Summary

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- $L = \sum_{a=1}^{N} \frac{1}{2} m_a v_a^2 - U(\vec{r}_1, \ldots, \vec{r}_N)$

$$= \sum_{ij} \frac{1}{2} a_{ij}(q) \dot{q}_i \dot{q}_j - U(q) \qquad \text{(generalized coord.'s).}$$

- E-L eqn.'s: $\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i}$

- cyclic coord.: $\frac{\partial L}{\partial q_j} = 0 \Rightarrow p_j \equiv \frac{\partial L}{\partial \dot{q}_j}$ is conserved. ($\dot{p}_j = 0$)

(q_j = "cyclic coord.").

- homog. of time: $L(q, \dot{q}, t) = L(q, \dot{q})$, $\frac{\partial L}{\partial t} = 0$.

$$\Rightarrow E = \sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L \quad \text{is conserved} \quad (\dot{E} = 0)$$

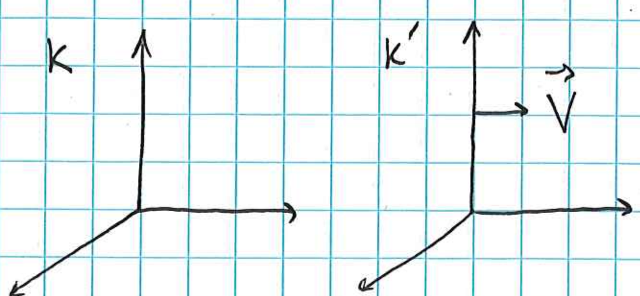
$$= T + U \quad \text{(energy)}$$

- homog. of space: $L(\vec{r}_1 + \vec{c}, \ldots, \vec{r}_N + \vec{c}, \vec{v}_1, \ldots, \vec{v}_N) = L(\vec{r}_1, \ldots, \vec{r}_N, \vec{v}_1, \ldots, \vec{v}_N)$

$$\Rightarrow \vec{P} = \sum_a m_a \vec{v}_a \quad \text{is conserved} \quad (\dot{\vec{P}} = 0)$$

(total momentum)

Wrap up cons. of $\vec{P}$. Transformation properties:



$\vec{v}_a = \vec{v}_a' + \vec{V}$ ($\vec{V}$ = const.)

$$\Rightarrow \vec{P} = \sum_a m_a \vec{v}_a$$

$$= \sum_a m_a \vec{v}_a' + \underbrace{\left(\sum_a m_a\right)}_{\equiv M} \vec{V}$$

$$= \vec{P}' + M\vec{V}, \quad M = \text{total mass.}$$

· choose $\vec{V} = \vec{P}/M \Rightarrow \vec{P}' = 0$. So, in frame K' system 02/05/24 is "at rest" as a whole."

· Can also write:

$$\vec{V} = \frac{1}{M}\sum_a m_a \vec{v}_a = \frac{d}{dt}\underbrace{\left(\frac{1}{M}\sum_a m_a \vec{r}_a\right)}_{\equiv \vec{R}}$$

$$\Rightarrow \vec{V} = \dot{\vec{R}}, \quad \vec{R} = \text{"center of mass" (COM)}$$

→ $\vec{R}(t)$ corresponds to motion of system "as a whole".

· Also leads to a useful decomposition of the energy:

"Internal energy" E_i = energy of a system which is at rest as a whole $(\vec{P} = 0)$.

→ then $E = \frac{1}{2}MV^2 + E_i$ where, $\vec{V}$ = velocity of COM.

Proof:

$$E = \sum_a \frac{1}{2} m_a v_a^2 + U$$

transform btwn. frames.

$$= \sum_a \frac{1}{2} m_a (\vec{v}_a' + \vec{V})^2 + U$$

$$= \underbrace{\sum_a \frac{1}{2} m_a v_a'^2 + U}_{E'} + \underbrace{\left(\sum_a m_a \vec{v}_a'\right)}_{\vec{P}'} \cdot \vec{V} + \frac{1}{2}\underbrace{\left(\sum_a m_a\right)}_{M} V^2$$

$$= E' + \vec{P}' \cdot \vec{V} + \frac{1}{2}MV^2$$

If primed frame = COM frame, then $\vec{P}' = 0$, $E' = E_i$

& $E = E_i + \frac{1}{2}MV^2$ ✓.

