Small Oscillations

- Motion near a point of stable equilibrium.

DOF= 1 (one dimension)

- For a system of DOF = 1, with potential $U(q)$:
  - **stable equilibrium** at $U(q)_{\min}$, upward parabola, where $F = -\frac{\mathrm{d}U}{\mathrm{d}q} = 0$
    - restoring force for small displacements $q - q_0$ is $F = -\frac{\mathrm{d}U(q-q_0)}{\mathrm{d}q}$
- **Unstable equilibrium** at $U(q)_{\max}$, downward parabola, where $F = -\frac{\mathrm{d}U}{\mathrm{d}q} = 0$ as well.
- Consider small deviation from point of stable equilibrium, we use taylor expansion to show that it is really a small displacement. that is,

$$U \approx U(q_0) + \frac{\mathrm{d}U(q_0)}{\mathrm{d}q}(q - q_0) + \frac{\mathrm{d}^2U(q_0)}{2\,\mathrm{d}q^2}(q - q_0)^2$$
$$\text{while } \frac{\mathrm{d}U(q_0)}{\mathrm{d}q}(q - q_0) = 0 \tag{1}$$

letting $x = q - q_0$, we have

$$\begin{cases} U(x) = U(q_0) + \left(\frac{1}{2}\right)\frac{\mathrm{d}^2U(q_0)}{\mathrm{d}q^2}x^2 \\ \text{putting into the form of } U(x) = U(x_0) + \left(\frac{1}{2}\right)kx^2. \end{cases}$$
$$\Rightarrow \boxed{k = \frac{\mathrm{d}^2U(q_0)}{\mathrm{d}q^2} > 0} \tag{2}$$

we get KE, while choosing $U(q_0) = 0$:

$$T = \frac{1}{2}a(q)^2\dot{q}^2 = \frac{1}{2}a(q_0 + x)\dot{x}^2 \approx \frac{1}{2}m\dot{x}^2, \overset{m=a(q_0)}{\Rightarrow}$$
$$\boxed{L = T - U = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2} \tag{3}$$

EOM for DOF = 1 small Oscillations

using EL on Equation 3, we can get the EOM for one dimensional small Oscillations:

$$m\ddot{x} = -kx$$
$$\Rightarrow \ddot{x} + \omega_0^2 x = 0, \text{where } \boxed{\omega_0 = \sqrt{\frac{k}{m}} \text{ freq of osc.}} \tag{4}$$

by magic of ODE, EOM reduces down to:

$$\boxed{x(t) = C_1\cos(\omega_0 t) + C_2\sin(\omega_0 t)}$$
$$\text{where } C_1, C_2 \text{ are constants} \tag{5}$$

by trig magic, this could also be written as

$$x(t) = a\cos(\omega_0 t + \alpha),$$

$$\text{where} \begin{cases} a = \sqrt{C_1^2 + C_2^2} & \text{amplitude of oscillation} \\ \omega_0 & \text{frequency of oscillation} \\ \tan\alpha = C_2/C_1 & \text{phase at t=0} \end{cases} \quad (6)$$

energy for 1D small Oscillation

checking $\frac{\partial L}{\partial t} = 0 \Rightarrow$ energy-conservation:

$$\begin{aligned} E = T + U &= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 \\ &= \frac{1}{2}ma^2\omega_0^2, [\text{ constant}] \end{aligned} \quad (7)$$

Damped 1D oscillation, and Complex representation

- when there is damping (friction, resistence, etc) $F_{\text{fric}} = -\beta\dot{x}$, the EOM becomes:

$$\begin{gathered} \ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0, \\ \text{where } 2\gamma = \frac{\beta}{m}, \omega_0 = \sqrt{\frac{k}{m}} \end{gathered} \quad (8)$$

with ansatz $x(t) = e^{rt}, \dot{x} = re^{rt}, \ddot{x} = r^2 e^{rt}$, the solution to Equation 8 is:

$$\begin{gathered} r^2 + 2\gamma r + \omega_0^2 = 0, \\ \text{which has solution } r_+, r_- = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2} \\ \Rightarrow x(t) = C_1 e^{r_+ t} + C_2 e^{r_- t}, \end{gathered} \quad (9)$$

notice the r subscripts here: r_+, r_-

underdamped, overdamped, and critically damped

Recall from your ODE class...

