

Brief Mechanics

Notes from Physics 311

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work in progress...

Contents

1 Equation of Motion: Lagrangian, Principle of Least Action, and E-L Equation	2
1.1 Lagrangian:	2
1.2 E-L equation	2
1.3 coordinate transformation:	3
2 Conservation Laws: Energy, Momentum, COM, and Angular Momentum	3
2.1 Energy:	3
2.2 General momentum:	3
2.3 Total momentum	3
2.4 Center of Mass	4
2.5 Conservation of angular momentum	4
3 Integration of the equations of motion: Connecting Energy with motion	4
3.1 Motion in 1 dimension	4
3.2 Turning points	4
3.3 Unbounded Motion:	5
3.4 Effective DOF=1 system	5
4 Two body problem	6
4.1 Problem setup	6
4.2 COM and relative coordinates, DOF= 6 -> DOF = 2	6
4.3 Conservation of Angular Momentum	6
4.4 2 body problem in gravitational field	6
4.5 Kepler's second Law	7
4.6 EOM for two body system	7
4.7 Shape of orbit	7
4.8 Effective potential and shape of orbit (Only for Attractive Potential)	8
5 The Kepler Problem: a special case of the two body problem	8
5.1 conditions	8
5.2 Conic section orbits	8
5.3 Classifications of orbits based on energy of system E	9
6 More Kepler: Period, Kepler's third law	9
6.1 Orbit of each body	9
6.2 Period of orbit	9
6.3 Conservation of Laplace-Runge-Lenz vector	10
7 Orbital Transfer	10
7.1 Instantaneous Change in velocity	10

7.2 Tangential thrust at perigee	10
7.3 changing between circular orbits	10
7.4 Verify:	11
8 Small Oscillations	11
8.1 DOF= 1 (one dimension)	11
8.1.1 EOM for DOF = 1 small Oscillations	12
8.1.2 energy for 1D small Oscillation	12
8.1.3 Damped 1D oscillation, and Complex representation	13
8.1.4 underdamped, overdamped, and critically damped	13
8.1.5 Forced Oscillations	14
8.1.6 reintroducing damping via external forcing	14
8.2 Oscillations DOF>1	16
8.2.1 Example: Oscillation with 2 mass and 3 springs	16
8.2.2 New Coords	16
8.2.3 General Coords	16
8.3 Normal Coords	18
9 Motion of Rigid Body	18
9.1 frames of reference	18
9.2 Lagrangian for Rigid Body	18
9.2.1 Inertial Tensor	18
9.3 Principle axis and principal moments of inertia	19
9.4 Parallel axis theorem	19
10 Appendix	21

1 Equation of Motion:

Lagrangian, Principle of Least Action, and E-L Equation

1.1 Lagrangian:

- Under the constraint of
 - 1) Space and time are homogenous, 2) time is isotropic, the
 Lagrangian for a system is given as

$$L = T - U(r), \text{ where } \begin{cases} T = \sum_{a=1}^N \frac{1}{2} m_a \dot{q}_a^2 \text{ sum of KE} \\ U: \text{ potential energy} \end{cases} \quad (1)$$

1.2 E-L equation

For a given functional,

$$S = \int_{t_1}^{t_2} L(q, \dot{q}, t) dt \quad (2)$$

we could optimize it using the Euler-Lagrange equation,

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0 \quad (3)$$

where each EL equation and its solution corresponds to a degree of freedom.

Upon applying the EL equation to a generalized lagrangian, we reveal Newton's second law

$$\frac{d}{dt} \frac{\partial(\frac{1}{2}mv^2 - U(r))}{\partial v} = \frac{\partial(\frac{1}{2}m\dot{q}^2 - U(r))}{\partial r}$$

$$\Rightarrow m\ddot{v} = -\frac{\partial U}{\partial q} \equiv \vec{F}(\text{force}) \quad (4)$$

1.3 coordinate transformation:

- In cartesian coordinates, $L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - U$
In cylindrical coordinates, $L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2) - U$
In spherical coordinates, $L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2(\theta)\dot{\phi}^2) - U$
- Note that when taking partial differentiations, we treat each variable and its derivative as two independent variables. Don't ask why... We are doing physics here

2 Conservation Laws:

Energy, Momentum, COM, and Angular Momentum

2.1 Energy:

- Energy is defined as the following, and when the Lagrangian is **homogeneity time**, the energy is conserved.

$$E \equiv \sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L \quad (5)$$

considering $L = T - U$, we have $E = T + U$

- Total energy is also given as

$$E = \frac{1}{2}\mu V^2 + E_i \quad (6)$$

where E_i is internal energy, and μ being the total mass

2.2 General momentum:

conservation of general momentum is from the following conservation

$$\frac{\partial L}{\partial q_j} = 0 \Rightarrow p_j \equiv \frac{\partial L}{\partial \dot{q}_j}, \quad (7)$$

where q_j is a cyclic coordinate, i.e. L is independent of q_j .

2.3 Total momentum

total momentum is defined as the following, and considering the **homogeneity of space**, the momentum is conserved in a closed system.

If the total momentum of a mechanical system in a given frame of reference is 0, then the said system is at rest relative to that frame. For simplicity's sake, we want to choose our frame of reference in which the total momentum is zero.

$$P \equiv \sum_a \frac{\partial L}{\partial \dot{q}_a} = \boxed{\sum_a m_a v_a}$$

force is also given by $F_j = \frac{\partial L}{\partial q_j}$ (8)

sum of all forces in a closed system is 0

2.4 Center of Mass

- Center of mass is defined so that, the velocity of the system as a whole, $V = P/(\sum m_a)$ is the time derivative of the center of mass. $R = \sum_a m_a r_a / (\sum m_a)$.

2.5 Conservation of angular momentum

Angular momentum characterizes the rotation of the system, and considering the **isotropy of space**, the angular momentum is conserved in a closed system.

$$\boxed{\vec{L} \equiv \sum_a r_a \times p_a}$$
 is conserved in a closed system (9)

- Angular momentum can be found by differentiating the lagrangian with respect to angular velocity, along the rotation axis z:

$$\overrightarrow{L}_z = \frac{\partial L}{\partial \dot{\varphi}_a} \quad (10)$$

3 Integration of the equations of motion: Connecting Energy with motion

3.1 Motion in 1 dimension

- For a system with DOF=1, and with $\frac{\partial L}{\partial t} = 0$ (lagrangian independent of time, i.e. energy conserved), we can write the lagrangian and total energy as

$$L = \frac{1}{2} m \dot{x}^2 - U(x), \quad (11)$$

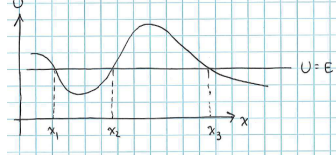
$$E = \frac{1}{2} m \dot{x}^2 + U(x) \quad (12)$$

Equation 12 is a differential equation of position and time. Solving this ODE for time gives:

$$t = \sqrt{\frac{m}{2}} \int \frac{dx}{E - U(x)} + C \quad (13)$$

when given $U(x)$, and by plugging it into Equation 12, we can solve for $x(t)$ by substitution. Tricks on sub: when $U(x)$ is of order 1, use u-sub; when it's of order 2, use trig-sub.

