Small Oscillations

- Motion near a point of stable equilibrium.

DOF= 1 (one dimension)

- For a system of DOF = 1, with potential $U(q)$:
  - **stable equilibrium** at $U(q)_{\text{min}}$, upward parabola, where $F = -\frac{\mathrm{d}U}{\mathrm{d}q} = 0$
    - restoring force for small displacements $q - q_0$ is $F = -\frac{\mathrm{d}U(q-q_0)}{\mathrm{d}q}$
- **Unstable equilibrium** at $U(q)_{\text{max}}$, downward parabola, where $F = -\frac{\mathrm{d}U}{\mathrm{d}q} = 0$ as well.
- Consider small deviation from point of stable equilibrium, we use taylor expansion to show that it is really a small displacement. that is,

$$U(q) \approx U(q_0) + \frac{\mathrm{d}U(q_0)}{\mathrm{d}q}(q - q_0) + \frac{\mathrm{d}^2U(q_0)}{2\,\mathrm{d}q^2}(q - q_0)^2 + \dots \quad \text{while } \frac{\mathrm{d}U(q_0)}{\mathrm{d}q}(q - q_0) = 0 \tag{1}$$

letting $x = q - q_0$, we have

$$\begin{cases} U(x) = U(q_0) + \left(\frac{1}{2}\right)\frac{\mathrm{d}^2U(q_0)}{\mathrm{d}q^2}x^2 \\ \text{putting into the form of } U(x) = U(x_0) + \left(\frac{1}{2}\right)kx^2. \end{cases} \tag{2}$$

$$\Rightarrow \boxed{k = \frac{\mathrm{d}^2U(q_0)}{\mathrm{d}q^2} > 0}$$

we get KE, while choosing $U(q_0) = 0$:

$$T = \frac{1}{2}a(q)^2\dot{q}^2 = \frac{1}{2}a(q_0 + x)\dot{x}^2 \approx \frac{1}{2}m\dot{x}^2, \overset{m=a(q_0)}{\Rightarrow} \boxed{L = T - U = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2} \tag{3}$$

EOM for DOF = 1 small Oscillations

using EL on Equation 3, we can get the EOM for one dimensional small Oscillations:

$$m\ddot{x} = -kx \quad \Rightarrow \ddot{x} + \omega_0^2 x = 0, \text{where } \boxed{\omega_0 = \sqrt{\frac{k}{m}} \text{ freq of osc.}} \tag{4}$$

by magic of ODE, EOM reduces down to:

$$\boxed{x(t) = C_1\cos(\omega_0 t) + C_2\sin(\omega_0 t)} \quad \text{where } C_1, C_2 \text{ are constants} \tag{5}$$

by trig magic, this could also be written as

$$x(t) = a\cos(\omega_0 t + \alpha), \quad \text{where} \begin{cases} a = \sqrt{C_1^2 + C_2^2} & \text{amplitude of oscillation} \\ \omega_0 & \text{frequency of oscillation} \\ \tan\alpha = C_2/C_1 & \text{phase at t=0} \end{cases} \tag{6}$$

energy for 1D small Oscillation

checking $\frac{\partial L}{\partial t} = 0 \Rightarrow$ energy-conservation:

$$\begin{aligned} E = T + U &= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 \\ &= \frac{1}{2}ma^2\omega_0^2, [\text{ constant}] \end{aligned} \tag{7}$$

Damped 1D oscillation, and Complex representation

- when there is damping (friction, resistence, etc) $F_{\text{fric}} = -\beta\dot{x}$, the EOM becomes:

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0, \quad \text{where } 2\gamma = \frac{\beta}{m}, \omega_0 = \sqrt{\frac{k}{m}} \tag{8}$$

with ansatz $x(t) = e^{rt}, \dot{x} = re^{rt}, \ddot{x} = r^2e^{rt}$, the solution to Equation 8 is:

$$r^2 + 2\gamma r + \omega_0^2 = 0, \quad \text{which has solution } r_+, r_- = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2} \quad \Rightarrow x(t) = C_1e^{r_+t} + C_2e^{r_-t}, \tag{9}$$

notice the r subscripts here: r_+, r_-

underdamped, overdamped, and critically damped

Recall from your ODE class…

Equation 9 has the following 3 cases, each with different physical interpretation:

1. underdamped:

$$\gamma < \omega_0 \Rightarrow 2 \text{ complex roots: } \begin{cases} r_\pm = -\gamma \pm i\sqrt{\omega_0^2 - \gamma^2} \\ = -\gamma \pm i\omega \\ \omega = \sqrt{\omega_0^2 - \gamma^2} \end{cases} \tag{10}$$

