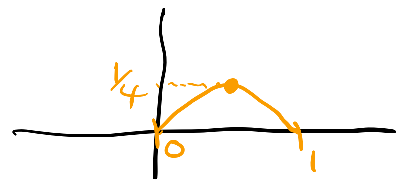
$= nx(1-x) \leq \frac{n}{4}.$



Theorem For every continuous function on $[0,1]$, $B_n f \to f$ uniformly on $[0,1]$.

Proof Assume f is not 0 everywhere. $M = \sup\{|f(x)| : x \in [0,1]\} > 0$.

Choose $\varepsilon > 0$, then $\exists \delta > 0$ s.t.

$$x, y \in [0,1] \text{ and } |x-y| < \delta \Rightarrow |f(x) - f(y)| < \frac{\varepsilon}{2}.$$

$$|B_n f(x) - f(x)| = \left| \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k} - f(x) \right|$$

$$\sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} = 1.$$

$$= \left| \sum_{k=0}^{n} \left[f\left(\frac{k}{n}\right) - f(x) \right] \binom{n}{k} x^k (1-x)^{n-k} \right|$$

$$\leq \sum_{k=0}^{n} \left| f\left(\frac{k}{n}\right) - f(x) \right| \binom{n}{k} x^k (1-x)^{n-k}$$

Divide the terms in the sum into two components.

If $\left| \frac{k}{n} - x \right| < \delta$ then $\left| f\left(\frac{k}{n}\right) - f(x) \right| < \frac{\varepsilon}{2}$

If $\left| \frac{k}{n} - x \right| \geq \delta$ then $\left| \frac{k - nx}{n} \right| \geq \delta$

$$\frac{(k-nx)^2}{n^2} \geq \delta^2.$$

$$|k - nx|^2 \geq \delta^2 n^2. \quad (*)$$

$$\sum_{k\in B} \left|f\left(\tfrac{k}{n}\right) - f(x)\right| \binom{n}{k} x^k (1-x)^{n-k}$$

$$\leq 2M \sum_{k\in B} \binom{n}{k} x^k (1-x)^{n-k}.$$

Let B be the set of indices where $(*)$ holds.

$$\leq \frac{2M}{n^2\delta^2} \sum_{k\in B} (k-xn)^2 \binom{n}{k} x^k (1-x)^{n-k}.$$

$$\leq \frac{2M}{n^2\delta^2}\left(\frac{n}{4}\right) = \frac{M}{2n\delta^2}$$

Consider for $n > N$

Choose $N = \dfrac{M}{\varepsilon\delta^2}$ $\quad \displaystyle\sum_{k\in B} \left|f\left(\tfrac{k}{n}\right) - f(x)\right| \binom{n}{k} x^k (1-x)^{n-k} < \frac{\varepsilon}{2}.$

Let A be the indices wher $\left|\frac{k}{n} - x\right| < \delta$

Then $\displaystyle\sum_{k\in A} \left|f\left(\tfrac{k}{n}\right) - f(x)\right| \binom{n}{k} x^k (1-x)^{n-k}$

$$< \sum_{k=0}^{n} \frac{\varepsilon}{2} \binom{n}{k} x^k (1-x)^{n-k} = \frac{\varepsilon}{2}$$

$$\Rightarrow \sum_{k=0}^{n} \left|f\left(\tfrac{k}{n}\right) - f(x)\right| \binom{n}{k} x^k (1-x)^{n-k} < \varepsilon.$$

Weierstraß approximation theorem

Every continuous function on a closed interval $[a,b]$ can be uniformly approximated by polynomials on $[a,b]$.

e.g. $g(x)$ on $[a,b]$.

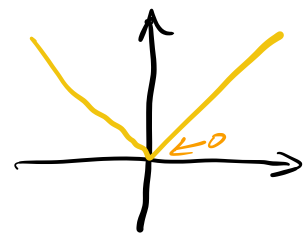
construct $h(x)$ on $[0,1]$ by

$$h(x) = g(a + (b-a)x).$$

Differentiation

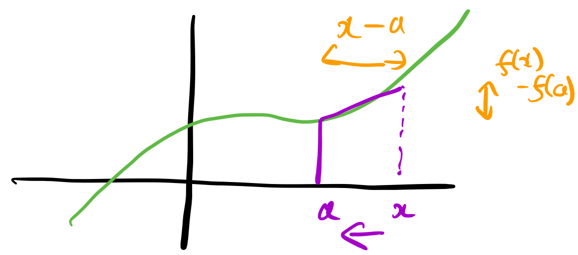
Let f be a real-valued function on $S \subseteq \mathbb{R}$. Define the derivative $f'(a)$ at $a \in S$ as

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$



if it exists and it is finite.

Example Suppose $f(x) = x^n$.



Then

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$= \lim_{x \to a} \frac{x^n - a^n}{x - a}$$

$$x^2 - a^2 = (x-a)(x+a)$$

$$x^3 - a^3 = (x-a)(x^2 + ax + a^2)$$

$$x^4 - a^4 = (x-a)(x^3 + ax^2 + a^2 x + a^3)$$

$$= \lim_{x \to a} \frac{x-a}{x-a} \sum_{j=0}^{n-1} x^j a^{n-1-j}$$

$$= \lim_{x \to a} \sum_{j=0}^{n-1} x^j a^{n-1-j} = \sum_{j=0}^{n-1} a^j a^{n-1-j} = n a^{n-1}.$$

Think of f' as function in its own right. Domain of f' is the set of points at which f is differentiable. $\mathrm{dom}(f') \subseteq \mathrm{dom}(f)$.

Theorem If f is differentiable at a point a, then f is continuous at a.

then f is continuous at a.

