HW 7, Harry Luo

5.8

We can solve for the probability density function by differentiating the cumulative distribution function. $X \in [-1,2] \Rightarrow X^2 \in [0,4]$. When $X^2 \in [0,4]$,

$$\begin{aligned} F_{Y(y)} &= P(Y \leq y) \\ &= P(X^2 \leq y) \\ &= P(-\sqrt{y} \leq X \leq \sqrt{y}) \\ &= F_X(\sqrt{y}) - F_X(-\sqrt{y}) \end{aligned} \tag{1}$$

Differentiating Equation 1, we get the probability density function as:

$$f_Y(y) = F'_{Y(y)} = \frac{1}{2\sqrt{y}} f_X(\sqrt{y}) + \frac{1}{2\sqrt{y}} f_X(-\sqrt{y}) \tag{2}$$

The probability density function of X is given as $f_{X(x)} = \frac{1}{3}$ when $x \in [-1,2]$ and zero otherwise.

Considering when $y \in [0,1]$, $\sqrt{y} \in [0,1]$ and $-\sqrt{y} = [-1,0]$, which are within the range of x,

so $f_X(\sqrt{y}) = f_X(-\sqrt{y}) = \frac{1}{3}$,

$$f_{Y(y)} = \frac{1}{2\sqrt{y}} * \frac{1}{3} + \frac{1}{2\sqrt{y}} * \frac{1}{3} = \frac{1}{3\sqrt{y}} \tag{3}$$

when $y \in [0,1]$.

When $y \in [1,4]$, $\sqrt{y} \in [1,2]$,but $-\sqrt{y} \in [-2,-1]$out of range

so $f_X(\sqrt{y}) = \frac{1}{3}, f_X(-\sqrt{y}) = 0$,

$$f_Y(y) = \frac{1}{6\sqrt{y}} \tag{4}$$

when $y \in [1,4]$.

Thus,

$$F_Y(y) = \begin{cases} 0 \text{ when } y < 0 \\ \frac{1}{3\sqrt{y}} \text{ when } 0 \leq y < 1 \\ \frac{1}{6\sqrt{y}} \text{ when } 1 \leq y < 4 \\ 1 \text{ when } y \geq 4 \end{cases} \tag{5}$$

5.28

We know $f_X(x) = \frac{1}{3}, x \in (-1,2)$ and 0 otherwise, while $Y = X^4 \in [0,16]$ thus

$$f_Y(y) = 0, y \notin [0,16] \tag{6}$$

When $y \in [0,16]$, we can find the probability density function by differentiating the cumulative distribution function.

$$F_Y(y) = P(Y \leq y) = P(X^4) \leq y = F_X(y^{1/4}) - F_X(-y^{1/4}) \tag{7}$$

Differentiating Equation 7, we get the probability density function as:

$$f_Y(y) = \frac{1}{4} y^{-\frac{3}{4}} f_X\left(y^{\frac{1}{4}}\right) + \frac{1}{4} y^{-\frac{3}{4}} f_X\left(-y^{\frac{1}{4}}\right) \tag{8}$$

When $y \in [0,1]$, $y^{\frac{1}{4}} \in [0,1]$ and $-y^{\frac{1}{4}} \in [-1,2]$, which are within the range of x, thus $f_X\left(y^{\frac{1}{4}}\right) = f_X\left(-y^{\frac{1}{4}}\right) = \frac{1}{3}$,

$$f_Y(y) = \frac{1}{6y^{\frac{3}{4}}} \tag{9}$$

When $y \in [1,16]$, $y^{\frac{1}{4}} \in [1,2]$ and $-y^{\frac{1}{4}} \in [-2,-1]$, which are within the range of x, thus $f_X\left(y^{\frac{1}{4}}\right) = f_X\left(-y^{\frac{1}{4}}\right) = \frac{1}{3}$,

$$f_Y(y) = \frac{1}{12y^{\frac{3}{4}}} \tag{10}$$

In summary,

$$f_Y(y) = \begin{cases} 0 \text{ when } y < 0 \\ \frac{1}{6y^{3/4}} \text{ when } 0 \leq y < 1 \\ \frac{1}{12y^{3/4}} \text{ when } 1 \leq y < 16 \\ 0 \text{ when } y \geq 16 \end{cases} \tag{11}$$

