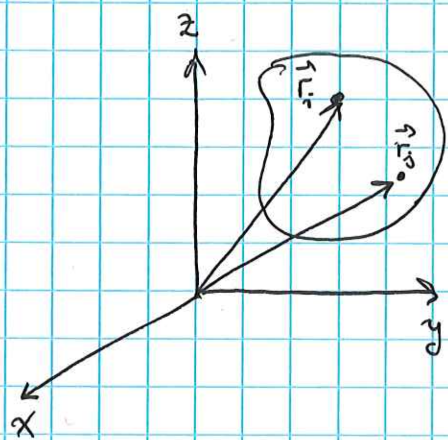
Motion of a Rigid Body

03/18/24

Rigid body = system of N particles s.t. distances btwn. particles are fixed.



$|\vec{r}_i - \vec{r}_j| = \text{const.}$

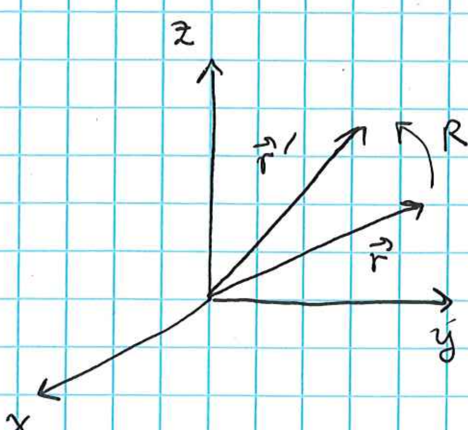
These constraints significantly reduce the # of DOF.

# of DOF : $3N \rightarrow 6 = 3$ DOF for COM

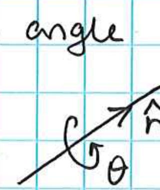
$+ 3$ DOF for motion rel. to COM.

Q: How do we choose generalized coord.'s to describe such motion?

Recall some facts about rotations:



- $\vec{r}\,' = R\vec{r}$, R = rotation s.t. $|\vec{r}\,'|^2 = |\vec{r}|^2$
- Rot.'n parametrized by axis of rotation $\hat{n}$ & angle of rotation $\theta \rightarrow R = R(\hat{n}, \theta)$



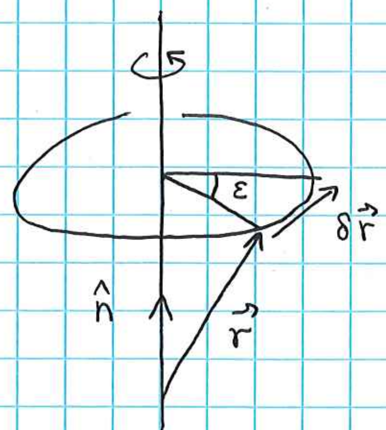
03/18/24

$(\hat{n}, \theta) =$ 3 #'s to specify rot.'n

= 2 #'s for direction $\hat{n}$

+ 1 # for angle θ.

· Infinitesimal rot.'n = rot.'n about axis $\hat{n}$ by small angle $\varepsilon \ll 1$.

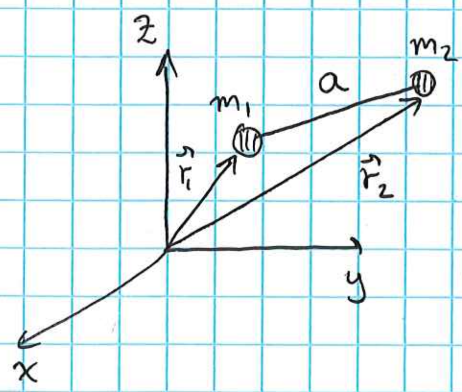


$$\delta \vec{r} = \varepsilon \, \hat{n} \times \vec{r}$$

$$\vec{r}\,' = R\vec{r} = \vec{r} + \delta\vec{r} \simeq \vec{r} + \varepsilon \hat{n} \times \vec{r}.$$

To address question of generalized coord.'s for rigid body motion, we'll start w/ an example.

