Summary

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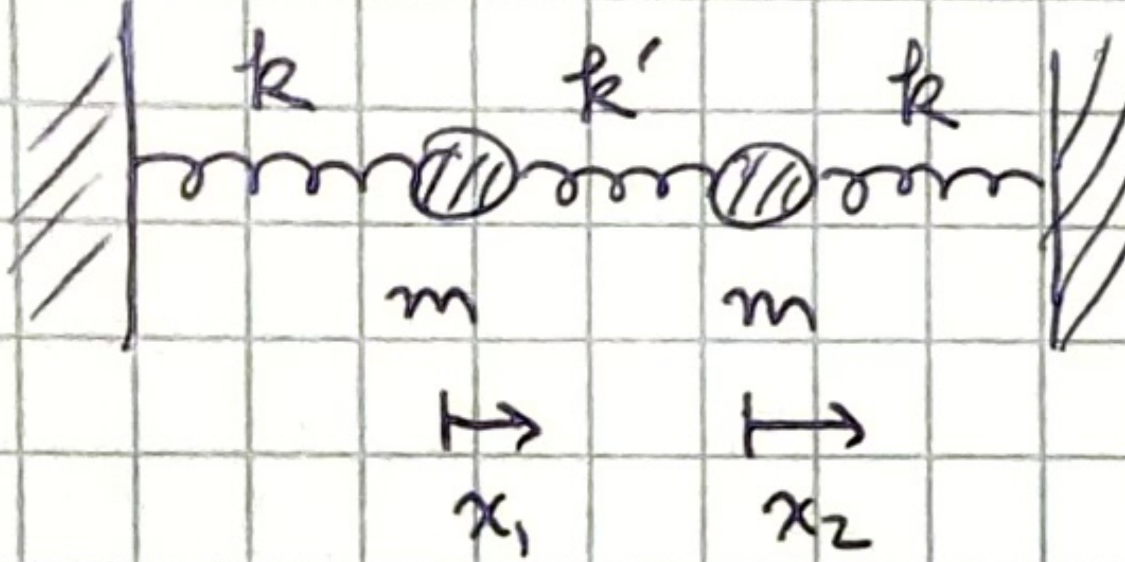
Studied in detail example of oscillations w/ 2 DOF:



$$L = \frac{1}{2} m \dot{x}_1^2 + \frac{1}{2} m \dot{x}_2^2 - \frac{1}{2} k x_1^2 - \frac{1}{2} k x_2^2 - \frac{1}{2} k' (x_1 - x_2)^2$$

EOM:

$$M \cdot \ddot{\vec{x}} + K \cdot \vec{x} = 0, \qquad \vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$M = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}, \quad K = \begin{pmatrix} k+k' & -k' \\ -k' & k+k' \end{pmatrix}.$$

$$\vec{x} = \mathrm{Re}\, \vec{z}, \quad \vec{z} = \vec{a} e^{i\omega t}$$

$$\to \quad (\omega^2 M - K) \cdot \vec{a} = 0 \quad \Rightarrow \quad \det(\omega^2 M - K) = 0.$$

$\to$ normal modes:

$$(1)\ \omega_-^2 = \frac{k}{m}, \quad \vec{x}_- = a_- \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos(\omega_- t + \delta_-)$$

$$(2)\ \omega_+^2 = \frac{k + 2k'}{m}, \quad \vec{x}_+ = a_+ \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos(\omega_+ t + \delta_+).$$

general sol.'n: $\vec{x}(t) = \vec{x}_-(t) + \vec{x}_+(t)$.

Our normal mode analysis also suggests a different choice of coord.'s, in which EOM will be particularly simple. Consider:

$$\begin{cases} Q_1 = \sqrt{\frac{m}{2}} (x_1 + x_2) \\ Q_2 = \sqrt{\frac{m}{2}} (x_1 - x_2) \end{cases}$$

(factor of $\sqrt{m/2}$ for later convenience)

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Plugging into L gives:

$$L = \frac{1}{2}\left(\dot{Q}_1^2 - \omega_-^2 Q_1\right) + \frac{1}{2}\left(\dot{Q}_2^2 - \omega_+^2 Q_2\right)$$

$\Rightarrow$ Lagrangian is that of decoupled oscillators w/ coord.'s Q_1 & Q_2.

EOM are especially simple in coord.'s Q_1 & Q_2:

$$Q_1: \quad \frac{d}{dt}\frac{\partial L}{\partial \ddot{Q}_1} = \frac{\partial L}{\partial Q_1} \quad \Rightarrow \quad \ddot{Q}_1 = -\omega_-^2 Q_1$$

$$Q_2: \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{Q}_2} = \frac{\partial L}{\partial Q_2} \quad \Rightarrow \quad \ddot{Q}_2 = -\omega_+^2 Q_2$$

Coord.'s $\{Q_1, Q_2\}$ are called "normal coord.'s", owing to simplicity of EOM. We'll see such coord.'s exist in any system executing small oscillations about pt. of stable equil.

Consider now general situation of system w/ n DOF.

