Math 521: Assignment 5 (due 5 PM, April 4)

1. A real-valued function f on an interval I is said to be *Lipschitz continuous* if there exists an $L > 0$ such that for all $x, y \in I$,
$$|f(x) - f(y)| \leq L|x - y|. \tag{1}$$
   (a) Show that if a function is Lipschitz continuous, then it is uniformly continuous.
   (b) Find an example of a function g defined on an interval I that is uniformly continuous but not Lipschitz continuous.

2. (a) Let S be a subset of , and let $f : S \to$ and $g :\to$ be uniformly continuous functions. Prove that the composition $g \circ f : S \to$ is uniformly continuous.
   (b) Let f and g be two uniformly continuous functions from S to . Prove that $f + g$ is uniformly continuous.
   (c) Show that there exist uniformly continuous functions f and g from S to such that the multiplication $f \cdot g$ is not uniformly continuous.

3. Let f be a uniformly continuous real-valued function on . Prove that there are constants A and B such that $|f(x)| \leq A + B|x|$ for all $x \in$.

4. (a) Sketch the function $f(x) = (x + 1)^{-2}(x - 2)^{-1}$.
   (b) Determine $\lim_{x\to 2^+} f(x)$, $\lim_{x\to 2^-} f(x)$, $\lim_{x\to -1^+} f(x)$, and $\lim_{x\to -1^-} f(x)$.
   (c) Determine $\lim_{x\to 2} f(x)$ and $\lim_{x\to -1} f(x)$ if they exist.

5. Suppose that the limits $L_1 = \lim_{x\to a^+} f_1(x)$ and $L_2 = \lim_{x\to a^+} f_2(x)$ exist.
   (a) Prove that if $f_1(x) \leq f_2(x)$ for some interval (a, b), then $L_1 \leq L_2$.
   (b) Suppose that $f_1(x) < f_2(x)$ for some interval (a, b). Is it always true that $L_1 < L_2$?

6. For each of the following power series, find the radius of convergence and determine the exact interval of convergence:
   (a) $\sum_n n^2 x^n$
   (b) $\sum_n \left(\frac{x}{n}\right)^n$
   (c) $\sum_n x^{n!}$
   (d) $\sum_n 5^n x^{2n+1}$

7. For $x \in [0, \infty)$, define $f_n(x) = \frac{x}{n}$.
   (a) Find $f(x) = \lim_{n\to\infty} f_n(x)$.
   (b) Determine whether $f_n \to f$ uniformly on $[0, 1]$.

(c) Determine whether $f_n \to f$ uniformly on $[0, \infty)$.

8. (a) Define a sequence of functions on as

$$f_n(x) = \begin{cases} 1 & \text{if } x = 1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \\ 0 & \text{otherwise,} \end{cases} \tag{2}$$

and let f be the pointwise limit of f_n. Is each f_n continuous at 0? Does $f_n \to f$ uniformly on ? Is f continuous at 0?

(b) Repeat part (a) for the sequence of functions

$$g_n(x) = \begin{cases} x & \text{if } x = 1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$