Equation 9 has the following 3 cases, each with different physical interpretation:

1. underdamped:

$$\gamma < \omega_0 \Rightarrow 2 \text{ complex roots: } \begin{cases} r_\pm = -\gamma \pm i\sqrt{\omega_0^2 - \gamma^2} \\ = -\gamma \pm i\omega \\ \omega = \sqrt{\omega_0^2 - \gamma^2} \end{cases} \quad (10)$$

   The EOM is thus a linear combination of two complex expoentials:

$$\begin{aligned} x(t) &= e^{-\gamma t}(C_1 e^{i\omega t} + C_2 e^{-i\omega t}) \\ &= e^{-\gamma t}(A\cos(\omega t) + B\sin(\omega t)) \\ &\text{-- where } \begin{cases} A = C_1 + C_2 \\ B = i(C_1 - C_2) \end{cases} \\ &= ae^{-\gamma t}\cos(\omega t + \alpha) \\ &a, \alpha \text{ are constants} \end{aligned} \quad (11)$$

"The solution is a damped oscillation with frequencyω, and amplitude expoentially decaying with time."

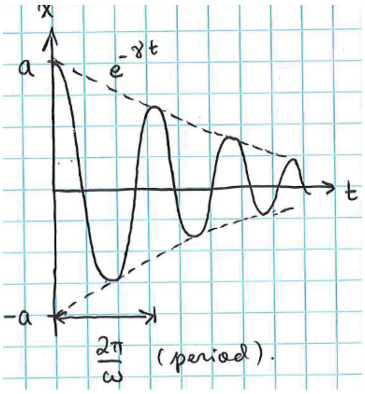
2. Overdameped

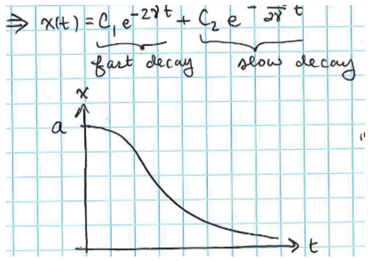
$$\begin{gathered} \gamma > \omega \Rightarrow x(t) = \\ c_1 e^{-\gamma + \sqrt{\gamma^2 - \omega^2} t} + c_2 e^{-\gamma - \sqrt{\gamma^2 - \omega^2} t} \end{gathered} \tag{12}$$

$$\text{when} \gamma \gg \omega_0, \Rightarrow \begin{cases} \gamma + \sqrt{\gamma^2 - \omega_0^2} \approx 2\gamma \\ \gamma - \sqrt{\gamma^2 - \omega^2} = \frac{\omega^2}{2\gamma} \end{cases} \tag{13}$$
$$x(t) = c_1 e^{-2\gamma t} + c_2 e^{(-\omega_0^2/2\gamma)t}$$

3. Critically damped

$$\gamma = \omega_0 \Rightarrow x(t) = c_1 e^{-\gamma t} + c_2 t e^{-\gamma t} \tag{14}$$



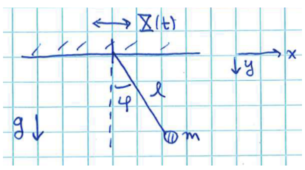


Forced Oscillations

When external force (F) is applied to the system, the largrangian becomes

$$\begin{gathered} L = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2 + F(t) x \\ \text{EL} \Rightarrow \ddot{x} + \omega_0^2 x = \frac{F(t)}{m}, \text{where } \omega_0 = \sqrt{\frac{k}{m}} \end{gathered} \tag{15}$$

- Example: Simple pendulum with moving pivot



$$\begin{gathered} \begin{cases} x = X + l \sin\varphi \\ y = l \cos\varphi \end{cases} \Rightarrow \begin{cases} \dot{x} = \dot{X} + l\dot{\varphi}\cos\varphi \\ \dot{y} = -l\dot{\varphi}\sin\varphi \end{cases} \\ \Rightarrow L = T - U \end{gathered} \tag{16}$$

$$\begin{gathered} L = \frac{1}{2} m l^2 \dot{\varphi}^2 - mgl(1 - \cos\varphi) - ml\ddot{X}\sin\varphi \\ \text{Expand ab. } \varphi = 0 \Rightarrow L = \frac{1}{2} m l^2 \dot{\varphi}^2 - \frac{1}{2} mgl\varphi^2 - ml\ddot{X}\varphi \\ \text{EL} \Rightarrow \boxed{\ddot{\varphi} + \omega_0^2 \varphi = -\frac{\ddot{X}}{l} \text{ ,where } \omega_0 = \sqrt{\frac{g}{l}}} \end{gathered} \tag{17}$$

reintroducing damping via external forcing

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = f(t), f(t) = \frac{F(t)}{m} \tag{18}$$

