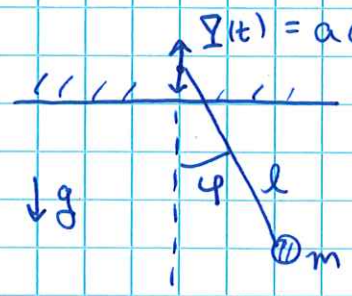
03/04/24

Summary

· Parametric resonance: $\ddot{x} + \omega^2(t)x = 0$, $\omega(t+T) = \omega(t)$ $(*)$

Q: under what conditions can we expect resonance phenomena? i.e., large amplitude oscillations

Examples of $(*)$:

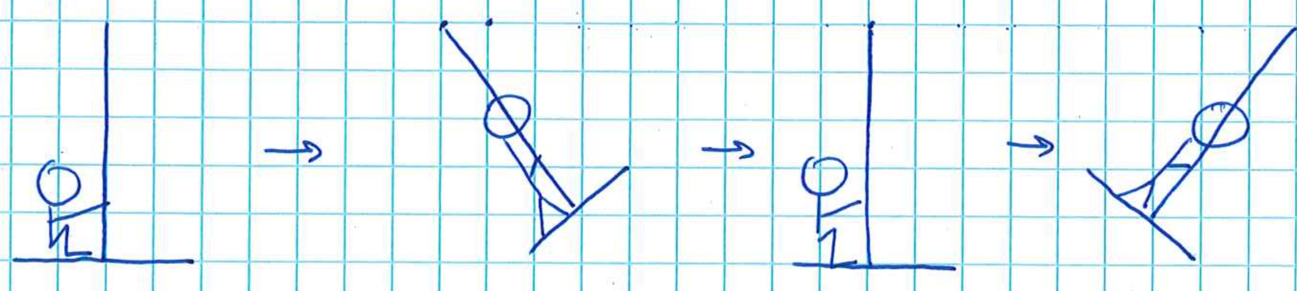


expanding near equil. at $\varphi = 0$:

$$L = \frac{1}{2} m l^2 \dot{\varphi}^2 - \frac{1}{2} m g l \left(1 + \frac{a\Omega^2}{g}\cos\Omega t\right)\varphi^2$$

$$\rightarrow \text{EOM:} \quad \ddot{\varphi} + \underbrace{\omega_0^2\left(1 + \frac{a\Omega^2}{g}\cos\Omega t\right)}_{\omega^2(t).}\varphi = 0$$

pumping a swing:



$$T_{pump} \simeq \frac{1}{2} T_{swing}.$$

Ex: Focus on specific ex.: $\omega^2(t) = \omega_0^2(1 + h\cos\Omega t)$, $h \ll 1$

$$\rightarrow \quad \ddot{x} + \omega_0^2(1 + h\cos\Omega t)\,x = 0$$

$$\Rightarrow \quad \ddot{x} + \omega_0^2 x = -\omega_0^2 h \cos(\Omega t)\,x \qquad (**)$$

C.f. $\ddot{x} + \omega_0^2 x = f_0 \cos\Omega t$ (forced oscillations).
In the present case, appearance of x on RHS dramatically changes behavior.

Idea: Expect strongest response when system oscillating near natural freq $\simeq \omega_0$. So suppose $x \sim \cos\omega_0 t$. Then RHS of $(**)$ is:

$$\cos(\Omega t)\cos(\omega_0 t) \sim \cos[(\Omega+\omega_0)t] + \cos[(\Omega-\omega_0)t]$$

→ this drives further oscillations at ω_0 if $\Omega \simeq 2\omega_0$, so that $\Omega - \omega_0 \simeq \omega_0$.

Since we anticipate resonance may be possible for $\Omega \simeq 2\omega_0$, we'll consider $\Omega = 2\omega_0 + \varepsilon$, $\varepsilon \ll \omega_0$ & seek soln.'s of the form:

$$x(t) = a(t)\cos\left[\left(\omega_0 + \tfrac{\varepsilon}{2}\right)t\right] + b(t)\sin\left[\left(\omega_0 + \tfrac{\varepsilon}{2}\right)t\right] \qquad (***)$$

Suppose a & b slowly varying w/ t in comparison to oscillatory part. (for $h=0$, a & b are simply const.'s).

Note: we know our ansatz (***) cannot be exact, as it ignores higher harmonics $\sim 3(\omega_0 + \frac{\varepsilon}{2})$, etc, which we know will be generated if we start w/ a sol.$^{\underline{n}}$ of the form (***). However, the amp. of such harmonics can be shown to be small in ~~prop~~ proportion to h. — see Landau & Lifshitz §27 Prob. 1.

