Summary

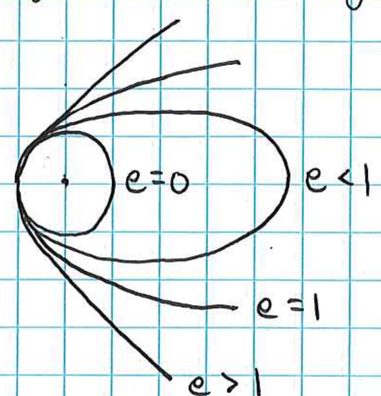
02/19/24

- Two-body problem: $L = \frac{1}{2} m_1 \dot{\vec{r}}_1^{\,2} + \frac{1}{2} m_2 \dot{\vec{r}}_2^{\,2} - U(|\vec{r}_1 - \vec{r}_2|)$
- Reduction to one-body problem: $L = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\varphi}^2) - U(r)$
- Kepler: $U(r) = -\dfrac{\alpha}{r}$

$\rightarrow$ shape of orbit $\quad r(\varphi) = \dfrac{p}{1 + e\cos\varphi}$ $\qquad p = \dfrac{L_z^2}{\mu\alpha}$

$$e = \sqrt{1 + \frac{2EL_z^2}{\mu\alpha^2}}$$

conic sections w/ one focus at origin:

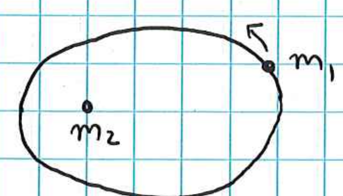


$$\left.\begin{aligned} \vec{r}_1 &= \frac{m_2}{m_1+m_2}\,\vec{r} \\ \vec{r}_2 &= -\frac{m_1}{m_1+m_2}\,\vec{r} \end{aligned}\right\}$$ orbit of each body

= conic section w/ focus at COM.

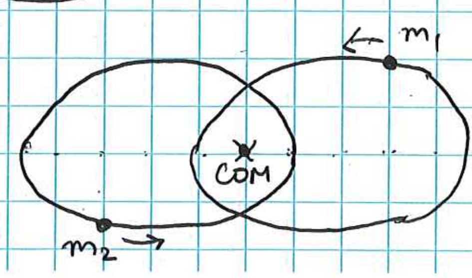
Ex: $m_2 \to \infty$. $\Rightarrow$ $\vec{r}_1 = \vec{r}$

$\vec{r}_2 = 0$



Ex: $m_1 = m_2$ $\Rightarrow$ $\vec{r}_1 = \vec{r}/2$

$\vec{r}_2 = -\vec{r}/2$



Period of (closed) Kepler orbit:

Recall: $L_z = 2\mu\dot{A}$, $\dot{A}$ = areal velocity = const. b/c of L_z conservation.

→ Integrate over a period (T):

$$A = \int_0^T dt\, \dot{A} = \frac{L_z T}{2\mu}$$

Area swept out by orbit is just area of ellipse:

$$A = \pi a b, \qquad \left.\begin{aligned} a &= \frac{p}{1-e^2} \\ b &= \frac{p}{\sqrt{1-e^2}} \end{aligned}\right\} \text{ semi-axes of ellipse.}$$

$$\Rightarrow \quad \pi a b = \frac{L_z T}{2\mu}$$

$$\Rightarrow \quad T = \frac{2\pi\mu}{L_z}\, ab$$

$b = \sqrt{pa}$

$$T = \frac{2\pi\mu}{L_z}\sqrt{p}\, a^{3/2}$$

$p = \frac{L_z^2}{\mu\alpha}$

$$T = 2\pi a^{3/2}\sqrt{\frac{\mu}{\alpha}}$$

→ i.e., square of period ~ cube of orbit size.

• One more remarkable fact about the Kepler problem: 02/19/24

→ we had to evaluate one integral to find orbits. In fact, Kepler problem possesses one more conserved quantity, allowing for <u>algebraic</u> sol'n of EOM:

$$\vec{A} = \vec{v} \times \vec{L} - \frac{\alpha \vec{r}}{r} \qquad \text{(Laplace-Runge-Lenz vector)}$$

• $\vec{A}$ is conserved:

$$\dot{\vec{A}} = \dot{\vec{v}} \times \vec{L} + \vec{v} \times \dot{\vec{L}} - \frac{\alpha \vec{v}}{r} + \frac{\alpha \dot{r} \vec{r}}{r^2}$$

$\dot{\vec{L}} = 0$, $\dot{r} = \frac{\vec{r}\cdot\vec{v}}{r}$

$$= \dot{\vec{v}} \times \vec{L} - \frac{\alpha \vec{v}}{r} + \alpha \frac{(\vec{r}\cdot\vec{v})\vec{r}}{r^3}$$

EOM: $\mu \dot{\vec{v}} = \vec{F} = -\frac{\alpha \vec{r}}{r^3}$

$$= -\frac{\alpha}{\mu r^3} \vec{r} \times \vec{L} - \frac{\alpha \vec{v}}{r} + \alpha \frac{(\vec{r}\cdot\vec{v})\vec{r}}{r^3}$$

