ISyE/Math/CS/Stat 525
Linear Optimization

3. The simplex method
part 1

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Based on the book *Introduction to Linear Optimization* by D. Bertsimas and J.N. Tsitsiklis

Outline

- From Theorem 2.7: If a LP problem in standard form has an optimal solution, then there exists a basic feasible solution that is optimal.

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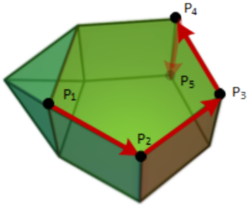
- The simplex method is based on this fact.
- It searches for an optimal solution by moving from one basic feasible solution to another, along the edges of the feasible set, always in a cost reducing direction.
- Eventually, a basic feasible solution is reached at which none of the available edges leads to a cost reduction.
- Such a basic feasible solution is optimal and the algorithm terminates.



Outline

Sec. 3.1 We present necessary and sufficient condition for a feasible solution to be optimal.

Sec. 3.2 We develop the simplex method.

Sec. 1.6 We review how to count the number of operations performed by algorithms.

Sec. 3.3 We discuss a few different implementations, including the simplex tableau and the revised simplex method.

Outline

- We consider the standard form problem

$$\begin{aligned} \text{minimize} \quad & c'x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0. \end{aligned}$$

- We let P be the corresponding feasible set.
- We assume that the dimensions of the matrix A are $m \times n$ and that its rows are linearly independent.
- We continue using our previous notation:
  - A_i is the ith column of the matrix A,
  - a_i' is the ith row of the matrix A.

3.1 Optimality conditions

Optimality conditions

Many optimization algorithms are structured as follows:

- Given a feasible solution, we search its neighborhood to find a nearby feasible solution with lower cost.
- If no nearby feasible solution leads to a cost improvement, the algorithm terminates and we have a locally optimal solution.

Optimality conditions

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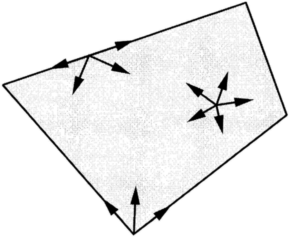
- Given a feasible solution, we search its neighborhood to find a nearby feasible solution with lower cost.
- If no nearby feasible solution leads to a cost improvement, the algorithm terminates and we have a locally optimal solution.

- For general optimization problems, a locally optimal solution need not be (globally) optimal.
- Fortunately, in LP, local optimality implies global optimality. You will show this in an exercise.
- In this section, we concentrate on the problem of searching for a direction of cost decrease in a neighborhood of a given basic feasible solution, and on the associated optimality conditions.

Optimality conditions

- Suppose that we are at a point $x \in P$ and that we contemplate moving away from x, in the direction of a vector $d \in \mathbb{R}^n$.
- Clearly, we should only consider those choices of d that do not immediately take us outside the feasible set.
- This leads to the following definition.

Definition 3.1
Let x be a point in a polyhedron P.
A vector $d \in \mathbb{R}^n$ is said to be a feasible direction at x, if there exists a positive scalar θ for which $x + \theta d \in P$.



Optimality conditions

- Let x be a basic feasible solution to the standard form problem.
- Let $B(1), \ldots, B(m)$ be the indices of the basic variables, and let

$$B = [A_{B(1)} \cdots A_{B(m)}]$$

  be the corresponding basis matrix.
- In particular, we have $x_i =$ for every nonbasic variable, while the vector $x_B = (x_{B(1)}, \ldots, x_{B(m)})$ of basic variables is given by

$$x_B =$$

Optimality conditions

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- Let $B(1), \ldots, B(m)$ be the indices of the basic variables, and let
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be the corresponding basis matrix.
- In particular, we have $x_i = 0$ for every nonbasic variable, while the vector $x_B = (x_{B(1)}, \ldots, x_{B(m)})$ of basic variables is given by
$$x_B = B^{-1} b.$$

Optimality conditions

- We consider the possibility of moving away from x, to a new vector
$$x + \theta d.$$
- We do so by selecting a nonbasic variable x_j (which is initially at zero level), and increasing it to a positive value θ, while keeping the remaining nonbasic variables at zero.
- Algebraically, $d_j = 1$, and $d_i = 0$ for every nonbasic index i other than j.
- At the same time, the vector x_B of basic variables changes to
$$x_B + \theta d_B,$$
where $d_B = (d_{B(1)}, d_{B(2)}, \dots, d_{B(m)})$ is the vector with those components of d that correspond to the basic variables.

Optimality conditions

- Given that we are only interested in feasible solutions, we require

$$A(x+\theta d) = b \quad \Leftrightarrow \quad \cancel{Ax} + \theta A d = \cancel{b} \quad \Leftrightarrow \quad \theta A d = 0,$$

  where we have used $Ax = b$, since x is feasible.
- Thus, for the equality constraints to be satisfied for $\theta > 0$, we need

$$Ad = 0.$$

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  where we have used $Ax = b$, since x is feasible.
- Thus, for the equality constraints to be satisfied for $\theta > 0$, we need

$$Ad = 0.$$

- Recall now that $d_j = 1$, and that $d_i = 0$ for all other nonbasic indices i.
- Then,

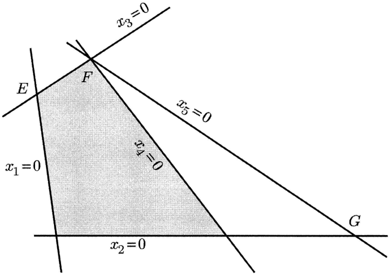
$$0 = Ad = \sum_{i=1}^{n} A_i d_i = \sum_{i=1}^{m} A_{B(i)} d_{B(i)} + A_j = B d_B + A_j.$$

- Since the basis matrix B is invertible, we obtain

$$d_B = -B^{-1} A_j.$$

Optimality conditions

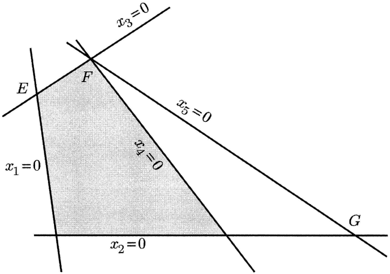
- The direction vector d that we have just constructed is called the jth basic direction.
- We have so far guaranteed that the equality constraints are respected by the vectors $x + \theta d$, for $\theta > 0$.



$n = 5,\ m = 3$

Optimality conditions

- How about the nonnegativity constraints?
- We recall that the variable x_j is increased, and all other nonbasic variables stay at zero level.
- Thus, we need only worry about the basic variables. We distinguish two cases:



$n = 5,\ m = 3$

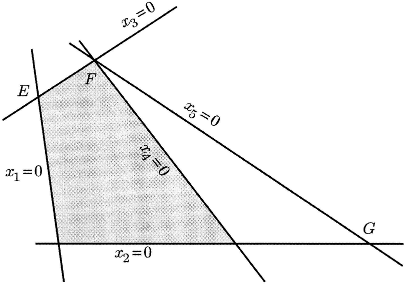
Optimality conditions

Case (a): x is a nondegenerate basic feasible solution.

- Then, $x_B > 0$, from which it follows that, when θ is sufficiently small,

$$x_B + \theta d_B \geq 0,$$

  and feasibility is maintained.
- In particular, d is a feasible direction.



$n = 5,\ m = 3$

Optimality conditions

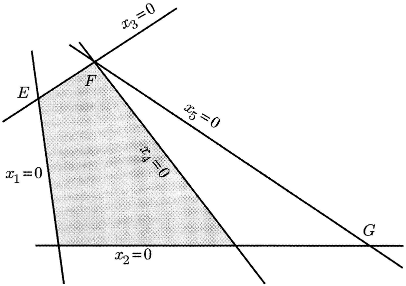
Case (b): x is a degenerate basic feasible solution.

- Then, d is not always a feasible direction.
- Indeed, it is possible that, for a basic variable,

$$x_{B(i)} = 0$$
$$d_{B(i)} < 0.$$

- In that case, for every positive θ we have

$$x_{B(i)} + \theta d_{B(i)} < 0.$$



$n = 5,\ m = 3$

Optimality conditions

We now study the effects on the cost function if we move along a basic direction.

- If d is the jth basic direction, the cost change is given by

$$c'(x + \theta d) - c'x = \theta c'd.$$

- The rate of cost change along the direction d is given by

$$c'd = \sum_{i=1}^{n} c_i d_i = \sum_{i=1}^{m} c_{B(i)} d_{B(i)} + c_j = c'_B d_B + c_j = c_j - c'_B B^{-1} A_j.$$

  where $c_B = (c_{B(1)}, \ldots, c_{B(m)})$, and where we have used $d_B = -B^{-1}A_j$.
- For an intuitive interpretation:
  - c_j is the cost per unit increase in the variable x_j,
  - $-c'_B B^{-1} A_j$ is the cost of the compensating change in the basic variables necessitated by the constraint $Ax = b$.

Optimality conditions

- The latter quantity is important enough to warrant a definition.

Definition 3.2
Let x be a basic solution, let B be an associated basis matrix, and let c_B be the vector of costs of the basic variables. For each j, we define the <u>reduced cost</u> $\bar{c}_j$ of the variable x_j according to the formula

$$\bar{c}_j = c_j - c_B' B^{-1} A_j.$$

- Note that the definition holds also if j is a basic index!

Example 3.1

Consider the LP problem

$$\begin{aligned} \text{minimize} \quad & c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4 \\ \text{subject to} \quad & x_1 + x_2 + x_3 + x_4 = 2 \\ & 2x_1 + 3x_3 + 4x_4 = 2 \\ & x_1, x_2, x_3, x_4 \geq 0. \end{aligned}$$

- We have $A_1 = (1, 2)$ and $A_2 = (1, 0)$.
- Since they are linearly independent, we can choose x_1 and x_2 as our basic variables.
- The corresponding basis matrix is

$$B = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}.$$

Example 3.1

Consider the LP problem

$$\begin{aligned} \text{minimize} \quad & c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4 \\ \text{subject to} \quad & x_1 + x_2 + x_3 + x_4 = 2 \\ & 2x_1 + 3x_3 + 4x_4 = 2 \\ & x_1, x_2, x_3, x_4 \geq 0. \end{aligned}$$

- We set $x_3 = x_4 = 0$, and solve for x_1, x_2, to obtain $x_1 = 1$ and $x_2 = 1$.
- We have thus obtained the nondegenerate basic feasible solution

$$(1, 1, 0, 0).$$

Example 3.1

Consider the LP problem

$$\begin{aligned} \text{minimize} \quad & c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4 \\ \text{subject to} \quad & x_1 + x_2 + x_3 + x_4 = 2 \\ & 2x_1 + 3x_3 + 4x_4 = 2 \\ & x_1, x_2, x_3, x_4 \geq 0. \end{aligned}$$

- The 3rd basic direction d is constructed as follows:
  - We have $d_3 = 1$ and $d_4 = 0$.
  - The vector $d_B = (d_1, d_2)$ is obtained using $d_B = -B^{-1}A_3$:

$$\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = -B^{-1}A_3 = -\begin{bmatrix} 0 & 1/2 \\ 1 & -1/2 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -3/2 \\ 1/2 \end{bmatrix}.$$

  - The 3rd basic direction is then the vector $d = (-\frac{3}{2}, \frac{1}{2}, 1, 0)$.

Example 3.1

Consider the LP problem

$$\begin{aligned} \text{minimize} \quad & c_1 x_1 + c_2 x_2 + c_3 x_3 + c_4 x_4 \\ \text{subject to} \quad & x_1 + x_2 + x_3 + x_4 = 2 \\ & 2x_1 + 3x_3 + 4x_4 = 2 \\ & x_1, x_2, x_3, x_4 \geq 0. \end{aligned}$$

- The rate of cost change along this basic direction is

$$c' d = -\frac{3}{2} c_1 + \frac{1}{2} c_2 + c_3.$$

- This is the same as the reduced cost of the variable x_3.

Optimality conditions

We now calculate the reduced cost

$$\bar{c}_j = c_j - c_B' B^{-1} A_j.$$

for the case of a basic variable ($j = B(i)$ for some $i \in \{1, \ldots, m\}$).

