Equation of Motion:
Lagrangian, Principle of Least Action, and E-L Equation

Larangian:

- Under the constraint of
  1)Space and time are homogenous, 2)time is isotropic, the Larangian for a system is given as

$$L = T - U(r), \text{where} \begin{cases} T = \sum_{a=1}^{N} \frac{1}{2} m_a \dot{q_a}^2 \text{ sum of KE} \\ \text{U: potential energy} \end{cases} \tag{1}$$

E-L equation

For a given functional,

$$S = \int_{t_1}^{t_2} L(q, \dot{q}, t)\,\mathrm{d}t \tag{2}$$

we could optimize it using the Euler-Lagrange equation,

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0 \tag{3}$$

where each EL equation and its solution corresponds to a degree of freedom.
Upon applying the El equation to a generalized lagrangian, we reveal Newton's second law

$$\begin{gathered} \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\left(\frac{1}{2}mv^2 - U(r)\right)}{\partial v} = \frac{\partial\left(\frac{1}{2}m\dot{q}^2 - U(r)\right)}{\partial r} \\ \Rightarrow m\dot{\vec{v}} = -\frac{\partial U}{\partial q} \equiv \vec{F}(\text{force}) \end{gathered} \tag{4}$$

coordinate transformation:

- In cartesian coordinates, $L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - U$
  In cylindrical coordinates, $L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2\right) - U$
  In spherical coordinates, $L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2(\theta)\dot{\varphi}^2\right) - U$
- Note that when taking partial differentiations, we treat each variable and its derivative as two independent variables. Don't ask why... We are doing physics here

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Conservation Laws:
Energy, Momentum, COM, and Angular Momentum

Energy:

- Energy is defined as the following, and when the Lagrangian is **homogeneity time**, the energy is conserved.

$$\begin{gathered} E \equiv \sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L \\ \text{considering } L = T - U, \text{we have } \boxed{\text{E = T + U}} \end{gathered} \tag{5}$$

- Total energy is also given as

$$\begin{gathered} E = \frac{1}{2}\mu V^2 + E_i \\ \text{where } E_i \text{ is internal energy, and } \mu \text{ being the total mass} \end{gathered} \tag{6}$$

General momentum:

conservation of general momentum is from the following conservation

$$\frac{\partial L}{\partial q_j} = 0 \Rightarrow p_j \equiv \frac{\partial L}{\partial \dot{q}_j}, \tag{7}$$

where $q_j$is a cyclic coordinate, i.e. L is independent of q_j.

Total momentum

total momentum is defined as the following, and considering the **homogeneity of space**, the momentum is conserved in a closed system.
If the total momentum of a mechanical system in a given frame of reference is 0, then the said system is at rest relative to that frame. For simplicity's sake, we want to chose our frame of reference in which the total momentum is zero.

$$\begin{gathered} P \equiv \sum_a \frac{\partial L}{\partial \dot{q}_a} = \boxed{\sum_a m_a v_a} \\ \text{force is also given by } F_j = \frac{\partial L}{\partial q_j} \\ \text{sum of all forces in a closed system is 0} \end{gathered} \tag{8}$$

Center of Mass

- Center of mass is defined so that, the velocity of the system as a whole, $V = P/(\sum m_a)$ is the time derivative of the center of mass. $R = \sum_a m_a r_a/(\sum m_a)$.

Conservation of angular momentum

Angular momentum caractorizes the rotation of the system, and considering the **isotropy of space**, the angular momentum is conserved in a closed system.

$$\boxed{\vec{L} \equiv \sum_a r_a \times p_a} \text{ is conserved in a closed system} \tag{9}$$

- Angular momentum can be found by differentiating the lagrangian with respect to angular velocity, along the rotation axis z:

$$\vec{L_z} = \frac{\partial L}{\partial \dot{\varphi}_a} \tag{10}$$

Integration of the equations of motion: Connetcting Energy with motion

Motion in 1 dimension

- For a system with DOF=1, and with $\frac{\partial L}{\partial t} = 0$ (largrangian independent of time, i.e. energy conserved) , we can write the largrangian and total energy as

$$L = \frac{1}{2}m\dot{x}^2 - U(x), \tag{11}$$

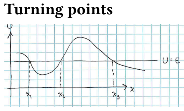
$$E = \frac{1}{2}m\dot{x}^2 + U(x) \tag{12}$$

Equation 12 is a differential equation of position and time. Solving this ODE for time gives:

$$t = \sqrt{\frac{m}{2}}\int \frac{\mathrm{d}x}{E - U(x)} + C \tag{13}$$

when given $U(x)$, and by plugging it into Equation 12, we can solve for $x(t)$ by substitution. Tricks on sub: when $U(x)$ is of order 1, use u-sub; when it's of order 2, use trig-sub.

Turning points



For a given potential function $U(x)$, the turning points are the points where the potential energy is equal to the total energy, i.e. $U(x) = E$. At turning points, the system is either just about to move, or just about to stop.
Only motion where potential is less or equal to total energy is allowed.
Bounded motion: $[x_1, x_2]$; unbounded motion: $x > x_3$

Unbounded Motion:

When there is a potential well, the system could go into periodic motion with potential energy moving back and forth in the well, and position between x_1, x_2.
We find period by doubling Equation 12:

$$T(E) = \sqrt{2m}\int_{x_1(E)}^{x_2(E)} \frac{\mathrm{d}x}{\sqrt{E - U(x)}} \tag{14}$$

where we represent $x_1(E), x_2(E)$ in terms of E.
When given $U(x)$, we can solve for $x_1(E), x_2(E)$, and then pluging in to Equation 14, we can solve for period by integration via subsitution.
Simple Pendulum in polar coord's has the following:

$$\begin{gathered} T = \frac{1}{2}ml^2\dot{\theta}^2 \\ U = mgl(1 - \cos(\theta)) \end{gathered} \tag{15}$$

It's period is given by Equation 14. Solving it gives us

$$\begin{gathered} T(E) = 4\sqrt{\frac{l}{g}}\int_0^{\frac{\pi}{2}} \frac{\mathrm{d}u}{\sqrt{1 - k^2\sin^2(u)}} \\ \text{where } k = \sin\left(\frac{\theta_0}{2}\right), \sin u = \frac{1}{k}\sin\left(\frac{\theta_0}{2}\right) \end{gathered} \tag{16}$$

Equation 16 can be simplified by small angle approx into

$$T(E) = 2\pi\sqrt{\frac{l}{g}}\left(1 + \left(\frac{\theta_0^2}{16}\right)\right) \tag{17}$$

Effective DOF=1 system

When the largrangian is of the form $L = f(\dot{x}) - g(x)$, we can see it as a system with effective potential $U_{\text{eff}(x)} = g(x)$, and effective kenetic energy $T_{\text{eff}(x)} = f(\dot{x})$. The effective energy is therefore $E = T_{\text{eff}} + U_{\text{eff}}$.

Two body problem

Problem setup

- The two body problem considers two interacting masses with an interacting potential $U(r_1, r_2) = U(|\vec{r_1} - \vec{r_2}|)$. The lagrangian is given by

$$L = \frac{1}{2}m_1\dot{\vec{r_1}}^2 + \frac{1}{2}m_2\dot{\vec{r_2}}^2 - U(|\vec{r_1} - \vec{r_2}|) \tag{18}$$

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COM and reletive coordinates, DOF= 6 -> DOF = 2

- Consider the following handy subsitution,

$$\begin{gathered} \text{Reduced mass } \mu = (m_1m_2)/(m_1 + m_2) = m_1m_2/M; \\ \text{Center of mass } R = (m_1r_1 + m_2r_2)/(M); \\ \text{relative positon } \vec{r} = \vec{r_1} - \vec{r_2} \end{gathered} \tag{19}$$

- Putting the two body system into relative coordinates, and represent masses with reduced mass and COM, we have the following lagrangian:

$$L = \frac{1}{2}M\dot{\vec{R}}^2 + \frac{1}{2}\mu\dot{\vec{r}}^2 - U(\vec{r}) \tag{20}$$

where the first term involves only the COM motion, and the second term involves only the relative motion.

