HW 6, Harry Luo

ex4.6

recall from lec the normal approximation formula, where

$$P(|\hat{p} - p| < \varepsilon) \geq 2\Phi\left(2\varepsilon\sqrt{n}\right) - 1 \tag{1}$$

For this problem, we have $\varepsilon = 0.02, 2\Phi\left(2\varepsilon\sqrt{n}\right) - 1 \geq 0.95$. We can solve for n when $n = n_{\min}$ with the following:

$$\begin{aligned} 2\Phi\left(2\varepsilon\sqrt{n}\right) - 1 &= 0.95 \\ \Phi\left(2 * 0.02\sqrt{n}\right) &= \frac{1.95}{2} \end{aligned} \tag{2}$$

accroding to the table of Phi values, we have

$$\begin{aligned} 0.04\sqrt{n} &= 1.96 \\ \Rightarrow &\boxed{n = 2401} \end{aligned} \tag{3}$$

therefore the smallest size should be 2401

ex4.8

Rolling a biased die can be modeled as a binomial distribution as either "rolling the number 6" or not. We denote an unknown probability of rolling a 6 as p, and denote the number of getting 6 as X. We write $X \sim \mathrm{Bin}(1000000, p)$. We want to find a confidence interval for p with 0.999 confidence. Using Equation 1, we have $n = 1000000$, $P|(\hat{p} - p| < \varepsilon) = 0.99$. We need to solve for ε at the lower bound, where:

$$\begin{aligned} 2\Phi\left(2\varepsilon\sqrt{n}\right) - 1 = 0.999 \\ \Rightarrow \Phi(2 * 1000\varepsilon) = 0.9995 \\ \Rightarrow 2000\varepsilon \approx 3.32 \\ \varepsilon = 0.00166 \end{aligned} \tag{4}$$

Since the number 6 shows up 180000 times when rolling 1000000 times, $\hat{p} = \frac{180000}{1000000} = 0.18$.

$\boxed{\text{Therefore, the confidence interval is } [\hat{p} - \varepsilon, \hat{p} + \varepsilon] = [0.1783, 0.1817]}$

ex4.10

We assume that scoring a goal in a certain game is a rare event for the player, we can approximate the r.v. X corresponding to the number of goals scored by the player as a Poisson distribution

$$P(X = k) = \frac{e^{-\lambda}\lambda^k}{k!} \tag{5}$$

probability of player scoring 0 goals is $P(X = 0) = e^{-\lambda}\frac{\lambda^0}{0!} = e^{-\lambda}$

Thus the probability of scoring at least 1 goal is $1 - e^{-\lambda} = 0.5 \Rightarrow \lambda = \ln(2) \approx 0.693$
We can now calculate the approximation for scroing 3 goals as

$$\boxed{P(X = 3) = \frac{e^{-\lambda}\lambda^k}{k!} = \frac{e^{-0.693}0.693^3}{3 * 2 * 1} = 0.028}$$

ex4.34

Assume that accidents happen rarely and independently. We can model the number of accidents happen in a week with a Poisson distribution. We denote the r.v. X as the number of accidents in a week, and we have $X \sim \mathrm{Poisson}(\lambda)$, where lambda is the average number of accidents in a week, given as $\lambda = 3$ Therefore, the probability of **at most** 2 accidents happening next week can be calculated as

$$P(X=1)+P(X=2)+P(X=0)=(e^{-3})\left(\frac{3^1}{1}+\frac{3^2}{2*1}+\frac{3^3}{3*2*1}\right)=\boxed{0.59744} \tag{6}$$

ex4.46

We can consider the series of trials of "flipping a coin 5 times each day for 30 days" as a binomial distribution, where we either get 5 tails each day or not. We denote the r.v. X as the number of days that we get 5 tails. The probability of having 5 tails in a day is $p=\frac{1}{2^5}=\frac{1}{32}$. Therefore, $X{\sim}\text{Bin}\left(30,\frac{1}{32}\right)$

Since $np(1-p)=\frac{465}{512}$, the normal approximation is not valid.

Poisson approximation is a bettor choice, especially when our $np=15/512$ is small.

