

Equation of Motion:
Lagrangian, Principle of Least Action, and E-L Equation

- Larangian:
Under the constraint of
1)Space and time are homogenous, 2)time is isotropic, the Larangian for a system is given as

L = T - U(r), where { T = sum\_{a=1}^N 1/2 m\_a q\_a^2 sum of KE
U: potential energy

E-L equation
For a given functional,

S = integral\_{t\_1}^{t\_2} L(q, q-dot, t) dt

we could optimize it using the Euler-Lagrange equation,
d/dt (dL/dq-dot) - dL/dq = 0

where each EL equation and its solution corresponds to a degree of freedom.
Upon applying the El equation to a generalized lagrangian, we reveal Newton's second law

d/dt (d(1/2 m v^2 - U(r))/dv) = d(1/2 m q-dot^2 - U(r))/dr
=> m v-dot = -dU/dq = F-dot(force)

coordinate transformation:
In cartesian coordinates, L = 1/2 m (x-dot^2 + y-dot^2 + z-dot^2) - U
In cylindrical coordinates, L = 1/2 m (r-dot^2 + r^2 theta-dot^2 + z-dot^2) - U
In spherical coordinates, L = 1/2 m (r-dot^2 + r^2 theta-dot^2 + r^2 sin^2(theta) phi-dot^2) - U
Note that when taking partial differentiations, we treat each variable and its derivative as two independent variables. Don't ask why... We are doing physics here

Conservation Laws:
Energy, Momentum, COM, and Angular Momentum

Energy:
Energy is defined as the following, and when the Lagrangian is homogeneity time, the energy is conserved.

E = sum\_i q\_i dL/dq\_i-dot - L

considering L = T - U, we have E = T + U

Total energy is also given as
E = 1/2 mu V^2 + E\_i
where E\_i is internal energy, and mu being the total mass

General momentum:
conservation of general momentum is from the following conservation

dL/dq\_j-dot = 0 => p\_j = dL/dq\_j-dot

where q\_j is a cyclic coordinate, i.e. L is independent of q\_j

Total momentum
total momentum is defined as the following, and considering the homogeneity of space, the momentum is conserved in a closed system.
If the total momentum of a mechanical system in a given frame of reference is 0, then the said system is at rest relative to that frame. For simplicity's sake, we want to chose our frame of reference in which the total momentum is zero.

P = sum\_a dL/dq\_a-dot = sum\_a m\_a v\_a
force is also given by F\_j = dL/dq\_j-dot
sum of all forces in a closed system is 0

Center of Mass
Center of mass is defined so that, the velocity of the system as a whole, V = P/(sum m\_a) is the time derivative of the center of mass. R = sum\_a m\_a r\_a / (sum m\_a).

Conservation of angular momentum
Angular momentum caracterizes the rotation of the system, and considering the isotropy of space, the angular momentum is conserved in a closed system.

L-dot = sum\_a r\_a x p\_a is conserved in a closed system

Angular momentum can be found by differentiating the lagrangian with respect to angular velocity, along the rotation axis z:

L\_z = dL/dphi\_a-dot

Integration of the equations of motion: Connetcting Energy with motion

Motion in 1 dimension
For a system with DOF=1, and with dL/dv = 0 (lagrangian independent of time, i.e. energy conserved), we can write the lagrangian and total energy as

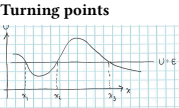
L = 1/2 m x-dot^2 - U(x),

E = 1/2 m x-dot^2 + U(x)

Equation 12 is a differential equation of position and time. Solving this ODE for time gives:

t = sqrt(m/2) integral dx / sqrt(E - U(x)) + C

when given U(x), and by plugging it into Equation 12, we can solve for x(t) by substitution. Tricks on sub: when U(x) is of order 1, use u-sub; when it's of order 2, use trig-sub.



For a given potential function U(x), the turning points are the points where the potential energy is equal to the total energy, i.e. U(x) = E. At turning points, the system is either just about to move, or just about to stop.
Only motion where potential is less or equal to total energy is allowed.
Bounded motion: [x\_1, x\_2]; unbounded motion: x > x\_3

Unbounded Motion:
When there is a potential well, the system could go into periodic motion with potential energy moving back and forth in the well, and position between x\_1, x\_2. We find period by doubling Equation 12:

T(E) = sqrt(2m) integral\_{x\_1(E)}^{x\_2(E)} dx / sqrt(E - U(x))

where we represent x\_1(E), x\_2(E) in terms of E.
When given U(x), we can solve for x\_1(E), x\_2(E), and then plugging in to Equation 14, we can solve for period by integration via subsitution.

Simple Pendulum in polar coord's has the following:

T = 1/2 m l^2 theta-dot^2
U = mgl(1 - cos(theta))

It's period is given by Equation 14. Solving it gives us
T(E) = 4 sqrt(l/g) integral\_0^{pi/2} du / sqrt(1 - k^2 sin^2(u))
where k = sin(theta\_0/2), sin u = 1/k sin(theta\_0/2)

Equation 16 can be simplified by small angle approx into
T(E) = 2pi sqrt(l/g) (1 + (theta\_0^2/16))

Effective DOF=1 system
When the lagrangian is of the form L = f(x) - g(x), we can see it as a system with effective potential U\_eff(x) = g(x), and effective kenetic energy T\_eff(x) = f(x). The effective energy is therefore E = T\_eff + U\_eff.

