Angular momentum of a rigid body

$\vec{L}$ in non-inertial frame

$$\vec{L} = \sum m(\vec{r} \times \vec{v}) = \sum m\left[\vec{\Omega} r^2 - \vec{r}\left(\vec{\Omega} \cdot \vec{r}\right)\right]$$
$$L_i = \boxed{I_{ij}\Omega_j} \quad \vec{L} = I * \vec{\Omega}$$
[1

If $(x_1 x_2 x_3)$ are principal axis, $L_1 = I_1\Omega_1$, $L_2 = I_2\Omega_2$, $L_3 = I_3\Omega_3$

Free motion of a rigid body

angular momentum is conserved if no external torque. Motion in inertial COM frame is simplier.

- *ex motion of a symmetric top* $I_1 = I_2 = I_3 = I, \quad \tilde{I} = I\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$\vec{L} = I\vec{\Omega} \to \dot{\vec{L}} = 0 \Rightarrow \dot{\vec{\Omega}} = 0$ Uniform rotation about fixed axis paralle to $\vec{L}$

- *ex rigid rotor* $I_1 = I_2 = \sum m x_3^2, \quad I_3 = 0$

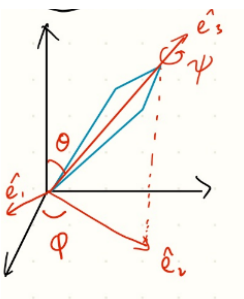
$\vec{L} = I\vec{\Omega}, \quad \vec{\Omega} \perp x_3$ by geometry We have $\dot{\vec{\Omega}} = 0 \Rightarrow$ Motion is unif in plane perp to $\vec{\Omega}$ and that it stays in that plane.

- *ex asymmetric top* $I_1 = I_2 = I_\perp \neq I_3 \Rightarrow \tilde{I} = \begin{pmatrix} I_\perp & 0 & 0 \\ 0 & I_\perp & 0 \\ 0 & 0 & I_3 \end{pmatrix}$ x_3 is symm. axis, for any orthogonal axes

Rigid body EOM

$$\begin{cases} \dot{\vec{p}} = \vec{F} \\ \dot{\vec{L}} = \vec{K} \text{ torque} \end{cases}$$
[2

Euler angles: ψ spin, θ nutation, φ precession



$(\theta \in [0, \pi], \varphi \in [0, 2\pi], \psi \in [0, 2\pi])$ in turns of rotation $R = R(\hat{z}, \varphi)R\left(\widehat{X}, \theta\right)R\left(\hat{Z}, \psi\right)$

The lagrangian in Euler angles

- First: $T = \frac{1}{2}\left(I_1\Omega_1^2 + I_2\Omega_2^2 + I_3\Omega_3^2\right)$
- Rotation in components:

$$\Omega_1 = \dot{\varphi}\sin\theta\sin\psi + \dot{\theta}\cos\psi$$
$$\Omega_2 = \dot{\varphi}\sin\theta\cos\psi - \dot{\theta}\sin\psi$$
$$\Omega_3 = \dot{\varphi}\cos\theta + \dot{\psi}$$
[3

- $T = \frac{1}{2}I_1\left(\dot{\varphi}\sin\theta\sin\psi + \dot{\theta}\cos\psi\right)^2 + \frac{1}{2}I_2\left(\dot{\varphi}\sin\theta\cos\psi - \dot{\theta}\sin\psi\right)^2 + \frac{1}{2}I_3\left(\dot{\varphi}\cos\theta + \dot{\psi}\right)^2$
- $L\left(\theta, \varphi, \psi, \dot{\theta}, \dot{\varphi}, \dot{\psi}\right) = T - U$

Free motion of symmetric top in Euler angles

$I_1 = I_2 = I_\perp \Rightarrow \quad T = \frac{1}{2}I_\perp\left(\dot{\theta}^2 + \dot{\varphi}^2\sin^2\theta\right) + \frac{1}{2}I_3\left(\dot{\varphi}\cos\theta + \dot{\psi}\right)^2$

$\Omega_\perp = L_z/I_\perp, \quad \Omega_3 = L_z\cos\theta/I_3$ E-L ->

$$\begin{aligned} \theta &: \frac{\mathrm{d}}{\mathrm{d}t}I_\perp\dot{\theta} = I_\perp\sin\theta\cos\theta\,\dot{\varphi}^2 - I_3\dot{\varphi}\sin\theta\left(\dot{\varphi}\cos\theta + \dot{\psi}\right) \\ \varphi &: \frac{\mathrm{d}}{\mathrm{d}t}\left(I_\perp\dot{\varphi}\sin^2\theta + I_3\cos\theta\left(\dot{\varphi}\cos\theta + \dot{\psi}\right)\right) = 0 \\ \psi &: \frac{\mathrm{d}}{\mathrm{d}t}I_3\left(\dot{\varphi}\cos\theta + \dot{\psi}\right) = 0 \end{aligned}$$ [4

choosing $\hat{z}$ along the angular momentum, we have $L_3 = L_z\cos\theta = I_3\Omega_3 = I_3\left(\dot{\varphi}\cos\theta + \dot{\psi}\right)$
$\Rightarrow \dot{L}_3 = \text{const} \Rightarrow \theta = \text{const} \quad \Omega_3 = \frac{L_z\cos\theta}{I_3} \quad \dot{\varphi} = \frac{L_3}{I_\perp\cos\theta} = \frac{L_z}{I_\perp} = \text{const}$

