Homework sheet 2 – Due 02/16/2025

ATTENTION: Version with solutions.

Comment: *Part of this exercise includes functions of matrices. These are defined by the Taylor series of the corresponding function, e.g.* $e^M = \sum_{k=0} \frac{M^k}{k!}$ *for any matrix* M*.*

Problem 1: Matrix Operations [1 + 2 + 1 + 2 + 2 + 1 +1 = 10 points]

In this exercise we prove some useful matrix identities.

a) For matrices A, B, C, prove

$$[A, BC] = ABC - BCA + BAC - BAC = B[A, C] + [A, B]C. \tag{1}$$

b) Prove the Bianchi identity for matrices A, B, C

$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0. \tag{2}$$

$$[A, [B, C]] = \underbrace{B[A, C] + [A, B]C}_{=-BCA+ABC} - \underbrace{(C[A, B] + [A, C]B)}_{=-CBA+ACB} \tag{3}$$

$$[B, [C, A]] = \underbrace{C[B, A] + [B, C]A}_{=-CAB+BCA} - \underbrace{(A[B, C] + [B, A]C)}_{=-ACB+BAC} \tag{4}$$

$$[C, [A, B]] = \underbrace{A[C, B] + [C, A]B}_{=-ABC+CAB} - \underbrace{(B[C, A] + [C, B]A)}_{=-BAC+CBA} \tag{5}$$

Clearly, all terms underneath the brackets sum up to zero.

c) Prove that

$$[A, B]^\dagger = -[A^\dagger, B^\dagger] \text{ and } [A, B]^T = -[A^T, B^T]. \tag{6}$$

$[A, B]^\dagger = (AB - BA)^\dagger = (B^\dagger A^\dagger - A^\dagger B^\dagger) = -[A^\dagger, B^\dagger]$ and analogously for transposition.

d) Consider two matrices A, B such that $[A, B] = C$ where $[A, C] = 0 = [B, C]$

Prove

$$e^{\alpha A} B e^{-\alpha A} = B + \alpha C \tag{7}$$

for arbitrary $\alpha \in \mathbb{C}$.

$$e^{\alpha A}B = B + \sum_{k=1}^{\infty} \alpha^k \frac{A^k}{k!} B = B + \sum_{k=1}^{\infty} \alpha^k \frac{kCA^{k-1} + BA^k}{k!} = (B + \alpha C)e^{\alpha A}, \quad (8)$$

or $e^{\alpha A}Be^{-\alpha A} = B + \alpha C$. At the second equality we used

$$\begin{aligned} A^k B &= A^{k-1}([A, B] + BA) = A^{k-1}(BA + \alpha C) \\ &= A^{k-2}(ABA + \alpha CA) && (9) \\ &= A^{k-2}((BA + \alpha C)A + \alpha CA) && (10) \\ &= \dots && (11) \\ &= BA^k + k\alpha CA^{k-1} && (12) \end{aligned}$$

e) Campbell-Baker-Hausdorff formula. Consider matrices A, B, C with the same properties as in the previous problem. Show

$$e^A e^B = e^{A+B+C/2}. \quad (13)$$

Hint: *Define a function $T(\alpha) = e^{\alpha A}e^{\alpha B}$ and first study its α-derivative. Use the result from part d) .*

$$\begin{aligned} \frac{\partial}{\partial \alpha} T(\alpha) &= Ae^{\alpha A}e^{\alpha B} + e^{\alpha A}Be^{\alpha B} && (14) \\ &= (A + e^{\alpha A}Be^{-\alpha A})T(\alpha) && (15) \\ &= (A + B + \alpha C)T(\alpha), && (16) \end{aligned}$$

or (multiplying the equation with $T^{-1}(\alpha)$ from the right

$$[\frac{\partial}{\partial \alpha} T(\alpha)]T^{-1}(\alpha) = \frac{\partial}{\partial \alpha} \ln[T(\alpha)] = (A + B + \alpha C). \quad (17)$$

(The second equation implicitly assumes that $\left[\ln[T(\alpha)], \ln[T(\alpha')]\right] = 0$ even at $\alpha \neq \alpha'$, something we can readily check at the end). We integrate over α and obtain

$$\ln[T(\alpha)] = \alpha[A + B + \frac{\alpha}{2}C] + \text{const}. \quad (18)$$

Taylor expansion of the defining equation $T(\alpha) = e^{\alpha A}e^{\alpha B} \simeq \mathbf{1} + \alpha(A + B)$ in small α fixes the integration constant to zero. The evaluation of Eq. (18) at $\alpha = 1$ is then the CBH formula.