Two body problem

Problem setup
The two body problem considers two interacting masses with an interacting potential U(r\_1, r\_2) = g(|r\_1 - r\_2|). The lagrangian is given by

L = 1/2 m\_1 r\_1-dot^2 + 1/2 m\_2 r\_2-dot^2 - U(|r\_1 - r\_2|)

Reduced mass mu = (m\_1 m\_2)/(m\_1 + m\_2) = m\_1 m\_2 / M;
Center of mass R = (m\_1 r\_1 + m\_2 r\_2)/(M);
relative positon r = r\_1 - r\_2

Putting the two body system into relative coordinates, and represent masses with reduced mass and COM, we have the following lagrangian:

L = 1/2 M R-dot^2 + 1/2 mu r-dot^2 - U(r)

where the first term involves only the COM motion, and the second term involves only the relative motion.
By choosing our frame with the COM at rest and the total momentum zero, our problem is simplified to an effective one body problem with DOF = 2, given by

L = 1/2 mu r-dot^2 - U(r)

Conservation of Angular Momentum
Angular momentum is defined as L = r x mu r-dot, and is conserved here.
Knowing r-dot . L = 0, the motion is in the plane perpendicular to L. We can use polar coordinates to describe the motion,

L = 1/2 mu (r-dot^2 + r^2 theta-dot^2) - U(r)

Using EL equation on Equation 22, we get
d/dt (dL/dtheta-dot) - dL/dtheta = 0
=> L\_z = mu r^2 theta-dot = constant
(conservation of angular momentum on z-axis)

2 body problem in gravitational field
L = 1/2 m\_1 r\_1-dot^2 + 1/2 m\_2 r\_2-dot^2 - [m\_1 g z\_1 + m\_2 g z\_2 + U(r)]
= [1/2 M R-dot^2 - MgZ +] + [1/2 mu r-dot^2 - U(r)]

where Z is the vertical coordinate of the CM position, Z = (m\_1 z\_1 + m\_2 z\_2)/M

Kepler's second Law
We calculate the differential of area swept by particle in polar coordinates,

dA = 1/2 r^2 dphi
=> dA/dt = 1/2 mu L\_z
L\_z = 2 mu A-dot(constant)

This is the Kepler's second law, which states that the area swept by the radius in a given time is constant.

EOM for two body system
The total energy:

E = T + U = 1/2 mu r-dot^2 + 1/2 mu r^2 phi-dot^2 + U(r)
= 1/2 mu r-dot^2 + U(r) + L\_z^2 / (2 mu r^2) (Notice L\_z = mu r^2 phi-dot)

solving this ODE by integration gives
t(r) = integral sqrt(2/mu) [E - U(r) - L\_z^2 / (2 mu r^2)] + C

Also from L\_z = mu r^2 phi-dot, by integrating with respect to time, we get
phi(t) = L\_z / mu integral dt / r^2(t) + C'

Equation 28 and Equation 26 describe the relative motion of the two body system in terms of constants {E, L\_z, C, C'}

Shape of orbit
Equation 26 skipped a step,

dr/dt = sqrt(2/mu) [E - U(r) - L\_z^2 / (2 mu r^2)]

this equation, combined with our beloved
L\_z = mu r^2 phi-dot => dphi = L\_z / (mu r^2) dt

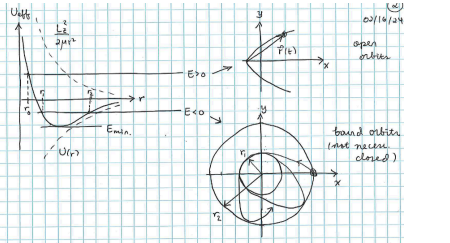
we get the equation of orbit:
dphi = L\_z / (sqrt(2 mu) r^2 sqrt(E - U(r) - L\_z^2 / (2 mu r^2))) dr
=> phi = L\_z / (sqrt(2 mu) integral dr / (r^2 sqrt(E - U(r) - L\_z^2 / (2 mu r^2))) + C

Effective potential and shape of orbit (Only for Attractive Potential)

U\_eff = U(r) + L\_z^2 / (2 mu r^2); E = 1/2 mu r-dot^2 + U\_eff(r)

When r -> inf, U\_eff -> U(r), and when r -> 0, U\_eff -> centrifutal potential L\_z^2 / (2 mu r^2).

by graphing the effective potential, and given constraint of total energy E, we can analyze the shape of the orbit:



- when E > 0, the orbit is unbounded, open orbit, hyperbola.
- when E < 0, the orbit is bounded into a potential well, although not necessarily closed.
- when E = E\_min, the orbit is circular, F = -mu v^2 / r

The Kepler Problem: a special case of the two body problem conditions

U(r) = -alpha/r; U\_eff = -alpha/r + L\_z^2 / (2 mu r^2)

Conic section orbits
We can proof that the orbit is a conic section given by

r(phi) = p / (1 + e cos(phi))
where { p = L\_z^2 / (mu alpha)
e = sqrt(1 + (2 E L\_z^2 / (mu alpha^2)))

Classifications of orbits based on energy of system E
When E > 0, e > 1, the orbit is unbounded, open orbit, hyperbola.

