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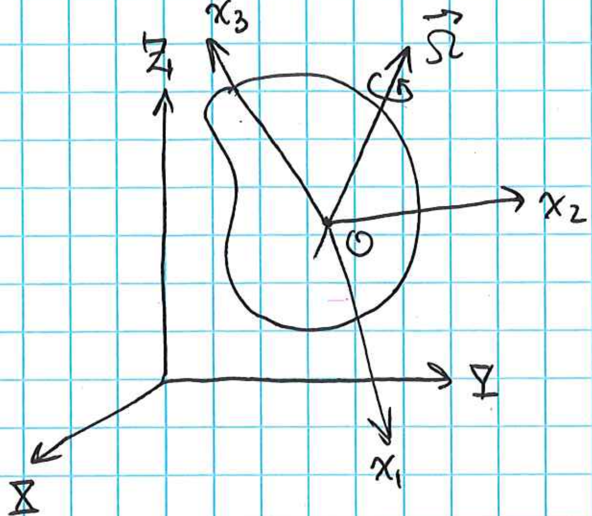
Summary



$(X\,Y\,Z)$ = fixed inertial frame.

$(x_1 x_2 x_3)$ = moving (body) frame.

$$\begin{cases} T = \frac{1}{2} M V^2 + \frac{1}{2} \sum_{ij} I_{ij} \Omega_i \Omega_j \\ I_{ij} = \sum_a m_a (\delta_{ij} r_a^2 - x_{a,i} x_{a,j}). \end{cases}$$ "inertia tensor".

$I_{ij} = I_{ji}$ (a symmetric matrix).

Review of eigenvalues & eigenvectors (Math).

- Consider $n \times n$ matrix A. components of $A = A_{ij}$.
- Suppose there is a vector $\vec{v}$ s.t.:

$$A\vec{v} = \lambda \vec{v}, \quad \lambda = \text{some number.}$$

Then $\vec{v}$ = "eigenvector" of matrix A & λ is the corresponding "eigenvalue". In components:

$$\sum_j A_{ij} v_j = \lambda v_i$$

When considering e.value problems, a special role is played by symmetric matrices, $A_{ij} = A_{ji}$. 03/22/24

~~Thm: Let A = symmetric (n×n matrix)~~

<u>Thm</u>: An $n \times n$ symmetric matrix A possesses n real eigenvalues & n associated real eigenvectors, which can be chosen to be orthogonal.

That is, there exist n vectors $(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n)$ s.t.:

$$A \vec{v}_i = \lambda_i \vec{v}_i \qquad \lambda_i \in \mathbb{R}.$$

$$\& \quad \vec{v}_i \cdot \vec{v}_j = \delta_{ij}$$

Now, since I_{ij} = symmetric matrix we use the stated properties of eigenvalues & eigenvectors of symmetric matrices to conclude that:

I_{ij} = diagonalizable w/ real eigenvalues (I_1, I_2, I_3) & orthonormal eigenvectors.

$\Rightarrow$ choose body axes $(x_1 x_2 x_3)$ to be e.vectors of I_{ij};

i.e., choose axes $(x_1 x_2 x_3)$ s.t. I_{ij} = diagonal matrix.

In such a frame:

$$T_{rot.} = \frac{1}{2} \sum_{ij} I_{ij} \Omega_i \Omega_j$$

$I_{ij} = I_i \delta_{ij}$ for diagonal matrix

$$= \frac{1}{2} \sum_{ij} I_i \delta_{ij} \Omega_i \Omega_j$$

$$= \frac{1}{2} \sum_i I_i \Omega_i^2$$

$$\Rightarrow T_{rot} = \frac{1}{2} \left(I_1 \Omega_1^2 + I_2 \Omega_2^2 + I_3 \Omega_3^2 \right)$$

(I_1, I_2, I_3) = "principal moments of inertia"

$(x_1\ x_2\ x_3)$ = "principal axes".

→ prefered choice of axes in the body frame.

Different cases:

- $I_1 = I_2 = I_3$ → "spherical top"

$$I = I_1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ → proportional to identity matrix. ⇒ any three ⊥ axes can be principal axes.

- $I_1 = I_2 \neq I_3$ → "symmetric top"

$$I = \left(\begin{array}{cc|c} I_1 & 0 & 0 \\ 0 & I_1 & 0 \\ \hline 0 & 0 & I_3 \end{array}\right)$$ → top 2×2 block prop. to identity ⇒ any two ⊥ axes in x_1x_2-plane are principal axes.

