Physics 415 - Lecture 2

January 24, 2025

Summary: Binomial Distribution

The binomial distribution gives the likelihood that an event with probability p occurs n times in N independent trials:

$$P_N(n) = \binom{N}{n} p^n q^{N-n} = \frac{N!}{n!(N-n)!} p^n (1-p)^{N-n}$$

where $q = 1 - p$.

- Mean: $\overline{n} = Np$
- Variance: $\sigma_n^2 = \overline{n^2} - (\overline{n})^2 = Npq$
- Relative width (RMS deviation / mean): $\frac{\Delta n_{rms}}{\overline{n}} = \frac{\sqrt{Npq}}{Np} = \sqrt{\frac{q}{p}} \frac{1}{\sqrt{N}}$

Important regime to understand is $N \gg 1$.

Gaussian Approximation to Binomial Distribution ($N \gg 1$)

It is convenient to work with $\ln P_N(n)$:

$$\ln P_N(n) = \ln N! - \ln n! - \ln(N-n)! + n \ln p + (N-n) \ln q$$

For $N \gg 1$, we use Stirling's formula (see Reif App. A.6 for derivation):

$$N! \approx \sqrt{2\pi N} \left(\frac{N}{e}\right)^N \quad (N \gg 1)$$

or more conveniently for logarithms:

$$\ln N! \approx N \ln N - N + \frac{1}{2} \ln(2\pi N)$$

Applying this to $\ln N!$, $\ln n!$, and $\ln(N-n)!$ (assuming $n \gg 1$ and $N - n \gg 1$):

$$\begin{aligned}
\ln P_N(n) \approx & \left(N \ln N - N + \tfrac{1}{2} \ln(2\pi N)\right) \\
& - \left(n \ln n - n + \tfrac{1}{2} \ln(2\pi n)\right) \\
& - \left((N-n) \ln(N-n) - (N-n) + \tfrac{1}{2} \ln(2\pi(N-n))\right) \\
& + n \ln p + (N-n) \ln q
\end{aligned}$$

Grouping terms:

$$\ln P_N(n) \approx \tfrac{1}{2} \ln \left(\frac{2\pi N}{2\pi n \cdot 2\pi(N-n)}\right) + N \ln N - n \ln n - (N-n) \ln(N-n) + n \ln p + (N-n) \ln q$$

Let $x = n/N$. Then $n = Nx$ and $N - n = N(1-x)$.

$$\begin{aligned}\ln P_N(n) \approx & \tfrac{1}{2}\ln\left(\frac{N}{2\pi n(N-n)}\right) \\ & + N\ln N - Nx\ln(Nx) - N(1-x)\ln(N(1-x)) \\ & + Nx\ln p + N(1-x)\ln q\end{aligned}$$

$$\begin{aligned}\ln P_N(n) \approx & \tfrac{1}{2}\ln\left(\frac{N}{2\pi n(N-n)}\right) \\ & + N\ln N - Nx(\ln N + \ln x) - N(1-x)(\ln N + \ln(1-x)) \\ & + Nx\ln p + N(1-x)\ln q\end{aligned}$$

$$\begin{aligned}\ln P_N(n) \approx & \tfrac{1}{2}\ln\left(\frac{N}{2\pi n(N-n)}\right) \\ & + (N - Nx - N(1-x))\ln N \\ & - N[x\ln x + (1-x)\ln(1-x)] \\ & + N[x\ln p + (1-x)\ln q]\end{aligned}$$

The $\ln N$ terms cancel. Let $f(x) = [x\ln x + (1-x)\ln(1-x)] - [x\ln p + (1-x)\ln q]$. Then $\ln P_N(n) \approx \frac{1}{2}\ln\left[\frac{N}{2\pi n(N-n)}\right] - Nf(n/N)$.

$$\implies P_N(n) \approx \sqrt{\frac{N}{2\pi n(N-n)}}e^{-Nf(n/N)} \quad (\text{for } n \gg 1, N-n \gg 1)$$

For N large, $P_N(n)$ will be sharply peaked near its maximum at $n = \tilde{n}$. We seek an approximation for $P_N(n)$ near $n = \tilde{n}$.

