

Brief Mechanics

Notes from Physics 311

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work in progress...

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1 Small Oscillations

- Motion near a point of stable equilibrium.

1.1 DOF= 1 (one dimension)

- For a system of DOF = 1, with potential $U(q)$:
 - **stable equilibrium** at $U(q)_{\min}$, upward parabola, where $F = -\frac{dU}{dq} = 0$
 - restoring force for small displacements $q - q_0$ is $F = -\frac{dU(q-q_0)}{dq}$
 - **Unstable equilibrium** at $U(q)_{\max}$, downward parabola, where $F = -\frac{dU}{dq} = 0$ as well.
- Consider small deviation from point of stable equilibrium, we use taylor expansion to show that it is really a small displacement. that is,

$$U(q) \approx U(q_0) + \frac{dU(q_0)}{dq}(q - q_0) + \left(\frac{1}{2}\right) \frac{d^2U(q_0)}{dq^2}(q - q_0)^2 + \dots \quad (1)$$

while $\frac{dU(q_0)}{dq}(q - q_0) = 0$

letting $x = q - q_0$, we have

$$\begin{cases} U(x) = U(q_0) + \left(\frac{1}{2}\right) \frac{d^2U(q_0)}{dq^2} x^2 \\ \text{putting into the form of } U(x) = U(x_0) + \left(\frac{1}{2}\right) kx^2. \end{cases} \quad (2)$$

$$\Rightarrow \boxed{k = \frac{d^2U(q_0)}{dq^2} > 0}$$

we get KE, while choosing $U(q_0) = 0$:

$$T = \frac{1}{2}a(q)^2\dot{q}^2 = \frac{1}{2}a(q_0 + x)\dot{x}^2 \approx \frac{1}{2}m\dot{x}^2, \text{ letting } m = a(q_0)$$

$$\Rightarrow \boxed{L = T - U = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2}$$
(3)

1.1.1 EOM for DOF = 1 small Oscillations

using EL on Equation 3, we can get the EOM for one dimensional small Oscillations:

$$m\ddot{x} = -kx$$

$$\Rightarrow \ddot{x} + \omega_0^2 x = 0, \text{ where } \boxed{\omega_0 = \sqrt{\frac{k}{m}} \text{ freq of osc.}}$$
(4)

by magic of ODE, EOM reduces down to:

$$\boxed{x(t) = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t)}$$
(5)

where C_1, C_2 are constants

by trig magic, this could also be written as

$$x(t) = a \cos(\omega_0 t + \alpha),$$

$$\text{where } \begin{cases} a = \sqrt{C_1^2 + C_2^2} & \text{amplitude of oscillation} \\ \omega_0 & \text{frequency of oscillation} \\ \tan \alpha = C_2/C_1 & \text{phase at } t=0 \end{cases}$$
(6)

1.1.2 energy for 1D small Oscillation

checking $\frac{\partial L}{\partial t} = 0 \Rightarrow$ energy-conservation:

$$E = T + U = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$$

$$= \frac{1}{2}ma^2\omega_0^2, [\text{constant}]$$
(7)

1.1.3 Damped 1D oscillation, and Complex representation

[I dont like the how the subscripts are used in this lecture but I guess this is what we are stuck with.]

- when there is damping (friction, resistance, etc) $F_{\text{fric}} = -\beta\dot{x}$, the EOM becomes:

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0,$$

$$\text{where } 2\gamma = \frac{\beta}{m}, \omega_0 = \sqrt{\frac{k}{m}}$$
(8)

with ansatz $x(t) = e^{rt}$, $\dot{x} = re^{rt}$, $\ddot{x} = r^2e^{rt}$, the solution to Equation 8 is:

$$r^2 + 2\gamma r + \omega_0^2 = 0,$$

$$\text{which has solution } r_+, r_- = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2} \quad (9)$$

$$\Rightarrow x(t) = C_1 e^{r_+ t} + C_2 e^{r_- t},$$

notice the r subscripts here: r_+, r_-

1.1.4 underdamped, overdamped, and critically damped

Recall from your ODE class...

