

Math 521: Solutions to sample final problems

1. For the first series

$$\beta = \limsup |a_n|^{1/n} = \limsup |n|^{3/n} = 1$$

so the radius of convergence is $R = 1/\beta = 1$. At $n = 1$, the series is

$$\sum_{n=1}^{\infty} \frac{1}{n^3}$$

which converges (and can be verified using the integral test). At $n = -1$, the series is

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$$

which converges by the alternating series test. Hence the exact interval of convergence is $[-1, 1]$. For the second series, only every third term is non-zero, and thus

$$\beta = \limsup |a_n|^{1/n} = \lim_{n \rightarrow \infty} |a_{3n}|^{1/3n} = \lim_{n \rightarrow \infty} \left| \frac{1}{2n} \right|^{1/3n} = 1$$

so the radius of convergence is 1. At $x = 1$, the series is

$$\sum_{n=1}^{\infty} \frac{1}{2n}$$

which diverges. At $x = -1$, the series is

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{2n}$$

which converges by the alternating series test. Hence the exact interval of convergence is $(-1, 1]$. For the third series

$$\beta = \limsup |a_n|^{1/n} = \lim_{n \rightarrow \infty} |a_{2n!}|^{1/2n!} = \lim_{n \rightarrow \infty} 1 = 1$$

and hence the radius of convergence is 1. At $x = 1$

$$\sum_{n=0}^{\infty} x^{2n!} = \sum_{n=0}^{\infty} 1$$

which diverges. Since the series is even, it diverges at $x = -1$ also. Hence the exact interval of convergence is $(-1, 1)$.

2. (a) Suppose that $s_n \rightarrow s$. Then for all $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that for all $n > N$,

$$|s_n - s| < \epsilon.$$

To show that $s_n^2 \rightarrow s^2$, consider any $\epsilon > 0$. First, note that

$$|s_n^2 - s^2| = |s_n - s| \cdot |s_n + s|.$$

There exists an $N_1 \in \mathbb{N}$ such that $n > N_1$ implies that $|s_n - s| < 1$, and hence that

$$|s_n + s| \leq |s_n| + |s| < (|s| + 1) + |s| = 2|s| + 1.$$

Similarly, there exists an $N_2 \in \mathbb{N}$ such that $n > N_2$ implies that

$$|s_n - s| < \frac{\epsilon}{2|s| + 1}.$$

Hence, if $N = \max\{N_1, N_2\}$, then $n > N$ implies

$$\begin{aligned} |s_n^2 - s^2| &\leq |s_n - s| \cdot |s_n + s| \\ &< \frac{\epsilon}{2|s| + 1} (2|s| + 1) = \epsilon. \end{aligned}$$

- (b) A function f is continuous at a point x if for all sequences (x_n) such that $\lim_{n \rightarrow \infty} x_n = x$, then $\lim_{n \rightarrow \infty} f(x_n) = f(x)$. By part (a), for all $x \in \mathbb{R}$, if $x_n \rightarrow x$, then $x_n^2 \rightarrow x^2$. Thus f is continuous for all $x \in \mathbb{R}$.

Alternatively, to use the ϵ - δ property, choose $\epsilon > 0$, and fix $x_0 \in \mathbb{R}$. Then

$$|f(x) - f(x_0)| = |x^2 - x_0^2| = |x - x_0| \cdot |x + x_0|.$$

If $|x - x_0| < 1$, then $|x + x_0| \leq |x| + |x_0| < 2|x_0| + 1$. Pick $\delta = \min\{\frac{\epsilon}{2|x_0| + 1}, 1\}$.

Then $|x - x_0| < \delta$ implies

$$|f(x) - f(x_0)| < \frac{\epsilon}{2|x_0| + 1} (2|x_0| + 1) = \epsilon$$

and hence f is continuous at x_0 . Since x_0 is arbitrary, f is continuous on \mathbb{R} .

3. By the mean value theorem, there exists a $c \in (r, s)$ such that

$$f'(c) = \frac{f(s) - f(r)}{s - r} = 0$$

and there exists and $d \in (s, t)$ such that

$$f'(d) = \frac{f(t) - f(s)}{t - s} = 0.$$

By construction, $d > c$. Hence there exists an $x \in (c, d) \subseteq (0, 1)$ such that

$$f''(x) = \frac{f'(d) - f'(c)}{d - c} = 0.$$

4. (a) If $\sum_{n=0}^{\infty} a_n$ converges to a limit a , then for all $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that $k > N$ implies

$$\left| a - \sum_{n=0}^k a_n \right| < \epsilon.$$

Now consider any $\epsilon > 0$ and define N as above. If $k > N/2$, then

$$\left| a - \sum_{n=0}^k b_n \right| = \left| a - \sum_{n=0}^{2k+1} a_n \right| < \epsilon.$$

Hence $\sum_{n=0}^{\infty} b_n$ converges.

- (b) Suppose $a_n = (-1)^n$. Then

$$\sum_{n=0}^N a_n = \begin{cases} 1 & \text{if } N \text{ is even,} \\ 0 & \text{if } N \text{ is odd;} \end{cases}$$

this series diverges. However $b_n = 0$ for all n , and thus $\sum_{n=0}^{\infty} b_n$ converges.

5. If $f(a)f(b) < 0$, then it follows that either $f(a) < 0$ and $f(b) > 0$, or $f(a) > 0$ and $f(b) < 0$. In either case, zero lies between $f(a)$ and $f(b)$, and thus the intermediate value theorem can be applied to show that there exists an $x \in (a, b)$ such that $f(x) = 0$.
6. Let f be a real-valued function defined on an interval $[0, b]$ as

$$f(x) = \begin{cases} x & \text{for } x \in \mathbb{Q}, \\ 0 & \text{for } x \notin \mathbb{Q}. \end{cases}$$

Consider a partition $P = \{0 = t_0 < t_1 < \dots < t_n = b\}$. First note that

$$m(f, [t_{k-1}, t_k]) = 0, \quad M(f, [t_{k-1}, t_k]) = t_k$$

for all $k = 1, \dots, n$. The first expression follows because any interval must contain an irrational number, whereas the second expression follows because the interval must take rational values arbitrarily close to t_k . Hence

$$U(f, P) = \sum_{k=1}^n (t_k - t_{k-1}) M(f, [t_{k-1}, t_k]) = \sum_{k=1}^n (t_k - t_{k-1}) t_k.$$

For a general partition this cannot be calculated explicitly, but it will always be strictly positive, since each term in the sum is strictly positive. The lower Darboux sum is given by

$$L(f, P) = \sum_{k=1}^n (t_k - t_{k-1}) m(f, [t_{k-1}, t_k]) = 0.$$

To show that f is not integrable on $[0, b]$, consider any partition P . Then

$$\begin{aligned} U(f, P) &= \sum_{k=1}^n (t_k - t_{k-1}) t_k \\ &\geq \sum_{k=1}^n (t_k - t_{k-1}) \frac{(t_k + t_{k-1})}{2} \\ &= \frac{1}{2} \sum_{k=1}^n (t_k^2 - t_{k-1}^2) \\ &= \frac{t_n^2 - t_0^2}{2} = \frac{b^2}{2}. \end{aligned}$$

Thus for any partition,

$$U(f, P) - L(f, P) \geq \frac{b^2}{2}$$

and thus f is not integrable on $[0, b]$.

7. Suppose that $\sum_{n=1}^{\infty} f(n)$ converges, and consider the integral

$$\int_1^b f(x) dx.$$

Let k be the largest integer such that $k < b$. Hence $b - k \leq 1$. Consider the partition $P = \{1 = t_0 < t_1 < \dots < t_{k-1} < t_k = b\}$, where $t_i = i + 1$ for $i = 0, \dots, k - 1$. Then

$$\begin{aligned} U(f, P) &= \sum_{j=1}^k M(f, [t_{j-1}, t_j]) (t_j - t_{j-1}) = \sum_{j=1}^k f(t_{j-1}) (t_j - t_{j-1}) \\ &= f(k)(b - k) + \sum_{j=1}^{k-1} f(j) \\ &\leq f(k) + \sum_{j=1}^{k-1} f(j) = \sum_{j=1}^k f(j) \end{aligned}$$

so

$$\int_1^b f \leq U(f, P) \leq \sum_{j=1}^k f(j) \leq \sum_{j=1}^{\infty} f(j).$$

Since the integral is an increasing function and is bounded above, it follows that it must converge.

Now suppose that $\int_1^x f$ converges, and consider $\sum_{n=1}^N f(n)$. Consider the partition

$P = \{1 = t_0 < t_1 < \dots < t_N - 1 = N\}$ where $t_i = i + 1$ for all $i = 0, \dots, N - 1$. Then

$$\begin{aligned} L(f, P) &= \sum_{j=1}^{N-1} m(f, [t_{j-1}, t_j])(t_j - t_{j-1}) = \sum_{j=1}^{N-1} m(f, [j, j+1]) \\ &= \sum_{j=1}^{N-1} f(j+1) = \sum_{j=2}^N f(j) \end{aligned}$$

and hence

$$\sum_{j=1}^N f(j) = f(1) + L(f, P) \leq f(1) + \int_1^N f \leq f(1) + \int_1^\infty f.$$

Since $\sum_{j=1}^N f(j)$ is increasing and bounded above, then it converges.

8. (a) From the definition, it is clear $d(x, y) \geq 0$ for all $x, y \in \mathbb{R}$, and $d(x, y) = 0$ if and only if $x = y$. In addition $d(x, y) = d(y, x)$. To prove the triangle inequality, consider any $x, y, z \in \mathbb{R}$. If $x = z$, then $d(x, z) = 0$, and since $d(x, y) + d(y, z) \geq 0$, the triangle inequality is satisfied. If $x \neq z$, then $d(x, z) = 1$. Either $y \neq z$ or $y = z$, and hence $d(x, y) + d(y, z) \geq 1$, so the triangle inequality is satisfied. Hence d is a metric.

- (b) The neighborhood of radius $1/2$ at 0 is

$$\{x \in \mathbb{R} : d(0, x) < 1/2\} = \{0\}.$$

It only contains zero, since all other points are a distance of 1 away.

- (c) Consider an arbitrary set $S \subseteq \mathbb{R}$. To show that S is open, consider any $x \in S$. Then, by the same argument as in (b), $N_{1/2}(x) = \{x\} \subseteq S$ so x is an interior point. Since all points are interior, it follows that S is open.

Consider an open cover \mathcal{S} of S . Suppose S has finitely many points, so that $S = \{s_1, s_2, \dots, s_n\}$. Then there exist sets $S_1, \dots, S_n \in \mathcal{S}$ such that $s_k \in S_k$ for $k = 1, \dots, n$, and it follows that $\{S_k : k = 1, \dots, n\}$ is a finite subcover. Hence S is compact.

Suppose S has infinitely many points. Consider the cover $\mathcal{S} = \{\{x\} : x \in S\}$. By above, the sets $\{x\}$ are open. Consider a subcover $\mathcal{T} \subseteq \mathcal{S}$. For any $x \in S$, $\{x\}$ must be in \mathcal{T} , since x is not an element of any other set in \mathcal{S} . Thus $\mathcal{T} = \mathcal{S}$. Hence \mathcal{S} has no finite subcover and S is not compact.