- *ex heavy symmertic top with one pt fixed* By paralle axis thm, $I'_{ij}I_{ij} + M\left(l^2\delta_{ij} - l_i l_j\right)$

  $\Rightarrow I'_\perp = I_\perp + Ml^2, \quad I'_3 = I_3, \quad U = mgZ = Mgl\cos\theta$

  $\Rightarrow L = T - U = \frac{1}{2}I'_\perp\left(\dot{\theta}^2 + \dot{\varphi}^2\sin^2\theta\right) + \frac{1}{2}I_3\left(\dot{\psi} + \varphi\cos\theta\right)^2 = Mgl\cos\theta$

  E-L :

$$\begin{aligned} L_z = p_\varphi &= (I'_\perp\sin^2\theta + I_3\cos^2\theta)\dot{\varphi} \quad \text{const} \\ L_3 = p_\psi &= I_3\left(\dot{\psi} + \varphi\cos\theta\right) \quad \text{const} \end{aligned}$$ [5

Considering energy conservation

$$E = T + U \Rightarrow \underbrace{E - \frac{L_3^2}{2I_3} - Mgl}_{E'} = \frac{1}{2}I'_\perp\dot{\theta}^2 + \underbrace{\frac{1}{2I'_\perp}\frac{(L_z - L_3\cos\theta)^2}{\sin^2\theta} - Mgl(1 - \cos\theta)}_{U_{\text{eff}}(\theta)}$$ [6

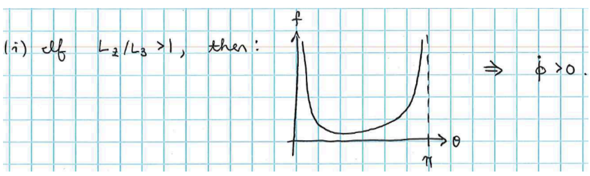
effective 1 dof problem. recognizing

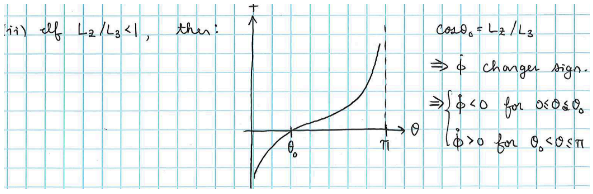
$$\dot{\theta} = \frac{\mathrm{d}\theta}{\mathrm{d}t} \Rightarrow \quad t = \int \frac{d\theta}{(\sqrt{2[E - U_{\text{eff}}(\theta)]/I'_\perp}}$$ [7

Considering U_eff: when $\theta = 0, L_z = L_3$when $\theta \approx 0 \Rightarrow \quad U_{\text{eff}} \approx \left(\frac{L_3^2}{8I'_\perp} - \frac{Mgl}{2}\right)\theta^2$

Motion about $\theta = 0$ stable if $L_3^2 > 4I'_\perp Mgl \Rightarrow \Omega_3^2 > 4I'_\perp Mgl/I_3^2$, or stable if sping ab. symm. axis is fast enough.

- Nutuation: cosider $\dot{\varphi} = \frac{L_3}{I'_\perp}\frac{(L_z/L_3) - (\cos\theta)}{\sin^2\theta} = \frac{L_3}{I'_\perp}f(\theta)$





considering the sign and trends of $f(\theta)$ given constrains on theta, we can differentiate different nutation motion. If θ_0 in graph 2 is out of range, the nutation is smooth; if θ_0 is in range, the nutation is oscillatory(will change sign and spin in spiral.); if θ_0 is on the endpoint of our constrained range, the nutation is spiky and "not smooth" at points.

Euler equations

set body frame $(X, Y, Z) = (\hat{e}_1^0, \hat{e}_2^0, \hat{e}_3^0$, space frame $(x_1, x_2, x_3) = (\hat{e}_1, \hat{e}_2, \hat{e}_3)$ Set any vector $\vec{A} = \sum A_i^0 \hat{e}_i^0 = \sum A_i \hat{e}_i$ By magic of vec analysis,

$$\left(\frac{\mathrm{d}\vec{A}}{\mathrm{d}t}\right)_{\text{Space}} = \left(\frac{\mathrm{d}\vec{A}}{\mathrm{d}t}\right)_{\text{Body}} + \vec{\Omega} \times \vec{A}_{\text{Space}} \tag{8}$$

When applied to $\left(\frac{\mathrm{d}\vec{L}}{\mathrm{d}t}\right)_{\text{Space}} = \vec{\mathrm{K}} = \left(\frac{\mathrm{d}\vec{L}}{\mathrm{d}t}\right)_{\text{body}} + \vec{\Omega} \times \vec{L}$, recognizing $L_i = I_i \Omega_i$:

$$\begin{aligned} I_1 \dot{\Omega}_1 + (I_3 - I_2)\Omega_2\Omega_3 &= K_1 \\ I_2 \dot{\Omega}_2 + (I_1 - I_3)\Omega_3\Omega_1 &= K_2 \\ I_3 \dot{\Omega}_3 + (I_2 - I_1)\Omega_1\Omega_2 &= K_3 \end{aligned} \tag{9}$$

$K_i = 0$ if $\vec{L}$ is conserved on i axis.

