HW 9, Harry Luo

8.6 (d)

$$
\begin{aligned}
E\left[(X+Y)^2\right] = E[X^2 + 2XY + Y^2] &= E[X^2] + E[XY] + E[Y^2] \\
&= \frac{2-p}{p^2} + \frac{2nr}{p} + n(n-1)r^2 + nr
\end{aligned} \tag{1}
$$

8.12

(a) By definition, the MGF can be found by

$$
\begin{aligned}
M_Z(t) = E(e^{tZ}) &= \int_0^\infty e^{tz}\lambda^2 z e^{-\lambda z}\,\mathrm{d}z \\
&= \lambda^2 \int_0^\infty z e^{(t-\lambda)z}\,\mathrm{d}z
\end{aligned} \tag{2}
$$

- if $\lambda - t > 0$, noticing the following expectation of an exp r.v. with parameter $\lambda - t$:

$$
\int_0^\infty z(\lambda - t)e^{-(\lambda - t)}z\,\mathrm{d}z = \frac{1}{\lambda - t} \tag{3}
$$

  Equation 2 becomes:

$$
M_Z(t) = \frac{\lambda^2}{(\lambda - t)^2} \tag{4}
$$

- if $\lambda - t \leq 0$, the integral converges to infinity.

$$
\Rightarrow M_Z = \begin{cases} \frac{\lambda}{\lambda - t} & t < \lambda \\ \infty & t \geq \lambda \end{cases} \tag{5}
$$

(b) for X and Y, example 5.6 on textbook suggests that

$$
M_X(t) = M_Y(t) = \begin{cases} \frac{\lambda}{\lambda - t} & t < \lambda \\ \infty & t \geq \lambda \end{cases} \tag{6}
$$

Given that X and Y are independent,

$$
M_{X+Y} = M_{X(t)}M_{Y(t)} = \begin{cases} \frac{\lambda^2}{(\lambda - t)^2} & t < \lambda \\ \infty & t \geq \lambda \end{cases} \tag{7}
$$

It is obvious that Equation 7 is the same as Equation 5, so the MGF of X+Y is the same as the MGF of Z. Thus X+Y has the same distribution as Z.

8.22

Considering the indicator method: let $I_j = 1 \text{ or } 0, j \in [1, 89]$ be the indicator of "among the five numbers, both j and j+1 are chosen".
Since every time j and $j+1$ are chosen, they are next to each other in the ordered sample, so $X = \sum_{j=1}^{89} I_j$

$$
E[X] = E\left[\sum_{j=1}^{89} I_j\right] = \sum_{j=1}^{89} E[I_j] \tag{8}
$$

recognizing that

$$
E[I_j] = P(j \text{ and } j+1 \text{ are chosen}) = \frac{\binom{88}{3}}{\binom{90}{5}} = \frac{2}{801} \tag{9}
$$

$$E[X] = \frac{89 * 2}{801} = \frac{20}{89} \tag{10}$$

8.23

(a) Denote the color of the ith pick as C_i, then $C_1, C_2, ..., C_{50}$ are exchangeable. Thus,

$$P(C_{28} \neq C_{29}) = P(C_{28} = R, C_{29} = G) = 2P(C_1 = R, C_2 = G) = 2 * \frac{20 * 30}{50 * 49} = \frac{24}{49} \tag{11}$$

(b) Let I_j be the indicator that $Y_j \neq Y_{j+1}, j = 1, 2, ..., 49$

$X = I_1 + ... + I_{49}$, thus,

$$E[X] = E[I_1] + ... + E[I_{49}] = 49 * P(C_1 \neq C_2) = 49 * \frac{24}{49} = 24 \tag{12}$$

8.36

(a) let I_j be the indicator that the number j appears at least once in the four rolls. Then $X = I_1 + ... + I_6$. By exchangeability,

$$E[X] = E[I_1] + ... + E[I_6] = 6 * E[I_1] \tag{13}$$

Since, $E[I_1] = P(1 \text{ appears in roll}) = 1 - P(\text{no rolls with } 1) = 1 - \left(\frac{5}{6}\right)^4$, we have

$$E[X] = 6 * \left(1 - \left(\frac{5}{6}\right)^4\right) = \frac{671}{216} \tag{14}$$

(b) noticing that $I_j^2 = 1$ since I is either 1 or zero, and using exchangeability,

$$\begin{aligned} E[X^2] &= E\left[(I_1 + ... + I_6)^2\right] = E[I_1^2 + ... + I_6^2 + 2I_1I_2 + ... + 2I_5I_6] \\ &= \sum_{j=1}^{6} E\left[I_j^2\right] + 2 \sum_{i<j\leq 6} E\left[I_iI_j\right] \\ &= 6E\left[I_1^2\right] + 30E[I_1I_2] \end{aligned} \tag{15}$$