We approximate the distribution of X with r.v. $Y{\sim}\text{Poisson}(\lambda)$where $\lambda=E(X)=np=\frac{30}{32}=0.9375$. Thus,

$$P(X=2)\approx P(Y=2)=\frac{e^{-0.9375}0.9375^2}{2}\approx\boxed{0.1721} \tag{7}$$

ex5.2

- (a)

Given the MGF, we can calculate its derivatives as

$$M'(t)=-\frac{4}{3}e^{-4t}+\frac{5}{6},M''(t)=\frac{16}{3}e^{-4t}\frac{25}{6}e^{5t} \tag{8}$$

We can get

$$\begin{aligned} E(X)=M'(0)=\frac{1}{2},E(X^2)=M''(0)=\frac{19}{2} \\ \Rightarrow \text{Var}(X)=E(X^2)-E(X)^2=\frac{37}{4} \end{aligned} \tag{9}$$

- (b)

Given the MGF, we observe that the possible values for r.v. are 0, −4, 5; and the corresponding probabilities are 1/2, 1/3,1/6. Thus the discrete probability mass function is $P(X=0)=\frac{1}{2},P(X=-4)=\frac{1}{3},P(X=5)=\frac{1}{6}$. From which we can calculate We can calculate

$$\begin{aligned} E(X)=-4*\frac{1}{3}+5*\frac{1}{6}=\frac{1}{2};E(X^2)=\frac{1}{3}*16+\frac{1}{6}*25=\frac{19}{2} \\ \text{Var}(X)=E(X^2)-E(X)^2=\frac{37}{4} \end{aligned} \tag{10}$$

As calculated in (a).

ex5.18

- (a)

Given $X{\sim}\text{Geom}(p)$, the probability mass function is $P(X=k)=p(1-p)^{k-1}$, where k=1,2,3,...

$$\begin{aligned} M_X(t)=E(e^{tX})=\sum_{k=1}^{\infty}e^{tk}P(X=k)=\sum_{k=1}^{\infty}e^{tk}p(1-p)^{k-1}=pe^t\sum_{k=1}^{\infty}\left(e^t(1-p)\right)^{k-1} \\ =pe^t\sum_{k=0}^{\infty}\left(e^t(1-p)\right)^k \end{aligned} \tag{11}$$

when $e^t(1-p)<1$, i.e. $t<\ln\left(\frac{1}{1-p}\right)$, the series converges, and

$$\boxed{M_X(t)=\frac{pe^t}{1-e^t(1-p)}} \tag{12}$$

while $t\geq\ln\left(\frac{1}{1-p}\right)$, the series diverges, and

$$M_X(t) = +\infty \tag{13}$$

- (b)

$$\begin{aligned} E(X) = M'_{X(0)} &= \frac{pe^t}{(1-e^t(1-p))^2}]_{t=0} = \frac{1}{p}. \\ E(X^2) = M''_X(0) &= \frac{pe^t}{(1-e^t(1-p))^2}|_{t=0} = \frac{2}{p^2} - \frac{1}{p} \\ \mathrm{Var}(X) = E(X^2) - E(X)^2 &= \frac{1}{p^2} - \frac{1}{p} \end{aligned} \tag{14}$$

ex 5.20

- (a) by def, we know

$$\begin{aligned} M_X(t) = \int_{-\infty}^{\infty} e^{tx} * \frac{1}{2} e^{-|x|} dx &= \frac{1}{2}\int_0^{\infty} e^{(-1-t)x} dx + \frac{1}{2}\int_{-\infty}^{0} e^{(t+1)x}\, \mathrm{d}x \\ &\qquad\qquad\qquad\qquad\qquad \text{x -> -x} \\ &= \frac{1}{2}\int_0^{\infty} e^{(-1-t)x} dx + \frac{1}{2}\int_0^{\infty} e^{-(t+1)x}\, \mathrm{d}x \end{aligned} \tag{15}$$

Noticing that $\int_0^\infty e^{-cx}\, \mathrm{d}x$ converges to $\frac{1}{c}$ iff c>0, we can get

$$\boxed{M_X(t) = \begin{cases} \frac{1}{2}\left(\frac{1}{1-t}\right) + \frac{1}{2}\left(\frac{1}{1+t}\right) = \frac{1}{2(1-t^2)} & \text{, when } -1 < t < 1 \\ \infty \text{ O.W.} \end{cases}} \tag{16}$$

- (b) Taylor expanding $M_X(t)$ at t=0 when $-1 < t < 1$, we have

$$M_X(t) = \frac{1}{2(1-t^2)} = \frac{1}{2} + \frac{t^2}{2} + \frac{t^4}{2} + \frac{t^6}{2} + ... = \sum_{k=0}^{\infty} \frac{1}{2} t^{2k} \tag{17}$$

Therefore,

$$\boxed{\text{odd-numbered moments are all 0, and the 2k-th moment is } \frac{1}{2} t^{2k}}$$