HW 4, Harry Luo

ex 3.4

Since X has uniform distribution on $[4, 10]$, it has a PDF of $f(x) = \frac{1}{6}$ for x in [4,10] and zero otherwise.

(a)

$$P(x < 6) = P(4 < X < 6) = \frac{6-4}{6} = \frac{1}{3} \tag{1}$$

(b)

$$\begin{aligned} P(|X-7| > 1) &= P(X < 6) + P(X > 8) \\ &= P(4 < X < 6) + P(8 < X < 10) \\ &= \frac{1}{3} + \frac{1}{3} = \frac{2}{3} \end{aligned} \tag{2}$$

(c)

$$\begin{aligned} P(X < t \mid X < 6) &= \frac{P(X < t, X < 6)}{P(X < 6)} \text{ for } 4 \le t \le 6 \\ &= \frac{P(X < t)}{P(X < 6)} \\ &= \frac{P(4 \le X < t)}{P(X < 6)} \\ &= \frac{t-4}{6-4} = \boxed{\frac{t-4}{2}} \end{aligned} \tag{3}$$

ex 3.5

possible values correspond to jumps in cdf, and the pmf is the size of the jump.

$$\begin{aligned} p_x(1) &= \frac{1}{3} \\ p_x\left(\frac{4}{3}\right) &= \frac{1}{2} - \frac{1}{3} = \frac{1}{6} \\ p_x\left(\frac{3}{2}\right) &= \frac{3}{4} - \frac{1}{2} = \frac{1}{4} \\ p_x\left(\frac{9}{5}\right) &= 1 - \frac{3}{4} = \frac{1}{4} \end{aligned} \tag{4}$$

ex 3.7

For a continuous random variable, the cdf is described as $F(x) = P(X \le x) = \int_{-\infty}^{x} f(t)\,\mathrm{d}t$

(a)

$$\begin{aligned} P(a \le X \le b) = F(b) - F(a) = 1 \\ \Rightarrow a \le \sqrt{2}; b \ge \sqrt{3} \\ \text{smallest interval} = \left[\sqrt{2}, \sqrt{3}\right] \end{aligned} \tag{5}$$

(b) for a continuous R.V., the pmf at any point is zero, so $P(X = 1.6) = 0$

(c)

$$
\begin{aligned}
P\left(1 \leq X \leq \frac{3}{2}\right) &= F\left(\frac{3}{2}\right) - F(1) \\
&= \left(\left(\frac{3}{2}\right)^2 - 2\right) - 0 \\
&= \boxed{\frac{1}{4}}
\end{aligned} \tag{6}
$$

(d) Noticing the fact that the cdf is continuous except at points $\sqrt{2}, \sqrt{3}$, the pdf could be retrieved by:

$$
f(x) = F'(x) = \begin{cases} 2x \text{ if sqrt}(2) <= \text{x} < \text{sqrt}(3) \\ 0 \text{ o.w.} \end{cases} \tag{7}
$$

ex 3.9

pdf was given by

$$
f(x) = \begin{cases} 3e^{-3x} \text{ if x} > 0 \\ 0 \text{ o.w.} \end{cases} \tag{8}
$$

(a) for a continous R.V., expectation is calculated by

$$
E(X) = \int_{-\infty}^{+\infty} x f(x)\,\mathrm{d}x = \int_0^{+\infty} 3xe^{-3x}\,\mathrm{d}x = \frac{1}{3} \tag{9}
$$

(b)

$$
E(e^{2X}) = \int_{-\infty}^{+\infty} e^{2x} f(x)\,\mathrm{d}x = \int_0^{+\infty} 3e^{-x}\,\mathrm{d}x = 3 \tag{10}
$$

ex 3.16

$$
\begin{aligned}
E(Z) = \int_{-\infty}^{+\infty} x f(x)\,\mathrm{d}x &= \int_1^2 \frac{1}{7}x\,\mathrm{d}x + \int_5^7 \frac{3}{7}x\,\mathrm{d}x \\
&= \frac{75}{14}
\end{aligned} \tag{11}
$$

$$
\begin{aligned}
\mathrm{Var}(Z) = E(Z^2) - (E(Z))^2 &= \int_1^2 \frac{1}{7}x^2\,\mathrm{d}x + \int_5^7 \frac{3}{7}x^2\,\mathrm{d}x - \left(\frac{75}{14}\right)^2 \\
&= \frac{1633}{588}
\end{aligned} \tag{12}
$$

ex 3.23

(a) Possible values of profit are

$$
\begin{gathered}
0 - 1 = -1 (\text{won nothing}); 2 - 1 = 1 (80 \text{ ppl who won } \$2); \\
100 - 1 = 99 (19 \text{ ppl who won } \$100); 7000 - 1 = 6999 \ (1 \text{ who won } \$7000).
\end{gathered} \tag{13}
$$

We can then represent the pmf as

$$
\begin{aligned}
P(X = -1) &= \frac{10000 - 100}{10000} = \frac{99}{100} \\
P(X = 1) &= \frac{80}{10000} = \frac{1}{125} \\
P(X = 99) &= \frac{19}{10000} \\
P(X = 6999) &= \frac{1}{10000}
\end{aligned} \tag{14}
$$

