Awesome applied analysis
Notes on MATH 321 Harry Luo

The course contents could be better had it been Fabien's class, but probably Trinh saved my GPA.

Contents

I Vector algebra

I.1 Coordinate Transformation

I.1.1 cylindical

$$x = \rho \cos \varphi$$
$$y = \rho \sin \varphi$$
$$z = z$$

reverse

$$\rho = \sqrt{x^2 + y^2}$$
$$\cos \varphi = \frac{x}{\rho}$$
$$\sin \varphi = \frac{y}{\rho}$$

I.1.2 spherical

$$x = \rho \sin \varphi \cos \theta$$
$$y = \rho \sin \varphi \sin \theta$$
$$z = \rho \cos \varphi$$

reverse

$$\rho = \sqrt{x^2 + y^2 + z^2}$$
$$\cos \varphi = \frac{z}{\rho}$$
$$\cos \theta = \frac{x}{r}$$
$$\sin \theta = \frac{y}{r}$$

I.2 Dot product

- commutative
- positive definite
- distributive
- cauchy-schwarz inequality

I.3 cross product

- anticommutative $\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$
- distributive $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} + \vec{w}$
- scalar mulipication
- triple scalar product $\vec{u} \cdot (\vec{v} \times \vec{w}) = (\vec{u} \times \vec{v} \cdot \vec{w})$
- triple vector product $\vec{a} \times \left(\vec{b} \times \vec{c}\right) = \left(\vec{b} \cdot \vec{a}\right)\vec{c} - (\vec{c} \cdot \vec{a})\vec{b}$

I.4 Projection

The projection of $\vec{a}$ onto $\vec{b}$ is given by

$$\boxed{\frac{\vec{a} \cdot \vec{b}}{\left\|\vec{b}\right\|^2} \vec{b} = \left(a \cdot \hat{b}\right)\hat{b}}$$

II Vector calculus

II.1 Are length

- Def: Given a curve $\vec{r}(u) = (x(u), y(u), z(u))$ for $a \leq t \leq b$ the length of the curve S, as a function of time is given by

$$\boxed{\begin{gathered} S(t) = \int_a^t \left\|r(\dot{u})\right\| \,\mathrm{d}u \\ \text{where } \|\dot{r}(u)\| = \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2} \end{gathered}}$$

- Curvature:

$$K(t) = \frac{\left\|\dot{T}(t)\right\|}{\|\dot{r}(t)\|} = \frac{\|(\dot{r}(t) \times \ddot{r}(t))\|}{(\|\dot{r}(t)\|)^3}, \text{where } T(t) = \frac{\dot{r}(t)}{\|\dot{r}(t)\|}$$

II.2 Line integration

- for curve $\vec{r}(t) = (x(t), y(t))$

$$\boxed{\int_a^b f(x, y) \,\mathrm{d}s = \int_a^b f[x(t), y(t)] \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2} \,\mathrm{d}t}$$

- center of mass $(\overline{x}, \overline{y}, \overline{z})$, where

$$\begin{cases} \overline{x} = \left(\frac{1}{M}\right) \int_C \rho(x, y, z) x ds \\ \overline{y} = \left(\frac{1}{M}\right) \int_C y \rho(x, y, z) ds \\ \overline{z} = \left(\frac{1}{M}\right) \int_C z \rho(x, y, z) ds \end{cases}$$

- Work done by force F along curve, $\vec{r}(t)$, which can be generalized into the formula for line integration,

$$W = \int_C F \cdot \mathrm{d}\vec{r} = \int_C \vec{F} \cdot \vec{T} \,\mathrm{d}s = \boxed{\int_a^b F[x(t), y(t)] \cdot (\dot{r}(t)) \,\mathrm{d}t}$$

- When vector field $\vec{F} = \vec{F}(x, y, z) = (P, Q, R)$,

$$\int_C \vec{F} \cdot \mathrm{d}\vec{r} = \int_C P dx + Q dy + R dz$$

II.3 Surface integration

- Parametric representation of surface:

$$\begin{cases} x = x(u, v) \\ y = y(u, v) \\ z = z(u, v) \end{cases}$$

- Use normal vector at a point (u_0, v_0) of surface to represent tangent plane.