The maximum of $P_N(n)$ corresponds to the minimum of $f(x)$. We find the minimum by setting $f'(x) = \frac{\mathrm{d}f}{\mathrm{d}x} = 0$.

$$f'(x) = [\ln x + 1 - \ln(1-x) - 1] - [\ln p - \ln q] = \ln\left(\frac{x}{1-x}\right) - \ln\left(\frac{p}{q}\right) = \ln\left(\frac{qx}{p(1-x)}\right)$$

Setting $f'(\tilde{x}) = 0 \implies \frac{q\tilde{x}}{p(1-\tilde{x})} = 1 \implies q\tilde{x} = p(1-\tilde{x}) \implies (q+p)\tilde{x} = p \implies \tilde{x} = p$. So the peak occurs at $\tilde{x} = \tilde{n}/N = p \implies \tilde{n} = Np$, which is the mean value.

Now expand $f(x)$ about $x = \tilde{x} = p$:

$$f(x) \approx f(\tilde{x}) + f'(\tilde{x})(x-\tilde{x}) + \frac{1}{2}f''(\tilde{x})(x-\tilde{x})^2$$

We have $f'(\tilde{x}) = 0$.

$$f(\tilde{x}) = p\ln p + q\ln q - (p\ln p + q\ln q) = 0$$

We need the second derivative:

$$f''(x) = \frac{\mathrm{d}}{\mathrm{d}x}\ln\left(\frac{qx}{p(1-x)}\right) = \frac{p(1-x)}{qx}\cdot\frac{qp(1-x) - qx(-p)}{(p(1-x))^2} = \frac{p(1-x)}{qx}\frac{pq}{(p(1-x))^2} = \frac{q}{x(1-x)p}$$

$$f''(x) = \frac{\mathrm{d}}{\mathrm{d}x}(\ln x - \ln(1-x) - (\ln p - \ln q)) = \frac{1}{x} - \frac{1}{1-x}(-1) = \frac{1}{x} + \frac{1}{1-x} = \frac{1}{x(1-x)}$$

At $x = \tilde{x} = p$: $f''(p) = \frac{1}{p(1-p)} = \frac{1}{pq}$.

So, $f(x) \approx \frac{1}{2}f''(p)(x-p)^2 = \frac{1}{2pq}(n/N-p)^2 = \frac{(n-Np)^2}{2N^2pq}$. The exponent becomes $-Nf(n/N) \approx -N\frac{(n-Np)^2}{2N^2pq} = -\frac{(n-Np)^2}{2Npq}$.

$$\implies P_N(n) \approx \sqrt{\frac{N}{2\pi n(N-n)}}e^{-\frac{(n-Np)^2}{2Npq}}$$

Finally, because the exponential factor is sharply peaked at $n = \tilde{n} = Np$, we may approximate the n dependence in the prefactor by replacing n with $\tilde{n} = Np$ and $N-n$ with $N - \tilde{n} = Nq$:

$$n(N-n) \approx (Np)(Nq) = N^2pq$$

$$\sqrt{\frac{N}{2\pi n(N-n)}} \approx \sqrt{\frac{N}{2\pi N^2pq}} = \sqrt{\frac{1}{2\pi Npq}} = \frac{1}{\sqrt{2\pi\sigma_n^2}}$$

where $\sigma_n^2 = Npq$ is the variance. Thus, the Gaussian approximation to the binomial distribution for $N \gg 1$ is:

$$P_N(n) \approx \frac{1}{\sqrt{2\pi Npq}}e^{-\frac{(n-Np)^2}{2Npq}}$$

This can be written as:

$$P_N(n) \approx \frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(n-\mu)^2}{2\sigma^2}}$$

where $\mu = Np$ (mean) and $\sigma^2 = Npq$ (variance). This is the Gaussian (or Normal) distribution. This result is an example of the "central limit theorem".

The Gaussian distribution can be shown to be properly normalized when treated as a continuous distribution (replacing sum by integral for large N):

$$\sum_{n=0}^{N} P_N(n) \approx \int_{-\infty}^{\infty} \mathrm{d}n P_N(n) = \int_{-\infty}^{\infty} \mathrm{d}n \frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(n-\mu)^2}{2\sigma^2}} = 1$$

(See Section 1 of notes for Gaussian integrals).

Distributions with Multiple Variables

Consider two random variables u and v (generalization to more variables is straightforward).

- Possible values: $\{u_i\}, i = 1, \ldots, M$; $\{v_j\}, j = 1, \ldots, L$.
- Joint probability: $P(u_i, v_j) =$ prob that $u = u_i$ and $v = v_j$.
- Normalization: $\sum_{i=1}^{M}\sum_{j=1}^{L} P(u_i, v_j) = 1$.
- "Unconditional" probability distributions (marginal distributions):
  - $P_u(u_i) = \sum_{j=1}^{L} P(u_i, v_j) =$ prob $u = u_i$, irrespective of v.
  - $P_v(v_j) = \sum_{i=1}^{M} P(u_i, v_j) =$ prob $v = v_j$, irrespective of u.

