Summary

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- State of mech. system described by n "generalized coord.'s"

$$q \equiv (q_1, q_2, \ldots, q_n) \qquad n \leq 3N, \quad N = \# \text{ of particles}$$

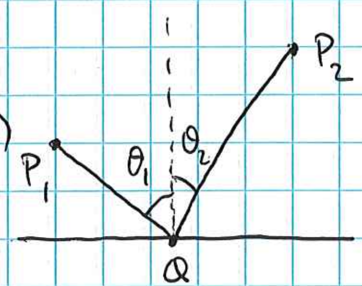
- Relation to Cartesian coord.'s:

$$\vec{r}_i = \vec{r}_i(q_1, \ldots, q_n) \qquad i = 1, \ldots, N.$$

- Goal: Reformulate dynamics in terms of "variational principle".

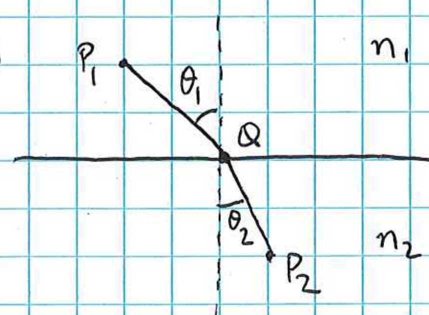
Ex: Variational principle in optics (Fermat's principle of least time).

(a)



$T_{P_1 Q P_2} = \min \Rightarrow \theta_1 = \theta_2$

(b)



$T_{P_1 Q P_2} = \min \Rightarrow n_1 \sin\theta_1 = n_2 \sin\theta_2$ (HW 1)

(c) In (a) & (b), T = fn. of position Q.

More generally, consider light travel through a medium w/ spatially varying index of refraction $n(\vec{r})$.

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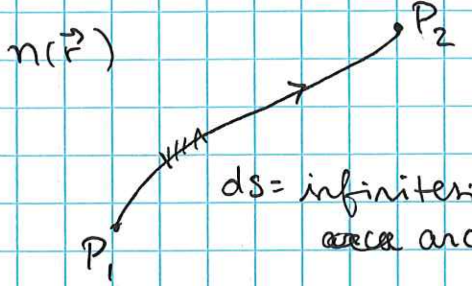


ds = infinitesimal ~~arca~~ arc-length

$$dt = \frac{ds}{v(\vec{r})} = \frac{1}{c}\, n(\vec{r})\, ds$$

$$\Rightarrow T_{P_1P_2} = \int dt = \frac{1}{c}\int ds\, n$$

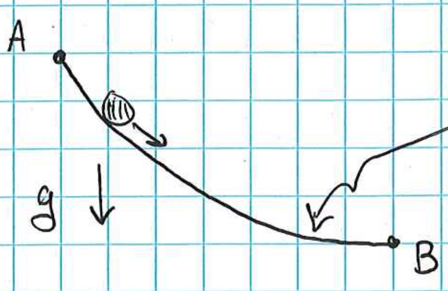
So, $T_{P_1P_2} = T_{P_1P_2}[\vec{r}(s)]$ is a <u>functional</u> of entire path $\vec{r}(s)$

arc-length parametrization.

[In fact, in (a) & (b) T was also a functional of the paths, but we were able to reduce it to a simple function b/c $n(\vec{r})$ was piece-wise constant]

<u>Ex</u>: The "brachistochrone" problem is another example of a variational problem in physics:

<u>Q</u>: Given two pt.'s A & B, find the path btwn. them which a particle sliding frictionlessly under influence of gravity will traverse in shortest time:



what is shape of this curve that minimizes time T?

(Bernoulli, 1696).

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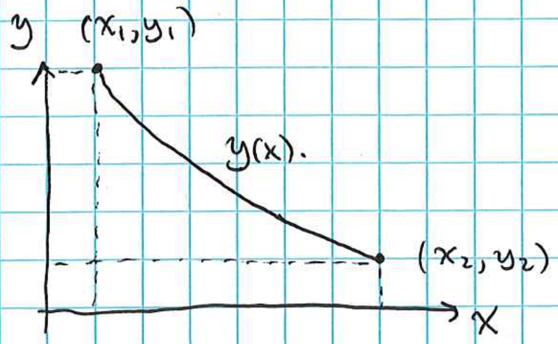

$dT = \frac{ds}{v}$, ds = infinitesimal arc-length

$$\rightarrow T[y(x)] = \int_{x_1}^{x_2} \frac{ds}{v}.$$

$$ds = \sqrt{dx^2 + dy^2} = dx\sqrt{1 + y'(x)^2}$$

· Speed v can be ~~fou~~ found from energy conservation:

$$\frac{1}{2}mv^2 + mgy(x) = mgy_1$$ (assume particle released from rest)

$$\Rightarrow \quad v = \sqrt{2g(y_1 - y)}$$

$$\Rightarrow \quad T[y(x)] = \int_{x_1}^{x_2} dx \sqrt{\frac{1+y'(x)^2}{y_1 - y(x)}}.$$

$$= \int_{x_1}^{x_2} dx \; L(y, y', x), \qquad L \equiv \sqrt{\frac{1+y'^2}{y_1 - y}}$$

So, the time T is a <u>functional</u> of path $y(x)$.