$$\vec{r_v} = \frac{\partial \vec{r}}{\partial v}(u_0, v_0), \vec{r_u} = \frac{\partial \vec{r}}{\partial u}(u_0, v_0)$$

$$\vec{N} = \vec{r_u} \times \vec{r_v}$$

- Surface area of a surface S with $(u, v) \in D$

$$A(S) = \iint_D \|\vec{r_u} \times \vec{r_v}\| \, \mathrm{d}u \, \mathrm{d}v$$

II.4 Jacobian

- Def: Given a transformation $(u, v) \in D \longrightarrow [x(u, v), y(u, v)] \in S$, the Jacobian is given by

$$\boxed{J(u, v) = \frac{\partial(x, y)}{\partial(u, v)} \equiv \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}}$$

- Jacobian in coordinate transformation

Upon evaluating an integral, we can change the coordinates of the integral from $\{x, y\} \to \{u, v\}$ by parametrize the variables:

$$x = x(u, v) \quad y = y(u, v)$$

Then the integral becomes

$$\iint_S f(x, y) \, \mathrm{d}A = \iint_D f(x(u, v), y(u, v)) \; |J(u, v)| \, \mathrm{d}u \, \mathrm{d}v$$

II.5 Gradient

- Nabla operation:

$$\boxed{\nabla = \frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}}$$

- Gradient in cartesian Scalar field $f = f(x, y, z)$

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$$

- Gradient in polar coordinates $f = f(r, \theta)$

$$\nabla f = \vec{e_r} \frac{\partial g}{\partial r} + \vec{e_\theta} \frac{1}{r} \frac{\partial g}{\partial \theta}$$

$$\text{where } \vec{e_r} = \frac{x}{\|x\|} = (\cos\theta, \sin\theta) \vec{e_\theta} = (-\sin\theta, \cos\theta)$$

$$\nabla = \vec{e_r} \partial_r + \vec{e_\theta} \frac{1}{r} \partial_\theta$$

- Gradient in spherical

$$\nabla f = \hat{\rho} \partial_\rho + \hat{\varphi} \frac{1}{\rho} \partial_\varphi + \hat{\theta} \frac{1}{\rho \sin\varphi} \partial_\theta$$

- Gradient of scalar field in spherical coordinates

Let $g(\rho, \phi, \theta) = f(x, y, z)$

$$\begin{cases} x = \rho \sin\phi \cos\theta \\ y = \rho \sin\phi \sin\theta \\ z = \rho \cos\phi \end{cases} \qquad \begin{bmatrix} \partial_\rho g \\ \partial_\phi g \\ \partial_\theta g \end{bmatrix} = \begin{bmatrix} \partial_\rho x & \partial_\rho y & \partial_\rho z \\ \partial_\phi x & \partial_\phi y & \partial_\phi z \\ \partial_\theta x & \partial_\theta y & \partial_\theta z \end{bmatrix} \begin{bmatrix} \partial_x f \\ \partial_y f \\ \partial_z f \end{bmatrix}$$

$$\begin{aligned} \hat{\rho} &= (\partial_\rho x, \partial_\rho y, \partial_\rho z) = \frac{(x, y, z)}{\rho} \\ \hat{\phi} &= \frac{1}{\rho}(\partial_\phi x, \partial_\phi y, \partial_\phi z) \\ \hat{\theta} &= \frac{1}{\rho \sin\phi}(\partial_\theta x, \partial_\theta y, \partial_\theta z) \end{aligned} \qquad \Rightarrow \begin{bmatrix} \partial_x f \\ \partial_y f \\ \partial_z f \end{bmatrix} = \begin{bmatrix} \hat{\rho}_1 & \hat{\phi}_1 & \hat{\theta}_1 \\ \hat{\rho}_2 & \hat{\phi}_2 & \hat{\theta}_2 \\ \hat{\rho}_3 & \hat{\phi}_3 & \hat{\theta}_3 \end{bmatrix} \begin{bmatrix} \partial_\rho g \\ \frac{1}{\rho} \partial_\phi g \\ \frac{1}{\rho \sin\phi} \partial_\theta g \end{bmatrix}$$