9. Let $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$ be in \mathbb{R}^2 . Consider the function

$$d(\mathbf{x}, \mathbf{y}) = |x_1 - y_1| + |x_2 - y_2|.$$

- (a) Consider the three properties of a metric:

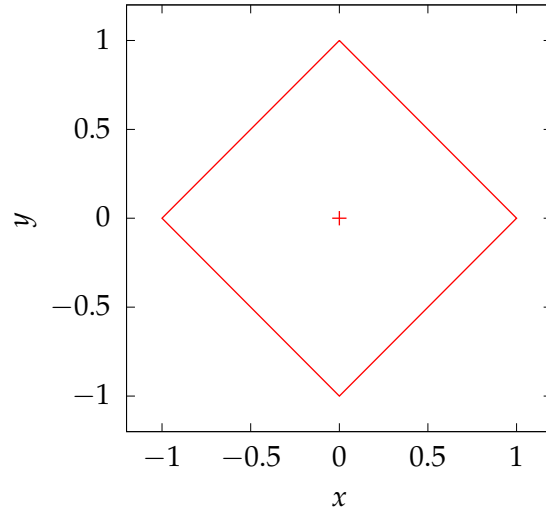


Figure 1: A plot of the neighborhood of radius 1 at $(0,0)$, considered in Problem 9(b).

M1. Note that

$$d(\mathbf{x}, \mathbf{x}) = |x_1 - x_1| + |x_2 - x_2| = 0$$

and if $d(\mathbf{x}, \mathbf{y}) = 0$, then

$$0 = |x_1 - y_1| + |x_2 - y_2|$$

from which it follows that $x_1 = y_1$ and $x_2 = y_2$, so $\mathbf{x} = \mathbf{y}$.

M2. For all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$,

$$d(\mathbf{x}, \mathbf{y}) = |x_1 - y_1| + |x_2 - y_2| = |y_1 - x_1| + |y_2 - x_2| = d(\mathbf{y}, \mathbf{x}),$$

and thus d is symmetric.

M3. For all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^2$,

$$\begin{aligned} d(\mathbf{x}, \mathbf{y}) + d(\mathbf{y}, \mathbf{z}) &= |x_1 - y_1| + |x_2 - y_2| + |y_1 - z_1| + |y_2 - z_2| \\ &\geq |x_1 - z_1| + |x_2 - z_2| \\ &= d(\mathbf{x}, \mathbf{z}), \end{aligned}$$

where the usual triangle inequality on \mathbb{R} has been applied twice. Hence d satisfies the triangle inequality.

(b) The neighborhood of radius 1 at $(0,0)$ is given by

$$N_1((0,0)) = \{\mathbf{x} \in \mathbb{R}^2 : d(\mathbf{x}, (0,0)) < 1\} = \{\mathbf{x} \in \mathbb{R}^2 : |x_1| + |x_2| < 1\}$$

Consider the quadrant where $x_1 \geq 0$ and $x_2 \geq 0$. Then $|x_1| + |x_2| = x_1 + x_2$, and thus the boundary of the neighborhood is given by $x_2 = 1 - x_1$. This is a

straight line which intercepts the x_2 axis at 1, and has slope -1 . By symmetry, it follows that the neighborhood is diamond, with corners at $(0, 1)$, $(0, -1)$, and $(-1, 0)$. It is plotted in Fig. 1.

10. Suppose that f is not constant. Then there exists $x < y$ such that $f(x) \neq f(y)$. For an arbitrary $n \in \mathbb{N}$ consider the points

$$t_k = x - \frac{(y-x)k}{n}$$

for $k = 0, \dots, n$. Then by the triangle inequality,

$$\begin{aligned} |f(x) - f(y)| &\leq \sum_{k=1}^n |f(t_k) - f(t_{k-1})| \\ &\leq \sum_{k=1}^n (t_k - t_{k-1})^2 \\ &= \sum_{k=1}^n \left(\frac{y-x}{n} \right)^2 = \frac{(y-x)^2}{n}. \end{aligned}$$

Since n is arbitrary, there exists an n such that

$$n > \frac{(y-x)^2}{|f(x) - f(y)|},$$

which leads to a contradiction. An alternative approach is to first note that for all $x, y \in \mathbb{R}$

$$\left| \frac{f(y) - f(x)}{y - x} \right| \leq \left| \frac{(y-x)^2}{y-x} \right| = |y-x|.$$

Since $|y-x| \rightarrow 0$ as $y \rightarrow x$, it follows that

$$\left| \frac{f(y) - f(x)}{y - x} \right|$$

as $y \rightarrow x$, and hence

$$\lim_{y \rightarrow x} \frac{f(y) - f(x)}{y - x} = 0$$

for all $x \in \mathbb{R}$. Thus f is differentiable on \mathbb{R} and $f'(x) = 0$ for all $x \in \mathbb{R}$. For any bounded interval $I \subseteq \mathbb{R}$, f must be constant. Hence f is constant on \mathbb{R} .

11. Suppose that f is differentiable on \mathbb{R} , and that $2 \leq f'(x) \leq 3$ for $x \in \mathbb{R}$. If $f(0) = 0$, prove that $2x \leq f(x) \leq 3x$ for all $x \geq 0$. Suppose $x > 0$. By the mean value theorem, there exists a $y \in (0, x)$ such that

$$\frac{f(x) - f(0)}{x - 0} = f'(y)$$

and hence

$$\frac{f(x)}{x} = f'(y).$$

The constraint on the derivative shows that

$$2 \leq \frac{f(x)}{x} \leq 3$$

and hence

$$2x \leq f(x) \leq 3x. \quad (1)$$

Since $f(0) = 0$, it follows that Eq. (1) holds for all $x \geq 0$.

12. Suppose that f is integrable on $[a, b]$. Then for all $\epsilon > 0$, there exists a partition P such that

$$U(f, P) - L(f, P) < \epsilon.$$

Consider the partition Q composed of the values a, c, d , and b ; in general this will have four points, but if $c = a$ or $d = b$ it may have fewer. Define a new partition $T = P \cup Q$. Then since T is a refinement,

$$U(f, T) - L(f, T) < \epsilon.$$

Write $T = \{a = t_0 < t_1 < \dots < t_n = b\}$. There exists i and j such that $t_i = c$ and $t_j = d$. Consider the partition $S = \{t_i < \dots < t_j\}$ of $[c, d]$. Then

$$\begin{aligned} U(f, S) - L(f, S) &= \sum_{k=i+1}^j (M(f, [t_{k-1}, t_k]) - m(f, [t_{k-1}, t_k])) \\ &\leq \sum_{k=1}^n (M(f, [t_{k-1}, t_k]) - m(f, [t_{k-1}, t_k])) \\ &= U(f, T) - L(f, T) \\ &< \epsilon. \end{aligned}$$

Since this is true for an arbitrary $\epsilon > 0$, it follows that f is integrable on $[c, d]$.

13. (a) Suppose that $r^{1/3}$ is rational. Then

$$r^{1/3} = \frac{p}{q}$$

for $p, q \in \mathbb{Z}$, where $q \neq 0$. Hence

$$r = \frac{p^3}{q^3}$$

and thus r is rational. Thus if r is irrational, then $r^{1/3}$ is irrational also. Similarly, if $r + 1$ is rational, then

$$r + 1 = \frac{p}{q}$$

for $p, q \in \mathbb{Z}$, where $q \neq 0$, and

$$r = \frac{p}{q} - 1 = \frac{p - q}{q}$$

so r is rational. Thus if r is irrational, then $r + 1$ is irrational also.

(b) First, consider the number $x = 5 + \sqrt{2}$. Then

$$\begin{aligned} x - 5 &= \sqrt{2}, \\ (x - 5)^2 &= 2, \\ x^2 - 10x + 25 &= 2, \\ x^2 - 10x + 23 &= 0. \end{aligned}$$

By the rational zeroes theorem, if x is rational, then $x = \pm 1$ or $x = \pm 23$. But $x > 5$ and $x < 5 + 2 = 7$, so none of these possibilities are valid, and thus x is irrational. By the results of part (a), $(5 + \sqrt{2})^{1/3}$ is irrational, and $(5 + \sqrt{2})^{1/3} + 1$ is irrational also.

14. For the first limit, L'Hôpital's rule can be applied once to show that

$$\lim_{x \rightarrow 0} \frac{x}{1 - e^{-x^2-3x}} = \lim_{x \rightarrow 0} \frac{1}{(2x + 3)e^{-x^2-3x}} = \frac{1}{3}.$$

The second limit can first be rewritten as

$$\lim_{x \rightarrow 0} \left(\frac{1}{\sin x} - \frac{1}{x} \right) = \lim_{x \rightarrow 0} \frac{x - \sin x}{x \sin x}$$

after which L'Hôpital's rule can be applied twice to show that

$$\lim_{x \rightarrow 0} \frac{x - \sin x}{x \sin x} = \lim_{x \rightarrow 0} \frac{1 - \cos x}{\sin x + x \cos x} = \lim_{x \rightarrow 0} \frac{\sin x}{2 \cos x + x \sin x} = \frac{0}{2} = 0.$$

For the third limit, L'Hôpital's rule can be applied three times to show that

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{x^3}{\sin x - x} &= \lim_{x \rightarrow 0} \frac{3x^2}{\cos x - 1} \\ &= \lim_{x \rightarrow 0} \frac{6x}{-\sin x} \\ &= \lim_{x \rightarrow 0} \frac{6}{-\cos x} \\ &= \frac{6}{-1} = -6. \end{aligned}$$

15. Suppose $a > 0$. Then there exists an $N \in \mathbb{N}$ such that $2^{-2N} < a$. Hence for $M > 2N$,

$$\sup\{s_n : n > M\} = a$$

and hence $\limsup s_n = a$. Now consider any $l > 0$. Then there exists an $N \in \mathbb{N}$ such that $2^{-n} < l$ for $n > 2N$. Hence l is not a lower bound for the set $\{s_n : n > M\}$ for any M . However, since 0 is lower bound, it follows that it must be the greatest lower bound, and hence $\limsup s_n = 0$.

Now suppose $a \leq 0$. Then $\inf\{s_n : n > M\} = a$ for all $M \in \mathbb{N}$, so $\liminf s_n = a$. Also,

$$\sup\{s_n : n > M\} = \begin{cases} 2^{-M-1} & \text{if } M \text{ is odd,} \\ 2^{-M-2} & \text{if } M \text{ is even.} \end{cases}$$

so $\limsup s_n = 0$.

If $a = 0$ then $\limsup s_n = \liminf s_n$ and hence the series converges with limit zero. Otherwise, $\limsup s_n \neq \liminf s_n$ and the series does not converge.