- I_1, I_2, I_3 → "asymmetric top".

$$I = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}.$$

In general, principal axes determined by solving eigenvalue eqn. However, determination of principal axes simplified in situations w/ symmetry.

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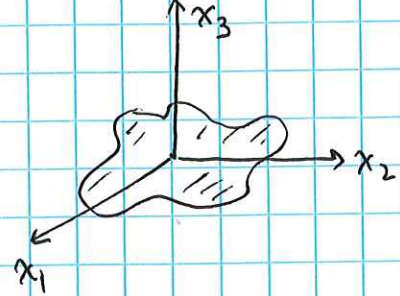
<u>Ex:</u> Coplanar system of particles.



→ Take plane of system as $x_1 x_2$-plane.

⇒ for each particle in the ~~a~~ rigid body $x_3 = 0$.

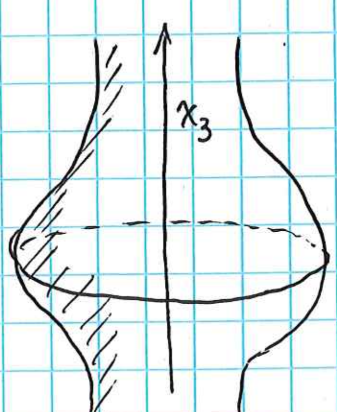
$$\Rightarrow I_{13} = I_{23} = 0. \Rightarrow x_3 = \text{principal axis}$$

$$\& \quad I_1 = \sum m x_2^2, \quad I_2 = \sum m x_1^2, \quad I_3 = \sum m (x_1^2 + x_2^2)$$

$$\Rightarrow I_3 = I_1 + I_2.$$

<u>Ex:</u> Continuous rigid body w/ axis of symmetry.

→ choose x_3 axis along symmetry axis



use cylindrical coord.'s $(r\, \varphi\, x_3)$

w/ $(r\varphi)$ = polar coord.'s in $x_1 x_2$-plane

symmetry $\Rightarrow \rho(\vec{r}) = \rho(r, x_3)$

i.e. ρ indep. of φ.

$$I_{ij} = \int dv\, \rho(\vec{r}) \left(r^2 \delta_{ij} - x_i x_j \right)$$

we have:

$$I_{13} = -\int dv\, \rho(\vec{r})\, x_1 x_3$$
$$= -\int dx_3\, r dr d\varphi\, \rho(r, x_3)\, r\cos\varphi\, x_3$$
$$= -\int dx_3\, r dr\, \rho(r, x_3)\, r x_3 \underbrace{\int_0^{2\pi} d\varphi\, \cos\varphi}_{=0}$$

& $I_{23} = 0$ by same analysis w/ $\cos\varphi \to \sin\varphi$.

$$\Rightarrow I = \begin{pmatrix} I_{11} & I_{12} & 0 \\ I_{21} & I_{22} & 0 \\ 0 & 0 & I_3 \end{pmatrix} \Rightarrow$$ x_3 = principal axis ;

i.e., symm. axis = princ. axis

we also have:

$$I_{12} = -\int dv\, \rho(\vec{r})\, x_1 x_2$$
$$= -\int dx_3\, r dr d\varphi\, \rho(r,z)\, r^2 \cos\varphi \sin\varphi$$
$$= -\int dx_3\, r dr d\varphi\, \rho(r,z)\, r^2 \underbrace{\int_0^{2\pi} d\varphi\, \cos\varphi \sin\varphi}_{=0}$$

$$\Rightarrow I = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix},$$ x_1, x_2, x_3 = principal axes.

Finally:

$$I_1 - I_2 = \int dv\, \rho(\vec{r})\, (x_2^2 - x_1^2)$$
$$= \int dx_3\, r dr\, \rho(r,z)\, r^2 \underbrace{\int_0^{2\pi} d\varphi\, (\sin^2\varphi - \cos^2\varphi)}_{=0}$$

$$\Rightarrow I_1 = I_2 \equiv I_\perp .$$

$$\Rightarrow I = \left(\begin{array}{cc|c} I_\perp & 0 & 0 \\ 0 & I_\perp & 0 \\ \hline 0 & 0 & I_3 \end{array}\right)$$ → any two $\perp$ axes in $x_1 x_2$-plane are principal axes.

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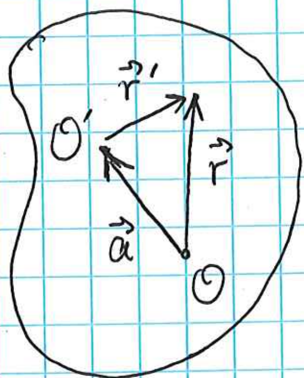
Parallel axis Thm.