Note that integrand is in terms of a fn. L that depends on both y & y'.

There are many examples of such variational problems in physics. You'll consider a few more examples in HW 1.

<u>Note</u>: We have not yet ~~lea~~ learned how to <u>solve</u> such a variational problem, that is, how to minimize such functionals.

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Back to mechanics:

$$S[q(t)] = \int_{t_1}^{t_2} dt \; L(q, \dot{q}, t) \qquad \text{action functional}$$

↑ Lagrangian.

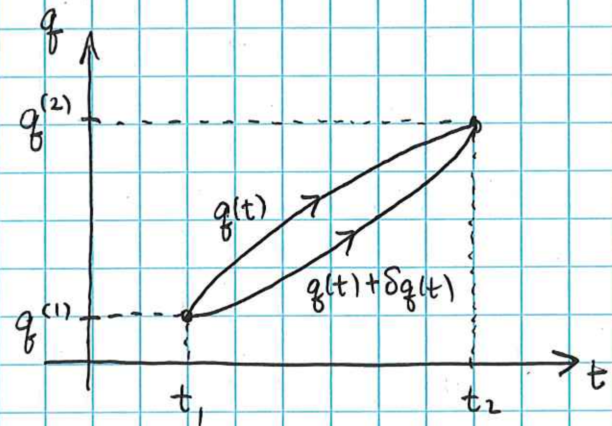
Variational principle: S is minimized* for physical paths $q(t)$.

→ "Principle of least action" / "Hamilton's principle".

· allowed paths are smooth paths s.t.

$q(t_1) = q^{(1)}$

$q(t_2) = q^{(2)}$



(1 DOF).

(c.f. fixed end pt.'s in Fermat's principle).

Q: Given L, how do we minimize S?

→ "Calculus of variations" (this is math, not physics).

Recall basic calc: 01/26/24

function $f(x)$.

- min. (or max) x_0 determined by $\frac{df}{dx}(x_0)=0$.

- Equivalently: Let $\delta x = x - x_0$ & expand about x_0:

$$f(x) = f(x_0) + \cancel{\frac{df}{dx}(x_0)}^{=0}\,\delta x + \frac{1}{2}\frac{d^2f}{dx^2}(x_0)\,\delta x^2 + \ldots$$

$$\Rightarrow \quad \delta f = f(x) - f(x_0) = \frac{1}{2}\frac{d^2f}{dx}(x_0)\,\delta x^2 + \ldots$$

So, small variations δx about min./max. give rise to second-order changes in the fn. $\delta f \sim \delta x^2$

or, $\delta f = 0$ to first-order in δx.

→ f is "stationary" w.r.t. small variations

x_0 is a "stationary pt."

Transfer this over to functionals:

Consider paths $q(t)$ & $q(t)+\delta q(t)$ s.t. $\delta q(t_1) = \delta q(t_2) = 0$

Let $\delta S = S[q(t)+\delta q(t)] - S[q(t)]$

→ physical paths ~~are~~ $q(t)$ are those s.t. $\delta S = 0$ to first-order in δq.

i.e., physical paths are those for which S is stationary (not necessarily a min.)

Consider 1 DOF for simplicity:

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$$\delta S = S[q+\delta q] - S[q]$$

$$= \int_{t_1}^{t_2} dt \left[L(q+\delta q, \dot{q}+\delta\dot{q}, t) - L(q,\dot{q},t) \right], \quad \delta\dot{q} \equiv \frac{d}{dt}\delta q$$

L is ordinary fn. $\Rightarrow$ series expansion

$$= \int_{t_1}^{t_2} dt \left(\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta\dot{q} \right)$$

integration by parts

$$= \int_{t_1}^{t_2} dt \left[\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) \right]\delta q + \underbrace{\left.\frac{\partial L}{\partial \dot{q}}\delta q\right|_{t_1}^{t_2}}_{=0 \text{ since } \delta q(t_1)=\delta q(t_2)=0.}$$

now, $\delta S = 0$ for arbitrary variations δq

$$\Rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q}$$

for more than 1 DOF:

$$\boxed{\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i}} \qquad i=1,\dots,n$$

"Euler-Lagrange eqn.'s".

Remarks: (1) E-L eqn.'s are n second-order diff eq.'s for n fn.'s $q_i(t)$ $\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} \rightarrow \ddot{q}_i \right)$

$\Rightarrow$ motion uniquely ~~specified~~ determined after specifying $2n$ constants $\rightarrow \begin{cases} n \text{ initial positions} \\ n \text{ initial velocities} \end{cases}$

(2) L is not unique.

$L(q,\dot{q},t)$ & $L'(q,\dot{q},t) = L(q,\dot{q},t) + \frac{d}{dt}f(q,t) \rightarrow$ same EOM (HW 1)