<u>Ex:</u> Two masses connected by rigid rod of length "a". (rotor)



$$L = \frac{1}{2} m_1 \dot{\vec{r}}_1^{\,2} + \frac{1}{2} m \dot{\vec{r}}_2^{\,2}$$

To implement constraint $|\vec{r}_1 - \vec{r}_2| = a$,

Change coords:

$$\left.\begin{aligned} \vec{r} &= \vec{r}_1 - \vec{r}_2 \\ \vec{R} &= \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2} \end{aligned}\right\} \qquad \left.\begin{aligned} \vec{r}_1 &= \vec{R} + m_2 \vec{r}/M \\ \vec{r}_2 &= \vec{R} - m_1 \vec{r}/M \end{aligned}\right\}$$

$M = m_1 + m_2$

03/18/24

$$\Rightarrow L = \frac{1}{2} M \dot{\vec{R}}^2 + \frac{1}{2}\mu \dot{\vec{r}}^2 \qquad \mu = m_1 m_2 / M.$$

express $\vec{r}$ in polar coord.'s: $(r\theta\varphi)$

$$\dot{\vec{r}}^2 = \dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2\theta\, \dot{\varphi}^2$$

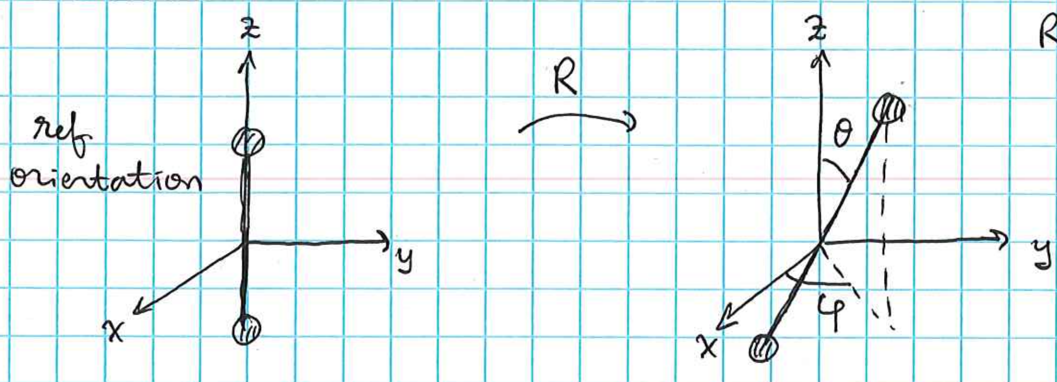
$r = a = \text{const.}$

$$= a^2(\dot{\theta}^2 + \sin^2\theta\,\dot{\varphi}^2).$$

$$\Rightarrow L = \frac{1}{2} M \dot{\vec{R}}^2 + \frac{1}{2}\mu a^2(\dot{\theta}^2 + \sin^2\theta\,\dot{\varphi}^2).$$

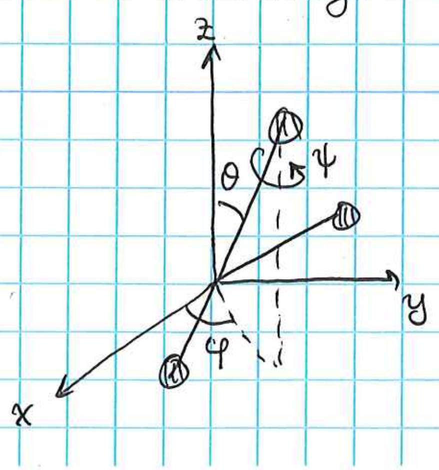
→ angles $(\theta\varphi)$ specify orientation of rotor relative to origin located at COM.

· Orientation specified by $(\theta\varphi)$ can be parametrized by a rotation that maps a "reference orientation" to the desired orientation:



$R(\theta,\varphi) = R(\hat{z},\varphi)R(\hat{y},\theta)$

· Now, rotor is special b/c rotation through its axis leaves rotor unchanged. Consider adding one more mass:



→ need one more angle ψ to put system in desired orientation

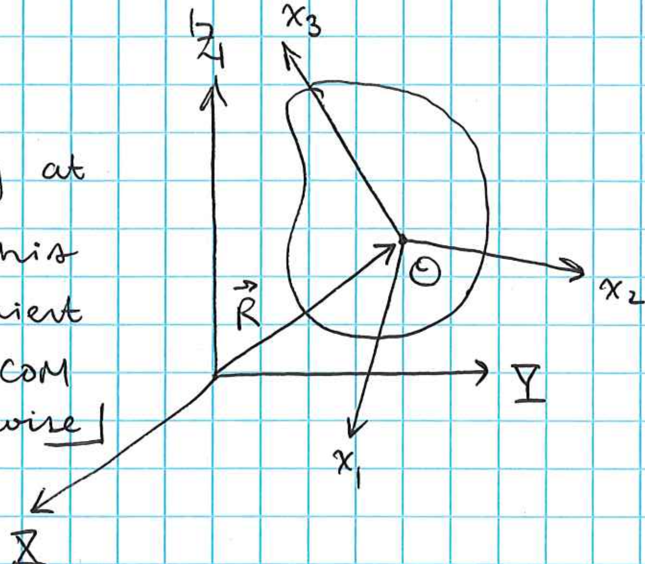
$$\Rightarrow R = R(\theta,\varphi,\psi).$$

03/18/24

The previous example captures essential features for a general rigid body:

Require 3 DoF to specify orientation of a frame moving w/ the body, relative to a fixed external frame.

[$\vec{R}$ not necessarily at COM, although this is often most convenient choice. Assume $\vec{R}$ = COM unless stated otherwise]



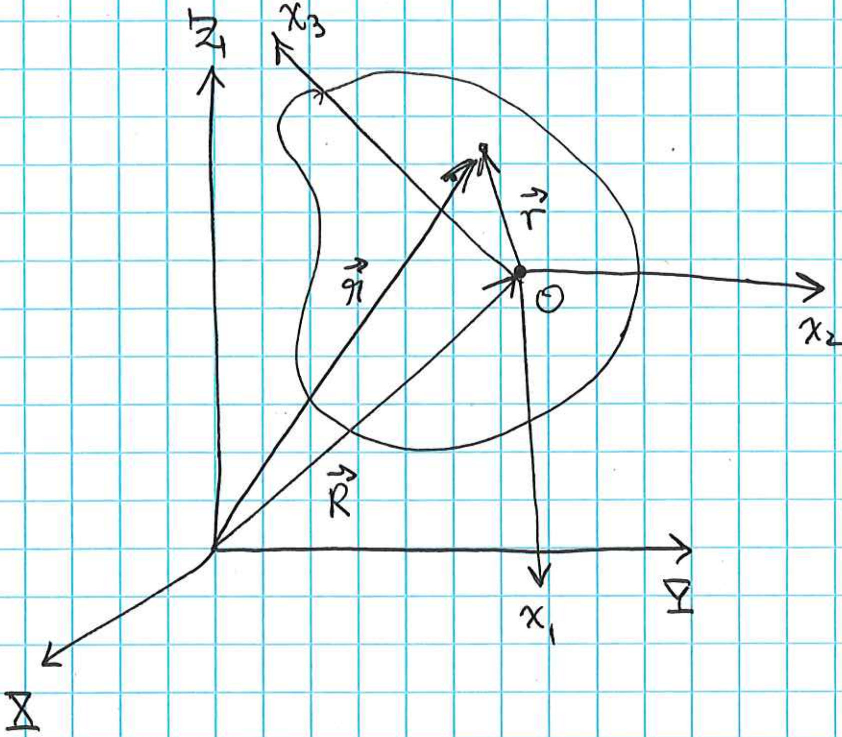
(XYZ) = fixed inertial frame. ("lab" or "space" frame).

$(x_1\, x_2\, x_3)$ = frame fixed in the body. This frame moving (translation + rotation) w/ the body.

$$(XYZ) \xrightarrow[R(\theta,\varphi,\psi)]{} (x_1\, x_2\, x_3)$$

specify this rotation later.

Q: How do we describe dynamics of the rigid body?



$\vec{\mathfrak{r}} = \vec{R} + \vec{r}$, $\vec{r}$ = position vector in the moving ("body") system

· Consider infinitesimal displacement of pt $\vec{\mathfrak{r}}$ in the rigid body:

$$\vec{\mathfrak{r}} \longrightarrow \vec{\mathfrak{r}} + d\vec{\mathfrak{r}}$$

$$d\vec{\mathfrak{r}} = d\vec{R} + d\vec{r}$$

· now, $|\vec{r}|$ = fixed in the moving frame (since we deal w/ a rigid body) & hence $d\vec{r}$ = infinitesimal rotation.

