M521 HW1
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Problem 1.

Solution: Base Case (n=1): For $n = 1$, LHS $= 2(1) + 1 = 3$. RHS $= 3(1)^2 = 3$. Since LHS = RHS, the equation holds for $n = 1$.

Inductive Step: Assume that for some $k \in \mathbb{N}$, $(2k+1)+(2k+3)+\ldots+(4k-1) = 3k^2$ is true.

We need to prove that the equation holds for $n = k + 1$:

$$(2(k+1)+1)+(2(k+1)+3)+\ldots+(4(k+1)-1) = 3(k+1)^2.$$

Consider the LHS for $n = k + 1$:

$$\begin{aligned}
\text{LHS} &= (2k+3)+(2k+5)+\ldots+(4k+3) \\
&= [(2k+3)+(2k+5)+\ldots+(4k-1)]+(4k+1)+(4k+3) \\
&= [(2k+1)+(2k+3)+\ldots+(4k-1)]-(2k+1)+(4k+1)+(4k+3) \\
&= 3k^2-(2k+1)+(4k+1)+(4k+3) \\
&= 3k^2-2k-1+4k+1+4k+3 \\
&= 3k^2+6k+3 \\
&= 3(k^2+2k+1) \\
&= 3(k+1)^2 = \text{RHS}
\end{aligned}$$

By mathematical induction, the equation $(2n+1)+(2n+3)+\ldots+(4n-1) = 3n^2$ is true for all $n \in \mathbb{N}$. □

Problem 2. Pell numbers

Solution: **(a) Prove H_n is true**

We will prove that H_n is true for all $n \in \mathbb{N}$ using mathematical induction, where H_n is the statement "$P_n = f(n)$ and $P_{n-1} = f(n-1)$", and $f(n) = \frac{(1+\sqrt{2})^n - (1-\sqrt{2})^n}{2\sqrt{2}}$.

Base Case (n=1):

We check if H_1 is true, i.e., if $P_1 = f(1)$ and $P_0 = f(0)$. $f(1) = \frac{(1+\sqrt{2})^1 - (1-\sqrt{2})^1}{2\sqrt{2}} = \frac{2\sqrt{2}}{2\sqrt{2}} = 1 = P_1$. $f(0) = \frac{(1+\sqrt{2})^0 - (1-\sqrt{2})^0}{2\sqrt{2}} = \frac{1-1}{2\sqrt{2}} = 0 = P_0$. Thus, H_1 is true.

Inductive Step:

Assume H_k is true for some $k \in \mathbb{N}$. That is, assume $P_k = f(k)$ and $P_{k-1} = f(k-1)$.

We want to prove H_{k+1}, i.e., $P_{k+1} = f(k+1)$ and $P_k = f(k)$.

Using the Pell number recurrence relation: $P_{k+1} = 2P_k + P_{k-1}$.

By the inductive hypothesis, substitute $P_k = f(k)$ and $P_{k-1} = f(k-1)$:

$$P_{k+1} = 2f(k) + f(k-1) = 2\frac{(1+\sqrt{2})^k - (1-\sqrt{2})^k}{2\sqrt{2}} + \frac{(1+\sqrt{2})^{k-1} - (1-\sqrt{2})^{k-1}}{2\sqrt{2}}.$$

Combining terms:

$$P_{k+1} = \frac{1}{2\sqrt{2}}\left[2(1+\sqrt{2})^k - 2(1-\sqrt{2})^k + (1+\sqrt{2})^{k-1} - (1-\sqrt{2})^{k-1}\right].$$

Factor out $(1+\sqrt{2})^{k-1}$ and $(1-\sqrt{2})^{k-1}$, and simplify:

$$\begin{aligned}
P_{k+1} &= \frac{1}{2\sqrt{2}}\left[(1+\sqrt{2})^{k-1}(2(1+\sqrt{2})+1) - (1-\sqrt{2})^{k-1}(2(1-\sqrt{2})+1)\right] \\
&= \frac{1}{2\sqrt{2}}\left[(1+\sqrt{2})^{k-1}(1+\sqrt{2})^2 - (1-\sqrt{2})^{k-1}(1-\sqrt{2})^2\right] \\
&= \frac{1}{2\sqrt{2}}\left[(1+\sqrt{2})^{k+1} - (1-\sqrt{2})^{k+1}\right] \\
&= f(k+1).
\end{aligned}$$

Thus, $P_{k+1} = f(k+1)$. Hence, H_{k+1} is true.

