Homework 5 Guide - Advanced Quantum Mechanics

Solution Strategy Guide

Due: 04/13/2025

Problem 1: Particle in a box

Note

Background Knowledge

This problem contrasts the behavior of a free particle with that of a particle confined to an infinite potential well (a "box").

- The dynamics are governed by the Time-Independent Schrödinger Equation (TISE): $\hat{H}\psi(x) = E\psi(x)$.
- For a 1D particle, the Hamiltonian is $\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x})$, with $\hat{p} = -i\hbar\frac{d}{dx}$.
- A free particle has $V(x) = 0$, leading to continuous energy eigenvalues $E > 0$.
- The infinite potential well is defined by $V(x) = 0$ for $|x| < L/2$ and $V(x) = \infty$ for $|x| \geq L/2$.
- The requirement that the wave function $\psi(x)$ be continuous, combined with the infinite potential, imposes boundary conditions: $\psi(x) = 0$ at the edges of the well ($x = \pm L/2$).
- These boundary conditions lead to energy quantization (discrete energy levels) and specific forms for the eigenstates (standing waves).
- We compare the results (energy spectrum, wave function nodes) to the quantum harmonic oscillator (QHO), characterized by $V(x) = \frac{1}{2}m\omega^2x^2$ and energy levels $E_n^{HO} = \hbar\omega(n + 1/2)$.

Key Equations

- Free particle TISE ($E > 0$): $-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi$.
- General solution for free particle ($E > 0$): $\psi(x) = A\sin(k_E x) + B\cos(k_E x)$, where $k_E = \sqrt{2mE}/\hbar$.
- Infinite potential well definition: $V(x) = \begin{cases} \infty, & |x| \geq L/2 \\ 0, & |x| < L/2 \end{cases}$.
- Boundary Conditions (BCs) for the well: $\psi(-L/2) = \psi(L/2) = 0$.
- Energy levels in the well: $E_n = \frac{\hbar^2\pi^2n^2}{2mL^2}$, for $n = 1, 2, 3, \ldots$.
- Normalized eigenstates in the well:

$$\psi_n(x) = \sqrt{\frac{2}{L}} \times \begin{cases} \cos(\frac{n\pi x}{L}) & \text{if } n \text{ is odd} \\ \sin(\frac{n\pi x}{L}) & \text{if } n \text{ is even} \end{cases} \quad \text{for } |x| < L/2, \text{ and 0 otherwise.}$$

- QHO Energy levels: $E_n^{HO} = \hbar\omega(n + 1/2)$, for $n = 0, 1, 2, \ldots$.
- Nodes: Points where $\psi(x) = 0$ (excluding boundaries for the box).

Solution Strategy

Part a): Free Particle

1. Write the TISE for $V(x) = 0$.
2. Substitute the proposed wave function $\psi(x) = A\sin(k_E x) + B\cos(k_E x)$ into the TISE.
3. Calculate the second derivative $\frac{d^2\psi}{dx^2}$.
4. Show that the TISE is satisfied if and only if $k_E^2 = 2mE/\hbar^2$, confirming the form of k_E.

Part b): Particle in a Box

(i) **Energy Spectrum and Eigenstates:**

- Inside the well ($|x| < L/2$), the potential is zero, so the general solution form $A\sin(kx) + B\cos(kx)$ still applies, with $k = \sqrt{2mE}/\hbar$.
- Apply the BC $\psi(L/2) = 0$: $A\sin(kL/2) + B\cos(kL/2) = 0$.
- Apply the BC $\psi(-L/2) = 0$: $-A\sin(kL/2) + B\cos(kL/2) = 0$.
- Solve this system of linear equations for A and B. Show that non-trivial solutions exist only if $A = 0$ and $\cos(kL/2) = 0$ (leading to even states) or $B = 0$ and $\sin(kL/2) = 0$ (leading to odd states).
- Combine these conditions to find the allowed values of k, denoted $k_n = n\pi/L$ for $n = 1, 2, 3, \ldots$.
- Derive the quantized energy levels $E_n = \frac{\hbar^2 k_n^2}{2m}$.
- Write down the corresponding unnormalized wave functions, distinguishing between odd n (cosine solutions, even parity) and even n (sine solutions, odd parity).
- Normalize the wave functions over the interval $[-L/2, L/2]$ by calculating $\int_{-L/2}^{L/2} |\psi_n(x)|^2 dx = 1$.

