HW 13 Harry Luo

1

recall cauchy integral formula

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0} \, dz \tag{1}$$

For our integral, $f(z) = z^3 + e^{z^2}$, $z_0 = 1 + i$ applying the CIF:

$$\int_C \frac{z^3 + e^{z^2}}{z - (1 + i)} \, dz = 2\pi i f(1 + i) = 2\pi i \left((1 + i)^3 + e^{(1+i)^2} \right)$$
$$= 2\pi i (e^{2i} + 2i - 2) \tag{2}$$

2

recall the CIF

$$f^n(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} \, dz \tag{3}$$

Here, $z_0 = 0, n = 1,$ let $f(z) = e^z + e^{z^3}$

$$\int_C \frac{f(z)}{z^2} \, dz = \frac{2\pi i}{1} f^{(1)}(0)$$
$$= 2\pi i \left(e^0 + 3(0)^2 e^0 \right) = \boxed{2\pi i} \tag{4}$$

3

Using the CIF, take $z_0 = -1, n = 2, f(z) = z^{2024} + 4z$

$$\int_C \frac{f(z)}{(z + 1)^3} \, dz = \frac{2\pi i}{2} f^{(2)}(-1) \tag{5}$$

where $f^{(2)}(-1) = 2024 * 2023(-1)^{2022} = 4094552$, Equation 5 becomes

$$\int_C \frac{z^{2024} + 4z}{(z + 1)^3} \, dz = \boxed{4094552\pi i} \tag{6}$$

4

Using the CIF, take $f(z) = \cos(z), z_0 = 0$

$$\int_{|z|=3} \frac{\cos(z)}{z^5} \, dz = 2\pi i \cos^{(4)}(0)$$
$$= 2\pi i \frac{\cos(0)}{4!} = \boxed{\frac{\pi i}{12}} \tag{7}$$

5

- (a)find poles

$$1 + z^2 = (z + i)(z - i) \Rightarrow z_1 = -i, z_2 = i$$
$$\text{Res}(f, z_1) = \lim_{z \to -i} \frac{z + i}{1 + z^2} = -\frac{1}{2i} \tag{8}$$
$$\text{Res}(f, z_2) = \lim_{z \to i} \frac{z - i}{1 + z^2} = \frac{1}{2i}$$

- (b)

Consider subsitution $f(z) = \frac{1}{1+x^2}$, we can use a semicircular contour in the upper half-plane, which will enclose only the pole at z = i.

$$\int_\gamma f(z)\,dz = \int_{-R}^R f(z)\,dz + \int_C f(z)\,dz = 2\pi i\ \text{Res}(f, z = i) + \underbrace{\int_C f(z)\,dz}_{(*)} \qquad 9$$

for the second term, parametrize $z = Re^{i\theta}$,

$$(*) = \int_C \frac{1}{1 + Re^{i2\theta}}\,d\theta, \quad \theta \in [0, \pi] \qquad 10$$

as $R \to \infty$, the second term becomes 0, and the integral becomes

$$\int_{-\infty}^{\infty} f(z)\,dz = 2\pi i\ \text{Res}(f, z = i)$$

$$= 2\pi i\left(\frac{1}{2i}\right) = \pi \qquad 11$$

Since $\frac{1}{1+x^2}$ is even,

$$\boxed{\int_0^\infty \frac{1}{1 + x^2}\,dx = \frac{\pi}{2}} \qquad 12$$

6
- (a)

Find all the poles

$$(1 + z^2)^2 = 0$$
$$z^2 = -1 \qquad 13$$
$$z = \pm i$$

Recall that we can find the residue by

$$\text{Res}(f, z_0) = \lim_{z \to z_0} \frac{1}{(n-1)!}\left(\frac{d}{dz}\right)^{n-1}\left[(z - z_0)^n f(z)\right] \qquad 14$$