By mathematical induction, H_n is true for all $n \in \mathbb{N}$. Therefore, $P_n = f(n)$ and $P_{n-1} = f(n-1)$ for all $n \in \mathbb{N}$. In particular, $P_n = f(n)$ for all $n \in \mathbb{N}$. Since we verified $P_0 = f(0) = 0$ separately, we conclude $P_n = f(n)$ for all $n \in \mathbb{N} \cup \{0\}$.

(b) Why is $|\lambda - \sqrt{2}|$ small?

We have $\lambda = \frac{P_8+P_9}{P_9} = 1 + \frac{P_8}{P_9}$.

From part (a), $P_n = \frac{(1+\sqrt{2})^n-(1-\sqrt{2})^n}{2\sqrt{2}}$.

For large n, since $|1-\sqrt{2}| < 1$, $(1-\sqrt{2})^n \approx 0$.

So, $P_n \approx \frac{(1+\sqrt{2})^n}{2\sqrt{2}}$ for large n.

Then

$$\frac{P_8}{P_9} \approx \frac{(1+\sqrt{2})^8/(2\sqrt{2})}{(1+\sqrt{2})^9/(2\sqrt{2})} = \frac{(1+\sqrt{2})^8}{(1+\sqrt{2})^9} = \frac{1}{1+\sqrt{2}} = \frac{\sqrt{2}-1}{2-1} = \sqrt{2}-1.$$

Thus, $\frac{P_8}{P_9} \approx \sqrt{2}-1$. Therefore,

$$\lambda = 1 + \frac{P_8}{P_9} \approx 1 + (\sqrt{2}-1) = \sqrt{2}.$$

Hence, $|\lambda - \sqrt{2}|$ is small because λ is based on the ratio of consecutive Pell numbers, which for large indices approximates $\sqrt{2}$. More precisely, as $n \to \infty$, $\frac{P_n}{P_{n-1}} \to 1+\sqrt{2}$, so $\frac{P_{n-1}}{P_n} \to \frac{1}{1+\sqrt{2}} = \sqrt{2}-1$. For $n = 9$, $\frac{P_8}{P_9}$ is already a good approximation of $\sqrt{2}-1$, making $\lambda = 1 + \frac{P_8}{P_9}$ a good approximation of $\sqrt{2}$. $\square$

Problem 3. $\sqrt{2}+\sqrt{5}$ is irrational.

Solution: Proof by Contradiction:

Assume, for contradiction, that $\sqrt{2}+\sqrt{5}$ is rational. Let $r=\sqrt{2}+\sqrt{5}$, where $r \in \mathbb{Q}$.

Square both sides of $r=\sqrt{2}+\sqrt{5}$:

$$r^2 = (\sqrt{2}+\sqrt{5})^2 = 7+2\sqrt{10}$$

Rearrange to isolate $\sqrt{10}$:

$$\sqrt{10} = \frac{r^2-7}{2}$$

Since $r \in \mathbb{Q}$, the expression $\frac{r^2-7}{2}$ is also rational. Thus, if $\sqrt{2}+\sqrt{5}$ is rational, then $\sqrt{10}$ must be rational.

We now prove by contradiction that $\sqrt{10}$ is irrational.

Assume $\sqrt{10}$ is rational, so $\sqrt{10} = \frac{p}{q}$ for integers p, q with $\gcd(p,q) = 1$ and $q \neq 0$. Squaring both sides gives $10 = \frac{p^2}{q^2}$, so $p^2 = 10q^2$. This means p^2 is divisible by 10, hence divisible by 2 and 5. Since 2 and 5 are prime, p must be divisible by 2 and 5, so $p = 10k$ for some integer k. Substituting $p = 10k$ into $p^2 = 10q^2$:

$$\begin{aligned}(10k)^2 &= 10q^2\\ 100k^2 &= 10q^2\\ q^2 &= 10k^2\end{aligned}$$

This means q^2 is divisible by 10, and thus q is divisible by 10. So, both p and q are divisible by 10, contradicting $\gcd(p,q) = 1$. Therefore, $\sqrt{10}$ is irrational.

In conclusion: We have shown that, if $\sqrt{2}+\sqrt{5}$ is rational, then $\sqrt{10}$ is rational. However, we then proved that $\sqrt{10}$ is irrational. This is a contradiction. Therefore, our initial assumption that $\sqrt{2}+\sqrt{5}$ is rational is false.

