Math Analysis Final Exam Review: Key Concepts and Theorems (Ross-Aligned)

Focus on Definitions and Theorems

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Introduction

This review outlines key concepts from Ross's *Elementary Analysis*, emphasizing precise definitions and theorems relevant to typical final exam questions. Mastery of ε-N and ε-δ formalism is essential. Citations refer to Ross.

1 Real Numbers and Set Properties (Ch 1 & 2)

Rational and Irrational Numbers (Sec 5)

Definition The set of rational numbers is $\mathbb{Q} = \{p/q \mid p \in \mathbb{Z}, q \in \mathbb{N}\}$. The set of irrational numbers is $\mathbb{R} \setminus \mathbb{Q}$.

Thm 5.2, Exer 5.4 Both $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ are dense in $\mathbb{R}$.

Remark Proofs of irrationality (Q13) typically use contradiction. This includes basic cases like $\sqrt{2}$ and derived cases like $r + 1$ or $r^{1/3}$ when r is irrational. Sometimes the rational root theorem is useful.

Supremum and Infimum (Sec 4)

Definition Let $S \subseteq \mathbb{R}$, $S \neq \emptyset$.

- u is an **upper bound** for S if $s \leq u$ for all $s \in S$. $\sup S$ (**supremum**) is the least upper bound.
- l is a **lower bound** for S if $l \leq s$ for all $s \in S$. $\inf S$ (**infimum**) is the greatest lower bound.

Completeness Axiom 4.4 Every non-empty subset of $\mathbb{R}$ that is bounded above has a least upper bound (supremum) that is a real number.

Thm 4.5 Every non-empty subset of $\mathbb{R}$ that is bounded below has a greatest lower bound (infimum) that is a real number.

Remark $\max S = \sup S$ if $\sup S \in S$. $\min S = \inf S$ if $\inf S \in S$. Determining sup/inf/max/min for specific sets is a common task (Q20). Continuity interacts with suprema (Q41): If f is continuous and strictly increasing, $\sup f(S) = f(\sup S)$. This property requires continuity.

2 Sequences of Real Numbers (Ch 2 & 3)

Convergence (Sec 7)

Def 7.1 A sequence (s_n) converges to $s \in \mathbb{R}$ if:

$$(\forall \varepsilon > 0)(\exists N \in \mathbb{N})(\forall n \in \mathbb{N})(n > N \implies |s_n - s| < \varepsilon).$$

Notation: $\lim_{n\to\infty} s_n = s$ or $s_n \to s$.

Remark Direct proofs using the ε-N definition are fundamental (Q2a, Q26). Key techniques involve algebraic manipulation of $|s_n - s|$, using the triangle inequality, and utilizing the boundedness of convergent sequences (Thm 9.1, used in Q2a proof).

Limit Superior and Limit Inferior (Sec 10)

Definition Let (s_n) be a sequence. Define $u_N = \sup\{s_k \mid k > N\}$ and $v_N = \inf\{s_k \mid k > N\}$. If (s_n) is bounded:

$$\limsup_{n\to\infty} s_n = \lim_{N\to\infty} u_N = \inf_N(\sup\{s_k \mid k > N\})$$
$$\liminf_{n\to\infty} s_n = \lim_{N\to\infty} v_N = \sup_N(\inf\{s_k \mid k > N\})$$

Handle unbounded cases appropriately ($\pm\infty$).

Thm 10.7 $\limsup s_n$ is the largest subsequential limit. $\liminf s_n$ is the smallest subsequential limit.

Based on Thm 10.7 A sequence (s_n) converges $\iff$ $\limsup s_n = \liminf s_n$. The limit is this common value.

Remark Calculating $\limsup, \liminf$ often involves finding limits of key subsequences (Q15, Q33). This is used to determine convergence (Q15, Q33). Continuity interacts with $\limsup$ (Q41b): For continuous increasing f, $\limsup f(a_n) = f(\limsup a_n)$.

Boundedness Criterion, cf. Q38a A sequence (s_n) is bounded $\iff$ $\limsup_{n\to\infty} |s_n| < \infty$.