Proof Given $\displaystyle\lim_{x \to a} \frac{f(x) - f(a)}{x-a}$ exists

$$f(x) = \underbrace{(x-a)}_{\text{limit is zero}} \underbrace{\frac{f(x) - f(a)}{x-a}}_{\text{limit is finite.}} + f(a).$$

$$\lim_{x \to a} f(x) = f(a).$$

Theorem Let f and g be functions that are differentiable at a. Let $c \in \mathbb{R}$. Then cf, $f+g$, $f \cdot g$ and f/g are all differentiable at a, except f/g when $g(a) = 0$.

$$(cf)'(a) = c\, f'(a).$$

$$(f+g)'(a) = f'(a) + g'(a).$$

$$(fg)'(a) = f(a)\, g'(a) + f'(a) g(a).$$

$$(f/g)'(a) = \frac{g(a) f'(a) - f(a) g'(a)}{[g(a)]^2}$$

Proof Look at product rule

$$\frac{(fg)(x) - (fg)(a)}{x-a} = \frac{\overbrace{f(x)g(x) - f(x)g(a)} + \overbrace{f(x)g(a) - f(a)g(a)}}{x-a}$$

$$= \underbrace{f(x) \frac{g(x) - g(a)}{x-a}} + \underbrace{g(a) \frac{f(x) - f(a)}{x-a}}$$

$$\lim_{x\to a} \frac{(fg)(x)-(fg)(a)}{x-a} = f(a)\,g'(a) + g(a)\,f'(a).$$

Inductive method to calculate $f'(x)$ for x^n.

$$f(x)=x^n \qquad f'(x)=nx^{n-1}$$
$$g(x)=x \qquad g'(x)=1.$$

$$(fg)'(x) = x^n(1) + nx^{n-1}x = (n+1)x^n$$

(proves the induction step)

Theorem (Chain rule)

Suppose that f is differentiable at a and that g is differentiable at $f(a)$. Then $g\circ f$ is differentiable at a, with derivative

$$g'(f(a))\,f'(a).$$

Proof Define $h(y) = \dfrac{g(y)-g(f(a))}{y-f(a)}.$ $y\in \operatorname{dom}(g)$, $y\neq f(a)$.

Also define

$$h(f(a)) = g'(f(a)).$$

$$\lim_{y\to f(a)} h(y) = h(f(a)) \left(= g'(f(a))\right)$$ $\Rightarrow$ h is continuous at $f(a)$.

Now $\overset{f(x)}{g(y)} - g(f(a)) = h(\underset{f(x)}{y})(\underset{f(x)}{y} - f(a))$ $\forall y\in\operatorname{dom}(g)$.

$g\circ f(x) - g\circ f(a) = h(f(x))(f(x)-f(a))$ $y=f(x)$, $x\in\operatorname{dom}(g\circ f)$

$$\frac{g\circ f(x) - g\circ f(a)}{x-a} = h(f(x))\,\frac{f(x)-f(a)}{x-a} \qquad x \in \operatorname{dom}(g\circ f),\ x \neq a.$$

Take limits

$$\lim_{x\to a} h(f(x)) = h(f(a)) = g'(f(a))$$

$$(g\circ f)'(a) = g'(f(a))\, f'(a).$$

Suppose $f(x) = \frac{1}{x}$

$$\frac{f(x)-f(a)}{x-a} = \frac{\frac{1}{x} - \frac{1}{a}}{x-a} = \frac{ax\left(\frac{1}{x}-\frac{1}{a}\right)}{ax(x-a)}$$

$$= \frac{a-x}{ax(x-a)} = \frac{-1}{ax} \qquad x \neq a.$$

$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} = -\frac{1}{a^2} \qquad a \neq 0.$$

Apply to chain rule with $g(x) = \frac{1}{x}$.

$$(g\circ f)'(a) = \left(\frac{1}{f}\right)'(a) = \frac{-1}{f(a)^2}\, f'(a).$$

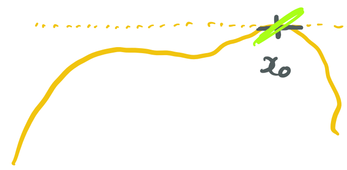
Use to derive the quotient rule

$$\left(\frac{f}{g}\right)'(a) = f'(a)\left(\frac{1}{g}\right)(a) + f(a)\left(\frac{1}{g}\right)'(a)$$

$$= f'(a)\left(\frac{1}{g}\right)(a) - f(a)\,\frac{g'(a)}{g(a)^2}$$

$$= \frac{f'(a)g(a) - f(a)g'(a)}{g(a)^2}.$$

Theorem If f is defined on an open interval containing x_0, and f assumes a minimum or maximum at x_0, and f is differentiable at x_0, then $f'(x_0) = 0$.



Proof Suppose f is defined on (a,b) where $a < x_0 < b$. Suppose it assumes its maximum.

If $f'(x_0) > 0$, then

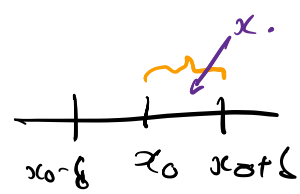
$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

Pick $\varepsilon = f'(x_0)$, $\exists \delta > 0$ s.t $0 < |x - x_0| < \delta$

then $\left| \frac{f(x) - f(x_0)}{x - x_0} - f'(x_0) \right| < f'(x_0)$.

$$\boxed{-f'(x_0) < \frac{f(x) - f(x_0)}{x - x_0} - f'(x_0)} < f'(x_0).$$

$$0 < \frac{f(x) - f(x_0)}{x - x_0}$$



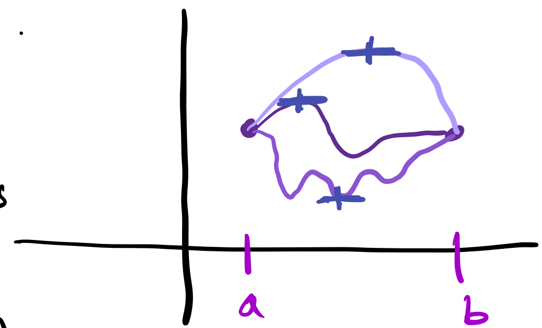
Choose x s.t $x_0 < x < x_0 + \delta$

$$f(x) - f(x_0) > 0.$$
$$f(x) > f(x_0)$$

Contradiction since x_0 is

a maximum.