3.2 Turning points



For a given potential function $U(x)$, the turning points are the points where the potential energy is equal to the total energy, i.e. $U(x) = E$. At turning points, the system is either just about to move, or just about to stop.

Only motion where potential is less or equal to total energy is allowed.

Bounded motion: $[x_1, x_2]$; unbounded motion: $x > x_3$

3.3 Unbounded Motion:

When there is a potential well, the system could go into periodic motion with potential energy moving back and forth in the well, and position between x_1, x_2 . We find period by doubling Equation 12:

$$T(E) = \sqrt{2m} \int_{x_1(E)}^{x_2(E)} \frac{dx}{\sqrt{E - U(x)}} \quad (14)$$

where we represent $x_1(E), x_2(E)$ in terms of E .

When given $U(x)$, we can solve for $x_1(E), x_2(E)$, and then plugging in to Equation 14, we can solve for period by integration via substitution.

Simple Pendulum in polar coord's has the following:

$$\begin{aligned} T &= \frac{1}{2} m l^2 \dot{\theta}^2 \\ U &= mgl(1 - \cos(\theta)) \end{aligned} \quad (15)$$

It's period is given by Equation 14. Solving it gives us

$$\begin{aligned} T(E) &= 4\sqrt{\frac{l}{g}} \int_0^{\frac{\pi}{2}} \frac{du}{\sqrt{1 - k^2 \sin^2(u)}} \\ \text{where } k &= \sin\left(\frac{\theta_0}{2}\right), \sin u = \frac{1}{k} \sin\left(\frac{\theta_0}{2}\right) \end{aligned} \quad (16)$$

Equation 16 can be simplified by small angle approx into

$$T(E) = 2\pi\sqrt{\frac{l}{g}} \left(1 + \left(\frac{\theta_0^2}{16} \right) \right) \quad (17)$$

3.4 Effective DOF=1 system

When the largrangian is of the form $L = f(\dot{x}) - g(x)$, we can see it as a system with effective potential $U_{\text{eff}(x)} = g(x)$, and effective kenetic energy $T_{\text{eff}(x)} = f(\dot{x})$. The effective energy is therefore $E = T_{\text{eff}} + U_{\text{eff}}$.

4 Two body problem

4.1 Problem setup

- The two body problem considers two interacting masses with an interacting potential $U(r_1, r_2) = U(|\vec{r}_1 - \vec{r}_2|)$. The lagrangian is given by

$$L = \frac{1}{2}m_1\dot{r}_1^2 + \frac{1}{2}m_2\dot{r}_2^2 - U(|\vec{r}_1 - \vec{r}_2|) \quad (18)$$

•

4.2 COM and reletive coordinates, DOF= 6 -> DOF = 2

- Consider the following handy substitution,

$$\begin{aligned} \text{Reduced mass } \mu &= (m_1 m_2) / (m_1 + m_2) = m_1 m_2 / M; \\ \text{Center of mass } R &= (m_1 r_1 + m_2 r_2) / (M); \\ \text{relative positon } \vec{r} &= \vec{r}_1 - \vec{r}_2 \end{aligned} \quad (19)$$

- Putting the two body system into relative coordinates, and represent masses with reduced mass and COM, we have the following lagrangian:

$$L = \frac{1}{2}M\dot{R}^2 + \frac{1}{2}\mu\dot{\vec{r}}^2 - U(\vec{r}) \quad (20)$$

where the first term involves only the COM motion, and the second term involves only the relative motion.

- By choosing our frame with the COM at rest and the total momentum zero, our problem is simplified to an **effective one body problem** with DOF = 2, given by

$$L = \frac{1}{2}\mu\dot{\vec{r}}^2 - U(\vec{r}) \quad (21)$$

4.3 Conservation of Angular Momentum

- Angular momentum is defined as $\vec{L} = \vec{r} \times \mu\dot{\vec{r}}$, and is conserved here.
- Knowing $\vec{r} \cdot \vec{L} = 0$, the motion is in the plane perpendicular to \vec{L} . We can use polar coordinates to describe the motion,

$$L = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) - U(r) \quad (22)$$

Using EL equation on Equation 22, we get

$$\begin{aligned} \frac{d}{dt} \frac{\partial L}{\partial \dot{\varphi}} &= \frac{\partial L}{\partial \varphi} \\ \Rightarrow \vec{L}_z &\equiv \mu r^2 \dot{\theta} = \text{constant} \end{aligned} \quad (23)$$

(conservation of angular momentum on z-axis)

4.4 2 body problem in gravitational field

$$\begin{aligned}
L &= \frac{1}{2}m_1\dot{r}_1^2 + \frac{1}{2}m_2\dot{r}_2^2 - [m_1gz_1 + m_2gz_2 + U(r)] \\
&= \left[\frac{1}{2}M\dot{R}^2 - MgZ + \right] + \left[\frac{1}{2}\mu\dot{r}^2 - U(r) \right]
\end{aligned} \tag{24}$$

where Z is the vertical coordinate of the CM position, $Z = \frac{m_1z_1 + m_2z_2}{M}$

4.5 Kepler's second Law

We calculate the differential of area swept by particle in polar coordinates,

$$\begin{aligned}
dA &= \frac{1}{2}r^2 d\varphi \\
\Rightarrow \frac{dA}{dt} &= \frac{1}{2\mu} \overrightarrow{L}_z \\
\overrightarrow{L}_z &= 2\mu\dot{A}(\text{constant})
\end{aligned} \tag{25}$$

This is the Kepler's second law, which states that the area swept by the radius in a given time is constant.

