Lecture Monday Feb 12

Review

Partition of Ω: events $B_1, ..., B_n$ that are pairwise disjoint: $B_1 \cup ... \cup B_n = \Omega$
Law of total probability $P(A) = P(A|B_1)P(B_1) + ... + P(A|B_n)P(B_n)$
Bayes' Theorem $P(B|A) = \frac{P(A|B)P(B)}{P(A)}$, for P(A) and P(B) > 0
Independence for independent events A and B

$$P(A|B) = P(A)$$
$$\Rightarrow P(A \cap B) = P(A)P(B)$$

Examples

- Suppose it rains 25% of the days. WHen it rains, it it cloudy at sunrise 50% of the time. When it doesn't rain, it is cloudy at sunrise 10% of the time. Find:

(a)P(cloudy at sunrise)
Considering the law of total probability,

$$\begin{aligned} &P(\text{cloudy at sunrise}) \\ &= P(\text{cloudy at sunrise}|\text{rain})P(\text{rain}) + P(\text{cloudy at sunrise}|\text{no rain})P(\text{no rain}) \\ &= 0.5 * 0.25 + 0.1 * 0.75 \\ &= \frac{1}{5} \end{aligned}$$

(b)P(rain|cloudy at sunrise)
Consider Bayes' Theorem,

$$\begin{aligned} &P(\text{rain}|\text{cloudy at sunrise}) \\ &= \frac{P(\text{cloudy at sunrise}|\text{rain})P(\text{rain})}{P(\text{cloudy at sunrise})} \\ &= \frac{0.5 * 0.25}{\frac{1}{5}} \\ &= \frac{5}{8} \end{aligned}$$

- Draw a single card from a standard deck of 52 cards. Let A be the event that the card is red, and B be the event that the card is a queen. Show: A and B are independent.

show $P(A \cap B) = P(A) * P(B)$
$P(A \cap B) = \frac{1}{52} + \frac{1}{52} = \frac{2}{52}$
$P(A) * P(B) = \frac{1}{2} * \frac{1}{13} = \frac{1}{26}$
$\frac{2}{52} = \frac{1}{26}$
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Claim: If A and B are independent, then $A^c \wedge B, A \wedge B^c, A^c \wedge B^c$ are independent

Proove:
$P(A^c \cap B) = P(B) - P(A \cap B)$
$= P(B) - P(A)P(B)$
$= P(B)(1 - P(A))$
$= P(B)P(A^c)$

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Independence of multiple events
Events $A_1, ..., A_n$ are mutually independent if for every collection of events $A_{\{i_1\}}, ..., A_{\{i_k\}}$,

$$P\left(A_{\{i_1\}} \cap ... \cap A_{\{i_k\}}\right) = P\left(A_{\{i_1\}}\right)...P\left(A_{\{i_k\}}\right)$$

for example, for events A,B,C, if

$$P(a \cap b) = P(A)P(B)$$
$$P(A \cap C) = P(A)P(C)$$
$$P(B \cap C) = P(B)P(C)$$
$$\Rightarrow P(A \cap B \cap C) = P(A)P(B)P(C)$$

We say, A,B,C are mutually independent.

=examples
Roll a fair 4-sided die. let

$$A = \{1, 2\}, B = \{1, 3\}, C = \{1, 4\}$$

,

$$P(A) = P(B) = P(C) = \frac{2}{4}$$

,

$$P(A \cap B) = P(A \cap C) = P(B \cap C) = P(\{1\}) = \frac{1}{4}$$

, we know

$$P(A \cap B \cap C) = P(\{1\}) = \frac{1}{4} \neq P(A)P(B)P(C) = \frac{1}{8}$$

Not mutually independent.

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Random varaibles in Independence

For random variables $X_1, X_2, ..., X_n$ on the same prob. space, we say they are independent if

$$P(X_1 \in B_1, X_2 \in B_2, ..., X_n \in B_n) = P(X_1 \in B_1)P(X_2 \in B_2)...P(X_n \in B_n)$$

for all choices of B_k (Diffidult to check)

- For discrete random variables, we can check independence if and only if $P(X_1 = x_1, X_2 = x_2, ..., X_n = x_n) = P(X_1 = x_1)P(X_2 = x_2)...P(X_n = x_n)$ for all choices of $x_1, x_2, ..., x_n$

examples

roll a red die and blue die, $\Omega = \{1, 2, ..., 6\} \times \{1, 2, ..., 6\}$