- Since $B = [A_{B(1)} \cdots A_{B(m)}]$, we have

$$B^{-1}[A_{B(1)} \cdots A_{B(m)}] = I,$$

  where I is the $m \times m$ identity matrix.
- In particular, $B^{-1}A_{B(i)}$ is the ith column of the identity matrix, which is the ith unit vector e_i.
- Therefore, for every basic variable $x_{B(i)}$, we have

$$\bar{c}_{B(i)} = c_{B(i)} - c_B' B^{-1} A_{B(i)} = c_{B(i)} - c_B' e_i = c_{B(i)} - c_{B(i)} = 0.$$

- Thus the reduced cost of every basic variable is zero.

Optimality conditions

- Our next result provides us with optimality conditions.
- Given our interpretation of the reduced costs as rates of cost change along certain directions, this result is intuitive.

Theorem 3.1
Consider a basic feasible solution x associated with a basis matrix B, and let $\bar{c}$ be the corresponding vector of reduced costs.

(a) If $\bar{c} \geq 0$, then x is optimal.

(b) If x is optimal and nondegenerate, then $\bar{c} \geq 0$.

Note that the contrapositive of (b) is:

(b') If $\bar{c}_j < 0$ for some j, then x is degenerate or not optimal.

Let's prove Theorem 3.1!

Optimality conditions

Theorem 3.1

Consider a basic feasible solution x associated with a basis matrix B, and let $\bar{c}$ be the corresponding vector of reduced costs.

(a) If $\bar{c} \geq 0$, then x is optimal.

(b) If x is optimal and nondegenerate, then $\bar{c} \geq 0$.

- Note that Theorem 3.1 allows the possibility that x is a (degenerate) optimal basic feasible solution, but that $\bar{c}_j < 0$ for some nonbasic index j.
- According to Theorem 3.1, in order to decide whether a nondegenerate basic feasible solution is optimal, we need only to check whether all reduced costs are nonnegative, which is the same as examining the $n - m$ basic directions.

Optimality conditions

Theorem 3.1
Consider a basic feasible solution x associated with a basis matrix B, and let $\bar{c}$ be the corresponding vector of reduced costs.

(a) If $\bar{c} \geq 0$, then x is optimal.

(b) If x is optimal and nondegenerate, then $\bar{c} \geq 0$.

- If x is a degenerate basic feasible solution, an equally simple computational test for determining whether x is optimal is not available.
- Fortunately, the simplex method manages to get around this difficulty.

Optimality conditions

- In order to use Theorem 3.1 and assert that a certain basic solution is optimal, we need to satisfy two conditions:
  (a) Feasibility,
  (b) Nonnegativity of the reduced costs.
- This leads us to the following definition.

Definition 3.3

A basis matrix B is said to be optimal if:

(a) $B^{-1}b \geq 0$, and

(b) $\bar{c}' = c' - c_B' B^{-1} A \geq 0'$.

Optimality conditions

- If an optimal basis is found, the corresponding basic solution is feasible, satisfies the optimality conditions, and is therefore optimal.
- On the other hand, it can happen that we have found a basis that is not optimal, and the corresponding basic solution is optimal.
- In this case at least one reduced cost is negative.
- Thus the basic solution is degenerate.

Definition 3.3

A basis matrix B is said to be optimal if:

(a) $B^{-1}b \geq 0$, and

(b) $\bar{c}' = c' - c'_B B^{-1} A \geq 0'$.

3.2 Development of the simplex method

Development of the simplex method

We now continue with the development of the simplex method.

- Our main task is to work out the details of how to move to a better basic feasible solution, whenever a profitable basic direction is discovered.
- Let us assume that every basic feasible solution is nondegenerate.
- This assumption will remain in effect until it is explicitly relaxed later in this section.

Development of the simplex method

- Suppose that we are at a basic feasible solution x and that we have computed the reduced costs $\bar{c}_j$ of the nonbasic variables.
- If all of them are nonnegative, Theorem 3.1 shows that we have an optimal solution, and we stop.
- Otherwise, the reduced cost $\bar{c}_j$ of a nonbasic variable x_j is negative, and the jth basic direction d is a feasible direction of cost decrease.
- While moving along this direction d, the nonbasic variable x_j becomes positive and all other nonbasic variables remain at zero.
- We describe this situation by saying that A_j (or x_j) <u>enters the basis</u>.

Development of the simplex method

- Once we start moving away from x along the direction d, we are tracing points of the form

$$x + \theta d, \qquad \text{where } \theta \geq 0.$$

- Since costs decrease along the direction d, it is desirable to move as far as possible.
- This takes us to the point $x + \theta^* d$, where

$$\theta^* = \max\{\theta \geq 0 \mid x + \theta d \in P\}.$$

- The resulting cost change is

$$\theta^* c' d = \theta^* \bar{c}_j.$$

Development of the simplex method

We now derive a formula for θ^*.

- Given that $Ad = 0$, we have

$$A(x + \theta d) = Ax = b \qquad \forall \theta \in \mathbb{R},$$

  and the equality constraints will never be violated.
- Thus, $x + \theta d$ can become infeasible only if one of its components becomes negative.

Development of the simplex method

We distinguish two cases:

(a) If $d \geq 0$, then $x + \theta d \geq 0$ for all $\theta \geq 0$, the vector $x + \theta d$ never becomes infeasible, and we let $\theta^* = \infty$.

(b) If $d_i < 0$ for some i, the constraint $x_i + \theta d_i \geq 0$ becomes

$$\theta \leq -\frac{x_i}{d_i}.$$

- This constraint on θ must be satisfied for every i with $d_i < 0$.
- Thus, the largest possible value of θ is

$$\theta^* = \min_{i \mid d_i < 0} \left(-\frac{x_i}{d_i} \right).$$

Development of the simplex method

$$\theta^* = \min_{i \mid d_i < 0} \left(-\frac{x_i}{d_i} \right).$$

- Recall that if x_i is a nonbasic variable, then either x_i is the entering variable and $d_i = 1$, or else $d_i = 0$.
- In either case, d_i is nonnegative.
- Thus, we only need to consider the basic variables and we have the equivalent formula

$$\theta^* = \min_{i=1,\ldots,m \mid d_{B(i)} < 0} \left(-\frac{x_{B(i)}}{d_{B(i)}} \right).$$

- Note that $\theta^* > 0$, because $x_{B(i)} > 0$ for all i, as a consequence of nondegeneracy.

Example 3.2

This is a continuation of Example 3.1.

$$
\begin{aligned}
\text{minimize} \quad & c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4 \\
\text{subject to} \quad & x_1 + x_2 + x_3 + x_4 = 2 \\
& 2x_1 + 3x_3 + 4x_4 = 2 \\
& x_1, x_2, x_3, x_4 \geq 0.
\end{aligned}
$$

- We again consider the basic feasible solution

$$x = (1, 1, 0, 0).$$

- The reduced cost $\bar{c}_3$ of the nonbasic variable x_3 was

$$\bar{c}_3 = -\frac{3}{2}c_1 + \frac{1}{2}c_2 + c_3.$$

- Suppose that $c = (2, 0, 0, 0)$, in which case, we have

$$\bar{c}_3 = -3.$$

Example 3.2

- Since $\bar{c}_3$ is negative, we form the 3rd basic direction, which is

$$d = \left(-\frac{3}{2}, \frac{1}{2}, 1, 0\right).$$

- We consider vectors of the form

$$x + \theta d, \qquad \text{with } \theta \geq 0.$$

- As θ increases, the only component of x that decreases is the first one (because $d_1 < 0$).
- The largest possible value of θ is given by

$$\theta^* = -\frac{x_1}{d_1} = \frac{2}{3}.$$

- This takes us to the point

$$y = x + \frac{2}{3}d = \left(0, \frac{4}{3}, \frac{2}{3}, 0\right).$$

Example 3.2

$$\begin{aligned} \text{minimize} \quad & c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4 \\ \text{subject to} \quad & x_1 + x_2 + x_3 + x_4 = 2 \\ & 2x_1 + 3x_3 + 4x_4 = 2 \\ & x_1, x_2, x_3, x_4 \geq 0. \end{aligned}$$

- Consider our new vector y

$$y = \left(0, \frac{4}{3}, \frac{2}{3}, 0\right).$$

- The columns corresponding to the nonzero variables at the new vector y are $A_2 = (1, 0)$ and $A_3 = (1, 3)$, and are linearly independent.
- Therefore, they form a basis and the vector y is the corresponding basic feasible solution.
- In particular, A_3 (or x_3) has entered the basis and A_1 (or x_1) has exited the basis.

Development of the simplex method

- Once θ^* is chosen, and assuming it is finite, we move to the new feasible solution

$$y = x + \theta^* d.$$

- Since $x_j = 0$ and $d_j = 1$, we have $y_j = \theta^* > 0$.
- Let ℓ be a minimizing index in the choice of θ^*, that is,

$$-\frac{x_{B(\ell)}}{d_{B(\ell)}} = \min_{i=1,\ldots,m \,|\, d_{B(i)}<0} \left(-\frac{x_{B(i)}}{d_{B(i)}} \right) = \theta^*.$$

- Hence

$$y_{B(\ell)} = x_{B(\ell)} + \theta^* d_{B(\ell)} = x_{B(\ell)} - \frac{x_{B(\ell)}}{\cancel{d_{B(\ell)}}} \cancel{d_{B(\ell)}} = 0.$$

Development of the simplex method

- The basic variable $x_{B(\ell)}$ has become zero, and the nonbasic variable x_j has become positive.
- This suggests that x_j should replace $x_{B(\ell)}$ in the basis.
- Accordingly, we take the old basis matrix B and replace $A_{B(\ell)}$ with A_j:

$$\bar{B} = \left[A_{B(1)} \cdots A_{B(\ell-1)} \; A_j \; A_{B(\ell+1)} \cdots A_{B(m)}\right].$$

- Equivalently, we are replacing the set $\{B(1), \ldots, B(m)\}$ of basic indices by a new set $\{\bar{B}(1), \ldots, \bar{B}(m)\}$ of indices given by

$$\bar{B}(i) = \begin{cases} B(i), & i \neq \ell, \\ j, & i = \ell. \end{cases}$$

Development of the simplex method

Theorem 3.2

(a) The columns $A_{\bar{B}(i)}$, $i = 1, \ldots, m$, are linearly independent and, therefore, $\bar{B}$ is a basis matrix.

(b) The vector $y = x + \theta^* d$ is a basic feasible solution associated with the basis matrix $\bar{B}$.

Let's prove it!

Development of the simplex method

- Since $\theta^* > 0$, the new basic feasible solution $x + \theta^* d$ is distinct from x.
- Since d is a direction of cost decrease, the cost of this new basic feasible solution is strictly smaller than the cost of x.
- We have therefore accomplished our objective of moving to a new basic feasible solution with lower cost.

Development of the simplex method

- We can now summarize a typical iteration of the simplex method, also known as a pivot.
- It is convenient to define a vector $u = (u_1, \ldots, u_m)$ by letting

$$u = -d_B = B^{-1}A_j,$$

  where A_j is the column that enters the basis.
- In particular,

$$u_i = -d_{B(i)}, \qquad \text{for } i = 1, \ldots, m.$$

Development of the simplex method

An iteration of the simplex method

1. We start with a basis consisting of the basic columns $A_{B(1)}, \ldots, A_{B(m)}$, and an associated basic feasible solution x.
2. Compute the reduced costs $\bar{c}_j = c_j - c'_B B^{-1} A_j$ for all nonbasic indices j.
   - If they are all nonnegative, the current basic feasible solution is optimal, and the algorithm terminates.
   - Else, choose some j for which $\bar{c}_j < 0$.
3. Compute $u = B^{-1} A_j$. If no component of u is positive, we have $\theta^* = \infty$, the optimal cost is $-\infty$, and the algorithm terminates.

Development of the simplex method

An iteration of the simplex method

4. If some component of u is positive, let
$$\theta^* = \min_{i=1,\ldots,m \,|\, u_i > 0} \frac{x_{B(i)}}{u_i}.$$
5. Let ℓ be such that
$$\theta^* = \frac{x_{B(\ell)}}{u_\ell}.$$
Form a new basis by replacing $A_{B(\ell)}$ with A_j. If y is the new basic feasible solution, the values of the new basic variables are $y_j = \theta^*$ and $y_{B(i)} = x_{B(i)} - \theta^* u_i$, $i \neq \ell$.

