HW 4, Harry Luo

ex 3.4

Since X has uniform distribution on $[4, 10]$, it has a PDF of $f(x) = \frac{1}{6}$ for x in $[4,10]$ and zero otherwise.

(a)

$$P(x < 6) = P(4 < X < 6) = \frac{6-4}{6} = \frac{1}{3} \tag{1}$$

(b)

$$\begin{aligned} P(|X - 7| > 1) &= P(X < 6) + P(X > 8) \\ &= P(4 < X < 6) + P(8 < X < 10) \\ &= \frac{1}{3} + \frac{1}{3} = \frac{2}{3} \end{aligned} \tag{2}$$

(c)

$$\begin{aligned} P(X < t \mid X < 6) &= \frac{P(X < t, X < 6)}{P(X < 6)} \quad \text{for } 4 \le t \le 6 \\ &= \frac{P(X < t)}{P(X < 6)} \\ &= \frac{P(4 \le X < t)}{P(X < 6)} \\ &= \frac{t - 4}{6 - 4} = \boxed{\frac{t - 4}{2}} \end{aligned} \tag{3}$$

ex 3.5

possible values correspond to jumps in cdf, and the pmf is the size of the jump.

$$\begin{aligned} p_x(1) &= \frac{1}{3} \\ p_x\left(\frac{4}{3}\right) &= \frac{1}{2} - \frac{1}{3} = \frac{1}{6} \\ p_x\left(\frac{3}{2}\right) &= \frac{3}{4} - \frac{1}{2} = \frac{1}{4} \\ p_x\left(\frac{9}{5}\right) &= 1 - \frac{3}{4} = \frac{1}{4} \end{aligned} \tag{4}$$

ex 3.7

For a continuous random variable, the cdf is described as $F(x) = P(X \le x) = \int_{-\infty}^{x} f(t)\,\mathrm{d}t$

(a)

$$\begin{aligned} P(a \le X \le b) &= F(b) - F(a) = 1 \\ &\Rightarrow a \le \sqrt{2}; b \ge \sqrt{3} \\ \text{smallest interval} &= \left[\sqrt{2}, \sqrt{3}\right] \end{aligned} \tag{5}$$

(b) for a continuous R.V., the pmf at any point is zero, so $P(X = 1.6) = 0$

(c)

$$P\left(1 \le X \le \frac{3}{2}\right) = F\left(\frac{3}{2}\right) - F(1)$$

$$= \left(\left(\frac{3}{2}\right)^2 - 2\right) - 0 \tag{6}$$

$$= \boxed{\frac{1}{4}}$$

(d) Noticing the fact that the cdf is continuous except at points $\sqrt{2}, \sqrt{3}$, the pdf could be retrieved by:

$$f(x) = F'(x) = \begin{cases} 2x & \text{if sqrt}(2) <= x < \text{sqrt}(3) \\ 0 & \text{o.w.} \end{cases} \tag{7}$$

ex 3.9

pdf was given by

$$f(x) = \begin{cases} 3e^{-3x} & \text{if x} > 0 \\ 0 & \text{o.w.} \end{cases} \tag{8}$$

(a) for a continous R.V., expectation is calculated by

$$E(X) = \int_{-\infty}^{+\infty} x f(x)\, \mathrm{d}x = \int_{0}^{+\infty} 3xe^{-3x}\, \mathrm{d}x = \frac{1}{3} \tag{9}$$

(b)

$$E(e^{2X}) = \int_{-\infty}^{+\infty} e^{2x} f(x)\, \mathrm{d}x = \int_{0}^{+\infty} 3e^{-x}\, \mathrm{d}x = 3 \tag{10}$$

ex 3.16

$$E(Z) = \int_{-\infty}^{+\infty} x f(x)\, \mathrm{d}x = \int_{1}^{2} \frac{1}{7}x\, \mathrm{d}x + \int_{5}^{7} \frac{3}{7}x\, \mathrm{d}x$$

$$= \frac{75}{14} \tag{11}$$

$$\mathrm{Var}(Z) = E(Z^2) - (E(Z))^2 = \int_{1}^{2} \frac{1}{7}x^2\, \mathrm{d}x + \int_{5}^{7} \frac{3}{7}x^2\, \mathrm{d}x - \left(\frac{75}{14}\right)^2$$

$$= \frac{1633}{588} \tag{12}$$

ex 3.23

(a) Possible values of profit are

$$0 - 1 = -1(\text{won nothing}); 2 - 1 = 1(80 \text{ ppl who won \$2});$$
$$100 - 1 = 99(19 \text{ ppl who won \$100}); 7000 - 1 = 6999 \ (1 \text{ who won \$7000}). \tag{13}$$

