Summary

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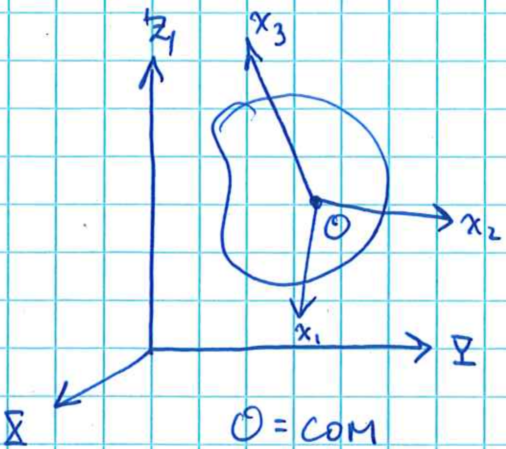

$O = \text{COM}$

- $T = \frac{1}{2} M V^2 + \frac{1}{2} \sum_{ij} I_{ij} \Omega_i \Omega_j$
- $I_{ij} = \sum m \left(r^2 \delta_{ij} - x_i x_j \right)$.
- $L_i = \sum_j I_{ij} \Omega_j$
  ↳ $\vec{L}$ = ang. mom. w.r.t. O.
- If $(x_1\, x_2\, x_3)$ = principal axes:

$$\begin{cases} T_{rot.} = \frac{1}{2} \left(I_1 \Omega_1^2 + I_2 \Omega_2^2 + I_3 \Omega_3^2 \right) \\ L_1 = I_1 \Omega_1, \quad L_2 = I_2 \Omega_2, \quad L_3 = I_3 \Omega_3. \end{cases}$$

Rigid-body EOM

Rigid body = mechanical system w/ 6 DOF.

$$\Rightarrow \text{need 6 EOM} \rightarrow \left. \begin{aligned} \dot{\vec{P}} &= \dots \\ \dot{\vec{L}} &= \dots \end{aligned} \right\} \text{6 EOM}$$

$\vec{P}$ = total mom.

$\vec{L}$ = total ang. mom. about COM.

- First EOM: $\vec{P} = \sum \vec{p}$ (sum over all particles of body).

$$\Rightarrow \dot{\vec{P}} = \sum \dot{\vec{p}} = \sum \vec{f} = \vec{F}$$

$\vec{f}$: individual forces acting on particles

$\vec{F}$: total force.

$$\Rightarrow \boxed{\dot{\vec{P}} = \vec{F}}$$

Note: $\vec{F}$ only includes external forces. Internal forces btwn. particles of body cancel.

Second EOM:

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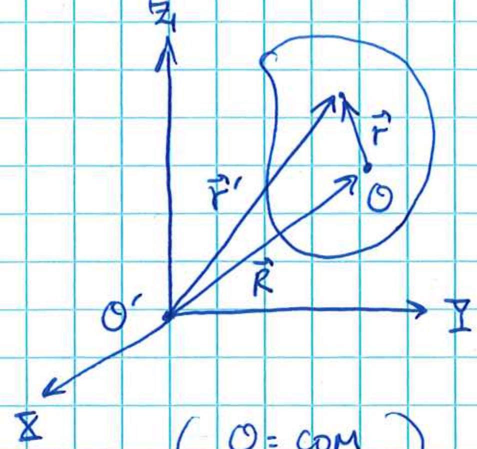

(O = COM)

$\vec{L} = \sum \vec{r} \times \vec{p}, \quad \vec{p} = m\dot{\vec{r}}$

= ang. mom. about COM O.

$\vec{L}' = \sum \vec{r}' \times \vec{p}', \quad \vec{p}' = m\dot{\vec{r}}'$

= ang. mom. about origin O' of fixed axis.