Development of the simplex method

- The simplex method is initialized with an arbitrary basic feasible solution, which, for feasible standard form problems, is guaranteed to exist (cf. Corollary 2.2).

Development of the simplex method

- The following theorem states that, in the nondegenerate case, the simplex method works correctly and terminates after a finite number of iterations.

Theorem 3.3

Assume that the feasible set is nonempty and that every basic feasible solution is nondegenerate. Then, the simplex method terminates after a finite number of iterations. At termination, there are the following two possibilities:

(a) We have an optimal basis B and an associated basic feasible solution which is optimal.

(b) We have found a vector d satisfying $Ad = 0$, $d \geq 0$, and $c'd < 0$, and the optimal cost is $-\infty$.

Let's prove it in the next slides!

Development of the simplex method

Proof (1/3):

- Case 1. The algorithm terminates in Step 2: $\bar{c}_j \geq 0$ for all nonbasic indices j.
- Then the optimality conditions in Theorem 3.1 have been met.
- B is an optimal basis, and the current basic feasible solution is optimal.

Development of the simplex method

Proof (2/3):

- Case 2. The algorithm terminates in Step 3: no component of u is positive.
- Then we are at a basic feasible solution x and we have discovered a nonbasic variable x_j s.t. $\bar{c}_j < 0$ and the jth basic direction d satisfies $Ad = 0$, $d \geq 0$.
- In particular, $x + \theta d \in P$ for all $\theta > 0$.
- Since $c'd = \bar{c}_j < 0$, by taking θ arbitrarily large, the cost can be made arbitrarily negative, and the optimal cost is $-\infty$.

Development of the simplex method

Proof (3/3):

- At each iteration, the algorithm moves by a positive amount θ^* along a direction d that satisfies $c'd < 0$.
- Therefore, the cost of every successive basic feasible solution visited by the algorithm is strictly less than the cost of the previous one.
- In particular, no basic feasible solution can be visited twice.
- Since there is a finite number of basic feasible solutions (Corollary 2.1), the algorithm must eventually terminate. $\square$

Development of the simplex method

- Theorem 3.3 provides an independent proof of some of the results of Chapter 2 for nondegenerate standard form problems.
- In particular, it shows that for standard form problems that are nondegenerate and feasible:
  - either the optimal cost is $-\infty$, or
  - there exists a basic feasible solution which is optimal.

  (cf. Theorem 2.8 in Section 2.6.)

The simplex method for degenerate problems

The simplex method for degenerate problems

- We have been working so far under the assumption that all basic feasible solutions are nondegenerate.
- Suppose now that the same algorithm is used in the presence of degeneracy.
- Then, the following new possibilities may be encountered in the course of the algorithm.

The simplex method for degenerate problems

(a) If the current basic feasible solution x is degenerate, θ^* can be equal to zero, in which case, the new basic feasible solution y is the same as x.

- This happens if some basic variable $x_{B(\ell)}$ is equal to zero and the corresponding component $d_{B(\ell)}$ of the direction vector d is negative.

The simplex method for degenerate problems

(a) If the current basic feasible solution x is degenerate, θ^* can be equal to zero, in which case, the new basic feasible solution y is the same as x.

- This happens if some basic variable $x_{B(\ell)}$ is equal to zero and the corresponding component $d_{B(\ell)}$ of the direction vector d is negative.
- Nevertheless, we can still define a new basis $\bar{B}$, by replacing $A_{B(\ell)}$ with A_j, and Theorem 3.2 is still valid.

Theorem 3.2

(a) The columns $A_{B(i)}$, $i \neq \ell$, and A_j are linearly independent and, therefore, $\bar{B}$ is a basis matrix.

(b) The vector $y = x + \cancel{\theta^*} d$ is a basic feasible solution associated with the basis matrix $\bar{B}$.

The simplex method for degenerate problems

(b) Even if θ^* is positive, it may happen that more than one of the original basic variables becomes zero at the new point $x + \theta^* d$.

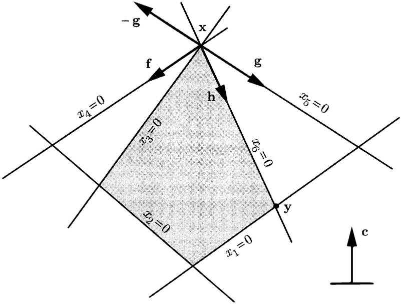
- Since only one of them exits the basis, the others remain in the basis at zero level, and the new basic feasible solution is degenerate.

The simplex method for degenerate problems

- Basis changes while staying at the same basic feasible solution are not in vain.
- A sequence of such basis changes may lead to the eventual discovery of a cost reducing feasible direction.

The simplex method for degenerate problems

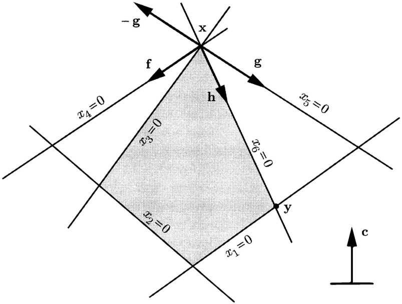
Example:



- The basic feasible solution x is degenerate.
- If x_4 and x_5 are the nonbasic variables, then
  - the 4th basic direction is g,
  - the 5th basic direction is f.
- For either of these two basic directions, we have $\theta^* = 0$.

The simplex method for degenerate problems

Example:



- However, if we perform a change of basis, with x_4 entering the basis and x_6 exiting, the new nonbasic variables are x_5 and x_6.
- Then
  - the 5th basic direction is h,
  - the 6th basic direction is $-g$.
- In particular, we can now follow direction h to reach a new basic feasible solution y with lower cost.

The simplex method for degenerate problems

- A sequence of basis changes might lead back to the initial basis, in which case the algorithm may loop indefinitely.
- This undesirable phenomenon is called cycling.
- It is sometimes maintained that cycling is an exceptionally rare phenomenon.
- However, for many highly structured LP problems, most basic feasible solutions are degenerate, and cycling is a real possibility.
- We will see that cycling can be avoided by judiciously choosing the variables that will enter or exit the basis.
- We now discuss the freedom available in this respect.

Pivot Selection

Pivot Selection

- The simplex algorithm, as we described it, has certain degrees of freedom:
  - In Step 2, we are free to choose any j whose reduced cost $\bar{c}_j$ is negative.
  - In Step 5, there may be several indices ℓ that attain the minimum in the definition of θ^*, and we are free to choose any one of them.
- Rules for making such choices are called pivoting rules.

Pivot Selection

Regarding the choice of the entering column, the following rules are some natural candidates:

(a) Choose a column A_j, with $\bar{c}_j < 0$, whose reduced cost is the most negative.

- Since the reduced cost is the rate of change of the cost function, this rule chooses a direction along which the cost decreases at the fastest rate.

Pivot Selection

Regarding the choice of the entering column, the following rules are some natural candidates:

(a) Choose a column A_j, with $\bar{c}_j < 0$, whose reduced cost is the most negative.

- Since the reduced cost is the rate of change of the cost function, this rule chooses a direction along which the cost decreases at the fastest rate.
- However, the actual cost decrease depends on how far we move along the chosen direction.
- This suggests the next rule.

Pivot Selection

(b) Choose a column with $\bar{c}_j < 0$ for which the corresponding cost decrease $\theta^* |\bar{c}_j|$ is largest.

- This rule offers the possibility of reaching optimality after a smaller number of iterations.
- On the other hand, the computational burden at each iteration is larger, because we need to compute θ^* for each column with $\bar{c}_j < 0$.
- The available empirical evidence suggests that the overall running time does not improve.

Pivot Selection

- For large problems, even rule (a) can be computationally expensive, because it requires the computation of the reduced cost of every variable.
- In practice, simpler rules are sometimes used, such as the smallest subscript rule, that chooses the smallest j for which $\bar{c}_j$ is negative.
- Under this rule, once a negative reduced cost is discovered, there is no reason to compute the remaining reduced costs.

Pivot Selection

- Regarding the choice of the exiting column, the simplest option is again the smallest subscript rule: out of all variables eligible to exit the basis, choose one with the smallest subscript.
- It turns out that by following the smallest subscript rule for both the entering and the exiting column, cycling can be avoided (cf. Section 3.4).

Implementations of the simplex method

- In the next section, we discuss some ways of carrying out the mechanics of the simplex method.
- But first, we review the conventions used in describing the computational requirements (operation count) of algorithms.

1.6 Algorithms and operation counts

Algorithms and operation counts

- An algorithm is a finite set of instructions of the type used in common programming languages.
- We are interested in comparing algorithms without having to examine the details of a particular implementation.
- As a first approximation, this can be accomplished by counting the number of arithmetic operations required by an algorithm.
- This approach is often adequate even though it ignores the fact that adding or multiplying large integers or high-precision floating point numbers is more demanding than adding or multiplying single-digit integers.

Example 1.9

(a) Inner product. Given vectors $a, b \in \mathbb{R}^n$, compute $a'b$.

$$a'b = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n.$$

- The natural algorithm requires n multiplications and $n - 1$ additions.
- Total number of arithmetic operations: $2n - 1$.

Example 1.9

(b) Matrix multiplication. Given matrices A, B of dimensions $n \times n$, compute $C = AB$.

$$c_{ij} = a_i' B_j, \qquad \forall i, j = 1, \ldots, n.$$

- There are n^2 entries of AB to be evaluated.
- To obtain each one, the natural algorithm forms the inner product of a row of A and a column of B.
- Total number of arithmetic operations: $n^2(2n-1)$.

Example 1.9

We estimate the rate of growth of the number of arithmetic operations:

(a) Inner product.

- Total number of arithmetic operations: $2n - 1$.
- It increases linearly with n.

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(a) Inner product.

- Total number of arithmetic operations: $2n - 1$.
- It increases linearly with n.

(b) Matrix multiplication.

- Total number of arithmetic operations: $n^2(2n - 1)$.
- It increases cubically with n.

Algorithms and operation counts

Definition 1.2

Let f and g be functions that map positive numbers to positive numbers.

(a) We write $f(n) = O(g(n))$ if there exist positive numbers n_0 and c such that

$$f(n) \leq cg(n) \quad \text{for all } n \geq n_0.$$

(b) We write $f(n) = \Omega(g(n))$ if there exist positive numbers n_0 and c such that

$$f(n) \geq cg(n) \quad \text{for all } n \geq n_0.$$

Example: $3n^3 + n^2 + 10 = O(n^3)$, $n \log n = O(n^2)$, $n \log n = \Omega(n)$.

Algorithms and operation counts

- The number of operations performed by an algorithm is called running time.
- Instead of trying to estimate the running time for each possible input, it is customary to estimate the running time for the worst possible input data in a given family.
- For example, if we have an algorithm for LP, we might be interested in estimating its worst-case running time over all problems with a given number of variables and constraints.
- In practice, the "average" running time of an algorithm might be more relevant than the "worst case" running time. However, the average running time is much more difficult to estimate.

Example 1.10

System of linear equations. Given a matrix A of dimension $n \times n$, and a vector $b \in \mathbb{R}^n$, either compute a solution or decide that no solution exists for the system of linear equations

$$Ax = b.$$

- The classical method that eliminates one variable at a time (Gaussian elimination) is known to require $O(n^3)$ arithmetic operations. (Exercise!)
- Practical methods for matrix inversion also require $O(n^3)$ arithmetic operations.

Polynomial time algorithms

Is the $O(n^3)$ running time of Gaussian elimination good or bad?

- Each time that technological advances lead to computer hardware that is faster by a factor of 2^3 (roughly every 3 years by Moore's Law), we can solve problems of twice the size than earlier possible:

$$(2n)^3 = 2^3 \cdot n^3.$$

- A similar argument applies to algorithms whose running time is $O(n^c)$ for some positive constant c:
  Roughly every c years we can solve problems of twice the size than earlier possible.
- Such algorithms are said to run in <u>polynomial time</u>.