II.6 Divergence

- div of vec field:

3D:

$$\nabla \cdot \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

- Div in polar 2D

$$\vec{U} = U_r \hat{r} + U_\theta \hat{\theta}, \text{where } U_r = U \cdot \hat{r}, U_\theta = U \cdot \hat{\theta}$$

$$\nabla \cdot U = \left(\frac{1}{r}\right) \frac{\partial(r U_r)}{\partial r} + \frac{\partial U_\theta}{\partial \theta}$$

- Div in sphereical coord

$$\vec{U} = U_\rho \hat{\rho} + U_\theta \hat{\theta} + U_\varphi \hat{\varphi},$$

$$\nabla \cdot \vec{U} = \frac{1}{\rho^2} \frac{\partial(\rho^2 U_\rho)}{\partial \rho} + \frac{1}{\rho} \sin\varphi \frac{\partial(U_\theta)}{\partial \theta} + \frac{1}{\rho \sin\varphi} \frac{\partial(U_\theta \sin\varphi)}{\partial \varphi})$$

II.7 Green's theorem

For $P(x,y), Q(x,y)$, and a simple closed curve C,

$$\boxed{\int_C P dx + Q dy = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dA = \iint_C \vec{F} \cdot \mathrm{d}\vec{r}}$$

II.8 Flux

- for a surface,

$$\vec{r}(u,v) = (x(u,v), y(u,v), z(u,v))$$
$$\Rightarrow \iint_S \vec{F} \cdot \mathrm{d}\vec{S} = \iint_S \vec{F} \cdot \hat{n}\, \mathrm{d}S = \iint_D \vec{F}(\vec{r}(u,v)) \cdot (\vec{r_u} \times \vec{r_v})\, \mathrm{d}A$$

- if the surface is a graph of a fucntion $z = g(x,y)$ where $(x,y) \in D, \vec{F} = (P,Q,R)$, then

$$\int_S \vec{F} \cdot \mathrm{d}\vec{s} = \iint_D (P,Q,R) \cdot \left(-\partial_x g, -\partial_y g, 1\right) \mathrm{d}A$$

II.9 Stokes' theorem

Let $F : R^3 \to R^3$ be a vector field on R^3 with any normal vector $\vec{n}$, and for a surface S with projection on $\{u,v\}$ being A, then

$$\boxed{\int_C \vec{F} \cdot \mathrm{d}\vec{r} = \iint_S \operatorname{curl}\left(\vec{F}\right) \cdot \hat{n}\, \mathrm{d}S = \iint_S \left(\nabla \times \vec{F}\right) \cdot \vec{n}\, \mathrm{d}A},$$
$$\text{where } \operatorname{curl}\left(\vec{F}\right) = \nabla \times \vec{F}$$

- Discussion on stokes theorem

  for a surface surface parametrized by $\vec{r}_u, \vec{r}_v$, we have

$$\mathrm{d}\vec{S} = \hat{n}\, \mathrm{d}S = \vec{n}\, \mathrm{d}A = \vec{n}\, \mathrm{d}u\, \mathrm{d}v$$

Therefore, when using stokes theorem, we cna either turn it into a surface integral with respect to actual surface S, with

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III Complex analysis

III.1 Complex numbers and basic operations

III.1.1 Definitions

- Def: $i^2 = -1$
- Complex number: $z = x + iy$
- Conjugate: $z = x - iy$
- Real part: $\Re(z) = x$, Imaginary part: $\Im(z) = y$
- Modulus/ Norm/ Magnitude: $|z| = \sqrt{x^2 + y^2}$
- Polar form: $z = |z|\ (\cos\theta + i\sin\theta) = re^{i\theta}$

