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Summary

· Damped harmonic oscillator:

$$m\ddot{x} = -kx - \beta\dot{x} \rightarrow \ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0, \qquad 2\gamma = \frac{\beta}{m}, \qquad \omega_0 = \sqrt{\frac{k}{m}}$$

→ homogeneous diff eq. for $x(t)$.

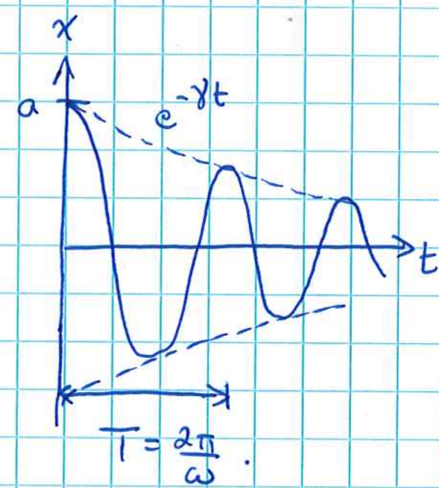
Sol.'n: $\quad x(t) = C_1 e^{r_+ t} + C_2 e^{r_- t}$

$(x = e^{rt})$ $\quad r_\pm = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2}$

Cases:

(i) $\gamma < \omega_0$: $\quad x(t) = a\, e^{-\gamma t} \cos(\omega t + \alpha)$

"underdamped".



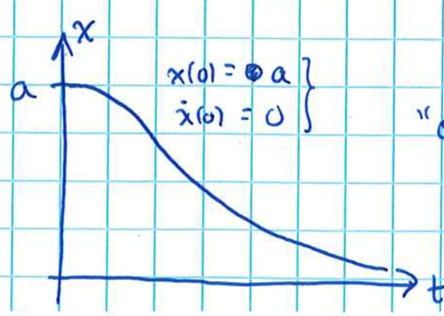
(ii) $\gamma > \omega_0$: $\quad x(t) = C_1 e^{-(\gamma + \sqrt{\gamma^2 - \omega_0^2})t} + C_2 e^{-(\gamma - \sqrt{\gamma^2 - \omega_0^2})t}$

→ exponentially decaying soln.'s.

To better understand sol.'n, consider $\gamma \gg \omega_0$:

$$\begin{cases} \gamma + \sqrt{\gamma^2 - \omega_0^2} \simeq 2\gamma \\ \gamma - \sqrt{\gamma^2 - \omega_0^2} = \gamma - \gamma\sqrt{1 - \omega_0^2/\gamma^2} \simeq \gamma - \gamma\left(1 - \frac{\omega_0^2}{2\gamma^2}\right) = \frac{\omega_0^2}{2\gamma} \end{cases}$$

$$\Rightarrow x(t) = \underbrace{C_1 e^{-2\gamma t}}_{\text{fast decay}} + \underbrace{C_2 e^{-\frac{\omega_0^2}{2\gamma}t}}_{\text{slow decay}}$$



$\left.\begin{matrix} x(0) = a \\ \dot{x}(0) = 0 \end{matrix}\right\}$

"overdamped motion".

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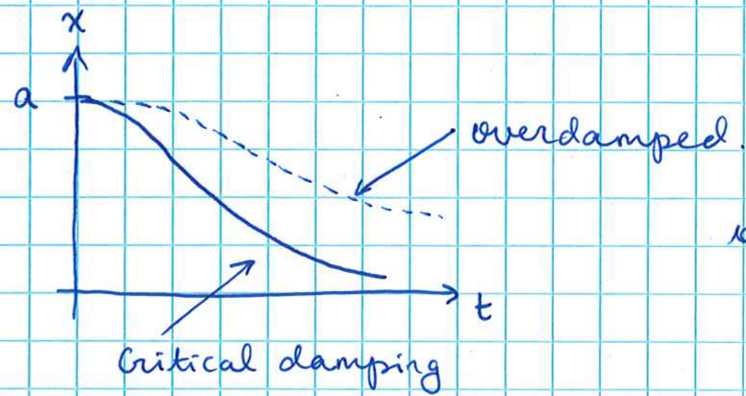
(iii) $\gamma = \omega_0 \longrightarrow \quad x(t) = C_1 e^{-\gamma t} + \underbrace{C_2 t e^{-\gamma t}}_{\text{check that this is a sol.'n!}}$

