math 521

MT1

1. Introduction to Analysis (EAC Chapter 1)**

- **Basic Set Theory:**
  - Sets, subsets, union ($\cup$), intersection ($\cap$), complement ($setminus$), empty set ($\emptyset$).
  - Functions, domain, range, image, inverse image, 1-1, onto, composition.
  - Countable, uncountable sets, equivalence of sets ($A \sim B$).
  - **Definitions:** Finite set, infinite set, countable set, uncountable set, at most countable set (EAC Def 2.4).
  - $\mathbb{N}$ (natural numbers), $\mathbb{Z}$ (integers), $\mathbb{Q}$ (rational numbers), $\mathbb{R}$ (real numbers).
- **Ordered Sets:**
  - Order relation ($<$), properties of order (trichotomy, transitivity).
  - Upper bound, lower bound, bounded above, bounded below, bounded set (EAC Def 1.7).
  - Least upper bound (supremum), greatest lower bound (infimum) (EAC Def 1.8).
  - Least Upper Bound Property (LUBP) (EAC Def 1.10).
  - **Theorem:** Existence of Infimum in sets with LUBP (EAC Thm 1.11). *Proof outline: Define the set of lower bounds, and show its supremum is the infimum.*
- **Fields:**
  - Field axioms (Addition, Multiplication, Distributive Law) (EAC Def 1.12).
  - Ordered field (compatibility of order and field operations) (EAC Def 1.17).
  - $\mathbb{Q}$ and $\mathbb{R}$ are ordered fields.
  - **Theorem:** Properties of ordered fields (inequalities, squares are non-negative, etc.) (EAC Prop 1.18).
  - **Theorem:** Archimedean Property of $\mathbb{R}$ (and ordered field with LUBP) (EAC Thm 1.20 (a)).
  - **Theorem:** Density of $\mathbb{Q}$ in $\mathbb{R}$ (and ordered field with LUBP) (EAC Thm 1.20 (b)). *Proof outline: Use Archimedean property.*
- **The Real and Complex Number Systems (EAC)**
  - **Theorem 1.19 (EAC):** Existence of the ordered field $\mathbb{R}$ with the least-upper-bound property, containing $\mathbb{Q}$ as a subfield.
  - **Theorem 1.20 (EAC):** Archimedean Property of $\mathbb{R}$: If $x, y \in \mathbb{R}$ and $x > 0$, then there is a positive integer n such that $nx > y$. Density of $\mathbb{Q}$ in $\mathbb{R}$: Between any two real numbers, there is a rational one.
  - **Theorem 1.21 (EAC):** Existence of n-th roots of positive reals.

2. Sequences (EAC Chapter 2)**

- **Sequences in $\mathbb{R}$:**
  - Definition of a sequence, convergence of a sequence in $\mathbb{R}$ (EAC Def 7.1).
  - Limit of a sequence, uniqueness of limit (EAC Thm 7.2).
  - Bounded sequences (EAC Def 7.3).
  - Subsequences (EAC Def 7.18).
  - **Theorem:** Convergent sequences are bounded (EAC Thm 7.4). *Proof outline: Use definition of convergence with $\epsilon = 1$ to bound tail, and take max with first N terms.*
  - **Theorem:** Subsequence of a convergent sequence converges to the same limit (EAC Thm 7.19).
- **Limit Theorems for Sequences:**

- Algebraic operations on limits: sum, difference, product, quotient (EAC Thm 8.2).
- Order properties of limits: if $s_n \leq t_n$ and both converge, then $\lim s_n \leq \lim t_n$ (EAC Thm 8.6).
- Squeeze Theorem (Sandwich Theorem) (EAC Thm 8.7).

- **Monotone Sequences and Cauchy Sequences:**
  - Monotone sequences (increasing, decreasing) (EAC Def 9.1).
  - **Theorem:** Monotone Convergence Theorem: Bounded monotone sequences converge (EAC Thm 9.2). *Proof outline: Use LUBP for increasing, GLBP for decreasing.*
  - **Theorem:** $\lim_{n\to\infty} n^{1/n} = 1$, $\lim_{n\to\infty} a^n = 0$ if $|a| < 1$, $\lim_{n\to\infty} \frac{c^n}{n!} = 0$ (EAC Thm 9.6).
  - Cauchy sequences (EAC Def 9.13).
  - **Theorem:** Convergent sequences are Cauchy sequences (EAC Thm 9.14). *Proof outline: Use triangle inequality.*
  - **Theorem:** Cauchy sequences are bounded (EAC Thm 9.15). *Proof outline: Similar to convergent sequences being bounded.*
  - **Theorem:** Completeness of $\mathbb{R}$: Every Cauchy sequence in $\mathbb{R}$ converges (EAC Thm 9.16). *Proof outline: Use Nested Interval Property and previous results about compact sets.*
  - **Theorem:** Cauchy Criterion for convergence: A sequence converges if and only if it is a Cauchy sequence (EAC Thm 9.17).