f) For an involutory matrix A (i.e. $A^2 = \mathbf{1}$), prove

$$e^{i\alpha A} = \cos(\alpha)\mathbf{1} + i\sin(\alpha)A \tag{19}$$

$$e^{i\alpha A} = \sum_{k=0}^{\infty} \frac{(i\alpha A)^k}{k!} = \sum_{k=0}^{\infty} (-1)^k \frac{(\alpha)^{2k}}{2k!}\mathbf{1} + iA\sum_{k=0}^{\infty} (-1)^k \frac{(\alpha)^{2k+1}}{(2k+1)!} = Q.E.D. \tag{20}$$

g) Prove that, for any diagonalizable matrix M

$$\ln(\det(M)) = \mathrm{tr}\,(\ln(M)). \tag{21}$$

We use diagonalizability to write $M = U^{-1}\Lambda U$, where $\Lambda = \mathrm{diag}(\lambda_i)$:

$$\begin{aligned}\mathrm{tr}\,(\ln(M)) &= -\sum_{k=0}^{\infty} \mathrm{tr}\left[\frac{(\mathbf{1}-M)^k}{k}\right] = -\sum_{k=0}^{\infty} \mathrm{tr}\left[U^{-1}\frac{(\mathbf{1}-\Lambda)^k}{k}U\right] \\ &\overset{\text{cycl. of tr}}{=} \sum_i \ln\lambda_i = \ln\prod_i \lambda_i = \ln(\det(M)).\end{aligned} \tag{22}$$

Problem 2: Single-qubit gates [1 +1 + 1 +2 + 2 + 2 + 1= 10 points]

In quantum information theory it is common practice to denote the Pauli gates as

$$X = \sigma_x, Y = \sigma_y, Z = \sigma_z. \tag{23}$$

Rotations about the x-axis are denoted $R_x(\theta_x) = e^{-i\theta_x X/2}$ (and analogously for y and z).

a) Show that, up to a phase, the $\pi/8$ gate $T = R_z(\pi/4)$.

$$R_z(\pi/4) = e^{-i\pi/8}\begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4}\end{pmatrix} \tag{24}$$

b) Show that, up to a phase, the Hadamard gate is a concatenation of $R_y(\theta_y)$ and $R_z(\theta_z)$. Determine the angles θ_y, θ_z.

$$H = \frac{1}{\sqrt{2}}[Z + X] = Z\frac{\mathbf{1}+iY}{\sqrt{2}} = -ie^{i\frac{\pi}{2}Z}e^{i\frac{\pi}{4}Y}. \tag{25}$$

so the angle are $\theta_z = \pi, \theta_y = \pi/2$. Note that the concatenation is not unique.

c) Show that, since $YXY = -X$,

$$YR_X(\theta_x)Y = R_X(-\theta_x),$$

(i.e. the direction of rotation is reversed by Y).

$$YR_X(\theta_x)Y = Y[\cos(\theta_x/2) + i\sin(\theta_x/2)X]Y = [\cos(\theta_x/2) - i\sin(\theta_x/2)X] = R_X(-\theta_x) \tag{26}$$

d) Show the following identities for single-qubit gates

$$HXH = Z, HYH = -Y, HZH = X. \tag{27}$$

We use $H = [X+Z]/\sqrt{2}, H^2 = 1$. Then $HYH = -Y$ follows trivially from the Pauli algebra.

$$HXH = \frac{1}{2}[X+Z]X[X+Z] = \frac{1}{2}X[X-Z][X+Z] = -iXY = Z \tag{28}$$

$$HZH = H^2XH^2 = X \tag{29}$$

e) Show that (up to a phase)

$$HTH = R_x(\pi/4). \tag{30}$$

$$\begin{aligned} HTH &= e^{i\pi/8}H[R_z(\pi/4)]H = e^{i\pi/8}H[\cos(\pi/8) - i\sin(\pi/8)Z]H \\ &= e^{i\pi/8}[\cos(\pi/8) - i\sin(\pi/8)X] = e^{i\pi/8}R_x(\pi/4). \end{aligned} \tag{31}$$

f) Calculate eigenstates of X, Y along with corresponding eigenvalues.

