

## Small Oscillations

- Motion near a point of stable equilibrium.

### DOF= 1 (one dimension)

- For a system of DOF = 1, with potential  $U(q)$ :
  - **stable equilibrium** at  $U(q)_{\min}$ , upward parabola, where  $F = -\frac{dU}{dq} = 0$ 
    - restoring force for small displacements  $q - q_0$  is  $F = -\frac{d^2U(q-q_0)}{dq}$
  - **Unstable equilibrium** at  $U(q)_{\max}$ , downward parabola, where  $F = -\frac{dU}{dq} = 0$  as well.
- Consider small deviation from point of stable equilibrium, we use Taylor expansion to show that it is really a small displacement. that is,

$$U(q) \approx U(q_0) + \frac{dU(q_0)}{dq}(q - q_0) + \frac{d^2U(q_0)}{2dq^2}(q - q_0)^2 + \dots$$

$$\text{while } \frac{dU(q_0)}{dq}(q - q_0) = 0$$
1

letting  $x = q - q_0$ , we have

$$\begin{cases} U(x) = U(q_0) + \left(\frac{1}{2}\right) \frac{d^2U(q_0)}{dq^2} x^2 \\ \text{putting into the form of } U(x) = U(x_0) + \left(\frac{1}{2}\right) kx^2. \end{cases}$$

$$\Rightarrow \boxed{k = \frac{d^2U(q_0)}{dq^2} > 0}$$
2

we get KE, while choosing  $U(q_0) = 0$ :

$$T = \frac{1}{2} a(q)^2 \dot{q}^2 = \frac{1}{2} a(q_0 + x) \dot{x}^2 \approx \frac{1}{2} m \dot{x}^2, \quad m=a(q_0) \Rightarrow$$

$$\boxed{L = T - U = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2}$$
3

### EOM for DOF = 1 small Oscillations

using EL on Equation 3, we can get the EOM for one dimensional small Oscillations:

$$m\ddot{x} = -kx$$

$$\Rightarrow \ddot{x} + \omega_0^2 x = 0, \text{ where } \boxed{\omega_0 = \sqrt{\frac{k}{m}} \text{ freq of osc.}}$$
4

by magic of ODE, EOM reduces down to:

$$\boxed{x(t) = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t)}$$

where  $C_1, C_2$  are constants

5

by trig magic, this could also be written as

$$x(t) = a \cos(\omega_0 t + \alpha),$$

$$\text{where } \begin{cases} a = \sqrt{C_1^2 + C_2^2} & \text{amplitude of oscillation} \\ \omega_0 & \text{frequency of oscillation} \\ \tan \alpha = C_2/C_1 & \text{phase at } t=0 \end{cases} \quad 6$$

### energy for 1D small Oscillation

checking  $\frac{\partial L}{\partial t} = 0 \Rightarrow$  energy-conservation:

$$E = T + U = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$$

$$= \frac{1}{2}ma^2\omega_0^2, [\text{constant}] \quad 7$$

### Damped 1D oscillation, and Complex representation

• when there is damping (friction, resistance, etc)  $F_{\text{fric}} = -\beta\dot{x}$ , the EOM becomes:

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0,$$

$$\text{where } 2\gamma = \frac{\beta}{m}, \omega_0 = \sqrt{\frac{k}{m}} \quad 8$$

with ansatz  $x(t) = e^{rt}$ ,  $\dot{x} = re^{rt}$ ,  $\ddot{x} = r^2 e^{rt}$ , the solution to Equation 8 is:

$$r^2 + 2\gamma r + \omega_0^2 = 0,$$

$$\text{which has solution } r_+, r_- = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2} \quad 9$$

$$\Rightarrow x(t) = C_1 e^{r_+ t} + C_2 e^{r_- t},$$

notice the r subscripts here:  $r_+, r_-$

### underdamped, overdamped, and critically damped

Recall from your ODE class...

Equation 9 has the following 3 cases, each with different physical interpretation:

1. underdamped:

$$\gamma < \omega_0 \Rightarrow 2 \text{ complex roots: } \begin{cases} r_{\pm} = -\gamma \pm i\sqrt{\omega_0^2 - \gamma^2} \\ = -\gamma \pm i\omega \\ \omega = \sqrt{\omega_0^2 - \gamma^2} \end{cases} \quad 10$$

The EOM is thus a linear combination of two complex exponentials:

$$x(t) = e^{-\gamma t} (C_1 e^{i\omega t} + C_2 e^{-i\omega t})$$

$$= e^{-\gamma t} (A \cos(\omega t) + B \sin(\omega t))$$

$$\text{-- where } \begin{cases} A = C_1 + C_2 \\ B = i(C_1 - C_2) \end{cases} \quad 11$$

$$= ae^{-\gamma t} \cos(\omega t + \alpha)$$

$a, \alpha$  are constants

“The solution is a damped oscillation with frequency  $\omega$ , and amplitude exponentially decaying with time.”