· Recall that we may write:

$$\vec{L}' = \vec{R} \times \vec{P} + \sum \vec{r} \times \vec{p}, \quad \vec{P} = M\dot{\vec{R}}$$

$$= \vec{R} \times \vec{P} + \vec{L}$$

$$\Rightarrow \dot{\vec{L}}' = \vec{R} \times \vec{F} + \dot{\vec{L}} \qquad (*)$$

On the other hand: $\dot{\vec{L}}' = \frac{d}{dt} \sum \vec{r}' \times \vec{p}' = \sum \underbrace{\dot{\vec{r}}' \times \vec{p}'}_{=0 \text{ since } \dot{\vec{r}}' \parallel \vec{p}} + \sum \underbrace{\vec{r}' \times \dot{\vec{p}}'}_{\vec{r}' \times \vec{f}}$

$\cancel{= \frac{d}{dt} \sum (\vec{R} + \vec{r}) \times \vec{p}}$

$$\Rightarrow \dot{\vec{L}}' = \sum \vec{r}' \times \vec{f} = \sum (\vec{R} + \vec{r}) \times \vec{f} = \vec{R} \times \vec{F} + \sum \vec{r} \times \vec{f} \qquad (**)$$

Equating $(*) = (**)$:

$$\dot{\vec{L}} = \sum \vec{r} \times \vec{f} \equiv \vec{K} \quad \text{"torque" about COM.}$$

$$\Rightarrow \boxed{\dot{\vec{L}} = \vec{K}}$$

Note: $\vec{K}$ only includes external forces. Internal torques cancel.

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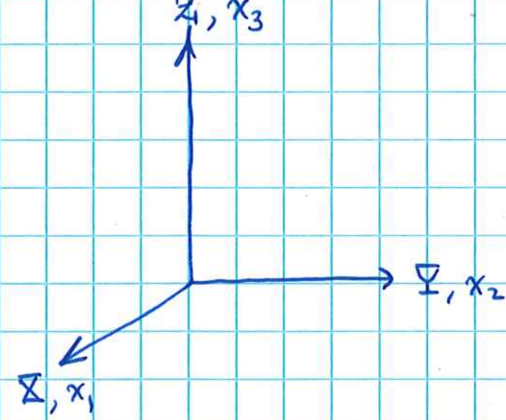
Euler angles

- Euler angles = convenient parametrization of rotn.'l coord.'s in terms of orientation of $(x_1 x_2 x_3)$ axes w.r.t. fixed (XYZ) axes.

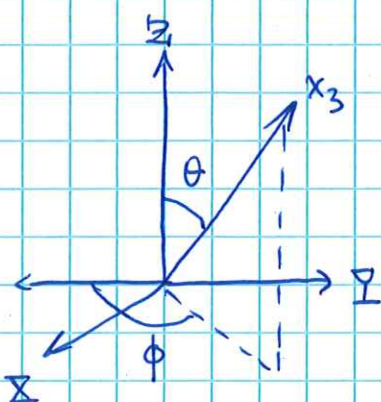
- Orient $x_1 x_2 x_3$ in steps, introducing intermediate frames

(0) In the zeroth step, ~~X~~ $x_1 x_2 x_3$ coincide w/ XYZ



Frame K_0

Next we get the orientation of x_3 axis:



* this definition of ϕ is different from standard azimuthal angle!

→ this can be achieved by composition of rotations:

$$\hat{x}_3 = R(\hat{Z}, \phi)\, R(\hat{X}, \theta)\, \hat{Z}$$

↳ axis-angle parametrization of rot.'n

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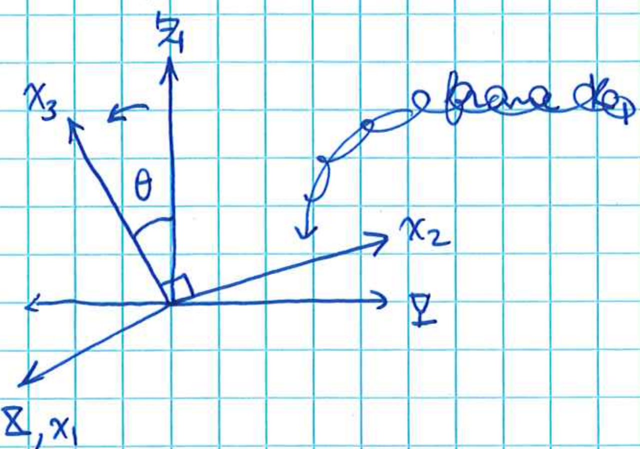
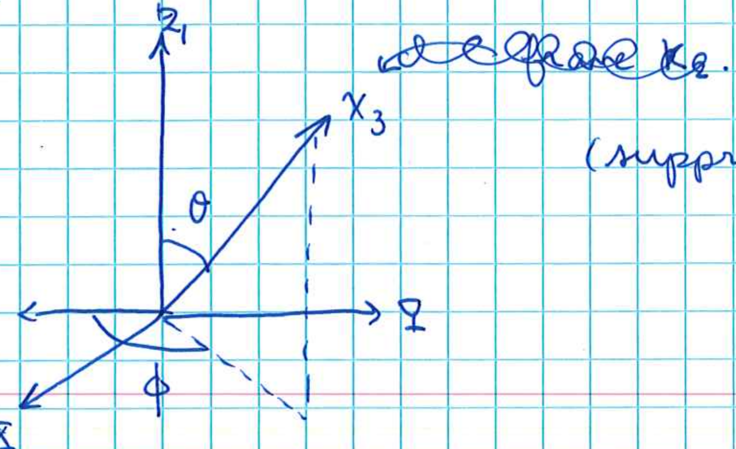
(1) First apply $R(\hat{X}, \theta)$:

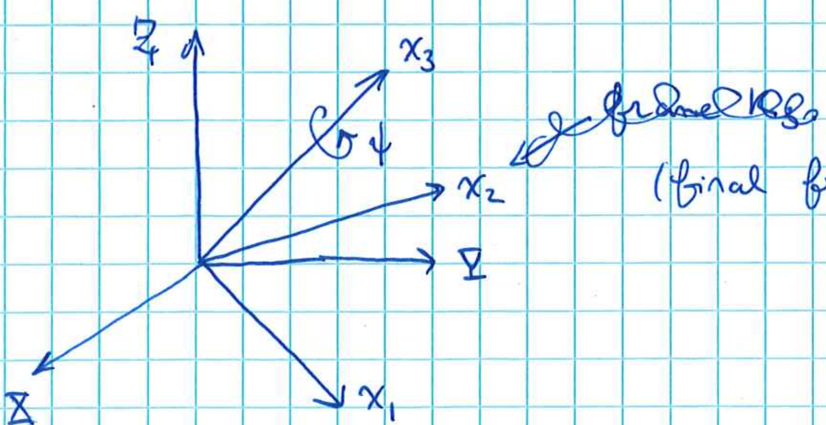


(2) Next apply $R(\hat{Z}, \phi)$:



(suppressing $x_1 x_2$ axes)

(3) Once x_3 is right, we have to properly orient $x_1 x_2$. This is achieved by appropriate rot.'n in $x_1 x_2$-plane; i.e., a rot.'n about x_3 axis $R(\hat{x}_3, \psi)$:



(final frame)

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$\Rightarrow$ $x_1 x_2 x_3$ properly oriented by sequence of rotations:

$$R = R(\hat{x}_3, \psi)\, R(\hat{z}_1, \phi)\, R(\hat{X}_1, \theta).$$

- we write R on LHS b/c composition of rotations is itself a rot.'n.

· For a rigid body in motion the angles become fn.'s of t; i.e. $R = R(t)$.

$$\boxed{\begin{array}{l} (\theta\ \phi\ \psi) = \text{Euler angles} \\ 0 \leq \theta \leq \pi \\ 0 \leq \phi < 2\pi \\ 0 \leq \psi < 2\pi \end{array}}$$

A pt. $\vec{r}$ in the rigid body is similarly specified $\vec{r}(t) = R(t)\vec{r}_0$ where $\vec{r}_0$ = location of that pt. in the reference ~~oo~~ orientation.