- By choosing our frame with the COM at rest and the total momentum zero, our problem is simplified to an **effective one body problem** with DOF = 2, given by

$$L = \frac{1}{2}\mu\dot{\vec{r}}^2 - U(\vec{r}) \tag{21}$$

Conservation of Angular Momentum

- Angular momentum is defiined as $\vec{L} = \vec{r} \times \mu\dot{\vec{r}}$, and is conserved here.
- Knowing $\vec{r} \cdot \vec{L} = 0$, the motion is in the plane perpendicular to $\vec{L}$. We can use polar coordinates to describe the motion,

$$L = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - U(r) \tag{22}$$

Using EL equation on Equation 22, we get

$$\begin{gathered} \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\varphi}} = \frac{\partial L}{\partial \varphi} \\ \Rightarrow \vec{L_z} \equiv \mu r^2\dot{\theta} = \text{constant} \\ \text{(conservation of angular momentum on z-axis)} \end{gathered} \tag{23}$$

2 body problem in gravitational field

$$\begin{aligned} L &= \frac{1}{2}m_1\dot{r_1}^2 + \frac{1}{2}m_2\dot{r_2}^2 - [m_1gz_1 + m_2gz_2 + U(r)] \\ &= \left[\frac{1}{2}M\dot{R}^2 - MgZ +\right] + \left[\frac{1}{2}\mu\dot{r}^2 - U(r)\right] \end{aligned} \tag{24}$$

where Z is the vertical coordinate of the CM position, $Z = \frac{m_1z_1 + m_2z_2}{M}$

Kepler's second Law

We calculate the differential of area swept by particle in polar coordinates,

$$\begin{gathered} \mathrm{d}A = \frac{1}{2}r^2\,\mathrm{d}\varphi \\ \Rightarrow \frac{\mathrm{d}A}{\mathrm{d}t} = \frac{1}{2\mu}\vec{L_z} \\ \vec{L_z} = 2\mu\dot{A}(\text{constant}) \end{gathered} \tag{25}$$

This is the Kepler's second law, which states that the area swept by the radius in a given time is constant.

EOM for two body system

- The total energy:

$$\begin{aligned} E = T + U &= \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\mu r^2\dot{\varphi}^2 + U(r) \\ &= \frac{1}{2}\mu\dot{r}^2 + U(r) + \frac{L_z^2}{2\mu r^2} \text{ (Notice } L_z = \mu r^2\dot{\varphi} \end{aligned} \tag{26}$$

solving this ODE by integration gives

$$t(r)) = \int \frac{\mathrm{d}r}{\sqrt{\frac{2}{\mu}\left[E - U(r) - \frac{L_z^2}{2\mu r^2}\right]}} + C \tag{27}$$

- Also from $L_z = \mu r^2\dot{\varphi}$, by integrating with respect to time, we get

$$\varphi(t) = \frac{L_z}{\mu}\int \frac{\mathrm{d}t}{r^2(t)} + C' \tag{28}$$

Equation 28 and Equation 26 describe the relative motion of the two body system in terms of constants $\{E, L_z, C, C'\}$

Shape of orbit

- Equation 26 skipped a step,

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \sqrt{\left(\frac{2}{\mu}\right)\left[E - U(r) - \frac{L_z^2}{2\mu r^2}\right]} \tag{29}$$

this equation, combined with our beloved

$$L_z = \mu r^2\dot{\varphi} \Rightarrow \mathrm{d}\varphi = \frac{L_z}{\mu r^2}\,\mathrm{d}t \tag{30}$$

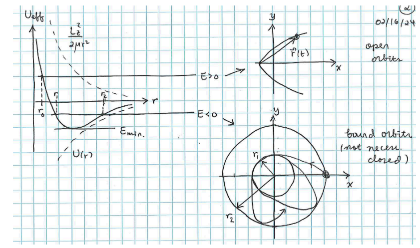
we get the equation of orbit:

$$\begin{gathered} \mathrm{d}\varphi = \frac{L_z}{\sqrt{2\mu}}\frac{\mathrm{d}r}{r^2\sqrt{E - U(r) - \frac{L_z^2}{2\mu r^2}}} \\ \Rightarrow \varphi = \frac{L_z}{\sqrt{2\mu}}\int \frac{\mathrm{d}r}{r^2\sqrt{E - U(r) - \frac{L_z^2}{2\mu r^2}}} + C \end{gathered} \tag{31}$$

Effective potential and shape of orbit (Only for Attractive Potential)

$$U_{\text{eff}} = U(r) + \frac{L_z^2}{2\mu r^2}; E = \frac{1}{2}\mu\dot{r}^2 + U_{\text{eff}(r)} \tag{32}$$

- When $r \to \infty$, $U_{\text{eff}} \to U(r)$, and when $r \to 0$, $U_{\text{eff}} \to$ centrifutal potential $\frac{L_z^2}{2\mu r^2}$.
- by graphing the effective potential, and given constraint of total energy E, we can analyze the shape of the orbit:



- when $E > 0$, the orbit is unbounded, open orbit, hyperbola.
- when $E < 0$, the orbit is bounded into a potential well, although not neccessarily closed.
- when $E = E_{\text{min}}$, the orbit is circular, $F = -\mu\frac{v^2}{r}$

The Kepler Problem: a special case of the two body problem conditions

$$U(r) = -\frac{\alpha}{r}; U_{\text{eff}} = -\frac{\alpha}{r} + \frac{L_z^2}{2\mu r^2} \tag{33}$$

Conic section orbits

We can proof that the orbit is a conic section given by

$$\boxed{r(\varphi) = \frac{p}{1 + e\cos(\varphi)}} \tag{35}$$

$$\text{where} \begin{cases} p = \frac{L_z^2}{\mu\alpha} \\ e = \sqrt{1 + \frac{2EL_z^2}{\mu\alpha^2}} \end{cases} \tag{34}$$

Classifications of orbits based on energy of system E

- When $E > 0, e > 1$, the orbit is unbounded, open orbit, hyperbola.

$$\begin{gathered} \frac{(x - c)^2}{a^2} - \frac{y^2}{b^2} = 1 \\ \begin{cases} a = \frac{p}{e^2 - 1}, b = \frac{p}{\sqrt{e^2 - 1}}, c = ae \\ r_{\text{min}} = \frac{p}{1 + e} \end{cases} \end{gathered} \tag{36}$$

- when $E = 0, e = 1$, the orbit is parabola.

$$\begin{gathered} y^2 = p^2 - 2xp, \\ r_{\text{min}} = \frac{p}{2} \end{gathered} \tag{37}$$

- when $E < 0, e < 1$, the orbit is closed, ellipse.

$$\begin{gathered} \frac{(x + c)^2}{a^2} + \frac{y^2}{b^2} = 1, \\ \begin{cases} a = \frac{p}{1 - e^2}, b = \frac{p}{\sqrt{1 - e^2}}, c = ae \\ r_{\text{min}} = \frac{p}{1 + e}\ ;\ r_{\text{max}} = \frac{p}{1 - e} \end{cases} \end{gathered} \tag{38}$$

- When $E = E_{\text{min}}, f = \frac{\mu\alpha^2}{2L_z^2}, e = 0$, orbit is circular. $r(\varphi) = p = \text{constant}$

More Kepler: Period, Kepler's third law

Orbit of each body

recall Equation 19, we can exprees the orbit of each body as such after some algebra:

$$\vec{r_1} = \frac{m_2}{m_1 + m_2}\vec{r}\ ;\vec{r_2} = -\frac{m_1}{m_1 + m_2}\vec{r} \tag{39}$$

- when $m_1 = m_2 \Rightarrow \vec{r_1} = \frac{\vec{r}}{2}, \vec{r_2} = -\frac{\vec{r}}{2}$, COM inside $r_1 \cap r_2$
- when $m_1 \gg m_2 \Rightarrow \vec{r_1} = \vec{r}, \vec{r_2} = 0$, m_1 is at rest, m_2 orbits m_1

Period of orbit

- $L_z = 2\mu\dot{A}$, areal vel. is constant
- Integrating $\dot{A}$ over a period,

$$A = \int_0^T \dot{A}\,\mathrm{d}t = \frac{L_zT}{2\mu} \tag{40}$$

Since area swept over a period is the area of the ellipse, we have

$$\begin{gathered} \pi ab = \frac{L_zT}{2\mu} \text{ ,letting: } b = \sqrt{pa}, p = \frac{L_z^2}{\mu\alpha} \\ \Rightarrow T = (2\pi a^{3/2})\sqrt{\frac{\mu}{\alpha}} \end{gathered} \tag{41}$$

Conservation of Laplace-Runge-Lenz vector

$\vec{A} = \vec{v} \times \vec{L} - (\alpha\vec{r})/(r)$ is conserved, and is perpendicular to the orbit plane. We can use it to verify : conic sections, eccentricity, and period.

