Summary

02/16/24

- Two-body problem: $L = \frac{1}{2} m_1 \dot{\vec{r}}_1^{\,2} + \frac{1}{2} m_2 \dot{\vec{r}}_2^{\,2} - U(|\vec{r}_1 - \vec{r}_2|)$

- Reduction to effective one-body problem:

$$L = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\varphi}^2) - U(r) \qquad \mu = \frac{m_1 m_2}{m_1 + m_2}.$$

6 DOF $\rightarrow$ 2 DOF.

- Conservation laws:

$$\left.\begin{aligned} L_z &= \mu r^2 \dot{\varphi} \\ E &= \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\varphi}^2) + U(r) \end{aligned}\right\} \qquad \begin{aligned} \dot{L}_z &= 0 \\ \dot{E} &= 0 \end{aligned}$$

- sol'n of EOM:

$$t = \int \frac{dr}{\sqrt{\frac{2}{\mu}\left[E - U(r) - \frac{L_z^2}{2\mu r^2}\right]}} + \text{const.}$$

↳ invert for $r(t)$.

$$\varphi(t) = \frac{L_z}{\mu}\int \frac{dt}{r^2(t)} + \text{const.}$$

shape of orbit: $\varphi = \frac{L_z}{\sqrt{2\mu}} \int \frac{dr}{r^2\sqrt{E - U(r) - \frac{L_z^2}{2\mu r^2}}} + \text{const.}$

- General prop.'s of orbits from effective potential:

$$E = \frac{1}{2}\mu\dot{r}^2 + U_{eff}(r)$$

$$U_{eff}(r) = U(r) + \frac{L_z^2}{2\mu r^2}$$

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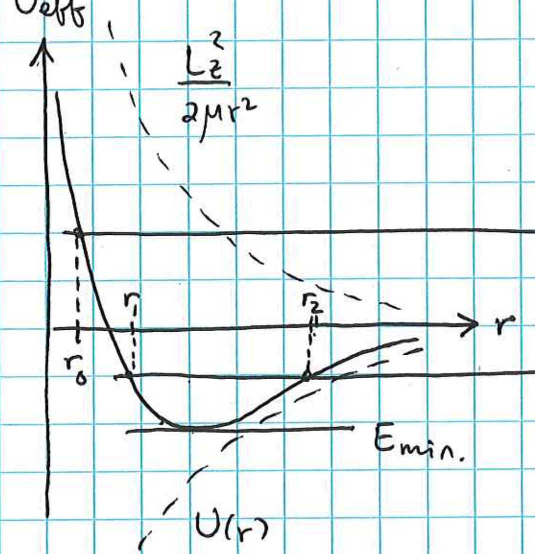
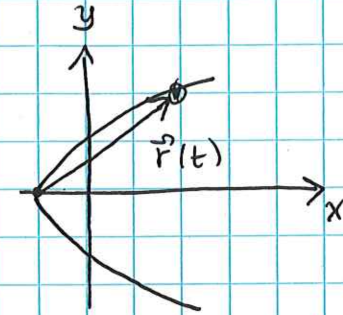
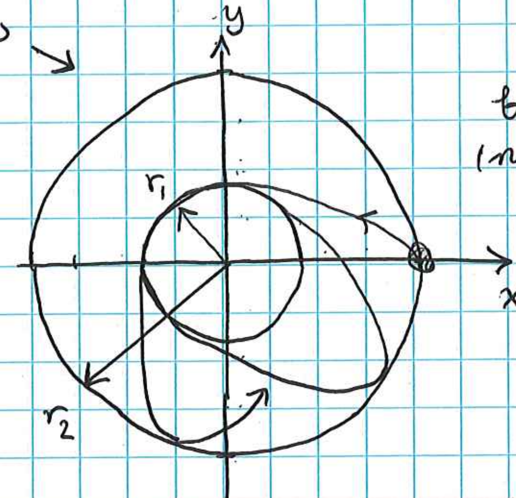




open orbits



bound orbits (not necess. closed)

• $E = E_{min.}$ $\quad r_1 = r_2$ & orbit in a circle $\dot{r} = 0$.

