

# Awesome applied analysis

## Notes on MATH 321 Harry Luo

The course contents could be better had it been Fabien's class, but probably Trinh saved my GPA.

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# I Vector algebra

## I.1 Coordinate Transformation

### I.1.1 cylindrical

$$x = \rho \cos \varphi$$

$$y = \rho \sin \varphi$$

$$z = z$$

reverse

$$\rho = \sqrt{x^2 + y^2}$$

$$\cos \varphi = \frac{x}{\rho}$$

$$\sin \varphi = \frac{y}{\rho}$$

### I.1.2 spherical

$$x = \rho \sin \varphi \cos \theta$$

$$y = \rho \sin \varphi \sin \theta$$

$$z = \rho \cos \varphi$$

reverse

$$\rho = \sqrt{x^2 + y^2 + z^2}$$

$$\cos \varphi = \frac{z}{\rho}$$

$$\cos \theta = \frac{x}{r}$$

$$\sin \theta = \frac{y}{r}$$

## I.2 Dot product

- commutative
- positive definite
- distributive
- cauchy-schwarz inequality

## I.3 cross product

- anticommutative  $\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$
- distributive  $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$
- scalar multiplication
- triple scalar product  $\vec{u} \cdot (\vec{v} \times \vec{w}) = (\vec{u} \times \vec{v}) \cdot \vec{w}$
- triple vector product  $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{b} \cdot \vec{a})\vec{c} - (\vec{c} \cdot \vec{a})\vec{b}$

## I.4 Projection

The projection of  $\vec{a}$  onto  $\vec{b}$  is given by

$$\frac{\vec{a} \cdot \vec{b}}{\|\vec{b}\|^2} \vec{b} = (a \cdot \hat{b}) \hat{b}$$

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## II Vector calculus

### II.1 Arc length

- Def: Given a curve  $\vec{r}(u) = (x(u), y(u), z(u))$  for  $a \leq t \leq b$  the length of the curve S, as a function of time is given by

$$S(t) = \int_a^t \|\dot{r}(u)\| du$$

where  $\|\dot{r}(u)\| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$

- Curvature:

$$K(t) = \frac{\|\dot{T}(t)\|}{\|\dot{r}(t)\|} = \frac{\|(\dot{r}(t) \times \ddot{r}(t))\|}{(\|\dot{r}(t)\|)^3}, \text{ where } T(t) = \frac{\dot{r}(t)}{\|\dot{r}(t)\|}$$

### II.2 Line integration

- for curve  $\vec{r}(t) = (x(t), y(t))$

$$\int_a^b f(x, y) ds = \int_a^b f[x(t), y(t)] \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

- center of mass  $(\bar{x}, \bar{y}, \bar{z})$ , where

$$\begin{cases} \bar{x} = \left(\frac{1}{M}\right) \int_C \rho(x, y, z) x ds \\ \bar{y} = \left(\frac{1}{M}\right) \int_C y \rho(x, y, z) ds \\ \bar{z} = \left(\frac{1}{M}\right) \int_C z \rho(x, y, z) ds \end{cases}$$

- Work done by force F along curve,  $\vec{r}(t)$ , which can be generalized into the formula for line integration,

$$W = \int_C \vec{F} \cdot d\vec{r} = \int_C \vec{F} \cdot \vec{T} ds = \int_a^b \vec{F}[x(t), y(t)] \cdot (\dot{r}(t)) dt$$

- When vector field  $\vec{F} = \vec{F}(x, y, z) = (P, Q, R)$ ,

$$\int_C \vec{F} \cdot d\vec{r} = \int_C P dx + Q dy + R dz$$

### II.3 Surface integration

- Parametric representation of surface:

$$\begin{cases} x = x(u, v) \\ y = y(u, v) \\ z = z(u, v) \end{cases}$$

- Use normal vector at a point  $(u_0, v_0)$  of surface to represent tangent plane.

$$\vec{r}_v = \frac{\partial \vec{r}}{\partial v}(u_0, v_0), \vec{r}_u = \frac{\partial \vec{r}}{\partial u}(u_0, v_0)$$

$$\vec{N} = \vec{r}_u \times \vec{r}_v$$

- Surface area of a surface S with  $(u, v) \in D$

$$A(S) = \iint_D \|\vec{r}_u \times \vec{r}_v\| \, du \, dv$$

## II.4 Jacobian

- Def: Given a transformation  $(u, v) \in D \longrightarrow [x(u, v), y(u, v)] \in S$ , the Jacobian is given by

$$J(u, v) = \frac{\partial(x, y)}{\partial(u, v)} \equiv \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$$