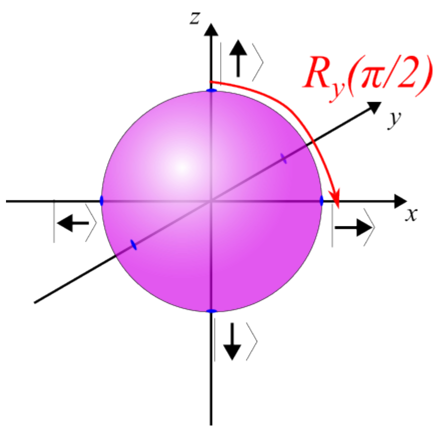
$$X\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad X\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = -\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \tag{32}$$

$$Y\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}, \qquad Y\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix} = -\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}. \tag{33}$$

g) Show that the eigenstate of X with eigenvalue $+1$ can be obtained by applying $R_y(\pi/2)$ on $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Illustrate this statement on the Bloch sphere.

$$R_y(\pi/2)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\mathbf{1} - iY}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{34}$$

which is the eigenstate of X with eigensvalue 1, as mentioned in the previous subsection. For illustration see below



Problem 3: Higher spin systems. Spin-1 systems [2 + 2 + 1 + 3 +2= 10 points]

a) In the lectures we found out that $\hat{J}_\pm |j,m\rangle = \hbar c_\pm |j, m \pm 1\rangle$, but did not determine the constant $c_\pm$. Calculate $c_\pm$ (we assume $c_\pm > 0$) when all $|j,m\rangle$ are orthonormalized.

Recall that

$$\hat{J}_\mp \hat{J}_\pm = \hat{\vec{J}}^2 - \hat{J}_z^2 \mp \hbar \hat{J}_z, \tag{35}$$

Hence

$$\begin{aligned} \hbar^2 |c_\pm|^2 &= \langle j,m|\hat{J}_\mp \hat{J}_\pm|j,m\rangle \\ &= \langle j,m|[\hat{\vec{J}}^2 - \hat{J}_z^2 \mp \hbar \hat{J}_z]|j,m\rangle \\ &= \hbar^2[j(j+1) - m^2 \mp m] \end{aligned} \tag{36}$$

$$\Rightarrow |c_\pm| = \sqrt{j(j+1) - m(m \pm 1)} \tag{37}$$

b) For spin-j systems consider the normalized magnetizations $\hat{m}_i = \hat{J}_i/[\hbar j]$ and calculate the Heisenberg-bound on the combined uncertainty of $\hat{m}_x$ and $\hat{m}_y$. Why is $j \to \infty$ sometimes called the semiclassical limit?

$$
\begin{aligned}
\Delta\hat{m}_x^2 \Delta\hat{m}_y^2 &= \frac{1}{\hbar^4 j^4}\Delta\hat{J}_x^2\Delta\hat{J}_y^2 \\
&\geq \frac{1}{\hbar^4 j^4}\frac{1}{4}|\langle[\hat{J}_x, \hat{J}_y]\rangle|^2 \\
&= \frac{1}{\hbar^4 j^4}\frac{\hbar^2}{4}|\langle\hat{J}_z\rangle|^2 \\
&= \frac{1}{4j^2}|\langle\hat{m}_z\rangle|^2
\end{aligned}
\tag{38}
$$

The lower bound vanishes for $j \to \infty$, hence the semiclassical behavior.

c) Explicitly present $\hat{J}_x, \hat{J}_y, \hat{J}_z$ for spin-1 systems (in the basis where $\hat{J}_z$ is diagonal).

Comment: *Exemplary spin-1 systems of relevance in atomic physics are ^{87}Rb and ^{23}Na (their groundstate forms a "hyperfine" triplet), which were the first cold atomic gases to display Bose-Einstein condensation (Nobel prize 2001).*

$$
\hat{J}_x = \hbar\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \hat{J}_x = \hbar\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \hat{J}_z = \hbar\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}
\tag{39}
$$

d) Convince yourself that spin nematicity operators (i.e. magnetic quadrupole operators)

$$
\hat{N}_{ij} = \frac{1}{2}\{\hat{J}_i, \hat{J}_j\} - \frac{1}{3}\delta_{ij}\hat{\vec{J}}^2, \quad i, j = x, y, z. \tag{40}
$$

vanish for spin-1/2 systems but they do exist for spin-1 systems. How many non-trivial $\hat{N}_{ij}$ are there for spin-1? Calculate them explicitly.

