Physics 311
Spring 2024
Final Exam Practice

Problem 1: (Constraints, E-L equations, small oscillations) A particle of mass m moves without friction on the inside wall of an axially symmetric vessel given by

$$z = \frac{b}{2}(x^2 + y^2) \tag{1}$$

where b is a constant z is the vertical direction, as shown in the Fig. 1.

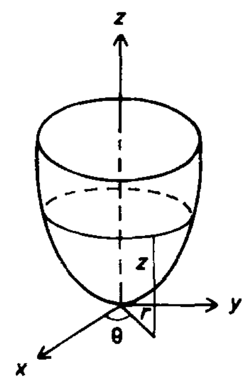

Figure 1: Setup for Problem 1.

a) Identify suitable generalized coordinates and write down the Lagrangian and associated Euler-Lagrange equations.

Solution: The symmetry of the problem suggests the use of cylindrical coordinates (ρ, θ, z). In these coordinates the kinetic energy is

$$T = \frac{m}{2}\mathbf{v}^2 = \frac{m}{2}(\dot{\rho}^2 + \rho^2\dot{\theta}^2 + \dot{z}^2). \tag{2}$$

In cylindrical coordinates the constraint is $z = b\rho^2/2$, so that $\dot{z} = b\dot{\rho}\rho$. Plugging this in we obtain the Lagrangian

$$L = \frac{m}{2}[\dot{\rho}^2(1 + b^2\rho^2) + \rho^2\dot{\theta}^2] - \frac{mgb}{2}\rho^2. \tag{3}$$

Equations of motion are

$$\rho: \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{\rho}} = \frac{\partial L}{\partial \rho} \quad \rightarrow \quad \ddot{\rho}(1 + b^2\rho^2) + b^2\dot{\rho}^2\rho - \rho\dot{\theta}^2 + gb\rho = 0. \tag{4}$$

$$\theta: \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} = \frac{\partial L}{\partial \phi} \quad \rightarrow \quad \frac{\partial L}{\partial \dot{\phi}} = m\rho^2\dot{\phi} = \text{const} \equiv M \tag{5}$$

The quantity M is the conserved angular momentum of the system.

b) The particle is moving on a circular orbit at a heigh $z = z_0$. Obtain its energy and angular momentum in terms of z_0, b, g, and m.

Solution: The energy is given by

$$E = \frac{\partial L}{\partial \dot{\rho}}\dot{\rho} + \frac{\partial L}{\partial \dot{\theta}}\dot{\theta} - L \tag{6}$$

$$= \frac{m}{2}[\dot{\rho}^2(1 + b^2\rho^2) + \rho^2\dot{\theta}^2] + \frac{mgb}{2}\rho^2. \tag{7}$$

The fixed height $z = z_0$ of the particle implies a fixed radius ρ_0 determined from $z_0 = b\rho^2/2$. Since the radius is fixed we can set all time derivatives of ρ in (7) and set $\rho = \rho_0$. This gives

$$E = \frac{m}{2}\rho_0^2\dot{\theta}^2 + \frac{mgb}{2}\rho_0^2 \tag{8}$$

To find $\dot{\theta}$ we look at the ρ EOM (4). Setting all time derivatives of ρ to zero and setting $\rho = \rho_0$, (4) becomes

$$\dot{\theta}^2 = gb. \tag{9}$$

Plugging this in gives

$$\boxed{E = 2mgz_0.} \tag{10}$$

To obtain M we also simply plug in for $\dot{\theta}$ and set $\rho = \rho_0$:

$$\boxed{M = 2mz_0\sqrt{\frac{g}{b}}.} \tag{11}$$

c) The particle in the horizontal circular orbit is poked downwards slightly. Obtain the frequency of oscillation about the unperturbed orbit for very small oscillation amplitude.

