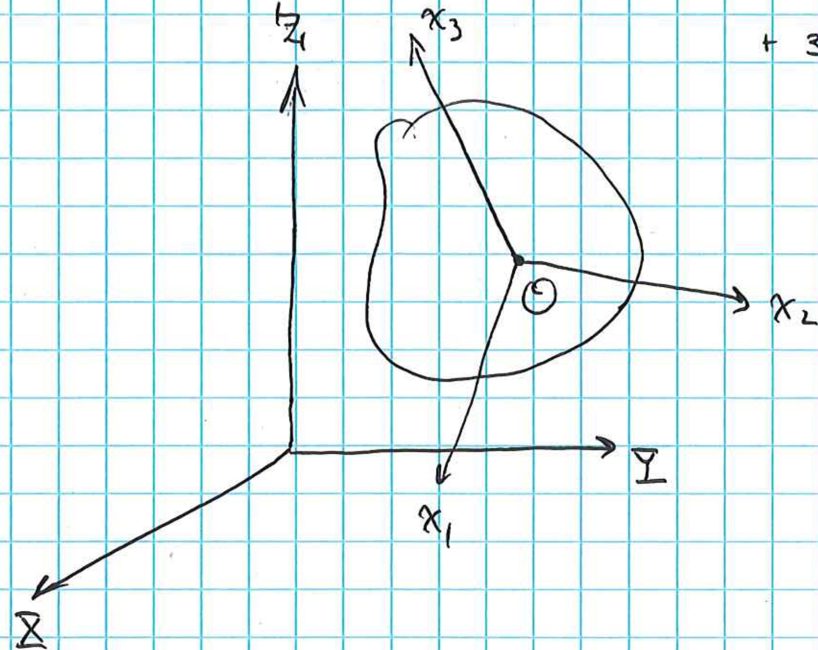
Summary

03/20/24

- Rigid body = mechanical system w/ 6 DOF = 3 DOF for COM + 3 DOF for orientation



$(X\,Y\,Z_1)$ = fixed inertial frame
$(x_1\,x_2\,x_3)$ = frame fixed in body

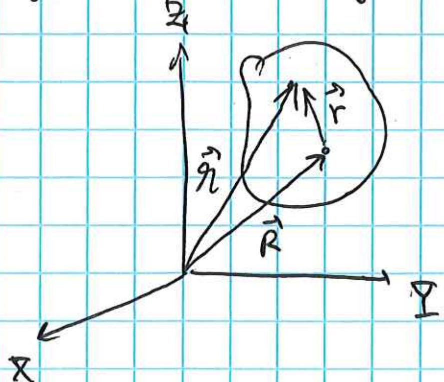
$$(X\,Y\,Z_1) \xrightarrow[\substack{R(\theta,\varphi,\psi) \\ =\text{rot.}^{\text{n}}}]{} (x_1\,x_2\,x_3)$$

→ 3 angles to specify rot.$^{\text{n}}$

- velocity of a pt. in the rigid body:

$$\vec{v} = \vec{V} + \vec{\Omega} \times \vec{r}$$

translational part, rotational part, $\vec{\Omega}$ = angular velocity vector



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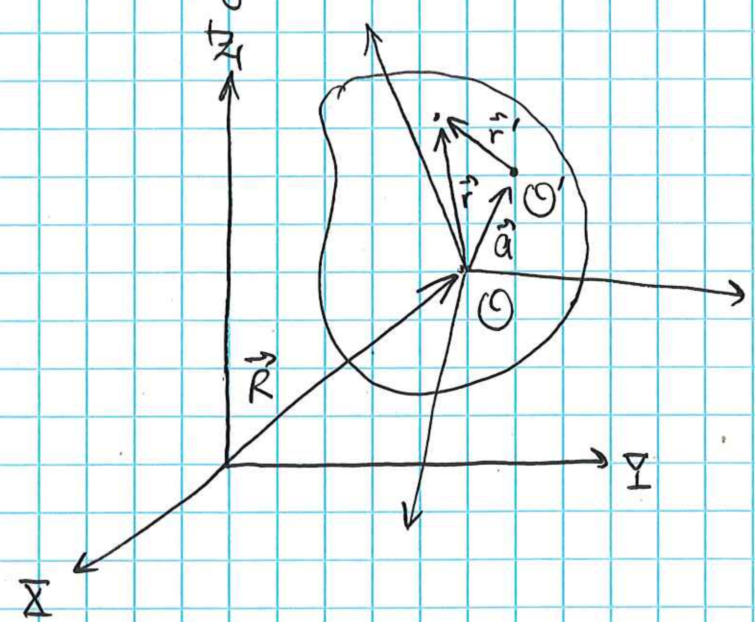
→ we considered situation where origin of the body frame O is situated at COM. This led to ang. velocity vector $\vec{\Omega}$. Is $\vec{\Omega}$ uniquely defined?

