Angular momentum of a rigid body

$\vec{L}$ in non-inertial frame

$$\vec{L} = \sum m(\vec{r} \times \vec{v}) = \sum m\left[\vec{\Omega} r^2 - \vec{r}\left(\vec{\Omega} \cdot \vec{r}\right)\right]$$

$$L_i = \boxed{I_{ij}\Omega_j} \qquad \vec{L} = I * \vec{\Omega}$$

If $(x_1 x_2 x_3)$ are principal axis, $L_1 = I_1\Omega_1$, $L_2 = I_2\Omega_2$, $L_3 = I_3\Omega_3$

Free motion of a rigid body

angular momentum is conserved if no external torque. Motion in inertial COM frame is simplier.

- *ex motion of a symmetric top* $I_1 = I_2 = I_3 = I, \quad \tilde{I} = I\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$\vec{L} = I\vec{\Omega} \to \dot{\vec{L}} = 0 \Rightarrow \dot{\vec{\Omega}} = 0$ Uniform rotation about fixed axis paralle to $\vec{L}$

- *ex rigid rotor* $I_1 = I_2 = \sum m x_3^2, \quad I_3 = 0$

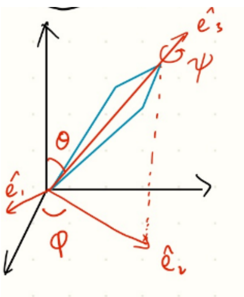
$\vec{L} = I\vec{\Omega}, \quad \vec{\Omega} \perp x_3$ by geometry We have $\dot{\vec{\Omega}} = 0 \Rightarrow$ Motion is unif in plane perp to $\vec{\Omega}$ and that it stays in that plane.

- *ex asymmetric top* $I_1 = I_2 = I_\perp \neq I_3 \Rightarrow \tilde{I} = \begin{pmatrix} I_\perp & 0 & 0 \\ 0 & I_\perp & 0 \\ 0 & 0 & I_3 \end{pmatrix}$ x_3 is symm. axis, for any orthogonal axes

Rigid body EOM

$$\begin{cases} \dot{\vec{p}} = \vec{F} \\ \dot{\vec{L}} = \vec{K} \text{ torque} \end{cases}$$

Euler angles: ψ spin, θ nutation, φ precession



$(\theta \in [0, \pi], \varphi \in [0, 2\pi], \psi \in [0, 2\pi])$ in turns of rotation $R = R(\hat{z}, \varphi)R\left(\widehat{X}, \theta\right)R\left(\hat{Z}, \psi\right)$

The lagrangian in Euler angles

- First: $T = \frac{1}{2}\left(I_1\Omega_1^2 + I_2\Omega_2^2 + I_3\Omega_3^2\right)$
- Rotation in components:

$$\Omega_1 = \dot{\varphi}\sin\theta\sin\psi + \dot{\theta}\cos\psi$$

$$\Omega_2 = \dot{\varphi}\sin\theta\cos\psi - \dot{\theta}\sin\psi$$

$$\Omega_3 = \dot{\varphi}\cos\theta + \dot{\psi}$$

- $T = \frac{1}{2}I_1\left(\dot\varphi\sin\theta\sin\psi + \dot\theta\cos\psi\right)^2 + \frac{1}{2}I_2\left(\dot\varphi\sin\theta\cos\psi - \dot\theta\sin\psi\right)^2 + \frac{1}{2}I_3\left(\dot\varphi\cos\theta + \dot\psi\right)^2$
- $L\left(\theta, \varphi, \psi, \dot\theta, \dot\varphi, \dot\psi\right) = T - U$

Free motion of symmetric top in Euler angles

$I_1 = I_2 = I_\perp \Rightarrow \quad T = \frac{1}{2}I_\perp\left(\dot\theta^2 + \dot\varphi^2\sin^2\theta\right) + \frac{1}{2}I_3\left(\dot\varphi\cos\theta + \dot\psi\right)^2$