Solution: The perturbation corresponds to setting $\rho = \rho_0 + \epsilon$, with $\epsilon/\rho_0 \ll 1$. Neglecting terms of order ϵ^2, the ρ EOM becomes

$$\ddot{\epsilon}(1 + b^2\rho_0^2) - \rho\dot{\theta}^2 + gb\rho_0 + gb\epsilon = 0. \tag{12}$$

The angular momentum of the system remains unchanged, from which we can find $\dot{\theta}$:

$$\rho\dot{\theta}^2 = \frac{M^2}{m^2\rho^3} \tag{13}$$

$$= \frac{M^2}{m^2\rho_0^3}\frac{1}{(1 + \epsilon/\rho_0)^3} \tag{14}$$

$$\approx \frac{M^2}{m^3\rho_0^4}(1 - 3\epsilon/\rho_0) \tag{15}$$

$$= \frac{M^2}{m^3\rho_0^4}(1 - 3\epsilon/\rho_0) \tag{16}$$

$$= b\rho_0 g - 3bg\epsilon. \tag{17}$$

To obtain the last line we have plugged in the explicit value of M in terms of g, b, and z_0. Plugging this into the ϵ EOM above, we find

$$\ddot{\epsilon}(1+2bz_0)+4gb\epsilon=0, \tag{18}$$

which can be rearranged as

$$\boxed{\ddot{\epsilon}=-\omega^2\epsilon, \quad \omega^2=\frac{4gb}{1+2bz_0}.} \tag{19}$$

Problem 2: (Conservation laws) Consider the dynamics of two particles with masses m_1 and m_2, electric charges q_1 and q_2, and position coordinates $\mathbf{r}_1$ and $\mathbf{r}_2$ moving in the interior of a parallel plate capacitor (taken to have infinite extent along $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ directions) as shown in the figure. The capacitor is charged such that there is a uniform electric field $\mathbf{E}=E_0\hat{\mathbf{z}}$. The particles additionally interact with each other though a potential $U(\mathbf{r}_1,\mathbf{r}_2)$ given by

$$U(\mathbf{r}_1,\mathbf{r}_2)=\frac{k}{|\mathbf{r}_1-\mathbf{r}_2|}\ \mathrm{e}^{-|\mathbf{r}_1-\mathbf{r}_2|/\lambda} \tag{20}$$

where k and λ are constants.

List all the conserved quantities and associate each with a specific symmetry of the problem. You should both name the conserved quantities and give explicit expressions for each.

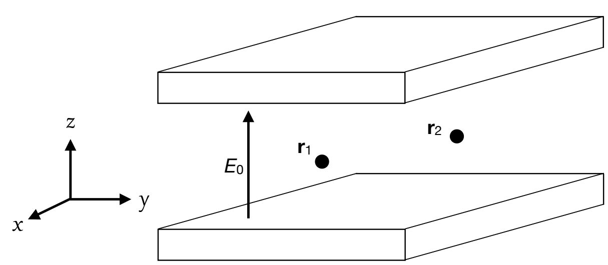

Figure 2: Setup for Problem 2.

Solution: The Lagrangian for the system is

$$L=\frac{1}{2}m_1\dot{\mathbf{r}}_1^2+\frac{1}{2}m_2\dot{\mathbf{r}}_2^2-U(\mathbf{r}_1,\mathbf{r}_2)+E_0(q_1z_1+q_2z_2) \tag{21}$$

Defining $\mathbf{r}=(x,y,z)=\mathbf{r}_1-\mathbf{r}_2$ and $\mathbf{R}=(X,Y,Z)=(m_1\mathbf{r}_1+m_2\mathbf{r}_2)/M$ where $M=m_1+m_2$, we see the interaction $U(\mathbf{r}_1,\mathbf{r}_2)=U(r)$, and

$$L=\left[\frac{1}{2}M\dot{\mathbf{R}}^2+(q_1+q_2)E_0Z\right]+\left[\frac{1}{2}\mu\dot{\mathbf{r}}^2-U(r)+\left(\frac{q_1m_2-q_2m_1}{M}\right)E_0z\right] \tag{22}$$

where $\mu=m_1m_2/M$ is the reduced mass. Unlike the problem of two particles interacting in a gravitational field, we see the potential due to the electric field cannot be written in terms of the center of mass $\mathbf{R}$ alone (except under special case $q_1m_2-q_2m_1=0$).

