Math 521: Homework 4 solutions[1]

1. F is bounded, since for any $p \in F$, $p^2 < 5$ and therefore $|p| < \sqrt{5}$. To show that F is closed, consider the complement $F^c = \{p \in \mathbb{Q} : p \neq F\}$. Choose $x \in F^c$. Then either $x^2 \leq 2$ or $x^2 \geq 5$. Since $\sqrt{2} \notin \mathbb{Q}$ and $\sqrt{5} \notin \mathbb{Q}$, it follows that either (a) $|x| \leq \sqrt{2}$ or (b) $|x| \geq \sqrt{5}$. Then $N_r(x) = \{y \in \mathbb{Q} : d(x,y) < r\} \subseteq F^c$ where $r = \sqrt{2} - |x|$ for case (a) and $r = |x| - \sqrt{5}$ for case (b). Therefore x is an interior point, and hence F^c is open, implying that F is closed.

   For $s > 2$, define the set $S(s) = \{p \in \mathbb{Q} : 2 < p^2 < s\}$. Then $S(s)$ is open for all choices of s, since for any point $x \in S(s)$, the neighborhood $N_r(x) \subseteq S(s)$ where $r = \max\{\sqrt{s} - |x|, |x| - \sqrt{2}\}$. Now consider the sets $G_i = S(5 - \frac{1}{i})$ for $i \in \mathbb{N}$; the collection $\{G_i\}$ will cover F because for any in $x \in F$, there exists an i such that $5 - \frac{1}{i} > x$, and hence $x \in G_i$.

   The collection $\{G_i\}$ does not contain a finite subcover. If it did, then there would be a G_I with maximum index I. But by the denseness of the rational numbers there exists in $x \in F$ such that $5 - \frac{1}{I} < x^2 < 5$, which will not be covered by the finite collection. Hence F is not compact. Furthermore, since $F = S(5)$, the previous argument shows that F is open.

   The Heine–Borel theorem states that a set $S \subseteq \mathbb{R}^n$ is compact if and only if it is closed and bounded. The set F considered in this question demonstrates that the Heine–Borel theorem only applies to subsets of $\mathbb{R}^n$, since here F is closed and bounded, but it is not compact.

2. Let $K_1, K_2, \ldots, K_n$ be compact subsets of S and define

$$K = \bigcup_{i=1}^{n} K_i. \tag{1}$$

   Let $\{G_\alpha\}$ be an open cover of K. Then for each $l = 1, 2, \ldots, m\}$ $\{G_\alpha\}$ is also an open cover of K_l, and there is a finite subcover $\{G_k^l\}$ where $k = 1, 2, \ldots, n_l$ for some constants n_l. Since there are a finite number of cases, the union of these collections must be finite also. Consider any point $z \in K$. Then there exists a j such that $z \in K_j$. But then $z \in \bigcup_{k=1}^{n_j} G_k^j$. Therefore the union of the collections $\{G_k^l\}$ is a finite open cover for K. Since every open cover has a finite subcover, it follows that K is compact.

   To show that this result does not generalize to infinite unions, consider the sets $K_i = [0, 2 - i^{-1}]$ for $i \in \mathbb{N}$ in the metric space $(\mathbb{R}, d)$ where d is the standard metric. Each of the K_i is a closed and bounded interval, so each set is compact by the Heine–Borel theorem. But $K = \bigcup_{i=1}^{\infty} K_i = [0, 2)$. The element 2 is a limit point of K, because the neighborhood $N_r(2)$ overlaps with K for all $r > 0$. Since $2 \notin K$, it follows that K is not closed. Therefore by the Heine–Borel theorem it is not bounded.

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3. Suppose that $K \times L$ is compact, and consider an open cover $\{G_\alpha\}$ of K. For each G_α, define a corresponding set H_α as

$$H_\alpha = \{(s,t) \,:\, s \in G_\alpha, t \in T\}. \tag{2}$$

The H_α are all open. To see this, consider any point $(s,t) \in H_\alpha$. Since G_α is open, there exists an $r > 0$ such that $\{s' \in S \,:\, d_S(s,s') < r\} \subseteq G_\alpha$. Consider the neighborhood

$$\begin{aligned} N &= \{(s',t') \in S \times T \,:\, d_{S\times T}((s',t'),(s,t)) < r\} \\ &= \{(s',t') \in S \times T \,:\, d_S(s,s') + d_T(t,t') < r\}. \end{aligned} \tag{3}$$

