Physics 415
Spring 2025
Homework 10
Due Friday, April 18, 2025

This assignment covers material in Chapter 9 of Reif. I recommend reading through the text and also Lectures Notes 30-33.

Problem 1: (Single-particle density of states) Calculate the energy-dependent density of states $\rho(\varepsilon)$, defined such that $\rho(\varepsilon)d\varepsilon$ is equal to the number of states in the energy range between ε and $\varepsilon + d\varepsilon$, in the following cases:

(a) Particle moving in two dimensions with energy $\varepsilon = p^2/2m = \hbar^2 k^2/2m$, where $k = |\mathbf{k}|$ is the magnitude of the two-dimensional wave-vector $\mathbf{k}$.

(b) Ultra-relativistic particle in three dimensions with single-particle energy $\varepsilon = cp = \hbar ck$, where where $k = |\mathbf{k}|$.

Problem 2: (Pressure of a quantum ideal gas, adapted from Reif 9.5) Consider a free particle in a cubic container of edge length L and volume $V = L^3$.

(a) Each quantum state r of this particle has a corresponding kinetic energy ε_r which depends on V. What is $\varepsilon_r(V)$?

(b) Find the contribution to the gas pressure $p_r = -\partial\varepsilon_r/\partial V$ of a particle in this state in terms of ε_r and V.

(c) Use this result to show that the mean pressure $\bar{p}$ of any ideal gas of non-interacting particles is always related to its mean total kinetic energy $\bar{E}$ by $\bar{p} = \frac{2}{3}\bar{E}/V$, irrespective of whether the gas obeys classical, FD, or BE statistics.

Problem 3: (Velocity distribution of a Fermi gas, adapted from Reif 9.16) An ideal Fermi gas is at rest at absolute zero and has a Fermi energy ε_F. The mass of each particle is m. If $\mathbf{v}$ denotes the velocity of a molecule, find $\overline{v_x}$ and $\overline{v_x^2}$.

Problem 4: (Virial expansion, ultra-relativistic particles) Consider a three-dimensional ultra-relativistic gas, with single-particle energy $\varepsilon = c|\mathbf{p}| = c\hbar|\mathbf{k}|$. Derive the first quantum correction to the equation of state $p(n, T)$ for this system, for both bosons and fermions. *Hint:* Follow the steps in Lecture Notes 31, using your result for the density of states in Problem 1b to write the thermodynamic quantities as integrals over energy.

Problem 5 (Two-dimensional Fermi gas) Consider an ideal (non-relativistic) Fermi gas in two dimensions, confined to a square area A[1].

[1] In semiconductor heterostructures, electrons can be confined in quantum wells, restricting their motion to two dimensions. At the simplest level, such a system can be modeled as a two-dimensional ideal Fermi gas

(a) Find the Fermi energy ε_F in terms of the density $n = N/A$ (where N is the total particle number), and show that the average total energy of the system at $T = 0$ is $E_0 = N\varepsilon_F/2$.

(b) Explicitly calculate the chemical potential μ as a function of temperature T and particle density n. Verify that μ has the expected behavior in the degenerate limit ($T \ll \varepsilon_F$) and non-degenerate limit ($T \gg \varepsilon_F$). Sketch μ as a function of T. *Hint:* Use your result for the density of states from Problem 1a to explicitly evaluate the energy integral relating the particle number and chemical potential.

(c) Calculate the heat capacity C_V of the gas in the limit $T \ll \varepsilon_F$. Also show that C_V has the expected behavior when $T \gg \varepsilon_F$. Sketch the heat capacity as a function of T.

Problem 6 (Pauli paramagnetism, adapted from Reif 9.21, 9.22) In this problem you analyze the paramagnetism of the conduction electrons in a metal, modeled as a degenerate Fermi gas.

(a) Following the qualitative approach in Lecture 32, make an estimate of the paramagnetic spin susceptibility χ of the conduction electrons.

