Lemma Π_1 and Π_2 are equivalent if and only if:

(i) For every feasible solution to Π_1, there exists a feasible solution to Π_2, with cost equal or lower, and

(ii) For every feasible solution to Π_2, there exists a feasible solution to Π_1, with cost equal or lower.

Proof.

"$\Rightarrow$" If Π_1, Π_2 both infeasible, there is nothing to prove.
Thus we assume Π_1, Π_2 have the same optimal cost.
(i) holds because the optimal solution to Π_2 has cost $\leq$ every feasible solution to Π_1.
(ii) holds symmetrically.

"$\Leftarrow$" If Π_1, Π_2 both infeasible, we are done. So assume one is feasible, wlog Π_1.
(i) implies that also Π_2 is feasible.
Assume by contradiction that one of the two problems has strictly lower optimal cost, wlog Π_1.

then, $\exists x^*$ feasible to Π_1 with cost strictly lower then any feasible solution to Π_2.
This contradicts (i).
Hence the optimal cost of the two problems is the same $\square$

Observation The two problems in Example 1.4 are equivalent.

$$\begin{aligned} \Pi_1 : \quad & \text{minimize} && 2x_1 + 4x_2 \\ & \text{subject to} && x_1 + x_2 \geq 3 \\ &&& 3x_1 + 2x_2 = 14 \\ &&& x_1 \geq 0, \end{aligned} \qquad \begin{aligned} \Pi_2 : \quad & \text{minimize} && 2x_1 + 4x_2^+ - 4x_2^- \\ & \text{subject to} && x_1 + x_2^+ - x_2^- - x_3 = 3 \\ &&& 3x_1 + 2x_2^+ - 2x_2^- = 14 \\ &&& x_1, x_2^+, x_2^-, x_3 \geq 0. \end{aligned}$$

Proof. We use the Lemma.

(i) Let (x_1, x_2) be any feasible solution to Π_1.

Define

$$x_2^+ := \begin{cases} x_2 & \text{if } x_2 \geq 0 \\ 0 & \text{if } x_2 < 0 \end{cases}, \quad x_2^- := \begin{cases} 0 & \text{if } x_2 \geq 0 \\ -x_2 & \text{if } x_2 < 0, \end{cases}$$

$x_3 := x_1 + x_2 - 3$.

Note that $x_2^+ - x_2^- = x_2$.

(x_1, x_2^+, x_2^-, x_3) is a feasible solution to Π_2 because

$$x_1 + (x_2^+ - x_2^-) - x_3 = \cancel{x_1} + \cancel{x_2} - (\cancel{x_1} + \cancel{x_2} - 3) = 3$$

$$3x_1 + (2x_2^+ - 2x_2^-) = 3x_1 + 2x_2 = 14$$

$$x_1, x_2^+, x_2^-, x_3 \geq 0$$

the cost of (x_1, x_2^+, x_2^-, x_3) is

$$2x_1 + (4x_2^+ - 4x_2^-) = 2x_1 + 4x_2$$

hence equal to the cost of (x_1, x_2).

(ii) Let (x_1, x_2^+, x_2^-, x_3) be any feasible solution to Π_2.

Define $x_2 := x_2^+ - x_2^-$.

(x_1, x_2) is a feasible solution to Π_1 because

$$x_1 + x_2 = x_1 + x_2^+ - x_2^- \geq 3$$

$$3x_1 + 2x_2 = 3x_1 + 2x_2^+ - 2x_2^- = 14$$

$$x_1 \geq 0$$

The cost of (x_1, x_2) is

$$2x_1 + 4x_2 = 2x_1 + 4x_2^+ - 4x_2^-,$$

hence equal to the cost of (x_1, x_2^+, x_2^-, x_3) $\square$