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now plug (***) into EOM & recall identities:

$$\cos\alpha \cos\beta = \frac{1}{2}\left[\cos(\alpha+\beta) + \cos(\alpha-\beta)\right]$$

$$\cos\alpha \sin\beta = \frac{1}{2}\left[\sin(\alpha+\beta) - \sin(\alpha-\beta)\right]$$

let $\varphi = (\omega_0 + \frac{\varepsilon}{2})t$, so that $\omega^2(t) = \omega_0^2(1 + h\cos(2\varphi))$

& $x(t) = a\cos\varphi + b\sin\varphi$.

(1) $$\omega^2(t)\,x(t) = \omega_0^2\left(a\cos\varphi + b\sin\varphi\right) + \frac{h\omega_0^2}{2}\left[a\left(\cos 3\varphi + \cos\varphi\right) + b\left(\sin 3\varphi - \sin\varphi\right)\right]$$

oscillating far off resonance at $3(\omega_0 + \frac{\varepsilon}{2})$ → ignore these terms.

$$\Rightarrow \omega^2(t)\,x(t) \simeq \omega_0^2\left[a\left(1 + \frac{h}{2}\right)\cos\varphi + b\left(1 \cancel{+} - \frac{h}{2}\right)\sin\varphi\right]$$

(2) $$\ddot{x} = \ddot{a}\cos\varphi - a\left(\omega_0 + \frac{\varepsilon}{2}\right)^2\cos\varphi - 2\dot{a}\left(\omega_0 + \frac{\varepsilon}{2}\right)\sin\varphi + \ddot{b}\sin\varphi - b\left(\omega_0 + \frac{\varepsilon}{2}\right)^2\sin\varphi + 2\dot{b}\left(\omega_0 + \frac{\varepsilon}{2}\right)\cos\varphi.$$

Now we use our assumption that a & b are slowly varying to drop $\ddot{a}$ & $\ddot{b}$ terms:

$$\ddot{x} \simeq \left[-a\left(\omega_0 + \frac{\varepsilon}{2}\right)^2 + 2\dot{b}\left(\omega_0 + \frac{\varepsilon}{2}\right) \right] \cos\varphi$$
$$+ \left[-b\left(\omega_0 + \frac{\varepsilon}{2}\right)^2 - 2\dot{a}\left(\omega_0 + \frac{\varepsilon}{2}\right) \right] \sin\varphi$$

now drop terms of order ε^2 & $\dot{a}\varepsilon$, $\dot{b}\varepsilon$ since a, b are slowly varying & ε = small:

$$\ddot{x} \simeq \left(-a\omega_0^2 - a\omega_0\varepsilon + 2\dot{b}\omega_0 \right) \cos\varphi$$
$$+ \left(-b\omega_0^2 - b\omega_0\varepsilon - 2\dot{a}\,\omega_0 \right) \sin\varphi.$$

Now we combine the results of (1) & (2):

$$\ddot{x} + \omega^2(t)\, x \simeq \omega_0\left(2\dot{b} - a\varepsilon + \frac{1}{2}h\omega_0 a \right) \cos\varphi$$
$$+ \omega_0\left(-2\dot{a} - b\varepsilon - \frac{1}{2}h\,\omega_0 b \right) \sin\varphi$$
$$= 0.$$

The ~~cos~~ coeff.'s of $\cos\varphi$ & $\sin\varphi$ must independently vanish, & we obtain coupled diff. eq.'s for a & b:

$$\begin{cases} 2\dot{a} = -\left(\varepsilon + \frac{1}{2}h\omega_0\right) b \\ 2\dot{b} = \left(\varepsilon - \frac{1}{2}h\omega_0\right) a \end{cases}$$

To solve this system, take another t-derivative:

$$2\ddot{a} = -\left(\varepsilon + \tfrac{1}{2}h\omega_0\right)\dot{b}$$

$$= -\frac{1}{2}\left[\varepsilon^2 - \left(\tfrac{1}{2}h\omega_0\right)^2\right] a$$

$$\Rightarrow \quad \ddot{a} = \frac{1}{4}\left[\left(\tfrac{1}{2}h\omega_0\right)^2 - \varepsilon^2\right] a.$$

Seek soln.'s of the form $a = e^{st}$, $\rightarrow$ $\ddot{a} = s^2 a$

$$\Rightarrow \quad s^2 = \frac{1}{4}\left[\left(\tfrac{1}{2}h\omega_0\right)^2 - \varepsilon^2\right]$$

For resonance, (i.e., exponentially growing a & b) need $s^2 > 0$, so that $s \in \mathbb{R}$. This is satisfied when $|\varepsilon| < \frac{1}{2}h\omega_0$. Thus we find resonance occurs in a range:

$$\boxed{-\frac{1}{2}h\omega_0 < \varepsilon < \frac{1}{2}h\omega_0}$$

$(\Omega = 2\omega_0 + \varepsilon)$