Coord.'s $q = (q_1, q_2, \ldots, q_n)$.

$$L = T - U = \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}(q)\,\dot{q}_i \dot{q}_j - U(q).$$

Suppose $U(q)$ has min. at $q^{(0)} = (q_1^{(0)}, \ldots, q_n^{(0)})$; i.e.,

$$\left.\frac{\partial U(q)}{\partial q_i}\right|_{q^{(0)}} = 0, \quad i = 1, \ldots, n.$$

Let $x_i = q_i - q_i^{(0)}$ & expand L in powers of x:

- $$U(q) = \underbrace{U(q^{(0)})}_{\text{set to zero.}} + \sum_{i=1}^{n} \underbrace{\left.\frac{\partial U(q)}{\partial q_i}\right|_{q^{(0)}}}_{=0 \text{ by assumption}} x_i + \frac{1}{2}\sum_{i,j=1}^{n} \underbrace{\left.\frac{\partial^2 U(q)}{\partial q_i \partial q_j}\right|_{q^{(0)}}}_{\equiv k_{ij}} x_i x_j + \dots$$

$$\Rightarrow \quad U(x) = \frac{1}{2}\sum_{ij} k_{ij} x_i x_j \, , \qquad k_{ij} = \left.\frac{\partial^2 U(q)}{\partial q_i \partial q_j}\right|_{q^{(0)}}.$$

Note: $k_{ij} = k_{ji}$ by symmetry of partial derivatives.

- The kinetic energy is:

$$T = \frac{1}{2}\sum_{ij} a_{ij}(q)\, \dot{q}_i \dot{q}_j$$

small x

$$= \frac{1}{2}\sum_{ij} m_{ij}\, \dot{x}_i \dot{x}_j$$

$$m_{ij} = m_{ji} = a_{ij}(q^{(0)}).$$

So, for small displacements from equil.:

$$\boxed{L = \frac{1}{2}\sum_{ij} m_{ij}\dot{x}_i\dot{x}_j - \frac{1}{2}\sum_{ij} k_{ij} x_i x_j}$$

Notation: x_i = components of vector $\vec{x}$

m_{ij} = " " matrix M
k_{ij} = " " matrix K } symmetric matrices

$$\Rightarrow \quad L = \frac{1}{2}\dot{\vec{x}}^T \cdot M \cdot \dot{\vec{x}} - \frac{1}{2}\vec{x}^T \cdot K \cdot \vec{x}.$$

• E-L. eqn.'s:
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} = \frac{\partial L}{\partial x_i} \qquad i=1,\dots,n$$

$$\frac{\partial L}{\partial \dot{x}_i} = \sum_j m_{ij}\dot{x}_j \qquad \frac{\partial L}{\partial x_i} = \sum_j k_{ij} x_j$$

$$\Rightarrow \sum_j m_{ij}\ddot{x}_j + \sum_j k_{ij} x_j = 0, \qquad i=1,\dots,n.$$

→ system of n coupled second-order diff. eq.'s.

• now ~~proce~~ proceed as in the problem w/ ~~two~~ 2 DOF. Consider $x_i \to z_i \in \mathbb{C}$ & auxiliary problem:

$$\sum_j m_{ij}\ddot{z}_j + \sum_j k_{ij} z_j = 0$$

→ solve this system & then set $x_i = \mathrm{Re}\, z_i$.

Consider ansatz $z_i(t) = a_i e^{i\omega t}$, $a_i \in \mathbb{C}$

$$\Rightarrow \sum_j (\omega^2 m_{ij} - k_{ij})\, a_j = 0. \qquad (*).$$

$(*)$ corresponds to n eqn.'s (one for each i) in n unknowns $(a_1, \dots, a_n)$.

→ trivial sol.'n: $a_i = 0$, $i=1,\dots,n$

→ non-trivial sol.'n when $\det(\omega^2 M - K) = 0$.

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$$\underbrace{\det(\omega^2 M - K)}_{} = 0$$

polynomial of degree n
in ~~var~~ variable ω^2.

$\Rightarrow$ n roots ω_α^2, $\alpha = 1, \dots, n$

i.e., there is a discrete set of frequencies s.t. $(*)$ has non-trivial sol.'n

⌈ can be shown that $\omega_\alpha^2 \geq 0$ ⌋

· Now suppose we have solved for the ω_α^2.

→ For each ω_α^2, plug back into $(*)$:

$$\sum_j \left(\omega_\alpha^2 m_{ij} - k_{ij}\right) a_j^{(\alpha)} = 0 \rightarrow \text{solve for } a_j^{(\alpha)}.$$

For each α, $a_j^{(\alpha)}$ has $n-1$ indep. comp.'s, since $\det(\omega_\alpha^2 M - K) = 0$ $\Rightarrow$ relation btwn. equations.

· General motion is a superposition:

$$z_i(t) = \sum_\alpha C_\alpha a_i^{(\alpha)} e^{i\omega_\alpha t}, \qquad C_\alpha \in \mathbb{C}$$

$$\Rightarrow \quad x_i(t) = \sum_\alpha a_i^{(\alpha)} \operatorname{Re}\left\{ C_\alpha e^{i\omega_\alpha t} \right\}.$$

→ ~~motion of system in each coord. is the superpos~~

$\Rightarrow$ motion of each coord. is superposition of n simple harmonic motions. or "normal modes"