Rolle's theorem Let f be a continuous function on $[a,b]$ that is differentiable on (a,b) and satisfies $f(a)=f(b)$. Then there exists at least one $x\in(a,b)$ such that $f'(x)=0$.



Proof By a previous theorem, f is bounded and achieves its bounds. $\exists x_0, y_0 \in [a,b]$ such that $f(x_0)\leq f(x)\leq f(y_0)$ for all $x\in[a,b]$.

If x_0 and y_0 are both endpoints then f is a constant function.

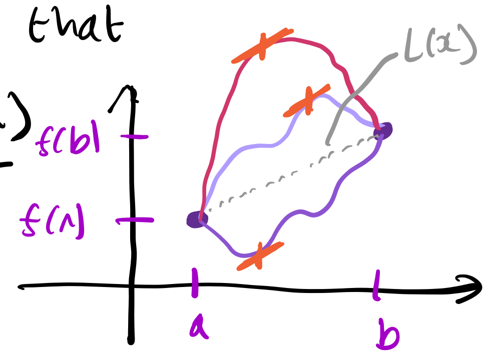
$\left(f(a) \leq f(x) \leq f(a). \quad f(x) = f(a). \right)$

Otherwise f assumes a maximum or minimum at a point x in (a,b) in which case $f'(x)=0$.

Mean value theorem

Let f be a continuous function on $[a,b]$ that is differentiable on (a,b). Then there exists at least one $x\in(a,b)$ such that

$$f'(x)=\frac{f(b)-f(a)}{b-a}$$



Proof

Define $L(x) = f(a) + \frac{(x-a)}{b-a}(f(b)-f(a))$

$L'(x) = \frac{f(b)-f(a)}{b-a}$. Define $g(x) = f(x) - L(x)$

$g(a) = 0 = g(b)$.

Rolle's theorem $\Rightarrow \exists x \in (a,b)$ s.t $g'(x)=0$.

$$f'(x) = L'(x) = \frac{f(b)-f(a)}{b-a}.$$

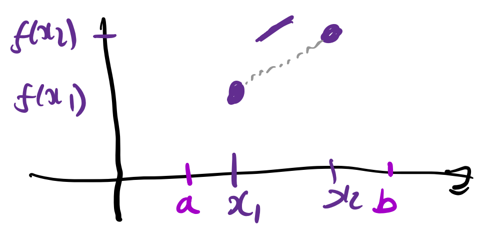
<u>Corollary</u>

Let f be a differentiable function on (a,b) such that $f'(x)=0$ for all $x \in (a,b)$. Then f is a constant function on (a,b).

<u>Proof</u> If it's not constant, $\exists x_1, x_2$ s.t.

$a < x_1 < x_2 < b$ & $f(x_1) \neq f(x_2)$.

Apply the mean value theorem (MVT) to (x_1, x_2)



$\exists x \in (x_1, x_2)$ s.t

$$f'(x) = \frac{f(x_1)-f(x_2)}{x_1 - x_2} \neq 0. \quad \text{Contradiction.}$$

<u>Corollary</u> If f and g are differentiable functions on (a,b) and $f'=g'$ on (a,b), then

$\exists c \in \mathbb{R}$ s.t. $f(x) = g(x) + c \quad \forall x \in (a,b)$.

<u>Proof</u> Apply previous corollary to $f-g$.

$f(x) - g(x) = c$ for some $c \in \mathbb{R}$

$f(x) = g(x) + c$.

<u>Definition</u>

A real-valued function f is strictly increasing on an interval I if

$x_1, x_2 \in I$ and $x_1 < x_2 \Rightarrow f(x_1) \overset{*}{<} f(x_2)$.

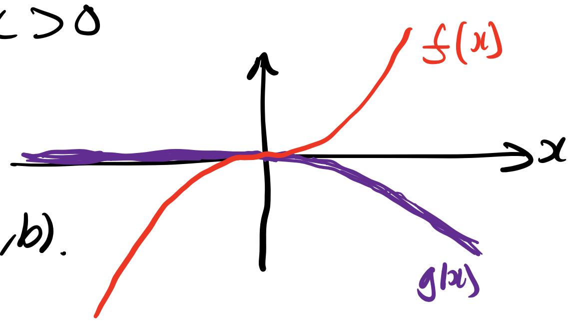
f is strictly decreasing if

$x_1, x_2 \in I$ and $x_1 < x_2 \Rightarrow f(x_1) \overset{*}{>} f(x_2)$

For increasing or decreasing functions, allow equality in the definitions $*$

<u>Examples</u> $f(x) = x^3$ strictly increasing on $\mathbb{R}$

$$g(x) = \begin{cases} 0 & x \leq 0 \\ -x^2 & x > 0 \end{cases} \quad \text{decreasing on } \mathbb{R}$$



<u>Corollary</u> Let f be a differentiable function on (a,b).

Then

- f is strictly increasing if $f'(x) > 0 \quad \forall x \in (a,b)$

• f is increasing if $f'(x) \geq 0$ $\forall x \in (a,b)$.

