Physics 415 - Lecture 36: Black-Body Radiation Thermodynamics

April 21, 2025

Summary

- Black-body radiation = equilibrium EM radiation $\approx$ photon gas.
- Single-particle photon state specified by $(\vec{k}, \alpha)$, where $\vec{k}$ is wave-vector and $\alpha = 1, 2$ is polarization $\perp \vec{k}$.
- Energy $\epsilon_k = \hbar ck = \hbar\omega$.
- Mean occupation number (Planck distribution): $\overline{n}_{k\alpha} = \frac{1}{e^{\beta\hbar\omega}-1}$ (since $\mu = 0$).
- Counting states: $\sum_r \to \sum_{\vec{k},\alpha} \to 2V \int \frac{d^3k}{(2\pi)^3} \to \int_0^\infty d\omega\, g(\omega)$, where $g(\omega) = \frac{V\omega^2}{\pi^2 c^3}$ is density of modes per frequency.
- Spectral energy density (Planck's formula): $u(\omega) = \frac{1}{V}\frac{dE}{d\omega} = g(\omega)/V \times \overline{n}(\omega) \times \hbar\omega = \frac{\hbar}{\pi^2 c^3}\frac{\omega^3}{e^{\beta\hbar\omega}-1}$.
- Total energy density (Stefan-Boltzmann Law): $u = \frac{E}{V} = \int_0^\infty u(\omega)d\omega = \frac{\pi^2 T^4}{15(\hbar c)^3}$. ($T$ in energy units, $k_B = 1$).

Thermodynamic Quantities for Black-Body Radiation

Free Energy (F)

Photons obey photon statistics (BE with $\mu = 0$). The Helmholtz free energy is $F = T\sum_r \ln(1 - e^{-\beta\epsilon_r})$. Convert sum to integral:

$$F = T \sum_{\alpha=1,2} V \int \frac{d^3k}{(2\pi)^3} \ln(1 - e^{-\beta\hbar ck})$$

$$F = 2TV \int_0^\infty \frac{4\pi k^2 dk}{(2\pi)^3} \ln(1 - e^{-\beta\hbar ck})$$

$$F = \frac{TV}{\pi^2} \int_0^\infty k^2 dk \ln(1 - e^{-\beta\hbar ck})$$

Change variable to frequency $\omega = ck$, $k = \omega/c$, $dk = d\omega/c$:

$$F = \frac{TV}{\pi^2 c^3} \int_0^\infty \omega^2 d\omega \ln(1 - e^{-\beta\hbar\omega})$$

Change variable to dimensionless $x = \beta\hbar\omega = \hbar\omega/T$. $\omega = Tx/\hbar$, $d\omega = Tdx/\hbar$.

$$F = \frac{TV}{\pi^2 c^3}\int_0^\infty \left(\frac{Tx}{\hbar}\right)^2 \left(\frac{Tdx}{\hbar}\right) \ln(1-e^{-x})$$

$$F = \frac{VT^4}{\pi^2(\hbar c)^3}\int_0^\infty dx\, x^2 \ln(1-e^{-x})$$

Integrate by parts: $u = \ln(1-e^{-x})$, $dv = x^2dx \implies v = x^3/3$. $\int_0^\infty udv = [uv]_0^\infty - \int_0^\infty vdu$. $du = \frac{-(-e^{-x})}{1-e^{-x}}dx = \frac{e^{-x}}{1-e^{-x}}dx = \frac{1}{e^x-1}dx$.

$$\int_0^\infty x^2\ln(1-e^{-x})dx = \left[\frac{x^3}{3}\ln(1-e^{-x})\right]_0^\infty - \int_0^\infty \frac{x^3}{3}\frac{1}{e^x-1}dx$$