- conserved quantity: $\vec{A}\cdot\vec{L}=0, \frac{A}{\alpha}=\sqrt{1+\frac{2EL_z^2}{\mu\alpha^2}}$

Orbital Transfer

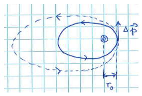
Instantaneous Change in velocity

$$\begin{aligned}(E,L_z)&\to(E',L_z')\\ \Rightarrow(e,p)&\to(e',p')\end{aligned}\tag{42}$$

if thrust occur when satellite is at angle φ_0, orbit orientation can change:

$$r(\varphi_0)=\frac{p}{1+e\cos\varphi_0}=\frac{p'}{1+e'\cos(\varphi_0-\delta)}\tag{43}$$

Tangential thrust at perigee



at $\varphi=0$, let $v=v_{\text{init}}, v'=v_{\text{right after}}, \lambda=v'/v$

$$\begin{aligned}L_z=\mu r_0 v&\Rightarrow L_z'=\mu r_0 v'=\lambda L_z\\ p'&=\lambda^2 p\end{aligned}\tag{44}$$

From Equation 43,

$$\frac{p}{1+e}=\lambda^2\frac{p}{1-e'}\Rightarrow e'=\lambda^2(1+e)-1\tag{45}$$

if $\lambda>1$, $e'>e$, the satellite is in a higher, more eliptical orbit. Unbound if λ big enough
if $\lambda<1$, $e'<e$, the satellite is in a lower orbit.

changing between circular orbits

- changing from R to R', two thrusts(λ_1,λ_2) are needed. There is also an intermediate orbit

$$\begin{aligned}r(\varphi)&=p'/(1+e'\cos\varphi),\\ \text{where } p'&=\lambda_1^2 p, e'=\lambda_1^2-1\end{aligned}\tag{46}$$

changed from intermetiade to final,

$$\begin{aligned}r(\varphi=\pi)=R'&=\lambda_2^2 R/(2-\lambda_1^2)\\ \Rightarrow\lambda_1&=\sqrt{\frac{2R'}{R+R'}}\end{aligned}\tag{47}$$

final orbit:

$$\begin{aligned}r(\varphi)&=R';e''=0,p''=R'\\ \Rightarrow p''&=\lambda_2^2 p'=p'/(1-e')\\ \Rightarrow\lambda_2&=\sqrt{\frac{R+R'}{2R'}}\end{aligned}\tag{48}$$

Small Oscillations

- Motion near a point of stable equilibrium.

DOF= 1 (one dimension)

- For a system of DOF = 1, with potential $U(q)$:
  - **stable equilibrium** at $U(q)_{\min}$, upward parabola, where $F=-\frac{dU}{dq}=0$
    - restoring force for small displacements $q-q_0$ is $F=-\frac{dU(q-q_0)}{dq}$
- **Unstable equilibrium** at $U(q)_{\max}$, downward parabola, where $F=-\frac{dU}{dq}=0$ as well.
- Consider small deviation from point of stable equilibrium, we use taylor expansion to show that it is really a small displacement. that is,

$$\begin{aligned}U\approx U(q_0)&+\frac{dU(q_0)}{dq}(q-q_0)+\frac{d^2U(q_0)}{2\,dq^2}(q-q_0)^2\\ \text{while }&\frac{dU(q_0)}{dq}(q-q_0)=0\end{aligned}\tag{49}$$

letting $x=q-q_0$, we have

$$\begin{aligned}&\begin{cases}U(x)=U(q_0)+\left(\frac{1}{2}\right)\frac{d^2U(q_0)}{dq^2}x^2\\ \text{putting into the form of } U(x)=U(x_0)+\left(\frac{1}{2}\right)kx^2.\end{cases}\\ &\Rightarrow\boxed{k=\frac{d^2U(q_0)}{dq^2}>0}\end{aligned}\tag{50}$$

we get KE, while choosing $U(q_0)=0$:

$$\begin{aligned}T=\frac{1}{2}a(q)^2\dot q^2&=\frac{1}{2}a(q_0+x)\dot x^2\approx\frac{1}{2}m\dot x^2,\overset{m=a(q_0)}{\Rightarrow}\\ &\boxed{L=T-U=\frac{1}{2}m\dot x^2-\frac{1}{2}kx^2}\end{aligned}\tag{51}$$

EOM for DOF = 1 small Oscillations

using EL on Equation 51, we can get the EOM for one dimensional small Oscillations:

$$\begin{aligned}m\ddot x&=-kx\\ \Rightarrow\ddot x+\omega_0^2x=0,\text{ where }&\boxed{\omega_0=\sqrt{\frac{k}{m}}\text{ freq of osc.}}\end{aligned}\tag{52}$$

by magic of ODE, EOM reduces down to:

$$\boxed{x(t)=C_1\cos(\omega_0 t)+C_2\sin(\omega_0 t)}\tag{53}$$

where C_1,C_2 are constants

by trig magic, this could also be written as

$$\begin{aligned}x(t)&=a\cos(\omega_0t+\alpha),\\ \text{where }&\begin{cases}a=\sqrt{C_1^2+C_2^2} & \text{amplitude of oscillation}\\ \omega_0 & \text{frequency of oscillation}\\ \tan\alpha=C_2/C_1 & \text{phase at t=0}\end{cases}\end{aligned}\tag{54}$$

energy for 1D small Oscillation

checking $\frac{\partial L}{\partial t}=0\Rightarrow$ energy-conservation:

$$\begin{aligned}E=T+U&=\frac{1}{2}m\dot x^2+\frac{1}{2}kx^2\\ &=\frac{1}{2}ma^2\omega_0^2,[\text{ constant}]\end{aligned}\tag{55}$$

Damped 1D oscillation, and Complex representation

- when there is damping (friction, resistence, etc) $F_{\text{fric}}=-\beta\dot x$, the EOM becomes:

$$\begin{aligned}\ddot x+2\gamma\dot x+\omega_0^2x&=0,\\ \text{where }2\gamma=\frac{\beta}{m},\omega_0&=\sqrt{\frac{k}{m}}\end{aligned}\tag{56}$$

with ansatz $x(t)=e^{rt},\dot x=re^{rt},\ddot x=r^2e^{rt}$, the solution to Equation 56 is:

$$\begin{aligned}r^2+2\gamma r+\omega_0^2&=0,\\ \text{which has solution } r_+,r_-&=-\gamma\pm\sqrt{\gamma^2-\omega_0^2}\\ \Rightarrow x(t)&=C_1e^{r_+t}+C_2e^{r_-t},\end{aligned}\tag{57}$$

notice the r subscripts here: r_+,r_-

underdamped, overdamped, and critically damped

Recall from your ODE class...

Equation 57 has the following 3 cases, each with different physical interpretation:

1. underdamped:

$$\gamma<\omega_0\Rightarrow 2\text{ complex roots: }\begin{cases}r_\pm=-\gamma\pm i\sqrt{\omega_0^2-\gamma^2}\\ =-\gamma\pm i\omega\\ \omega=\sqrt{\omega_0^2-\gamma^2}\end{cases}\tag{58}$$

The EOM is thus a linear combination of two complex expoentials:

$$\begin{aligned}x(t)&=e^{-\gamma t}(C_1e^{i\omega t}+C_2e^{-i\omega t})\\ &=e^{-\gamma t}(A\cos(\omega t)+B\sin(\omega t))\\ &\text{-- where }\begin{cases}A=C_1+C_2\\ B=i(C_1-C_2)\end{cases}\\ &=ae^{-\gamma t}\cos(\omega t+\alpha)\\ &a,\alpha\text{ are constants}\end{aligned}\tag{59}$$

"The solution is a damped oscillation with frequencyω, and amplitude expoentially decaying with time."

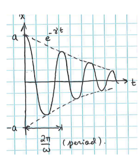
2. Overdameped

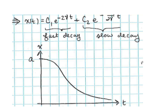
$$\begin{aligned}\gamma>\omega&\Rightarrow x(t)=\\ c_1e^{-\gamma+\sqrt{\gamma^2-\omega^2}t}&+c_2e^{-\gamma-\sqrt{\gamma^2-\omega^2}t}\end{aligned}\tag{60}$$

$$\begin{aligned}\text{when}\gamma\gg\omega_0,\Rightarrow&\begin{cases}\gamma+\sqrt{\gamma^2-\omega_0^2}\approx 2\gamma\\ \gamma-\sqrt{\gamma^2-\omega^2}=\frac{\omega^2}{2\gamma}\end{cases}\\ x(t)&=c_1e^{-2\gamma t}+c_2e^{(-\omega_0^2/2\gamma)t}\end{aligned}\tag{61}$$

3. Critically damped

$$\gamma=\omega_0\Rightarrow x(t)=c_1e^{-\gamma t}+c_2te^{-\gamma t}\tag{62}$$



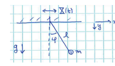


Forced Oscillations

When external force (F) is applied to the system, the largrangian becomes

$$\begin{aligned}L&=\frac{1}{2}m\dot x^2-\frac{1}{2}kx^2+F(t)x\\ \text{EL}\Rightarrow\ddot x+\omega_0^2x&=\frac{F(t)}{m},\text{ where }\omega_0=\sqrt{\frac{k}{m}}\end{aligned}\tag{63}$$

- Example: Simple pendulum with moving pivot



$$\begin{cases}x=X+l\sin\varphi\\ y=l\cos\varphi\end{cases}\Rightarrow\begin{cases}\dot x=\dot X+l\dot\varphi\cos\varphi\\ \dot y=-l\dot\varphi\sin\varphi\end{cases}\tag{64}$$

$$\begin{aligned}&\Rightarrow L=T-U\\ L&=\frac{1}{2}ml^2\dot\varphi^2-mgl(1-\cos\varphi)-ml\ddot X\sin\varphi\\ \text{Expand ab. }\varphi=0\Rightarrow L&=\frac{1}{2}ml^2\dot\varphi^2-\frac{1}{2}mgl\varphi^2-ml\ddot X\varphi\\ \text{EL}\Rightarrow&\boxed{\ddot\varphi+\omega_0^2\varphi=-\frac{\ddot X}{l}\text{ ,where }\omega_0=\sqrt{\frac{g}{l}}}\end{aligned}\tag{65}$$

reintroducing damping via external forcing

$$\ddot x+2\gamma\dot x+\omega_0^2x=f(t),f(t)=\frac{F(t)}{m}\tag{66}$$

When damping $f(t)=f_0\cos(\Omega t)$, solution via complex number:

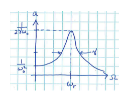
$$\begin{aligned}\ddot z+2\gamma\dot z+\omega_0^2&=f_0e^{i\Omega t}\\ \text{ansatz } z(t)=z_0e^{i\Omega t}\Rightarrow z_0&=\frac{f_0}{\omega_0^2+2i\gamma\Omega+\Omega_0^2}\\ \boxed{z_0=a(\Omega)\cos(\Omega t+\delta(\Omega))f_0}&\text{ is a partcular solution,where}\\ &\begin{cases}a(\Omega)=\frac{1}{\sqrt{(\omega_0^2-\Omega^2)^2+(2\gamma\Omega)^2}}\\ \delta(\Omega)=\arctan\left(2\gamma\frac{\Omega}{\omega_0^2-\Omega^2}\right)\end{cases}\end{aligned}\tag{67}$$

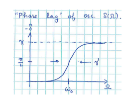
We can study the properties of the system by looking at the amplitude and phase of the solution.