When damping $f(t) = f_0 \cos(\Omega t)$, solution via complex number:

$$\ddot{z} + 2\gamma\dot{z} + \omega_0^2 = f_0 e^{i\Omega t}$$

$$\text{ansatz } z(t) = z_0 e^{i\Omega t} \Rightarrow z_0 = \frac{f_0}{\omega_0^2 + 2i\gamma\Omega + \Omega_0^2}$$

$$\boxed{z_0 = a(\Omega)\cos(\Omega t + \delta(\Omega)) f_0} \text{ is a partcular solution,where} \tag{19}$$

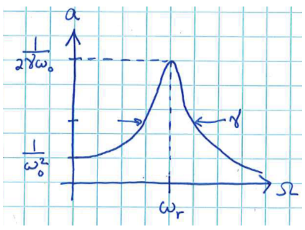
$$\begin{cases} a(\Omega) = \frac{1}{\sqrt{\left(\omega_0^2 - \Omega^2\right)^2 + (2\gamma\Omega)^2}} \\ \delta(\Omega) = \arctan\left(2\gamma\frac{\Omega}{\omega_0^2 - \Omega^2}\right) \end{cases}$$

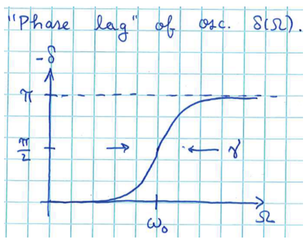
We can study the properties of the system by looking at the amplitude and phase of the solution.

- Amplitude:

$$a_{(\Omega)} = \frac{1}{\sqrt{\left(\omega_0^2 - \Omega^2\right)^2 + (2\gamma\Omega)^2}} \tag{20}$$

, when $\gamma \ll \omega_0$, response strongest and amplitude largest when $\omega_r = \omega_0$.





- Phase lag: $\tan\delta(\Omega) = 2\gamma\frac{\Omega}{\Omega^2 - \omega_0^2}$

  in phase as $\Omega \to 0$, and out of phase as $\Omega \to \omega_0$.

- Genral solution to sinusoidal forcing:

$$\begin{aligned} x(t) &= a(\Omega) f_0 \cos(\Omega t + \delta(\Omega)) + a_0 e^{-\gamma t}\cos(\omega t + \alpha) \\ &\overset{t > \frac{1}{r}}{\to} a(\Omega) f_0 \cos(\Omega t + \delta(\Omega)) \end{aligned} \tag{21}$$

  Forgets initial condition after time.

- Power obsorbed by oscillation

  $p = F\dot{x} = mf\dot{x}$

  Avg power of oscillation

$$\begin{gathered} P_{\text{avg}} = \frac{1}{T}\int_0^T mf\dot{x}\,\mathrm{d}t = -\frac{1}{2}mf_0 a(\Omega)\Omega\sin\delta(\Omega) \\ \text{simplifies to } P_{\text{avg}}(\Omega) = \gamma m f_0^2 \Omega^2 a_{(\Omega)}^2 \end{gathered} \tag{22}$$

  Absorption around resonance frequency $\Omega = \omega_0 + \varepsilon$ is maximum:

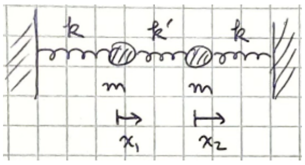
$$P = \frac{\gamma m f_0^2}{4(\varepsilon^2 + \gamma^2)} \approx \frac{m f_0^2}{4\gamma} \tag{23}$$

Oscillations DOF>1

For a system with n DOF: $q = (q_1, q_2, ..., q_n), \text{PE} = U(q)$

- Stable equilibrium $\frac{\partial U(q)}{\partial q_i}|_{q=0}$

Example: Oscillation with 2 mass and 3 springs



$$L = \frac{1}{2} m \dot{x_1} + \frac{1}{2} m \dot{x_2} - \frac{1}{2} k x_1^2$$
$$- \frac{1}{2} k x_2^2 - \frac{1}{2} k' (x_1 - x_2)^2$$

