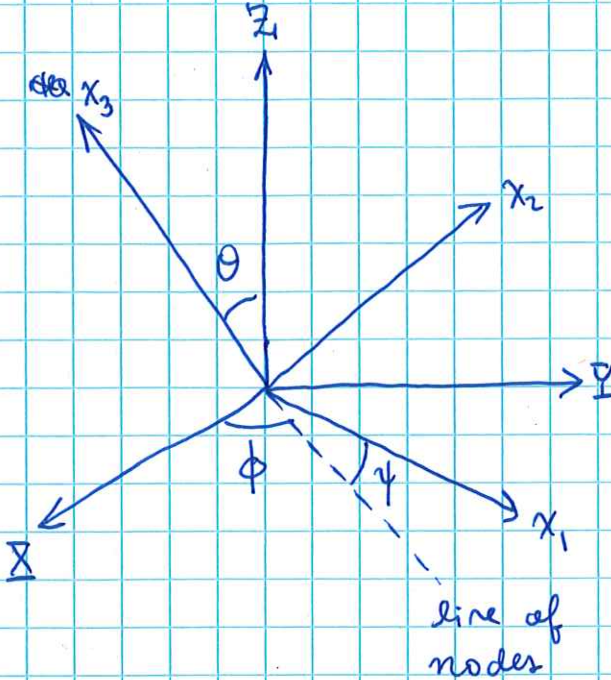
Summary 04/05/24

Rotational DOF of rigid body described by "Euler angles"



$(X\,Y\,Z) \xrightarrow[R(t)]{} (x_1\, x_2\, x_3)$.

$R(t) = R(\hat{Z}, \phi(t))\, R(\hat{X}, \theta(t))\, R(\hat{Z}, \psi(t))$.

· line of nodes = intersection of XY- & $x_1 x_2$-planes.

parallel to $\vec{n} = \hat{Z} \times \hat{x}_3$.

$$\vec{\Omega} = \dot{\theta}\, \underset{\substack{\uparrow \\ \parallel \text{ to line of nodes.}}}{\hat{n}} + \dot{\phi}\, \hat{Z} + \dot{\psi}\, \hat{x}_3$$

Goal: Express Lagrangian L in terms of gen. coord.'s $q = (\theta\, \phi\, \psi)$. Start w/ $T = \frac{1}{2}(I_1 \Omega_1^2 + I_2 \Omega_2^2 + I_3 \Omega_3^2)$.

· From def.'n of $(\theta \phi \psi)$ we find: $\hat{Z} = \cos\theta\, \hat{x}_3 + \sin\theta \left(\sin\psi\, \hat{x}_1 + \cos\psi\, \hat{x}_2\right)$.

$$\Rightarrow \Omega_1 = \hat{x}_1 \cdot \vec{\Omega} = \dot{\theta}\, \hat{x}_1 \cdot \hat{n} + \dot{\phi}\, \hat{x}_1 \cdot \hat{Z} \qquad (\hat{x}_1 \cdot \hat{x}_3 = 0).$$

$$\Rightarrow \boxed{\begin{aligned} \Omega_1 &= \dot{\theta} \cos\psi + \dot{\phi} \sin\theta \sin\psi \\ \Omega_2 &= -\dot{\theta} \sin\psi + \dot{\phi} \sin\theta \cos\psi \\ \Omega_3 &= \dot{\phi} \cos\theta + \dot{\psi} \end{aligned}}$$

similarly for Ω_2, Ω_3:

We thus have: 04/05/24

$$T = \frac{1}{2} I_1 (\dot\phi \sin\theta \sin\psi + \dot\theta \cos\psi)^2 + \frac{1}{2} I_2 (\dot\phi \sin\theta \cos\psi - \dot\theta \sin\psi)^2 + \frac{1}{2} I_3 (\dot\phi \cos\theta + \dot\psi)^2$$

$U = U(\theta, \phi, \psi)$ (depends on the problem).

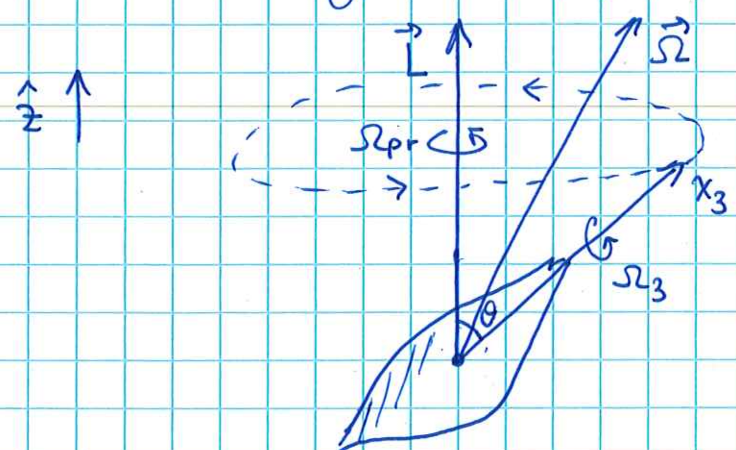
$$\rightarrow L(\underbrace{\theta, \phi, \psi}_{q}, \underbrace{\dot\theta, \dot\phi, \dot\psi}_{\dot q}) = T - U$$

