Small Oscillations

- Motion near a point of stable equilibrium.

DOF= 1 (one dimension)

- For a system of DOF = 1, with potential $U(q)$:
  - **stable equilibrium** at $U(q)_{\text{min}}$, upward parabola, where $F = -\frac{dU}{dq} = 0$
    - restoring force for small displacements $q - q_0$ is $F = -\frac{dU(q-q_0)}{dq}$
- **Unstable equilibrium** at $U(q)_{\text{max}}$, downward parabola, where $F = -\frac{dU}{dq} = 0$ as well.
- Consider small deviation from point of stable equilibrium, we use taylor expansion to show that it is really a small displacement. that is,

$$\begin{gathered} U(q) \approx U(q_0) + \frac{dU(q_0)}{dq}(q - q_0) + \left(\frac{1}{2}\right)\frac{d^2U(q_0)}{dq^2}(q - q_0)^2 + \dots \\ \text{while } \frac{dU(q_0)}{dq}(q - q_0) = 0 \end{gathered} \tag{1}$$

letting $x = q - q_0$, we have

$$\begin{gathered} \begin{cases} U(x) = U(q_0) + \left(\frac{1}{2}\right)\frac{d^2U(q_0)}{dq^2}x^2 \\ \text{putting into the form of } U(x) = U(x_0) + \left(\frac{1}{2}\right)kx^2. \end{cases} \\ \Rightarrow \boxed{k = \frac{d^2U(q_0)}{dq^2} > 0} \end{gathered} \tag{2}$$

we get KE, while choosing $U(q_0) = 0$:

$$\begin{gathered} T = \frac{1}{2}a(q)^2\dot{q}^2 = \frac{1}{2}a(q_0 + x)\dot{x}^2 \approx \frac{1}{2}m\dot{x}^2, \text{letting } m = a(q_0) \\ \Rightarrow \boxed{L = T - U = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2} \end{gathered} \tag{3}$$

EOM for DOF = 1 small Oscillations

using EL on Equation 3, we can get the EOM for one dimensional small Oscillations:

$$\begin{gathered} m\ddot{x} = -kx \\ \Rightarrow \ddot{x} + \omega_0^2 x = 0, \text{where } \boxed{\omega_0 = \sqrt{\frac{k}{m}} \text{ freq of osc.}} \end{gathered} \tag{4}$$

by magic of ODE, EOM reduces down to:

$$\begin{gathered} \boxed{x(t) = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t)} \\ \text{where } C_1, C_2 \text{ are constants} \end{gathered} \tag{5}$$

by trig magic, this could also be written as

$$
x(t) = a\cos(\omega_0 t + \alpha),
$$
$$
\text{where} \begin{cases} a = \sqrt{C_1^2 + C_2^2} & \text{amplitude of oscillation} \\ \omega_0 & \text{frequency of oscillation} \\ \tan\alpha = C_2/C_1 & \text{phase at t=0} \end{cases} \tag{6}
$$

energy for 1D small Oscillation
checking $\frac{\partial L}{\partial t} = 0 \Rightarrow$ energy-conservation:

$$
\begin{aligned} E = T + U &= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 \\ &= \frac{1}{2}ma^2\omega_0^2, [\text{ constant}] \end{aligned} \tag{7}
$$

Damped 1D oscillation, and Complex representation
[*I dont like the how the subscripts are used in this lecture but I guess this is what we are stuck with.*]

- when there is damping (friction, resistence, etc) $F_{\text{fric}} = -\beta\dot{x}$, the EOM becomes:

$$
\begin{gathered} \ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0, \\ \text{where } 2\gamma = \frac{\beta}{m}, \omega_0 = \sqrt{\frac{k}{m}} \end{gathered} \tag{8}
$$

with ansatz $x(t) = e^{rt}, \dot{x} = re^{rt}, \ddot{x} = r^2 e^{rt}$, the solution to Equation 8 is:

$$
\begin{gathered} r^2 + 2\gamma r + \omega_0^2 = 0, \\ \text{which has solution } r_+, r_- = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2} \\ \Rightarrow x(t) = C_1 e^{r_+ t} + C_2 e^{r_- t}, \end{gathered} \tag{9}
$$

notice the r subscripts here: r_+, r_-

underdamped, overdamped, and critically damped
Recall from your ODE class...

