Power series

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of complex number. The formal series

$$S(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

is called a power series in z centered at z_0

Theorem (Existence and uniqueness of radius of convergence)

Let $S(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$

There is a unique number $R \in [0, \infty)$

such that

$S(z)$ converges if $|z - z_0| < R$

$S(z)$ diverges if $|z - z_0| > R$

Some typical examples

1) geometric series

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z} \qquad (R = 1)$$

for all $|z| < 1$

2) exponential function

$$\sum_{n=0}^{\infty} \frac{z^n}{n!} = e^z$$

for all $z \in \mathbb{C}$

$(R = +\infty)$

Theorem Suppose that the power series

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$ is

convergent in $|z - z_0| < R$ and R is the radius of convergence. Then $f'(z)$ is also a power series, with

$$f'(z) = \sum_{n=0}^{\infty} n\, a_n z^{n-1}$$

and f' has the same radius of convergence as f

Example $f(z) = \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \quad |z|<1$

$$f'(z) = \frac{1}{(1-z)^2} = \sum_{n=0}^{\infty} n z^{n-1}$$

$$= 1 + 2z + 3z^2 + \dots$$

for $|z| < 1$

In other words, a power series is infinitely complex differentiable in its disc of convergence, and their higher derivatives are also power series obtained by differentiating term by term.

Analytic function Let $\Omega \subseteq \mathbb{C}$ be an open set, a function $f : \Omega \to \mathbb{C}$ is said to be analytic at a point $z_0 \in \Omega$ if f can be expanded as

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

for all z in a open neighborhood of z_0

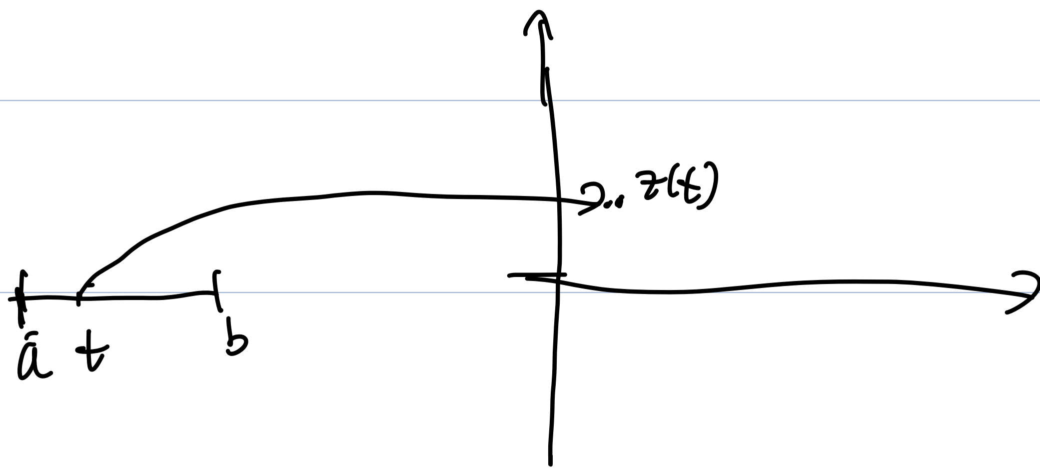
If f can be expanded as a power series for all points in Ω, we say that f is analytic in Ω.

Curve in $\mathbb{C}$

A parametrized curve in $\mathbb{C}$ is a function

$$z : [a,b] \longrightarrow \mathbb{C}$$

where a, b are given real numbers



A parametrized curve $z(t)$ is called <u>smooth</u>

if $z'(t)$ exists and continuous on $[a,b]$

A curve is closed if $z(a) = z(b)$

Integrating along curves

given a smooth curve γ in $\mathbb{C}$

parametrized by $z : [a,b] \longrightarrow \mathbb{C}$ and f

is a continuous function on γ, we define

the integral of f along γ by

$$\int_\gamma f(z)\,dz = \int_a^b f(z(t))\, z'(t)\, dt$$

Note that this is NOT the same as the line integral when f is viewed as $\mathbb{R}^2 \to \mathbb{R}$

Theorem 1) Integrations of continuous functions along curves is linear, namely for all $\alpha, \beta \in \mathbb{C}$

$$\int_\gamma (\alpha f + \beta g)\, dz = \alpha \int_\gamma f\, dz + \beta \int_\gamma g\, dz$$