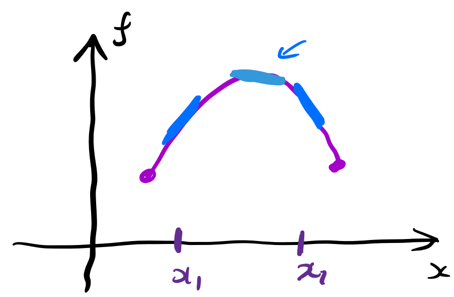
Proof Consider x_1, x_2 where $a < x_1 < x_2 < b$.

MVT $\exists x$ s.t. $\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(x) \geq 0$.

$$\Rightarrow f(x_2) \geq f(x_1)$$

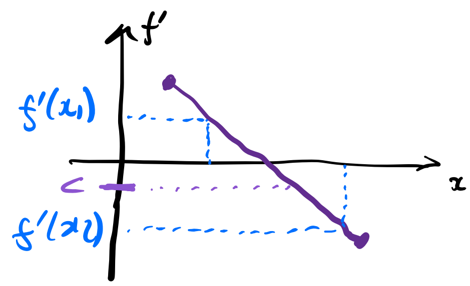
Theorem (Intermediate value theorem for derivatives)

Let f be a differentiable function on (a,b). Whenever $a < x_1 < x_2 < b$ and c is between $f'(x_1)$ and $f'(x_2)$. Then $\exists x \in (x_1, x_2)$ such that $f'(x) = c$.



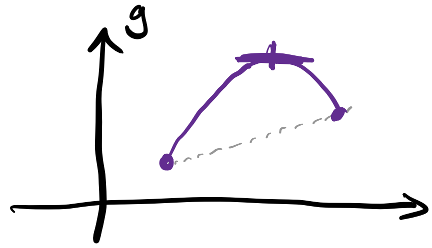
Proof Assume $f'(x_1) < c < f'(x_2)$.

Define $g(x) = f(x) - cx$ for $x \in (a,b)$. Then $g'(x_1) < 0 < g'(x_2)$.



Prev. theorem: g assumes its minimum on $x_0 \in [x_1, x_2]$.

$$g'(x_1) = \lim_{y \to x_1} \frac{g(y) - g(x_1)}{y - x_1} < 0.$$



$g(y) - g(x_1) < 0$ close to x_1

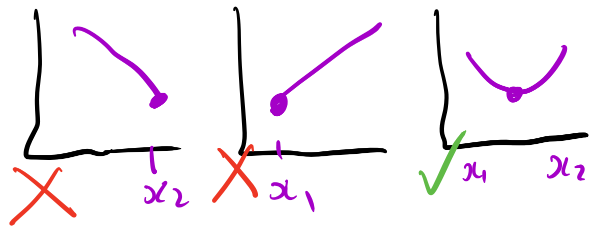
$g(y) < g(x_1)$ close to x_1

$\Rightarrow x_1$ is not a minimum.

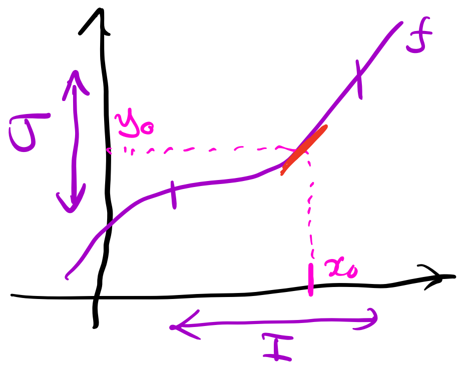
Similarly $\exists y_2 \in (x_1, x_2)$ s.t $g(y_2) < g(x_2)$.

$g'(x_0) = 0$ by previous theorem.

$$f'(x_0) = g'(x_0) + c = c$$



Let f be a one-to-one differentiable function on an open interval I. $f(I) = J$.

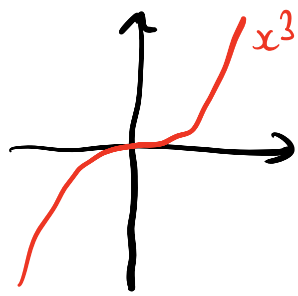


$f^{-1}: J \to I$ $\qquad f^{-1} \circ f(x) = x$

$$1 = (f^{-1})'(f(x))\, f'(x)$$

$x_0 \in I$

$y_0 = f(x_0)$ $\qquad (f^{-1})'(y_0) = \dfrac{1}{f'(x_0)}$



Theorem Let f be a one-to-one continuous function on an open interval I, and let $J = f(I)$. If f is differentiable at $x_0 \in I$ and is $f'(x_0) \neq 0$ then f^{-1} is differentiable at $y_0 = f(x_0)$ and

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)}.$$

Proof J is also an open interval.

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0).$$

Choose $\varepsilon > 0$, $\exists \delta > 0$ s.t.

$$0 < |x - x_0| < \delta \Rightarrow \left| \frac{x - x_0}{f(x) - f(x_0)} - \frac{1}{f'(x_0)} \right| < \varepsilon$$

Since $f'(x_0) \neq 0$ and $f(x) \neq f(x_0)$ for $x \neq x_0$

Let $g = f^{-1}$. g is continuous at y_0. $\exists \eta > 0$ such that

$$0 < |y - y_0| < \eta \Rightarrow |g(y) - g(y_0)| < \delta$$

$$|g(y) - x_0| < \delta$$

Hence $$\left| \frac{g(y) - x_0}{f(g(y)) - f(x_0)} - \frac{1}{f'(x_0)} \right| < \varepsilon$$

$$\left| \frac{g(y) - g(y_0)}{y - y_0} - \frac{1}{f'(x_0)} \right| < \varepsilon.$$

f^{-1}

$$\lim_{y \to y_0} \frac{g(y) - g(y_0)}{y - y_0} = \frac{1}{f'(x_0)}.$$

L'Hôpital's rule

Interested in

Lipschitz continuity

$L > 0$ s.t. $\forall x, y$

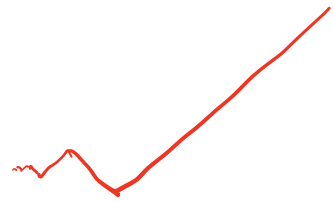
$$\lim_{x \to s} \frac{f(x)}{g(x)} \qquad s = a, a^+, a^-, -\infty, \infty$$

in cases where $f(x)$ and $g(x)$ both have limit zero.