Remark Note that $\limsup s_n$ being finite does not imply (s_n) is bounded (Q38b). Boundedness requires considering $|s_n|$.

Key Theorems and Concepts for Sequences (Sec 9, 10, 11)

Thm 9.1 Convergent sequences are bounded.

Thm 10.2: Monotone Convergence Theorem If (s_n) is monotone and bounded, then (s_n) converges.

Thm 11.5: Bolzano-Weierstrass Theorem Every bounded sequence in $\mathbb{R}$ has a convergent subsequence.

Subsequences, Thm 11.2, 11.3 If $s_n \to s$, then every subsequence $s_{n_k} \to s$. If subsequences covering all terms converge to the same limit L, then $s_n \to L$ (Q32).

Recursive Sequences ($s_{n+1} = f(s_n)$) Analysis typically involves finding fixed points ($s = f(s)$), proving monotonicity and boundedness (often via induction), and applying the Monotone Convergence Theorem (Q19, Q27).

3 Series of Real Numbers (Sec 14, 15)

Convergence (Sec 14)

Def 14.1 A series $\sum_{n=m}^{\infty} a_n$ converges to S if its sequence of partial sums (S_N), where $S_N = \sum_{k=m}^{N} a_k$, converges to S.

Thm 14.4: Cauchy Criterion $\sum a_n$ converges $\iff$ $(\forall \varepsilon > 0, \exists N \text{ s.t. } m > n \geq N \implies |\sum_{k=n+1}^{m} a_k| < \varepsilon)$.

Thm 14.5: Term Test If $\sum a_n$ converges, then $\lim a_n = 0$. (Contrapositive is the Divergence Test).

Remark Grouping terms (Q4): If $\sum a_n$ converges, then $\sum(a_{2n} + a_{2n+1})$ converges (to the same sum). The converse is false (e.g., $a_n = (-1)^n$).

Convergence Tests (Sec 14, 15)

Key tests include:

- Comparison Test (Thm 14.6) and Limit Comparison Test (Exer 14.10).
- Ratio Test (Thm 15.1) and Root Test (Thm 15.2).
- Integral Test (Thm 15.3): Connects convergence of $\sum f(n)$ and $\int_1^{\infty} f(x)dx$ for positive, decreasing, continuous f (Q7 asks for proof, Q34b applies it).
- Alternating Series Test (AST, Thm 15.4): For series with alternating signs and terms decreasing to 0 (Q37a).

Absolute vs. Conditional Convergence $\sum a_n$ converges **absolutely** if $\sum |a_n|$ converges. Absolute convergence implies convergence (Thm 14.7). $\sum a_n$ converges **conditionally** if $\sum a_n$ converges but $\sum |a_n|$ diverges (Q37a).

4 Continuity (Ch 4)

Definitions (Sec 17)

Let $f : D \to \mathbb{R}$, $D \subseteq \mathbb{R}$.

Def 17.1: ε-δ Definition f is continuous at $x_0 \in D$ if:

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x \in D)(|x - x_0| < \delta \implies |f(x) - f(x_0)| < \varepsilon).$$

Thm 17.2: Sequential Definition f is continuous at $x_0 \in D \iff$ for every sequence $(x_n) \subseteq D$:

$$(x_n \to x_0 \implies f(x_n) \to f(x_0)).$$

Remark Be fluent in both definitions. ε-δ is often needed for direct proofs (Q2b, Q16). Sequential is useful for proving discontinuity or relating limits.

Fundamental Theorems for Continuous Functions (Sec 18)

Assume $f : [a, b] \to \mathbb{R}$ is continuous.

Thm 18.1: Extreme Value Theorem (EVT) f is bounded on $[a, b]$ and attains its maximum and minimum values.

Thm 18.2: Intermediate Value Theorem (IVT) If y is strictly between $f(a)$ and $f(b)$, then $\exists c \in (a, b)$ such that $f(c) = y$.