Since $d(s',s) < r$ for all points $(s',t') \in N$, it follows that $N \subseteq H_\alpha$. Since the same argument holds for all points $(s,t) \in H_\alpha$, it follows that H_α is open. Therefore $\{H_\alpha\}$ form an open cover of $K \times L$, and it has a finite subcover $\{H_i\}$ for $i = 1, \ldots, n$. Any point $(s,t) \in K \times L$ must be contained within a particular H_j, and therefore $s \in G_j$. Therefore the $\{G_i\}$ for $i = 1, \ldots, n$ form a finite open cover for K. Hence K is compact. Since the question is equivalent if K and L are switched, it follows that L is compact also.

The proof of the converse is more difficult. It is a consequence of the tube lemma.

4. Consider a point $x \in A = (-1,1] \cap \mathbb{Q}$. For x to be in the interior of A, we must have that $N_r(x) = (x-r, x+r) \subseteq \mathbb{Q}$. But from question 10 on homework 1, the interval $(x-r, x+r)$ will contain an irrational number $p \notin \mathbb{Q}$. Hence $p \notin A$, and therefore $N_r(x)$ is not contained within A for any r. Therefore the interior of A is the empty set.

Now consider the closure of A. Pick any point $x \in [-1,1]$. Define a sequence (x_n) contained in A, such that $|x_n - x| < \frac{1}{n}$. This must be possible since $[-1,1]$ and $N_{1/n}(x)$ will overlap in a finite interval I, and by the denseness of $\mathbb{Q}$ there will be a rational number x_n in I. Since the $\lim_{n\to\infty} x_n = x$, x must be in $\bar{A}$. Now consider $x > 1$, and define $r = x - 1 > 0$. The $N_r(x) \cup A = \emptyset$ and therefore x is not a limit point. Similarly, if $x < -1$, define $r = -1 - x > 0$, and $N_r(x) \cup A = \emptyset$. Therefore $\bar{A} = [-1,1]$.

Consider a point $y \in B = (-1,1] \cup \mathbb{Q}$. If y is in the interior of B, then there exists an $r > 0$ such that $N_r(y) = (y-r, y+r)$. If $y \in (-1,1)$, then choosing $r = \min\{1 - y, 1 - y\}$ will ensure that $N_r(y) \subseteq (-1,1) \subseteq B$. Now suppose $y \geq 1$ and consider $N_r(y) = (y-r, y+r)$. The range $(y, y+r)$ will contain an irrational number p that is not in B, and thus y is not an interior point. Similarly if $y \leq -1$ then $N_r(y) = (y-r, y+r)$, and the range $(y-r, y)$ will contain an irration number not in B. Therefore the interior of B is $(0,1)$.

Consider any point $x \in \mathbb{R}$, and let (x_n) be the sequence of decimal expansions of x truncated to n decimal places. Then all of the x_n are rational and hence in B, and $\lim_{n\to\infty} x_n = x$. Thus the closure $\bar{B} = \mathbb{R}$.

5. First, examine the point at $x = 1$ and choose $\epsilon > 0$. Then

$$|h(x) - h(1)| = |h(x)| \leq |1 - x^2| = |1 - x||1 + x|. \tag{4}$$

Suppose now that $\delta = \min\{1, \frac{\epsilon}{2}\}$. Then if $|x - 1| < \delta$,

$$|1 - x||1 + x| < \frac{\epsilon}{2}2 = \epsilon \tag{5}$$

and therefore the function is continuous at $x = 1$. Since the function is the invariant under the switch $x \leftrightarrow -x$, it must be continuous at $x = -1$ also.

Consider any other point when $x \neq \pm 1$. Then $1 - x^2 \neq 0$. Construct a sequence (y_n) of rational numbers such that $x - \frac{1}{n} < y_n < x + \frac{1}{n}$ for all n. By the denseness of $\mathbb{Q}$, such numbers can always be found. Then $\lim_{n\to\infty} y_n = x$, and $\lim_{n\to\infty} h(y_n) = 1 - x^2$. Similarly, construct a sequence (z_n) of irrational numbers such that $x - \frac{1}{n} < z_n < x + \frac{1}{n}$ for all n, which was shown to always be possible on a previous homework question. Then $\lim_{n\to\infty} z_n = x$ and $\lim_{n\to\infty} h(z_n) = 0$. Since $1 - x^2 \neq 0$, it shows that $\lim_{n\to\infty} h(x_n)$ is not always equal to $h(x)$ for all sequences (x_n) converging to x. Hence h is not continuous at x.