Take $n = 2, z_0 = i$, we have

$$\text{Res}(f, i) = \lim_{z \to i}\frac{d}{dz}\left[(z - i)^2 f(z)\right] = \lim_{z \to i}\frac{d}{dz}\left[\frac{(z - i)^2}{(z - i)^2(z + i)^2}\right]$$

$$= \lim_{z \to i}\frac{d}{dz}\left[\frac{1}{(z + i)^2}\right]$$

$$= \lim_{z \to i} -\frac{2}{(z + i)^3} \qquad 15$$

$$\boxed{= -\frac{i}{4}}$$

similarly, take $n = 2, z_0 = -i$, we have

$$\text{Res}(f, -i) = \lim_{z \to -i} \frac{d}{dz}\left[\frac{(z-z_0)^2}{(z^2-i^2)^2}\right] = \lim_{z \to -i} \frac{d}{dz}\left[\frac{(z+i)^2}{(z+i)^2(z-i)^2}\right]$$

$$= \lim_{z \to -i} \frac{d}{dz}\left[(z-i)^{-2}\right]$$

$$= \lim_{z \to -i}\left(-2(z-i)^{-3}\right)$$
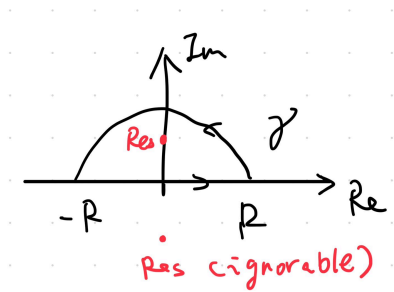
$$\boxed{= \frac{i}{4}}$$

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- (b)

recall Cauchy residue thm,

$$\int_\gamma f(z)\,dz = 2\pi i \sum_{k=1}^{n} \text{Res}(f, z_k)$$

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Consider subsitution $f(z) = \frac{1}{(1+z^2)^2}$, we can use a semicircular contour in the upper half-plane, which will enclose only the pole at z = i.

$$\int_\gamma f(z)\,dz = \int_{-R}^{R} f(z)\,dz + \int_C f(z)\,dz = 2\pi i\,\text{Res}(f, z=i)$$

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$$\Rightarrow \int_{-R}^{R} f(z)\,dz = 2\pi i\,\text{Res}(f, z=i) - \int_C f(z)\,dz$$

$$= 2\pi i\left(-\frac{i}{4}\right) - \underbrace{\int_C f(z)\,dz}_{(*)}$$

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for $(*)$, parametrize $z = Re^{i\theta}$,

$$(*) = \int_C \frac{1}{1+Re^{i\theta}}\,d\theta, \quad \theta \in [0, \pi]$$

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When $R \to \infty$, Equation 19 becomes

$$\int_{-\infty}^{\infty} f(z)\,dz = 2\pi i\left(-\frac{i}{4}\right) - \underbrace{\int_C \frac{1}{1+Re^{i\theta}}\,d\theta}_{\to 0}$$

$$= \frac{\pi}{2}$$

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Since $\frac{1}{(1+x^2)^2}$ is even,

$$\int_0^{\infty} \frac{1}{(1+x^2)^2}\,dx = \boxed{\frac{\pi}{4}}$$

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$$f(z) = \frac{z^2}{(z^2 + 1)(z^2 + 4)}$$

- (a)

$$\frac{z^2}{(z+i)(z-i)(z+2i)(z-2i)} \Rightarrow z_1 = i, z_2 = -i, z_3 = 2i, z_4 = -2i$$

$$\operatorname{Res}(f, z_1) = \lim_{z \to i} \frac{z^2}{(z+i)(z+2i)(z-2i)} = -\frac{1}{6i}$$

$$\operatorname{Res}(f, z_2) = \lim_{z \to -i} \frac{z^2}{(z-i)(z+2i)(z-2i)} = \frac{1}{6i}$$

$$\operatorname{Res}(f, z_3) = \lim_{z \to 2i} \frac{z^2}{(z-i)(z+i)(z+2i)} = \frac{1}{3i}$$

$$\operatorname{Res}(f, z_4) = \lim_{z \to -2i} \frac{z^2}{(z-i)(z+i)(z-2i)} = -\frac{1}{3i}$$

- (b)

Similarly to 6(b), we can use a semicircular contour in the upper half-plane, which will enclose only the poles at z = i, 2i. The contour integration over the complex arc is still 0 as $R \to \infty$

$$\int_\gamma f(z)\, \mathrm{d}z = \int_{-R}^{R} f(z)\, \mathrm{d}z + \int_C f(z)\, \mathrm{d}z$$

$$\Rightarrow \int_{-\infty}^{\infty} f(z)\, \mathrm{d}z = 2\pi i(\operatorname{Res}(f, z_1) + \operatorname{Res}(f, z_3)) = \frac{\pi}{3}$$

Since $f(x)$ is even,

$$\int_0^R f(x)\, \mathrm{d}x = \frac{1}{2} \int_{-\infty}^{\infty} f(x)\, \mathrm{d}x = \boxed{\frac{\pi}{6}}$$