4.6 EOM for two body system

- The total energy:

$$\begin{aligned}
E = T + U &= \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\mu r^2 \dot{\varphi}^2 + U(r) \\
&= \frac{1}{2}\mu\dot{r}^2 + U(r) + \frac{L_z^2}{2\mu r^2} \quad (\text{Notice } L_z = \mu r^2 \dot{\varphi})
\end{aligned} \tag{26}$$

solving this ODE by integration gives

$$t(r) = \int \frac{dr}{\sqrt{\frac{2}{\mu} \left[E - U(r) - \frac{L_z^2}{2\mu r^2} \right]}} + C \tag{27}$$

- Also from $L_z = \mu r^2 \dot{\varphi}$, by integrating with respect to time, we get

$$\varphi(t) = \frac{L_z}{\mu} \int \frac{dt}{r^2(t)} + C' \tag{28}$$

Equation 28 and Equation 26 describe the relative motion of the two body system in terms of constants $\{E, L_z, C, C'\}$

4.7 Shape of orbit

- Equation 26 skipped a step,

$$\frac{dr}{dt} = \sqrt{\left(\frac{2}{\mu}\right) \left[E - U(r) - \frac{L_z^2}{2\mu r^2} \right]} \tag{29}$$

this equation, combined with our beloved

$$L_z = \mu r^2 \dot{\varphi} \Rightarrow d\varphi = \frac{L_z}{\mu r^2} dt \quad (30)$$

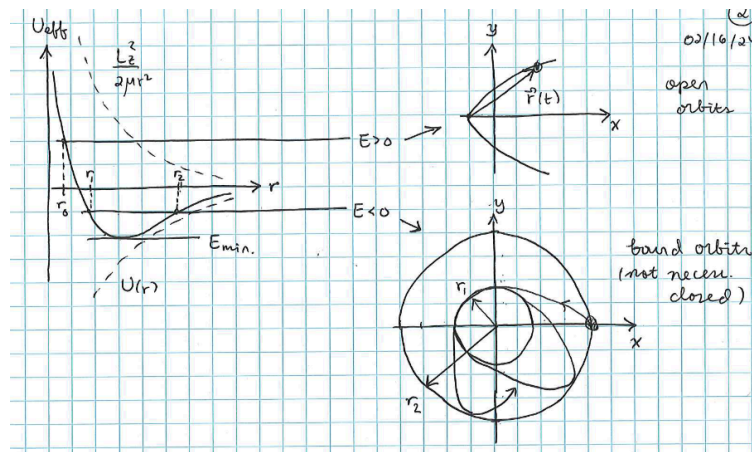
we get the equation of orbit:

$$\begin{aligned} d\varphi &= \frac{L_z}{\sqrt{2\mu}} \frac{dr}{r^2 \sqrt{E - U(r) - \frac{L_z^2}{2\mu r^2}}} \\ \Rightarrow \varphi &= \frac{L_z}{\sqrt{2\mu}} \int \frac{dr}{r^2 \sqrt{E - U(r) - \frac{L_z^2}{2\mu r^2}}} + C \end{aligned} \quad (31)$$

4.8 Effective potential and shape of orbit (Only for Attractive Potential)

$$U_{\text{eff}} = U(r) + \frac{L_z^2}{2\mu r^2}; E = \frac{1}{2}\mu \dot{r}^2 + U_{\text{eff}}(r) \quad (32)$$

- When $r \rightarrow \infty$, $U_{\text{eff}} \rightarrow U(r)$, and when $r \rightarrow 0$, $U_{\text{eff}} \rightarrow$ centrifugal potential $\frac{L_z^2}{2\mu r^2}$.
- by graphing the effective potential, and given constraint of total energy E , we can analyze the shape of the orbit:



- when $E > 0$, the orbit is unbounded, open orbit, hyperbola.
- when $E < 0$, the orbit is bounded into a potential well, although not necessarily closed.
- when $E = E_{\text{min}}$, the orbit is circular, $F = -\mu \frac{v^2}{r}$

5 The Kepler Problem: a special case of the two body problem

5.1 conditions

$$U(r) = -\frac{\alpha}{r}; U_{\text{eff}} = -\frac{\alpha}{r} + \frac{L_z^2}{2\mu r^2} \quad (33)$$

5.2 Conic section orbits

We can prove that the orbit is a conic section given by

$$r(\varphi) = \frac{p}{1 + e \cos(\varphi)} \quad (35)$$

$$\text{where } \begin{cases} p = \frac{L_z^2}{\mu\alpha} \\ e = \sqrt{1 + \frac{2EL_z^2}{\mu\alpha^2}} \end{cases} \quad (34)$$

5.3 Classifications of orbits based on energy of system E

- When $E > 0, e > 1$, the orbit is unbounded, open orbit, hyperbola.

$$\begin{aligned} \frac{(x-c)^2}{a^2} - \frac{y^2}{b^2} &= 1 \\ \begin{cases} a = \frac{p}{e^2-1}, b = \frac{p}{\sqrt{e^2-1}}, c = ae \\ r_{\min} = \frac{p}{1+e} \end{cases} \end{aligned} \quad (36)$$

- when $E = 0, e = 1$, the orbit is parabola.

$$\begin{aligned} y^2 &= p^2 - 2xp, \\ r_{\min} &= \frac{p}{2} \end{aligned} \quad (37)$$

- when $E < 0, e < 1$, the orbit is closed, ellipse.

$$\begin{aligned} \frac{(x+c)^2}{a^2} + \frac{y^2}{b^2} &= 1, \\ \begin{cases} a = \frac{p}{1-e^2}, b = \frac{p}{\sqrt{1-e^2}}, c = ae \\ r_{\min} = \frac{p}{1+e}; r_{\max} = \frac{p}{1-e} \end{cases} \end{aligned} \quad (38)$$

- When $E = E_{\min}, f = \frac{\mu\alpha^2}{2L_z^2}, e = 0$, orbit is circular. $r(\varphi) = p =$ constant

6 More Kepler: Period, Kepler's third law

6.1 Orbit of each body

recall Equation 19, we can express the orbit of each body as such after some algebra:

$$\vec{r}_1 = \frac{m_2}{m_1 + m_2} \vec{r}; \vec{r}_2 = -\frac{m_1}{m_1 + m_2} \vec{r} \quad (39)$$

- when $m_1 = m_2 \Rightarrow \vec{r}_1 = \frac{\vec{r}}{2}, \vec{r}_2 = -\frac{\vec{r}}{2}$, COM inside $r_1 \cap r_2$
- when $m_1 \gg m_2 \Rightarrow \vec{r}_1 = \vec{r}, \vec{r}_2 = 0$, m_1 is at rest, m_2 orbits m_1

6.2 Period of orbit

- $L_z = 2\mu\dot{A}$, areal vel. is constant
- Integrating \dot{A} over a period,

$$A = \int_0^T \dot{A} dt = \frac{L_z T}{2\mu} \quad (40)$$

Since area swept over a period is the area of the ellipse, we have

$$\pi ab = \frac{L_z T}{2\mu}, \text{ letting: } b = \sqrt{pa}, p = \frac{L_z^2}{\mu\alpha}$$

$$\Rightarrow T = (2\pi a^{3/2}) \sqrt{\frac{\mu}{\alpha}} \quad (41)$$

6.3 Conservation of Laplace-Runge-Lenz vector

$\vec{A} = \vec{v} \times \vec{L} - (\alpha \vec{r})/(r)$ is conserved, and is perpendicular to the orbit plane. We can use it to verify : conic sections, eccentricity, and period.

