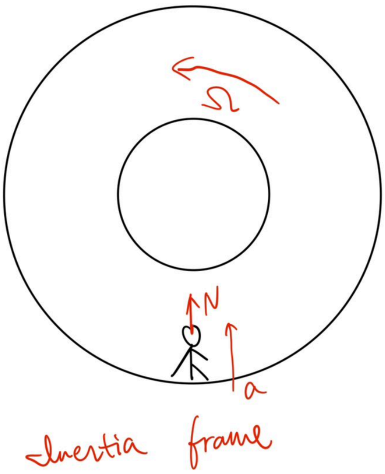
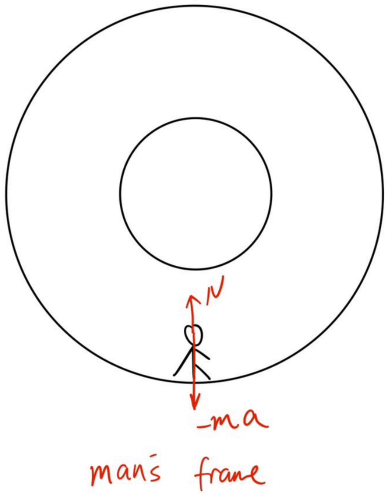
1





- (a) (b) as graphed above. In order to simulate gravity,

$$A = \omega^2 R = g \Rightarrow \omega = \sqrt{\frac{g}{R}} = 0.5\ \text{rad}/s \tag{1}$$

- (c) effective gravity is

$$\begin{aligned} g_{\text{eff}} &= \omega^2 R \propto R \\ \Rightarrow \frac{g_{\text{head}} - g_{\text{feet}}}{g_{\text{feet}}} &= \frac{R_{\text{head}} - R_{\text{feet}}}{R_{\text{feet}}} = \frac{-2}{40} = -5\% \end{aligned} \tag{2}$$

2

- (a) recall EOM for a particle in rotating frame with angular velocity Ω,

$$m\ddot{\vec{r}} = -m\Omega^2\rho + \overbrace{2m\dot{\vec{r}} \times \vec{\Omega}}^{F_{\text{cor}}} + \overbrace{m\vec{\Omega} \times \left(\vec{r} \times \vec{\Omega}\right)}^{F_{\text{centripetal}}} + F_{\text{buoyancy}} \tag{3}$$

  noticing that the droplet is relatively stationary with respect to the rotation frame, $F_{\text{cor}} = 0, \quad F_{\text{centripetal}} \neq 0$

- (b) (c) The direction of $F_g + F_{\text{cent}}$ is downward, tengential to the water surface.

  The water level is a equipotential surface of the combined potential of gravity and centrifugal force. So for a given height z,

$$\begin{aligned} U &= mgz + U_{\text{cent}} \\ \text{where } U_{\text{cent}} &= -\int F_{\text{cent}}\,\mathrm{d}\rho = -\int m\Omega^2\rho\,\mathrm{d}\rho = -\frac{1}{2}m\Omega^2\rho^2 \\ \Rightarrow U &= mgz - \frac{1}{2}m\Omega^2\rho^2 = \text{constant} \\ \Rightarrow z(\rho) &= \frac{\Omega^2\rho^2}{2g} + \text{const} \end{aligned} \tag{4}$$

It is obvious that $z(\rho)$, i.e. water surface, is a parabola.

we choose the frame to be the following: $\{\hat{x}, \hat{y}, \hat{z}\}$, where $\hat{x}$ points eastward, $\hat{y}$ points northward, and $\hat{z}$ points upward. Let particle above earth surface has position $\vec{r}$, whose angle with earth's rotation axis is θ.

EOM of particle

$$\begin{cases} \dot{\vec{v}} = -g\hat{z} + 2\vec{v} \times \vec{\Omega} \\ \vec{\Omega} = \Omega \sin\theta \hat{y} + \Omega \cos\theta \hat{z} \\ \vec{v} = v_x \hat{x} + v_y \hat{y} + v_z \hat{z} \end{cases} \Rightarrow \dot{\vec{v}} = -g\hat{z} + 2\Omega\big[(v_y \cos\theta - v_z \sin\theta)\hat{x} - v_x \cos\theta \hat{y} + v_x \sin\theta \hat{z}\big] \quad (5)$$

let $\vec{v}(t) = \vec{v_1} + \vec{v_2}$, with the first term being velocity neglecting rotation, and 2nd term is the correction with rotation.

for $\vec{v}_1, \Omega = 0$.

