02/14/2014

# Summary

- Two-body problem: $L = \frac{1}{2} m_1 \dot{\vec{r}}_1^{\,2} + \frac{1}{2} m_2 \dot{\vec{r}}_2^{\,2} - U(|\vec{r}_1 - \vec{r}_2|)$
- Reduction:
  - decomp. into COM + relative motion:

$$\left.\begin{aligned} \vec{r} &= \vec{r}_1 - \vec{r}_2 \\ \vec{R} &= \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{M} \end{aligned}\right\} \quad L = \underbrace{\frac{1}{2} M \dot{\vec{R}}^2}_{L_{COM}} + \underbrace{\frac{1}{2} \mu \dot{\vec{r}}^2 - U(r)}_{L_{rel.}}$$

$M = m_1 + m_2$, $\mu = \frac{m_1 m_2}{M}$

$L_{COM}$ removed by changing to frame moving w/ COM.

$\rightarrow$ effective one-body problem $L = \frac{1}{2} \mu \dot{\vec{r}}^2 - U(r)$

6 DOF $\rightarrow$ 2 DOF.

- Cons. of $\vec{L}$:
  - $\vec{L} = \vec{r} \times \vec{p}$ conserved by rotational symm.

$\vec{r} \cdot \vec{L} = 0 \Rightarrow$ motion confined to plane $\perp \vec{L}$

$\Rightarrow L = \frac{1}{2} \mu (\dot{r}^2 + r^2 \dot{\varphi}^2) - U(r)$ (polar coord.'s in plane $\perp \vec{L}$).

  - $\frac{\partial L}{\partial \varphi} = 0 \Rightarrow \mu r^2 \dot{\varphi} = \text{const.} \leftrightarrow$ conservation of mag. of $\vec{L}$

& $\mu r^2 \dot{\varphi} = L_z = z$-comp. of $\vec{L}$

($=$ mag. of $\vec{L}$)

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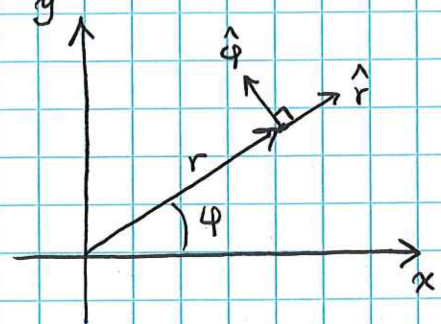
Polar coord. review.




$$\hat{r} = \cos\varphi\,\hat{x} + \sin\varphi\,\hat{y}$$

$$\hat{\varphi} = -\sin\varphi\,\hat{x} + \cos\varphi\,\hat{x}$$

$$\vec{r} = r\hat{r}$$

$$\dot{\vec{r}} = \frac{d}{dt}(r\hat{r}) = \dot{r}\hat{r} + r\dot{\varphi}\hat{\varphi}$$

$$\ddot{\vec{r}} = (\ddot{r} - r\dot{\varphi}^2)\hat{r} + (r\ddot{\varphi} + 2\dot{r}\dot{\varphi})\hat{\varphi}$$

(will be useful when we consider motion in the plane expressed in polar coord.'s).

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We have not yet considered EOM for r.

Instead of writing out E-L. eqn. for r, note we have not yet used cons. of energy (a consequence of $\frac{\partial L}{\partial t} = 0$).

$$\rightarrow\ E = T + U = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\varphi}^2) + U(r) = \text{const.}$$

Combining w/ cons. of $L_z = \mu r^2\dot{\varphi}$ :

$$\rightarrow\ E = \frac{1}{2}\mu\dot{r}^2 + U(r) + \frac{1}{2}\mu r^2\left(\frac{L_z}{\mu r^2}\right)^2$$

$$\rightarrow\ E = \frac{1}{2}\mu\dot{r}^2 + U(r) + \frac{L_z^2}{2\mu r^2}$$

~~(cf. analysis for 1D motion)~~

This eqn. may be integrated: 02/14/2024

$$\dot{r} = \frac{dr}{dt} = \sqrt{\frac{2}{\mu}\left[E - U(r) - \frac{L_z^2}{2\mu r^2}\right]}$$

$$\Rightarrow t = \int \frac{dr}{\sqrt{\frac{2}{\mu}\left[E - U(r) - \frac{L_z^2}{2\mu r^2}\right]}} + \underset{\substack{\uparrow \\ \text{const.}}}{C} \qquad (*).$$

→ implicit eqn. for $r(t)$.