Remark IVT is crucial for existence proofs, like finding roots ($f(c) = 0$) when $f(a)$ and $f(b)$ have opposite signs (Q5). It's also used to show properties based on the sign of functions (Q23a: if $\int(f - g) = 0$ and $f - g$ is continuous, it must be zero somewhere).

Uniform Continuity (Sec 19)

Def 19.1 $f : D \to \mathbb{R}$ is uniformly continuous on D if:

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x, y \in D)(|x - y| < \delta \implies |f(x) - f(y)| < \varepsilon).$$

Crucially, δ depends only on ε, not on x, y.

Thm 19.2 If f is continuous on a closed bounded interval $[a, b]$ (a compact set), then f is uniformly continuous on $[a, b]$.

Remark Uniform continuity can fail if the domain is not compact or if the function's slope is unbounded (Q29c). To show non-uniform continuity, often negate the definition: $\exists \varepsilon > 0$ s.t. $\forall \delta > 0, \exists x, y \in D$ with $|x - y| < \delta$ but $|f(x) - f(y)| \geq \varepsilon$.

5 Differentiation (Ch 5)

Definition (Sec 28)

Def 28.1 Let f be defined on an open interval containing x_0. f is differentiable at x_0 if the limit exists and is finite:

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

Remark Using the definition is necessary when standard rules don't apply, e.g., with absolute values (Q40b) or piecewise functions. Conditions like $|f(x) - f(y)| \leq (x - y)^2$ imply $f'(x) = 0$ everywhere via the definition (Q10). The limit defining the second symmetric derivative relates to $f''(x)$ (Q31b), often via L'Hôpital's rule, but existence of the limit doesn't guarantee existence of $f''(x)$ (Q31c, e.g., $x|x|$ at $x = 0$).

Mean Value Theorems (Sec 29)

Assume f continuous on $[a, b]$, differentiable on (a, b).

Thm 29.2: Rolle's Theorem If $f(a) = f(b)$, then $\exists c \in (a, b)$ s.t. $f'(c) = 0$.

Thm 29.3: Mean Value Theorem (MVT) $\exists c \in (a, b)$ s.t. $f(b) - f(a) = f'(c)(b - a)$.

Remark Rolle's Theorem can be applied iteratively to higher derivatives (Q3). MVT is key for relating function values to derivative bounds (Q11) and proving functions are constant if $f' = 0$ (Q10, Cor 29.4).

L'Hôpital's Rule (Sec 30)

Thm 30.1, 30.2 For indeterminate forms $0/0$ or ∞/∞, if $\lim f'(x)/g'(x)$ exists, then $\lim f(x)/g(x)$ exists and equals it (subject to conditions like $g'(x) \neq 0$).

Remark Verify conditions before applying. Useful for various indeterminate limits (Q14, Q31a). May require algebraic manipulation first (e.g., $\infty - \infty \to 0/0$).

Taylor's Theorem (Sec 31)

Thm 31.3 If f has $N+1$ derivatives, $f(x) = P_N(x) + R_N(x)$, where $P_N(x) = \sum_{n=0}^{N} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$ is the Taylor polynomial and $R_N(x) = \frac{f^{(N+1)}(c)}{(N+1)!}(x-x_0)^{N+1}$ is the Lagrange remainder (for some c between x, x_0).

Remark Used for polynomial approximation and constructing Taylor series $\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$. The series converges to $f(x)$ if $\lim_{N\to\infty} R_N(x) = 0$. (Q24, Q40c).

6 Integration (Ch 6)

Darboux Integrability (Sec 32)

Let $f : [a, b] \to \mathbb{R}$ bounded, P a partition. Define Upper/Lower sums $U(f, P), L(f, P)$ using sup/inf on subintervals. Define Upper/Lower integrals $U(f) = \inf U(f, P), L(f) = \sup L(f, P)$.

Def 32.1 f is integrable on $[a, b]$ if $L(f) = U(f)$. The value is $\int_a^b f = L(f) = U(f)$.

Thm 32.5: Integrability Criterion f is integrable $\iff \forall \varepsilon > 0, \exists P$ s.t. $U(f, P) - L(f, P) < \varepsilon$.