- conserved quantity: $\vec{A} \cdot \vec{L} = 0, \frac{A}{\alpha} = \sqrt{1 + \frac{2EL_z^2}{\mu\alpha^2}}$

7 Orbital Transfer

7.1 Instantaneous Change in velocity

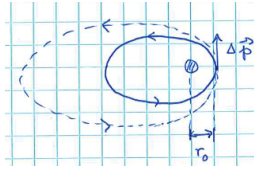
$$(E, L_z) \rightarrow (E', L'_z)$$

$$\Rightarrow (e, p) \rightarrow (e', p') \quad (42)$$

if thrust occur when satellite is at angle φ_0 , orbit orientation can change:

$$r(\varphi_0) = \frac{p}{1 + e \cos \varphi_0} = \frac{p'}{1 + e' \cos(\varphi_0 - \delta)} \quad (43)$$

7.2 Tangential thrust at perigee



at $\varphi = 0$, let $v = v_{\text{init}}, v' = v_{\text{right after}}, \lambda = v'/v$

$$L_z = \mu r_0 v \Rightarrow L'_z = \mu r_0 v' = \lambda L_z$$

$$p' = \lambda^2 p \quad (44)$$

From Equation 43,

$$\frac{p}{1 + e} = \lambda^2 \frac{p}{1 - e'} \Rightarrow e' = \lambda^2(1 + e) - 1 \quad (45)$$

if $\lambda > 1, e' > e$, the satellite is in a higher, more elliptical orbit.

Unbound if λ big enough

if $\lambda < 1, e' < e$, the satellite is in a lower orbit.

7.3 changing between circular orbits

- changing from R to R' , two thrusts(λ_1, λ_2) are needed. There is also an intermediate orbit

$$r(\varphi) = p'/(1 + e' \cos \varphi),$$

$$\text{where } p' = \lambda_1^2 p, e' = \lambda_1^2 - 1 \quad (46)$$

changed from indermetiade to final,

$$\begin{aligned} r(\varphi = \pi) &= R' = \lambda_2^2 R / (2 - \lambda_1^2) \\ \Rightarrow \lambda_1 &= \sqrt{\frac{2R'}{R + R'}} \end{aligned} \quad (47)$$

final orbit:

$$\begin{aligned} r(\varphi) &= R'; e'' = 0, p'' = R' \\ \Rightarrow p'' &= \lambda_2^2 p' = p' / (1 - e') \\ \Rightarrow \lambda_2 &= \sqrt{\frac{R + R'}{2R'}} \end{aligned} \quad (48)$$

7.4 Verify:

using Newton 2nd,

$$\begin{aligned} \frac{\alpha}{R^2} &= \mu \frac{v_R^2}{R} \\ \Rightarrow v_R &= \sqrt{\frac{\alpha}{\mu} R} \\ \Rightarrow \frac{V_{R'}}{V_R} &= \sqrt{\frac{R}{R'}} \end{aligned} \quad (49)$$

verify using conservation of angular momentum,

$$\begin{aligned} Rv(\varphi = 0) &= R'v(\varphi = \pi), \\ V_{R'} &= \lambda v(\varphi = \pi) = \lambda_2 \frac{v(\varphi = \pi)}{v(\varphi = 0)} \lambda_1 v_R \\ \Rightarrow \frac{v_{R'}}{v_R} &= \sqrt{\frac{R}{R'}} \end{aligned} \quad (50)$$

agrees with Equation 49 !!

8 Small Oscillations

- Motion near a point of stable equilibrium.

8.1 DOF= 1 (one dimension)

- For a system of DOF = 1, with potential $U(q)$:
 - **stable equilibrium** at $U(q)_{\min}$, upward parabola, where $F = -\frac{dU}{dq} = 0$
 - restoring force for small displacements $q - q_0$ is $F = -\frac{d^2U(q-q_0)}{dq^2}$
 - **Unstable equilibrium** at $U(q)_{\max}$, downward parabola, where $F = -\frac{dU}{dq} = 0$ as well.
- Consider small deviation from point of stable equilibrium, we use taylor expansion to show that it is really a small displacement. that is,

$$\begin{aligned}
U &\approx U(q_0) + \frac{dU(q_0)}{dq}(q - q_0) + \frac{d^2U(q_0)}{2dq^2}(q - q_0)^2 \\
&\text{while } \frac{dU(q_0)}{dq}(q - q_0) = 0
\end{aligned}
\tag{51}$$

letting $x = q - q_0$, we have

$$\begin{aligned}
&\begin{cases} U(x) = U(q_0) + \left(\frac{1}{2}\right) \frac{d^2U(q_0)}{dq^2} x^2 \\ \text{putting into the form of } U(x) = U(x_0) + \left(\frac{1}{2}\right) kx^2. \end{cases} \\
&\Rightarrow \boxed{k = \frac{d^2U(q_0)}{dq^2} > 0}
\end{aligned}
\tag{52}$$

we get KE, while choosing $U(q_0) = 0$:

$$\begin{aligned}
T &= \frac{1}{2} a(q)^2 \dot{q}^2 = \frac{1}{2} a(q_0 + x) \dot{x}^2 \approx \frac{1}{2} m \dot{x}^2, \quad \overset{m=a(q_0)}{\Rightarrow} \\
&\boxed{L = T - U = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2}
\end{aligned}
\tag{53}$$

