Small Oscillations

- Motion near a point of stable equilibrium.

DOF= 1 (one dimension)

- For a system of DOF = 1, with potential $U(q)$:
  - **stable equilibrium** at $U(q)_{\text{min}}$, upward parabola, where $F=-\frac{\mathrm{d}U}{\mathrm{d}q}=0$
    - restoring force for small displacements $q-q_0$ is $F=-\frac{\mathrm{d}U(q-q_0)}{\mathrm{d}q}$
- **Unstable equilibrium** at $U(q)_{\text{max}}$, downward parabola, where $F=-\frac{\mathrm{d}U}{\mathrm{d}q}=0$ as well.
- Consider small deviation from point of stable equilibrium, we use taylor expansion to show that it is really a small displacement. that is,

$$\begin{gathered}U\approx U(q_0)+\frac{\mathrm{d}U(q_0)}{\mathrm{d}q}(q-q_0)+\frac{\mathrm{d}^2U(q_0)}{2\,\mathrm{d}q^2}(q-q_0)^2\\ \text{while } \frac{\mathrm{d}U(q_0)}{\mathrm{d}q}(q-q_0)=0\end{gathered} \tag{1}$$

letting $x=q-q_0$, we have

$$\begin{cases}U(x)=U(q_0)+\left(\frac12\right)\frac{\mathrm{d}^2U(q_0)}{\mathrm{d}q^2}x^2\\ \text{putting into the form of } U(x)=U(x_0)+\left(\frac12\right)kx^2.\end{cases} \tag{2}$$

$$\Rightarrow \boxed{k=\frac{\mathrm{d}^2U(q_0)}{\mathrm{d}q^2}>0}$$

we get KE, while choosing $U(q_0)=0$:

$$\begin{gathered}T=\frac12 a(q)^2\dot q^2=\frac12 a(q_0+x)\dot x^2\approx\frac12 m\dot x^2, \overset{m=a(q_0)}{\Rightarrow}\\ \boxed{L=T-U=\frac12 m\dot x^2-\frac12 kx^2}\end{gathered} \tag{3}$$

EOM for DOF = 1 small Oscillations

using EL on Equation 3, we can get the EOM for one dimensional small Oscillations:

$$\begin{gathered}m\ddot x=-kx\\ \Rightarrow \ddot x+\omega_0^2x=0,\text{ where }\boxed{\omega_0=\sqrt{\frac{k}{m}}\text{ freq of osc.}}\end{gathered} \tag{4}$$

by magic of ODE, EOM reduces down to:

$$\begin{gathered}\boxed{x(t)=C_1\cos(\omega_0t)+C_2\sin(\omega_0t)}\\ \text{where } C_1, C_2 \text{ are constants}\end{gathered} \tag{5}$$

by trig magic, this could also be written as

$$\begin{gathered}x(t)=a\cos(\omega_0t+\alpha),\\ \text{where}\begin{cases}a=\sqrt{C_1^2+C_2^2} & \text{amplitude of oscillation}\\ \omega_0 & \text{frequency of oscillation}\\ \tan\alpha=C_2/C_1 & \text{phase at t=0}\end{cases}\end{gathered} \tag{6}$$

energy for 1D small Oscillation

checking $\frac{\partial L}{\partial t}=0\Rightarrow$ energy-conservation:

$$\begin{aligned}E=T+U&=\frac12 m\dot x^2+\frac12 kx^2\\ &=\frac12 ma^2\omega_0^2,[\text{ constant}]\end{aligned} \tag{7}$$

Damped 1D oscillation, and Complex representation

- when there is damping (friction, resistence, etc) $F_{\text{fric}}=-\beta\dot x$, the EOM becomes:

$$\begin{gathered}\ddot x+2\gamma\dot x+\omega_0^2x=0,\\ \text{where } 2\gamma=\frac{\beta}{m},\omega_0=\sqrt{\frac{k}{m}}\end{gathered} \tag{8}$$

with ansatz $x(t)=e^{rt},\dot x=re^{rt},\ddot x=r^2e^{rt}$, the solution to Equation 8 is:

$$\begin{gathered}r^2+2\gamma r+\omega_0^2=0,\\ \text{which has solution } r_+,r_-=-\gamma\pm\sqrt{\gamma^2-\omega_0^2}\\ \Rightarrow x(t)=C_1e^{r_+t}+C_2e^{r_-t},\end{gathered} \tag{9}$$

notice the r subscripts here: r_+, r_-

underdamped, overdamped, and critically damped

Recall from your ODE class...