For spin-1/2 we use $\hat{\vec{J}}^2 = \hbar^2 3/4$ and $\{\hat{J}_i, \hat{J}_j\} = \hbar^2\delta_{ij}/2$ to see that $\hat{N}_{ij} = 0$.
For spin-1 we find the following 6 matrices (clearly $\hat{N}_{ij} = \hat{N}_{ji}$ so there are a total of

9 $\hat{N}_{ij}$ and their trace explicitly)

$$\hat{N}_{xy} = \hbar^2 \begin{pmatrix} 0 & 0 & -\frac{i}{2} \\ 0 & 0 & 0 \\ \frac{i}{2} & 0 & 0 \end{pmatrix} \qquad \hat{N}_{zz} = \hbar^2 \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & -\frac{2}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix} \tag{41}$$

$$\hat{N}_{yz} = \hbar^2 \begin{pmatrix} 0 & -\frac{i}{2\sqrt{2}} & 0 \\ \frac{i}{2\sqrt{2}} & 0 & \frac{i}{2\sqrt{2}} \\ 0 & -\frac{i}{2\sqrt{2}} & 0 \end{pmatrix} \qquad \hat{N}_{xx} = \hbar^2 \begin{pmatrix} -\frac{1}{6} & 0 & \frac{1}{2} \\ 0 & \frac{1}{3} & 0 \\ \frac{1}{2} & 0 & -\frac{1}{6} \end{pmatrix} \tag{42}$$

$$\hat{N}_{zx} = \hbar^2 \begin{pmatrix} 0 & \frac{1}{2\sqrt{2}} & 0 \\ \frac{1}{2\sqrt{2}} & 0 & -\frac{1}{2\sqrt{2}} \\ 0 & -\frac{1}{2\sqrt{2}} & 0 \end{pmatrix} \qquad \hat{N}_{yy} = \hbar^2 \begin{pmatrix} -\frac{1}{6} & 0 & -\frac{1}{2} \\ 0 & \frac{1}{3} & 0 \\ -\frac{1}{2} & 0 & -\frac{1}{6} \end{pmatrix} \tag{43}$$

e) Which $\hat{N}_{jk}$ and $\hat{J}_i$ are compatible? Calculate $[\hat{J}_i, \hat{N}_{jk}]$ to find out.

$$[\hat{J}_i, \hat{N}_{jk}] = \frac{i}{2}\epsilon_{ijl}(\hat{J}_l\hat{J}_k + \hat{J}_k\hat{J}_l) + j \leftrightarrow k = i\epsilon_{ijl}\hat{N}_{lk} + i\epsilon_{ikl}\hat{N}_{lj}. \tag{44}$$

The commutator vanishes if $i = j = k$, otherwise it's non-zero.

Problem 4: Lie Algebra for special unitary group SU(N)[2+3 + 2 + 3 + 2 + 3= 10 points + 5 bonus points.]

For a Lie group G, the Lie algebra $\mathfrak{g}$ is given by the d_G-dimensional real vector space of generators λ_a of the group supplemented with the Lie-bracket

$$[\lambda_a, \lambda_b] = i\sum_c f_{abc}\lambda_c. \tag{45}$$

The exponential map relates Lie algebra and Lie group

$$\exp : \mathfrak{g} \to G, \alpha \mapsto e^{i\alpha}, \tag{46}$$

In this exercise we consider the special unitary group $G = SU(N)$ of $N \times N$ unitary matrices U with unit determinant $\det(U) = 1$. Elements of the Lie algebra $\alpha \in \mathfrak{su}(N)$ are $N \times N$ matrices, the Lie bracket is just the matrix commutator and the exponential map is just the matrix exponential. This is called the "fundamental representation" of the Lie algebra.

a) Prove that $\mathfrak{su}(N)$ is spanned by traceless, Hermitian matrices.

Hint: *Eq.* (21).

The relationship of det and tr implies tracelessness and $[e^{i\alpha}]^\dagger = [e^{i\alpha}]^{-1}$ only iff $\alpha = \alpha^\dagger$.

The Lie algebra can be equipped with an inner product

$$\langle \alpha, \beta \rangle = \frac{1}{2} \mathrm{tr}\,[\alpha\beta], \tag{47}$$

and we assume the $\{\lambda_a\}_{a=1}^{d_G}$ to be orthonormal with respect to this inner product, hence elements $\alpha \in \mathfrak{su}(N)$ can be expanded as $\alpha = \sum_{a=1}^{d_{SU(N)}} \alpha_a \lambda_a$.

b) Prove that $d_{SU(N)} = N^2 - 1$ and use the results from homework sheet 1 to convince yourself that the Pauli matrices form an orthonormal basis for the fundamental representation of $\mathfrak{su}(2)$. Which physical spin does the fundamental representation of SU(2) correspond to?

The number linearly independent hermitian $N \times N$ matrices is $d_{SU(N)} = N^2 - 1$. We saw on sheet one that any hermitian 2x2 matrix can be expanded in $2^1 - 1 = 3$ Pauli matrices and that they are orthonormalized to the above norm. The fundamental representation corresponds to spin 1/2.

c) For general N, use the orthonormal basis of the fundamental representation to show that the "structure factors" f_{abc} are real, totally antisymmetric tensors.

