lim_{x\to a^+} h(x) = \lim_{x\to a^+} g(x) = g(a)$ (since g is continuous at a).

4. Thus, $f(a) = g(a)$. This is the first condition.
5. Since h is differentiable at a, the limit $h'(a) = \lim_{x\to a} \frac{h(x)-h(a)}{x-a}$ exists.
6. The left-hand and right-hand limits of the difference quotient must exist and be equal.
7. Left-hand derivative:
$$\lim_{x\to a^-} \frac{h(x)-h(a)}{x-a} = \lim_{x\to a^-} \frac{f(x)-g(a)}{x-a}.$$
Using $f(a) = g(a)$:
$$\lim_{x\to a^-} \frac{f(x)-f(a)}{x-a} = f'(a)$$
(since f is differentiable at a).
8. Right-hand derivative:
$$\lim_{x\to a^+} \frac{h(x)-h(a)}{x-a} = \lim_{x\to a^+} \frac{g(x)-g(a)}{x-a} = g'(a)$$
(since g is differentiable at a).
9. For $h'(a)$ to exist, the one-sided derivatives must be equal: $f'(a) = g'(a)$. This is the second condition.

(⇐) Assume $f(a) = g(a)$ and $f'(a) = g'(a)$.

1. We need to show that $h'(a) = \lim_{x\to a} \frac{h(x)-h(a)}{x-a}$ exists.
2. Examine the left-hand and right-hand limits of the difference quotient.
3. Left-hand derivative:
$$\lim_{x\to a^-} \frac{h(x)-h(a)}{x-a} = \lim_{x\to a^-} \frac{f(x)-g(a)}{x-a}.$$
Using $f(a) = g(a)$:
$$\lim_{x\to a^-} \frac{f(x)-f(a)}{x-a} = f'(a).$$
4. Right-hand derivative:
$$\lim_{x\to a^+} \frac{h(x)-h(a)}{x-a} = \lim_{x\to a^+} \frac{g(x)-g(a)}{x-a} = g'(a).$$
5. We are given $f'(a) = g'(a)$. Let $L = f'(a) = g'(a)$.
6. The left-hand derivative is L and the right-hand derivative is L.
7. Since the left-hand and right-hand limits exist and are equal, the limit exists:
$$\lim_{x\to a} \frac{h(x)-h(a)}{x-a} = L.$$
8. Therefore, h is differentiable at a and $h'(a) = L$.

Problem 4

Suppose f is differentiable at a. Define

$$L_1(a,h) = \frac{f(a+h) - f(a-h)}{2h}, \quad L_2(a,h) = \frac{-f(a+2h) + 8f(a+h) - 8f(a-h) + f(a-2h)}{12h}.$$

(a) Prove that $\lim_{h\to 0} L_i(a,h) = f'(a)$ for $i = 1, 2$. **(b)** Consider $f(x) = x^5$. How does $|L_i(a,h) - f'(a)|$ behave as $h \to 0$?

Relevant Definitions and Theorems

Definition (Differentiability at a Point [10]). *f is differentiable at a if $f'(a) = \lim_{k\to 0} \frac{f(a+k)-f(a)}{k}$ exists. This implies $f(a+k) = f(a) + f'(a)k + \phi(k)$ where $\lim_{k\to 0} \frac{\phi(k)}{k} = 0$.*

Theorem (Taylor's Theorem [11]). *If f has $n+1$ derivatives in an interval around a, then for x in that interval,*

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + R_n(x),$$

where $R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$ for some c between a and x.