Radius of circle determined by $\frac{dU_{eff}}{dr} = 0$.

$$\Rightarrow \frac{dU}{dr} = -\frac{d}{dr}\left(\frac{L_z^2}{2\mu r^2}\right) = \frac{L_z^2}{\mu r^3} = \mu r \dot{\varphi}^2 = \frac{\mu v_\varphi^2}{r} \qquad (v_\varphi = r\dot{\varphi})$$

$$F = -\frac{dU}{dr} \Rightarrow F = -\frac{\mu v_\varphi^2}{r}$$

$$\text{or } |F| = \frac{\mu v_\varphi^2}{r} \qquad (F<0)$$

→ applied force = centripetal accel.

Behavior we have found for orbits is generic for an attractive potential (open, bounded, circular) so long as: 02/16/24

(1) $U(r)$ dominates over centrifugal potential as $r \to \infty$

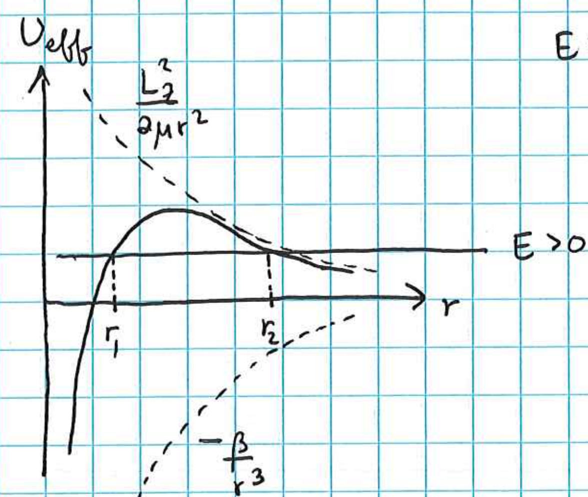
$\Rightarrow U(r)$ falls off slower than $1/r^2$ as $r \to \infty$.

(2) Centrif. pot. dominates over $U(r)$ as $r \to 0$.

$\Rightarrow U(r)$ diverges slower than $1/r^2$ as $r \to 0$.

In other cases, if (1) or (2) not satisfied, orbits will be qualitatively different.

<u>Ex:</u> $U = -\frac{\beta}{r^3}$



$E \geq U_{eff}(r)$

- If ~~particle~~ initially $r \leq r_1$, then $0 \leq r \leq r_1$ $\forall t$ & particle eventually "falls" into force center.
- If initially $r \geq r_2$ then orbit is open & particle never gets inside "potential hole"

Kepler problem

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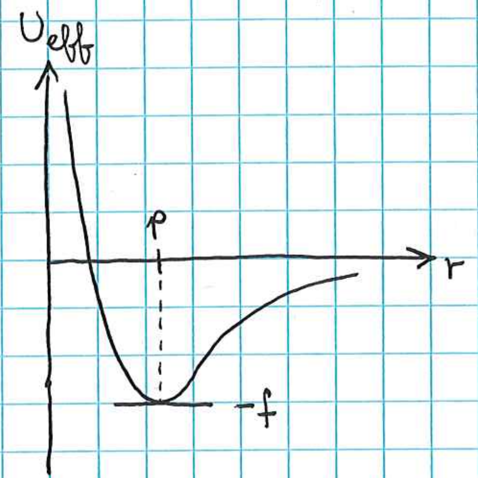
$$U(r) = -\frac{\alpha}{r} \qquad U_{eff}(r) = -\frac{\alpha}{r} + \frac{L_z^2}{2\mu r^2} \qquad (\alpha > 0).$$



· Consider first the shape of the orbit:

$$\varphi = \frac{L_z}{\sqrt{2\mu}} \int \frac{dr}{r^2\sqrt{E + \frac{\alpha}{r} - \frac{L_z^2}{2\mu r^2}}}$$ (set const. = 0 → choice of origin for φ).