Equation 9 has the following 3 cases, each with different physical interpretation:

1. underdamped:

$$\gamma < \omega_0 \Rightarrow 2 \text{ complex roots: } \begin{cases} r_{\pm} = -\gamma \pm i\sqrt{\omega_0^2 - \gamma^2} = -\gamma \pm i\omega \\ \omega = \sqrt{\omega_0^2 - \gamma^2} \end{cases} \quad (10)$$

The EOM is thus a linear combination of two complex exponentials:

$$\begin{aligned} x(t) &= e^{-\gamma t} (C_1 e^{i\omega t} + C_2 e^{-i\omega t}) \\ &= e^{-\gamma t} (A \cos(\omega t) + B \sin(\omega t)) \\ &\text{-- where } \begin{cases} A = C_1 + C_2 \\ B = i(C_1 - C_2) \end{cases} \quad (11) \\ &= a e^{-\gamma t} \cos(\omega t + \alpha) \\ &a, \alpha \text{ are constants} \end{aligned}$$

“The solution is a damped oscillation with frequency ω , and amplitude exponentially decaying with time.”

2. Overdamped

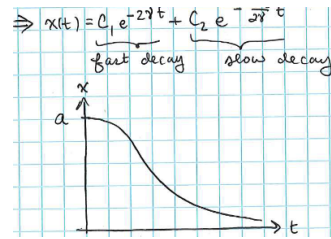
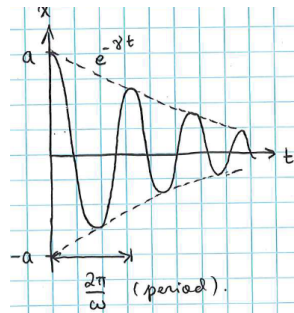
$$\gamma > \omega \Rightarrow x(t) = c_1 e^{-\gamma + \sqrt{\gamma^2 - \omega^2} t} + c_2 e^{-\gamma - \sqrt{\gamma^2 - \omega^2} t} \quad (12)$$

When

$$\begin{aligned} \gamma \gg \omega_0, \Rightarrow \begin{cases} \gamma + \sqrt{\gamma^2 - \omega_0^2} \approx 2\gamma \\ \gamma - \sqrt{\gamma^2 - \omega_0^2} \approx \frac{\omega_0^2}{2\gamma} \end{cases} \quad (13) \\ x(t) = c_1 e^{-2\gamma t} + c_2 e^{(-\omega_0^2/2\gamma)t} \end{aligned}$$

3. Critically damped

$$\gamma = \omega_0 \Rightarrow x(t) = c_1 e^{-\gamma t} + c_2 t e^{-\gamma t} \quad (14)$$



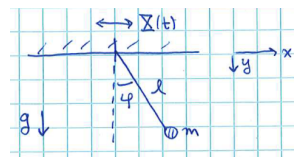
1.1.5 Forced Oscillations

When external force (F) is applied to the system, the lagrangian becomes

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 + F(t)x \quad (15)$$

$$\text{EL} \Rightarrow \ddot{x} + \omega_0^2 x = \frac{F(t)}{m}, \text{ where } \omega_0 = \sqrt{\frac{k}{m}}$$

- Example: Simple pendulum with moving pivot



$$\begin{cases} x = X + l \sin \varphi \\ y = l \cos \varphi \end{cases} \Rightarrow \begin{cases} \dot{x} = \dot{X} + l\dot{\varphi} \cos \varphi \\ \dot{y} = -l\dot{\varphi} \sin \varphi \end{cases}$$

$$\Rightarrow L = T - U$$

$$L = \frac{1}{2}ml^2\dot{\varphi}^2 - mgl(1 - \cos \varphi) - ml\ddot{X} \sin \varphi$$

$$\text{Expand ab. } \varphi = 0 \Rightarrow L = \frac{1}{2}ml^2\dot{\varphi}^2 - \frac{1}{2}mgl\varphi^2 - ml\ddot{X}\varphi \quad (17)$$

$$\text{EL} \Rightarrow \ddot{\varphi} + \omega_0^2 \varphi = -\frac{\ddot{X}}{l}, \text{ where } \omega_0 = \sqrt{\frac{g}{l}}$$

