

HW 13 Harry Luo

1

recall cauchy integral formula

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0} dz \quad 1$$

For our integral, $f(z) = z^3 + e^{z^2}$, $z_0 = 1 + i$ applying the CIF:

$$\begin{aligned} \int_C \frac{z^3 + e^{z^2}}{z - (1 + i)} dz &= 2\pi i f(1 + i) = 2\pi i \left((1 + i)^3 + e^{(1+i)^2} \right) \\ &= 2\pi i (e^{2i} + 2i - 2) \end{aligned} \quad 2$$

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recall the CIF

$$f^n(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} dz \quad 3$$

Here, $z_0 = 0$, $n = 1$, let $f(z) = e^z + e^{z^3}$

$$\begin{aligned} \int_C \frac{f(z)}{z^2} dz &= \frac{2\pi i}{1} f^{(1)}(0) \\ &= 2\pi i (e^0 + 3(0)^2 e^0) = \boxed{2\pi i} \end{aligned} \quad 4$$

3

Using the CIF, take $z_0 = -1$, $n = 2$, $f(z) = z^{2024} + 4z$

$$\int_C \frac{f(z)}{(z + 1)^3} dz = \frac{2\pi i}{2} f^{(2)}(-1) \quad 5$$

where $f^{(2)}(-1) = 2024 * 2023(-1)^{2022} = 4094552$, Equation 5 becomes

$$\int_C \frac{z^{2024} + 4z}{(z + 1)^3} dz = \boxed{4094552\pi i} \quad 6$$

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Using the CIF, take $f(z) = \cos(z)$, $z_0 = 0$

$$\begin{aligned} \int_{|z|=3} \frac{\cos(z)}{z^5} dz &= 2\pi i \cos^{(4)}(0) \\ &= 2\pi i \frac{\cos(0)}{4!} = \boxed{\frac{\pi i}{12}} \end{aligned} \quad 7$$

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• (a) find poles

$$1 + z^2 = (z + i)(z - i) \Rightarrow z_1 = -i, z_2 = i$$

$$\text{Res}(f, z_1) = \lim_{z \rightarrow -i} \frac{z + i}{1 + z^2} = -\frac{1}{2i} \quad 8$$

$$\text{Res}(f, z_2) = \lim_{z \rightarrow i} \frac{z - i}{1 + z^2} = \frac{1}{2i}$$

• (b)

Consider substitution $f(z) = \frac{1}{1+z^2}$, we can use a semicircular contour in the upper half-plane, which will enclose only the pole at $z = i$.

$$\int_{\gamma} f(z) dz = \int_{-R}^R f(z) dz + \underbrace{\int_C f(z) dz}_{(*)} = 2\pi i \operatorname{Res}(f, z = i) + \int_C f(z) dz \quad 9$$

for the second term, parametrize $z = Re^{i\theta}$,

$$(*) = \int_C \frac{1}{1 + Re^{i2\theta}} d\theta, \quad \theta \in [0, \pi] \quad 10$$

as $R \rightarrow \infty$, the second term becomes 0, and the integral becomes

$$\begin{aligned} \int_{-\infty}^{\infty} f(z) dz &= 2\pi i \operatorname{Res}(f, z = i) \\ &= 2\pi i \left(\frac{1}{2i} \right) = \pi \end{aligned} \quad 11$$

Since $\frac{1}{1+x^2}$ is even,

$$\boxed{\int_0^{\infty} \frac{1}{1+x^2} dx = \frac{\pi}{2}} \quad 12$$

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• (a)

Find all the poles

$$\begin{aligned} (1+z^2)^2 &= 0 \\ z^2 &= -1 \\ z &= \pm i \end{aligned} \quad 13$$

Recall that we can find the residue by

$$\operatorname{Res}(f, z_0) = \lim_{z \rightarrow z_0} \frac{1}{(n-1)!} \left(\frac{d}{dz} \right)^{n-1} [(z - z_0)^n f(z)] \quad 14$$

Take $n = 2, z_0 = i$, we have

$$\begin{aligned} \operatorname{Res}(f, i) &= \lim_{z \rightarrow i} \frac{d}{dz} [(z - i)^2 f(z)] = \lim_{z \rightarrow i} \frac{d}{dz} \left[\frac{(z - i)^2}{(z - i)^2 (z + i)^2} \right] \\ &= \lim_{z \rightarrow i} \frac{d}{dz} \left[\frac{1}{(z + i)^2} \right] \\ &= \lim_{z \rightarrow i} -\frac{2}{(z + i)^3} \\ &= \boxed{-\frac{i}{4}} \end{aligned} \quad 15$$

similarly, take $n = 2, z_0 = -i$, we have

$$\begin{aligned}
\text{Res}(f, -i) &= \lim_{z \rightarrow -i} \frac{d}{dz} \left[\frac{(z - z_0)^2}{(z^2 - i^2)^2} \right] = \lim_{z \rightarrow -i} \frac{d}{dz} \left[\frac{(z + i)^2}{(z + i)^2 (z - i)^2} \right] \\
&= \lim_{z \rightarrow -i} \frac{d}{dz} [(z - i)^{-2}] \\
&= \lim_{z \rightarrow -i} (-2(z - i)^{-3}) \\
&= \boxed{\frac{i}{4}}
\end{aligned}$$

