Summary

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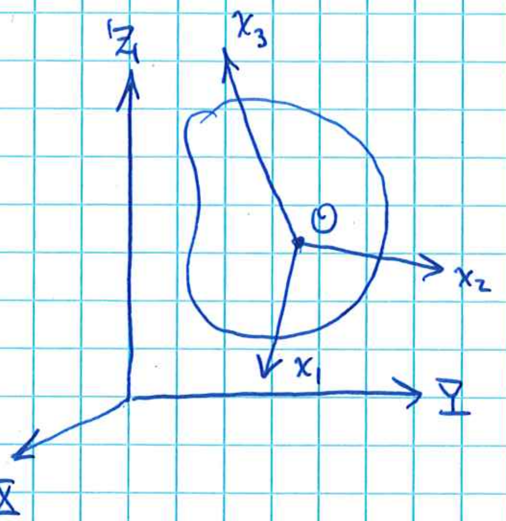


(XYZ) = fixed inertial frame.

$(x_1 x_2 x_3)$ = moving frame fixed in body.

O = COM:

- $T = \frac{1}{2} M V^2 + \frac{1}{2} \sum_{ij} I_{ij} \Omega_i \Omega_j$

  $I_{ij} = \sum m (r^2 \delta_{ij} - x_i x_j)$

- principal axes diagonalize I_{ij}:

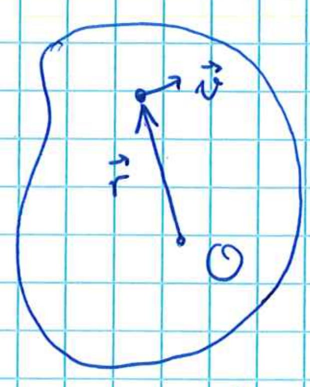
  $I_{ij} \rightarrow (I_1, I_2, I_3)$

  $T_{rot.} = \frac{1}{2} (I_1 \Omega_1^2 + I_2 \Omega_2^2 + I_3 \Omega_3^2)$.

  $\rightarrow$ preferred axes in the body frame.

Angular momentum of a rigid body

- Consider angular momentum $\vec{L}$ defined w.r.t. O at COM.



motion w.r.t. O = pure rotation $\Rightarrow \vec{v}_o = \vec{\Omega} \times \vec{r}$

$$\vec{L} = \sum m (\vec{r} \times \vec{v})$$

(sum over all particles in the body)

$\vec{v} = \vec{\Omega} \times \vec{r}$

$$= \sum m [\vec{r} \times (\vec{\Omega} \times \vec{r})]$$

$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$

$$= \sum m [\vec{\Omega} r^2 - \vec{r}(\vec{\Omega} \cdot \vec{r})]$$

in components: $L_i = \sum m [\Omega_i r^2 - x_i (\sum_j x_j \Omega_j)]$

B/c $\vec{\Omega}$ is the same for all terms in the sum, we'd like to pull out common factor of Ω_j:

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$$L_i = \sum m \left[\sum_j \delta_{ij} \Omega_j r^2 - x_i \sum_j (x_j \Omega_j) \right]$$

$$= \sum_j \underbrace{\left[\sum m \left(r^2 \delta_{ij} - x_i x_j \right) \right]}_{I_{ij}} \Omega_j$$

$$\Rightarrow \boxed{L_i = I_{ij} \Omega_j}$$

relation ~~b~~ btwn. ang. mom. & ang. vel. of rigid body (c.f. $\vec{p} = m\vec{v}$).

In matrix-vector notation:

$$\vec{L} = \tilde{I} \vec{\Omega}, \qquad \tilde{I} = \text{matrix w/ comp.'s } I_{ij}$$

matrix acting on vector $\vec{\Omega}$.

If $(x_1 x_2 x_3)$ = principal axes:

$$L_1 = I_1 \Omega_1, \quad L_2 = I_2 \Omega_2, \quad L_3 = I_3 \Omega_3.$$

We see that, in general, the angular velocity & ang. momentum of a rigid body are in <u>different</u> directions. Only $\parallel$ if $\vec{\Omega}$ along a principal axis

$\Rightarrow$ even free motion of a rigid body can be non-trivial.

turn to this now.

Free motion of a rigid body

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· In absence of external forces, total momentum $\vec{P}$ is conserved. ⇒ COM moves w/ const. velocity & we can go to inertial COM frame to analyze motion.

· Absence of ext. forces ⇒ ang. mom. $\vec{L}$ = conserved.

However, even when $\vec{L}$ = const. $\vec{\Omega}$ will in general vary throughout the motion.

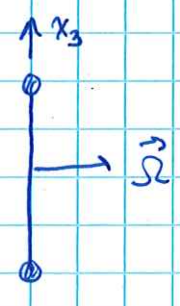
Ex: spherical top $\quad I_1 = I_2 = I_3 \equiv I \rightarrow \tilde{I} = I \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

$$\Rightarrow \vec{L} = \underbrace{I}_{\text{number.}} \vec{\Omega}$$

$\dot{\vec{L}} = I \dot{\vec{\Omega}}$, so $\dot{\vec{L}} = 0 \Rightarrow \dot{\vec{\Omega}} = 0$ & $\vec{\Omega}$ = const.

⇒ for spherical top free rotation = uniform rot.'n about fixed axis ∥ $\vec{L}$.

Ex: rigid rotor $\quad I_1 = I_2 = \sum m x_3^2 \equiv I, \quad I_3 = 0$



$\vec{L} = I \vec{\Omega}$ & $\vec{\Omega} \perp x_3$.

⇒ free rot.'n of rigid rotor = uniform rot.'n ~~in~~ in plane $\perp$ to $\vec{\Omega}$.