· Sometimes more convenient to compute I_{ij} about an origin different from COM.



$O =$ COM.

$$\vec{r} = \vec{r}\,' + \vec{a}$$

Let $I'_{ij} = \sum m(r'^2 \delta_{ij} - x'_i x'_j)$

(dropping "a" subscript for notational convenience).

We have: $$I'_{ij} = \sum m\left[(r^2 - 2\vec{r}\cdot\vec{a} + a^2)\delta_{ij} - (x_i - a_i)(x_j - a_j) \right]$$

$$= \sum m(r^2\delta_{ij} - x_i x_j) + M(a^2\delta_{ij} - a_i a_j)$$

$$- 2\,\cancel{\vec{a}}\,\delta_{ij}\,\vec{a}\cdot\underbrace{\left(\sum m\vec{r}\right)}_{=0} + a_i\underbrace{\sum m x_j}_{=0} + a_j\underbrace{\sum m x_i}_{=0}$$

since $O =$ COM.

$$\Rightarrow \boxed{I_{ij} = I'_{ij} - M(a^2\delta_{ij} - a_i a_j)} \quad (*)$$

→ sometimes easier to compute I' about convenient origin O' & translate to I using Eq. (*).

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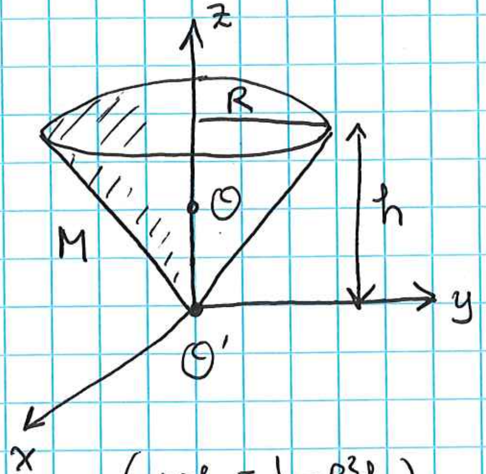
Ex: (Circular Cone)



($\text{vol.} = \frac{1}{3}\pi R^2 h$)

• To calculate I about COM, convenient to calculate first I' about O' & then use ||-axis thm.

• First, rotn.'l symm. about $\hat{z}$:

$$I' = \begin{pmatrix} I_\perp' & 0 & 0 \\ 0 & I_\perp' & 0 \\ 0 & 0 & I_z' \end{pmatrix}$$

$$\Rightarrow I_z' = \int dv\, \rho\, (x^2+y^2)$$

cylindrical coord.'s

$$= \int dz\, r dr\, d\varphi\, \rho r^2$$

$$\frac{r}{z} = \frac{R}{h}$$

$$= 2\pi\rho \int_0^h dz \int_0^{zR/h} dr\, r^3$$

$$= 2\pi\rho \int_0^h dz\, \frac{1}{4}\left(\frac{zR}{h}\right)^4$$

$$= \frac{\pi}{10}\rho R^4 h$$

$$M = \rho \times \frac{1}{3}\pi R^2 h$$

$$= \frac{3}{10} M R^2$$

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$$\Rightarrow \quad I'_\perp = \frac{1}{2}\left(I_{xx} + I_{yy}\right)$$

$$= \frac{1}{2}\int dv \; \rho \left(x^2+y^2+2z^2\right)$$

$$= \frac{1}{2} I'_z + \underbrace{\int dv \; \rho \, z^2}$$

$$= 2\pi\rho \int_0^h dz \, z^2 \int_0^{zR/h} dr \; r \qquad \text{(cylindrical coord.'s)}.$$

$$= \pi \rho \frac{R^2}{h^2} \int_0^h dz \, z^4$$

$$= \frac{1}{5}\pi\rho R^2 h^3$$

$$= \frac{3}{5} M h^2$$

$$\Rightarrow \quad I'_z = \frac{3}{10} M R^2, \qquad I'_\perp = \frac{3}{20} M R^2 + \frac{3}{5} M h^2.$$

$$\vec{a} = a\hat{z} \quad \Rightarrow \quad I_z = I'_z \;, \qquad I_\perp = I'_\perp - M a^2$$

$$\& \quad a = \frac{3}{4}h \quad \Rightarrow \quad I_z = \frac{3}{10} M R^2 \qquad I_\perp = \frac{3}{20} M \left(R^2 + \frac{1}{4} h^2\right).$$