8.1.1 EOM for DOF = 1 small Oscillations

using EL on Equation 53, we can get the EOM for one dimensional small Oscillations:

$$\begin{aligned}
&m\ddot{x} = -kx \\
&\Rightarrow \ddot{x} + \omega_0^2 x = 0, \text{ where } \boxed{\omega_0 = \sqrt{\frac{k}{m}} \text{ freq of osc.}}
\end{aligned}
\tag{54}$$

by magic of ODE, EOM reduces down to:

$$\begin{aligned}
&\boxed{x(t) = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t)} \\
&\text{where } C_1, C_2 \text{ are constants}
\end{aligned}
\tag{55}$$

by trig magic, this could also be written as

$$\begin{aligned}
&x(t) = a \cos(\omega_0 t + \alpha), \\
&\text{where } \begin{cases} a = \sqrt{C_1^2 + C_2^2} & \text{amplitude of oscillation} \\ \omega_0 & \text{frequency of oscillation} \\ \tan \alpha = C_2/C_1 & \text{phase at } t=0 \end{cases}
\end{aligned}
\tag{56}$$

8.1.2 energy for 1D small Oscillation

checking $\frac{\partial L}{\partial t} = 0 \Rightarrow$ energy-conservation:

$$\begin{aligned}
E = T + U &= \frac{1}{2} m \dot{x}^2 + \frac{1}{2} k x^2 \\
&= \frac{1}{2} m a^2 \omega_0^2, [\text{constant}]
\end{aligned}
\tag{57}$$

8.1.3 Damped 1D oscillation, and Complex representation

- when there is damping (friction, resistance, etc) $F_{\text{fric}} = -\beta\dot{x}$, the EOM becomes:

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0,$$

$$\text{where } 2\gamma = \frac{\beta}{m}, \omega_0 = \sqrt{\frac{k}{m}} \quad 58$$

with ansatz $x(t) = e^{rt}$, $\dot{x} = re^{rt}$, $\ddot{x} = r^2 e^{rt}$, the solution to Equation 58 is:

$$r^2 + 2\gamma r + \omega_0^2 = 0,$$

$$\text{which has solution } r_{\pm} = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2} \quad 59$$

$$\Rightarrow x(t) = C_1 e^{r_+ t} + C_2 e^{r_- t},$$

notice the r subscripts here: r_+ , r_-

8.1.4 underdamped, overdamped, and critically damped

Recall from your ODE class...

Equation 59 has the following 3 cases, each with different physical interpretation:

1. underdamped:

$$\gamma < \omega_0 \Rightarrow 2 \text{ complex roots: } \begin{cases} r_{\pm} = -\gamma \pm i\sqrt{\omega_0^2 - \gamma^2} \\ = -\gamma \pm i\omega \\ \omega = \sqrt{\omega_0^2 - \gamma^2} \end{cases} \quad 60$$

The EOM is thus a linear combination of two complex exponentials:

$$x(t) = e^{-\gamma t} (C_1 e^{i\omega t} + C_2 e^{-i\omega t})$$

$$= e^{-\gamma t} (A \cos(\omega t) + B \sin(\omega t))$$

$$\text{-- where } \begin{cases} A = C_1 + C_2 \\ B = i(C_1 - C_2) \end{cases} \quad 61$$

$$= a e^{-\gamma t} \cos(\omega t + \alpha)$$

a, α are constants

“The solution is a damped oscillation with frequency ω , and amplitude exponentially decaying with time.”

2. Overdamped

$$\gamma > \omega \Rightarrow x(t) =$$

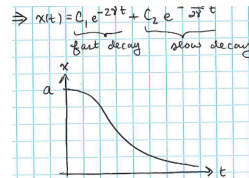
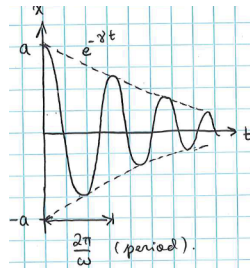
$$c_1 e^{-\gamma + \sqrt{\gamma^2 - \omega^2} t} + c_2 e^{-\gamma - \sqrt{\gamma^2 - \omega^2} t} \quad 62$$

$$\text{when } \gamma \gg \omega_0, \Rightarrow \begin{cases} \gamma + \sqrt{\gamma^2 - \omega_0^2} \approx 2\gamma \\ \gamma - \sqrt{\gamma^2 - \omega_0^2} = \frac{\omega_0^2}{2\gamma} \end{cases} \quad 63$$

$$x(t) = c_1 e^{-2\gamma t} + c_2 e^{(-\omega_0^2/2\gamma)t}$$

3. Critically damped

$$\gamma = \omega_0 \Rightarrow x(t) = c_1 e^{-\gamma t} + c_2 t e^{-\gamma t} \quad 64$$



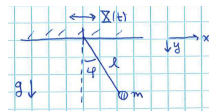
8.1.5 Forced Oscillations

When external force (F) is applied to the system, the lagrangian becomes

$$L = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2 + F(t)x \quad 65$$

$$\text{EL} \Rightarrow \ddot{x} + \omega_0^2 x = \frac{F(t)}{m}, \text{ where } \omega_0 = \sqrt{\frac{k}{m}}$$

- Example: Simple pendulum with moving pivot



$$\begin{cases} x = X + l \sin \varphi \\ y = l \cos \varphi \end{cases} \Rightarrow \begin{cases} \dot{x} = \dot{X} + l \dot{\varphi} \cos \varphi \\ \dot{y} = -l \dot{\varphi} \sin \varphi \end{cases}$$

$$\Rightarrow L = T - U$$

$$L = \frac{1}{2} m l^2 \dot{\varphi}^2 - mgl(1 - \cos \varphi) - ml\ddot{X} \sin \varphi$$

$$\text{Expand ab. } \varphi = 0 \Rightarrow L = \frac{1}{2} m l^2 \dot{\varphi}^2 - \frac{1}{2} mgl\varphi^2 - ml\ddot{X}\varphi \quad 67$$

$$\text{EL} \Rightarrow \ddot{\varphi} + \omega_0^2 \varphi = -\frac{\ddot{X}}{l}, \text{ where } \omega_0 = \sqrt{\frac{g}{l}}$$

8.1.6 reintroducing damping via external forcing

$$\ddot{x} + 2\gamma \dot{x} + \omega_0^2 x = f(t), f(t) = \frac{F(t)}{m} \quad 68$$

When damping $f(t) = f_0 \cos(\Omega t)$, solution via complex number:

$$\ddot{z} + 2\gamma\dot{z} + \omega_0^2 = f_0 e^{i\Omega t}$$

$$\text{ansatz } z(t) = z_0 e^{i\Omega t} \Rightarrow z_0 = \frac{f_0}{\omega_0^2 + 2i\gamma\Omega + \Omega_0^2}$$

$z_0 = a(\Omega) \cos(\Omega t + \delta(\Omega)) f_0$ is a particular solution, where

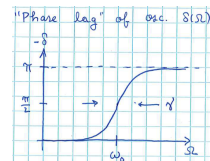
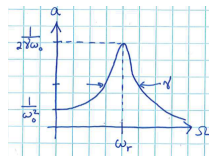
$$\begin{cases} a(\Omega) = \frac{1}{\sqrt{(\omega_0^2 - \Omega^2)^2 + (2\gamma\Omega)^2}} \\ \delta(\Omega) = \arctan\left(2\gamma \frac{\Omega}{\omega_0^2 - \Omega^2}\right) \end{cases}$$

We can study the properties of the system by looking at the amplitude and phase of the solution.

