Math 431 hw2

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- ex 1.15

(a) Following the definition of events W, G, R, we describe the event of "not all three colors are present" as $W \cap G \cap R$. Then, according to inclusion-exclusion principle, we have

$$\begin{aligned} P(\text{did not see all three colors}) &= P(W \cap G \cap R) \\ &= P(W) + P(G) + P(R) \\ &\quad -P(WG) - P(WR) - P(GR) \\ &\quad +P(WGR) \end{aligned} \tag{1}$$

Note that, for each draw, we are picking 3 balls (excluding one white ball) from 4, so after 3 draws, $P(W) = \left(\frac{3}{4}\right)^2$. Similarly, $P(G) = \left(\frac{3}{4}\right)^2$, $P(R) = \left(\frac{2}{4}\right)^2$.
Notice that if neither white nor green is drawn, then the only possibility is that we draw 3 red balls. If we label the balls to differenciate the 2 red balls in urn, we have $P(WG) = \left(\frac{2}{4}\right)^2$.
Similarly, $P(WR) = \left(\frac{1}{4}\right)^2$, $P(GR) = \left(\frac{1}{4}\right)^2$.
Finally, $P(WGR) = 0$, since it is impossible to draw 3 balls without drawing any of the 3 colors.
Combining all of the above,

$$\begin{aligned} P(W \cap G \cap R) &= \left(\frac{3}{4}\right)^2 + \left(\frac{3}{4}\right)^2 + \left(\frac{2}{4}\right)^2 - \left(\frac{2}{4}\right)^2 - \left(\frac{1}{4}\right)^2 - \left(\frac{1}{4}\right)^2 + 0 \\ &= \frac{13}{16} \end{aligned} \tag{2}$$

(b) We can represent the complement of the event "did not see all three colors" as "saw all three colors". Then, $P(\text{did not see all three colors}) = 1 - P(\text{saw all three colors})$
The event "saw all three colors" can be analyzed as follows: the three colors are picked in order, with $3 \neq 6$ number of ways to pick the colors. Consider there are two red balls, we have in total 2 ∗ 6 = 12 number of ways to pick the balls. Then,

$$P(\text{saw all three colors}) = \frac{12}{4^3} = \frac{3}{16} \tag{3}$$

Therefore, $P(\text{did not see all three colors}) = 1 - \frac{3}{16} = \frac{13}{16}$.

- ex 1.18

The sample space of X is $\{3, 4, 5\}$. The probability mass function of each value is:

$$\begin{aligned} P(X=3) &= \frac{\text{letters in ARE}}{\text{total letters}} = \frac{3}{16} \\ P(X=4) &= \frac{\text{letters in SOME \& DOGS}}{\text{total letters}} = \frac{8}{16} = \frac{1}{2} \\ P(X=5) &= \frac{\text{letters in BROWN}}{\text{total letters}} = \frac{5}{16} \end{aligned} \tag{4}$$

- ex 1.40

  We denote the event "at least one color is repeated exactly twice" as T, where $T = G \cap R \cap Y \cap W$. Therefore, the probability of this event is

$$
\begin{aligned}
P(T) &= P(G \cap R \cap Y \cap W) \\
&= P(G) + P(R) + P(Y) + P(W) && \text{by inclusion-exclusion principle} \\
&\quad - P(G \cap R) - P(G \cap Y) - P(G \cap W) - P(R \cap Y) - P(R \cap W) - P(Y \cap W) \\
&\quad + P(G \cap R \cap Y) + P(G \cap R \cap W) + P(G \cap Y \cap W) + P(R \cap Y \cap W) \\
&\quad - P(G \cap R \cap Y \cap W)
\end{aligned} \tag{5}
$$

  When exactly two balls are of the same color, we are picking 2 balls from the 4 spots to be the same color, and then pick the remaining 2 spots randomly from the urn. Since the total number of events is 4^4, We can calculate

$$
P(G) = P(R) = P(Y) = P(W) = \frac{\binom{4}{2} * 3 * 3}{4^4} = \frac{27}{128} \tag{6}
$$

  We denote each term in the third line in Equation 5 as $P(A \cap B)$, where $A, B \in \{G, R, Y, W\}$. The magnitude of the set $A \cap B$ is $\binom{4}{2}$, which is the number of ways to pick 2 colors from 4. Then, we have

$$
\begin{aligned}
P(G \cap R) = P(G \cap Y) = P(G \cap W) = P(R \cap Y) = P(R \cap W) &= P(Y \cap W) \\
&= \left(\frac{\binom{4}{2}}{4^4} \right) \\
&= \frac{3}{128}
\end{aligned} \tag{7}
$$

  Since we cannot have 3 colors or 4 colors all appearing twice in the 4 draws, we know that the last two rows in Equation 5 are 0. Therefore,

$$
P(T) = 3 * \frac{27}{128} - 6 * \frac{3}{128} = \frac{45}{64} \tag{8}
$$

- ex 2.4

  We mark the event of "picking the second urn" as A, and A^c for "picking the first urn" and the event of "picking the ball labeled 5" as B. The probability of B could be given as

$$
\begin{aligned}
P(B) &= P(BA) + P(BA^c) \\
&= P(B|A)P(A) + P(B|A^c)P(A^c)
\end{aligned} \tag{9}
$$