6. Suppose that $\alpha > 0$, and choose $\epsilon > 0$. Then

$$|f(x) - f(0)| = \left| |x|^\alpha \sin \tfrac{1}{x} - 0 \right| \leq |x|^\alpha. \tag{6}$$

Choose $\delta = \epsilon^{1/\alpha}$. Then if $|x - 0| < \delta$,

$$|f(x) - f(0)| < \delta^\alpha = \left(\epsilon^{1/\alpha}\right)^\alpha = \epsilon \tag{7}$$

and hence f is continuous at 0. Suppose now that $\alpha \leq 0$, and consider the sequence

$$x_n = \frac{1}{(2n + \frac{1}{2})\pi)}. \tag{8}$$

Then $\lim_{n\to\infty} x_n = 0$ and $f(x_n) = |x_n|^\alpha \sin(2\pi n + \frac{\pi}{2}) = x_n^\alpha$. Therefore

$$\begin{aligned}
\lim_{n\to\infty} f(x_n) &= \lim_{n\to\infty} x_n^\alpha \\
&= \lim_{n\to\infty} \left[(2n + \tfrac{1}{2})\pi\right]^{-\alpha} \\
&= \begin{cases} \infty & \text{if } \alpha < 0, \\ 1 & \text{if } \alpha = 0. \end{cases}
\end{aligned} \tag{9}$$

Therefore for all $\alpha \leq 0$, $\lim_{n\to\infty} f(x_n)$ does not equal $f(0) = 0$, and hence f is not continuous at 0.

7. (a) Let $p \in (a, b)$ be an irrational number. Define a sequence (x_n) of terms in (a, b) where each $x_n \in \mathbb{Q}$ and $p - \frac{1}{n} < x_n < p + \frac{1}{n}$. By the denseness of $\mathbb{Q}$, such a sequence can be found. Then $\lim_{n\to\infty} x_n = p$ and $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} 0 = 0$. Since f is continuous, $f(p) = 0$. Therefore $f(x) = 0$ for all $x \in (a, b)$.

   (b) Define the function $h(x) = f(x) - g(x)$, which is continuous on (a, b). In addition $h(q) = 0$ at all $q \in \mathbb{Q}$, and thus it satisfies the same conditions as the function in part (a). Therefore $h(x) = 0$ for all $x \in (a, b)$ and $f(x) = g(x)$ for all $x \in (a, b)$.

   (c) The construction of the sequence (x_n) can still be done even at the end points of a closed interval $[a, b]$ (assuming $a < b$). Therefore the results of part (a) and part (b) still hold.

8. This is false. Consider the function

$$f(x) = \begin{cases} 0 & \text{if } x \neq 0, \\ 1 & \text{if } x = 1. \end{cases} \tag{10}$$

   Consider any sequence (a_n) that converges to zero. Since f is even, $f(0 + a_n) - f(0 - a_n) = 0$, so $\lim_{n\to\infty}[f(0 + a_n) - f(0 - a_n)] = 0$. However, f is not continuous at $x = 0$. To verify this, consider the sequence $a_n = 1/n$. Then $f(a_n) = 0$ for all n and hence $\lim f(a_n) = 0$. However, $\lim_{n\to\infty} a_n = 0$, and $f(0) = 1$.

9. First note that if $a = b$, then $f(a)f(b) \geq 0$. Hence $a \neq b$. By considering $f(-x)$ instead of $f(x)$, we can assume that $a < b$. If $f(a)f(b) < 0$ then either $f(a) < 0 < f(b)$ or $f(b) < 0 < f(a)$. Then the intermediate value theorem states that there exists an $x \in (a, b)$ such that $f(x) = 0$ as required.

10. Consider $x, y \in [a, b]$ where $x < y$. Define $S(w) = \{f(z) : a \leq z \leq w\}$. Since $S(x) \subseteq S(y)$,

$$f^*(x) = \sup S(x) \leq \sup S(y) = f^*(y) \tag{11}$$

   and therefore f^* is an increasing function. To show that it is continuous, pick any point $y \in [a, b]$ and choose an $\epsilon > 0$. Then there exists a $\delta > 0$ such that if $x \in [a, b]$ and $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon/2$.