$\Rightarrow$ write $d\vec{r} = d\vec{\varphi} \times \vec{r}$ w/ $d\vec{\varphi}$ along instantaneous axis of rotation & $\cancel{|d\vec{\varphi}|}$.

$|d\vec{\varphi}|$ = infinitesimal rot.'n angle.

$\cancel{\Rightarrow d\vec{\mathfrak{r}} = d\vec{R} + d\vec{\varphi} \times \vec{r}}$.

$$\Rightarrow d\vec{\mathfrak{r}} = d\vec{R} + d\vec{\varphi} \times \vec{r}.$$

Dividing through by dt: 03/18/24

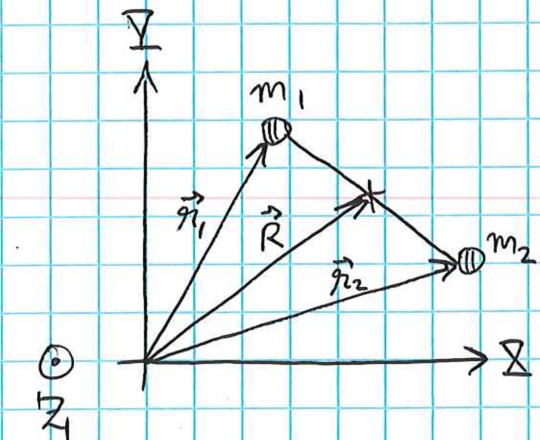
$$\boxed{\vec{v} = \frac{d\vec{r}}{dt} = \vec{V} + \vec{\Omega}\times\vec{r}} \qquad (*)$$

$\vec{V} = \dfrac{d\vec{R}}{dt}$ = velocity of COM $\leftrightarrow$ translational motion

$\vec{\Omega} = \dfrac{d\vec{\varphi}}{dt}$ = "angular velocity" $\leftrightarrow$ rotational motion

$\Rightarrow$ motion of a pt. in the rigid body = translational motion of COM + rotational motion about an axis through the COM.

<u>Ex:</u> (rotor in the plane − checking (*))



$\vec{r} = \vec{r}_1 - \vec{r}_2$

$\vec{R} = \dfrac{m_1\vec{r}_1 + m_2\vec{r}_2}{M}$, $\quad M = m_1 + m_2$

$$\Rightarrow \begin{cases} \vec{r}_1 = \vec{R} + \vec{r}_1, & \vec{r}_1 = m_2\vec{r}/M \\ \vec{r}_2 = \vec{R} + \vec{r}_2 & \vec{r}_2 = -m_1\vec{r}/M. \end{cases}$$

$$\Rightarrow \vec{v}_1 = \dot{\vec{r}}_1 = \dot{\vec{R}} + \dot{\vec{r}}_1, \qquad \dot{\vec{r}}_1 = \frac{d}{dt}(r_1\hat{r}) = \dot{r}_1\hat{r} + r_1\dot{\hat{r}}$$

r_1 = const.

$$= r_1\dot{\hat{r}}$$

$\dot{\hat{r}} = \dot{\varphi}\hat{\varphi}$.

$$= r_1\dot{\varphi}\hat{\varphi}$$

$$\Rightarrow \vec{v}_1 = \dot{\vec{R}} + r_1\dot{\varphi}\hat{\varphi}.$$

Check against (*): $\vec{\Omega} = \dot{\varphi}\hat{z} \Rightarrow \vec{\Omega}\times\vec{r}_1 = \dot{\varphi}\,\hat{z}\times(r_1\hat{r}) = r_1\dot{\varphi}(\hat{z}\times\hat{r}) = r_1\dot{\varphi}\hat{\varphi}$

$$\Rightarrow \vec{v}_1 = \vec{V} + \vec{\Omega}\times\vec{r}_1 = \dot{\vec{R}} + r_1\dot{\varphi}\hat{\varphi} \quad \checkmark.$$