· It is sometimes more convenient to express R in terms of rotations purely about fixed space axes. Using properties of rotation matrices it can be shown that:

$$R = R(\hat{z}_1, \phi) R(\hat{X}, \theta) R(\hat{z}_1, \psi).$$

"ZXZ" parametrization. Note: there are (many) other parametrizations in the literature.

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• Now we would like to express the Lagrangian L in terms of the generalized coord.'s $q = (\theta\ \phi\ \psi)$ & generalized velocities $\dot{q} = (\dot\theta\ \dot\phi\ \dot\psi)$.

Start w/ kinetic energy:

$$T = \frac{1}{2}\left(I_1 \Omega_1^2 + I_2 \Omega_2^2 + I_3 \Omega_3^2\right).$$

→ express comp.'s of ang. velocity $(\Omega_1\ \Omega_2\ \Omega_3)$ in terms of Euler angles.

• To do this we'll need two facts about rotations:

(1) for infinitesimal rot.'n $R(\hat{n}, \varepsilon)$ $(\varepsilon \ll 1)$:

~~$R(\hat{n},\varepsilon)\vec{r} =$~~

$$R(\hat{n}, \varepsilon)\,\vec{r} = \vec{r} + \varepsilon\,\hat{n}\times\vec{r}$$

(2) $R(\vec{a}\times\vec{b}) = (R\vec{a})\times(R\vec{b})$.

i.e., rot.'n of cross product of two vec.'s = cross product of the rotated vec.'s.

[note: Only true for "proper rotations" w/ $\det R = +1$]

- Now consider the motion of a pt. $\vec{r}(t) = R(t)\vec{r}_0$ in the rigid body.

Suppose in time dt angle ψ changes $\psi(t+dt) = \psi(t) + \dot{\psi}\, dt$.

$$\Rightarrow \quad \vec{r}(t+dt) = R(t+dt)\vec{r}_0 = R(\hat{z},\phi)\, R(\hat{x},\theta)\, \underbrace{R(\hat{z},\psi+\dot{\psi}\,dt)\,\vec{r}_0}$$

$$= R(\hat{z},\psi)\, R(\hat{z},\dot{\psi}\,dt)\,\vec{r}_0 \qquad \text{(composition of two rotations)}.$$

$$= R(\hat{z},\psi)\left(\vec{r}_0 + \dot{\psi}\,dt\ \hat{z}\times\vec{r}_0\right) \qquad \dot{\psi}\,dt = \text{small angle}.$$

$$\Rightarrow \quad \vec{r}(t+dt) = R(\hat{z},\phi)\, R(\hat{x},\theta)\, R(\hat{z},\psi)\left(\vec{r}_0 + \dot{\psi}\,dt\ \hat{z}\times\vec{r}_0\right) \qquad \text{Prop. (2)}$$

$$= \vec{r}(t) + \dot{\psi}\,dt\left[\ \cancel{R(\hat{z},\phi)R(\hat{x}}\ (R\hat{z})\times\vec{r}\ \right] \qquad R\hat{z} = \hat{x}_3$$

$$= \vec{r}(t) + \dot{\psi}\,dt\ \hat{x}_3\times\vec{r}(t).$$

$$\& \quad \vec{r}(t+dt) = \vec{r}(t) + \vec{v}\,dt$$

$$\Rightarrow \quad \vec{v} = \dot{\psi}\ \hat{x}_3\times\vec{r}$$

And recall the ~~angle~~ angular vel. $\vec{\Omega}$ was defined according to:

$$\vec{v} = \vec{\Omega}\times\vec{r}$$

$$\Rightarrow \quad \vec{\Omega} = \dot{\psi}\,\hat{x}_3$$

- The same exercise can be repeated for θ & ϕ. The result is that the ang. vel. is the vector sum:

$$\boxed{\vec{\Omega} = \dot{\phi}\,\hat{z} + \dot{\theta}\ R(\hat{z},\phi)\hat{x} + \dot{\psi}\,\hat{x}_3}$$

