

Angular momentum of a rigid body

\vec{L} in non-inertial frame

$$\vec{L} = \sum m(\vec{r} \times \vec{v}) = \sum m[\vec{\Omega} r^2 - \vec{r}(\vec{\Omega} \cdot \vec{r})]$$

$$L_i = \boxed{I_{ij}\Omega_j} \quad \vec{L} = I * \vec{\Omega}$$

If $(x_1 x_2 x_3)$ are principal axis, $L_1 = I_1 \Omega_1$, $L_2 = I_2 \Omega_2$, $L_3 = I_3 \Omega_3$

Free motion of a rigid body

angular momentum is conserved if no external torque. Motion in inertial COM frame is simpler.

- *ex motion of a symmetric top* $I_1 = I_2 = I_3 = I$, $\tilde{I} = I \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$\vec{L} = I\vec{\Omega} \rightarrow \dot{\vec{L}} = 0 \Rightarrow \dot{\vec{\Omega}} = 0$ Uniform rotation about fixed axis parallel to \vec{L}

- *ex rigid rotor* $I_1 = I_2 = \sum m x_3^2$, $I_3 = 0$

$\vec{L} = I\vec{\Omega}$, $\vec{\Omega} \perp x_3$ by geometry We have $\dot{\vec{\Omega}} = 0 \Rightarrow$ Motion is unif in plane perp to $\vec{\Omega}$ and that it stays in that plane.

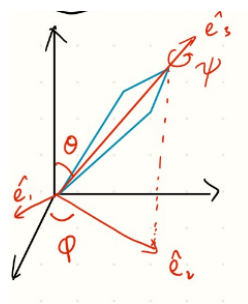
- *ex asymmetric top* $I_1 = I_2 = I_{\perp} \neq I_3 \Rightarrow \tilde{I} = \begin{pmatrix} I_{\perp} & 0 & 0 \\ 0 & I_{\perp} & 0 \\ 0 & 0 & I_3 \end{pmatrix}$ x_3 is symm. axis, for any orthogonal

axes

Rigid body EOM

$$\begin{cases} \dot{\vec{p}} = \vec{F} \\ \dot{\vec{L}} = \vec{K} \text{ torque} \end{cases}$$

Euler angles: ψ spin, θ nutation, φ precession



$(\theta \in [0, \pi], \varphi \in [0, 2\pi], \psi \in [0, 2\pi])$ in turns of rotation $R = R(\hat{z}, \varphi)R(\hat{X}, \theta)R(\hat{Z}, \psi)$

The lagrangian in Euler angles

- First: $T = \frac{1}{2}(I_1 \Omega_1^2 + I_2 \Omega_2^2 + I_3 \Omega_3^2)$
- Rotation in components:

$$\Omega_1 = \dot{\varphi} \sin \theta \sin \psi + \dot{\theta} \cos \psi$$

$$\Omega_2 = \dot{\varphi} \sin \theta \cos \psi - \dot{\theta} \sin \psi$$

$$\Omega_3 = \dot{\varphi} \cos \theta + \dot{\psi}$$

- $T = \frac{1}{2}I_1(\dot{\varphi} \sin \theta \sin \psi + \dot{\theta} \cos \psi)^2 + \frac{1}{2}I_2(\dot{\varphi} \sin \theta \cos \psi - \dot{\theta} \sin \psi)^2 + \frac{1}{2}I_3(\dot{\varphi} \cos \theta + \dot{\psi})^2$
- $L(\theta, \varphi, \psi, \dot{\theta}, \dot{\varphi}, \dot{\psi}) = T - U$