(x-c)^2/a^2 - y^2/b^2 = 1
{ a = p/(e^2-1), b = p/sqrt(e^2-1), c = ae
r\_min = p/(1+e)

- when E = 0, e = 1, the orbit is parabola.
y^2 = p^2 - 2xp,
r\_min = p/2
- when E < 0, e < 1, the orbit is closed, ellipse.
(x+c)^2/a^2 + y^2/b^2 = 1,
{ a = p/(1-e^2), b = p/sqrt(1-e^2), c = ae
r\_min = p/(1+e); r\_max = p/(1-e)

When E = E\_min, f = mu alpha^2 / (2 L\_z^2), e = 0, orbit is circular. r(phi) = p = constant

More Kepler: Period, Kepler's third law

Orbit of each body
recall Equation 19, we can exrees the orbit of each body as such after some algebra:

r\_1 = (m\_2 / (m\_1 + m\_2)) r; r\_2 = -(m\_1 / (m\_1 + m\_2)) r

- when m\_1 = m\_2 => r\_1 = r\_2 = -r/2, COM inside r\_1 cap r\_2
- when m\_1 >> m\_2 => r\_1 = r, r\_2 = 0, m\_1 is at rest, m\_2 orbits m\_1

Period of orbit
L\_z = 2 mu A-dot areal vel. is constant
Integrating A over a period,

A = integral\_0^T A-dot dt = L\_z T / (2 mu)

Since area swept over a period is the area of the ellipse, we have

pi ab = L\_z T / (2 mu), letting: b = sqrt(pa), p = L\_z^2 / (mu alpha)
=> T = (2 pi a^3 / 2) sqrt(mu / alpha)

Conservation of Laplace-Runge-Lenz vector
A = v x L - (alpha r)/r is conserved, and is perpendicular to the orbit plane. We can use it to verify : conic sections, eccentricity, and period.

• conserved quantity:  $\vec{A} \cdot \vec{L} = 0, \frac{A}{\alpha} = \sqrt{1 + \frac{2EL^2}{\mu\alpha^2}}$

### Orbital Transfer

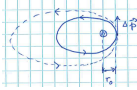
#### Instantaneous Change in velocity

$$\begin{aligned} (E, L_x) &\rightarrow (E', L'_x) \\ \Rightarrow (e, p) &\rightarrow (e', p') \end{aligned} \tag{42}$$

if thrust occur when satellite is at angle  $\varphi_0$ , orbit orientation can change:

$$r(\varphi_0) = \frac{p}{1 + e \cos \varphi_0} = \frac{p'}{1 + e' \cos(\varphi_0 - \delta)} \tag{43}$$

#### Tangential thrust at perigee



at  $\varphi = 0$ , let  $v = v_{\text{init}}$ ,  $v' = v_{\text{right after}}$ ,  $\lambda = v' / v$

$$\begin{aligned} L_x = \mu r_0 v &\Rightarrow L'_x = \mu r_0 v' = \lambda L_x \\ p' &= \lambda^2 p \end{aligned} \tag{44}$$

From Equation 43,

$$\frac{p}{1 + e} = \lambda^2 \frac{p}{1 - e'} \Rightarrow e' = \lambda^2(1 + e) - 1 \tag{45}$$

if  $\lambda > 1$ ,  $e' > e$ , the satellite is in a higher, more elliptical orbit. Unbound if  $\lambda$  big enough

if  $\lambda < 1$ ,  $e' < e$ , the satellite is in a lower orbit.

#### changing between circular orbits

- changing from  $R$  to  $R'$ , two thrusts( $\lambda_1, \lambda_2$ ) are needed. There is also an intermediate orbit

$$\begin{aligned} r(\varphi) &= p' / (1 + e' \cos \varphi), \\ \text{where } p' &= \lambda_1^2 p, e' = \lambda_1^2 - 1 \end{aligned} \tag{46}$$

changed from indermetiade to final,

$$\begin{aligned} r(\varphi = \pi) &= R' = \lambda_2^2 R / (2 - \lambda_1^2) \\ \Rightarrow \lambda_1 &= \sqrt{\frac{2R'}{R + R'}} \end{aligned} \tag{47}$$

final orbit:

$$\begin{aligned} r(\varphi) &= R'; e'' = 0, p'' = R' \\ \Rightarrow p'' &= \lambda_2^2 p' = p' / (1 - e') \\ \Rightarrow \lambda_2 &= \sqrt{\frac{R + R'}{2R'}} \end{aligned} \tag{48}$$

#### Small Oscillations

- Motion near a point of stable equilibrium.

#### DOF= 1 (one dimension)

- For a system of DOF = 1, with potential  $U(q)$ :

- stable equilibrium** at  $U(q)_{\min}$ , upward parabola, where  $F = -\frac{dU}{dq} = 0$   
- restoring force for small displacements  $q - q_0$  is  $F = -\frac{d^2U(q - q_0)}{dq}$

- Unstable equilibrium** at  $U(q)_{\max}$ , downward parabola, where  $F = -\frac{dU}{dq} = 0$  as well.