TO find $E[I_1I_2]$, we use the inclusion-disclusion principle,

$$\begin{aligned} E[I_1I_2] &= P(1 \text{ and } 2 \text{ both show up at least onece}) \\ &= 1 - P(1 \text{ does not show up}) - P(2 \text{ does not show up}) + P(1 \text{ and } 2 \text{ both do not show up}) \\ &= 1 - \left(\left(\frac{5}{6}\right)^4 + \left(\frac{5}{6}\right)^4 - \left(\frac{4}{6}\right)^4\right) = \frac{151}{648} \end{aligned} \tag{16}$$

Thus,

$$\begin{aligned} E[X^2] &= \frac{671}{216} + \frac{151}{648} = \frac{541}{162} \\ \text{var}[X] &= E[X^2] - E[X]^2 \approx 0.447 \end{aligned} \tag{17}$$

8.40 (a)

Using indiator method, let I_k is the indicator for the event thta hte number k is drawn at least once in the 4 weeks. Then $X = I_1 + ... + I_{90}$. By exchangeability,

$$E[X] = E[I_1] + ... + E[I_{90}] = 90E[I_1] \tag{18}$$

Considering that

$$E[I_1] = P(1 \text{ is drawn at least once in 4 weeks}) = 1 - P(1 \text{ is not drawn in 4 weeks}) = 1 - \left(\frac{85}{90}\right)^4 \tag{19}$$

$$\Rightarrow E[X] = 90\left(1-\left(\frac{85}{90}\right)^4\right) \approx 18.39 \tag{20}$$

8.42

the fourth moment can be found as

$$E\left[\overline{X_n^4}\right] = E\left[\left(\frac{X_1+X_2+...+X_n}{n}\right)^4\right] = \frac{1}{n^3}E\left[(X_1+...X_n)^4\right] \tag{21}$$

According to binomial theorem,

$$\begin{aligned} E\left[\overline{X_n^4}\right] &= \frac{1}{n^4}E\left[\sum_k = 1^n X_k^4 + 24\sum_{i<j<k<l} X_iX_jX_kX_l + 12\sum_{k<l,j\neq k,j\neq l} X_j^2X_k^2 + 4\sum_{j\neq k} X_j^3X_k\right] \\ &= \frac{1}{n^3}E[X_1^4] + 24\binom{n}{4}E[X_1X_2X_3X_4] + 12\binom{n}{3}E[X_1^2X_2X_3] + 4\binom{n}{2}E[X_1^3X_2] + 6\binom{n}{2}E[X_1^2X_2^2] \end{aligned} \tag{22}$$

by independence, we know

$$\begin{aligned} E[X_1X_2X_3X_4] = E[X_1]E[X_2]E[X_3]E[X_4] = (E[X_1])^4 = 0 \\ E[X_1^2X_2X_3] = E[X_1^2]E[X_2]E[X_3] = 0 \\ E[X_1^3X_2] = E[X_1^3]E[X_2] = 0 \\ E[X_1^2X_2^2] = E[X_1^2]E[X_2^2] = E[X_1^2]^2 \end{aligned} \tag{23}$$

Therefore,

$$\begin{aligned} E\left[\overline{X_n^4}\right] &= \frac{1}{n^3}E[X_1^4] + \frac{3n(n-1)}{n^4}E[X_1^2]^2 \\ &\boxed{= \frac{c}{n^3} + \frac{3(n-1)a^2}{n^3}} \end{aligned} \tag{24}$$

8.48

We can graph the joint pmf as follows:

X\Y	0	1	2
1	9/100	0	0
2	81/100	9/100	0
3	0	0	1/100

From which we can find the marginals:

$$\begin{aligned} p_X(1) = \frac{9}{100}, p_X(2) = \frac{90}{100}, p_X(3) = \frac{1}{100} \\ p_Y(0) = \frac{90}{100}, p_Y(1) = \frac{9}{100}, p_Y(2) = \frac{1}{100} \end{aligned} \tag{25}$$

The expectance can be naturally found as

$$
\begin{aligned}
E[X] = \frac{48}{25}, E[Y] = \frac{11}{100}, E[XY] = \frac{6}{25} \\
\text{Cov }(X, Y) = E[XY] - E[X]E[Y] = \frac{18}{625}
\end{aligned}
\tag{26}
$$

8.54

(a)

$$
\begin{aligned}
\text{var }(X) = E[X^2] - E[X]^2 = 5 - 2^2 = 1 \\
\text{var }(Y) = E[Y^2] - E[Y]^2 = 10 - 1 = 9 \\
\text{Cov }(X, Y) = E[XY] - E[X]E[Y] = 1 - 2 = -1 \\
\Rightarrow \text{Corr }(X, Y) = \text{Cov } \frac{X, Y}{\sqrt{\text{var }(X) \text{ var }(Y)}} = -\frac{1}{3}
\end{aligned}
\tag{27}
$$

(b) Noticing

$$
\text{Cov }(X, X + cY) = \text{Var }(X) + c \text{ Cov}(X, Y) = 1 + c \tag{28}
$$

so when c = −1, the covariance is 0, and the r.v. X and $X + cY$ are not correlated