Physics 311
Spring 2024
Final Exam Practice

Problem 1: (Constraints, E-L equations, small oscillations) A particle of mass m moves without friction on the inside wall of an axially symmetric vessel given by

$$z = \frac{b}{2}(x^2 + y^2) \tag{1}$$

where b is a constant z is the vertical direction, as shown in the Fig. 1.

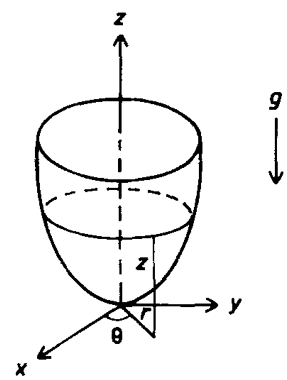

Figure 1: Setup for Problem 1.

a) Identify suitable generalized coordinates and write down the Lagrangian and associated Euler-Lagrange equations.

b) The particle is moving on a circular orbit at a heigh $z = z_0$. Obtain its energy and angular momentum in terms of z_0, b, g, and m.

c) The particle in the horizontal circular orbit is poked downwards slightly. Obtain the frequency of oscillation about the unperturbed orbit for very small oscillation amplitude.

Problem 2: (Conservation laws) Consider the dynamics of two particles with masses m_1 and m_2, electric charges q_1 and q_2, and position coordinates $\mathbf{r}_1$ and $\mathbf{r}_2$ moving in the interior of a parallel plate capacitor (taken to have infinite extent along $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ directions) as shown in the figure. The capacitor is charged such that there is a uniform electric field $\mathbf{E} = E_0\hat{\mathbf{z}}$. The particles additionally interact with each other though a potential $U(\mathbf{r}_1, \mathbf{r}_2)$ given by

$$U(\mathbf{r}_1, \mathbf{r}_2) = \frac{k}{|\mathbf{r}_1 - \mathbf{r}_2|}\, \mathrm{e}^{-|\mathbf{r}_1 - \mathbf{r}_2|/\lambda} \tag{2}$$

where k and λ are constants.

List all the conserved quantities and associate each with a specific symmetry of the problem. You should both name the conserved quantities and give explicit expressions for each.

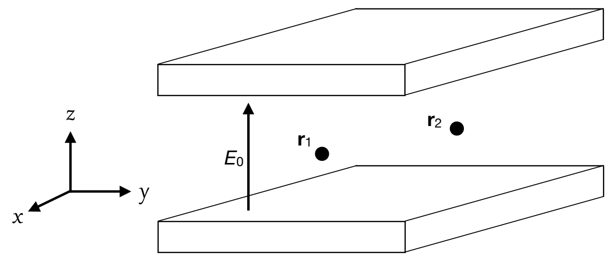

Figure 2: Setup for Problem 2.

Problem 3: (Normal modes) A system of N particles, $i = 1, 2, 3, \ldots, N$, with mass m_i, moves around a circle of radius a. The particles are at angles θ_i on the circle. The interaction potential for the system is

$$U = \frac{k}{2} \sum_{j=1}^{N} (\theta_{j+1} - \theta_j)^2, \tag{3}$$

where k is a positive constant and $\theta_{N+1} = \theta_1 + 2\pi$. The Lagrangian for the system is

$$L = \frac{a^2}{2} \sum_{j=1}^{N} m_j \dot{\theta}_j^2 - U. \tag{4}$$

The situation is depicted in Fig. 3.

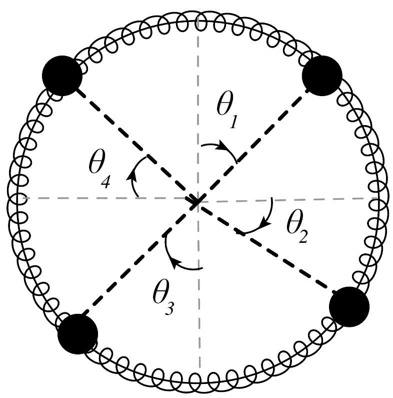

Figure 3: Setup for Problem 3 for the case $N = 4$. Light dashed lines denote equilibrium positions of the masses and angles θ_i denote displacement from equilibrium. The springs connecting them represent the harmonic potential $k(\theta_{i+1} - \theta_i)^2/2$

a) Write down the equation of motion for particle i and show that the system is in equilibrium when the particles are equally spaced around the circle.

b) Show further that the system always has a normal mode of oscillation with zero frequency. What is the form of the motion associated with this?

c) Find all the normal modes when $N = 2, m_1 = km/a^2$ and $m_2 = 2km/a^2$, where m is a constant.