- Consider small deviation from point of stable equilibrium, we use taylor expansion to show that it is really a small displacement. that is,

$$\begin{aligned} U &\approx U(q_0) + \frac{dU(q_0)}{dq}(q - q_0) + \frac{d^2U(q_0)}{2dq^2}(q - q_0)^2 \\ \text{while } \frac{dU(q_0)}{dq}(q - q_0) &= 0 \end{aligned} \tag{49}$$

letting  $x = q - q_0$ , we have

$$\begin{aligned} \left\{ \begin{aligned} U(x) &= U(q_0) + \left(\frac{1}{2}\right) \frac{d^2U(q_0)}{dq^2} x^2 \\ \text{putting into the form of } U(x) &= U(x_0) + \left(\frac{1}{2}\right) kx^2. \end{aligned} \right. \\ \Rightarrow \quad k = \frac{d^2U(q_0)}{dq^2} > 0 \end{aligned} \tag{50}$$

we get KE, while choosing  $U(q_0) = 0$ :

$$\begin{aligned} T &= \frac{1}{2} a(q)^2 \dot{q}^2 = \frac{1}{2} a(q_0 + x) \dot{x}^2 \approx \frac{1}{2} m \dot{x}^2, \quad m = a(q_0) \\ L &= T - U = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} kx^2 \end{aligned} \tag{51}$$

#### EOM for DOF = 1 small Oscillations

using EL on Equation 51, we can get the EOM for one dimensional small Oscillations:

$$\begin{aligned} m\ddot{x} &= -kx \\ \Rightarrow \ddot{x} + \omega_0^2 x &= 0, \text{ where } \omega_0 = \sqrt{\frac{k}{m}} \text{ freq of osc.} \end{aligned} \tag{52}$$

by magic of ODE, EOM reduces down to:

$$\begin{aligned} x(t) &= C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t) \\ \text{where } C_1, C_2 &\text{ are constants} \end{aligned} \tag{53}$$

by trig magic, this could also be written as

$$\begin{aligned} x(t) &= a \cos(\omega_0 t + \alpha), \\ \text{where } \begin{cases} a = \sqrt{C_1^2 + C_2^2} & \text{amplitude of oscillation} \\ \omega_0 & \text{frequency of oscillation} \\ \tan \alpha = C_2 / C_1 & \text{phase at } t=0 \end{cases} \end{aligned} \tag{54}$$

#### energy for 1D small Oscillation

checking  $\frac{\partial L}{\partial t} = 0 \Rightarrow$  energy-conservation:

$$\begin{aligned} E &= T + U = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} kx^2 \\ &= \frac{1}{2} m a^2 \omega_0^2 [\text{constant}] \end{aligned} \tag{55}$$

#### Damped 1D oscillation, and Complex representation

- when there is damping (friction, resistance, etc)  $F_{\text{ric}} = -\beta \dot{x}$ , the EOM becomes:

$$\begin{aligned} \ddot{x} + 2\gamma \dot{x} + \omega_0^2 x &= 0, \\ \text{where } 2\gamma &= \frac{\beta}{m}, \omega_0 = \sqrt{\frac{k}{m}} \end{aligned} \tag{56}$$

with ansatz  $x(t) = e^{rt}$ ,  $\dot{x} = r e^{rt}$ ,  $\ddot{x} = r^2 e^{rt}$ , the solution to Equation 56 is:

$$\begin{aligned} r^2 + 2\gamma r + \omega_0^2 &= 0, \\ \text{which has solution } r_+, r_- &= -\gamma \pm \sqrt{\gamma^2 - \omega_0^2} \\ \Rightarrow x(t) &= C_1 e^{r_+ t} + C_2 e^{r_- t}, \end{aligned} \tag{57}$$

notice the r subscripts here:  $r_+, r_-$

#### underdamped, overdamped, and critically damped

Recall from your ODE class...

Equation 57 has the following 3 cases, each with different physical interpretation:

$$\begin{aligned} 1. \text{ underdamped: } \quad \gamma < \omega_0 &\Rightarrow 2 \text{ complex roots: } \begin{cases} r_{\pm} = -\gamma \pm i \sqrt{\omega_0^2 - \gamma^2} \\ = -\gamma \pm i\omega \\ = \sqrt{\omega_0^2 - \gamma^2} \end{cases} \end{aligned} \tag{58}$$

The EOM is thus a linear combination of two complex exponentials:

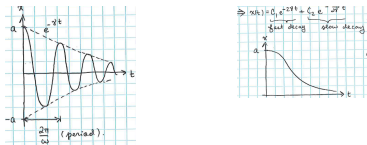
$$\begin{aligned} x(t) &= e^{-\gamma t} (C_1 e^{i\omega t} + C_2 e^{-i\omega t}) \\ &= e^{-\gamma t} (A \cos(\omega t) + B \sin(\omega t)) \\ \text{-- where } \begin{cases} A = C_1 + C_2 \\ B = i(C_1 - C_2) \end{cases} \\ &= ae^{-\gamma t} \cos(\omega t + \alpha) \\ a, \alpha &\text{ are constants} \end{aligned} \tag{59}$$

“The solution is a damped oscillation with frequency $\omega$ , and amplitude exponentially decaying with time.”