16. Consider any $\epsilon > 0$. Then

$$\begin{aligned} |f(x_1, x_2) - f(0, 0)| &= \left| \frac{1}{x_1^2 + x_2^2 + 1} - \frac{1}{1} \right| \\ &= \left| \frac{x_1^2 + x_2^2}{x_1^2 + x_2^2 + 1} \right| \\ &\leq |x_1^2 + x_2^2|. \end{aligned}$$

Suppose $d((x_1, x_2), (0, 0)) < \delta$ where $\delta = \sqrt{\epsilon}$. Then

$$|f(x_1, x_2) - f(0, 0)| \leq (d((x_1, x_2), (0, 0)))^2 < \delta^2 = \epsilon.$$

Now consider the point $(0, 1)$:

$$\begin{aligned} |f(x_1, x_2) - f(0, 1)| &= \left| \frac{1}{x_1^2 + x_2^2 + 1} - \frac{1}{2} \right| \\ &= \left| \frac{x_1^2 + x_2^2 - 1}{2(x_1^2 + x_2^2 + 1)} \right| \\ &\leq \frac{|x_1^2 + x_2^2 - 1|}{2} \\ &= \frac{|x_1^2 + (x_2 - 1)(x_2 + 1)|}{2} \\ &\leq \frac{|x_1|^2 + |x_2 - 1| \cdot |x_2 + 1|}{2}. \end{aligned}$$

Suppose $d((x_1, x_2), (0, 1)) < 1$, so that

$$x_1^2 + (x_2 - 1)^2 < 1.$$

from which it follows that $|x_1| < 1$ and $|x_2 - 1| < 1$. Hence $|x_2 + 1| \leq |x_2| + 1 \leq 3$. Define $\delta = \min\{1, \frac{\epsilon}{3}\}$. Then $\delta < \frac{\epsilon}{3}$ and $\delta < \sqrt{\epsilon}$. Hence $|x_1| < \sqrt{\epsilon}$ and $|x_2 - 1| < \frac{\epsilon}{3}$. Then

$$|f(x_1, x_2) - f(0, 1)| \leq \frac{|x_1|^2 + |x_2 - 1| \cdot |x_2 + 1|}{2} < \frac{(\sqrt{\epsilon})^2 + 3\frac{\epsilon}{3}}{2} = \epsilon.$$

Hence f is continuous at $(0, 1)$.

17. (a) For $p \neq 1$, the improper integral can be written as

$$\begin{aligned} \int_0^1 x^{-p} dx &= \lim_{c \rightarrow 0^+} \int_c^1 x^{-p} dx \\ &= \lim_{c \rightarrow 0^+} \left[\frac{x^{-p+1}}{-p+1} \right]_c^1 \\ &= \lim_{c \rightarrow 0^+} \frac{1 - c^{1-p}}{1-p}. \end{aligned}$$

Then if $0 < p < 1$, the exponent is positive, so

$$\int_0^1 x^{-p} dx = \frac{1}{1-p}$$

whereas if $p > 1$, the exponent is negative, so

$$\int_0^1 x^{-p} dx = \infty.$$

(b) The improper integral can be written as

$$\int_0^1 x^{-p} dx = \lim_{c \rightarrow 0^+} \int_c^1 x^{-p} dx + \lim_{d \rightarrow \infty} \int_1^d x^{-p} dx.$$

Since both terms are non-negative, it follows that if one term is ∞ , then the integral must ∞ also. By the result above, if $p > 1$, then the first term is ∞ . Now suppose $p = 1$. Then the second term is

$$\begin{aligned} \lim_{d \rightarrow \infty} \int_1^d x^{-1} dx &= \lim_{d \rightarrow \infty} [\log x]_1^d \\ &= \lim_{d \rightarrow \infty} \log d = \infty. \end{aligned}$$

If $0 < p < 1$, then second term is

$$\begin{aligned}\lim_{d \rightarrow \infty} \int_1^d x^{-p} dx &= \lim_{d \rightarrow \infty} \left[\frac{x^{-p+1}}{1-p} \right]_1^d \\ &= \lim_{d \rightarrow \infty} \frac{d^{-p+1} - 1}{1-p} = \infty.\end{aligned}$$

Hence in all cases, $\int_0^1 x^{-p} dx = \infty$.

18. Since f is integrable on $[a, b]$, it is bounded, so there exists a $B > 0$ such that $f(x) < B$ for all $x \in [a, b]$. Assume that if f is integrable on $[a, b]$ then it is integrable on any interval $[c, d] \subseteq [a, b]$; for full details see Problem 12.

To show the above limit, consider any $\epsilon > 0$, and examine $c \in (b - \delta, b)$ where $\delta = \epsilon/B$. Then

$$\left| \int_a^c f - \int_a^b f \right| = \left| \int_c^b f \right| \leq \int_c^b |f| \leq (b - c)B < \delta B = \epsilon.$$

Hence

$$\lim_{d \rightarrow b^-} \int_a^d f(x) dx = \int_a^b f(x) dx.$$

19. If $\lambda = 0$, then $s_n = 0$ for all n , so s_n is a constant convergent sequence. Similarly if $\lambda = 1$, then $s_n = 1$ for all n , so s_n is also constant and convergent. In general,

$$s_n = \lambda^{2^{n-1}}.$$

If $0 < \lambda < 1$, then since $0 < \lambda^{2^{n-1}} \leq \lambda^n$ for $n \in \mathbb{N}$ and $\lambda^n \rightarrow 0$ as $n \rightarrow \infty$ it follows that $s_n \rightarrow 0$. If $\lambda > 1$, then since $\lambda^{2^{n-1}} \geq \lambda^n$ for $n \in \mathbb{N}$ and $\lambda^n \rightarrow \infty$ as $n \rightarrow \infty$, it follows that $s_n \rightarrow \infty$.

Finally, suppose $\lambda < 0$. Then for $n \geq 2$, the resulting sequence (s_n) will be the same as the case for $-\lambda$, and will therefore have the same convergence properties. Hence (s_n) converges if and only if $|\lambda| \leq 1$.

20. The minimum element of $[0, \sqrt{2}]$ is 0, and since this is also in A it follows that $\min A = 0$. The maximum element of $[0, \sqrt{2}]$ is $\sqrt{2}$, but this is not in A . Since there are rational numbers arbitrarily close to $\sqrt{2}$, it follows that A does not have a maximum. The infimum is just $\inf A = \min A = 0$. $\sqrt{2}$ is an upper bound for A . For any $\epsilon > 0$, there exist elements in A which are greater than $\sqrt{2} - \epsilon$, and thus $\sqrt{2}$ is the least upper bound. Hence $\sup A = \sqrt{2}$.

For B , note that

$$x^2 + x - 1 = (x + 1/2)^2 - 5/4.$$

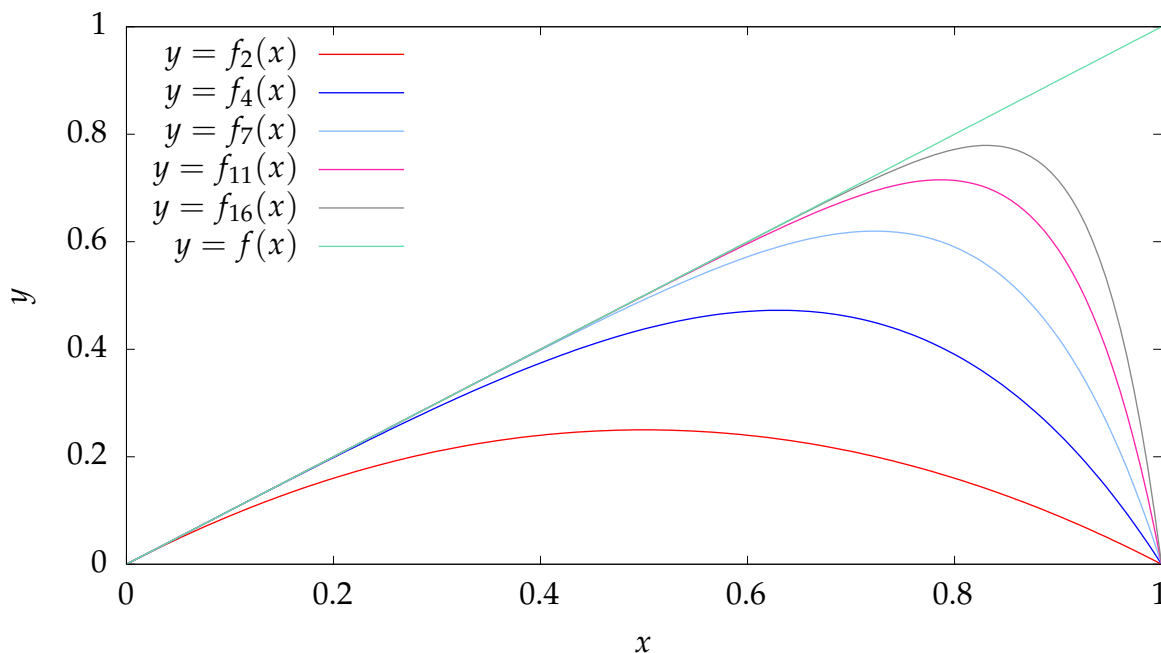


Figure 2: Graph for Problem 21, on pointwise and uniform convergence.

Since the first term can take any positive value, it follows that $B = [-5/4, \infty)$. Hence $\min B = \inf B = -5/4$, the maximum does not exist, and $\sup B = \infty$.

By completing the square, above equation can be written as

$$x^2 + x - 1 = \left(x + \frac{1 - \sqrt{5}}{2}\right) \left(x + \frac{1 + \sqrt{5}}{2}\right).$$

The quadratic will be strictly negative when one of these two factors is strictly negative and the other is strictly positive. Hence $C = (-1/2 - \sqrt{5}/2, -1/2 + \sqrt{5}/2)$, so the minimum and maximum do not exist, $\inf C = -1/2 - \sqrt{5}/2$ and $\sup C = -1/2 + \sqrt{5}/2$.

21. (a) Suppose $0 \leq x < 1$. Then

$$\lim_{n \rightarrow \infty} f_n(x) = \lim_{n \rightarrow \infty} (x - x^n) = x$$

since if $|x| < 1$, then $x^n \rightarrow 0$ as $n \rightarrow \infty$. For $x = 1$,

$$\lim_{n \rightarrow \infty} f_n(x) = \lim_{n \rightarrow \infty} (1 - 1^n) = 0.$$

Hence f_n converges pointwise to a limit f on $[0, 1]$ given by

$$f(x) = \begin{cases} x & \text{if } 0 \leq x < 1, \\ 0 & \text{if } x = 1. \end{cases}$$

Graphs of several of the f_n and the limit f are shown in Fig. 2.

- (b) As can be seen from Fig. 2, the convergence does not appear to be uniform on $[0, 1]$, since it does not appear that the f_n will ever lie within a strip of a fixed width ϵ around f . To see this mathematically, for a given n , consider the point $x = (1/2)^{1/n}$. Then

$$\begin{aligned} |f_n(x) - f(x)| &= |x - x^n - x| \\ &= |x^n| \\ &= 1/2 \end{aligned}$$

Hence if $\epsilon = 1/2$, there does not exist an N such that $n > N$ implies that $|f_n(x) - f(x)| < \epsilon$ for all $x \in [0, 1]$.