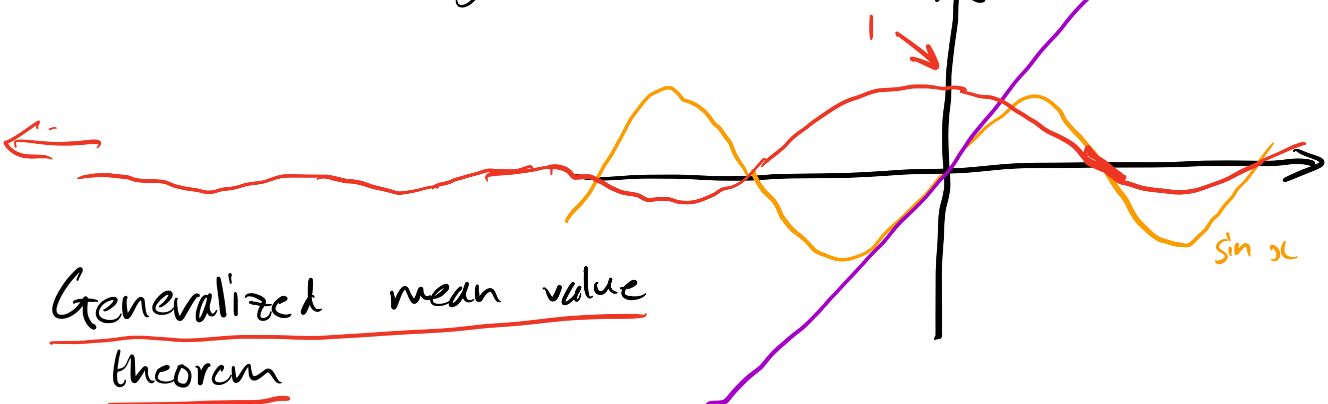
e.g. $\operatorname{sinc} x = \dfrac{\sin x}{x}$.

$$|f(x) - f(y)| \le |x - y| L$$

$$\frac{|f(x) - f(y)|}{|x - y|} \le L.$$



$x \to \infty$



Generalized mean value theorem

Let f and g be continuous functions on $[a,b]$ and differentiable on (a,b).

Then $\exists x \in (a,b)$ such that

$$f'(x)\,[g(b) - g(a)] = g'(x)\{f(b) - f(a)\}.$$

Proof Define

$$h(x) = f(x)[g(b) - g(a)] - g(x)\{f(b) - f(a)\}.$$

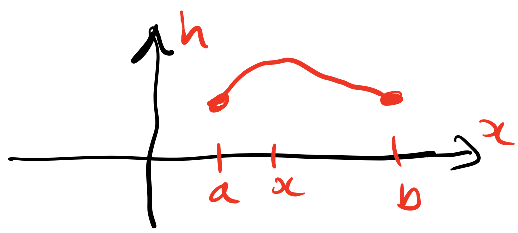
Suppose $g(x) = x$.
$g'(x) = 1$.

$$f'(x) = \frac{f(b) - f(a)}{b - a}$$

$$h(a) = f(a)g(b) - \cancel{f(a)g(a)} - f(b)g(a) + \cancel{g(a)f(a)}$$
$$= f(a)g(b) - f(b)g(a).$$

$$h(b) = \cancel{f(b)g(b)} - f(b)g(a) - \cancel{g(b)f(b)} + g(b)f(a)$$
$$= f(a)g(b) - f(b)g(a) = h(a).$$

$\exists x \in (a,b)$ such that $h'(x) = 0$.



$$\Rightarrow f'(x)\,[g(b) - g(a)] = g'(x)\,[f(b) - f(a)].$$

L'Hôpital's rule: Suppose that f, g are differentiable and $s = a, a^+, a^-, \pm\infty$, and the limit

$$\lim_{x \to s} \frac{f'(x)}{g'(x)} = L \text{ exists.}$$

If $\lim_{x \to s} f(x) = \lim_{x \to s} g(x) = 0$,

or $\lim_{x \to s} |g(x)| = \infty$

the $\lim_{x \to s} \frac{f(x)}{g(x)} = L$.

$$\boxed{\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\cos x}{1}}$$

(Implicit assumptions: there is

an interval where f' and g' exist. g' is non-zero near c)

Proof Consider $\lim_{x \to a^+}$ or $\lim_{x \to -\infty}$. Take $a \in \mathbb{R}$ or $a = -\infty$.

We will show.

If $-\infty \leq L < \infty$ and $L_1 > L$ $\exists \alpha_1 > a$ such that $a < x < \alpha_1 \Rightarrow \frac{f(x)}{g(x)} < L_1$

1

If $-\infty < L \leq \infty$ and $L_2 < L$, $\exists \alpha_2 > a$ such that $a < x < \alpha_2 \Rightarrow \frac{f(x)}{g(x)} > L_2$.

2

suppose L is finite.

