Angular momentum of a rigid body

$\vec{L}$ in non-inertial frame

$$\vec{L} = \sum m(\vec{r} \times \vec{v}) = \sum m\left[\vec{\Omega} r^2 - \vec{r}\left(\vec{\Omega} \cdot \vec{r}\right)\right]$$

$$L_i = \boxed{I_{ij}\Omega_j} \qquad \vec{L} = I * \vec{\Omega}$$

If $(x_1 x_2 x_3)$ are principal axis, $L_1 = I_1\Omega_1$, $L_2 = I_2\Omega_2$, $L_3 = I_3\Omega_3$

Free motion of a rigid body

angular momentum is conserved if no external torque. Motion in inertial COM frame is simplier.

- *ex motion of a symmetric top* $I_1 = I_2 = I_3 = I, \quad \tilde{I} = I\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$\vec{L} = I\vec{\Omega} \to \dot{\vec{L}} = 0 \Rightarrow \dot{\vec{\Omega}} = 0$ Uniform rotation about fixed axis paralle to $\vec{L}$

- *ex rigid rotor* $I_1 = I_2 = \sum m x_3^2, \quad I_3 = 0$

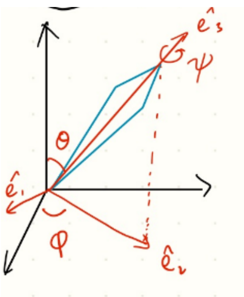
$\vec{L} = I\vec{\Omega}, \quad \vec{\Omega} \perp x_3$ by geometry We have $\dot{\vec{\Omega}} = 0 \Rightarrow$ Motion is unif in plane perp to $\vec{\Omega}$ and that it stays in that plane.

- *ex asymmetric top* $I_1 = I_2 = I_\perp \neq I_3 \Rightarrow \tilde{I} = \begin{pmatrix} I_\perp & 0 & 0 \\ 0 & I_\perp & 0 \\ 0 & 0 & I_3 \end{pmatrix}$ x_3 is symm. axis, for any orthogonal axes

Rigid body EOM

$$\begin{cases} \dot{\vec{p}} = \vec{F} \\ \dot{\vec{L}} = \vec{K} \text{ torque} \end{cases}$$

Euler angles: ψ spin, θ nutation, φ precession



$(\theta \in [0, \pi], \varphi \in [0, 2\pi], \psi \in [0, 2\pi])$ in turns of rotation $R = R(\hat{z}, \varphi) R\left(\widehat{X}, \theta\right) R\left(\hat{Z}, \psi\right)$

The lagrangian in Euler angles

- First: $T = \frac{1}{2}\left(I_1\Omega_1^2 + I_2\Omega_2^2 + I_3\Omega_3^2\right)$
- Rotation in components:

$$\Omega_1 = \dot{\varphi}\sin\theta\sin\psi + \dot{\theta}\cos\psi$$

$$\Omega_2 = \dot{\varphi}\sin\theta\cos\psi - \dot{\theta}\sin\psi$$

$$\Omega_3 = \dot{\varphi}\cos\theta + \dot{\psi}$$

- $T = \frac{1}{2}I_1\left(\dot{\varphi}\sin\theta\sin\psi + \dot{\theta}\cos\psi\right)^2 + \frac{1}{2}I_2\left(\dot{\varphi}\sin\theta\cos\psi - \dot{\theta}\sin\psi\right)^2 + \frac{1}{2}I_3\left(\dot{\varphi}\cos\theta + \dot{\psi}\right)^2$
- $L\left(\theta, \varphi, \psi, \dot{\theta}, \dot{\varphi}, \dot{\psi}\right) = T - U$

Free motion of symmetric top in Euler angles

$I_1 = I_2 = I_\perp \Rightarrow \quad T = \frac{1}{2}I_\perp\left(\dot{\theta}^2 + \dot{\varphi}^2\sin^2\theta\right) + \frac{1}{2}I_3\left(\dot{\varphi}\cos\theta + \dot{\psi}\right)^2$