Exponential time algorithms

- Algorithms also exist whose running time is $\Omega(2^n)$, where n is a parameter representing problem size; these are said to take at least exponential time.
- For such algorithms, each time that computer hardware becomes faster by a factor of 2 (roughly every year by Moore's Law), we can increase the value of n that we can handle only by 1:
$$2^{n+1} = 2 \cdot 2^n.$$
- A similar argument applies to algorithms whose running time is $\Omega(2^{cn})$ for some positive constant c:
Roughly every c years we can increase the value of n that we can handle only by 1.
- It is then reasonable to expect that no matter how much technology improves, problems with truly large values of n will always be difficult to handle.

Polynomial vs Exponential time algorithms

Example 1.11

Suppose that we have a choice of two algorithms:

- The running time of the first is $10^n/100$ (exponential).
- The running time of the second is $10n^3$ (polynomial).

Polynomial vs Exponential time algorithms

Example 1.11

Suppose that we have a choice of two algorithms:

- The running time of the first is $10^n/100$ (exponential).
- The running time of the second is $10n^3$ (polynomial).

- For very small n, e.g., for $n = 3$, the exponential time algorithm is preferable:

$$10^3/100 = 10 \quad < \quad 10 \cdot 3^3 = 270.$$

Polynomial vs Exponential time algorithms

Example 1.11

Suppose that we have a choice of two algorithms:

- The running time of the first is $10^n/100$ (exponential).
- The running time of the second is $10n^3$ (polynomial).

- Suppose that we have access to a workstation that can execute 10^7 arithmetic operations per second and that we are willing to let it run for 1000 seconds ($\sim$17 minutes).
- Let us figure out what size problems can each algorithm handle within this time frame:
  - The equation $10^n/100 = 10^7 \times 1000$ yields $n = 12$.
  - The equation $10n^3 = 10^7 \times 1000$ yields $n = 1000$.
- Thus the polynomial time algorithm allows us to solve much larger problems.

Algorithms and operation counts

As a first cut, it is useful to juxtapose polynomial and exponential time algorithms:

- Polynomial time algorithms are viewed as fast and efficient.
- Exponential time algorithms are viewed as slow.

3.3 Implementations of the simplex method

Implementations of the simplex method

Let's now get back to the simplex method.

- It should be clear from the statement of the algorithm that the vectors $B^{-1}A_j$ play a key role.
- If these vectors are available, then we can easily compute:
  - The reduced costs

$$\bar{c}_j = c_j - c_B' B^{-1} A_j.$$

  - The direction of motion

$$-u = -B^{-1} A_j.$$

  - The stepsize

$$\theta^* = \min_{i=1,\ldots,m \,|\, u_i > 0} \frac{x_{B(i)}}{u_i}.$$

- The main difference between alternative implementations lies in:
  - The way that the vectors $B^{-1}A_j$ are computed,
  - The amount of related information that is carried from one iteration to the next.

Implementations of the simplex method

When comparing different implementations, it is important to keep the following facts in mind (see Section 1.6).

- Let B be a given $m \times m$ matrix, and let $b, p \in \mathbb{R}^m$ be given vectors.
- Computing the inverse of B or solving a linear system of the form $Bx = b$ takes $O(m^3)$ arithmetic operations.
- Computing a matrix-vector product Bb takes $O(m^2)$ operations.
- Computing an inner product $p'b$ takes $O(m)$ arithmetic operations.

Naive implementation

Naive implementation

We start by describing the most straightforward implementation.

- At the beginning of a typical iteration, we have the indices $B(1), \ldots, B(m)$ of the current basic variables.
- We form the basis matrix B and solve the linear system $p'B = c'_B$ to compute

$$p' = c'_B B^{-1}.$$

  This vector $p \in \mathbb{R}^m$ is called the vector of simplex multipliers associated with the basis B.

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  This vector $p \in \mathbb{R}^m$ is called the vector of simplex multipliers associated with the basis B.
- The reduced cost $\bar{c}_j = c_j - c'_B B^{-1} A_j$ of any variable x_j is then obtained according to the formula

$$\bar{c}_j = c_j - p' A_j.$$

- Regardless of the pivoting rule employed, we may have to compute all of the reduced costs.

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- The reduced cost $\bar{c}_j = c_j - c'_B B^{-1} A_j$ of any variable x_j is then obtained according to the formula

$$\bar{c}_j = c_j - p' A_j. \qquad [n \cdot O(m) = O(mn) \text{ operations}]$$

- Regardless of the pivoting rule employed, we may have to compute all of the reduced costs.

Naive implementation

- Once a column A_j is selected to enter the basis, we solve the linear system $Bu = A_j$ in order to determine the vector

$$u = B^{-1}A_j.$$

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$$u = B^{-1}A_j. \qquad [O(m^3) \text{ operations}]$$

- At this point, we can form the direction along which we will be moving away from the current basic feasible solution.
- We finally determine

$$\theta^* = \min_{i=1,\ldots,m \,|\, u_i > 0} \frac{x_{B(i)}}{u_i}$$

and the variable that will exit the basis, and construct the new basic feasible solution.

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and the variable that will exit the basis, and construct the new basic feasible solution.

Naive implementation: running time

- Thus, the total computational effort per iteration is
$$O(m^3 + mn + m^3 + m) = O(m^3 + mn).$$
- We will see shortly that alternative implementations require only
$$O(m^2 + mn)$$
arithmetic operations.
- Therefore, the naive implementation is rather inefficient.

Revised simplex method

Revised simplex method

- Much of the computational burden in the naive implementation is due to the need for solving the two linear systems of equations

$$p'B = c'_B \qquad \text{and} \qquad Bu = A_j.$$

- In an alternative implementation, the matrix B^{-1} is made available at the beginning of each iteration, and the vectors

$$p' = c'_B B^{-1} \qquad \text{and} \qquad u = B^{-1} A_j$$

are computed by a matrix-vector multiplication.

- For this approach to be practical, we need an efficient method for updating the matrix B^{-1} each time that we effect a change of basis.

Revised simplex method

- Let

$$B = [A_{B(1)} \cdots A_{B(m)}]$$

be the basis matrix at the beginning of an iteration and let

$$\bar{B} = [A_{B(1)} \cdots A_{B(\ell-1)}\ A_j\ A_{B(\ell+1)} \cdots A_{B(m)}]$$

be the basis matrix at the beginning of the next iteration.

- These two basis matrices have the same columns except that the ℓth column $A_{B(\ell)}$ has been replaced by A_j.
- It is then reasonable to expect that B^{-1} contains information that can be exploited in the computation of $\bar{B}^{-1}$.

Revised simplex method

IDEA:

- $B^{-1}\bar{B}$ is not very different from the identity matrix.
- We can compute Q such that $QB^{-1}\bar{B} = I$.
- Then $\bar{B}^{-1} = QB^{-1}$.

How can we do this efficiently?

- Since $B^{-1}B = I$, we see that $B^{-1}A_{B(i)} = e_i$. We have:

$$B^{-1}\bar{B} = [e_1 \cdots e_{\ell-1} \; u \; e_{\ell+1} \cdots e_m] \quad = \begin{bmatrix} 1 & & u_1 & & \\ & \ddots & \vdots & & \\ & & u_\ell & & \\ & & \vdots & \ddots & \\ & & u_m & & 1 \end{bmatrix},$$

  where $u = B^{-1}A_j$. (!!)
- Q only needs to change the above matrix to the identity.

Revised simplex method

Definition 3.4
Given a matrix, the operation of adding a constant multiple of one row to the same or to another row is called an elementary row operation.

- Performing an elementary row operation on a matrix C is equivalent to forming the matrix QC, where Q is a suitably constructed square matrix.

Example 3.3

Let

$$Q = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix},$$

and note that

$$QC = \begin{bmatrix} 1 + 2 \cdot 5 & 2 + 2 \cdot 6 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} 11 & 14 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}.$$

- Multiplication from the left by the matrix Q has the effect of multiplying the third row of C by two and adding it to the first row.

Revised simplex method

Let's generalize Example 3.3.

- Multiplying the jth row by β and adding it to the ith row (for $i \neq j$) is the same as left-multiplying by the matrix

$$Q = I + D_{ij},$$

  where D_{ij} is a matrix with all entries equal to zero, except for the (i, j)th entry which is equal to β.
- The determinant of such a matrix Q is equal to 1 and, therefore, Q is invertible.

Revised simplex method

- Suppose now that we apply a sequence of K elementary row operations and that the kth such operation corresponds to left-multiplication by a certain invertible matrix Q_k.
- Then, the sequence of these elementary row operations is the same as left-multiplication by the invertible matrix

$$Q_K Q_{K-1} \cdots Q_2 Q_1.$$

- We conclude that performing a sequence of elementary row operations on a given matrix is equivalent to left-multiplying that matrix by a certain invertible matrix.
- We will now see how we can use elementary row operations to compute of $\bar{B}^{-1}$ exploiting B^{-1}.

Revised simplex method

- Recall that, since $B^{-1}B = I$, $B^{-1}A_{B(i)} = e_i$.

$$B^{-1}\bar{B} = [e_1 \cdots e_{\ell-1} \ u \ e_{\ell+1} \cdots e_m]$$
$$= \begin{bmatrix} 1 & & u_1 & & \\ & \ddots & \vdots & & \\ & & u_\ell & & \\ & & \vdots & \ddots & \\ & & u_m & & 1 \end{bmatrix},$$

  where $u = B^{-1}A_j$. (!!)

- Our goal is to change the above matrix to the identity matrix
  We apply the following sequence of elementary row operations:
  (a) For each $i \neq \ell$, we add the ℓth row times $-u_i/u_\ell$ to the ith row. (Recall that $u_\ell > 0$. Why?) This replaces u_i by zero.
  (b) We divide the ℓth row by u_ℓ. Why is this an elementary row operation? This replaces u_ℓ by one.

Revised simplex method

- This sequence of elementary row operations replaces the ℓth column u by the ℓth unit vector e_ℓ.
- Furthermore, it is equivalent to left-multiplying $B^{-1}\bar{B}$ by a certain invertible matrix Q.
- Since the result is the identity, we have

$$QB^{-1}\bar{B} = I \qquad \Rightarrow \qquad QB^{-1} = \bar{B}^{-1}.$$

- The last equation shows that if we apply the same sequence of row operations to the matrix B^{-1}, we obtain $\bar{B}^{-1}$.
- We conclude that all it takes to generate $\bar{B}^{-1}$, is to start with B^{-1} and apply the sequence of elementary row operations described above.
- Total number of arithmetic operations:

Revised simplex method

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- Furthermore, it is equivalent to left-multiplying $B^{-1}\bar{B}$ by a certain invertible matrix Q.
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- The last equation shows that if we apply the same sequence of row operations to the matrix B^{-1}, we obtain $\bar{B}^{-1}$.
- We conclude that all it takes to generate $\bar{B}^{-1}$, is to start with B^{-1} and apply the sequence of elementary row operations described above.
- Total number of arithmetic operations: $O(m^2)$.

Example 3.4

$$B^{-1} = \begin{bmatrix} 1 & 2 & 3 \\ -2 & 3 & 1 \\ 4 & -3 & -2 \end{bmatrix}, \qquad u = \begin{bmatrix} -4 \\ 2 \\ 2 \end{bmatrix}, \qquad \ell = 3.$$

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$$B^{-1} = \begin{bmatrix} 1 & 2 & 3 \\ -2 & 3 & 1 \\ 4 & -3 & -2 \end{bmatrix}, \qquad u = \begin{bmatrix} -4 \\ 2 \\ 2 \end{bmatrix}, \qquad \ell = 3.$$

- We have

$$B^{-1}\bar{B} = [e_1 \ e_2 \ u] = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}.$$

- Thus, our objective is to transform the vector u to the unit vector $e_3 = (0, 0, 1)$.

Example 3.4

$$B^{-1} = \begin{bmatrix} 1 & 2 & 3 \\ -2 & 3 & 1 \\ 4 & -3 & -2 \end{bmatrix}, \qquad u = \begin{bmatrix} -4 \\ 2 \\ 2 \end{bmatrix}, \qquad \ell = 3.$$

- We multiply the third row by 2 and add it to the first row.
- We multiply the third row by -1 and add it to the second.
- We divide the third row by 2.