(ii) **Plot and Compare Spectrum:**

- Sketch the energy levels $E_n \propto n^2$ for the infinite well.
- Sketch the energy levels $E_n^{HO} \propto (n + 1/2)$ for the QHO.
- Compare the dependence on the quantum number n and the spacing between adjacent energy levels. Note that the ground state index is $n = 1$ for the box and $n = 0$ for the HO.

(iii) **Compare Sign Changes (Nodes):**

- For the box, determine the number of points within the open interval $(-L/2, L/2)$ where $\psi_n(x) = 0$ for the ground state ($n = 1$) and the first few excited states ($n = 2, 3, \ldots$). Identify the general pattern relating the number of nodes to n.
- Recall (or derive using results from Problem 4) the number of nodes for the QHO ground state ($n = 0$) and the first few excited states ($n = 1, 2, \ldots$).
- Compare the number of nodes for the k-th excited state in both systems (remembering that the k-th excited state corresponds to $n = k + 1$ for the box and $n = k$ for the QHO).

Problem 2: Spherical Harmonics

Note

Background Knowledge

Spherical harmonics $Y_l^m(\theta, \phi)$ are fundamental functions in quantum mechanics for systems with spherical symmetry.

- They are eigenfunctions of the angular momentum operators $\hat{L}^2$ and $\hat{L}_z$.
- They form a complete orthonormal basis on the surface of a sphere.
- The indices l (orbital) and m (magnetic) are integers with $l \geq 0$ and $-l \leq m \leq l$.

- They are composed of an azimuthal part $e^{im\phi}$ and a polar part involving associated Legendre polynomials $P_l^m(\cos\theta)$.
- Legendre polynomials $P_l(x)$ ($x = \cos\theta$) correspond to the case $m = 0$. They are orthogonal on the interval $[-1, 1]$.

Key Equations

- Spherical Harmonic definition: $Y_l^m(\theta, \phi) = \mathcal{N}_{lm} e^{im\phi} P_l^m(\cos(\theta))$.
- Normalization constant: $\mathcal{N}_{lm} = \sqrt{\frac{(2l+1)}{4\pi}\frac{(l-m)!}{(l+m)!}}$.
- Associated Legendre Polynomial (definition from problem): $P_l^m(x) = (-1)^m \left(1 - x^2\right)^{m/2} \frac{d^m}{dx^m} P_l(x)$ (for $m \geq 0$).
- Legendre Polynomial Recursion: $(l + 1)P_{l+1}(x) = (2l + 1)xP_l(x) - lP_{l-1}(x)$, $P_0(x) = 1, P_1(x) = x$.
- Spherical Harmonic Orthogonality:
$$\int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi [Y_l^m(\theta, \phi)]^* Y_{l'}^{m'}(\theta, \phi) = \int_{-1}^1 dx \int_0^{2\pi} d\phi [Y_l^m]^* Y_{l'}^{m'} = \delta_{ll'}\delta_{mm'}$$
(with $x = \cos\theta$).
- Legendre Polynomial Orthogonality: $\int_{-1}^1 dx P_l(x)P_{l'}(x) = \frac{2}{2l+1}\delta_{ll'}$.

Solution Strategy

Part a): Orthogonality in m

1. Write the orthogonality integral in terms of θ and ϕ, then change variables to $x = \cos\theta$.
2. Substitute the definition $Y_l^m(\theta, \phi) = \mathcal{N}_{lm} e^{im\phi} P_l^m(x)$ into the integral.
3. Separate the integral into a part over x (involving P_l^m) and a part over ϕ (involving $e^{im\phi}$).
4. Evaluate the ϕ integral: $\int_0^{2\pi} d\phi (e^{im\phi})^* (e^{im'\phi}) = \int_0^{2\pi} d\phi e^{i(m'-m)\phi}$.
5. Show that this integral equals 2π if $m = m'$ and 0 if $m \neq m'$.
6. Conclude that the entire integral is zero when $m \neq m'$.

Part b): Legendre Polynomials Orthogonality

1. Use the Legendre recursion relation, starting with $P_0(x) = 1$ and $P_1(x) = x$, to explicitly calculate $P_2(x)$ and $P_3(x)$.
2. Select pairs (l, l') from $\{0, 1, 2, 3\}$ with $l \neq l'$ and explicitly calculate the integral $\int_{-1}^1 dx P_l(x)P_{l'}(x)$. Show it evaluates to zero (e.g., calculate $\int P_0 P_1 dx$, $\int P_1 P_2 dx$, $\int P_0 P_3 dx$, etc.).
3. Select pairs with $l = l'$ and calculate $\int_{-1}^1 dx [P_l(x)]^2$. Show it is non-zero.
4. Relate these results to the general orthogonality property $\int_{-1}^1 dx P_l(x)P_{l'}(x) \propto \delta_{ll'}$. You can explicitly check the proportionality constant $2/(2l + 1)$ for the calculated cases.