Free motion of symmetric top in Euler angles

$$I_1 = I_2 = I_{\perp} \Rightarrow T = \frac{1}{2}I_{\perp}(\dot{\theta}^2 + \dot{\varphi}^2 \sin^2 \theta) + \frac{1}{2}I_3(\dot{\varphi} \cos \theta + \dot{\psi})^2$$

$$\Omega_{\perp} = L_z/I_{\perp}, \quad \Omega_3 = L_z \cos \theta/I_3 \quad \text{E-L} \rightarrow$$

$$\theta : \frac{d}{dt}I_{\perp}\dot{\theta} = I_{\perp} \sin \theta \cos \theta \dot{\varphi}^2 - I_3 \dot{\varphi} \sin \theta (\dot{\varphi} \cos \theta + \dot{\psi})$$

$$\varphi : \frac{d}{dt}(I_{\perp} \dot{\varphi} \sin^2 \theta + I_3 \cos \theta (\dot{\varphi} \cos \theta + \dot{\psi})) = 0$$

$$\psi : \frac{d}{dt}I_3(\dot{\varphi} \cos \theta + \dot{\psi}) = 0$$

choosing \hat{z} along the angular momentum, we have $L_3 = L_z \cos \theta = I_3 \Omega_3 = I_3(\dot{\varphi} \cos \theta + \dot{\psi})$
 $\Rightarrow \dot{L}_3 = \text{const} \Rightarrow \theta = \text{const} \quad \Omega_3 = \frac{L_z \cos \theta}{I_3} \quad \dot{\varphi} = \frac{L_3}{I_{\perp} \cos \theta} = \frac{L_z}{I_{\perp}} = \text{const}$

- ex heavy symmetric top with one pt fixed By parallel axis thm, $I'_{ij}I_{ij} + M(l^2 \delta_{ij} - l_i l_j)$

$$\Rightarrow I'_{\perp} = I_{\perp} + Ml^2, \quad I'_3 = I_3, \quad U = mgZ = Mgl \cos \theta$$

$$\Rightarrow L = T - U = \frac{1}{2}I'_{\perp}(\dot{\theta}^2 + \dot{\varphi}^2 \sin^2 \theta) + \frac{1}{2}I_3(\dot{\psi} + \dot{\varphi} \cos \theta)^2 = Mgl \cos \theta$$

E-L :

$$L_z = p_{\varphi} = (I'_{\perp} \sin^2 \theta + I_3 \cos^2 \theta) \dot{\varphi} \quad \text{const}$$

$$L_3 = p_{\psi} = I_3(\dot{\psi} + \dot{\varphi} \cos \theta) \quad \text{const}$$

Considering energy conservation

$$E = T + U \Rightarrow \underbrace{E - \frac{L_3^2}{2I_3} - Mgl}_{E'} = \frac{1}{2}I'_{\perp}\dot{\theta}^2 + \underbrace{\frac{1}{2I'_{\perp}} \frac{(L_z - L_3 \cos \theta)^2}{\sin^2 \theta} - Mgl(1 - \cos \theta)}_{U_{\text{eff}}(\theta)}$$

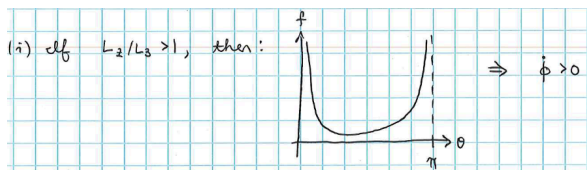
effective 1 dof problem. recognizing

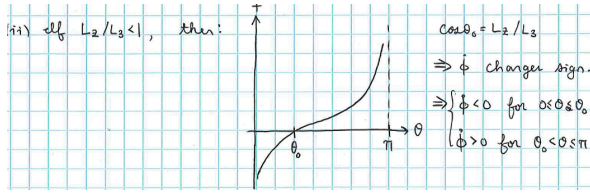
$$\dot{\theta} = \frac{d\theta}{dt} \Rightarrow t = \int \frac{d\theta}{(\sqrt{2[E - U_{\text{eff}}(\theta)]/I'_{\perp}})}$$

Considering U_{eff} : when $\theta = 0$, $L_z = L_3$ when $\theta \approx 0 \Rightarrow U_{\text{eff}} \approx \left(\frac{L_3^2}{8I'_{\perp}} - \frac{Mgl}{2}\right)\theta^2$

Motion about $\theta = 0$ stable if $L_3^2 > 4I'_{\perp}Mgl \Rightarrow \Omega_3^2 > 4I'_{\perp}Mgl/I_3^2$, or stable if spinning ab. symm. axis is fast enough.

- Nutation: consider $\dot{\varphi} = \frac{L_3}{I'_{\perp}} \frac{(L_z/L_3) - (\cos \theta)}{\sin^2 \theta} = \frac{L_3}{I'_{\perp}} f(\theta)$





considering the sign and trends of $f(\theta)$ given constraints on theta, we can differentiate different nutation motion. If θ_0 in graph 2 is out of range, the nutation is smooth; if θ_0 is in range, the nutation is oscillatory (will change sign and spin in spiral.); if θ_0 is on the endpoint of our constrained range, the nutation is spiky and “not smooth” at points.

Euler equations

set body frame $(X, Y, Z) = (\hat{e}_1^0, \hat{e}_2^0, \hat{e}_3^0)$, space frame $(x_1, x_2, x_3) = (\hat{e}_1, \hat{e}_2, \hat{e}_3)$ Set any vector $\vec{A} = \sum A_i^0 \hat{e}_i^0 = \sum A_i \hat{e}_i$ By magic of vec analysis,

$$\left(\frac{d\vec{A}}{dt} \right)_{\text{Space}} = \left(\frac{d\vec{A}}{dt} \right)_{\text{Body}} + \vec{\Omega} \times \vec{A}_{\text{Space}}$$

When applied to $\left(\frac{d\vec{L}}{dt} \right)_{\text{Space}} = \vec{K} = \left(\frac{d\vec{L}}{dt} \right)_{\text{body}} + \vec{\Omega} \times \vec{L}$, recognizing $L_i = I_i \Omega_i$:

$$I_1 \dot{\Omega}_1 + (I_3 - I_2) \Omega_2 \Omega_3 = K_1$$

$$I_2 \dot{\Omega}_2 + (I_1 - I_3) \Omega_3 \Omega_1 = K_2$$

$$I_3 \dot{\Omega}_3 + (I_2 - I_1) \Omega_1 \Omega_2 = K_3$$

$K_i = 0$ if \vec{L} is conserved on i axis.

- ex symmetric top $I_1 = I_2 = I$, $\vec{K} = 0$ $\left(\dot{\Omega}_1 + \frac{I_3 - I_1}{I_+} \Omega_2 \Omega_3 = 0; \dot{\Omega}_2 + \frac{I_1 - I_3}{I_+} \Omega_3 \Omega_1 = 0; \dot{\Omega}_3 = 0 \right)$
let $\omega = ((I_3 - I_+)/I_+) \Omega_3 \Rightarrow \boxed{\left(\Omega_1 = A \cos \omega t; \Omega_2 = -\frac{1}{\omega} \dot{\Omega}_1 = +A \sin \omega t \right)}$

Motion in non-inertial frame

- Set non-inertial frame with velocity $\vec{V}(t)$, $\vec{A} = \dot{\vec{V}}$, $\vec{v} = \vec{v}' + \vec{V}(t)$ where \vec{v}' is velocity w.r.t. non-inertial frame.

lagrangian $L' = \frac{1}{2} m v'^2 - m \vec{r}' \cdot \vec{A} - U$, using E-L eq: $m \dot{\vec{v}}' = -\frac{\partial U}{\partial \vec{r}'} - m \vec{A}$

- ex pendulum in acc. car $m \ddot{\vec{r}} = \vec{T} + m \vec{g} - m \vec{A}$,

finding equil. angle: $\vec{T} = -m(\vec{g} - \vec{A}) = -m \vec{g}_{\text{eff}}$, then use geometry between \vec{g} , $-\vec{A} \Rightarrow \tan \varphi_0 = \frac{A}{g}$. Oscillation freq. $\omega = \sqrt{g_{\text{eff}}/l}$

Motion in rotating frame

Set rotation with $\vec{\Omega}$, $L = \frac{1}{2} m v^2 + \vec{m} \vec{v} \cdot (\vec{\Omega} \times \vec{r}) + \frac{1}{2} m (\vec{\Omega} \times \vec{r})^2 - m \vec{r} \cdot \vec{A} - U$

Using E-L, $\boxed{m \dot{\vec{v}} = -\frac{\partial U}{\partial \vec{r}} - m \vec{A} + 2m(\vec{v} \times \vec{\Omega}) + m \vec{\Omega} \times (\vec{r} \times \vec{\Omega}) + m \vec{r} \times \dot{\vec{\Omega}}}$

- Namely,

$$m \dot{\vec{v}} = -\frac{\partial U}{\partial \vec{r}} + \vec{F}_{\text{cor}} + \vec{F}_{\text{cent}}$$

$$\vec{F}_{\text{Cor}} = 2m(\vec{v} \times \vec{\Omega}), \quad \vec{F}_{\text{cent}} = m \vec{\Omega} \times (\vec{r} \times \vec{\Omega}) = m(\vec{\Omega} \times \vec{r}) \times \vec{\Omega}$$

- ex free fall on earth, centrifugal force $\vec{F} = \vec{g}_0 + m\Omega^2 R \sin \theta \hat{\rho} \Rightarrow \vec{g}_{\text{eff}} = \vec{g}_0 + \Omega^2 R \sin \theta \hat{\rho}$
- ex free fall, coriolis force $\dot{\vec{v}} = \vec{g} + 2\vec{v} \times \vec{\Omega}$, $\vec{\Omega} = \Omega \sin \theta \hat{y} + \Omega \cos \theta \hat{z}$

In components,

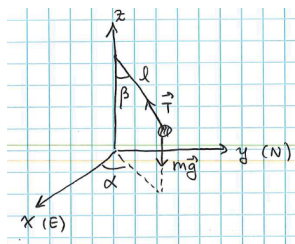
$$\vec{v}_x = 2\Omega(v_y \cos \theta - v_z \sin \theta)$$

$$\vec{v}_y = -2\Omega v_x \cos \theta$$

$$\vec{v}_z = 2\Omega v_x \sin \theta - g$$

Free fall EOM: $\vec{R} = \int v \, dr$, consider $\vec{v} = \vec{v}_1 + \vec{v}_2 = -\vec{g} + 2\vec{v}_1 \times \vec{\Omega} + 2\vec{v}_2 \times \vec{\Omega}$ where approximately, $\vec{v}_2 = 2(\vec{v}_0 - gt\hat{z}) \times \vec{\Omega}$. If no initial velocity, integrating velocity in x components gives, $x(t) = \frac{1}{3}g\Omega\left(\frac{2h}{g}\right)^{3/2} \sin \theta$

- ex foucaults pendulum EOM



$$\vec{r} = l \sin \beta \cos \alpha \hat{x} + l \sin \beta \sin \alpha \hat{y} + (l - l \cos \beta) \hat{z}$$

$$\vec{T} = -T \sin \beta \cos \alpha \hat{x} - T \sin \beta \sin \alpha \hat{y} + T \cos \beta \hat{z}$$

$$\vec{\Omega} = \Omega \sin \theta \hat{y} + \Omega \cos \theta \hat{z}$$

$$\begin{cases} T = mg \\ m\ddot{x} = T_x + 2m\hat{x} \cdot (\dot{\vec{r}} \times \vec{\Omega}) = -\frac{mgx}{l} + 2m\Omega\dot{y} \cos \theta \\ m\ddot{y} = -\frac{mgy}{l} - 2m\Omega\dot{x} \cos \theta \end{cases}$$

letting $\omega^2 = \frac{g}{l}$, $\Omega_z = \Omega \cos \theta$, $\boxed{\eta = x + iy = e^{i\gamma t}}$

$$\ddot{x} + \omega^2 x = 2\Omega_z \dot{y}, \ddot{y} + \omega^2 y = -2\Omega_z \dot{x}$$

$$\gamma = -\Omega_z \pm \sqrt{\omega^2 - \Omega_z^2}$$

$$\eta(t) = ae^{-i\Omega_z t} \cos \omega t$$

$$\Rightarrow \begin{cases} x = a \cos \Omega_z t \cos \omega t \\ y = a \sin \Omega_z t \cos \omega t \end{cases}$$