$$\begin{aligned} 2. \text{ Overdamped } \quad \gamma > \omega &\Rightarrow x(t) = \\ c_1 e^{-\gamma + \sqrt{\gamma^2 - \omega^2} t} + c_2 e^{-\gamma - \sqrt{\gamma^2 - \omega^2} t} \end{aligned} \tag{60}$$

$$\begin{aligned} \text{when } \gamma \gg \omega_0, &\Rightarrow \begin{cases} \gamma + \sqrt{\gamma^2 - \omega_0^2} \approx 2\gamma \\ \gamma - \sqrt{\gamma^2 - \omega^2} = \frac{\omega^2}{2\gamma} \end{cases} \\ x(t) &= c_1 e^{-2\gamma t} + c_2 e^{-(\omega^2/2\gamma)t} \end{aligned} \tag{61}$$

$$\begin{aligned} 3. \text{ Critically damped } \quad \gamma &= \omega_0 \Rightarrow x(t) = c_1 e^{-\gamma t} + c_2 t e^{-\gamma t} \end{aligned} \tag{62}$$



#### Forced Oscillations

When external force (F) is applied to the system, the largrangian becomes

$$L = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} kx^2 + F(t)x \tag{63}$$

$$\text{EL} \Rightarrow \ddot{x} + \omega_0^2 x = \frac{F(t)}{m}, \text{ where } \omega_0 = \sqrt{\frac{k}{m}}$$

- Example: Simple pendulum with moving pivot

$$\begin{aligned} \begin{cases} \dot{x} = X + l \sin \varphi \\ y = l \cos \varphi \end{cases} &\Rightarrow \begin{cases} \dot{x} = \dot{X} + l \dot{\varphi} \cos \varphi \\ \dot{y} = -l \dot{\varphi} \sin \varphi \end{cases} \\ &\Rightarrow L = T - U \end{aligned} \tag{64}$$

$$L = \frac{1}{2} m l^2 \dot{\varphi}^2 - mgl(1 - \cos \varphi) - ml\dot{X} \sin \varphi$$

$$\text{Expand ab. } \varphi = 0 \Rightarrow L = \frac{1}{2} m l^2 \dot{\varphi}^2 - \frac{1}{2} mgl\varphi^2 - ml\dot{X}\varphi \tag{65}$$

$$\text{EL} \Rightarrow \quad \ddot{\varphi} + \omega_0^2 \varphi = -\frac{\ddot{X}}{l}, \text{ where } \omega_0 = \sqrt{\frac{g}{l}}$$

#### reintroducing damping via external forcing

$$\ddot{x} + 2\gamma \dot{x} + \omega_0^2 x = f(t), f(t) = \frac{F(t)}{m} \tag{66}$$

When damping  $f(t) = f_0 \cos(\Omega t)$ , solution via complex number:

$$\ddot{z} + 2\gamma \dot{z} + \omega_0^2 z = f_0 e^{i\Omega t}$$

$$\text{ansatz } z(t) = z_0 e^{i\Omega t} \Rightarrow z_0 = \frac{f_0}{\omega_0^2 + 2i\gamma\Omega + \Omega_0^2}$$

$$\begin{aligned} \boxed{z_0 = a(\Omega) \cos(\Omega t + \delta(\Omega))} f_0 \end{aligned} \tag{67}$$

is a particular solution,where

$$\begin{cases} a(\Omega) = \frac{1}{\sqrt{(\omega_0^2 - \Omega^2)^2 + (2\gamma\Omega)^2}} \\ \delta(\Omega) = \arctan\left(2\gamma \frac{\Omega}{\omega_0^2 - \Omega^2}\right) \end{cases}$$

We can study the properties of the system by looking at the amplitude and phase of the solution.

- Amplitude:

$$a(\Omega) = \frac{1}{\sqrt{(\omega_0^2 - \Omega^2)^2 + (2\gamma\Omega)^2}} \tag{68}$$

, when  $\gamma \ll \omega_0$ , response strongest and amplitude largest when  $\omega_r = \omega_0$ .



- Phase lag:  $\tan \delta(\Omega) = 2\gamma \frac{\Omega}{\Omega^2 - \omega_0^2}$   
in phase as  $\Omega \rightarrow 0$ , and out of phase as  $\Omega \rightarrow \omega_0$ .
- Genral solution to sinusoidal forcing:

$$\begin{aligned} x(t) &= a(\Omega) f_0 \cos(\Omega t + \delta(\Omega)) + a_0 e^{-\gamma t} \cos(\omega t + \alpha) \\ &\xrightarrow{t \gg \frac{1}{\gamma}} a(\Omega) f_0 \cos(\Omega t + \delta(\Omega)) \end{aligned} \tag{69}$$

Forgets initial condition after time.

- Power obsorbed by oscillation

$$\begin{aligned} p &= F \dot{x} = m f \dot{x} \\ \text{Avg power of oscillation } P_{\text{avg}} &= \frac{1}{T} \int_0^T m f \dot{x} dt = -\frac{1}{2} m f_0 a(\Omega) \Omega \sin \delta(\Omega) \\ \text{simplifies to } P_{\text{avg}}(\Omega) &= \gamma m f_0^2 \Omega^2 a_{(\Omega)}^2 \end{aligned} \tag{70}$$

Absorption around resonance frequency  $\Omega = \omega_0 + \varepsilon$  is maximum:

$$P = \frac{\gamma m f_0^2}{4(\varepsilon^2 + \gamma^2)} \approx \frac{m f_0^2}{4\gamma} \tag{71}$$

