Summary

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- Cyclic coord.'s: $p_i = \dfrac{\partial L}{\partial \dot{q}_i}$ $\quad F_i = \dfrac{\partial L}{\partial q_i}$

$$\Rightarrow \dot{p}_i = F_i$$

If $F_k = \dfrac{\partial L}{\partial q_k} = 0$, then p_k = conserved $(\dot{p}_k = 0)$

q_k = "cyclic coord."

- Symmetry & conservation laws.

(1) homog. of time $(t \to t+\varepsilon)$ $\Rightarrow$ $E = \sum_i \dot{q}_i \dfrac{\partial L}{\partial \dot{q}_i} - L$ conserved.

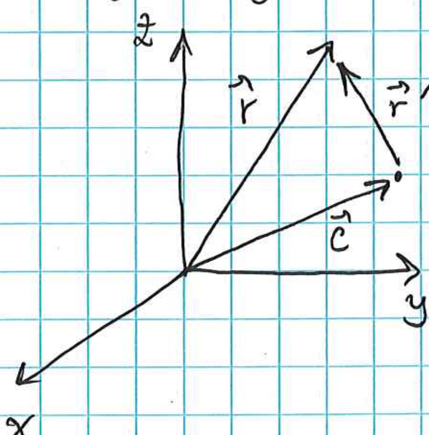
$$= T + U$$

(2) homog. of space $(\vec{r}_a \to \vec{r}_a + \vec{\varepsilon})$ $\Rightarrow$ $\vec{P} = \sum_a m_a \vec{v}_a$ " "

(3) isotropy of space $\begin{pmatrix} \vec{r}_a \to \vec{r}_a + \varepsilon \hat{n} \times \vec{r}_a \\ \vec{v}_a \to \vec{v}_a + \varepsilon \hat{n} \times \vec{v}_a \end{pmatrix}$ $\Rightarrow$ $\vec{L} = \sum_a \vec{r}_a \times \vec{p}_a$ " "

Wrap up from last time. Transformation properties of $\vec{L}$.

- choice of origin: $\vec{r}_a = \vec{r}_a\,' + \vec{c}$ $\Rightarrow$ $\vec{L} = \sum_a \vec{r}_a \times \vec{p}_a$



$$= \sum_a \vec{r}_a\,' \times \vec{p}_a + \vec{c} \times \sum_a \vec{p}_a$$

$$= \vec{L}\,' + \vec{c} \times \vec{P}$$

$\Rightarrow \vec{L}$ indep. of origin when $\vec{P} = 0$ (COM frame).

· transform.'n btwn. inertial frames

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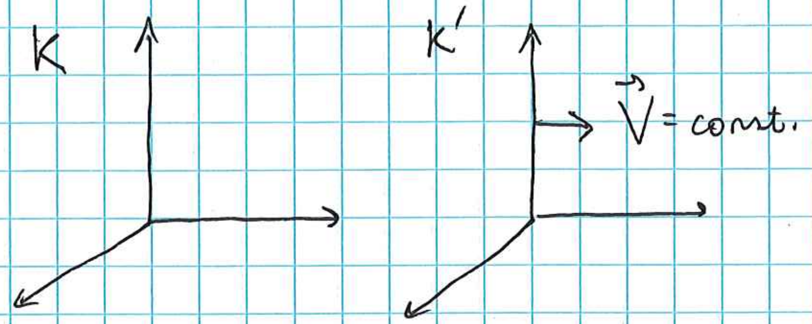

$\vec{v}_a = \vec{v}_a' + \vec{V}$

(suppose origins of K & K' coincide at instant considered)

$$\vec{L} = \sum_a \vec{r}_a \times \vec{p}_a = \sum_a \vec{r}_a \times m_a \vec{v}_a = \sum_a \vec{r}_a \times m_a (\vec{v}_a' + \vec{V}).$$

$$\Rightarrow \vec{L} = \sum_a \vec{r}_a \times m_a \vec{v}_a' + \left(\sum_a m_a \vec{r}_a\right) \times \vec{V}$$

$$\vec{R} = \frac{1}{M} \sum_a m_a \vec{r}_a$$

$$= \vec{L}' + M \vec{R} \times \vec{V} \qquad M = \sum_a m_a.$$

$\vec{L}'$: ang. mom. in frame K'.

If K' = COM frame ~~then~~, where $\vec{P}' = 0$, then $\vec{P} = M\vec{V}$

$$\Rightarrow \vec{L}^{\circ} = \vec{L}' + \vec{R} \times \vec{P}$$

$\vec{L}'$: "intrinsic ang. mom."

