Small Oscillations

- Motion near a point of stable equilibrium.

DOF= 1 (one dimension)

- For a system of DOF = 1, with potential $U(q)$:
  - **stable equilibrium** at $U(q)_{\min}$ where $F = -\frac{dU}{dq} = 0$
    - restoring force for small displacements $q - q_0$ is $F = -\frac{dU(q-q_0)}{dq}$
- **Unstable equilibrium** at $U(q)_{\max}$ where $F = -\frac{dU}{dq} = 0$ as well.
- Consider small deviation from point of stable equilibrium, we use taylor expansion to show that it is really a small displacement. that is,

$$\begin{gathered} U(q) \approx U(q_0) + \frac{dU(q_0)}{dq}(q - q_0) + \left(\frac{1}{2}\right)\frac{d^2U(q_0)}{dq^2}(q - q_0)^2 + \dots \\ \text{while } \frac{dU(q_0)}{dq}(q - q_0) = 0 \end{gathered} \tag{1}$$

letting $x = q - q_0$, we have

$$\begin{cases} U(x) = U(q_0) + \left(\frac{1}{2}\right)\frac{d^2U(q_0)}{dq^2}x^2 \\ \text{also } U(x) = \left(\frac{1}{2}\right)kx^2. \end{cases} \Rightarrow \boxed{k = \frac{d^2U(q_0)}{dq^2} > 0} \tag{2}$$

we get KE, while choosing $U(q_0) = 0$:

$$\begin{gathered} T = \frac{1}{2}a(q)^2\dot{q}^2 = \frac{1}{2}a(q_0 + x)\dot{x}^2 \approx \frac{1}{2}m\dot{x}^2, \text{letting } \boxed{m = a(q_0)} \\ \Rightarrow L = T - U = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 \end{gathered} \tag{3}$$