& from cons. of $L_z$:

$$\dot{\varphi} = \frac{d\varphi}{dt} = \frac{L_z^{\cancel{2}}}{\cancel{2}\mu r^2}$$

$$\Rightarrow \varphi(t) = \frac{L_z}{\mu} \int \frac{dt}{\underset{\substack{\uparrow \\ \text{from sol.'n of } (*)}}{r^2(t)}} + \underset{\substack{\uparrow \\ \text{const.}}}{C'} \qquad (**).$$

So, (*) + (**) give a complete sol.'n to the EOM for the two-body problem, that is, a sol.'n for $(r(t), \varphi(t))$ in terms of 4 const.'s $\{E, L_z, C, C'\}$. These const.'s may equivalently be expressed in terms of, e.g., $\{r(0), \dot{r}(0), \varphi(0), \dot{\varphi}(0)\}$

We can also compute something different. Rather than the trajectory $\vec{r}(t)$, we may also solve for shape of the orbit $r(\varphi)$.

Cons of $L_z$: $d\varphi = \frac{L_z}{\mu r^2}\, dt$

Cons. of $E$: $dt = \dfrac{dr}{\sqrt{\frac{2}{\mu}\left[E - U(r) - \frac{L_z^2}{2\mu r^2}\right]}}$

$$\Rightarrow d\varphi = \frac{L_z}{\sqrt{2\mu}} \frac{dr}{r^2 \sqrt{E - U(r) - \frac{L_z^2}{2\mu r^2}}}$$

$$\Rightarrow \varphi = \frac{L_z}{\sqrt{2\mu}} \int \frac{dr}{r^2 \sqrt{E - U(r) - \frac{L_z^2}{2\mu r^2}}} + \text{const.}$$

→ invert to find $r(\varphi)$, shape of the orbit (we'll come back to this).

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Before putting in a particular form for $U(r)$, we can draw a number of general conclusions:

write: $E = \frac{1}{2}\mu \dot{r}^2 + U_{eff}(r)$, $U_{eff}(r) = U(r) + \frac{L_z^2}{2\mu r^2}$

"Centrifugal potential"

→ 1D problem for particle in effective potential $U_{eff}(r)$

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Q: why do we call it "centrifugal potential"?

→ Recall "centrifugal force" for circular motion:

$$\frac{mv_\varphi^2}{r} = \frac{m(r\dot{\varphi})^2}{r} = mr\dot{\varphi}^2 = \frac{L_z^2}{mr^3} = \frac{-d}{dr}\left(\frac{L_z^2}{2mr^2}\right)$$

→ centrifugal forces arises from precisely what we called centrifugal potential

Of course, it is neither a real potential or a real force, arising from the kinetic energy T of the system.

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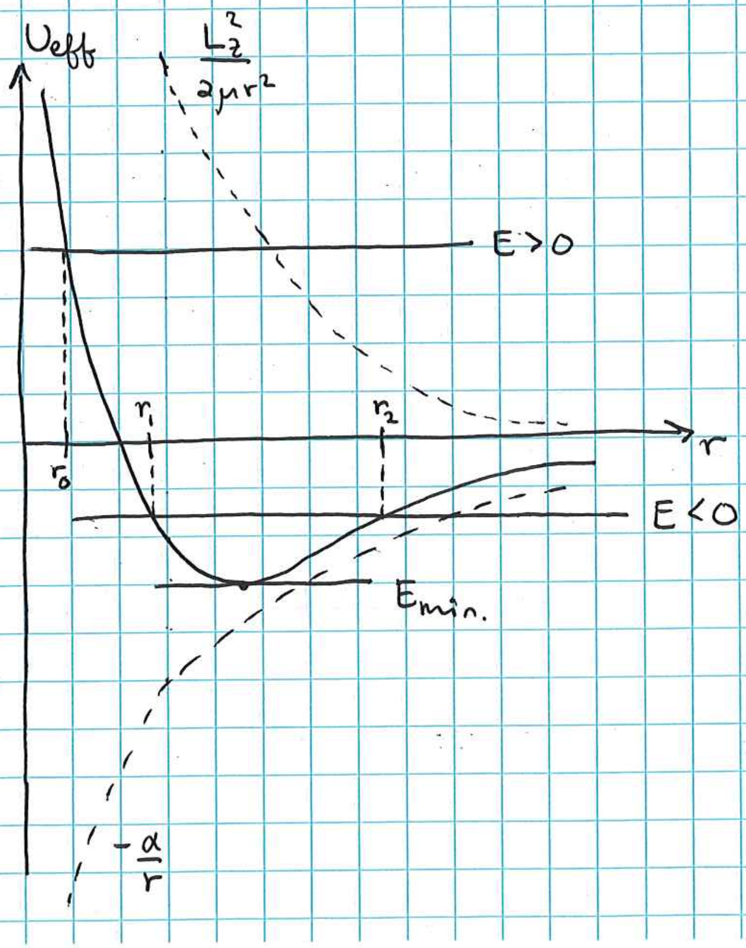
Given the analogy w/ 1D motion, we can carry out a similar analysis to what we did in that case

