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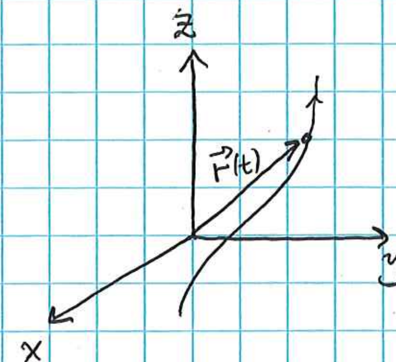
Reformulating Classical Mechanics

- Newtonian $\rightarrow$ Lagrangian
- Discuss advantages (& some disadvantages) of such a reformulation as we go along.

Setting for particle mech.:



- single particle: trajectory $\vec{r}(t)$

$\vec{r} = (x, y, z)$ (3 coord.'s)

velocity: $\vec{v} = \dfrac{d\vec{r}}{dt} \equiv \dot{\vec{r}}$

accel. : $\vec{a} = \dfrac{d^2\vec{r}}{dt^2} \equiv \ddot{\vec{r}}$

- N particles: $(\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_N) = (x_1, y_1, z_1, \ldots, x_N, y_N, z_N)$

$\rightarrow$ 3N-dim. space to describe motion.

In practice, such a description of particle motion in terms of Cartesian coord.'s is not the most convenient.

Ex: In problems w/ spherical symmetry, polar ~~coord~~ coord.'s may be useful: $(x, y, z) \rightarrow (r, \theta, \varphi)$

$\rightarrow$ such changes of coord.'s are ~~aw~~ awkward in Newtonian mech.

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# of quantities needed to specify config. of system

= # of "degrees of freedom" (DOF).

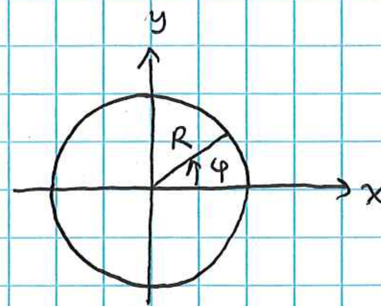
- simplest case: # of DOF = $3N$ (N particles).
- If particles subject to <u>constraints</u>, # of DOF = $n \leq 3N$

$\Rightarrow$ change coord.'s: $(\vec{r}_1, \dots, \vec{r}_N) \longrightarrow (q_1, \dots, q_n)$.

<u>Ex:</u> particle on a ring.

constraint $x^2+y^2=R^2$

$s=1$ $\quad q=\varphi$

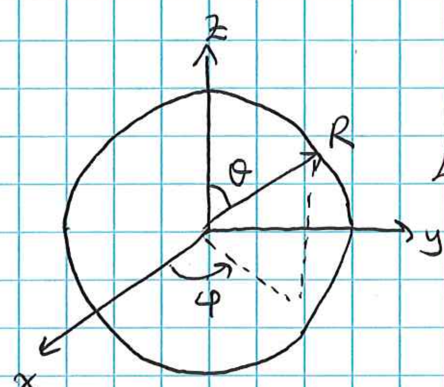


$$\begin{cases} x(\varphi) = R\cos\varphi \\ y(\varphi) = R\sin\varphi \end{cases}$$

<u>Ex:</u> particle on a sphere

constraint $x^2+y^2+z^2=R^2$

$s=2$ $\quad (q_1, q_2) = (\theta, \varphi)$



$$\begin{cases} x(\theta,\varphi) = R\sin\theta\cos\varphi \\ y(\theta,\varphi) = R\sin\theta\sin\varphi \\ z(\theta,\varphi) = R\cos\theta. \end{cases}$$

In general, write $\vec{r}_i = \vec{r}_i(q_1, \dots, q_n)$ $\quad i=1,\dots,N$

$(q_1, \dots, q_n)$ = "generalized coord.'s"

$\equiv q.$

$(\dot{q}_1, \dots, \dot{q}_n)$ = "generalized velocities"

$\equiv \dot{q}$

- So, in particle mech. we will in general describe state of the system by n generalized coord.'s $(q_1, \ldots, q_n)$
- Dynamics (time evolution, $q(t)$) is determined by the "equations of motion" (EOM).

→ Newtonian approach to dynamics:

specify all forces F → $m\ddot{\vec{r}} = \vec{F}(\vec{r})$ (single particle)

$m_i\ddot{\vec{r}}_i = \vec{F}_i(\vec{r}_1, \ldots, \vec{r}_N)$ (N-particles)

+ initial conditions $\vec{r}_i(t=0)$ & $\dot{\vec{r}}_i(t=0)$.

☆ we will develop an alternative procedure for determining EOM based on a <u>variational principle</u>

→ Lagrangian mech. (equivalent to Newton).

<u>Why?</u> (1) Deal much more easily w/ different kinds of coord.'s. In particular, easier to deal w/ constrained systems.

(2) Consequences of symmetry will be manifest → conservation laws.

(3) Connection to quantum mech.

→ path integrals & Hamiltonian ($\hat{H}$, $[\hat{x}, \hat{p}] = i\hbar$)

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Variational principles & Calculus of variations

Roughly: A "variational principle" states that a physical quantity can be obtained by minimizing* a certain "functional"

functional = object that takes as input a function & returns a number.

Ex: Definite integral is a functional, taking as input the function $f(x)$ & returning the number $= \int_a^b f(x)\,dx$.