Solution Outline

(a) *For $L_1(a,h)$:*

1. Rewrite $L_1(a,h)$:
$$L_1(a,h) = \frac{f(a+h) - f(a) - (f(a-h) - f(a))}{2h} = \frac{1}{2}\left[\frac{f(a+h) - f(a)}{h} - \frac{f(a-h) - f(a)}{h}\right].$$
2. Let $k = -h$. Then $\frac{f(a-h)-f(a)}{h} = \frac{f(a+k)-f(a)}{-k} = -\frac{f(a+k)-f(a)}{k}$.
3. So, $L_1(a,h) = \frac{1}{2}\left[\frac{f(a+h)-f(a)}{h} + \frac{f(a+(-h))-f(a)}{-h}\right]$.
4. Take the limit as $h \to 0$. Since f is differentiable at a:
$$\lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = f'(a), \quad \lim_{h\to 0} \frac{f(a+(-h)) - f(a)}{-h} = f'(a).$$
5. Therefore, $\lim_{h\to 0} L_1(a,h) = \frac{1}{2}[f'(a) + f'(a)] = f'(a)$.

For $L_2(a,h)$:

1. Use the definition $f(a+k) = f(a) + f'(a)k + \phi(k)$, where $\phi(k)/k \to 0$ as $k \to 0$.
2. Substitute into the numerator of $L_2(a,h)$ for $k = 2h, h, -h, -2h$.
3. Sum the terms:
   - $f(a)$ terms: $-1 + 8 - 8 + 1 = 0$.
   - $f'(a)k$ terms: $f'(a)[-2h + 8h - 8(-h) + (-2h)] = f'(a)[-2h + 8h + 8h - 2h] = 12f'(a)h$.
   - Remainder terms: $R(h) = -\phi(2h) + 8\phi(h) - 8\phi(-h) + \phi(-2h)$.
4. Numerator $= 12f'(a)h + R(h)$.
5. $L_2(a,h) = \frac{12f'(a)h+R(h)}{12h} = f'(a) + \frac{R(h)}{12h}$.

[10] See Definition 28.1 in Ross, K. A. *Elementary Analysis: The Theory of Calculus*. Springer, 2013.
[11] See Theorem 30.2 in Ross, K. A. *Elementary Analysis: The Theory of Calculus*. Springer, 2013.

6. We need to show $\lim_{h\to 0} \frac{R(h)}{12h} = 0$.

$$\frac{R(h)}{12h} = \frac{1}{12}\left[-\frac{\phi(2h)}{h} + 8\frac{\phi(h)}{h} - 8\frac{\phi(-h)}{h} + \frac{\phi(-2h)}{h}\right]$$
$$= \frac{1}{12}\left[-2\frac{\phi(2h)}{2h} + 8\frac{\phi(h)}{h} + 8\frac{\phi(-h)}{-h} - 2\frac{\phi(-2h)}{-2h}\right].$$

7. As $h \to 0$, each term $\frac{\phi(k)}{k} \to 0$.
8. Thus, $\lim_{h\to 0} \frac{R(h)}{12h} = \frac{1}{12}[-2(0) + 8(0) + 8(0) - 2(0)] = 0$.
9. Therefore, $\lim_{h\to 0} L_2(a,h) = f'(a) + 0 = f'(a)$.

(b)

