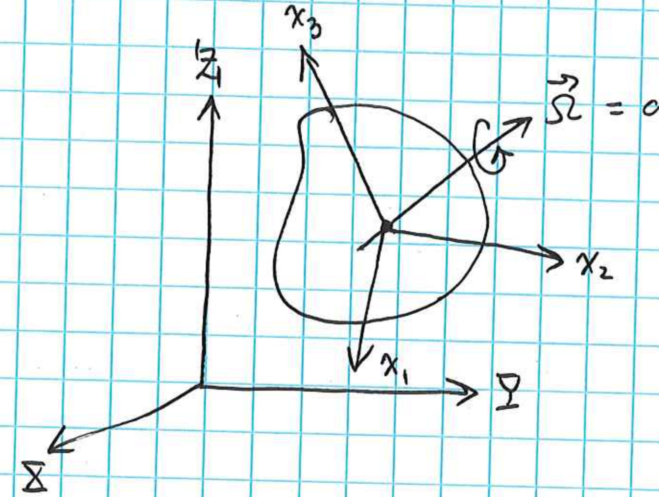
Summary



(XYZ) = fixed space frame

$(x_1 x_2 x_3)$ = moving body frame.

Euler equations

· A convenient way to analyze rigid body motion is in the frame co-moving w/ the body — e.g., principal axis frame. This is not an inertial frame.

· Writing EOM in moving frame → "Euler equations".

First, recall rigid body EOM in an inertial frame:

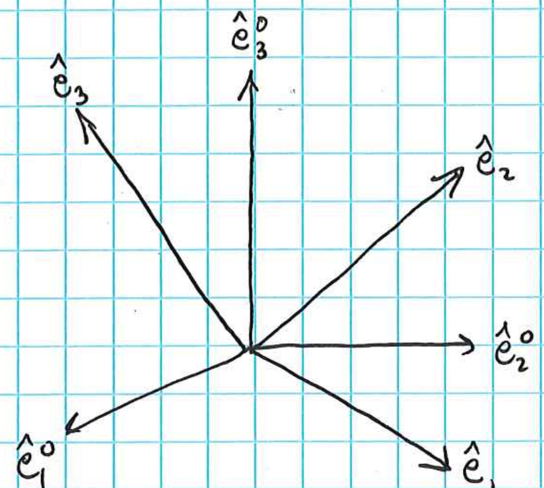
$$\begin{cases} \dot{\vec{P}} = \vec{F}, & \vec{F} = \sum \vec{f} = \text{net ext. force.} \\ \dot{\vec{L}} = \vec{K}, & \vec{K} = \sum \vec{r} \times \vec{f} = \text{" " torque.} \end{cases}$$

(2) 04/12/24

To get to Euler equations useful to change notation:

$$(X\,Y\,Z) \longrightarrow (\hat{e}_1^{\,o}, \hat{e}_2^{\,o}, \hat{e}_3^{\,o})$$

$$(x_1\, x_2\, x_3) \longrightarrow (\hat{e}_1, \hat{e}_2, \hat{e}_3).$$



(focus on rotn.'l motion → take origins of frames to coincide).

· Also important to get straight time dependence of various quantities:

→ If $\hat{e}_i$ = principal axes, then

$$\tilde{I}\,\hat{e}_i = I_i\,\hat{e}_i \,, \qquad \tilde{I} = \text{inertia tensor}$$

matrix-vector multiplication

principal moment of inertia, ordinary #.

when rigid body in motion: $\tilde{I} = \tilde{I}(t)$, $\hat{e}_i = \hat{e}_i(t)$.

$$\rightarrow \tilde{I}(t)\,\hat{e}_i(t) = I_i\,\hat{e}_i(t)$$

I_i = time-indep. #, characteristic of the body.

04/12/24

The t-dep. motion of $\hat{e}_i$ is:

$$\hat{e}_i(t) = R(t)\,\hat{e}_i^{\,\circ}$$

$R(t)$ = t-dep. rot.'n that maps fixed space frame to principal axes at time t.

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now, a general vector can be decomposed in comp.'s relative to either the fixed frame or the moving frame:

$$\vec{A} = \sum_i A_i^{\circ}\,\hat{e}_i^{\,\circ} = \sum_i A_i\,\hat{e}_i$$

(e.g., $\vec{A} \rightarrow \vec{L}, \vec{\Omega}$, etc.).