- Amplitude:

$$a_{(\Omega)}=\frac{1}{\sqrt{\left(\omega_0^2-\Omega^2\right)^2+\left(2\gamma\Omega\right)^2}}\tag{68}$$

, when $\gamma\ll\omega_0$, response strongest and amplitude largest when $\omega_r=\omega_0$.





- Phase lag: $\tan\delta(\Omega)=2\gamma\frac{\Omega}{\Omega^2-\omega_0^2}$

  in phase as $\Omega\to 0$, and out of phase as $\Omega\to\omega_0$.

- Genral solution to sinusoidal forcing:

$$\begin{aligned}x(t)&=a(\Omega)f_0\cos(\Omega t+\delta(\Omega))+a_0e^{-\gamma t}\cos(\omega t+\alpha)\\ &\overset{t\gg\frac{1}{\gamma}}{\to}a(\Omega)f_0\cos(\Omega t+\delta(\Omega))\end{aligned}\tag{69}$$

Forgets initial condition after time.

- Power obsorbed by oscillation

  $p=F\dot x=mf\dot x$

  Avg power of oscillation

$$\begin{aligned}P_{\text{avg}}=\frac{1}{T}\int_0^T mf\dot x\,dt&=-\frac{1}{2}mf_0a(\Omega)\Omega\sin\delta(\Omega)\\ \text{simplifies to }P_{\text{avg}}(\Omega)&=\gamma mf_0^2\Omega^2a_{(\Omega)}^2\end{aligned}\tag{70}$$

Absorption around resonance frequency $\Omega=\omega_0+\varepsilon$ is maximum:

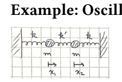
$$P=\frac{\gamma mf_0^2}{4(\varepsilon^2+\gamma^2)}\approx\frac{mf_0^2}{4\gamma}\tag{71}$$

Oscillations DOF>1

For a system with n DOF: $q=(q_1,q_2,\ldots,q_n)$, PE $=U(q)$

- Stable equilibrium $\frac{\partial U(q)}{\partial q_i}|_{q=0}$

Example: Oscillation with 2 mass and 3 springs



$$L=\frac{1}{2}m\dot x_1+\frac{1}{2}m\dot x_2-\frac{1}{2}kx_1^2-\frac{1}{2}kx_2^2-\frac{1}{2}k'(x_1-x_2)^2$$

EOM:

$$\begin{aligned}M\cdot\ddot{\vec x}=-K\vec x\quad&,\text{ where }M=\begin{pmatrix}m&0\\0&m\end{pmatrix},\\ \vec x=\begin{pmatrix}x_1\\x_2\end{pmatrix},K&=\begin{pmatrix}k+k'&-k'\\-k'&k+k'\end{pmatrix}\end{aligned}\tag{72}$$

ansatz: $\vec x=\text{Re}[\vec a e^{i\omega t}]$ Then the EOM eq becomes solving the eigenvalue problem:

$$\begin{aligned}\det(\omega^2M-K)&=0\\ \Rightarrow\begin{cases}\omega_-^2=\frac{k}{m}\\ \omega_+^2=\frac{k+2k'}{m}\end{cases}&\begin{cases}\overrightarrow{x_-}=a_-\begin{pmatrix}1\\1\end{pmatrix}\cos(\omega_-t+\delta_-)\\ \overrightarrow{x_+}=a_+\begin{pmatrix}1\\-1\end{pmatrix}\cos(\omega_+t+\delta_+)\end{cases}\end{aligned}\tag{73}$$

with constants $a_-,a_+,\delta_-,\delta_+$.

New Coords

$$\begin{aligned}&\begin{cases}Q_1=\sqrt{\frac{m}{2}}(x_1+x_2)\\ Q_2=\sqrt{\frac{m}{2}}(x_1-x_2)\end{cases}\\ \Rightarrow L&=\frac{1}{2}\left(\dot Q_1^{\,2}+\dot Q_2^{\,2}\right)-\frac{1}{2}\left(\omega_-^2Q_1^2+\omega_+^2Q_2^2\right)\\ &\overset{\text{E-L}}{\Rightarrow}\ddot Q_1=-\omega_-^2Q_1,\ddot Q_2=-\omega_+^2Q_2\end{aligned}\tag{74}$$

Decoupled oscillators with coords Q_1,Q_2.

General Coords

for general coords q_i, let $x_i=q_i-q_i^{(0)}$

$$\begin{aligned}U&=\frac{1}{2}\sum_{i,j}k_{ij}x_ix_j,\quad k_{ij}=k_{ji}=\frac{\partial^2U(q)}{\partial q_i\partial q_j}\text{ symm mat}\\ T&=\frac{1}{2}\sum_{i,j}m_{ij}\dot x_i\dot x_j,\quad m_{ij}=m_{ji}=a_{ij}(q^{(0)})\end{aligned}\tag{75}$$

the largrangian, in Matix form:

$$L=\frac{1}{2}\dot{\vec x}^T\cdot M\cdot\dot{\vec x}-\frac{1}{2}\vec x^T\cdot K\vec x\overset{\text{EL}}{\Longrightarrow}(\omega^2M-K)\cdot\vec a=0\tag{76}$$

$\Rightarrow\det(\omega^2M-K)=0$ Solving the det for omega gives the normal freq (Eigenvalues)of system ω_α^2 . plug in Evalue into Equation 76 for eigenvec(normal modes) $\overrightarrow{a^\alpha}$ of system.

- General motion

$$x_i(t)=\sum_\alpha a_i^\alpha\text{Re}[C_\alpha e^{i\omega_\alpha t}]\tag{77}$$

- EXAMPLE: Normal freq is given

$$\begin{aligned}\omega&=\left\{0,\sqrt{2}\omega_0,\sqrt{3}\omega_0\right\}.\\ \omega=\sqrt{2}\omega_0&\Rightarrow a_1=-a_3=-a_2=ae^{i\delta}\Rightarrow\\ \vec\theta&=a(1\ -1\ -1)^T\cos\left(\sqrt{2}\omega_0t+\delta\right)\\ \omega=\sqrt{3}\omega_0&\Rightarrow a_1=0,a_2=-a_3=ae^{i\delta}\Rightarrow\\ \vec\theta&=a(0\ 1\ -1)^T\cos\left(\sqrt{3}\omega_0t+\delta\right)\end{aligned}\tag{78}$$

- EXAMPLE: double pendulum

$$\begin{cases}x_1=l_1\sin\varphi_1 & y_1=-l_1\cos\varphi_1\\ x_2=l_1\sin\varphi_1+l_2\sin\varphi_2 & y_2=l_1\cos\varphi_1+l_2\cos\varphi_2\end{cases}\tag{79}$$

$$\begin{aligned}\Rightarrow T&=\frac{1}{2}m_1l_1\dot\varphi^2+\frac{1}{2}m_2(l_1^2\dot\varphi_1{}^2+l_2^2\dot\varphi_2{}^2\\ &+2l_1l_2\dot\varphi_1\dot\varphi_2\cos(\varphi_1-\varphi_2))\\ U&=-m_1gl_1\cos\varphi_1-m_2g(l_1\cos\varphi_1+l_2\cos\varphi_2)\end{aligned}\tag{80}$$

using $\cos\varphi\approx 1-\frac{\varphi^2}{2}$

$$\begin{aligned}L&=\frac{1}{2}(\dot\varphi_1\ \ \dot\varphi_2)\begin{pmatrix}(m_1+m_2)l_1^2 & m_2l_1l_2\\ m_2l_1l_2 & m_2l_2^2\end{pmatrix}(\dot\varphi_1\ \ \dot\varphi_2)\\ &-\frac{1}{2}(\varphi_1\ \ \varphi_2)\begin{pmatrix}(m_1+m_2)l_1g & 0\\ 0 & m_2gl_2\end{pmatrix}(\varphi_1\ \ \varphi_2)\\ &=\frac{1}{2}\dot{\vec\varphi}^TM\cdot\dot{\vec\varphi}-\frac{1}{2}\vec\varphi^TK\vec\varphi\end{aligned}\tag{81}$$

$$\begin{aligned}\text{When }m_1=m_2=m,\quad l_1=l_2=l\Rightarrow\quad M&=ml^2\begin{pmatrix}2&1\\1&1\end{pmatrix},K=mgl\begin{pmatrix}2&0\\0&1\end{pmatrix}\\ \det\left((\omega^2M-K)\right)=0&\Rightarrow\omega^2=\left(2\pm\sqrt{2}\omega_0^2\right)\\ \begin{pmatrix}a_1^-\\a_2^-\end{pmatrix}=C_-\begin{pmatrix}1\\\sqrt{2}\end{pmatrix},&\quad\begin{pmatrix}a_1^+\\a_2^+\end{pmatrix}=C_+\begin{pmatrix}1\\-\sqrt{2}\end{pmatrix}\end{aligned}\tag{82}$$