2) If γ^- is γ with reverse orientation, then

$$\int_{\gamma^-} f\, dz = -\int_\gamma f\, dz$$

3)

$$\left| \int_\gamma f\, dz \right| \le \max_{z \in \gamma} |f(z)| \, \mathrm{length}(\gamma)$$

Example Evaluate $\int_\gamma \frac{1}{z}\,dz$

where γ is the unit circle in the counter clockwise direction

Ans: γ can be parametrized by

$$z(t) = e^{it} \qquad 0 \le t \le 2\pi$$

By def,

$$\int_\gamma f\,dz = \int_a^b f(z(t))\, z'(t)\,dt$$

$$= \int_0^{2\pi} e^{-it}\left(ie^{it}\right) dt$$

$$= \int_0^{2\pi} i\,dt$$

Hence $\int_\gamma \frac{1}{z}\,dz = 2\pi i$

Cauchy theorem: If f is holomorphic in an open set Ω that has no holes (simply connected) and $\gamma \subseteq \Omega$ is a close curve in Ω then

$$\int_\gamma f\,dz = 0$$

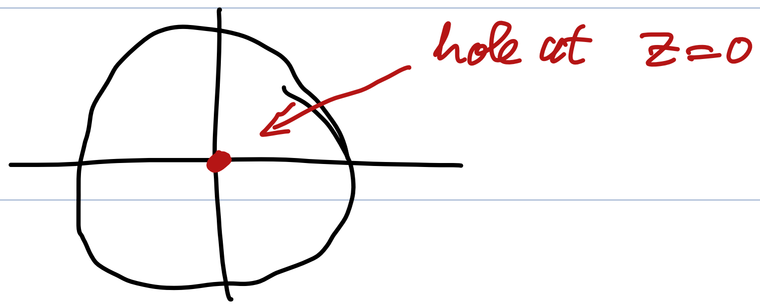
Note To apply the Cauchy theorem,

we need Ω to have no holes (simply connected)

For example, $f(z) = \frac{1}{z}$ is holomorphic

in $\{0 < |z| < 1\}$ but

$$\int_{C_r} \frac{1}{z}\,dz = 2\pi i$$ for all circles with radius $r < 1$



To show the Cauchy theorem, we need

several steps/lemmas/theorems in between

We will cover the general ideas (formal)

Theorem If f has an antiderivative in Ω

Namely $F'(z) = f(z)$ for all $z \in \Omega$

then $\int_\gamma f(z)\, dz = F(w_2) - F(w_1)$

for any curve in Ω that begins at w_1

and ends at w_2

Proof: If $z(t) : [a, b] \longrightarrow \mathbb{C}$

and $z(a) = w_1$, $z(b) = w_2$.

$$\int_\gamma f(z)\,dz = \int_a^b f(z(t))\, z'(t)\,dt$$

$$= \int_a^b F'(z(t))\, z'(t)\,dt$$

$$= \frac{d}{dt}\left(\int_a^b F(z(t))\,dt\right)$$

$$= F(z(b)) - F(z(a))$$

$$= F(w_2) - F(w_1)$$

Example $\quad \int_C (z^2 + e^z)\,dz = 0$

for any close curve C

$$\left(\frac{z^3}{3} + e^z\right)' = z^2 + e^z$$

Theorem Every holomorphic function f on a simply connected Ω has an antiderivative on Ω

Cauchy Integral Formula

Let f be holomorphic on Ω that contains the closure of a disc D.

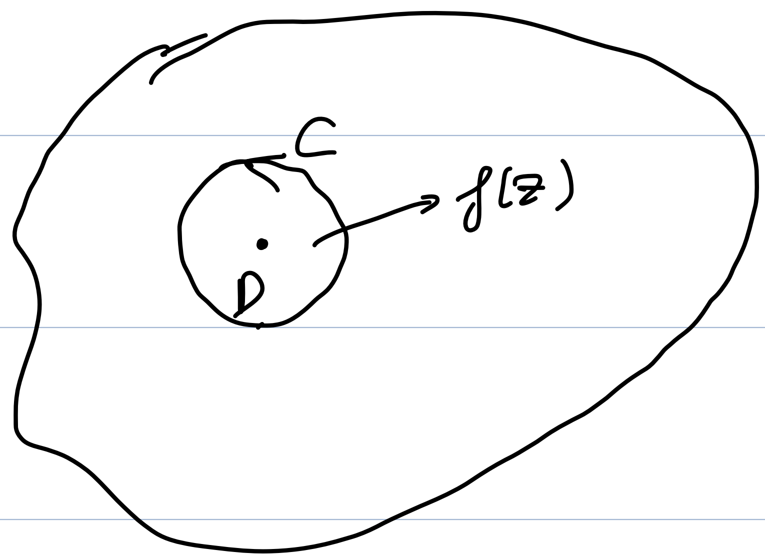
If C denotes the boundary circle of this disc with counter clockwise direction, then