Equation 9 has the following 3 cases, each with different physical interpretation:

1. underdamped:

$$\gamma<\omega_0\Rightarrow 2\text{ complex roots:}\begin{cases}r_\pm=-\gamma\pm i\sqrt{\omega_0^2-\gamma^2}\\ =-\gamma\pm i\omega\\ \omega=\sqrt{\omega_0^2-\gamma^2}\end{cases} \tag{10}$$

The EOM is thus a linear combination of two complex expoentials:

$$\begin{aligned}x(t)&=e^{-\gamma t}(C_1e^{i\omega t}+C_2e^{-i\omega t})\\ &=e^{-\gamma t}(A\cos(\omega t)+B\sin(\omega t))\\ &\text{-- where}\begin{cases}A=C_1+C_2\\ B=i(C_1-C_2)\end{cases}\\ &=ae^{-\gamma t}\cos(\omega t+\alpha)\\ &a,\alpha\text{ are constants}\end{aligned} \tag{11}$$

"The solution is a damped oscillation with frequencyω, and amplitude expoentially decaying with time."

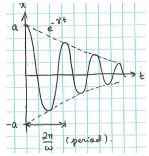
2. Overdameped

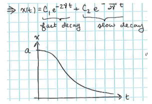
$$\begin{gathered}\gamma>\omega\Rightarrow x(t)=\\ c_1e^{-\gamma+\sqrt{\gamma^2-\omega^2}t}+c_2e^{-\gamma-\sqrt{\gamma^2-\omega^2}t}\end{gathered} \tag{12}$$

$$\begin{gathered}\text{when}\gamma\gg\omega_0,\Rightarrow\begin{cases}\gamma+\sqrt{\gamma^2-\omega_0^2}\approx 2\gamma\\ \gamma-\sqrt{\gamma^2-\omega^2}=\frac{\omega^2}{2\gamma}\end{cases}\\ x(t)=c_1e^{-2\gamma t}+c_2e^{(-\omega_0^2/2\gamma)t}\end{gathered} \tag{13}$$

3. Critically damped

$$\gamma=\omega_0\Rightarrow x(t)=c_1e^{-\gamma t}+c_2te^{-\gamma t} \tag{14}$$



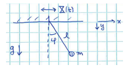


Forced Oscillations

When external force (F) is applied to the system, the largrangian becomes

$$\begin{gathered}L=\frac12 m\dot x^2-\frac12 kx^2+F(t)x\\ \text{EL}\Rightarrow \ddot x+\omega_0^2x=\frac{F(t)}{m},\text{where }\omega_0=\sqrt{\frac{k}{m}}\end{gathered} \tag{15}$$

- Example: Simple pendulum with moving pivot



$$\begin{gathered}\begin{cases}x=X+l\sin\varphi\\ y=l\cos\varphi\end{cases}\Rightarrow\begin{cases}\dot x=\dot X+l\dot\varphi\cos\varphi\\ \dot y=-l\dot\varphi\sin\varphi\end{cases}\\ \Rightarrow L=T-U\end{gathered} \tag{16}$$

$$\begin{gathered}L=\frac12 ml^2\dot\varphi^2-mgl(1-\cos\varphi)-ml\ddot X\sin\varphi\\ \text{Expand ab. }\varphi=0\Rightarrow L=\frac12 ml^2\dot\varphi^2-\frac12 mgl\varphi^2-ml\ddot X\varphi\\ \text{EL}\Rightarrow\boxed{\ddot\varphi+\omega_0^2\varphi=-\frac{\ddot X}{l}\text{ ,where }\omega_0=\sqrt{\frac{g}{l}}}\end{gathered} \tag{17}$$

reintroducing damping via external forcing

$$\ddot x+2\gamma\dot x+\omega_0^2x=f(t),f(t)=\frac{F(t)}{m} \tag{18}$$

When damping $f(t)=f_0\cos(\Omega t)$, solution via complex number:

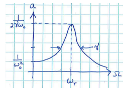
$$\begin{gathered}\ddot z+2\gamma\dot z+\omega_0^2=f_0e^{i\Omega t}\\ \text{ansatz } z(t)=z_0e^{i\Omega t}\Rightarrow z_0=\frac{f_0}{\omega_0^2+2i\gamma\Omega+\Omega_0^2}\\ \boxed{z_0=a(\Omega)\cos(\Omega t+\delta(\Omega))f_0}\text{ is a partcular solution, where}\\ \begin{cases}a(\Omega)=\frac{1}{\sqrt{(\omega_0^2-\Omega^2)^2+(2\gamma\Omega)^2}}\\ \delta(\Omega)=\arctan\left(2\gamma\frac{\Omega}{\omega_0^2-\Omega^2}\right)\end{cases}\end{gathered} \tag{19}$$

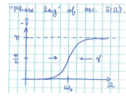
We can study the properties of the system by looking at the amplitude and phase of the solution.

- Amplitude:

$$a_{(\Omega)}=\frac{1}{\sqrt{(\omega_0^2-\Omega^2)^2+(2\gamma\Omega)^2}} \tag{20}$$

, when $\gamma\ll\omega_0$, response strongest and amplitude largest when $\omega_r=\omega_0$.





- Phase lag: $\tan\delta(\Omega)=2\gamma\frac{\Omega}{\Omega^2-\omega_0^2}$

  in phase as $\Omega\to 0$, and out of phase as $\Omega\to\omega_0$.
- Genral solution to sinusoidal forcing:

$$\begin{gathered}x(t)=a(\Omega)f_0\cos(\Omega t+\delta(\Omega))+a_0e^{-\gamma t}\cos(\omega t+\alpha)\\ \overset{t>\frac1r}{\to}a(\Omega)f_0\cos(\Omega t+\delta(\Omega))\end{gathered} \tag{21}$$

  Forgets initial condition after time.
- Power obsorbed by oscillation

  $p=F\dot x=mf\dot x$

  Avg power of oscillation

$$\begin{gathered}P_{\text{avg}}=\frac1T\int_0^T mf\dot x\,\mathrm{d}t=-\frac12 mf_0a(\Omega)\Omega\sin\delta(\Omega)\\ \text{simplifies to } P_{\text{avg}}(\Omega)=\gamma mf_0^2\Omega^2a_{(\Omega)}^2\end{gathered} \tag{22}$$

  Absorption around resonance frequency $\Omega=\omega_0+\varepsilon$ is maximum:

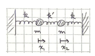
$$P=\frac{\gamma mf_0^2}{4(\varepsilon^2+\gamma^2)}\approx\frac{mf_0^2}{4\gamma} \tag{23}$$

Oscillations DOF>1

For a system with n DOF: $q=(q_1,q_2,...,q_n)$, PE $=U(q)$

- Stable equilibrium $\frac{\partial U(q)}{\partial q_i}\Big|_{q=0}$

Example: Oscillation with 2 mass and 3 springs



$$\begin{aligned}L=&\tfrac12 m\dot x_1+\tfrac12 m\dot x_2-\tfrac12 kx_1^2\\ &-\tfrac12 kx_2^2-\tfrac12 k'(x_1-x_2)^2\end{aligned}$$