Normal Coords

$\{x_i\}=\{Q_\alpha\}$, where $x_i=\sum_{\alpha=1}^nA_{i\alpha}Q_\alpha\Rightarrow$
$\sum_j(\omega_\alpha^2m_{ij}-k_{ij}A_{j\alpha})=0$

$\Rightarrow L=\frac{1}{2}\sum_{\alpha=1}^n\left(\dot Q^2{}_\alpha-\omega_\alpha^2Q_\alpha^2\right)\overset{\text{EL}}{\Longrightarrow}\ddot Q_\alpha+\omega_\alpha^2Q_\alpha=0$

Motion of Rigid Body

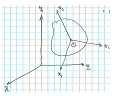
- EXample: rotor

rotation with constraint $|\vec r_i-\vec r_j|$ .COM coords are useful here

$$\begin{cases}\vec r = \vec r_1 - \vec r_2\\ \vec R = \frac{m_1\vec r_1 + m_2\vec r_2}{m_1+m_2}\end{cases} \Rightarrow \begin{cases}\vec r_1 = \vec R + m_2\vec r/M\\ \vec r_2 = \vec R - m_1\vec r/M\end{cases} \tag{83}$$

$$L = \frac{1}{2}M\dot{\vec R}^2 + \mu\dot{\vec r}^2, \quad \mu = m_1\frac{m_2}{m_1+m_2}$$
$$\overset{\text{polar}}{\Longrightarrow} L = \frac{1}{2}M\dot{\vec R}^2 + \frac{1}{2}\mu a^2\left(\dot\theta^2 + \dot\varphi^2\sin^2\theta\right) \tag{84}$$

frames of reference



$$(XYZ) \overset{R(\theta,\varphi,\psi)}{\Longrightarrow} (x_1, x_2, x_3)$$

Velocity of pt in body: $\vec v = \vec V + \vec\Omega\times\vec r$, where V is Translational vel, Omega is angular vel, r is position vector.

Largrangian for Rigid Body

$$T = \frac{1}{2}MV^2 + \frac{1}{2}\sum_a m_a\left[\Omega^2 r_a^2 - \left(\vec\Omega\cdot\vec r_a\right)^2\right)]$$
$$T\text{translational} + T\text{rotational} \tag{85}$$

consider rotation,

$$\Omega^2 = \sum_i \Omega_i^2, \quad \vec\Omega\cdot\vec r_a = \sum_i \Omega_i x_{a,i}$$
$$\Rightarrow T_{\text{rot}} = \frac{1}{2}\sum_{\text{i,j}}\Omega_i\Omega_j I_{\text{i,j}}. \quad I_{ij} \equiv \sum_a m_{a(\delta_{ij}r_a^2 - x_{a,i}x_{a,j})} \tag{86}$$

$\Rightarrow L = \frac{1}{2}MV^2 + \frac{1}{2}\sum_{i,j} I_{i,j}\Omega_i\Omega_j - U$

Inertial Tensor

- Discrete

$$I = \begin{pmatrix}\sum m(y^2+z^2) & -\sum mxy & -\sum mxz\\ -\sum mxy & \sum m(x^2+z^2) & -\sum myz\\ -\sum mxz & -\sum myz & \sum m(x^2+y^2)\end{pmatrix}) \tag{87}$$

- Continuous

$$I_{ij} = \int\rho(x)\left(\delta_{ij}r^2 - x_i x_j\right)\mathrm{d}V$$
$$I_{xx} = \int\rho(x)(y^2+z^2)\,\mathrm{d}V, I_{xy} = I_{yx} = -\int\rho(x)xy\,\mathrm{d}V$$
$$I_{yy} = \int\rho(x)(x^2+z^2)\,\mathrm{d}V, I_{yz} = I_{zy} = -\int\rho(x)yz\,\mathrm{d}V \tag{88}$$
$$I_{zz} = \int\rho(x)(x^2+y^2)\,\mathrm{d}V, I_{zx} = I_{xz} = -\int\rho(x)zx\,\mathrm{d}V$$

example:

$I_{xx} = \rho\iint[b^2\hat y^2 + c^2\hat z^2]\, abc\, d\hat x\, d\hat y\, d\hat z$

$= abc\rho\int_V(b^2\hat y^2 + c^2\hat z^2)\, d\hat x\, d\hat y\, d\hat z$

transform into spherical coord:

$\boxed{I_{xx}} = abc\rho\int_V(b^2r^2\sin^2\varphi\sin^2\theta + c^2r^2\cos^2\theta)\, r^2\sin\theta\, dr\, d\varphi\, d\theta$

$= abc\rho\left[b^2\int_0^{2\pi}\sin^2\varphi\, d\varphi\int_0^\pi\sin^3\theta\, d\theta\int_0^1 r^4 dr\right.$
$\left.+ c^2\int_0^{2\pi}d\varphi\int_0^\pi\cos^2\theta\sin\theta\, d\theta\int_0^1 r^4 dr\right]$

$= \boxed{4/15\, abc\pi\rho[b^2+c^2]}$

- Example: coplanar system principal axis: Z $\Rightarrow I_{13} = I_{23} = 0$
  $I_3 = I_1 + I_2$

Principle axis and principal moments of inertia

In the principal frame:

$$T_{\text{rot}} = \frac{1}{2}\left(I_1\Omega_1^2 + I_2\Omega_2^2 + I_3\Omega_3^2\right) \tag{89}$$

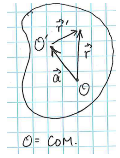
- spherical top $I_1 = I_2 = I_3$
- Symmetric top $I_1 = I_2 \neq I_3$
- Asymmetric top $I_1 \neq I_2 \neq I_3$
- EXample:

$$\det(I - \lambda\mathbb{1}) = 0 \Rightarrow \lambda \text{ prncp. mom.}$$
$$\vec v = \text{eigenvec.} = \text{prncp. axis} \tag{90}$$

- EXample: continuous with axis of symmetry $\rho(\vec r) = \rho = (r, x_3) \Rightarrow I_{ij} = \int\rho(\vec r)(r^2\delta - x_i x_j)\,\mathrm{d}V$

Parallel axis theorem

when changing Origin diff. from COM(O),



$$I_{ij} = I'_{ij} - M\left(a^2\delta_{ij} - a_i a_j\right)$$

For a cube, when finding I at corner, first find I at COM, and

$$I'_{xx} = I_{xx} + M(b^2+c^2) = \frac{4}{3}M(b^2) + c^2$$
$$I'_{yy} = I_{yy} + M(a^2+c^2) = \frac{4}{3}M(a^2+c^2) \tag{91}$$
$$I'_{zz} = I_{zz} + M(a^2+b^2) = \frac{4}{3}M(a^2+b^2)$$

$I_{13} = -\int dv\,\rho(\vec r)\, x_1 x_3$ 03/22/24

$= -\int dx_3\, r\,dr\,d\varphi\,\rho(r, x_3)\, r\cos\varphi\, x_3$

$= -\int dx_3\, r\,dr\,\rho(r,x_3)\, r x_3 \underbrace{\int_0^{2\pi}d\varphi\,\cos\varphi}_{=0}$

& $I_{23} = 0$ by same analysis w/ $\cos\varphi \to \sin\varphi$.

$\Rightarrow I = \begin{pmatrix}I_{11} & I_{12} & 0\\ I_{21} & I_{22} & 0\\ 0 & 0 & I_3\end{pmatrix} \Rightarrow x_3 =$ principal axis;

i.e., symm. axis = princ. axis

$I_{12} = -\int dv\,\rho(\vec r)\, x_1 x_2$

$= -\int dx_3\, r\,dr\,d\varphi\,\rho(r,z)\, r^2\cos\varphi\sin\varphi$

$= -\int dx_3\, r\,dr\,d\varphi\,\rho(r,z)\, r^2 \underbrace{\int_0^{2\pi}d\varphi\,\cos\varphi\sin\varphi}_{=0}$

$\Rightarrow I = \begin{pmatrix}I_{11} & 0 & 0\\ 0 & I_{22} & 0\\ 0 & 0 & I_3\end{pmatrix}$, $x_1, x_2, x_3 =$ principal axes.