- Jacobian in coordinate transformation

Upon evaluating an integral, we can change the coordinates of the integral from  $\{x, y\} \rightarrow \{u, v\}$  by parametrize the variables:

$$x = x(u, v) \quad y = y(u, v)$$

Then the integral becomes

$$\iint_S f(x, y) \, dA = \iint_D f(x(u, v), y(u, v)) |J(u, v)| \, du \, dv$$

## II.5 Gradient

- Nabla operation:

$$\nabla = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}$$

- Gradient in cartesian Scalar field  $f = f(x, y, z)$

$$\nabla f = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$$

- Gradient in polar coordinates  $f = f(r, \theta)$

$$\nabla f = \vec{e}_r \frac{\partial g}{\partial r} + \vec{e}_\theta \frac{1}{r} \frac{\partial g}{\partial \theta}$$

$$\text{where } \vec{e}_r = \frac{x}{\|x\|} = (\cos \theta, \sin \theta) \vec{e}_\theta = (-\sin \theta, \cos \theta)$$

$$\nabla = \vec{e}_r \partial_r + \vec{e}_\theta \frac{1}{r} \partial_\theta$$

- Gradient in spherical

$$\nabla f = \hat{\rho} \partial_\rho + \hat{\varphi} \frac{1}{\rho} \partial_\varphi + \hat{\theta} \frac{1}{\rho \sin \varphi} \partial_\theta$$

- Gradient of scalar field in spherical coordinates

$$\text{Let } g(\rho, \varphi, \theta) = f(x, y, z)$$

$$\begin{cases} x = \rho \sin \varphi \cos \theta \\ y = \rho \sin \varphi \sin \theta \\ z = \rho \cos \varphi \end{cases} \quad \begin{bmatrix} \partial_\rho g \\ \partial_\varphi g \\ \partial_\theta g \end{bmatrix} = \begin{bmatrix} \partial_\rho x & \partial_\rho y & \partial_\rho z \\ \partial_\varphi x & \partial_\varphi y & \partial_\varphi z \\ \partial_\theta x & \partial_\theta y & \partial_\theta z \end{bmatrix} \begin{bmatrix} \partial_x f \\ \partial_y f \\ \partial_z f \end{bmatrix}$$

$$\begin{aligned} \hat{\rho} &= (\partial_\rho x, \partial_\rho y, \partial_\rho z) = \frac{(x, y, z)}{\rho} \\ \hat{\varphi} &= \frac{1}{\rho} (\partial_\varphi x, \partial_\varphi y, \partial_\varphi z) \\ \hat{\theta} &= \frac{1}{\rho \sin \varphi} (\partial_\theta x, \partial_\theta y, \partial_\theta z) \end{aligned} \quad \Rightarrow \quad \begin{bmatrix} \partial_x f \\ \partial_y f \\ \partial_z f \end{bmatrix} = \begin{bmatrix} \hat{\rho}_1 & \hat{\varphi}_1 & \hat{\theta}_1 \\ \hat{\rho}_2 & \hat{\varphi}_2 & \hat{\theta}_2 \\ \hat{\rho}_3 & \hat{\varphi}_3 & \hat{\theta}_3 \end{bmatrix} \begin{bmatrix} \partial_\rho g \\ \frac{1}{\rho} \partial_\varphi g \\ \frac{1}{\rho \sin \varphi} \partial_\theta g \end{bmatrix}$$

## II.6 Divergence

- div of vec field:

3D:

$$\nabla \cdot \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

- Div in polar 2D

$$\vec{U} = U_r \hat{r} + U_\theta \hat{\theta}, \text{ where } U_r = U \cdot \hat{r}, U_\theta = U \cdot \hat{\theta}$$

$$\nabla \cdot U = \left( \frac{1}{r} \right) \frac{\partial(r U_r)}{\partial r} + \frac{\partial U_\theta}{\partial \theta}$$

- Div in spherical coord

$$\vec{U} = U_\rho \hat{\rho} + U_\theta \hat{\theta} + U_\varphi \hat{\varphi},$$

$$\nabla \cdot \vec{U} = \frac{1}{\rho^2} \frac{\partial(\rho^2 U_\rho)}{\partial \rho} + \frac{1}{\rho} \sin \varphi \frac{\partial(U_\theta)}{\partial \theta} + \frac{1}{\rho \sin \varphi} \frac{\partial(U_\theta \sin \varphi)}{\partial \varphi}$$