The EOM is thus a linear combination of two complex expoentials:

$$\begin{aligned} x(t) &= e^{-\gamma t}(C_1e^{i\omega t} + C_2e^{-i\omega t}) \\ &= e^{-\gamma t}(A\cos(\omega t) + B\sin(\omega t)) \\ &\text{-- where } \begin{cases} A = C_1 + C_2 \\ B = i(C_1 - C_2) \end{cases} \\ &= ae^{-\gamma t}\cos(\omega t + \alpha) \\ &a, \alpha \text{ are constants} \end{aligned} \tag{11}$$

"The solution is a damped oscillation with frequencyω, and amplitude expoentially decaying with time."

2. Overdameped

$$\gamma > \omega \Rightarrow x(t) = c_1e^{-\gamma + \sqrt{\gamma^2 - \omega^2}t} + c_2e^{-\gamma - \sqrt{\gamma^2 - \omega^2}t} \tag{12}$$

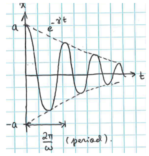
When

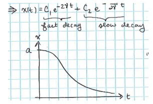
$$\gamma \gg \omega_0, \Rightarrow \begin{cases} \gamma + \sqrt{\gamma^2 - \omega_0^2} \approx 2\gamma \\ \gamma - \sqrt{\gamma^2 - \omega^2} = \frac{\omega^2}{2\gamma} \end{cases} \tag{13}$$

$$x(t) = c_1e^{-2\gamma t} + c_2e^{(-\omega_0^2/2\gamma)t}$$

3. Critically damped

$$\gamma = \omega_0 \Rightarrow x(t) = c_1e^{-\gamma t} + c_2te^{-\gamma t} \tag{14}$$



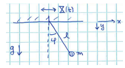


Forced Oscillations

When external force (F) is applied to the system, the largrangian becomes

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 + F(t)x \quad \text{EL} \Rightarrow \ddot{x} + \omega_0^2 x = \frac{F(t)}{m}, \text{where } \omega_0 = \sqrt{\frac{k}{m}} \tag{15}$$

- Example: Simple pendulum with moving pivot



$$\begin{cases} x = X + l\sin\varphi \\ y = l\cos\varphi \end{cases} \Rightarrow \begin{cases} \dot{x} = \dot{X} + l\dot{\varphi}\cos\varphi \\ \dot{y} = -l\dot{\varphi}\sin\varphi \end{cases} \tag{16}$$

$$\Rightarrow L = T - U$$

$$L = \frac{1}{2}ml^2\dot{\varphi}^2 - mgl(1 - \cos\varphi) - ml\ddot{X}\sin\varphi$$

$$\text{Expand ab. } \varphi = 0 \Rightarrow L = \frac{1}{2}ml^2\dot{\varphi}^2 - \frac{1}{2}mgl\varphi^2 - ml\ddot{X}\varphi \tag{17}$$

$$\text{EL} \Rightarrow \boxed{\ddot{\varphi} + \omega_0^2\varphi = -\frac{\ddot{X}}{l} \text{ ,where } \omega_0 = \sqrt{\frac{g}{l}}}$$

reintroducing damping via external forcing

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = f(t), f(t) = \frac{F(t)}{m} \tag{18}$$

When damping $f(t) = f_0\cos(\Omega t)$, solution via complex number:

$$\ddot{z} + 2\gamma\dot{z} + \omega_0^2 = f_0e^{i\Omega t}$$

$$\text{ansatz } z(t) = z_0e^{i\Omega t} \Rightarrow z_0 = \frac{f_0}{\omega_0^2 + 2i\gamma\Omega + \Omega_0^2}$$

$$\boxed{z_0 = a(\Omega)\cos(\Omega t + \delta(\Omega))f_0} \text{ is a partcular solution, where} \tag{19}$$

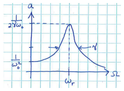
$$\begin{cases} a(\Omega) = \frac{1}{\sqrt{(\omega_0^2 - \Omega^2)^2 + (2\gamma\Omega)^2}} \\ \delta(\Omega) = \arctan\left(2\gamma\frac{\Omega}{\omega_0^2 - \Omega^2}\right) \end{cases}$$

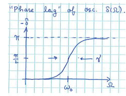
We can study the properties of the system by looking at the amplitude and phase of the solution.

- Amplitude:

$$a_{(\Omega)} = \frac{1}{\sqrt{(\omega_0^2 - \Omega^2)^2 + (2\gamma\Omega)^2}} \tag{20}$$

, when $\gamma \ll \omega_0$, response strongest and amplitude largest when $\omega_r = \omega_0$.





