Integration of EOM.

02/09/24

- Lagrangian formalism = efficient way to write down EOM

→ hard part is still to solve them.

- Consider some examples where EOM can be integrated

→ possibility to do this facilitated by symmetries.

(& hence conservation laws).

Motion with 1 DOF:

$$L = \frac{1}{2} a(q)\dot{q}^2 - U(q) \qquad q = \text{generalized coord.}$$

If $q = x$ is a Cartesian coord, $L = \frac{1}{2} m\dot{x}^2 - U(x)$.

$\frac{\partial L}{\partial t} = 0 \Rightarrow$ energy conserved.

$$\rightarrow E = \dot{x}\frac{\partial L}{\partial \dot{x}} - L = m\dot{x}^2 - \left(\frac{1}{2} m\dot{x}^2 - U(x)\right) = \frac{1}{2} m\dot{x}^2 + U(x) = \text{const.}$$

$$\Rightarrow \left(\frac{dx}{dt}\right)^2 = \frac{2}{m}\left[E - U(x)\right]$$

$$\Rightarrow dt = \sqrt{\frac{m}{2}} \frac{dx}{\sqrt{E - U(x)}}$$

$$\Rightarrow \boxed{t = \sqrt{\frac{m}{2}} \int \frac{dx}{\sqrt{E - U(x)}} + C}$$

→ E & C = two arbitrary const.'s, as expected.

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Ex: $U(x) = mgx$

$$t = \sqrt{\frac{m}{2E}} \int \frac{dx}{\sqrt{1 - \frac{mgx}{E}}} + C$$

$x = \frac{E}{mg} u$

$dx = \frac{E}{mg} du.$

$$= \sqrt{\frac{m}{2E}} \times \frac{E}{mg} \int \frac{du}{\sqrt{1-u}} + C$$

$$= -\sqrt{\frac{2E}{mg^2}} \sqrt{1-u} + C$$

$C = t_0$

$$\Rightarrow (t-t_0)^2 = \frac{2E}{mg^2}(1-u)$$

$$\Rightarrow u = 1 - \frac{mg^2}{2E}(t-t_0)^2$$

$$\Rightarrow x(t) = \frac{E}{mg} - \frac{1}{2} g (t-t_0)^2 \quad ✓$$

E & t_0 = arbitrary const.'s

Ex: $U(x) = \frac{1}{2} k x^2$

$$t = \sqrt{\frac{m}{2E}} \int \frac{dx}{\sqrt{1 - \frac{kx^2}{2E}}} + C$$

$x = \sqrt{\frac{2E}{k}} \sin\theta$

$dx = \sqrt{\frac{2E}{k}} \cos\theta \, d\theta$

$$= \sqrt{\frac{m}{k}} \int \frac{\cos\theta \, d\theta}{\sqrt{1-\sin^2\theta}} + C$$

$1 - \sin^2\theta = \cos^2\theta.$

$$= \sqrt{\frac{m}{k}} \int d\theta + C$$

$\omega = \sqrt{k/m}$

$\theta_0 = \omega C = \text{const}.$

$$\Rightarrow t = \frac{1}{\omega}(\theta + \theta_0),$$

$$\Rightarrow \sin(\omega t - \theta_0) = \sin\theta$$

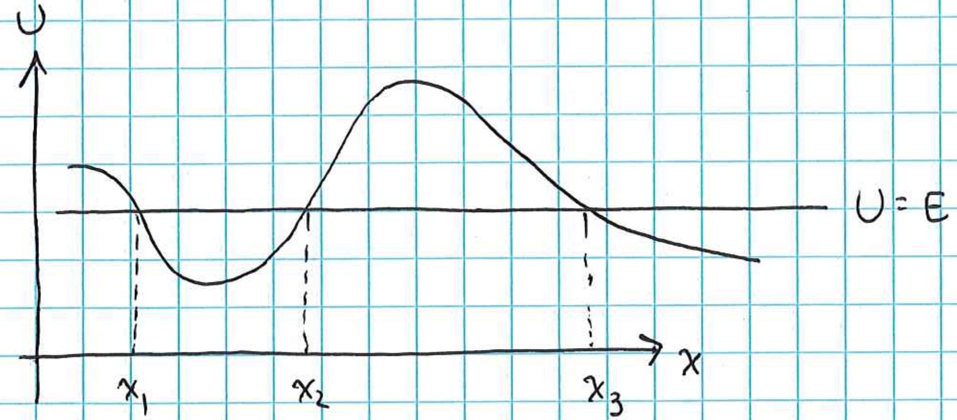
$$\Rightarrow x(t) = \sqrt{\frac{2E}{k}} \sin(\omega t - \theta_0)$$

E & θ_0 = arbitrary const.'s

Immediate inference about motion 02/09/24

$E = T + U \geq U$, since $T \geq 0$.