- (c) Consider the interval $[0, 1/2]$. Then for any x in this interval

$$|f_n(x) - f(x)| = |x^n| \leq 2^{-n}$$

Consider any $\epsilon > 0$. Then there exists an N such that $n > N$ implies $2^{-n} < \epsilon$, and thus $|f_n(x) - f(x)| < \epsilon$. Hence $f_n \rightarrow f$ uniformly on $[0, 1/2]$.

- (d) Since continuous functions are integrable, it follows immediately that f_n is integrable for all $n \in \mathbb{N}$. For a specific n ,

$$\int_0^1 f_n = \int_0^1 (x - x^n) dx = \left[\frac{x^2}{2} - \frac{x^{n+1}}{n+1} \right]_0^1 = \frac{1}{2} - \frac{1}{n+1} = \frac{n-1}{2(n+1)}.$$

To show that f is integrable, consider the function

$$g(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1, \\ 1 & \text{if } x = 1. \end{cases}$$

Choose any $\epsilon > 0$, and examine the partition $P = \{0 = t_0 < t_1 < t_2 = 1\}$ where $t_1 = 1 - \epsilon/2$. Then

$$L(f, P) = \sum_{k=1}^2 (t_k - t_{k-1}) m(f, [t_{k-1}, t_k]) = \sum_{k=1}^2 0 = 0$$

and

$$U(f, P) = \sum_{k=1}^2 (t_k - t_{k-1}) M(f, [t_{k-1}, t_k]) = (t_1 - t_0) \cdot 0 + (t_2 - t_1) 1 = \frac{\epsilon}{2}.$$

Thus $U(f, P) - L(f, P) < \epsilon$. Since a partition such as this can be constructed for an arbitrary $\epsilon > 0$, it follows that g is integrable and $\int_0^1 g = 0$. Since

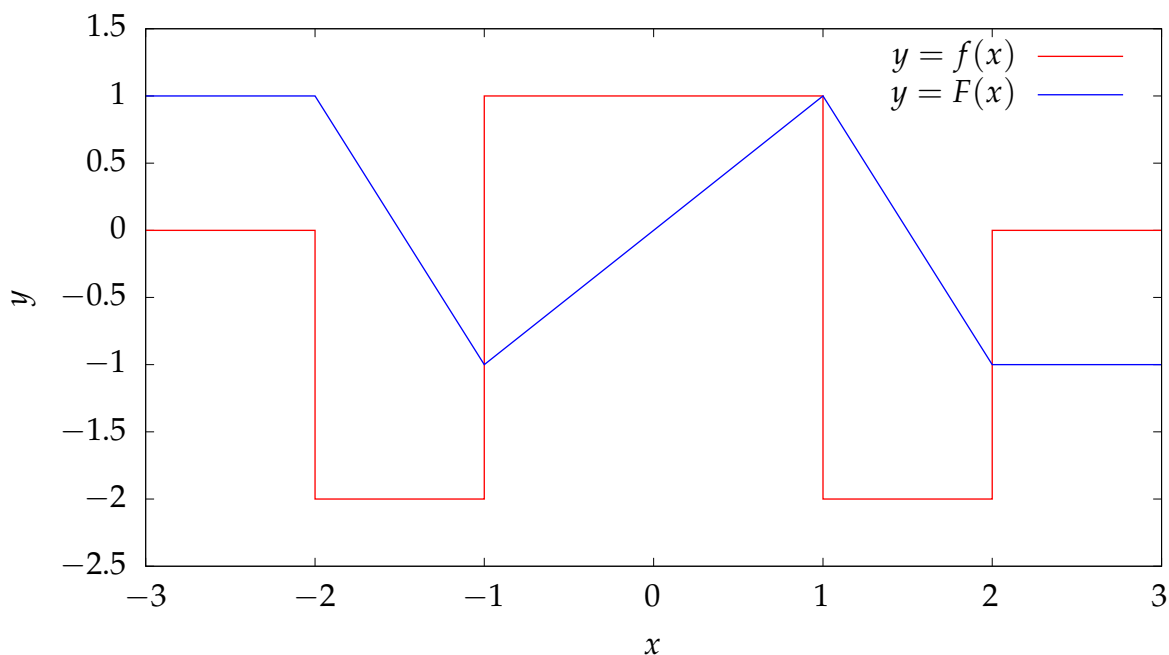


Figure 3: Graph for Problem 22, on the second fundamental theorem of calculus.

$f(x) = x - g(x)$ and both x and g are integrable, it follows that f is integrable and $\int_0^1 f = \int_0^1 x - \int_0^1 g = 1/2$. Note that

$$\lim_{n \rightarrow \infty} \int_0^1 f_n = \lim_{n \rightarrow \infty} \frac{n-1}{2(n+1)} = \frac{1}{2} = \int_0^1 f.$$

22. (a) If $0 \leq x \leq 1$, then

$$F(x) = \int_0^x f = \int_0^x 1 \, dt = x.$$

If $1 < x \leq 2$, then

$$F(x) = \int_0^1 f + \int_1^x = 1 - 2(x-1) = 3 - 2x.$$

If $x > 2$, then

$$F(x) = F(2) + \int_2^x 0 = F(2) = -1.$$

For negative values of x , note that f is an even function, and thus

$$F(-x) = \int_0^{-x} f(t) \, dt = \int_0^x f(-s)(-ds) = -\int_0^x f(s) \, ds = -F(x)$$

so F is odd. Hence

$$F(x) = \begin{cases} 1 & \text{if } x < -2, \\ -3 - 2x & \text{if } -2 \leq x < 1, \\ x & \text{if } -1 \leq x \leq 1, \\ 3 - 2x & \text{if } 1 < x \leq 2, \\ -1 & \text{if } x > 2. \end{cases}$$

(b) The functions f and F are plotted in Fig. 3.

(c) By the second fundamental theorem of calculus, if f is continuous at x , and then F is differentiable at x and $F'(x) = f(x)$. Thus the only points where F may not be defined are $x = \pm 1, \pm 2$. Since

$$\lim_{x \rightarrow 1^-} \frac{F(x) - F(1)}{x - 1} = \lim_{x \rightarrow 1^-} \frac{x - 1}{x - 1} = 1$$

but

$$\lim_{x \rightarrow 1^+} \frac{F(x) - F(1)}{x - 1} = \lim_{x \rightarrow 1^+} \frac{3 - 2x - 1}{x - 1} = -2$$

so F is not differentiable at 1. Similarly

$$\lim_{x \rightarrow 2^-} \frac{F(x) - F(2)}{x - 2} = \lim_{x \rightarrow 2^-} \frac{3 - 2x - (-1)}{x - 2} = 1$$

but

$$\lim_{x \rightarrow 2^+} \frac{F(x) - F(2)}{x - 2} = \lim_{x \rightarrow 2^+} \frac{0}{x - 2} = 0$$

so F is not differentiable at 2. Since F is odd, it follows that F is not differentiable at -1 and -2 also. Hence F' is defined on $\mathbb{R} \setminus \{-2, -1, 1, 2\}$ and

$$F'(x) = \begin{cases} 0 & \text{if } x < -2, \\ -2 & \text{if } -2 < x < 1, \\ 1 & \text{if } -1 < x < 1, \\ -2 & \text{if } 1 < x < 2, \\ 0 & \text{if } x > 2. \end{cases}$$

23. (a) If $\int_a^b f = \int_a^b g$, then if $h(x) = f(x) - g(x)$, then $\int_a^b h = 0$. Consider the partition $P = \{a = t_0 < t_1 = b\}$. Then

$$0 = \int_a^b h \leq U(h, P) = (b - a)M(h, [a, b])$$

and

$$0 = \int_a^b h \geq L(h, P) = (b - a)m(h, [a, b]).$$

Since a continuous function on a closed interval achieves its bounds, there exist x_1 and x_2 such that $h(x_1) = M(h, [a, b])$ and $h(x_2) = m(h, [a, b])$. Either $h(x_1) = 0$ or $h(x_2) = 0$, or otherwise $h(x_1) > 0$ and $h(x_2) < 0$. In the latter case, the intermediate value theorem can be applied to show that there exists an x_3 between x_1 and x_2 such that $h(x_3) = 0$. In all cases there exists an x such that $h(x) = 0$ and hence $f(x) = g(x)$.

(b) On the interval $[-1, 1]$, define

$$f(x) = \begin{cases} -1 & \text{if } x < 0, \\ 1 & \text{if } x \geq 0, \end{cases}$$

and let $g(x) = -f(x)$. By construction $f(x) \neq g(x)$ for all $x \in [-1, 1]$. To find the integral of f , choose $\epsilon > 0$ and consider the partition $P = \{a = -1 < t_1 < t_2 < t_3 = 1\}$ where $t_1 = -\epsilon/5$ and $t_2 = \epsilon/5$. Then

$$\begin{aligned} L(f, P) &= \sum_{k=1}^3 (t_k - t_{k-1})m(f, [t_{k-1}, t_k]) \\ &= \left(1 - \frac{\epsilon}{5}\right)(-1) + \frac{2\epsilon(-1)}{5} + \left(1 - \frac{\epsilon}{5}\right)(1) \\ &= -\frac{2\epsilon}{5}. \end{aligned}$$

Similarly

$$\begin{aligned} U(f, P) &= \sum_{k=1}^3 (t_k - t_{k-1})M(f, [t_{k-1}, t_k]) \\ &= \left(1 - \frac{\epsilon}{5}\right)(-1) + \frac{2\epsilon(1)}{5} + \left(1 - \frac{\epsilon}{5}\right)(1) \\ &= \frac{2\epsilon}{5}. \end{aligned}$$

Then $U(f, P) - L(f, P) = 4\epsilon/5 < \epsilon$, and since ϵ is arbitrary it follows that f is integrable, and that $\int_{-1}^1 f = 0$. In addition, so $\int_{-1}^1 g = \int_{-1}^1 (-f) = -\int_{-1}^1 f = 0$. Thus $\int_{-1}^1 f = \int_{-1}^1 g$ but $f(x) \neq g(x)$ for all $x \in [-1, 1]$.

24. (a) For $x > 0$,

$$f(x) = \frac{1}{1 + \frac{1}{x}}$$

and since $1/x \rightarrow 0$ as $x \rightarrow \infty$, it follows that $f(x) \rightarrow 0$ as $x \rightarrow \infty$. On the interval under consideration $0 \leq x < 1 + x$, so $0 \leq f(x) < 1$.

