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1. recall stokes theorem $\iint_S \left(\nabla \times \vec{F}\right) \cdot \vec{n} \,\mathrm{d}S = \oint_C \vec{F} \cdot \mathrm{d}\vec{r}$.
   Let the surface bound by C be a simple disk $x^2 + y^2 \leq 4, z = 7$. Unit normal vector of this surface is $\vec{n} = (0, 0, 1)$

$$\nabla \times \vec{F} = \det \begin{pmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x \ln(z) & 2yz^2 & \sqrt{xy + e^x} \end{pmatrix} = \begin{pmatrix} \frac{x}{2\sqrt{xy+e^x}} - 4yz \\ \frac{3x}{z} - \frac{y+e^x}{2\sqrt{xy+e^x}} \\ 0 \end{pmatrix}$$

   Thus, $\left(\nabla \times \vec{F}\right) \cdot \vec{n} = 0$. This implies that $\iint_S \left(\nabla \times \vec{F}\right) \cdot \vec{n} \,\mathrm{d}S = 0$

   $\boxed{\text{Thus the result of the given line integral is } 0}$

2. Recall stokes' theorem,

$$\int_C \vec{F} \cdot \mathrm{d}\vec{r} = \iint_S \operatorname{curl}\left(\vec{F}\right) \cdot \vec{n} \,\mathrm{d}A = \iint_S \operatorname{curl}\left(\vec{F}\right) \cdot \hat{n} \,\mathrm{d}S$$

   for A s.t. $\mathrm{d}\vec{S} = \|\vec{r}_u \times \vec{r}_v\| \,\mathrm{d}A$, A being the projection of S onto the $\{u, v\}$ plane.

   Choose S as the triangle with vertices $(1, 0, 0), (0, 1, 0), (0, 0, 1)$ as suggested, and S follows the equation $x + y + z = 1$.

   Normal vector to S is $\vec{n} = \vec{r}_x \times \vec{r}_y = (1, 1, 1)$ Unit normal vector to S is $\hat{n} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

$$\nabla \times \vec{F} = \det \begin{pmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x + y^2 & y + z^2 & z + x^2 \end{pmatrix} = \begin{pmatrix} -2z \\ -2x \\ -2y \end{pmatrix}$$

   Thus, $\left(\nabla \times \vec{F}\right) \cdot \vec{n} = -2(x + y + z)$.

$$\begin{aligned} \iint_S \left(\nabla \times \vec{F}\right) \cdot \vec{n} \,\mathrm{d}A &= -2 \iint_S x + y + z \,\mathrm{d}A \\ &= -2 \iint_S x + y + 1 - x - y \,\mathrm{d}A \\ &= -2 \iint_S 1 \,\mathrm{d}A \\ &= -2 \times \text{Area: projection of 3d triangle on the xy plane} \\ &= -2 \times \frac{1}{2} \\ &= -1 \end{aligned}$$

Alternatively, we can find this without doing the projection of S onto the x, y plane:

$\left(\nabla \times \vec{F}\right) \cdot \hat{n} = -\frac{2}{\sqrt{3}}(x + y + z)$

$$
\begin{aligned}
\iint_S \operatorname{curl}\left(\vec{F}\right)\cdot\hat{n}\,\mathrm{d}S &= \iint_S -\frac{2}{\sqrt{3}}(x+y+z)\,\mathrm{d}S \\
&= -\frac{2}{\sqrt{3}}\iint_S (x+y+1-x-y)\,\mathrm{d}S \\
&= -\frac{2}{\sqrt{3}}\iint_S 1\,\mathrm{d}S \\
&= -\frac{2}{\sqrt{3}} * \text{Area: 3d triangle} \\
&= -\frac{2}{\sqrt{3}}\cdot\left(\frac{1}{2} * \|(1,0,-1)\times(-1,1,0)\|\right) \\
&= -1
\end{aligned}
$$

1. $\boxed{\mathrm{Re}(z)=\sqrt{2};\quad \mathrm{Im}=-\pi}$

2. (a)

$$(2+i)+\left(\sqrt{3}+8i\right)=\boxed{\left(2+\sqrt{3}\right)+9i}$$

(b)

$$(3-6-6i)=\boxed{(-3-6i)}$$

(c)

$$16+2i-24i-3i^2=\boxed{(19-22i)}$$

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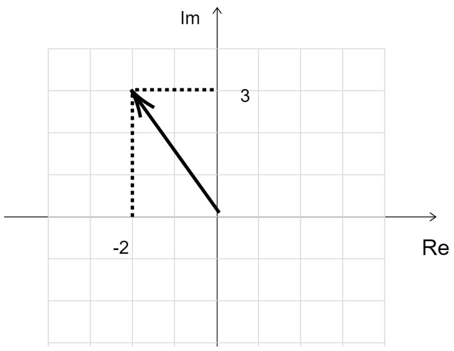
3. $\boxed{|z|=\sqrt{(-2)^2+3^2}=\sqrt{13}}$



4. recall triangle inequality $|z_1+z_2|\le|z_1|+|z_2|$.

$$|3+\cos(5)i|\le|3|+|\cos(5)i|=3+|\cos(5)|\le 4$$
$$|3+\cos(5)i|=\sqrt{3^2+\cos^2(5)}\ge\sqrt{9-1}=\sqrt{8}\ge 2$$

$\boxed{\text{therefore } 2\le|3+\cos(5)i|\le 4}$

5. $\boxed{z^* = 3 - 8i}$

6. (a)

$$\frac{(1-i)(1-i)}{(1-i)(1+i)} = \frac{1-2i-1}{2} = \boxed{0 + i(-1)}$$

(b)

$$\frac{(1+i)\left(1-\sqrt{2}i\right)}{\left(1+\sqrt{2}i\right)\left(1-\sqrt{2}i\right)} = \frac{1-i\sqrt{2}+i+\sqrt{2}}{3} = \frac{1+\sqrt{2}+\left(1-\sqrt{2}\right)i}{3} \boxed{= \frac{1+\sqrt{2}}{3} + i\frac{1-\sqrt{2}}{3}}$$

(c)

$$-i - 1 - 4 = \boxed{-5 + i(-1)}$$

7. let $z = a + bi; w = c + di$. It follows that

$$\begin{aligned} (zw)^* &= (ac - bd) - i(ad + bc) \\ z^* w^* &= (a - bi)(c - di) = (ac - bd) - i(ad + bc) \\ &\boxed{\Rightarrow (zw)^* = z^* w^*} \end{aligned}$$