Solutions to sample midterm questions

1. To show that f is uniformly continuous, choose $\epsilon > 0$. Since $f_n \to f$ uniformly, there exists an N such that $n > N$ implies that
$$|f_n(x) - f(x)| < \frac{\epsilon}{3} \tag{1}$$
for all $x \in (a, b)$. Now consider f_{N+1}: since this is uniformly continuous, there exists a $\delta > 0$ such that if $x, y \in (a, b)$ and $|x - y| < \delta$, then
$$|f_{N+1}(x) - f_{N+1}(y)| < \frac{\epsilon}{3}. \tag{2}$$
Now, for any $x, y \in (a, b)$ with $|x - y| < \delta$,
$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - f_{N+1}(x)| + |f_{N+1}(x) - f_{N+1}(y)| + |f_{N+1}(y) - f(y)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \end{aligned}$$
and hence f is uniformly continuous.

2. (a) For f_1,
$$\begin{aligned} \beta &= \limsup |n^{-2}|^{1/n} \\ &= \limsup \frac{1}{n^{2/n}} = 1 \end{aligned}$$
Hence the radius of convergence is $R = 1$.
For f_2, since some terms are zero, the radius of convergence can be evaluated by computing the limit of the non-zero terms:
$$\begin{aligned} \beta &= \limsup |a_n|^{1/n} \\ &= \lim_{k\to\infty} |a_{2k}|^{1/2k} \\ &= \lim_{k\to\infty} |2^{-k}|^{1/2k} \\ &= \lim_{k\to\infty} 2^{-1/2} \\ &= \frac{1}{\sqrt{2}}. \end{aligned}$$
Hence $R = \sqrt{2}$.

(b) Define $x = y/(1 + y^2)$. Then $f_3(y) = f_1(x)$. If $|y| \leq 1$, then $|y| < 1 + y^2$, and hence $|x| < 1$. If $|y| > 1$, then $|y| < y^2$ and so $|y| < 1 + y^2$, so $|x| < 1$ also. Hence for all $y \in \mathbb{R}$, $|x| < 1$, and since $f_1(x)$ converges for x in this range, $f_3(y)$ must converge also.
This question can also be answered using the Weierstraß M-test, by showing that the nth term in the series is bounded by $1/n^2$, and $\sum |1/n^2|$ converges.

3. To show that $f_n(x_n)$ converges to $f(x)$, consider any $\epsilon > 0$. Since the f_n are continuous and converge uniformly to f, then f must be continuous also. Furthermore, since the interval is closed the limit point x must be within $[a, b]$. Hence, since f is continuous at x, then $f(x_n) \to f(x)$ and hence there exists $N_1 \in \mathbb{N}$ such that $n > N_1$ implies
$$|f(x_n) - f(x)| < \frac{\epsilon}{2}. \tag{3}$$
In addition, since f_n converges uniformly to f, then there exists an $N_2 \in \mathbb{N}$ such that $n > N_2$ implies
$$|f_n(y) - f(y)| < \frac{\epsilon}{2} \tag{4}$$
for all $y \in [a, b]$. Hence if $N = \max\{N_1, N_2\}$, then
$$|f_n(x_n) - f(x)| \le |f_n(x_n) - f(x_n)| + |f(x_n) - f(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \tag{5}$$
Therefore $\lim_{n\to\infty} f_n(x_n) = f(x)$.

4. (a) Note that $1/\sqrt{n^2-1} > 1/n$ for all $n \ge 2$. Since $\sum_n 1/n$ diverges, we can conclude that $\sum_{n=2}^{\infty} 1/\sqrt{n^2-1}$ diverges by the comparison test.
   The terms $1/\sqrt{n^2-1}$ form a decreasing sequence and $\lim_{n\to\infty} 1/\sqrt{n^2-1} = 0$. Hence $\sum_{n=2}^{\infty} (-1)^n/\sqrt{n^2-1}$ converges by the alternating series test.

   (b) Putting $x = 1/5$ into the sum gives
$$\sum_{n=2}^{\infty} \frac{5^n 5^{-n}}{\sqrt{n^2-1}} = \sum_{n=2}^{\infty} \frac{1}{\sqrt{n^2-1}}. \tag{6}$$
   Since this diverges by part (a) the radius of convergence R must satisfy $R \le 1/5$. Putting $x = -1/5$ into the sum gives
$$\sum_{n=2}^{\infty} \frac{5^n (-5)^{-n}}{\sqrt{n^2-1}} = \sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n^2-1}}. \tag{7}$$
   Since this converges by part (a), then $R \ge 1/5$. Hence $R = 1/5$.

5. Consider the function $g(x) = f(x) - f(x+1)$. Then
$$g(0) = f(0) - f(1) \tag{8}$$
and
$$g(1) = f(1) - f(2) = f(1) - f(0) = -g(0). \tag{9}$$
If $g(0) = 0$, then $f(0) = f(1)$ and setting $(x, y) = (0, 1)$ satisfies $f(x) = f(y)$ as required. Otherwise $g(0) \ne 0$, in which case $g(0)$ and $g(1)$ have opposite sign. Applying the intermediate value theorom to the range $[0, 1]$ shows that there exists a $c \in (0, 1)$ such that $g(c) = f(c) - f(c+1) = 0$. Setting $(x, y) = (c, c+1)$ satisfies $f(x) = f(y)$ as required.

6. Choose $\epsilon > 0$. Since $f_n \to f$ uniformly on S, there exists an N such that $n > N$ implies that
$$|f_n(x) - f(x)| < \frac{\epsilon}{2} \tag{10}$$
for all $x \in S$. Consider an arbitrary $m > N$ and $n > N$. Then, using the triangle inequality
$$|f_n(x) - f_m(x)| \leq |f_n(x) - f(x)| + |f_m(x) - f(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \tag{11}$$
and hence (f_n) is uniformly Cauchy.

7. (a) If f_n converges uniformly to f, then there exists an N such that $n > N$ implies that
$$|f_n(x) - f(x)| < 1 \tag{12}$$
for all $x \in S$. By using the triangle inequality,
$$|f(x)| < |f_{N+1}(x)| + 1. \tag{13}$$
Since f_{N+1} is bounded, $|f_{N+1}(x)| < M$ for all x and for some $M \geq 0$, and thus $|f(x)| < M + 1$ for all x. Hence f is bounded.

(b) Consider the set $S = (0, 1]$ and the functions
$$f_n(x) = \begin{cases} n & \text{if } x \leq 1/n \\ 1/x & \text{if } x > 1/n \end{cases} \tag{14}$$
These functions converge pointwise to $f(x) = 1/x$. To prove this, consider a fixed $x \in S$. For all $n > 1/x$, $f_n(x) = f(x)$ and therefore becomes a constant sequence. Hence $f_n \to f$ pointwise.

Each f_n satisfies $|f_n(x)| < n + 1$ for all x, and is therefore bounded. But the function f is not bounded.