Remark Q6 involves $f(x) = x$ on $\mathbb{Q}$, 0 else. For any partition P, $L(f, P) = 0$, but $U(f, P) \geq \int_0^b x dx/2 > 0$ (approx). Since $L(f) = 0 \neq U(f)$, it's not integrable. Continuous (Thm 32.7) and monotone (Thm 32.8) functions are integrable.

Fundamental Theorems of Calculus (FTC) (Sec 33)

Thm 33.1: FTC Part 1 If f integrable on $[a, b]$, then $F(x) = \int_a^x f(t)dt$ is continuous on $[a, b]$. If f is continuous at x_0, then F is differentiable at x_0 and $F'(x_0) = f(x_0)$.

Thm 33.3: FTC Part 2 If f is integrable on $[a, b]$ and F is an antiderivative ($F' = f$), then $\int_a^b f(x)dx = F(b) - F(a)$.

Remark FTC1 describes properties of $F(x) = \int_a^x f$ (Q22, Q29). FTC2 is the main tool for evaluating integrals. Note the hypotheses carefully (integrability of f, existence of antiderivative).

Properties and Integrability Conditions (Sec 32, 33)

- Integrability holds on subintervals (Q12, Exer 32.7). Proof uses refinement of partitions.
- Linearity, Additivity, Monotonicity, Triangle Inequality ($|\int f| \leq \int |f|$) hold (Thm 33.2).
- Positivity (Q35a): If f is *continuous*, $f \geq 0$, and $\int_a^b f = 0$, then $f \equiv 0$. Proof uses continuity to argue $f > 0$ on an interval if $f > 0$ at a point. Fails if f is merely integrable (Q23b, Q35c).

Improper Integrals (Sec 34)

Definition Integrals over infinite intervals or with unbounded integrands, defined as limits: Type I: $\int_a^\infty f(x)dx = \lim_{b\to\infty} \int_a^b f(x)dx$. Type II: $\int_a^b f(x)dx = \lim_{c\to a^+} \int_c^b f(x)dx$ (if unbounded near a). Convergence requires the limit to be finite.

Remark Know behavior of p-integrals $\int_0^1 x^{-p}dx$ (conv iff $p < 1$) and $\int_1^\infty x^{-p}dx$ (conv iff $p > 1$) (Q17). Use Comparison Tests (Thm 34.2). If f is integrable on $[a, b]$, then $\lim_{d\to b^-} \int_a^d f = \int_a^b f$ by continuity of $F(d) = \int_a^d f$ (FTC1) (Q18). The Integral Test connects $\int_1^\infty f$ and $\sum f(n)$ (Q7).

7 Sequences and Series of Functions (Ch 7)

Pointwise and Uniform Convergence (Sec 24)

Let $f_n : S \to \mathbb{R}$.

Def 24.1: Pointwise Convergence $f_n \to f$ pointwise on S if $\forall x \in S, \lim_{n\to\infty} f_n(x) = f(x)$.

Def 24.2: Uniform Convergence $f_n \to f$ uniformly on S if $\lim_{n\to\infty} \sup_{x\in S} |f_n(x) - f(x)| = 0$. ($\iff \forall\varepsilon > 0, \exists N$ (independent of x) s.t. $n > N \implies |f_n(x) - f(x)| < \varepsilon$ for all $x \in S$).

Remark Uniform convergence implies pointwise. Disprove uniform convergence by showing $\sup |f_n(x) - f(x)| \not\to 0$ (Q21, Q30, Q42). Check if the limit function inherits properties like continuity.

Consequences of Uniform Convergence (Sec 24, 25)

Uniform convergence allows interchange of limits:

Thm 24.3 Limit of continuous functions is continuous: If $f_n \to f$ uniformly and f_n are continuous, then f is continuous.

Thm 25.2 Limit of integrals is integral of limit: If $f_n \to f$ uniformly on $[a, b]$ and f_n are integrable, then f is integrable and $\lim \int_a^b f_n = \int_a^b f$.

