Summary 03/13/24

- Small oscillations w/ $n \geq 1$ DOF:

$$L = \frac{1}{2}\sum_{i,j=1}^{n} m_{ij}\dot{x}_i\dot{x}_j - \frac{1}{2}\sum_{i,j=1}^{n} k_{ij}x_ix_j$$

x = displacement from equil.

- normal coord.'s:

$$\{x_i\}_{i=1,\dots,n} \longrightarrow \{Q_\alpha\}_{\alpha=1,\dots,n}$$

$$x_i = \sum_{\alpha=1}^{n} A_{i\alpha}Q_\alpha$$

w/ $\sum_j (\omega_\alpha^2 m_{ij} - k_{ij})A_{j\alpha} = 0$ (previously $A_{j\alpha} = a_j^{(\alpha)}$)

$$\Rightarrow L = \frac{1}{2}\sum_{\alpha=1}^{n}\left(\dot{Q}_\alpha^2 - \omega_\alpha^2 Q_\alpha^2\right)$$

E-L. eqn.'s: $\frac{d}{dt}\frac{\partial L}{\partial \dot{Q}_\alpha} = \frac{\partial L}{\partial Q_\alpha} \Rightarrow \ddot{Q}_\alpha + \omega_\alpha^2 Q_\alpha = 0.$

→ description of dynamics in terms of n decoupled oscillators.

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as an example of the use of normal coord.'s, consider the problem of forced oscillations:

$$\to U_{ext}(q_1,\dots,q_n,t) = U_{ext}(q_1^{(0)}+x_1,\dots,q_n^{(0)}+x_n,t)$$

$$\simeq U_{ext}(q^{(0)},t) + \sum_{i=1}^{n}\left.\frac{\partial U_{ext}(q)}{\partial q_i}\right|_{q^{(0)}} x_i$$

small x

03/13/24

Let $F_i(t) = -\left.\dfrac{\partial U_{ext}(q)}{\partial q_i}\right|_{q^{(0)}}$

$$\Rightarrow U_{ext} = \underbrace{U_{ext}(q^{(0)}, t)}_{\text{const., drop from } L} - \sum_{i=1}^{n} F_i(t)\, x_i$$

$$\Rightarrow L = L_0 + \sum_{i=1}^{n} F_i(t)\, x_i$$

L_0: free oscillations

In normal coord.'s $x_i = \sum_{\alpha=1}^{n} A_{i\alpha} Q_\alpha$:

$$L = \frac{1}{2}\sum_{\alpha=1}^{n}\left(\dot{Q}_\alpha^2 - \omega_\alpha^2 Q_\alpha^2\right) + \sum_{\alpha=1}^{n} Q_\alpha \underbrace{\left(\sum_{i=1}^{n} A_{i\alpha} F_i(t)\right)}_{\equiv f_\alpha(t)}$$

$$\Rightarrow L = \frac{1}{2}\sum_{\alpha=1}^{n}\left(\dot{Q}_\alpha^2 - \omega_\alpha^2 Q_\alpha^2\right) + \sum_{\alpha=1}^{n} f_\alpha(t)\, Q_\alpha$$

E-L eqn.'s: $\ddot{Q}_\alpha + \omega_\alpha^2 Q_\alpha = f_\alpha(t)$

$\rightarrow$ n decoupled, driven oscillators

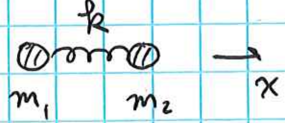
Now we will consider the consequences of symmetry for properties of normal modes. Start from:

$$L = \frac{1}{2}\sum_{ij} m_{ij}\dot{x}_i\dot{x}_j - \frac{1}{2}\sum_{ij} k_{ij} x_i x_j$$

As a concrete example, suppose we have a symmetry of the system $x_i \rightarrow x_i + a$, $a = \text{const.}$

03/13/24

Ex:



(diatomic molecule)

$$L = \tfrac{1}{2} m_1 \dot{x}_1^2 + \tfrac{1}{2} m \dot{x}_2^2 - \tfrac{1}{2} k (x_1 - x_2)^2$$

→ symmetry: $x_1 \to x_1 + a$, $x_2 \to x_2 + a$

$\Rightarrow L \to L$.

In the general case, take $a = \varepsilon$ = small:

$$L \to L' = \frac{1}{2} \sum_{ij} m_{ij} \dot{x}_i \dot{x}_j - \frac{1}{2} \sum_{ij} k_{ij} (x_i + \varepsilon)(x_j + \varepsilon)$$

$k_{ij} = k_{ji}$

$$\simeq \underbrace{\frac{1}{2} \sum_{ij} m_{ij} \dot{x}_i \dot{x}_j - \frac{1}{2} \sum_{ij} k_{ij} x_i x_j}_{=L} - \varepsilon \sum_{ij} k_{ij} x_j$$

If transform'n is symmetry then $L' = L$ &

$$\sum_{ij} k_{ij} x_j = 0. \qquad (*)$$

Consider implication of this for EOM:

$$\sum_j m_{ij} \ddot{x}_j = - \sum_j k_{ij} x_j$$

sum over i: $\sum_{ij} m_{ij} \ddot{x}_j = - \sum_{ij} k_{ij} x_j = 0$

by (*).

$$\Rightarrow \frac{d}{dt} \underbrace{\left(\sum_{ij} m_{ij} \dot{x}_j \right)}_{= \text{const.}} = 0$$

As we may have anticipated, the symmetry $x_i \to x_i + a$ has led us to a conservation law:

$$\sum_{ij} m_{ij} \dot{x}_j = \text{const.}$$

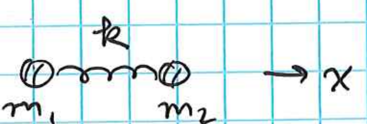
Existence of ~~the~~ this conservation law. leads to a "zero mode": Let $Q_0 = \sum_{ij} m_{ij} x_j$, then:

$$\ddot{Q}_0 = \sum_{ij} m_{ij} \ddot{x}_j = 0$$

$\Rightarrow$ Q_0 is a ~~no~~ normal coord w/ freq. $\omega^2 = 0$.

~~These is a particular example of a more general result: continuous symmetries lead to zero modes~~

<u>Ex</u>: (diatomic molecule)



$$Q_0 = m_1 x_1 + m_2 x_2$$

$\Rightarrow$ ~~normal~~ zero mode = motion of COM.

What we have found ~~a~~ is a particular instance of a more general result: continuous symmetries lead to normal modes w/ zero freq. (zero modes), corresponding to collective motions of the system that don't require any energy.