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Motion in a rapidly oscillating field. (Bonus material).

· Consider EOM:

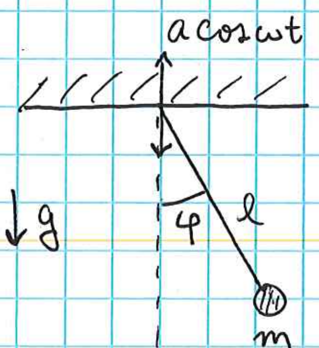
$$m\ddot{x} = -\frac{dU}{dx} + f(x,t)$$

w/ $U(x)$ = time-indep. potential

$f(x,t) = f_0(x)\cos\omega t$ = time-dep. force.

· Consider the situation where $\cos\omega t$ is rapidly oscillating. That is, $\omega \gg 1/\tau$, where τ = time scale of the motion when $f=0$.

Ex: (Pendulum w/ support oscillating at high frequency)



$$L = \frac{1}{2}ml^2\dot{\varphi}^2 + mgl\cos\varphi + mla\omega^2\cos\varphi\cos\omega t.$$

E-L eqn: $\frac{d}{dt}\frac{\partial L}{\partial\dot{\varphi}} = \frac{\partial L}{\partial\varphi}$

$$\Rightarrow ml^2\ddot{\varphi} = \underbrace{-mgl\sin\varphi}_{-\frac{\partial U}{\partial\varphi}} \underbrace{- mla\omega^2\sin\varphi\cos\omega t}_{f(\varphi,t).}$$

Simplifying: $\ddot{\varphi} = -\omega_0^2\sin\varphi - \left(\frac{a\omega^2}{l}\sin\varphi\right)\cos\omega t$, $\omega_0^2 = \frac{g}{l}$.

We are interested in the limit $\omega \gg \omega_0$.

- We can't solve the problem exactly, so we will ~~[illegible]~~ seek an approximate sol.'n in the large ω limit.

- Decompose motion: $x(t) = X(t) + \xi(t)$.

$$\begin{cases} X(t) = \text{smooth, slowly varying} \\ \xi(t) = \text{rapidly oscillating near freq. } \omega. \end{cases}$$

& suppose ξ = small (~~[illegible]~~ although f not necessarily small).

- Useful to consider quantities averaged over period $T = \frac{2\pi}{\omega}$.

$$\bar{\xi}(t) = \frac{1}{T}\int_t^{t+T} dt'\, \xi(t') \simeq 0 \quad (\xi \text{ rapidly varying})$$

$$\bar{X}(t) = \frac{1}{T}\int_t^{t+T} dt'\, X(t') \simeq X(t) \quad (X \text{ slowly varying})$$

$\Rightarrow \bar{x}(t) = X(t) \rightarrow X(t)$ describes smoothed motion, averaged over rapid osc.

EOM: $m\ddot{X} + m\ddot{\xi} = -\frac{dU(X+\xi)}{dx} + f(X+\xi, t)$ ↲ ξ = small.

$$\simeq -\frac{dU}{dX} - \xi\frac{d^2U}{dX^2} + f(X,t) + \xi\frac{\partial f}{\partial X}$$

· EOM contains fast & slow terms that must seperately be equal:

— = slow.
— = fast.

$$m\ddot{X} + m\ddot{\xi} = -\frac{dU}{dX} - \xi\frac{d^2U}{dX^2} + f(X,t) + \xi\frac{\partial f}{\partial X}$$

fast × fast → contains slow part & fast part

$(\cos\varphi_1 \times \cos\varphi_1 \sim \cos(\varphi_1+\varphi_2) + \cos(\varphi_1-\varphi_2))$.

· Isolate slow part by T-avg. $(\bar{\xi}=0)$.

$$\Rightarrow m\ddot{X} = -\frac{dU}{dX} + \overline{\xi\frac{\partial f}{\partial X}} \qquad \text{(slow)}.$$

& subtract this from original EOM to isolate fast part

$$\Rightarrow m\ddot{\xi} = -\xi\frac{d^2U}{dX^2} + f(X,t) + \left(\xi\frac{\partial f}{\partial X} - \overline{\xi\frac{\partial f}{\partial X}}\right). \qquad \text{(fast)}.$$

now, ξ assumed small. But $\ddot{\xi}$ can be large, since $\ddot{\xi} \sim \omega^2\xi$ & ω = large. So we approx. fast eq:

$$m\ddot{\xi} \simeq f(X,t) \qquad (\text{dropping terms} \propto \xi).$$

$f(X,t) = f_0(X)\cos\omega t \Rightarrow \xi(t) \tilde{\simeq} -\frac{1}{m\omega^2} f(X,t)$

(neglects time-dependence of X).