In these coordinates, $L(\mathbf{r}, \dot{\mathbf{r}}, \mathbf{R}, \dot{\mathbf{R}}) = L(r, z, \dot{r}, Z, \dot{R})$ such that L is invariant under spatial translations $X \to X + \epsilon$ and $Y \to Y + \epsilon$. Momenta $P_x = \partial L/\partial \dot{X}$ and $P_y = \partial L/\partial \dot{Y}$ are conserved.

Invariance under time translation $t \to t + \tau$ gives a conserved energy $E = (\partial L/\partial \dot{\mathbf{R}})\dot{\mathbf{R}} + (\partial L/\partial \dot{\mathbf{r}})\dot{\mathbf{r}} - L$.

The total angular momentum $\mathbf{L}_{tot} = \mathbf{r}_1 \times \mathbf{p}_1 + \mathbf{r}_2 \times \mathbf{p}_2 = M\mathbf{R} \times \dot{\mathbf{R}} + \mu \mathbf{r} \times \dot{\mathbf{r}} = \mathbf{R} \times \mathbf{P} + \mathbf{r} \times \mathbf{p}$. Invariance under rotations about $\hat{\mathbf{z}}$: $\mathbf{R} \to \mathbf{R} + \epsilon \hat{\mathbf{z}} \times \mathbf{R}$ and $\mathbf{r} \to \mathbf{r} + \epsilon \hat{\mathbf{z}} \times \mathbf{r}$ give conserved z-components of c.o.m angular momentum $L_z = [\mathbf{R} \times \mathbf{P}]_z$ and relative angular momentum $l_z = [\mathbf{r} \times \mathbf{p}]_z$.

Problem 3: (Normal modes) A system of N particles, $i = 1, 2, 3, \ldots, N$, with mass m_i, moves around a circle of radius a. The particles are at angles θ_i on the circle. The interaction potential for the system is

$$U = \frac{k}{2} \sum_{j=1}^{N} (\theta_{j+1} - \theta_j)^2, \tag{23}$$

where k is a positive constant and $\theta_{N+1} = \theta_1 + 2\pi$. The Lagrangian for the system is

$$L = \frac{a^2}{2} \sum_{j=1}^{N} m_j \dot{\theta}_j^2 - U. \tag{24}$$

The situation is depicted in Fig. 3.

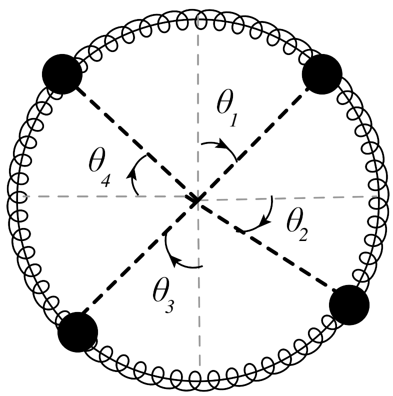

Figure 3: Setup for Problem 3 for the case $N = 4$. Light dashed lines denote equilibrium positions of the masses and angles θ_i denote displacement from equilibrium. The springs connecting them represent the harmonic potential $k(\theta_{i+1} - \theta_i)^2/2$

a) Write down the equation of motion for particle i and show that the system is in equilibrium when the particles are equally spaced around the circle.

Solution: The Lagrangian is

$$L = \frac{a^2}{2} \sum_{j=i}^{N} m_j \dot{\theta_j}^2 - \frac{k}{2} \sum_{j=1}^{N} (\theta_{j+1} - \theta_j)^2. \tag{25}$$

The set up is illustrated in Fig. 3 for the case $N = 4$. Equation of motion for θ_i is

$$a^2 m_i \ddot{\theta}_i = k(\theta_{i+1} - \theta_i) - k(\theta_i - \theta_{i-1}) = -k[2\theta_i - (\theta_{i+1} + \theta_{i-1})]. \tag{26}$$

When all particles are equally spaced around the circle the θ_i we have $\theta_i = \frac{2\pi i}{N}$. From (26) this implies $\ddot{\theta}_i = 0$ for all i and we see the system will be in equilibrium.

b) Show further that the system always has a normal mode of oscillation with zero frequency. What is the form of the motion associated with this?