#### Oscillations DOF>1

For a system with n DOF:  $q = (q_1, q_2, ..., q_n)$ , PE =  $U(q)$

- Stable equilibrium  $\frac{\partial U(q)}{\partial q_i}|_{q=0}$

#### Example: Oscillation with 2 mass and 3 springs

$$\begin{aligned} L &= \frac{1}{2} m \dot{x}_1 + \frac{1}{2} m \dot{x}_2 - \frac{1}{2} kx_1^2 \\ &\quad - \frac{1}{2} kx_2^2 - \frac{1}{2} k'(x_1 - x_2)^2 \end{aligned}$$

EOM:

$$\begin{aligned} M \cdot \ddot{x} &= -K \ddot{x} \quad , \text{ where } M = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}, \\ \ddot{x} &= \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, K = \begin{pmatrix} k + k' & -k' \\ -k' & k + k' \end{pmatrix} \end{aligned} \tag{72}$$

ansatz:  $\ddot{x} = \text{Re}[\tilde{a} e^{i\omega t}]$  Then the EOM eq becomes solving the eigenvalue problem:

$$\begin{aligned} \det(\omega^2 M - K) &= 0 \\ \Rightarrow \begin{cases} \omega_-^2 = \frac{k}{m} \\ \omega_+^2 = \frac{k+2k'}{m} \end{cases} \begin{cases} \vec{x}_- = a_- \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos(\omega_- t + \delta_-) \\ \vec{x}_+ = a_+ \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos(\omega_+ t + \delta_+) \end{cases} \end{aligned} \tag{73}$$

with constants  $a_-, a_+, \delta_-, \delta_+$ .

#### New Coords

$$\begin{aligned} \begin{cases} Q_1 = \frac{\sqrt{m}}{2}(x_1 + x_2) \\ Q_2 = \frac{\sqrt{m}}{2}(x_1 - x_2) \end{cases} \\ \Rightarrow L = \frac{1}{2} (\dot{Q}_1^2 + \dot{Q}_2^2) - \frac{1}{2} (\omega_-^2 Q_1^2 + \omega_+^2 Q_2^2) \\ \xrightarrow{\text{E.L.}} \dot{Q}_1 = -\omega_- Q_1, \dot{Q}_2 = -\omega_+ Q_2 \end{aligned} \tag{74}$$

Decoupled oscillators with coords  $Q_1, Q_2$ .

#### General Coords

for general coords  $q_i$ , let  $x_i = q_i - q_i^{(0)}$

$$\begin{aligned} U &= \frac{1}{2} \sum_{i,j} k_{ij} x_i x_j, \quad k_{ij} = k_{ji} = \frac{\partial^2 U(q)}{\partial q_i \partial q_j} \text{ symm mat} \\ T &= \frac{1}{2} \sum_{i,j} m_{ij} \dot{x}_i \dot{x}_j, \quad m_{ij} = m_{ji} = a_{ij}(q^{(0)}) \end{aligned} \tag{75}$$

the largrangian, in Matix form:

$$L = \frac{1}{2} \dot{\vec{x}}^T \cdot M \cdot \dot{\vec{x}} - \frac{1}{2} \vec{x}^T \cdot K \vec{x} \xrightarrow{\text{E.L.}} (\omega^2 M - K) \cdot \vec{a} = 0 \tag{76}$$

$\Rightarrow \det(\omega^2 M - K) = 0$  Solving the det for omega gives the normal freq (Eigenvalues)of system  $\omega_n^2$ , plug in Evalute into Equation 76 for eigenvec(normal modes)  $\vec{a}^n$  of system.

- General motion

$$x_i(t) = \sum_n a_n^i \text{Re}[C_n e^{i\omega_n t}] \tag{77}$$

- EXAMPLE: Normal freq is given

$$\begin{aligned} \omega &= \{0, \sqrt{2}\omega_0, \sqrt{3}\omega_0\}. \\ \omega &= \sqrt{2}\omega_0 \Rightarrow a_1 = -a_3 = -a_2 = ae^{i\delta} \Rightarrow \\ \vec{\theta} &= a(1 \quad -1 \quad -1)^T \cos(\sqrt{2}\omega_0 t + \delta) \\ \omega &= \sqrt{3}\omega_0 \Rightarrow a_1 = 0, a_2 = -a_3 = ae^{i\delta} \Rightarrow \\ \vec{\theta} &= a(0 \quad 1 \quad -1)^T \cos(\sqrt{3}\omega_0 t + \delta) \end{aligned} \tag{78}$$

- EXAMPLE: double pendulum

$$\begin{cases} \dot{x}_1 = l_1 \sin \varphi_1 & y_1 = -l_1 \cos \varphi_1 \\ \dot{x}_2 = l_1 \sin \varphi_1 + l_2 \sin \varphi_2 & y_2 = l_1 \cos \varphi_1 + l_2 \cos \varphi_2 \end{cases} \tag{79}$$

$$\begin{aligned} \Rightarrow T &= \frac{1}{2} m_1 l_1 \dot{\varphi}^2 + \frac{1}{2} m_2 (l_1^2 \dot{\varphi}_1^2 + l_2^2 \dot{\varphi}_2^2 \\ &\quad + 2l_1 l_2 \dot{\varphi}_1 \dot{\varphi}_2 \cos(\varphi_1 - \varphi_2)) \end{aligned} \tag{80}$$

$$U = -m_1 g l_1 \cos \varphi_1 - m_2 g (l_1 \cos \varphi_1 + l_2 \cos \varphi_2)$$