- Argument(angle) : $\arg(z) = \theta$ such that $z = |z| \ (\cos\theta + i\sin\theta)$. Angle between vector (x, y) with real axis

III.1.2 operations

- addition: $z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$
- multiplication: $z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$
  (normal multiplication with $i^2 = 1$)
- Division:

$$\frac{z_1}{z_2} = \frac{z_1 z_1^*}{z_2 z_2^*} = \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + i\frac{x_2 y_1 - x_1 y_2}{x_2^2 + y_2^2}$$

- Commutativity: $z_1 z_2 = z_2 z_1 \quad z_1 + z_2 = z_2 + z_1$
- associativity: $(z_1 z_2) z_3 = z_1 (z_2 z_3) \quad (z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$
- distributivity: $z_1 (z_2 + z_3) = z_1 z_2 + z_1 z_3$
- Trig inequality: $|z_1 + z_2| \leq |z_1| + |z_2|$

III.2 Differentiation

III.2.1 open sets in $\mathbb{C}$

- Def: Let $z_0 \in \mathbb{C}, r > 0$. Disk $B_{r(z_0)} = \{z \in \mathbb{C} | \ |z - z_0| < r\}$ It is very important to note that it's not "less or equal"

  Given a set $\Omega \in \mathbb{C}$, A point $z_0 \in \Omega$ is called an interior point of Ω if there exists $r > 0$ s.t. $B_{r(z_0)} \subset \Omega$.
  A set Ω is **open** if every point of Ω is an interior poinot of Ω. In other words, there are no points on the boundary of Ω that are included in Ω.

III.2.2 Holomorphic function

Let Ω be an open set in $\mathbb{C}$, A function $f : \Omega \to \mathbb{C}$ is called **holomorphic** at $z_0 \in \Omega$ if the limit

$$f'(z_0) = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} (h \in \mathbb{C}, h \neq 0)$$

exists.

- The said function $f(z)$ is holomorphic on Ω if it is holomorphic on every point of Ω.
- In the special case that f is holomorphic on $\mathbb{C}$, f is an **entire** function.
- Holomorphic in 1st order guarantees holomorphic and analytic in any order and thus continous.

III.2.3 Differentiation operations

If f and g are holomorphic on Ω, then

- $f + g$ is holomorphic on Ω,

  $$(f + g)' = f' + g'$$

- fg is analytic on Ω,

$$(fg)' = f'g + fg'$$

- $\frac{f}{g}$ is analytic and, if $g(z) \neq 0$,

$$\frac{f}{g} = \frac{f'g - fg'}{g^2}$$

III.2.4 Cauchy-Riemann equations

for complex function $f : \Omega \to \mathbb{C}$, $f(z) = u(x, y) + iv(x, y)$ that is holomorphic at $z_0 = x_0 + iy_0$, then the partial derivatives of u and v exist and satisfy the Cauchy-Riemann equations:

$$\boxed{\partial_x u = \partial_y v \quad \partial_y u = -\partial_x}$$

Conversly, if u and v are continuously differentiable on an open set Ω and satisfy the Cauchy-Riemann equations, then $f(z) = u(x, y) + iv(x, y)$ is holomorphic on Ω.

In the language of logic, let C be "satisfying cauchy-riemann equations", and H be "function is holomorphic", then $H \to C$. If D is "u and v have continuous partial derivatives with respect to x and y", then $(C \& D) \leftrightarrow H$

III.3 Cauchy's integral theorem (closed loop)

For a closed curve C in an open set Ω and a holomorphic function $f : \Omega \to \mathbb{C}$, then

$$\oint_C f(z) \, \mathrm{d}z = 0$$

III.4 Fundemental theorem of calculus for complex analysis

If f is holomorphic on an open set Ω and $a, b \in \Omega$, and for $f(z) = F'(z)$, and a, b are the start and end points of curve C, we have

$$\boxed{\int_C f(z) \, \mathrm{d}z = F(b) - F(a)}$$

III.5 Cauchy's integral formula

This relates the value of a contour integration to the value of its derivatives on a curve.

