HW 8, Harry Luo

- 6.28

Let $p = 1 - q$. Need to find $P(V = k, W = l), k \geq 1, l = 0, 1, 2$ Noticing the independence of X and Y, we have

$$\begin{aligned} P(V = k, W = 0) &= P(\min(X, Y) = k, X < Y) \\ &= P(X = k, k < Y) \\ &= P(X = k)P(k < Y) \\ &= pq^{k-1}q^k = pq^{2k-1} \end{aligned} \tag{1}$$

Similarly, $P(V = k, W = 2) = pq^{2k-1}$ Thus,

$$\begin{aligned} P(V = k, W = 1) &= P(\min(X, Y) = k, X = Y) \\ &= P(X = k, Y = k) = p^2 q^{2k-2} \end{aligned} \tag{2}$$

Need to check if $P(V = k, W = 1)$ is the product of the marginal probabilities.

$V \sim$ Geom$(1 - q^2) \Rightarrow P(V = k) = (1 - (1 - q^2))^{k-1}(1 - q^2) = q^{2k-2}(1 - q^2)$

By argument of symmetry, $P(W = 0) = P(X < Y) = P(Y < X) = P(W = 2)$ Noticing again the independence of X and Y, we have

$$\begin{aligned} P(W = 1) = P(X = Y) &= \sum_{k=1}^{\infty} P(X = k)P(Y = k) \\ &= \sum_{k=1}^{\infty} pq^{k-1}pq^{k-1} = \frac{p^2}{1 - q^2} \end{aligned} \tag{3}$$

Using the fact that $P(W = 0) + P(W = 1) + P(W = 2) = 1$, we have

$$P(W = 0) = P(W = 2) = \frac{1}{2}(1 - P(W = 1)) = \frac{1 - p}{2 - p} \tag{4}$$

To check independence of V and W:

$$\begin{aligned} P(V = k, W = 0) &= pq^{2k-1} \\ P(V = k)P(W = 0) &= q^{2k-2}(1 - q^2)\frac{1 - p}{2 - p} \end{aligned} \tag{5}$$

Noticing $\frac{1-q^2}{2-p} = \frac{(1-q)(1+q)}{1+q} = p \Rightarrow P(V = k, W = 0) = P(V = k)P(W = 0)$

Similarly, $P(V = k, W = 1) = P(V = k)P(W = 1)$ and $P(V = k, W = 2) = P(V = k)P(W = 2)$

Thus, we have shown that for all k>=1, l=0,1,2, $P(V = k, W = l) = P(V = k)P(W = l)$, which implies independence of V and W.

- 6.30

The joint pmf of X and Y, for $k \geq 1, l \geq 0$ is

$$P(X = k, Y = l) = (1 - p)^{k-1} p \times e^{-\lambda}(\lambda^l)/(l!) \tag{6}$$

Noticing that $\{X = Y + 1\}$ can be expressed in $\cup_{k=0}^{\infty \{X=k+1, Y=k\}}$. It follows that

$$
\begin{aligned}
P(X = Y + 1) &= \sum_{k=0}^{\infty} P(X = k + 1, Y = k) \\
&= \sum_{k=0}^{\infty} (1-p)^k p e^{-\lambda} (\lambda^k)/(k!) \\
&= P e^{-\lambda} \sum_{k=0}^{\infty} (\lambda(1-p))^k/(k!) \\
&= p e^{-\lambda} e^{\lambda(1-p)} \boxed{= p e^{-p\lambda}}
\end{aligned} \tag{7}
$$

- 7.2

As suggested, we find the probability mass function of X+Y to represent its distribution.

Since $X, Y \in \{0, 1\} \Rightarrow X + Y = \{0, 1, 2\}$

When $X = 0, Y = 0, X + Y = 0$. By independence,

$$P(X + Y = 0) = P(X = 0, Y = 0) = P(X = 0)P(Y = 0) = (1-p)(1-r) \tag{8}$$

WHen $X + Y = 2, X = 1, Y = 1$. Similarly,

$$P(X + Y = 2) = P(X = 1, Y = 1) = P(X = 1)P(Y = 1) = pr \tag{9}$$

Considering the complement of $P(X + Y) = 1$,

$$P(X + Y = 1) = 1 - P(X + Y = 0) - P(X + Y = 2) = p + r - 2pr \tag{10}$$

Thus, the probability mass function of X+Y is

$$
\begin{aligned}
&P(X + Y = 0) = (1-p)(1-r) \\
&P(X + Y = 1) = p + r - 2pr \\
&P(X + Y = 2) = pr
\end{aligned} \tag{11}
$$

- 7.16

$$
\begin{gathered}
p_X(k) = \frac{\lambda^k}{k!} e^{-\lambda}, k = 0, 1, 2, \dots \\
P_Y(0) = 1 - p, p_Y(1) = p
\end{gathered} \tag{12}
$$

By convolution,

$$p_{X+Y}(n) = \sum_{k=0}^{n} p_X(k) p_Y(n-k) \tag{13}$$

Since $X + Y \in \{0, 1, 2, \dots\}$, we need to consider only $n \geq 0$ Equation 13 becomes

$$p_{X+Y}(n) = p_X(n)p_Y(0) + p_X(n-1)p_Y(1) \tag{14}$$

- when $n = 0, p_X(n-1) = 0$,

$$p_{X+Y}(0) = p_X(0)p_Y(0) = e^{-\lambda}(1-p) \tag{15}$$

- when $n > 0$,

$$
\begin{aligned}
p_{X+Y}(n) &= p_X(n)p_Y(0) + p_X(n-1)p_Y(1) \\
&= (1-p)\frac{\lambda^n}{n!} e^{-\lambda} + \frac{p(\lambda^{n-1})}{(n-1)!} e^{-\lambda} \\
&= \frac{\lambda^{n-1} e^{-\lambda} (\lambda(1-p) + np)}{n!}
\end{aligned} \tag{16}
$$