- *ex symmetric top* $I_1 = I_2 = I, \vec{K} = 0 \quad \left(\dot{\Omega}_1 + \frac{I_3 - I_1}{I_\perp}\Omega_2\Omega_3 = 0; \dot{\Omega}_2 + \frac{I_1 - I_3}{I_\perp}\Omega_3\Omega_1 = 0; \dot{\Omega}_3 = 0\right)$
  let $\omega = ((I_3 - I_\perp)/(I_\perp))\Omega_3 \Rightarrow \boxed{\left(\Omega_1 = A\cos\omega t; \Omega_2 = -\frac{1}{\omega}\dot{\Omega}_1 = +A\sin\omega t\right)}$

Motion in non-inertial frame

- Set non-inertial frame with velocity $\vec{V}(t)$, $\vec{A} = \dot{\vec{V}}, \quad \vec{v} = \vec{v}' + \vec{V}(t)$ where $\vec{v}'$ is velocity w.r.t. non-inertial frame.

lagrangian $L' = \frac{1}{2}m{v'}^2 - m\vec{r}' \cdot \vec{A} - U$, using E-L eq: $m\dot{\vec{v}}' = -\frac{\partial U}{\partial \vec{r}'} - m\vec{A}$

- *ex pendulum in acc. car* $\quad m\ddot{\vec{r}} = \vec{T} + m\vec{g} - m\vec{A}$,

  finding equil. angle: $\vec{T} = -m\left(\vec{g} - \vec{A}\right) = -m\vec{g}_{\text{eff}}$, then use geometry between $\vec{g}, -\vec{A} \Rightarrow \tan\varphi_0 = \frac{A}{g}$. Oscillation freq. $\omega = \sqrt{g_{\text{eff}}/l}$

Motion in rotating frame

Set rotation with $\vec{\Omega}$, $L = \frac{1}{2}mv^2 + \overline{m\vec{v}} \cdot \left(\vec{\Omega} \times \vec{r}\right) + \frac{1}{2}m\left(\vec{\Omega} \times \vec{r}\right)^2 - m\vec{r} \cdot \vec{A} - U$

Using E-L, $\boxed{m\dot{\vec{v}} = -\frac{\partial U}{\partial \vec{r}} - m\vec{A} + 2m\left(\vec{v} \times \vec{\Omega}\right) + m\vec{\Omega} \times \left(\vec{r} \times \vec{\Omega}\right) + m\vec{r} \times \dot{\vec{\Omega}}}$

- Namely,

$$\begin{gathered} m\dot{\vec{v}} = -\frac{\partial U}{\partial \vec{r}} + \vec{F}_{\text{cor}} + \vec{F}_{\text{cent}} \\ \vec{F}_{\text{Cor}} = 2m\left(\vec{v} \times \vec{\Omega}\right), \quad \vec{F}_{\text{cent}} = m\vec{\Omega} \times \left(\vec{r} \times \vec{\Omega}\right) = m\left(\vec{\Omega} \times \vec{r}\right) \times \vec{\Omega} \end{gathered} \tag{10}$$

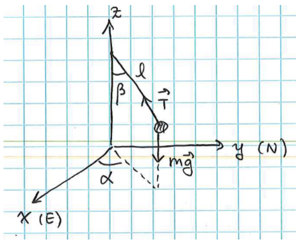
- *ex free fall on earth, centrifugal force* $\vec{F} = \vec{g}_0 + m\Omega^2 R \sin\theta \hat{\rho} \Rightarrow \vec{g}_{\text{eff}} = \vec{g}_0 + \Omega^2 R \sin\theta \hat{\rho}$
- *ex free fall, coriolis force* $\dot{\vec{v}} = \vec{g} + 2\vec{v} \times \vec{\Omega}, \quad \vec{\Omega} = \Omega \sin\theta \hat{y} + \Omega \cos\theta \hat{z}$

In components,

$$\begin{aligned} \vec{v_x} &= 2\Omega\left(v_y \cos\theta - v_z \sin\theta\right) \\ \vec{v_y} &= -2\Omega v_x \cos\theta \\ \vec{v_z} &= 2\Omega v_x \sin\theta - g \end{aligned} \tag*{[11}$$

Free fall EOM: $\vec{R} = \int v \, \mathrm{d}r$, consider $\vec{v} = \vec{v_1} + \vec{v_2} = -\vec{g} + 2\vec{v_1} \times \vec{\Omega} + 2\vec{v_2} \times \vec{\Omega}$ where approximately, $\vec{v_2} = 2(\vec{v_0} - gt\hat{z}) \times \vec{\Omega}$. If no initial velocity, integrating velocity in x components gives, $x(t) = \frac{1}{3} g\Omega \left(\frac{2h}{g}\right)^{3/2} \sin\theta$

- *ex foucaults pendulum* EOM



$$\begin{aligned} \vec{r} &= l \sin\beta \cos\alpha \hat{x} + l \sin\beta \sin\alpha \hat{y} + (l - l\cos\beta)\hat{z} \\ \vec{T} &= -T \sin\beta \cos\alpha \hat{x} - T \sin\beta \sin\alpha \hat{y} + T \cos\beta \hat{z} \\ \vec{\Omega} &= \Omega \sin\theta \hat{y} + \Omega \cos\theta \hat{z} \end{aligned}$$

$$\begin{cases} T = mg \\ m\ddot{x} = T_x + 2m\hat{x} \cdot \left(\dot{\vec{r}} \times \vec{\Omega}\right) = -\frac{mgx}{l} + 2m\Omega\dot{y}\cos\theta \\ m\ddot{y} = -\frac{mgy}{l} - 2m\Omega\dot{x}\cos\theta \end{cases} \tag*{[12}$$

letting $\omega^2 = \frac{g}{l}, \Omega_z = \Omega\cos\theta$, $\boxed{\eta = x + iy = e^{i\gamma t}}$

$$\begin{aligned} \ddot{x} + \omega^2 x = 2\Omega_z \dot{y}, \ddot{y} + \omega^2 y &= -2\Omega_z \dot{x} \\ \gamma &= -\Omega_z \pm \sqrt{\omega^2 - \Omega_z^2} \\ \eta(t) &= ae^{-i\Omega_z t} \cos\omega t \\ &\Rightarrow \begin{cases} x = a \cos\Omega_z t \cos\omega t \\ y = a \sin\Omega_z t \cos\omega t \end{cases} \end{aligned} \tag*{[13}$$

Hamiltonian Mechanics

$H(q, p, t) = \sum_{j=1}^{n} p_j \dot{q}_j - L(q, \dot{q}, t)$ 1D: $H = \frac{p^2}{2m} + U(x)$

- Hamilton's equation $\dot{q}_i = \frac{\partial H}{\partial p_i} \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}$
- *ex particle in polar*

$$L = T - U = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\varphi}^2) - U(r, \varphi) \Rightarrow \quad p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}, p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = mr^2\dot{\varphi} \tag*{[14}$$

$$H = p_r \dot{r} + p_\varphi \dot{\varphi} - L = \frac{p_r^2}{2m} + \frac{p_\varphi^2}{2mr^2} \Rightarrow \quad \dot{r} = \frac{\partial H}{\partial p_r} = \frac{p_r}{m}, \quad \dot{\varphi} = \frac{\partial H}{\partial p_\varphi} = \frac{p_\varphi}{mr^2}$$
$$\dot{p}_r = -\frac{\partial H}{\partial r} = \frac{p_\varphi^2}{mr^3} - \frac{\partial U}{\partial r}, \quad \dot{p}_\varphi = -\frac{\partial H}{\partial \varphi} = -\frac{\partial U}{\partial \varphi} \quad [15$$

Phase space

- *ex harmonic oscillator* $H = \frac{p^2}{2m} + \left(\frac{1}{2}\right) m\omega^2 x^2, \quad \omega = \sqrt{\frac{k}{m}}$

$$\left\{ \dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m}, \quad \dot{p} = -\frac{\partial H}{\partial x} = -m\omega^2 x \right\} \Rightarrow \quad \left\{ \dot{q} = \frac{p}{m}, \quad \dot{p} = -m\omega^2 x \right\} \quad [16$$

$q(t_0 + \delta t) = q(t_0) + \dot{q}\delta t = q_0 + \frac{p}{m}\delta t; \quad p(t_0 + \delta t) = p(t_0) + \dot{p}\delta t = p_0 - m\omega^2 q \delta t$ parametric ellipse in phase space.

Liouville's thm

volume of a region op phase space is conserved under time evolution, when boundary of volume and all pts inside move along their orit for some amount of time.

Poisson bracket

Time evolution of an observable $A(q, p, t)$:

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{\partial A}{\partial t} + \underbrace{\sum_{i=1}^{n} \frac{\partial A}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial H}{\partial q_i}}_{\equiv \{A, H\}} \quad [17$$

More generally, for $A(q, p, t), \quad B(q, p, t)$

$$\{A, B\} = \sum_i \frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i} \quad [18$$

notice, $\{A, p_i\} = \frac{\partial A}{\partial q_i}, \{A, q_i\} = -\frac{\partial A}{\partial p_i}$

- When

$$\frac{\mathrm{d}C}{\mathrm{d}t} = \frac{\partial C}{\partial t} + \{C, H\} = 0 \quad [19$$

  then $C(q, p, t)$ is conserved.

Cononical transformation

consider transformation $q_i \to Q_i(q, t)$ the transformation is canonical iff the transformation leave the form of Hamilton's eq. unchanged.

$$\begin{cases} \dot{q} = \frac{\partial H}{\partial p} \\ \dot{p} = -\frac{\partial H}{\partial q} \end{cases} \Rightarrow \texttt{cases}\ \dot{Q} = \frac{\partial K}{\partial P}, \dot{P} = -\frac{\partial K}{\partial Q} \quad [20$$

where $K(Q, P, t)$ new Hamiltonian.

Exerpts from practice problems

constraints, small Oscillations

A particle of massmmoves withoutfriction on the inside wall of an axially symmetric vessel given by $z = b^2(x^2 + y^2)$

- KE in cylindrical coords:

$$T = \frac{1}{2}m\left(\dot{\rho}^2 + \rho^2\dot{\theta}^2 + \dot{z}^2\right), \quad \dot{z} = b\dot{\rho}\rho \Rightarrow$$
$$L = \frac{m}{2}\left[\dot{\rho}^2(1 + b^2\rho^2) + \rho^2\dot{\theta}^2\right] - \frac{mgb}{2}\rho^2 \tag{21}$$

E-L:

$$\ddot{\rho}(1 + b^2\rho^2) + b^2\dot{\rho}^2\rho - \rho\dot{\theta}^2 + gb\rho = 0$$
$$m\rho^2\dot{\theta} = \text{const} \equiv M \;\; \text{conserved angular momentum} \tag{22}$$

- energy and angular momentum given z_0, b, g, m

$$E = \frac{m}{2}\left[\dot{\rho}^2(1 + b^2\rho^2) + \rho^2\dot{\theta}^2\right] + \frac{mgb}{2}\rho^2 \tag{23}$$

For a fixed z_0, ρ_0 is the equilibrium position, and $\dot{\rho} = 0$, then

$$E = \frac{m}{2}\rho_0^2\dot{\theta}^2 + mgb\frac{\rho_0^2}{2}$$
$$\dot{\theta}^2 = gb$$
$$\Rightarrow E = 2mgz_0 \tag{24}$$

plugging in $\dot{\theta}$, $\rho = \rho_0$, we have $M = 2mz_0\sqrt{\frac{g}{b}}$

- frequency of small oscillations about equilibrium perturbation: $\rho = \rho_0 + \varepsilon$, neglecting anything with ε^2 , EOM of rho is

$$\ddot{\varepsilon}(1 + b^2\rho_0^2) - \rho\dot{\theta}^2 + gb\rho_0 + gb\varepsilon = 0 \tag{25}$$

want to know $\rho\dot{\theta}^2$, can be found from θ EOM

$$\rho\dot{\theta}^2 = \frac{M^2}{m^2\rho^3} = \frac{M^2}{m^2\rho_0^3}\left(\frac{1}{\left(1 + \frac{\varepsilon}{\rho_0}\right)^3}\right) \approx \frac{M^2}{m^3\rho_0^4}\left(1 - 3\frac{\varepsilon}{\rho_0}\right)$$
$$= b\rho_0 g - 3bg\varepsilon \tag{26}$$

Plugging in to rho EOM, we have

$$\ddot{\varepsilon}(1 + 2bz_0) + 4gb\varepsilon = 0$$
$$\ddot{\varepsilon} = -\omega^2\varepsilon, \Omega^2 = \frac{4gb}{1 + 2bz_0} \tag{27}$$

Conservation laws

two particles of $\{m_1, q_1, \vec{r}_1\}, \{m_2, q_2, \vec{r}_2$ in capacitor with $\vec{E} = E_0\hat{z}$, particles interact with $U(r_1, r_2) = \frac{k}{|\vec{r_1} - \vec{r_2}|} e^{-\frac{|\vec{r}_1 - \vec{r}_2|}{\lambda}}$. List all conserved quantities and associate each with a specific symmetry of the problem.

- lagrangian $L = \frac{1}{2}m_1\dot{\vec{r}}_1^2 + \frac{1}{2}m_2\dot{\vec{r}}_2^2 - U + E_0(q_1z_1 + q_2z_2)$. Setting $\vec{r} = (x, y, z) = \vec{r}_1 - \vec{r}_2, \vec{R} = (X, Y, Z) = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{M}, \mu = \frac{m_1m_1}{M}$, we can have

$$L = \left[\frac{1}{2}M\dot{\vec{R}}^2 + (q_1 + q_2)E_0Z\right] + \left[\frac{1}{2}\mu\dot{\vec{r}}^2 - U(r) + \frac{q_1m_2 - q_2m_2}{M}\,E_0z\right] \tag{28}$$