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Now plug this result for $\xi(t)$ back into the slow eqn:

$$m\ddot{X} = -\frac{dU}{dX} - \frac{1}{m\omega^2}\,\overline{f\frac{\partial f}{\partial X}}$$

$$= f_0\frac{\partial f_0}{\partial X}\underbrace{\overline{\cos^2\omega t}}_{=\frac{1}{2}} = \frac{1}{2}f_0\frac{\partial f_0}{\partial X} = \frac{1}{4}\frac{d}{dX}f_0^2$$

total derivative since f_0 only a fn. of X.

$$\Rightarrow m\ddot{X} = -\frac{dU}{dX} - \frac{1}{4m\omega^2}\frac{d}{dX}f_0^2$$

$$= -\frac{d}{dX}\underbrace{\left(U + \frac{1}{4m\omega^2}f_0^2\right)}_{\equiv U_{eff}(X)}$$

$$\Rightarrow m\ddot{X} = -\frac{dU_{eff}}{dX}$$

→ "smooth" motion is that of particle of mass m in an effective <u>static</u> potential

Note: $\dot{\xi} = \frac{f_0}{m\omega}\sin\omega t \Rightarrow \overline{\dot{\xi}^2} = \frac{f_0^2}{2m^2\omega^2}$

$$\Rightarrow U_{eff} = U + \underbrace{\frac{1}{2}m\overline{\dot{\xi}^2}}_{\text{avg. kinetic energy of fast motion}}$$

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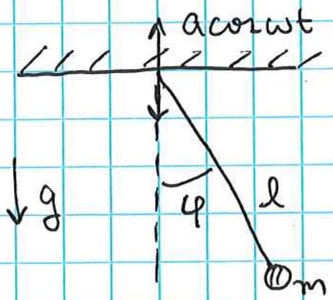
Ex: (pendulum w/ vertically oscillating support)



$$m l^2 \ddot{\varphi} = -mgl \sin\varphi - mla\omega^2 \sin\varphi \cos\omega t.$$

$$\Rightarrow U_{eff} = U + \frac{1}{4m\omega^2}\left(ma\omega^2 \sin\varphi\right)^2$$

$$= mgl\left(-\cos\varphi + \frac{a^2\omega^2}{4gl}\sin^2\varphi\right)$$

· Now, w/o osc. force $U_{eff} = U$ & $\varphi = 0$ is stable $\left(U''\big|_{\varphi=0} > 0\right)$ & $\varphi = \pi$ is unstable $\left(U''\big|_{\varphi=\pi} < 0\right)$.

· But now consider what happens w/ oscillating ~~osc~~ force:

$$\frac{dU_{eff}}{d\varphi} = mgl\left(\sin\varphi + \frac{a^2\omega^2}{2gl}\sin\varphi\cos\varphi\right) = 0$$

$$\Rightarrow \sin\varphi = 0 \quad \text{or} \quad \cos\varphi = -\frac{2gl}{a^2\omega^2}.$$

$\hookrightarrow \varphi = 0$ or π.

For stability check $\dfrac{d^2U_{eff}}{d\varphi^2}$:

$$\frac{d^2U_{eff}}{d\varphi^2} = mgl\left(\cos\varphi + \frac{a^2\omega^2}{2gl}\left(\cos^2\varphi - \sin^2\varphi\right)\right)$$

(i) $\varphi = 0$: $\displaystyle \frac{d^2U_{eff}}{d\varphi^2}\bigg|_{\varphi=0} = mgl\left(1 + \frac{a^2\omega^2}{2gl}\right) > 0.$

$\Rightarrow \varphi = 0$ always stable.

(ii) $\varphi = \pi$: $\displaystyle \frac{d^2U_{eff}}{d\varphi^2}\bigg|_{\varphi=\pi} = mgl\left(-1 + \frac{a^2\omega^2}{2gl}\right) > 0$ if $\dfrac{a^2\omega^2}{2gl} > 0$.

$\Rightarrow \varphi = \pi$ becomes <u>stable</u> equil.!

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