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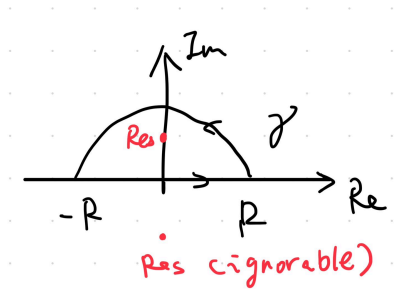
• (b)

recall Cauchy residue thm,

$$\int_{\gamma} f(z) dz = 2\pi i \sum_{k=1}^n \text{Res}(f, z_k)$$

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Consider substitution $f(z) = \frac{1}{(1+z^2)^2}$, we can use a semicircular contour in the upper half-plane, which will enclose only the pole at $z = i$.



$$\int_{\gamma} f(z) dz = \int_{-R}^R f(z) dz + \int_C f(z) dz = 2\pi i \text{Res}(f, z = i)$$

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$$\begin{aligned}
\Rightarrow \int_{-R}^R f(z) dz &= 2\pi i \text{Res}(f, z = i) - \underbrace{\int_C f(z) dz}_{(*)} \\
&= 2\pi i \left(-\frac{i}{4} \right) - \underbrace{\int_C f(z) dz}_{(*)}
\end{aligned}$$

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for $(*)$, parametrize $z = Re^{i\theta}$,

$$(*) = \int_C \frac{1}{1 + Re^{i\theta}} d\theta, \quad \theta \in [0, \pi]$$

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When $R \rightarrow \infty$, Equation 19 becomes

$$\begin{aligned}
\int_{-\infty}^{\infty} f(z) dz &= 2\pi i \left(-\frac{i}{4} \right) - \underbrace{\int_C \frac{1}{1 + Re^{i\theta}} d\theta}_{\rightarrow 0} \\
&= \frac{\pi}{2}
\end{aligned}$$

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Since $\frac{1}{(1+x^2)^2}$ is even,

$$\int_0^{\infty} \frac{1}{(1+x^2)^2} dx = \boxed{\frac{\pi}{4}}$$

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$$f(z) = \frac{z^2}{(z^2 + 1)(z^2 + 4)} \quad 23$$

• (a)

$$\begin{aligned} \frac{z^2}{(z+i)(z-i)(z+2i)(z-2i)} &\Rightarrow z_1 = i, z_2 = -i, z_3 = 2i, z_4 = -2i \\ \text{Res}(f, z_1) &= \lim_{z \rightarrow i} \frac{z^2}{(z+i)(z+2i)(z-2i)} = -\frac{1}{6i} \\ \text{Res}(f, z_2) &= \lim_{z \rightarrow -i} \frac{z^2}{(z-i)(z+2i)(z-2i)} = \frac{1}{6i} \\ \text{Res}(f, z_3) &= \lim_{z \rightarrow 2i} \frac{z^2}{(z-i)(z+i)(z+2i)} = \frac{1}{3i} \\ \text{Res}(f, z_4) &= \lim_{z \rightarrow -2i} \frac{z^2}{(z-i)(z+i)(z-2i)} = -\frac{1}{3i} \end{aligned} \quad 24$$

• (b)

Similarly to 6(b), we can use a semicircular contour in the upper half-plane, which will enclose only the poles at $z = i, 2i$. The contour integration over the complex arc is still 0 as $R \rightarrow \infty$

$$\begin{aligned} \int_{\gamma} f(z) dz &= \int_{-R}^R f(z) dz + \int_C f(z) dz \\ \Rightarrow \int_{-\infty}^{\infty} f(z) dz &= 2\pi i (\text{Res}(f, z_1) + \text{Res}(f, z_3)) = \frac{\pi}{3} \end{aligned} \quad 25$$

Since $f(x)$ is even,

$$\int_0^R f(x) dx = \frac{1}{2} \int_{-\infty}^{\infty} f(x) dx = \boxed{\frac{\pi}{6}} \quad 26$$