HW 7, Harry Luo

5.8

We can solve for the probability density function by differentiating the cumulative distribution function. $X \in [-1, 2] \Rightarrow X^2 \in [0, 4]$. When $X^2 \in [0, 4]$,

$$\begin{aligned} F_{Y(y)} &= P(Y \leq y) \\ &= P(X^2 \leq y) \\ &= P(-\sqrt{y} \leq X \leq \sqrt{y}) \\ &= F_X(\sqrt{y}) - F_X(-\sqrt{y}) \end{aligned} \tag{1}$$

Differentiating Equation 1, we get the probability density function as:

$$f_Y(y) = F'_{Y(y)} = \frac{1}{2\sqrt{y}} f_X(\sqrt{y}) + \frac{1}{2\sqrt{y}} f_X(-\sqrt{y}) \tag{2}$$

The probability density function of X is given as $f_{X(x)} = \frac{1}{3}$ when $x \in [-1, 2]$ and zero otherwise.

Considering when $y \in [0, 1]$, $\sqrt{y} \in [0, 1]$ and $-\sqrt{y} = [-1, 0]$, which are within the range of x,

so $f_X(\sqrt{y}) = f_X(-\sqrt{y}) = \frac{1}{3}$,

$$f_{Y(y)} = \frac{1}{2\sqrt{y}} * \frac{1}{3} + \frac{1}{2\sqrt{y}} * \frac{1}{3} = \frac{1}{3\sqrt{y}} \tag{3}$$

when $y \in [0, 1]$.

When $y \in [1, 4]$, $\sqrt{y} \in [1, 2]$,but $-\sqrt{y} \in [-2, -1]$out of range

so $f_X(\sqrt{y}) = \frac{1}{3}, f_X(-\sqrt{y}) = 0$,

$$f_Y(y) = \frac{1}{6\sqrt{y}} \tag{4}$$

when $y \in [1, 4]$.

Thus,

$$F_Y(y) = \begin{cases} 0 \text{ when } y < 0 \\ \frac{1}{3\sqrt{y}} \text{ when } 0 \leq y < 1 \\ \frac{1}{6\sqrt{y}} \text{ when } 1 \leq y < 4 \\ 1 \text{ when } y \geq 4 \end{cases} \tag{5}$$

5.28

We know $f_X(x) = \frac{1}{3}, x \in (-1, 2)$ and 0 otherwise, while $Y = X^4 \in [0, 16]$ thus

$$f_Y(y) = 0, y \notin [0, 16] \tag{6}$$

When $y \in [0, 16]$, we can find the probability density function by differentiating the cumulative distribution function.

$$F_Y(y) = P(Y \leq y) = P(X^4) \leq y = F_X(y^{1/4}) - F_X(-y^{1/4}) \tag{7}$$

Differentiating Equation 7, we get the probability density function as:

$$f_Y(y) = \frac{1}{4} y^{-\frac{3}{4}} f_X\left(y^{\frac{1}{4}}\right) + \frac{1}{4} y^{-\frac{3}{4}} f_X\left(-y^{\frac{1}{4}}\right) \tag{8}$$

When $y \in [0, 1]$, $y^{\frac{1}{4}} \in [0, 1]$ and $-y^{\frac{1}{4}} \in [-1, 2]$, which are within the range of x, thus $f_X\left(y^{\frac{1}{4}}\right) = f_X\left(-y^{\frac{1}{4}}\right) = \frac{1}{3}$,

$$f_Y(y) = \frac{1}{6y^{\frac{3}{4}}} \tag{9}$$

When $y \in [1, 16]$, $y^{\frac{1}{4}} \in [1, 2]$ and $-y^{\frac{1}{4}} \in [-2, -1]$, which are within the range of x, thus $f_X\left(y^{\frac{1}{4}}\right) = f_X\left(-y^{\frac{1}{4}}\right) = \frac{1}{3}$,

$$f_Y(y) = \frac{1}{12y^{\frac{3}{4}}} \tag{10}$$

In summary,

$$f_Y(y) = \begin{cases} 0 \text{ when } y < 0 \\ \dfrac{1}{6y^{3/4}} \text{ when } 0 \leq y < 1 \\ \dfrac{1}{12y^{3/4}} \text{ when } 1 \leq y < 16 \\ 0 \text{ when } y \geq 16 \end{cases} \tag{11}$$