Problem 4: (Non-inertial frames) Consider a pendulum suspended inside a railroad car that is being forced to accelerate with a constant acceleration A.

a) Write down the Lagrangian for the system and the equation of motion for the angle ϕ the pendulum makes with the vertical.

b) Find the equilibrium angle ϕ at which the pendulum can remain fixed (relative to the car) as the car accelerates. Demonstrate that the equilibrium is stable and compute the associated frequency of small oscillations.

Problem 5: (Hamiltonian mechanics) Find the Hamiltonian for a particle in a uniformly rotating frame of reference. Notice the Coriolis force does not appear in the Hamiltonian. Explain why you might have expected this to be the case.

Problem 6: (Hamiltonian mechanics and conservation laws) Consider a one dimensional system with the Hamiltonian

$$H = \frac{p^2}{2} - \frac{1}{2q^2}. \tag{5}$$

Show that the quantity

$$D = \frac{pq}{2} - Ht \tag{6}$$

is a constant of the motion, that is, D is conserved.

Problem 7: (Hamiltonian of a rigid body) The Lagrangian for a heavy symmetric top of mass M, pinned at point O which is a distance ℓ from the centre of mass is

$$L = \frac{I_\perp}{2}\left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta\right) + \frac{I_3}{2}\left(\dot{\psi} + \dot{\phi}\cos\theta\right)^2 - Mg\ell\cos\theta. \tag{7}$$

Obtain the momenta p_θ, p_ψ, p_ψ and the Hamiltonian H. Derive Hamilton's equations for this system. Identify the three conserved quantities and explain their physical meaning.

Problem 8: (Dynamics in a magnetic field) In this problem you consider the motion of a charged particle (charge q) in the presence of external electric and magnetic fields, $\mathbf{E}$ and $\mathbf{B}$. It turns out the Lagrangian in this case is

$$L = \frac{1}{2}mv^2 - q\phi(\mathbf{r}, t) + q\mathbf{A}(\mathbf{r}, t)\cdot\mathbf{v}. \tag{8}$$

Here ϕ and $\mathbf{A}$ are the scalar and vector potentials, related to the electric and magnetic fields by

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}, \quad \mathbf{B} = \nabla \times \mathbf{A}, \tag{9}$$

written in units with $c = 1$. This Lagrangian is different from many of the ones we have discussed previously, as it will lead to a velocity dependent force (it is, however, similar to the Lagrangian in a rotating frame).

a) Write out the Euler-Lagrange equations for the charged particle based on the Lagrangian (8). Express your result in terms of $\mathbf{E}$ and $\mathbf{B}$ and verify you recover the Lorentz force law.

b) Recall the scalar and vector potentials are not unique. The gauge transformation

$$\phi(\mathbf{r}, t) \to \phi(\mathbf{r}, t) - \frac{\partial f(\mathbf{r}, t)}{\partial t}, \quad \mathbf{A}(\mathbf{r}, t) \to \mathbf{A}(\mathbf{r}, t) + \nabla f(\mathbf{r}, t), \tag{10}$$

leaves the fields $\mathbf{E}$ and $\mathbf{B}$ unchanged (as you may verify from Eq. (9)). Thus the scalar and vector potentials contain an "unphysical" component related to this gauge redundancy. You might then be worried that these unphysical fields appear in the Lagrangian. Compute the change in the Lagrangian (8) under such a gauge transformation and explain why the gauge redundancy is not a cause for concern.

c) Calculate the momentum conjugate to $\mathbf{r}$ from the Lagrangian (8). Recall the conjugate momentum is $\mathbf{p} = \partial L/\partial \mathbf{v}$. Use this to compute the Hamiltonian associated to the Lagrangian (8) via the usual Legendre transform.

d) Compute the Poisson brackets between the different components of the "kinetic momentum" $\mathbf{k} = m\mathbf{v}$ (which is different from the canonical momentum $\mathbf{p} = \partial L/\partial \mathbf{v}$).