We can then represent the pmf as

$$P(X = -1) = \frac{10000 - 100}{10000} = \frac{99}{100}$$

$$P(X = 1) = \frac{80}{10000} = \frac{1}{125}$$

$$P(X = 99) = \frac{19}{10000} \tag{14}$$

$$P(X = 6999) = \frac{1}{10000}$$

(b)

$$P(X \geq 100) = P(X = 6999) = \frac{1}{10000} \tag{15}$$

(c)For X as a discrete R.V., the expectation is

$$E(X) = \sum_k kP(X = k) = -1 \times \frac{99}{100} + \frac{1}{125} + 99 \times \frac{19}{10000} + 6999 \times \frac{1}{10000} = -0.094$$

$$E(X^2) = \sum_k k^2 P(X = k) = (-1)^2 \times \frac{99}{100} + 1^2 \times \frac{1}{125} + 99^2 \times \frac{19}{10000} + 6999^2 \times \frac{1}{10000} = 4918.22 \tag{16}$$

$$\Rightarrow \text{Var}(X) = E(X^2) - (E(X))^2 = 4918.22 - (-0.094)^2 = 4918.22 - 0.008836 = 4918.211$$

ex 3.28

(a)X has possible values of 1,2 3. This is a sampling without replacement trial, where

$$\text{price in the first open box: } P(X = 1) = \frac{3}{5}$$

$$\text{price in the second open box } P(X = 2) = \frac{2}{5} \times \frac{3}{4} = \frac{3}{10} \tag{17}$$

$$\text{price in the third open box } P(X = 3) = \frac{2}{5} \times \frac{1}{4} \times \frac{3}{3} = \frac{1}{10}$$

(b)$E(X) = 1 \cdot \frac{3}{5} + 2 \cdot \frac{3}{10} + 3 \cdot \frac{1}{10} = 3/2$

(c)$E(X^2) = 1^2 \cdot \frac{3}{5} + 2^2 \cdot \frac{3}{10} + 3^2 \cdot \frac{1}{10} = \frac{27}{10}$
$\Rightarrow \text{Var}(X) = E(X^2) - E(X)^2 = 9/20$

(d) We represent the loss or gain of the game as $Y = g(X)$.

$$Y = g(X) = \begin{cases} 100 & \text{if X} = 1 \\ 0 & \text{if X} = 2 \\ -100 & \text{if X} = 3. \end{cases} \tag{18}$$

$$E(Y) = 100 \times \frac{3}{5} + 0 \times \frac{3}{10} - 100 \times \frac{1}{10} = 50$$

ex 3.31

(a)

$$1 = \int_{-\infty}^{+\infty} f(x)\,dx = \int_1^{\infty} cx^{-4}\,dx = \frac{c}{3} \tag{19}$$

$$\Rightarrow c = 3$$

(c)

$$P(0.5 < X < 2) = \int_{0.5}^2 3x^{-4}\,dx = \frac{7}{8} \tag{20}$$

(e)

$$\text{when x} \geq 1: F(x) = P(X \leq x) = \int_1^x 3t^{-4}\,dt = 1 - x^{-3} \tag{21}$$

$$\text{when x} < 1: \quad F(x) = P(X \leq x) = 0$$

(f)

$$E(x) = \int_{-\infty}^{+\infty} xf(x)\,dx = \int_1^{\infty} 3x^{-3}\,dx = \frac{3}{2} \tag{22}$$

(g)

$$E(5X^2 + 3X) = \int_{-\infty}^{\infty} 3x^{-2}\,\mathrm{d}x + 3 \times \frac{3}{2} = \frac{15}{2} + \frac{9}{2} = 12 \tag{23}$$

ex 3.47

$$E(5X^2 + 3X) = \int_{-\infty}^{\infty} 3x^{-2}\,\mathrm{d}x + 3 \times \frac{3}{2} = \frac{15}{2} + \frac{9}{2} = 12 \tag{23}$$

ex 3.47