5.32

Given $X \in (0,1)$, possible values for Y is the interval $(1, \infty)$ THerefore, when $t < 1$, $f_Y(t) = 0$ and when $t \geq 1$, we can find the probability density function by differentiating the mass function.

$$\begin{aligned} P(Y \leq t) &= P\left(\frac{1}{x} \leq t\right) = P\left(X \geq \frac{1}{t}\right) = 1 - \frac{1}{t} \\ f_Y(t) &= \frac{\mathrm{d}}{\mathrm{d}t} P(Y \leq t) = \frac{1}{t^2} \text{ when } t \geq 1 \end{aligned} \tag{12}$$

6.6

- (a) Marginal of X, when x > 0, is

$$f_X(x) = \int_0^\infty x e^{-x(1+y)} \,\mathrm{d}x = e^{-x} \tag{13}$$

and zero otherwise.

The marginal of Y when y >0, similarly, is

$$f_Y(y) = \int_0^\infty x e^{-x(1+y)} \,\mathrm{d}x = \frac{1}{(1+y)^2} \tag{14}$$

and zero otherwise.

- (b) Expectation:

$$\begin{aligned} E[XY] &= \int_0^\infty \int_0^\infty xy \times f(x,y) \,\mathrm{d}x \,\mathrm{d}y \\ &= \int_0^\infty \int_0^\infty x^2 y e^{-x(1+y)} \,\mathrm{d}x \,\mathrm{d}y \\ &= \int_0^\infty e^{-y} \,\mathrm{d}y \\ &= 1 \end{aligned} \tag{15}$$

- (c) Expectation:

$$\begin{aligned} E\left[\frac{X}{1+Y}\right] &= \int_0^\infty \int_0^\infty \frac{x}{1+y} x e^{-x(1+y)} \,\mathrm{d}x \,\mathrm{d}y \\ &= \int_0^\infty \frac{1}{1+y} \frac{2}{(1+y)^3} \,\mathrm{d}y = 2 \int_0^\infty \frac{1}{(1+y)^4} \,\mathrm{d}y \\ &= \frac{2}{3} \end{aligned} \tag{16}$$

6.18

- (a) We can write the pmf as a table:

X\Y	1	2	3	4
1	1/4	0	0	0
2	1/8	1/8	0	0
3	1/12	1/12	1/12	0
4	1/16	1/16	1/16	1/16

we can confirm that the terms are non negative, and the sum of all terms is 1. This certifies that $p_{X,Y}$ is a **valid pmf**.

- (b) Marginal of X and Y can be found by summing the rows and columns:

$$\begin{gathered} X: \\ P(X=1)=\frac{1}{4}, P(X=2)=\frac{1}{4}, P(X=3)=\frac{1}{4}, P(X=4)=\frac{1}{4} \\ Y: \\ P(Y=1)=\frac{25}{48}, P(Y=2)=\frac{13}{48} P(Y=3)=\frac{7}{48}, P(Y=4)=\frac{1}{16} \end{gathered} \tag{17}$$

- (c)

$$\begin{aligned} P(X=Y+1) &= P(X=2,Y=1)+P(x=3,Y=2)+P(X=4,Y=3) \\ &= \frac{1}{8}+\frac{1}{12}+\frac{1}{16} \\ &= \frac{13}{48} \end{aligned} \tag{18}$$

6.24

We can use binomial distribution with n = 3 and p = 1/4. The probability of of having exactly two balls are green and one is not green is

$$\binom{3}{2}\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)=\frac{9}{64} \tag{19}$$

By the same logic, the probability of having exactly 2 R ball, 2 Y balll or 2 W balls ae also 9/64.

So the probability of having exactly 2 balls of the same color is

$$\boxed{\frac{9}{64}\times 4=\frac{9}{16}}$$

6.34

Consider a random point (X, Y) uniformly distributed over the quadrilateral region D with vertices at $(0, 0)$, $(2, 0)$, $(1, 1)$, and $(0, 1)$.