$\Rightarrow$ restriction on regions of allowed motion



only $x_1 \leq x \leq x_2$ or $x \geq x_3$ allowed.

· points x s.t. $U(x) = E$ are "turning pt.'s", since $v = 0$.

· motion can be "bounded" $(x_1 \leq x \leq x_2)$ or "unbounded" $(x \geq x_3)$

Focus on bounded motion: $x_1(E) \leq x \leq x_2(E)$.

Associated motion is periodic (oscillation in potential well)

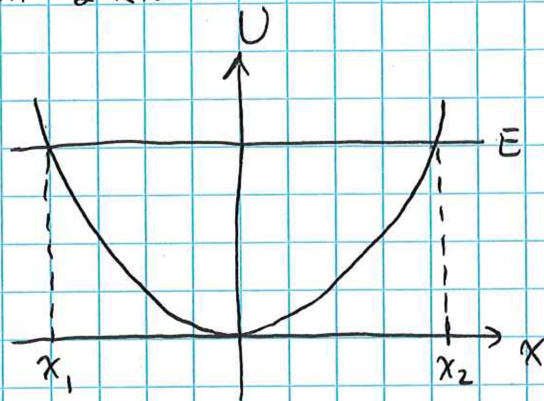
Period of the motion:

$$\boxed{T(E) = \sqrt{2m} \int_{x_1(E)}^{x_2(E)} \frac{dx}{\sqrt{E - U(x)}}}$$

where $U(x_1) = E$ & $U(x_2) = E$.

→ determines period of motion as fn. of total energy E.

Ex: $U(x) = \frac{1}{2}kx^2$ 02/09/24



$U(x) = E$

$\Rightarrow \frac{1}{2}kx^2 = E$

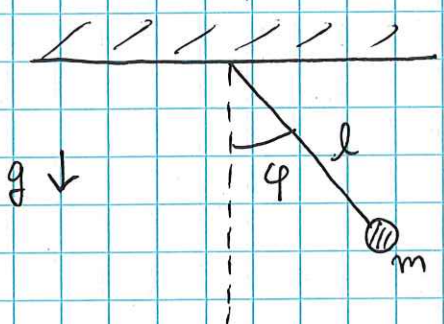
$$\Rightarrow \left.\begin{array}{l} x_1 = -\sqrt{2E/k} \\ x_2 = +\sqrt{2E/k} \end{array}\right\}$$

$$\Rightarrow \quad T(E) = \sqrt{\frac{2m}{E}} \int_{-\sqrt{2E/k}}^{+\sqrt{2E/k}} \frac{dx}{\sqrt{1 - \frac{k}{2E}x^2}} = \frac{2}{\omega}\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} d\theta = \frac{2\pi}{\omega}$$

$x = \sqrt{\frac{2E}{k}}\sin\theta, \quad \omega = \sqrt{k/m}$

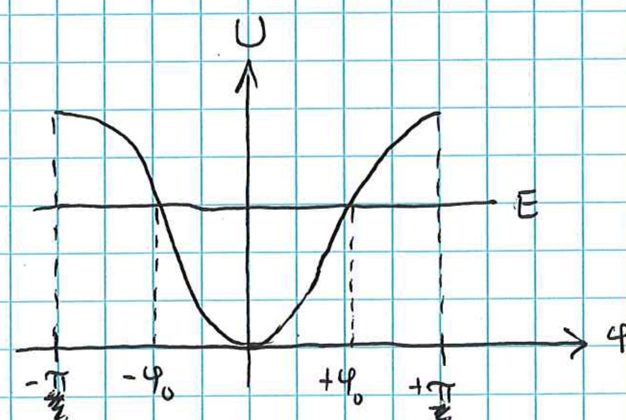
$\Rightarrow T = \frac{2\pi}{\omega}$ independent of E!