<u>Ex</u>: Free motion ($U=0$) of a symmetric top.

symm. top $\Rightarrow$ $I_1 = I_2 \equiv I_\perp$.

$$\Rightarrow T = \frac{1}{2} I_\perp (\dot\theta^2 + \dot\phi^2 \sin^2\theta) + \frac{1}{2} I_3 (\dot\phi \cos\theta + \dot\psi)^2 .$$

<u>Recall</u>: we have found motion of free symm. top using cons. of $\vec{L}$.



motion = precession of symm. axis x_3 about fixed dir. of $\vec{L}$ + rot.^{n} about x_3.

$$\Omega_{pr} = L_z / I_\perp \qquad (|\vec{L}| = L_z)$$

$$\Omega_3 = L_3 / I_3 = L_z \cos\theta / I_3 .$$

E-L. eqn.'s $(L = T)$: 04/05/24

θ: $\frac{d}{dt}\frac{\partial L}{\partial \dot\theta} = \frac{\partial L}{\partial \theta} \Rightarrow \frac{d}{dt}\left(I_\perp \dot\theta\right) = I_\perp \sin\theta\cos\theta\,\dot\phi^2 - I_3\dot\phi\sin\theta\left(\dot\phi\cos\theta + \dot\psi\right).$

ϕ: $\frac{d}{dt}\frac{\partial L}{\partial \dot\phi} = \frac{\partial L}{\partial \phi} \Rightarrow \frac{d}{dt}\left(I_\perp\dot\phi\sin^2\theta + I_3\cos\theta\left(\dot\phi\cos\theta + \dot\psi\right)\right) = 0.$

ψ: $\frac{d}{dt}\frac{\partial L}{\partial \dot\psi} = \frac{\partial L}{\partial \psi} \Rightarrow \frac{d}{dt}\left(I_3\left(\dot\phi\cos\theta + \dot\psi\right)\right) = 0.$

- now choose $\hat{z}$ along fixed ang. mom. $\vec{L}$.

$$\Rightarrow L_3 = L_z\cos\theta \quad (L_z = |\vec{L}|).$$

& we also have $L_3 = I_3\Omega_3 = I_3(\dot\phi\cos\theta + \dot\psi)$. Thus we recognize that ψ EOM is simply $\frac{d}{dt}L_3 = 0$.

Since $L_3 = L_z\cos\theta$ & $L_z = |\vec{L}| = $ const, $\dot L_3 = 0 \Rightarrow \cos\theta = $ const.

$\Rightarrow \theta = $ const. & $L_z\cos\theta = I_3\Omega_3 \Rightarrow \Omega_3 = \frac{L_z\cos\theta}{I_3}$ (as we found before)

- next consider θ EOM. & use $\dot\theta = 0$:

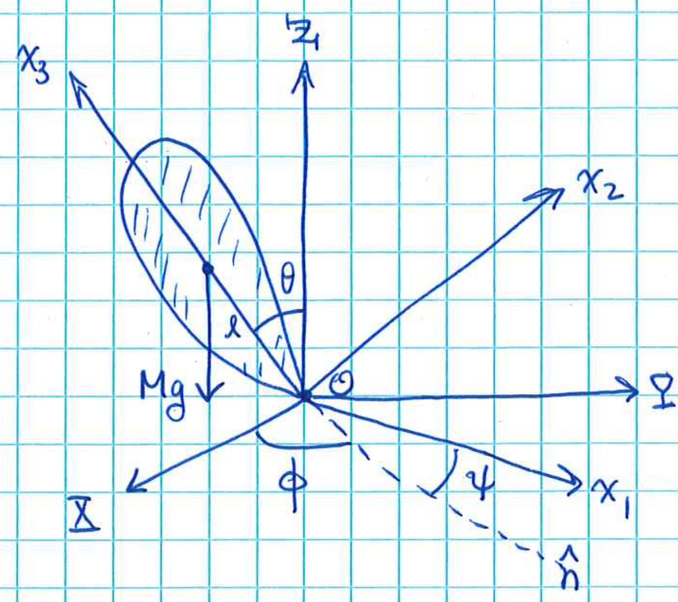
$$0 = \dot\phi\sin\theta\left(I_\perp\cos\theta\dot\phi - L_3\right)$$

If $\dot\phi\sin\theta \neq 0$, we obtain $\dot\phi = \frac{L_3}{I_\perp\cos\theta} = \frac{L_z}{I_\perp}$

→ same as precession freq. Ω_{pr} we found before.

- Finally, ϕ EOM gives $\ddot\phi = 0$; i.e., $\dot\phi = $ const.

