Physics 415 - Lecture 35: Black-Body Radiation

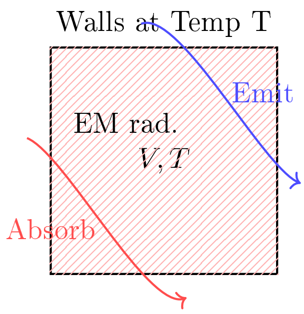
April 18, 2025

Summary: Photon Statistics

- Applies to bosonic particles whose total particle number is not conserved (e.g., photons, phonons).
- Equivalent to BE statistics with chemical potential $\mu = 0$.
- Mean occupation number: $\overline{n}_r = \frac{1}{e^{\beta\epsilon_r}-1}$ (Planck distribution).
- Canonical partition function (unrestricted sum): $Z = \prod_r \frac{1}{1-e^{-\beta\epsilon_r}}$.
- Helmholtz free energy: $F = -T\ln Z = T\sum_r \ln(1-e^{-\beta\epsilon_r})$.

Black-Body Radiation

"Black-body radiation" = electromagnetic (EM) radiation in thermal equilibrium within a cavity (volume V) whose walls are maintained at temperature T.



The walls continually absorb & emit radiation. In equilibrium, the properties of the EM radiation depend only on T.

Wave-particle duality of QM $\implies$ EM radiation can be described as waves (classical EM) or particles ("photons").

- Photons are bosons.
- Photon number is not conserved (can be absorbed/emitted by walls).
- Interactions between photons are negligible (because Maxwell's equations are linear).

Therefore, we may treat the equilibrium EM radiation as an ideal gas of photons obeying photon statistics ($\mu = 0$).

Single-Particle States for Photons

Need to specify the states r and energies ϵ_r. From Maxwell's equations in vacuum, the electric field $\vec{E}$ satisfies the wave equation:

$$\nabla^2\vec{E} = \frac{1}{c^2}\frac{\partial[}{\partial 2}]\vec{E}t^2$$

Plane wave solutions $\vec{E} = \mathrm{Re}\{\vec{E}_0 e^{i(\vec{k}\cdot\vec{x}-\omega t)}\}$ require the dispersion relation $\omega = c|\vec{k}| = ck$. Also, $\nabla\cdot\vec{E} = 0 \implies \vec{k}\cdot\vec{E}_0 = 0$, so $\vec{E}$ is transverse to $\vec{k}$. For each $\vec{k}$, there are two linearly independent modes of oscillation (polarization states) perpendicular to $\vec{k}$.

In the particle description (QM):

- Photon energy: $\epsilon = \hbar\omega = \hbar ck$.
- Photon momentum: $\vec{p} = \hbar\vec{k}$.
- Energy-momentum relation: $\epsilon = c|\vec{p}| = cp$. (Massless particle).

A single-photon state r is specified by its wave-vector $\vec{k}$ (or momentum $\vec{p}$) and its polarization state s (two possibilities, $s = 1, 2$). The energy $\epsilon_k = \hbar ck$ is independent of polarization.

Photon Gas Thermodynamics

The mean number of photons in a state $(\vec{k}, s)$ is given by the Planck distribution ($\mu = 0$):

$$\overline{n}_{\vec{k},s} = \frac{1}{e^{\beta\epsilon_k}-1} = \frac{1}{e^{\beta\hbar ck}-1}$$

Counting States: Use Periodic Boundary Conditions (PBC) on a large box $V = L_xL_yL_z$. Allowed wave-vectors: $k_i = 2\pi n_i/L_i$. Sum over states $\sum_r$ becomes sum over $(\vec{k}, s)$. Spin degeneracy $g = 2$ for polarization.

$$\sum_r \to \sum_{\vec{k},s=1,2} \to gV\int\frac{d^3k}{(2\pi)^3} = 2V\int\frac{d^3k}{(2\pi)^3}$$

Convert integral to frequency $\omega = ck$. Use spherical coordinates in k-space ($d^3k = 4\pi k^2 dk$).

$$\sum_r \to 2V\int_0^\infty \frac{4\pi k^2 dk}{(2\pi)^3} = \frac{V}{\pi^2}\int_0^\infty k^2 dk$$

Since $k = \omega/c$, $dk = d\omega/c$:

$$\sum_r \to \frac{V}{\pi^2}\int_0^\infty \left(\frac{\omega}{c}\right)^2\left(\frac{d\omega}{c}\right) = \int_0^\infty d\omega\left(\frac{V\omega^2}{\pi^2c^3}\right)$$

We define the density of modes (states) per unit frequency range $g(\omega)$:

$$g(\omega)d\omega = \frac{V\omega^2}{\pi^2c^3}d\omega$$

$$g(\omega) = \frac{V\omega^2}{\pi^2c^3}$$

Planck's Formula

Use this result to investigate the mean energy of the photon gas.

$$E = \sum_r \overline{n}_r \epsilon_r = \int_0^\infty d\omega\, g(\omega)\overline{n}(\omega)\epsilon(\omega)$$

where $\overline{n}(\omega) = 1/(e^{\beta\hbar\omega} - 1)$ and $\epsilon(\omega) = \hbar\omega$.

$$E = \int_0^\infty d\omega \left(\frac{V\omega^2}{\pi^2 c^3}\right)\left(\frac{1}{e^{\beta\hbar\omega} - 1}\right)(\hbar\omega)$$

$$E = \frac{V\hbar}{\pi^2 c^3}\int_0^\infty d\omega \frac{\omega^3}{e^{\beta\hbar\omega} - 1}$$

Write this as $E = \int_0^\infty d\omega(\frac{dE}{d\omega})$. The spectral energy density (mean energy per unit volume per unit frequency range) is $u(\omega) = \frac{1}{V}\frac{dE}{d\omega}$.

