1 Sums of independent r.v.& Symmetry

1.1 Convolution of two distributions

given two independent r.v. X and Y, the distribution of $Z = X + Y$ is the convolution of the distributions of X and Y.

1. when X and Y are both discrete, the pmf of $X + Y$ is given by

$$p_{X+Y}(n) = p_X * p_Y(n) = \sum_k p_{X(k)} p_{Y(n-k)} = \sum_k p_{X(n-k)} p_{Y(k)} \tag{1}$$

2. when X and Y are both continuous, the pdf of $X + Y$ is given by

$$f_{X+Y}(z) = f_X * f_Y(z) = \int_{-\infty}^{\infty} f_X(z-y) f_Y(y) dy = \int_{-\infty}^{\infty} f_X(z-y) f_Y(y) dy \tag{2}$$

- *example: convolution of geometric random variables*

  let X and Y be independent geometric random variables with the same suvvess parameter $p < 1$, find the distrbution of $Z = X + Y$.

  We know $p_X(k) = p_Y(k) = p(1-p)^{k-1}$ $k \geq 1$ r.v. $Z = X + Y$ takes on values $n = 2, 3, \ldots$. Via the convolution magic promised above, we have

$$\begin{aligned} P(X+Y=n) &= \sum_{k=-\infty}^{\infty} P(X=k) P(Y=n-k) \\ &= \sum_{k=1}^{n-1} p(X=k) P(Y=n-k) \\ &= \sum_{k=1}^{n-1} p(1-p)^{k-1} p(1-p)^{n-k-1} \\ &= \sum_{k=1}^{n-1} p^2 (1-p)^{n-2} \\ &= (n-1) p^2 (1-p)^{n-2} \end{aligned} \tag{3}$$

1.2 Negative binomial distribution

Coming off from the geometric distribution, we have the negative binomial distribution, which is the distribution of the number of trials needed to get r successes in a sequence of independent Bernoulli trials with success probability p. Its distribution, i.e. pmf, is given by

$$P(X=n) = \binom{-1}{k-1} p^k (1-p)^{n-k} \quad (n \geq k) \tag{4}$$

abbriviate this by $X \sim$ Negbin (k, p) , where the geometric is a special case with $k = 1$.

1.3 Collection of normal distributed r.v.s

For $X_i \sim \mathcal{N}(\mu_i, \sigma_i^2)$, $X = \sum_i a_i X_i$, we know

$$\begin{gathered} X \sim \mathcal{N}(\mu, \sigma^2) \\ \text{where } \mu = \sum_i a_i \mu_i, \sigma^2 = \sum_i a_i^2 \sigma_i^2 \end{gathered} \tag{5}$$

in other words, the sum of normal distributed r.v.s is also normal distributed.

1.4 Exchangeable r.v.s

a sequence of r.v.s $X_1, X_2, X_3, ..., X_n$ is **exchangable** if the following condition holds: for any permutation (k_1, k_2, k_3) of $(1, 2, ..., n)$, we have

$$(X_1, X_2, ..., X_n) \stackrel{d}{=} \left(X_{k_1}, X_{k_2}, ..., X_{k_n}\right) \tag{6}$$

- **How to check exchangability**

  "it just works" method: check if the r.v. are identically distributed, i.e. if marginal pdf or pmf is the same.

  Suppose $X_1, X_2, ..., X_n$are discrete random variables with joint probability mass function p. Then these random variables are exchangeable if and only if p is a symmetric function.

  Suppose $X_1, X_2, ..., X_n$ are jointly continuous random variables with joint density function f.Then these random variables are exchangeable if and only if f is a symmetric function.

  If the expectation is conserved under permutations of our set of r.v.s.

Importantly, if the r.v.s are independent and identically distributed, they are also exchangeable.

remarks:
1. r.v. denoting outcomes of sampling without replacement $X_1, X_2, ...X_n$ are exchangeable.
2. For any function g dependent on , the r.v.s $g(X_1), g(X_2), ..., g(X_n)$ are exchangeable.