Part c): Specific Y_l^m Calculation

1. **Calculate $Y_3^0(\theta, \phi)$:**
   - Identify $l = 3, m = 0$. Use $P_3^0(x) = P_3(x)$ found in part (b).
   - Calculate the normalization constant $\mathcal{N}_{30}$.
   - Combine the parts: $Y_3^0 = \mathcal{N}_{30} e^{i0\phi} P_3(\cos\theta)$. Substitute $x = \cos\theta$.
2. **Calculate $Y_3^3(\theta, \phi)$:**
   - Identify $l = 3, m = 3$. Use the formula $P_l^m(x) = (-1)^m \left(1 - x^2\right)^{m/2} \frac{d^m}{dx^m} P_l(x)$ to find $P_3^3(x)$.
   - This requires calculating the third derivative of $P_3(x)$.
   - Calculate the normalization constant $\mathcal{N}_{33}$. Remember $0! = 1$.
   - Combine the parts: $Y_3^3 = \mathcal{N}_{33} e^{i3\phi} P_3^3(\cos\theta)$. Substitute $x = \cos\theta$ and $1 - x^2 = \sin^2\theta$. Simplify the resulting expression. Be careful with signs coming from $(-1)^m$.

Problem 3: Algebra of harmonic oscillator

Note

Background Knowledge

The algebraic method provides an elegant way to solve the QHO using ladder operators ($\hat{a}$ and $\hat{a}^\dagger$).

- These operators are constructed from $\hat{x}$ and $\hat{p}$.
- Their commutation relation $[\hat{a}, \hat{a}^\dagger] = 1$ is fundamental.
- The number operator $\hat{N} = \hat{a}^\dagger \hat{a}$ commutes with the Hamiltonian and has eigenstates $|n\rangle$ with integer eigenvalues $n = 0, 1, 2, \ldots$.
- $\hat{a}$ lowers the eigenvalue n by 1 ($\hat{a}|n\rangle \propto |n-1\rangle$), while $\hat{a}^\dagger$ raises it by 1 ($\hat{a}^\dagger|n\rangle \propto |n+1\rangle$).
- The ground state $|0\rangle$ is defined by $\hat{a}|0\rangle = 0$.
- The uncertainty principle $\Delta x \Delta p \geq \hbar/2$ provides a lower bound on the product of uncertainties.

Key Equations

- Operators: $\hat{a} = \frac{1}{\sqrt{2}\ell}\left(\hat{x} + \frac{i\ell^2}{\hbar}\hat{p}\right)$, $\quad \hat{a}^\dagger = \frac{1}{\sqrt{2}\ell}\left(\hat{x} - \frac{i\ell^2}{\hbar}\hat{p}\right)$, with $\ell = \sqrt{\frac{\hbar}{m\omega}}$.
- Number operator: $\hat{N} = \hat{a}^\dagger \hat{a}$.
- CCR: $[\hat{x}, \hat{p}] = i\hbar$.
- Inverse relations: $\hat{x} = \frac{\ell}{\sqrt{2}}(\hat{a} + \hat{a}^\dagger)$, $\quad \hat{p} = \frac{\hbar}{i\ell\sqrt{2}}(\hat{a} - \hat{a}^\dagger)$.
- Key commutators: $[\hat{a}, \hat{a}^\dagger] = 1$, $[\hat{a}, \hat{N}] = \hat{a}$, $[\hat{a}^\dagger, \hat{N}] = -\hat{a}^\dagger$.
- Action on number states: $\hat{a}|n\rangle = \sqrt{n}|n-1\rangle$, $\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$.
- Orthonormality: $\langle m|n\rangle = \delta_{mn}$.
- Standard deviation: $\Delta O = \sqrt{\langle \hat{O}^2 \rangle - \langle \hat{O} \rangle^2}$.