EOM:

$$\begin{gathered}M\cdot\ddot{\vec x}=-K\vec x\ \text{ , where } M=\begin{pmatrix}m&0\\0&m\end{pmatrix},\\ \vec x=\begin{pmatrix}x_1\\x_2\end{pmatrix},K=\begin{pmatrix}k+k'&-k'\\-k'&k+k'\end{pmatrix}\end{gathered} \tag{24}$$

ansatz: $\vec x=\mathrm{Re}[\vec a e^{i\omega t}]$ Then the EOM eq becomes solving the eigenvalue problem:

$$\det(\omega^2 M - K) = 0$$

$$\Rightarrow \begin{cases} \omega_-^2 = \frac{k}{m} \\ \omega_+^2 = \frac{k+2k'}{m} \end{cases} \begin{cases} \overrightarrow{x_-} = a_- \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos(\omega_- t + \delta_-) \\ \overrightarrow{x_+} = a_+ \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos(\omega_+ t + \delta_+) \end{cases} \quad 25$$

with constants $a_-, a_+, \delta_-, \delta_+$.

New Coords

$$\begin{cases} Q_1 = \sqrt{\frac{m}{2}}(x_1 + x_2) \\ Q_2 = \sqrt{\frac{m}{2}}(x_1 - x_2) \end{cases}$$

$$\Rightarrow L = \frac{1}{2}\left(\dot{Q_1}^2 + \dot{Q_2}^2\right) - \frac{1}{2}\left(\omega_-^2 Q_1^2 + \omega_+^2 Q_2^2\right) \quad 26$$

$$\overset{\text{E-L}}{\Rightarrow} \ddot{Q}_1 = -\omega_-^2 Q_1, \ddot{Q}_2 = -\omega_+^2 Q_2$$

Decoupled oscillators with coords Q_1, Q_2.

General Coords

for general coords q_i, let $x_i = q_i - q_i^{(0)}$

$$U = \frac{1}{2}\sum_{i,j} k_{ij} x_i x_j, \quad k_{ij} = k_{ji} = \frac{\partial^2 U(q)}{\partial q_i \partial q_j} \text{ symm mat}$$

$$T = \frac{1}{2}\sum_{i,j} m_{ij} \dot{x}_i \dot{x}_j, \quad m_{ij} = m_{ji} = a_{ij}\left(q^{(0)}\right) \quad 27$$

the largrangian, in Matix form:

$$L = \frac{1}{2}\dot{\vec{x}}^T \cdot M \cdot \dot{\vec{x}} - \frac{1}{2}\vec{x}^T \cdot K\vec{x} \overset{\text{EL}}{\Longrightarrow} \left(\omega^2 M - K\right) \cdot \vec{a} = 0 \quad 28$$

$\Rightarrow \det(\omega^2 M - K) = 0$ Solving the det for omega gives the normal freq (Eigenvalues)of system ω_α^2 . plug in Evalue into Equation 28 for eigenvec(normal modes) $\overrightarrow{a^\alpha}$ of system.

- General motion

$$x_i(t) = \sum_\alpha a_i^\alpha \mathrm{Re}\left[C_\alpha e^{i\omega_\alpha t}\right] \quad 29$$

- EXAMPLE: Normal freq is given

$$\omega = \left\{0, \sqrt{2}\omega_0, \sqrt{3}\omega_0\right\}.$$

$$\omega = \sqrt{2}\omega_0 \Rightarrow a_1 = -a_3 = -a_2 = ae^{i\delta} \Rightarrow$$

$$\vec{\theta} = a(1 \ -1 \ -1)^T \cos\left(\sqrt{2}\omega_0 t + \delta\right) \quad 30$$

$$\omega = \sqrt{3\omega_0} \Rightarrow a_1 = 0, a_2 = -a_3 = ae^{i\delta} \Rightarrow$$

$$\vec{\theta} = a(0 \ 1 \ -1)^T \cos\left(\sqrt{3}\omega_0 t + \delta\right)$$

- EXAMPLE: double pendulum

$$\begin{cases} x_1 = l_1 \sin\varphi_1 \quad y_1 = -l_1 \cos\varphi_1 \\ x_2 = l_1 \sin\varphi_1 + l_2 \sin\varphi_2 \quad y_2 = l_1 \cos\varphi_1 + l_2 \cos\varphi_2 \end{cases} \quad 31$$

$$\Rightarrow T = \frac{1}{2}m_1 l_1 \dot{\varphi}^2 + \frac{1}{2}m_2\left(l_1^2 \dot{\varphi_1}^2 + l_2^2 \dot{\varphi_2}^2\right.$$
$$\left. + 2l_1 l_2 \dot{\varphi}_1 \dot{\varphi}_2 \cos(\varphi_1 - \varphi_2)\right) \quad 32$$
$$U = -m_1 g l_1 \cos\varphi_1 - m_2 g(l_1 \cos\varphi_1 + l_2 \cos\varphi_2)$$

using $\cos\varphi \approx 1 - \frac{\varphi^2}{2}$

$$L = \frac{1}{2}(\dot{\varphi_1} \ \dot{\varphi_2}) \begin{pmatrix} (m_1+m_2)l_1^2 & m_2 l_1 l_2 \\ m_2 l_1 l_2 & m_2 l_2^2 \end{pmatrix} (\dot{\varphi_1} \ \dot{\varphi_2})$$
$$-\frac{1}{2}(\varphi_1 \ \varphi_2) \begin{pmatrix} (m_1+m_2)l_1 g & 0 \\ 0 & m_2 g l_2 \end{pmatrix} (\varphi_1 \ \varphi_2) \quad 33$$
$$= \frac{1}{2}\dot{\vec{\varphi}}^T M \cdot \dot{\vec{\varphi}} - \frac{1}{2}\vec{\varphi}^T K \vec{\varphi}$$

When $m_1 = m_2 = m, \quad l_1 = l_2 = l \Rightarrow \quad M = ml^2\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, K = mgl\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$

$$\det((\omega^2 M - K)) = 0 \Rightarrow \omega^2 = \left(2 \pm \sqrt{2}\omega_0^2\right)$$

$$\begin{pmatrix} a_1^- \\ a_2^- \end{pmatrix} = C_-\begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}, \quad \begin{pmatrix} a_1^+ \\ a_2^+ \end{pmatrix} = C_+\begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix} \quad 34$$

Normal Coords

$\{x_i\} = \{Q_\alpha\}$, where $x_i = \sum_{\alpha=1}^n A_{i\alpha} Q_\alpha \Rightarrow$

$\sum_j \left(\omega_\alpha^2 m_{ij} - k_{ij} A_{j\alpha}\right) = 0$

$\Rightarrow L = \frac{1}{2}\sum_{\alpha=1}^n \left(\dot{Q}_\alpha^2 - \omega_\alpha^2 Q_\alpha^2\right) \overset{\text{EL}}{\Longrightarrow} \ddot{Q}_\alpha + \omega_\alpha^2 Q_\alpha = 0$

Motion of Rigid Body

- EXample: rotor

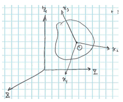
rotation with constraint $\left|\vec{r_i} - \vec{r_j}\right|$ .COM coords are useful here

$$\begin{cases} \vec{r} = \vec{r_1} - \vec{r_2} \\ \vec{R} = \frac{m_1\vec{r_1} + m_2\vec{r_2}}{m_1+m_2} \end{cases} \Rightarrow \begin{cases} \vec{r_1} = \vec{R} + m_2\vec{r}/M \\ \vec{r_2} = \vec{R} - m_1\vec{r}/M \end{cases} \quad 35$$

$$L = \frac{1}{2}M\dot{\vec{R}}^2 + \mu\dot{\vec{r}}^2, \quad \mu = m_1\frac{m_2}{m_1+m_2}$$

$$\overset{\text{polar}}{\Longrightarrow} L = \frac{1}{2}M\dot{\vec{R}}^2 + \frac{1}{2}\mu a^2\left(\dot{\theta}^2 + \dot{\varphi}^2\sin^2\theta\right) \quad 36$$

frames of reference



$$(XYZ) \overset{R(\theta,\varphi,\psi)}{\Longrightarrow} (x_1, x_2, x_3)$$

Velocity of pt in body: $\vec{v} = \vec{V} + \vec{\Omega} \times \vec{r}$, where V is Translational vel, Omega is angular vel, r is position vector.