& consider the time rate of change of $\vec{A}$:

$$\frac{d\vec{A}}{dt} = \sum_i \frac{dA_i^{\circ}}{dt}\,\hat{e}_i^{\,\circ} = \sum_i \left(\frac{dA_i}{dt}\,\hat{e}_i + A_i \frac{d\hat{e}_i}{dt} \right)$$

$$= \underbrace{\sum_i \left(\frac{dA_i}{dt} \right) \hat{e}_i}_{\left(\frac{d\vec{A}}{dt}\right)_{body}} + \sum_i A_i \frac{d\hat{e}_i}{dt}$$

= time rate of change of $\vec{A}$ ~~body~~ as viewed from moving body frame (where $\frac{d\hat{e}_i}{dt} = 0$).

what is $\frac{d\hat{e}_i}{dt}$? Write $\hat{e}_i(t) = R(t)\,\hat{e}_i^{\,0}$ & consider 04/12/24

a change in time $t \to t+\delta t$. In the elapsed time δt, $\hat{e}_i(t)$ undergoes a ~~small~~ rot.'n by a small angle $\delta\varphi = \Omega\,\delta t$, ($\Omega$ = magnitude of instantaneous angular velocity) about the axis $\hat{\Omega}$ (~~so~~ direction of $\vec{\Omega}$ is along instantaneous axis of rot.'n).

$$\Rightarrow \underbrace{\hat{e}_i(t+\delta t)}_{\hat{e}_i(t) + \frac{d\hat{e}_i}{dt}\delta t} = \hat{e}_i(t) + \underbrace{(\Omega\,\delta t)\,\hat{\Omega}}_{\vec{\Omega}\,\delta t} \times \hat{e}_i(t)$$

$$\Rightarrow \boxed{\frac{d\hat{e}_i}{dt} = \vec{\Omega} \times \hat{e}_i(t)}$$

$$\Rightarrow \frac{d\vec{A}}{dt} = \left(\frac{d\vec{A}}{dt}\right)_{body} + \underbrace{\sum_i A_i\,\vec{\Omega} \times \hat{e}_i(t)}_{\vec{\Omega} \times \left(\sum_i A_i \hat{e}_i(t)\right) = \vec{\Omega} \times \vec{A}}$$

$$\Rightarrow \boxed{\left(\frac{d\vec{A}}{dt}\right)_{space} = \left(\frac{d\vec{A}}{dt}\right)_{body} + \vec{\Omega} \times \vec{A}}$$

→ relates time rate of change of vector viewed from fixed (inertial) space frame to that observed in moving (non-inertial) body frame.

04/12/24

Apply this to rigid body EOM (focus on rotational part)

$$\left(\frac{d\vec{L}}{dt}\right)_{space} = \vec{K}$$

$$\Rightarrow \left(\frac{d\vec{L}}{dt}\right)_{body} + \vec{\Omega} \times \vec{L} = \vec{K} .$$

Now express this in components. Let $\dot{L}_i = \hat{e}_i \cdot \left(\frac{d\vec{L}}{dt}\right)_{body}$:

$$\Rightarrow \dot{L}_i + \hat{e}_i \cdot (\vec{\Omega} \times \vec{L}) = K_i \qquad (i = 1, 2, 3).$$

w/ $\vec{\Omega} = \sum_i \Omega_i \hat{e}_i$, $\vec{L} = \sum_i L_i \hat{e}_i$.

$$\Rightarrow \dot{L}_1 + (\Omega_2 L_3 - \Omega_3 L_2) = K_1$$

$$\dot{L}_2 + (\Omega_3 L_1 - \Omega_1 L_3) = K_2$$

$$\dot{L}_3 + (\Omega_1 L_2 - \Omega_2 L_1) = K_3 .$$

& using $L_i = I_i \Omega_i$:

$$\boxed{\begin{aligned} I_1 \dot{\Omega}_1 + (I_3 - I_2)\Omega_2 \Omega_3 &= K_1 \\ I_2 \dot{\Omega}_2 + (I_1 - I_3)\Omega_3 \Omega_1 &= K_2 \\ I_3 \dot{\Omega}_3 + (I_2 - I_1)\Omega_1 \Omega_2 &= K_3 \end{aligned}}$$

"Euler equations".

→ set of non-linear coupled ODE's for $\vec{\Omega}$.