$\Omega_\perp = L_z/I_\perp, \quad \Omega_3 = L_z\cos\theta/I_3$ E-L ->

$$\theta : \frac{\mathrm{d}}{\mathrm{d}t}I_\perp\dot{\theta} = I_\perp\sin\theta\cos\theta\,\dot{\varphi}^2 - I_3\dot{\varphi}\sin\theta\left(\dot{\varphi}\cos\theta + \dot{\psi}\right)$$

$$\varphi : \frac{\mathrm{d}}{\mathrm{d}t}\left(I_\perp\dot{\varphi}\sin^2\theta + I_3\cos\theta\left(\dot{\varphi}\cos\theta + \dot{\psi}\right)\right) = 0$$

$$\psi : \frac{\mathrm{d}}{\mathrm{d}t}I_3\left(\dot{\varphi}\cos\theta + \dot{\psi}\right) = 0$$

choosing $\hat{z}$ along the angular momentum, we have $L_3 = L_z\cos\theta = I_3\Omega_3 = I_3\left(\dot{\varphi}\cos\theta + \dot{\psi}\right)$
$\Rightarrow \dot{L}_3 = \text{const} \Rightarrow \theta = \text{const} \quad \Omega_3 = \frac{L_z\cos\theta}{I_3} \quad \dot{\varphi} = \frac{L_3}{I_\perp\cos\theta} = \frac{L_z}{I_\perp} = \text{const}$

- *ex heavy symmertic top with one pt fixed* By paralle axis thm, $I'_{ij}I_{ij} + M\left(l^2\delta_{ij} - l_i l_j\right)$

  $\Rightarrow I'_\perp = I_\perp + Ml^2, \quad I'_3 = I_3, \quad U = mgZ = Mgl\cos\theta$

  $\Rightarrow L = T - U = \frac{1}{2}I'_\perp\left(\dot{\theta}^2 + \dot{\varphi}^2\sin^2\theta\right) + \frac{1}{2}I_3\left(\dot{\psi} + \dot{\varphi}\cos\theta\right)^2 = Mgl\cos\theta$

  E-L :

$$L_z = p_\varphi = \left(I'_\perp\sin^2\theta + I_3\cos^2\theta\right)\dot{\varphi} \quad \text{const}$$

$$L_3 = p_\psi = I_3\left(\dot{\psi} + \varphi\cos\theta\right) \quad \text{const}$$

Considering energy conservation

$$E = T + U \Rightarrow \quad \underbrace{E - \frac{L_3^2}{2I_3} - Mgl}_{E'} = \frac{1}{2}I'_\perp\dot{\theta}^2 + \underbrace{\frac{1}{2I'_\perp}\frac{\left(L_z - L_3\cos\theta\right)^2}{\sin^2\theta} - Mgl(1 - \cos\theta)}_{U_{\text{eff}}(\theta)}$$

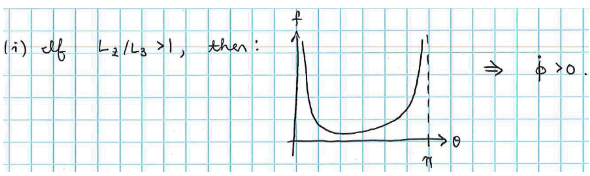
effective 1 dof problem. recognizing

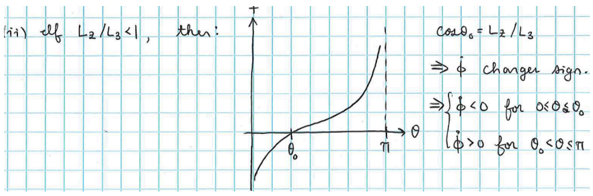
$$\dot{\theta} = \frac{\mathrm{d}\theta}{\mathrm{d}t} \Rightarrow \quad t = \int \frac{d\theta}{(\sqrt{2[E - U_{\text{eff}}(\theta)]/I'_\perp}}$$

Considering U_eff: when $\theta = 0, L_z = L_3$when $\theta \approx 0 \Rightarrow \quad U_{\text{eff}} \approx \left(\frac{L_3^2}{8I'_\perp} - \frac{Mgl}{2}\right)\theta^2$

Motion about $\theta = 0$ stable if $L_3^2 > 4I'_\perp Mgl \Rightarrow \Omega_3^2 > 4I'_\perp Mgl/I_3^2$, or stable if sping ab. symm. axis is fast enough.

- Nutuation: cosider $\dot{\varphi} = \frac{L_3}{I'_\perp}\frac{(L_z/L_3) - (\cos\theta)}{\sin^2\theta} = \frac{L_3}{I'_\perp}f(\theta)$





considering the sign and trends of $f(\theta)$ given constrains on theta, we can differentiate different nutation motion. If θ_0 in graph 2 is out of range, the nutation is smooth; if θ_0 is in range, the nutation is oscillatory(will change sign and spin in spiral.); if θ_0 is on the endpoint of our constrained range, the nutation is spiky and "not smooth" at points.

Euler equations