Thm 25.4 Limit of derivatives: If $f_n \to f$ pointwise, f_n differentiable, and $f_n' \to g$ **uniformly**, then f is differentiable and $f' = g$.

Remark Failure of uniform convergence often breaks these interchanges (Q30d). Sometimes interchange holds even without uniform convergence (e.g., via Dominated Convergence Theorem, not in Ross, but see Q21d). Q42 explores integral convergence related to approximate identities. Q39a shows uniform convergence of f_n to 0, but Q39b asks if $g_n = \sup |f_n'|$ is bounded. The example $f_n(x) = x^{n^2}/n$ shows $f_n \to 0$ uniformly but $\sup |f_n'| = n$ is unbounded. Uniform convergence of f_n does not imply boundedness or convergence of $\sup |f_n'|$.

Dini's Theorem, Exer 24.10 On a compact set K, if continuous $f_n \to f$ pointwise, f is continuous, and (f_n) is monotone, then $f_n \to f$ uniformly (Q36). Fails if K not compact (Q36c).

Power Series (Sec 23, 26)

Series of the form $\sum a_n(x-c)^n$.

Thm 23.1 Every power series has a radius of convergence $R \in [0, \infty]$. Let $\beta = \limsup_{n\to\infty} |a_n|^{1/n}$. Then $R = 1/\beta$ (with $1/0 = \infty, 1/\infty = 0$). The series converges absolutely for $|x-c| < R$ and diverges for $|x-c| > R$. (If $L = \lim |a_{n+1}/a_n|$ exists, $R = 1/L$).

Remark The interval of convergence is $(c-R, c+R)$ plus potentially the endpoints $x = c \pm R$, which must be tested separately using standard series tests (Q1, Q25, Q37b). The $\limsup$ formula for β is the most general.

Thm 26.1, 26.4, 26.5 Convergence is uniform on any closed interval $[c-r, c+r]$ with $r < R$. Power series can be differentiated and integrated term-by-term within $(-R, R)$.

8 Metric Spaces and Basic Topology (Ch 13)

Metrics (Sec 13)

Definition A metric d on X satisfies non-negativity, positive definiteness ($d(x, y) = 0 \iff x = y$), symmetry, and the triangle inequality $d(x, z) \leq d(x, y) + d(y, z)$.

Remark Verify these axioms for given functions (Q8a, Q9a, Q28b, Q35b). Show failure by counterexample (Q28a, Q35c). Common metrics: Euclidean, taxicab (d_1), max (d_∞), discrete, integral (L^1). Note $d(f, g) = \int |f - g|$ is not a metric on integrable functions because $d(f, g) = 0$ does not imply $f = g$ everywhere (Q35c).

Topological Concepts (Sec 13)

Definition Key concepts:

- **Neighborhood**: $B(x, r) = \{y \in X \mid d(x, y) < r\}$.
- **Open Set**: Contains a neighborhood around each point.
- **Closed Set**: Complement is open; contains all its limit points ($C' \subseteq C$).
- **Limit Point**: Every neighborhood contains another point from the set.
- **Closure** $\bar{S} = S \cup S'$. **Interior** S°. **Boundary** $\partial S = \bar{S} \setminus S^\circ$.
- **Compact Set**: Every open cover has a finite subcover.

Thm 13.6: Heine-Borel In $\mathbb{R}^n$, compact $\iff$ closed and bounded.

Remark Properties depend on the metric. In discrete metric, all sets are open and closed; compact iff finite (Q8c). Calculate closure, interior, boundary (Q43a). Understand limit points in relation to boundaries (Q43b). Sketch neighborhoods (Q9b, Q28b).

Continuity in Metric Spaces (cf. Def 17.1)

Definition $f : (X, d_X) \to (Y, d_Y)$ is continuous at x_0 if $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $d_X(x, x_0) < \delta \implies d_Y(f(x), f(x_0)) < \varepsilon$.

Remark Continuity depends on the chosen metrics d_X, d_Y. Equivalence of metrics (or one being stronger, e.g., $d_B \leq 2d_E$ in Q28c) can imply continuity w.r.t. one metric from continuity w.r.t. another.