Statistical Independence

An important special case is when the probability that one variable assumes a certain value is independent of the value assumed by the other variable. The variables are "statistically independent" or "uncorrelated". In this case:

$$P(u_i, v_j) = P_u(u_i)P_v(v_j) \quad \text{(for statistically independent } u, v)$$

Mean Values

The mean of a function $F(u, v)$ is:

$$\overline{F(u,v)} = \sum_{i=1}^{M} \sum_{j=1}^{L} P(u_i, v_j) F(u_i, v_j)$$

A special case is the mean of a product $f(u) \times g(v)$ when u and v are statistically independent:

$$\begin{aligned}
\overline{f(u)g(v)} &= \sum_{i,j} P(u_i, v_j) f(u_i) g(v_j) \\
&= \sum_{i,j} P_u(u_i) P_v(v_j) f(u_i) g(v_j) \quad \text{(using independence)} \\
&= \left(\sum_{i=1}^{M} P_u(u_i) f(u_i) \right) \times \left(\sum_{j=1}^{L} P_v(v_j) g(v_j) \right) \\
&= \overline{f(u)} \times \overline{g(v)}
\end{aligned}$$

$$\implies \overline{f(u)g(v)} = \overline{f(u)} \times \overline{g(v)} \quad \text{if } u, v \text{ are statistically independent}$$

The average "factorizes". (This is not true in general!)

Continuous Probability Distributions

We will often encounter "continuous" probability distributions, where a random variable X can assume a continuous range of values, e.g., $a_1 < x < a_2$. Example: Gaussian distribution $\mathcal{P}(()x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ for $-\infty < x < \infty$.

For a continuous distribution, it does not make sense to consider the probability of X taking any particular value (which would be vanishingly small). Rather, we consider the probability that the random variable lies in a small range between x and $x + \mathrm{d}x$.

- $\mathcal{P}(()x)\mathrm{d}x$ = probability to find the random variable in the range $(x, x + \mathrm{d}x)$.
- $\mathcal{P}(()x)$ is the "probability density".
- Normalization: $\int_{a_1}^{a_2} \mathrm{d}x \mathcal{P}(()x) = 1$. (cf. $\sum P(x_i) = 1$)
- Mean of a function $f(x)$: $\overline{f(x)} = \int_{a_1}^{a_2} \mathrm{d}x \mathcal{P}(()x) f(x)$. (cf. $\overline{f(x)} = \sum P(x_i) f(x_i)$)

Examples of important continuous probability distributions:

- Gaussian: $\mathcal{P}(()x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/(2\sigma^2)}$
- Dirac delta: $\mathcal{P}(()x) = \delta(x - x_0)$
- Lorentzian: $\mathcal{P}(()x) = \frac{1}{\pi} \frac{\gamma}{\gamma^2 + (x - x_0)^2}$

Transformation of Continuous Distributions

For continuous probability distributions, it's important to know how to transform from one random variable x to another random variable $y = f(x)$. That is, given the distribution $\mathcal{P}()x(x)$, what is the distribution $\mathcal{P}()y(y)$ of $y = f(x)$?

Consider the probability conservation: the probability that y falls in the range $(y, y + \mathrm{d}y)$ must be equal to the probability that x falls in the corresponding range(s) $(x_i, x_i + \mathrm{d}x_i)$. In general, there may be multiple points x_i such that $y = f(x_i)$.

From the diagram: $\mathcal{P}()y(y)|\mathrm{d}y| = \sum_i \mathcal{P}()x(x_i)|\mathrm{d}x_i|$, where the sum is over all x_i such that $f(x_i) = y$. Since $\mathrm{d}y = \frac{\mathrm{d}f}{\mathrm{d}x}\mathrm{d}x = f'(x)\mathrm{d}x$, we have $|\mathrm{d}x_i| = \frac{|\mathrm{d}y|}{|f'(x_i)|} = |\frac{dx}{dy}|_{x=x_i}|\mathrm{d}y|$.

$$
\begin{aligned}
\mathcal{P}()y(y)|\mathrm{d}y| &= \sum_i \mathcal{P}()x(x_i) \left|\frac{dx}{dy}\right|_{x=x_i} |\mathrm{d}y| \\
\implies \mathcal{P}()y(y) &= \sum_i \mathcal{P}()x(x_i) \left|\frac{dx}{dy}\right|_{x=x_i}
\end{aligned}
$$

where the sum is over all roots x_i of $y = f(x)$ for a fixed y.

