Math 521: Assignment 7 (due 5 PM, May 2)

At this point in the course, being able to sketch functions and determine their properties is an important skill, which greatly helps in understanding concepts such as continuity, differentiability, and uniform convergence. The first section of this homework is devoted to this topic.

For questions 1–3 and 4(a), no detailed proofs are required, although you will need to provide some discussion in words about what is going on. To begin, I would like you to try and draw the graphs by hand. There are many ways to do this, such as looking at the behavior as $x \to \pm\infty$, calculating a few specific points and drawing a line through them, using calculus, or searching for zeroes of the function. After this, you can confirm your results using a plotting program if you wish. There are many free ones available, such as *Gnuplot* (`http://gnuplot.info`) or *Matplotlib* (`https://matplotlib.org`).

1. Consider the function

$$f(x) = \begin{cases} 1 - |x-1| & \text{if } 0 \le x \le 2 \\ 0 & \text{if } x > 2 \end{cases}$$

defined on the interval $[0, \infty)$[2]. Draw $f(x)$.

   (a) Draw $f(x/2)$, $f(x/3)$, and $f(x/4)$, and explain how the shapes of these curves relate to $f(x)$.

   (b) Draw $2f(x)$, $f(x + {}^1\!/\!_2)$, $f(x) - {}^1\!/\!_2$ and explain how the shapes of these curves relate to $f(x)$.

   (c) Draw $|f(x) - {}^1\!/\!_2|$. Is this function continuous? Is it differentiable everywhere?

   (d) Draw $f(x^2)$ and $f(x)^2$.

2. Consider the sequence of functions

$$f_n(x) = \frac{nx^2}{1 + nx^2}$$

defined on the interval $[0, \infty)$.

   (a) Begin by considering $f_1(x)$. How does it behave as $x \to \infty$? How does it look close to $x = 0$? Use these facts to draw $f_1(x)$.

   (b) Show that $f_n(x) = f_1(\sqrt{n}x)$. By considering question 1(a), use this fact to draw several of the $f_n(x)$.

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[2] If you try *Gnuplot*, you can define this function by typing `f(x)=x>2?0:1-abs(x-1)`. It can then be plotted with `plot [0:4] f(x)`.

(c) It can be shown that f_n converges pointwise to a function f defined on $[0, \infty)$ as

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0. \end{cases}$$

Draw $f(x)$ and draw a strip of width $\epsilon = 1/4$ around $f(x)$. If $f_n \to f$ uniformly, then there exists an N such that $n > N$ implies that f_n lies wholly within this strip. Use the graph to explain in words why no such N exists, so that f_n does not converge uniformly to f.

3. Consider the sequence of functions defined on $\mathbb{R}$ as

$$f_n(x) = \begin{cases} x^n \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

(a) Draw the sequence of functions $f_0(x)$, $f_1(x)$ and $f_2(x)$. Which of the functions are continuous at $x = 0$? Which of them are differentiable at $x = 0$?

(b) Consider the functions f_n on the interval $[-1/2, 1/2]$, and define $f(x) = 0$. By considering a strip of width ϵ around $f(x)$, explain why f_n will converge uniformly to f on this interval.

4. Consider the function $g_0(x) = |x|$ on $\mathbb{R}$. For $n \in \mathbb{N}$, define $g_n(x) = |g_{n-1}(x) - 2^{1-n}|$.

(a) Draw $g_0(x)$, $g_1(x)$, $g_2(x)$, and $g_3(x)$.

(b) Prove that the functions g_n converge uniformly to a limit g on $\mathbb{R}$.

5. Suppose f is a continuous function on $[a, b]$, and $f(x) \geq 0$ for all $x \in [a, b]$. Prove that if $\int_a^b f = 0$, then $f(x) = 0$ for all $x \in [a, b]$.