- Amplitude:

$$a(\Omega) = \frac{1}{\sqrt{(\omega_0^2 - \Omega^2)^2 + (2\gamma\Omega)^2}} \quad 70$$

, when $\gamma \ll \omega_0$, response strongest and amplitude largest when $\omega_r = \omega_0$.



- Phase lag: $\tan \delta(\Omega) = 2\gamma \frac{\Omega}{\Omega^2 - \omega_0^2}$

in phase as $\Omega \rightarrow 0$, and out of phase as $\Omega \rightarrow \omega_0$.

- Genral solution to sinusoidal forcing:

$$\begin{aligned} x(t) &= a(\Omega) f_0 \cos(\Omega t + \delta(\Omega)) + a_0 e^{-\gamma t} \cos(\omega t + \alpha) \\ &\xrightarrow{t > \frac{1}{\gamma}} a(\Omega) f_0 \cos(\Omega t + \delta(\Omega)) \end{aligned} \quad 71$$

Forgets initial condition after time.

- Power obsorbed by oscillation

$$p = F\dot{x} = m f \dot{x}$$

Avg power of oscillation

$$\begin{aligned} P_{\text{avg}} &= \frac{1}{T} \int_0^T m f \dot{x} dt = -\frac{1}{2} m f_0 a(\Omega) \Omega \sin \delta(\Omega) \\ &\text{simplifies to } P_{\text{avg}}(\Omega) = \gamma m f_0^2 \Omega^2 a_{(\Omega)}^2 \end{aligned} \quad 72$$

Absorption around resonance frequency $\Omega = \omega_0 + \varepsilon$ is maximum:

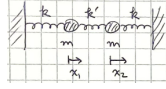
$$P = \frac{\gamma m f_0^2}{4(\varepsilon^2 + \gamma^2)} \approx \frac{m f_0^2}{4\gamma} \quad 73$$

8.2 Oscillations DOF>1

For a system with n DOF: $q = (q_1, q_2, \dots, q_n)$, PE = $U(q)$

- Stable equilibrium $\left. \frac{\partial U(q)}{\partial q_i} \right|_{q=0}$

8.2.1 Example: Oscillation with 2 mass and 3 springs



$$L = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2 - \frac{1}{2}kx_1^2 - \frac{1}{2}kx_2^2 - \frac{1}{2}k'(x_1 - x_2)^2$$

EOM:

$$M \cdot \ddot{\vec{x}} = -K\vec{x} \quad , \quad \text{where } M = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix},$$

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, K = \begin{pmatrix} k + k' & -k' \\ -k' & k + k' \end{pmatrix} \quad 74$$

ansatz: $\vec{x} = \text{Re}[\vec{a}e^{i\omega t}]$ Then the EOM eq becomes solving the eigenvalue problem:

$$\det(\omega^2 M - K) = 0$$

$$\Rightarrow \begin{cases} \omega_-^2 = \frac{k}{m} \\ \omega_+^2 = \frac{k+2k'}{m} \end{cases} \begin{cases} \vec{x}_- = a_- \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos(\omega_- t + \delta_-) \\ \vec{x}_+ = a_+ \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos(\omega_+ t + \delta_+) \end{cases} \quad 75$$

with constants $a_-, a_+, \delta_-, \delta_+$.

8.2.2 New Coords

$$\begin{cases} Q_1 = \sqrt{\frac{m}{2}}(x_1 + x_2) \\ Q_2 = \sqrt{\frac{m}{2}}(x_1 - x_2) \end{cases}$$

$$\Rightarrow L = \frac{1}{2}(\dot{Q}_1^2 + \dot{Q}_2^2) - \frac{1}{2}(\omega_-^2 Q_1^2 + \omega_+^2 Q_2^2) \quad 76$$

$$\stackrel{\text{E-L}}{\Rightarrow} \ddot{Q}_1 = -\omega_-^2 Q_1, \ddot{Q}_2 = -\omega_+^2 Q_2$$

Decoupled oscillators with coords Q_1, Q_2 .

8.2.3 General Coords

for general coords q_i , let $x_i = q_i - q_i^{(0)}$

$$U = \frac{1}{2} \sum_{i,j} k_{ij} x_i x_j, \quad k_{ij} = k_{ji} = \frac{\partial^2 U(q)}{\partial q_i \partial q_j} \text{ symm mat} \quad 77$$

$$T = \frac{1}{2} \sum_{i,j} m_{ij} \dot{x}_i \dot{x}_j, \quad m_{ij} = m_{ji} = a_{ij}(q^{(0)})$$

the largrangian, in Matix form:

$$L = \frac{1}{2} \dot{\vec{x}}^T \cdot M \cdot \dot{\vec{x}} - \frac{1}{2} \vec{x}^T \cdot K \vec{x} \stackrel{\text{EL}}{\Rightarrow} (\omega^2 M - K) \cdot \vec{a} = 0 \quad 78$$

$\Rightarrow \det(\omega^2 M - K) = 0$ Solving the det for omega gives the normal freq (Eigenvalues) of system ω_α^2 . plug in Eval into Equation 78 for eigenvec(normal modes) \vec{a}^α of system.