$I_1 - I_2 = \int dv\,\rho(\vec r)\,(x_2^2 - x_1^2)$

$= \int dx_3\, r\,dr\,\rho(r,z)\, r^2 \underbrace{\int_0^{2\pi}d\varphi\,(\sin^2\varphi - \cos^2\varphi)}_{=0}$

$\Rightarrow I_1 = I_2 \equiv I_\perp$.

$\Rightarrow I = \left(\begin{array}{cc|c}I_\perp & 0 & 0\\ 0 & I_\perp & 0\\ \hline 0 & 0 & I_3\end{array}\right) \to$ any two $\perp$ axes in x_1x_2-plane are principal axes.

Angular momentum of a rigid body

$\vec L$ in non-inertial frame

$$\vec L = \sum m(\vec r\times\vec v) = \sum m\left[\vec\Omega r^2 - \vec r\left(\vec\Omega\cdot\vec r\right)\right]$$
$$L_i = \boxed{I_{ij}\Omega_j} \quad \vec L = I * \vec\Omega \tag{92}$$

If $(x_1 x_2 x_3)$ are principal axis, $L_1 = I_1\Omega_1$, $L_2 = I_2\Omega_2$, $L_3 = I_3\Omega_3$

Free motion of a rigid body

angular momentum is conserved if no external torque. Motion in inertial COM frame is simplier.

- *ex motion of a symmetric top* $I_1 = I_2 = I_3 = I, \quad \tilde I = I\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$

$\vec L = I\vec\Omega \to \dot{\vec L} = 0 \Rightarrow \dot{\vec\Omega} = 0$ Uniform rotation about fixed axis paralle to $\vec L$

- *ex rigid rotor* $I_1 = I_2 = \sum mx_3^2, \quad I_3 = 0$

$\vec L = I\vec\Omega, \quad \vec\Omega\perp x_3$ by geometry We have $\dot{\vec\Omega} = 0 \Rightarrow$ Motion is unif in plane perp to $\vec\Omega$ and that it stays in that plane.

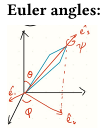
- *ex asymmetric top* $I_1 = I_2 = I_\perp \neq I_3 \Rightarrow \tilde I = \begin{pmatrix}I_\perp & 0 & 0\\ 0 & I_\perp & 0\\ 0 & 0 & I_3\end{pmatrix}$ x_3 is symm. axis,

  for any orthogonal axes

Rigid body EOM

$$\begin{cases}\dot{\vec p} = \vec F\\ \dot{\vec L} = \vec K \text{ torque}\end{cases} \tag{93}$$

Euler angles: ψ spin, θ nutation, φ precession



$(\theta\in[0,\pi], \varphi\in[0,2\pi], \psi\in[0,2\pi])$ in turns of rotation $R = R(\hat z,\varphi)R\left(\hat X,\theta\right)R\left(\hat Z,\psi\right)$

The lagrangian in Euler angles

- First: $T = \frac{1}{2}\left(I_1\Omega_1^2 + I_2\Omega_2^2 + I_3\Omega_3^2\right)$
- Rotation in components:

$$\Omega_1 = \dot\varphi\sin\theta\sin\psi + \dot\theta\cos\psi$$
$$\Omega_2 = \dot\varphi\sin\theta\cos\psi - \dot\theta\sin\psi \tag{94}$$
$$\Omega_3 = \dot\varphi\cos\theta + \dot\psi$$

- $T = \frac{1}{2}I_1\left(\dot\varphi\sin\theta\sin\psi + \dot\theta\cos\psi\right)^2 + \frac{1}{2}I_2\left(\dot\varphi\sin\theta\cos\psi - \dot\theta\sin\psi\right)^2 + \frac{1}{2}I_3\left(\dot\varphi\cos\theta + \dot\psi\right)^2$
- $L\left(\theta,\varphi,\psi,\dot\theta,\dot\varphi,\dot\psi\right) = T - U$

Free motion of symmetric top in Euler angles

$I_1 = I_2 = I_\perp \Rightarrow \quad T = \frac{1}{2}I_\perp\left(\dot\theta^2 + \dot\varphi^2\sin^2\theta\right) + \frac{1}{2}I_3\left(\dot\varphi\cos\theta + \dot\psi\right)^2$

$\Omega_\perp = L_z/I_\perp, \quad \Omega_3 = L_z\cos\theta/I_3 \quad$ E-L ->

$$\theta: \frac{\mathrm{d}}{\mathrm{d}t}I_\perp\dot\theta = I_\perp\sin\theta\cos\theta\,\dot\varphi^2 - I_3\dot\varphi\sin\theta\left(\dot\varphi\cos\theta + \dot\psi\right)$$
$$\varphi: \frac{\mathrm{d}}{\mathrm{d}t}\left(I_\perp\dot\varphi\sin^2\theta + I_3\cos\theta\left(\dot\varphi\cos\theta + \dot\psi\right)\right) = 0 \tag{95}$$
$$\psi: \frac{\mathrm{d}}{\mathrm{d}t}I_3\left(\dot\varphi\cos\theta + \dot\psi\right) = 0$$

choosing $\hat z$ along the angular momentum, we have $L_3 = L_z\cos\theta = I_3\Omega_3 = I_3\left(\dot\varphi\cos\theta + \dot\psi\right)$

$\Rightarrow \dot L_3 = \text{const} \Rightarrow \theta = \text{const} \quad \Omega_3 = \frac{L_z\cos\theta}{I_3} \quad \dot\varphi = \frac{L_3}{I_\perp\cos\theta} = \frac{L_z}{I_\perp} = \text{const}$

- *ex heavy symmetric top with one pt fixed* By paralle axis thm, $I'_{ij}I_{ij} + M\left(l^2\delta_{ij} - l_i l_j\right)$

  $\Rightarrow I'_\perp = I_\perp + Ml^2, \quad I'_3 = I_3, \quad U = mgZ = Mgl\cos\theta$

  $\Rightarrow L = T - U = \frac{1}{2}I'_\perp\left(\dot\theta^2 + \dot\varphi^2\sin^2\theta\right) + \frac{1}{2}I_3\left(\dot\psi + \dot\varphi\cos\theta\right)^2 = Mgl\cos\theta$

  E-L :

$$L_z = p_\varphi = \left(I'_\perp\sin^2\theta + I_3\cos^2\theta\right)\dot\varphi \quad \text{const}$$
$$L_3 = p_\psi = I_3\left(\dot\psi + \varphi\cos\theta\right) \quad \text{const} \tag{96}$$

Considering energy conservation

$$E = T + U \Rightarrow \underbrace{E - \frac{L_3^2}{2I_3} - Mgl}_{E'} = \frac{1}{2}I'_\perp\dot\theta^2 + \underbrace{\frac{1}{2I'_\perp}\frac{(L_z - L_3\cos\theta)^2}{\sin^2\theta} - Mgl(1-\cos\theta)}_{U_{\text{eff}}(\theta)} \tag{97}$$

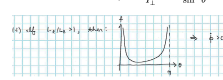
effective 1 dof problem. recognizing

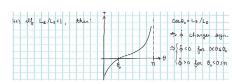
$$\dot\theta = \frac{\mathrm{d}\theta}{\mathrm{d}t} \Rightarrow \quad t = \int\frac{d\theta}{\left(\sqrt{2[E - U_{\text{eff}}(\theta)]/I'_\perp}\right.} \tag{98}$$

Considering U_eff: when $\theta = 0, L_z = L_3$when $\theta\approx 0 \Rightarrow \quad U_{\text{eff}} \approx \left(\frac{L_3^2}{8I'_\perp} - \frac{Mgl}{2}\right)\theta^2$

Motion about $\theta = 0$ stable if $L_3^2 > 4I'_\perp Mgl \Rightarrow \Omega_3^2 > 4I'_\perp Mgl/I_3^2$, or stable if sping ab. symm. axis is fast enough.

- Nutuation: cosider $\dot\varphi = \frac{L_3}{I'_\perp}\frac{(L_z/L_3) - (\cos\theta)}{\sin^2\theta} = \frac{L_3}{I'_\perp}f(\theta)$





considering the sign and trends of $f(\theta)$ given constrains on theta, we can differentiate different nutation motion. If θ_0 in graph 2 is out of range, the nutation is smooth; if θ_0 is in range, the nutation is oscillatory(will change sign and spin in spiral.); if θ_0 is on the endpoint of our constrained range, the nutation is spiky and "not smooth" at points.

