Brief Theory of Probability, Part 1
Survey of main ideas and equations

Contents

1 Vector algebra

1.a Coordinate Transformation

1.a.a cylindical

$$x = \rho \cos \varphi$$
$$y = \rho \sin \varphi$$
$$z = z$$

reverse

$$\rho = \sqrt{x^2 + y^2}$$
$$\cos \varphi = \frac{x}{\rho}$$
$$\sin \varphi = \frac{y}{\rho}$$

1.a.b spherical

$$x = \rho \sin \varphi \cos \theta$$
$$y = \rho \sin \varphi \sin \theta$$
$$z = \rho \cos \varphi$$

reverse

$$\rho = \sqrt{x^2 + y^2 + z^2}$$
$$\cos \varphi = \frac{z}{\rho}$$
$$\cos \theta = \frac{x}{r}$$
$$\sin \theta = \frac{y}{r}$$

1.b Dot product

- commutative
- positive definite
- distributive
- cauchy-schwarz inequality

1.c cross product

- anticommutative $\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$
- distributive $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} + \vec{w}$
- scalar mulipication
- triple scalar product $\vec{u} \cdot (\vec{v} \times \vec{w}) = (\vec{u} \times \vec{v} \cdot \vec{w})$
- triple vector product $\vec{a} \times \left(\vec{b} \times \vec{c}\right) = \left(\vec{b} \cdot \vec{a}\right)\vec{c} - (\vec{c} \cdot \vec{a})\vec{b}$

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2 Vector calculus

2.a Are length

- Def: Given a curve $\vec{r}(u) = (x(u), y(u), z(u))$ for $a \leq t \leq b$ the length of the curve S, as a function of time is given by

$$\boxed{\begin{gathered} S(t) = \int_a^t \left\|r(\dot{u})\right\| \mathrm{d}u \\ \text{where } \|\dot{r}(u)\| = \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2} \end{gathered}}$$

- Curvature:

$$K(t) = \frac{\left\|\dot{T}(t)\right\|}{\|\dot{r}(t)\|} = \frac{\|(\dot{r}(t) \times \ddot{r}(t))\|}{(\|\dot{r}(t)\|)^3}, \text{where } T(t) = \frac{\dot{r}(t)}{\|\dot{r}(t)\|}$$

2.b Line integration

- for curve $\vec{r}(t) = (x(t), y(t))$

$$\int_C f(x(t), y(t)) \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2} \, \mathrm{d}t$$

- center of mass $(\overline{x}, \overline{y}, \overline{z})$, where

$$\begin{cases} \overline{x} = \left(\frac{1}{M}\right) \int_C \rho(x,y,z) x ds \\ \overline{y} = \left(\frac{1}{M}\right) \int_C y\rho(x,y,z) ds \\ \overline{z} = \left(\frac{1}{M}\right) \int_C z\rho(x,y,z) ds \end{cases}$$

- Work done by force F along curve, $\vec{r}(t)$, which can be generalized into the formula for line integration,

$$W = \int_C F \cdot \mathrm{d}\vec{r} = \int_C \vec{F} \cdot \vec{T} \,\mathrm{d}s = \boxed{\int_a^b F[x(t), y(t)] \cdot (\dot{r}(t)) \,\mathrm{d}t}$$

- When vector field $\vec{F} = \vec{F}(x, y, z) = (P, Q, R)$,

$$\int_C \vec{F} \cdot \mathrm{d}\vec{r} = \int_C P dx + Q dy + R dz$$

2.c Surface integration

- Parametric representation of surface:

$$\begin{cases} x = x(u,v) \\ y = y(u,v) \\ z = z(u,v) \end{cases}$$

- Use normal vector at a point (u_0, v_0) of surface to represent tangent plane.

$$\vec{r_v} = \frac{\partial \vec{r}}{\partial v}(u_0, v_0), \vec{r_u} = \frac{\partial \vec{r}}{\partial u}(u_0, v_0)$$

$$\vec{N} = \vec{r_u} \times \vec{r_v}$$

- Surface area of a surface S with $(u, v) \in D$

$$A(S) = \iint_D \|\vec{r_u} \times \vec{r_v}\| \,\mathrm{d}u \,\mathrm{d}v$$

2.d Jacobian

- Def: Given a transformation $(u, v) \in D \longrightarrow [x(u, v), y(u, v)] \in S$, the Jacobian is given by

$$J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} \equiv \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$$

- Jacobian in coordinate transformation

$$\iint_S f(x,y) \,\mathrm{d}A = \iint_D f(x(u,v), y(u,v)) \; |J(u,v)| \,\mathrm{d}u \,\mathrm{d}v$$

2.e Gradient, Divergence, Curl

- Nabla operation:

$$\nabla = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}$$

- Gradient in 2D cartisian (r, θ) :

$$\nabla f = \frac{\partial f}{\partial r} i + \left(\frac{1}{r}\right) \frac{\partial f}{\partial \theta} j$$

- Gradient in polar

$$\nabla f = \frac{\partial f}{\partial \rho} i + \left(\frac{1}{\rho}\right) \frac{\partial f}{\partial \varphi} j$$

- Gradient in spherical

$$\nabla f = \partial_\rho \hat{\rho} + \hat{\varphi} \frac{1}{\rho} \partial_\varphi + \hat{\theta} \frac{1}{\rho \sin \varphi} \partial_\theta$$

2.f Green's theorem

2.g Stokes' theorem