$\vec{L} = \mu(\vec{r} \times \vec{v})$

$$= -\frac{\alpha}{r^3} \underbrace{\vec{r} \times (\vec{r} \times \vec{v})}_{\vec{r}(\vec{r}\cdot\vec{v}) - \vec{v}\, r^2} - \frac{\alpha \vec{v}}{r} + \alpha \frac{(\vec{r}\cdot\vec{v})\vec{r}}{r^3}$$

$$= \cancel{-\frac{\alpha \vec{r}(\vec{r}\cdot\vec{v})}{r^3}} + \cancel{\frac{\alpha \vec{v}}{r}} - \cancel{\frac{\alpha \vec{v}}{r}} + \cancel{\frac{\alpha \vec{r}(\vec{r}\cdot\vec{v})}{r^3}}$$

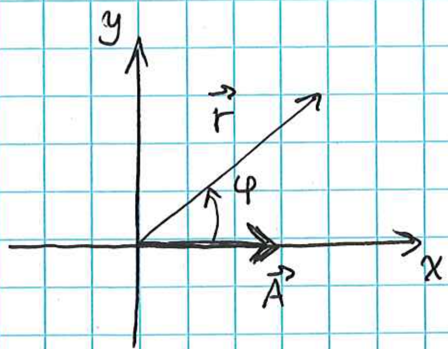
$$= 0 \quad \checkmark$$

• Note also: $\vec{A}\cdot\vec{L} = (\vec{v} \times \vec{L})\cdot\vec{L} - \alpha \frac{\vec{r}\cdot\vec{L}}{r} = 0$

$$\Rightarrow \vec{A} \perp \vec{L} \Rightarrow \vec{A} = \text{conserved vector}$$

which lies in orbital plane

Orient $\hat{x}$ along fixed direction of $\vec{A}$:   02/19/24



$$\vec{A}\cdot\vec{r} = Ar\cos\varphi.$$

$$\& \quad \vec{A}\cdot\vec{r} = (\vec{v}\times\vec{L})\cdot\vec{r} - \alpha\frac{\vec{r}\cdot\vec{r}}{r}$$

$$= \vec{r}\cdot(\vec{v}\times\vec{L}) - \alpha r$$

$$= \vec{L}\cdot(\vec{r}\times\vec{v}) - \alpha r \qquad |\vec{L}| = L_z$$

$$= \frac{1}{\mu}L_z^2 - \alpha r$$

$$\Rightarrow \quad Ar\cos\varphi = \frac{L_z^2}{\mu} - \alpha r$$

$$\Rightarrow \quad r(\varphi) = \frac{L_z^2/\mu\alpha}{1 + \frac{A}{\alpha}\cos\varphi} \quad \rightarrow \quad \text{conic section!}$$

& $\vec{A}$ points toward "perihelion" (= pt. of closest approach).

Last check: $$A^2 = \vec{A}\cdot\vec{A} = \left(\vec{v}\times\vec{L} - \frac{\alpha\vec{r}}{r}\right)\cdot\left(\vec{v}\times\vec{L} - \frac{\alpha\vec{r}}{r}\right)$$

$$= \underbrace{(\vec{v}\times\vec{L})\cdot(\vec{v}\times\vec{L})}_{v^2L_z^2 - (\cancel{\vec{v}\cdot\vec{L}})^2} + \alpha^2 - \frac{2\alpha}{r}\underbrace{\vec{r}\cdot(\vec{v}\times\vec{L})}_{\vec{L}\cdot(\vec{r}\times\vec{v}) = \frac{1}{\mu}L_z^2}$$

$$= \left(v^2 - \frac{2\alpha}{\mu r}\right)L_z^2 + \alpha^2 \qquad E = \frac{1}{2}\mu v^2 - \frac{\alpha}{r}$$

$$= \frac{2EL_z^2}{\mu} + \alpha^2$$

$$\Rightarrow \quad \frac{A}{\alpha} = \sqrt{1 + \frac{2EL_z^2}{\mu\alpha^2}} = e \quad \checkmark$$

agree w/ what we found before

Counting conserved quantities: 02/19/24

$$1\,E + 3\,\vec{L} + 3\,\vec{A} = 7 \text{ conserved quantities.}$$

→ but the effective one-body problem has 3 DOF

⇒ only 6 arbitrary const.'s allowed.

→ resolution: there are relations btwn. ~~con~~ the conserved quantities:

$$\vec{A}\cdot\vec{L} = 0 \quad \& \quad \frac{A}{\alpha} = \sqrt{1 + \frac{2EL_z^2}{\mu\alpha^2}}$$

⇒ 2 relations & hence $7-2=5$ indep. conserved quantities

(this is in fact the maximal allowed number, since initial time will always be ~~least~~ one arbitrary const.)

As an application of what we've learned, ~~so~~ we'll consider problem of changes of orbit. For example, we might ask: How does a satellite change from one orbit to another?

<u>Ex</u>: