HW 6, Harry Luo

ex4.6

recall from lec the normal approximation formula, where

$$P(|\hat{p} - p| < \varepsilon) \geq 2\Phi\left(2\varepsilon\sqrt{n}\right) - 1 \tag{1}$$

For this problem, we have $\varepsilon = 0.02, 2\Phi\left(2\varepsilon\sqrt{n}\right) - 1 \geq 0.95$. We can solve for n when $n = n_{\min}$ with the following:

$$\begin{aligned} 2\Phi\left(2\varepsilon\sqrt{n}\right) - 1 &= 0.95 \\ \Phi\left(2 * 0.02\sqrt{n}\right) &= \frac{1.95}{2} \end{aligned} \tag{2}$$

accroding to the table of Phi values, we have

$$\begin{gathered} 0.04\sqrt{n} = 1.96 \\ \Rightarrow n = 2401 \end{gathered} \tag{3}$$

therefore the smallest size should be 2401

ex4.8

Rolling a biased die can be modeled as a binomial distribution as either "rolling the number 6" or not. We denote an unknown probability of rolling a 6 as p, and denote the number of getting 6 as X. We write $X \sim \mathrm{Bin}(1000000, p)$. We want to find a confidence interval for p with 0.999 confidence. Using Equation 1, we have $n = 1000000$, $P|(\hat{p} - p| < \varepsilon) = 0.99$. We need to solve for ε at the lower bound, where:

$$\begin{gathered} 2\Phi\left(2\varepsilon\sqrt{n}\right) - 1 = 0.999 \\ \Rightarrow \Phi(2 * 1000\varepsilon) = 0.9995 \\ \Rightarrow 2000\varepsilon \approx 3.32 \\ \varepsilon = 0.00166 \end{gathered} \tag{4}$$

Since the number 6 shows up 180000 times when rolling 1000000 times, $\hat{p} = \frac{180000}{1000000} = 0.18$.

$\boxed{\text{Therefore, the confidence interval is } [\hat{p} - \varepsilon, \hat{p} + \varepsilon] = [0.1783, 0.1817]}$

ex4.10

We assume that scoring a goal in a certain game is a rare event for the player, we can approximate the r.v. X corresponding to the number of goals scored by the player as a Poisson distribution

$$P(X = k) = \frac{e^{-\lambda}\lambda^k}{k!} \tag{5}$$

probability of player scoring 0 goals is $P(X = 0) = e^{-\lambda}\frac{\lambda^0}{0!} = e^{-\lambda}$

Thus the probability of scoring at least 1 goal is $1 - e^{-\lambda} = 0.5 \Rightarrow \lambda = \ln(2) \approx 0.693$
We can now calculate the approximation for scroing 3 goals as

$$\boxed{P(X = 3) = \frac{e^{-\lambda}\lambda^k}{k!} = \frac{e^{-0.693}0.693^3}{3 * 2 * 1} = 0.028}$$

ex4.34

Assume that accidents happen rarely and independently. We can model the number of accidents happen in a week with a Poisson distribution. We denote the r.v. X as the number of accidents in a week, and we have $X \sim \mathrm{Poisson}(\lambda)$, where lambda is the average number of accidents in a week, given as $\lambda = 3$ Therefore, the probability of **at most** 2 accidents happening next week can be calculated as

$$P(X = 1) + P(X = 2) + P(X = 0) = \left(e^{-3}\right)\left(\frac{3^1}{1} + \frac{3^2}{2 * 1} + \frac{3^3}{3 * 2 * 1}\right) = 0.59744 \tag{6}$$

ex4.46

We can consider the series of trials of "flipping a coin 5 times each day for 30 days" as a binomial distribution, where we either get 5 tails each day or not. We denote the r.v. X as the number of days that we get 5 tails. The probability of having 5 tails in a day is $p = \frac{1}{2^5} = \frac{1}{32}$. Therefore, $X \sim \text{Bin}\left(30, \frac{1}{32}\right)$

Since $np(1-p) = \frac{465}{512}$, the normal approximation is not valid.