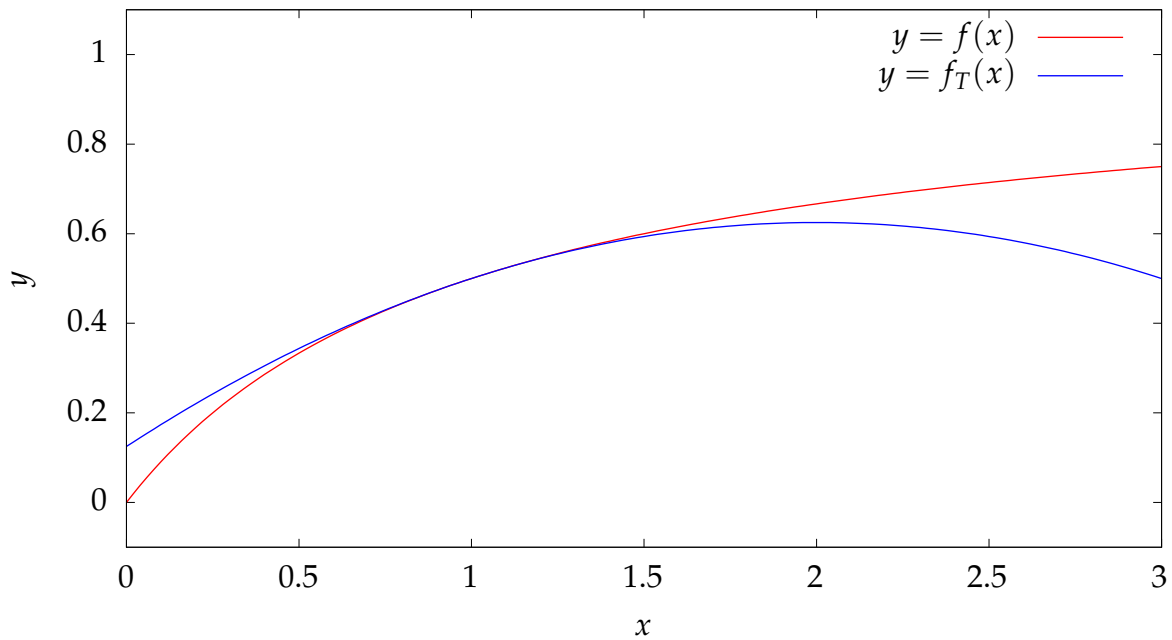


Figure 4: A graph of the function f and a Taylor series approximation f_T at $x = 1$, discussed in Problem 24.

(b) The function f is shown in Fig. 4.

(c) Since

$$f(x) = 1 - \frac{1}{x+1},$$

it follows that the derivatives are

$$f'(x) = \frac{1}{(x+1)^2}$$

and

$$f''(x) = -\frac{2}{(x+1)^3}.$$

Hence $f(1) = 1/2$, $f'(1) = 1/4$, $f''(1) = -1/4$ and thus

$$\begin{aligned} f_T(x) &= \sum_{n=0}^2 \frac{(x-1)^n f^{(n)}(1)}{n!} \\ &= f(1) + f'(1)(x-1) + f''(1) \frac{(x-1)^2}{2} \\ &= \frac{1}{2} + \frac{x-1}{4} - \frac{(x-1)^2}{8}. \end{aligned}$$

(d) f_T can be rewritten as

$$\begin{aligned} f_T(x) &= \frac{1}{2} + \frac{x}{4} - \frac{1}{4} - \frac{x^2}{8} + \frac{x}{4} - \frac{1}{8} \\ &= \frac{1}{8} + \frac{x}{2} - \frac{x^2}{8} \end{aligned}$$

which is a quadratic.

(e) The function f_T is shown in Fig. 4. By construction, the curves intersect at $x = 1$, and have the same slope and curvature there.

25. For the first series,

$$\beta = \limsup |a_n|^{1/n} = \limsup \left| \frac{a_{n+1}}{a_n} \right| = \limsup \sqrt{\frac{n^2 + 1}{(n+1)^2 + 1}} = 1,$$

so the radius of convergence is $R = 1$. At $x = -1$, the series is

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n^2 + 1}},$$

which converges by the alternating series theorem. At $x = 1$, the series is

$$\sum_{n=0}^{\infty} \frac{1}{\sqrt{n^2 + 1}}.$$

Since $\sqrt{n^2 + 1} \leq \sqrt{n^2 + 2n + 1} = n + 1$ and $1/(n + 1)$ diverges, it follows that the series diverges at $x = 1$. Hence the exact interval of convergence is $[-1, 1)$.

For the second series, by looking at the limit of positive coefficients,

$$\beta = \limsup |a_n|^{1/n} = \lim_{n \rightarrow \infty} |a_{2n}|^{1/2n} = \lim_{n \rightarrow \infty} \left| \frac{(-2)^n}{n^2} \right|^{1/2n} = \lim_{n \rightarrow \infty} (\sqrt{2})^{1/n} = \sqrt{2},$$

and thus the radius of convergence is $R = 2^{-1/2}$. At $x = R$, the series is

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2},$$

which converges by the alternating series theorem. Since the series is even, the sum converges at $x = -R$ also. Hence the interval of convergence is $[-2^{-1/2}, 2^{-1/2}]$.

26. If a sequence a_n converges to a limit a , then for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $n > N$ implies that $|a_n - a| < \epsilon$.

Consider any $\epsilon > 0$. Since $s_n \rightarrow s$, there exists an N_1 such that $n > N_1$ implies that

$$|s_n - s| < \frac{\epsilon}{4}$$

and since $t_n \rightarrow t$, there exists an N_2 such that $n > N_2$ implies that

$$|t_n - t| < \frac{\epsilon}{4}.$$

Define $N = \max\{N_1, N_2\}$. Then for $n > N$,

$$\begin{aligned} |(3s_n + t_n) - 3s - t| &\leq 3|s_n - s| + |t_n - t| \\ &< \frac{3\epsilon}{4} + \frac{\epsilon}{4} = \epsilon \end{aligned}$$

and thus $3s_n + t_n \rightarrow 3s + t$ as $n \rightarrow \infty$.

27. (a) By making use of the definition,

$$\varphi^2 = \frac{(1 + \sqrt{5})^2}{4} = \frac{6 + 2\sqrt{5}}{4} = 1 + \frac{1 + \sqrt{5}}{2} = 1 + \varphi.$$

(b) First consider the induction hypothesis H_1 :

$$f(0) = \frac{\varphi^0 - (1 - \varphi)^0}{\sqrt{5}} = 0, \quad f(1) = \frac{\varphi - (1 - \varphi)}{\sqrt{5}} = \frac{2\varphi - 1}{\sqrt{5}} = 1.$$

Now consider the induction step. Suppose H_n is true, and consider H_{n+1} . Then $F_n = f(n)$ and

$$\begin{aligned} F_{n+1} &= F_n + F_{n-1} = \frac{\varphi^n - (1 - \varphi)^n + \varphi^{n-1} - (1 - \varphi)^{n-1}}{\sqrt{5}} \\ &= \frac{\varphi^{n-1}(1 + \varphi) - (1 - \varphi)^{n-1}(1 + (1 - \varphi))}{\sqrt{5}}. \end{aligned}$$

Since $(1 - \varphi)^2 = 1 - 2\varphi + \varphi^2 = 1 - 2\varphi + (1 + \varphi) = 2 - \varphi$, it follows that

$$F_{n+1} = \frac{\varphi^{n+1} - (1 - \varphi)^{n+1}}{\sqrt{5}} = f(n + 1)$$

so H_{n+1} is true. Hence by mathematical induction H_n is true for all n , so $F_n = f(n)$ for all $n \in \mathbb{N} \cup \{0\}$.

(c) By using the explicit formula for F_n ,

$$\frac{F_{n+1}}{F_n} = \frac{\varphi^{n+1} - (1-\varphi)^{n+1}}{\varphi^n - (1-\varphi)^n} = \varphi \frac{1 - \left(\frac{1-\varphi}{\varphi}\right)^{n+1} \varphi^{-1}}{1 - \left(\frac{1-\varphi}{\varphi}\right)^n}.$$

Since $|\varphi| > 1$ and $|1-\varphi| = \frac{\sqrt{5}-1}{2} < 1$, then

$$\lim_{n \rightarrow \infty} \frac{F_{n+1}}{F_n} = \varphi \frac{1-0}{1-0} = \varphi.$$

28. (a) $d_A((0,1), (0,0)) = \min\{0,2\} = 0$, which violates the property that $d_A(\mathbf{x}, \mathbf{y}) = 0$ if and only if $\mathbf{x} = \mathbf{y}$.

(b) Consider the three properties of a metric:

M1. Note that $d_B(\mathbf{x}, \mathbf{x}) = \max\{0,0\} = 0$. If $d_B(\mathbf{x}, \mathbf{y}) = 0$, then $\max\{|x_1 - y_1|, 2|x_2 - y_2|\} = 0$ so $|x_1 - y_1| = 0$ and $|x_2 - y_2| = 0$, and $x_1 = y_1$ and $x_2 = y_2$, so $\mathbf{x} = \mathbf{y}$.

M2. $d_B(\mathbf{x}, \mathbf{y}) = \max\{|x_1 - y_1|, 2|x_2 - y_2|\} = \max\{|y_1 - x_1|, 2|y_2 - x_2|\} = d_B(\mathbf{y}, \mathbf{x})$.

M3. For any $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}$,

$$\begin{aligned} d_B(\mathbf{x}, \mathbf{y}) + d_B(\mathbf{y}, \mathbf{z}) &= \max\{|x_1 - y_1|, 2|x_2 - y_2|\} + \max\{|y_1 - z_1|, 2|y_2 - z_2|\} \\ &\geq \max\{|x_1 - y_1| + |y_1 - z_1|, 2|x_2 - y_2| + 2|y_2 - z_2|\} \\ &\geq \max\{|x_1 - z_1|, 2|x_2 - z_2|\} \end{aligned}$$

where on the final line, the usual triangle inequality has been applied.

Hence d_B is a metric. The neighborhood of radius 1 at $(0,0)$ is shown in Fig. 5.

(c) By the given continuity property, for all $x \in X$, and all $\epsilon > 0$, there exists a $\delta > 0$ such that $d(y, x) < \delta$ implies $d_E(f(x), f(y)) < \epsilon/2$, so that

$$\sqrt{(f(x_1) - f(y_1))^2 + (f(x_2) - f(y_2))^2} < \frac{\epsilon}{2}$$

and hence $|f(x_1) - f(y_1)| < \epsilon/2$ and $|f(x_2) - f(y_2)| < \epsilon/2$. Then

$$d_B(f(x), f(y)) = \max\{|f(x_1) - f(y_1)|, 2|f(x_2) - f(y_2)|\} < 2\frac{\epsilon}{2} = \epsilon.$$

Hence f is continuous with respect to (X, d) and (\mathbb{R}^2, d_B) also.

29. (a) For $x \in [0, 1]$, $F(x) = \int_0^x 0 = 0$. For $x > 1$,

$$F(x) = \int_1^x f(t)dt = \left[\frac{t^2}{2}\right]_1^x = \frac{x^2 - 1}{2}.$$

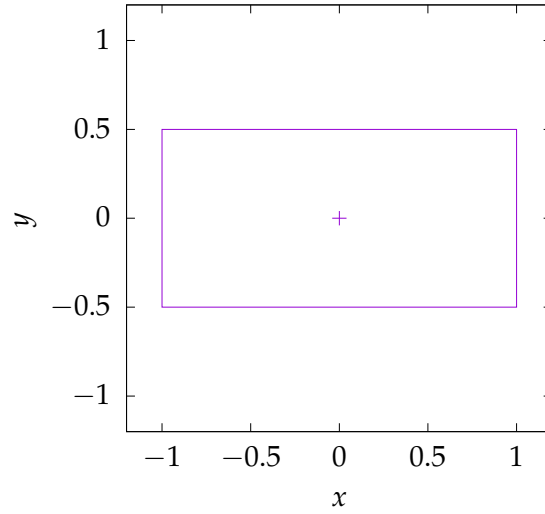


Figure 5: Graph of the neighborhood of radius 1 at $(0,0)$ with respect to d_B .