For us: physical quantity = trajectory $q(t)$

functional = "action"

$$S[q(t)] = \int_{t_1}^{t_2} dt \; L(q, \dot{q}, t)$$

L = "Lagrangian function".

Variational principles are ubiquitous in physics.

Ex: Geometric optics

Fermat's principle: path traveled by light ray btwn. two fixed pt.'s is that of least time.

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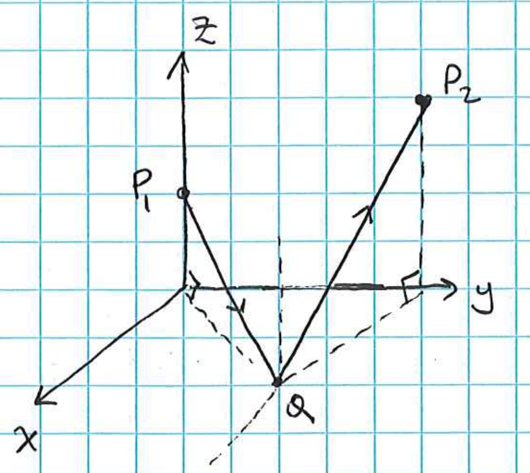
(a) Reflection from a mirror



(mirror in xy-plane)

$T_{P_1QP_2} = \text{min.} \Rightarrow$ Q lies in same plane as P_1 & P_2 & $\theta_1 = \theta_2$

$P_1 = (0, 0, z_1)$

$P_2 = (0, y_2, z_2)$

$Q = (x, y, 0)$.

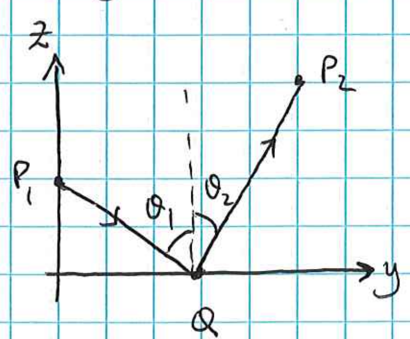
$$\Rightarrow T_{P_1QP_2} = \frac{1}{c}\left(\sqrt{x^2+y^2+z_1^2} + \sqrt{x^2+(y_2-y)^2+z_2^2}\right).$$

So, $T_{P_1QP_2} = T_{P_1QP_2}(x, y)$. To minimize: $\frac{\partial T_{P_1QP_2}}{\partial x} = \frac{\partial T_{P_1QP_2}}{\partial y} = 0$.

- $\frac{\partial T_{P_1QP_2}}{\partial x} = 0 \quad \Rightarrow \quad \frac{x}{\sqrt{x^2+y^2+z_1^2}} + \frac{x}{\sqrt{x^2+(y_2-y)^2+z_2^2}} = 0.$

  $\Rightarrow \quad x = 0$, so Q lies in the same plane as P_1 & P_2. ✓

- $\frac{\partial T_{P_1QP_2}}{\partial y} = 0 \quad \Rightarrow \quad \frac{y}{\sqrt{y^2+z_1^2}} - \frac{(y_2-y)}{\sqrt{(y_2-y)^2+z_2^2}} = 0.$



$\Rightarrow \sin\theta_1 = \sin\theta_2$ or $\theta_1 = \theta_2$ ✓

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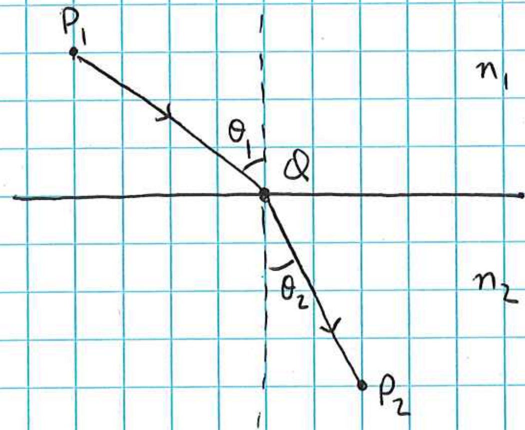
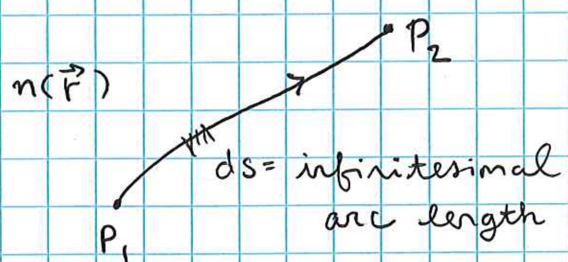
(b) Refraction



$T_{P_1 Q P_2} = \text{min.} \Rightarrow \quad n_1 \sin\theta_1 = n_2 \sin\theta_2$ (Snell's law)

→ see HW 1.

(c) In (a) & (b), T = <u>function</u> of position Q. More generally, consider light propagating through medium w/ spatially varying index of refraction $n(\vec{r})$



$$dt = \frac{ds}{v(\vec{r})} = \frac{1}{c}\, n(\vec{r})\, ds$$

$$\Rightarrow T_{P_1P_2} = \int dt = \frac{1}{c}\int ds\; n$$

$\Rightarrow T_{P_1P_2} = T_{P_1P_2}[\vec{r}(s)]$ is a <u>functional</u> of the entire path $\vec{r}(s)$

↑

"arc-length parametrization."