- Phase lag: $\tan\delta(\Omega) = 2\gamma\frac{\Omega}{\Omega^2 - \omega_0^2}$

  in phase as $\Omega \to 0$, and out of phase as $\Omega \to \omega_0$.
- Genral solution to sinusoidal forcing:

$$\begin{aligned} x(t) &= a(\Omega)f_0\cos(\Omega t + \delta(\Omega)) + a_0e^{-\gamma t}\cos(\omega t + \alpha) \\ &\overset{t>\frac{1}{r}}{\rightarrow} a(\Omega)f_0\cos(\Omega t + \delta(\Omega)) \end{aligned} \tag{21}$$

  Forgets initial condition after time.
- Power obsorbed by oscillation

  $$p = F\dot{x} = mf\dot{x}$$

  Avg power of oscillation

$$P_{\text{avg}} = \frac{1}{T}\int_0^T mf\dot{x}\,\mathrm{d}t = -\frac{1}{2}mf_0a(\Omega)\Omega\sin\delta(\Omega) \quad \text{simplifies to } P_{\text{avg}}(\Omega) = \gamma mf_0^2\Omega^2a_{(\Omega)}^2 \tag{22}$$

  Absorption around resonance frequency $\Omega = \omega_0 + \varepsilon$ is maximum:

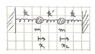
$$P = \frac{\gamma mf_0^2}{4(\varepsilon^2 + \gamma^2)} \approx \frac{mf_0^2}{4\gamma} \tag{23}$$

Oscillations DOF>1

For a system with n DOF: $q = (q_1, q_2, \dots, q_n), \mathrm{PE} = U(q)$

- Stable equilibrium $\frac{\partial U(q)}{\partial q_i}\Big|_{q=0}$

Example: Oscillation with 2 mass and 3 springs



$$\begin{aligned} L = &\tfrac{1}{2}m\dot{x}_1 + \tfrac{1}{2}m\dot{x}_2 - \tfrac{1}{2}kx_1^2 \\ &-\tfrac{1}{2}kx_2^2 - \tfrac{1}{2}k'(x_1 - x_2)^2 \end{aligned}$$

EOM:

$$\begin{aligned} &M \cdot \ddot{\vec{x}} = -K\vec{x} \ \text{, where } M = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}, \\ &\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, K = \begin{pmatrix} k + k' & -k' \\ -k' & k + k' \end{pmatrix} \end{aligned} \tag{24}$$

ansatz: $\vec{x} = \mathrm{Re}\left[\vec{a}e^{i\omega t}\right]$ Then the EOM eq becomes solving the eigenvalue problem:

$$\det(\omega^2 M - K) = 0$$

$$\Rightarrow \begin{cases}\omega_-^2 = \frac{k}{m} \\ \omega_+^2 = \frac{k+2k'}{m}\end{cases} \begin{cases}\overrightarrow{x_-} = a_-\begin{pmatrix}1\\1\end{pmatrix}\cos(\omega_- t+\delta_-) \\ \overrightarrow{x_+} = a_+\begin{pmatrix}1\\-1\end{pmatrix}\cos(\omega_+ t+\delta_+)\end{cases} \tag{25}$$

with constants $a_-, a_+, \delta_-, \delta_+$.

New Coords

$$\begin{cases}Q_1 = \sqrt{\frac{m}{2}}(x_1+x_2)\\ Q_2 = \sqrt{\frac{m}{2}}(x_1-x_2)\end{cases}$$

$$\Rightarrow L = \frac{1}{2}\left(\dot{Q_1}^2+\dot{Q_2}^2\right) - \frac{1}{2}(\omega_-^2 Q_1^2 + \omega_+^2 Q_2^2) \tag{26}$$

$$\overset{\text{E-L}}{\Rightarrow} \ddot{Q}_1 = -\omega_-^2 Q_1, \ddot{Q}_2 = -\omega_+^2 Q_2$$

Decoupled oscillators with coords Q_1, Q_2.