Solution: To solve the equations of motion (26) we organize things into a vector equation

$$\mathbf{M} \cdot \ddot{\boldsymbol{\theta}} = -\mathbf{K} \cdot \boldsymbol{\theta}, \qquad M_{ij} = a^2 m_i \delta_{ij}, \quad K_{ij} = k(2\delta_{i,j} - \delta_{i,j+1} - \delta_{i,j-1}), \tag{27}$$

and $\boldsymbol{\theta}$ is a vector of the θ_i. We make the replacement $\boldsymbol{\theta} \to \mathbf{z}$ where $\mathbf{z}$ is a complex vector and then make the usual ansatz $\mathbf{z} = \mathbf{b}e^{i\omega t}$ for constant vector $\mathbf{b}$. This yields

$$\omega^2 \mathbf{M} \cdot \mathbf{b} = \mathbf{K} \cdot \mathbf{b}. \tag{28}$$

Let us look for a zero frequency solution $\omega = 0$. From the form of matrix $\mathbf{K}$ in (27) this will be consistent if all entries of vector $\mathbf{b}$ are equal, $b_i = b$ for all i. Furthermore b can be taken to be a function of time $b = \Theta(t)$ and the equation (28) remains satisfied. To find the form of $\Theta(t)$ we can go back to the original system of equations in (26). We find

$$\ddot{\Theta} = 0 \quad \Rightarrow \Theta(t) = \Theta_0 + \Theta_1 t. \tag{29}$$

Thus the trajectory of each θ_i is $\theta_i(t) = \Theta_0 + \Theta_1 t$. This corresponds to all masses rotating together with angular speed Θ_1, starting from equal initial displacements Θ_0.

c) Find all the normal modes when $N = 2, m_1 = km/a^2$ and $m_2 = 2km/a^2$, where m is a constant.

Solution: Now consider the case $N = 2$. We do this two ways. First we do a standard normal mode analysis and then we solve the problem using the techniques you learned when solving the two body problem with central potential.

For $N = 2$ the matrix equation (28) becomes

$$\begin{pmatrix} a^2\omega^2 m_1 - 2k & 2k \\ 2k & a^2\omega^2 m_2 - 2k \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = 0 \tag{30}$$

For this to have a solution with nontrivial b_1, b_2 the matrix must have zero determinant. This condition yields the equation

$$a^4\omega^4 m_1 m_2 - 2ka^2\omega^2(m_1 + m_2) = 0 \qquad \Rightarrow \omega^2 = 0, \quad \omega^2 = \frac{2k}{a^2}\frac{m_1 + m_2}{m_1 m_2}. \tag{31}$$

There is a zero frequency solution we argued above must always exist. Setting $m_2 = 2m_1 = km/a^2$ the other solution is

$$\omega^2 = \frac{3}{m}. \tag{32}$$

The corresponding normal mode is found by plugging this into (30)

$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = 0 \qquad \Rightarrow b_1 = -2b_2 \equiv Ae^{-i\delta}. \tag{33}$$

Taking the real part we find the solution

$$\begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1/2 \end{pmatrix} A\cos(\omega t - \delta) \tag{34}$$

This corresponds to the two masses oscillating exactly out of phase, with the m_2 oscillating with half the amplitude. We could have guessed this, since m_2 is twice as massive as m_1. We conclude that any motion is a superposition of uniform rotation (the zero frequency mode) and out of phase oscillation.

We now solve the problem a slightly different way. Start from the Lagrangian

$$L = \frac{a^2}{2}(m_1\dot{\theta}_1^2 + m_2\dot{\theta}_2^2) - k(\theta_1 - \theta_2)^2. \tag{35}$$

Note the potential only depends on the relative angle $\theta_1 - \theta_2$. Let us now change coordinates

$$\Theta = \frac{m_1\theta_1 + m_2\theta_2}{m_1 + m_2}, \tag{36}$$

$$\theta = \theta_1 - \theta_2. \tag{37}$$

These are the analogues of center of mass and relative coordinates $\mathbf{R}, \mathbf{r}$ you saw before. Letting $M = m_1 + m_2$ and $\mu = m_1 m_2/(m_1 + m_2)$, the Lagrangian (35) in these coordinates is

$$L = \frac{1}{2}M\dot{\Theta}^2 + \frac{1}{2}\mu\dot{\theta}^2 - k\theta^2. \tag{38}$$