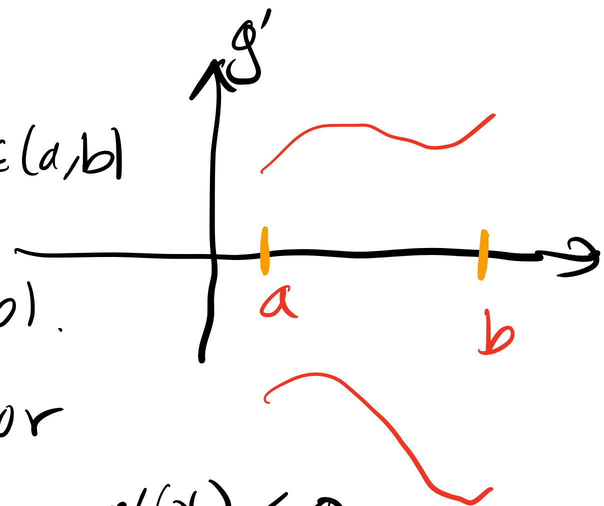
$$a < x < \alpha_1 \Rightarrow \frac{f(x)}{g(x)} < L + \varepsilon$$

$$a < x < \alpha_2 \Rightarrow \frac{f(x)}{g(x)} > L - \varepsilon$$

Define $\alpha = \min\{\alpha_1, \alpha_2\}$

$$a < x < \alpha \Rightarrow \left| \frac{f(x)}{g(x)} - L \right| < \varepsilon.$$

Examine ①. Let (a,b) be an interval on which f and g are differentiable and on which g' never vanishes.



Either $g'(x) > 0 \;\forall x \in (a,b)$ or $g'(x) < 0 \;\forall x \in (a,b)$. using the IVT for derivatives. Assume $g'(x) < 0$ $\forall x \in (a,b)$. g is strictly decreasing and one-to-one. $g(x) = 0$ for at most one x in (a,b). Choose b smaller than this to ensure g never vanishes.

Choose K s.t $L < K < L_1$. $\exists \alpha$ s.t

$$a < x < \alpha \Rightarrow \frac{f'(x)}{g'(x)} < K$$

If $a < x < y < \alpha$,

$$\frac{f(x) - f(y)}{g(x) - g(y)} = \frac{f'(z)}{g'(z)} < K^{*} \quad \text{for some } z \in (x,y)$$

$g(x) - g(y) \qquad g'(t) \qquad (\ldots y)$

Generalized MVT

Case 1 $\lim_{x \to a^+} f(x) = \lim_{x \to a^+} g(x) = 0.$

$$\lim_{x \to a^+} \frac{f(x) - f(y)}{g(x) - g(y)} = \frac{f(y)}{g(y)} \qquad \frac{f(y)}{g(y)} \leq K < L,$$

Case 2 $\lim_{x \to a^+} g(x) = \infty$

Multiply both sides of * by $\dfrac{g(x) - g(y)}{g(x)}$.

$$\frac{f(x) - f(y)}{g(x)} < K \frac{g(x) - g(y)}{g(x)}$$

$$\frac{f(x)}{g(x)} < K + \frac{f(y) - Kg(y)}{g(x)}.$$

For y fixed $\displaystyle \lim_{x \to a^+} \frac{f(y) - Kg(y)}{\underset{\to \infty}{g(x)}} = 0.$

$\exists \alpha_2 > a$ s.t.

$$a < x < \alpha_2 \qquad \frac{f(x)}{g(x)} < L_1.$$

In both cases ① is true. Same for ② to complete the proof.

Theorem Suppose the series $\sum_{n=0}^{\infty} c_n x^n$ converges for $|x|<R$ and define $f(x)=\sum_{n=0}^{\infty} c_n x^n$ for $|x|<R$.

[Uniform convergence for $[-R+\varepsilon, R-\varepsilon]$. Continuous and differentiable on $(-R,R)$ and $f'(x)=\sum_{n=1}^{\infty} n c_n x^{n-1}$.

Proof $\limsup_{n\to\infty} |c_n n|^{1/n} = \limsup_{n\to\infty} |c_n|^{1/n}$

Theorem Suppose $\{f_n\}$ is a sequence of functions differentiable on $[a,b]$ s.t. $\{f_n(x_0)\}$ converges for some point x_0 in $[a,b]$. If $\{f_n'\}$ converges uniformly on $[a,b]$ then f_n converges uniformly on $[a,b]$ to f and $f'(x)=\lim_{n\to\infty} f_n'(x)$.

Proof Choose $\varepsilon>0$. Choose N s.t $m,n>N \Rightarrow$

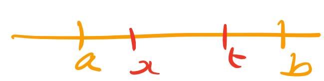
$$|f_n(x_0)-f_m(x_0)|<\frac{\varepsilon}{2} \qquad \text{(Cauchy series)}$$

$$\text{and} \quad |f_n'(t)-f_m'(t)|<\frac{\varepsilon}{2(b-a)} \qquad \text{Unif. convergence of } f'$$

$x,t\in[a,b]$.

Apply the mean value theorem to f_n-f_m $\exists y_0$ between x and t s.t.

$x\neq t$
$$\frac{|f_n(x)-f_m(x)-f_n(t)+f_m(t)|}{|x-t|} = |f_n'(y_0)-f_m'(y_0)|.$$



$$|f_n(x)-f_m(x)-f_n(t)+f_m(t)| \leq \frac{|x-t|\,\varepsilon}{2(b-a)} < \frac{\varepsilon}{2}.$$

$t=x_0$.

$$|f_n(x) - f_m(x)| \le |f_n(x) - f_m(x) - f_n(x_0) + f_m(x_0)| + |f_n(x_0) - f_m(x_0)|$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}.$$

f_n converges uniformly. Let $f(x) = \lim_{n\to\infty} f_n(x)$.