using  $\cos \varphi \approx 1 - \frac{\varphi^2}{2}$

$$\begin{aligned} L &= \frac{1}{2} (\dot{\varphi}_1 \quad \dot{\varphi}_2) \begin{pmatrix} (m_1 + m_2) l_1^2 & m_2 l_1 l_2 \\ m_2 l_1 l_2 & m_2 l_2^2 \end{pmatrix} (\varphi_1 \quad \varphi_2) \\ -\frac{1}{2} (\varphi_1 \quad \varphi_2) \begin{pmatrix} (m_1 + m_2) l_1 g & 0 \\ 0 & m_2 g l_2 \end{pmatrix} (\varphi_1 \quad \varphi_2) \\ &= \frac{1}{2} \dot{\vec{\varphi}}^T M \cdot \dot{\vec{\varphi}} - \frac{1}{2} \vec{\varphi}^T K \vec{\varphi} \end{aligned} \tag{81}$$

When  $m_1 = m_2 = m, \quad l_1 = l_2 = l \Rightarrow \quad M = ml^2 \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, K = mgl \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$

$$\det((\omega^2 M - K)) = 0 \Rightarrow \omega^2 = \left(2 \pm \sqrt{2}\omega_0^2\right)$$

$$\begin{pmatrix} a_1^- \\ a_2^- \end{pmatrix} = C_- \cdot \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}, \quad \begin{pmatrix} a_1^+ \\ a_2^+ \end{pmatrix} = C_+ \cdot \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix} \tag{82}$$

#### Normal Coords

$\{x_i\} = \{Q_\alpha\}$ , where  $x_i = \sum_{\alpha=1}^n A_{i\alpha} Q_\alpha \Rightarrow$

$$\begin{aligned} \sum_j (\omega_\alpha^2 m_{ij} - k_{ij} A_{j\alpha}) &= 0 \\ \Rightarrow L = \frac{1}{2} \sum_{\alpha=1}^n (\dot{Q}_\alpha^2 - \omega_\alpha^2 Q_\alpha^2) \xrightarrow{\text{E.L.}} \dot{Q}_\alpha + \omega_\alpha^2 Q_\alpha = 0 \end{aligned}$$

#### Motion of Rigid Body

- Example: rotor

rotation with constraint  $|\vec{r}_i - \vec{r}_j| = r_{ij}$  ,COM coords are useful here

$$\begin{cases} \vec{r} = \vec{r}_1 - \vec{r}_2 \\ \vec{R} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2} \end{cases} \Rightarrow \begin{cases} \vec{r}_1 = \vec{R} + m_2 \vec{r} / M \\ \vec{r}_2 = \vec{R} - m_1 \vec{r} / M \end{cases}$$

$$L = \frac{1}{2} M \dot{R}^2 + \mu \dot{r}^2, \quad \mu = m_1 \frac{m_2}{m_1 + m_2}$$

$$\stackrel{\text{polar}}{\Rightarrow} L = \frac{1}{2} M \dot{R}^2 + \frac{1}{2} \mu a^2 (\dot{\theta}^2 + \dot{\varphi}^2 \sin^2 \theta)$$

### frames of reference

$$(XYZ) \stackrel{R(\theta, \varphi, \psi)}{\Rightarrow} (x_1, x_2, x_3)$$

Velocity of pt in body:  $\vec{v} = \vec{V} + \vec{\Omega} \times \vec{r}$ , where V is Translational vel, Omega is angular vel, r is position vector.

### Largrangian for Rigid Body

$$T = \frac{1}{2} M V^2 + \frac{1}{2} \sum_a m_a [\Omega^2 r_a^2 - (\vec{\Omega} \cdot \vec{r}_a)^2]$$

$$T_{\text{translational}} + T_{\text{rotational}}$$

consider rotation,

$$\Omega^2 = \sum_i \Omega_i^2, \quad \vec{\Omega} \cdot \vec{r}_a = \sum_i \Omega_i x_{a,i}$$

$$\Rightarrow T_{\text{rot}} = \frac{1}{2} \sum_{i,j} \Omega_i \Omega_j I_{ij}, \quad I_{ij} = \sum_a m_a (\delta_{ij} r_a^2 - x_{a,i} x_{a,j})$$

$$\Rightarrow L = \frac{1}{2} M V^2 + \frac{1}{2} \sum_{i,j} I_{i,j} \Omega_i \Omega_j - U$$

### Inertial Tensor

• Discrete

$$I = \begin{pmatrix} \sum m(y^2 + z^2) & -\sum mxy & -\sum mxz \\ -\sum mxy & \sum m(x^2 + z^2) & -\sum myz \\ -\sum mxz & -\sum myz & \sum m(x^2 + y^2) \end{pmatrix}$$