Note: In presence of external field, certain components of $\vec{P}$ may still be conserved if ext. pot. does not depend on particular coord.

Ex: $U = mgz$ → then $\dot{P}_z \neq 0$ but $\dot{P}_x = \dot{P}_y = 0$

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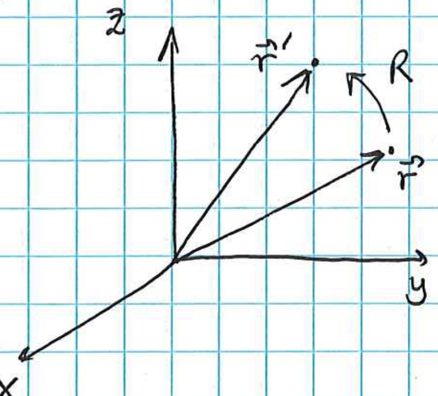
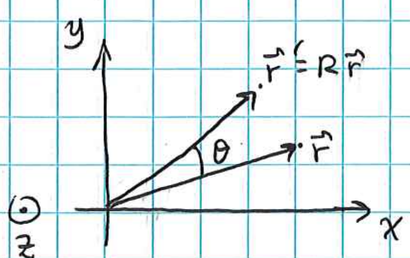
(3) Isotropy of space & angular momentum.

First, we make digression into math of rotations (will also be relevant for rigid-body dynamics & motion in non-inertial frames).



· Rotation R maps old pt. $\vec{r}$ into a new pt. $\vec{r}' = R\vec{r}$ such that $|\vec{r}'|^2 = |\vec{r}|^2$; i.e., rotations preserve lengths of vectors.

Ex: Consider $\vec{r} \in$ xy-plane & a rotation about $\hat{z}$ by angle θ:

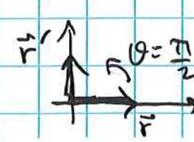


$$\vec{r} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$\vec{r}' = \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta\, x & -\sin\theta\, y \\ \sin\theta\, x & +\cos\theta\, y \end{pmatrix}$$

Check: $\vec{r} = \begin{pmatrix} x \\ 0 \end{pmatrix}$ & $\theta = \frac{\pi}{2}$: $\quad \vec{r}' = \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 \\ x \end{pmatrix}$



✓

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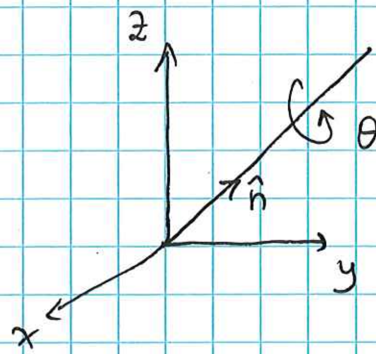
· More generally, rotations may be parametrized by axis of rotation $\hat{n}$ & angle of rotation θ:

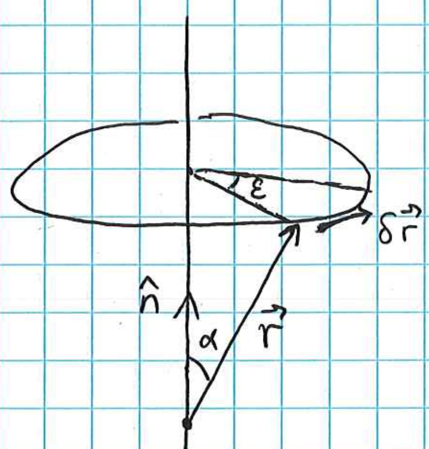
$$R = R(\hat{n}, \theta)$$



(direction of θ determined by right-hand rule)

(In previous ex., $\hat{n} \to \hat{z}$).

· Now consider an infinitesimal rotation R, that is, a rotation by a small angle $\theta = \varepsilon \ll 1$.



$|\delta\vec{r}| = (r \sin\alpha)\, \varepsilon$.

& $\delta\vec{r} \perp \vec{r}$ & $\hat{n}$.

$\Rightarrow \delta\vec{r} = \varepsilon\, \hat{n} \times \vec{r}$.