- General motion

$$x_i(t) = \sum_{\alpha} a_i^{\alpha} \text{Re}[C_{\alpha} e^{i\omega_{\alpha} t}] \quad 79$$

- EXAMPLE: Normal freq is given

$$\begin{aligned} \omega &= \{0, \sqrt{2}\omega_0, \sqrt{3}\omega_0\}. \\ \omega = \sqrt{2}\omega_0 &\Rightarrow a_1 = -a_3 = -a_2 = ae^{i\delta} \Rightarrow \\ \vec{\theta} &= a(1 \ -1 \ -1)^T \cos(\sqrt{2}\omega_0 t + \delta) \\ \omega = \sqrt{3}\omega_0 &\Rightarrow a_1 = 0, a_2 = -a_3 = ae^{i\delta} \Rightarrow \\ \vec{\theta} &= a(0 \ 1 \ -1)^T \cos(\sqrt{3}\omega_0 t + \delta) \end{aligned} \quad 80$$

- EXAMPLE: double pendulum

$$\begin{cases} x_1 = l_1 \sin \varphi_1 & y_1 = -l_1 \cos \varphi_1 \\ x_2 = l_1 \sin \varphi_1 + l_2 \sin \varphi_2 & y_2 = l_1 \cos \varphi_1 + l_2 \cos \varphi_2 \end{cases} \quad 81$$

$$\begin{aligned} \Rightarrow T &= \frac{1}{2} m_1 l_1 \dot{\varphi}_1^2 + \frac{1}{2} m_2 (l_1^2 \dot{\varphi}_1^2 + l_2^2 \dot{\varphi}_2^2 \\ &\quad + 2l_1 l_2 \dot{\varphi}_1 \dot{\varphi}_2 \cos(\varphi_1 - \varphi_2)) \end{aligned} \quad 82$$

$$U = -m_1 g l_1 \cos \varphi_1 - m_2 g (l_1 \cos \varphi_1 + l_2 \cos \varphi_2)$$

using $\cos \varphi \approx 1 - \frac{\varphi^2}{2}$

$$\begin{aligned} L &= \frac{1}{2} (\dot{\varphi}_1 \ \dot{\varphi}_2) \begin{pmatrix} (m_1 + m_2)l_1^2 & m_2 l_1 l_2 \\ m_2 l_1 l_2 & m_2 l_2^2 \end{pmatrix} (\dot{\varphi}_1 \ \dot{\varphi}_2) \\ &\quad - \frac{1}{2} (\varphi_1 \ \varphi_2) \begin{pmatrix} (m_1 + m_2)l_1 g & 0 \\ 0 & m_2 g l_2 \end{pmatrix} (\varphi_1 \ \varphi_2) \\ &= \frac{1}{2} \dot{\vec{\varphi}}^T M \cdot \dot{\vec{\varphi}} - \frac{1}{2} \vec{\varphi}^T K \vec{\varphi} \end{aligned} \quad 83$$

When $m_1 = m_2 = m$, $l_1 = l_2 = l \Rightarrow M = ml^2 \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, K = mgl \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$

$$\begin{aligned} \det((\omega^2 M - K)) &= 0 \Rightarrow \omega^2 = (2 \pm \sqrt{2}\omega_0^2) \\ \begin{pmatrix} a_1^- \\ a_2^- \end{pmatrix} &= C_- \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}, \quad \begin{pmatrix} a_1^+ \\ a_2^+ \end{pmatrix} = C_+ \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix} \end{aligned} \quad 84$$

8.3 Normal Coords

$\{x_i\} = \{Q_\alpha\}$, where $x_i = \sum_{\alpha=1}^n A_{i\alpha} Q_\alpha \Rightarrow$

$$\sum_j (\omega_\alpha^2 m_{ij} - k_{ij} A_{jx}) = 0$$

$$\Rightarrow L = \frac{1}{2} \sum_{\alpha=1}^n (\dot{Q}_\alpha^2 - \omega_\alpha^2 Q_\alpha^2) \xrightarrow{\text{EL}} \ddot{Q}_\alpha + \omega_\alpha^2 Q_\alpha = 0$$

9 Motion of Rigid Body

- EXample: rotor

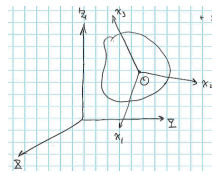
rotation with constraint $|\vec{r}_i - \vec{r}_j|$. COM coords are useful here

$$\begin{cases} \vec{r} = \vec{r}_1 - \vec{r}_2 \\ \vec{R} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2} \end{cases} \Rightarrow \begin{cases} \vec{r}_1 = \vec{R} + m_2 \vec{r} / M \\ \vec{r}_2 = \vec{R} - m_1 \vec{r} / M \end{cases} \quad 85$$

$$L = \frac{1}{2} M \dot{\vec{R}}^2 + \mu \dot{\vec{r}}^2, \quad \mu = m_1 \frac{m_2}{m_1 + m_2}$$

$$\xrightarrow{\text{polar}} L = \frac{1}{2} M \dot{\vec{R}}^2 + \frac{1}{2} \mu a^2 (\dot{\theta}^2 + \dot{\varphi}^2 \sin^2 \theta) \quad 86$$

9.1 frames of reference



$$(XYZ) \xrightarrow{R(\theta, \varphi, \psi)} (x_1, x_2, x_3)$$

Velocity of pt in body: $\vec{v} = \vec{V} + \vec{\Omega} \times \vec{r}$, where V is Translational vel, Omega is angular vel, r is position vector.

9.2 Lagrangian for Rigid Body

$$T = \frac{1}{2} M V^2 + \frac{1}{2} \sum_a m_a [\Omega^2 r_a^2 - (\vec{\Omega} \cdot \vec{r}_a)^2] \quad 87$$

$T_{\text{translational}} + T_{\text{rotational}}$

consider rotation,

$$\Omega^2 = \sum_i \Omega_i^2, \quad \vec{\Omega} \cdot \vec{r}_a = \sum_i \Omega_i x_{a,i}$$

$$\Rightarrow T_{\text{rot}} = \frac{1}{2} \sum_{i,j} \Omega_i \Omega_j I_{i,j}, \quad I_{i,j} \equiv \sum_a m_a (\delta_{ij} r_a^2 - x_{a,i} x_{a,j}) \quad 88$$

$$\Rightarrow L = \frac{1}{2} M V^2 + \frac{1}{2} \sum_{i,j} I_{i,j} \Omega_i \Omega_j - U$$

9.2.1 Inertial Tensor

- Discrete

$$I = \begin{pmatrix} \sum m(y^2 + z^2) & -\sum mxy & -\sum mxz \\ -\sum mxy & \sum m(x^2 + z^2) & -\sum myz \\ -\sum mxz & -\sum myz & \sum m(x^2 + y^2) \end{pmatrix} \quad 89$$