$$\boxed{f^n(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} \, \mathrm{d}z}$$

Often times, we are concernnd in finding the value of a function of the form

$$\int_C \frac{f(z)}{(z - z_0)^{n+1}} \, \mathrm{d}z,$$

so we would like to take the nth derivative of the function $f(z)$ at z_0, and find the desired integral by

$$\frac{2\pi i}{n!} f^n(z_0)$$

III.6 Cauchy's residue theorem

III.6.1 Poles

Simply find where the fraction is not defined, i.e. where the denominator is 0. This is normally done by first using $(a^2 + z^2) = (z + ai)(z - ai)$ to factor the denominator, and then setting the denominator to 0 to find poles z_i .

III.6.2 Residue

If the factored denominator has the form $(z + ai)(z + bi)$, then it has two poles of order 1. If it has the form $(z + ai)^2(z + bi)^2$, then it has 2 poles of order 2.

If has poles of order one, for each pole z_i , find residue by

$$\operatorname{Res}(f, z_i) = \lim_{z \to z_i} (z - z_i) f(z)$$

If has pole of order n, for each pole z_0, find res by

$$\operatorname{Res}(f, z_0) = \lim_{z \to z_0} \frac{1}{(n-1)!} \left(\frac{d}{dz}\right)^{n-1} ((z - z_0)^n f(z))$$

III.6.3 Cauchy's residue theorem

For a simple closed curve C in an open set Ω and a holomorphic function $f : \Omega \to \mathbb{C}$, then

$$\oint_C f(z) \, \mathrm{d}z = 2\pi i \sum_{k=1}^{n} \operatorname{Res}(f, z_k)$$

where z_k are the poles of f in C.

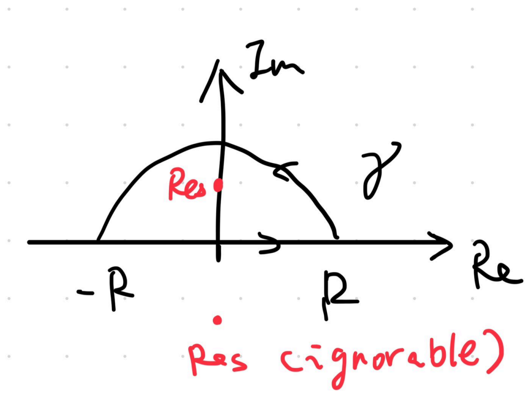
Often times, we want to find the value of the integral

$$\int_0^\infty f(x) \, \mathrm{d}x$$

to which we are clueless to solve in the real domain. Cauchy suggests that we can take a detour via the complex domain by using the subsitution $f(z) = f(x)$ where $z \in \mathbb{C}$. By residue theorem we have

$$\oint_C f(z) \, \mathrm{d}z = \int_{-R}^{R} f(z) \, \mathrm{d}z + \int_\gamma f(z) \, \mathrm{d}z = 2\pi i \sum_{k=1}^{n} \operatorname{Res}(f, z_k)$$

Normally, this looks like



where γ is the semicircle in the complex domain

We thus get

$$\int_{-R}^{R} f(z)\,\mathrm{d}z = 2\pi i \sum_{k=1}^{n} \mathrm{Res}(f, z_k) - \underbrace{\int_{\gamma} f(z)\,\mathrm{d}z}_{*}$$

we notice that $(*) \leq \max_{|z|\ =R}[f(z)] * \text{length of } \gamma = f(R) * \pi R \overset{R \to 0}{=} 0.$

In english this means $(*)$ is smaller than the product of maximal value of $f(z)$ on the semicircle and the length of the semicircle, which goes to 0 as R goes to infinity.

Thus the above integral becomes

$$\int_{-R}^{R} f(z)\,\mathrm{d}z = 2\pi i \sum_{k=1}^{n} \mathrm{Res}(f, z_k)$$