$\Omega_\perp = L_z/I_\perp, \quad \Omega_3 = L_z\cos\theta/I_3$ E-L ->

$$\theta : \frac{\mathrm{d}}{\mathrm{d}t}I_\perp\dot\theta = I_\perp\sin\theta\cos\theta\,\dot\varphi^2 - I_3\dot\varphi\sin\theta\left(\dot\varphi\cos\theta + \dot\psi\right)$$

$$\varphi : \frac{\mathrm{d}}{\mathrm{d}t}\left(I_\perp\dot\varphi\sin^2\theta + I_3\cos\theta\left(\dot\varphi\cos\theta + \dot\psi\right)\right) = 0$$

$$\psi : \frac{\mathrm{d}}{\mathrm{d}t}I_3\left(\dot\varphi\cos\theta + \dot\psi\right) = 0$$

choosing $\hat z$ along the angular momentum, we have $L_3 = L_z\cos\theta = I_3\Omega_3 = I_3\left(\dot\varphi\cos\theta + \dot\psi\right)$
$\Rightarrow \dot L_3 = \text{const} \Rightarrow \theta = \text{const} \quad \Omega_3 = \frac{L_z\cos\theta}{I_3} \quad \dot\varphi = \frac{L_3}{I_\perp\cos\theta} = \frac{L_z}{I_\perp} = \text{const}$

- *ex heavy symmertic top with one pt fixed* By paralle axis thm, $I'_{ij}I_{ij} + M\left(l^2\delta_{ij} - l_i l_j\right)$

  $\Rightarrow I'_\perp = I_\perp + Ml^2, \quad I'_3 = I_3, \quad U = mgZ = Mgl\cos\theta$

  $\Rightarrow L = T - U = \frac{1}{2}I'_\perp\left(\dot\theta^2 + \dot\varphi^2\sin^2\theta\right) + \frac{1}{2}I_3\left(\dot\psi + \dot\varphi\cos\theta\right)^2 = Mgl\cos\theta$

  E-L :

$$L_z = p_\varphi = (I'_\perp\sin^2\theta + I_3\cos^2\theta)\dot\varphi \quad \text{const}$$

$$L_3 = p_\psi = I_3\left(\dot\psi + \varphi\cos\theta\right) \quad \text{const}$$

Considering energy conservation

$$E = T + U \Rightarrow \quad \underbrace{E - \frac{L_3^2}{2I_3} - Mgl}_{E'} = \frac{1}{2}I'_\perp\dot\theta^2 + \underbrace{\frac{1}{2I'_\perp}\frac{\left(L_z - L_3\cos\theta\right)^2}{\sin^2\theta} - Mgl(1 - \cos\theta)}_{U_{\text{eff}}(\theta)}$$

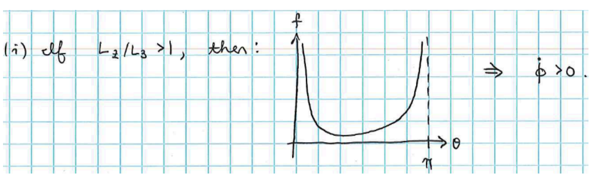
effective 1 dof problem. recognizing

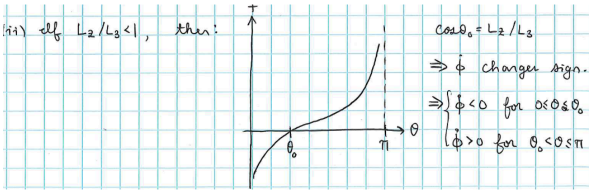
$$\dot\theta = \frac{\mathrm{d}\theta}{\mathrm{d}t} \Rightarrow \quad t = \int\frac{d\theta}{(\sqrt{2[E - U_{\text{eff}}(\theta)]/I'_\perp}}$$

Considering U_eff: when $\theta = 0, L_z = L_3$when $\theta \approx 0 \Rightarrow \quad U_{\text{eff}} \approx \left(\frac{L_3^2}{8I'_\perp} - \frac{Mgl}{2}\right)\theta^2$

Motion about $\theta = 0$ stable if $L_3^2 > 4I'_\perp Mgl \Rightarrow \Omega_3^2 > 4I'_\perp Mgl/I_3^2$, or stable if sping ab. symm. axis is fast enough.