Euler equations

set body frame $(X,Y,Z) = (\hat e_1^0, \hat e_2^0, \hat e_3^0)$, space frame $(x_1,x_2,x_3) = (\hat e_1, \hat e_2, \hat e_3)$ Set any vector $\vec A = \sum A_i^0\hat e_i^0 = \sum A_i\hat e_i$ By magic of vec analysis,

$$\left(\frac{\mathrm{d}\vec A}{\mathrm{d}t}\right)_{\text{Space}} = \left(\frac{\mathrm{d}\vec A}{\mathrm{d}t}\right)_{\text{Body}} + \vec\Omega\times\vec A_{\text{Space}} \tag{99}$$

When applied to $\left(\frac{\mathrm{d}\vec L}{\mathrm{d}t}\right)_{\text{Space}} = \vec{\mathrm{K}} = \left(\frac{\mathrm{d}\vec L}{\mathrm{d}t}\right)_{\text{body}} + \vec\Omega\times\vec L$, recognizing $L_i = I_i\Omega_i$:

$$I_1\dot\Omega_1 + (I_3 - I_2)\Omega_2\Omega_3 = K_1$$
$$I_2\dot\Omega_2 + (I_1 - I_3)\Omega_3\Omega_1 = K_2 \tag{100}$$
$$I_3\dot\Omega_3 + (I_2 - I_1)\Omega_1\Omega_2 = K_3$$

$K_i = 0$ if $\vec L$ is conserved on i axis.

- *ex symmetric top* $I_1 = I_2 = I, \vec K = 0 \quad \left(\dot\Omega_1 + \frac{I_3 - I_1}{I_\perp}\Omega_2\Omega_3 = 0; \dot\Omega_2 + \frac{I_1 - I_3}{I_\perp}\Omega_3\Omega_1 = 0; \dot\Omega_3 = 0\right)$ let $\omega = ((I_3 - I_\perp)/(I_\perp))\Omega_3 \Rightarrow$
  $\boxed{\left(\Omega_1 = A\cos\omega t; \Omega_2 = -\frac{1}{\omega}\dot\Omega_1 = +A\sin\omega t\right)}$

Motion in non-inertial frame

- Set non-inertial frame with velocity $\vec V(t)$, $\vec A = \dot{\vec V}$, $\quad \vec v = \vec v' + \vec V(t)$ where $\vec v'$ is velocity w.r.t. non-inertial frame.

lagrangian $L' = \frac{1}{2}mv'^2 - m\vec r'\cdot\vec A - U$, using E-L eq: $m\dot{\vec v}' = -\frac{\partial U}{\partial\vec r'} - m\vec A$

- *ex pendulum in acc. car* $\quad m\ddot{\vec r} = \vec T + m\vec g - m\vec A$,

  finding equil. angle: $\vec T = -m\left(\vec g - \vec A\right) = -m\vec g_{\text{eff}}$, then use geometry between $\vec g, -\vec A \Rightarrow \tan\varphi_0 = \frac{A}{g}$. Oscillation freq. $\omega = \sqrt{g_{\text{eff}}/l}$

Motion in rotating frame

Set rotation with $\vec\Omega$, $L = \frac{1}{2}mv^2 + \overline{m\vec v}\cdot\left(\vec\Omega\times\vec r\right) + \frac{1}{2}m\left(\vec\Omega\times\vec r\right)^2 - m\vec r\cdot\vec A - U$

Using E-L,

$$\boxed{m\dot{\vec v} = -\frac{\partial U}{\partial\vec r} - m\vec A + 2m\left(\vec v\times\vec\Omega\right) + m\vec\Omega\times\left(\vec r\times\vec\Omega\right) + m\vec r\times\dot{\vec\Omega}}$$

- Namely,

$$m\dot{\vec v} = -\frac{\partial U}{\partial\vec r} + \vec F_{\text{cor}} + \vec F_{\text{cent}}$$
$$\vec F_{\text{Cor}} = 2m\left(\vec v\times\vec\Omega\right), \quad \vec F_{\text{cent}} = m\vec\Omega\times\left(\vec r\times\vec\Omega\right) = m\left(\vec\Omega\times\vec r\right)\times\vec\Omega \tag{101}$$

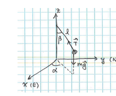
- *ex free fall on earth, centrifugal force* $\vec F = \vec g_0 + m\Omega^2R\sin\theta\hat\rho \Rightarrow \vec g_{\text{eff}} = \vec g_0 + \Omega^2R\sin\theta\hat\rho$
- *ex free fall, coriolis force* $\quad \dot{\vec v} = \vec g + 2\vec v\times\vec\Omega, \quad \vec\Omega = \Omega\sin\theta\hat y + \Omega\cos\theta\hat z$

In components,

$$\dot{\vec v_x} = 2\Omega\left(v_y\cos\theta - v_z\sin\theta\right)$$
$$\dot{\vec v_y} = -2\Omega v_x\cos\theta \tag{102}$$
$$\dot{\vec v_z} = 2\Omega v_x\sin\theta - g$$

Free fall EOM: $\vec R = \int v\,\mathrm{d}r$, consider $\vec v = \vec v_1 + \vec v_2 = -\vec g + 2\vec v_1\times\vec\Omega + 2\vec v_2\times\vec\Omega$ where approximately, $\vec v_2 = 2(\vec v_0 - gt\hat z)\times\vec\Omega$. If no initial velocity, integrating velocity in x components gives, $x(t) = \frac{1}{3}g\Omega\left(\frac{2h}{g}\right)^{3/2}\sin\theta$

- *ex foucaults pendulum* EOM



$$\vec r = l\sin\beta\cos\alpha\hat x + l\sin\beta\sin\alpha\hat y + (l - l\cos\beta)\hat z$$
$$\vec T = -T\sin\beta\cos\alpha\hat x - T\sin\beta\sin\alpha\hat y + T\cos\beta\hat z$$
$$\vec\Omega = \Omega\sin\theta\hat y + \Omega\cos\theta\hat z$$

$$\begin{cases}T = mg\\ m\ddot x = T_x + 2m\hat x\cdot\left(\dot{\vec r}\times\vec\Omega\right) = -\frac{mgx}{l} + 2m\Omega\dot y\cos\theta\\ m\ddot y = -\frac{mgy}{l} - 2m\Omega\dot x\cos\theta\end{cases} \tag{103}$$

letting $\omega^2 = \frac{g}{l}, \Omega_z = \Omega\cos\theta$, $\quad \boxed{\eta = x + iy = e^{i\gamma t}}$

$$\ddot x + \omega^2 x = 2\Omega_z\dot y, \ddot y + \omega^2 y = -2\Omega_z\dot x$$
$$\gamma = -\Omega_z \pm \sqrt{\omega^2 - \Omega_z^2}$$
$$\eta(t) = ae^{-i\Omega_z t}\cos\omega t \tag{104}$$
$$\Rightarrow \begin{cases}x = a\cos\Omega_z t\cos\omega t\\ y = a\sin\Omega_z t\cos\omega t\end{cases}$$

Hamiltonian Mechanics

$H(q,p,t) = \sum_{j=1}^n p_j\dot q_j - L(q,\dot q,t) \quad$ 1D: $H = \frac{p^2}{2m} + U(x)$

- Hamilton's equation $\dot q_i = \frac{\partial H}{\partial p_i} \quad \dot p_i = -\frac{\partial H}{\partial q_i}$
- *ex particle in polar*

$$L = T - U = \frac{1}{2}m\left(\dot r^2 + r^2\dot\varphi^2\right) - U(r,\varphi) \Rightarrow \quad p_r = \frac{\partial L}{\partial\dot r} = m\dot r, p_\varphi = \frac{\partial L}{\partial\dot\varphi} = mr^2\dot\varphi \tag{105}$$

$$H = p_r\dot r + p_\varphi\dot\varphi - L = \frac{p_r^2}{2m} + \frac{p_\varphi^2}{2mr^2} \Rightarrow \quad \dot r = \frac{\partial H}{\partial p_r} = \frac{p_r}{m}, \quad \dot\varphi = \frac{\partial H}{\partial p_\varphi} = \frac{p_\varphi}{mr^2}$$
$$\dot p_r = -\frac{\partial H}{\partial r} = \frac{p_\varphi^2}{mr^3} - \frac{\partial U}{\partial r}, \quad \dot p_\varphi = -\frac{\partial H}{\partial\varphi} = -\frac{\partial U}{\partial\varphi} \tag{106}$$

Phase space

- *ex harmonic oscillator* $H = \frac{p^2}{2m} + \left(\frac{1}{2}\right)m\omega^2x^2, \quad \omega = \sqrt{\frac{k}{m}}$

$$\left\{\dot x = \frac{\partial H}{\partial p} = \frac{p}{m}, \quad \dot p = -\frac{\partial H}{\partial x} = -m\omega^2 x\right\} \Rightarrow \quad \left\{\dot q = \frac{p}{m}, \quad \dot p = -m\omega^2 q\right\} \tag{107}$$

$q(t_0 + \delta t) = q(t_0) + \dot q\delta t = q_0 + \frac{p}{m}\delta t; \quad p(t_0 + \delta t) = p(t_0) + \dot p\delta t = p_0 - m\omega^2 q\delta t$ parametric ellipse in phase space.

Liouville's thm

volume of a region op phase space is conserved under time evolution, when boundary of volume and all pts inside move along their orit for some amount of time.