## 2. Overdamped

$$\gamma > \omega \Rightarrow x(t) = c_1 e^{-\gamma + \sqrt{\gamma^2 - \omega^2} t} + c_2 e^{-\gamma - \sqrt{\gamma^2 - \omega^2} t} \quad 12$$

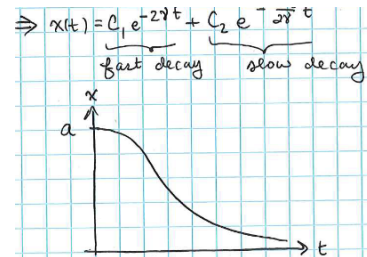
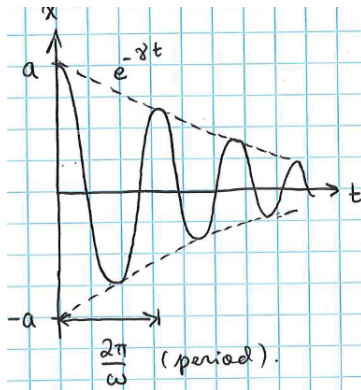
When

$$\gamma \gg \omega_0, \Rightarrow \begin{cases} \gamma + \sqrt{\gamma^2 - \omega_0^2} \approx 2\gamma \\ \gamma - \sqrt{\gamma^2 - \omega_0^2} = \frac{\omega^2}{2\gamma} \end{cases} \quad 13$$

$$x(t) = c_1 e^{-2\gamma t} + c_2 e^{(-\omega_0^2/2\gamma)t}$$

## 3. Critically damped

$$\gamma = \omega_0 \Rightarrow x(t) = c_1 e^{-\gamma t} + c_2 t e^{-\gamma t} \quad 14$$



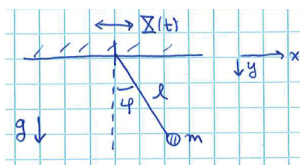
## Forced Oscillations

When external force ( $F$ ) is applied to the system, the lagrangian becomes

$$L = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2 + F(t)x \quad 15$$

$$\text{EL} \Rightarrow \ddot{x} + \omega_0^2 x = \frac{F(t)}{m}, \text{ where } \omega_0 = \sqrt{\frac{k}{m}}$$

- Example: Simple pendulum with moving pivot



$$\begin{cases} x = X + l \sin \varphi \\ y = l \cos \varphi \end{cases} \Rightarrow \begin{cases} \dot{x} = \dot{X} + l \dot{\varphi} \cos \varphi \\ \dot{y} = -l \dot{\varphi} \sin \varphi \end{cases} \quad 16$$

$$\Rightarrow L = T - U$$

$$L = \frac{1}{2} m l^2 \dot{\varphi}^2 - m g l (1 - \cos \varphi) - m l \ddot{X} \sin \varphi$$

$$\text{Expand ab. } \varphi = 0 \Rightarrow L = \frac{1}{2} m l^2 \dot{\varphi}^2 - \frac{1}{2} m g l \varphi^2 - m l \ddot{X} \varphi \quad 17$$

$$\text{EL} \Rightarrow \ddot{\varphi} + \omega_0^2 \varphi = -\frac{\ddot{X}}{l}, \text{ where } \omega_0 = \sqrt{\frac{g}{l}}$$

## reintroducing damping via external forcing

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = f(t), f(t) = \frac{F(t)}{m} \quad 18$$

When damping  $f(t) = f_0 \cos(\Omega t)$ , solution via complex number:

$$\ddot{z} + 2\gamma\dot{z} + \omega_0^2 z = f_0 e^{i\Omega t}$$

$$\text{ansatz } z(t) = z_0 e^{i\Omega t} \Rightarrow z_0 = \frac{f_0}{\omega_0^2 + 2i\gamma\Omega + \Omega^2}$$

$$\boxed{z_0 = a(\Omega) \cos(\Omega t + \delta(\Omega)) f_0} \text{ is a particular solution, where} \quad 19$$

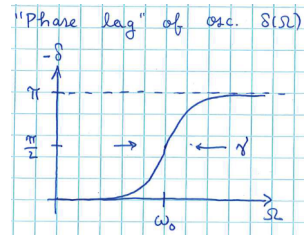
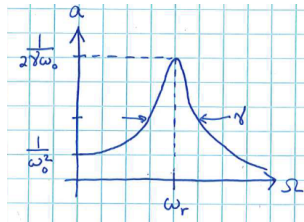
$$\begin{cases} a(\Omega) = \frac{1}{\sqrt{(\omega_0^2 - \Omega^2)^2 + (2\gamma\Omega)^2}} \\ \delta(\Omega) = \arctan\left(2\gamma\frac{\Omega}{\omega_0^2 - \Omega^2}\right) \end{cases}$$

We can study the properties of the system by looking at the amplitude and phase of the solution.