· The integral is a bit messy. → useful to introduce "natural units" for the problem to simplify.

In our case, a natural unit for length is the radius p of the circular orbit & the corresponding value of the energy $-f = U_{eff}(p)$

$$\frac{dU_{eff}}{dr} = 0 \quad \Rightarrow \quad p = \frac{L_z^2}{\mu\alpha}$$

$$\Rightarrow \quad f = -U_{eff}(p) = \frac{\mu\alpha^2}{2L_z^2} = \frac{\alpha}{2p}$$

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Setting $s = r/p$:

$$\varphi = \int \frac{ds}{s^2\sqrt{\frac{E}{f} + \frac{2}{s} - \frac{1}{s^2}}} \qquad s = \frac{1}{u}$$

$$= -\int \frac{du}{\sqrt{\frac{E}{f} + 2u - u^2}}$$

$$= \cos^{-1}\left(\frac{u-1}{\sqrt{\frac{E}{f}+1}}\right)$$

$$\Rightarrow \quad \cos\varphi = \frac{u-1}{\sqrt{\frac{E}{f}+1}}$$

$$\Rightarrow \quad u = 1 + \sqrt{1+E/f}\,\cos\varphi$$

Setting $u = p/r$ & defining $e = \sqrt{1+E/f}$:

$$\boxed{r(\varphi) = \frac{p}{1 + e\cos\varphi}}$$

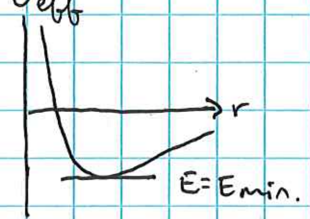
$$\left.\begin{aligned} p &= \frac{L_z^2}{\mu\alpha} \\ e &= \sqrt{1 + \frac{2EL_z^2}{\mu\alpha^2}} \end{aligned}\right\}$$

This is eqn. for conic section (circle, ellipse, parabola, hyperbola) w/ "eccentricity" e & "latus rectum" p & one focus at the origin.

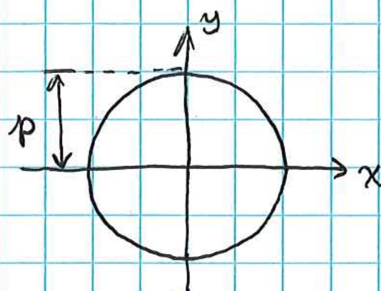
$\Rightarrow$ Kepler orbits are conic sections!

Classification of orbits: (based on energy E). 02/16/24

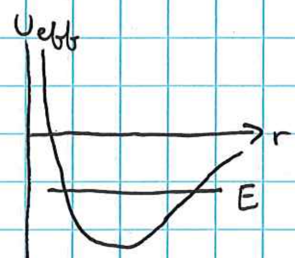
• $E = E_{min.} = -f = -\frac{\mu\alpha^2}{2L_z^2}$ $\Rightarrow$ $e = 0$

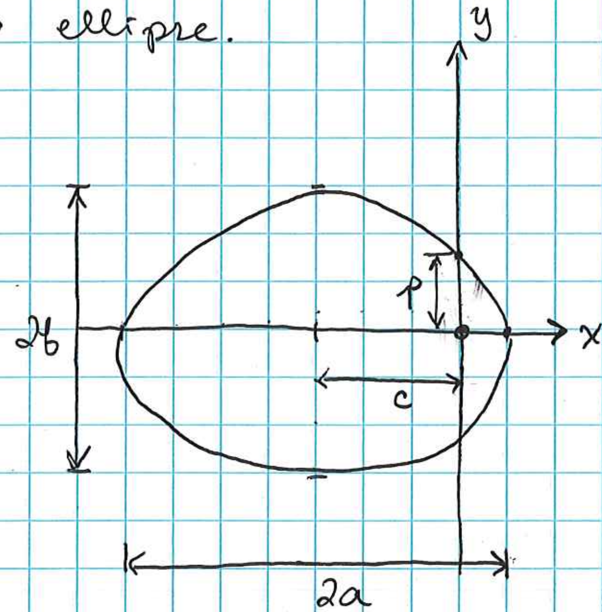


$\Rightarrow$ $r(\varphi) = p = \text{const.}$; i.e. a circle.



• $E < 0 \Rightarrow e < 1 \longleftrightarrow$ ellipse.





$$
\begin{cases}
\dfrac{(x+c)^2}{a^2} + \dfrac{y^2}{b^2} = 1 \\[2ex]
a = \dfrac{p}{1-e^2}, \quad b = \dfrac{p}{\sqrt{1-e^2}}, \quad c = ae \\[2ex]
r_{min.} = \dfrac{p}{1+e} \qquad r_{max.} = \dfrac{p}{1-e}
\end{cases}
$$

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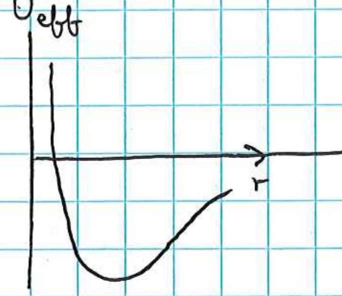
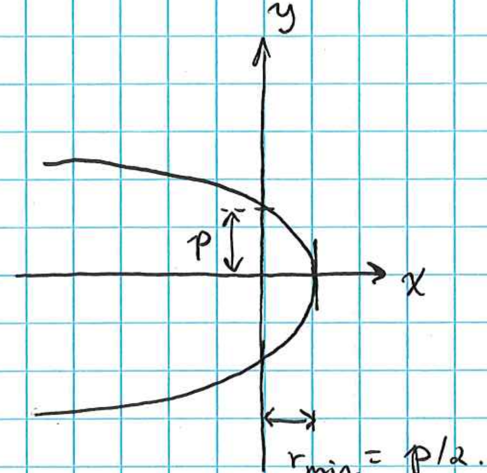
- $E=0 \Rightarrow e=1 \longleftrightarrow$ parabola.

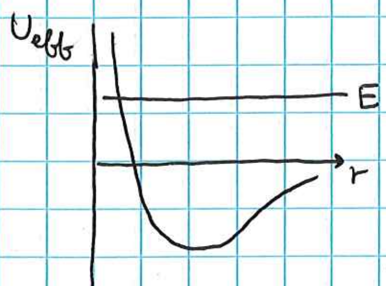


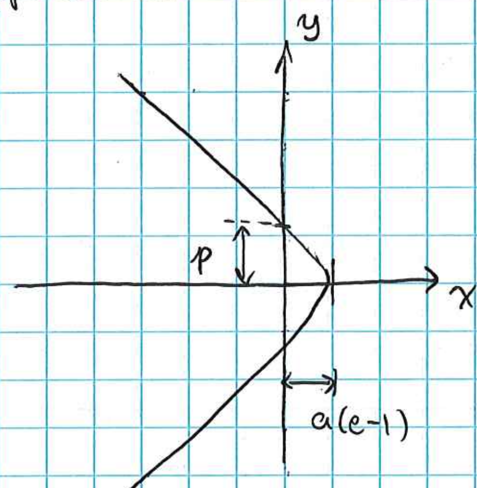


→ corresponds to particle starting from rest at ∞

$$\begin{cases} y^2 = p^2 - 2xp. \\ r_{min.} = p/2. \end{cases}$$

- $E>0 \Rightarrow e>1 \longleftrightarrow$ hyperbola.





$$\begin{cases} \dfrac{(x-c)^2}{a^2} - \dfrac{y^2}{b^2} = 1 \\ a = \dfrac{p}{e^2-1}, \quad b = \dfrac{p}{\sqrt{e^2-1}}, \quad c = ae \\ r_{min.} = \dfrac{p}{1+e} \end{cases}$$