- Nutuation: cosider $\dot\varphi = \frac{L_3}{I'_\perp}\frac{(L_z/L_3) - (\cos\theta)}{\sin^2\theta} = \frac{L_3}{I'_\perp}f(\theta)$





considering the sign and trends of $f(\theta)$ given constrains on theta, we can differentiate different nutation motion. If θ_0 in graph 2 is out of range, the nutation is smooth; if θ_0 is in range, the nutation is oscillatory(will change sign and spin in spiral.); if θ_0 is on the endpoint of our constrained range, the nutation is spiky and "not smooth" at points.

Euler equations

set body frame $(X, Y, Z) = (\hat{e}_1^0, \hat{e}_2^0, \hat{e}_3^0$, space frame $(x_1, x_2, x_3) = (\hat{e}_1, \hat{e}_2, \hat{e}_3)$ Set any vector $\vec{A} = \sum A_i^0 \hat{e}_i^0 = \sum A_i \hat{e}_i$ By magic of vec analysis,

$$\left(\frac{\mathrm{d}\vec{A}}{\mathrm{d}t}\right)_{\text{Space}} = \left(\frac{\mathrm{d}\vec{A}}{\mathrm{d}t}\right)_{\text{Body}} + \vec{\Omega} \times \vec{A}_{\text{Space}}$$

When applied to $\left(\frac{\mathrm{d}\vec{L}}{\mathrm{d}t}\right)_{\text{Space}} = \vec{\mathrm{K}} = \left(\frac{\mathrm{d}\vec{L}}{\mathrm{d}t}\right)_{\text{body}} + \vec{\Omega} \times \vec{L}$, recognizing $L_i = I_i \Omega_i$:

$$I_1 \dot{\Omega}_1 + (I_3 - I_2)\Omega_2\Omega_3 = K_1$$
$$I_2 \dot{\Omega}_2 + (I_1 - I_3)\Omega_3\Omega_1 = K_2$$
$$I_3 \dot{\Omega}_3 + (I_2 - I_1)\Omega_1\Omega_2 = K_3$$

$K_i = 0$ if $\vec{L}$ is conserved on i axis.

- *ex symmetric top* $I_1 = I_2 = I, \vec{K} = 0$ $\quad \left(\dot{\Omega}_1 + \frac{I_3 - I_1}{I_\perp}\Omega_2\Omega_3 = 0; \dot{\Omega}_2 + \frac{I_1 - I_3}{I_\perp}\Omega_3\Omega_1 = 0; \dot{\Omega}_3 = 0\right)$
  let $\omega = ((I_3 - I_\perp)/(I_\perp))\Omega_3 \Rightarrow \boxed{\left(\Omega_1 = A\cos\omega t; \Omega_2 = -\frac{1}{\omega}\dot{\Omega}_1 = +A\sin\omega t\right)}$

Motion in non-inertial frame

- Set non-inertial frame with velocity $\vec{V}(t)$, $\vec{A} = \dot{\vec{V}}$, $\quad \vec{v} = \vec{v}' + \vec{V}(t)$ where $\vec{v}'$ is velocity w.r.t. non-inertial frame.

lagrangian $L' = \frac{1}{2}mv'^2 - m\vec{r}' \cdot \vec{A} - U$, using E-L eq: $m\dot{\vec{v}}' = -\frac{\partial U}{\partial \vec{r}'} - m\vec{A}$

- *ex pendulum in acc. car* $\quad m\ddot{\vec{r}} = \vec{T} + m\vec{g} - m\vec{A}$,

  finding equil. angle: $\vec{T} = -m\left(\vec{g} - \vec{A}\right) = -m\vec{g}_{\text{eff}}$, then use geometry between $\vec{g}, -\vec{A} \Rightarrow \tan\varphi_0 = \frac{A}{g}$. Oscillation freq. $\omega = \sqrt{g_{\text{eff}}/l}$

Motion in rotating frame

Set rotation with $\vec{\Omega}$, $L = \frac{1}{2}mv^2 + m\vec{v} \cdot \left(\vec{\Omega} \times \vec{r}\right) + \frac{1}{2}m\left(\vec{\Omega} \times \vec{r}\right)^2 - m\vec{r} \cdot \vec{A} - U$

Using E-L, $\boxed{m\dot{\vec{v}} = -\frac{\partial U}{\partial \vec{r}} - m\vec{A} + 2m\left(\vec{v} \times \vec{\Omega}\right) + m\vec{\Omega} \times \left(\vec{r} \times \vec{\Omega}\right) + m\vec{r} \times \dot{\vec{\Omega}}}$

- Namely,

$$m\dot{\vec{v}} = -\frac{\partial U}{\partial \vec{r}} + \vec{F}_{\text{cor}} + \vec{F}_{\text{cent}}$$

$$\vec{F}_{\text{Cor}} = 2m\left(\vec{v} \times \vec{\Omega}\right), \quad \vec{F}_{\text{cent}} = m\vec{\Omega} \times \left(\vec{r} \times \vec{\Omega}\right) = m\left(\vec{\Omega} \times \vec{r}\right) \times \vec{\Omega}$$

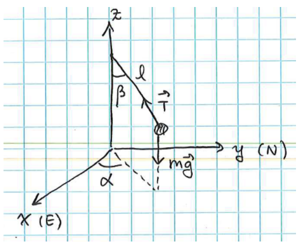
- *ex free fall on earth, centrifugal force* $\vec{F} = \vec{g}_0 + m\Omega^2 R \sin\theta\hat{\rho} \Rightarrow \vec{g}_{\text{eff}} = \vec{g}_0 + \Omega^2 R \sin\theta\hat{\rho}$
- *ex free fall, coriolis force* $\quad \dot{\vec{v}} = \vec{g} + 2\vec{v} \times \vec{\Omega}, \quad \vec{\Omega} = \Omega\sin\theta\hat{y} + \Omega\cos\theta\hat{z}$

In components,

$$\vec{v_x} = 2\Omega\left(v_y\cos\theta - v_z\sin\theta\right)$$
$$\vec{v_y} = -2\Omega v_x\cos\theta$$
$$\vec{v_z} = 2\Omega v_x\sin\theta - g$$

Free fall EOM: $\vec{R} = \int v\,\mathrm{d}r$, consider $\vec{v} = \vec{v_1} + \vec{v_2} = -\vec{g} + 2\vec{v_1} \times \vec{\Omega} + 2\vec{v_2} \times \vec{\Omega}$ where approximately, $\vec{v_2} = 2(\vec{v_0} - gt\hat{z}) \times \vec{\Omega}$. If no initial velocity, integrating velocity in x components gives, $x(t) = \frac{1}{3}g\Omega\left(\frac{2h}{g}\right)^{3/2}\sin\theta$

- *ex foucaults pendulum* EOM



$$\vec{r} = l\sin\beta\cos\alpha\hat{x} + l\sin\beta\sin\alpha\hat{y} + (l - l\cos\beta)\hat{z}$$
$$\vec{T} = -T\sin\beta\cos\alpha\hat{x} - T\sin\beta\sin\alpha\hat{y} + T\cos\beta\hat{z}$$
$$\vec{\Omega} = \Omega\sin\theta\hat{y} + \Omega\cos\theta\hat{z}$$

$$\begin{cases} T = mg \\ m\ddot{x} = T_x + 2m\hat{x}\cdot\left(\dot{\vec{r}} \times \vec{\Omega}\right) = -\frac{mgx}{l} + 2m\Omega\dot{y}\cos\theta \\ m\ddot{y} = -\frac{mgy}{l} - 2m\Omega\dot{x}\cos\theta \end{cases}$$

letting $\omega^2 = \frac{g}{l}, \Omega_z = \Omega\cos\theta$, $\boxed{\eta = x + iy = e^{i\gamma t}}$

$$\ddot{x} + \omega^2 x = 2\Omega_z\dot{y}, \ddot{y} + \omega^2 y = -2\Omega_z\dot{x}$$
$$\gamma = -\Omega_z \pm \sqrt{\omega^2 - \Omega_z^2}$$
$$\eta(t) = ae^{-i\Omega_z t}\cos\omega t$$
$$\Rightarrow \begin{cases} x = a\cos\Omega_z t\cos\omega t \\ y = a\sin\Omega_z t\cos\omega t \end{cases}$$

Hamiltonian Mechanics

$H(q,p,t) = \sum_{j=1}^{n} p_j\dot{q}_j - L(q,\dot{q},t) \quad$ 1D: $H = \frac{p^2}{2m} + U(x)$

- Hamilton's equation $\dot{q}_i = \frac{\partial H}{\partial p_i} \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}$
- *ex particle in polar*

$$L = T - U = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\varphi}^2) - U(r,\varphi) \Rightarrow \quad p_r = \frac{\partial L}{\partial\dot{r}} = m\dot{r}, p_\varphi = \frac{\partial L}{\partial\dot{\varphi}} = mr^2\dot{\varphi}$$

$$H = p_r \dot{r} + p_\varphi \dot{\varphi} - L = \frac{p_r^2}{2m} + \frac{p_\varphi^2}{2mr^2} \Rightarrow \quad \dot{r} = \frac{\partial H}{\partial p_r} = \frac{p_r}{m}, \quad \dot{\varphi} = \frac{\partial H}{\partial p_\varphi} = \frac{p_\varphi}{mr^2}$$

$$\dot{p}_r = -\frac{\partial H}{\partial r} = \frac{p_\varphi^2}{mr^3} - \frac{\partial U}{\partial r}, \quad \dot{p}_\varphi = -\frac{\partial H}{\partial \varphi} = -\frac{\partial U}{\partial \varphi}$$

Phase space

- *ex harmonic oscillator* $H = \frac{p^2}{2m} + \left(\frac{1}{2}\right) m\omega^2 x^2, \quad \omega = \sqrt{\frac{k}{m}}$

$$\left\{ \dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m}, \quad \dot{p} = -\frac{\partial H}{\partial x} = -m\omega^2 x \right\} \Rightarrow \quad \left\{ \dot{q} = \frac{p}{m}, \quad \dot{p} = -m\omega^2 x \right\}$$

$q(t_0 + \delta t) = q(t_0) + \dot{q}\delta t = q_0 + \frac{p}{m}\delta t; \quad p(t_0 + \delta t) = p(t_0) + \dot{p}\delta t = p_0 - m\omega^2 q \delta t$ parametric ellipse in phase space.

Liouville's thm

volume of a region op phase space is conserved under time evolution, when boundary of volume and all pts inside move along their orit for some amount of time.

Poisson bracket

Time evolution of an observable $A(q, p, t)$:

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{\partial A}{\partial t} + \underbrace{\sum_{i=1}^{n} \frac{\partial A}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial H}{\partial q_i}}_{\equiv \{A, H\}}$$

More generally, for $A(q, p, t), \quad B(q, p, t)$

$$\{A, B\} = \sum_i \frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i}$$

notice, $\{A, p_i\} = \frac{\partial A}{\partial q_i}, \{A, q_i\} = -\frac{\partial A}{\partial p_i}$

- When

  $$\frac{\mathrm{d}C}{\mathrm{d}t} = \frac{\partial C}{\partial t} + \{C, H\} = 0$$

  then $C(q, p, t)$ is conserved.