Observe: momenta $P_x = \frac{\partial L}{\partial \dot{X}}, P_y = \frac{\partial L}{\partial \dot{Y}}$ are conserved. Invariance under time translation gives conserved energy

$$E = \frac{\partial L}{\partial \dot{R}}\dot{R} + \frac{\partial L}{\partial \dot{r}}\dot{r} - L \tag{29}$$

Angular momentum $L_{\text{ttl}} = \vec{r_1} \times \vec{p_1} + \vec{r_2} \times \vec{p_2} = M\vec{R} \times \dot{\vec{R}} + \mu\vec{r} \times \dot{\vec{r}} = \vec{R} \times \vec{P} + \vec{r} \times \vec{p}$. Invariance under rotation about $\hat{z} : R \to R + \varepsilon\hat{z} \times R, \quad r \to \vec{r} + \varepsilon\hat{z} \times \vec{r}$ gives conserved $L_z = \left(\vec{R} \times \vec{P}\right)_z \quad l_z = [\vec{r} \times \vec{p}]_z$.

Normal modes

A system of N particles with masses m_i moves around a circle of radius a, with position angle θ_i. Interaction potential $U = \frac{k}{2}\sum_1^N \left(\theta_{j+1} - \theta_j\right)^2$., with $\theta_{N+1} = \theta_1 + 2\pi$. lagrangian of system is $\frac{a^2}{2}\sum_1^N m_j\dot{\theta}_j^2 - U$

- show Largrangian for particle i, show system in equalibrium when particles are equally spaced.

$$L = \frac{a^2}{2}\sum_1^N m_j\dot{\theta}_j^2 - \frac{k}{2}\sum_1^N \left(\theta_{j+1} - \theta_j\right)^2 \tag{30}$$

  E-L for $\theta_i: \quad a^2m_i\ddot{\theta}_i = k(\theta_{i+1} - \theta_i) - k(\theta_i - \theta_{i-1}) = -k[2\theta_i - (\theta_{i+1} + \theta_{i-1})]$ When eqaully spaced, $\theta_i = \frac{2\pi i}{N}$, thus $\ddot{\theta}_i = 0$ for all particles, thus equalibrium.

- show the system always has a normal mode of osc. with 0 freq.

$$\mathbb{M} \cdot \ddot{\vec{\theta}} = -\mathbb{K} \cdot \vec{\theta}, \quad M_{ij} = a^2m_i\delta_{ij}, \quad K_{ij} = k\left(2\delta_{i,j} - \delta_{i,j+1} - \delta_{i,j-1}\right) \tag{31}$$

  take anstaz susbsitution $\vec{\theta} \to \vec{z} = \vec{b}e^{i\omega t}$ gives $\omega^2\mathbb{M} \cdot \vec{b} = \mathbb{K} \cdot \vec{b}$, where $\vec{b}$ is a constant vec. Look for a 0 freq $\omega = 0$, $\mathbb{K} \cdot \vec{b} = 0$ holds, so $b_i = b$. let $b = \Theta(t)$, knowing $\ddot{\Theta} = 0$ recall our subsitution, the time evo of $\theta_{i(t)} = \Theta_0 + \Theta_1 t$ i.e. trajectory is all masses rotating at same rate Θ_1

- find all normal modes when $N = 2, M_1 = km/a^2, m_2 = 2km/a^2$. Using standard normal mode analysis, for $N = 2, \quad \omega^2\mathbb{M} \cdot \vec{b} = \mathbb{K} \cdot \vec{b}$ becomes

$$\begin{pmatrix} a^2\omega^2m_1 - 2k & 2k \\ 2k & a^2\omega^2m_2 - -2k \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = 0 \tag{32}$$

  zero det gives

$$a^4\omega^4m_1m_2 - 2ka^2\omega^2(m_1 + m_2) = 0 \quad \Rightarrow \omega^2 = 0 \text{ or } \frac{2k(m_1 + m_2)}{a^2m_1m_2} \tag{33}$$

  setting $m_2 = 2m_1 = km/a^2$, the second sol becomes $\omega^2 = \frac{3}{m}$

Corresponding normal mode is pound by plugging ω into Equation 32

$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = 0 \quad \Rightarrow b_1 = -2b_2 \equiv Ae^{-i\delta} \qquad [34$$

taking the real part, we find the SOLUTION

$$\begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix} A\cos(\omega t - \delta) \qquad [35$$

two masses osc. exactly out of phase, with m2 osc. with half the amplitude.

non-inertial frame

a pendulum suspended inside a car, accelerated at cosntant $\vec{A}$.