The boundary term is zero at $x = 0$ (since $x^3 = 0$) and at $x = \infty$ (since $\ln(1) = 0$). The remaining integral is $\int_0^\infty \frac{x^3}{e^x-1}dx = \frac{\pi^4}{15}$. So, $\int_0^\infty x^2\ln(1-e^{-x})dx = -\frac{1}{3}\frac{\pi^4}{15} = -\frac{\pi^4}{45}$. Substitute back into F:

$$F = \frac{VT^4}{\pi^2(\hbar c)^3}\left(-\frac{\pi^4}{45}\right) = -V\frac{\pi^2T^4}{45(\hbar c)^3}$$

Check Energy and Pressure

- Energy E: Use $E = F + TS$. First find $S = -(\partial F/\partial T)_V$.

$$S = -\frac{\partial}{\partial T}\left(-V\frac{\pi^2T^4}{45(\hbar c)^3}\right)_V = V\frac{\pi^2}{45(\hbar c)^3}(4T^3)$$

$$E = F + TS = -V\frac{\pi^2T^4}{45(\hbar c)^3} + T\left(V\frac{4\pi^2T^3}{45(\hbar c)^3}\right)$$

$$E = V\frac{\pi^2T^4}{(\hbar c)^3}\left(-\frac{1}{45}+\frac{4}{45}\right) = V\frac{\pi^2T^4}{(\hbar c)^3}\frac{3}{45} = V\frac{\pi^2T^4}{15(\hbar c)^3}$$

  This matches the Stefan-Boltzmann law $u = E/V = \frac{\pi^2T^4}{15(\hbar c)^3}$. ✓

- Pressure p: Use $p = -(\partial F/\partial V)_T$.

$$p = -\frac{\partial}{\partial V}\left(-V\frac{\pi^2T^4}{45(\hbar c)^3}\right)_T = \frac{\pi^2T^4}{45(\hbar c)^3}$$

  Note the relation between pressure and energy density $u = E/V$:

$$p = \frac{1}{3}\left(\frac{\pi^2T^4}{15(\hbar c)^3}\right) = \frac{1}{3}u$$

  So, $pV = \frac{1}{3}E$. This is the equation of state for a photon gas ("radiation pressure"). (Compare with $pV = \frac{2}{3}E$ for non-relativistic ideal gas).

Total Number of Photons

The total number of photons N is not fixed, but we can calculate its average value.

$$N = \overline{N} = \sum_r \overline{n}_r = \int_0^\infty d\omega\, g(\omega)\overline{n}(\omega)$$

$$N = \int_0^\infty d\omega\left(\frac{V\omega^2}{\pi^2c^3}\right)\frac{1}{e^{\beta\hbar\omega}-1}$$

Let $x = \beta\hbar\omega$. $\omega = Tx/\hbar$, $d\omega = Tdx/\hbar$.

$$N = \frac{V}{\pi^2 c^3} \int_0^\infty \left(\frac{Tx}{\hbar}\right)^2 \frac{1}{e^x - 1} \left(\frac{Tdx}{\hbar}\right) = \frac{VT^3}{\pi^2 (\hbar c)^3} \int_0^\infty dx \frac{x^2}{e^x - 1}$$

The integral is $\int_0^\infty \frac{x^{s-1}}{e^x-1} dx = \Gamma(s)\zeta(s)$. Here $s - 1 = 2 \implies s = 3$. Integral $= \Gamma(3)\zeta(3) = 2!\zeta(3) = 2\zeta(3)$, where $\zeta(3) \approx 1.202$.

$$N = \frac{VT^3}{\pi^2 (\hbar c)^3} (2\zeta(3))$$

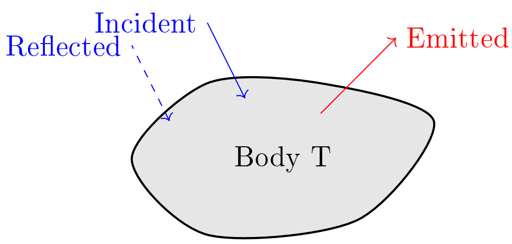
The number density of photons is $n = N/V$:

$$n = \frac{2\zeta(3)}{\pi^2} \left(\frac{T}{\hbar c}\right)^3 \approx 0.244 \left(\frac{T}{\hbar c}\right)^3$$

In equilibrium, the density of photons is determined solely by the temperature T.

