Summary

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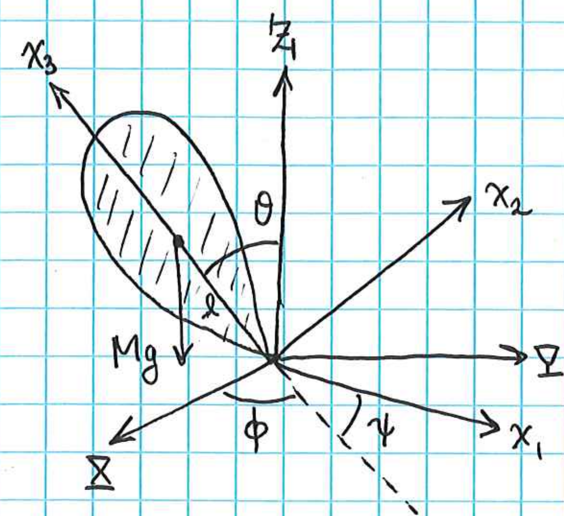
· Heavy symmetric top – sol.'n w/ Euler angles.



$$L = T - U$$

$$= \frac{1}{2} I_1' (\dot\theta^2 + \dot\phi^2 \sin^2\theta) + \frac{1}{2} I_3 (\dot\psi + \dot\phi\cos\theta)^2 - Mgl\cos\theta.$$

E-L eqn.'s:

· ϕ: $\frac{d}{dt}\frac{\partial L}{\partial\dot\phi} = \frac{\partial L}{\partial\phi} \Rightarrow L_z \equiv (I_1' \sin^2\theta + I_3\cos^2\theta)\dot\phi + I_3\cos\theta\,\dot\psi = \text{const.}$

· ψ: $\frac{d}{dt}\frac{\partial L}{\partial\dot\psi} = \frac{\partial L}{\partial\psi} \Rightarrow L_3 = I_3(\dot\psi + \dot\phi\cos\theta) = \text{const.}$

· Instead of using θ EOM, recall that time-translation symmetry of $L \Rightarrow$ energy = third conserved quantity.

$$E = T + U = \frac{1}{2} I_1' (\dot\theta^2 + \dot\phi^2\sin^2\theta) + \frac{1}{2} I_3 (\dot\psi + \dot\phi\cos\theta)^2 + Mgl\cos\theta.$$

$\dot\phi$ & $\dot\psi$ eliminated using L_z & L_3:

$$\begin{cases} L_z = I_1' \sin^2\theta\dot\phi + L_3\cos\theta & \Rightarrow \quad \dot\phi = \dfrac{L_z - L_3\cos\theta}{I_1'\sin^2\theta} \\ L_3 = I_3(\dot\psi + \dot\phi\cos\theta) & \Rightarrow \quad \dot\psi = \dfrac{L_3}{I_3} - \dfrac{\cos\theta(L_z - L_3\cos\theta)}{I_1'\sin^2\theta} \end{cases}$$

plugging into E:

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$$E = \frac{1}{2} I_1' \dot{\theta}^2 + \frac{1}{2I_1'} \frac{(L_z - L_3\cos\theta)^2}{\sin^2\theta} + \frac{1}{2I_3} L_3^2 + Mgl\cos\theta.$$

$$\Rightarrow \underbrace{E - \frac{L_3^2}{2I_3} - Mgl}_{E'} = \frac{1}{2} I_1' \dot{\theta}^2 + \underbrace{\frac{1}{2I_1'} \frac{(L_z - L_3\cos\theta)^2}{\sin^2\theta} - Mgl(1-\cos\theta)}_{U_{eff}(\theta)}.$$

$$\Rightarrow E' = \frac{1}{2} I_1' \dot{\theta}^2 + U_{eff}(\theta).$$

→ effective 1 DOF problem (in θ).

as before, this can be integrated.

$$\dot{\theta} = \frac{d\theta}{dt} \Rightarrow \qquad t = \int \frac{d\theta}{\sqrt{2[E - U_{eff}(\theta)]/I_1'}} \qquad (0 \le \theta \le \pi).$$

→ full sol.'n of problem.