EOM:

$$\begin{gathered} M \cdot \ddot{\vec{x}} = -K \vec{x} \quad \text{, where } M = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}, \\ \vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, K = \begin{pmatrix} k + k' & -k' \\ -k' & k + k' \end{pmatrix} \end{gathered} \tag{24}$$

ansatz: $\vec{x} = \text{Re}\left[\vec{a} e^{i\omega t}\right]$ Then the EOM eq becomes solving the eigenvalue problem:

$$\det(\omega^2 M - K) = 0$$

$$\Rightarrow \begin{cases} \omega_-^2 = \frac{k}{m} \\ \omega_+^2 = \frac{k+2k'}{m} \end{cases} \begin{cases} \overrightarrow{x_-} = a_- \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos(\omega\_t + \delta\_) \\ \overrightarrow{x_+} = a_+ \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos(\omega_+ t + \delta_+) \end{cases} \tag{25}$$

with constants $a_-, a_+, \delta_-, \delta_+$.

New Coords

$$\begin{gathered} \begin{cases} Q_1 = \sqrt{\frac{m}{2}}(x_1 + x_2) \\ Q_2 = \sqrt{\frac{m}{2}}(x_1 - x_2) \end{cases} \\ \Rightarrow L = \frac{1}{2}\left(\dot{Q_1}^2 + \dot{Q_2}^2\right) - \frac{1}{2}\left(\omega_-^2 Q_1^2 + \omega_+^2 Q_2^2\right) \\ \overset{\text{E-L}}{\Rightarrow} \ddot{Q}_1 = -\omega_-^2 Q_1, \ddot{Q}_2 = -\omega_+^2 Q_2 \end{gathered} \tag{26}$$

Decoupled oscillators with coords Q_1, Q_2.

General Coords

for general coords q_i, let $x_i = q_i - q_i^{(0)}$

$$\begin{gathered} U = \frac{1}{2} \sum_{i,j} k_{ij} x_i x_j, \quad k_{ij} = k_{ji} = \frac{\partial^2 U(q)}{\partial q_i \partial q_j} \text{ symm mat} \\ T = \frac{1}{2} \sum_{i,j} m_{ij} \dot{x_i} \dot{x_j}, \quad m_{ij} = m_{ji} = a_{ij}\left(q^{(0)}\right) \end{gathered} \tag{27}$$

the largrangian, in Matix form:

$$L = \frac{1}{2}\dot{\vec{x}}^T \cdot M \cdot \dot{\vec{x}} - \frac{1}{2}\vec{x}^T \cdot K\vec{x} \overset{\text{EL}}{\Longrightarrow} (\omega^2 M - K) \cdot \vec{a} = 0 \tag{28}$$

$\Rightarrow \det(\omega^2 M - K) = 0$ Solving the det for omega gives the normal freq (Eigenvalues)of system ω_α^2 . plug in Evalue into Equation 28 for eigenvec(normal modes) $\overrightarrow{a^\alpha}$ of system.

- General motion

$$x_i(t) = \sum_\alpha a_i^\alpha \mathrm{Re}[C_\alpha e^{i\omega_\alpha t}] \tag{29}$$

- EXAMPLE: Normal freq is given

$$\begin{gathered}\omega = \left\{0, \sqrt{2}\omega_0, \sqrt{3}\omega_0\right\}. \\ \omega = \sqrt{2}\omega_0 \Rightarrow a_1 = -a_3 = -a_2 = ae^{i\delta} \Rightarrow \\ \vec{\theta} = a(1\ \ -1\ \ -1)^T \cos\left(\sqrt{2}\omega_0 t + \delta\right) \\ \omega = \sqrt{3\omega_0} \Rightarrow a_1 = 0, a_2 = -a_3 = ae^{i\delta} \Rightarrow \\ \vec{\theta} = a(0\ \ 1\ \ -1)^T \cos\left(\sqrt{3}\omega_0 t + \delta\right)\end{gathered} \tag{30}$$