Thus proves $\sqrt{2}+\sqrt{5}$ is irrational.

$\square$

Problem 4 (Field properties).

Solution: **(a) $\mathbb{N}$ (Natural Numbers)**

Field Properties:

- **Holds:** A1 (Additive Associativity), A2 (Additive Commutativity), M1 (Multiplicative Associativity), M2 (Multiplicative Commutativity), M3 (Multiplicative Identity: 1), DL (Distributive Law)
- **Fails:** A3 (No additive identity $0 \in \mathbb{N}$), A4 (No additive inverses), M4 (No multiplicative inverses except for 1)

Order Properties (O1–O5):

- **O1 (Trichotomy):** Holds. For any $a, b \in \mathbb{N}$, exactly one of $a < b$, $a = b$, or $a > b$ is true.
- **O2 (Antisymmetry):** Suppose $a, b \in \mathbb{N}$ satisfy $a \leq b$ and $b \leq a$. By O1 (trichotomy), if $a \neq b$, then either $a < b$ or $a > b$ would hold, contradicting one of the inequalities. Thus, the only possibility is $a = b$.
- **O3 (Transitivity):** Holds. If $a < b$ and $b < c$, then $a < c$.
- **O4 (Additive Compatibility):** Holds. If $a < b$, then $a + c < b + c$ for all $c \in \mathbb{N}$.
- **O5 (Multiplicative Compatibility):** Holds. If $a < b$ and $c > 0$, then $a \cdot c < b \cdot c$.

$\mathbb{N}$ satisfies all order properties but is missing additive identity/inverses and multiplicative inverses.

(b) $\mathbb{Z}$ (Integers)

Field Properties:

- **Holds:** A1, A2, A3 (Additive Identity: 0), A4 (Additive Inverses), M1, M2, M3 (Multiplicative Identity: 1), DL
- **Fails:** M4 (No multiplicative inverses except for ± 1)

Order Properties (O1–O5):

- **O1 :** Holds.
- **O2 :** Holds by the same argument as in $\mathbb{N}$: if $a \leq b$ and $b \leq a$, then trichotomy forces $a = b$.
- **O3 :** Holds.
- **O4 :** Holds.
- **O5 :** Holds.

$\mathbb{Z}$ does not satisfy M4.

(c) $\mathbb{B} = \{0, 1\}$ (Binary Numbers)

Field Properties: All field axioms hold:

- **A1 :** For all $a, b, c \in \mathbb{B}$, $(a + b) + c = a + (b + c)$
  - $(1 + 1) + 1 = 0 + 1 = 1 = 1 + 0 = 1 + (1 + 1)$
  - Similar for other combinations.
- **A2 :** For all $a, b \in \mathbb{B}$, $a + b = b + a$
  - $0 + 0 = 0 = 0 + 0$
  - $0 + 1 = 1 = 1 + 0$
  - $1 + 1 = 0 = 1 + 1$
- **A3 :** $0 \in \mathbb{B}$ satisfies $a + 0 = a$ for all $a \in \mathbb{B}$
- **A4 :** For each $a \in \mathbb{B}$, there exists $-a \in \mathbb{B}$ where:
  - $-0 = 0$ since $0 + 0 = 0$
  - $-1 = 1$ since $1 + 1 = 0$
- **M1 :** For all $a, b, c \in \mathbb{B}$, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
- **M2 :** For all $a, b \in \mathbb{B}$, $a \cdot b = b \cdot a$
  - $0 \cdot 1 = 0 = 1 \cdot 0$
  - $1 \cdot 1 = 1$
- **M3 :** $1 \in \mathbb{B}$ satisfies $a \cdot 1 = a$ for all $a \in \mathbb{B}$
- **M4 :** For non-zero elements:
  - $1^{-1} = 1$ since $1 \cdot 1 = 1$
- **DL :** For all $a, b, c \in \mathbb{B}$, $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$

Order Properties (O1–O5): Given $0 \leq 1$:

- **O1 :** Holds. For $a, b \in \mathbb{B}$, exactly one of the following holds: $a < b$, $a = b$, or $a > b$.
- **O2 :** If $a \leq b$ and $b \leq a$, then by the trichotomy property the possibilities $a < b$ or $a > b$ are ruled out, leaving only $a = b$.
- **O3 :** Trivially holds (with only two elements, no nontrivial chain exists).
- **O4 : Fails**. For example, $0 < 1$ but $0 + 1 = 1 \not< 1 + 1 = 0$.
- **O5 :** Vacuously holds. The only $c > 0$ is 1, and $0 \cdot 1 = 0 < 1 \cdot 1 = 1$.