Solution Strategy

Part a): Basic Algebra

i) Calculate $[\hat{a}, \hat{a}^\dagger] = \hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}$. Substitute the definitions of $\hat{a}, \hat{a}^\dagger$ in terms of $\hat{x}, \hat{p}$. Expand the product and use $[\hat{x}, \hat{p}] = i\hbar$. Simplify the result to 1.

ii) Calculate $\{\hat{a}, \hat{a}^\dagger\} = \hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a}$. Express $\hat{x}$ and $\hat{p}$ in terms of $\hat{a}, \hat{a}^\dagger$. Substitute these into $\frac{\hat{x}^2}{\ell^2} + \frac{\ell^2\hat{p}^2}{\hbar^2}$. Expand the terms $\hat{x}^2$ and $\hat{p}^2$ using $\hat{a}, \hat{a}^\dagger$. Show that the sum simplifies to $\hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a}$. (Alternatively, express $\{\hat{a}, \hat{a}^\dagger\}$ in terms of $\hat{x}, \hat{p}$ and simplify).

iii) Calculate $[\hat{a}, \hat{N}] = [\hat{a}, \hat{a}^\dagger\hat{a}]$. Use the commutator identity $[A, BC] = [A, B]C + B[A, C]$ and the known values $[\hat{a}, \hat{a}^\dagger] = 1$ and $[\hat{a}, \hat{a}] = 0$. Simplify the result to $\hat{a}$.

Part b): Action on Number States

i) To show $\hat{a}|n\rangle = \sqrt{n}|n-1\rangle$:

- First, show that $\hat{a}|n\rangle$ is an eigenstate of $\hat{N}$ with eigenvalue $n-1$. Use $[\hat{a}, \hat{N}] = \hat{a}$ to write $\hat{N}\hat{a} = \hat{a}\hat{N} - \hat{a}$ and apply this to $|n\rangle$.
- Second, calculate the norm squared of $\hat{a}|n\rangle$: $||\hat{a}|n\rangle||^2 = \langle n|\hat{a}^\dagger\hat{a}|n\rangle = \langle n|\hat{N}|n\rangle = n$.
- Conclude that $\hat{a}|n\rangle$ must be $c|n-1\rangle$ where $|c|^2 = n$. Use the standard phase convention $c = \sqrt{n}$. Address the $n = 0$ case separately: $\hat{a}|0\rangle = 0$.

ii) To show $\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$:

- Show that $\hat{a}^\dagger|n\rangle$ is an eigenstate of $\hat{N}$ with eigenvalue $n+1$. Calculate $[\hat{a}^\dagger, \hat{N}] = -\hat{a}^\dagger$ (similar to part a.iii), which gives $\hat{N}\hat{a}^\dagger = \hat{a}^\dagger\hat{N} + \hat{a}^\dagger$. Apply this to $|n\rangle$.
- Calculate the norm squared: $||\hat{a}^\dagger|n\rangle||^2 = \langle n|\hat{a}\hat{a}^\dagger|n\rangle$. Use $[\hat{a}, \hat{a}^\dagger] = 1 \implies \hat{a}\hat{a}^\dagger = \hat{a}^\dagger\hat{a} + 1 = \hat{N} + 1$. Evaluate $\langle n|\hat{N} + 1|n\rangle = n + 1$.
- Conclude $\hat{a}^\dagger|n\rangle = d|n+1\rangle$ with $|d|^2 = n+1$. Use convention $d = \sqrt{n+1}$.

iii) Deduce orthogonality $\langle m|n\rangle = 0$ for $n > m$.

- Use the fact that $|n\rangle$ and $|m\rangle$ are eigenstates of the Hermitian operator $\hat{N}$ with different eigenvalues n and m. Recall the theorem that eigenstates of a Hermitian operator corresponding to distinct eigenvalues are orthogonal.
- Alternatively, use $|n\rangle = \frac{(\hat{a}^\dagger)^n}{\sqrt{n!}}|0\rangle$ and $|m\rangle = \frac{(\hat{a}^\dagger)^m}{\sqrt{m!}}|0\rangle$. Write $\langle m|n\rangle = \frac{1}{\sqrt{n!m!}}\langle 0|(\hat{a})^m(\hat{a}^\dagger)^n|0\rangle$. Use $\hat{a}|0\rangle = 0$ and $[\hat{a}, \hat{a}^\dagger] = 1$ to commute the m operators $\hat{a}$ to the right until they hit $|0\rangle$. Since $n > m$, show that there will always be remaining $\hat{a}^\dagger$ operators, and the expression evaluates to $\langle 0|(\hat{a}^\dagger)^{n-m}|0\rangle \times$ constant. However, this needs careful handling of commutations. The eigenvalue argument is simpler. The structure $(\hat{a}^\dagger)^n|0\rangle$ confirms they are eigenstates. Focus on the eigenvalue argument.

