Summary

02/08/24

- Obtaining EOM in the Lagrangian formalism:

1) Identify the number n of DOF & choose suitable generalized coord.'s $(q_1, q_2, \dots, q_n)$.

2) Express Cartesian coord.'s in terms of q's:

$$\begin{cases} \vec{r}_a = \vec{r}_a(q_1, \dots, q_n) & a = 1, \dots, N = \#\text{ of particles.} \\ \dot{\vec{r}}_a = \sum_{i=1}^{n} \frac{\partial \vec{r}_a}{\partial q_i} \dot{q}_i & \end{cases}$$

Find $T = \sum_a \frac{1}{2} m_a v_a^2 = \sum_{ij} a_{ij}(q) \dot{q}_i \dot{q}_j$

& $U = U(q)$.

→ Write the Lagrangian $L(q, \dot{q}, t) = T - U$.

3) Write down the E-L. eqn.'s:

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i} \qquad i = 1, \dots, n.$$

Conservation Laws.

- A conserved quantity $A(q, \dot{q}, t)$ is a quantity that is constant throughout the motion:

$$\frac{dA}{dt} = 0 \quad \text{if } q = q(t) \text{ is a physical trajectory.}$$

Start w/ cyclic coordinates: 02/02/24

Consider $L(q, \dot{q}, t)$, $q = (q_1, q_2, \ldots, q_n)$

& suppose L does not depend explicitly

on a gen. coord. q_j, that is, $\dfrac{\partial L}{\partial q_j} = 0$.

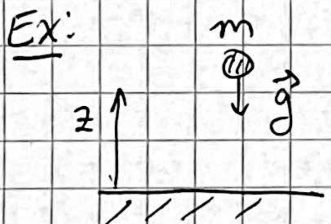
The E-L. eqn. for q_j is then:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} = \frac{\partial L}{\partial q_j} = 0.$$

$$\Rightarrow \frac{\partial L}{\partial \dot{q}_j} = \text{conserved}.$$

q_j = "cyclic" or "ignorable" coord.

Ex:



$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

$$U = mgz.$$

$$\Rightarrow L = T - U = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz.$$

x: $$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x}.$$

$$\Rightarrow \frac{d}{dt}(m\dot{x}) = 0 \quad \Rightarrow \quad m\dot{x} = p_x = \text{cons}.$$

y: $$\frac{d}{dt}\frac{\partial L}{\partial \dot{y}} = \frac{\partial L}{\partial y}$$

$$\Rightarrow \frac{d}{dt}(m\dot{y}) = 0 \quad \Rightarrow \quad m\dot{y} = p_y = \text{cons}.$$

Motivates following notation: 02/02/24

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \quad \text{"generalized momentum"}$$

$$F_i = \frac{\partial L}{\partial q_i} \quad \text{"generalized force"}.$$

$\rightarrow$ E-L. eqn's $\dot{p}_i = F_i$

So, if L = indep. of a gen. coord. q_j, then

$F_j = \frac{\partial L}{\partial q_j} = 0$ & $\dot{p}_j = 0$; i.e., ~~as~~ associated gen.

momentum is conserved. ~~(q_j ≡ "cyclic coordinate")~~

Note: p_i won't necessarily have units of linear

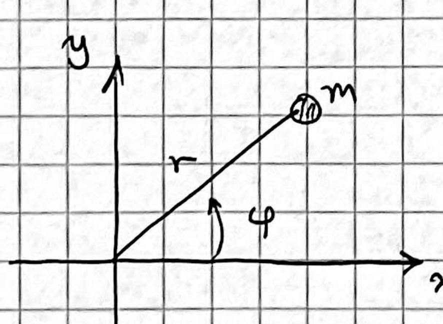
momentum (mass × velocity) & F_i won't

necess. have units of force (Newtons).

$\rightarrow$ this will depend on nature of gen. coord. q_i.

Ex: particle moving in 2D plane under influence

of potential $U(x,y) = U(\sqrt{x^2+y^2})$.



$$T = \frac{1}{2}m(\dot{x}^2+\dot{y}^2) = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\varphi}^2)$$

$$U = U(\sqrt{x^2+y^2}) = U(r)$$

$$\Rightarrow L = T - U$$

$$= \frac{1}{2}m(\dot{r}^2 + r^2\dot{\varphi}^2)$$

$x = r\cos\varphi$
$y = r\sin\varphi$

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$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}$$

$$p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = mr^2\dot{\varphi}$$ = angular momentum

(different units from

linear momentum!)

$$F_\varphi = \frac{\partial L}{\partial \varphi} = 0 \Rightarrow p_\varphi = mr^2\dot{\varphi} = \text{conserved}$$

→ conservation of

angular momentum.

Note: F_φ (were it non-zero) would have

units of torque (length × force).

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· Now we'll explore the connection btwn. conservation laws & symmetry

Roughly: Symmetry = transform.'n of a system that leaves its properties unchanged.

Ex: homogeneity in time $\Rightarrow$ symmetry of physical phenomena under time translations.

That is, if we carry out an experiment now or a week from now, the results will be the same (so long as conditions of experiment are same).

