

Brief Theory of Probability: Notes from MATH 431

Compiled by Harry Luo

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1 Sample Spaces, collection of events, probability measure

- Sample space Ω : set of all possible outcomes of an experiment. Comes in n-tuples where n represents number of repeated trials.
 - Collection of events \mathcal{F} : subset of state space to which we assign a probability.
 - Probability measure: function that assigns a probability to each event. $P : \mathcal{F} \rightarrow \mathbb{R}$.
 - Range is $[0, 1]$.
 - $P(\Omega) = 1$ and $P(\emptyset) = 0$
 - For pairwise disjoint events A_1, A_2, \dots ,
 $P(A_1 \cup A_2 \cup \dots) = P(A_1) + P(A_2) + \dots$
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2 Sampling: Uniform, Replacement, Order

- uniform sampling: each outcome is equally likely
- Binomial coeff

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \quad (1)$$

2.1 Replacement

- ex: sample K distinct marked balls from N balls in a box, **with** Replacement

$$\begin{aligned} \Omega &= \{1, 2, 3, \dots, N\}^K \\ \|\Omega\| &= N^K \end{aligned} \quad (2)$$

$$P(\text{none of the balls is marked 1}) = \frac{(N-1)^K}{N^K}$$

- ex: sample K distinct marked balls from N balls in a box, **without** Replacement

$$\begin{aligned} \Omega &= \{(i_1, i_2, \dots, i_K) \mid i_1, \dots, i_K \in \{1, 2, \dots, N\}, \text{distinct}\} \\ \|\Omega\| &= \binom{N-1}{K} \\ P(\text{none of the balls is marked 1}) &= \frac{\binom{N-1}{K}}{\binom{N}{K}} = \frac{N-K}{N} \end{aligned} \quad (3)$$

2.2 Order

- order matters: $A_n^k = \frac{n!}{(n-k)!}$
 - order doesn't matter: $\binom{n}{k} = C_n^k = \frac{n!}{k!(n-k)!}$
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3 Infinite Sample Spaces

3.1 discrete

$$\Omega = \{\infty, 1, 2, \dots\}$$

3.2 continuous

$$P([a', b']) = \frac{\text{length of } [a', b']}{\text{length of } [a, b]} \quad (4)$$

$$\text{single point, or sets of points: } P(\{x\}) = P(\cup_{i=1}^{\infty} \{x_i\}) = 0$$

- Complements: $P(A) = 1 - P(A^C)$
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4 Conditionial Probability, Law of Total Prob., Bayes' Theorem, Independence

4.1 Conditional prob.

$$P(A|B) = \frac{|A \cap B|}{|B|} \Rightarrow P(AB) = P(B)P(A|B) \quad (5)$$

(new sample space is B, total number of outcomes is $A \cap B$)

4.2 Law of total probability:

Given partitions B_1, B_2, \dots of Ω ,

$$P(A) = \sum_i P(A|B_i)P(B_i) \quad (6)$$

4.3 Bayes' Theorem:

Given events A, B, $P(A)$ and $P(B) > 0$,

$$P(B_i|A) = \frac{P(A|B_i)P(B_i)}{P(A)} \quad (7)$$

Considering the law of total prob., the generalized form, when B_i are partitions, is given as:

$$P(B_i|A) = \frac{P(A|B_i)P(B_i)}{\sum_j P(A|B_j)P(B_j)} \quad (8)$$

4.4 Independence:

$$P(AB) = P(A)P(B) \Leftrightarrow P(B|A) = P(B) \quad (9)$$

Note: By virtue of conventions, we write $A \cap B$ as AB in Probability.

If A,B,C,D are independent, it follows that $P(ABCD) = P(A)P(B)P(C)P(D)$; however, the inverse is not always true.

- Independence of Random Variables (messy as hell...)

Given 2 random variables

$$\begin{aligned} X_1 &\in \{x_{11}, x_{12}, x_{13}, \dots, x_{1m}\} \\ X_2 &\in \{x_{21}, x_{22}, x_{23}, \dots, x_{2n}\} \end{aligned} \quad (10)$$

Random variables X_1 and X_2 are independent \Leftrightarrow

$$P(X_1 = x_{1i}, X_2 = x_{2j}) = P(X_1 = x_{1i})P(X_2 = x_{2j})$$

Need to check $n \cdot m$ equations to verify independence.

4.5 Conditional Independence:

For events A_1, A_2, \dots, A_n, B , any set of events in A: A_{i1}, A_{i2}, A_{i3} , they are conditionally independent given B if

$$P(A_{i1}A_{i2}A_{i3}|B) = P(A_{i1}|B) * P(A_{i2}|B) * P(A_{i3}|B) \quad (11)$$

5 Independent Trials, Distributions

5.1 Bernoulli distribution:

a single trial, with success probability p, and failure probability 1-p. Parameter being the success probability.

$$X \sim \text{Ber}(p) \Rightarrow P(X = x) = p^x * (1 - p)^{1-x}, x \in \{0, 1\} \quad (12)$$

5.2 Binomial Distribution:

multiple independent Bernoulli trials, with success probability p, and failure probability 1-p. Parameters being the number of trials n and the success probability p .

$$X \sim \text{Bin}(n, p) \Rightarrow P(X = k) = \binom{n}{k} p^k * (1 - p)^{n-k}, k \in \{0, 1, \dots, n\} \quad (13)$$

5.3 Geometric distribution:

multiple independent Bernoulli trials with success probability p , while stopping the experiment at the first success.

$$X \sim \text{Geom}(p) = p * (1 - p)^{k-1}, k \in \{1, 2, \dots\} \quad (14)$$

5.4 Hypergeometric distribution:

There are N objects of type A, and $N_A - N$ objects of type B. Pick n objects without replacement. Denote number of A objects we picked as k . Parameters are N, N_A, n .

$$P(X = k) = \frac{\binom{N_A}{k} \binom{N - N_A}{n - k}}{\binom{N}{n}} \quad (15)$$

choose k from N_A , choose $n - k$ from $N - N_A$, divide by total number of ways to choose n from N

6 Probability Analysis: Probability Density Function

$$P(X \leq b) = \int_{-\infty}^b f(x) dx \quad (16)$$