Part d): Uncertainty Product

1. Calculate the expectation values $\langle\hat{x}\rangle_n = \langle n|\hat{x}|n\rangle$ and $\langle\hat{p}\rangle_n = \langle n|\hat{p}|n\rangle$. Express $\hat{x}$ and $\hat{p}$ in terms of $\hat{a}, \hat{a}^\dagger$. Use orthogonality $\langle n|n \pm 1\rangle = 0$.
2. Calculate $\langle\hat{x}^2\rangle_n = \langle n|\hat{x}^2|n\rangle$ and $\langle\hat{p}^2\rangle_n = \langle n|\hat{p}^2|n\rangle$. Express $\hat{x}^2$ and $\hat{p}^2$ in terms of $\hat{a}, \hat{a}^\dagger$. Expand the products (e.g., $(\hat{a} + \hat{a}^\dagger)^2 = \hat{a}^2 + \hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a} + (\hat{a}^\dagger)^2$). Use $\langle n|\hat{a}^2|n\rangle = 0$, $\langle n|(\hat{a}^\dagger)^2|n\rangle = 0$, $\langle n|\hat{a}\hat{a}^\dagger|n\rangle = n+1$, $\langle n|\hat{a}^\dagger\hat{a}|n\rangle = n$.
3. Calculate the variances $(\Delta x)^2_n = \langle\hat{x}^2\rangle_n - \langle\hat{x}\rangle^2_n$ and $(\Delta p)^2_n = \langle\hat{p}^2\rangle_n - \langle\hat{p}\rangle^2_n$.
4. Find the standard deviations Δx_n and Δp_n.
5. Calculate the product $\Delta x_n \Delta p_n$.
6. Compare the result with the Heisenberg Uncertainty Principle inequality $\Delta x \Delta p \geq \hbar/2$. Check if the QHO eigenstates satisfy it, and identify if/when they saturate the bound.

Problem 4: Hermite polynomials and the harmonic oscillator

Note

Background Knowledge

This problem connects the abstract algebraic solution ($|n\rangle$ states) to the concrete wave functions $\psi_n(x) = \langle x|n\rangle$ by solving the TISE in the position representation.

- The TISE for the QHO is a second-order ordinary differential equation.
- Changing to a dimensionless coordinate $y = x/\ell$ simplifies the equation.
- The asymptotic behavior of the solutions ($x \to \pm\infty$) suggests factoring out a Gaussian term $e^{-y^2/2}$.
- The remaining part satisfies Hermite's differential equation.
- The polynomial solutions to Hermite's equation are the Hermite polynomials $H_n(y)$.
- Hermite polynomials can be defined via Rodrigues' formula and are orthogonal with respect to the weight function e^{-y^2} over $(-\infty, \infty)$.

Key Equations

- QHO Hamiltonian: $\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2}{2}\hat{x}^2$.
- TISE: $\hat{H}\psi_n(x) = E_n\psi_n(x)$ with $E_n = \hbar\omega(n + 1/2)$.
- Position representation: $\hat{p} = -i\hbar\frac{d}{dx}$.
- Dimensionless variable: $y = x/\ell$, $\ell = \sqrt{\hbar/m\omega}$.
- Transformed TISE: $\left[y^2 - \frac{d^2}{dy^2}\right]\phi_n(y) = (2n+1)\phi_n(y)$, where $\phi_n(y) = \psi_n(x)$.
- Ansatz: $\phi_n(y) = H_n(y)e^{-y^2/2}$.
- Hermite Equation: $H_n''(y) - 2yH_n'(y) + 2nH_n(y) = 0$.
- Rodrigues' formula: $H_n(y) = (-1)^n e^{y^2}\frac{d^n}{dy^n}e^{-y^2}$.
- Hermite Polynomial Orthogonality: $\int_{-\infty}^{\infty} dy e^{-y^2} H_n(y)H_m(y) \propto \delta_{nm}$.

