Small Oscillations

- Motion near a point of stable equilibrium.

DOF= 1 (one dimension)

- For a system of DOF = 1, with potential $U(q)$:
  - **stable equilibrium** at $U(q)_{\text{min}}$ where $F = -\frac{\mathrm{d}U}{\mathrm{d}q} = 0$
    - restoring force for small displacements $q - q_0$ is $F = -\frac{\mathrm{d}U(q-q_0)}{\mathrm{d}q}$
- **Unstable equilibrium** at $U(q)_{\text{max}}$ where $F = -\frac{\mathrm{d}U}{\mathrm{d}q} = 0$ as well.
- Consider small deviation from point of stable equilibrium, we use taylor expansion to show that it is really a small displacement. that is,

$$\begin{gathered} U(q) \approx U(q_0) + \frac{\mathrm{d}U(q_0)}{\mathrm{d}q}(q - q_0) + \left(\frac{1}{2}\right)\frac{\mathrm{d}^2U(q_0)}{\mathrm{d}q^2}(q - q_0)^2 + \ldots \\ \text{while } \frac{\mathrm{d}U(q_0)}{\mathrm{d}q}(q - q_0) = 0 \end{gathered} \tag{1}$$

letting $x = q - q_0$, we have

$$\begin{cases} U(x) = U(q_0) + \left(\frac{1}{2}\right)\frac{\mathrm{d}^2U(q_0)}{\mathrm{d}q^2}x^2 \\ \text{also } U(x) = \left(\frac{1}{2}\right)kx^2. \end{cases} \Rightarrow \boxed{k = \frac{\mathrm{d}^2U(q_0)}{\mathrm{d}q^2} > 0} \tag{2}$$

we get KE, while choosing $U(q_0) = 0$:

$$\begin{gathered} T = \frac{1}{2}a(q)^2\dot{q}^2 = \frac{1}{2}a(q_0 + x)\dot{x}^2 \approx \frac{1}{2}m\dot{x}^2, \text{letting } \boxed{m = a(q_0)} \\ \Rightarrow L = T - U = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 \end{gathered} \tag{3}$$

EOM for DOF = 1 small Oscillations

using EL on Equation 3, we can get the EOM for one dimensional small Oscillations:

$$\begin{gathered} m\ddot{x} = -kx \\ \Rightarrow \ddot{x} + \omega_0^2 x = 0, \text{where } \boxed{\omega_0 = \sqrt{\frac{k}{m}}} \end{gathered} \tag{4}$$

by magic of ODE, EOM reduces down to:

$$x(t) = C_1\cos(\omega_0 t) + C_2\sin(\omega_0 t) \text{ ,where } \boxed{C_1, C_2 \text{ are constants}} \tag{5}$$

by trig magic, this could also be written as

$$\begin{gathered} x(t) = a\cos(\omega_0 t + \varphi) \\ \text{where } \Big[a = \sqrt{C_1^2 + C_2^2}, \text{amplitude of oscillation} \\ \omega_0 = \text{frequency of oscillation} \\ \text{and } \tan\varphi = C_2/C_1, \text{phase corresponding to origin of time}\Big] \end{gathered} \tag{6}$$

- checking $\frac{\partial L}{\partial t} = 0$ this is an energy-conserved system, meaning that:

$$
\begin{aligned}
E = T + U = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 = \frac{1}{2}ma^2\omega_0^2[\ \text{ constant}] \\
\omega = \sqrt{\frac{k}{m}} = \text{frequency of oscillation}
\end{aligned}
\tag{7}
$$