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1 (a)

$$\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \boxed{-2}$$

(b)

$$\mathrm{d}x\,\mathrm{d}y = |\frac{\partial(x,y)}{\partial(u,v)}|\,\mathrm{d}u\,\mathrm{d}v = \boxed{2\,\mathrm{d}u\,\mathrm{d}v}$$

2.(a)

$$\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{pmatrix} v & u \\ \frac{1}{v} & -\frac{u}{v^2} \end{pmatrix} = \boxed{\frac{-2u}{v}}$$

(b)

$$\mathrm{d}x\,\mathrm{d}y = |\frac{\partial(x,y)}{\partial(u,v)}|\,\mathrm{d}u\,\mathrm{d}v = \boxed{\frac{2u}{v}\,\mathrm{d}u\,\mathrm{d}v}$$

3.

For $T(u,v) = (u^2 - v^2, 2uv), u \in [0,1], v \in [0,1]$, we know:

$x = u^2 - v^2, y = 2uv$, and the Jacobian is $\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{pmatrix} 2u & -2v \\ 2v & 2u \end{pmatrix} = 4u^2 + 4v^2$

Thus,

$$\begin{aligned}
\int\int_R (x+y)\,\mathrm{d}x\,\mathrm{d}y &= \int_0^1 \int_0^1 (u^2 - v^2 + 2uv) \cdot 4(u^2 + v^2)\,\mathrm{d}u\,\mathrm{d}v \\
&= 4\int_0^1 \int_0^1 u^4 - v^4 + 2u^3v + 2uv^3\,\mathrm{d}u\,\mathrm{d}v \\
&= 4\int_0^1 \frac{1}{5} + \frac{v}{2} + v^3 - v^4\,\mathrm{d}v \\
&= 4 * \frac{1}{2} = \boxed{2}
\end{aligned}$$

4.

Considering the change of variables $x = au, y = bv, z = cw$,

The Jacobian is $\frac{\partial(x,y,z)}{\partial(u,v,w)} = \det\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} = abc$

therefore,

$$\begin{aligned}
\int\int\int_E 1\,\mathrm{d}V = \int\int\int_D abc\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w &\overset{\text{from symmetry}}{=} 8\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^1 abc\rho^2 \sin\varphi\,\mathrm{d}\rho\,\mathrm{d}\varphi\,\mathrm{d}\theta \\
&= 8abc \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{1}{3}\rho^3 \sin\varphi \Big|_{\rho=0}^{\rho=1} \mathrm{d}\varphi\,\mathrm{d}\theta \\
&= 8abc \int_0^{\frac{\pi}{2}} \left(-\frac{1}{3}\cos\varphi\right)\Big|_0^{\frac{\pi}{2}} \mathrm{d}\theta \\
&= 8abc \cdot \int_0^{\frac{\pi}{2}} \frac{1}{3}\,\mathrm{d}\theta \\
&= 8abc \cdot \frac{1}{3} \cdot \frac{\pi}{2} = \boxed{4\frac{\pi}{3}abc}
\end{aligned}$$