We have separated the Lagrangian into independent Lagrangians for the 'center of mass' and relative motion. Equation of motion are

$$\Theta: \qquad \frac{d}{dt}\frac{\partial L}{\partial \dot{\Theta}} = \frac{\partial L}{\partial \Theta} \qquad \Rightarrow \ddot{\Theta} = 0 \qquad \Rightarrow \Theta(t) = \Theta_0 + \Theta_1 t. \tag{39}$$

$$\theta: \qquad \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = \frac{\partial L}{\partial \theta} \qquad \Rightarrow \mu\ddot{\theta} = -2k\theta \qquad \Rightarrow \theta(t) = A\cos(\omega t - \delta), \quad \omega = \sqrt{\frac{2k}{\mu}}. \tag{40}$$

From $m_2 = 2m_1$ we find $\omega^2 = 3k/m_1$. These are the same results we found above but now we understand the motion as decoupled into constant angular speed 'center of mass' motion (Θ) and oscillating relative motion (θ) with frequency $\omega^2 = 3k/m_1$.

Problem 4: (Non-inertial frames) Consider a pendulum suspended inside a railroad car that is being forced to accelerate with a constant acceleration A.

a) Write down the Lagrangian for the system and the equation of motion for the angle ϕ the pendulum makes with the vertical.

Solution: Take as coordinates

$$x = X + \ell \sin\phi, \quad y = \ell\cos\phi \tag{41}$$

where X is the coordinate of the pendulum support, moving with acceleration A. The kinetic energy is

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2}m\ell^2\dot{\phi}^2 + m\ell\dot{X}\dot{\phi}\cos\phi + \frac{1}{2}m\dot{X}^2 \tag{42}$$

The potential is

$$U = -mgy = -mg\ell\cos\phi. \tag{43}$$

The Lagrangian is, up to total time derivatives,

$$L = \frac{1}{2}m\ell^2\dot{\phi}^2 + mg\ell\cos\phi - mA\ell\sin\phi. \tag{44}$$

EOM are:

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\phi}} = \frac{\partial L}{\partial\phi} \quad \Rightarrow \ell\ddot{\phi} = -g\sin\phi - A\cos\phi \tag{45}$$

b) Find the equilibrium angle ϕ at which the pendulum can remain fixed (relative to the car) as the car accelerates. Demonstrate that the equilibrium is stable and compute the associated frequency of small oscillations.

Solution: The equilibrium condition is that the force vanishes:

$$-g\sin\phi_0 - A\cos\phi_0 = 0 \quad \Rightarrow \tan\phi_0 = -A/g. \tag{46}$$

Writing $\phi = \phi_0 + \delta\phi$ and expanding:

$$\ell\delta\ddot{\phi} = (-g\cos\phi_0 + A\sin\phi_0)\delta\phi = -\sqrt{g^2 + A^2}\delta\phi \tag{47}$$

$$\Rightarrow \delta\ddot{\phi} = -\omega^2\delta\phi, \quad \omega^2 = \frac{g^2 + A^2}{\ell} \tag{48}$$

Problem 5: (Hamiltonian mechanics) Find the Hamiltonian for a particle in a uniformly rotating frame of reference. Notice the Coriolis force does not appear in the Hamiltonian. Explain why you might have expected this to be the case.