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now compute components. Notation $\begin{cases} R_1 = R(\hat{Z}_1, \phi) \\ R_2 = R(\hat{X}, \theta) \\ R_3 = R(\hat{Z}_1, \psi) \end{cases}$

$$\Omega_1 = \hat{x}_1 \cdot \vec{\Omega} = \dot{\phi}\ \hat{x}_1 \cdot \hat{Z}_1 + \dot{\theta}\ \hat{x}_1 \cdot (R_1 \hat{X}) \qquad (\hat{x}_1 \cdot \hat{x}_3 = 0)$$

$$= \dot{\phi}\ (R_1 R_2 R_3 \hat{X}) \cdot \hat{Z}_1 + \dot{\theta}\ (R_1 R_2 R_3 \hat{X}) \cdot (R_1 \hat{X}) \qquad (R\vec{a}) \cdot \vec{b} = \vec{a} \cdot (R^{-1} \vec{b})$$

$$= \dot{\phi}\ (R_2 R_3 \hat{X}) \cdot \hat{Z}_1 + \dot{\theta}\ (R_2 R_3 \hat{X}) \cdot \hat{X}$$

$$= \dot{\phi}\ (R_3 \hat{X}) \cdot (R_2^{-1} \hat{Z}_1) + \dot{\theta}\ (R_3 \hat{X}) \cdot (R_2^{-1} \hat{X})$$

$$= \dot{\phi}\ (\cos\psi\, \hat{X} + \sin\psi\, \hat{Y}) \cdot (\cos\theta\, \hat{Z}_1 + \sin\theta\, \hat{Y})$$

$$+ \dot{\theta}\ (\cos\psi\, \hat{X} + \sin\psi\, \hat{Y}) \cdot \hat{X}.$$

$$= \dot{\phi} \sin\theta \sin\psi + \dot{\theta} \cos\psi$$

$$\Omega_2 = \hat{x}_2 \cdot \vec{\Omega} = \dot{\phi}\ \hat{x}_2 \cdot \hat{Z}_1 + \dot{\theta}\ \hat{x}_2 \cdot (R_1 \hat{X}).$$

$$= \dot{\phi}\ (R_2 R_3 \hat{Y}) \cdot \hat{Z}_1 + \dot{\theta}\ (R_2 R_3 \hat{Y}) \cdot \hat{X}.$$

$$= \dot{\phi}\ (\cos\psi\, \hat{Y} - \sin\psi\, \hat{X}) \cdot (\cos\theta\, \hat{Z}_1 + \sin\theta\, \hat{Y})$$

$$+ \dot{\theta}\ (\cos\psi\, \hat{Y} - \sin\psi\, \hat{X}) \cdot \hat{X}$$

$$= \dot{\phi} \sin\theta \cos\psi - \dot{\theta} \sin\psi$$

$$\Omega_3 = \hat{x}_3 \cdot \vec{\Omega} = \dot{\phi}\ \hat{x}_3 \cdot \hat{Z}_1 + \dot{\theta}\ \hat{x}_3 \cdot (R_1 \hat{X}) + \dot{\psi}$$

$$= \dot{\phi}\ (R_2 \hat{Z}_1) \cdot \hat{Z}_1 + \dot{\theta} (R_2 \hat{Z}_1) \cdot \hat{X} + \dot{\psi}$$

$$= \dot{\phi} \cos\theta + \dot{\psi}$$

$$\Rightarrow \boxed{\begin{aligned} \Omega_1 &= \dot{\phi} \sin\theta \sin\psi + \dot{\theta} \cos\psi \\ \Omega_2 &= \dot{\phi} \sin\theta \cos\psi - \dot{\theta} \sin\psi \\ \Omega_3 &= \dot{\phi} \cos\theta + \dot{\psi} \end{aligned}} \rightarrow T = T(\theta\phi\psi, \dot{\theta}\dot{\phi}\dot{\psi}) = T(q, \dot{q})$$