Poisson bracket

Time evolution of an observable $A(q,p,t)$:

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{\partial A}{\partial t} + \underbrace{\sum_{i=1}^n\frac{\partial A}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial H}{\partial q_i}}_{\equiv\{A,H\}} \tag{108}$$

More generally, for $A(q,p,t), \quad B(q,p,t)$

$$\{A, B\} = \sum_i \frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i} \tag{109}$$

notice, $\{A, p_i\} = \frac{\partial A}{\partial q_i}, \{A, q_i\} = -\frac{\partial A}{\partial p_i}$

- When

$$\frac{\mathrm{d}C}{\mathrm{d}t} = \frac{\partial C}{\partial t} + \{C, H\} = 0 \tag{110}$$

then $C(q,p,t)$ is conserved.

Cononical transformation

consider transformation $q_i \to Q_i(q,t)$ the transformation is canonical iff the transformation leave the form of Hamilton's eq. unchanged.

$$\begin{cases} \dot{q} = \frac{\partial H}{\partial p} \\ \dot{p} = -\frac{\partial H}{\partial q} \end{cases} \Rightarrow \texttt{cases}\ \dot{Q} = \frac{\partial K}{\partial P}, \dot{P} = -\frac{\partial K}{\partial Q} \tag{111}$$

where $K(Q,P,t)$ new Hamiltonian.

Exerts from practice problems

constraints, small Oscillations

A particle of massmmoves withoutfriction on the inside wall of an axially symmetric vessel given by $z = b^2(x^2+y^2)$

- KE in cylindrical coords:

$$\begin{aligned} T &= \frac{1}{2}m\left(\dot{\rho}^2 + \rho^2\dot{\theta}^2 + \dot{z}^2\right), \quad \dot{z} = b\dot{\rho}\rho \Rightarrow \\ L &= \frac{m}{2}\left[\dot{\rho}^2(1+b^2\rho^2) + \rho^2\dot{\theta}^2\right] - \frac{mgb}{2}\rho^2 \end{aligned} \tag{112}$$

E-L:

$$\begin{aligned} \ddot{\rho}(1+b^2\rho^2) + b^2\dot{\rho}^2\rho - \rho\dot{\theta}^2 + gb\rho &= 0 \\ m\rho^2\dot{\theta} = \text{const} \equiv M &\quad \text{conserved angular momentum} \end{aligned} \tag{113}$$

- energy and angular momentum given z_0, b, g, m

$$E = \frac{m}{2}\left[\dot{\rho}^2(1+b^2\rho^2) + \rho^2\dot{\theta}^2\right] + \frac{mgb}{2}\rho^2 \tag{114}$$

For a fixed z_0, ρ_0 is the equilibrium position, and $\dot{\rho} = 0$, then

$$\begin{aligned} E &= \frac{m}{2}\rho_0^2\dot{\theta}^2 + mgb\frac{\rho_0^2}{2} \\ \dot{\theta}^2 &= gb \\ \Rightarrow E &= 2mgz_0 \end{aligned} \tag{115}$$

plugging in $\dot{\theta}, \rho = \rho_0$, we have $M = 2mz_0\sqrt{\frac{g}{b}}$

- frequency of small oscillations about equilibrium perturbation: $\rho = \rho_0 + \varepsilon$, neglecting anything with ε^2 , EOM of rho is

$$\ddot{\varepsilon}(1+b^2\rho_0^2) - \rho\dot{\theta}^2 + gb\rho_0 + gb\varepsilon = 0 \tag{116}$$

want to know $\rho\dot{\theta}^2$, can be found from θ EOM

$$\begin{aligned} \rho\dot{\theta}^2 = \frac{M^2}{m^2\rho^3} = \frac{M^2}{m^2\rho_0^3}\left(\frac{1}{\left(1+\frac{\varepsilon}{\rho_0}\right)^3}\right) &\approx \frac{M^2}{m^3\rho_0^4}\left(1 - 3\frac{\varepsilon}{\rho_0}\right) \\ &= b\rho_0 g - 3bg\varepsilon \end{aligned} \tag{117}$$

Plugging in to rho EOM, we have

$$\begin{aligned} \ddot{\varepsilon}(1+2bz_0) + 4gb\varepsilon &= 0 \\ \ddot{\varepsilon} = -\omega^2\varepsilon, \Omega^2 &= \frac{4gb}{1+2bz_0} \end{aligned} \tag{118}$$

Conservation laws

two particles of $\{m_1, q_1, \vec{r}_1\}, \{m_2, q_2, \vec{r}_2$ in capacitor with $\vec{E} = E_0\hat{z}$, particles interact with $U(r_1, r_2) = \frac{k}{|\vec{r_1} - \vec{r_2}|}e^{-\frac{|\vec{r_1}-\vec{r_2}|}{\lambda}}$. List all conserved quantities and associate each with a specific symmetry of the problem.

- lagrangian $L = \frac{1}{2}m_1\dot{\vec{r}}_1^2 + \frac{1}{2}m_2\dot{\vec{r}}_2^2 - U + E_0(q_1z_1 + q_2z_2)$. Setting $\vec{r} = (x,y,z) = \vec{r}_1 - \vec{r}_2, \vec{R} = (X,Y,Z) = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{M}, \mu = \frac{m_1m_1}{M}$, we can have

$$L = \left[\frac{1}{2}M\dot{\vec{R}}^2 + (q_1+q_2)E_0Z\right] + \left[\frac{1}{2}\mu\dot{\vec{r}}^2 - U(r) + \frac{q_1m_2 - q_2m_2}{M}\ E_0z\right] \tag{119}$$

Observe: momenta $P_x = \frac{\partial L}{\partial \dot{X}}, P_y = \frac{\partial L}{\partial \dot{Y}}$ are conserved. Invariance under time translation gives conserved energy

$$E = \frac{\partial L}{\partial \dot{R}}\dot{R} + \frac{\partial L}{\partial \dot{r}}\dot{r} - L \tag{120}$$

Angular momentum $L_{\text{ttl}} = \vec{r_1} \times \vec{p_1} + \vec{r_2} \times \vec{p_2} = M\vec{R} \times \dot{\vec{R}} + \mu\vec{r} \times \dot{\vec{r}} = \vec{R} \times \vec{P} + \vec{r} \times \vec{p}$. Invariance under rotation about $\hat{z} : R \to R + \varepsilon\hat{z} \times R, \quad r \to \vec{r} + \varepsilon\hat{z} \times \vec{r}$ gives conserved $L_z = \left(\vec{R} \times \vec{P}\right)_z \quad l_z = [\vec{r} \times \vec{p}]_z$.

Normal modes

A system of N particles with masses m_i moves around a circle of radius a, with position angle θ_i. Interaction potential $U = \frac{k}{2}\sum_1^N\left(\theta_{j+1} - \theta_j\right)^2$., with $\theta_{N+1} = \theta_1 + 2\pi$. lagrangian of system is $\frac{a^2}{2}\sum_1^N m_j\dot{\theta}_j^2 - U$

- show Largrangian for particle i, show system in equalibrium when particles are equally spaced.

$$L = \frac{a^2}{2}\sum_1^N m_j\dot{\theta}_j^2 - \frac{k}{2}\sum_1^N\left(\theta_{j+1} - \theta_j\right)^2 \tag{121}$$

E-L for θ_i : $\quad a^2m_i\ddot{\theta}_i = k(\theta_{i+1} - \theta_i) - k(\theta_i - \theta_{i-1}) = -k[2\theta_i - (\theta_{i+1} + \theta_{i-1})]$
When eqaully spaced, $\theta_i = \frac{2\pi i}{N}$, thus $\ddot{\theta}_i = 0$ for all particles, thus equalibrium.

- show the system always has a normal mode of osc. with 0 freq.

$$\mathbb{M} \cdot \ddot{\vec{\theta}} = -\mathbb{K} \cdot \vec{\theta}, \quad M_{ij} = a^2m_i\delta_{ij}, \quad K_{ij} = k\left(2\delta_{i,j} - \delta_{i,j+1} - \delta_{i,j-1}\right) \tag{122}$$

take anstaz susbsitution $\vec{\theta} \to \vec{z} = \vec{b}e^{i\omega t}$ gives $\omega^2\mathbb{M} \cdot \vec{b} = \mathbb{K} \cdot \vec{b}$, where $\vec{b}$ is a constant vec. Look for a 0 freq $\omega = 0$, $\mathbb{K} \cdot \vec{b} = 0$ holds, so $b_i = b$. let $b = \Theta(t)$, knowing $\ddot{\Theta} = 0$ recall our subsitution, the time evo of $\theta_{i(t)} = \Theta_0 + \Theta_1 t$ i.e. trajectory is all masses rotating at same rate Θ_1

- find all normal modes when $N = 2, M_1 = km/a^2, m_2 = 2km/a^2$. Using standard normal mode analysis, for $N = 2$,

Appendix

1. Taylor expansion:

$$f(x)|_0 \approx f(a) + f'(a)(x-a) + f''(a)\frac{(x-a)^2}{2} \qquad 123$$

2. small angle approximation:

$$\sin(\theta) \approx \theta \quad \cos(\theta) \approx 1 - \frac{\theta^2}{2} \qquad 124$$