

Small Oscillations

- Motion near a point of stable equilibrium.

DOF= 1 (one dimension)

- For a system of DOF = 1, with potential $U(q)$:
 - **stable equilibrium** at $U(q)_{\min}$, upward parabola, where $F = -\frac{dU}{dq} = 0$
 - restoring force for small displacements $q - q_0$ is $F = -\frac{d^2U(q-q_0)}{dq}$
 - **Unstable equilibrium** at $U(q)_{\max}$, downward parabola, where $F = -\frac{dU}{dq} = 0$ as well.
- Consider small deviation from point of stable equilibrium, we use Taylor expansion to show that it is really a small displacement. that is,

$$U(q) \approx U(q_0) + \frac{dU(q_0)}{dq}(q - q_0) + \left(\frac{1}{2}\right) \frac{d^2U(q_0)}{dq^2}(q - q_0)^2 + \dots$$

$$\text{while } \frac{dU(q_0)}{dq}(q - q_0) = 0$$
1

letting $x = q - q_0$, we have

$$\begin{cases} U(x) = U(q_0) + \left(\frac{1}{2}\right) \frac{d^2U(q_0)}{dq^2} x^2 \\ \text{putting into the form of } U(x) = U(x_0) + \left(\frac{1}{2}\right) kx^2. \end{cases}$$

$$\Rightarrow \boxed{k = \frac{d^2U(q_0)}{dq^2} > 0}$$
2

we get KE, while choosing $U(q_0) = 0$:

$$T = \frac{1}{2} a(q)^2 \dot{q}^2 = \frac{1}{2} a(q_0 + x) \dot{x}^2 \approx \frac{1}{2} m \dot{x}^2, \text{ letting } m = a(q_0)$$

$$\Rightarrow \boxed{L = T - U = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} kx^2}$$
3

EOM for DOF = 1 small Oscillations

using EL on Equation 3, we can get the EOM for one dimensional small Oscillations:

$$m\ddot{x} = -kx$$

$$\Rightarrow \ddot{x} + \omega_0^2 x = 0, \text{ where } \boxed{\omega_0 = \sqrt{\frac{k}{m}} \text{ freq of osc.}}$$
4

by magic of ODE, EOM reduces down to:

$$\boxed{x(t) = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t)}$$

where C_1, C_2 are constants

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by trig magic, this could also be written as

$$x(t) = a \cos(\omega_0 t + \alpha),$$

$$\text{where } \begin{cases} a = \sqrt{C_1^2 + C_2^2} & \text{amplitude of oscillation} \\ \omega_0 & \text{frequency of oscillation} \\ \tan \alpha = C_2/C_1 & \text{phase at } t=0 \end{cases} \quad 6$$

energy for 1D small Oscillation

checking $\frac{\partial L}{\partial t} = 0 \Rightarrow$ energy-conservation:

$$E = T + U = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$$

$$= \frac{1}{2}ma^2\omega_0^2, [\text{constant}] \quad 7$$

Damped 1D oscillation, and Complex representation

[I dont like the how the subscripts are used in this lecture but I guess this is what we are stuck with.]

- when there is damping (friction, resistance, etc) $F_{\text{fric}} = -\beta\dot{x}$, the EOM becomes:

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0,$$

$$\text{where } 2\gamma = \frac{\beta}{m}, \omega_0 = \sqrt{\frac{k}{m}} \quad 8$$

with ansatz $x(t) = e^{rt}$, $\dot{x} = re^{rt}$, $\ddot{x} = r^2 e^{rt}$, the solution to Equation 8 is:

$$r^2 + 2\gamma r + \omega_0^2 = 0,$$

$$\text{which has solution } r_+, r_- = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2} \quad 9$$

$$\Rightarrow x(t) = C_1 e^{r_+ t} + C_2 e^{r_- t},$$

notice the r subscripts here: r_+, r_-

underdamped, overdamped, and critically damped

Recall from your ODE class...