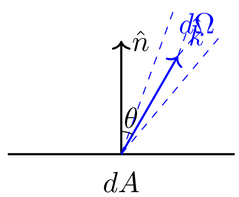
Radiation Emitted by a Body at Temperature T

So far, considered equilibrium radiation *inside* an enclosure at temperature T. Now consider the radiation *emitted* by a body whose surface is at temperature T. A body in thermal equilibrium with black-body radiation continually reflects & absorbs incident photons, and also emits new photons. In equilibrium, these processes balance, and the photon distribution $\overline{n}_{k\alpha}$ remains unchanged.



Question: How much power does the body emit by radiation as a function of frequency ω?

Let $J(\omega, \theta) d\omega d\Omega =$ Power emitted per unit area of the body surface into a small solid angle $d\Omega$ around direction $\hat{k}$, within frequency range $(\omega, \omega + d\omega)$. The angle θ is between $\hat{k}$ and the surface normal $\hat{n}$ ($\cos\theta = \hat{k} \cdot \hat{n}$). Summed over polarization.



First consider radiation incident on the body from the equilibrium field at temperature T. Energy density of photons in $(\omega, \omega + d\omega)$ is $u_\omega d\omega = \frac{\hbar}{\pi^2 c^3} \frac{\omega^3 d\omega}{e^{\beta\hbar\omega} - 1}$. Energy incident on dA from $d\Omega$ about $\hat{k}$ in time dt: $dE_{inc} = (u_\omega d\omega) \times (\frac{d\Omega}{4\pi}) \times (cdt \cos\theta dA)$. ($\frac{d\Omega}{4\pi}$ = fraction of isotropic photons in $d\Omega$. $cdt\cos\theta dA$ = volume containing photons hitting dA). Incident power per unit area from $d\Omega, d\omega$:

$$\frac{dP_{inc}}{dA} = \frac{dE_{inc}}{dAdt} = \frac{cu_\omega \cos\theta}{4\pi} d\omega d\Omega$$

$$\frac{dP_{inc}}{dA} = \frac{\hbar\omega^3}{4\pi^3 c^2}\frac{\cos\theta}{e^{\beta\hbar\omega}-1}d\omega d\Omega$$

In general, only a fraction $a(\omega,\theta)$ (absorptivity) of incident radiation is absorbed, the rest is reflected. Power absorbed per unit area from $d\Omega, d\omega$:

$$\frac{dP_{abs}}{dA} = a(\omega,\theta)\frac{dP_{inc}}{dA} = a(\omega,\theta)\frac{cu_\omega\cos\theta}{4\pi}d\omega d\Omega$$

In equilibrium, the energy of the body is unchanged, so radiated power must balance absorbed power for each $(\omega,\theta,d\Omega,d\omega)$:

$$J(\omega,\theta)d\omega d\Omega = \frac{dP_{abs}}{dA} = a(\omega,\theta)\frac{cu_\omega\cos\theta}{4\pi}d\omega d\Omega$$

$$\implies J(\omega,\theta) = a(\omega,\theta)\frac{cu_\omega\cos\theta}{4\pi}$$

The functions $J(\omega,\theta)$ (emissivity) and $a(\omega,\theta)$ (absorptivity) are determined solely by properties of the body at temperature T. But their ratio is a universal function of ω, T, θ, independent of the body:

$$\frac{J(\omega,\theta)}{a(\omega,\theta)} = \frac{cu_\omega\cos\theta}{4\pi} = \frac{\hbar\omega^3}{4\pi^3c^2}\frac{\cos\theta}{e^{\beta\hbar\omega}-1}$$

This is **Kirchhoff's Law**: A good emitter of radiation is also a good absorber (and vice-versa).