Ex: Free symmetric top (again). 04/12/24

$\vec{K} = 0, \qquad I_1 = I_2 \equiv I_\perp .$

$$\begin{cases} \dot{\Omega}_1 + \left(\dfrac{I_3 - I_\perp}{I_\perp}\right)\Omega_2\Omega_3 = 0. \\ \dot{\Omega}_2 + \left(\dfrac{I_\perp - I_3}{I_\perp}\right)\Omega_3\Omega_1 = 0. \\ \dot{\Omega}_3 = 0. \end{cases}$$

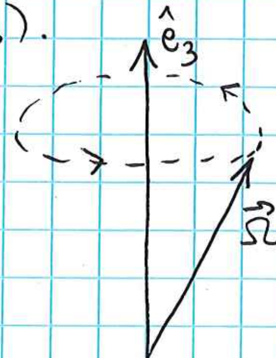
$$\Rightarrow \Omega_3 = \text{const.} \quad \& \quad \text{let} \quad \omega = \left(\frac{I_3 - I_\perp}{I_\perp}\right)\Omega_3 .$$

$$\Rightarrow \begin{cases} \dot{\Omega}_1 + \omega\Omega_2 = 0 \\ \dot{\Omega}_2 - \omega\Omega_1 = 0 \end{cases}$$

$$\Rightarrow \ddot{\Omega}_1 = -\omega\dot{\Omega}_2 = -\omega^2\Omega_1 \Rightarrow \Omega_1 = A\cos\omega t.$$

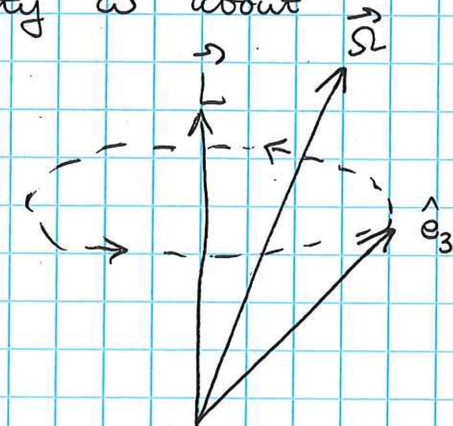
$$\& \quad \Omega_2 = -\frac{1}{\omega}\dot{\Omega}_1 = +A\sin\omega t.$$

So we find $\vec{\Omega}$ rotates w/ ang. velocity ω about axis of top ($\hat{e}_3$).



(body frame)

c.f.



(space frame)

$L_i = I_i\Omega_i \Rightarrow \vec{L}$ executes similar motion

Ex: When is rot.'n about a principal axis stable? (torque free)

Consider first uniform rot.'n about a principal axis: $\vec{\Omega} = \Omega_0 \hat{e}_3$.

→ this is a sol.'n to Euler eqn.'s (check it!).

Now perturb: $\vec{\Omega} = \Omega_0 \hat{e}_3 + \delta\vec{\Omega}$, $\delta\Omega \ll \Omega_0$.

Euler eqn.'s:
$$I_1 \delta\dot{\Omega}_1 = (I_2 - I_3)\delta\Omega_2(\Omega_0 + \delta\Omega_3) \simeq (I_2 - I_3)\Omega_0 \delta\Omega_2$$
$$I_2 \delta\dot{\Omega}_2 = (I_3 - I_1)(\Omega_0 + \delta\Omega_3)\delta\Omega_1 \simeq (I_3 - I_1)\Omega_0 \delta\Omega_1$$
$$I_3 \delta\dot{\Omega}_3 = (I_1 - I_2)\delta\Omega_1 \delta\Omega_2 \simeq 0.$$

$$\Rightarrow \delta\Omega_3 \simeq \text{const.}$$

$$\& \quad \delta\ddot{\Omega}_1 = \frac{(I_2 - I_3)}{I_1}\Omega_0 \delta\dot{\Omega}_2$$
$$= \frac{(I_2 - I_3)(I_3 - I_1)}{I_1 I_2}\Omega_0^2 \delta\Omega_1$$

$$\& \quad \delta\ddot{\Omega}_2 = \frac{(I_3 - I_1)(I_2 - I_3)}{I_2 I_1}\Omega_0^2 \delta\Omega_2.$$

$$\Rightarrow \delta\ddot{\Omega}_1 = -\omega^2 \delta\Omega_1, \quad \delta\ddot{\Omega}_2 = -\omega^2 \delta\Omega_2$$
$$\omega^2 = \frac{(I_3 - I_1)(I_3 - I_2)}{I_1 I_2}\Omega_0^2.$$

- If $\omega^2 > 0$ the motion is harmonic precession about $\hat{e}_3$. This occurs if $I_3 > I_1$ & $I_3 > I_2$ or if $I_3 < I_1$ & $I_3 < I_2$; i.e., if I_3 is either largest or smallest moment of inertia.
- If I_3 = middle moment ($I_1 < I_3 < I_2$ or $I_2 < I_3 < I_1$) then $\omega^2 < 0$ & motion is unstable.