Example: X-component of a Random 2D Vector

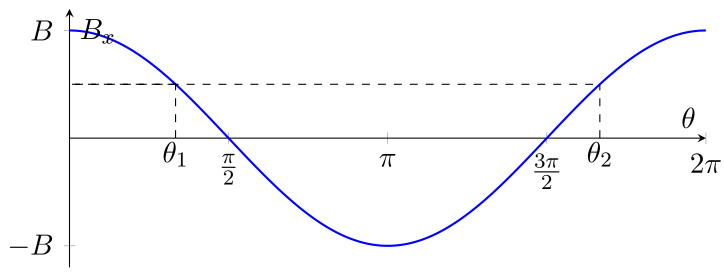
Consider a 2D vector $\vec{B}$ with fixed length $B = |\vec{B}|$, equally likely to point in any direction θ (where θ is the angle with the x-axis). The probability distribution for the angle θ is uniform:

$$\mathcal{P}()\theta(\theta) = \frac{1}{2\pi}, \quad 0 \le \theta < 2\pi$$

Let $y = B_x$ be the x-component of $\vec{B}$. We have the relation:

$$B_x(\theta) = B\cos\theta$$

We want to find the probability distribution $\mathcal{P}()B_x(B_x)$. Note that $-B \le B_x \le B$.



For a given value of B_x (where $-B < B_x < B$), there are two angles θ_1 and $\theta_2 = 2\pi - \theta_1$ such that $B_x = B\cos\theta_1 = B\cos\theta_2$. (Let $\theta_1 = \arccos(B_x/B)$). We use the general formula: $\mathcal{P}()B_x(B_x) = \sum_{i=1,2} \mathcal{P}()\theta(\theta_i) \left|\frac{\mathrm{d}\theta}{\mathrm{d}B_x}\right|_{\theta=\theta_i}$. We need the derivative $\frac{\mathrm{d}\theta}{\mathrm{d}B_x}$. It's easier to compute $\frac{\mathrm{d}B_x}{\mathrm{d}\theta}$:

$$\frac{\mathrm{d}B_x}{\mathrm{d}\theta} = \frac{\mathrm{d}}{\mathrm{d}\theta}(B\cos\theta) = -B\sin\theta$$

So, $\left|\frac{\mathrm{d}\theta}{\mathrm{d}B_x}\right| = \frac{1}{|-B\sin\theta|} = \frac{1}{B|\sin\theta|}$. Since $\sin^2\theta + \cos^2\theta = 1$, $|\sin\theta| = \sqrt{1-\cos^2\theta} = \sqrt{1-(B_x/B)^2} = \frac{\sqrt{B^2-B_x^2}}{B}$. Therefore, $\left|\frac{\mathrm{d}\theta}{\mathrm{d}B_x}\right| = \frac{1}{B(\sqrt{B^2-B_x^2}/B)} = \frac{1}{\sqrt{B^2-B_x^2}}$. This derivative is the same for θ_1 and θ_2 since $|\sin(\theta_1)| = |\sin(2\pi - \theta_1)|$.

Now apply the formula:

$$
\begin{aligned}
\mathcal{P}()B_x(B_x) &= \mathcal{P}()\theta(\theta_1)\left|\frac{\mathrm{d}\theta}{\mathrm{d}B_x}\right|_{\theta_1} + \mathcal{P}()\theta(\theta_2)\left|\frac{\mathrm{d}\theta}{\mathrm{d}B_x}\right|_{\theta_2} \\
&= \left(\frac{1}{2\pi}\right)\frac{1}{\sqrt{B^2-B_x^2}} + \left(\frac{1}{2\pi}\right)\frac{1}{\sqrt{B^2-B_x^2}} \\
&= 2 \times \frac{1}{2\pi}\frac{1}{\sqrt{B^2-B_x^2}}
\end{aligned}
$$

$$\mathcal{P}(_)B_x(B_x) = \frac{1}{\pi\sqrt{B^2 - B_x^2}} \quad \text{for } -B < B_x < B$$

This distribution is peaked near $B_x = \pm B$ and minimum at $B_x = 0$.