- (b) By the second fundamental theorem of calculus, if f is continuous at a given x , then F is differentiable at x and $F'(x) = f(x)$. Hence the only value to check is $x = 1$ where f is not continuous:

$$\lim_{y \rightarrow 1^+} \frac{F(y) - F(1)}{y - 1} = \lim_{y \rightarrow 1^+} \frac{y^2 - 1}{2(y - 1)} = \lim_{y \rightarrow 1^+} \frac{y + 1}{2} = 1$$

and

$$\lim_{y \rightarrow 1^-} \frac{F(y) - F(1)}{y - 1} = 0$$

so F is not differentiable at 1. Hence F' exists on $[0, 1) \cup (1, \infty)$, and

$$F'(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1, \\ x & \text{if } x > 1. \end{cases}$$

- (c) Since $\lim_{x \rightarrow 1} f(x) = \lim_{x \rightarrow 1} x = 1$, but $f(1) = 0$, f is not continuous, and hence it is not uniformly continuous. To show that F is not uniformly continuous, consider $\epsilon = 1/2$, and for a $\delta > 0$, put $\eta = \min\{\delta, 1\}$, and examine $x = \eta^{-1}$ and $y = \eta^{-1} + \eta/2$. Then $|x - y| < \delta$, but

$$|f(x) - f(y)| = \frac{|\eta^{-2} - \eta^{-2} - 1 - \frac{\eta^2}{2}|}{2} > \frac{1}{2}.$$

Since an x and y with this property can be found for any $\delta > 0$, it follows that F is not uniformly continuous.

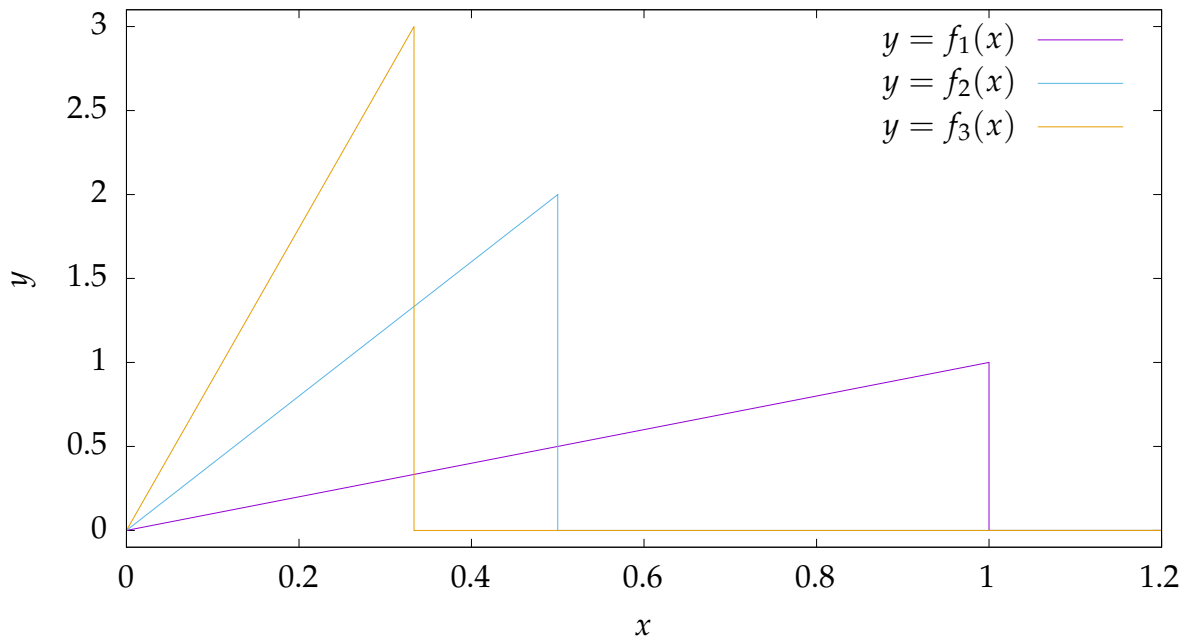


Figure 6: Graph of the functions considered in question 30 on pointwise and uniform convergence.

30. (a) The functions are plotted in Fig. 6.

(b) Since $f_n(0) = 0$ for all n , then $\lim_{n \rightarrow \infty} f_n(0) = 0$. Consider any $x > 0$. Then there exists an $n \in \mathbb{N}$ such that $Nx > 1$. But then for all $n > N$, $f_n(x) = ng(nx) = 0$, so $\lim_{n \rightarrow \infty} f_n(x) = 0$. Hence f_n converges pointwise to f , where $f(x) = 0$ for all $x \in [0, 1]$.

(c) For any $n \in \mathbb{N}$,

$$|f_n(1/n) - f(1/n)| = |n - 0| = n \geq 1.$$

Hence there does not exist any $N \in \mathbb{N}$ such that $n > N$ implies that $|f_n(x) - f(x)| < 1$ for all $x \in [0, 1]$. Hence f_n does not converge uniformly to f .

(d) The integrals are

$$\int_0^1 f_n = \int_0^{1/n} n^2 x \, dx = \left[n^2 \frac{x^2}{2} \right]_0^{1/n} = \frac{1}{2}.$$

Since f is identically zero, $\int_0^1 f = 0$. Hence the integrals of f_n do not converge to f .

31. (a) By applying L'Hôpital's rule once,

$$\lim_{x \rightarrow 0} \frac{x}{e^x - e^{-x}} = \lim_{x \rightarrow 0} \frac{1}{e^x + e^{-x}} = \frac{1}{2}.$$

By applying L'Hôpital's rule twice,

$$\lim_{x \rightarrow 0} \frac{\sin^2 x}{x^2} = \lim_{x \rightarrow 0} \frac{2 \cos x \sin x}{2x} = \lim_{x \rightarrow 0} \frac{2 \cos^2 x - 2 \sin^2 x}{2} = 1.$$

(b) L'Hôpital's rule can be applied to show that

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{f(x+h) + f(x-h) - 2f(x)}{h^2} &= \lim_{h \rightarrow 0} \frac{f'(x+h) - f'(x-h)}{2h} \\ &= \lim_{h \rightarrow 0} \frac{f'(x+h) - f'(x)}{2h} \\ &\quad - \lim_{h \rightarrow 0} \frac{f'(x-h) - f'(x)}{2h} \\ &= \frac{f''(x) + f''(x)}{2} = f''(x). \end{aligned}$$

(c) Consider the function

$$f(x) = \begin{cases} 1 & \text{if } 0 < x < 1, \\ 0 & \text{if } x = 0, \\ -1 & \text{if } -1 < x < 0, \end{cases}$$

defined on $(-1, 1)$. This is discontinuous at 0, and thus it is not twice differentiable there.

$$\lim_{h \rightarrow 0} \frac{f(x+h) + f(x-h) - 2f(x)}{h^2} = \lim_{h \rightarrow 0} \frac{1 + (-1) - 0}{h^2} = \lim_{h \rightarrow 0} \frac{0}{h^2} = 0.$$

Any odd function that is not twice differentiable will give the same result.

32. Define $\lim_{n \rightarrow \infty} x_{2n} = a$, $\lim_{n \rightarrow \infty} x_{2n+1} = b$, and $\lim_{n \rightarrow \infty} x_{7n} = c$. Consider the subsequence (x_{14n}) : since it is a subsequence of (x_{2n}) it must converge to a , and since it is a subsequence of (x_{7n}) it must converge to c . Therefore $a = c$. Now consider the subsequence (x_{14n+7}) : since it is a subsequence of (x_{2n+1}) it must converge to b , and since it is a subsequence of (x_{7n}) it must converge to c . Therefore $b = c$, and hence $a = b$.

Now consider any $\epsilon > 0$. There exists an N_1 such that for $2n > N_1$, $|x_{2n} - a| < \epsilon$, and there exists an N_2 such that for $2n+1 > N_2$, $|x_{2n+1} - a| < \epsilon$. Define $N = \max\{N_1, N_2\}$. Then for any $m > N$, it will be either odd or even, and in both cases $|x_m - a| < \epsilon$. Thus (x_n) converges to a .

33. When n is even $a_n = c + n^{-1}$, and when n is odd $a_n = -c - n^{-1}$. If $c \geq 0$, then the odd terms will be negative and the even terms will be positive. Therefore

$$u_N = \inf\{a_n : n > N\} = \begin{cases} -c - (N+2)^{-1} & \text{if } N \text{ is odd,} \\ -c - (N+1)^{-1} & \text{if } N \text{ is even,} \end{cases}$$

and

$$v_N = \sup\{a_n : n > N\} = \begin{cases} c + (N+1)^{-1} & \text{if } N \text{ is odd,} \\ c + (N+2)^{-1} & \text{if } N \text{ is even.} \end{cases}$$

Hence $\liminf a_n = \lim_{N \rightarrow \infty} u_N = -c$ and $\limsup a_n = \lim_{N \rightarrow \infty} v_N = c$. If $c < 0$ then $\liminf a_n = c$ since the set $\{a_n : n > N\}$ always has terms arbitrarily close to c . Similarly $\limsup a_n = -c$.

Hence for a general c , $\liminf a_n = -|c|$ and $\limsup a_n = |c|$. When $c = 0$, $\liminf a_n = \limsup a_n = 0$ and hence (a_n) converges to 0 in this case.

34. (a) If $f(x) = \log(\log(x))$, then

$$f'(x) = \frac{1}{\log x} \frac{d}{dx}(\log x) = \frac{1}{x \log x}.$$

(b) For $x > 1$, the function $1/(x \log(x))$ is decreasing, and hence

$$\begin{aligned} \sum_{n=2}^N \frac{1}{n \log n} &\geq \int_2^{N+1} \frac{1}{x \log x} dx \\ &= [\log(\log(x))]_2^{N+1} = \log(\log(N+1)) - \log(\log(2)). \end{aligned}$$

Since $\log(\log(x)) \rightarrow \infty$ as $x \rightarrow \infty$, the series must diverge to ∞ .

(c) For an m digit number, the maximum that s can be is $9m$, if the number is made of all nines. If a number n has m digits, then $n \geq 10^{m-1}$, so $\log_{10} n \geq m-1$ and $m \leq 1 + \log_{10} n$. For $n \geq 10$, $\log_{10} n \geq 1$, and hence $m \leq 2 \log_{10} n = C \log n$ for some constant $C > 0$. Thus for $n \geq 10$,

$$s(n) \leq 9m \leq 9C \log n.$$

Hence for $n \geq 10$,

$$\frac{1}{n s(n)} \geq \frac{1}{n 9C \log n}.$$

A finite number of terms have no effect on the convergence properties. Hence since each term in the series is larger than $1/(9Cn \log n)$ for $n \geq 10$, and this series diverges, the given series must diverge also.

