6.34

Consider a random point (X, Y) uniformly distributed over the quadrilateral region D with vertices at $(0,0)$, $(2,0)$, $(1,1)$, and $(0,1)$.

(a) Given that the area of D equals $\frac{3}{2}$, the joint probability density function is:

$$f_{X,Y}(x,y) = \begin{cases} \frac{2}{3} & \text{for } (x,y) \in D, \\ 0 & \text{otherwise.} \end{cases}$$

Note that the boundary of D includes a line segment from $(1,1)$ to $(2,0)$, described by $y = 2 - x$. We can derive the marginal density functions as follows:

For the marginal density of X:

$$f_X(x) = \begin{cases} 0, & x \leq 0 \text{ or } x \geq 2, \quad (1) \\ \displaystyle\int_0^1 \frac{2}{3}\, dy = \frac{2}{3}, & 0 < x \leq 1, \quad (2) \\ \displaystyle\int_0^{2-x} \frac{2}{3}\, dy = \frac{4}{3} - \frac{2}{3}x, & 1 < x < 2. \quad (3) \end{cases}$$

To verify that this is a valid density function:

$$\begin{aligned} \int_{-\infty}^{\infty} f_X(x)\, dx &= \int_0^1 \frac{2}{3}\, dx + \int_1^2 \left(\frac{4}{3} - \frac{2}{3}x\right) dx \\ &= 1, \end{aligned}$$

confirming that $f_X(x)$ is indeed a proper density function.

Similarly, for the marginal density of Y:

$$f_Y(y) = \begin{cases} 0, & y \leq 0 \text{ or } y \geq 1, \\ \int_0^{2-y} \frac{2}{3}\, dx = \frac{4}{3} - \frac{2}{3}y, & 0 < y < 1. \end{cases}$$

(b) To find $\mathbb{E}[X]$ and $\mathbb{E}[Y]$:

$$\begin{aligned} \mathbb{E}[X] &= \int_{-\infty}^{\infty} x f_X(x)\, dx \\ &= \int_0^1 \frac{2}{3}x\, dx + \int_1^2 \left(\frac{4}{3}x - \frac{2}{3}x^2\right) dx \\ &= \frac{7}{9}, \end{aligned}$$

and

$$\begin{aligned} \mathbb{E}[Y] &= \int_{-\infty}^{\infty} y f_Y(y)\, dy \\ &= \int_0^1 \left(\frac{4}{3}y - \frac{2}{3}y^2\right) dy \\ &= \frac{4}{9}. \end{aligned}$$

6.36

Suppose that the random variables X and Y have the joint probability density function

$$f(x,y) = ce^{-\frac{x^2}{2}-\frac{(x-y)^2}{2}}, \qquad x,y \in (-\infty,\infty),$$

where c is a constant.

(a) To determine the value of c, we require that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dxdy = 1.$$

Evaluating this double integral:

$$\begin{aligned}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} ce^{-\frac{x^2}{2}-\frac{(x-y)^2}{2}}\,dxdy &= c\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{2}}\,dydx \\ &= \sqrt{2\pi}c\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\,dx = 2\pi c.\end{aligned}$$

Setting this equal to 1 yields $c = \frac{1}{2\pi}$.

(b) The marginal density function of X is given by

$$\begin{aligned}f_X(x) &= \int_{-\infty}^{\infty} \frac{1}{2\pi}e^{-\frac{x^2}{2}-\frac{(x-y)^2}{2}}\,dy \\ &= \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{(x-y)^2}{2}}\,dy \\ &= \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}},\end{aligned}$$

which is the density of a standard normal random variable.

Similarly, for the marginal density of Y:

$$\begin{aligned}f_Y(y) &= \int_{-\infty}^{\infty} \frac{1}{2\pi}e^{-\frac{x^2}{2}-\frac{(x-y)^2}{2}}\,dx \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}-\frac{(x-y)^2}{2}}\,dx.\end{aligned}$$

Completing the square in the exponent and simplifying:

$$\begin{aligned}f_Y(y) &= \frac{1}{\sqrt{4\pi}}e^{-y^2/4}\int_{-\infty}^{\infty}\frac{1}{\sqrt{\pi}}e^{-(x-y/2)^2}\,dx \\ &= \frac{1}{\sqrt{4\pi}}e^{-y^2/4}.\end{aligned}$$

The last step uses the fact that $\frac{1}{\sqrt{\pi}}e^{-(x-y/2)^2}$ is the pdf of an $N(y/2,1)$ distributed random variable. Thus, $Y \sim N(0,2)$.

In summary, $X \sim N(0,1)$ and $Y \sim N(0,2)$.