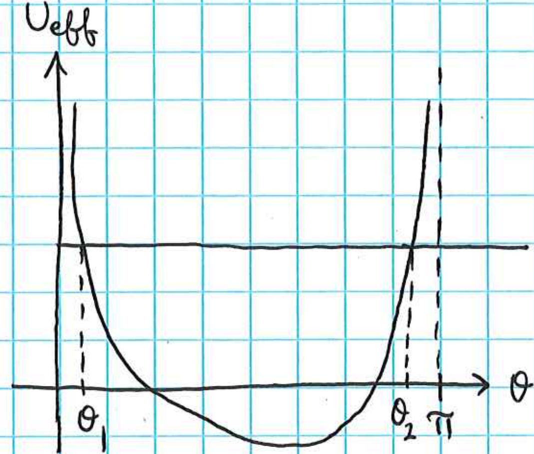
invert $t(\theta) \to \theta(t)$ & plug into

(*) + (**) for $\phi(t)$ & $\psi(t)$.

· We can learn much about the motion by considering prop.'s of $U_{eff}(\theta)$.

- First, note $U_{eff} \to +\infty$ as $\theta \to 0$ or π.
- It can also be shown $U_{eff}(\theta)$ has a single minimum for $0 \le \theta \le \pi$ (see end of notes for proof).

$\Rightarrow U_{eff}$ has the following generic form: 04/08/24



- For given value of E', there are two turning pt.'s: $U_{eff}(\theta_{1,2}) = E'$ $(\dot{\theta} = 0)$.

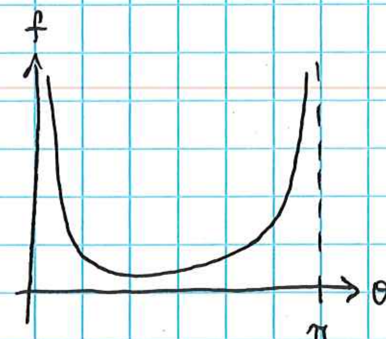
$\Rightarrow$ motion restricted: $\theta_1 \leq \theta \leq \theta_2$.

What else can we say about the motion in general?

$\rightarrow$ consider just motion of the symmetry axis x_3, which is described by θ & ϕ.

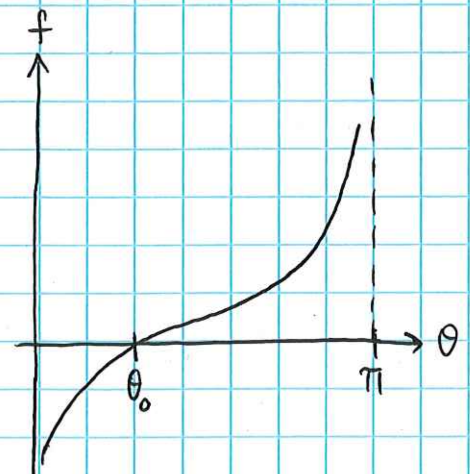
$$\rightarrow \dot{\phi} = \frac{L_z - L_3 \cos\theta}{I_1' \sin^2\theta} = \frac{L_3}{I_1'} \frac{(L_z/L_3 - \cos\theta)}{\sin^2\theta} \equiv \frac{L_3}{I_1'} f(\theta).$$

(i) If $L_z/L_3 > 1$, then:



$\Rightarrow \dot{\phi} > 0$.

(ii) If $L_z/L_3 < 1$, then:



$\cos\theta_0 = L_z/L_3$

$\Rightarrow \dot{\phi}$ changes sign.

$$\Rightarrow \begin{cases} \dot{\phi} < 0 & \text{for } 0 \leq \theta \leq \theta_0 \\ \dot{\phi} > 0 & \text{for } \theta_0 < \theta \leq \pi \end{cases}$$

