HW 13 Harry Luo

1

recall cauchy integral formula

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0} \, \mathrm{d}z \qquad (1)$$

For our integral, $f(z) = z^3 + e^{z^2}$, $z_0 = 1 + i$ applying the CIF:

$$\begin{aligned} \int_C \frac{z^3 + e^{z^2}}{z - (1+i)} \, \mathrm{d}z = 2\pi i f(1+i) &= 2\pi i \left((1+i)^3 + e^{(1+i)^2} \right) \\ &= 2\pi i (e^{2i} + 2i - 2) \end{aligned} \qquad (2)$$

2

recall the CIF

$$f^n(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} \, \mathrm{d}z \qquad (3)$$

Here, $z_0 = 0, n = 1, \quad \text{let } f(z) = e^z + e^{z^3}$

$$\begin{aligned} \int_C \frac{f(z)}{z^2} \, \mathrm{d}z &= \frac{2\pi i}{1} f^{(1)}(0) \\ &= 2\pi i \left(e^0 + 3(0)^2 e^0 \right) = \boxed{2\pi i} \end{aligned} \qquad (4)$$

3

Using the CIF, take $z_0 = -1, n = 2, f(z) = z^{2024} + 4z$

$$\int_C \frac{f(z)}{(z+1)^3} \, \mathrm{d}z = \frac{2\pi i}{2} f^{(2)}(-1) \qquad (5)$$

where $f^{(2)}(-1) = 2024 * 2023(-1)^{2022} = 4094552$, Equation 5 becomes

$$\int_C \frac{z^{2024} + 4z}{(z+1)^3} \, \mathrm{d}z = \boxed{4094552\pi i} \qquad (6)$$

4

Using the CIF, take $f(z) = \cos(z), z_0 = 0$

$$\begin{aligned} \int_{|z|=3} \frac{\cos(z)}{z^5} \, \mathrm{d}z &= 2\pi i \cos^{(4)}(0) \\ &= 2\pi i \frac{\cos(0)}{4!} = \boxed{\frac{\pi i}{12}} \end{aligned} \qquad (7)$$

5

- (a)find poles

$$\begin{gathered} 1 + z^2 = (z+i)(z-i) \Rightarrow z_1 = -i, z_2 = i \\ \operatorname{Res}(f, z_1) = \lim_{z \to -i} \frac{z+i}{1+z^2} = -\frac{1}{2i} \\ \operatorname{Res}(f, z_2) = \lim_{z \to i} \frac{z-i}{1+z^2} = \frac{1}{2i} \end{gathered} \qquad (8)$$

- (b)

Consider subsitution $f(z) = \frac{1}{1+x^2}$, we can use a semicircular contour in the upper half-plane, which will enclose only the pole at z = i.

$$\int_\gamma f(z)\,\mathrm{d}z = \int_{-R}^{R} f(z)\,\mathrm{d}z + \int_C f(z)\,\mathrm{d}z = 2\pi i\ \mathrm{Res}(f, z = i) + \underbrace{\int_C f(z)\,\mathrm{d}z}_{(*)} \tag{9}$$

for the second term, parametrize $z = Re^{i\theta}$,

$$(*) = \int_C \frac{1}{1 + Re^{i2\theta}}\,\mathrm{d}\theta, \quad \theta \in [0, \pi] \tag{10}$$

as $R \to \infty$, the second term becomes 0, and the integral becomes

$$\begin{aligned} \int_{-\infty}^{\infty} f(z)\,\mathrm{d}z &= 2\pi i\ \mathrm{Res}(f, z = i) \\ &= 2\pi i \left(\frac{1}{2i}\right) = \pi \end{aligned} \tag{11}$$

Since $\frac{1}{1+x^2}$ is even,

$$\boxed{\int_0^{\infty} \frac{1}{1 + x^2}\,\mathrm{d}x = \frac{\pi}{2}} \tag{12}$$

6

- (a)

Find all the poles

$$\begin{aligned} \left(1 + z^2\right)^2 &= 0 \\ z^2 &= -1 \\ z &= \pm i \end{aligned} \tag{13}$$

Recall that we can find the residue by

$$\mathrm{Res}(f, z_0) = \lim_{z \to z_0} \frac{1}{(n-1)!} \left(\frac{d}{dz}\right)^{n-1} [(z - z_0)^n f(z)] \tag{14}$$