Solution: The Lagrangian is

$$L = \frac{1}{2}mv^2 + m\mathbf{v}\cdot(\mathbf{\Omega}\times\mathbf{r}) + \frac{1}{2}m(\mathbf{\Omega}\times\mathbf{r})^2 - U. \tag{49}$$

The canonical momentum is

$$\mathbf{p} = \frac{\partial L}{\partial\mathbf{v}} = m\mathbf{v} + m\mathbf{\Omega}\times\mathbf{r}. \tag{50}$$

The Hamiltonian is then

$$H = \mathbf{p}\cdot\mathbf{v} - L = \frac{p^2}{2m} - \mathbf{\Omega}\cdot(\mathbf{r}\times\mathbf{p}) + U. \tag{51}$$

Although we have expressed H in terms of the momentum, notice that we may also write

$$H = \frac{1}{2}mv^2 - \frac{1}{2}m(\mathbf{\Omega} \times \mathbf{r})^2 + U \tag{52}$$

The first and third term combine to give the usual kinetic plus potential energy, while the second term is the centrifugal potential. There is no term linear in the velocity from the Coriolis force. We may understand this as a consequence of the fact that the Coriolis force is perpendicular to the velocity and hence does no work (thereby not effecting the energy).

Problem 6: (Hamiltonian mechanics and conservation laws) Consider a one dimensional system with the Hamiltonian

$$H = \frac{p^2}{2} - \frac{1}{2q^2}. \tag{53}$$

Show that the quantity

$$D = \frac{pq}{2} - Ht \tag{54}$$

is a constant of the motion, that is, D is conserved.

Solution: The equations of motion are

$$\dot{q} = \frac{\partial H}{\partial p} = p \tag{55}$$

$$\dot{p} = -\frac{\partial H}{\partial q} = -\frac{1}{q^3}. \tag{56}$$

Now let's write out

$$\frac{dD}{dt} = \frac{p\dot{q}}{2} + \frac{\dot{p}q}{2} - H \tag{57}$$

$$= \frac{p^2}{2} - \frac{1}{2q^2} - H \tag{58}$$

$$= 0 \quad \checkmark \tag{59}$$

where in the first line we have used the fact that $\dot{H} = 0$.

Alternatively we can use the fact that

$$\frac{dD}{dt} = \{H, D\} + \frac{\partial D}{\partial t} \tag{60}$$

$$= \{H, \frac{pq}{2}\} - H \tag{61}$$

$$= \left(p \times \frac{p}{2} - \frac{1}{q^3} \times \frac{q}{2}\right) - \frac{p^2}{2} + \frac{1}{2q^2} \tag{62}$$

$$= 0. \tag{63}$$

Problem 7: (Hamiltonian of a rigid body) The Lagrangian for a heavy symmetric top of mass M, pinned at point O which is a distance ℓ from the centre of mass is

$$L = \frac{I_\perp}{2}\left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta\right) + \frac{I_3}{2}\left(\dot{\psi} + \dot{\phi}\cos\theta\right)^2 - Mg\ell\cos\theta. \tag{64}$$

Obtain the momenta p_θ, p_ψ, p_ψ and the Hamiltonian H. Derive Hamilton's equations for this system. Identify the three conserved quantities and explain their physical meaning.

Solution: Conjugate momenta are

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = I_\perp \dot{\theta}, \tag{65}$$

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = I_3 \cos\theta(\dot{\psi} + \dot{\phi}\cos\theta) + I_\perp \dot{\phi} \sin^2\theta, \tag{66}$$

$$p_\psi = \frac{\partial L}{\partial \dot{\psi}} = I_3(\dot{\psi} + \dot{\phi}\cos\theta) \tag{67}$$

The Hamiltonian is given by

$$H = p_\theta \dot{\theta} + p_\phi \dot{\phi} + p_\psi \dot{\psi} - L \tag{68}$$

Solving the above equations for velocities in terms of conjugate momenta we obtain

$$H = \frac{p_\theta^2}{2I_\perp} + \frac{p_\psi^2}{2I_3} + \frac{(p_\phi - p_\psi \cos\theta)^2}{2I_\perp \sin^2\theta} + Mg\ell\cos\theta. \tag{69}$$

Hamilton's equations are

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{I_\perp} \tag{70}$$

$$\dot{\phi} = \frac{\partial H}{\partial p_\phi} = \frac{p_\phi - p_\psi \cos\theta}{I_\perp \sin^2\theta} \tag{71}$$

$$\dot{\psi} = \frac{\partial H}{\partial p_\psi} = \frac{p_\psi}{I_3} - \frac{\cos\theta(p_\phi - p_\psi\cos\theta)}{I_\perp \sin^2\theta} \tag{72}$$