$(r_+ = r_-)$

"critical damping"

→ fastest decay

$\begin{cases} x(0) = a \\ \dot{x}(0) = 0 \end{cases}$



Forced oscillations

· w/o driving force we have "free" oscillations. now consider the situation w/ an external force → "forced" osc.

$$L = \frac{1}{2} m\dot{x}^2 - \frac{1}{2} k x^2 - U_{ext}(x,t)$$

expand for small x: $U_{ext}(x,t) \simeq \underbrace{U_{ext}(0,t)}_{\text{purely a fn. of } t \text{ \& may be omitted from } L.} + \underbrace{\left.\frac{\partial U_{ext}}{\partial x}\right|_{x=0}}_{= -F_{ext}(t)} x + \ldots$

a t-dep. force

$$\Rightarrow L = \frac{1}{2} m\dot{x}^2 - \frac{1}{2} k x^2 + F_{ext}(t)\, x.$$

E-L. eqn.'s: $\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x} \longrightarrow \ddot{x} + \omega_0^2 x = \frac{F(t)}{m}, \quad \omega_0 = \sqrt{\frac{k}{m}}$

<u>inhomogeneous</u> diff. eq. for $x(t)$

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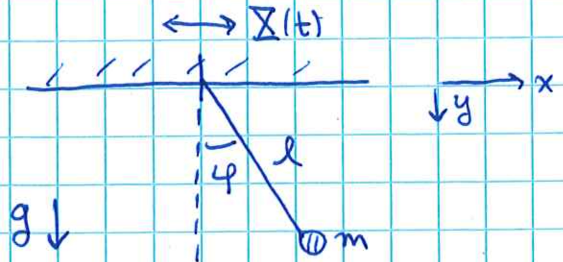
Ex: simple pendulum w/ moving pivot



$$\begin{cases} x = X + l\sin\varphi. \\ y = l\cos\varphi \end{cases}$$

$$\begin{cases} \dot{x} = \dot{X} + l\dot{\varphi}\cos\varphi \\ \dot{y} = \cancel{\dot{X}} - l\dot{\varphi}\sin\varphi. \end{cases}$$

$$\Rightarrow \quad L = T - U = \tfrac{1}{2}m\left[(\dot{X} + l\dot{\varphi}\cos\varphi)^2 + l^2\dot{\varphi}^2\sin^2\varphi\right] \mp mgl(1-\cos\varphi).$$

$$= \tfrac{1}{2}m\left(\dot{X}^2 + l^2\dot{\varphi}^2 + 2l\dot{X}\dot{\varphi}\cos\varphi\right) \mp mgl(1-\cos\varphi).$$

can be dropped from L

$$= +2l\dot{X}\frac{d}{dt}\sin\varphi = +2l\underbrace{\frac{d}{dt}\left(\dot{X}\sin\varphi\right)}_{\text{total t-der. may be dropped from L.}} \mp 2l\ddot{X}\sin\varphi$$

$$\Rightarrow L = \tfrac{1}{2}ml^2\dot{\varphi}^2 \mp mgl(1-\cos\varphi) \mp \underbrace{ml\ddot{X}\sin\varphi}_{U_{ext}(\varphi,t)}.$$

- expand about stable equil. at $\varphi = 0$:

$$L = \tfrac{1}{2}ml^2\dot{\varphi}^2 - \tfrac{1}{2}mgl\varphi^2 \mp \underbrace{ml\ddot{X}\varphi}_{=F(t)} \qquad \left(\begin{matrix}\cos\varphi \simeq 1 - \varphi^2/2 \\ \sin\varphi \simeq \varphi\end{matrix}\right)$$

E-L. eqn.'s: $\dfrac{d}{dt}\dfrac{\partial L}{\partial\dot{\varphi}} = \dfrac{\partial L}{\partial\varphi} \longrightarrow ml^2\ddot{\varphi} = -mgl\varphi - ml\ddot{X}$

$$\Rightarrow \ddot{\varphi} + \omega_0^2\varphi = -\frac{\ddot{X}}{l}, \qquad \omega_0 = \sqrt{\frac{g}{l}}.$$