Hamiltonian Mechanics

$$H(q, p, t) = \sum_{j=1}^n p_j \dot{q}_j - L(q, \dot{q}, t) \quad 1D: H = \frac{p^2}{2m} + U(x)$$

• Hamilton's equation $\dot{q}_i = \frac{\partial H}{\partial p_i}$ $\dot{p}_i = -\frac{\partial H}{\partial q_i}$

- ex particle in polar

$$L = T - U = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\varphi}^2) - U(r, \varphi) \Rightarrow p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}, p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = mr^2\dot{\varphi}$$

$$H = p_r \dot{r} + p_\varphi \dot{\varphi} - L = \frac{p_r^2}{2m} + \frac{p_\varphi^2}{2mr^2} \Rightarrow \dot{r} = \frac{\partial H}{\partial p_r} = \frac{p_r}{m}, \quad \dot{\varphi} = \frac{\partial H}{\partial p_\varphi} = \frac{p_\varphi}{mr^2}$$

$$\dot{p}_r = -\frac{\partial H}{\partial r} = -\frac{p_\varphi^2}{mr^3} - \frac{\partial U}{\partial r}, \quad \dot{p}_\varphi = -\frac{\partial H}{\partial \varphi} = -\frac{\partial U}{\partial \varphi}$$

Phase space

- ex harmonic oscillator $H = \frac{p^2}{2m} + \left(\frac{1}{2}\right)m\omega^2 x^2$, $\omega = \sqrt{\frac{k}{m}}$

$$\left\{ \dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m}, \quad \dot{p} = -\frac{\partial H}{\partial x} = -m\omega^2 x \right\} \Rightarrow \left\{ \dot{q} = \frac{p}{m}, \quad \dot{p} = -m\omega^2 x \right\}$$

$q(t_0 + \delta t) = q(t_0) + \dot{q}\delta t = q_0 + \frac{p}{m}\delta t$; $p(t_0 + \delta t) = p(t_0) + \dot{p}\delta t = p_0 - m\omega^2 q\delta t$ parametric ellipse in phase space.

Liouville's thm

volume of a region of phase space is conserved under time evolution, when boundary of volume and all pts inside move along their orbit for some amount of time.

Poisson bracket

Time evolution of an observable $A(q, p, t)$:

$$\frac{dA}{dt} = \frac{\partial A}{\partial t} + \underbrace{\sum_{i=1}^n \frac{\partial A}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial A}{\partial p_i} \frac{\partial H}{\partial q_i}}_{\equiv \{A, H\}}$$

More generally, for $A(q, p, t)$, $B(q, p, t)$

$$\{A, B\} = \sum_i \frac{\partial A}{\partial q_i} \frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i} \frac{\partial B}{\partial q_i}$$

notice, $\{A, p_i\} = \frac{\partial A}{\partial q_i}$, $\{A, q_i\} = -\frac{\partial A}{\partial p_i}$

- When

$$\frac{dC}{dt} = \frac{\partial C}{\partial t} + \{C, H\} = 0$$

then $C(q, p, t)$ is conserved.

Cononical transformation

consider transformation $q_i \rightarrow Q_i(q, t)$ the transformation is canonical iff the transformation leave the form of Hamilton's eq. unchanged.

$$\begin{cases} \dot{q} = \frac{\partial H}{\partial p} \\ \dot{p} = -\frac{\partial H}{\partial q} \end{cases} \Rightarrow \text{cases } \dot{Q} = \frac{\partial K}{\partial P}, \dot{P} = -\frac{\partial K}{\partial Q}$$

where $K(Q, P, t)$ new Hamiltonian.