We multiply the definition of the the Lie bracket, Eq. (45), with $-i\lambda_d/2$ and take the trace. Orthonormality implies that we can write

$$-i\mathrm{tr}\,[[\lambda_a, \lambda_b]\lambda_d]/2 = f_{abc}\mathrm{tr}\,[\lambda_c\lambda_d]/2 = f_{abd}. \tag{48}$$

Since $-i[\lambda_a, \lambda_b]$ is Hermitian, f_{abd} is real. To see the total antisymmetry we use this expression

$$f_{abc} = -i\mathrm{tr}\,[[\lambda_a, \lambda_b]\lambda_c]/2 = -i\mathrm{tr}\,[\lambda_a\lambda_b\lambda_c - \lambda_b\lambda_a\lambda_c]/2, \tag{49}$$

which is manifestly totally antisymmetric as a consequence of the cyclicity of the trace.

A "faithful representation" of a Lie algebra is an injective map $D : \alpha \mapsto D(\alpha)$, where $\alpha \in \mathfrak{g}$ and $D(\alpha)$ is a $d_{D(\mathfrak{g})} \times d_{D(\mathfrak{g})}$ dimensional matrix and the matrices $\{D(\lambda_a)\}_{a=1}^{d_G}$ fulfill the same Lie algebra as $\{\lambda_a\}_{a=1}^{d_G}$

$$[D(\lambda_a), D(\lambda_b)] = i \sum_c f_{abc} D(\lambda_c). \tag{50}$$

d) Use the Bianchi identity to show that the $d_G \times d_G$ matrices $[T_a]_{bc} = -if_{abc}$ fulfill the Lie algebra (they form the "adjoint representation" $D(\lambda_a) = T_a$).

(We here use Einstein summation convention)

$$[[\lambda_a, \lambda_b], \lambda_c] = -f_{abd} f_{dce} \lambda_e = \underbrace{(T_a)_{bd}(T_e)_{dc}}_{1} \lambda_e \tag{51}$$

$$[[\lambda_b, \lambda_c], \lambda_a] = -f_{bcd} f_{dae} \lambda_e = \underbrace{f_{aed}[-i(T_d)_{bc}]}_{2} \lambda_e \tag{52}$$

$$[[\lambda_c, \lambda_a], \lambda_b] = -f_{cad} f_{dbe} \lambda_e = \underbrace{(T_e)_{db}(T_a)_{dc}}_{3} \lambda_e \tag{53}$$

Since the λ_a form an ONB we conclude

$$\begin{aligned} 0 &= 1 + 3 + 2 \\ &= [T_a T_e]_{bc} - [T_e T_a]_{bc} - i f_{aed}(T_d)_{bc} \quad \text{QED.} \end{aligned} \tag{54}$$

e) Write down the matrices of the adjoint representation of SU(2). Diagonalize one of the matrices. Which spin does this representation correspond to?

The matrices are

$$T_1 = -i \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad T_2 = -i \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad T_3 = -i \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{55}$$

In its eigenbasis $T_3 = \mathrm{diag}(1, 0, -1)$, corresponding to spin-1.

f) Write down the matrices of the orthonormal basis for the fundamental representation of SU(3) and determine a set of matrices which form an SU(2) sub-algebra.

The orthonormal basis is given by Gell-Mann matrices [link]. $\lambda_{1,2,3}$ form an SU(2) subalgebra.

g) In the energy window $E \lesssim 900 MeV$ only three quarks are relevant for quantum chromodynamics. They are distinguished by their flavor quantum number: up ($|u\rangle$), down ($|d\rangle$), strange ($|s\rangle$) with an approximate $SU(3)$ symmetry between them.

Comment: *Before this energy range was reached, aspects of particle physics could be understood by means of Heisenberg's SU(2) isospin, essentially acting in $|u\rangle$, $|d\rangle$ space. Once experiments surpassed the energy of the strange-quark rest mass $\sim 95 MeV/c^2$, SU(2) isospin flavor symmetry had to be extended to SU(3).*

i) Discuss the dimension of fundamental and adjoint SU(3) representations

ii) Based on the newly acquired knowledge on Lie algebras, explain the appearance of an octet of mesons (=quark-antiquark boundstates) for $E \lesssim 900 MeV$. Why is there only a triplet for $E \lesssim 200 MeV$?

The dimensions are 3 and $3^2 - 1 = 8$ for fundamental and adjoint representions. The meson-octet is made of quantum states which transform under the adjoint representation of SU(3). For smaller energies there is only SU(2) and we get a meson-isospin-triplet of pions.