1.5 Expectation and Varience of Multivariable r.v.

1.5.1 Expectation: linear

$$E[g_1(X_1) + g_2(X_2) + ... + g_n(X_n)] = E[g_1(X_1)] + E[g_2(X_2)] + ... + E[g_n(X_n)] \tag{7}$$

$$E[X_1 + X_2 + ... + X_n] = E[X_1] + E[X_2] + ... + E[X_n] \tag{8}$$

Expectation of a sum is always the sum of expectations.

1.5.2 Varience: sum of independent r.v., linear

$$\mathrm{Var}(X_1 + X_2 + ... + X_n) = \mathrm{Var}(X_1) + \mathrm{Var}(X_2) + ... + \mathrm{Var}(X_n) \tag{9}$$

1.5.3 The indicator method

- *example* We draw five cards from a deck of 52 without replaceement. Let X denote the number of Aces among the chosen cards. Find the expected value of X.

  Two ways to solve this:
  1. Since order does not matter in our draw of 5, by argument of exchangability, we can construct the following inditator:

     $$I_i = \begin{cases} 1 & \text{if the ith card is an ace} \\ 0 & \text{otherwise} \end{cases} \tag{10}$$

     Since X is the number of Aces among our 5 cards, we have

     $$X = I_1 + I_2 + I_3 + I_4 + I_5 \tag{11}$$

     Recall the linearity of expectation, we can rephrase the expected value as

     $$E[X] = E[I_1] + E[I_2] + E[I_3] + E[I_4] + E[I_5] \tag{12}$$

     Since r.v. I_i are exchangeable, we have

$$E[I_1] = E[I_2] = E[I_3] = E[I_4] = E[I_5] \tag{13}$$

Equation 12 becomes

$$5 * E[I_1] = 5 * P(I_1 = 1) = 5 * \frac{4}{52} = \frac{5}{13} \tag{14}$$

2. We can also label the Aces in the total deck as 1,2,3,4, and have our indicators j_1, j_2, j_3, j_4 indicating if the ith Ace is in our draw or not. The number of Aces in our draw is then $X = j_1 + j_2 + j_3 + j_4$. By similar arguements of exchangability, we have $E[X] = 4E[J_1] = 4P(\text{one of the ace is among the 5 cards})$. Notice that

$$\begin{aligned} P(\text{one of the ace is among the 5 cards}) = \frac{\binom{1}{1}, \binom{51}{4}}{\binom{52}{5}} = \frac{5}{52} \\ \Rightarrow E[X] = \frac{5}{13} \end{aligned} \tag{15}$$

1.5.4 Expectation of multiple products

let X_1, X_2, X_3 be independend r.v., when for all function g_1, g_2, g_3

$$E\left[\prod_{i=1}^{3} g_{i(X_i)}\right] = \prod_{i=1}^{3} E\left[g_{i(X_i)}\right] \tag{16}$$

1.6 Moment generating function with sums of r.v.

For independent r.v. X, Y, and mgf $M_X(t)$, $M_Y(t)$,

$$M_{X+Y}(t) = M_X(t) M_Y(t) \tag{17}$$

1.7 Covariance and correlation

1.7.1 Covariance

$$\text{Cov}(X, Y) = E[(X - E[X])(Y - E[Y])] = E[XY] - E[X]E[Y] \tag{18}$$

- X&Y are
  - positively correlated if $\text{Cov}(X, Y) > 0$
  - negatively correlated if $\text{Cov}(X, Y) < 0$
  - uncorrelated if $\text{Cov}(X, Y) = 0$

1.7.2 Properties of Covariance

- $\text{COV}(X, Y) = \text{COV } (Y, X)$
- $\text{COV } (aX + b, Y) = a \text{ COV}(X, Y)$
- for any r.v. X_i, Y_j and real numbers a_i, b_j:

$$\text{COV}\left(\sum_{i=1}^{n} a_i X_i, \sum_{j=1}^{m} b_j Y_j\right) = \sum_{i=1}^{n} \sum_{j=1}^{m} a_i b_j \text{ COV}(X_i, Y_j) \tag{19}$$

