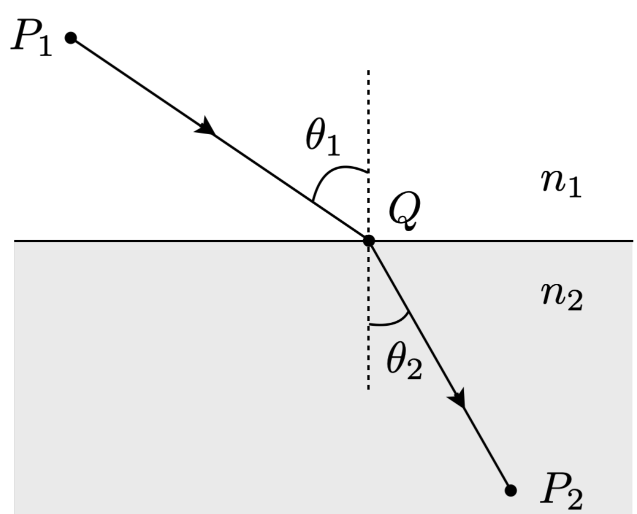
Physics 311
Spring 2024
Homework 1
Due Friday, February 2, 2024

This assignment covers material in Chapter 6 of Taylor. I recommend reading through the text and also Lectures Notes 1, 2, and 3.

Problem 1: (Snell's law) A ray of light travels from point P_1 in a medium with refractive index n_1 to P_2 in a medium of index n_2, by way of the point Q on the plane interface between the two media, as shown in the Figure below. Show that Fermat's principle implies that, on the actual path followed, Q lies in the same vertical plane as P_1 and P_2 and obeys Snell's law: $n_1 \sin\theta_1 = n_2 \sin\theta_2$.



Problem 2: (Geodesics) Use the calculus of variations to show that the shortest path between two points in the plane is a straight line. *Hint:* Start by writing down the integral expression for the arc length of a curve $y(x)$ in the 2D xy plane. The resulting expression for the arc length is a *functional* of the curve $y(x)$, which has to be minimized.

Problem 3: (Coordinate invariance of the E-L equations) Let $q_1, \ldots, q_n$ be a set of generalized coordinates. They obey the Euler-Lagrange equations following from a Lagrangian $L(q, \dot{q}, t)$. Now consider a change of coordinates, in which we express these coordinates in terms of a new set $r_1, \ldots, r_n$:

$$q_i = q_i(r_1, \ldots, r_n, t)\,. \tag{1}$$

Note that we have allowed the coordinate transformation to be time dependent. We can now consider the Lagrangian as a function of the r coordinates, that is, we write $\tilde{L}(r, \dot{r}, t) = L(q(r,t), \frac{d}{dt}q(r,t), t)$.

Using appropriate chain rules for differentiation, show that the Euler-Lagrange equations are also obeyed for $\tilde{L}$:

$$\frac{\partial \tilde{L}}{\partial r_i} - \frac{d}{dt}\frac{\partial \tilde{L}}{\partial \dot{r}_i} = 0\,. \tag{2}$$

Part of the power of the Euler-Lagrange equations is that the form of the equations is the same in all coordinates. (The same is not true for Newton's equations, where various 'fictitious forces' may arise due to changing frames). Note this problem is also meant to be an exercise in the chain rule and partial differentiation.

Problem 4: (Total Derivatives in the Action) In this problem you will show, starting from the Euler-Lagrange equations, that shifting the Lagrangian by a total derivative leaves the equations of motion unchanged. This fact will be useful when we take up the topic of symmetries in Lagrangian mechanics.

a) As a preliminary step, write out the expression for $\frac{d}{dt}f(q(t),t)$ in terms of partial derivatives.

b) Write down the Euler-Lagrange equations for the Lagrangian $L(q,\dot{q},t)$.

c) Consider a new Lagrangian $\tilde{L}$, related to L by a total derivative:

$$\tilde{L}(q,\dot{q},t) = L(q,\dot{q},t) + G(q,\dot{q},t)$$

where $G(q,\dot{q},t) = \frac{d}{dt}F(q,t)$. Using your result from (a), show the G dependent terms in the Euler-Lagrange equations for $\tilde{L}$ drop out. Thus, $\tilde{L}$ and L give rise to the same equations of motion.

Problem 5: (Higher derivative Lagrangian) Consider the action

$$S[q(t)] = \int_{t_1}^{t_2} dt\ L(q,\dot{q},\ddot{q}). \tag{3}$$

where, in contrast to what we have considered in class, the Lagrangian is allowed to depend on the higher derivative $\ddot{q}$. Compute the variation δS, subject to the usual condition that the variation at the endpoints vanish $\delta q(t_1) = \delta q(t_2) = 0$, as well as the additional condition that the time derivative of the variation vanishes $\delta\dot{q}(t_1) = \delta\dot{q}(t_2) = 0$. Deduce the equations of motion from the condition that the action is stationary, $\delta S = 0$. What is the order of the resulting EOM? What initial data is required to uniquely determine a physical trajectory?