Summary

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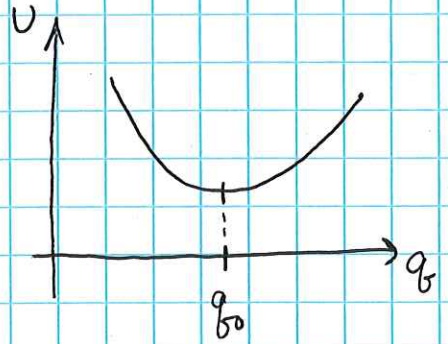
- Small oscillations in one dimension

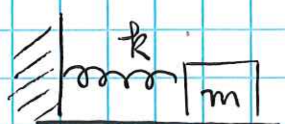


$$L = T - U = \frac{1}{2}\, a(q)\, \dot{q}^2 - U(q).$$

→ small deviations about q_0

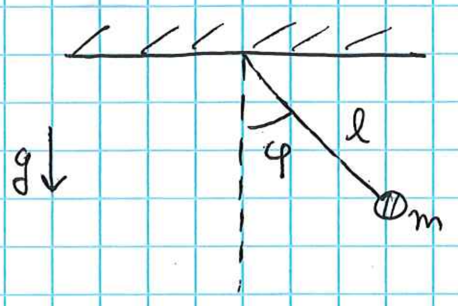
$$\Rightarrow L = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2, \qquad \begin{cases} m = a(q_0) \\ k = \left.\dfrac{d^2U}{dq^2}\right|_{q_0} > 0 \end{cases}$$

Ex: $U(x) = \frac{1}{2} k x^2$



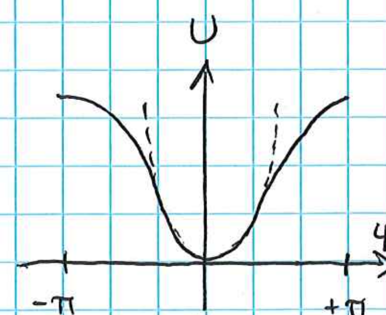
$$L = T - U = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2.$$

Ex: (simple pendulum).



$(q \to \varphi)$

$$\begin{cases} U(\varphi) = mgl(1 - \cos\varphi) \\ T = \frac{1}{2} m l^2 \dot{\varphi}^2 \end{cases}$$



φ near zero: $U(\varphi) \simeq \frac{1}{2} mgl\varphi^2$

$(\cos\varphi \simeq 1 - \frac{1}{2}\varphi^2)$

$$\Rightarrow L = T - U = \frac{1}{2} m l^2 \dot{\varphi}^2 - \frac{1}{2} mgl\varphi^2$$

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Coming back to $L = \frac{1}{2} m\dot{x}^2 - \frac{1}{2} k x^2$

E-L eqn.'s: $\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x}$

$$\Rightarrow \quad m\ddot{x} = -kx$$

$$\Rightarrow \boxed{\ddot{x} + \omega_0^2 x = 0}, \qquad \omega_0 = \sqrt{k/m}$$

· Soln.'s of EOM:

$$x_1(t) = \cos\omega_0 t$$

$$\rightarrow \ddot{x}_1 = -\omega_0^2 \cos\omega_0 t = -\omega_0^2 x_1 \Rightarrow \ddot{x}_1 + \omega_0^2 x_1 = 0 \quad ✓$$

$$x_2(t) = \sin\omega_0 t$$

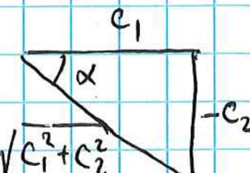
$$\rightarrow \ddot{x}_2 = -\omega_0^2 \sin\omega_0 t = -\omega_0^2 x_2 \Rightarrow \ddot{x}_2 + \omega_0^2 x_2 = 0$$

$\Rightarrow$ General sol.'n is a superposition:

$$\boxed{x(t) = C_1 \cos\omega_0 t + C_2 \sin\omega_0 t,} \qquad C_1, C_2 \text{: arbitrary const.'s}$$

Alternative parametrization:

$$x(t) = \sqrt{C_1^2 + C_2^2}\left[\underbrace{\frac{C_1}{\sqrt{C_1^2+C_2^2}}}_{\cos\alpha} \cos\omega_0 t + \underbrace{\frac{C_2}{\sqrt{C_1^2+C_2^2}}}_{-\sin\alpha} \sin\omega_0 t\right]$$



$$\Rightarrow \boxed{x(t) = a\cos(\omega_0 t + \alpha)}$$

$$\begin{cases} a = \sqrt{C_1^2 + C_2^2} \\ \tan\alpha = C_2/C_1 \end{cases}$$

$\begin{cases} a = \text{amplitude of oscillation} \\ \alpha = \text{phase corresponding to origin of time} \end{cases}$ $\qquad \omega_0$ = freq. of osc.