- Amplitude:

$$a(\Omega) = \frac{1}{\sqrt{(\omega_0^2 - \Omega^2)^2 + (2\gamma\Omega)^2}} \quad 20$$

, when  $\gamma \ll \omega_0$ , response strongest and amplitude largest when  $\omega_r = \omega_0$ .



- Phase lag:  $\tan \delta(\Omega) = 2\gamma \frac{\Omega}{\Omega^2 - \omega_0^2}$

in phase as  $\Omega \rightarrow 0$ , and out of phase as  $\Omega \rightarrow \omega_0$ .

- Genral solution to sinusoidal forcing:

$$x(t) = a(\Omega) f_0 \cos(\Omega t + \delta(\Omega)) + a_0 e^{-\gamma t} \cos(\omega t + \alpha)$$

$$\xrightarrow{t > \frac{1}{\gamma}} a(\Omega) f_0 \cos(\Omega t + \delta(\Omega)) \quad 21$$

Forgets initial condition after time.

- Power obsorbed by oscillation

$$p = F\dot{x} = m f \dot{x}$$

Avg power of oscillation

$$P_{\text{avg}} = \frac{1}{T} \int_0^T m f \dot{x} dt = -\frac{1}{2} m f_0 a(\Omega) \Omega \sin \delta(\Omega)$$

$$\text{simplifies to } P_{\text{avg}}(\Omega) = \gamma m f_0^2 \Omega^2 a_{(\Omega)}^2 \quad 22$$

Absorption around resonance frequency  $\Omega = \omega_0 + \varepsilon$  is maximum:

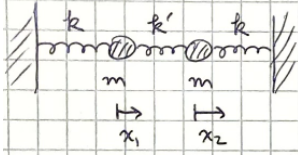
$$P = \frac{\gamma m f_0^2}{4(\varepsilon^2 + \gamma^2)} \approx \frac{m f_0^2}{4\gamma} \quad 23$$

## Oscillations DOF>1

For a system with n DOF:  $q = (q_1, q_2, \dots, q_n)$ , PE =  $U(q)$

- Stable equilibrium  $\left. \frac{\partial U(q)}{\partial q_i} \right|_{q=0}$

### Example: Oscillation with 2 mass and 3 springs



$$L = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2 - \frac{1}{2}kx_1^2 - \frac{1}{2}kx_2^2 - \frac{1}{2}k'(x_1 - x_2)^2$$

EOM:

$$M \cdot \ddot{\vec{x}} = -K\vec{x}, \text{ where } M = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}, \quad \vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, K = \begin{pmatrix} k+k' & -k' \\ -k' & k+k' \end{pmatrix} \quad 24$$

ansatz:  $\vec{x} = \text{Re}[\vec{a}e^{i\omega t}]$  Then the EOM eq becomes solving the eigenvalue problem:

$$\det(\omega^2 M - K) = 0 \Rightarrow \begin{cases} \omega_-^2 = \frac{k}{m} \\ \omega_+^2 = \frac{k+2k'}{m} \end{cases} \begin{cases} \vec{x}_- = a_- \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos(\omega_- t + \delta_-) \\ \vec{x}_+ = a_+ \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos(\omega_+ t + \delta_+) \end{cases} \quad 25$$

with constants  $a_-, a_+, \delta_-, \delta_+$ .

### New Coords

$$\begin{cases} Q_1 = \sqrt{\frac{m}{2}}(x_1 + x_2) \\ Q_2 = \sqrt{\frac{m}{2}}(x_1 - x_2) \end{cases} \Rightarrow L = \frac{1}{2}(\dot{Q}_1^2 + \dot{Q}_2^2) - \frac{1}{2}(\omega_-^2 Q_1^2 + \omega_+^2 Q_2^2) \quad 26$$

$$\stackrel{\text{E-L}}{\Rightarrow} \ddot{Q}_1 = -\omega_-^2 Q_1, \ddot{Q}_2 = -\omega_+^2 Q_2$$

Decoupled oscillators with coords  $Q_1, Q_2$ .