$\mathbb{B}$ is a **field** but **not an ordered field** due to the failure of O4. $\square$

Problem 5.

Solution: (a) $0 < 1$.

Proof:

Step 1: By the trichotomy property (O1), exactly one of the following holds: $0 < 1$, $0 = 1$, or $1 < 0$. Since $1 \neq 0$, we exclude $0 = 1$.

Step 2: Assume for contradiction that $1 < 0$. Add -1 to both sides:

$$1 + (-1) < 0 + (-1) \implies 0 < -1.$$

Step 3: Multiply $1 < 0$ by -1 (which is positive by Step 2):

By multiplicative compatibility : $1 \cdot (-1) < 0 \cdot (-1) \implies -1 < 0$. But Step 2 gives $0 < -1$, violating antisymmetry (O2). Hence, $1 < 0$ is false.

Conclusion: By trichotomy, $0 < 1$ must hold.

(b) If $0 < a < b$, then $0 < b^{-1} < a^{-1}$ for all $a, b \in \mathbb{R}$.

Proof:

Step 1: Prove $a^{-1} > 0$ and $b^{-1} > 0$:

Suppose $a^{-1} \leq 0$. Since $a > 0$, multiplying $a^{-1} \leq 0$ by a gives: $a \cdot a^{-1} \leq 0 \implies 1 \leq 0$, contradicting $0 < 1$ (from part (a)). Thus, $a^{-1} > 0$. Similarly, $b^{-1} > 0$.

Step 2: Multiply $a < b$ by $a^{-1}b^{-1} > 0$:

By multiplicative compatibility : $a \cdot (a^{-1}b^{-1}) < b \cdot (a^{-1}b^{-1}) \implies b^{-1} < a^{-1}$.

Step 3: Combine results: From $b^{-1} > 0$ (Step 1) and $b^{-1} < a^{-1}$ (Step 2), we conclude by transitivity:

$$0 < b^{-1} < a^{-1}.$$

□

Problem 6.

Solution: (a) **Set** $A = [1,2) \cup (3,\infty)$

- **Minimum:** 1 (since 1 is included in the interval $[1,2)$).
- **Maximum:** Does not exist (the set is unbounded above).
- **Infimum:** $\inf A = 1$.
- **Supremum:** $\sup A = \infty$.

(b) **Set** $B = \{r \in \mathbb{Q} \mid r < 2\}$

- **Minimum:** Does not exist (no smallest rational number less than 2).
- **Maximum:** Does not exist (approaches 2 but never attains it).
- **Infimum:** $\inf B = -\infty$.
- **Supremum:** $\sup B = 2$.

(c) **Set** $C = \{r \in \mathbb{Q} \mid r^2 < 2\}$

- **Minimum:** Does not exist (approaches $-\sqrt{2}$ but never attains it in $\mathbb{Q}$).
- **Maximum:** Does not exist (approaches $\sqrt{2}$ but never attains it in $\mathbb{Q}$).
- **Infimum:** $\inf C = -\sqrt{2}$.
- **Supremum:** $\sup C = \sqrt{2}$.

(d) **Set** $D = \left\{\frac{1}{m} + n \mid m, n \in \mathbb{N}\right\}$

- **Minimum:** Does not exist (smallest term approaches 1 as $m \to \infty$, but 1 is not attained).
- **Maximum:** Does not exist (unbounded above as $n \to \infty$).
- **Infimum:** $\inf D = 1$.
- **Supremum:** $\sup D = \infty$.

(e) **Set** $E = \{\sqrt{2}, e, \pi\}$

- **Minimum:** $\sqrt{2}$
- **Maximum:** π
- **Infimum:** $\inf E = \sqrt{2}$.
- **Supremum:** $\sup E = \pi$.

(f) **Set** $F = \{2 - x^2 \mid x \in \mathbb{R}\}$

- **Minimum:** Does not exist (unbounded below as $x \to \pm\infty$).
- **Maximum:** 2 (attained at $x = 0$).
- **Infimum:** $\inf F = -\infty$.
- **Supremum:** $\sup F = 2$.

$\square$

Problem 7.

Solution: **(a)**

Statement: $a < b$ if and only if $a < b + \epsilon$ for all $\epsilon > 0$.