Applying the same row operations to B^{-1} we obtain

$$\bar{B}^{-1} = \begin{bmatrix} 9 & -4 & -1 \\ -6 & 6 & 3 \\ 2 & -1.5 & -1 \end{bmatrix}.$$

- When the matrix B^{-1} is updated in the manner we have described, we obtain an implementation of the simplex method known as the revised simplex method.

Revised simplex method

An iteration of the revised simplex method

1. We start with a basis consisting of the basic columns $A_{B(1)}, \ldots, A_{B(m)}$, an associated basic feasible solution x, and the inverse B^{-1} of the basis matrix.
2. Compute the row vector $p' = c'_B B^{-1}$ and then compute the reduced costs $\bar{c}_j = c_j - p' A_j$.
   - If they are all nonnegative, the current basic feasible solution is optimal, and the algorithm terminates.
   - Else, choose some j for which $\bar{c}_j < 0$.
3. Compute $u = B^{-1} A_j$. If no component of u is positive, the optimal cost is $-\infty$, and the algorithm terminates.

Revised simplex method

An iteration of the revised simplex method

4. If some component of u is positive, let

$$\theta^* = \min_{i=1,\ldots,m \mid u_i > 0} \frac{x_{B(i)}}{u_i}.$$

5. Let ℓ be such that $\theta^* = x_{B(\ell)}/u_\ell$. Form a new basis by replacing $A_{B(\ell)}$ with A_j. If y is the new basic feasible solution, the values of the new basic variables are $y_j = \theta^*$ and $y_{B(i)} = x_{B(i)} - \theta^* u_i$, $i \neq \ell$.
6. Form the $m \times (m+1)$ matrix $[B^{-1} \mid u]$. Add to each one of its rows a multiple of the ℓth row to make the last column equal to the unit vector e_ℓ. The first m columns of the result is the matrix $\bar{B}^{-1}$.

Revised simplex method: implementation

- At the beginning of a typical iteration, we have the indices $B(1), \ldots, B(m)$ of the current basic variables, and the inverse B^{-1} of the basis matrix.
- We compute

$$p' = c'_B B^{-1}.$$

Revised simplex method: implementation

- At the beginning of a typical iteration, we have the indices $B(1), \ldots, B(m)$ of the current basic variables, and the inverse B^{-1} of the basis matrix.
- We compute

$$p' = c'_B B^{-1}. \qquad [O(m^2) \text{ operations}]$$

Revised simplex method: implementation

- At the beginning of a typical iteration, we have the indices $B(1), \ldots, B(m)$ of the current basic variables, and the inverse B^{-1} of the basis matrix.
- We compute

$$p' = c_B' B^{-1}. \qquad [O(m^2) \text{ operations}]$$

- The reduced cost $\bar{c}_j = c_j - c_B' B^{-1} A_j$ of any variable x_j is then obtained according to the formula

$$\bar{c}_j = c_j - p' A_j.$$

- Regardless of the pivoting rule employed, we may have to compute all of the reduced costs.

Revised simplex method: implementation

- At the beginning of a typical iteration, we have the indices $B(1), \ldots, B(m)$ of the current basic variables, and the inverse B^{-1} of the basis matrix.
- We compute

$$p' = c_B' B^{-1}. \qquad [O(m^2) \text{ operations}]$$

- The reduced cost $\bar{c}_j = c_j - c_B' B^{-1} A_j$ of any variable x_j is then obtained according to the formula

$$\bar{c}_j = c_j - p' A_j. \qquad [O(mn) \text{ operations}]$$

- Regardless of the pivoting rule employed, we may have to compute all of the reduced costs.

Revised simplex method: implementation

- Once a column A_j is selected to enter the basis, we compute the vector

$$u = B^{-1} A_j.$$

Revised simplex method: implementation

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Revised simplex method: implementation

- Once a column A_j is selected to enter the basis, we compute the vector

$$u = B^{-1}A_j. \qquad [O(m^2) \text{ operations}]$$

- We determine

$$\theta^* = \min_{i=1,\dots,m \,|\, u_i>0} \frac{x_{B(i)}}{u_i}$$

and the variable that will exit the basis, and construct the new basic feasible solution, and the new basis matrix $\bar{B}$.

Revised simplex method: implementation

- Once a column A_j is selected to enter the basis, we compute the vector

$$u = B^{-1} A_j. \qquad [O(m^2) \text{ operations}]$$

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$$\theta^* = \min_{i=1,\ldots,m \,|\, u_i > 0} \frac{x_{B(i)}}{u_i} \qquad [O(m) \text{ operations}]$$

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  and the variable that will exit the basis, and construct the new basic feasible solution, and the new basis matrix $\bar{B}$.
- We construct the inverse $\bar{B}^{-1}$ of $\bar{B}$.

Revised simplex method: implementation

- Once a column A_j is selected to enter the basis, we compute the vector

$$u = B^{-1} A_j. \qquad [O(m^2) \text{ operations}]$$

- We determine

$$\theta^* = \min_{i=1,\ldots,m \,|\, u_i > 0} \frac{x_{B(i)}}{u_i} \qquad [O(m) \text{ operations}]$$

  and the variable that will exit the basis, and construct the new basic feasible solution, and the new basis matrix $\bar{B}$.
- We construct the inverse $\bar{B}^{-1}$ of $\bar{B}$. $\qquad [O(m^2)$ operations$]$

Revised simplex method: running time

- Thus, the total number of operations per iteration is

$$O(m^2 + mn + m^2 + m + m^2) = O(m^2 + mn) = O(mn).$$

- Therefore, the revised simplex method is more efficient than the naive implementation, which required

$$O(m^3 + mn)$$

arithmetic operations.

The full tableau implementation

The full tableau implementation

We now describe the implementation of the simplex method in terms of the so-called full tableau.

- Here, instead of maintaining and updating the matrix B^{-1}, we maintain and update the $m \times (n+1)$ matrix

$$B^{-1}[b \mid A]$$

with columns $B^{-1}b, B^{-1}A_1, \ldots, B^{-1}A_n$.

- This matrix is called the simplex tableau.

The full tableau implementation

- The column $B^{-1}b$ is called the zeroth column and contains the values of the basic variables.
- The column $B^{-1}A_i$ is called the ith column of the tableau.
- The column $u = B^{-1}A_j$ corresponding to the variable that enters the basis is called the pivot column.

$$
\begin{array}{|c|ccccc|}
\hline
x_{B(1)} & (B^{-1}A_1)_1 & \dots & u_1 & \dots & (B^{-1}A_n)_1 \\
\vdots & \vdots & & \vdots & & \vdots \\
x_{B(\ell)} & (B^{-1}A_1)_\ell & \dots & u_\ell & \dots & (B^{-1}A_n)_\ell \\
\vdots & \vdots & & \vdots & & \vdots \\
x_{B(m)} & (B^{-1}A_1)_m & \dots & u_m & \dots & (B^{-1}A_n)_m \\
\hline
\end{array}
$$

The full tableau implementation

- If the ℓth basic variable exits the basis, the ℓth row of the tableau is called the pivot row.
- The element belonging to both the pivot row and the pivot column is called the pivot element.
- Note that the pivot element is u_ℓ and is always positive (unless $u \leq 0$, in which case the algorithm has met the termination condition in Step 3).

$x_{B(1)}$	$(B^{-1}A_1)_1$	$\dots$	u_1	$\dots$	$(B^{-1}A_n)_1$
$\vdots$	$\vdots$		$\vdots$		$\vdots$
$x_{B(\ell)}$	$(B^{-1}A_1)_\ell$	$\dots$	u_ℓ	$\dots$	$(B^{-1}A_n)_\ell$
$\vdots$	$\vdots$		$\vdots$		$\vdots$
$x_{B(m)}$	$(B^{-1}A_1)_m$	$\dots$	u_m	$\dots$	$(B^{-1}A_n)_m$

The full tableau implementation

The information contained in the rows of the tableau

$$B^{-1}[b \mid A]$$

admits the following interpretation.

- The equality constraints are initially given to us in the form

$$b = Ax.$$

- Given the current basis matrix B, these equality constraints can also be expressed in the equivalent form

$$B^{-1}b = B^{-1}Ax.$$

- The tableau provides us with the coefficients of these equality constraints.

The full tableau implementation

- At the end of each iteration, we need to update the tableau $B^{-1}[b \mid A]$ and compute

$$\bar{B}^{-1}[b \mid A].$$

- This can be accomplished by left-multiplying the simplex tableau with a matrix Q satisfying $QB^{-1} = \bar{B}^{-1}$.
- As explained earlier, this is the same as performing those elementary row operations that turn B^{-1} to $\bar{B}^{-1}$.
- That is, we add to each row a multiple of the pivot row to set all entries of the pivot column to zero, with the exception of the pivot element which is set to one.

The full tableau implementation

Regarding the determination of the exiting column $A_{B(\ell)}$ and the stepsize θ^*, Steps 4 and 5 of the simplex method amount to:

- $\frac{x_{B(i)}}{u_i}$ is the ratio of the ith entry in the zeroth column of the tableau to the ith entry in the pivot column of the tableau.
- We only consider those i for which u_i is positive.
- The smallest ratio is equal to θ^* and determines ℓ.

$$
\begin{array}{|c|ccccc|}
\hline
x_{B(1)} & (B^{-1}A_1)_1 & \dots & u_1 & \dots & (B^{-1}A_n)_1 \\
\vdots & \vdots & & \vdots & & \vdots \\
x_{B(\ell)} & (B^{-1}A_1)_\ell & \dots & u_\ell & \dots & (B^{-1}A_n)_\ell \\
\vdots & \vdots & & \vdots & & \vdots \\
x_{B(m)} & (B^{-1}A_1)_m & \dots & u_m & \dots & (B^{-1}A_n)_m \\
\hline
\end{array}
$$

The zeroth row

It is customary to augment the simplex tableau by including a top row, to be referred to as the zeroth row.

$x_{B(1)}$	$(B^{-1}A_1)_1$	$\dots$	u_1	$\dots$	$(B^{-1}A_n)_1$
$\vdots$	$\vdots$		$\vdots$		$\vdots$
$x_{B(\ell)}$	$(B^{-1}A_1)_\ell$	$\dots$	u_ℓ	$\dots$	$(B^{-1}A_n)_\ell$
$\vdots$	$\vdots$		$\vdots$		$\vdots$
$x_{B(m)}$	$(B^{-1}A_1)_m$	$\dots$	u_m	$\dots$	$(B^{-1}A_n)_m$

The zeroth row

It is customary to augment the simplex tableau by including a top row, to be referred to as the <u>zeroth row</u>.

- The entry at the top left corner contains the negative of the current cost:
$$-c_B' x_B = -c_B' B^{-1} b.$$
- The reason for the minus sign is that it allows for a simple update rule.

$-c_B' x_B$					
$x_{B(1)}$	$(B^{-1}A_1)_1$	$\dots$	u_1	$\dots$	$(B^{-1}A_n)_1$
$\vdots$	$\vdots$		$\vdots$		$\vdots$
$x_{B(\ell)}$	$(B^{-1}A_1)_\ell$	$\dots$	u_ℓ	$\dots$	$(B^{-1}A_n)_\ell$
$\vdots$	$\vdots$		$\vdots$		$\vdots$
$x_{B(m)}$	$(B^{-1}A_1)_m$	$\dots$	u_m	$\dots$	$(B^{-1}A_n)_m$

The zeroth row

It is customary to augment the simplex tableau by including a top row, to be referred to as the zeroth row.

- The rest of the zeroth row is the row vector of reduced costs, that is, the vector

$$\bar{c}' = c' - c_B' B^{-1} A.$$

$-c_B' x_B$	$\bar{c}_1$	$\ldots$	$\bar{c}_j$	$\ldots$	$\bar{c}_n$
$x_{B(1)}$	$(B^{-1}A_1)_1$	$\ldots$	u_1	$\ldots$	$(B^{-1}A_n)_1$
$\vdots$	$\vdots$		$\vdots$		$\vdots$
$x_{B(\ell)}$	$(B^{-1}A_1)_\ell$	$\ldots$	u_ℓ	$\ldots$	$(B^{-1}A_n)_\ell$
$\vdots$	$\vdots$		$\vdots$		$\vdots$
$x_{B(m)}$	$(B^{-1}A_1)_m$	$\ldots$	u_m	$\ldots$	$(B^{-1}A_n)_m$

The zeroth row

- The rule for updating the zeroth row turns out to be identical to the rule used for the other rows of the tableau:

  Add a multiple of the pivot row to the zeroth row to set the reduced cost of the entering variable to zero.