So, $\vec{r}\,' = R\vec{r} \simeq \vec{r} + \delta\vec{r} = \vec{r} + \varepsilon\, \hat{n} \times \vec{r}$ (for $\varepsilon \ll 1$)

Ex: In previous ex. of rotation about $\hat{z}$:

$$\vec{r}\,' = \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta\, x - \sin\theta\, y \\ \sin\theta\, x + \cos\theta\, y \end{pmatrix} \underset{\substack{\uparrow \\ \theta = \varepsilon \ll 1 \\ \cos\varepsilon \simeq 1,\ \sin\varepsilon \simeq \varepsilon}}{\simeq} \begin{pmatrix} x - \varepsilon y \\ \varepsilon x + y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \varepsilon \begin{pmatrix} -y \\ x \end{pmatrix}$$

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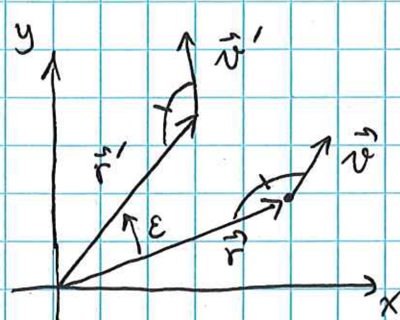
So, for a small rotation: $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -y \\ x \end{pmatrix}$

$$\rightarrow \quad \vec{r}\,' = \vec{r} + \delta\vec{r}, \qquad \delta\vec{r} = \begin{pmatrix} -y \\ x \end{pmatrix}.$$

Check: $\delta\vec{r} = \varepsilon\,\hat{z}\times\vec{r} = \varepsilon\,\hat{z}\times\left(x\hat{x} + y\hat{y}\right) = \varepsilon\left(x\hat{y} - y\hat{x}\right)$

$$= \varepsilon\begin{pmatrix} -y \\ x \end{pmatrix} \quad \checkmark$$

Now consider rotation of mechanical system:



rotate positions & velocities:

$$\vec{r} \rightarrow \vec{r}\,' = R\vec{r} \simeq \vec{r} + \varepsilon\,\hat{n}\times\vec{r}$$

$$\vec{v} \rightarrow \vec{v}\,' = R\vec{v} \simeq \vec{v} + \varepsilon\,\hat{n}\times\vec{v}$$

isotropy of space $\Rightarrow$ rotated Lagrangian must be same as unrotated Lagrangian:

$$L(\vec{r}_1 + \varepsilon\,\hat{n}\times\vec{r}_1, \ldots, \vec{v}_1 + \varepsilon\,\hat{n}\times\vec{v}_1, \ldots) = L(\vec{r}_1, \ldots, \vec{v}_1, \ldots)$$

$$\Rightarrow \quad \delta L = \sum_a \left(\frac{\partial L}{\partial \vec{r}_a}\cdot\delta\vec{r}_a + \frac{\partial L}{\partial \vec{v}_a}\cdot\delta\vec{v}_a \right) = 0$$

$$\Rightarrow \quad \varepsilon\left[\sum_a \left(\frac{\partial L}{\partial \vec{r}_a}\cdot(\hat{n}\times\vec{r}_a) + \frac{\partial L}{\partial \vec{v}_a}\cdot(\hat{n}\times\vec{v}_a) \right) \right] = 0.$$

use EOM: $\frac{\partial L}{\partial \vec{r}_a} = \dot{\vec{p}}_a$ & $\frac{\partial L}{\partial \vec{v}_a} = \vec{p}_a$:

$$\Rightarrow \varepsilon\left[\sum_a \dot{\vec{p}}_a\cdot(\hat{n}\times\vec{r}_a) + \vec{p}_a\cdot(\hat{n}\times\vec{v}_a) \right] = 0.$$

Use cyclic prop. of triple product: $\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{B}\cdot(\vec{C}\times\vec{A})$:

$$\varepsilon\left[\sum_a \hat{n}_a\cdot(\vec{r}_a\times\dot{\vec{p}}_a) + \hat{n}\cdot(\dot{\vec{r}}_a\times\vec{p}_a)\right] = 0$$

$$\Rightarrow \varepsilon\,\hat{n}\cdot\left[\sum_a\Big((\vec{r}_a\times\dot{\vec{p}}_a) + (\dot{\vec{r}}_a\times\vec{p}_a)\Big)\right] = 0.$$

$$\Rightarrow \varepsilon\,\hat{n}\cdot\frac{d}{dt}\left(\sum_a \vec{r}_a\times\vec{p}_a\right) = 0.$$