General Coords

for general coords q_i, let $x_i = q_i - q_i^{(0)}$

$$(q) = \frac{1}{2}\sum_{i,j} k_{ij}x_i x_j, \quad k_{ij}=k_{ji} = \frac{\partial^2 U(q)}{\partial q_i \partial q_j} \text{ symmetric matrix}$$

$$T = \frac{1}{2}\sum_{i,j} m_{ij}\dot{x}_i \dot{x}_j, \quad m_{ij}=m_{ji}=a_{ij}(q^{(0)}) \tag{27}$$

the largrangian, in Matix form:

$$L = \frac{1}{2}\dot{\vec{x}}^T\cdot M\cdot\dot{\vec{x}} - \frac{1}{2} \vec{x}^T\cdot K\vec{x} \overset{\text{EL}}{\Longrightarrow} (\omega^2 M - K)\cdot\vec{a} = 0 \tag{28}$$

$\Rightarrow \det(\omega^2 M - K) = 0$ Solving the det for omega gives the normal freq (Eigenvalues)of system ω_α^2 . plug in Evalue into Equation 28 for eigenvec(normal modes) $\overrightarrow{a^\alpha}$ of system.

- General motion

$$x_i(t) = \sum_\alpha a_i^\alpha \mathrm{Re}[C_\alpha e^{i\omega_\alpha t}] \tag{29}$$

- EXAMPLE: Normal freq is given

$$\omega = \left\{0, \sqrt{2}\omega_0, \sqrt{3}\omega_0\right\}.$$

$$\omega = \sqrt{2}\omega_0 \Rightarrow a_1 = -a_3 = -a_2 = ae^{i\delta} \Rightarrow$$

$$\vec{\theta} = a(1\ -1\ -1)^T\cos\left(\sqrt{2}\omega_0 t+\delta\right) \tag{30}$$

$$\omega = \sqrt{3\omega_0} \Rightarrow a_1=0, a_2=-a_3=ae^{i\delta}\Rightarrow$$

$$\vec{\theta} = a(0\ 1\ -1)^T\cos\left(\sqrt{3}\omega_0 t+\delta\right)$$

- EXAMPLE: double pendulum

$$\begin{cases}x_1 = l_1\sin\varphi_1 \quad y_1 = -l_1\cos\varphi_1\\ x_2 = l_1\sin\varphi_1 + l_2\sin\varphi_2 \quad y_2 = l_1\cos\varphi_1 + l_2\cos\varphi_2\end{cases}$$

$$T = \frac{1}{2}m_1 l_1\dot{\varphi}^2 + \frac{1}{2}m_2(l_1^2\dot{\varphi_1}^2 + l_2^2\dot{\varphi_2}^2 + 2l_1l_2\dot{\varphi_1}\dot{\varphi_2}\cos(\varphi_1-\varphi_2)) \tag{31}$$

$$U = -m_1gl_1\cos\varphi_1 - m_2g(l_1\cos\varphi_1 + l_2\cos\varphi_2)$$

using $\cos\varphi \approx 1 - \frac{\varphi^2}{2}$

$$L = \frac{1}{2}(\dot{\varphi}_1\ \ \dot{\varphi}_2)\begin{pmatrix}(m_1+m_2)l_1^2 & m_2l_1l_2\\ m_2l_1l_2 & m_2l_2^2\end{pmatrix}(\dot{\varphi}_1\ \ \dot{\varphi}_2)$$

$$-\frac{1}{2}(\varphi_1\ \ \varphi_2)\begin{pmatrix}(m_1+m_2)l_1 g & 0\\0 & m_2gl_2\end{pmatrix}(\varphi_1\ \ \varphi_2)$$

$$= \frac{1}{2}\dot{\vec{\varphi}}^T M\cdot\dot{\vec{\varphi}} - \frac{1}{2}\vec{\varphi}^T K\vec{\varphi}$$

When $m_1=m_2=m, \quad l_1=l_2=l \Rightarrow \quad M = ml^2\begin{pmatrix}2&1\\1&1\end{pmatrix}, K = mgl\begin{pmatrix}2&0\\0&1\end{pmatrix}$

$$\det((\omega^2M-K)) = 0 \Rightarrow \omega^2 = \left(2\pm\sqrt{2}\omega_0^2\right)$$

$$\begin{pmatrix}a_1^-\\a_2^-\end{pmatrix} = C_-\begin{pmatrix}1\\\sqrt{2}\end{pmatrix}, \quad \begin{pmatrix}a_1^+\\a_2^+\end{pmatrix} = C_+\begin{pmatrix}1\\-\sqrt{2}\end{pmatrix} \tag{32}$$

Normal Coords

$\{x_i\} = \{Q_\alpha\}$, where $x_i = \sum_{\alpha=1}^n A_{i\alpha}Q_\alpha \Rightarrow$

