Summary

01/31/24

- $S[q(t)] = \int_{t_1}^{t_2} dt \; L(q, \dot{q}, t)$

$$\delta S = 0 \Rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i} \qquad i = 1, \dots, n$$

- $$L = \underbrace{\sum_{a=1}^{N} \frac{1}{2} m_a v_a^2}_{\text{Galilean invariance}} - U(\vec{r}_1, \dots, \vec{r}_N) = T - U$$

E-L. eqn.'s $\Rightarrow$ $m\dot{\vec{v}}_a = -\dfrac{\partial U(\vec{r}_1, \dots, \vec{r}_N)}{\partial \vec{r}_a}$

$= \vec{F}_a$ (force) $\rightarrow$ comments on dissipative forces.

External fields:

- The Lagrangian $L = T - U$ also applies when U contains contribution from external field, that is, $U_{ext} = U_{ext}(\vec{r}, t)$ is a prescribed fn. of t.

Ex: single particle in ext. field $U(\vec{r}, t)$:

$$L = \frac{1}{2} m v^2 - U(\vec{r}, t)$$

E-L eqn.'s: $\dfrac{d}{dt}\dfrac{\partial L}{\partial \vec{v}} = \dfrac{\partial L}{\partial \vec{r}}$

$$\Rightarrow m\dot{\vec{v}} = -\frac{\partial U}{\partial \vec{r}}$$

Lagrangian in generalized coord.'s $(q_1, q_2, \dots, q_n)$: 01/31/24

$$\vec{r}_a = \vec{r}_a(q_1, \dots, q_n) \quad , \quad a = 1, \dots, N$$

$N = \#$ of particles

$$\dot{\vec{r}}_a = \sum_{i=1}^{n} \frac{\partial \vec{r}_a}{\partial q_i} \dot{q}_i$$

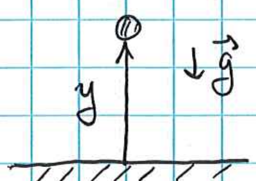
$$\Rightarrow T = \sum_{a=1}^{N} \frac{1}{2} m_a v_a^2 = \sum_{ij} \frac{1}{2} a_{ij}(q) \dot{q}_i \dot{q}_j$$

$$\Rightarrow L = T - U = \sum_{ij} \frac{1}{2} a_{ij}(q) \dot{q}_i \dot{q}_j - U(q)$$

also applies to constrained systems w/ $n < 3N$. when constraints can be expressed as relations btwn. coord.'s → "holonomic constraints"

Work through various examples, starting w/ $n=1$ DOF.

Ex: motion in a uniform g-field.



$$q \to y$$

$$T = \frac{1}{2} m v^2 = \frac{1}{2} m \dot{y}^2$$

$$U = mgy$$

$$\Rightarrow L = T - U = \frac{1}{2} m \dot{y}^2 - mgy.$$

E-L. eqn: $\frac{d}{dt} \frac{\partial L}{\partial \dot{y}} = \frac{\partial L}{\partial y}$

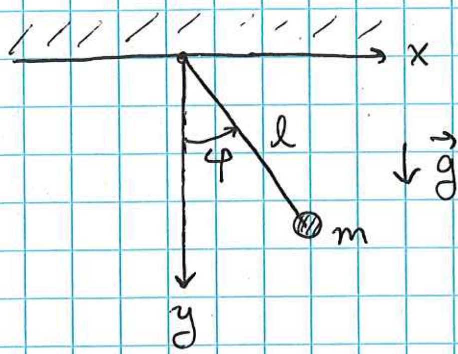
$$\Rightarrow \frac{d}{dt}(m\dot{y}) = -mg$$

$$\Rightarrow \ddot{y} = -g \quad \checkmark.$$

01/31/24

Ex: Simple planar pendulum (ex. w/ constraints)

$$(x,y) \to q \equiv \varphi$$



$\cancel{x(\varphi)=R\cos\varphi \quad \& \quad y(\varphi)=R\sin\varphi}$

$$\begin{cases} x(\varphi) = l\sin\varphi & y(\varphi) = l\cos\varphi \\ \dot{x}(\varphi) = l\dot{\varphi}\cos\varphi & \dot{y}(\varphi) = -l\dot{\varphi}\sin\varphi. \end{cases}$$

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{x}^2+\dot{y}^2)$$

$$= \frac{1}{2}m\left(l^2\dot{\varphi}^2\cos^2\varphi + l^2\dot{\varphi}^2\sin^2\varphi\right)$$

$$= \frac{1}{2}ml^2\dot{\varphi}^2$$

$$U = -mgy = -mgl\cos\varphi.$$

(for sign, check $F_y = -\frac{\partial U}{\partial y} = +mg$.