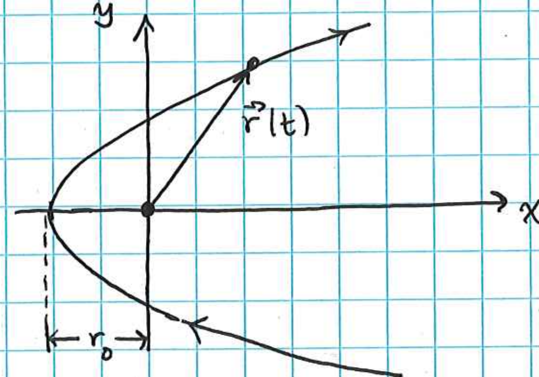
Ex: $U = -\frac{\alpha}{r}$



- $E > 0$ $\quad E \geq U_{eff}(r)$ restricts allowed motion.

  min. distance $r_0$ s.t. $E = U_{eff}(r_0)$ & $\dot{r} = 0$

  $\rightarrow$ $r \geq r_0$ & $r_0$ = "turning pt." at which distance from force center $r(t)$ goes from decreasing to increasing

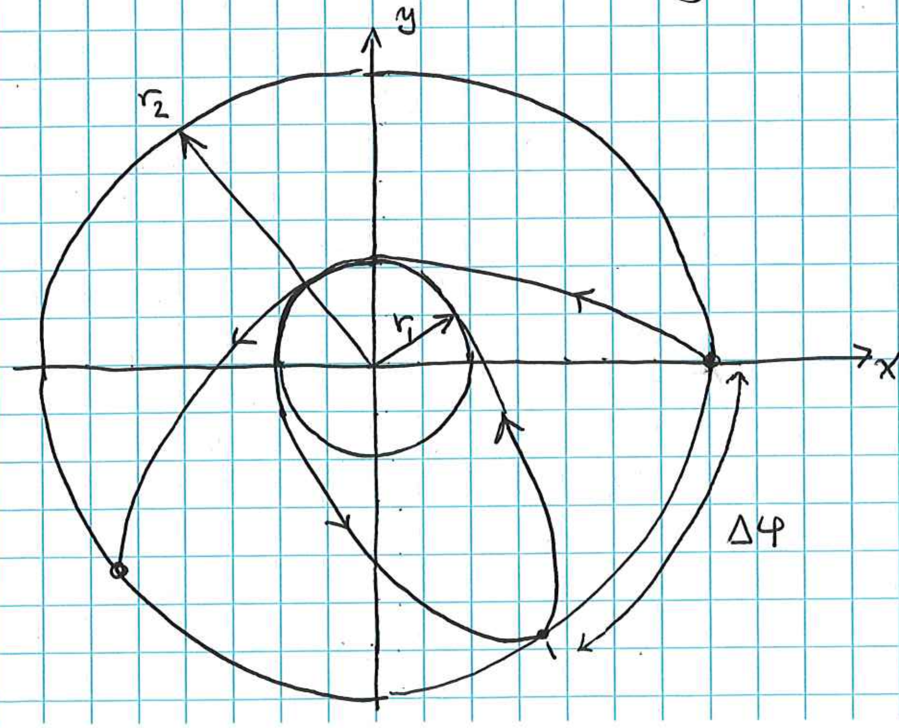



- $E < 0$ $\quad E \geq U_{eff}(r)$ restricts $r_1 \leq r \leq r_2$ w/

  $E = U_{eff}(r_1) = U_{eff}(r_2)$ & $\dot{r} = 0$, $r_1, r_2$ = turning pt.'s.

  $\rightarrow$ motion confined to annular region $r_1 \leq r \leq r_2$

  However, orbit not necessarily closed.



Consider path $r_2 \to r_1 \to r_2$

$$\Rightarrow \quad \Delta\varphi = \frac{2L_z}{\sqrt{2\mu}} \int_{r_1}^{r_2} \frac{dr}{r^2\sqrt{E - U(r) - \frac{L_z^2}{2\mu r^2}}}$$

If $\Delta\varphi = 2\pi \frac{m}{n}$ ($m, n =$ integers), then after $n$ periods, $n\,\Delta\varphi = 2\pi m$ & orbit closes.

In general, orbit doesn't close & as $t \to \infty$ orbit covers entire annulus

(as we'll see later, for Kepler orbits are closed).

• $E = E_{min}$. $r_1 = r_2$ & orbit is a closed circle w/ $\dot{r} = 0$ for all $t$.

Radius of circular orbit determined by

$$\frac{dU_{eff}}{dr} = 0 \Rightarrow \frac{dU}{dr} = -\frac{d}{dr}\left(\frac{L_z^2}{2\mu r^2}\right) = \frac{L_z^2}{\mu r^3} = \frac{\mu v_\varphi^2}{r}$$

$$\& \; F = -\frac{dU}{dr} \Rightarrow F = -\frac{m v_\varphi^2}{r}$$

$$\text{or} \quad |F| = \frac{m v_\varphi^2}{r} \qquad (F < 0)$$

→ applied force = centripetal accel.