→ Consider another frame in the body w/ origin displaced by $\vec{a}$:



$$\vec{r} = \vec{r}\,' + \vec{a}$$

$$\vec{v} = \vec{V} + \vec{\Omega} \times \vec{r}$$

$$= \vec{V} + \vec{\Omega} \times (\vec{r}\,' + \vec{a})$$

$$= (\vec{V} + \vec{\Omega} \times \vec{a}) + \vec{\Omega} \times \vec{r}\,'$$

$\Rightarrow$ shift of origin of body frame modifies translational comp. of vel. but $\vec{\Omega}$ unchanged.

$\Rightarrow \vec{\Omega}$ of body uniquely defined.

Q: what is the Lagrangian for a rigid body?

→ start w/ kinetic energy.

$$T = \sum_a \frac{1}{2} m_a v_a^2 \qquad a = \text{label of particle in rigid body}$$

$$= \sum_a \frac{1}{2} m_a (\vec{V} + \vec{\Omega} \times \vec{r}_a)^2$$

$$= \sum_a \frac{1}{2} m_a V^2 + \sum_a m_a \vec{V} \cdot (\vec{\Omega} \times \vec{r}_a) + \sum_a \frac{1}{2} m_a (\vec{\Omega} \times \vec{r}_a)^2$$

now we use the following vector identities: 03/20/24

$$(i) \quad \vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{B}\cdot(\vec{C}\times\vec{A}) = \vec{C}\cdot(\vec{A}\times\vec{B}).$$

$$(ii) \quad (\vec{A}\times\vec{B})\cdot(\vec{C}\times\vec{D}) = (\vec{A}\cdot\vec{C})(\vec{B}\cdot\vec{D}) - (\vec{A}\cdot\vec{D})(\vec{B}\cdot\vec{C}).$$

$$\rightarrow \quad \vec{V}\cdot(\vec{\Omega}\times\vec{r}_a) = \vec{r}_a\cdot(\vec{V}\times\vec{\Omega}) \quad \text{by (i)}$$

$$(\vec{\Omega}\times\vec{r}_a)^2 = (\vec{\Omega}\times\vec{r}_a)\cdot(\vec{\Omega}\times\vec{r}_a) = \Omega^2 r_a^2 - (\vec{\Omega}\cdot\vec{r}_a)^2 \quad \text{by (ii)}$$

$$\Rightarrow T = \frac{1}{2}\underbrace{\left(\sum_a m_a\right)}_{\substack{M=\text{total}\\ \text{mass}}} V^2 + \underbrace{\left(\sum_a m_a\vec{r}_a\right)}_{\substack{=M\vec{R}\\ =0 \text{ if } O \text{ at COM}}}\cdot(\vec{V}\times\vec{\Omega}) + \frac{1}{2}\sum_a m_a\left[\Omega^2 r_a^2 - (\vec{\Omega}\cdot\vec{r}_a)^2\right]$$

$$\Rightarrow \boxed{T = \underbrace{\frac{1}{2}MV^2}_{\substack{\text{translational}\\ \text{motion of COM.}}} + \underbrace{\frac{1}{2}\sum_a m_a\left[\Omega^2 r_a^2 - (\vec{\Omega}\cdot\vec{r}_a)^2\right]}_{\substack{\text{rotation w/ } \vec{\Omega} \text{ about}\\ \text{axis through COM.}}}}$$

$$= T_{\text{translational}} + T_{\text{rotational}}.$$

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now, $T_{rot.}$ can be rewritten in a more useful way:

$\vec{\Omega} \longrightarrow \Omega_i$ = components of $\vec{\Omega}$ in body frame (projections onto x_1, x_2, x_3 axes)

$\vec{r}_a \longrightarrow x_{a,i}$ = comp.'s of $\vec{r}_a$ in body frame.