1. Let $f(x) = x^5$. Derivatives: $f'(x) = 5x^4$, $f''(x) = 20x^3$, $f'''(x) = 60x^2$, $f^{(4)}(x) = 120x$, $f^{(5)}(x) = 120$, $f^{(k)}(x) = 0$ for $k \geq 6$.
2. Use Taylor expansion around a:
$$f(a+h) = f(a) + f'(a)h + \frac{f''(a)}{2}h^2 + \frac{f'''(a)}{6}h^3 + \frac{f^{(4)}(a)}{24}h^4 + \frac{f^{(5)}(a)}{120}h^5.$$
For $L_1(a,h)$:
3. Calculate $f(a+h) - f(a-h)$ using the expansion. Odd power terms add, even power terms cancel.
$$f(a+h) - f(a-h) = 2f'(a)h + 2\frac{f'''(a)}{6}h^3 + 2\frac{f^{(5)}(a)}{120}h^5 = 2f'(a)h + \frac{f'''(a)}{3}h^3 + \frac{f^{(5)}(a)}{60}h^5.$$
4. $L_1(a,h) = \frac{f(a+h)-f(a-h)}{2h} = f'(a) + \frac{f'''(a)}{6}h^2 + \frac{f^{(5)}(a)}{120}h^4$.
5. $L_1(a,h) - f'(a) = \frac{f'''(a)}{6}h^2 + \frac{f^{(5)}(a)}{120}h^4$.
6. Substitute derivatives of x^5: $f'''(a) = 60a^2$, $f^{(5)}(a) = 120$.
$$L_1(a,h) - f'(a) = \frac{60a^2}{6}h^2 + \frac{120}{120}h^4 = 10a^2h^2 + h^4.$$
7. $|L_1(a,h) - f'(a)| = \left|10a^2h^2 + h^4\right| = O(h^2)$ as $h \to 0$ (unless $a = 0$, then $O(h^4)$). *For $L_2(a,h)$:*
8. Calculate $-f(a+2h) + 8f(a+h) - 8f(a-h) + f(a-2h)$ using Taylor expansions.
9. Coefficients of $h^j/j!$ for $j = 0, 2, 3, 4$ cancel out.
10. Coefficient of $h^1/1!$: $f'(a)[-2 + 8 - 8(-1) + (-2)] = 12f'(a)$.
11. Coefficient of $h^5/5!$: $f^{(5)}(a)[-(2)^5 + 8(1)^5 - 8(-1)^5 + (-2)^5] = f^{(5)}(a)[-32 + 8 + 8 - 32] = -48f^{(5)}(a)$.
12. Numerator $= 12f'(a)h + \frac{-48f^{(5)}(a)}{120}h^5 = 12f'(a)h - \frac{2}{5}f^{(5)}(a)h^5$.
13. $L_2(a,h) = \frac{12f'(a)h - \frac{2}{5}f^{(5)}(a)h^5}{12h} = f'(a) - \frac{1}{30}f^{(5)}(a)h^4$.
14. $L_2(a,h) - f'(a) = -\frac{1}{30}f^{(5)}(a)h^4$.
15. Substitute $f^{(5)}(a) = 120$:
$$L_2(a,h) - f'(a) = -\frac{1}{30}(120)h^4 = -4h^4.$$
16. $|L_2(a,h) - f'(a)| = \left|-4h^4\right| = 4h^4 = O(h^4)$ as $h \to 0$. *Comparison:*
17. The error for L_1 is $O(h^2)$, while the error for L_2 is $O(h^4)$.
18. L_2 converges to $f'(a)$ faster than L_1 as $h \to 0$.

Problem 5

For a real-valued function f, x is a fixed point if $f(x) = x$. Show that if f is differentiable on an interval with $f'(x) \neq 1$, then f can have at most one fixed point.

Relevant Definitions and Theorems

Definition (Fixed Point)**.** *A point x such that $f(x) = x$.*

Theorem (Mean Value Theorem (MVT) [12])**.** *If g is continuous on $[a, b]$ and differentiable on (a, b), then there exists $c \in (a, b)$ such that $g(b) - g(a) = g'(c)(b - a)$.*

Theorem (Rolle's Theorem [13])**.** *If g is continuous on $[a, b]$, differentiable on (a, b), and $g(a) = g(b)$, then there exists $c \in (a, b)$ such that $g'(c) = 0$.*

Solution Outline

Method 1: Using MVT directly

1. Assume for contradiction that f has two distinct fixed points, x_1 and x_2, with $x_1 < x_2$.
2. By definition, $f(x_1) = x_1$ and $f(x_2) = x_2$.
3. f is differentiable on an interval I containing x_1, x_2. Thus f is continuous on $[x_1, x_2]$ and differentiable on (x_1, x_2).
4. Apply MVT to f on $[x_1, x_2]$. There exists $c \in (x_1, x_2)$ such that
$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$
5. Substitute $f(x_1) = x_1$ and $f(x_2) = x_2$:
$$f'(c) = \frac{x_2 - x_1}{x_2 - x_1}.$$
6. Since $x_1 \neq x_2$, $x_2 - x_1 \neq 0$. Therefore, $f'(c) = 1$.
7. This contradicts the hypothesis that $f'(x) \neq 1$ for all $x \in I$. Since $c \in (x_1, x_2) \subseteq I$, we must have $f'(c) \neq 1$.
8. The assumption of two distinct fixed points leads to a contradiction.
9. Therefore, f can have at most one fixed point.