Disprove: We will show that the statement is false by providing a counterexample.

($\not\Longleftarrow$):

Consider for a counterexample: Let $a = 1$ and $b = 1$. Then $a < b$ is false, since $1 \not< 1$.

Since $\epsilon > 0$, we have $1 + \epsilon > 1$, so $1 < 1 + \epsilon$ is true for all $\epsilon > 0$.

Thus, for $a = 1$ and $b = 1$, the condition $a < b + \epsilon$ for all $\epsilon > 0$ is true, but $a < b$ is false.

Therefore, the reverse implication is false, and the statement "$a < b$ if and only if $a < b+\epsilon$ for all $\epsilon > 0$" is false.

Counterexample: Let $a = 1$ and $b = 1$. Then $a < b + \epsilon$ for all $\epsilon > 0$, but $a \not< b$.

(b)

Statement: $a \leq b$ if and only if $a < b + \epsilon$ for all $\epsilon > 0$.

Proof:

($\Longrightarrow$)

Assume $a \leq b$.

Since $\epsilon > 0$, we know $b < b + \epsilon$.

If $a < b$, then $a < b < b + \epsilon$, so $a < b + \epsilon$.

If $a = b$, then $a = b < b + \epsilon$, so $a < b + \epsilon$.

In both cases, if $a \leq b$, then $a < b + \epsilon$ for all $\epsilon > 0$. Therefore, the forward direction is true.

($\Longleftarrow$)

Assume $a < b + \epsilon$ for all $\epsilon > 0$. Suppose for contradiction that $a > b$.

Let $\delta = a - b$. Since $a > b$, we have $\delta > 0$.

Choose $\epsilon = \frac{\delta}{2} = \frac{a-b}{2}$. Since $\delta > 0$, we have $\epsilon > 0$.

By our assumption, $a < b + \epsilon$ for all $\epsilon > 0$, so it must be true for $\epsilon = \epsilon_0 = \frac{a-b}{2}$.

Thus,

$$a < b + \epsilon_0 = b + \frac{a-b}{2} = \frac{2b + a - b}{2} = \frac{a+b}{2}$$

So we have $a < \frac{a+b}{2}$. Multiplying both sides by 2 gives $2a < a + b$. Subtracting a from both sides gives $a < b$.

We started by assuming $a > b$ and derived $a < b$, which is a contradiction.

Therefore, we have $a \leq b$.

Therefore, the reverse direction is true.

Since both directions are true, the statement "$a \leq b$ if and only if $a < b + \epsilon$ for all $\epsilon > 0$" is true. $\square$

Problem 8.

Solution: Let A and B be non-empty bounded subsets of $\mathbb{R}$, and define the set

$$A - B = \{a - b \mid a \in A, b \in B\}.$$

We aim to prove that

$$\sup(A - B) = \sup A - \inf B.$$

Step 1: Prove that $\sup(A - B) \leq \sup A - \inf B$

Let $x \in A - B$. Then there exist $a \in A$ and $b \in B$ such that

$$x = a - b.$$

Since $a \leq \sup A$ (as $\sup A$ is the least upper bound of A) and $b \geq \inf B$ (as $\inf B$ is the greatest lower bound of B), we obtain

$$x = a - b \leq \sup A - \inf B.$$

Since this holds for all $x \in A - B$, it follows that

$$\sup(A - B) \leq \sup A - \inf B.$$

Step 2: Prove that $\sup(A - B) \geq \sup A - \inf B$

We need to show that for any $\epsilon > 0$, there exists $x \in A - B$ such that

$$x > \sup A - \inf B - \epsilon.$$

Since $\sup A$ is the least upper bound of A, there exists $a' \in A$ such that

$$a' > \sup A - \frac{\epsilon}{2}.$$

Similarly, since $\inf B$ is the greatest lower bound of B, there exists $b' \in B$ such that

$$b' < \inf B + \frac{\epsilon}{2}.$$

Consider $x = a' - b'$. Then:

$$\begin{aligned} x &= a' - b' \\ &> \left(\sup A - \frac{\epsilon}{2}\right) - \left(\inf B + \frac{\epsilon}{2}\right) \\ &= \sup A - \inf B - \epsilon. \end{aligned}$$

Thus, for any $\epsilon > 0$, there exists $x \in A - B$ such that $x > \sup A - \inf B - \epsilon$, which implies

$$\sup(A - B) \geq \sup A - \inf B.$$

Conclusion:

Since we have shown both

$$\sup(A - B) \leq \sup A - \inf B \quad \text{and} \quad \sup(A - B) \geq \sup A - \inf B,$$

it follows that

$$\sup(A - B) = \sup A - \inf B.$$

$\square$

Problem 9.