35. (a) Suppose that $\int_a^b f = 0$ but that there exists $z \in [a, b]$ such that $f(z) > 0$. If $z = a$ or $z = b$, then by continuity, there exists $\delta > 0$ such that if $|x - z| < \delta$ then $|f(x) - f(z)| < f(z)/2$, and hence $f(x) > f(z)/2 > 0$. Hence, f is strictly positive over a small interval surrounding z . Therefore in all cases there exists $y \in (a, b)$ such that $f(y) > 0$.

Now choose $\delta > 0$ such that $(y - \delta, y + \delta) \subseteq [a, b]$ and if $|x - y| < \delta$ then $|f(x) - f(y)| < f(y)/2$, so $f(x) > f(y)/2 > 0$. Consider the partition $P = \{a <$

$y - \delta/2 < y + \delta/2 < b\}$. Since $f(x) < f(y)/2$ for all $x \in (y - \delta/2, y + \delta/2)$ it follows that $L(f, P) \geq \delta f(y)/2 > 0$, and therefore $\int_a^b f > 0$, which is a contradiction. Hence $f(x) = 0$ for all x .

(b) Consider the three properties of a metric:

M1. If $g(x) = h(x)$ for all $x \in [a, b]$, then

$$d(g, h) = \int_a^b |g(x) - h(x)| dx = \int_a^b 0 dx = 0.$$

Note that for any $g, h \in X$, the function $|g(x) - h(x)|$ is continuous. Therefore using part (a), if

$$\int_a^b |g(x) - h(x)| dx = 0,$$

then $g(x) - h(x) = 0$ for all x , and $g = h$. Since $|g(x) - h(x)| \geq 0$, $d(g, h) \geq 0$ for all $g, h \in X$.

M2. From the definition,

$$d(g, h) = \int_a^b |g(x) - h(x)| dx = \int_a^b |h(x) - g(x)| dx = d(h, g)$$

and therefore d is symmetric.

M3. For any $f, g, h \in X$

$$\begin{aligned} d(f, g) + d(g, h) &= \int_a^b |f(x) - g(x)| dx + \int_a^b |g(x) - h(x)| dx \\ &= \int_a^b (|f(x) - g(x)| + |g(x) - h(x)|) dx \\ &\leq \int_a^b |f(x) - h(x)| dx = d(f, h) \end{aligned}$$

and hence the triangle inequality is satisfied.

Hence d is a metric.

(c) Consider the interval $[-1, 1]$ and the functions

$$f(x) = 0, \quad g(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{if } x \neq 0. \end{cases}$$

Both of these functions are in Y , the set of integrable functions. To see that g is integrable, consider the partition $P = \{-1 < -c < c < 1\}$. Then $L(g, P) = 0$ and $U(g, P) = 2c$; taking c arbitrarily small shows that $\int_a^b g = L(g) = U(g) = 0$. But then $d(f, g) = \int_a^b |f(x) - g(x)| dx = \int_a^b g = 0$. This violates the M1 property since f and g are not equal.

36. (a) For a given $\epsilon > 0$, consider the set $E_n = \{x \in S : f_n(x) < \epsilon\}$. Take a point $y \in E_n$. Then $f_n(y) < \epsilon$. Define $r = \epsilon - f_n(y) > 0$. Since f_n is continuous, there exists a $\delta > 0$ such that for all $z \in S$ with $|y - z| < \delta$ then $|f_n(y) - f_n(z)| < r$. Hence $f_n(z) < r + f_n(y) = \epsilon$ and thus $z \in E_n$. Hence the neighborhood $N_\delta(y) = \{z \in S : |y - z| < \delta\} \subseteq E_n$ and y is an interior point. Therefore E_n is open.

Consider a point $y \in E_n$. Then $f_n(y) < \epsilon$, and $f_{n+1}(y) \leq f_n(y) < \epsilon$, so $y \in E_{n+1}$. Hence $E_n \subseteq E_{n+1}$.

Consider any point $y \in S$. Then since $f_n(y)$ converges pointwise to zero, there exists an N such that $n > N$ implies $|f_n(y) - 0| < \epsilon$, so $f_n(y) < \epsilon$. Therefore $y \in E_n$. Since the same procedure can be applied for all $y \in S$, it follows that $\{E_n\}_{n \in \mathbb{N}}$ is an open cover for S .

- (b) By the Heine–Borel theorem, $S = [0, 1]$ is closed and bounded, therefore it is compact. Hence the open cover $\{E_n\}_{n \in \mathbb{N}}$ must have a finite subcover. Let E_P be the set in the finite subcover with maximum index. Since $E_n \subseteq E_P$ for all $n \leq P$, it follows that $S \subseteq E_P$. From the definition of E_P , $f_P(x) < \epsilon$ for all $x \in S$, and therefore $|f_m(x) - 0| < \epsilon$ for all $m \geq P$. Since ϵ is arbitrary, this procedure shows that (f_n) converges uniformly to zero.
- (c) Consider the functions $f_n(x) = x^n$. Then $f_{n+1}(x) \leq f_n(x)$ for all $x \in (0, 1)$ and $n \in \mathbb{N}$. Furthermore $\lim_{n \rightarrow \infty} f_n(x) = \lim_{n \rightarrow \infty} x^n = 0$ for all $x \in (0, 1)$, so (f_n) converges pointwise to zero. But $f_n(2^{-1/n}) = (2^{-1/n})^n = \frac{1}{2}$. Thus for all $n \in \mathbb{N}$, there exists $x \in (0, 1)$ such that $|f_n(x) - 0| \geq \frac{1}{2}$, so (f_n) does not converge uniformly to zero.
37. (a) Since $1/3n+1 \rightarrow 0$ is a decreasing sequence that converges to zero, the alternating series test can be applied to show that $\sum_{n=0}^{\infty} (-1)^n/3n+1$ converges. Note that for $n \geq 1$, $3n + 1 < 4n$ and hence

$$\frac{1}{3n+1} > \frac{1}{4n}.$$

Since $\sum_{n=1}^{\infty} 1/4n$ diverges, it follows that $\sum_{n=1}^{\infty} 1/3n+1$ diverges also by the comparison test. Hence $\sum_{n=0}^{\infty} 1/3n+1$ diverges since the extra term for $n = 0$ has no effect on the convergence properties.

- (b) Let R be the radius of convergence. When $x = 1/2$ the series matches the first one from part (a), which converges. Hence $R \geq 1/2$. When $x = -1/2$ the series matches the second one from part (a), which diverges. Hence $R \leq 1/2$. Therefore $R = 1/2$. The exact interval of convergence is $(1/2, 1/2]$.
38. (a) Suppose that $\limsup |s_n| < \infty$, so that $\limsup |s_n| = s > 0$. Define $v_N = \sup\{|s_n| : n > N\}$. Since $\lim_{N \rightarrow \infty} v_N = s$ there exists an Q such that $v_Q < s + 1$, and hence $|s_n| < s + 1$ for all $n > Q$. Now define

$$M = \max\{|s_1|, |s_2|, \dots, |s_Q|, s + 1\}.$$

By construction $|s_n| < M$ for all $n \in \mathbb{N}$, and therefore (s_n) is bounded. Suppose now that (s_n) is bounded, so that there exists an $M > 0$ such that $|s_n| < M$ for all $n \in \mathbb{N}$. Then $v_N = \sup\{|s_n| : n > N\} \leq M$, and hence $\limsup |s_n| = \lim_{N \rightarrow \infty} v_N \leq M < \infty$.

(b) Consider the sequence

$$s_n = \begin{cases} 0 & \text{if } n \text{ even,} \\ -n & \text{if } n \text{ odd.} \end{cases}$$

Then $v_N = \sup\{s_n : n > N\} = 0$, and therefore $\limsup s_n = \lim_{N \rightarrow \infty} v_N = 0$, which is a real number. But the subsequence of odd terms of (s_n) diverges to $-\infty$, and therefore (s_n) is unbounded.

39. (a) Choose $\epsilon > 0$. Then there exists an N such that $n > N$ implies $|f_n(1) - 0| < \epsilon$, and hence $f_n(1) < \epsilon$. Consider any $x \in [0, 1]$. Since f_n is increasing, it follows that

$$0 = f_n(0) \geq f_n(x) \geq f_n(1) < \epsilon$$

and therefore $-\epsilon < f_n(x) < \epsilon$, so $|f_n(x)| < \epsilon$. Hence (f_n) converges uniformly to the zero function.

(b) (g_n) does not need to be bounded. Consider the functions

$$f_n(x) = \frac{x^{n^2}}{n},$$

which satisfy $f_n(0) = 0$ and

$$\lim_{n \rightarrow \infty} f_n(1) = \lim_{n \rightarrow \infty} \frac{1}{n} = 0.$$

The functions have derivatives

$$f'_n(x) = nx^{n^2-1}.$$

Since f'_n is a non-negative increasing function on $[0, 1]$ it follows that $g_n = f'_n(1) = n$, and therefore $\lim_{n \rightarrow \infty} g_n = \infty$, so (g_n) is not bounded.

40. (a) The functions are plotted in Fig. 7.

(b) Since

$$\lim_{x \rightarrow 0^+} \frac{g(x) - g(0)}{x - 0} = \lim_{x \rightarrow 0^+} \frac{|x^2(1-x)|}{x} = \lim_{x \rightarrow 0^+} x(1-x) = 0,$$

and

$$\lim_{x \rightarrow 0^-} \frac{g(x) - g(0)}{x - 0} = \lim_{x \rightarrow 0^-} \frac{|x^2(1-x)|}{x} = \lim_{x \rightarrow 0^-} x(1-x) = 0,$$

so the two limits agree then g is differentiable at $x = 0$. However

$$\lim_{x \rightarrow 1^+} \frac{g(x) - g(1)}{x - 1} = \lim_{x \rightarrow 1^+} \frac{|x^2(1 - x)|}{x - 1} = \lim_{x \rightarrow 1^+} x^2 = 1,$$

and

$$\lim_{x \rightarrow 1^-} \frac{g(x) - g(1)}{x - 1} = \lim_{x \rightarrow 1^-} \frac{|x^2(1 - x)|}{x - 1} = \lim_{x \rightarrow 1^-} -x^2 = -1,$$

so g is not differentiable at $x = 1$.

- (c) For $x \geq 1$, $g(x) = -f(x) = x^3 - x^2$, and the derivatives are $g'(x) = 3x^2 - 2x$, $g''(x) = 6x - 2$, $g'''(x) = 6$, and $g^{(n)}(x) = 0$ for $n \geq 4$. Hence, the Taylor series of g at $x = 2$ is

$$\begin{aligned} g(x) &= g(2) + g'(2)(x - 2) + \frac{g''(2)}{2!}(x - 2)^2 + \frac{g'''(2)}{3!}(x - 2)^3 \\ &= 4 + 8(x - 2) + 5(x - 2)^2 + (x - 2)^3 \\ &= 4 + 8x - 16 + 5x^2 - 20x + 20 + x^3 - 6x^2 + 12x - 8 \\ &= x^3 - x^2 \end{aligned}$$

which is equal to $-f(x)$ for all x . Alternatively, Taylor's theorem states that the remainder $R_n(x)$ between $-f$ and the Taylor series with n terms is given in terms of $-f^{(n)}$. Since the $-f^{(n)}$ are identically zero for $n \geq 4$, it follows that $R_n(x) = 0$ and hence the Taylor series must agree with $-f$.

