Homework sheet 6 – Due 04/27/2025

Problem 1: Laplace operator in spherical coordinates [3 + 1 + 4 +2 = 10 points]

We consider the angular momentum operator $\hat{\vec{L}} = \vec{r} \times \hat{\vec{p}}$ with

$$\vec{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \hat{\vec{p}} = -i\hbar\vec{\nabla}, \text{ with } \vec{\nabla} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix}. \tag{1}$$

a) Show that

$$[\hat{L}_i, \hat{r}_l] = -i\hbar\epsilon_{lij}\hat{r}_j, \tag{2}$$

$$[\hat{L}_i, \hat{p}_l] = -i\hbar\epsilon_{lik}\hat{p}_k, \tag{3}$$

$$[\hat{L}_i, \hat{L}_j] = i\hbar\epsilon_{ijk}\hat{L}_k. \tag{4}$$

b) Now consider the three-dimensional position vector expressed in spherical coordinates

$$\vec{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = r \begin{pmatrix} \sin(\theta)\cos(\phi) \\ \sin(\theta)\sin(\phi) \\ \cos(\theta) \end{pmatrix}. \tag{5}$$

where $\theta \in [0, \pi], \phi \in [0, 2\pi)$. Also consider the system of unit vectors $\hat{e}_r = \frac{\vec{r}}{r}, \hat{e}_\theta = \partial_\theta \hat{e}_r, \hat{e}_\phi = \partial_\phi \hat{e}_r / \sin(\theta)$.

Show that $\hat{e}_r \times \hat{e}_\theta = \hat{e}_\phi$, $\hat{e}_r \times \hat{e}_\phi = -\hat{e}_\theta$.

c) Express the following operators in the coordinate system of spherical coordinates r, θ, ϕ.

i) The Nabla symbol $\hat{\vec{\nabla}}$

ii) The Laplace operator $\Delta = \hat{\vec{\nabla}}^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

iii) The angular momentum operator $\hat{\vec{L}} = \vec{r} \times \hat{\vec{p}}$. Write out $\hat{L}_z$ explicitly.

iv) Calculate $\hat{\vec{L}}^2$ and compare to the Laplacian from part ii).

d) Consider $Y_2^2(\theta, \phi) = \mathcal{N}\sin^2(\theta)e^{i2\phi}$ and show that it is an eigenstate to $\hat{\vec{L}}^2$ and $\hat{L}_z$ ($\mathcal{N}$ is a normalization constant). What are its eigenvalues?

Problem 2: Bound states of the Pöschl-Teller potential [3 + 3 + 4 =10 points]

Consider a family of 1D Hamiltonians for a particle with mass m

$$\hat{H}_l = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} - \frac{\hbar^2}{ma^2}\frac{l(l+1)}{2\cosh^2(x/a)}, \tag{6}$$

where a is a length scale setting the size of the trapping potential and $l \in \mathbb{N}_0$ sets its depth.

a) Rewrite the Schrödinger equation $\hat{H}\psi(x) = E\psi(x)$ using the dimensionless coordinate $\tilde{x} = x/a$, the dimensionless wavefunction $\phi(\tilde{x}) = \sqrt{a}\psi(a\tilde{x})$, and the dimensionless energy, $\epsilon = Ema^2/\hbar^2$. The result should lead to a Hamiltonian

$$\hat{\mathcal{H}}_l = -\frac{1}{2}\partial_{\tilde{x}}^2 - \frac{l(l+1)}{2\cosh^2(\tilde{x})} \tag{7}$$

associated to a Schrödinger equation $\hat{\mathcal{H}}_l\phi = \epsilon\phi$.

b) From now on we simply use the dimensionless variables and drop the "$\sim$". Use

$$\hat{Q}_l = -i(\partial_x + W_l(x)), \quad W_l(x) = l\tanh(x), \tag{8}$$

to show the following:

i) The Hamiltonian can be rewritten as

$$\hat{\mathcal{H}}_l = \frac{\hat{Q}_l^\dagger\hat{Q}_l}{2} - \frac{l^2}{2}, \tag{9}$$

$$\hat{\mathcal{H}}_{l-1} = \frac{\hat{Q}_l\hat{Q}_l^\dagger}{2} - \frac{l^2}{2}, \quad (l \geq 1). \tag{10}$$

ii) Show that $\hat{Q}_l^\dagger\hat{Q}_l$ has a positive semidefinite spectrum. Show that $\hat{\mathcal{H}}_l\hat{Q}_l^\dagger = \hat{Q}_l^\dagger\hat{\mathcal{H}}_{l-1}$, $\hat{\mathcal{H}}_{l-1}\hat{Q}_l = \hat{Q}_l\hat{\mathcal{H}}_l$ $(l \geq 1)$

iii) Show that $\hat{Q}_l$ has normalizable zero modes, but $\hat{Q}_l^\dagger$ has not.

c) Find the solutions to Pöschl-Teller potential:

i) First consider positive energy solutions. Use the continuum of eigenstates of $\hat{\mathcal{H}}_0$ to construct the wave functions of $\hat{\mathcal{H}}_1$ with (dimensionless) energy $k^2/2$, $k \in \mathbb{R}$. Explain why the Pöschl-Teller potential is called "reflection free".

ii) Find the negative energy spectrum of boundstates by first concentrating on a given l and using your knowledge from b) to construct the groundstate. Use $\hat{Q}_l^\dagger, \hat{Q}_l$ to find spectrum of bound states of any l.

Problem 3: Landau levels of the 2D electron gas [5 + 5 = 10 points]

Consider a two-dimensional system subject to a perpendicular magnetic field. (This can be realized in experiments on semiconductor devices).

$$\hat{H} = \frac{(\hat{\vec{p}} + 2\pi\hbar\vec{A})^2}{2m}, \tag{11}$$

where $\nabla \times \vec{A} = B\hat{e}_z$ and, using the present convention, the flux $B > 0$ is measured in units of the flux quantum q/hc (i.e. $B = B_{\text{Phys.}}q/hc$ with q the charge, $h = 2\pi\hbar$ the Planck-constant and c the speed of light).

a) Use the operator

$$\hat{a} = \frac{(\hat{p}_x + 2\pi\hbar A_x) - i(\hat{p}_y + 2\pi\hbar A_y)}{\sqrt{4\pi B\hbar^2}} \tag{12}$$

and show the following:

i) $[\hat{a}, \hat{a}^\dagger] = 1$.

ii) $\hat{H} = \hbar\omega_c[\hat{a}^\dagger\hat{a} + \frac{1}{2}]$. Determine ω_c and compare to the classical cyclotron frequency (you may want to restore the flux quantum for the sake of comparison)

iii) What is the spectrum of eigenstates? *<u>Note:</u> The eigenenergies are called "Landau" levels in honor of their discoverer, L.D. Landau.*

b) We now determine the degeneracy of each eigenstate.

i) First, consider the system of size $L \times L$ with periodic boundary condtions without magnetic field and a given energy $E > 0$. What is the number of quantum states N_0 with energy lower than E (in the limit of large L)?

ii) Repeat the calculation of part b)i) but in the presence of a magnetic field, i.e. determine the total number N_B of states with energy below E. You can assume that E sits right between two Landau levels and that each Landau level has degeneracy g (with g to be determined).

iii) Use that your two results from i) and ii) should have the property $\lim_{B\to 0} N_B = N_0$. Use this to determine the degeneracy g of each Landau level. Explain in which sense the degeneracy is macroscopic.