$\varepsilon\hat{n}$ = arbitrary $\Rightarrow$ $\frac{d}{dt}\underbrace{\left(\sum_a \vec{r}_a\times\vec{p}_a\right)}_{\text{Const. vector in time.}} = 0.$

$\Rightarrow \vec{L} \equiv \sum_a \vec{r}_a\times\vec{p}_a$ is conserved. $(\dot{\vec{L}}=0)$

in a closed system.

"angular momentum".

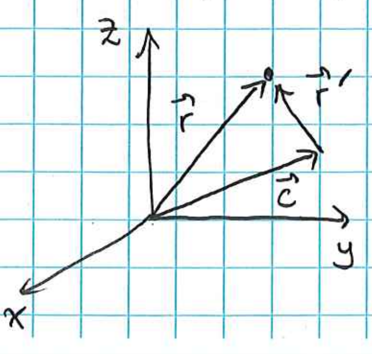
Note: $\vec{L} = \sum_a \vec{L}_a$, $\vec{L}_a = \vec{r}_a\times\vec{p}_a$. ; i.e., angular momentum is additive.

Transformation properties of $\vec{L}$:

· choice of origin: $\vec{r}_a = \vec{r}_a\,' + \vec{c}$ $\Rightarrow$ $\vec{L} = \sum_a \vec{r}_a\times\vec{p}_a$

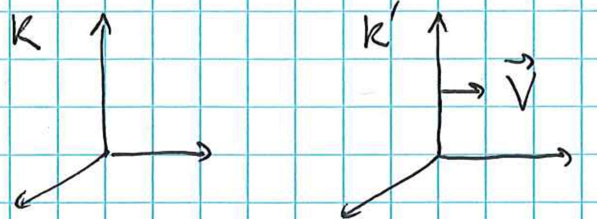
$$= \sum_a \vec{r}_a\,'\times\vec{p}_a + \vec{c}\times\sum_a\vec{p}_a$$

$$= \vec{L}\,' + \vec{c}\times\vec{P}$$



$\Rightarrow \vec{L}$ indep. of origin when $\vec{P}=0$ (COM frame)

• transform.^{n} btwn. inertial frames: 02/05/24



$$\vec{v}_a = \vec{v}_a{}' + \vec{V}$$

$$\vec{L} = \sum_a \vec{r}_a \times \vec{p}_a = \sum_a \vec{r}_a \times m_a \vec{v}_a = \sum_a \vec{r}_a \times m_a (\vec{v}_a{}' + \vec{V})$$

$$\Rightarrow \vec{L} = \sum_a \vec{r}_a \times m_a \vec{v}_a{}' + \left(\sum_a m_a \vec{r}_a\right) \times \vec{V}$$

$$\vec{R} = \frac{1}{M} \sum_a m_a \vec{r}_a$$

$$= \vec{L}' + M\vec{R} \times \vec{V}$$

$$M = \sum_a m_a$$

$\vec{L}'$: ang. mom. in frame K'.

If K' = COM frame, where $\vec{P}' = 0$, then $\vec{P} = M\vec{V}$ &

$$\vec{L} = \vec{L}' + \vec{R} \times \vec{P}$$

$\vec{L}'$: "intrinsic ang. mom." $\vec{R} \times \vec{P}$: motion "as a whole".

Conservation of $\vec{L}$ in ext. fields:

• Motion in central field: $U = U(r)$, $r = |\vec{r}|$.

under rotation: $\vec{r} \to \vec{r}\,' = R \cdot \vec{r}$, $U_{ext}(r) \to U_{ext}(r') = U_{ext}(r)$.

$\Rightarrow$ Lagrangian invariant under rotations & $\vec{L}$ conserved.

• More generally, components of $\vec{L}$ conserved if there is axis of symmetry.

Ex: $U = U(z) \Rightarrow$ inv. under rot. about $\hat{z}$ & $L_z = \hat{z} \cdot \vec{L}$ conserved.