Physics 415 - Lecture 32: Fermi Gas

April 11, 2025

Summary

- Fermi-Dirac (FD) statistics: Mean occupation number $\overline{n}_r = \frac{1}{e^{\beta(\epsilon_r - \mu)}+1}$. Grand Potential $\Phi = -T\sum_r \ln(1+e^{-\beta(\epsilon_r-\mu)})$.
- Density of States (DOS) for free particles (spin J, degeneracy $g = 2J+1$) in volume V:

$$\sum_r \to g\int d\epsilon\, \rho(\epsilon)$$

where $\rho(\epsilon) = \frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\sqrt{\epsilon}$. $\rho(\epsilon)d\epsilon = \#$ of spatial states in energy range $(\epsilon, \epsilon + d\epsilon)$.

So far, we developed the general theory of ideal QM gases. Now focus on the specific case of FD statistics.

Fermi Gas (FG)

A system of non-interacting fermions. Examples: conduction electrons (e^-) in a metal, white dwarf stars, neutron stars, liquid ^{3}He, ...

The mean occupation number $\overline{n}(\epsilon)$ plays a central role:

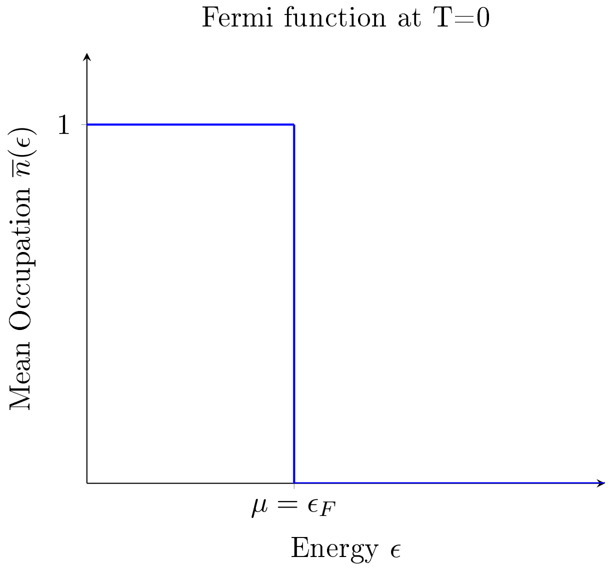
$$\overline{n}(\epsilon) = \frac{1}{e^{\beta(\epsilon-\mu)}+1} \quad \text{(Fermi function)}$$

Fermi Gas at T=0

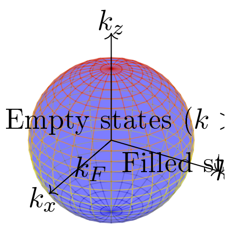
At $T = 0$ ($\beta \to \infty$), the FG will be in its QM ground state. Due to the Pauli exclusion principle, fermions successively fill up the single-particle states starting from the lowest energy. The Fermi function becomes a step function:

$$\lim_{T\to 0}\overline{n}(\epsilon) = \begin{cases} 1 & \text{if } \epsilon < \mu(T=0) \\ 0 & \text{if } \epsilon > \mu(T=0) \end{cases}$$

The chemical potential at $T = 0$ is called the **Fermi energy**, denoted by $\epsilon_F \equiv \mu(T=0)$. It is the energy of the highest occupied state at $T = 0$.



The filled states with $\epsilon < \epsilon_F$ form the **"Fermi sea"**. In $\vec{k}$-space, the Fermi sea corresponds to filling all $\vec{k}$ states up to a limiting wave-vector k_F (Fermi wave-vector). The surface $|\vec{k}| = k_F$ is the **Fermi surface**. Relation: $\epsilon_F = \frac{\hbar^2 k_F^2}{2m}$. The Fermi momentum is $p_F = \hbar k_F$.