6. Consider the function

$$f(x) = \begin{cases} 1 & \text{if } x = 2^{-n} \text{ for } n \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Prove that f is integrable on $[0, 1]$ and that $\int_0^1 f = 0$.

7. Construct an example of a function where $f(x)^2$ is integrable on $[0, 1]$ but $f(x)$ is not.

8. Construct an example of a sequence of functions (f_n) on $[0, 1]$ such that $f_n \to 0$ pointwise, but the sequence $s_n = \int_0^1 f_n$ diverges.

9. (a) Suppose that g is integrable on $[0, 1]$ and continuous at 0. Prove that

$$\lim_{n \to \infty} \int_0^1 g(x^n) dx = g(0). \tag{2}$$

(b) Show that the condition in part (a) that g is continuous is necessary for Eq. (2) to be true.

10. Suppose that f is continuous on (a, b), where a may be $-\infty$ and b may be ∞. If $\int_a^b |f(x)| dx < \infty$, show that the integral $\int_a^b f(x) dx$ exists and is finite.

11. (a) Suppose that the real-valued function f satisfies $|f(x)| \leq M$ for all $x \in [a, b]$. Show that

$$\left| f(x)^2 - f(y)^2 \right| \leq 2M |f(x) - f(y)|. \tag{3}$$

(b) Prove that if f is integrable on $[a, b]$, then so is f^2.

(c) Suppose that f and g are integrable on $[a, b]$. By considering $(f + g)^2$, or otherwise, show that fg is also integrable on $[a, b]$.

12. (a) For any two numbers $u, v \in \mathbb{R}$, prove that $uv \leq (u^2 + v^2)/2$. Let f and g be two integrable functions on $[a, b]$. Prove that if $\int_a^b f^2 = 1$ and $\int_a^b g^2 = 1$ then

$$\int_a^b fg \leq 1. \tag{4}$$

(b) Prove the Schwarz inequality, that for any two integrable functions f and g on an interval $[a, b]$,

$$\left| \int_a^b fg \right| \leq \left(\int_a^b f^2 \right)^{1/2} \left(\int_a^b g^2 \right)^{1/2}. \tag{5}$$

(c) Let X be the set of all continuous functions on the interval $[a, b]$. For any $f, g \in X$, define

$$d(f, g) = \left(\int_a^b |f - g|^2 \right)^{1/2}. \tag{6}$$

Prove that d is a metric.

Further optional exercises

13. **Optional.** Define the sequence of functions on $[0,1]$ as $h_0(x) = |x - 1/2|$ and $h_n(x) = |h_{n-1}(x) - 3^{-n}|$. Draw several of the h_n. Prove that h_n converges uniformly to a limit h. *Difficult:* where is h differentiable?

14. **Optional.** Given a function f on $[a,b]$ define the *total variation* to be

$$Vf = \sup \left\{ \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| \right\}, \tag{7}$$

where the supremum is taken over all partitions $P = \{a = x_0 < x_1 < \ldots < x_n = b\}$ of $[a,b]$.

(a) If f is continuously differentiable use the fundamental theorem of calculus to show $Vf \leq \int_a^b |f'|$.

(b) Use the mean value theorem to show that $Vf \geq \int_a^b |f'|$ and hence that $Vf = \int_a^b |f'|$.

15. **Optional.**

(a) By using simple properties of $\sin x$ and $\cos x$, show how to define the function $\tan : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$. Prove that it is differentiable, strictly increasing, and neither bounded above nor below.

(b) By using inverse function theorems, define $\tan^{-1} : \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$ and show that

$$(\tan^{-1})'(x) = \frac{1}{1+x^2}. \tag{8}$$

(c) Prove that for $|x| < 1$,

$$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}. \tag{9}$$

(d) By making use of Abel's theorem, or otherwise, show that

$$\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}. \tag{10}$$

(e) Calculate $(5+i)^4(239-i)$ and use it to prove Machin's formula

$$\frac{\pi}{4} = 4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239}. \tag{11}$$