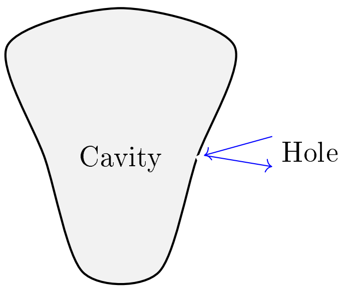
Black Body Emission

Important special case: $a(\omega,\theta) = 1$ for all ω,θ. The body absorbs all incident radiation. Such a body is called a "black body" (at $T = 0$ K, it would look black). For a black body:

$$J_{bb}(\omega,\theta) = \frac{cu_\omega\cos\theta}{4\pi} = \frac{\hbar\omega^3}{4\pi^3c^2}\frac{\cos\theta}{e^{\beta\hbar\omega}-1}$$

The emitted power is entirely determined by the equilibrium radiation field at temperature T. In particular, the emitted power is the same for all black bodies at the same T.

Model of a Black Body: A cavity with highly absorbing walls and a small hole.



Any radiation incident on the hole enters the cavity and has negligible probability of escaping after repeated reflections from the (absorbing) cavity walls. The hole effectively absorbs all incident radiation and acts as a black body surface. The radiation emerging from the hole is characteristic of the equilibrium radiation inside at temperature T.

Total Radiated Power

Compute the total power per unit area radiated by a black body in frequency range $(\omega, \omega + d\omega)$, denoted $J_\omega d\omega$. Integrate $J_{bb}(\omega, \theta)$ over the outgoing hemisphere ($d\Omega = \sin\theta d\theta d\phi$, $\theta \in [0, \pi/2]$).

$$J_\omega d\omega = \left(\int_{\text{hemisphere}} J_{bb}(\omega, \theta) d\Omega \right) d\omega$$

$$J_\omega d\omega = \left(\int_0^{2\pi} d\phi \int_0^{\pi/2} d\theta \sin\theta \, a(\omega, \theta) \frac{c u_\omega \cos\theta}{4\pi} \right) d\omega$$

Assuming $a = 1$ for black body:

$$J_\omega = \frac{c u_\omega}{4\pi} \int_0^{2\pi} d\phi \int_0^{\pi/2} d\theta \sin\theta \cos\theta$$

The angular integral is $(2\pi) \times [\frac{1}{2}\sin^2\theta]_0^{\pi/2} = (2\pi)(1/2) = \pi$.

$$J_\omega = \frac{c u_\omega}{4\pi}(\pi) = \frac{c}{4} u_\omega$$

$$J_\omega = \frac{c}{4} \frac{\hbar}{\pi^2 c^3} \frac{\omega^3}{e^{\beta\hbar\omega} - 1} = \frac{\hbar}{4\pi^2 c^2} \frac{\omega^3}{e^{\beta\hbar\omega} - 1}$$

This is the Planck law for the spectral distribution of power emitted by a black body.

Total power per unit area radiated over all frequencies, J:

$$J = \int_0^\infty J_\omega d\omega = \int_0^\infty \frac{c}{4} u_\omega d\omega = \frac{c}{4} \int_0^\infty u_\omega d\omega$$

The integral is the total energy density $u = \frac{\pi^2 T^4}{15(\hbar c)^3}$.

$$J = \frac{c}{4} \left(\frac{\pi^2 T^4}{15(\hbar c)^3} \right) = \frac{\pi^2 T^4}{60\hbar^3 c^2}$$

If T is in Kelvin, use $T_{energy} = k_B T_{Kelvin}$:

$$J = \frac{\pi^2 (k_B T)^4}{60\hbar^3 c^2} = \left(\frac{\pi^2 k_B^4}{60\hbar^3 c^2} \right) T^4$$

$$J = \sigma T^4$$

This is the **Stefan-Boltzmann Law** for black-body emission. The Stefan-Boltzmann constant is $\sigma = \frac{\pi^2 k_B^4}{60\hbar^3 c^2}$.