- We will now verify that this update rule produces the correct results for the zeroth row.

The zeroth row

- At the beginning of a typical iteration, the zeroth row is of the form

$$[-c_B'B^{-1}b \mid c' - c_B'B^{-1}A] = [0 \mid c'] - \underbrace{c_B'B^{-1}}_{\text{a row vector}} [b \mid A].$$

- Hence, it is equal to $[0 \mid c']$ plus a linear combination of the rows of $[b \mid A]$.
- Let column j be the pivot column, and row ℓ be the pivot row.
- Note that the pivot row is of the form

$$h'[b \mid A],$$

where the vector h' is the ℓth row of B^{-1}.
- Hence, after a multiple of the pivot row is added to the zeroth row, that row is again equal to $[0 \mid c']$ plus a (different) linear combination of the rows of $[b \mid A]$, and is of the form

$$[0 \mid c'] - p'[b \mid A],$$

for some vector p.

The zeroth row

Beginning of iteration:	$[0 \mid c'] - c'_B B^{-1}[b \mid A]$
End of iteration:	$[0 \mid c'] - p'[b \mid A]$

- We now calculate the vector p using our update rule.
- We should obtain
$$p' = c'_{\bar{B}} \bar{B}^{-1}.$$

The zeroth row

a) Consider the column $\bar{B}(\ell)$ of the tableau.

$-c_B' x_B$	$\bar{c}_1$	$\ldots$	$\bar{c}_j$	$\ldots$	$\bar{c}_n$
$x_{B(1)}$	$(B^{-1}A_1)_1$	$\ldots$	u_1	$\ldots$	$(B^{-1}A_n)_1$
$\vdots$	$\vdots$		$\vdots$		$\vdots$
$x_{B(\ell)}$	$(B^{-1}A_1)_\ell$	$\ldots$	u_ℓ	$\ldots$	$(B^{-1}A_n)_\ell$
$\vdots$	$\vdots$		$\vdots$		$\vdots$
$x_{B(m)}$	$(B^{-1}A_1)_m$	$\ldots$	u_m	$\ldots$	$(B^{-1}A_n)_m$

- Recall that $\bar{B}(\ell) = j$, thus this is the pivot column.
- Our update rule is such that the pivot column entry of the zeroth row becomes zero.
- We obtain

$$c_{\bar{B}(\ell)} - p' A_{\bar{B}(\ell)} = 0.$$

The zeroth row

b) Consider the column $\bar{B}(i)$ of the tableau, for $i \neq \ell$.

$-c'_B x_B$	$\bar{c}_1$	$\dots$	$\bar{c}_j$	$\dots$	$\bar{c}_{\bar{B}(i)}$	$\dots$	$\bar{c}_n$
$x_{B(1)}$	$(B^{-1}A_1)_1$	$\dots$	u_1	$\dots$	$(B^{-1}A_{\bar{B}(i)})_1$	$\dots$	$(B^{-1}A_n)_1$
$\vdots$	$\vdots$		$\vdots$		$\vdots$		$\vdots$
$x_{B(\ell)}$	$(B^{-1}A_1)_\ell$	$\dots$	u_ℓ	$\dots$	$(B^{-1}A_{\bar{B}(i)})_\ell$	$\dots$	$(B^{-1}A_n)_\ell$
$\vdots$	$\vdots$		$\vdots$		$\vdots$		$\vdots$
$x_{B(i)}$	$(B^{-1}A_1)_i$	$\dots$	u_i	$\dots$	$(B^{-1}A_{\bar{B}(i)})_i$	$\dots$	$(B^{-1}A_n)_i$
$\vdots$	$\vdots$		$\vdots$		$\vdots$		$\vdots$
$x_{B(m)}$	$(B^{-1}A_1)_m$	$\dots$	u_m	$\dots$	$(B^{-1}A_{\bar{B}(i)})_m$	$\dots$	$(B^{-1}A_n)_m$

- This is a column corresponding to a basic variable that stays in the basis. Thus $\bar{B}(i) = B(i)$.
- The zeroth row entry of that column is zero, before the change of basis, since it is the reduced cost of a basic variable.

The zeroth row

b) Consider the column $\bar{B}(i)$ of the tableau, for $i \neq \ell$.

$-c'_B x_B$	$\bar{c}_1$	$\dots$	$\bar{c}_j$	$\dots$	$\bar{c}_{B(i)}$	$\dots$	$\bar{c}_n$
$x_{B(1)}$	$(B^{-1}A_1)_1$	$\dots$	u_1	$\dots$	$(B^{-1}A_{B(i)})_1$	$\dots$	$(B^{-1}A_n)_1$
$\vdots$	$\vdots$		$\vdots$		$\vdots$		$\vdots$
$x_{B(\ell)}$	$(B^{-1}A_1)_\ell$	$\dots$	u_ℓ	$\dots$	$(B^{-1}A_{B(i)})_\ell$	$\dots$	$(B^{-1}A_n)_\ell$
$\vdots$	$\vdots$		$\vdots$		$\vdots$		$\vdots$
$x_{B(i)}$	$(B^{-1}A_1)_i$	$\dots$	u_i	$\dots$	$(B^{-1}A_{B(i)})_i$	$\dots$	$(B^{-1}A_n)_i$
$\vdots$	$\vdots$		$\vdots$		$\vdots$		$\vdots$
$x_{B(m)}$	$(B^{-1}A_1)_m$	$\dots$	u_m	$\dots$	$(B^{-1}A_{B(i)})_m$	$\dots$	$(B^{-1}A_n)_m$

- This is a column corresponding to a basic variable that stays in the basis. Thus $\bar{B}(i) = B(i)$.
- The zeroth row entry of that column is zero, before the change of basis, since it is the reduced cost of a basic variable.

The zeroth row

b) Consider the column $\bar{B}(i)$ of the tableau, for $i \neq \ell$.

$-c'_B x_B$	$\bar{c}_1$	$\ldots$	$\bar{c}_j$	$\ldots$	0	$\ldots$	$\bar{c}_n$
$x_{B(1)}$	$(B^{-1}A_1)_1$	$\ldots$	u_1	$\ldots$	$(B^{-1}A_{B(i)})_1$	$\ldots$	$(B^{-1}A_n)_1$
$\vdots$	$\vdots$		$\vdots$		$\vdots$		$\vdots$
$x_{B(\ell)}$	$(B^{-1}A_1)_\ell$	$\ldots$	u_ℓ	$\ldots$	$(B^{-1}A_{B(i)})_\ell$	$\ldots$	$(B^{-1}A_n)_\ell$
$\vdots$	$\vdots$		$\vdots$		$\vdots$		$\vdots$
$x_{B(i)}$	$(B^{-1}A_1)_i$	$\ldots$	u_i	$\ldots$	$(B^{-1}A_{B(i)})_i$	$\ldots$	$(B^{-1}A_n)_i$
$\vdots$	$\vdots$		$\vdots$		$\vdots$		$\vdots$
$x_{B(m)}$	$(B^{-1}A_1)_m$	$\ldots$	u_m	$\ldots$	$(B^{-1}A_{B(i)})_m$	$\ldots$	$(B^{-1}A_n)_m$

- This is a column corresponding to a basic variable that stays in the basis. Thus $\bar{B}(i) = B(i)$.
- The zeroth row entry of that column is zero, before the change of basis, since it is the reduced cost of a basic variable.

The zeroth row

b) Consider the column $\bar{B}(i)$ of the tableau, for $i \neq \ell$.

$-c'_B x_B$	$\bar{c}_1$	$\dots$	$\bar{c}_j$	$\dots$	0	$\dots$	$\bar{c}_n$
$x_{B(1)}$	$(B^{-1}A_1)_1$	$\dots$	u_1	$\dots$	$(B^{-1}A_{B(i)})_1$	$\dots$	$(B^{-1}A_n)_1$
$\vdots$	$\vdots$		$\vdots$		$\vdots$		$\vdots$
$x_{B(\ell)}$	$(B^{-1}A_1)_\ell$	$\dots$	u_ℓ	$\dots$	$(B^{-1}A_{B(i)})_\ell$	$\dots$	$(B^{-1}A_n)_\ell$
$\vdots$	$\vdots$		$\vdots$		$\vdots$		$\vdots$
$x_{B(i)}$	$(B^{-1}A_1)_i$	$\dots$	u_i	$\dots$	$(B^{-1}A_{B(i)})_i$	$\dots$	$(B^{-1}A_n)_i$
$\vdots$	$\vdots$		$\vdots$		$\vdots$		$\vdots$
$x_{B(m)}$	$(B^{-1}A_1)_m$	$\dots$	u_m	$\dots$	$(B^{-1}A_{B(i)})_m$	$\dots$	$(B^{-1}A_n)_m$

- Before the operation, the $B(i)$th column is $B^{-1}A_{B(i)}$, thus it is the ith unit vector.
- Since $i \neq \ell$, the entry in the pivot row for that column is equal to zero.

The zeroth row

b) Consider the column $\bar{B}(i)$ of the tableau, for $i \neq \ell$.

$-c'_B x_B$	$\bar{c}_1$	$\dots$	$\bar{c}_j$	$\dots$	0	$\dots$	$\bar{c}_n$
$x_{B(1)}$	$(B^{-1}A_1)_1$	$\dots$	u_1	$\dots$	0	$\dots$	$(B^{-1}A_n)_1$
$\vdots$	$\vdots$		$\vdots$		$\vdots$		$\vdots$
$x_{B(\ell)}$	$(B^{-1}A_1)_\ell$	$\dots$	u_ℓ	$\dots$	0	$\dots$	$(B^{-1}A_n)_\ell$
$\vdots$	$\vdots$		$\vdots$		$\vdots$		$\vdots$
$x_{B(i)}$	$(B^{-1}A_1)_i$	$\dots$	u_i	$\dots$	1	$\dots$	$(B^{-1}A_n)_i$
$\vdots$	$\vdots$		$\vdots$		$\vdots$		$\vdots$
$x_{B(m)}$	$(B^{-1}A_1)_m$	$\dots$	u_m	$\dots$	0	$\dots$	$(B^{-1}A_n)_m$

- Before the operation, the $B(i)$th column is $B^{-1}A_{B(i)}$, thus it is the ith unit vector.
- Since $i \neq \ell$, the entry in the pivot row for that column is equal to zero.

The zeroth row

b) Consider the column $\bar{B}(i)$ of the tableau, for $i \neq \ell$.

$-c'_B x_B$	$\bar{c}_1$	$\ldots$	$\bar{c}_j$	$\ldots$	0	$\ldots$	$\bar{c}_n$
$x_{B(1)}$	$(B^{-1}A_1)_1$	$\ldots$	u_1	$\ldots$	0	$\ldots$	$(B^{-1}A_n)_1$
$\vdots$	$\vdots$		$\vdots$		$\vdots$		$\vdots$
$x_{B(\ell)}$	$(B^{-1}A_1)_\ell$	$\ldots$	u_ℓ	$\ldots$	0	$\ldots$	$(B^{-1}A_n)_\ell$
$\vdots$	$\vdots$		$\vdots$		$\vdots$		$\vdots$
$x_{B(i)}$	$(B^{-1}A_1)_i$	$\ldots$	u_i	$\ldots$	1	$\ldots$	$(B^{-1}A_n)_i$
$\vdots$	$\vdots$		$\vdots$		$\vdots$		$\vdots$
$x_{B(m)}$	$(B^{-1}A_1)_m$	$\ldots$	u_m	$\ldots$	0	$\ldots$	$(B^{-1}A_n)_m$

- Hence, adding a multiple of the pivot row to the zeroth row of the tableau does not affect the zeroth row entry of that column, which is left at zero.
- Thus for $i \neq \ell$ we have $c_{\bar{B}(i)} - p' A_{\bar{B}(i)} = 0$.