$\sum_j(\omega_\alpha^2 m_{ij} - k_{ij}A_{jx}) = 0$

$\Rightarrow L = \frac{1}{2}\sum_{\alpha=1}^n\left(\dot{Q}^2{}_\alpha - \omega_\alpha^2 Q_\alpha^2\right) \overset{\text{EL}}{\Longrightarrow} \ddot{Q}_\alpha + \omega_\alpha^2 Q_\alpha = 0$

Motion of Rigid Body

- EXample: rotor

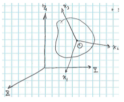
rotation with constraint $\left|\vec{r}_i - \vec{r}_j\right|$ .COM coords are useful here

$$\begin{cases}\vec{r} = \vec{r}_1 - \vec{r}_2\\ \vec{R} = \frac{m_1\vec{r_1}+m_2\vec{r_2}}{m_1+m_2}\end{cases}\Rightarrow\begin{cases}\vec{r}_1 = \vec{R} + m_2\vec{r}/M\\ \vec{r}_2 = \vec{R} - m_1\vec{r}/M\end{cases} \tag{33}$$

$$L = \frac{1}{2}M\dot{\vec{R}}^2 + \mu\dot{\vec{r}}^2, \quad \mu = m_1\frac{m_2}{m_1+m_2}$$

$$\overset{\text{polar}}{\Longrightarrow} L = \frac{1}{2}M\dot{\vec{R}}^2 + \frac{1}{2}\mu a^2\left(\dot{\theta}^2 + \dot{\varphi}^2\sin^2\theta\right) \tag{34}$$

frames of reference



$$(XYZ) \overset{R(\theta,\varphi,\psi)}{\Longrightarrow} (x_1,x_2,x_3)$$

Velocity of pt in body: $\vec{v} = \vec{V} + \vec{\Omega}\times\vec{r}$, where V is Translational vel, Omega is angular vel, r is position vector.

Largrangian for Rigid Body

$$T = \frac{1}{2}MV^2 + \frac{1}{2}\sum_a m_a\left[\Omega^2 r_a^2 - \left((\vec{\Omega})\vec{r_a}\right)^2)\right] \tag{35}$$

$$T\text{translational} + T\text{rotational}$$

consider rotation,

$$\Omega^2 = \sum_i \Omega_i^2, \quad \vec{\Omega}\cdot\vec{r_a} = \sum_i\Omega_i x_{a,i}$$

$$\Rightarrow T_{\text{rot}} = \frac{1}{2}\sum_{\text{i,j}}\Omega_i\Omega_j I_{\text{i,j}}. \quad I_{ij}\equiv\sum_a m_{a(\delta_{ij}r_a^2 - x_{a,i}x_{a,j})} \tag{36}$$

$\Rightarrow L = \frac{1}{2}MV^2 + \frac{1}{2}\sum_{i,j}I_{i,j}\Omega_i\Omega_j - U$

Inertial Tensor

- Discrete

$$I = \begin{pmatrix}\sum m(y^2+z^2) & -\sum mxy & -\sum mxz\\ -\sum mxy & \sum m(x^2+z^2) & -\sum myz\\ -\sum mxz & -\sum myz & \sum m(x^2+y^2)\end{pmatrix} \tag{37}$$

- Continuous

$$I_{ij} = \int\rho(x)\left(\delta_{ij}r^2 - x_ix_j\right)\mathrm{d}V$$

$$I_{xx} = \int\rho(x)(y^2+z^2)\,\mathrm{d}V, I_{xy} = I_{yx} = -\int\rho(x)xy\,\mathrm{d}V$$

$$I_{yy} = \int\rho(x)(x^2+z^2)\,\mathrm{d}V, I_{yz} = I_{zy} = -\int\rho(x)yz\,\mathrm{d}V \tag{38}$$

$$I_{zz} = \int\rho(x)(x^2+y^2)\,\mathrm{d}V, I_{zx} = I_{xz} = -\int\rho(x)zx\,\mathrm{d}V$$

Principle axis and principal moments of inertia

In the principal frame:

$$T_{\text{rot}} = \frac{1}{2}(I_1\Omega_1^2 + I_2\Omega_2^2 + I_3\Omega_3^2) \tag{39}$$

- spherical top $I_1 = I_2 = I_3$
- Symmetric top $I_1 = I_2 \neq I_3$
- Asymmetric top $I_1 \neq I_2 \neq I_3$

Parallel axis theorem

when changing

Appendix

1. Taylor expansion:

$$f(x)|_0 \approx f(a) + f'(a)(x-a) + f''(a)\frac{(x-a)^2}{2} \tag{40}$$

2. small angle approximation:

$$\sin(\theta)\approx\theta \quad \cos(\theta)\approx 1-\frac{\theta^2}{2} \tag{41}$$