Ex: Simple pendulum



$T = \frac{1}{2}ml^2\dot{\varphi}^2$ ($m \to ml^2$ in previous formulas)

$U = mgl(1-\cos\varphi)$.



turning pt.'s: $U(\varphi_0) = E$

$\Rightarrow \quad E = mgl(1-\cos\varphi_0)$

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$$\Rightarrow T(E) = 2\sqrt{2ml^2} \int_0^{\varphi_0} \frac{d\varphi}{\sqrt{E - U(\varphi)}}$$

$$= 4\sqrt{\frac{l}{2g}} \int_0^{\varphi_0} \frac{d\varphi}{\sqrt{\cos\varphi - \cos\varphi_0}} \qquad (\text{recall } \varphi_0 = \varphi_0(E))$$

$$\cos\varphi = 1 - 2\sin^2\frac{\varphi}{2}$$

$$= 2\sqrt{\frac{l}{g}} \int_0^{\varphi_0} \frac{d\varphi}{\sqrt{\sin^2\frac{\varphi_0}{2} - \sin^2\frac{\varphi}{2}}}$$

$$k \equiv \sin\frac{\varphi_0}{2}$$

$$= 2\sqrt{\frac{l}{g}} \int_0^{\varphi_0} \frac{d\varphi}{\sqrt{k^2 - \sin^2\frac{\varphi}{2}}}$$

$$\sin u = \frac{1}{k}\sin\frac{\varphi}{2}$$

$$\Rightarrow T(E) = 4\sqrt{\frac{l}{g}} \int_0^{\frac{\pi}{2}} \frac{du}{\sqrt{1 - k^2\sin^2 u}}$$

• The last expression is useful b/c it is a standard integral expressible in terms of a special fn. ("complete elliptic integral of the first kind"). For us what's more interesting is that it allows for a systematic small angle expansion: $\varphi_0 \ll 1 \Rightarrow k \ll 1$.

→ Expanding: $$T(E) \simeq 4\sqrt{\frac{l}{g}} \int_0^{\frac{\pi}{2}} du \left(1 + \tfrac{1}{2}k^2\sin^2 u\right)$$

$$\int_0^{\frac{\pi}{2}} du \sin^2 u = \frac{\pi}{4}$$

$$= 2\pi\sqrt{\frac{l}{g}}\left(1 + \frac{k^2}{4}\right)$$

$$k \simeq \varphi_0/2, \quad \varphi_0 \ll 1$$

$$\simeq 2\pi\sqrt{\frac{l}{g}}\left(1 + \frac{\varphi_0^2}{16}\right)$$

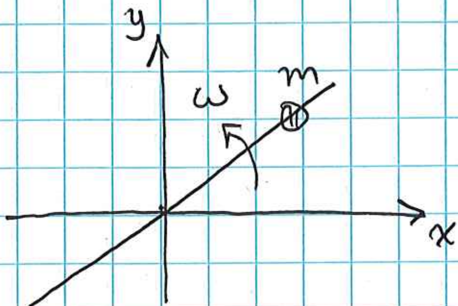
→ first term is just usual formula for pendulum in small angle approx. But now we have also obtained the first correction.

The 1D problem is not as restrictive as it seems. 02/09/24

There are certain cases in which a problem can be reduced to an "effective 1D problem".

This can happen ~~so~~ when symmetries reduce the # of DOF (we'll see this explicitly when we take up the two-body problem) or with constraints (like the pendulum ex.). Here is another such ex.:

<u>Ex:</u> (Bead on rotating rod).



$$\left.\begin{aligned} x(r) &= r\cos\omega t \\ y(r) &= r\sin\omega t \end{aligned}\right\} \text{ constraint.}$$

$$T = \frac{1}{2}m(\dot{x}^2+\dot{y}^2) = \frac{1}{2}m(\dot{r}^2+r^2\omega^2).$$

$$U = 0.$$

$$\Rightarrow L = T - U = T = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\omega^2.$$

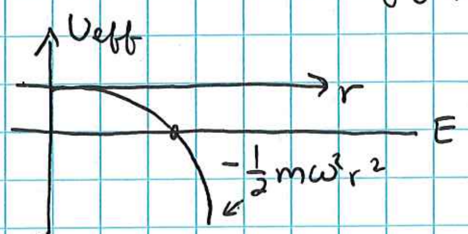
Now, $\frac{\partial L}{\partial t} = 0 \Rightarrow \tilde{E} = \dot{r}\frac{\partial L}{\partial \dot{r}} - L$ = conserved (explain "~" soon)

$$= \underbrace{\frac{1}{2}m\dot{r}^2}_{T_{eff}} \underbrace{- \frac{1}{2}mr^2\omega^2}_{U_{eff}}.$$

$$= T_{eff} + U_{eff}.$$

→ like a particle in an "effective potential" $U_{eff} = -\frac{1}{2}mr^2\omega^2$

Note $\tilde{E} \neq$ mechanical energy, which is simply $= T$.



→ immediately see motion is always unbounded.