w/ our choice of coord.'s $F_y = +mg$

gives a downward force ✓)

$$\Rightarrow \quad L = T - U = \frac{1}{2}ml^2\dot{\varphi}^2 + mgl\cos\varphi.$$

E-L. eqn: $\quad \frac{d}{dt}\frac{\partial L}{\partial \dot{\varphi}} = \frac{\partial L}{\partial \varphi}$

$$\Rightarrow \quad \frac{d}{dt}(ml^2\dot{\varphi}) = -mgl\sin\varphi$$

$$\Rightarrow ml^2\ddot{\varphi} = -mgl\sin\varphi$$

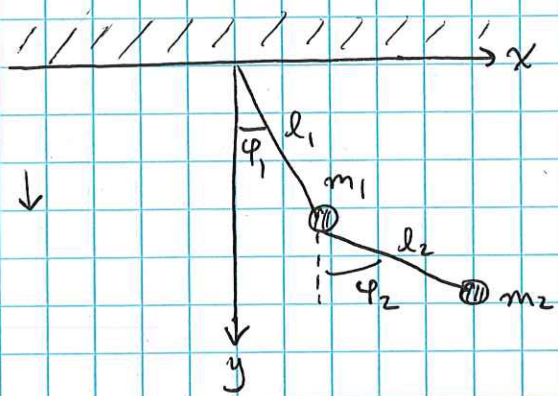
Check: Recall relation btwn. torque Γ & ang. accel. $\ddot{\varphi}$:

$\Gamma = I\ddot{\varphi}$, I = moment of inertia

In this ex., $I = ml^2$ & $\Gamma = -mgl\sin\varphi$ ✓

Now we consider an example w/ $n=2$ DOF: 01/31/24

<u>Ex</u>: Coplanar double pendulum.



$$(x_1, y_1, x_2, y_2) \longrightarrow (q_1, q_2) = (\varphi_1, \varphi_2)$$

$$\begin{cases} x_1(\varphi_1) = l_1 \sin\varphi_1 & y_1(\varphi_1) = l_1 \cos\varphi_1 \\ x_2(\varphi_1, \varphi_2) = l_1 \sin\varphi_1 + l_2 \sin\varphi_2 & y_2(\varphi_1, \varphi_2) = l_1\cos\varphi_1 + l_2 \cos\varphi_2 \end{cases}$$

$$\begin{cases} \dot{x}_1(\varphi_1) = l_1 \dot{\varphi}_1 \cos\varphi_1 & \dot{y}_1(\varphi_1) = -l_1\dot{\varphi}_1 \sin\varphi_1 \\ \dot{x}_2(\varphi_1, \varphi_2) = l_1 \dot{\varphi}_1 \cos\varphi_1 + l_2 \dot{\varphi}_2 \cos\varphi_2 & \dot{y}_2(\varphi_1, \varphi_2) = -l_1\dot{\varphi}_1 \sin\varphi_1 - l_2\dot{\varphi}_2 \sin\varphi_2 \end{cases}$$

$$\Rightarrow T = T_1 + T_2 = \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2)$$

$$= \frac{1}{2} m_1 l_1^2 \dot{\varphi}_1^2 + \frac{1}{2} m_2 \left[l_1^2 \dot{\varphi}_1^2 + l_2^2 \dot{\varphi}_2^2 + 2 l_1 l_2 \dot{\varphi}_1 \dot{\varphi}_2 \cos(\varphi_1 - \varphi_2) \right]$$

$$U = U_1 + U_2 = -m_1 g y_1 - m_2 g y_2$$

$$= -m_1 g l_1 \cos\varphi_1 - m_2 g (l_1 \cos\varphi_1 + l_2 \cos\varphi_2)$$

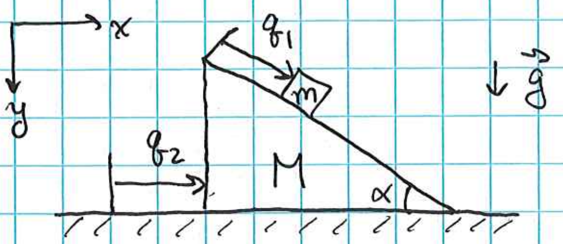
$$\Rightarrow L = T - U = \frac{1}{2} m_1 l_1^2 \dot{\varphi}_1^2 + \frac{1}{2} m_2 \left[l_1^2 \dot{\varphi}_1^2 + l_2 \dot{\varphi}_2^2 + 2 l_1 l_2 \dot{\varphi}_1 \dot{\varphi}_2 \cos(\varphi_1 - \varphi_2) \right]$$

$$+ (m_1 + m_2) g l_1 \cos\varphi_1 + m_2 g l_2 \cos\varphi_2$$

E-L. eqn.'s: $\quad \varphi_1 : \quad \dfrac{d}{dt} \dfrac{\partial L}{\partial \dot{\varphi}_1} = \dfrac{\partial L}{\partial \varphi_1}$