Solution Strategy

Part a): Derivation of Hermite's Equation

i) Transform the TISE to dimensionless form:

- Start with the TISE in x coordinates: $[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2x^2]\psi_n(x) = E_n\psi_n(x)$.
- Substitute $x = \ell y$, $\frac{d}{dx} = \frac{1}{\ell}\frac{d}{dy}$, $\frac{d^2}{dx^2} = \frac{1}{\ell^2}\frac{d^2}{dy^2}$, $\psi_n(x) = \phi_n(y)$, and $E_n = \hbar\omega(n+1/2)$.
- Use $\ell^2 = \hbar/m\omega$ to simplify the coefficients of $\frac{d^2}{dy^2}$ and y^2.
- Divide the entire equation by $\hbar\omega/2$ and rearrange to obtain $\left[y^2 - \frac{d^2}{dy^2}\right]\phi_n(y) = (2n+1)\phi_n(y)$.

ii) Derive Hermite's Equation from the Ansatz:

- Substitute the Ansatz $\phi_n(y) = H_n(y)e^{-y^2/2}$ into the equation found in (i), specifically into the form $\frac{d^2\phi_n}{dy^2} = [y^2 - (2n+1)]\phi_n(y)$.
- Calculate the first derivative $\phi_n'(y)$ using the product rule.
- Calculate the second derivative $\phi_n''(y)$ using the product rule again.
- Substitute ϕ_n'' and ϕ_n into the transformed TISE.
- Cancel the common factor $e^{-y^2/2}$ (which is non-zero).
- Simplify the resulting algebraic equation involving H_n, H_n', H_n'' to show it is equivalent to the Hermite equation $H_n'' - 2yH_n' + 2nH_n = 0$.

Part b): Solutions to Hermite's Equation

1. Verify Rodrigues' Formula:
   - Follow the hint: Show $H_0(y)$ satisfies Hermite's equation for $n = 0$.
   - **Inductive Step Approach:** Assume $H_n(y)$ given by Rodrigues' formula satisfies the n-th Hermite equation. Use properties derivable from Rodrigues' formula (like recurrence relations, e.g., $H_n' = 2nH_{n-1}$ and $H_{n+1} = 2yH_n - 2nH_{n-1}$) to show that $H_{n+1}(y)$ constructed via Rodrigues' formula satisfies the $(n+1)$-th Hermite equation. This involves deriving those recurrence relations from Rodrigues' formula first, or taking them as known properties.
   - **Alternative (Direct Substitution):** Substitute $H_n(y)$ from Rodrigues' formula directly into the Hermite equation $H_n'' - 2yH_n' + 2nH_n = 0$. This requires careful computation of derivatives and simplification; it's a standard but potentially lengthy proof.
2. Calculate First Three Hermite Polynomials:
   - Use Rodrigues' formula $H_n(y) = (-1)^n e^{y^2} \frac{d^n}{dy^n} e^{-y^2}$ to compute $H_0(y)$, $H_1(y)$, and $H_2(y)$ explicitly.
3. Order of $H_n(y)$:
   - Observe the results for H_0, H_1, H_2.
   - Argue from Rodrigues' formula that $\frac{d^n}{dy^n} e^{-y^2}$ is of the form $P_n(y)e^{-y^2}$ where $P_n(y)$ is a polynomial of degree n. Multiplying by e^{y^2} leaves $H_n(y)$ as a polynomial of degree n. Note the leading term arises from differentiating e^{-y^2} n times yielding $(-2y)^n e^{-y^2}$.

Part c): Orthogonality of Hermite Polynomials

1. State the integral to be proven zero for $n > m$: $I_{nm} = \int_{-\infty}^{\infty} dy e^{-y^2} H_n(y)H_m(y)$.
2. Substitute Rodrigues' formula for $H_n(y)$: $I_{nm} = \int_{-\infty}^{\infty} dy e^{-y^2}[(-1)^n e^{y^2}(\partial_y^n e^{-y^2})]H_m(y)$. Simplify by canceling $e^{-y^2}e^{y^2}$.
3. Integrate by parts n times. In each step, differentiate $H_m(y)$ and integrate the derivative of e^{-y^2}. Let $u = H_m(y)$ (or its derivatives) and $dv = (\partial_y^k e^{-y^2})dy$.
4. Show that the boundary terms at $\pm\infty$ vanish at each step because the exponential term e^{-y^2} decays faster than any polynomial term grows.
5. After n integrations, the integral becomes $I_{nm} = (-1)^{2n} \int_{-\infty}^{\infty} dy e^{-y^2}(\partial_y^n H_m(y))$.
6. Use the fact that $H_m(y)$ is a polynomial of degree m.
7. Since $n > m$, the n-th derivative $\partial_y^n H_m(y)$ is identically zero.
8. Conclude that the integral I_{nm} is zero for $n > m$. By symmetry, it is also zero for $m > n$, thus proving orthogonality for $n \neq m$.