Math 521: Midterm 1

March 12th, 2025 9:57am–10:42am

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Instructions

- Do not open the exam until instructed to do so.
- The exam is closed book—no textbooks, notes, or smartphones allowed.
- Write your answers in the spaces provided. If you run out of space, use the reverse side of the paper. Extra paper is available on request.
- There are three questions, each of which will be graded out of ten points.

$\frac{17}{30}$

1. Consider the set $A = \{2^{-n} : n \in \mathbb{N}\} \subseteq \mathbb{R}$ with the standard metric $d(x,y) = |x-y|$.

   (a) Determine min A and max A if they exist.

   (b) Determine inf A and sup A.

   (c) Is A open?

   (d) Is A closed?

   (e) Is A compact? *Note: you may use the Heine–Borel theorem.*

$A = \frac{1}{2^n}$

[a] as $\frac{1}{2^n}$ diverges to 0. min A D.N.E ✓, max $A = \{\frac{1}{2^n} : n=1\} = \frac{1}{2}$. ✓

[b] $\inf A = 0$, ✓ $\sup A = \max A = \frac{1}{2}$. ✓

[c] for any $x, y \in A$ and $x \neq y$:

$d(x,y) = |x-y| = |\frac{1}{2^{n_1}} - \frac{1}{2^{n_2}}|$ (for $\exists\, n_1 \in \mathbb{N},\ n_2 \in \mathbb{N}$)

Since $\frac{1}{2^n}$ diverges, $|\frac{1}{2^{n_1}} - \frac{1}{2^{n_2}}| < \varepsilon$ for any $\varepsilon > 0$

Not clear how you are defining openness.

−2 So for any pt $x \in A$, we have

$B(x,\varepsilon) = \{ d(x,y) < \varepsilon : x \in A, y \in A, x \neq y \}$

and thus A is open ✗

?

[d] Consider complement of A: $\mathbb{R} \setminus A =: C$ −1

then for any $x \in C$. $\nexists\, n \in \mathbb{N}$ s.t. $x = \frac{1}{2^n}$

so for any $x, y \in C$, $d(x,y) = |x-y|$

Not clear what ~~an~~ exterior point is. we never defined this

~~open~~ But there are exterior pts on $x = \frac{1}{2^n}$ where $x \notin C$

not closed (✓)

e] As $\inf A = 0$, $\sup A = \frac{1}{2}$, A is bounded.

A not closed

By Heine-Borel's, A is not compact ✓

7

2. Let S and T be two non-empty subsets of $(0, \infty)$. Define $R = \{st : s \in S, t \in T\}$ to be the set of all products of elements from S and T.

(a) Suppose that S and T are bounded. Prove that

$$\sup R = (\sup S)(\sup T).$$

(b) Prove that if $\sup S = \infty$ then $\sup R = \infty$.

[a] Pf: · consider arbitrary element $r \in R$. $r = st$ $\forall s \in S$, $\forall t \in T$.

Since S, T are bounded,

$s \leq \sup S$ ($\forall s \in S$), $t \leq \sup T$ ($\forall t \in T$) ✓

so $r = st \leq \sup S \cdot \sup T$. $\sup S \cdot \sup T$ is an upper bound

· consider arbitrary upper bound m of R. $r \leq m$

Suppose for contradiction $m < \sup S \cdot \sup T$.

then $\exists \varepsilon > 0$ s.t. $m = \dfrac{\sup S \cdot \sup T}{\varepsilon}$ ← The details of this don't work. You probably wanted $m = \sup S \sup T - \varepsilon$.

chose $t \in T$ s.t. $t > \sup T / 2\varepsilon$

chose $s \in S$ s.t. $s > \sup S / 2\varepsilon$

$st > \dfrac{\sup S . \sup T}{4\varepsilon}$ in your solution but this not sufficient

-2 $m = rt = \left(\dfrac{\sup T}{2\varepsilon}\right)\left(\dfrac{\sup S}{2\varepsilon}\right) \geq \dfrac{1}{4\varepsilon}(\sup S)(\sup T) < \dfrac{1}{\varepsilon} \sup S \sup T$ #

to show $st > m$ and obtain a contradiction.

$\Rightarrow m \geq \sup S \cdot \sup T$

$\Rightarrow \sup R = (\sup S)(\sup T)$ □

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[b] given $\sup S = \infty$.

case 1 T bound, then $\sup T > 0$ as $T \subseteq (0, \infty)$

then by (a), $\sup R = \sup S \sup T = \infty$

← This result from (a) was only true for bounded sets. It does not apply here.

case 2 T not bounded $\sup T = \infty$

then by (a) $\sup R = \sup S \sup T = \infty$

-2

6

∞ is a useful symbol to indicate limiting behavior But you can't do arithmetic with it.

3. Let $s \in \mathbb{R}$, and let (s_n) be a sequence where $s_n \neq s$ for all n.

(a) Suppose that $\limsup \left|\frac{s_{n+1}-s}{s_n-s}\right| = L$, where $L \in [0,1)$. Prove that $\lim_{n\to\infty} s_n = s$.

(b) Suppose that $\liminf \left|\frac{s_{n+1}-s}{s_n-s}\right| = 0$. Does (s_n) always converge to s? Either prove the result or find a counterexample.

(a) to prove $\lim_{n\to\infty} S_n = s$, suffice to prove $|S_n - s| < \varepsilon$ $(\forall n, \exists \varepsilon > 0)$

or $\lim_{n\to\infty} s_n - s = 0$

given $\limsup \left|\frac{S_{n+1}-s}{S_n-s}\right| = L$, where $0 \leq L < 1$

the sequence $S_n - s$ converges ← True, but more detail needed about how you know this.

-3

and so $\lim_{n\to\infty} S_n - s = 0$ as wanted

↑ why does it converge to 0?

(b) consider $(S_n) = (-1)^n \frac{1}{n}$ and $s = 0$

$\liminf \left|\frac{S_{n+1}-s}{S_n-s}\right| = \liminf \left|\frac{(-1)^{n+1}\frac{1}{n} - 1}{(-1)^n \frac{1}{n} - 1}\right| = 0$ ← Not true. ✓

<u>But</u> $(-1)^n \frac{1}{n}$ is an alternating sequence and does not converge to 0; an counterexample

↑ Not true. It converges to 0.

-3

4