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = -\frac{p_\psi(p_\phi - p_\psi\cos\theta)}{I_\perp \sin\theta} + \frac{\cos\theta(p_\phi - p_\psi\cos\theta)^2}{I_\perp \sin^3\theta} + Mg\ell\sin\theta \tag{73}$$

$$\dot{p}_\phi = -\frac{\partial H}{\partial \phi} = 0 \tag{74}$$

$$\dot{p}_\psi = -\frac{\partial H}{\partial \psi} = 0. \tag{75}$$

No explicit time dependence means the energy is conserved. The energy is simply equal to the Hamiltonian, evaluated on physical trajectories:

$$E = H(q(t), p(t)). \tag{76}$$

From Hamilton's equations,

$$\dot{p}_\phi = -\frac{\partial H}{\partial \phi} = 0, \qquad \dot{p}_\psi = -\frac{\partial H}{\partial \psi} = 0, \tag{77}$$

and we see the momenta p_ϕ and p_ψ, conjugate to variables ϕ and ψ, are also conserved. Momentum p_ϕ is the z-component of the angular momentum, that is, $p_\phi = \hat{\mathbf{z}} \cdot \mathbf{L}$. Conservation of p_ϕ is due to the fact that there is no z-component to the gravitational torque. Momentum p_ψ is the component of the angular momentum along the symmetry axis of the top, $p_\psi = \hat{\mathbf{x}}_3 \cdot \mathbf{L}$. Conservation of p_ψ is due to the fact that there is no x_3-component to the gravitational torque

Problem 8: (Dynamics in a magnetic field) In this problem you consider the motion of a charged particle (charge q) in the presence of external electric and magnetic fields, $\mathbf{E}$ and $\mathbf{B}$. It turns out the Lagrangian in this case is

$$L = \frac{1}{2}mv^2 - q\phi(\mathbf{r}, t) + q\mathbf{A}(\mathbf{r}, t) \cdot \mathbf{v}. \tag{78}$$

Here ϕ and $\mathbf{A}$ are the scalar and vector potentials, related to the electric and magnetic fields by

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}, \quad \mathbf{B} = \nabla \times \mathbf{A}, \tag{79}$$

written in units with $c = 1$. This Lagrangian is different from many of the ones we have discussed previously, as it will lead to a velocity dependent force (it is, however, similar to the Lagrangian in a rotating frame).

a) Write out the Euler-Lagrange equations for the charged particle based on the Lagrangian (78). Express your result in terms of $\mathbf{E}$ and $\mathbf{B}$ and verify you recover the Lorentz force law.

Solution: We write this solution using indices and Einstein summation convention – doubled indices are assumed to be summed. We also write ∂_i as a shorthand for $\frac{\partial}{\partial x_i}$ and ∂_t for $\frac{\partial}{\partial t}$. (As such the letter t will never be used as an index.) If you aren't totally comfortable with this notation, it may be a good exercise to follow through this solution checking every step.

The Lagrangian for a particle in an electromagnetic field is given by

$$L = \frac{1}{2}m\dot{x}_i{}^2 - q\phi(t, x) + qA_i(t, x)\dot{x}_i. \tag{80}$$

The Euler-Lagrange equations are given by

$$\frac{\partial L}{\partial x_i} = \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} \tag{81}$$

which when applied to (80) give

$$-q\partial_i\phi + q(\partial_i A_j)\dot{x}_j = \frac{d}{dt}(m\dot{x}_i + qA_i). \tag{82}$$

We relabelled the dummy index i in (80) as j in (82) to not confuse with the free index i.