Method 2: Using Rolle's Theorem

1. Define $g(x) = f(x) - x$. Fixed points of f are roots of g.
2. g is differentiable on I since $f(x)$ and x are.
3. $g'(x) = f'(x) - 1$.
4. The hypothesis $f'(x) \neq 1$ implies $g'(x) = f'(x) - 1 \neq 0$ for all $x \in I$.
5. Assume for contradiction that f has two distinct fixed points, x_1 and x_2, with $x_1 < x_2$.
6. Then $g(x_1) = f(x_1) - x_1 = 0$ and $g(x_2) = f(x_2) - x_2 = 0$.
7. g is continuous on $[x_1, x_2]$, differentiable on (x_1, x_2), and $g(x_1) = g(x_2) = 0$.

[12] See Theorem 29.3 in Ross, K. A. *Elementary Analysis: The Theory of Calculus*. Springer, 2013.
[13] See Theorem 29.2 in Ross, K. A. *Elementary Analysis: The Theory of Calculus*. Springer, 2013.

8. By Rolle's Theorem, there exists $c \in (x_1, x_2)$ such that $g'(c) = 0$.
9. This contradicts the fact that $g'(x) \neq 0$ for all $x \in I$.
10. The assumption of two distinct fixed points leads to a contradiction.
11. Therefore, f can have at most one fixed point.

Problem 6

Find the following limits if they exist: (a) $\lim_{x\to 0} \frac{x^3}{\sin x - x}$ (b) $\lim_{x\to 0} \frac{\tan x - x}{x^3}$ (c) $\lim_{x\to 0} \left(\frac{1}{\sin x} - \frac{1}{x}\right)$ (d) $\lim_{x\to 0} \frac{1-\cos x}{e^{3x}-3x-1}$

Relevant Theorems

Theorem (L'Hôpital's Rule [14])**.** *If* $\lim f(x) = \lim g(x) = 0$ *(or* $\pm\infty$*) and* $\lim \frac{f'(x)}{g'(x)}$ *exists, then* $\lim \frac{f(x)}{g(x)} = \lim \frac{f'(x)}{g'(x)}$.

Taylor Series Expansions around 0:

- $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$
- $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$
- $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \dots$
- $e^u = 1 + u + \frac{u^2}{2!} + \frac{u^3}{3!} + \dots$

Solution Outline

(a) $\lim_{x\to 0} \frac{x^3}{\sin x - x}$

- Taylor: $\sin x - x = (x - \frac{x^3}{6} + O(x^5)) - x = -\frac{x^3}{6} + O(x^5)$.

$$\lim_{x\to 0} \frac{x^3}{-\frac{x^3}{6} + O(x^5)} = \lim_{x\to 0} \frac{1}{-\frac{1}{6} + O(x^2)} = -6.$$

- L'Hôpital (form $\frac{0}{0}$):

$$\lim_{x\to 0} \frac{3x^2}{\cos x - 1} \stackrel{L'H}{=} \lim_{x\to 0} \frac{6x}{-\sin x} \stackrel{L'H}{=} \lim_{x\to 0} \frac{6}{-\cos x} = \frac{6}{-1} = -6.$$

(b) $\lim_{x\to 0} \frac{\tan x - x}{x^3}$

- Taylor: $\tan x - x = (x + \frac{x^3}{3} + O(x^5)) - x = \frac{x^3}{3} + O(x^5)$.