(b) Verify your estimate by explicitly calculating χ at $T = 0$ in the following way: Suppose the metal has a density of n conduction electrons per unit volume, each electron having an associated magnetic moment μ_m (not to be confused with the chemical potential), and that the metal is placed in a small external magnetic field H. The total energy of the conduction electrons in the presence of the field H must be as small as possible. Use this fact to find an explicit expression for χ due to the spin magnetic moments of these conduction electrons.

Problem 7 (Chandrasekhar limit) When stars exhaust their fuel, the temperature $T \to 0$ and they have to rely on the Pauli exclusion principle to support themselves through degeneracy pressure. Such stars, supported by electron degeneracy pressure, are called *white dwarfs*. In this problem, you will investigate conditions for the stability of a white dwarf.

(a) A crude non-relativistic model of a white dwarf star consists of a sphere of radius R of free electrons at zero temperature, together with a sufficient number of positive ions to make the star electrically neutral. Determine the energy E_{el} of all the electrons. Assuming the gravitational energy of the star is given by $E_{\text{grav}} = -\gamma GM^2/R$, where M is the total mass of the star and G is Newton's constant, show that if the state of equilibrium of the star is obtained by minimizing the total energy $E = E_{\text{grav}} + E_{\text{el}}$, then $R \propto M^{-1/3}$. The elements that can compose a white dwarf: helium, carbon or oxygen, all have roughly equal numbers of neutrons and protons, so that we can approximate the total mass of the star as $M = 2Nm_p$, where N is the number of electrons and m_p is the proton mass. For concreteness, you may set $\gamma = 3/5$, which corresponds to a simplified model in which the mass density of the star is uniform (more realistic modeling is also possible).

(b) According to part (a), a heavier mass star will have a smaller radius. As R decreases, the Fermi energy ε_F grows until it becomes comparable to the electron rest mass energy $m_e c^2$, at which point the non-relativistic approximation breaks down. (Alternatively, we may say that the Fermi velocity v_F grows to be of order the speed of light c, where v_F is defined via $\varepsilon_F = \frac{1}{2}m_e v_F^2$.) We account for this by using the relativistic energy-momentum relation:

$$\varepsilon(p) = \sqrt{p^2c^2 + m_e^2c^4}. \tag{1}$$

When the electrons become ultra-relativistic $pc \gg m_e c^2$, this may be approximated as

$$\varepsilon(p) \approx cp + \frac{1}{2}\frac{m_e^2 c^3}{p}. \tag{2}$$

Use this approximate form of $\varepsilon(p)$ to compute the zero temperature electronic energy $E_{\rm el}$ in the ultra-relativistic limit. *Hint:* Use the de Broglie relation to replace $p = \hbar k$, with k the magnitude of the electron wave-vector, and compute the total energy through an integration over k-space.

(c) Using your result from (b), show that the total energy $E = E_{\rm el} + E_{\rm grav}$ in the ultra-relativistic limit now takes the form

$$E(R) \approx \frac{A - B}{R} + CR, \tag{3}$$

and determine the coefficients A, B, and C.

(d) As in part (a), the equilibrium radius is determined by minimizing the energy $E(R)$ in Eq. (3). What is the condition on the coefficients A and B for such a minimum to exist? Express this condition as a bound on the mass M of the white dwarf in the form $M < M_C$, and determine M_C. Compare M_C to the solar mass $M_\odot$.

The limiting mass M_C is known as the *Chandrasekhar limit*, after S. Chandrasekhar, who first derived it in 1931. A more realistic calculation, which does not assume a constant mass density of the star, yields $M_C = 1.4 M_\odot$. If a star's mass exceeds the Chandrasekhar limit, it cannot become a white-dwarf when its nuclear fuel is exhausted. Instead, it must continue to collapse, becoming either a neutron star or a black hole.

For those interested, a more detailed discussion of stellar evolution, white dwarfs, and neutron stars at the level of our course can be found here: `https://schwartz.scholars.harvard.edu/sites/g/files/omnuum7046/files/schwartz/files/15-stars_0.pdf`.