The Fermi wave-vector k_F (and hence ϵ_F) is determined by the total particle number N:

$$N = \sum_r \overline{n}_r \xrightarrow{T=0} \sum_{r \text{ with } \epsilon_r < \epsilon_F} (g \times 1)$$

Using the integral form:

$$N = gV \int_{k<k_F} \frac{d^3k}{(2\pi)^3}$$

The integral is the volume of a sphere of radius k_F in k-space, which is $\frac{4}{3}\pi k_F^3$.

$$N = gV \frac{1}{(2\pi)^3} \left(\frac{4}{3}\pi k_F^3 \right) = \frac{gVk_F^3}{6\pi^2}$$

The particle density is $n = N/V = \frac{gk_F^3}{6\pi^2}$.

$$k_F = \left(\frac{6\pi^2 n}{g} \right)^{1/3}$$

The Fermi energy is:

$$\epsilon_F = \frac{\hbar^2 k_F^2}{2m} = \frac{\hbar^2}{2m} \left(\frac{6\pi^2 n}{g} \right)^{2/3}$$

(For electrons, spin $J = 1/2$, so $g = 2J + 1 = 2$. $k_F = (3\pi^2 n)^{1/3}$).

Alternatively, use the DOS $\rho(\epsilon)$:

$$N = g\int_0^{\epsilon_F} \rho(\epsilon)d\epsilon \quad (\text{since } \overline{n} = 1 \text{ for } \epsilon < \epsilon_F, 0 \text{ otherwise})$$

$$N = g\left(\frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\right)\int_0^{\epsilon_F}\sqrt{\epsilon}d\epsilon = \frac{gV}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\left[\frac{2}{3}\epsilon^{3/2}\right]_0^{\epsilon_F}$$

$$N = \frac{gV}{6\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\epsilon_F^{3/2}$$

Solving for ϵ_F: $\epsilon_F^{3/2} = \frac{6\pi^2 N}{gV}\left(\frac{\hbar^2}{2m}\right)^{3/2}$.

$$\epsilon_F = \left(\frac{6\pi^2 n}{g}\right)^{2/3}\left(\frac{\hbar^2}{2m}\right) \checkmark$$

It is useful to express the DOS at the Fermi energy, $\rho(\epsilon_F)$, in terms of N and ϵ_F. From $N = \frac{2}{3}g(AV\epsilon_F^{1/2})\epsilon_F$ where $AV\epsilon_F^{1/2} = \rho(\epsilon_F)$: $N = \frac{2}{3}g\rho(\epsilon_F)\epsilon_F$.

$$\rho(\epsilon_F) = \frac{3N}{2g\epsilon_F}$$

Total Energy at T=0 (E_0):

$$E_0 = \sum_r \overline{n}_r\epsilon_r = g\int_0^{\epsilon_F}\rho(\epsilon)\epsilon d\epsilon$$

$$E_0 = g\left(\frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\right)\int_0^{\epsilon_F}\epsilon^{3/2}d\epsilon$$

$$E_0 = gAV\left[\frac{2}{5}\epsilon^{5/2}\right]_0^{\epsilon_F} = \frac{2}{5}gAV\epsilon_F^{5/2}$$

Substitute $gAV = \frac{3N}{(2/3)\epsilon_F^{3/2}}$? No, substitute $gAV = \frac{3N}{(2/3)\epsilon_F^{3/2}} \times \frac{2}{3} = \frac{3N}{\epsilon_F^{3/2}}$. $E_0 = \frac{2}{5}\left(\frac{3N}{\epsilon_F^{3/2}}\right)\epsilon_F^{5/2} = \frac{3}{5}N\epsilon_F$. The total energy $E_0 = \frac{3}{5}N\epsilon_F$ is non-zero even at $T=0$. This is purely a QM effect due to the Pauli principle. (Classical ideal gas has $E=0$ at $T=0$).

Pressure at T=0 (p_0): Using the general relation $pV = \frac{2}{3}E$ for non-relativistic free particles:

$$p_0 = \frac{2}{3}\frac{E_0}{V} = \frac{2}{3V}\left(\frac{3}{5}N\epsilon_F\right) = \frac{2}{5}\frac{N}{V}\epsilon_F = \frac{2}{5}n\epsilon_F$$

The pressure of a Fermi gas is non-zero even at $T=0$. This is the "degeneracy pressure", arising from the kinetic energy forced upon the fermions by the exclusion principle. (Classical ideal gas $p = nT \to 0$ as $T \to 0$).