Equation 9 has the following 3 cases, each with different physical interpretation:

1. underdamped:

$$
\gamma < \omega_0 \Rightarrow 2 \text{ complex roots: } \begin{cases} r_\pm = -\gamma \pm i\sqrt{\omega_0^2 - \gamma^2} = -\gamma \pm i\omega \\ \omega = \sqrt{\omega_0^2 - \gamma^2} \end{cases} \tag{10}
$$

The EOM is thus a linear combination of two complex expoentials:

$$
\begin{aligned} x(t) &= e^{-\gamma t}(C_1 e^{i\omega t} + C_2 e^{-i\omega t}) \\ &= e^{-\gamma t}(A\cos(\omega t) + B\sin(\omega t)) \\ &\text{-- where } \begin{cases} A = C_1 + C_2 \\ B = i(C_1 - C_2) \end{cases} \\ &= ae^{-\gamma t}\cos(\omega t + \alpha) \\ &a, \alpha \text{ are constants} \end{aligned} \tag{11}
$$

"The solution is a damped oscillation with frequencyω, and amplitude expoentially decaying with time."

2. Overdameped

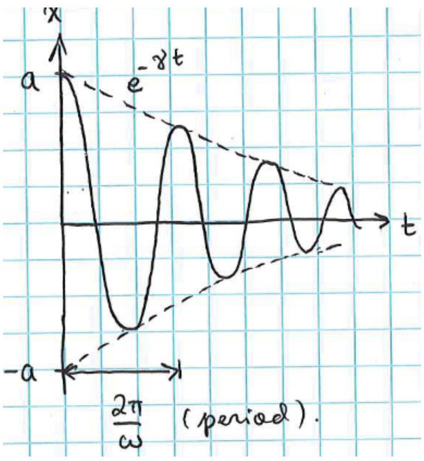
$$\gamma > \omega \Rightarrow x(t) = c_1 e^{-\gamma + \sqrt{\gamma^2 - \omega^2} t} + c_2 e^{-\gamma - \sqrt{\gamma^2 - \omega^2} t} \tag{12}$$

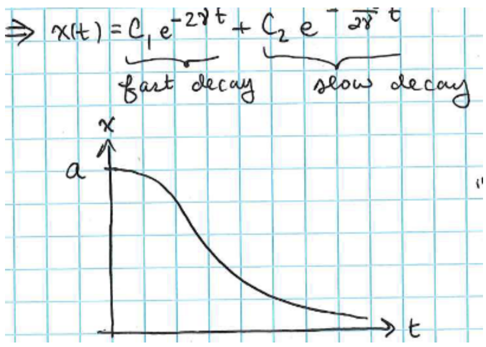
When

$$\gamma \gg \omega_0, \Rightarrow \begin{cases} \gamma + \sqrt{\gamma^2 - \omega_0^2} \approx 2\gamma \\ \gamma - \sqrt{\gamma^2 - \omega^2} = \frac{\omega^2}{2\gamma} \end{cases}$$
$$x(t) = c_1 e^{-2\gamma t} + c_2 e^{(-\omega_0^2/2\gamma)t} \tag{13}$$

3. Critically damped

$$\gamma = \omega_0 \Rightarrow x(t) = c_1 e^{-\gamma t} + c_2 t e^{-\gamma t} \tag{14}$$



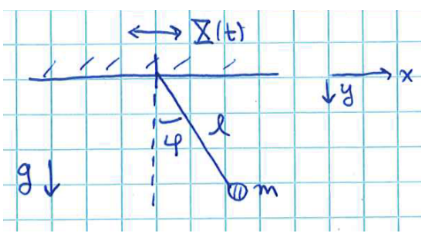


Forced Oscillations

When external force (F) is applied to the system, the largrangian becomes

$$L = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2 + F(t) x$$
$$\text{EL} \Rightarrow \ddot{x} + \omega_0^2 x = \frac{F(t)}{m}, \text{where } \omega_0 = \sqrt{\frac{k}{m}} \tag{15}$$