Consider $x, t \in [a,b]$ $x \neq t$.

$$\phi_n(t) = \frac{f_n(t) - f_n(x)}{t - x} \qquad \phi(t) = \frac{f(t) - f(x)}{t - x}.$$

$$\lim_{t\to x} \phi_n(t) = f_n'(x).$$

Since $f_n \to f$ uniformly, ϕ_n converges uniformly.

$\exists N$ s.t. $n, m > N \Rightarrow$
$$|\phi_n(t) - \phi_m(t)| < \frac{\varepsilon}{2(b-a)}.$$

$$\lim_{n\to\infty} \phi_n(t) = \phi(t). \qquad \lim_{t\to x} \phi(t) = \lim_{n\to\infty} f_n'(x)$$
$$= f'(x).$$

Taylor series

Consider power series $f(x) = \sum_{k=0}^{\infty} a_k x^k$

$$f'(x) = \sum_{k=1}^{\infty} k a_k x^{k-1}$$

$$f^{(n)}(x) = \sum_{k=n}^{\infty} k(k-1)(k-2)\ldots(k-n+1)\, a_k x^{k-n}$$

$$f^{(n)}(0) = n(n-1)(n-2)\ldots 1\, a_n = n!\, a_n.$$

So power series has the form

$$f(x) = \sum^{\infty} \frac{f^{(k)}(0)\, x^k}{\ldots}$$

$k=0 \qquad k!$

we call this the Taylor series of f about 0.

The remainder is defined as

$$R_n(x) = f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} x^k .$$

Important because

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k \iff \lim_{n\to\infty} R_n(x) = 0.$$

<u>Example</u>

$$f(x) = \frac{1}{1+x} = (1+x)^{-1}$$

$$f'(x) = -(1+x)^{-2}$$

$$f^{(n)}(x) = (1+x)^{-n-1}(-1)^n n!$$

Taylor series

$$f(x) = \sum_{k=0}^{\infty} \frac{(-1)^k k!}{k!} x^k = \sum_{k=0}^{\infty} (-1)^k x^k$$

Radius of convergence $=1$.

can show $\lim_{n\to\infty} R_n(x) = 0$ for $x \in (-1,1)$.

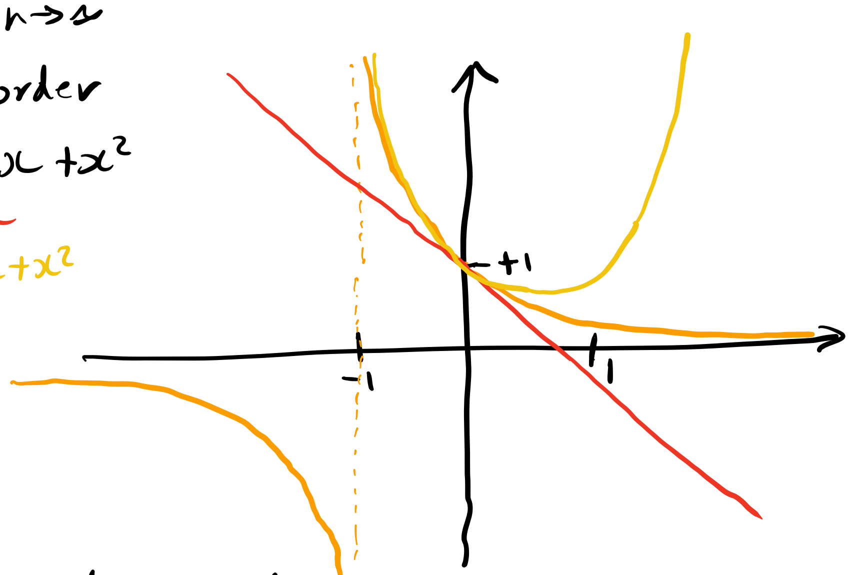
To quadratic order

$$f(x) \simeq 1 - x + x^2$$

$1-x$

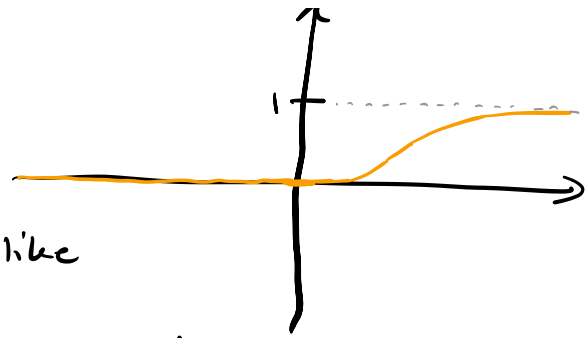
$1-x+x^2$

$f(x) = \dfrac{1}{1+x}$.



<u>Example</u> where it doesn't work

$$f(x) = \begin{cases} 0 & x \le 0 \\ e^{-1/x^2} & x > 0 \end{cases}$$



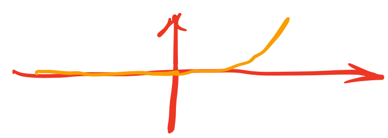
Derivatives for $x>0$ look like
$$p(x)e^{-1/x^2}$$
for some polynomial p. Since e^{-1/x^2} dominates, this goes to zero at $x=0$.

Taylor series $$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = \sum_{k=0}^{\infty} 0 x^k = 0.$$

$$\lim_{n\to\infty} R_n(x) \neq 0$$

Taylor series does not match the function

$$g(x) = \begin{cases} 0 & x \le 0 \\ x^4 & x > 0 \end{cases}$$



$$g''''(x) = \begin{cases} 0 & x \le 0 \\ 12 & x > 0 \end{cases}$$

Taylor's theorem

Let f be defined on (a,b) where $a<0<b$ and suppose the nth derivative $f^{(n)}$ exists on (a,b). Then for each non-zero x in (a,b), $\exists y$ between 0 and x such that
$$R_n(x) = \frac{f^{(n)}(y)x^n}{n!}$$

Proof Fix an $x \neq 0$. Assume that $x>0$. Let M be the unique solution of
$$f(x) = \sum^{n-1} \underline{f^{(k)}(0)}\, x^k + \underline{Mx^n}$$

just linear in M.