$$\lim_{x\to 0} \frac{\frac{x^3}{3} + O(x^5)}{x^3} = \lim_{x\to 0} (\frac{1}{3} + O(x^2)) = \frac{1}{3}.$$

- L'Hôpital (form $\frac{0}{0}$):

$$\lim_{x\to 0} \frac{\sec^2 x - 1}{3x^2} \stackrel{L'H}{=} \lim_{x\to 0} \frac{2\sec^2 x \tan x}{6x} = \lim_{x\to 0} \frac{\sec^2 x}{3} \cdot \frac{\tan x}{x} = \frac{1}{3} \cdot 1 = \frac{1}{3}.$$

[14] See Theorems 30.3, 30.4 in Ross, K. A. *Elementary Analysis: The Theory of Calculus*. Springer, 2013.

(c) $\lim_{x\to 0}\left(\frac{1}{\sin x} - \frac{1}{x}\right)$

- Combine: $\lim_{x\to 0}\frac{x-\sin x}{x\sin x}$. (Form $\frac{0}{0}$).
- Taylor: Numerator is $\frac{x^3}{6} + O(x^5)$. Denominator is $x(x - \frac{x^3}{6} + \dots) = x^2 - \frac{x^4}{6} + \dots$.

$$\lim_{x\to 0}\frac{\frac{x^3}{6} + O(x^5)}{x^2 + O(x^4)} = \lim_{x\to 0}\frac{x(\frac{1}{6} + O(x^2))}{1 + O(x^2)} = 0 \cdot \frac{1/6}{1} = 0.$$

- L'Hôpital:

$$\lim_{x\to 0}\frac{1-\cos x}{\sin x + x\cos x} \overset{L'H}{=} \lim_{x\to 0}\frac{\sin x}{\cos x + \cos x - x\sin x} = \frac{0}{1+1-0} = 0.$$

(d) $\lim_{x\to 0}\frac{1-\cos x}{e^{3x}-3x-1}$

- (Form $\frac{0}{0}$).
- Taylor: Numerator is $\frac{x^2}{2} + O(x^4)$. Denominator is $(1 + 3x + \frac{(3x)^2}{2} + O(x^3)) - 3x - 1 = \frac{9x^2}{2} + O(x^3)$.

$$\lim_{x\to 0}\frac{\frac{x^2}{2} + O(x^4)}{\frac{9x^2}{2} + O(x^3)} = \lim_{x\to 0}\frac{\frac{1}{2} + O(x^2)}{\frac{9}{2} + O(x)} = \frac{1/2}{9/2} = \frac{1}{9}.$$

- L'Hôpital:

$$\lim_{x\to 0}\frac{\sin x}{3e^{3x}-3} \overset{L'H}{=} \lim_{x\to 0}\frac{\cos x}{9e^{3x}} = \frac{1}{9}.$$

Problem 7

(a) Let $f : \mathbb{R} \to \mathbb{R}$ be twice differentiable, $f(0) = 0$, $f''(x) \geq 0$ for $x > 0$. Prove $g(x) = f(x)/x$ is increasing for $x > 0$.

(b) If $f : \mathbb{R} \to \mathbb{R}$ is twice differentiable, $f(0) = 0$ and $g(x) = f(x)/x$ is increasing for $x > 0$, show $f''(x) \geq 0$ for *some* $x > 0$, but not necessarily for *all* $x > 0$.

Relevant Definitions and Theorems

Definition (Increasing Function [15])**.** *g is increasing on an interval if $x_1 < x_2$ implies $g(x_1) \leq g(x_2)$.*

Theorem (Derivative Test for Increasing Function [16])**.** *If g is differentiable on (a, b) and $g'(x) \geq 0$ for all $x \in (a, b)$, then g is increasing on (a, b).*

Solution Outline

(a)

1. Define $g(x) = f(x)/x$ for $x > 0$. g is differentiable for $x > 0$.
2. We show $g'(x) \geq 0$ for $x > 0$.
3. Calculate $g'(x)$ using the quotient rule:

$$g'(x) = \frac{xf'(x) - f(x)}{x^2}.$$

4. Let $N(x) = xf'(x) - f(x)$. We need to show $N(x) \geq 0$ for $x > 0$.

[15] See Definition 29.1 in Ross, K. A. *Elementary Analysis: The Theory of Calculus*. Springer, 2013.
[16] See Corollary 29.7 in Ross, K. A. *Elementary Analysis: The Theory of Calculus*, 2nd ed. Springer, 2013.