### General Coords

for general coords  $q_i$ , let  $x_i = q_i - q_i^{(0)}$

$$U(q) = \frac{1}{2} \sum_{i,j} k_{ij} x_i x_j, \quad k_{ij} = k_{ji} = \frac{\partial^2 U(q)}{\partial q_i \partial q_j} \text{ symmetric matrix} \quad 27$$

$$T = \frac{1}{2} \sum_{i,j} m_{ij} \dot{x}_i \dot{x}_j, \quad m_{ij} = m_{ji} = a_{ij}(q^{(0)})$$

the largrangian, in Matix form:

$$L = \frac{1}{2} \dot{\vec{x}}^T \cdot M \cdot \dot{\vec{x}} - \frac{1}{2} \dot{\vec{x}}^T \cdot K \vec{x} \stackrel{\text{EL}}{\implies} (\omega^2 M - K) \cdot \vec{a} = 0 \quad 28$$

$\implies \det(\omega^2 M - K) = 0$  Solving the det for omega gives the normal freq (Eigenvalues) of system  $\omega_\alpha^2$ .  
plug in Eval into Equation 28 for eigenvec(normal modes)  $\vec{a}^\alpha$  of system.

- General motion

$$x_i(t) = \sum_{\alpha} a_i^\alpha \text{Re}[C_\alpha e^{i\omega_\alpha t}] \quad 29$$

- EXAMPLE: Normal freq is given

$$\omega = \{0, \sqrt{2}\omega_0, \sqrt{3}\omega_0\}.$$

$$\omega = \sqrt{2}\omega_0 \Rightarrow a_1 = -a_3 = -a_2 = ae^{i\delta} \Rightarrow \vec{\theta} = a(1 \ -1 \ -1)^T \cos(\sqrt{2}\omega_0 t + \delta) \quad 30$$

$$\omega = \sqrt{3}\omega_0 \Rightarrow a_1 = 0, a_2 = -a_3 = ae^{i\delta} \Rightarrow \vec{\theta} = a(0 \ 1 \ -1)^T \cos(\sqrt{3}\omega_0 t + \delta)$$

- EXAMPLE: double pendulum

$$\begin{cases} x_1 = l_1 \sin \varphi_1 & y_1 = -l_1 \cos \varphi_1 \\ x_2 = l_1 \sin \varphi_1 + l_2 \sin \varphi_2 & y_2 = l_1 \cos \varphi_1 + l_2 \cos \varphi_2 \end{cases} \quad 31$$

$$\Rightarrow T = \frac{1}{2} m_1 l_1 \dot{\varphi}^2 + \frac{1}{2} m_2 (l_1^2 \dot{\varphi}_1^2 + l_2^2 \dot{\varphi}_2^2 + 2l_1 l_2 \dot{\varphi}_1 \dot{\varphi}_2 \cos(\varphi_1 - \varphi_2))$$

$$U = -m_1 g l_1 \cos \varphi_1 - m_2 g (l_1 \cos \varphi_1 + l_2 \cos \varphi_2)$$

$$L = \frac{1}{2} (\varphi_1 \ \varphi_2) \begin{pmatrix} (m_1+m_2)l_1^2 & m_2 l_1 l_2 \\ m_2 l_1 l_2 & m_2 l_2^2 \end{pmatrix} (\varphi_1 \ \varphi_2)$$

$$-\frac{1}{2} (\varphi_1 \ \varphi_2) \begin{pmatrix} (m_1+m_2)l_1 g & 0 \\ 0 & m_2 g l_2 \end{pmatrix} (\varphi_1 \ \varphi_2)$$

$$\text{using } \cos \varphi \approx 1 - \frac{\varphi^2}{2} = \frac{1}{2} \vec{\varphi}^T M \cdot \dot{\vec{\varphi}} - \frac{1}{2} \vec{\varphi}^T K \vec{\varphi}$$

$$\text{When } m_1 = m_2 = m, \ l_1 = l_2 = l \Rightarrow M = ml^2 \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, K = mgl \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\det((\omega^2 M - K)) = 0 \Rightarrow \omega^2 = (2 \pm \sqrt{2}\omega_0^2)$$

$$\begin{pmatrix} a_1^- \\ a_2^- \end{pmatrix} = C_- \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}, \quad \begin{pmatrix} a_1^+ \\ a_2^+ \end{pmatrix} = C_+ \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix} \quad 32$$

## Appendix

1. Taylor expansion:

$$f(x)|_0 \approx f(a) + f'(a)(x-a) + f''(a) \frac{(x-a)^2}{2} \quad 33$$

2. small angle approximation:

$$\sin(\theta) \approx \theta \quad \cos(\theta) \approx 1 - \frac{\theta^2}{2} \quad 34$$