Notice that

$$
\begin{aligned}
P(B|A^c) &= 0 \\
P(A) = P(A^c) &= \frac{1}{2} \\
P(B|A) &= \frac{1}{3}
\end{aligned} \tag{10}
$$

Therefore,

$$
P(B) = \frac{1}{3} * \frac{1}{2} = \frac{1}{6} \tag{11}
$$

- ex 2.10

We mark the event of "pulled out a x-sided die" as D_x, and the event of "outcome of the roll is 4" as F. According to the law of total probability, we have

$$P(D_6|F) = \frac{P(D_6F)}{P(F)} = P(F|D_6)\frac{P(D_6)}{P(F)} = \frac{\frac{1}{6} * \frac{1}{3}}{P(F)} \tag{12}$$

Notice that

$$\begin{aligned} P(F) &= P(F|D_4)P(D_4) + P(F|D_6)P(D_6) + P(F|D_{12})P(D_{12}) \\ &= \frac{1}{4} * \frac{1}{3} + \frac{1}{6} * \frac{1}{3} + \frac{1}{12} * \frac{1}{3} \\ &= \frac{1}{12} + \frac{1}{18} + \frac{1}{36} \\ &= \frac{1}{6} \end{aligned} \tag{13}$$

Therefore, Combining equation 12 and 13, we have $P(D_6|F) = \frac{\frac{1}{18}}{\frac{1}{6}} = \frac{1}{3}$

- ex 2.32

(a) Mark boy as B, girl as G. Sample space
$\Omega = \{BBB, BBG, BGB, GBB, BGG, GBG, GGB, GGG\}$.
Probability measure for each sample point would be $\left(\frac{1}{2}\right)^3 = \frac{1}{8}$.

(b) Mark the event of there is a boy amongst the children as M, and the event of 2 of the children are girls as N.Then, $M = \{BBB, BBG, BGB, GBB, BGG, GBG, GGB, \}$. Probability of the child not seen given that two are girls,

$$\begin{aligned} P(M|N) &= \frac{P(M \cap N)}{P(N)} \\ &= \frac{P(\{BGG, GBG, GGB\})}{P(\{BGG, GBG, GGB, GGG\})} \\ &= \frac{\frac{3}{8}}{\frac{4}{8}} \\ &= \frac{3}{4} \end{aligned} \tag{14}$$

(c) Similar to (b), we mark the event of there is a boy amongst the children as M, and the event of the two yougest children are girls as N.

$$P(M|N) = \frac{P(M \cap N)}{P(N)} = \frac{P(\{GGB\})}{P(\{GGB, GGG\})} = \frac{1}{2} \tag{15}$$

- ex 2.34
  Suppose we put the marked ball in urn 1. Denote the event of "Friend picked the marked ball" as A, and the event "Friend chose urn k" as B_k. We also denote that there are m balls in urn 2 $(0 \leq m \leq 2)$, and 3-m balls in urn 1.
  Since there are in total 3 arrangements, and considering that $P(A|B_2) = 0$, we can list the following:

$$
\begin{aligned}
\text{All 3 balls in urn 1:} P(A) = P(A|B_1)P(B_1) = \frac{1}{3} * \frac{1}{2} = \frac{1}{6} \\
\text{2 balls in urn 1:} P(A) = P(A|B_1)P(B_1) = \frac{1}{2} * \frac{1}{2} = \frac{1}{4} \\
\text{only the marked ball in urn 1:} P(A) = P(A|B_1)P(B_1) = 1 * \frac{1}{2} = \frac{1}{2}
\end{aligned} \tag{16}
$$

(a) Therefore, to minimize $P(A)$, we should put all three balls in one urn.
(b) To maximize $P(A)$, we should put the marked ball in one urn, and the other two balls in the other urn.
(C) when there are one marked ball amongst n balls, we denote the following:
There are m balls in urn 1 put together with the marked ball $(0 \leq m \leq n)$, and $n - m$ balls in urn 2.

$$
P(A) = P(A|B_1)P(B_1) = \frac{1}{m+1} * \frac{1}{2} = \frac{1}{2m+2} \tag{17}
$$

So, to **minimize** $P(A)$, we should put all the balls in urn 1.
To **maximize** $P(A)$, we should put the marked ball in one urn, and the other $n - 1$ balls in the other urn.

- ex 2.38

(a) We denote: the event of "the chosen letter is R" is R, and the event of "the kth word is chosen" is W_k. Then, we have

$$
\begin{aligned}
P(R) &= P(R|W_1)P(W_1) + P(R|W_2)P(W_2) + P(R|W_3)P(W_3) + P(R|W_4)P(W_4) \\
&= 0 + 0 + \frac{1}{3} * \frac{1}{4} + \frac{1}{5} * \frac{1}{4} \\
&= \frac{2}{15}
\end{aligned} \tag{18}
$$

(b) There are 4 words in total: 1 of which X=3, 2 of which X=4, 1 of which X=5. So,

$$
\begin{aligned}
P(X = 3) = \frac{1}{4} \\
P(X = 4) = \frac{2}{4} \\
P(X = 5) = \frac{1}{4}
\end{aligned} \tag{19}
$$