- EXAMPLE: double pendulum

$$\begin{cases} x_1 = l_1 \sin\varphi_1 \quad y_1 = -l_1 \cos\varphi_1 \\ x_2 = l_1 \sin\varphi_1 + l_2 \sin\varphi_2 \quad y_2 = l_1 \cos\varphi_1 + l_2 \cos\varphi_2 \end{cases} \tag{31}$$

$$\begin{gathered}\Rightarrow T = \frac{1}{2}m_1 l_1 \dot{\varphi}^2 + \frac{1}{2}m_2(l_1^2 \dot{\varphi_1}^2 + l_2^2 \dot{\varphi_2}^2 \\ + 2l_1 l_2 \dot{\varphi}_1 \dot{\varphi}_2 \cos(\varphi_1 - \varphi_2)) \\ U = -m_1 g l_1 \cos\varphi_1 - m_2 g(l_1 \cos\varphi_1 + l_2 \cos\varphi_2)\end{gathered} \tag{32}$$

using $\cos\varphi \approx 1 - \frac{\varphi^2}{2}$

$$\begin{aligned} L &= \frac{1}{2}(\dot{\varphi_1}\ \ \dot{\varphi_2})\begin{pmatrix}(m_1 + m_2)l_1^2 & m_2 l_1 l_2 \\ m_2 l_1 l_2 & m_2 l_2^2\end{pmatrix}(\dot{\varphi_1}\ \ \dot{\varphi_2}) \\ &-\frac{1}{2}(\varphi_1\ \ \varphi_2)\begin{pmatrix}(m_1 + m_2)l_1 g & 0 \\ 0 & m_2 g l_2\end{pmatrix}(\varphi_1\ \ \varphi_2) \\ &= \frac{1}{2}\dot{\vec{\varphi}}^T M \cdot \dot{\vec{\varphi}} - \frac{1}{2}\vec{\varphi}^T K \vec{\varphi}\end{aligned} \tag{33}$$

When $m_1 = m_2 = m, \quad l_1 = l_2 = l \Rightarrow \quad M = ml^2\begin{pmatrix}2 & 1 \\ 1 & 1\end{pmatrix}, K = mgl\begin{pmatrix}2 & 0 \\ 0 & 1\end{pmatrix}$

$$\begin{gathered}\det((\omega^2 M - K)) = 0 \Rightarrow \omega^2 = \left(2 \pm \sqrt{2}\omega_0^2\right) \\ \begin{pmatrix}a_1^- \\ a_2^-\end{pmatrix} = C_-\begin{pmatrix}1 \\ \sqrt{2}\end{pmatrix}, \quad \begin{pmatrix}a_1^+ \\ a_2^+\end{pmatrix} = C_+\begin{pmatrix}1 \\ -\sqrt{2}\end{pmatrix}\end{gathered} \tag{34}$$

Normal Coords
$\{x_i\} = \{Q_\alpha\}$, where $x_i = \sum_{\alpha=1}^{n} A_{i\alpha} Q_\alpha \Rightarrow$

$\sum_j \left(\omega_\alpha^2 m_{ij} - k_{ij} A_{jx}\right) = 0$

$\Rightarrow L = \frac{1}{2} \sum_{\alpha=1}^{n} \left(\dot{Q}^2{}_\alpha - \omega_\alpha^2 Q_\alpha^2\right) \overset{\text{EL}}{\Longrightarrow} \ddot{Q}_\alpha + \omega_\alpha^2 Q_\alpha = 0$

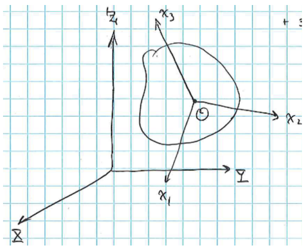
Motion of Rigid Body
- EXample: rotor

rotation with constraint $\left|\vec{r}_i - \vec{r}_j\right|$ .COM coords are useful here

$$\begin{cases} \vec{r} = \vec{r}_1 - \vec{r}_2 \\ \vec{R} = \frac{m_1 \vec{r_1} + m_2 \vec{r_2}}{m_1 + m_2} \end{cases} \Rightarrow \begin{cases} \vec{r}_1 = \vec{R} + m_2 \vec{r}/M \\ \vec{r}_2 = \vec{R} - m_1 \vec{r}/M \end{cases} \quad (35)$$