5.32

Given $X \in (0, 1)$, possible values for Y is the interval $(1, \infty)$ THerefore, when $t < 1$, $f_Y(t) = 0$ and when $t \geq 1$, we can find the probability density function by differentiating the mass function.

$$\begin{aligned} P(Y \leq t) = P\left(\frac{1}{x} \leq t\right) = P\left(X \geq \frac{1}{t}\right) = 1 - \frac{1}{t} \\ f_Y(t) = \frac{\mathrm{d}}{\mathrm{d}t} P(Y \leq t) = \frac{1}{t^2} \text{ when } t \geq 1 \end{aligned} \tag{12}$$

6.6

- (a) Marginal of X, when x > 0, is

$$f_X(x) = \int_0^\infty x e^{-x(1+y)} \,\mathrm{d}x = e^{-x} \tag{13}$$

and zero otherwise.

The marginal of Y when y >0, similarly, is

$$f_Y(y) = \int_0^\infty x e^{-x(1+y)} \,\mathrm{d}x = \frac{1}{(1+y)^2} \tag{14}$$

and zero otherwise.

- (b) Expectation:

$$\begin{aligned} E[XY] &= \int_0^\infty \int_0^\infty xy \times f(x, y) \,\mathrm{d}x \,\mathrm{d}y \\ &= \int_0^\infty \int_0^\infty x^2 y e^{-x(1+y)} \,\mathrm{d}x \,\mathrm{d}y \\ &= \int_0^\infty e^{-y} \,\mathrm{d}y \\ &= 1 \end{aligned} \tag{15}$$

- (c) Expectation:

$$\begin{aligned} E\left[\frac{X}{1+Y}\right] &= \int_0^\infty \int_0^\infty \frac{x}{1+y} x e^{-x(1+y)} \,\mathrm{d}x \,\mathrm{d}y \\ &= \int_0^\infty \frac{1}{1+y} \frac{2}{(1+y)^3} \,\mathrm{d}y = 2 \int_0^\infty \frac{1}{(1+y)^4} \,\mathrm{d}y \\ &= \frac{2}{3} \end{aligned} \tag{16}$$

6.18

- (a) We can write the pmf as a table:

X\Y	1	2	3	4
1	1/4	0	0	0
2	1/8	1/8	0	0
3	1/12	1/12	1/12	0
4	1/16	1/16	1/16	1/16

we can confirm that the terms are non negative, and the sum of all terms is 1. This certifies that $p_{X,Y}$ is a **valid pmf**.

- (b) Marginal of X and Y can be found by summing the rows and columns:

$$\begin{gathered} X: \\ P(X=1)=\frac{1}{4}, P(X=2)=\frac{1}{4}, P(X=3)=\frac{1}{4}, P(X=4)=\frac{1}{4} \\ Y: \\ P(Y=1)=\frac{25}{48}, P(Y=2)=\frac{13}{48} P(Y=3)=\frac{7}{48}, P(Y=4)=\frac{1}{16} \end{gathered} \tag{17}$$

- (c)

$$\begin{aligned} P(X=Y+1) &= P(X=2, Y=1)+P(x=3, Y=2)+P(X=4, Y=3) \\ &= \frac{1}{8}+\frac{1}{12}+\frac{1}{16} \\ &= \frac{13}{48} \end{aligned} \tag{18}$$

6.24

We can use binomial distribution with n = 3 and p = 1/4. The probability of of having exactly two balls are green and one is not green is

$$\binom{3}{2}\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)=\frac{9}{64} \tag{19}$$

By the same logic, the probability of having exactly 2 R ball, 2 Y balll or 2 W balls ae also 9/64.

So the probability of having exactly 2 balls of the same color is

$$\boxed{\frac{9}{64} \times 4=\frac{9}{16}}$$