5. Calculate $N'(x)$:
$$N'(x) = (1 \cdot f'(x) + xf''(x)) - f'(x) = xf''(x).$$

6. Given $f''(x) \geq 0$ for $x > 0$. Since $x > 0$, $N'(x) = xf''(x) \geq 0$ for $x > 0$.

7. Since $N'(x) \geq 0$, $N(x)$ is increasing on $(0, \infty)$.

8. Consider the limit of $N(x)$ as $x \to 0^+$:
$$\lim_{x \to 0^+} N(x) = \lim_{x \to 0^+} (xf'(x) - f(x)).$$
Since f is differentiable at 0 and $f(0) = 0$, $f'(0) = \lim_{x \to 0} f(x)/x$. Since f'' exists, f' is continuous.
$$\lim_{x \to 0^+} N(x) = (0 \cdot f'(0)) - f(0) = 0 - 0 = 0.$$

9. Since $N(x)$ is increasing on $(0, \infty)$ and $\lim_{x \to 0^+} N(x) = 0$, we have $N(x) \geq 0$ for all $x > 0$.

10. Therefore, $g'(x) = N(x)/x^2 \geq 0$ for $x > 0$.

11. By the theorem, $g(x) = f(x)/x$ is increasing on $(0, \infty)$.

(b)

1. Assume f is twice differentiable, $f(0) = 0$, and $g(x) = f(x)/x$ is increasing for $x > 0$.

2. From (a), $g'(x) = \frac{xf'(x) - f(x)}{x^2} \geq 0$ for $x > 0$.

3. Let $N(x) = xf'(x) - f(x)$. Then $N(x) \geq 0$ for $x > 0$.

4. Also $N'(x) = xf''(x)$ and $\lim_{x \to 0^+} N(x) = 0$.

5. **Show $f''(x) \geq 0$ for some $x > 0$:** Assume for contradiction that $f''(x) < 0$ for all $x > 0$. Then $N'(x) = xf''(x) < 0$ for all $x > 0$. This implies $N(x)$ is strictly decreasing on $(0, \infty)$. Since $\lim_{x \to 0^+} N(x) = 0$, this would mean $N(x) < 0$ for all $x > 0$. This contradicts $N(x) \geq 0$. Therefore, the assumption is false. There must exist some $x_0 > 0$ such that $f''(x_0) \geq 0$.

6. **Show $f''(x)$ is not necessarily ≥ 0 for all $x > 0$:** Consider $f(x) = x(1 - e^{-x}) = x - xe^{-x}$.
   - $f(0) = 0$. f is twice differentiable.
   - $g(x) = f(x)/x = 1 - e^{-x}$ for $x \neq 0$.
   - $g'(x) = e^{-x} > 0$ for all x. So $g(x)$ is increasing for $x > 0$.
   - $f'(x) = 1 - e^{-x} + xe^{-x}$.
   - $f''(x) = e^{-x} + (e^{-x} - xe^{-x}) = 2e^{-x} - xe^{-x} = (2 - x)e^{-x}$.
   - If $x > 2$, then $2 - x < 0$, so $f''(x) < 0$. For example, $f''(3) = -e^{-3} < 0$.
   - Thus, $f''(x)$ is not non-negative for all $x > 0$.

7. This example satisfies the conditions but $f''(x)$ is negative for $x > 2$. It does satisfy $f''(x) \geq 0$ for some $x > 0$ (e.g., for $x \in (0, 2]$).