$\varphi_2 : \quad \dfrac{d}{dt} \dfrac{\partial L}{\partial \dot{\varphi}_2} = \dfrac{\partial L}{\partial \varphi_2}$

Ex: (Block sliding on wedge). 01/31/24



· block of mass m slides on wedge of mass M, which is also free to slide.

$$(x_1, y_1, x_2) \longrightarrow (q_1, q_2)$$

$$\begin{cases} x_1(q_1, q_2) = q_2 + q_1 \cos\alpha & y_1(q_1,q_2) = q_1 \sin\alpha & x_2(q_2) = q_2 \\ \dot{x}_1(q_1, q_2) = \dot{q}_2 + \dot{q}_1 \cos\alpha & \dot{y}_1(q_1,q_2) = \dot{q}_1 \sin\alpha & \dot{x}_2(q_2) = \dot{q}_2 \end{cases}$$

$$T = T_1 + T_2 = \frac{1}{2} m (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} M \dot{x}_2^2$$
$$= \frac{1}{2} m \left[(\dot{q}_2 + \dot{q}_1 \cos\alpha)^2 + \dot{q}_1^2 \sin^2\alpha \right] + \frac{1}{2} M \dot{q}_2^2$$
$$= \frac{1}{2} m \left(\dot{q}_1^2 + \dot{q}_2^2 + 2\dot{q}_1 \dot{q}_2 \cos\alpha \right) + \frac{1}{2} M \dot{q}_2^2$$

$$U = U_1 + U_2 = -mgy_1$$
$$= -mg q_1 \sin\alpha$$

$$\Rightarrow L = T - U = \frac{1}{2} m (\dot{q}_1^2 + \dot{q}_2^2 + 2\dot{q}_1 \dot{q}_2 \cos\alpha) + \frac{1}{2} M \dot{q}_2^2 + mg q_1 \sin\alpha.$$

E-L. eqn.'s:

$$q_1: \quad \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_1} = \frac{\partial L}{\partial q_1}$$

$$q_2: \quad \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_2} = \frac{\partial L}{\partial q_2}$$

01/31/24

q_2 eqn. first:

$$\frac{\partial L}{\partial q_2} = 0 \quad \Rightarrow \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_2}\right) = 0.$$

$$\Rightarrow \quad \frac{\partial L}{\partial \dot{q}_2} = \text{const.}$$

$$\Rightarrow \underbrace{m(\dot{q}_1 \cos\alpha + \dot{q}_2) + M\dot{q}_2}_{= P_x,\ \text{total momentum in } x\text{-dir.}} = \text{const.}$$

→ conservation of total $\vec{P}$ along x.

q_1 eqn:

$$\frac{d}{dt}\left[m(\dot{q}_1 + \dot{q}_2\cos\alpha)\right] = mg\sin\alpha$$

$$\Rightarrow \cancel{m}(\ddot{q}_1 + \cancel{m}\ddot{q}_2\cos\alpha) = g\sin\alpha.$$

from q_2 EOM, $\ddot{q}_2 = -\dfrac{m\cos\alpha}{M+m}\ddot{q}_1$

$$\Rightarrow \left(1 - \frac{m\cos^2\alpha}{m+M}\right)\ddot{q}_1 = g\sin\alpha$$

$$\Rightarrow \ddot{q}_1 = \frac{g\sin\alpha}{1 - \frac{m\cos^2\alpha}{m+M}}.$$

Check limits: (i) $\alpha = \pi/2 \Rightarrow \ddot{q}_1 = g$ ✓

(ii) $M \to \infty \Rightarrow \ddot{q}_1 = g\sin\alpha$ (stationary wedge) ✓

(iii) $M \to 0 \Rightarrow \ddot{q}_1 = g/\sin\alpha \Rightarrow \left.\begin{matrix}\ddot{x}_1 = 0 \\ \ddot{y}_1 = g\end{matrix}\right\}$ (falls straight down.)

✓