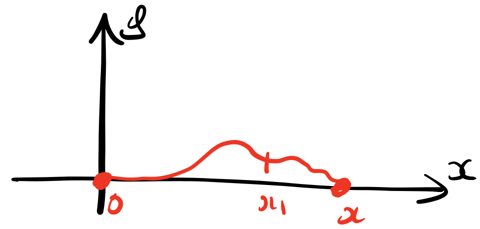
$n=0 \qquad k! \qquad\qquad n!$

Put $g(t) = \displaystyle\sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} t^k + \frac{Mt^n}{n!} - f(t).$

$g(0) = f(0) - f(0) = 0.$

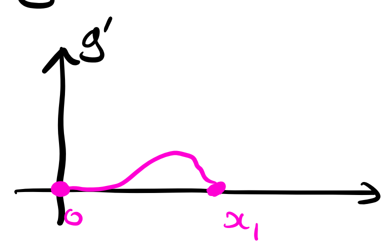
$g^{(k)}(0) = 0$ for $k<n.$

In addition, $g(x) = 0.$



Rolle's theorem: $\exists x_1 \in (0,x)$ s.t $g'(x_1) = 0$

$\exists x_2 \in (0, x_1)$ s.t $g''(x_2) = 0$



Recursively apply to find

$x_n \in (0, x_{n-1})$ s.t $g^{(n)}(x_n) = 0.$

$g^{(n)}(t) = \dfrac{n!\,M}{n!} - f^{(n)}(t).$ $\qquad f^{(n)}(x_n) = M.$

which proves the result.

Corollary

Let f be defined on (a,b) where $a<0<b$.

If all derivatives $f^{(n)}$ exist on (a,b) and are bounded by a single C, then

$$\lim_{n\to\infty} R_n(x) = 0. \qquad \forall x \in (a,b).$$

Proof $\quad |R_n(x)| \le \dfrac{C}{n!}|x|^n \qquad \displaystyle\lim_{n\to\infty} \frac{|x^n|}{n!} = 0$

$$\Rightarrow \lim_{n\to\infty} R_n(x) = 0.$$

$\cdots - \infty$

Can define exponential function

$$e^x = \sum_{k=0}^{\infty} \frac{1}{k!} x^k \qquad R = \infty$$

Can show $\frac{d}{dx} e^x = e^x$.

Shifted Taylor Series

Let f be a function defined on some interval containing $x_0 \in \mathbb{R}$. If f has derivatives of all order at x_0, then the series

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k$$

is called a Taylor series of f about x_0.

Riemann integration

Consider a bounded function on a closed interval $[a,b]$. For $S \subseteq [a,b]$, define

$$M(f,S) = \sup \{ f(x) \mid x \in S \}$$
$$m(f,S) = \inf \{ f(x) \mid x \in S \}$$

Define a partition of $[a,b]$ as any finite ordered subset

$$P = \{ a = t_0 < t_1 < \dots < t_n = b \}$$

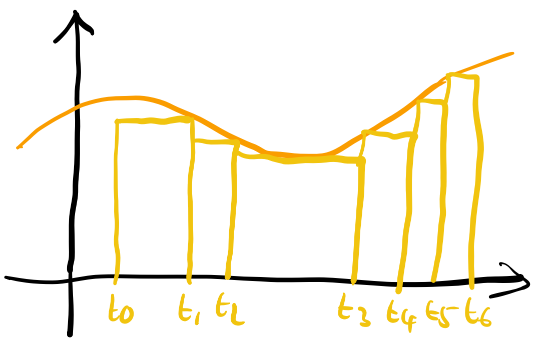
$U(f,P)$ of f with

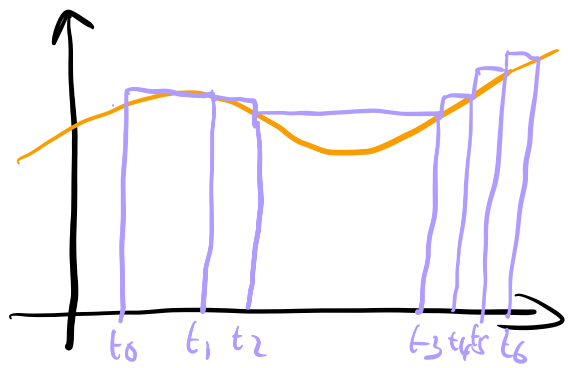
The upper Darboux sum $U(f,P)$ of f with respect to P is

$$U(f,P) = \sum_{k=1}^{n} M(f,[t_{k-1},t_k])(t_k - t_{k-1})$$

The lower Darboux sum is

$$L(f,P) = \sum_{k=1}^{n} m(f,[t_{k-1},t_k])(t_k - t_{k-1}).$$





Bounds on sums

$$U(f,P) \leq \sum_{k=1}^{n} M(f,[a,b])(t_k - t_{k-1})$$
$$= M(f,[a,b])(b-a).$$

So

$$m(f,[a,b])(b-a) \leq L(f,P) \leq U(f,P) \leq M(f,[a,b])(b-a)$$

Can define integrals

$$U(f) = \inf\{U(f,P) \mid P \text{ is a partition of } [a,b]\}$$
$$L(f) = \sup\{L(f,P) \mid P \text{ is a partition of } [a,b]\}$$