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

for all $z \in D$

Moreover $$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(\xi)}{(\xi - z)^{n+1}} \, d\xi$$

for all $z \in D$



The proof can be found in a standard complex analysis book, we skip covering the proof.

<u>Theorem</u> (holomorphic $\Longleftrightarrow$ analytic)

If f is holomorphic on an open set Ω,

Let $z_0 \in \Omega$, then as long as

z lying inside a disk inside Ω, we have

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

where $a_n = \frac{1}{n!} f^{(n)}(z_0)$

$$\boxed{a_n = \frac{1}{2\pi i} \int_{C_r(z_0)} \frac{f(\xi)}{(\xi - z_0)^{n+1}} \, d\xi}$$

Cauchy inequality

If f is holomorphic in an open set Ω that contains the closure of a disc D centered at z_0 and of radius R, then

$$|f^{(n)}(z_0)| \leq \frac{n! \; \|f\|_C}{R^n}$$

where $\|f\|_C = \max_{z \in C} |f(z)|$ is the maximum value of $|f(z)|$ on the boundary circle $C = \partial D$

(Liouville theorem)

<u>Corollary</u> Any entire and bounded function must be constant.

<u>Proof</u>

$$|f'(z_0)| \leq \frac{1! \max_{z \in B_R} |f(z)|}{R}$$

$$\leq \frac{\max_{z \in \mathbb{C}} |f(z)|}{R}$$

Let $R \to \infty$, we have $f'(z_0) = 0$

This means $f' \equiv 0 \Rightarrow f = \text{const.}$

Analytic continuation (theorem) (OMIT, not covered)

Let Ω be an open and simply connected set of $\mathbb{C}$

Suppose that f, g are holomorphic in Ω

and $f = g$ on a sequence of points

that has limit in Ω, then

$$f(z) = g(z) \text{ for all } z \in \Omega$$

Order of a zero of complex holomorphic function

Let f be holomorphic in a connected open set of Ω. A point $z_0 \in \Omega$ is called zero of f if $f(z_0) = 0$.

Theorem Suppose that f is holomorphic in a connected open set Ω, and $f(z_0) = 0$ and f does not vanish identically on Ω, then there exist an open disk centered at z_0 in Ω and a non-vanishing function g on that disk and a unique $n \in \mathbb{N}$, $n \geq 1$ such that

$$\begin{cases} f(z) = (z - z_0)^n g(z) \text{ for all } z \in B_r(z_0) \\ g(z) \neq 0 \text{ for all } z \in B_r(z_0) \end{cases}$$

Types of singularities of complex functions

There are 3 types of singularity that a complex function can have, increasing by the order of severity

1. Removable singularity (skip)
2. Poles
3. Essential singularitys (skip)

Removable singularity

$z = z_0 \in \Omega$ is called a removable singularity of f if there exists $A \in \mathbb{C}$

such that the function

$$\tilde{f}(z) = \begin{cases} A & \text{if } z = z_0 \\ f(z) & \text{if } z \neq z_0 \end{cases}$$

is holomorphic on Ω

<u>Example</u> $f(z) = \dfrac{z^2 - 1}{z - 1}$ has a

removable singularity at $z = 1$

One can define $\tilde{f}(z) = z + 1$ for $z \in \mathbb{C}$

Poles

Let Ω be a open set of $\mathbb{C}$

The point $z = z_0$ is called a pole of order n of f if $\frac{1}{f}$ has a zero of order n at $z = z_0$

Namely, there exists $n \in \mathbb{N}, n \geq 1$ and a non vanishing holomorphic function h such that

$$f(z) = (z - z_0)^{-n} h(z)$$

Characterization of complex functions at a pole

(Laurent expansion)