$$\Rightarrow \begin{cases} \dot{v}_x = \dot{v}_y = 0 \\ \dot{v}_z = -g \end{cases} \Rightarrow \quad \begin{cases} \vec{v_1}(t) = \vec{v_0} - gt\hat{z} \\ \vec{r}(t) = \vec{r_0} + \vec{v_0}t - \frac{1}{2}gt^2\hat{z} \end{cases} \quad (6)$$

Therefore, when recalling the coriolis force, $\dot{\vec{v}} = -g\hat{z} + 2\vec{v} \times \vec{\Omega}$, we have

$$\begin{aligned} \dot{\vec{v_1}} + \dot{\vec{v_2}} &= -g\hat{z} + 2\vec{v_1} \times \vec{\Omega} + 2\vec{v_2} \times \vec{\Omega} \\ & \qquad\qquad\qquad\qquad \text{recognizing } \Omega = \text{small}, v_2 = \text{small} \\ \vec{v_2} &\approx 2(\vec{v_0} - gt\hat{z}) \times \vec{\Omega} \end{aligned} \quad (7)$$

Now consider:

1. free fall from height h, with $v_0 = 0$, over a time lapse of $t = \sqrt{\frac{2h}{g}}$

$$\begin{gathered} \dot{\vec{v}} = 2gt\Omega \sin\theta \hat{x} \Rightarrow \quad \vec{v} = \int \dot{\vec{v}}\,\mathrm{d}t = gt^2\Omega \sin\theta \hat{x} \\ \vec{v} = -gt\hat{z} + gt^2\Omega \sin\theta \hat{x} \end{gathered} \quad (8)$$

$$\vec{r} = \int \vec{v}\,\mathrm{d}t = \begin{pmatrix} \frac{1}{3}g\Omega t^3 \sin\theta \\ 0 \\ -\frac{1}{2}gt^2 \end{pmatrix} \quad (9)$$

Noticing that $t = \sqrt{\frac{2h}{g}}$, displacement on the x-y plane is

$$x_f = \frac{1}{3}g\Omega \left(\sqrt{\frac{2h}{g}}\right)^3 \sin\theta \quad (10)$$

2. compare with particle thrown from ground, with $\vec{v_0} = v_0\hat{z}, \quad t = 2\sqrt{\frac{2h}{g}} = \frac{2v_0}{g}$

$$\begin{aligned} \dot{\vec{v_2}} &= 2(v_0 - gt)\hat{z} \times \vec{\Omega} \\ &= 2(v_0 - gt)\hat{z} \times (\Omega \sin\theta \hat{y} + \Omega \cos\theta \hat{z}) \\ &= \hat{x}[-2(v_0 - gt)\Omega \sin\theta] \\ &= (-2v_0\Omega \sin\theta + 2gt\Omega \sin\theta)\hat{x} \end{aligned} \quad (11)$$

$$\vec{v_2} = \int \vec{v_2}\,\mathrm{d}t = (-2v_0\Omega\sin\theta t + g\Omega\sin\theta t^2)\hat{x}$$

$$\Rightarrow \vec{v} = \vec{v_1} + \vec{v_2} = \begin{pmatrix} -2v_0\Omega\sin\theta t + g\Omega\sin\theta t^2 \\ 0 \\ v_0 - gt \end{pmatrix} \tag{12}$$

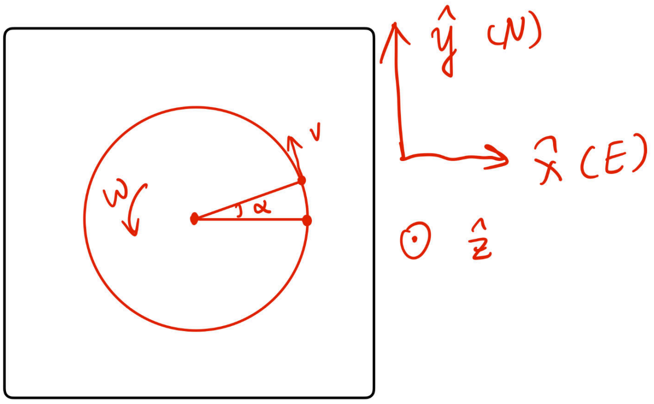
Considering $t = 2\sqrt{\frac{2h}{g}} = \frac{2v_0}{g}$

$$\begin{aligned} \Rightarrow x_t = \int v_x\,\mathrm{d}t = -v_0\Omega\sin\theta t^2 + \frac{1}{3}g\Omega\sin\theta t^3 \\ = -v_0\Omega\sin\theta\left(2\frac{v_0}{g}\right)^2 + \frac{1}{3}g\Omega\sin\theta\left(2\frac{v_0}{g}\right)^3 \\ = -\frac{4}{3}g\Omega\left(\sqrt{\frac{2h}{g}}\right)^3\sin\theta \end{aligned} \tag{13}$$

Comparing Equation 10 and Equation 13, we have $(x_t)/(x_f) = -4$, thus showing that the deflection is indeed opposite and 4 times larger when the particle is thrown from the ground.