Poisson approximation is a bettor choice, especially when our $np = 15/512$ is small.

We approximate the distribution of X with r.v. $Y \sim \text{Poisson}(\lambda)$where $\lambda = E(X) = np = \frac{30}{32} = 0.9375$. Thus,

$$P(X = 2) \approx P(Y = 2) = \frac{e^{-0.9375} 0.9375^2}{2} \approx 0.1721 \tag{7}$$

ex5.2

- (a)

Given the MGF, we can calculate its derivatives as

$$M'(t) = -\frac{4}{3}e^{-4t} + \frac{5}{6}, M''(t) = \frac{16}{3}e^{-4t}\frac{25}{6}e^{5t} \tag{8}$$

We can get

$$\begin{gathered} E(X) = M'(0) = \frac{1}{2}, E(X^2) = M''(0) = \frac{19}{2} \\ \Rightarrow \text{Var}(X) = E(X^2) - E(X)^2 = \frac{37}{4} \end{gathered} \tag{9}$$

- (b)

Given the MGF, we observe that the possible values for r.v. are 0, −4, 5; and the corresponding probabilities are 1/2, 1/3,1/6. Thus the discrete probability mass function is $P(X = 0) = \frac{1}{2}, P(X = -4) = \frac{1}{3}, P(X = 5) = \frac{1}{6}$. From which we can calculate We can calculate

$$\begin{gathered} E(X) = -4 * \frac{1}{3} + 5 * \frac{1}{6} = \frac{1}{2}; E(X^2) = \frac{1}{3} * 16 + \frac{1}{6} * 25 = \frac{19}{2} \\ \text{Var}(X) = E(X^2) - E(X)^2 = \frac{37}{4} \end{gathered} \tag{10}$$

As calculated in (a).

ex5.18

- (a)

Given $X \sim \text{Geom}(p)$, the probability mass function is $P(X = k) = p(1-p)^{k-1}$, where k=1,2,3,...

$$\begin{aligned} M_X(t) = E(e^{tX}) &= \sum_{k=1}^{\infty} e^{tk} P(X = k) = \sum_{k=1}^{\infty} e^{tk} p(1-p)^{k-1} = pe^t \sum_{k=1}^{\infty} \left(e^t(1-p)\right)^{k-1} \\ &= pe^t \sum_{k=0}^{\infty} \left(e^t(1-p)\right)^k \end{aligned} \tag{11}$$

when $e^t(1-p) < 1$, i.e. $t < \ln\left(\frac{1}{1-p}\right)$, the series converges, and

$$M_X(t) = \frac{pe^t}{1 - e^t(1-p)} \tag{12}$$

while $t \geq \ln\left(\frac{1}{1-p}\right)$, the series diverges, and

$$M_X(t) = +\infty \tag{13}$$

- (b)

$$
\begin{aligned}
E(X) = M'_{X(0)} &= \frac{pe^t}{\left(1 - e^t(1-p)\right)^2}]_{t=0} = \frac{1}{p}. \\
E(X^2) = M''_X(0) &= \frac{pe^t}{\left(1 - e^t(1-p)\right)^2}|_{t=0} = \frac{2}{p^2} - \frac{1}{p} \\
\mathrm{Var}(X) &= E(X^2) - E(X)^2 = \frac{1}{p^2} - \frac{1}{p}
\end{aligned}
\tag{14}
$$

ex 5.20

- (a) by def, we know

$$
M_X(t) = \int_{-\infty}^{\infty} e^{tx} * \frac{1}{2} e^{-|x|} dx = \frac{1}{2} \int_0^{\infty} e^{(-1-t)x} dx + \frac{1}{2} \int_{-\infty}^{0} e^{(t+1)x} dx \tag{15}
$$