Equation 9 has the following 3 cases, each with different physical interpretation:

1. underdamped:

$$\gamma < \omega_0 \Rightarrow 2 \text{ complex roots: } \begin{cases} r_{\pm} = -\gamma \pm i\sqrt{\omega_0^2 - \gamma^2} = -\gamma \pm i\omega \\ \omega = \sqrt{\omega_0^2 - \gamma^2} \end{cases} \quad 10$$

The EOM is thus a linear combination of two complex exponentials:

$$x(t) = e^{-\gamma t} (C_1 e^{i\omega t} + C_2 e^{-i\omega t})$$

$$= e^{-\gamma t} (A \cos(\omega t) + B \sin(\omega t))$$

$$\text{-- where } \begin{cases} A = C_1 + C_2 \\ B = i(C_1 - C_2) \end{cases} \quad 11$$

$$= ae^{-\gamma t} \cos(\omega t + \alpha)$$

a, α are constants

“The solution is a damped oscillation with frequency ω , and amplitude exponentially decaying with time.”

2. Overdamped

$$\gamma > \omega \Rightarrow x(t) = c_1 e^{-\gamma + \sqrt{\gamma^2 - \omega^2} t} + c_2 e^{-\gamma - \sqrt{\gamma^2 - \omega^2} t} \quad 12$$

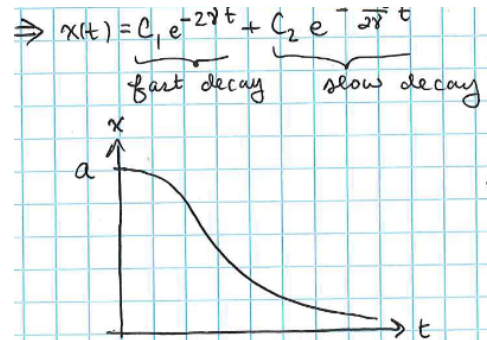
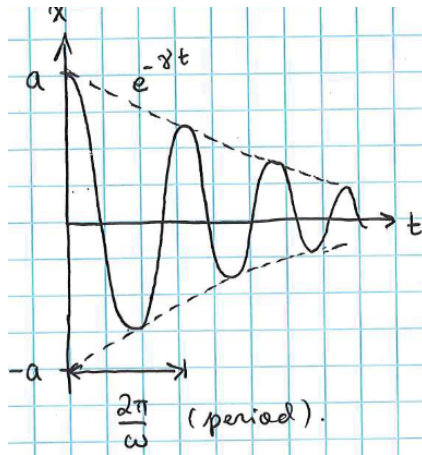
When

$$\gamma \gg \omega_0, \Rightarrow \begin{cases} \gamma + \sqrt{\gamma^2 - \omega_0^2} \approx 2\gamma \\ \gamma - \sqrt{\gamma^2 - \omega_0^2} = \frac{\omega_0^2}{2\gamma} \end{cases} \quad 13$$

$$x(t) = c_1 e^{-2\gamma t} + c_2 e^{(-\omega_0^2/2\gamma)t}$$

3. Critically damped

$$\gamma = \omega_0 \Rightarrow x(t) = c_1 e^{-\gamma t} + c_2 t e^{-\gamma t} \quad 14$$



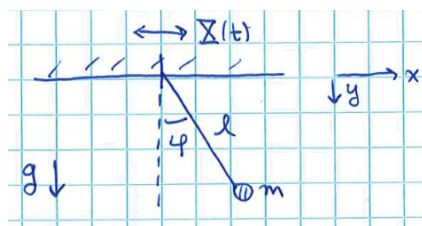
Forced Oscillations

When external force (F) is applied to the system, the lagrangian becomes

$$L = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2 + F(t)x \quad 15$$

$$\text{EL} \Rightarrow \ddot{x} + \omega_0^2 x = \frac{F(t)}{m}, \text{ where } \omega_0 = \sqrt{\frac{k}{m}}$$

- Example: Simple pendulum with moving pivot



$$\begin{cases} x = X + l \sin \varphi \\ y = l \cos \varphi \end{cases} \Rightarrow \begin{cases} \dot{x} = \dot{X} + l \dot{\varphi} \cos \varphi \\ \dot{y} = -l \dot{\varphi} \sin \varphi \end{cases} \quad 16$$

$$\Rightarrow L = T - U$$

$$L = \frac{1}{2}ml^2\dot{\varphi}^2 - mgl(1 - \cos \varphi) - ml\ddot{X} \sin \varphi$$

$$\text{Expand ab. } \varphi = 0 \Rightarrow L = \frac{1}{2}ml^2\dot{\varphi}^2 - \frac{1}{2}mgl\varphi^2 - ml\ddot{X}\varphi \quad 17$$

$$\text{EL} \Rightarrow \boxed{\ddot{\varphi} + \omega_0^2\varphi = -\frac{\ddot{X}}{l}, \text{ where } \omega_0 = \sqrt{\frac{g}{l}}}$$

reintroducing damping via external forcing

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2x = f(t), f(t) = \frac{F(t)}{m} \quad 18$$