$$\Rightarrow \begin{cases} \Omega^2 = \sum_i \Omega_i^2, \quad \cancel{r_a^2 = \sum_i x_{a,i}^2} \\ \vec{\Omega}\cdot\vec{r}_a = \sum_i \Omega_i x_{a,i} \end{cases}$$

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Then we write:

$$T_{rot.} = \frac{1}{2} \sum_a m_a \left[\Omega^2 r_a^2 - (\vec{\Omega} \cdot \vec{r}_a)^2 \right]$$

$$= \frac{1}{2} \sum_a m_a \left[\sum_i \Omega_i \Omega_i \, r_a^2 - \sum_{ij} (\Omega_i x_{a,i})(\Omega_j x_{a,j}) \right]$$

$$= \frac{1}{2} \sum_a m_a \left[\sum_{ij} \Omega_i \Omega_j \delta_{ij} r_a^2 - \sum_{ij} \Omega_i \Omega_j x_{a,i} x_{a,j} \right]$$

$$= \frac{1}{2} \sum_{ij} \Omega_i \Omega_j \sum_a m_a \left[\delta_{ij} r_a^2 - x_{a,i} x_{a,j} \right]$$

$$\Rightarrow \boxed{T_{rot.} = \frac{1}{2} \sum_{ij} \Omega_i \Omega_j I_{ij}, \quad I_{ij} \equiv \sum_a m_a (\delta_{ij} r_a^2 - x_{a,i} x_{a,j})}$$

"moment of inertia tensor"

$$\Rightarrow T = \frac{1}{2} \overset{M}{\cancel{m}} V^2 + \frac{1}{2} \sum_{ij} I_{ij} \Omega_i \Omega_j$$

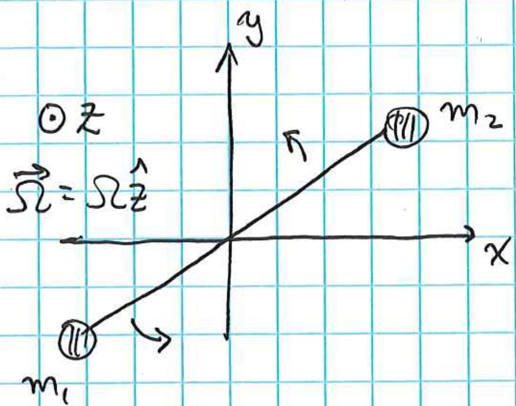
$$\Rightarrow L = T - U = \frac{1}{2} \overset{M}{\cancel{m}} V^2 + \frac{1}{2} \sum_{ij} I_{ij} \Omega_i \Omega_j - U$$

U = fn. of 3 COM coord.'s

+ 3 coord.'s to specify orientation.

(have not yet figured out how to choose coord.'s for orientation).

Ex: Rigid rotor in the plane. 03/20/24



focus on rotational motion

→ coord. system w/ origin at COM.

$$\vec{\Omega} = \Omega \hat{z} \longrightarrow T_{rot.} = \frac{1}{2} I_{zz} \Omega_z \Omega_z .$$

$$I_{zz} = \sum_a m_a \Big(r_a^2 - \underbrace{\overset{z_a\, z_a}{\cancel{x_a x_a}}}_{} \Big)$$

=0, since masses only in ~~z~~ xy-plane.

$$= m_1 r_1^2 + m_2 r_2^2$$

$$\Rightarrow T_{rot.} = \frac{1}{2} \left(m_1 r_1^2 + m_2 r_2^2 \right) \Omega_z^2 .$$

Properties of inertia tensor:

- $I_{ij} = I_{ji}$ → symmetric matrix
- ~~ch~~ Components:

$$I = \begin{pmatrix} \sum m(y^2+z^2) & -\sum m\, xy & -\sum m\, xz \\ -\sum m\, xy & \sum m(x^2+z^2) & -\sum m\, yz \\ -\sum m\, xz & -\sum m\, yz & \sum m(x^2+y^2) \end{pmatrix}$$

- ~~I~~ Inertia tensor = $\sum$ moments of inertia of individual parts
- Continuous body: $m_a \longrightarrow dm = \rho\, dv$, ρ = mass density, dv = infinitesimal vol.

$$\Rightarrow I_{ij} = \int dv\, \rho(\vec{r}) \left(r^2 \delta_{ij} - x_i x_j \right) .$$