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(1) Homog. of time & energy

$\rightarrow$ in a closed system (no external forces) time is homogeneous.

$$\Rightarrow \quad L(q,\dot{q},t) = L(q,\dot{q}) \quad ; \text{ i.e., } \frac{\partial L}{\partial t} = 0.$$

Now compute:

$$\frac{dL}{dt} = \sum_i \left(\frac{\partial L}{\partial q_i} \dot{q}_i + \frac{\partial L}{\partial \dot{q}_i} \ddot{q}_i \right)$$

use E-L. eqn.'s $\frac{\partial L}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i} \right)$

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$$\Rightarrow \quad \frac{dL}{dt} = \sum_i \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\dot{q}_i + \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i \right]$$

$$= \frac{d}{dt}\sum_i \left(\frac{\partial L}{\partial \dot{q}_i}\dot{q}_i\right)$$

$$\Rightarrow \quad \frac{d}{dt}\Big(\underbrace{\sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L}_{=\text{ const. in time (conserved).}}\Big) = 0.$$

$$\Rightarrow \quad E \equiv \sum_i \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L$$

"energy" E = conserved as a consequence of homog. of time.

<u>Q</u>: Why do I call this the energy?

Consider $L = \sum_a \frac{1}{2} m_a v_a^2 - U(\vec{r}_1, \ldots, \vec{r}_N) = T - U$

Then $E = \sum_a \frac{\partial L}{\partial \vec{v}_a}\cdot\vec{v}_a - (T-U)$

$$= \sum_a m_a \vec{v}_a \cdot \vec{v}_a - (T-U)$$

$$= 2T - (T-U)$$

$$= T + U$$

so, energy E is indeed the sum of kinetic (T) + potential (U) energies.

(The same proof goes through in gen. coord.'s q as well).

Note: we only used $\frac{\partial L}{\partial t} = 0$. 02/02/24

$\Rightarrow$ conservation of energy remains valid for a system in a time-indep. ext field (not necessarily closed).

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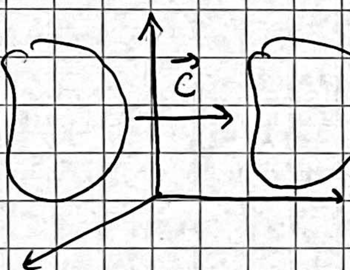
(2) Homog. of space & momentum.

$\rightarrow$ In a closed system, space is homogeneous.



$\vec{r}_a \rightarrow \vec{r}_a + \vec{c}$, uniform translation of all particles by vector $\vec{c}$

· homog. of space means L is invariant:

$$L(\vec{r}_1 + \vec{c}, \ldots, \vec{r}_N + \vec{c}, \vec{v}_1, \ldots, \vec{v}_N) = L(\vec{r}_1, \ldots, \vec{r}_N, \vec{v}_1, \ldots, \vec{v}_N).$$

· now consider a small translation $\vec{c} = \vec{\varepsilon}$:

the change in L is:
$$\delta L = \sum_a \frac{\partial L}{\partial \vec{r}_a} \cdot \vec{\varepsilon}$$
$$= \vec{\varepsilon} \cdot \left(\sum_a \frac{\partial L}{\partial \vec{r}_a} \right)$$
$$= 0$$

$$\Rightarrow \sum_a \frac{\partial L}{\partial \vec{r}_a} = 0 \qquad (\text{since } \vec{\varepsilon} = \text{arbitrary})$$

For physical trajectories satisfying E-L eqn.'s: 02/02/24

$$\frac{\partial L}{\partial \vec{r}_a} = \frac{d}{dt}\frac{\partial L}{\partial \vec{v}_a}$$

$$\Rightarrow 0 = \sum_a \frac{\partial L}{\partial \vec{r}_a} = \sum_a \frac{d}{dt}\left(\frac{\partial L}{\partial \vec{v}_a}\right) = \frac{d}{dt}\left(\sum_a \frac{\partial L}{\partial \vec{v}_a}\right)$$

$$\Rightarrow \sum_a \frac{\partial L}{\partial \vec{v}_a} = \text{const. in time.}$$

So we find $\vec{P} \equiv \sum_a \frac{\partial L}{\partial \vec{v}_a}$ = cons. vector in a closed system. $(\dot{\vec{P}} = 0)$.

"momentum"

that is, homog. of space in a closed system implies total momentum $\vec{P}$ is conserved.

• Furthermore, since $\frac{\partial L}{\partial \vec{v}_a} = m_a \vec{v}_a$:

$$\vec{P} = \sum_a m_a \vec{v}_a = \sum_a \vec{p}_a, \quad \vec{p}_a = m_a \vec{v}_a$$

is momentum of particle "a".

→ momentum is additive.

• Finally, $\frac{\partial L}{\partial \vec{r}_a} = -\frac{\partial U}{\partial \vec{r}_a} \Rightarrow 0 = \sum_a \frac{\partial L}{\partial \vec{r}_a} = \sum_a -\frac{\partial U}{\partial \vec{r}_a} = \sum_a \vec{F}_a$

→ sum of all forces in a closed system is zero.

In particular, for a system of 2 particles, $\vec{F}_1 + \vec{F}_2 = 0$

i.e., forces equal & opposite (Newton's third law).