(a) Given that the area of D equals $\frac{3}{2}$, the joint probability density function is:

$$f_{X,Y}(x, y) = \begin{cases} \frac{2}{3} & \text{for } (x, y) \in D, \\ 0 & \text{otherwise.} \end{cases}$$

Note that the boundary of D includes a line segment from $(1, 1)$ to $(2, 0)$, described by $y = 2-x$. We can derive the marginal density functions as follows:

For the marginal density of X:

$$f_X(x) = \begin{cases} 0, & x \leq 0 \text{ or } x \geq 2, & (1) \\ \displaystyle\int_0^1 \frac{2}{3}\, dy = \frac{2}{3}, & 0 < x \leq 1, & (2) \\ \displaystyle\int_0^{2-x} \frac{2}{3}\, dy = \frac{4}{3} - \frac{2}{3}x, & 1 < x < 2. & (3) \end{cases}$$

To verify that this is a valid density function:

$$\begin{aligned} \int_{-\infty}^{\infty} f_X(x)\, dx &= \int_0^1 \frac{2}{3}\, dx + \int_1^2 \left(\frac{4}{3} - \frac{2}{3}x\right) dx \\ &= 1, \end{aligned}$$

confirming that $f_X(x)$ is indeed a proper density function.

Similarly, for the marginal density of Y:

$$f_Y(y) = \begin{cases} 0, & y \leq 0 \text{ or } y \geq 1, \\ \int_0^{2-y} \frac{2}{3}\, dx = \frac{4}{3} - \frac{2}{3}y, & 0 < y < 1. \end{cases}$$

(b) To find $\mathbb{E}[X]$ and $\mathbb{E}[Y]$:

$$\begin{aligned} \mathbb{E}[X] &= \int_{-\infty}^{\infty} x f_X(x)\, dx \\ &= \int_0^1 \frac{2}{3}x\, dx + \int_1^2 \left(\frac{4}{3}x - \frac{2}{3}x^2\right) dx \\ &= \frac{7}{9}, \end{aligned}$$

and

$$\begin{aligned} \mathbb{E}[Y] &= \int_{-\infty}^{\infty} y f_Y(y)\, dy \\ &= \int_0^1 \left(\frac{4}{3}y - \frac{2}{3}y^2\right) dy \\ &= \frac{4}{9}. \end{aligned}$$

6.36

Suppose that the random variables X and Y have the joint probability density function

$$f(x, y) = ce^{-\frac{x^2}{2} - \frac{(x-y)^2}{2}}, \qquad x, y \in (-\infty, \infty),$$

where c is a constant.

(a) To determine the value of c, we require that

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)\, dx dy = 1.$$

Evaluating this double integral:

$$\begin{aligned}
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} ce^{-\frac{x^2}{2} - \frac{(x-y)^2}{2}}\, dx dy &= c \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{2}}\, dy dx \\
&= \sqrt{2\pi} c \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\, dx = 2\pi c.
\end{aligned}$$

Setting this equal to 1 yields $c = \frac{1}{2\pi}$.

(b) The marginal density function of X is given by

$$\begin{aligned}
f_X(x) &= \int_{-\infty}^{\infty} \frac{1}{2\pi} e^{-\frac{x^2}{2} - \frac{(x-y)^2}{2}}\, dy \\
&= \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-y)^2}{2}}\, dy \\
&= \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}},
\end{aligned}$$

which is the density of a standard normal random variable.

Similarly, for the marginal density of Y:

$$\begin{aligned}
f_Y(y) &= \int_{-\infty}^{\infty} \frac{1}{2\pi} e^{-\frac{x^2}{2} - \frac{(x-y)^2}{2}}\, dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2} - \frac{(x-y)^2}{2}}\, dx.
\end{aligned}$$

Completing the square in the exponent and simplifying:

$$\begin{aligned}
f_Y(y) &= \frac{1}{\sqrt{4\pi}} e^{-y^2/4} \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} e^{-(x-y/2)^2}\, dx \\
&= \frac{1}{\sqrt{4\pi}} e^{-y^2/4}.
\end{aligned}$$

The last step uses the fact that $\frac{1}{\sqrt{\pi}} e^{-(x-y/2)^2}$ is the pdf of an $N(y/2, 1)$ distributed random variable. Thus, $Y \sim N(0, 2)$.

In summary, $X \sim N(0, 1)$ and $Y \sim N(0, 2)$.