(b)

$$P(X \geq 100) = P(X = 6999) = \frac{1}{10000} \tag{15}$$

(c)For X as a discrete R.V., the expectation is

$$\begin{aligned}
E(X) &= \sum_k kP(X=k) = -1 \times \frac{99}{100} + \frac{1}{125} + 99 \times \frac{19}{10000} + 6999 \times \frac{1}{10000} = -0.094 \\
E(X^2) &= \sum_k k^2 P(X=k) = (-1)^2 \times \frac{99}{100} + 1^2 \times \frac{1}{125} + 99^2 \times \frac{19}{10000} + 6999^2 \times \frac{1}{10000} = 4918.22 \\
&\Rightarrow \mathrm{Var}(X) = E(X^2) - (E(X))^2 = 4918.22 - (-0.094)^2 = 4918.22 - 0.008836 = 4918.211
\end{aligned} \tag{16}$$

ex 3.28

(a)X has possible values of 1,2 3. This is a sampling without replacement trial, where

$$\begin{aligned}
\text{price in the first open box: } P(X=1) &= \frac{3}{5} \\
\text{price in the second open box } P(X=2) &= \frac{2}{5} \times \frac{3}{4} = \frac{3}{10} \\
\text{price in the third open box } P(X=3) &= \frac{2}{5} \times \frac{1}{4} \times \frac{3}{3} = \frac{1}{10}
\end{aligned} \tag{17}$$

(b)$E(X) = 1 \cdot \frac{3}{5} + 2 \cdot \frac{3}{10} + 3 \cdot \frac{1}{10} = 3/2$

(c)$E(X^2) = 1^2 \cdot \frac{3}{5} + 2^2 \cdot \frac{3}{10} + 3^2 \cdot \frac{1}{10} = \frac{27}{10}$
$\Rightarrow \mathrm{Var}(X) = E(X^2) - E(X)^2 = 9/20$

(d) We represent the loss or gain of the game as $Y = g(X)$.

$$\begin{aligned}
Y = g(X) &= \begin{cases} 100 & \text{if X} = 1 \\ 0 & \text{if X} = 2 \\ -100 & \text{if X} = 3. \end{cases} \\
E(Y) &= 100 \times \frac{3}{5} + 0 \times \frac{3}{10} - 100 \times \frac{1}{10} = 50
\end{aligned} \tag{18}$$

ex 3.31

(a)

$$\begin{aligned}
1 = \int_{-\infty}^{+\infty} f(x)\,\mathrm{d}x &= \int_1^{\infty} cx^{-4}\,\mathrm{d}x = \frac{c}{3} \\
&\Rightarrow c = 3
\end{aligned} \tag{19}$$

(c)

$$P(0.5 < X < 2) = \int_{0.5}^{2} 3x^{-4}\,\mathrm{d}x = \frac{7}{8} \tag{20}$$

(e)

$$\begin{aligned}
&\text{when x >= 1: } F(x) = P(X \leq x) = \int_1^x 3t^{-4}\,\mathrm{d}t = 1 - x^{-3} \\
&\text{when x < 1:} \quad F(x) = P(X \leq x) = 0
\end{aligned} \tag{21}$$

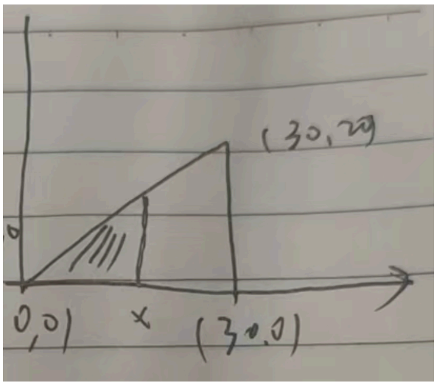
(f)

$$E(x) = \int_{-\infty}^{+\infty} xf(x)\,\mathrm{d}x = \int_1^{\infty} 3x^{-3}\,\mathrm{d}x = \frac{3}{2} \tag{22}$$

(g)

$$E(5X^2 + 3X) = \int_1^{\infty} (5x^{-2} + 3x)3x^{-4} \, \mathrm{d}x = \frac{39}{2} \tag{23}$$

ex 3.47



(a)when X is below 0, cdf is 0; when X is above 30, cdf is 1.
when X is between 0 and 30, the set of points in the triangle with $X \leq x$ would have vertices $(0,0)$, $(x,0)$, $\left(x, 2\frac{x}{3}\right)$, with area $\frac{1}{3}x^2$, while the original triangle has size 300.

$$F(X) = \begin{cases} 0 \text{ X <0} \\ \left(\frac{1}{3}x^2\right)/300 = \frac{x^2}{900} \text{ 0 <= x < 30} \\ 1 \text{ x >= 30.} \end{cases} \tag{24}$$

(b)Since F(x) is continous and differentiable(besides x = 30), the pdf is the derivative of F(x), which is

$$f(x) = \begin{cases} 2\frac{x}{900} = \frac{x}{450} \text{ 0 <= x < 30} \\ 0 \text{ o.w.} \end{cases} \tag{25}$$

(c) Notice X is continuous,

$$E(X) = \int_{-\infty}^{+\infty} x f(x) \, \mathrm{d}x = \int_0^{30} \frac{x^2}{450} \, \mathrm{d}x = 20 \tag{26}$$