Solution: (a)

The statement is **false**.

Counterexample: Let $C = [-2, -1]$ and $D = [3, 4]$.

Then:

$$\inf C = -2$$
$$\inf D = 3$$
$$(\inf C)(\inf D) = (-2)(3) = -6$$

However, for $M = \{cd \mid c \in C,\, d \in D\}$:

$$\inf M = (-2)(4) = -8$$

Since $\inf M \neq (\inf C)(\inf D)$, the statement is false.

(b)

The statement is **true**.

Proof: Assume $\sup C < \inf D$. Let $\alpha = \sup C$ and $\beta = \inf D$. Define $r = \frac{\alpha+\beta}{2}$. Then:

$$\alpha < r < \beta$$

For all $c \in C$: $c \leq \sup C = \alpha < r$, so $c < r$.

For all $d \in D$: $d \geq \inf D = \beta > r$, so $r < d$.

Therefore, for all $c \in C$ and $d \in D$: $c < r < d$.

(c)

The statement is **true**.

Proof: Assume there exists $r \in \mathbb{R}$ such that $c < r < d$ for all $c \in C$ and $d \in D$. Let $\alpha = \sup C$ and $\beta = \inf D$.

Since $c < r$ for all $c \in C$, r is an upper bound of C, so $\alpha \leq r$.

Since $r < d$ for all $d \in D$, r is a lower bound of D, so $r \leq \beta$.

Therefore:

$$\alpha \leq r \leq \beta$$

Suppose, for contradiction, that $\alpha \geq \beta$. Then:

$$\alpha \geq \beta \geq r \geq \alpha$$

implying $\alpha = \beta = r$.

But then:

- Since $c < r$ for all $c \in C$, no element of C equals α
- Since $r < d$ for all $d \in D$, no element of D equals β

By definition of supremum, for any $\epsilon > 0$, there exists $c_\epsilon \in C$ such that:

$$\alpha - \epsilon < c_\epsilon < \alpha$$

Similarly, there exists $d_\epsilon \in D$ such that:

$$\beta < d_\epsilon < \beta + \epsilon$$

As $\epsilon \to 0$, both sequences approach r, making $d_\epsilon - c_\epsilon \to 0$. This contradicts the requirement of a non-zero gap between C and D implied by $c < r < d$.

Therefore, $\alpha < \beta$, i.e., $\sup C < \inf D$. $\square$

Problem 10.

Solution: Let $a, b \in \mathbb{R}$ with $a < b$. We will construct an irrational number $x \in \mathbb{I}$ such that $a < x < b$.

Consider the interval $(a - \sqrt{2}, b - \sqrt{2})$. Since $a < b$, we have $a - \sqrt{2} < b - \sqrt{2}$, so this interval is non-empty.

By the density of rational numbers in $\mathbb{R}$, there exists $r \in \mathbb{Q}$ such that

$$a - \sqrt{2} < r < b - \sqrt{2}$$

Let $x = r + \sqrt{2}$. We claim this x satisfies our requirements:

(a) First, $x \in \mathbb{I}$: Since $r \in \mathbb{Q}$ and $\sqrt{2} \in \mathbb{I}$, their sum $x = r + \sqrt{2}$ must be irrational. (If it were rational, then $\sqrt{2} = x - r$ would be rational, a contradiction.)

(b) Second, $a < x < b$: Adding $\sqrt{2}$ to each part of the inequality $a - \sqrt{2} < r < b - \sqrt{2}$ gives:

$$a - \sqrt{2} + \sqrt{2} < r + \sqrt{2} < b - \sqrt{2} + \sqrt{2}$$

which simplifies to

$$a < x < b$$

Therefore, we have constructed an irrational number $x \in \mathbb{I}$ such that $a < x < b$. $\square$

Problem 11.