## II.7 Green's theorem

For  $P(x, y)$ ,  $Q(x, y)$ , and a simple closed curve  $C$ ,

$$\int_C P dx + Q dy = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dA = \iint_C \vec{F} \cdot d\vec{r}$$

## II.8 Flux

- for a surface,

$$\vec{r}(u, v) = (x(u, v), y(u, v), z(u, v))$$

$$\Rightarrow \iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \vec{n} dS = \iint_D \vec{F}(\vec{r}(u, v)) \cdot (\vec{r}_u \times \vec{r}_v) dA$$

- if the surface is a graph of a function  $z = g(x, y)$  where  $(x, y) \in D$ ,  $\vec{F} = (P, Q, R)$ , then

$$\int_S \vec{F} \cdot d\vec{s} = \iint_D (P, Q, R) \cdot (-\partial_x g, -\partial_y g, 1) dA$$

## II.9 Stokes' theorem

Let  $F : R^3 \rightarrow R^3$  be a vector field on  $R^3$  with any normal vector  $\vec{n}$ , and for a surface  $S$  with projection on  $\{u, v\}$  being  $A$ , then

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl}(\vec{F}) \cdot \hat{n} dS = \iint_S (\nabla \times \vec{F}) \cdot \vec{n} dA,$$

$$\text{where } \text{curl}(\vec{F}) = \nabla \times \vec{F}$$

- Discussion on stokes theorem

for a surface surface parametrized by  $\vec{r}_u, \vec{r}_v$ , we have

$$d\vec{S} = \hat{n} dA = \vec{n} du dv$$

Therefore, when using stokes theorem, we can either turn it into a surface integral with respect to actual surface S, with

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## III Complex analysis

### III.1 Complex numbers and basic operations

#### III.1.1 Definitions

- Def:  $i^2 = -1$
- Complex number:  $z = x + iy$
- Conjugate:  $z = x - iy$
- Real part:  $\Re(z) = x$ , Imaginary part:  $\Im(z) = y$
- Modulus/ Norm/ Magnitude:  $|z| = \sqrt{x^2 + y^2}$
- Polar form:  $z = |z| (\cos \theta + i \sin \theta) = re^{i\theta}$
- Argument(angle) :  $\arg(z) = \theta$  such that  $z = |z| (\cos \theta + i \sin \theta)$ . Angle between vector  $(x, y)$  with real axis

#### III.1.2 operations

- addition:  $z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$

- multiplication:  $z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$   
(normal multiplication with  $i^2 = -1$ )
- Division:

$$\frac{z_1}{z_2} = \frac{z_1 z_2^*}{z_2 z_2^*} = \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + i \frac{x_2 y_1 - x_1 y_2}{x_2^2 + y_2^2}$$

- Commutativity:  $z_1 z_2 = z_2 z_1$      $z_1 + z_2 = z_2 + z_1$
- associativity:  $(z_1 z_2) z_3 = z_1 (z_2 z_3)$      $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$
- distributivity:  $z_1 (z_2 + z_3) = z_1 z_2 + z_1 z_3$
- Trig inequality:  $|z_1 + z_2| \leq |z_1| + |z_2|$

## III.2 Differentiation

### III.2.1 open sets in $\mathbb{C}$

- Def: Let  $z_0 \in \mathbb{C}$ ,  $r > 0$ . Disk  $B_{r(z_0)} = \{z \in \mathbb{C} \mid |z - z_0| < r\}$  It is very important to note that it's not "less or equal"

Given a set  $\Omega \in \mathbb{C}$ , A point  $z_0 \in \Omega$  is called an interior point of  $\Omega$  if there exists  $r > 0$  s.t.  $B_{r(z_0)} \subset \Omega$ .

A set  $\Omega$  is **open** if every point of  $\Omega$  is an interior point of  $\Omega$ . In other words, there are no points on the boundary of  $\Omega$  that are included in  $\Omega$ .

### III.2.2 Holomorphic function

Let  $\Omega$  be an open set in  $\mathbb{C}$ , A function  $f : \Omega \rightarrow \mathbb{C}$  is called **holomorphic** at  $z_0 \in \Omega$  if the limit

$$f'(z_0) = \lim_{h \rightarrow 0} \frac{f(z_0 + h) - f(z_0)}{h} \quad (h \in \mathbb{C}, h \neq 0)$$

exists.