Expanding out the RHS gives

$$-q\partial_i\phi + q(\partial_i A_j)\dot{x}_j = m\ddot{x}_i + q\partial_t A_i + q\partial_j A_i\dot{x}_j. \tag{83}$$

Collecting terms we finally get

$$m\ddot{x}_i = q(-\partial_i\phi - \partial_t A_i) + q\dot{x}_j(\partial_i A_j - \partial_j A_i). \tag{84}$$

or

$$F_i = q(-\partial_i \phi - \partial_t A_i) + q v_j (\partial_i A_j - \partial_j A_i). \tag{85}$$

We recognize the $-\partial_i\phi - \partial_t A_i$ term as simply E_i. To match the familiar Lorentz force, we need to check that $v_j(\partial_i A_j - \partial_j A_i)$ is indeed $(\mathbf{v}\times\mathbf{B})_i$. This is easily checked with the definition of the cross product:

$$\begin{aligned}
(\mathbf{v}\times\mathbf{B})_i &= \epsilon_{ijk} v_j B_k \\
&= \epsilon_{ijk} v_j (\nabla \times \mathbf{A})_k \\
&= \epsilon_{ijk} v_j \epsilon_{klm} \partial_l A_m \\
&= \epsilon_{kij} \epsilon_{klm} v_j \partial_l A_m \\
&= (\delta_{i,l}\delta_{j,m} - \delta_{i,m}\delta_{j,l}) v_j \partial_l A_m \\
&= v_j (\partial_i A_j - \partial_j A_i)
\end{aligned} \tag{86}$$

where we have used cyclicity of the ϵ_{ijk} symbol, and the identity $\epsilon_{kij}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$. Putting everything together we then get

$$\boxed{m\ddot{\mathbf{r}} = q\mathbf{E} + q(\mathbf{v}\times\mathbf{B})}. \tag{87}$$

b) Recall the scalar and vector potentials are not unique. The gauge transformation

$$\phi(\mathbf{r},t) \to \phi(\mathbf{r},t) - \frac{\partial f(\mathbf{r},t)}{\partial t}, \quad \mathbf{A}(\mathbf{r},t) \to \mathbf{A}(\mathbf{r},t) + \nabla f(\mathbf{r},t), \tag{88}$$

leaves the fields $\mathbf{E}$ and $\mathbf{B}$ unchanged (as you may verify from Eq. (79)). Thus the scalar and vector potentials contain an "unphysical" component related to this gauge redundancy. You might then be worried that these unphysical fields appear in the Lagrangian. Compute the change in the Lagrangian (78) under such a gauge transformation and explain why the gauge redundancy is not a cause for concern.

Solution: Let's see how the Lagrangian under a gauge transformation. From (80) we the change δL to be

$$\delta L = -q\partial_t f + q(-\partial_i f)\dot{x}_i. \tag{89}$$

But this is simply the total time derivative of f.

$$\boxed{\delta L = -q\frac{df}{dt}}. \tag{90}$$

Thus the gauge transformation does not change the equations of motion.

c) Calculate the momentum conjugate to $\mathbf{r}$ from the Lagrangian (78). Recall the conjugate momentum is $\mathbf{p} = \partial L/\partial \mathbf{v}$. Use this to compute the Hamiltonian associated to the Lagrangian (78) via the usual Legendre transform.

Solution: We have

$$\mathbf{p} = \frac{\partial L}{\partial \mathbf{v}} = m\mathbf{v} + q\mathbf{A} \quad \Rightarrow \quad \mathbf{v} = \frac{1}{m}(\mathbf{p} - q\mathbf{A}) \tag{91}$$

The Hamiltonian is then

$$H = \mathbf{p} \cdot \mathbf{v} - L = \frac{(\mathbf{p} - q\mathbf{A})^2}{2m} + q\phi(\mathbf{r}, t). \tag{92}$$

d) Compute the Poisson brackets between the different components of the "kinetic momentum" $\mathbf{k} = m\mathbf{v}$ (which is different from the canonical momentum $\mathbf{p} = \partial L/\partial \mathbf{v}$).

Solution: From (c) we have $k_i = p_i - qA_i$. The Poisson brackets are

$$\begin{aligned} \{k_i, k_j\} &= \{p_i - qA_i, p_j - qA_j\} && (93) \\ &= q(\{A_i, p_j\} - \{A_j, p_i\}) && (94) \\ &= q\left(\frac{\partial A_i}{\partial x_j} - \frac{\partial A_j}{\partial x_i}\right) && (95) \\ &= -q\epsilon_{ijk} B_k. && (96) \end{aligned}$$

The Poisson bracket of the components of the kinetic momentum is thus non-zero in a magnetic field.