- Example: Simple pendulum with moving pivot



$$\begin{cases} x = X + l \sin\varphi \\ y = l \cos\varphi \end{cases} \Rightarrow \begin{cases} \dot{x} = \dot{X} + l\dot{\varphi}\cos\varphi \\ \dot{y} = -l\dot{\varphi}\sin\varphi \end{cases} \tag{16}$$
$$\Rightarrow L = T - U$$

$$
\begin{aligned}
L &= \frac{1}{2} m l^2 \dot{\varphi}^2 - mgl(1 - \cos\varphi) - ml\ddot{X}\sin\varphi \\
\text{Expand ab. } \varphi = 0 \Rightarrow L &= \frac{1}{2} m l^2 \dot{\varphi}^2 - \frac{1}{2} mgl\varphi^2 - ml\ddot{X}\varphi \\
\text{EL} \Rightarrow & \boxed{\ddot{\varphi} + \omega_0^2 \varphi = -\frac{\ddot{X}}{l} \text{ ,where } \omega_0 = \sqrt{\frac{g}{l}}}
\end{aligned} \tag{17}
$$

reintroducing damping via external forcing

$$
\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = f(t), f(t) = \frac{F(t)}{m} \tag{18}
$$

When damping $f(t) = f_0 \cos(\Omega t)$, solution via complex number:

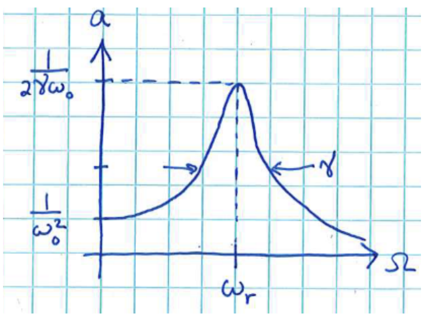
$$
\begin{aligned}
\ddot{z} + 2\gamma\dot{z} + \omega_0^2 &= f_0 e^{i\Omega t} \\
\text{ansatz } z(t) = z_0 e^{i\Omega t} \Rightarrow z_0 &= \frac{f_0}{\omega_0^2 + 2i\gamma\Omega + \Omega_0^2} \\
\boxed{z_0 = a(\Omega)\cos(\Omega t + \delta(\Omega)) f_0} & \text{ is a partcular solution,where} \\
& \begin{cases} a(\Omega) = \frac{1}{\sqrt{\left(\omega_0^2 - \Omega^2\right)^2 + (2\gamma\Omega)^2}} \\ \delta(\Omega) = \arctan\left(2\gamma \frac{\Omega}{\omega_0^2 - \Omega^2}\right) \end{cases}
\end{aligned} \tag{19}
$$

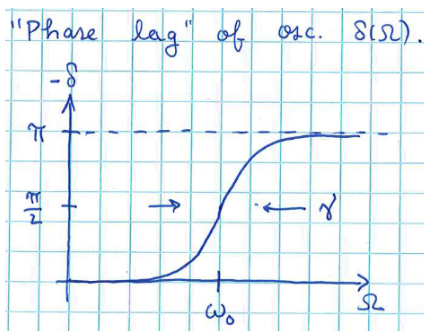
We can study the properties of the system by looking at the amplitude and phase of the solution.

- Amplitude:

$$
a(\Omega) = \frac{1}{\sqrt{\left(\omega_0^2 - \Omega^2\right)^2 + (2\gamma\Omega)^2}} \tag{20}
$$

  , when $\gamma \ll \omega_0$, response strongest and amplitude largest when $\omega_r = \omega_0$.





- Phase lag: $\tan\delta(\Omega) = 2\gamma\frac{\Omega}{\Omega^2 - \omega_0^2}$

  in phase as $\Omega \to 0$, and out of phase as $\Omega \to \omega_0$.

- Genral solution to sinusoidal forcing:

$$
\begin{aligned}
x(t) &= a(\Omega) f_0 \cos(\Omega t + \delta(\Omega)) + a_0 e^{-\gamma t}\cos(\omega t + \alpha) \\
&\overset{t > \frac{1}{r}}{\to} a(\Omega) f_0 \cos(\Omega t + \delta(\Omega))
\end{aligned} \tag{21}
$$

  Forgets initial condition after time.

- Power absorbed by oscillation

  $p = F\dot{x} = mf\dot{x}$ Avg power

$$P_{\text{avg}} = \frac{1}{T}\int_0^T mf\dot{x}\,\mathrm{d}t = -\frac{1}{2}mf_0 a(\Omega)\Omega\sin\delta(\Omega)$$
$$P(\Omega) = \gamma m f_0^2 \Omega^2 a^2(\Omega) \tag{22}$$

  Absorption around resonance frequency $\Omega = \omega_0 + \varepsilon$ is maximum:

$$P = \frac{\gamma m f_0^2}{4(\varepsilon^2 + \gamma^2)} \approx \frac{m f_0^2}{4\gamma} \tag{23}$$