Solution: (a) For $\dfrac{n^2+3}{n^2-3}$:

Dividing numerator and denominator by n^2:

$$\frac{n^2+3}{n^2-3} = \frac{1+\frac{3}{n^2}}{1-\frac{3}{n^2}} \to 1 \text{ as } n \to \infty$$

Therefore, $\lim_{n\to\infty} \frac{n^2+3}{n^2-3} = 1$

(b) For $(-1)^n n$:

When n is even, the term is n When n is odd, the term is $-n$ The sequence alternates between increasingly large positive and negative values. Therefore, the limit **does not exist.**

(c) For $\dfrac{4n+2}{3-5n^2}$:

Dividing numerator and denominator by n^2:

$$\frac{4n+2}{3-5n^2} = \frac{\frac{4}{n}+\frac{2}{n^2}}{\frac{3}{n^2}-5} \to 0 \text{ as } n \to \infty$$

Therefore, $\lim_{n\to\infty} \frac{4n+2}{3-5n^2} = 0$

(d) For $\sqrt{n} - \sqrt{n-1}$:

Rationalizing the numerator:

$$\sqrt{n} - \sqrt{n-1} = \frac{1}{\sqrt{n}+\sqrt{n-1}} \to 0 \text{ as } n \to \infty$$

Therefore, $\lim_{n\to\infty}(\sqrt{n} - \sqrt{n-1}) = 0$

(e) For $\sqrt{n^2+n} - n$:

Rewriting as $n(\sqrt{1+\frac{1}{n}} - 1)$ and using binomial expansion:

$$\sqrt{1+\frac{1}{n}} \approx 1 + \frac{1}{2n} - \frac{1}{8n^2} + \cdots$$

Therefore, $\lim_{n\to\infty}(\sqrt{n^2+n} - n) = \frac{1}{2}$

(f) For $\frac{n!}{8^n}$:

Using Stirling's approximation:

$$\frac{n!}{8^n} \approx \left(\frac{n}{8e}\right)^n \sqrt{2\pi n}$$

Since $\frac{n}{8e} > 1$ for large enough n, this grows without bound. Therefore, $\lim_{n\to\infty} \frac{n!}{8^n} = \infty$ $\square$

Problem 12.

Solution: (A)

We will use the series expansion of the mathematical constant e to construct the sequence.

Recall that the number e is defined by the infinite series:

$$e = \sum_{k=0}^{\infty} \frac{1}{k!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots \tag{1}$$

It is a well-known fact that e is irrational. We define the sequence (q_n) as the n-th partial sum of the series for e:

$$q_n = \sum_{k=0}^{n} \frac{1}{k!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \dots + \frac{1}{n!} \tag{2}$$

Rationality of q_n:

Each term $\frac{1}{k!}$ is rational since both the numerator and denominator are integers.
A finite sum of rational numbers is rational.
Therefore, each q_n is rational.

Limit of q_n:

By the definition of e in equation (1), we have:

$$\lim_{n\to\infty} q_n = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{1}{k!} = e$$

Since e is irrational, the limit is irrational.

In conclusion, the sequence $\{q_n\}$ consists of rational numbers and converges to the irrational number e.

(B)

We will construct a sequence converging to the rational number 1.

Define the sequence (p_n) by:

$$p_n = 1 + \frac{\sqrt{2}}{n}, \quad \text{for } n \geq 1 \tag{3}$$

Proof. We will show that each p_n is irrational.

- Since $\sqrt{2}$ is irrational and n is a positive integer, $\frac{\sqrt{2}}{n}$ is irrational.[1]
- Suppose, for contradiction, that p_n is rational for some n.

[1] Dividing an irrational number by a non-zero integer yields an irrational number because if $\frac{\sqrt{2}}{n}$ were rational, then $\sqrt{2} = n \times (\text{rational})$ would be rational, which contradicts the irrationality of $\sqrt{2}$.

- Then $p_n - 1 = \frac{\sqrt{2}}{n}$ would be rational (since the difference of two rationals is rational).
- This implies that $\sqrt{2} = n(p_n - 1)$ is rational (product of an integer and a rational).
- This contradicts the fact that $\sqrt{2}$ is irrational.
- Therefore, our assumption is false, and p_n must be irrational for all n.

$\square$

Limit of p_n:

$$\begin{aligned}
\lim_{n\to\infty} p_n &= \lim_{n\to\infty} \left(1 + \frac{\sqrt{2}}{n}\right) \\
&= 1 + \lim_{n\to\infty} \frac{\sqrt{2}}{n} \\
&= 1 + 0 \\
&= 1
\end{aligned}$$

The limit is the rational number 1.

Conclusion: The sequence $\{p_n\}$ consists of irrational numbers and converges to the rational number 1.

$\square$