1.1.6 reintroducing damping via external forcing

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = f(t), f(t) = \frac{F(t)}{m} \quad (18)$$

When damping $f(t) = f_0 \cos(\Omega t)$, solution via complex number:

$$\ddot{z} + 2\gamma\dot{z} + \omega_0^2 z = f_0 e^{i\Omega t}$$

$$\text{ansatz } z(t) = z_0 e^{i\Omega t} \Rightarrow z_0 = \frac{f_0}{\omega_0^2 + 2i\gamma\Omega + \Omega^2}$$

$z_0 = a(\Omega) \cos(\Omega t + \delta(\Omega)) f_0$ is a particular solution, where

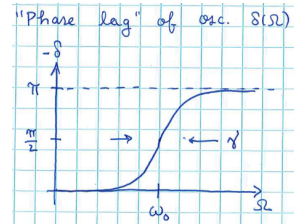
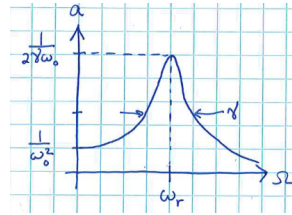
$$\begin{cases} a(\Omega) = \frac{1}{\sqrt{(\omega_0^2 - \Omega^2)^2 + (2\gamma\Omega)^2}} \\ \delta(\Omega) = \arctan\left(2\gamma\frac{\Omega}{\omega_0^2 - \Omega^2}\right) \end{cases}$$

We can study the properties of the system by looking at the amplitude and phase of the solution.

- Amplitude:

$$a(\Omega) = \frac{1}{\sqrt{(\omega_0^2 - \Omega^2)^2 + (2\gamma\Omega)^2}} \quad (20)$$

, when $\gamma \ll \omega_0$, response strongest and amplitude largest when $\omega_r = \omega_0$.



- Phase lag: $\tan \delta(\Omega) = 2\gamma \frac{\Omega}{\Omega^2 - \omega_0^2}$

in phase as $\Omega \rightarrow 0$, and out of phase as $\Omega \rightarrow \omega_0$.

- Genral solution to sinusoidal forcing:

$$\begin{aligned} x(t) &= a(\Omega)f_0 \cos(\Omega t + \delta(\Omega)) + a_0 e^{-\gamma t} \cos(\omega t + \alpha) \\ &\xrightarrow{t > \frac{1}{\gamma}} a(\Omega)f_0 \cos(\Omega t + \delta(\Omega)) \end{aligned} \quad (21)$$

Forgets initial condition after time.

- Power obsorbed by oscillation

$p = F\dot{x} = m f \dot{x}$ Avg power

$$\begin{aligned} P_{\text{avg}} &= \frac{1}{T} \int_0^T m f \dot{x} dt = -\frac{1}{2} m f_0 a(\Omega) \Omega \sin \delta(\Omega) \\ P(\Omega) &= \gamma m f_0^2 \Omega^2 a^2(\Omega) \end{aligned} \quad (22)$$

Absorption around resonance frequency $\Omega = \omega_0 + \varepsilon$ is maximum:

$$P = \frac{\gamma m f_0^2}{4(\varepsilon^2 + \gamma^2)} \approx \frac{m f_0^2}{4\gamma} \quad (23)$$

1.2 Oscillations DOF>1

For a system with n DOF: $q = (q_1, q_2, \dots, q_n)$, PE = $U(q)$

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2 Appendix

1. Taylor expansion:

$$f(x)|_0 \approx f(a) + f'(a)(x-a) + f''(a) \frac{(x-a)^2}{2} \quad 24$$

2. small angle approximation:

$$\sin(\theta) \approx \theta \quad \cos(\theta) \approx 1 - \frac{\theta^2}{2} \qquad 25$$