- lagrangian, and EOM for angle θ, the angle from vertical. set X be coord of the moving support with $\vec{A}$

$x = X + l\sin\varphi, \quad y = l\cos\varphi$

$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2}ml^2\dot{\varphi}^2 + ml\dot{X}\dot{\varphi}\cos\varphi + \frac{1}{2}m\dot{X}^2$, $U = -mgy = -mgl\cos\varphi$

$L = T - U = \frac{1}{2}ml^2\dot{\varphi}^2 + mgl\cos\varphi - mAl\sin\varphi$ feeding into EL: $l\ddot{\varphi} = -g\sin\varphi - A\cos\varphi$

- Find equilibrium, show it is stable, and find freq. the equilibrium condition is that the forcee vanishes

$$-g\sin\varphi_0 - A\cos\varphi_0 = 0 \quad \Rightarrow \tan\varphi_0 = -A/g \qquad [36$$

to find equil. take $\varphi = \varphi_0 + \delta\varphi$, expanding the above

$$\begin{gathered} l\delta\ddot{\varphi} = (-g\cos\varphi_0 + A\sin\varphi_0)\delta\varphi = -\delta\varphi\sqrt{g^2 + A^2} \\ \Rightarrow \delta\ddot{\varphi} = -\omega^2\delta\varphi, \quad \omega^2 = \frac{g^2 +^2}{l} \end{gathered} \qquad [37$$

Hamiltonian of particle in rotating frame

find H of said particle, and show coriolis force does not appear in hamiltonian

Largrangian: $L = \frac{1}{2}mv^2 + m \cdot \left(\vec{\Omega} \times \vec{r}\right) + \frac{1}{2}m\left(\vec{\Omega} \times \vec{r}\right)^2 - U$ Do conical transformation, the momentum is $\vec{P} = \frac{\partial L}{\partial v} = m\vec{v} + m\Omega \times \vec{r}$

Hamiltonian $H = \vec{p} \cdot \vec{v} - L = \frac{p^2}{2m} - \vec{\Omega} \cdot (\vec{r} \times \vec{p}) + U$ THis can also be $\frac{1}{2}mv^2 - \frac{1}{2}m\left(\vec{\Omega} \times \vec{r}^2\right) + U$

Observe that there is no term linaer in velocity from centrifugal force, therefore no coriolis force in Hamiltonian.

conservation laws in hamiltonian

1D system with $H = \frac{p^2}{2} - \frac{1}{2q^2}$, show that $D = \frac{pq}{2} - Ht$ is conserved.

- EOM:

$$\dot{q} = \frac{\partial H}{\partial p} = p \quad \dot{p} = -\frac{\partial H}{\partial q} = -\frac{1}{q^3} \qquad [38$$

now write $\frac{\mathrm{d}D}{\mathrm{d}t} = \frac{p\dot{q}}{2} + \frac{\dot{p}q}{2} - H = \frac{p^2}{2} - \frac{1}{2q^2} - H = 0$ as wanted.

  - or use possion braket:

$$
\begin{aligned}
\frac{\mathrm{d}D}{\mathrm{d}t} &= \{H, D\} + \frac{\partial D}{\partial t} = \left\{H, \frac{pq}{2}\right\} - H \\
&= \left(p * \frac{p}{2} - \frac{1}{q^3} * \frac{q}{2}\right) - \frac{p^2}{2} + \frac{1}{2q^2} = 0
\end{aligned}
\tag{39}
$$

Hamiltonian of a rigid body

lagrangian of heavy symm top of mass M, at pt O with distance l from the center of mass is

$$
L = \frac{I_\perp}{2}\left(\dot{\theta}^2 + \dot{\varphi}^2 \sin^2\theta\right) + \frac{I_3}{2}\left(\dot{\psi} + \dot{\varphi}\cos\theta\right)^2 - Mgl\cos\theta \tag{40}
$$

Observe momenta, and Hamiltonian H. find ham's eqn for this system. Identify the three conserved quantities and explain their physical meaning.

$$
\begin{gathered}
p_\theta = \frac{\partial L}{\partial \dot{\theta}} = I_\perp \dot{\theta} \\
p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = I_3 \cos\theta\left(\dot{\psi} + \dot{\varphi}\cos\theta\right) + I_\perp \dot{\varphi}\sin^2\theta \\
p_\psi = \frac{\partial L}{\partial \dot{\psi}} = I_3\left(\dot{\psi} + \dot{\varphi}\cos\theta\right)
\end{gathered}
\tag{41}
$$

and the Hamiltonian is $H = p_\theta\dot{\theta} + p_\varphi\dot{\varphi} + p_\psi\dot{\psi} - L$, plugging in gives

$$
H = \frac{p_\theta^2}{2I_\perp} + \frac{p_\psi^2}{2I_3} + \frac{\left(p_\varphi - p_\psi\cos\theta\right)^2}{2I_\perp\sin^2\theta} + Mgl\cos\theta \tag{42}
$$

