Oscillations of systems w/ more than one DOF

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- Consider a system w/ n DOF & generalized coord.'s $q = (q_1, q_2, \ldots, q_n)$. The potential is $U(q)$.

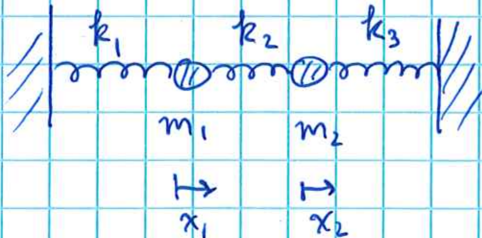
- Suppose $U(q)$ has a min. at $q^{(0)} = (q_1^{(0)}, q_2^{(0)}, \ldots, q_n^{(0)})$; i.e.,

$$\left.\frac{\partial U(q)}{\partial q_i}\right|_{q^{(0)}} = 0 \quad \text{for} \quad i = 1, \ldots, n$$

<u>Q</u>: How do we describe the motion of small deviations $x_i = q_i - q_i^{(0)}$ away from equil.?

Before going into formalism, start w/ an example.

<u>Ex</u>:



x_1, x_2 measure deviations away from equil.

$$T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2$$

$$U = \frac{1}{2} k_1 x_1^2 + \frac{1}{2} k_2 (x_1 - x_2)^2 + \frac{1}{2} k_3 x_2^2$$

$$\Rightarrow L = T - U = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 - \frac{1}{2} k_1 x_1^2 - \frac{1}{2} k_2 (x_1 - x_2)^2 - \frac{1}{2} k_3 x_2^2 .$$

E-L. eqn.'s:

$$x_1: \quad \frac{d}{dt} \frac{\partial L}{\partial \dot{x}_1} = \frac{\partial L}{\partial x_1} \quad \Rightarrow \quad m_1 \ddot{x}_1 = -k_1 x_1 - k_2 (x_1 - x_2)$$

$$x_2: \quad \frac{d}{dt} \frac{\partial L}{\partial \dot{x}_2} = \frac{\partial L}{\partial x_2} \quad \Rightarrow \quad m_2 \ddot{x}_2 = -k_3 x_2 - k_2 (x_2 - x_1)$$

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→ we have a coupled pair of diff. eq.'s:

$$\begin{cases} m_1 \ddot{x}_1 + (k_1+k_2)x_1 - k_2 x_2 = 0 \\ m_2 \ddot{x}_2 + (k_2+k_3)x_2 - k_2 x_1 = 0 \end{cases}$$

· Rewrite this in a more compact matrix-vector notation:

$$\underbrace{\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}}_{M} \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} + \underbrace{\begin{pmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2+k_3 \end{pmatrix}}_{K} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0.$$

$M, K = 2 \times 2$ matrices.

& let $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $\ddot{\vec{x}} = \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix}$

→ $M \cdot \ddot{\vec{x}} + K \cdot \vec{x} = 0$ [c.f. $m\ddot{x} + kx = 0$ for oscillator in one DOF]

· As we did for harmonic oscillations in one DOF, consider an auxiliary problem:

$$(*) \quad M \cdot \ddot{\vec{z}} + K \cdot \vec{z} = 0, \qquad \vec{z} = \text{complex 2-component vec.}$$

→ if we can solve $(*)$, we obtain $\vec{x}$ as:

$$\vec{x} = \operatorname{Re} \vec{z}$$

· Attempt to solve $(*)$ by seeking soln of the form:

$$\vec{z} = \vec{a} e^{i\omega t}, \qquad \vec{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \quad a_1, a_2 \in \mathbb{C}.$$

note $\ddot{\vec{z}} = -\omega^2 \vec{a} e^{i\omega t}$

plugging into (*):

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$$-\omega^2 M \cdot \vec{a}\, e^{i\omega t} + K \cdot \vec{a}\, e^{i\omega t} = 0$$

$$\Rightarrow \quad (\omega^2 M - K) \cdot \vec{a} = 0. \qquad (**).$$

→ we are led to an algebraic matrix-vector eqn.

· One sol.'n of (**) is $\vec{a}=0$, which is not very interesting. To obtain a non-trivial sol.'n for $\vec{a}$, the following condition must hold:

$$\det(\omega^2 M - K) = 0$$

determinant

$$\Rightarrow \det\left[\omega^2\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} - \begin{pmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2+k_3 \end{pmatrix}\right] = 0.$$

· At this point we'll simplify & set $m_1 = m_2 = m$ & $k_1 = k_3 = k$, $k_2 = k'$:

$$\det\begin{pmatrix} \omega^2 m - (k+k') & k' \\ k' & \omega^2 m - (k+k') \end{pmatrix} = 0.$$

$$\Rightarrow \quad \left[\omega^2 m - (k+k')\right]^2 - k'^2 = 0.$$

→ this is a quadratic eqn. for ω^2. ⇒ <u>two</u> soln.'s.

$$\Rightarrow \quad \omega^2 = \frac{k+k'}{m} \pm \frac{k'}{m}$$

The two soln.'s are thus: 03/06/24

$$\omega_-^2 = \frac{k}{m} \quad \& \quad \omega_+^2 = \frac{k+2k'}{m}.$$

Associated to $\omega_\pm^2$ we have two soln.'s for the motion: $\vec{z}_\pm = \vec{a}_\pm e^{i\omega_\pm t}$

still have to determine $\vec{a}_\pm$.