• Continuous

$$I_{ij} = \int \rho(x) (\delta_{ij} r^2 - x_i x_j) dV$$

$$I_{xx} = \int \rho(x) (y^2 + z^2) dV, I_{xy} = I_{yx} = - \int \rho(x) xy dV$$

$$I_{yy} = \int \rho(x) (x^2 + z^2) dV, I_{yz} = I_{zy} = - \int \rho(x) yz dV$$

$$I_{zz} = \int \rho(x) (x^2 + y^2) dV, I_{zx} = I_{xz} = - \int \rho(x) zx dV$$

example:

$$\begin{aligned} I_{zz} &= \int [b^2 \hat{y}^2 + c^2 \hat{x}^2] abc d\hat{x} d\hat{y} d\hat{z} \\ &= abc \int \int \int (b^2 \hat{y}^2 + c^2 \hat{x}^2) d\hat{x} d\hat{y} d\hat{z} \end{aligned}$$

change from into spherical coord :

$$\begin{aligned} I_{zz} &= abc \int \int \int (b^2 r^2 \sin^2 \theta \cos^2 \phi + c^2 r^2 \cos^2 \theta) r^2 \sin \theta dr d\theta d\phi \\ &= abc \int \int \int [b^2 \int_0^{2\pi} \cos^2 \phi d\phi \int_0^\pi \sin^3 \theta d\theta \int_0^R r^4 dr \\ &\quad + c^2 \int_0^{2\pi} d\phi \int_0^\pi \cos^2 \theta \sin \theta d\theta \int_0^R r^4 dr] \\ &= \frac{1}{5} abc R^5 [b^2 + c^2] \end{aligned}$$

• Example: coplanar system principal axis: Z  $\Rightarrow I_{13} = I_{23} = 0$   
 $I_3 = I_1 + I_2$

### Principle axis and principal moments of inertia

In the principal frame:

$$T_{\text{rot}} = \frac{1}{2} (I_1 \Omega_1^2 + I_2 \Omega_2^2 + I_3 \Omega_3^2)$$

- spherical top  $I_1 = I_2 = I_3$
- Symmetric top  $I_1 = I_2 \neq I_3$
- Asymmetric top  $I_1 \neq I_2 \neq I_3$
- EXample:

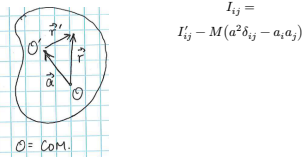
$$\det(I - \lambda \mathbf{1}) = 0 \Rightarrow \lambda \text{ prncp. mom.}$$

$$\vec{v} = \text{eigenvec.} = \text{prncp. axis}$$

• EXample: continuous with axis of symmetry  $\rho(\vec{r}) = \rho = (r, x_3) \Rightarrow I_{ij} = \int \rho(\vec{r}) (\vec{r}^2 \delta_{ij} - x_i x_j) dV$

### Parallel axis theorem

when changing Origin diff. from COM(O),



$$I_{ij} = M(a^2 \delta_{ij} - a_i a_j)$$

For a cube, when finding I at corner, first find I at COM, and

$$I'_{xx} = I_{xx} + M(b^2 + c^2) = \frac{4}{3} M(b^2 + c^2)$$

$$I'_{yy} = I_{yy} + M(a^2 + c^2) = \frac{4}{3} M(a^2 + c^2)$$

$$I'_{zz} = I_{zz} + M(a^2 + b^2) = \frac{4}{3} M(a^2 + b^2)$$

$$\begin{aligned} I_{33} &= - \int dV g(r^2) x_1 x_3 \\ &= - \int dx_3 dx_1 r dr d\varphi g(r, x_3) r \cos \varphi x_3 \\ &= - \int dx_3 r dr g(r, x_3) r x_3 \underbrace{\int_0^{2\pi} d\varphi \cos \varphi}_{=0} \end{aligned}$$

∴  $I_{23} = 0$  by same analysis w/  $\cos \varphi \rightarrow \sin \varphi$ .

$$\Rightarrow I = \begin{pmatrix} I_{11} & I_{12} & 0 \\ I_{21} & I_{22} & 0 \\ 0 & 0 & I_{33} \end{pmatrix} \Rightarrow x_3 = \text{principal axis};$$

i.e., symm. axis = princ. axis

$$\begin{aligned} I_{12} &= - \int dV g(r^2) x_1 x_2 \\ &= - \int dx_3 r dr d\varphi g(r, z) r^2 \cos \varphi \sin \varphi \\ &= - \int dx_3 r dr d\varphi g(r, z) r^2 \underbrace{\int_0^{2\pi} d\varphi \cos \varphi \sin \varphi}_{=0} \end{aligned}$$

$$\Rightarrow I = \begin{pmatrix} I_{11} & 0 & 0 \\ 0 & I_{22} & 0 \\ 0 & 0 & I_{33} \end{pmatrix}, \quad x_1, x_2, x_3 = \text{principal axes.}$$

$$\begin{aligned} I_1 - I_2 &= \int dV g(r^2) (x_1^2 - x_2^2) \\ &= \int dx_3 r dr d\varphi g(r, z) r^2 \underbrace{\int_0^{2\pi} d\varphi (\sin^2 \varphi - \cos^2 \varphi)}_{=0} \end{aligned}$$

$$\Rightarrow I_1 = I_2 = I_{11}$$

$$\Rightarrow I = \begin{pmatrix} I_{11} & 0 & 0 \\ 0 & I_{11} & 0 \\ 0 & 0 & I_{33} \end{pmatrix} \rightarrow \text{any two } \perp \text{ axes in } x_1 x_2 \text{ plane are principal axes.}$$

### Appendix

1. Taylor expansion:

$$f(x)|_0 \approx f(a) + f'(a)(x-a) + f''(a) \frac{(x-a)^2}{2}$$

2. small angle approximation:

$$\sin(\theta) \approx \theta \quad \cos(\theta) \approx 1 - \frac{\theta^2}{2}$$