Theorem If f has a pole of order n at z_0

then

$$f(z) = \frac{a_{-n}}{(z-z_0)^n} + \frac{a_{-n+1}}{(z-z_0)^{n-1}} + \cdots + \frac{a_{-1}}{(z-z_0)} + G(z)$$

where $G(z)$ is a analytic function

in an open ball around z_0

(here $a_{-n}, a_{-n+1}, \dots, a_{-1}$ are complex numbers)

The sum

$$\frac{a_{-n}}{(z-z_0)^n} + \frac{a_{-n+1}}{(z-z_0)^{n-1}} + \cdots + \frac{a_{-1}}{z-z_0}$$

is called the principal part of f at the pole z_0, and the coefficient a_{-1} is called the residue of f at the pole z_0. The importance of the residue comes from the fact that

$$\frac{1}{2\pi i} \int_C f(z)\,dz = a_{-1}$$

for any circle C centered at $z = z_0$

oriented in the positive direction.

The pole z_0 is called a simple pole if

$$f(z) = \frac{a_{-1}}{z - z_0} + G(z)$$

where G is an analytic function in a neighborhood of z_0

In order words, $a_{-n} = -a_{-n+1} = \ldots = a_{-2} = 0$ in the Laurent expansion

Theorem If f has a simple pole at z_0, then the residue a_{-1} of f is

$$\operatorname{Res}(f, z_0) = \lim_{z \to z_0} (z - z_0) f(z)$$

Proof

$$f(z) = \frac{a_{-1}}{z - z_0} + G(z)$$

$$(z - z_0) f(z) = a_{-1} + G(z)(z - z_0)$$

Let $z \to z_0$, we get

$$\lim_{z \to z_0} (z - z_0) f(z) = a_{-1} + 0$$

Example

Let $f(z) = \dfrac{z}{(z^2-1)(z^2+4)}$

Determine all the poles of f, and find their residues

Ans

$$f(z) = \frac{z}{(z^2-1)(z^2+4)}$$

$$= \frac{z}{(z-1)(z+1)(z+2i)(z-2i)}$$

Poles of f: $z_1 = 1$

$z_2 = -1$

$z_3 = -2i$

$z_4 = 2i$

all are simple poles

$$\operatorname{Res}(f, z_1) = \lim_{z \to 1} (z-1) f(z)$$

$$= \lim_{z \to 1} \frac{z}{(z+1)(z^2+4)} = \frac{1}{2 \times 5}$$

$$= \frac{1}{10}$$

$$\operatorname{Res}(f, z_2) = \lim_{z \to -1} \frac{z}{(z-1)(z^2+4)} = \frac{-1}{(-2)5}$$

$$= \frac{1}{10}$$

$$\operatorname{Res}(f, z_3) = \lim_{z \to -2i} \frac{z}{(\cancel{z}-2i)(z^2-1)}$$

$$= \frac{-2i}{(-4i)(4i^2-1)}$$

$$= \frac{1}{2(-4-1)} = \frac{-1}{10}$$

$$\operatorname{Res}(f, z_4) = \lim_{z \to 2i} \frac{z}{(z+2i)(z^2-1)}$$

$$= \frac{2i}{4i\,(4i^2-1)}$$

$$= \frac{1}{2(-4-1)} = \frac{-1}{10}$$

Theorem If f has a pole of order $n \geq 2$ at z_0, then

$$\operatorname{Res}(f, z_0) = \lim_{z \to z_0} \frac{1}{(n-1)!} \left(\frac{d}{dz}\right)^{n-1} \left[(z-z_0)^n f(z) \right]$$

Proof:

$$f(z) = \frac{a_{-n}}{(z-z_0)^n} + \dots + \frac{a_{-1}}{z-z_0} + \underset{\text{analytic}}{G(z)}$$

$$(z-z_0)^n f(z) = a_{-n} + a_{-n+1}(z-z_0) + \dots$$
$$+ a_{-1}(z-z_0)^{n-1} + G(z)(z-z_0)^n$$

At the same time

$$(z-z_0)^n f(z) = \sum_{k=0}^{\infty} \frac{\partial_z^k \left[(z-z_0)^n f(z)\right]\Big|_{z=z_0}}{k!}$$