Largrangian for Rigid Body

$$T = \frac{1}{2}MV^2 + \frac{1}{2}\sum_a m_a\left[\Omega^2 r_a^2 - \left(\vec{\Omega}\right)\vec{r_a}\right)^2\right)] \quad 37$$

$$T\text{translational} + T\text{rotational}$$

consider rotation,

$$\Omega^2 = \sum_i \Omega_i^2, \quad \vec{\Omega} \cdot \vec{r_a} = \sum_i \Omega_i x_{a,i} \quad 38$$

$$\Rightarrow T_{\text{rot}} = \frac{1}{2}\sum_{\text{i,j}} \Omega_i \Omega_j I_{\text{i,j}}. \quad I_{ij} \equiv \sum_a m_{a\left(\delta_{ij} r_a^2 - x_{a,i} x_{a,j}\right)}$$

$\Rightarrow L = \frac{1}{2}MV^2 + \frac{1}{2}\sum_{i,j} I_{i,j}\Omega_i\Omega_j - U$

Inertial Tensor

- Discrete

$$I = \begin{pmatrix} \sum m(y^2+z^2) & -\sum mxy & -\sum mxz \\ -\sum mxy & \sum m(x^2+z^2) & -\sum myz \\ -\sum mxz & -\sum myz & \sum m(x^2+y^2) \end{pmatrix} \quad 39$$

- Continuous

$$I_{ij} = \int \rho(x)\left(\delta_{ij}r^2 - x_i x_j\right) \mathrm{d}V$$

$$I_{xx} = \int \rho(x)(y^2+z^2)\,\mathrm{d}V, I_{xy} = I_{yx} = -\int \rho(x)xy\,\mathrm{d}V$$
$$I_{yy} = \int \rho(x)(x^2+z^2)\,\mathrm{d}V, I_{yz} = I_{zy} = -\int \rho(x)yz\,\mathrm{d}V \quad 40$$
$$I_{zz} = \int \rho(x)(x^2+y^2)\,\mathrm{d}V, I_{zx} = I_{xz} = -\int \rho(x)zx\,\mathrm{d}V$$

example:

$$I_{xx} = \rho\int\left[b^2\hat{y}^2 + c^2\hat{z}^2\right] abc\, d\hat{x}\, d\hat{y}\, d\hat{z}$$
$$= abc\rho\int_V\left(b^2\hat{y}^2 + c^2\hat{z}^2\right) d\hat{x}\, d\hat{y}\, d\hat{z}$$

transform into spherical coord:

$$\boxed{I_{xx}} = abc\rho\int_V\left(b^2 r^2\sin^2\varphi\sin^2\theta + c^2r^2\cos^2\theta\right) r^2\sin\theta\, dr\, d\varphi\, d\theta$$
$$= abc\rho\left[b^2\int_0^{2\pi}\sin^2\varphi\, d\varphi\int_0^\pi \sin^3\theta\, d\theta\int_0^1 r^4 dr\right.$$
$$\left. + c^2\int_0^{2\pi}d\varphi\int_0^\pi\cos^2\theta\sin\theta\, d\theta\int_0^1 r^4 dr\right]$$
$$= \boxed{\frac{4}{15}abc\pi\rho\left[b^2+c^2\right]}$$

- Example: coplanar system principal axis: Z $\Rightarrow I_{13} = I_{23} = 0$
  $I_3 = I_1 + I_2$