$\vec{R} \times \vec{P}$: motion "as a whole".

Conservation of $\vec{L}$ in ext. fields:

· motion in central potential $U(r)$, $r = |\vec{r}|$.

under rotation: $\vec{r} \to \vec{r}' = R \cdot \vec{r}$, $U(r) \to U(r') = U(r)$

$\Rightarrow$ Lagrangian invariant & $\vec{L}$ = conserved.

more generally, components of $\vec{L}$ conserved when there is an axis of symmetry.

Ex: $U = U(z) \Rightarrow$ inv. under rot. about $\hat{z} \Rightarrow L_z = \hat{z} \cdot \vec{L}$ conserved.

Explicitly, if $U = U(z)$, we use cylindrical coord.'s & write:

$$L = T - U = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\phi}^2 + \dot{z}^2\right) - U(z).$$

E-L eqn.'s: ϕ: $\quad \frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} = \frac{\partial L}{\partial \phi} = 0$

$$\Rightarrow \frac{\partial L}{\partial \dot{\phi}} = mr^2\dot{\phi} = \text{conserved}.$$

$$\text{\&} \quad L_z = \hat{z}\cdot\vec{L} = mr^2\dot{\phi}$$

check: $L_z = x p_y - y p_x$ $\quad x = r\cos\phi \quad \dot{x} = \dot{r}\cos\phi - r\dot{\phi}\sin\phi$; $\quad y = r\sin\phi \quad \dot{y} = \dot{r}\sin\phi + r\dot{\phi}\cos\phi$

$$= m\left(r\dot{r}\cos\phi\sin\phi + r^2\dot{\phi}\cos^2\phi - r\dot{r}\cos\phi\sin\phi + r^2\dot{\phi}\sin^2\phi\right)$$

$$= mr^2\dot{\phi} \quad \checkmark$$

Mechanical Similarity

(This is a "bonus" topic. You will not be tested on this material).

From symmetries of the Lagrangian L we have deduced a number of conservation laws. We'll now exploit one more transformation property of L to obtain some non-trivial consequences regarding the motion:

- Rescaling $L \to L' = \alpha L$ does not affect EOM.

$$\left(\frac{d}{dt}\frac{\partial L'}{\partial \dot{q}} = \alpha\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = \alpha\frac{\partial L}{\partial q} = \frac{\partial L'}{\partial q}\right)$$

• Keeping this rescaling property of L in mind, consider (02/07/24)

a rescaling of the coord.'s:

$$\vec{r} \longrightarrow \vec{r}\,' = \alpha \vec{r}$$

$$\Rightarrow \; T = \sum_a \frac{1}{2} m_a \dot{\vec{r}}_a^{\,2} \longrightarrow T' = \sum_a \frac{1}{2} m_a (\dot{\vec{r}}_a\,')^2$$

$$= \sum_a \frac{1}{2} m_a \alpha^2 \dot{\vec{r}}_a^{\,2}$$

$$= \alpha^2 T$$

$$\Rightarrow T' = \alpha^2 T.$$

we say T is a "homogeneous" fn., which means the value of the fn. is simply rescaled upon rescaling its arguments.

• Now suppose the potential is also a homogeneous fn:

$$\vec{r} \rightarrow \vec{r}\,' = \alpha \vec{r}$$

$$\Rightarrow \quad U \rightarrow U' = U(\alpha \vec{r}_1, \ldots, \alpha \vec{r}_N) = \alpha^k U(\vec{r}_1, \ldots, \vec{r}_N)$$

U = "homog. fn. of degree k"

(T = homog. fn. of degree 2).

<u>Ex:</u> Uniform force: $U \sim z \Rightarrow k = 1$

Harmonic force: $U \sim r^2 \Rightarrow k = 2$.

Newton / Coulomb: $U \sim 1/r \Rightarrow k = -1$

• So, if U = homog. fn. of deg. k, then upon rescaling:

$$L \rightarrow L' = T' - U' = \alpha^2 T - \alpha^k U.$$

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· now also make a rescaling of time:

$$t \to t' = \beta t$$

$$\Rightarrow \quad \vec{v} = \frac{d\vec{r}}{dt} \to \vec{v}' = \frac{d\vec{r}'}{dt'} = \frac{\alpha}{\beta}\frac{d\vec{r}}{dt} = \frac{\alpha}{\beta}\vec{v}$$

$$\Rightarrow \quad T \to T' = \frac{\alpha^2}{\beta^2} T.$$

$$\& \quad L \to L' = \frac{\alpha^2}{\beta^2} T - \alpha^k U.$$

· Now we choose $\beta^2 = \alpha^{2-k}$; then we get:

$$L' = \frac{\alpha^2}{\alpha^{2-k}} T - \alpha^k U$$

$$= \alpha^k T - \alpha^k U$$

$$= \alpha^k L$$

· Thus we conclude that, under the transform.'n:

$$\vec{r}(t) \to \vec{r}'(\underbrace{t'}_{=\beta t}) = \alpha \vec{r}(t)$$

L is simply rescaled & EOM unchanged.