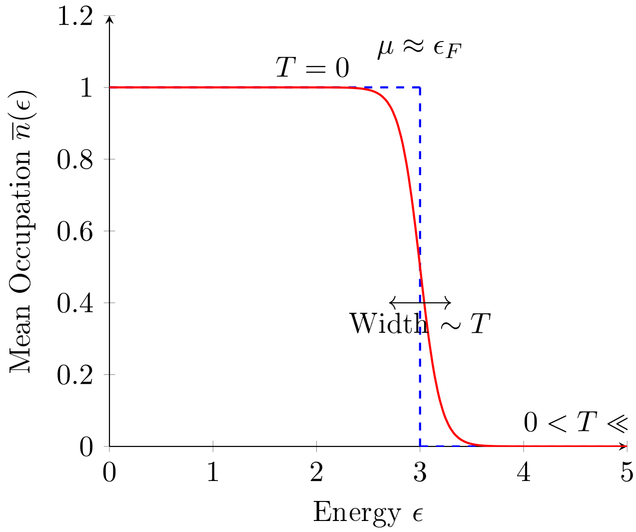
Fermi Gas at T>0

The important temperature scale is the Fermi energy ϵ_F (or Fermi temperature $T_F = \epsilon_F/k_B$).

- Low-T "degenerate" regime: $T \ll T_F$. Quantum effects dominate. Condition $n\lambda_{th}^3 \gg 1$.
- High-T "non-degenerate" regime: $T \gg T_F$. Classical limit applies. Condition $n\lambda_{th}^3 \ll 1$.

For electrons in metals, $T_F \sim 10^4 - 10^5$ K. So at room temperature ($T \approx 300$ K), they are highly degenerate and QM effects dominate.

Consider the Fermi function $\overline{n}(\epsilon)$ for $0 < T \ll T_F$. The step function at $T = 0$ broadens into a smooth curve over an energy range of order T (or $k_B T$) around $\epsilon = \mu$. For $T \ll T_F$, the chemical potential μ is only slightly different from ϵ_F.



Qualitatively: When $T \ll T_F$, only particles within an energy range of order T near the Fermi surface ($\epsilon \approx \epsilon_F$) can be thermally excited into available empty states just above ϵ_F. States deep in the Fermi sea ($\epsilon \ll \epsilon_F$) are "blocked" by occupied states above them (Pauli exclusion).

The "effective" number of particles that participate in thermal activity is approximately:

$$N_{eff} \approx (\text{Density of states near } \epsilon_F) \times (\text{Energy range})$$

$$N_{eff} \sim \rho(\epsilon_F) \times T$$

Using $\rho(\epsilon_F) \sim N/\epsilon_F$:

$$N_{eff} \sim \frac{N}{\epsilon_F} T = N \left(\frac{T}{T_F} \right)$$

Since $T \ll T_F$, $N_{eff} \ll N$. Only a small fraction of fermions contribute to thermal properties like heat capacity at low T.

Low Temperature Thermodynamics (Estimates)

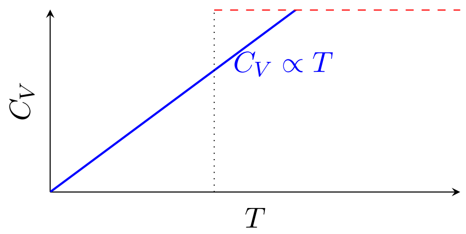
Estimate the energy $E(T)$ at low T: $E(T) \approx E_0 +$ (Energy added to excited particles). Each of the N_{eff} particles gains roughly energy T.

$$E(T) \approx E_0 + N_{eff} \times T \sim E_0 + N \left(\frac{T}{T_F} \right) T = E_0 + \frac{NT^2}{T_F}$$

(Here T, T_F are in energy units). Heat capacity C_V:

$$C_V = \left(\frac{\partial E}{\partial T} \right)_V \sim \frac{\partial}{\partial T} \left(E_0 + \frac{NT^2}{T_F} \right) = \frac{2NT}{T_F} \sim N \left(\frac{T}{T_F} \right)$$

So $C_V \propto T$ at low temperatures.



This linear dependence $C_V \propto T$ is characteristic of degenerate Fermi gases (e.g., electron contribution in metals) and contrasts sharply with the classical result ($C_V = \frac{3}{2}Nk_B$, constant) or lattice vibrations ($C_V \propto T^3$).

To make these estimates more precise requires evaluating integrals of the form

$$I = \int_0^\infty d\epsilon\, f(\epsilon)\overline{n}(\epsilon) = \int_0^\infty d\epsilon \frac{f(\epsilon)}{e^{\beta(\epsilon-\mu)} + 1}$$

in the degenerate regime $T \ll T_F$. This involves the Sommerfeld expansion (next lecture).