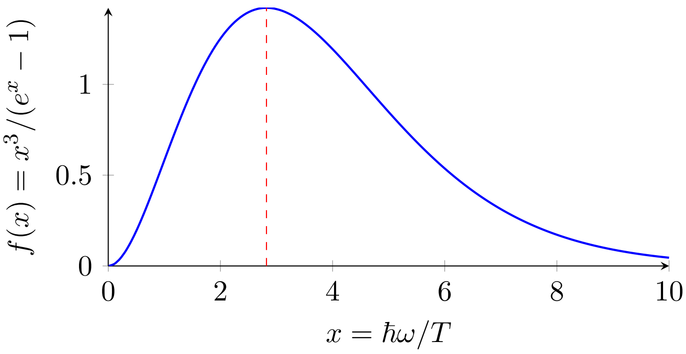
$$u(\omega) = \frac{\hbar}{\pi^2 c^3}\frac{\omega^3}{e^{\beta\hbar\omega} - 1}$$

This is **Planck's formula** for black-body radiation.

Plot $u(\omega)$ vs ω. It is useful to use a dimensionless variable $x = \beta\hbar\omega = \hbar\omega/T$. $\omega = Tx/\hbar$, $d\omega = Tdx/\hbar$.

$$u(\omega)d\omega = \frac{\hbar}{\pi^2 c^3}\frac{(Tx/\hbar)^3}{e^x - 1}\left(\frac{T}{\hbar}dx\right) = \frac{T^4}{\pi^2(\hbar c)^3}\frac{x^3}{e^x - 1}dx$$

Let $f(x) = \frac{x^3}{e^x - 1}$.



The function $f(x)$ peaks at $x_{max} \approx 2.821$. This means the peak frequency ω_{max} in the spectrum satisfies $\hbar\omega_{max}/T \approx 2.821$.

$$\hbar\omega_{max} \approx 2.821T \quad \text{or} \quad \omega_{max} \approx \frac{2.821T}{\hbar}$$

As T increases, the maximum of the spectral distribution shifts to higher frequencies, proportional to T. This is **Wien's displacement law**.

Low Frequency Limit ($\hbar\omega \ll T$, or $x \ll 1$): $e^x \approx 1 + x$. $u(\omega) \approx \frac{\hbar}{\pi^2 c^3}\frac{\omega^3}{(1+x)-1} = \frac{\hbar}{\pi^2 c^3}\frac{\omega^3}{x} = \frac{\hbar}{\pi^2 c^3}\frac{\omega^3}{\beta\hbar\omega} = \frac{T}{\pi^2 c^3}\omega^2$.

$$u(\omega) \approx \frac{T\omega^2}{\pi^2 c^3} \quad \text{(Rayleigh-Jeans formula)}$$

This classical result follows from equipartition: each EM mode (oscillator) has average energy T. $T \times$ (number of modes per frequency per volume) gives $T \times (\omega^2/(\pi^2 c^3))$. Note: The temperature

T drops out of the average energy per mode $\bar{\epsilon} = \hbar\omega\bar{n}(\omega) \approx \hbar\omega/(\beta\hbar\omega) = T$, consistent with equipartition.

The RJ formula predicts $u(\omega) \propto \omega^2$, which diverges at high frequencies ($\omega \to \infty$). If integrated, it gives infinite total energy density. This is the **"ultraviolet catastrophe"** of classical physics. Planck's resolution was to introduce quantized energies $E = n\hbar\omega$, leading to the $e^{\beta\hbar\omega} - 1$ denominator which suppresses high frequencies. This was a first success of quantum theory.

Total Energy Density (Stefan-Boltzmann Law)

Compute the total energy density $u = E/V$.

$$u = \int_0^\infty u(\omega)d\omega = \frac{T^4}{\pi^2(\hbar c)^3}\int_0^\infty dx \frac{x^3}{e^x - 1}$$

The definite integral is $\int_0^\infty \frac{x^3}{e^x-1}dx = \Gamma(4)\zeta(4) = 3!(\pi^4/90) = 6\pi^4/90 = \pi^4/15$.

$$u = \frac{T^4}{\pi^2(\hbar c)^3}\frac{\pi^4}{15} = \frac{\pi^2}{15(\hbar c)^3}T^4$$

Energy density $u \propto T^4$. This is the **Stefan-Boltzmann Law**. $u = \sigma_E T^4$, where $\sigma_E = \frac{\pi^2}{15(\hbar c)^3}$. (If using T in Kelvin, $\sigma_E = \frac{\pi^2 k_B^4}{15\hbar^3 c^3}$).

Qualitatively: Each thermally relevant photon mode has energy $\sim T$. The relevant modes are those with $\hbar\omega \sim T$, or $\hbar ck \sim T$, so $k \lesssim k_{max} \sim T/(\hbar c)$. The number of modes per volume up to k_{max} is $\sim k_{max}^3 \sim (T/(\hbar c))^3 \propto T^3$. Total energy density $u \sim$ (# modes / vol) $\times$ (avg energy per mode) $\propto T^3 \times T = T^4$.