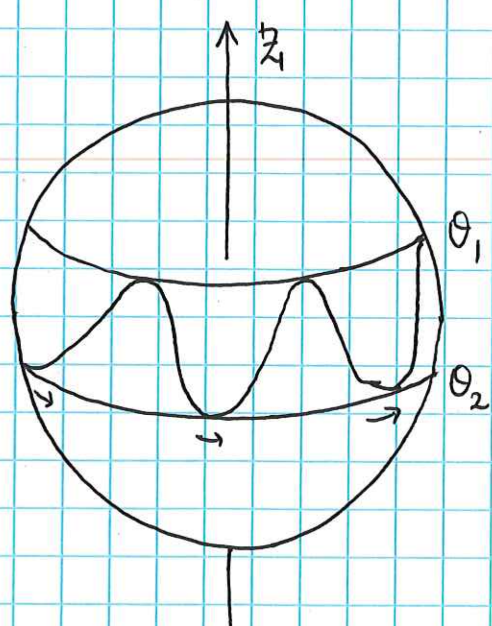
Now, since we know motion restricted to $\theta_1 \leq \theta \leq \theta_2$, we may distinguish different cases:

(1)  $\Rightarrow \dot{\phi} > 0$ for $\theta_1 \leq \theta \leq \theta_2$.

 $\Rightarrow \dot{\phi} < 0$ " ".

(note: in case (i) above $\dot{\phi} > 0$).

In this situation where $\dot{\phi}$ has single sign throughout motion, x_3 precesses monotonically about $\hat{z}$ axis (ϕ motion), while also oscillating up & down. (θ motion).



"nutation".

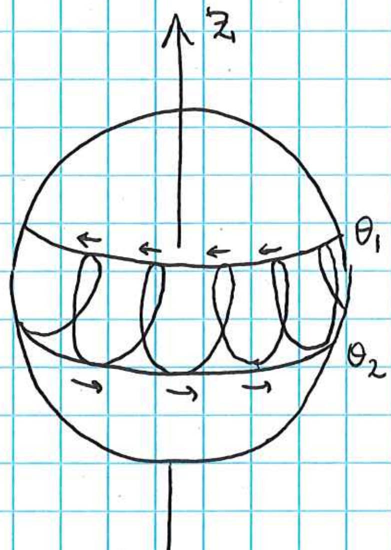
$\leftarrow$ trace of x_3 axis on surface of sphere centered at the origin.

(2) 

$\Rightarrow$ $\dot{\phi} < 0$ for $\theta_1 \leq \theta < \theta_0$

$\dot{\phi} > 0$ for $\theta_0 < \theta \leq \theta_2$.

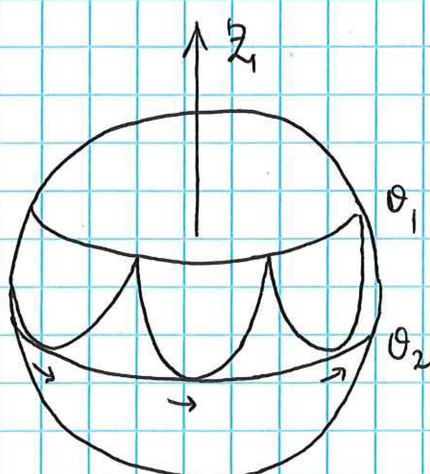
In this case, $\dot{\phi}$ changes sign throughout the motion. & signs are opposite on limiting circles θ_1 & θ_2



(3) 

$\theta_1 = \theta_0$ or $\theta_2 = \theta_0$

$\Rightarrow$ $\dot{\phi}$ & $\dot{\theta}$ vanish simultaneously.



$(\theta_1 = \theta_0)$

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Ex: near vertical oscillation

Q: under what conditions is $\theta = 0$ a pt. of stable equilibrium?

$$U_{eff}(\theta) = \frac{(L_z - L_3\cos\theta)^2}{2I_1'\sin^2\theta} - Mgl(1-\cos\theta).$$

First, observe that when $\theta = 0$ $L_z = L_3$, since z & x_3 axes coincide. More explicitly:

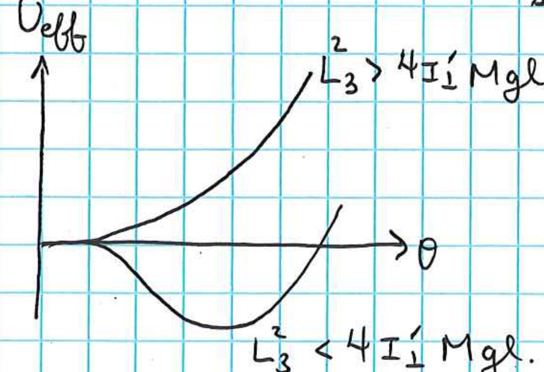
$$L_z = (I_1'\sin^2\theta + I_3\cos^2\theta)\dot\phi + I_3\dot\psi\cos\theta \xrightarrow{\theta=0} I_3(\dot\phi+\dot\psi)$$

$$L_3 = I_3(\dot\phi\cos\theta + \dot\psi) \xrightarrow{\theta=0} I_3(\dot\phi+\dot\psi)$$

$=$

$$L_z = L_3 \Rightarrow U_{eff}(\theta) = \frac{L_3^2}{2I_1'}\frac{(1-\cos\theta)^2}{\sin^2\theta} - Mgl(1-\cos\theta).$$

θ near zero: $\left.\begin{array}{l}\cos\theta \simeq 1-\theta^2/2 \\ \sin\theta \simeq \theta\end{array}\right\}$ $U_{eff}(\theta) \simeq \underbrace{\left(\frac{L_3^2}{8I_1'} - \frac{Mgl}{2}\right)}_{\text{stable if } >0}\theta^2.$



$\Rightarrow$ motion about $\theta = 0$ stable if $L_3^2 > 4I_1'Mgl$.

or $\Omega_3^2 > 4I_1'Mgl/I_3^2$ $(L_3 = I_3\Omega_3)$, that is, ~~motion~~ $\theta=0$ is stable if spin about symm. axis is fast enough.

Appendix

· proof that $V_{eff}(\theta)$ in problem of heavy symmetric top has at most two turning pt.'s in range $0 \le \theta \le \pi$.

$$E' = \frac{1}{2} I_1' \dot{\theta}^2 + V_{eff}(\theta).$$

$$V_{eff}(\theta) = \frac{1}{2I_1'} \frac{(L_z - L_3 \cos\theta)^2}{\sin^2\theta} - Mgl(1-\cos\theta).$$

· let $u = \cos\theta \rightarrow \dot{u} = -\dot{\theta}\sin\theta$. $(-1 \le u \le 1)$.

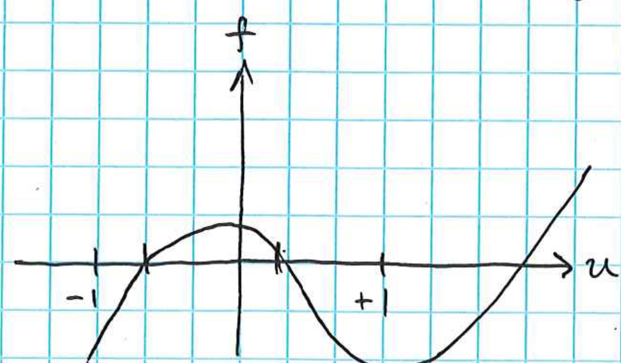
$$\Rightarrow \dot{u}^2 = \frac{2E'}{I_1'}(1-u^2) - \left(\frac{L_z - L_3 u}{I_1'}\right)^2 + \frac{2Mgl}{I_1'}(1-u)(1-u^2) \equiv f(u)$$

$\Rightarrow$ turning pt's $\dot{u} = 0$ correspond to zeros of polynomial $f(u) = 0$.

· now, $f(u)$ = cubic poly. s.t. $f(u) \xrightarrow{u \to \infty} \frac{2Mgl}{I_1'} u^3$.

& $f(\pm 1) = -\left(\frac{L_z \mp L_3}{I_1'}\right)^2 < 0$

$\Rightarrow$ at least one root for $u > 1$. Since $f(u)$ = cubic, there are at most two roots for $-1 \le u \le 1$. (& these roots correspond to turning pt.'s).



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