   Since $f(y) \in S(y)$, $f^*(y) \geq f(y)$. For all $x \in (y, y + \delta) \cap [a, b]$, $|f(y) - f(x)| < \epsilon/2$ and therefore $f(x) < f(y) + \epsilon/2$. Define $T(x) = \{f(w) : w \in (y, x) \cap [a, b]\}$ and then $\sup T(x) \leq f(y) + \epsilon/2 \leq f^*(y) + \epsilon/2$. Hence for all $x \in (y, y + \delta) \cap [a, b]$,

$$\begin{aligned} f^*(x) &= \sup(S(y) \cup T(x)) \\ &= \max\{\sup S(y), \sup T(x)\} \\ &= \max\{f^*(y), \sup T(x)\} \\ &\leq f^*(y) + \frac{\epsilon}{2} < f^*(y) + \epsilon. \end{aligned} \tag{12}$$

Since f^* is increasing, then $|f^*(x) - f^*(y)| < \epsilon$.

Now consider $x \in (y-\delta, y) \cap [a,b]$. Then $|f(x) - f(y)| < \epsilon/2$ and hence $f(y) < f(x) + \epsilon/2$. Define $U(x) = \{f(w) : w \in (x,y) \cap [a,b]\}$. For each $w \in (y-x, y) \cap [a,b]$, $f(w) < f(y)$ because f is increasing and hence $f(w) < f(x) + \epsilon/2$. Therefore $\sup U(x) \leq f(x) + \epsilon/2 \leq f^*(x) + \epsilon/2$. Hence for all $x \in (y-\delta, y) \cap [a,b]$,

$$\begin{aligned} f^*(y) &= \sup(S(x) \cup U(x)) \\ &= \max\{\sup S(x), \sup U(x)\} \\ &= \max\{f^*(x), \sup U(x)\} \\ &\leq f^*(x) + \frac{\epsilon}{2} < f^*(x) + \epsilon. \end{aligned} \tag{13}$$

Since f^* is increasing, then $|f^*(x) - f^*(y)| < \epsilon$. Therefore for all $x \in [a,b]$ where $|x-y| < \delta$, $|f^*(x) - f^*(y)| < \epsilon$ and f^* is continuous.

11. Choose a point $x_0 \in (a,b)$. Then there exists a Δ such that $N = (x_0 - 2\Delta, x_0 + 2\Delta) \subseteq (a,b)$. Now define $p = f(x_0 - \Delta)$ and $q = f(x_0 + \Delta)$, and consider $x \in [x_0 - \Delta, x_0]$. By using the convexity property applied to the point x between $x_0 - \Delta$ and x_0,

$$f(x) \geq \frac{f(x_0)(x - x_0 + \Delta) + (x_0 - x)p}{\Delta} = f(x_0) + \frac{(x - x_0)(f(x_0) - p)}{\Delta} \tag{14}$$

and hence

$$f(x) - f(x_0) \geq \frac{(x - x_0)(f(x_0) - p)}{\Delta}. \tag{15}$$

Applying the convexity property to the point x_0 between x and $x_0 + \Delta$ shows that

$$f(x_0) \geq \frac{f(x)((x_0 + \Delta) - x_0) + (x_0 - x)q}{x_0 + \Delta - x} \tag{16}$$

and hence

$$f(x_0)(x_0 + \Delta - x) \geq f(x)\Delta + (x_0 - x)q \tag{17}$$

so

$$f(x) - f(x_0) \leq \frac{(x - x_0)(q - f(x_0))}{\Delta}. \tag{18}$$

If $K = \max\{(f(x_0) - p)/\Delta, (q - f(x_0))/\Delta\}$, then combining Eqs. (15) & (18) shows that

$$|f(x) - f(x_0)| \leq K|x - x_0|. \tag{19}$$

By symmetry, the same arguments can be applied to show that this inequality also holds for $x \in [x_0, x_0 + \Delta]$. From here, it can be seen that for any $\epsilon > 0$, if $\delta = \epsilon/K$, then $|x - x_0| < \delta$ implies that

$$|f(x) - f(x_0)| < K\delta = K\frac{\epsilon}{K} = \epsilon. \tag{20}$$

Hence f must be continuous at any interior point. To prove that f is continuous at a given x_0, the above proof requires that function values are available on both sides of x_0, which will be not be true at the end points. An example of a convex function that is not continuous at the end points is

$$f(x) = \begin{cases} 1 & \text{if } x \neq a \text{ and } x \neq b, \\ 0 & \text{if } x = a \text{ or } x = b. \end{cases} \tag{21}$$