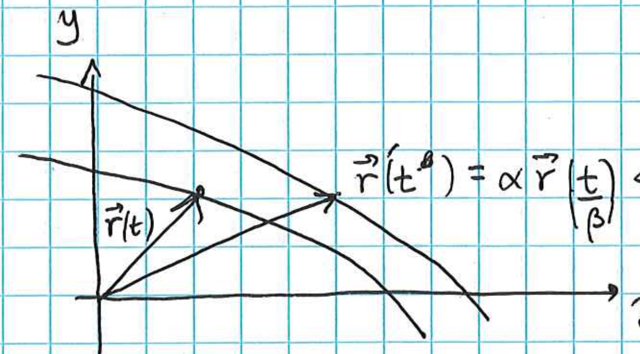
· written a slightly different way:

$$\vec{r}'(\beta t) = \alpha \vec{r}(t) \quad \text{set } t \to t/\beta$$

$$\vec{r}'(t) = \alpha \vec{r}(t/\beta)$$

$$= \alpha \vec{r}\left(\alpha^{\frac{k}{2}-1} t\right)$$

So, if U = homog. fn. of deg. k, we can generate entire (02/07/24) families of geometrically similar paths (solutions to EOM) from one solution $\vec{r}(t)$.

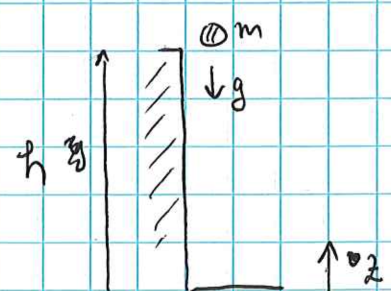


← new solution to EOM: rescaling of old soln. ($\vec{r}(t)$) w/ a diff. time of motion through trajectory.

Some consequences of this:

Q: Suppose we drop a ball, initially at rest, from height h & it takes time T to reach the ground. How long will it take to reach ground if dropped from height $2h$?

- straight-forward approach:



$$m\ddot{z} = -mg$$

$$\Rightarrow z(t) = h - \tfrac{1}{2}gt^2$$

$$0 = z(T) = h - \tfrac{1}{2}gT^2 \Rightarrow T = \sqrt{2h/g}$$

Thus, if $h \to 2h$, then $T \to \sqrt{2} \times T$.

Note: to get this result, we had to solve EOM

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- alternatively, use the fact that $U = mgz$ is a homog. fn. of deg. $k=1$

$\Rightarrow$ if there is a solution in which particle travels distance h in time T, then there is another solution in which particle travels $h' = \alpha h$ in time $T' = \beta T = \alpha^{1-k/2} T = \sqrt{\alpha} \times T$

So, if $h' = 2h$, then $T' = \sqrt{2}\, T$ ✓

→ note we didn't have to solve EOM!

More generally, suppose U = homog. fn. of deg. k & particle travels distance l in time t. Then there is another trajectory (sol.'n of EOM) in which particle travels l' in time t' w/ $l' = \alpha l$, $t' = \beta t = \alpha^{1-\frac{k}{2}} t$.

$$\Rightarrow \left(\frac{t'}{t}\right) = \alpha^{1-k/2} = \left(\frac{l'}{l}\right)^{1-k/2}$$

Ex:
- uniform force, $k=1$ $\quad \frac{t'}{t} = \sqrt{\frac{l'}{l}} \Rightarrow$ time of fall ~square root of initial height.
- harmonic force, $k=2$ $\quad \frac{t'}{t} = \alpha^0 = \text{const.} \Rightarrow$ period of oscillation indep. of amplitude

• Newton / Coulomb, $k=-1$ $\quad \frac{t'}{t} = \left(\frac{l'}{l}\right)^{3/2}$

$\Rightarrow$ for orbit of size L & period T, we have $T^2 \sim L^3$. That is, orbital period varies in proportion to the cube of the size of orbit

→ Kepler's third law of planetary motion.