Hence $$a_{-1} = \frac{1}{(n-1)!} \left(\frac{d}{dz}\right)^{n-1} \left[(z-z_0)^n f(z)\right]\Bigg|_{z=z_0}$$

Example Let

$$f(z) = \frac{z^3 + e^z}{(z+2i)^2}$$

Find $\mathrm{Res}(f, -2i)$

Write your answer as Real parts + i Imaginary part

Ans $z_0 = -2i$, $n = 2$

$$\operatorname{Res}(f, z_0) = \lim_{z \to z_0} \frac{1}{(n-1)!} \left(\frac{d}{dz}\right)^{n-1} \left[(z-z_0)^n f(z)\right]$$

$$(z+2i)^2 f(z) = z^3 + e^z$$

$$\frac{d}{dz}\left[(z+2i)^2 f(z)\right] = 3z^2 + e^z$$

$$\lim_{z \to -2i} (3z^2 + e^z) = 3(-2i)^2 + e^{-2i}$$

$$= 3(4i^2) + e^{-2i}$$

$$= -12 + e^{-2i}$$

$$= -12 + (\cos 2 - i\sin 2)$$

$$= (-12 + \cos 2) - i\sin 2$$

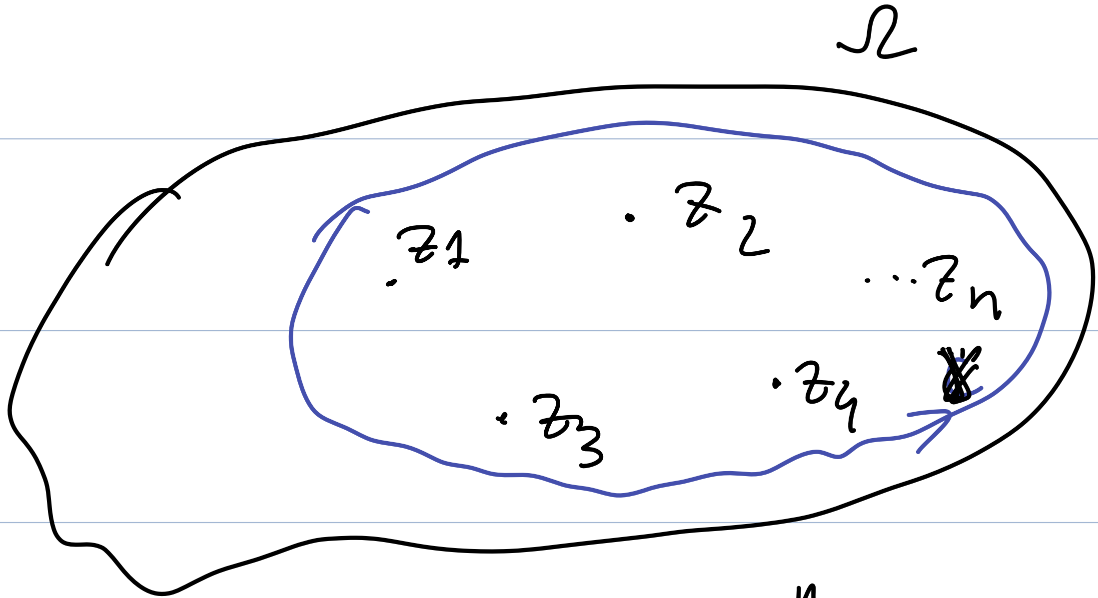
Cauchy Residue theorem

Let Ω be an open set in $\mathbb{C}$

Assume that the function f is analytic inside Ω, except for a finite number of singular points $z_1, z_2, \dots, z_n \in \Omega$.