Cononical transformation

consider transformation $q_i \to Q_i(q, t)$ the transformation is canonical iff the transformation leave the form of Hamilton's eq. unchanged.

$$\begin{cases} \dot{q} = \frac{\partial H}{\partial p} \\ \dot{p} = -\frac{\partial H}{\partial q} \end{cases} \Rightarrow \text{cases } \dot{Q} = \frac{\partial K}{\partial P}, \dot{P} = -\frac{\partial K}{\partial Q}$$

where $K(Q, P, t)$ new Hamiltonian.

Exerts from practice problems

constraints, small Oscillations

A particle of massmmoves withoutfriction on the inside wall of an axially symmetric vessel given by $z = b^2(x^2 + y^2)$

- KE in cylindrical coords:

$$T = \frac{1}{2}m\left(\dot{\rho}^2 + \rho^2\dot{\theta}^2 + \dot{z}^2\right), \quad \dot{z} = b\dot{\rho}\rho \Rightarrow$$
$$L = \frac{m}{2}\left[\dot{\rho}^2(1 + b^2\rho^2) + \rho^2\dot{\theta}^2\right] - \frac{mgb}{2}\rho^2$$

  E-L:

$$\ddot{\rho}(1 + b^2\rho^2) + b^2\dot{\rho}^2\rho - \rho\dot{\theta}^2 + gb\rho = 0$$
$$m\rho^2\dot{\theta} = \text{const} \equiv M \quad \text{conserved angular momentum}$$

- energy and angular momentum given z_0, b, g, m

$$E = \frac{m}{2}\left[\dot{\rho}^2(1 + b^2\rho^2) + \rho^2\dot{\theta}^2\right] + \frac{mgb}{2}\rho^2$$

  For a fixed z_0, ρ_0 is the equilibrium position, and $\dot{\rho} = 0$, then

$$E = \frac{m}{2}\rho_0^2\dot{\theta}^2 + mgb\frac{\rho_0^2}{2}$$
$$\dot{\theta}^2 = gb$$
$$\Rightarrow E = 2mgz_0$$

  plugging in $\dot{\theta}$, $\rho = \rho_0$, we have $M = 2mz_0\sqrt{\frac{g}{b}}$

- frequency of small oscillations about equilibrium perturbation: $\rho = \rho_0 + \varepsilon$, neglecting anything with ε^2 , EOM of rho is

$$\ddot{\varepsilon}(1 + b^2\rho_0^2) - \rho\dot{\theta}^2 + gb\rho_0 + gb\varepsilon = 0$$

  want to know $\rho\dot{\theta}^2$, can be found from θ EOM

$$\rho\dot{\theta}^2 = \frac{M^2}{m^2\rho^3} = \frac{M^2}{m^2\rho_0^3}\left(\frac{1}{\left(1 + \frac{\varepsilon}{\rho_0}\right)^3}\right) \approx \frac{M^2}{m^3\rho_0^4}\left(1 - 3\frac{\varepsilon}{\rho_0}\right)$$
$$= b\rho_0 g - 3bg\varepsilon$$

  Plugging in to rho EOM, we have

$$\ddot{\varepsilon}(1 + 2bz_0) + 4gb\varepsilon = 0$$
$$\ddot{\varepsilon} = -\omega^2\varepsilon, \Omega^2 = \frac{4gb}{1 + 2bz_0}$$

Conservation laws

two particles of $\{m_1, q_1, \vec{r}_1\}, \{m_2, q_2, \vec{r}_2$ in capacitor with $\vec{E} = E_0 \hat{z}$, particles interact with $U(r_1, r_2) = \frac{k}{|\vec{r_1} - \vec{r_2}|} e^{-\frac{|\vec{r_1} - \vec{r_2}|}{\lambda}}$. List all conserved quantities and associate each with a specific symmetry of the problem.