When damping $f(t) = f_0 \cos(\Omega t)$, solution via complex number:

$$\ddot{z} + 2\gamma\dot{z} + \omega_0^2z = f_0e^{i\Omega t}$$

$$\text{ansatz } z(t) = z_0e^{i\Omega t} \Rightarrow z_0 = \frac{f_0}{\omega_0^2 + 2i\gamma\Omega + \Omega^2}$$

$$\boxed{z_0 = a(\Omega) \cos(\Omega t + \delta(\Omega))f_0} \text{ is a particular solution, where} \quad 19$$

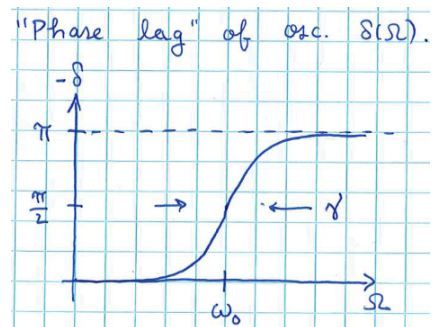
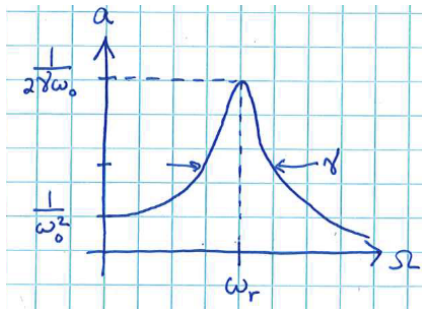
$$\begin{cases} a(\Omega) = \frac{1}{\sqrt{(\omega_0^2 - \Omega^2)^2 + (2\gamma\Omega)^2}} \\ \delta(\Omega) = \arctan\left(2\gamma\frac{\Omega}{\omega_0^2 - \Omega^2}\right) \end{cases}$$

We can study the properties of the system by looking at the amplitude and phase of the solution.

- Amplitude:

$$a(\Omega) = \frac{1}{\sqrt{(\omega_0^2 - \Omega^2)^2 + (2\gamma\Omega)^2}} \quad 20$$

, when $\gamma \ll \omega_0$, response strongest and amplitude largest when $\omega_r = \omega_0$.



- Phase lag: $\tan \delta(\Omega) = 2\gamma\frac{\Omega}{\omega_0^2 - \Omega^2}$

in phase as $\Omega \rightarrow 0$, and out of phase as $\Omega \rightarrow \omega_0$.

- Genral solution to sinusoidal forcing:

$$x(t) = a(\Omega)f_0 \cos(\Omega t + \delta(\Omega)) + a_0e^{-\gamma t} \cos(\omega t + \alpha)$$

$$\xrightarrow{t > \frac{1}{\gamma}} a(\Omega)f_0 \cos(\Omega t + \delta(\Omega))$$

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Forgets initial condition after time.

- Power absorbed by oscillation

$$p = F\dot{x} = mf\dot{x} \text{ Avg power}$$

$$P_{\text{avg}} = \frac{1}{T} \int_0^T mf\dot{x} dt = -\frac{1}{2}mf_0a(\Omega)\Omega \sin \delta(\Omega) \quad 22$$

$$P(\Omega) = \gamma mf_0^2\Omega^2a^2(\Omega)$$

Absorption around resonance frequency $\Omega = \omega_0 + \varepsilon$ is maximum:

$$P = \frac{\gamma mf_0^2}{4(\varepsilon^2 + \gamma^2)} \approx \frac{mf_0^2}{4\gamma} \quad 23$$

Oscillations DOF>1

For a system with n DOF: $q = (q_1, q_2, \dots, q_n)$, PE = $U(q)$

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