The zeroth row

- a) and b) imply that the vector p satisfies

$$c_{\bar{B}(i)} - p' A_{\bar{B}(i)} = 0 \qquad i = 1, \ldots, m.$$

- In matrix form we have

$$c'_{\bar{B}} - p'\bar{B} = 0 \qquad \Longleftrightarrow \qquad p' = c'_{\bar{B}}\bar{B}^{-1}.$$

- Hence, with our update rule, the updated zeroth row of the tableau is equal to

$$[0 \mid c'] - p'[b \mid A] = [0 \mid c'] - c'_{\bar{B}}\bar{B}^{-1}[b \mid A],$$

as desired.

The full tableau implementation

We can now summarize the mechanics of the full tableau implementation.

An iteration of the full tableau implementation

1. A typical iteration starts with the tableau associated with a basis matrix B and the corresponding basic feasible solution x.
2. Examine the reduced costs in the zeroth row of the tableau.
   - If they are all nonnegative, the current basic feasible solution is optimal, and the algorithm terminates.
   - Else, choose some j for which $\bar{c}_j < 0$.
3. Consider the vector $u = B^{-1}A_j$, which is the jth column (the pivot column) of the tableau. If no component of u is positive, the optimal cost is $-\infty$, and the algorithm terminates.

The full tableau implementation

An iteration of the full tableau implementation

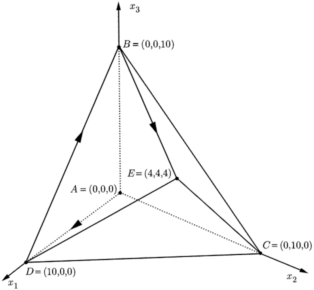
4. For each i for which u_i is positive, compute the ratio

$$\frac{x_{B(i)}}{u_i}.$$

   Let ℓ be the index of a row that corresponds to the smallest ratio. The column $A_{B(\ell)}$ exits the basis and the column A_j enters the basis.

5. Add to each row of the tableau a constant multiple of the ℓth row (the pivot row) so that u_ℓ (the pivot element) becomes one and all other entries of the pivot column become zero.

Example 3.5



$$
\begin{aligned}
\text{minimize} \quad & -10x_1 - 12x_2 - 12x_3 \\
\text{subject to} \quad & x_1 + 2x_2 + 2x_3 \leq 20 \\
& 2x_1 + x_2 + 2x_3 \leq 20 \\
& 2x_1 + 2x_2 + x_3 \leq 20 \\
& x_1, x_2, x_3 \geq 0.
\end{aligned}
$$

Example 3.5

After introducing slack variables, we obtain the standard form problem

$$
\begin{aligned}
\text{minimize} \quad & -10x_1 - 12x_2 - 12x_3 \\
\text{subject to} \quad & x_1 + 2x_2 + 2x_3 + x_4 = 20 \\
& 2x_1 + x_2 + 2x_3 + x_5 = 20 \\
& 2x_1 + 2x_2 + x_3 + x_6 = 20 \\
& x_1, x_2, x_3, x_4, x_5, x_6 \geq 0.
\end{aligned}
$$

$$
\begin{aligned}
\text{minimize} \quad & -10x_1 - 12x_2 - 12x_3 \\
\text{subject to} \quad & x_1 + 2x_2 + 2x_3 \leq 20 \\
& 2x_1 + x_2 + 2x_3 \leq 20 \\
& 2x_1 + 2x_2 + x_3 \leq 20 \\
& x_1, x_2, x_3 \geq 0.
\end{aligned}
$$

Example 3.5

After introducing slack variables, we obtain the standard form problem

$$\begin{aligned} \text{minimize} \quad & -10x_1 - 12x_2 - 12x_3 \\ \text{subject to} \quad & x_1 + 2x_2 + 2x_3 + x_4 = 20 \\ & 2x_1 + x_2 + 2x_3 + x_5 = 20 \\ & 2x_1 + 2x_2 + x_3 + x_6 = 20 \\ & x_1, x_2, x_3, x_4, x_5, x_6 \geq 0. \end{aligned}$$

- Note that $x = (0, 0, 0, 20, 20, 20)$ is a basic feasible solution and can be used to start the algorithm.
- Let accordingly, $B(1) = 4$, $B(2) = 5$, and $B(3) = 6$.
- The corresponding basis matrix is the identity matrix I.
- To obtain the zeroth row of the initial tableau, we note that $c_B = 0$ and, therefore, $c_B' x_B = 0$ and $\bar{c}' = c' - c_B' B^{-1} A = c'$.
- Hence, we have the following initial tableau:

Example 3.5: Initial tableau

		x_1	x_2	x_3	x_4	x_5	x_6
	0	-10	-12	-12	0	0	0
$x_4 =$	20	1	2	2	1	0	0
$x_5 =$	20	2	1	2	0	1	0
$x_6 =$	20	2	2	1	0	0	1

Example 3.5: Initial tableau

$$
\begin{array}{r|r|rrrrrr|}
 & & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\
\hline
 & 0 & -10 & -12 & -12 & 0 & 0 & 0 \\
\hline
x_4 = & 20 & 1 & 2 & 2 & 1 & 0 & 0 \\
x_5 = & 20 & 2 & 1 & 2 & 0 & 1 & 0 \\
x_6 = & 20 & 2 & 2 & 1 & 0 & 0 & 1 \\
\hline
\end{array}
$$

We note a few conventions in the format of the above tableau:

- The label x_i on top of the ith column indicates the variable associated with that column.
- The labels "$x_i =$" to the left of the tableau tell us which are the basic variables and in what order:
  - $x_{B(1)} = x_4 = 20$,
  - $x_{B(2)} = x_5 = 20$,
  - $x_{B(3)} = x_6 = 20$.

Example 3.5: Initial tableau

		x_1	x_2	x_3	x_4	x_5	x_6
	0	-10	-12	-12	0	0	0
$x_4 =$	20	1	2	2	1	0	0
$x_5 =$	20	2	1	2	0	1	0
$x_6 =$	20	2	2	1	0	0	1

We note a few conventions in the format of the above tableau:

- These labels are not necessary.
- We know that the column in the tableau associated with the first basic variable must be the first unit vector.
- Once we observe that the column associated with the variable x_4 is the first unit vector, it follows that x_4 is the first basic variable.

Example 3.5: Initial tableau

		x_1	x_2	x_3	x_4	x_5	x_6
	0	-10	-12	-12	0	0	0
$x_4 =$	20	1	2	2	1	0	0
$x_5 =$	20	2	1	2	0	1	0
$x_6 =$	20	2	2	1	0	0	1

We continue with our example.

- The reduced cost of x_1 is negative and we let that variable enter the basis.
- The pivot column is $u = (1, 2, 2)$.
- We form the ratios $x_{B(i)}/u_i$, $i = 1, 2, 3$:
  - $x_{B(1)}/u_1 = 20/1 = 20$,
  - $x_{B(2)}/u_2 = 20/2 = 10$,
  - $x_{B(3)}/u_3 = 20/2 = 10$.
- The smallest ratio corresponds to $i = 2$ and $i = 3$.
- We break this tie by choosing $\ell = 2$.

Example 3.5: Initial tableau

		x_1	x_2	x_3	x_4	x_5	x_6
	0	-10	-12	-12	0	0	0
$x_4 =$	20	1	2	2	1	0	0
$x_5 =$	20	2	1	2	0	1	0
$x_6 =$	20	2	2	1	0	0	1

- The second basic variable $x_{B(2)}$, which is x_5, exits the basis. This determines the pivot row and the pivot element.
- The new basis is given by $\bar{B}(1) = 4$, $\bar{B}(2) = 1$, and $\bar{B}(3) = 6$.

Example 3.5: Initial tableau

		x_1	x_2	x_3	x_4	x_5	x_6
	0	-10	-12	-12	0	0	0
$x_4 =$	20	1	2	2	1	0	0
$x_5 =$	20	2	1	2	0	1	0
$x_6 =$	20	2	2	1	0	0	1

- We multiply the pivot row by 5 and add it to the zeroth row.
- We multiply the pivot row by $1/2$ and subtract it from the first row.
- We subtract the pivot row from the third row.
- Finally, we divide the pivot row by 2.
- This leads us to the new tableau:

Example 3.5: First pivot

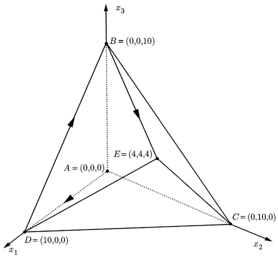
$$
\begin{array}{r|r|rrrrrr|}
 & & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\
\cline{2-8}
 & 100 & 0 & -7 & -2 & 0 & 5 & 0 \\
\cline{2-8}
x_4 = & 10 & 0 & 1.5 & 1 & 1 & -0.5 & 0 \\
x_1 = & 10 & 1 & 0.5 & 1 & 0 & 0.5 & 0 \\
x_6 = & 0 & 0 & 1 & -1 & 0 & -1 & 1 \\
\cline{2-8}
\end{array}
$$

- The cost has been reduced to -100.
- The corresponding basic feasible solution is $x = (10, 0, 0, 10, 0, 0)$.
- Note that this is a degenerate basic feasible solution, because the basic variable x_6 is equal to zero.

Example 3.5: First pivot

		x_1	x_2	x_3	x_4	x_5	x_6
	100	0	-7	-2	0	5	0
$x_4 =$	10	0	1.5	1	1	-0.5	0
$x_1 =$	10	1	0.5	1	0	0.5	0
$x_6 =$	0	0	1	-1	0	-1	1

- In terms of the original variables x_1, x_2, x_3, we have moved to the degenerate solution $D = (10, 0, 0)$.



Example 3.5: First pivot

$$\begin{array}{r|r|rrrrrr|}
 & & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\
\hline
 & 100 & 0 & -7 & -2 & 0 & 5 & 0 \\
\hline
x_4 = & 10 & 0 & 1.5 & 1 & 1 & -0.5 & 0 \\
x_1 = & 10 & 1 & 0.5 & 1 & 0 & 0.5 & 0 \\
x_6 = & 0 & 0 & 1 & -1 & 0 & -1 & 1 \\
\hline
\end{array}$$

- We have mentioned earlier that the rows of the tableau (other than the zeroth row) amount to a representation of the equality constraints $B^{-1}Ax = B^{-1}b$, which are equivalent to the original constraints $Ax = b$.
- In our current example, the tableau indicates that the equality constraints can be written in the equivalent form:

$$\begin{array}{rrrrrrr}
10 = & & 1.5x_2 & +x_3 & +x_4 & -0.5x_5 & \\
10 = & x_1 & +0.5x_2 & +x_3 & & +0.5x_5 & \\
0 = & & x_2 & -x_3 & & -x_5 & +x_6.
\end{array}$$

Example 3.5: First pivot

		x_1	x_2	x_3	x_4	x_5	x_6
	100	0	-7	-2	0	5	0
$x_4 =$	10	0	1.5	1	1	-0.5	0
$x_1 =$	10	1	0.5	1	0	0.5	0
$x_6 =$	0	0	1	-1	0	-1	1

- We now return to the simplex method.
- With the current tableau, the variables x_2 and x_3 have negative reduced costs.
- We choose x_3 to be the one that enters the basis.
- The pivot column is $u = (1, 1, -1)$.
- Since $u_3 < 0$, we only form the ratios $x_{B(i)}/u_i$, for $i = 1, 2$:
  - $x_{B(1)}/u_1 = 10/1 = 10$,
  - $x_{B(2)}/u_2 = 10/1 = 10$.
- There is again a tie, which we break by letting $\ell = 1$.