To conclude,

$$
\begin{aligned}
p_{X+Y}(n) &= \frac{\lambda^{n-1} e^{-\lambda}(\lambda(1-p)+np)}{n!}, n = 0, 1, 2, ... \\
p_{X+Y}(n) &= (1-p)e^{-\lambda}, n = 0
\end{aligned} \tag{17}
$$

- 7.20

(a) By independence of X and Y,

$$
\begin{aligned}
f_X(x) f_Y(y) &= f_{X,Y}(x, y) \\
f_X(x) &= \begin{cases} 2x & x \in (0,1) \\ 0 & o.w. \end{cases} \\
f_Y(y) &= \begin{cases} 1 & y \in (1,2) \\ 0 & o.w. \end{cases}
\end{aligned} \tag{18}
$$

For $P\left(Y - X \geq \frac{3}{2}\right)$, we need to integrate the joint density function over the region $y - x \geq \frac{3}{2}$. Since pdf is only positive on $x \in (0,1), y \in (1,2)$, we only need to consider the reagion of $\left\{(x,y) |\ y - x \geq \frac{3}{2}\right\} \cap \{(x,y) |\ x \in (0,1), y \in (1,2)\} = \left\{(x,y) |\ x \in \left(0, \frac{1}{2}, y \in \left(x + \frac{3}{2}, 2\right)\right)\right\}$ Therefore,

$$
\begin{aligned}
P\left(Y - X \geq \frac{3}{2}\right) &= \int_0^{\frac{1}{2}} \int_{x+\frac{3}{2}}^{2} 2x \,\mathrm{d}y \,\mathrm{d}x \\
&= \frac{1}{24}
\end{aligned} \tag{19}
$$

(b) since $X \in (0,1), Y \in (1,2) \Rightarrow X + Y \in (1,3)$

$$
\begin{aligned}
f_{X+Y}(z) &= \int_{-\infty}^{\infty} f_X(x) f_Y(z-x) \,\mathrm{d}x \\
&= \int_0^1 f_X(x) f_Y(z-x) \,\mathrm{d}x
\end{aligned} \tag{20}
$$

Considering $f_X(x) f_Y(z-x) \neq 0$ iff $z - x \in (1,2)$

WE want $x \in (0,1) s.t. f_X(x) \neq 0$ Therefore, $f_X(x) f_Y(z-x)$ is non zero if and only if $\max(0, z-2) < x < \min(1, z-1)$

$$
f_{X+Y}(z) = \int_{\max(0,z-2)}^{\min(1,z-1)} 2x \,\mathrm{d}x = \min(1, z-1)^2 - \max(0, z-2)^2 \tag{21}
$$

$$
f_{X+Y}(z) = \begin{cases} (z-1)^2 & z \in (1,2) \\ 1 - (z-2)^2 & z \in (2,3) \\ 0 & o.w. \end{cases} \tag{22}
$$

- 7.24

Denote $\mathrm{Var}(X) = \sigma_X^2, \mathrm{Var}(Y) = \sigma_Y^2, \mathrm{Var}(Z) = \sigma_Z^2$
Notice:

$$
\begin{aligned}
X + 2Y - 3Z &\sim N(0, \sigma_X^2 + 4\sigma_Y^2 + 9\sigma_Z^2), \\
\frac{X + 2Y - 3Z}{\sqrt{\sigma_X^2 + 4\sigma_Y^2 + 9\sigma_Z^2}} &\sim N(0,1)
\end{aligned} \tag{23}
$$

It follows that

$$
P(X + 2Y - 3Z \leq 0) = P\left(\frac{X + 2Y - 3Z}{\sqrt{\sigma_X^2 + 4\sigma_Y^2 + 9\sigma_Z^2}} \leq 0\right) = 1 - \Phi(0) = \frac{1}{2} \tag{24}
$$

- 7.28

Because of joint continous of X_1, X_2, X_3, prob. of any pairs of them being equal is 0. Therefore, $P(X_1 \neq X_2 \neq X_3) = 0$

By exchangeability

$$P(X_1 < X_2 < X_3) = P(X_2 < X_1 < X_3) = P(X_3 < X_1 < X_2) = ... \tag{25}$$

There are 6 different permutations, each has same prob, and since they are mutually exclusive,

$$P(X_1 < X_2 < X_3) = \frac{1}{6} \tag{26}$$

- 7.30 (a) By exchangeability,

$$P(\text{card } 2 = A, \text{card } 4 = K) = P(\text{card } 1 = A, \text{card } 2 = K) = \frac{4 * 4}{52 * 51} = \frac{4}{663} \tag{27}$$

  (b) By exchangeability,

$$P(\text{card } 1 = S, \text{card } 5 = S) = P(\text{card } 1 = S, \text{card } 2 = S) = \frac{\binom{13}{2}}{\binom{52}{2}} = \frac{1}{17} \tag{28}$$

  (c) By exchangeability,

$$\begin{aligned} P(\text{card } 2 = K | \text{ last 2 cards are A}) &= \frac{P(\text{card } 2 = K, \text{last 2 cards are A})}{P(\text{last 2 cards are A})} \\ &= \frac{P(\text{card } 3 = K, \text{card } 1 = \text{card } 2 = A)}{P(\text{card 1 card } 2 = A)} \\ &= P(\text{3rd card is K}|\text{first 2 cards are A}) \\ &= \frac{4}{50} \end{aligned} \tag{29}$$