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1.

$$
\begin{aligned}
&\int_C \left(y^2+\sin x\right)\mathrm{d}x+\left(3xy+y^4\right)\mathrm{d}y, \quad D=\left\{(r,\theta)|\ r\in[0,2],\theta\in\left[0,\frac{\pi}{2}\right]\right\}\\
&=\int_D \left(\frac{\partial(3xy+y^4)}{\partial x}-\frac{\partial y^2+\sin x}{\partial y}\right)\mathrm{d}A\\
&=\int_D (3y-2y)\,\mathrm{d}A\\
&=\int_D y\,\mathrm{d}A \qquad \text{polar transform: }\begin{cases}\mathrm{d}A=r\,\mathrm{d}r\,\mathrm{d}\theta\\ r\in[0,2]\\ \theta\in\left[0,\frac{\pi}{2}\right]\end{cases}\\
&=\int_0^{\frac{\pi}{2}}\int_0^2 r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\\
&=\int_0^{\frac{\pi}{2}}\sin\theta\,\mathrm{d}\theta\int_0^2 r^2\,\mathrm{d}r\\
&=\int_0^{\frac{\pi}{2}}\frac{8}{3}\sin\theta\,\mathrm{d}\theta\\
&=\frac{8}{3}
\end{aligned}
$$

2

$$
\begin{aligned}
&\int_C (x^5+y^3)dx-(x^3+y^5)dy, C: x^2+y^2=4\\
&=\int_D \frac{\partial(-x^3-y^3)}{\partial x}-\frac{\partial(x^5+y^3)}{\partial y})\,\mathrm{d}A\\
&=\int_D \left(-3x^2-3y^2\right)\mathrm{d}A\\
&=\int_D -3(x^2+y^2)\,\mathrm{d}A \qquad T:\theta\in[0,2\pi], r\in[0,2]\\
&=\int_0^{2\pi}\int_0^2 -3r^2r\,\mathrm{d}r\,\mathrm{d}\theta\\
&=\int_0^2 -6\pi r^3\,\mathrm{d}r\\
&=-24\pi
\end{aligned}
$$

3

$$\frac{\partial M}{\partial x} = \frac{\partial\left(7y + \sqrt{y^3+1}\right)}{\partial x} = 0$$

$$\frac{\partial N}{\partial y} = \frac{\partial\left(3y - e^{\sin(x^2)}\right)}{\partial y} = 3$$

$$\text{by greens} \int_D (-3)\,\mathrm{d}A = -3\int_0^{2\pi}\int_0^1 r\,\mathrm{d}r\,\mathrm{d}\theta = -3\pi$$

4

$$\int_C \left(e^{\sqrt{x+1}} + y^2 + 1\right)\mathrm{d}x + \sin(y^2-1) + x^2\,\mathrm{d}y$$

$$\frac{\partial(\sin(y^2-1)+x^2)}{\partial x} = 2x$$

$$\frac{\partial\left(e^{\sqrt{x+1}}\right) + y^2 + 1}{\partial y} = 2y$$

$$\text{by greens} \int_D (2y-2x)\,\mathrm{d}A$$

$$= \int_0^2\int_0^{1-\frac{x}{2}} 2x - 2y\,\mathrm{d}x\,\mathrm{d}y$$

$$= \int_0^2 -\frac{5}{4}x^2 + 3x - 1\,\mathrm{d}x = \frac{2}{3}$$

5

Let's calculate $\frac{\partial M}{\partial x}$ and $\frac{\partial L}{\partial y}$:

$$\frac{\partial M}{\partial x} = \frac{\partial}{\partial x}(2x + \cos(y^2)) = 2$$
$$\frac{\partial L}{\partial y} = \frac{\partial}{\partial y}(y + e^{\sqrt{x}}) = 1$$

Therefore,

$$\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y} = 2 - 1 = 1.$$

$$\begin{aligned}
\int_C (y + e^{\sqrt{x}})dx + (2x + \cos(y^2))dy &= \iint_D \left(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}\right) dA \\
&= \iint_D 1\,dA \\
&= \int_0^1 \int_{x^2}^{\sqrt{x}} 1\,dy\,dx \\
&= \int_0^1 (\sqrt{x} - x^2)\,dx \\
&= \left(\frac{2}{3}x^{3/2} - \frac{1}{3}x^3\right)\Big|_0^1 \\
&= \frac{2}{3} - \frac{1}{3} \\
&= \frac{1}{3}.
\end{aligned}$$

6

$$\frac{\partial \vec{r}}{\partial u} = (1, 1, 2), \quad \frac{\partial \vec{r}}{\partial v} = (1, -1, 1)$$

$$\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} = (3, 1, -2)$$

$$\int_D \vec{F}\, \mathrm{d}S = \int_D \vec{F} \cdot \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\, \mathrm{d}u\, \mathrm{d}v$$

$$= \int_D (u + v, u - v, 1 + 2u + v) \cdot (3, -1, -2)\, \mathrm{d}u\, \mathrm{d}v$$

$$= \int_D (3u + 3v - u + v - 2 - 4u - 2v) dudv$$

$$= \int_D (-2u + 2v - 2) dudv$$

$$= \int_0^1 \int_0^1 -2u + 2v - 2)\, \mathrm{d}u\, \mathrm{d}v$$

$$= -2$$

7

$$\int_D (y, x^2 + y^2, x^2) \cdot (-2x, -2y, 1)\, \mathrm{d}A$$

$$= \int_0^1 \int_0^1 (-2xy - 2x^2 y - 2y^3 + x^2)\, \mathrm{d}x\, \mathrm{d}y$$

$$= \int_0^1 \left(-\frac{5}{3} y - 2y^3 + \frac{1}{3} \right) \mathrm{d}y$$

$$= -1$$