$\vec{a}_\pm$ determined by the original eqn:

$$(\omega_\pm^2 M - K)\cdot \vec{a}_\pm = 0.$$

(1) Consider first ω_-:

$$(\omega_-^2 M - K)\cdot \vec{a}_- = 0$$

$$\Rightarrow \begin{pmatrix} \omega_-^2 m - (k+k') & k' \\ k' & \omega_-^2 m - (k+k') \end{pmatrix} \begin{pmatrix} a_{-,1} \\ a_{-,2} \end{pmatrix} = 0.$$

$$\omega_-^2 = \frac{k}{m} \Rightarrow \begin{pmatrix} -k' & k' \\ k' & -k' \end{pmatrix} \begin{pmatrix} a_{-,1} \\ a_{-,2} \end{pmatrix} = 0$$

$$\Rightarrow a_{-,2} = a_{-,1}$$

$$\Rightarrow \vec{a}_- = \underbrace{a_- e^{i\delta_-}}_{\text{complex const.}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\Rightarrow \vec{z}_- = \vec{a}_- e^{i\omega_- t} = a_- e^{i\delta_-} \cancel{e^{i\omega}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{i\omega_- t}$$

$$\Rightarrow \vec{x}_- = \mathrm{Re}\, \vec{z}_- = a_- \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos(\omega_- t + \delta_-)$$

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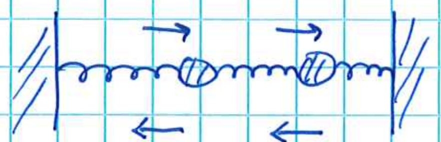
more explicitly, we have found a sol.'n to the coupled EOM:

$$\begin{cases} x_1(t) = a_- \cos(\omega_- t + \delta_-) \\ x_2(t) = a_- \cos(\omega_- t + \delta_-) \end{cases}$$

→ sol.'n contains two arbitrary const.'s a_- & δ_-.

→ motion corresponds to both masses oscillating <u>in phase</u> w/ equal amplitude



Note: Since $x_1 = x_2$, middle spring not compressed. & each mass ~~os~~ oscillates as if it were attached to a single spring of stiffness k. This is why the freq. is simply $\omega_- = \sqrt{k/m}$.

· The sol.'n $\{x_1(t), x_2(t)\}$ we have found is called a "normal mode".

(2) ~~ω_+~~ Consider now ω_+:

$$(\omega_+^2 M - K)\cdot \vec{a}_+ = 0$$

$$\Rightarrow \begin{pmatrix} \omega_+^2 m - (k+k') & k' \\ k' & \omega_+^2 m - (k+k') \end{pmatrix} \begin{pmatrix} a_{+,1} \\ a_{+,2} \end{pmatrix} = 0$$

$$\omega_+^2 = \frac{k+2k'}{m} \Rightarrow \begin{pmatrix} k' & k' \\ k' & k' \end{pmatrix} \begin{pmatrix} a_{+,1} \\ a_{+,2} \end{pmatrix} = 0$$

$$\Rightarrow a_{+,2} = -a_{+,1}$$

$$\Rightarrow \vec{a}_+ = \underbrace{a_+ e^{i\delta_+}}_{\text{complex const.}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\Rightarrow \vec{z}_+ = \vec{a}_+ e^{i\omega_+ t} = a_+ e^{i\delta_+} \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{i\omega_+ t}$$

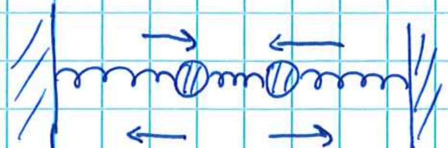
$$\Rightarrow \vec{x}_+ = \mathrm{Re}\,\vec{z}_+ = a_+ \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos(\omega_+ t + \delta_+)$$

more explicitly:

$$\left.\begin{aligned} x_1(t) &= a_+ \cos(\omega_+ t + \delta_+) \\ x_2(t) &= -a_+ \cos(\omega_+ t + \delta_+) \end{aligned}\right\} \text{another "normal mode".}$$

→ two more arbitrary const.'s a_+ & δ_+

→ motion in masses oscillating <u>out of phase</u> w/ equal amplitude:



- Now, the general sol.'n to the EOM is a superposition of the $\pm$ soln.'s we have found:

$$\vec{x}(t) = a_-\begin{pmatrix}1\\1\end{pmatrix}\cos(\omega_- t+\delta_-) + a_+\begin{pmatrix}1\\-1\end{pmatrix}\cos(\omega_+ t+\delta_+).$$

→ note: general soln. contains four arbitrary const.'s $\{a_-, \delta_-, a_+, \delta_+\}$, as it should.

- unlike the normal modes, the motion in the general case is complicated, being a sum of two oscillating components.
- The normal modes describe the simplest "elementary" motions of the system. Notice that these elementary motions do not correspond to the individual motions of the masses but rather to <u>collective</u> motions.