- Practically,

$$\begin{aligned} \text{Cov}(Y_1 + Y_2, Z) = \text{Cov}(Y_1, Z) + \text{Cov}(Y_2, Z) \\ \text{Cov}(X, X) = \text{Var}(X) \end{aligned} \tag{20}$$

1.7.3 Variance of sum of r.v.s

$$\text{Var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \text{Var}\ (X_i) + 2 \sum_{i \leq i < j \leq n} \text{Cov}(X_i, X_j)) \tag{21}$$

For two r.v.s, this comes down to

$$\text{Var}\ (X+Y) = \text{Var}\ (x) + \ \text{Var}\ (Y) + 2\ \text{Cov}(X,Y) \tag{22}$$

For three r.v.s, this is uglier...

$$\begin{aligned} &\text{Var}\ (X+Y+Z) \\ &= \text{Var}\ (X) + \text{Var}\ (Y) + \text{Var}\ (Z) + 2\ \text{Cov}(X,Y) + 2\ \text{Cov}(X,Z) + 2\ \text{Cov}(Y,Z) \end{aligned} \tag{23}$$

You dont want to compute this for four or more...

1.7.4 Correlation

$$\text{Corr}(X,Y) = \frac{\text{Cov}(X,Y)}{\sqrt{\text{Var}(X)\ \text{Var}(Y)}} \tag{24}$$

2 Tail bounds and limit theorems

2.1 Markov's inequality

For any non-negative r.v. X and any $a > 0$, we have

$$P(X \geq a) \leq \frac{E[X]}{a} \tag{25}$$

2.2 Chebyshev's inequality

For any r.v. X with finite mean and variance, and any $k > 0$, we have

$$P(|X - E[X]| \geq k) \leq \frac{\text{Var}(X)}{k^2} \tag{26}$$

normally used to find $P(X \geq c + \mu) \leq \frac{\sigma^2}{c^2}$ and $\quad P(X \leq \mu - c) \leq \frac{\sigma^2}{c^2}$

2.3 generalized Law of large numbers

For a sequence of iid r.v.s $X_1, X_2, ..., X_n$ with finite mean $E[X_i] = \mu$ and finite variance $\text{Var}\ [X_i] = \sigma^2$, letting $S_n = X_1 + X_2 + ... + X_n$, for any $\varepsilon > 0$, we have

$$\lim_{n \to \infty} P\left(|\frac{S_n}{n} - \mu| < \varepsilon\right) = 1 \tag{27}$$

2.4 Generalized Central Limit Theorem

For a sequence of iid r.v.s $X_1, X_2, ..., X_n$, where n is the sample size, with finite mean $E[X_i] = \mu$ and finite variance $\text{Var}\ [X_i] = \sigma^2$, letting $S_n = X_1 + X_2 + ... + X_n$, we have

$$\lim_{n \to \infty} P\left(a \leq \frac{S_n - n\mu}{\sigma\sqrt{n}} \leq b\right) = \Phi(b) - \Phi(a) \tag{28}$$

More practically, we use

$$P(S \geq k) = P\left(\frac{S_n - \mu}{\sqrt{n\sigma^2}} \geq \frac{k - n\mu}{\sqrt{n\sigma^2}}\right) = 1 - \Phi\left(\frac{k - n\mu}{\sqrt{n\sigma^2}}\right) \tag{29}$$

3 Conditional distribution

A combination of conditional probability and marginal distribution.