Ex: (Heavy symmetric top w/ one pt. fixed) 04/05/24



(M = total mass)

- Take origin O of moving system at fixed pt. of the body (not COM) & same origin for space frame.
- COM distance l along x_3 axis.

- Since we take O at ~~fixed~~ a pt. in the body that is fixed, motion is a pure rotation (no translational component).
- Given moments of inertia $I_\perp$ & I_3 about COM, use $\parallel$-axis thm. to find moments of inertia about O:

$$I'_{ij} = I_{ij} + M\,(l^2 \delta_{ij} - l_i l_j)$$

$$\vec{l} = l\,\hat{x}_3 \Rightarrow I'_\perp = I_\perp + Ml^2\,, \qquad I'_3 = I_3$$

- Finally, potential energy is:

$$U = \sum mgz \quad \text{(sum over particles of body)}$$

$$= Mg z_1\,, \quad z_1 = z\text{-coord. of COM.}$$

$$= Mgl\cos\theta.$$

$$\Rightarrow L = T - U = \tfrac{1}{2} I'_\perp\,(\dot\theta^2 + \dot\phi^2 \sin^2\theta) + \tfrac{1}{2} I_3 (\dot\psi + \dot\phi\cos\theta)^2 - Mgl\cos\theta.$$

E-L eqn.'s (3 DOF, $\theta\phi\psi$). 04/05/24

ϕ: $\frac{d}{dt}\frac{\partial L}{\partial \dot\phi} = \frac{\partial L}{\partial \phi} \Rightarrow \frac{d}{dt}\left[I_1' \dot\phi \sin^2\theta + I_3 \cos\theta(\dot\psi + \dot\phi\cos\theta)\right] = 0$

$\Rightarrow p_\phi = (I_1' \sin^2\theta + I_3\cos^2\theta)\dot\phi + I_3\cos\theta\dot\psi = \text{const.}$

$\equiv L_z$.

gen. momentum

ψ: $\frac{d}{dt}\frac{\partial L}{\partial\dot\psi} = \frac{\partial L}{\partial\psi} \Rightarrow \frac{d}{dt}\left[I_3(\dot\psi + \dot\phi\cos\theta)\right] = 0$

$\Rightarrow p_\psi = I_3(\dot\psi + \dot\phi\cos\theta) = \text{const.} \equiv L_3$.

$\Rightarrow L_z$ & L_3 are two conserved quantities

[check: torque $\vec{K} = -Mg\vec{R}\times\hat{z}_1$, $\vec{R}$ = COM

$= -Mgl\,\hat{x}_3\times\hat{z}_1$

$\Rightarrow K_z = K_3 = 0 \Rightarrow L_z, L_3 =$ conserved]

· Instead of writing θ EOM, recall that time-translation symmetry of L $\Rightarrow$ energy is a third conserved quantity.

$$E = T + U = \frac{1}{2}I_1'(\dot\theta^2 + \dot\phi^2\sin^2\theta) + \frac{1}{2}I_3(\dot\psi + \dot\phi\cos\theta)^2 + Mgl\cos\theta.$$

$\dot\phi$ & $\dot\psi$ may be eliminated using L_z & L_3:

$$\begin{cases} L_z = I_1'\sin^2\theta\,\dot\phi + \cos\theta L_3 \Rightarrow \dot\phi = \dfrac{L_z - L_3\cos\theta}{I_1'\sin^2\theta} & (*) \\[2ex] L_3 = I_3(\dot\psi + \dot\phi\cos\theta) \Rightarrow \dot\psi = \dfrac{L_3}{I_3} - \dfrac{\cos\theta(L_z - L_3\cos\theta)}{I_1'\sin^2\theta} & (**) \end{cases}$$

04/05/24

plugging into E:

$$E = \frac{1}{2} I_1' \dot{\theta}^2 + \frac{1}{2I_1'} \frac{(L_z - L_3\cos\theta)^2}{\sin^2\theta} + \frac{1}{2I_3} L_3^2 + Mgl\cos\theta.$$

$$\Rightarrow \underbrace{E - \frac{L_3^2}{2I_3} - Mgl}_{= E'} = \frac{1}{2} I_1' \dot{\theta}^2 + \underbrace{\frac{1}{2I_1'} \frac{(L_z - L_3\cos\theta)^2}{\sin^2\theta} - Mgl(1-\cos\theta)}_{U_{eff}(\theta).}$$

$$\Rightarrow E' = \frac{1}{2} I_1' \dot{\theta}^2 + U_{eff}(\theta).$$

→ effective 1 DOF problem.

as before, this can be integrated

$$\dot{\theta} = \frac{d\theta}{dt} \Rightarrow t = \int \frac{d\theta}{\sqrt{2[E' - U_{eff}(\theta)]/I_1'}}$$

→ full sol.'n of problem:

invert $t(\theta) \rightarrow \theta(t)$ & plug into

(*) + (**) for $\phi(t)$ & $\psi(t)$.

· Rather than evaluating the integral, we can learn much more about the motion by studying the effective potential.