- Continuous

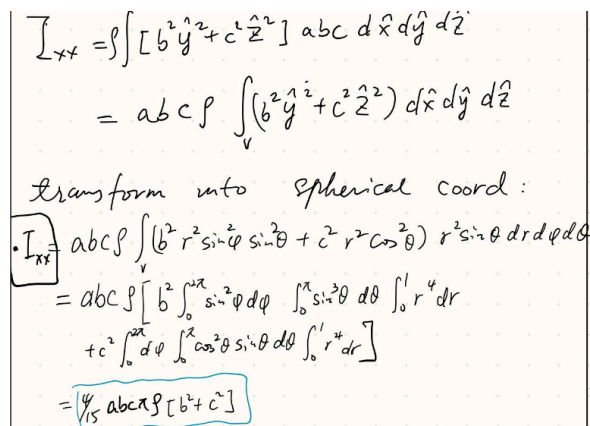
$$I_{ij} = \int \rho(x) (\delta_{ij} r^2 - x_i x_j) dV$$

$$I_{xx} = \int \rho(x) (y^2 + z^2) dV, I_{xy} = I_{yx} = - \int \rho(x) xy dV$$

$$I_{yy} = \int \rho(x) (x^2 + z^2) dV, I_{yz} = I_{zy} = - \int \rho(x) yz dV$$

$$I_{zz} = \int \rho(x) (x^2 + y^2) dV, I_{zx} = I_{xz} = - \int \rho(x) zx dV$$

example:



$$I_{xx} = \int [b^2 \hat{y}^2 + c^2 \hat{z}^2] abc d\hat{x} d\hat{y} d\hat{z}$$

$$= abc \int (b^2 \hat{y}^2 + c^2 \hat{z}^2) d\hat{x} d\hat{y} d\hat{z}$$

transform into spherical coord:

$$I_{xx} = abc \int (b^2 r^2 \sin^2 \varphi \sin^2 \theta + c^2 r^2 \cos^2 \theta) r^2 \sin \theta dr d\varphi d\theta$$

$$= abc \int [b^2 \int_0^{2\pi} \sin^2 \varphi d\varphi \int_0^\pi \sin^3 \theta d\theta \int_0^1 r^4 dr + c^2 \int_0^{2\pi} d\varphi \int_0^\pi \cos^2 \theta \sin \theta d\theta \int_0^1 r^4 dr]$$

$$= \frac{1}{15} abc \pi [b^2 + c^2]$$

- Example: coplanar system principal axis: Z $\Rightarrow I_{13} = I_{23} = 0$
 $I_3 = I_1 + I_2$

9.3 Principle axis and principal moments of inertia

In the principal frame:

$$T_{\text{rot}} = \frac{1}{2} (I_1 \Omega_1^2 + I_2 \Omega_2^2 + I_3 \Omega_3^2)$$

- spherical top $I_1 = I_2 = I_3$
- Symmetric top $I_1 = I_2 \neq I_3$
- Asymmetric top $I_1 \neq I_2 \neq I_3$
- EXample:

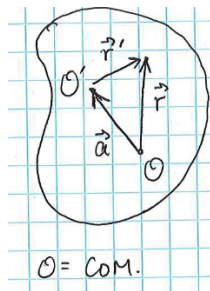
$$\det(I - \lambda \mathbf{1}) = 0 \Rightarrow \lambda \text{ prncp. mom.}$$

$$\vec{v} = \text{eigenvec.} = \text{prncp. axis}$$

- EXample: continuous with axis of symmetry $\rho(\vec{r}) = \rho = \rho(r, x_3) \Rightarrow I_{ij} = \int \rho(\vec{r}) (r^2 \delta_{ij} - x_i x_j) dV$

9.4 Parallel axis theorem

when changing Origin diff. from COM(O),



$$I'_{ij} = I_{ij} - M(a^2 \delta_{ij} - a_i a_j)$$

For a cube, when finding I at corner, first find I at COM, and

$$\begin{aligned} I'_{xx} &= I_{xx} + M(b^2 + c^2) = \frac{4}{3}M(b^2) + c^2 \\ I'_{yy} &= I_{yy} + M(a^2 + c^2) = \frac{4}{3}M(a^2 + c^2) \\ I'_{zz} &= I_{zz} + M(a^2 + b^2) = \frac{4}{3}M(a^2 + b^2) \end{aligned} \quad 93$$

$$\begin{aligned} I_{13} &= - \int dV \rho(\vec{r}) x_1 x_3 \\ &= - \int dx_3 \int r dr d\phi \rho(r, x_3) r \cos\phi x_3 \\ &= - \int dx_3 \int r dr \rho(r, x_3) r x_3 \underbrace{\int_0^{2\pi} d\phi \cos\phi}_{=0} \\ \& \quad I_{23} = 0 \quad \text{by same analysis w/ } \cos\phi \rightarrow \sin\phi. \\ \Rightarrow I &= \begin{pmatrix} I_{11} & I_{12} & 0 \\ I_{21} & I_{22} & 0 \\ 0 & 0 & I_3 \end{pmatrix} \Rightarrow x_3 = \text{principal axis;} \\ & \quad \text{i.e., symm. axis} = \text{princ. axis} \end{aligned} \quad 03/22/24$$

$$\begin{aligned} I_{12} &= - \int dV \rho(\vec{r}) x_1 x_2 \\ &= - \int dx_3 \int r dr d\phi \rho(r, z) r^2 \cos\phi \sin\phi \\ &= - \int dx_3 \int r dr d\phi \rho(r, z) r^2 \underbrace{\int_0^{2\pi} d\phi \cos\phi \sin\phi}_{=0} \\ \Rightarrow I &= \begin{pmatrix} I_{11} & 0 & 0 \\ 0 & I_{22} & 0 \\ 0 & 0 & I_3 \end{pmatrix}, \quad x_1, x_2, x_3 = \text{principal axes.} \end{aligned}$$

$$\begin{aligned}
 I_1 - I_2 &= \int dV \rho(\vec{r}) (x_i^2 - x_j^2) \\
 &= \int dx_3 \, r dr \, dz \, \rho(r, z) r^2 \underbrace{\int_0^{2\pi} d\varphi (\sin^2\varphi - \cos^2\varphi)}_{=0} \\
 \Rightarrow I_1 &= I_2 \equiv I_{\perp} \\
 \Rightarrow I &= \left(\begin{array}{cc|c} I_{\perp} & 0 & 0 \\ 0 & I_{\perp} & 0 \\ \hline 0 & 0 & I_3 \end{array} \right) \rightarrow \text{any two } \perp \text{ axes in } x_i x_i \text{-plane} \\
 &\quad \text{are principal axes.}
 \end{aligned}$$

10 Appendix

1. Taylor expansion:

$$f(x)|_0 \approx f(a) + f'(a)(x-a) + f''(a) \frac{(x-a)^2}{2} \quad 94$$

2. small angle approximation:

$$\sin(\theta) \approx \theta \quad \cos(\theta) \approx 1 - \frac{\theta^2}{2} \quad 95$$