3.1 Discrete conditional distribution

recall the conditional probability

$$P(A|B) = \frac{P(A \cap B)}{B}, \text{for } P(B) > 0 \tag{30}$$

When A is now a r.v., we have the conditional distribution

$$p_{X|B}(k) = P(X = k|B) = \frac{P(\{X = k\} \cap B)}{P(B)} \tag{31}$$

3.1.1 Conditional expectation of X, given event B

$$E[X|B] = \sum_k kP(X = k|B) \tag{32}$$

3.1.2 Unconditiond pmf of X

$$p_X(k) = \sum_{i=1}^{n} p_{X|B_i}(k)\, P(B_i) \tag{33}$$

- From Equation 32 and Equation 33 we get

$$E[X] = \sum_{i=1}^{n} E[X|B_i]P(B_i) \tag{34}$$

3.1.3 Conditioning on r.v.

When both X and Y are r.v.s, we can have the following two-variable function

$$p_{X|Y}(k|j) = P(X = k|Y = j) = \frac{P(\{X = k\}, \{Y = j\})}{P(Y = j)} = \frac{p_{X,Y}(k, j)}{p_Y(j)} \tag{35}$$

3.1.4 Conditional expectation of X, given Y=Y

$$E[X|Y = j] = \sum_k kP(X = k|Y = j) = \sum_k kp_{X|Y}(k|j) \tag{36}$$

3.1.5 Unconditioned pmf with 2 r.v.S

$$p_X(k) = \sum_i p_{X|Y}(k|j)\, p_Y(j) \tag{37}$$

- From this, we can derive the unconditioned expectation of X and Y

$$E[X] = \sum_k E[X|Y = j]\, p_Y(j) \tag{38}$$

3.1.6 Joint pmf with 2 r.v.s

$$p_{X,Y}(k, j) = p_{X|Y}(k|j)p_Y(j) = p_{Y|X}(j|k)p_X(k) \tag{39}$$

3.2 Continuous conditional distribution

For continuous r.v.s, with both X, Y random variables, we have the conditional pdf of X given Y = y as

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} \tag{40}$$

3.2.1 Conditional probability and expectation

$$P(X \in A|Y = y) = \int_A f_{X|Y}(t|y)dt \tag{41}$$

The conditoinal expectation of $g(X)$

$$E[g(X)|Y = y] = \int_{-\infty}^{\infty} g(t)\ f_{X|Y}(t|y)dt \tag{42}$$

3.2.2 The unconditioned pdf and expectation of X

Given the conditional pdf $f_{X|Y}(x|y)$, we can derive the unconditioned pdf of X as

$$f_X(x) = \int_{-\infty}^{\infty} f_{X|Y}(x|y)f_Y(y)dy \tag{43}$$

$$E[g(X)] = \int_{-\infty}^{\infty} E[g(X)|Y = y]\ f_Y(y)\,\mathrm{d}y \tag{44}$$

3.3 Conditional expectation

3.3.1 conditional expectation as a r.v.

Let X and Y jointly continuous r.v., The conditional expectatino of X given Y is a new random variable dependent on Y $v(Y)$

$$v(Y) = E[X|Y = y] \tag{45}$$

3.3.2 Conditioning and independence

recall that

- Discrete r.v. two discrete r.v.s are only independent iff

$$p_{X,Y}(x,y) = p_X(x)p_Y(y) \tag{46}$$

- Continuous r.v. two continuous r.v.s are only independent iff

$$f_{X,Y}(x,y) = f_X(x)f_Y(y) \tag{47}$$

now, If given pmf or pdf of X given Y, we now have the joint pmf

$$p_{X,Y}(x,y) = p_{X|Y}(x|y)p_Y(y) \tag{48}$$

and the joint pdf

$$f_{X,Y}(x,y) = f_{X|Y}(x|y)f_Y(y) \tag{49}$$

3.3.3 Independency of X and Y

discrete r.v. X and Y are independent iff

$$p_{X|Y}(x|y) = p_X(x) \tag{50}$$

continuous r.v. X and Y are independent iff

$$f_{X|Y}(x|y) = f_X(x) \tag{51}$$

3.4 Conditioning on the random variable

3.4.1 Conditioning X on y

for independent r.v. X and Y, we can condition on Y and have the conditional expectation of X given Y = y as

$$E[g(X)|Y = y] = E[g(X)] \quad \text{and } E[g(X)|Y = y] = E[g(X)] \tag{52}$$

3.4.2 COnditioning X on X

For a r.v. X, we can condition on X itself, and have the conditional expectation of X given X = x as

$$E[g(X)|X = x] = g(X) \tag{53}$$