Math 431 - Intro to Probability
Comprehensive Review Notes

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Probability Spaces and Basic Properties

Definition 1. *A* ***probability space*** *is a triple* $(\Omega, \mathcal{F}, P)$ *where:*

- Ω *is the sample space*
- $\mathcal{F}$ *is a* σ*-algebra of subsets of* Ω *(the events)*
- P *is a probability measure on* $(\Omega, \mathcal{F})$*, satisfying: (i)* $P(\Omega) = 1$*, (ii)* $P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$ *for disjoint* $A_i \in \mathcal{F}$*.*

Property 1 (Probability Axioms)**.** *For events* $A, B \in \mathcal{F}$*:*

1. $P(A^c) = 1 - P(A)$
2. *If* $B \subset A$ *then* $P(B) \leq P(A)$
3. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Discrete Probability Spaces

Definition 2. *For a discrete sample space* Ω*, a* ***probability mass function*** *is a function* $p : \Omega \to [0, 1]$ *satisfying* $\sum_{x \in \Omega} p(x) = 1$*. Then* $P(A) = \sum_{x \in A} p(x)$*.*

Continuous Probability Spaces

Definition 3. *A random variable* X *has a* ***probability density function*** f *if* $f \geq 0$*,* $\int_{-\infty}^{\infty} f(x)\, dx = 1$*, and* $P(a \leq X \leq b) = \int_a^b f(x)\, dx$*.*

Conditional Probability and Independence

Definition 4. *The **conditional probability** of A given B is*

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} \text{ when } P(B) > 0$$

Theorem 1 (Multiplication Rule)**.**

$$P(A_1 \cap A_2 \cap \cdots \cap A_n) = P(A_1)P(A_2 \mid A_1)P(A_3 \mid A_2 \cap A_1) \ldots P(A_n \mid A_{n-1} \cap \cdots \cap A_1)$$

Theorem 2 (Law of Total Probability)**.** *If $\{B_1, \ldots, B_n\}$ is a partition of Ω with $P(B_i) > 0$ for all i, then*

$$P(A) = \sum_{i=1}^{n} P(A \mid B_i)P(B_i)$$

Theorem 3 (Bayes' Theorem)**.**

$$P(B_i \mid A) = \frac{P(A \mid B_i)P(B_i)}{\sum_{j=1}^{n} P(A \mid B_j)P(B_j)}$$

Definition 5. *Events A and B are **independent** if $P(A \cap B) = P(A)P(B)$.*

Important Discrete Distributions

- **Bernoulli(p)**: $P(X = 1) = p$, $P(X = 0) = 1 - p$
- **Binomial(n, p)**: $P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}$, $k = 0, 1, \ldots, n$
- **Geometric(p)**: $P(X = k) = (1 - p)^{k-1} p$, $k = 1, 2, \ldots$
- **Negative Binomial(r, p)**: $P(X = k) = \binom{k-1}{r-1} p^r (1 - p)^{k-r}$, $k = r, r + 1, \ldots$
- **Hypergeometric(N, K, n)**: $P(X = k) = \frac{\binom{K}{k}\binom{N-K}{n-k}}{\binom{N}{n}}$, $\max(0, n - N + K) \leq k \leq \min(n, K)$
- **Poisson(λ)**: $P(X = k) = e^{-\lambda} \frac{\lambda^k}{k!}$, $k = 0, 1, 2, \ldots$

Important Continuous Distributions

- **Uniform**(a, b): $f(x) = \frac{1}{b-a}$, $a \le x \le b$
- **Exponential**(λ): $f(x) = \lambda e^{-\lambda x}$, $x \ge 0$
- **Normal**(μ, σ^2): $f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$, $x \in \mathbb{R}$

Joint Distributions

Definition 6. *The **joint probability mass function** of discrete random variables X and Y is $p_{X,Y}(x, y) = P(X = x, Y = y)$.*

Definition 7. *The **joint probability density function** of continuous random variables X and Y is a function $f_{X,Y}$ satisfying*

$$P((X, Y) \in A) = \iint_A f_{X,Y}(x, y)\, dy\, dx$$

Definition 8. *The **marginal pmf** of X is $p_X(x) = \sum_y p_{X,Y}(x, y)$. The **marginal pdf** of X is $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)\, dy$.*

Definition 9. *The **conditional pmf** of Y given $X = x$ is $p_{Y|X}(y|x) = \frac{p_{X,Y}(x,y)}{p_X(x)}$. The **conditional pdf** of Y given $X = x$ is $f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$.*

Theorem 4. *X and Y are **independent** if and only if $p_{X,Y}(x, y) = p_X(x)p_Y(y)$ (discrete case) or $f_{X,Y}(x, y) = f_X(x)f_Y(y)$ (continuous case).*

Functions of Random Variables

Definition 10. *The **moment generating function** of X is $M_X(t) = \mathbf{E}(e^{tX})$.*

Theorem 5. *If X is discrete with pmf p_X and $Y = g(X)$, then the pmf of Y is*

$$p_Y(y) = \sum_{x:g(x)=y} p_X(x)$$

Theorem 6. *If X is continuous with pdf f_X and $Y = g(X)$, where g is monotone, then the pdf of Y is*

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right|$$

Theorem 7. *If X is continuous with cdf F_X, then $F_X(X) \sim$ Uniform$(0, 1)$.*

Sums of Independent Random Variables

Theorem 8 (Convolution). *If X and Y are independent, then:*

- *Discrete case:* $p_{X+Y}(z) = \sum_x p_X(x)p_Y(z-x)$
- *Continuous case:* $f_{X+Y}(z) = \int_{-\infty}^{\infty} f_X(x)f_Y(z-x)\,dx$

Property 2 (Memoryless Property).

- *If* $X \sim$ *Geometric*(p)*, then* $P(X > m+n \mid X > m) = P(X > n)$.
- *If* $X \sim$ *Exponential*(λ)*, then* $P(X > s+t \mid X > s) = P(X > t)$.

Limit Theorems

Theorem 9 (Law of Large Numbers). *If $X_1, X_2, \ldots$ are iid with $\mathbf{E}(X_i) = \mu$, then $\bar{X}_n \to \mu$ as $n \to \infty$.*

Theorem 10 (Central Limit Theorem). *If $X_1, X_2, \ldots$ are iid with $\mathbf{E}(X_i) = \mu$ and $Var(X_i) = \sigma^2 < \infty$, then*

$$\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \xrightarrow{d} N(0,1) \quad \text{as } n \to \infty$$

Example 1 (Poisson Limit of Binomial). *If $X_n \sim Bin(n, \lambda/n)$ and λ is fixed, then $X_n \xrightarrow{d} Poisson(\lambda)$ as $n \to \infty$.*

Conclusion

This comprehensive review covers all the key topics from the course:

- Probability spaces and basic properties
- Conditional probability, Bayes' theorem, independence
- Discrete and continuous distributions
- Joint, marginal, and conditional distributions
- Functions of random variables, transformations
- Sums of independent random variables, convolution
- Memoryless property of Geometric and Exponential

- Limit theorems: Law of Large Numbers, Central Limit Theorem

  With a solid grasp of these concepts, you are well-prepared for further studies in probability theory and statistics. Keep practicing problems, exploring connections between ideas, and appreciating the power and beauty of probabilistic reasoning. All the best!