Covered topics

Math 431 - Lecture 2

Spring 2024

Disclaimer: although the plan is to have a fairly detailed list of the covered topics, the list below might not cover everything that we discussed in class.

Week 1

W 1/24: probability space (sample space, collection of events, probability measure)
what is an event?
Axioms of probability (properties of the probability measure)
Examples: coin flip, several coin flips
Direct product of sets

Fr 1/26: Experiments with equally likely outcomes
Sampling (with replacement with order, without replacement with/without order)

Reading materials:
Sections 1.1-1.3

Week 2

M 1/29: Sampling examples
Probability spaces with infinitely many outcomes
The probability of eventually getting tails in a sequence of coin flips is 1.
Decomposing an event as the disjoint union of simpler events: the probability of getting heads in an even number of coin flips
Uniformly chosen point from the interval [0,1].

W 1/31: $P(A^c) = 1 - P(A)$
If $B \subset A$ then $P(B) \leq P(A)$.
How to prove $P(\text{eventually we will get heads with a fair coin}) = 1$ using the monotonicity property of the probability measure

Fr 2/2: Inclusion-exclusion to compute the probability of union of events
Two events: $P(A \cup B) = P(A) + P(B) - P(AB)$
Inclusion-exclusion formula for three and n events
Inclusion-exclusion examples
Definition of a random variable

distribution of a random variable
Discrete random variable, probability mass function

Reading materials:
Sections 1.4-1.5

Week 3

M 2/5: Conditional probability of A given B (with $P(B) > 0$): $P(A|B) = \frac{P(AB)}{P(B)}$
$P(\cdot|B)$ is a probability measure for any fixed B with $P(B) > 0$.
The multiplication rule for conditional probabilities

$$P(AB) = P(A|B)P(B), \quad P(A_1A_2 \cdots A_n) = P(A_1)P(A_2|A_1)P(A_3|A_2A_1) \cdots P(A_n|A_{n-1} \cdots A_1)$$

W 2/7: Definition of a partition
If $B_1, \ldots, B_n$ is a partition with $P(B_i) > 0$ then

$$P(A) = \sum_{i=1}^{n} P(A|B_i)P(B_i)$$

Bayes' formula:

$$P(B|A) = \frac{P(A|B)P(B)}{P(A|B)P(B) + P(A|B^c)P(B^c)}$$

Similarly with a partition $B_1, \ldots, B_n$.

Fr 2/9: Applications of Bayes' formula
Independence of two events A and B:

$$P(AB) = P(A)P(B)$$

If A and B are independent then the same is true for (i) A and B^c (ii) A^c and B (iii) A^c and B^c.

Reading materials:
Sections 2.1-2.3

Week 4

M 2/12: independence on n events, various characterizations
Independence of events constructed from independent events
E.g.: if A, B and C are independent then $A \cup B$ and C^c are independent.

Independence of the random variables $X_1, \dots, X_n$:
for any collection of sets $B_1, \dots, B_n \in \mathbb{R}$:

$$P(X_1 \in B_1, \dots, X_n \in B_n) = \prod_{i=1}^{n} P(X_i \in B_i)$$

Equivalent definition for the independence of discrete random variables $X_1, \dots, X_n$:

$$P(X_1 = x_1, \dots, X_n = x_n) = \prod_{i=1}^{n} P(X_i = x_i)$$

for any $x_1, \dots, x_n$.
Example: sampling with and without independence

W 2/14: The probability space of repeated independent trials with the same success probability.
Named distributions constructed from a sequence of independent trials with the same success probability

- Bernoulli with parameter p (the outcome of a single trial)
- Binomial with parameters n and p (the number of successes out of n trials)

Fr 2/16: Geometric distribution with parameter p (the number of trials needed for the first success)
The hypergeometric distribution.
Conditional independence of events.

Reading materials:
Sections 2.4-2.5

Week 5 (planned)

M 2/19: The birthday problem: exact formula, and how to estimate it using the fact that $e^{-x} \approx 1 - x$ for small x.
Continuous distributions: the definition of the probability density function

Basic properties of the probability density function, how to identify a p.d.f.

How to compute probabilities using a p.d.f.

The uniform distribution on $[a, b]$

W 2/21: The cumulative distribution function of a random variable, definition, basic properties

How does the c.d.f. of a discrete and a continuous random variable look like.

Fr 2/23: How to identify the probability mass function from the c.d.f. and vice versa.

How to compute the p.d.f. of a continuous random variable from the c.d.f.

The expected value of a random variable as the weighted average of possible values.

Computing the expected value for discrete and continuous random variables.

The definition of an indicator random variable, expected value of an indicator random variable.

Reading materials:
Sections 3.1-3.3