$\frac{\partial L}{\partial t} = 0 \Rightarrow$ energy conserved: $E = T + U$

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$$= \frac{1}{2} m\dot{x}^2 + \frac{1}{2} kx^2$$

$$= \frac{1}{2} m\left[-a\omega \sin(\omega t+\alpha)\right]^2 + \frac{1}{2} k\left[a\cos(\omega t+\alpha)\right]^2$$

$$= \frac{1}{2} ma^2\omega^2 \sin^2(\omega t+\alpha) + \frac{1}{2} ma^2\omega^2 \cos^2(\omega t+\alpha) \qquad \left(\omega^2 = \frac{k}{m}\right)$$

$$\Rightarrow E = \frac{1}{2} ma^2\omega^2$$

Complex parametrization: $z(t) = Ae^{i\omega t}$, $A = ae^{i\alpha} \in \mathbb{C}$

= "complex amplitude".

$$\Rightarrow x(t) = \mathrm{Re}\, z(t)$$

"real part".

→ often easier to work w/ $z(t)$ & take Re at the end.

Ex: $\dot{z} = i\omega z$, $\ddot{z} = -\omega^2 z$, etc...

→ $\dot{x} = \mathrm{Re}\,\dot{z} = -\omega\, \mathrm{Im}\, z = -a\omega \sin(\omega t+\alpha)$

"imaginary part".

→ $\ddot{x} = \mathrm{Re}\,\ddot{z} = -\omega^2 \mathrm{Re}\, z = -\omega^2 x = -\omega^2 \cos(\omega t+\alpha)$

In reality, motion is affected by surrounding environment. This interaction tends to retard the motion, as energy is dissipated to the environment. → "friction".

For motion w/ small velocities, model $F_{fr.} = -\beta\dot{x}$, $\beta > 0$.

⇒ EOM for "damped" harmonic oscillator become: $m\ddot{x} = -kx - \beta\dot{x}$

$$\rightarrow \boxed{\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0}, \quad 2\gamma = \frac{\beta}{m}, \quad \omega_0 = \sqrt{\frac{k}{m}}$$

Sol.^{n} to EOM: Consider $x(t) = e^{rt}$

$$\rightarrow \dot{x} = re^{rt} \quad \& \quad \ddot{x} = r^2 e^{rt}$$

$$\Rightarrow \ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = \left(r^2 + 2\gamma r + \omega_0^2\right)e^{rt} = 0$$

$$\Rightarrow r^2 + 2\gamma r + \omega_0^2 = 0$$

$$\Rightarrow r = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2}$$

2 sol.^{n}s for r.

$$\Rightarrow x(t) = C_1 e^{r_+ t} + C_2 e^{r_- t}, \qquad r_\pm = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2}$$

both decay as $t \to \infty$ since $\gamma > 0$

To understand better nature of sol.^{n}, consider cases.

(i) $\gamma < \omega_0 \Rightarrow r_\pm = -\gamma \pm i\sqrt{\omega_0^2 - \gamma^2}$

oscillating soln.^{n}s.

$$\Rightarrow x(t) = C_1 e^{-\gamma t} e^{i\omega t} + C_2 e^{-\gamma t} e^{-i\omega t}, \qquad \omega = \sqrt{\omega_0^2 - \gamma^2}$$

· The sol.^{n} should be real; i.e., $x^* = x$

complex conjugate

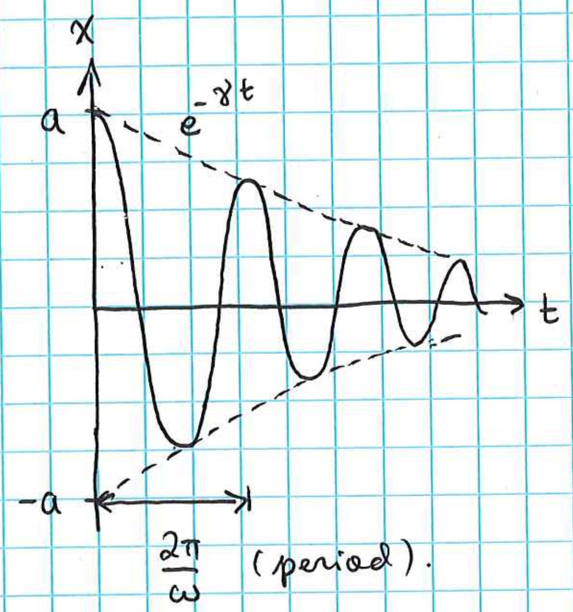
$$\Rightarrow x^*(t) = C_1^* e^{-\gamma t} e^{-i\omega t} + C_2^* e^{-\gamma t} e^{i\omega t} = x(t)$$

$$\Rightarrow C_2 = C_1^* \quad \& \text{ write } \quad C_1 = \frac{A}{2} = \frac{a}{2} e^{i\alpha}$$