Then for any closed curve γ that circumscribe $z_1, z_2, \dots, z_n$ in Ω, we have

$$\int_\gamma f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname{Res}(f, z_k)$$



$$\int_\gamma f(z)\,dz = 2\pi i \sum_{k=1}^{n} \mathrm{Res}(f, z_k)$$

Applications of Cauchy Residue theorem in evaluating some improper integrals

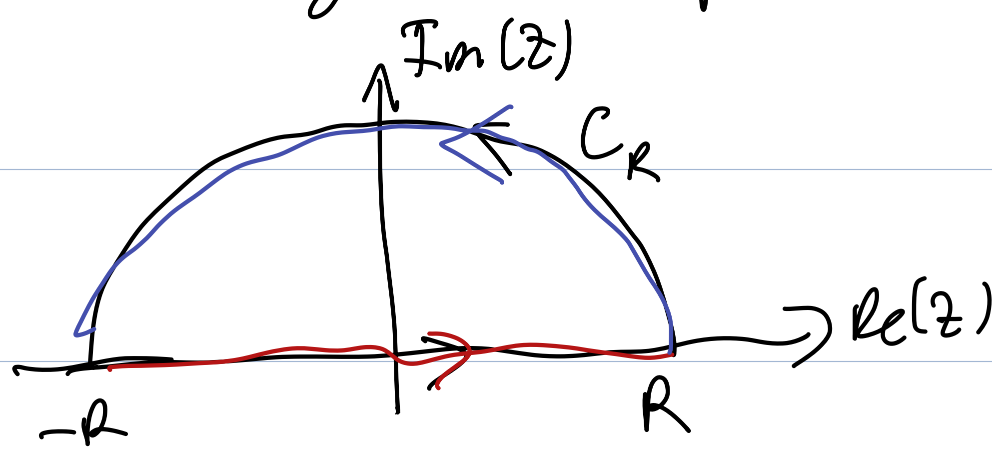
<u>Example</u> Evaluate the improper integral

$$\int_0^\infty \frac{x^2}{(x^2+16)(x^2+9)}\,dx$$

Ans Define

$$f(z) = \frac{z^2}{(z^2+16)(z^2+9)}$$

and the curve γ_R to be the contour define by the picture



Namely γ_R consists of the segment from $(-R)$ to R, then the circle C_R parametrized by

$$z(t) = Re^{it} \qquad 0 \leq t \leq \pi$$

We have

$$\int_{\gamma_R} f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_{C_R} f(z)\,dz$$

First, for R large, γ_R will circulate the poles $z = 4i$, $z = 3i$

of $f(z)$. Hence by the

Cauchy residue theorem,

$$\int_{\gamma_R} f(z)\,dz = 2\pi i\left(\operatorname{Res}(f, 4i) + \operatorname{Res}(f, 3i)\right)$$

$$= 2\pi i\left[\lim_{z\to 4i} \frac{z^2}{(z+4i)(z^2+9)} + \lim_{z\to 3i} \frac{z^2}{(z+3i)(z^2+16)}\right]$$

$$= 2\pi i\left[\frac{16i^2}{8i(16i^2+9)} + \frac{9i^2}{6i(9i^2+16)}\right]$$

$$= 2\pi i\left(\frac{-16}{8i(-16+9)} + \frac{-9}{6i(-9+16)}\right)$$

$$= 2\pi\left[\frac{-2}{(-7)} + \frac{-3}{2(7)}\right]$$

$$= 2\pi\left(\frac{2}{7} - \frac{3}{14}\right) = 2\pi\,\frac{(4-3)}{14} = 2\pi \times \frac{1}{14}$$

$$= \frac{\pi}{7}$$

Hence $\frac{\pi}{7} = \int_{-R}^{R} f(x)\,dx + \int_{C_R} f(z)\,dz \quad (*)$

Now

$$\int_{C_R} f(z)\,dz \leq \max_{|z|=R} |f(z)|\,\pi R \quad (1)$$

$$f(z) = \frac{z^2}{(z^2+16)(z^2+9)}$$

As $|z| = R$ and $R \to \infty$ (R large)

we have $|f(z)| \leq \frac{R^2}{(R^2-16)(R^2-9)}$ (2)

From (1) and (2), we have

$$\left|\int_{C_R} f(z)\,dz\right| \leq \frac{R^2}{(R^2-16)(R^2-9)}\,\pi R$$

$$\leq \frac{\pi R^3}{(R^2-16)(R^2-9)}$$

Hence $\lim_{R\to\infty} \left|\int_{C_R} f(z)\,dz\right| = 0$ (**)

From (*) and (**)

$$\pi = \lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx$$

$$= \lim_{R\to\infty} 2\int_0^R f(x)\,dx$$

(since $f(x)$ is an even function)

$$\Rightarrow \int_0^\infty \frac{x^2}{(x^2+16)(x^2+9)}\,dx = \frac{\pi}{14}$$