Take $n = 2, z_0 = i$, we have

$$\begin{aligned} \mathrm{Res}(f, i) = \lim_{z \to i} \frac{\mathrm{d}}{\mathrm{d}z} \left[(z - i)^2 f(z)\right] &= \lim_{z \to i} \frac{\mathrm{d}}{\mathrm{d}z} \left[\frac{(z - i)^2}{(z - i)^2 (z + i)^2}\right] \\ &= \lim_{z \to i} \frac{\mathrm{d}}{\mathrm{d}z} \left[\frac{1}{(z + i)^2}\right] \\ &= \lim_{z \to i} -\frac{2}{(z + i)^3} \\ &\boxed{= -\frac{i}{4}} \end{aligned} \tag{15}$$

similarly, take $n = 2, z_0 = -i$, we have

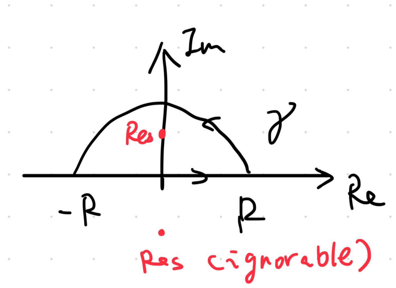
$$\begin{aligned}\operatorname{Res}(f,-i) &= \lim_{z\to -i}\frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{(z-z_0)^2}{(z^2-i^2)^2}\right] = \lim_{z\to -i}\frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{(z+i)^2}{(z+i)^2(z-i)^2}\right] \\ &= \lim_{z\to -i}\frac{\mathrm{d}}{\mathrm{d}z}\left[(z-i)^{-2}\right] \\ &= \lim_{z\to -i}\left(-2(z-i)^{-3}\right) \\ &\boxed{=\frac{i}{4}}\end{aligned} \tag{16}$$

- (b)

recall Cauchy residue thm,

$$\int_\gamma f(z)\,\mathrm{d}z = 2\pi i\sum_{k=1}^{n}\operatorname{Res}(f,z_k) \tag{17}$$

Consider subsitution $f(z)=\frac{1}{(1+z^2)^2}$, we can use a semicircular contour in the upper half-plane, which will enclose only the pole at z = i.



$$\int_\gamma f(z)\,\mathrm{d}z = \int_{-R}^{R} f(z)\,\mathrm{d}z + \int_C f(z)\,\mathrm{d}z = 2\pi i\ \operatorname{Res}(f,z=i) \tag{18}$$

$$\begin{aligned}\Rightarrow \int_{-R}^{R} f(z)\,\mathrm{d}z &= 2\pi i\ \operatorname{Res}(f,z=i) - \int_C f(z)\,\mathrm{d}z \\ &= 2\pi i\left(-\frac{i}{4}\right) - \underbrace{\int_C f(z)\,\mathrm{d}z}_{(*)}\end{aligned} \tag{19}$$

for $(*)$, parametrize $z = Re^{i\theta}$,

$$(*) = \int_C \frac{1}{1+Re^{i\theta}}\,\mathrm{d}\theta,\quad \theta\in[0,\pi] \tag{20}$$

When $R\to\infty$, Equation 19 becomes

$$\begin{aligned}\int_{-\infty}^{\infty} f(z)\,\mathrm{d}z &= 2\pi i\left(-\frac{i}{4}\right) - \underbrace{\int_C \frac{1}{1+Re^{i\theta}}\,\mathrm{d}\theta}_{\to 0} \\ &= \frac{\pi}{2}\end{aligned} \tag{21}$$

Since $\frac{1}{(1+x^2)^2}$ is even,

$$\int_0^{\infty}\frac{1}{(1+x^2)^2}\,\mathrm{d}x = \boxed{\frac{\pi}{4}} \tag{22}$$

$$f(z) = \frac{z^2}{(z^2+1)(z^2+4)} \tag{23}$$

- (a)

$$\begin{gathered}
\frac{z^2}{(z+i)(z-i)(z+2i)(z-2i)} \Rightarrow z_1 = i, z_2 = -i, z_3 = 2i, z_4 = -2i \\
\operatorname{Res}(f, z_1) = \lim_{z \to i} \frac{z^2}{(z+i)(z+2i)(z-2i)} = -\frac{1}{6i} \\
\operatorname{Res}(f, z_2) = \lim_{z \to -i} \frac{z^2}{(z-i)(z+2i)(z-2i)} = \frac{1}{6i} \\
\operatorname{Res}(f, z_3) = \lim_{z \to 2i} \frac{z^2}{(z-i)(z+i)(z+2i)} = \frac{1}{3i} \\
\operatorname{Res}(f, z_4) = \lim_{z \to -2i} \frac{z^2}{(z-i)(z+i)(z-2i)} = -\frac{1}{3i}
\end{gathered} \tag{24}$$

- (b)

Similarly to 6(b), we can use a semicircular contour in the upper half-plane, which will enclose only the poles at z = i, 2i. The contour integration over the complex arc is still 0 as $R \to \infty$

$$\begin{gathered}
\int_\gamma f(z)\,\mathrm{d}z = \int_{-R}^{R} f(z)\,\mathrm{d}z + \int_C f(z)\,\mathrm{d}z \\
\Rightarrow \int_{-\infty}^{\infty} f(z)\,\mathrm{d}z = 2\pi i(\operatorname{Res}(f, z_1) + \operatorname{Res}(f, z_3)) = \frac{\pi}{3}
\end{gathered} \tag{25}$$

Since $f(x)$ is even,

$$\int_0^R f(x)\,\mathrm{d}x = \frac{1}{2}\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = \boxed{\frac{\pi}{6}} \tag{26}$$