$$L = \frac{1}{2} M \dot{\vec{R}}^2 + \mu \dot{\vec{r}}^2, \quad \mu = m_1 \frac{m_2}{m_1 + m_2}$$
$$\overset{\text{polar}}{\Longrightarrow} L = \frac{1}{2} M \dot{\vec{R}}^2 + \frac{1}{2} \mu a^2 \left(\dot{\theta}^2 + \dot{\varphi}^2 \sin^2\theta\right) \quad (36)$$

frames of reference



$$(XYZ) \overset{R(\theta,\varphi,\psi)}{\Longrightarrow} (x_1, x_2, x_3)$$

Velocity of pt in body: $\vec{v} = \vec{V} + \vec{\Omega} \times \vec{r}$, where V is Translational vel, Omega is angular vel, r is position vector.

Largrangian for Rigid Body

$$T = \frac{1}{2} M V^2 + \frac{1}{2} \sum_a m_a \left[\Omega^2 r_a^2 - \left(\vec{\Omega}\right) \vec{r_a}\right)^2)]$$
$$T\text{translational} + T\text{rotational} \quad (37)$$

consider rotation,

$$\Omega^2 = \sum_i \Omega_i^2, \quad \vec{\Omega} \cdot \vec{r_a} = \sum_i \Omega_i x_{a,i}$$
$$\Rightarrow T_{\text{rot}} = \frac{1}{2} \sum_{\text{i,j}} \Omega_i \Omega_j I_{\text{i,j}}. \quad I_{ij} \equiv \sum_a m_{a \left(\delta_{ij} r_a^2 - x_{a,i} x_{a,j}\right)} \quad (38)$$

$\Rightarrow L = \frac{1}{2} M V^2 + \frac{1}{2} \sum_{i,j} I_{i,j} \Omega_i \Omega_j - U$

Inertial Tensor
- Discrete

$$I = \begin{pmatrix} \sum m(y^2+z^2) & -\sum mxy & -\sum mxz \\ -\sum mxy & \sum m(x^2+z^2) & -\sum myz \\ -\sum mxz & -\sum myz & \sum m(x^2+y^2) \end{pmatrix}) \tag{39}$$

- Continuous

$$\begin{gathered} I_{ij} = \int \rho(x)\left(\delta_{ij} r^2 - x_i x_j\right) \mathrm{d}V \\ I_{xx} = \int \rho(x)\left(y^2+z^2\right) \mathrm{d}V, I_{xy} = I_{yx} = -\int \rho(x) xy \, \mathrm{d}V \\ I_{yy} = \int \rho(x)\left(x^2+z^2\right) \mathrm{d}V, I_{yz} = I_{zy} = -\int \rho(x) yz \, \mathrm{d}V \\ I_{zz} = \int \rho(x)\left(x^2+y^2\right) \mathrm{d}V, I_{zx} = I_{xz} = -\int \rho(x) zx \, \mathrm{d}V \end{gathered} \tag{40}$$

example:

$$I_{xx} = \rho \int [b^2 \hat{y}^2 + c^2 \hat{z}^2] \, abc \, d\hat{x} \, d\hat{y} \, d\hat{z}$$

$$= abc\rho \int_V (b^2 \hat{y}^2 + c^2 \hat{z}^2) \, d\hat{x} \, d\hat{y} \, d\hat{z}$$

transform into spherical coord:

$$\boxed{I_{xx}} = abc\rho \int_V (b^2 r^2 \sin^2\varphi \sin^2\theta + c^2 r^2 \cos^2\theta) \, r^2 \sin\theta \, dr \, d\varphi \, d\theta$$

$$= abc\rho \Big[b^2 \int_0^{2\pi} \sin^2\varphi \, d\varphi \int_0^{\pi} \sin^3\theta \, d\theta \int_0^1 r^4 dr + c^2 \int_0^{2\pi} d\varphi \int_0^{\pi} \cos^2\theta \sin\theta \, d\theta \int_0^1 r^4 dr \Big]$$

$$= \boxed{\tfrac{4}{15} abc\pi\rho [b^2 + c^2]}$$

- Example: coplanar system principal axis: Z $\Rightarrow I_{13} = I_{23} = 0$
  $I_3 = I_1 + I_2$