- lagrangian $L = \frac{1}{2} m_1 \dot{\vec{r}}_1^2 + \frac{1}{2} m_2 \dot{\vec{r}}_2^2 - U + E_0 (q_1 z_1 + q_2 z_2)$. Setting $\vec{r} = (x, y, z) = \vec{r}_1 - \vec{r}_2, \vec{R} = (X, Y, Z) = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{M}, \mu = \frac{m_1 m_1}{M}$, we can have

$$L = \left[\frac{1}{2} M \dot{\vec{R}}^2 + (q_1 + q_2) E_0 Z\right] + \left[\frac{1}{2} \mu \dot{\vec{r}}^2 - U(r) + \frac{q_1 m_2 - q_2 m_2}{M} E_0 z\right]$$

Observe: momenta $P_x = \frac{\partial L}{\partial \dot{X}}, P_y = \frac{\partial L}{\partial \dot{Y}}$ are conserved. Invariance under time translation gives conserved energy

$$E = \frac{\partial L}{\partial \dot{R}} \dot{R} + \frac{\partial L}{\partial \dot{r}} \dot{r} - L$$

Angular momentum $L_{\text{ttl}} = \vec{r_1} \times \vec{p_1} + \vec{r_2} \times \vec{p_2} = M \vec{R} \times \dot{\vec{R}} + \mu \vec{r} \times \dot{\vec{r}} = \vec{R} \times \vec{P} + \vec{r} \times \vec{p}$. Invariance under rotation about $\hat{z} : R \to R + \varepsilon \hat{z} \times R, \quad r \to \vec{r} + \varepsilon \hat{z} \times \vec{r}$ gives conserved $L_z = \left(\vec{R} \times \vec{P}\right)_z \quad l_z = [\vec{r} \times \vec{p}]_z$.

Normal modes

A system of N particles with masses m_i moves around a circle of radius a, with position angle θ_i. Interaction potential $U = \frac{k}{2} \sum_1^N \left(\theta_{j+1} - \theta_j\right)^2$., with $\theta_{N+1} = \theta_1 + 2\pi$. lagrangian of system is $\frac{a^2}{2} \sum_1^N m_j \dot{\theta}_j^2 - U$

- show Largrangian for particle i, show system in equalibrium when particles are equally spaced.

$$L = \frac{a^2}{2} \sum_1^N m_j \dot{\theta}_j^2 - \frac{k}{2} \sum_1^N \left(\theta_{j+1} - \theta_j\right)^2$$

  E-L for $\theta_i : \quad a^2 m_i \ddot{\theta}_i = k(\theta_{i+1} - \theta_i) - k(\theta_i - \theta_{i-1}) = -k[2\theta_i - (\theta_{i+1} + \theta_{i-1})]$ When eqaully spaced, $\theta_i = \frac{2\pi i}{N}$, thus $\ddot{\theta}_i = 0$ for all particles, thus equalibrium.

- show the system always has a normal mode of osc. with 0 freq.

$$\mathbb{M} \cdot \ddot{\vec{\theta}} = -\mathbb{K} \cdot \vec{\theta}, \quad M_{ij} = a^2 m_i \delta_{ij}, \quad K_{ij} = k\left(2\delta_{i,j} - \delta_{i,j+1} - \delta_{i,j-1}\right)$$

  take anstaz susbsitution $\vec{\theta} \to \vec{z} = \vec{b} e^{i\omega t}$ gives $\omega^2 \mathbb{M} \cdot \vec{b} = \mathbb{K} \cdot \vec{b}$, where $\vec{b}$ is a constant vec. Look for a 0 freq $\omega = 0$, $\mathbb{K} \cdot \vec{b} = 0$ holds, so $b_i = b$. let $b = \Theta(t)$, knowing $\ddot{\Theta} = 0$ recall our subsitution, the time evo of $\theta_{i(t)} = \Theta_0 + \Theta_1 t$ i.e. trajectory is all masses rotating at same rate Θ_1

- find all normal modes when $N = 2, M_1 = km/a^2, m_2 = 2km/a^2$. Using standard normal mode analysis, for $N = 2$,