- The said function  $f(z)$  is holomorphic on  $\Omega$  if it is holomorphic on every point of  $\Omega$ .
- In the special case that  $f$  is holomorphic on  $\mathbb{C}$ ,  $f$  is an **entire** function.
- Holomorphic in 1st order guarantees holomorphic and analytic in any order and thus continuous.

### III.2.3 Differentiation operations

If  $f$  and  $g$  are holomorphic on  $\Omega$ , then

- $f + g$  is holomorphic on  $\Omega$ ,

$$(f + g)' = f' + g'$$

- $fg$  is analytic on  $\Omega$ ,

$$(fg)' = f'g + fg'$$

- $\frac{f}{g}$  is analytic and, if  $g(z) \neq 0$ ,

$$\frac{f}{g} = \frac{f'g - fg'}{g^2}$$

### III.2.4 Cauchy-Riemann equations

for complex function  $f : \Omega \rightarrow \mathbb{C}$ ,  $f(z) = u(x, y) + iv(x, y)$  that is holomorphic at  $z_0 = x_0 + iy_0$ , then the partial derivatives of  $u$  and  $v$  exist and satisfy the Cauchy-Riemann equations:

$$\partial_x u = \partial_y v \quad \partial_y u = -\partial_x v$$

Conversely, if  $u$  and  $v$  are continuously differentiable on an open set  $\Omega$  and satisfy the Cauchy-Riemann equations, then  $f(z) = u(x, y) + iv(x, y)$  is holomorphic on  $\Omega$ .

In the language of logic, let  $C$  be “satisfying cauchy-riemann equations”, and  $H$  be “function is holomorphic”, then  $H \rightarrow C$ . If  $D$  is “ $u$  and  $v$  have continuous partial derivatives with respect to  $x$  and  $y$ ”, then  $(C \& D) \leftrightarrow H$

### III.3 Cauchy’s integral theorem (closed loop)

For a closed curve  $C$  in an open set  $\Omega$  and a holomorphic function  $f : \Omega \rightarrow \mathbb{C}$ , then

$$\oint_C f(z) dz = 0$$

### III.4 Fundamental theorem of calculus for complex analysis

If  $f$  is holomorphic on an open set  $\Omega$  and  $a, b \in \Omega$ , and for  $f(z) = F'(z)$ , we have

$$\int_C f(z) dz = F(b) - F(a)$$

### III.5 Poles and singularities

### III.6 Cauchy’s integral formula

This relates the value of a holomorphic function at a point to the value of its derivatives on a curve.

$$f^n(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} dz$$

Often times, we are concerned in finding the value of a function of the form

$$\int_C \frac{f(z)}{(z - z_0)^{n+1}} dz,$$

so we would like to take the  $n$ th derivative of the function  $f(z)$  at  $z_0$ , and find the desired integral by

$$\frac{2\pi i}{n!} f^n(z_0)$$

### III.7 Cauchy’s residue theorem

#### III.7.1 Poles

Simply find where the fraction is not defined, i.e. where the denominator is 0. This is normally done by first using  $(a^2 + z^2) = (z + ai)(z - ai)$  to factor the denominator, and set it to 0.



### III.7.2 Residue

If the factored denominator has the form  $(z + ai)(z + bi)$ , then it has two poles of order 1. If it has the form  $(z + ai)^2(z + bi)^2$ , then it has 2 poles of order 2.

If has poles of order one, for each pole  $z_0$ , find residue by

$$\text{Res}(f, z_0) = \lim_{z \rightarrow z_0} (z - z_0) f(z)$$

If has pole of order n, for each pole  $z_0$ , find res by

$$\text{Res}(f, z_0) = \lim_{z \rightarrow z_0} \frac{1}{(n-1)!} \left( \frac{d}{dz} \right)^{n-1} ((z - z_0)^n f(z))$$

### III.7.3 Cauchy's residue theorem

For a simple closed curve  $C$  in an open set  $\Omega$  and a holomorphic function  $f : \Omega \rightarrow \mathbb{C}$ , then

$$\oint_C f(z) dz = 2\pi i \sum_{k=1}^n \text{Res}(f, z_k)$$

where  $z_k$  are the poles of  $f$  in  $C$ .

Often times, we want to find the value of the integral

$$\int f(z) dz$$