Ham's eqn are

$$
\begin{aligned}
\dot{\theta} &= \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{I_\perp} \\
\dot{\phi} &= \frac{\partial H}{\partial p_\phi} = \frac{p_\phi - p_\psi\cos\theta}{I_\perp\sin^2\theta} \\
\dot{\psi} &= \frac{\partial H}{\partial p_\psi} = \frac{p_\psi}{I_3} - \frac{\cos\theta(p_\phi - p_\psi\cos\theta)}{I_\perp\sin^2\theta} \\
\dot{p}_\theta &= -\frac{\partial H}{\partial\theta} = -\frac{p_\psi(p_\phi - p_\psi\cos\theta)}{I_\perp\sin\theta} + \frac{\cos\theta(p_\phi - p_\psi\cos\theta)^2}{I_\perp\sin^3\theta} + Mg\ell\sin\theta \\
\dot{p}_\phi &= -\frac{\partial H}{\partial\phi} = 0 \\
\dot{p}_\psi &= -\frac{\partial H}{\partial\psi} = 0.
\end{aligned}
$$

No explicit time dependence means the energy is conserved. The energy is now hamiltoninan, $E = H(q(t), p(t))$ From ham's eqn, we see

$$
\dot{p}_\varphi = -\frac{\partial H}{\partial\varphi} = 0, \quad \dot{p}_\psi = -\frac{\partial H}{\partial\psi} = 0 \tag{43}
$$

momentum on the φ is conserved, due to the fact that there is no z-component to the gravitational torque. momentum on ψ is conserved, due to the fact that there is nox3-component to the gravitational torque

Dynamics in a magnetic field

consider motion of a charged particle q in the presence of B and E field. Lagrangian of particle is

$$L = \frac{1}{2}mv^2 - q\varphi(\vec{r}, t) + q\vec{A}(\vec{r}, t) \cdot \vec{v} \qquad [44$$

where $\varphi, \vec{A}$ are the scalar and vector potentials, related to the electric and magnetic fields by

$$\mathbb{E} = -\nabla\varphi - \frac{\partial \vec{A}}{\partial t}, \quad \mathbb{B} = \nabla \times \vec{A} \qquad [45$$

- write E-L, express results in terms of E and B, verify that this is lorentz force law.

$$-q\partial_i\varphi + q(\partial_i A_j)\dot{x}_j = \frac{\mathrm{d}}{\mathrm{d}t}(m\dot{x}_i + qA_i) \qquad [46$$

expanding gets us

$$m\ddot{x}_i = q(-\partial_i\varphi - \partial_t A_i) + q\dot{x}_j(\partial_i A_j - \partial_j A_i) \qquad [47$$

algebra magic tells us that $\vec{v} \times \mathbb{B} = v_j(\partial_i A_j - \partial_j A_i) \quad E_i = -\partial_i\varphi - \partial_t A_i$, so this turns out to be

$$m\ddot{r} = q\left(\vec{E} + \vec{v} \times \vec{B}\right) \qquad [48$$

- show lagrangian is invariant under gauge transformation

b) Recall the scalar and vector potentials are not unique. The gauge transformation

$$\phi(\mathbf{r}, t) \to \phi(\mathbf{r}, t) - \frac{\partial f(\mathbf{r}, t)}{\partial t}, \quad \mathbf{A}(\mathbf{r}, t) \to \mathbf{A}(\mathbf{r}, t) + \nabla f(\mathbf{r}, t), \qquad (88)$$

leaves the fields **E** and **B** unchanged (as you may verify from Eq. (79)). Thus the scalar and vector potentials contain an "unphysical" component related to this gauge redundancy. You might then be worried that these unphysical fields appear in the Lagrangian. Compute the change in the Lagrangian (78) under such a gauge transformation and explain why the gauge redundancy is not a cause for concern.

Solution: Let's see how the Lagrangian under a gauge transformation. From (80) we the change δL to be

$$\delta L = -q\partial_t f + q(-\partial_i f)\dot{x}_i. \qquad (89)$$

But this is simply the total time derivative of f.

$$\boxed{\delta L = -q\frac{df}{dt}}. \qquad (90)$$

Thus the gauge transformation does not change the equations of motion.

- find $p = \frac{\partial L}{\partial v}$ from lagrangian and from which recover the hamiltonian.

$$
\begin{aligned}
\vec{p} &= \frac{\partial L}{\partial v} = mv + q\vec{A} \quad \Rightarrow v = \frac{1}{m}\left(\vec{p} - q\vec{A}\right) \\
H &= \vec{p} \cdot \vec{v} - L = \frac{\left(\vec{p} - q\vec{A}\right)^2}{2m} + q\varphi(\vec{r}, t)
\end{aligned}
\quad [49
$$

- Compute the prosson brackets between the different components of the kenetic momentum $\vec{k} = m\vec{v}$ from the above answer we have $k_i = p_i - qA_i$ use poisson brackets

$$
\begin{aligned}
\left\{k_i, k_j\right\} &= \left\{p_i - qA_i, p_j - qA_j\right\} \\
&= q(\left\{A_i, p_j\right\} - \left\{A_j, p_i\right\} \\
&= q\left(\frac{\partial A_i}{\partial x_j} - \frac{\partial A_j}{\partial x_i}\right) \\
&= -q\varepsilon_{ijk} B_k
\end{aligned}
\quad [50
$$

the poisson brackets of the components of the kinetic momentum is thus non-zero in a magnetic field.