4

We propose the following frame: $\{\hat{x}, \hat{y}, \hat{z}\}$, where $\hat{x}$ points eastward, $\hat{y}$ points northward, and $\hat{z}$ points upward. Picture a hoop lying on ground with the following illustration:



Consider the rotation of an infinitesimal element of the hoop over an infinitesimal time interval, across small angle α. Find coriolis force

$$dF_{\text{cor}} = 2\,\mathrm{d}m\left(\vec{v} \times \vec{\Omega}\right) \tag{14}$$

where

$$\vec{v} = \omega r \begin{pmatrix} -\sin\alpha \\ \cos\alpha \\ 0 \end{pmatrix}, \quad \vec{\Omega} = \Omega \begin{pmatrix} 0 \\ \sin\theta \\ \cos\theta \end{pmatrix}, \quad \mathrm{d}m = m\frac{\mathrm{d}\alpha}{2\pi} \tag{15}$$

Torque on said infinitesimal element can be found by

$$\begin{gathered} d\tau = \vec{r} \times \mathrm{d}F_{\text{cor}}, \quad \vec{r} = r\begin{pmatrix} \cos\alpha \\ \sin\alpha \\ 0 \end{pmatrix} \\ \Rightarrow d\tau = 2\,\mathrm{d}m\left(\vec{v}\left(\vec{r}\cdot\vec{\Omega} - \vec{\Omega}(\vec{r}\cdot\vec{v})\right)\right) = 2\omega r^2\Omega\sin\theta\,\mathrm{d}m\begin{pmatrix} -\sin^2\alpha \\ \sin\alpha\cos\alpha \\ 0 \end{pmatrix} \end{gathered} \tag{16}$$

noticing $\mathrm{d}m = m\frac{\mathrm{d}\alpha}{2\pi}$, integrating the above w.r.t. α from 0 to 2π,

$$\tau = -(m\omega\Omega r^2\sin\theta)\hat{x} \tag{17}$$

The above shows that the torque due to coriolis force is westward with magnitude $m\omega\Omega r^2\sin\theta$

5

According to the hints in problem statement of Taylor 9.34, we write the puck's position vector, relative to the earth's center O, as $\vec{R} + \vec{r}$, where $\vec{R}$ is the position of thi point P and $\vec{r} = (x, y, 0)$ is the puck's position relative to P.

Ignoring centrifugal force, we can write the EOM as the following

$$\ddot{\vec{r}} = g_0(r) + 2\dot{\vec{r}} \times \vec{\Omega} \tag{18}$$

$$\text{where } g_0(r) = -GM\frac{\vec{R} + \vec{r}}{\left\|\left(\vec{R} + \vec{r}\right)\right\|^3} = -GM\frac{\vec{R} + \vec{r}}{R^3}\left(1 + \frac{r^2}{R^2}\right)^{-\frac{3}{2}} \tag{19}$$

recognizing that $r \ll R$, expanding the above function gives an approximation:

$$g(r) = -GM\frac{\vec{R} + \vec{r}}{R^3} = \vec{g}(0) + g(0)\frac{\vec{r}}{R} \tag{20}$$

putting the above into Equation 18, we have

$$\begin{aligned}\ddot{\vec{r}} &= \vec{g}(0) + g(0)\frac{\vec{r}}{R} + 2\dot{\vec{r}} \times \vec{\Omega} \\ &= g(0)\begin{pmatrix}0\\0\\1\end{pmatrix} - \frac{g}{R}\begin{pmatrix}x\\y\\0\end{pmatrix} + 2\begin{pmatrix}\dot{y}\Omega\cos\theta - \dot{z}\Omega\sin\theta\\ -\dot{x}\Omega\cos\theta\\ \dot{x}\Omega\sin\theta\end{pmatrix}\end{aligned} \tag{21}$$

The above is set of 2nd order ODEs:

$$\begin{aligned}\ddot{x} &= -g\frac{x}{R} + 2\dot{y}\Omega\cos\theta \\ \ddot{y} &= -g\frac{y}{R} - 2\dot{x}\Omega\cos\theta\end{aligned} \tag{22}$$

This is the same as Foucault pendulum equation with length of pendulum being R.

The oscillation frequency is $\omega_0 = \sqrt{\frac{g}{R}} \approx 1.24e(-3)\ s^{-1}$, frequency of Foucault precession, $\Omega_z = \Omega\cos\theta$