Example 3.5: First pivot

		x_1	x_2	x_3	x_4	x_5	x_6
	100	0	-7	-2	0	5	0
$x_4 =$	10	0	1.5	1	1	-0.5	0
$x_1 =$	10	1	0.5	1	0	0.5	0
$x_6 =$	0	0	1	-1	0	-1	1

- The first basic variable, x_4, exits the basis.
  This determines the pivot row and the pivot element.
- We multiply the pivot row by 2 and add it to the zeroth row.
- We subtract the pivot row from the second row.
- Finally, we add the pivot row to the third row.
- We obtain the following new tableau:

Example 3.5: Second pivot

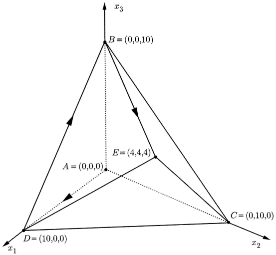
$$
\begin{array}{r|r|rrrrrr|}
 & & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\
\hline
 & 120 & 0 & -4 & 0 & 2 & 4 & 0 \\
\hline
x_3 = & 10 & 0 & 1.5 & 1 & 1 & -0.5 & 0 \\
x_1 = & 0 & 1 & -1 & 0 & -1 & 1 & 0 \\
x_6 = & 10 & 0 & 2.5 & 0 & 1 & -1.5 & 1 \\
\hline
\end{array}
$$

- The cost has been reduced to -120.
- The corresponding basic feasible solution is $x = (0, 0, 10, 0, 0, 10)$.

Example 3.5: Second pivot

		x_1	x_2	x_3	x_4	x_5	x_6
	120	0	-4	0	2	4	0
$x_3 =$	10	0	1.5	1	1	-0.5	0
$x_1 =$	0	1	-1	0	-1	1	0
$x_6 =$	10	0	2.5	0	1	-1.5	1

- In terms of the original variables x_1, x_2, x_3, we have moved to point $B = (0, 0, 10)$.



Example 3.5: Second pivot

		x_1	x_2	x_3	x_4	x_5	x_6
	120	0	-4	0	2	4	0
$x_3 =$	10	0	1.5	1	1	-0.5	0
$x_1 =$	0	1	-1	0	-1	1	0
$x_6 =$	10	0	2.5	0	1	-1.5	1

- At this point, x_2 is the only variable with negative reduced cost.
- We bring x_2 into the basis.
- The pivot column is $u = (1.5, -1, 2.5)$.
- Since $u_2 < 0$, we only form the ratios $x_{B(i)}/u_i$, for $i = 1, 3$:
  - $x_{B(1)}/u_1 = 10/1.5 = 6.\bar{6}$,
  - $x_{B(3)}/u_3 = 10/2.5 = 4$.
- We obtain $\ell = 3$, and the third basic variable, x_6 exits the basis.

Example 3.5: Second pivot

		x_1	x_2	x_3	x_4	x_5	x_6
	120	0	-4	0	2	4	0
$x_3 =$	10	0	1.5	1	1	-0.5	0
$x_1 =$	0	1	-1	0	-1	1	0
$x_6 =$	10	0	2.5	0	1	-1.5	1

- This determines the pivot row and the pivot element.
- We multiply the pivot row by $4/2.5$ and add it to the zeroth row.
- We multiply the pivot row by $1.5/2.5$ and subtract it to the first row.
- We multiply the pivot row by $1/2.5$ and add it to the second row.
- Finally, we divide the pivot row by 2.5.
- We obtain the following new tableau:

Example 3.5: Third pivot

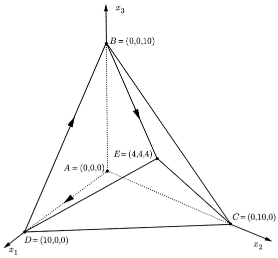
		x_1	x_2	x_3	x_4	x_5	x_6
	136	0	0	0	3.6	1.6	1.6
$x_3 =$	4	0	0	1	0.4	0.4	-0.6
$x_1 =$	4	1	0	0	-0.6	0.4	0.4
$x_2 =$	4	0	1	0	0.4	-0.6	0.4

- The cost has been reduced to -136.
- The corresponding basic feasible solution is $x = (4, 4, 4, 0, 0, 0)$.

Example 3.5: Third pivot

		x_1	x_2	x_3	x_4	x_5	x_6
	136	0	0	0	3.6	1.6	1.6
$x_3 =$	4	0	0	1	0.4	0.4	-0.6
$x_1 =$	4	1	0	0	-0.6	0.4	0.4
$x_2 =$	4	0	1	0	0.4	-0.6	0.4

- In terms of the original variables x_1, x_2, x_3, we have moved to point $E = (4, 4, 4)$.

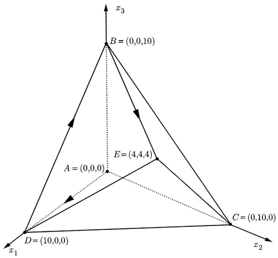


Example 3.5: Third pivot

		x_1	x_2	x_3	x_4	x_5	x_6
	136	0	0	0	3.6	1.6	1.6
$x_3 =$	4	0	0	1	0.4	0.4	−0.6
$x_1 =$	4	1	0	0	−0.6	0.4	0.4
$x_2 =$	4	0	1	0	0.4	−0.6	0.4

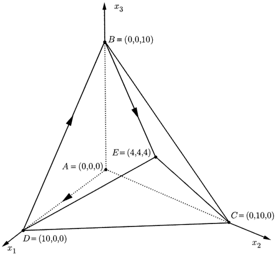
- The optimality of this solution is confirmed by observing that all reduced costs are nonnegative.

Example 3.5



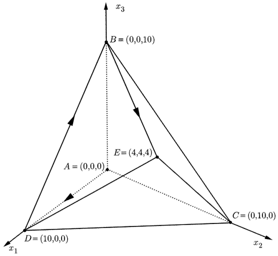
- In this example, the simplex method took three changes of basis to reach the optimal solution, and it traced the path $A - D - B - E$.
- With different pivoting rules, a different path would have been traced.

Example 3.5



- Question: Could the simplex method have solved the problem by tracing the path $A - D - E$, which involves only two edges, with only two iterations?

Example 3.5



- Question: Could the simplex method have solved the problem by tracing the path $A - D - E$, which involves only two edges, with only two iterations? The answer is no.
- The initial and final bases differ in three columns, and therefore at least three basis changes are required.
- In particular, if the method were to trace the path $A - D - E$, there would be a degenerate change of basis at point D (with no edge being traversed), which would again bring the total to three.

The full tableau implementation: running time

What is the total computational effort per iteration?

- The full tableau method requires a constant (and small) number of arithmetic operations for updating each entry of the tableau.
- Thus, the amount of computation per iteration is proportional to the size of the tableau, which is

$$O(mn).$$

- Therefore, the full tableau method is as efficient as the revised simplex method.

Comparison of the full tableau and the revised simplex methods

Comparison of the full tableau and the revised simplex

- Consider a LP problem in standard form

$$\begin{aligned} \text{minimize} \quad & c'x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0. \end{aligned}$$

- Let us pretend that the problem is changed to

$$\begin{aligned} \text{minimize} \quad & c'x + 0'y \\ \text{subject to} \quad & Ax + Iy = b \\ & x, y \geq 0. \end{aligned}$$

Comparison of the full tableau and the revised simplex

- Consider a LP problem in standard form

$$\begin{aligned} \text{minimize} \quad & c'x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0. \end{aligned}$$

- Let us pretend that the problem is changed to

$$\begin{aligned} \text{minimize} \quad & c'x + 0'y \\ \text{subject to} \quad & Ax + Iy = b \\ & x, y \geq 0. \end{aligned}$$

- We implement the simplex method on this new problem, except that we never allow any of the components of the vector y to become basic.
- Then, we always have $y = 0$, and the simplex method performs basis changes as if the vector y were entirely absent.

Comparison of the full tableau and the revised simplex

- The equality constraints of our new standard form problem are $Ax + Iy = b$, thus the new constraint matrix is

$$[A \mid I].$$

- The vector of reduced costs in the augmented problem is

$$[c' \mid 0'] - c_B' B^{-1}[A \mid I] = [\bar{c}' \mid -c_B' B^{-1}].$$

- Thus, the simplex tableau for the augmented problem is

$-c_B' B^{-1} b$	$\bar{c}'$	$-c_B' B^{-1}$
$B^{-1} b$	$B^{-1} A$	B^{-1}

- If we follow the mechanics of the full tableau implementation on the above tableau, the inverse basis matrix B^{-1} is updated at each iteration.

Comparison of the full tableau and the revised simplex

$-c_B'B^{-1}b$	$\bar{c}'$	$-c_B'B^{-1}$
$B^{-1}b$	$B^{-1}A$	B^{-1}

- The revised simplex method is essentially the full tableau method applied to the above augmented problem, except that the part of the tableau containing $\bar{c}'$ and $B^{-1}A$ is never formed explicitly.
- If the revised simplex method also updates the zeroth row entries that lie on top of B^{-1} (by the usual elementary operations), the simplex multipliers $p' = c_B'B^{-1}$ become available, thus eliminating the need for computing $p' = c_B'B^{-1}$ at each iteration.

Comparison of the full tableau and the revised simplex

$-c_B' B^{-1} b$	$\bar{c}'$	$-c_B' B^{-1}$
$B^{-1} b$	$B^{-1} A$	B^{-1}

- We can apply the revised simplex method with a pivoting rule that evaluates one reduced cost at a time, until a negative reduced cost is found.
- Once the entering variable x_j is chosen, the pivot column $B^{-1} A_j$ is computed on the fly.

Comparison of the full tableau and the revised simplex

$-c_B' B^{-1} b$	$\bar{c}'$	$-c_B' B^{-1}$
$B^{-1} b$	$B^{-1} A$	B^{-1}

We now discuss the relative merits of the two methods.

- The full tableau method updates all the tableau at each iteration.
- So the computational requirements per iteration are

$$O(mn).$$

Comparison of the full tableau and the revised simplex

$-c_B' B^{-1} b$	$\bar{c}'$	$-c_B' B^{-1}$
$B^{-1} b$	$B^{-1} A$	B^{-1}

- The revised simplex method updates B^{-1} and $p' = c_B' B^{-1}$. [$O(m^2)$ operations]
- In addition, the reduced cost of each variable x_j is computed as $p' A_j$, requiring $O(m)$ operations. In the worst case, the reduced cost of every variable is computed. [$O(mn)$ operations]
- Once the entering variable x_j is chosen, the pivot column $B^{-1} A_j$ is computed on the fly as a matrix-vector product. [$O(m^2)$ operations]
- Since $m \leq n$, the worst-case computational effort per iteration is
$$O(mn + m^2) = O(mn).$$

Comparison of the full tableau and the revised simplex

$-c'_B B^{-1} b$	$\bar{c}'$	$-c'_B B^{-1}$
$B^{-1} b$	$B^{-1} A$	B^{-1}

- On the other hand, a typical iteration of the revised simplex method might require a lot less work.
- In the best case, if the first reduced cost computed is negative, and the corresponding variable is chosen to enter the basis, the total computational effort is only

$$O(m^2).$$

- The conclusion is that the revised simplex method cannot be slower than the full tableau method, and could be much faster during most iterations.

Comparison of the full tableau and the revised simplex

$-c_B' B^{-1} b$	$\bar{c}'$	$-c_B' B^{-1}$
$B^{-1} b$	$B^{-1} A$	B^{-1}

- Another important element in favor of the revised simplex method is that memory requirements are reduced from $O(mn)$ to $O(m^2)$. Why?
- As n is often much larger than m, this effect can be quite significant.

Comparison of the full tableau and the revised simplex

$-c_B'B^{-1}b$	$\bar{c}'$	$-c_B'B^{-1}$
$B^{-1}b$	$B^{-1}A$	B^{-1}

- It could be counterargued that the memory requirements of the revised simplex method are also $O(mn)$ because of the need to store the matrix A.
- However, in most large scale problems that arise in applications, the matrix A is very sparse (has many zero entries) and can be stored compactly.
- The sparsity of A does not usually help in the storage of the full simplex tableau because even if A and B are sparse, $B^{-1}A$ is not sparse, in general.

Comparison of the full tableau and the revised simplex

We summarize this discussion in the following table:

	Full tableau	Revised simplex
Worst-case time	$O(mn)$	$O(mn)$
Worst-case memory	$O(mn)$	$O(mn)$
Best-case time	$O(mn)$	$O(m^2)$
Best-case memory	$O(mn)$	$O(m^2)$