Principle axis and principal moments of inertia

In the principal frame:

$$T_{\text{rot}} = \frac{1}{2}\left(I_1\Omega_1^2 + I_2\Omega_2^2 + I_3\Omega_3^2\right) \quad 41$$

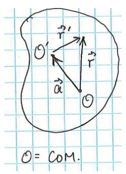
- spherical top $I_1 = I_2 = I_3$
- Symmetric top $I_1 = I_2 \neq I_3$
- Asymmetric top $I_1 \neq I_2 \neq I_3$
- EXample:

$$\det(I - \lambda\mathbf{1}) = 0 \Rightarrow \lambda \text{ prncp. mom.}$$
$$\vec{v} = \text{eigenvec.} = \text{prncp. axis} \quad 42$$

- EXample: continuous with axis of symmetry $\rho(\vec{r}) = \rho = (r, x_3) \Rightarrow I_{ij} = \int \rho(\vec{r})\left(r^2\delta - x_i x_j\right) \mathrm{d}V$

Parallel axis theorem

when changing Origin diff. from COM(O),



$$I_{ij} = I'_{ij} - M\left(a^2\delta_{ij} - a_i a_j\right)$$

For a cube, when finding I at corner, first find I at COM, and

$$I'_{xx} = I_{xx} + M(b^2+c^2) = \frac{4}{3}M(b^2) + c^2$$
$$I'_{yy} = I_{yy} + M(a^2+c^2) = \frac{4}{3}M(a^2+c^2) \quad 43$$
$$I'_{zz} = I_{zz} + M(a^2+b^2) = \frac{4}{3}M(a^2+b^2)$$

$I_{13} = -\int dv\, \rho(\vec{r})\, x_1 x_3$ 03/22/2

$= -\int dx_3\, r dr d\varphi\, \rho(r, x_3)\, r\cos\varphi\, x_3$

$= -\int dx_3\, r dr\, \rho(r, x_3)\, r x_3 \underbrace{\int_0^{2\pi} d\varphi \cos\varphi}_{=0}$

& $I_{23} = 0$ by same analysis w/ $\cos\varphi \to \sin\varphi$.

$\Rightarrow I = \begin{pmatrix} I_{11} & I_{12} & 0 \\ I_{21} & I_{22} & 0 \\ 0 & 0 & I_3 \end{pmatrix} \Rightarrow$ x_3 = principal axis;

i.e., symm. axis = princ. ax

$I_{12} = -\int dv\, \rho(\vec{r})\, x_1 x_2$

$= -\int dx_3\, r dr d\varphi\, \rho(r, z)\, r^2\cos\varphi\sin\varphi$

$= -\int dx_3\, r dr d\varphi\, \rho(r, z)\, r^2 \underbrace{\int_0^{2\pi} d\varphi \cos\varphi\sin\varphi}_{=0}$

$\Rightarrow I = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}$, x_1, x_2, x_3 = principal axes

$I_1 - I_2 = \int dv\, \rho(\vec{r})\left(x_2^2 - x_1^2\right)$

$= \int dx_3\, r dr\, \rho(r, z)\, r^2 \underbrace{\int_0^{2\pi} d\varphi \left(\sin^2\varphi - \cos^2\varphi\right)}_{=0}$

$\Rightarrow I_1 = I_2 \equiv I_\perp$.

$\Rightarrow I = \left(\begin{array}{cc|c} I_\perp & 0 & 0 \\ 0 & I_\perp & 0 \\ \hline 0 & 0 & I_3 \end{array}\right)$ → any two $\perp$ axes in x_1x_2-plane are principal axes.

Appendix

1. Taylor expansion:

$$f(x)\big|_0 \approx f(a) + f'(a)(x-a) + f''(a)\frac{(x-a)^2}{2} \quad 44$$

2. small angle approximation:

$$\sin(\theta) \approx \theta \quad \cos(\theta) \approx 1 - \frac{\theta^2}{2} \quad 45$$