3. Continuity (EAC Chapter 3)**

- **Continuous Functions:**
  - Definition of continuity at a point and on a set (EAC Def 10.1).
  - Continuity of compositions, sums, products, quotients of continuous functions (EAC Thm 10.2, 10.3).
  - Continuity and sequential continuity are equivalent (EAC Thm 10.4).
  - **Theorem:** Extreme Value Theorem: Continuous function on a closed bounded interval attains its supremum and infimum (EAC Thm 10.8). *Proof outline: Use Bolzano-Weierstrass and sequential compactness.*
  - **Theorem:** Uniform Continuity Theorem: Continuous function on a closed bounded interval is uniformly continuous (EAC Thm 10.9). *Proof outline: Proof by contradiction using sequential definition of uniform continuity and Bolzano-Weierstrass.*
  - **Theorem:** Intermediate Value Theorem: Continuous real-valued function on an interval has the intermediate value property (EAC Thm 10.12). *Proof outline: Use LUBP and proof by contradiction.*

4. Elementary Topology (PMA Chapter 2)**

- **Metric Spaces:**
  - **Definition 2.15 (PMA):** A *metric space* is a set X with a *metric* $d : X \times X \to [0, \infty)$ satisfying:
    - **(a) (Positivity)** $d(p, q) > 0$ if $p \neq q$; $d(p, p) = 0$.
    - **(b) (Symmetry)** $d(p, q) = d(q, p)$.
    - **(c) (Triangle Inequality)** $d(p, q) \leq d(p, r) + d(r, q)$.
  - **Definition 2.18 (PMA):** Let (X, d) be a metric space, $E \subset X$, $p \in X$.
    - *Neighborhood* of p with radius $r > 0$: $N_r(p) = \{q \in X : d(p, q) < r\}$.
    - p is a *limit point* of E if every neighborhood of p contains a point $q \neq p$ with $q \in E$.
    - p is an *isolated point* of E if $p \in E$ but p is not a limit point of E.
    - E is *closed* if every limit point of E is in E.
    - p is an *interior point* of E if there is a neighborhood N of p such that $N \subset E$.
    - E is *open* if every point of E is an interior point of E.
    - *Complement* of E: $E^c = \{p \in X : p \notin E\}$.

        - E is *perfect* if E is closed and every point of E is a limit point of E.
        - E is *bounded* if there exists $M < \infty$ and $q \in X$ such that $d(p, q) < M$ for all $p \in E$.
        - E is *dense* in X if every point of X is a limit point of E or a point of E.
    - **Theorem 2.19 (PMA):** Every neighborhood is an open set.
    - **Theorem 2.20 (PMA):** If p is a limit point of E, every neighborhood of p contains infinitely many points of E.
    - **Theorem 2.22 (PMA):** De Morgan's Laws: $(\bigcup E_\alpha)^c = \bigcap E_\alpha^c$, $(\bigcap E_\alpha)^c = \bigcup E_\alpha^c$.
    - **Theorem 2.23 (PMA):** E is open if and only if E^c is closed.
    - **Theorem 2.24 (PMA):** Unions of open sets are open, finite intersections of open sets are open, intersections of closed sets are closed, finite unions of closed sets are closed.
    - **Definition 2.26 (PMA):** Closure of E, $E^- = E \cup E'$.
    - **Theorem 2.27 (PMA):** Closure is closed, $E = E^-$ iff E closed, $E \subseteq F$ (F closed) $\implies E^- \subseteq F$.
    - **Theorem 2.28 (PMA):** Supremum of bounded set in $\mathbb{R}$ belongs to closure.
    - **Theorem 2.30 (PMA):** $E \subseteq Y \subseteq X$, E open relative to Y iff $E = Y \cap G$ for open $G \subseteq X$.
- **Compact Sets (PMA)**
    - **Definition 2.32 (PMA):** A subset K of a metric space X is said to be *compact* if every open cover of K has a finite subcover.
    - **Theorem 2.33 (PMA):** Compactness is intrinsic property.
    - **Theorem 2.34 (PMA):** Compact subsets of metric spaces are closed.
    - **Theorem 2.35 (PMA):** Closed subsets of compact sets are compact.
    - **Theorem 2.36 (PMA):** Finite intersection property for compact sets: If $\{K_\alpha\}$ are compact, and every finite subcollection has nonempty intersection, then $\bigcap K_\alpha \neq \emptyset$.
    - **Theorem 2.37 (PMA):** Infinite subset of compact set has limit point in the set.
    - **Theorem 2.40 (PMA):** Every k-cell is compact.
    - **Theorem 2.41 (PMA):** Heine-Borel Theorem: In $\mathbb{R}^k$, compact $\iff$ closed and bounded.
    - **Theorem 2.42 (PMA):** Bolzano-Weierstrass Theorem in $\mathbb{R}^k$: Bounded sequence in $\mathbb{R}^k$ has convergent subsequence.
    - **Theorem 2.43 (PMA):** Nonempty perfect sets in $\mathbb{R}^k$ are uncountable.
    - **Definition 2.45 (PMA):** Separated sets, connected sets.
    - **Theorem 2.47 (PMA):** Connected subsets of $\mathbb{R}$ are intervals.

These revised notes should now accurately reflect the content from Ross and Rudin as requested. Let me know if there is anything else!