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we may also reintroduce damping:

$$\boxed{\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = f(t)}\ , \qquad f(t) = \frac{F(t)}{m}$$

General sol.'n has form $x(t) = x_p(t) + x_h(t)$

x_p = "particular sol.'n"

x_h = "homogeneous sol.'n"

$\rightarrow$ x_h satisfies: $\ddot{x}_h + 2\gamma\dot{x}_h + \omega_0^2 x_h = 0.$

An especially important example is a sinusoidal driving force: $f(t) = f_0\cos(\Omega t).$

$$\rightarrow \ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = f_0\cos(\Omega t).$$

To solve this eqn., introduce complex fn. $z(t)$ & consider:

$$\ddot{z} + 2\gamma\dot{z} + \omega_0^2 z = f_0 e^{i\Omega t}.$$

Seek sol.'n of the form $z(t) = z_0 e^{i\Omega t}$.

$$\rightarrow \dot{z} = i\Omega z, \qquad \ddot{z} = -\Omega^2 z.$$

$$\Rightarrow z_0\left(-\Omega^2 + 2i\gamma\Omega + \omega_0^2\right)\cancel{e^{i\Omega t}} = f_0\cancel{e^{i\Omega t}}$$

$$\Rightarrow z_0 = \frac{f_0}{\omega_0^2 + 2i\gamma\Omega - \Omega^2}$$

useful to express this as $z_0 = a(\Omega)e^{i\delta(\Omega)}f_0$, $\left\{\begin{array}{l} a(\Omega) = \dfrac{1}{\sqrt{(\omega_0^2-\Omega^2)^2 + 4\gamma^2\Omega^2}} \\ \tan\delta(\Omega) = \dfrac{2\gamma\Omega}{\Omega^2 - \omega_0^2} \end{array}\right.$

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Prove this relation as follows:

write $$z_0 = \frac{f_0}{\omega_0^2 - \Omega^2 + 2i\gamma\Omega} = \frac{f_0}{\omega_0^2 - \Omega^2 + 2i\gamma\Omega} \times \frac{\omega_0^2 - \Omega^2 - 2i\gamma\Omega}{\omega_0^2 - \Omega^2 - 2i\gamma\Omega}$$

$$= \frac{f_0(\omega_0^2 - \Omega^2)}{(\omega_0^2 - \Omega^2)^2 + 4\gamma^2\Omega^2} - i\,\cancel{2}\,\cancel{i}\,\frac{2 f_0 \gamma \Omega}{(\omega_0^2 - \Omega^2)^2 + 4\gamma^2\Omega^2}$$

next we'd like to express $z_0 = c - id$ as amplitude $\times$ phase.

$$\to \quad z_0 = |z_0| e^{i\delta}$$

$$= |z_0| (\cos\delta + i\sin\delta).$$

$$\Rightarrow \quad c = |z_0|\cos\delta \quad \& \quad d = -|z_0|\sin\delta.$$

$$\Rightarrow \quad |z_0|^2 = c^2 + d^2 = \frac{1}{(\omega_0^2 - \Omega^2)^2 + 4\gamma^2\Omega^2}$$

$$\& \quad \tan\delta = -\frac{d}{c} = -\frac{2\gamma\Omega}{\omega_0^2 - \Omega^2}$$

So, we have found $z(t) = z_0 e^{i\Omega t} = a(\Omega) f_0 e^{i(\Omega t + \delta(\Omega))}$

Taking real part of $\ddot{z} + 2\gamma\dot{z} + \omega_0^2 z = f_0 e^{i\Omega t}$:

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = \cancel{e} f_0 \cos \cancel{\cos} \Omega t, \qquad x = \mathrm{Re}\, z.$$

$\Rightarrow$ we have found a particular sol.'n $x_p(t) = \mathrm{Re}\, z(t)$

$$\boxed{x_p(t) = a(\Omega) f_0 \cos(\Omega t + \delta(\Omega))}$$