41. (a) Define $\alpha = \sup S$. Since α is an upper bound for S , then $\alpha \geq s$ for all $s \in S$. Hence $f(\alpha) \geq f(s)$ for all $s \in S$, so $f(\alpha) \geq t$ for all $t \in T$, so $f(\alpha)$ is an upper bound for T .

Since f is a continuous strictly increasing function there is an inverse f^{-1} . Suppose $\beta < f(\alpha)$ is an upper bound for T . Then $f^{-1}(\beta) < \alpha$, and since α is the supremum of S , there exists an $s \in S$ such that $s > f^{-1}(\beta)$. But then $f(s) > \beta$, so there exists a $t \in T$ such that $t > \beta$ which is a contradiction. Hence $f(\alpha)$ is the least upper bound, so $\sup B = f(\sup A)$.

- (b) This follows from the result in part (a) and the fact that f is continuous:

$$\begin{aligned} \limsup b_n &= \lim_{N \rightarrow \infty} (\sup\{b_n : n > N\}) \\ &= \lim_{N \rightarrow \infty} (\sup\{f(a_n) : n > N\}) \\ &= \lim_{N \rightarrow \infty} f(\sup\{a_n : n > N\}) \\ &= f\left(\lim_{N \rightarrow \infty} \sup\{a_n : n > N\}\right) \\ &= f(\limsup a_n). \end{aligned}$$

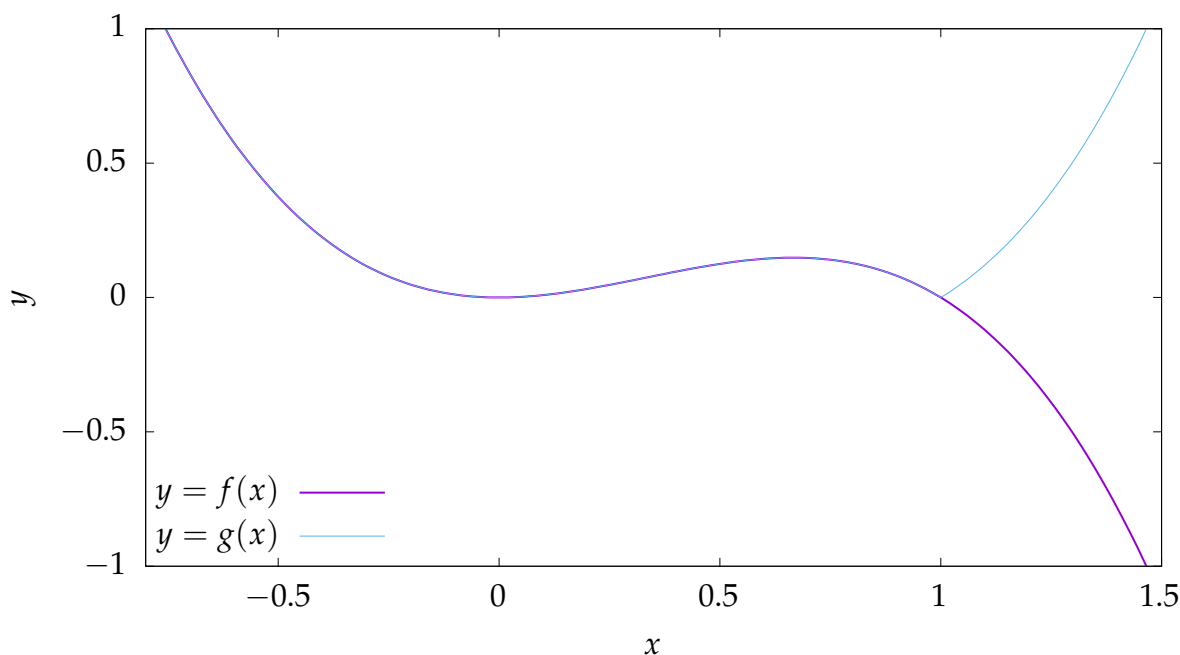


Figure 7: Graph of the functions considered in question 4 on Taylor series.

(c) Let $S = (-1, 0)$ and

$$f(x) = \begin{cases} x & \text{if } x < 0, \\ x + 1 & \text{if } x \geq 0. \end{cases}$$

Then $\sup S = 0$, so $f(\sup S) = 1$. However, $T = (-1, 0)$ and $\sup T = 0 \neq 1$.

42. (a) The functions h_1 , h_2 , and h_3 are plotted in Fig. 8
- (b) Consider any $x \neq 0$, and $\epsilon > 0$. Then there exists an $N \in \mathbb{N}$ such that $1/N < 2|x|$. Then if $n > N$, $h_n(x) = 0$. Hence $\lim_{n \rightarrow \infty} h_n(x) = 0$. At $x = 0$,

$$h_n(x) = n$$

which tends to ∞ as $n \rightarrow \infty$.

- (c) Consider any $\epsilon > 0$. Then since f is continuous, there exists a $\delta > 0$ such that $|x| < \delta$ implies $|f(x) - f(0)| < \epsilon/2$. Then there exists an N such that $1/2N < \delta$. By using the definition of an improper integral,

$$\begin{aligned} \int_{-\infty}^{\infty} h_n f &= \lim_{a \rightarrow -\infty} \int_a^0 h_n f + \lim_{b \rightarrow \infty} \int_0^b h_n f \\ &= \int_{-1/2n}^0 n f + \int_0^{1/2n} n f \\ &= n \int_{-1/2n}^{1/2n} f. \end{aligned}$$

By using the bound on f due to continuity,

$$n \int_{-1/2n}^{1/2n} \left(f(0) - \frac{\epsilon}{2} \right) dx \leq \int_{-\infty}^{\infty} h_n f \leq n \int_{-1/2n}^{1/2n} \left(f(0) + \frac{\epsilon}{2} \right) dx$$

so

$$f(0) - \frac{\epsilon}{2} \leq \int_{-\infty}^{\infty} h_n f \leq f(0) + \frac{\epsilon}{2}.$$

Hence

$$\left| f(0) - \int_{-\infty}^{\infty} h_n f \right| \leq \frac{\epsilon}{2} < \epsilon$$

so

$$\lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} h_n f = f(0).$$

(d) Consider

$$g(x) = \begin{cases} 0 & \text{if } x < 0, \\ 1 & \text{if } x \geq 0. \end{cases}$$

Then for any $n \in \mathbb{N}$,

$$\int_{-\infty}^{\infty} h_n g = \int_{-1/2n}^{1/2n} n g = \int_{-1/2n}^0 0 + \int_0^{1/2n} n = \frac{1}{2}$$

and hence

$$\lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} h_n g = \frac{1}{2}$$

which does not equal $g(0) = 1$.

43. (a) The sequence $s_n = 3 + \frac{1}{n}$ for $n \geq 2$ is contained in B and converges to 3. Hence 0 is a limit point of B . Similarly $s_n = 4 - \frac{1}{n}$ for $n \geq 2$ is contained in B and converges to 4. Hence $\bar{B} = [1, 2] \cup [3, 4]$.

No neighborhood of the form $N_r(1)$ or $N_r(2)$ is contained within B for all $r > 0$. All other points in B are interior points, since they are within the open intervals $(1, 2)$ or $(3, 4)$. Hence $B^\circ = (1, 2) \cup (3, 4)$. The boundary is $\partial B = \bar{B} \setminus B^\circ = \{1, 2, 3, 4\}$.

- (b) Since each x_n is an element of $A_{n+1} \subseteq A$, it follows that each $x_n \in A$. Therefore either $x \in A$ or x is a limit point of A . Hence $x \in \bar{A}$.

Suppose that $x \in A^\circ$. Then $x \in A_n$ for some n , and since A_n is open there exists a neighborhood $N_r(x) \subseteq A_n$ for some $r > 0$. Since $\lim_{n \rightarrow \infty} x_n = x$, there exists an $N \in \mathbb{N}$ such that $m > N$ implies $d(x_m, x) < r$. Choose a specific $m > n$; since $x_m \in A_{m+1} \setminus A_m$, then $x_m \notin A_m$. Since $A_n \subseteq A_m$, then $x_m \notin A_n$, which implies that $x_m \notin N_r(x)$ and $d(x_m, x) \geq r$, which is a contradiction. Hence $x \notin A^\circ$ and $x \in \partial A$.

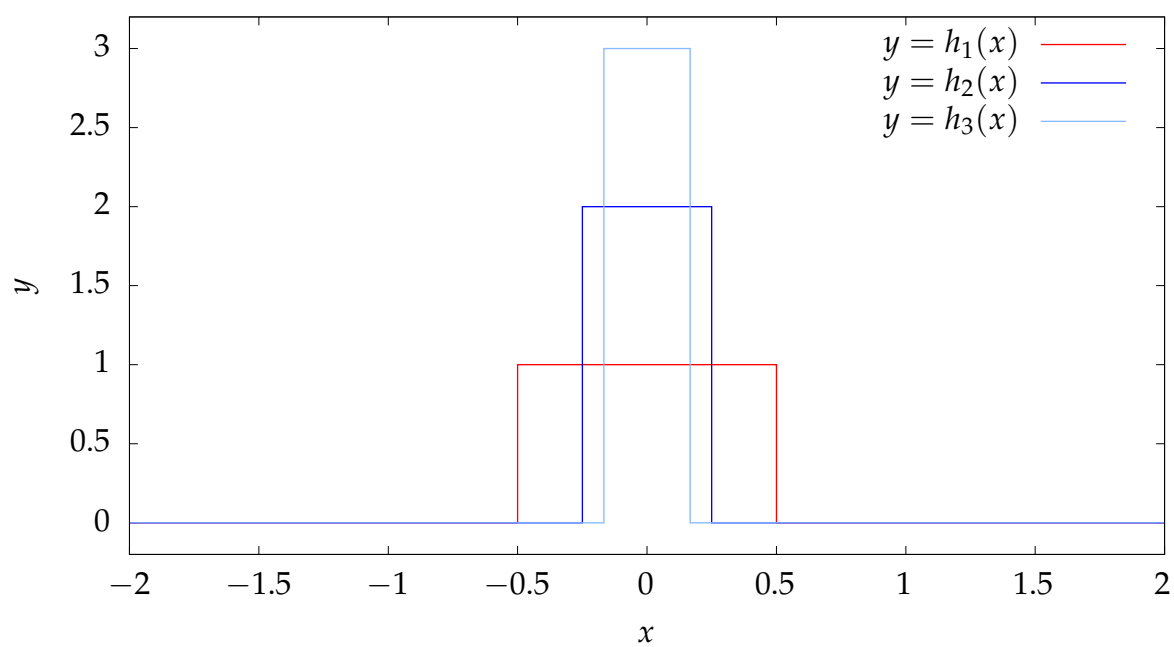


Figure 8: Graphs of several functions h_n used in question 6 on integration limits.