Principle axis and principal moments of inertia

In the principal frame:

$$T_{\text{rot}} = \frac{1}{2}\left(I_1 \Omega_1^2 + I_2 \Omega_2^2 + I_3 \Omega_3^2\right) \tag{41}$$

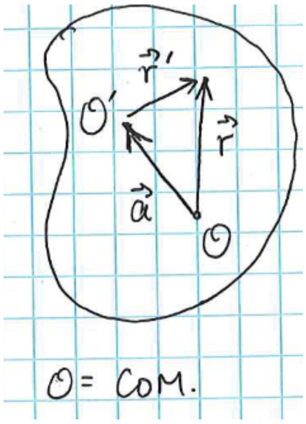
- spherical top $I_1 = I_2 = I_3$
- Symmetric top $I_1 = I_2 \neq I_3$
- Asymmetric top $I_1 \neq I_2 \neq I_3$
- EXample:

$$\begin{gathered} \det(I - \lambda \mathbf{1}) = 0 \Rightarrow \lambda \text{ prncp. mom.} \\ \vec{v} = \text{eigenvec.} = \text{prncp. axis} \end{gathered} \tag{42}$$

- EXample: continuous with axis of symmetry $\rho(\vec{r}) = \rho = (r, x_3) \Rightarrow I_{ij} = \int \rho(\vec{r})(r^2\delta - x_i x_j)\, \mathrm{d}V$

Parallel axis theorem

when changing Origin diff. from COM(O),



$$I_{ij} = I'_{ij} - M(a^2\delta_{ij} - a_i a_j)$$

For a cube, when finding I at corner, first find I at COM, and

$$I'_{xx} = I_{xx} + M(b^2 + c^2) = \frac{4}{3}M(b^2) + c^2$$

$$I'_{yy} = I_{yy} + M(a^2 + c^2) = \frac{4}{3}M(a^2 + c^2)$$ 43

$$I'_{zz} = I_{zz} + M(a^2 + b^2) = \frac{4}{3}M(a^2 + b^2)$$

$I_{13} = -\int dv\, \rho(\vec{r})\, x_1 x_3$ 03/22/24

$= -\int \cancel{dz}\, dx_3\, r\,dr\,d\varphi\, \rho(r, x_3)\, r\cos\varphi\, x_3$

$= -\int dx_3\, r\,dr\, \rho(r, x_3)\, r\, x_3 \underbrace{\int_0^{2\pi} d\varphi\, \cos\varphi}_{=0}$

& $I_{23} = 0$ by same analysis w/ $\cos\varphi \to \sin\varphi$.

$\Rightarrow I = \begin{pmatrix} I_{11} & I_{12} & 0 \\ I_{21} & I_{22} & 0 \\ 0 & 0 & I_3 \end{pmatrix} \Rightarrow x_3 =$ principal axis;

i.e., symm. axis = princ. axis

$$I_{12} = -\int dv\, \rho(\vec{r})\, x_1 x_2$$

$$= -\int dx_3\, r\,dr\, d\varphi\, \rho(r,z)\, r^2 \cos\varphi \sin\varphi$$

$$= -\int dx_3\, r\,dr\, \cancel{d\varphi}\, \rho(r,z)\, r^2 \underbrace{\int_0^{2\pi} d\varphi\, \cos\varphi \sin\varphi}_{=0}$$

$$\Rightarrow I = \begin{pmatrix} I_{11} & 0 & 0 \\ 0 & I_{22} & 0 \\ 0 & 0 & I_3 \end{pmatrix}, \quad x_1, x_2, x_3 = \text{principal axes.}$$

$$I_1 - I_2 = \int dv\, \rho(\vec{r})\, (x_2^2 - x_1^2)$$

$$= \int dx_3\, r\,dr\, \cancel{d\varphi}\, \rho(r,z)\, r^2 \underbrace{\int_0^{2\pi} d\varphi\, (\sin^2\varphi - \cos^2\varphi)}_{=0}$$

$$\Rightarrow I_1 = I_2 \equiv I_\perp.$$

$$\Rightarrow I = \left(\begin{array}{cc|c} I_\perp & 0 & 0 \\ 0 & I_\perp & 0 \\ \hline 0 & 0 & I_3 \end{array}\right)$$

→ any two ⊥ axes in x_1x_2-plane are principal axes.