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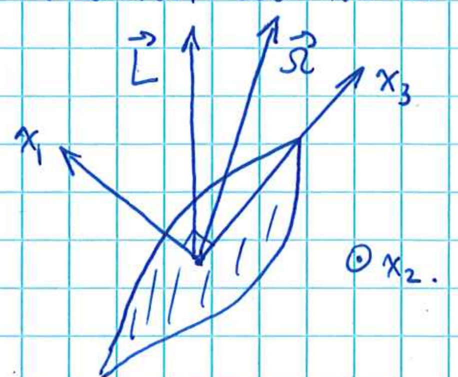
Ex: symmetric top $I_1 = I_2 \neq I_3$ $\rightarrow$ $\tilde{I} = \begin{pmatrix} I_\perp & 0 & 0 \\ 0 & I_\perp & 0 \\ 0 & 0 & I_3 \end{pmatrix}$.

$I_1 \equiv I_\perp$

x_3 = symm. axis

$x_1 x_2$ = any orthogonal axes $\perp x_3$.

· Consider motion at some moment in time:



· choose principal axes as shown:

$x_1 \in$ plane spanned by $\vec{L}$ & x_3

Since $L_2 = I_2 \Omega_2$, $L_2 = 0 \Rightarrow \Omega_2 = 0$

$\Rightarrow \vec{\Omega} \in$ plane spanned by $\vec{L}$ & x_3.

In fact, $\vec{\Omega}, \vec{L}, x_3 \in$ common plane $\forall t$. We can also demonstrated this more geometrically:

Consider triple product $\vec{L} \cdot (\hat{x}_3 \times \vec{\Omega})$:

$$
\begin{aligned}
\vec{L} \cdot (\hat{x}_3 \times \vec{\Omega}) &= (\tilde{I} \vec{\Omega}) \cdot (\hat{x}_3 \times \vec{\Omega}) \\
&= \vec{\Omega} \cdot [\tilde{I} (\hat{x}_3 \times \vec{\Omega})] \\
&= \vec{\Omega} \cdot [I_\perp (\hat{x}_3 \times \vec{\Omega}) \\
&= I_\perp \vec{\Omega} \cdot (\hat{x}_3 \times \vec{\Omega}) \\
&= 0
\end{aligned}
$$

) $\tilde{I}$ = symm. matrix

) $\hat{x}_3 \times \vec{\Omega} \perp \hat{x}_3$ & any axis $\perp \hat{x}_3$ = principal axis w/ moment $I_\perp$

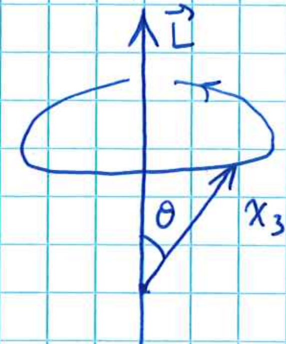
$\Rightarrow \vec{L}, x_3, \vec{\Omega} \in$ common plane.

This implies that motion of x_3 axis = precession about about fixed direction of $\vec{L}$. 04/01/24

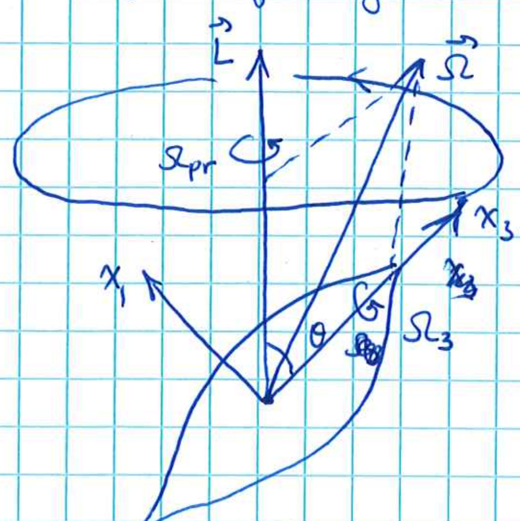
To see this explicitly consider pt. $\vec{r}$ along x_3 axis:

$$\frac{d}{dt}(\vec{r}\cdot\vec{L}) = \vec{v}\cdot\vec{L} + \vec{r}\cdot\cancelto{0}{\dot{\vec{L}}} \qquad \vec{v} = \vec{\Omega}\times\vec{r}$$
$$= (\vec{\Omega}\times\vec{r})\cdot\vec{L}$$
$$= 0 \quad \text{since } \vec{L}, \vec{r}, \vec{\Omega} \in \text{common plane.}$$

$\Rightarrow$ angle θ btwn. x_3 & $\vec{L}$ = const., corresponding to precession:



The free motion of the symmetric top thus corresponds to precession about fixed vector $\vec{L}$ together w/ rotation about axis of symmetry x_3:



· The angular speed about the symm. axis is: 04/01/24

$$\Omega_3 = \frac{L_3}{I_3} = \frac{L\cos\theta}{I_3}.$$

· To find angular speed of precession Ω_{pr} need $\vec{\Omega}\cdot\hat{L}$; i.e., component of $\vec{\Omega}$ along $\vec{L}$.

$$\vec{\Omega} = \Omega_{pr}\,\hat{L} + \Omega_3\,\hat{x}_3.$$

$$\left.\begin{aligned} \Rightarrow\ \Omega_1 &= \hat{x}_1\cdot\vec{\Omega} = \Omega_{pr}\,\hat{x}_1\cdot\hat{L} \\ \&\ \Omega_1 &= \frac{L_1}{I_\perp} = \frac{\hat{x}_1\cdot\vec{L}}{I_\perp} \end{aligned}\right\} \Rightarrow \Omega_{pr} = \frac{L}{I_\perp}$$