$a, \alpha \in \mathbb{R}$ & $a \geq 0$.

$$\Rightarrow \boxed{x(t) = \mathrm{Re}\, \underbrace{A e^{-\gamma t} e^{i\omega t}}_{z(t)} = a e^{-\gamma t} \cos(\omega t + \alpha)}$$

The sol.'n $x(t) = a e^{-\gamma t} \cos(\omega_0 t + \alpha)$ describes "damped" oscillations; i.e., osc. w/ exponentially decaying amplitude



γ = "decay parameter".

"underdamped motion".

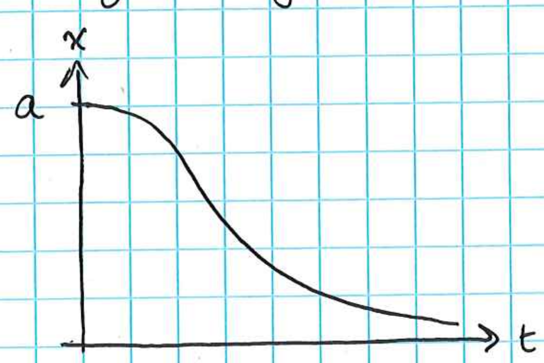
(ii) $\gamma > \omega_0 \Rightarrow x(t) = C_1 e^{-(\gamma + \sqrt{\gamma^2 - \omega_0^2})t} + C_2 e^{-(\gamma - \sqrt{\gamma^2 - \omega_0^2})t}$

$\rightarrow$ exponentially decaying soln.'s.

To understand better character of sol.'n, consider $\gamma \gg \omega_0$:

$$\begin{cases} \gamma + \sqrt{\gamma^2 - \omega_0^2} \simeq 2\gamma \\ \gamma - \sqrt{\gamma^2 - \omega_0^2} = \gamma - \gamma\sqrt{1 - \frac{\omega_0^2}{\gamma^2}} \simeq \gamma - \gamma\left(1 - \frac{\omega_0^2}{2\gamma^2}\right) \\ \qquad = \frac{\omega_0^2}{2\gamma}. \end{cases}$$

$$\Rightarrow x(t) = \underbrace{C_1 e^{-2\gamma t}}_{\text{fast decay}} + \underbrace{C_2 e^{-\frac{\omega_0^2}{2\gamma} t}}_{\text{slow decay}}$$



"overdamped motion"

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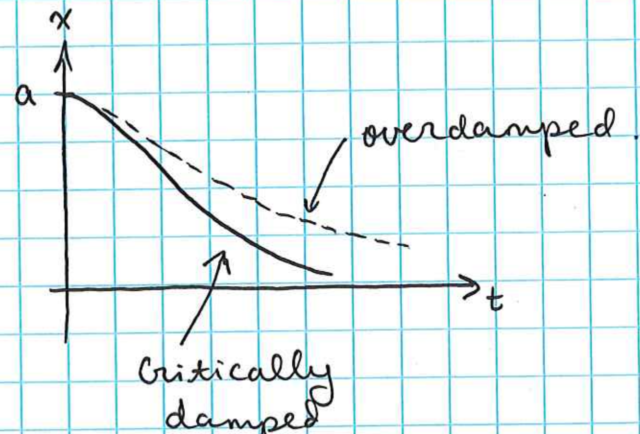
(iii) $\gamma = \omega_0 \rightarrow \quad x(t) = C_1 e^{-\gamma t} + \underbrace{C_2 t e^{-\gamma t}}$

Check that this is a sol'n.

"critical damping"

→ fastest decay.



Forced oscillations.

- w/o driving force we have "free" oscillations. Now consider the case of an external force → "forced" osc.

$$\rightarrow L = \frac{1}{2} m\dot{x}^2 - \frac{1}{2} kx^2 - U_{ext}(x,t)$$

external potential

expand for small x: $U_{ext}(x,t) = \underbrace{U_{ext}(0,t)}_{} + \underbrace{\left.\frac{\partial U_{ext}}{\partial x}\right|_{x=0}}_{} x + \ldots$

purely a fn. of t & may be omitted from L

$= -F_{ext}(t)$ a t-dep. force

$$\Rightarrow L = \frac{1}{2} m\dot{x}^2 - \frac{1}{2} kx^2 + F_{ext}(t)\, x$$

E-L eqn.'s: $\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x} \rightarrow \quad \ddot{x} + \omega_0^2 x = \frac{F(t)}{m}, \quad \omega_0 = \sqrt{k/m}$

→ inhomogeneous diff eq. for $x(t)$.