03/11/24

Summary

• Small oscillations w/ $n \geq 1$ DOF:

$q = (q_1, q_2, \ldots, q_n)$ & potential $U(q)$.

equil. pt: $q^{(0)} = (q_1^{(0)}, q_2^{(0)}, \ldots, q_n^{(0)})$ s.t. $\left.\frac{\partial U(q)}{\partial q_i}\right|_{q^{(0)}} = 0$, $i = 1, \ldots, n$

$x_i = q_i - q_i^{(0)}$ = small displacement from equil.

$$\rightarrow L = \frac{1}{2}\sum_{i,j=1}^{n} m_{ij}\dot{x}_i\dot{x}_j - \frac{1}{2}\sum_{i,j=1}^{n} k_{ij}x_i x_j$$

• E-L. eqn.'s: $\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} = \frac{\partial L}{\partial x_i} \Rightarrow \sum_{j=1}^{n} m_{ij}\ddot{x}_j + \sum_{j=1}^{n} k_{ij}x_j = 0$, $i = 1, \ldots, n$

→ system of n coupled second order diff eq.'s.

• ~~Sol~~ Solving EOM: $x_i = \mathrm{Re}\, a_i e^{i\omega t}$

$$\rightarrow (\omega^2 M - K)\cdot \vec{a} = 0$$

$$\Rightarrow \det(\omega^2 M - K) = 0.$$

$\rightarrow \omega_\alpha^2 \quad \alpha = 1, \ldots, n$ "normal mode frequencies".

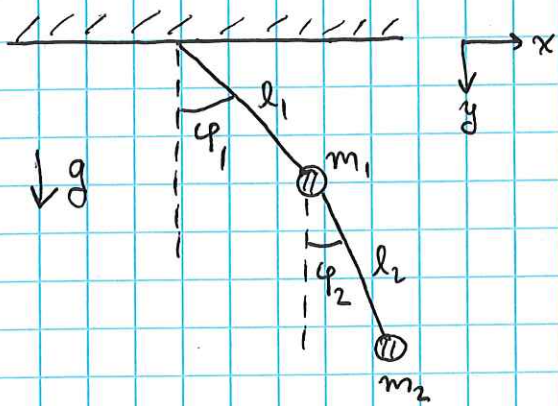
& $\sum_j (\omega_\alpha^2 m_{ij} - k_{ij})\, a_j^{(\alpha)} = 0$

$a_j^{(\alpha)}$ → "normal mode".

$$\Rightarrow x_i(t) = \sum_{\alpha=1}^{n} a_i^{(\alpha)}\, \mathrm{Re}\{C_\alpha e^{i\omega_\alpha t}\}$$

$C_\alpha \in \mathbb{C}$

Ex: (Double pendulum, $n=2$ DOF).



$$\begin{cases} x_1 = \ell_1 \sin\varphi_1 & y_1 = \ell_1 \cos\varphi_1 \\ x_2 = \ell_1 \sin\varphi_1 + \ell_2 \sin\varphi_2, & y_2 = \ell_1 \cos\varphi_1 + \ell_2 \cos\varphi_2 \end{cases}$$

$$\Rightarrow T = \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2)$$

$$= \frac{1}{2} m_1 \ell_1 \dot{\varphi}_1^2 + \frac{1}{2} m_2 (\ell_1^2 \dot{\varphi}_1^2 + \ell_2^2 \dot{\varphi}_2^2 + 2\ell_1 \ell_2 \dot{\varphi}_1 \dot{\varphi}_2 \cos(\varphi_1 - \varphi_2)).$$

$$U = -m_1 g y_1 - m_2 g y_2$$

$$= -m_1 g \ell_1 \cos\varphi_1 - m_2 g (\ell_1 \cos\varphi_1 + \ell_2 \cos\varphi_2).$$

Stable equil.: $\varphi_1 = \varphi_2 = 0$. Expanding for small φ $(\cos\varphi \simeq 1 - \varphi^2/2)$

$$T = \frac{1}{2} m_1 \ell_1^2 \dot{\varphi}_1^2 + \frac{1}{2} m_2 (\ell_1^2 \dot{\varphi}_1^2 + \ell_2^2 \dot{\varphi}_2^2 + 2\ell_1 \ell_2 \dot{\varphi}_1 \dot{\varphi}_2)$$

$$U = \text{const} + \frac{1}{2} m_1 g \ell_1 \varphi_1^2 + \frac{1}{2} m_2 g \ell_1 \varphi_1^2 + \frac{1}{2} m_2 g \ell_2 \varphi_2^2$$

$$\Rightarrow L = T - U = \frac{1}{2} (\dot{\varphi}_1 \ \dot{\varphi}_2) \begin{pmatrix} (m_1 + m_2)\ell_1^2 & m_2 \ell_1 \ell_2 \\ m_2 \ell_1 \ell_2 & m_2 \ell_2^2 \end{pmatrix} \begin{pmatrix} \dot{\varphi}_1 \\ \dot{\varphi}_2 \end{pmatrix}$$

$$- \frac{1}{2} (\varphi_1 \ \varphi_2) \begin{pmatrix} (m_1 + m_2) g \ell_1 & 0 \\ 0 & m_2 g \ell_2 \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}$$

$$= \frac{1}{2} \dot{\vec{\varphi}}^T \cdot M \cdot \dot{\vec{\varphi}} - \frac{1}{2} \vec{\varphi}^T \cdot K \cdot \vec{\varphi}$$

Simplifying case: $m_1 = m_2 = m$, $l_1 = l_2 = l$. 03/11/24

$$\Rightarrow \begin{cases} M = ml^2 \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \\ K = mgl \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \end{cases}$$

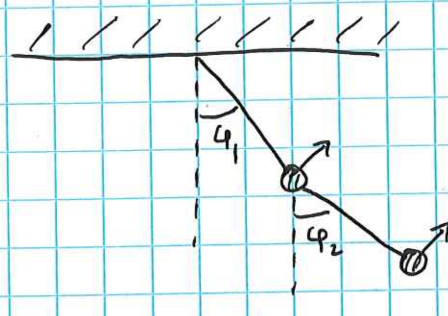
Normal modes:

$$\omega^2 M - K = ml^2 \begin{pmatrix} 2\omega^2 - 2\omega_0^2 & \omega^2 \\ \omega^2 & \omega^2 - \omega_0^2 \end{pmatrix}, \qquad \omega_0^2 = \frac{g}{l}.$$

$$\det(\omega^2 M - K) = 0 \Rightarrow \omega^2 = (2 \pm \sqrt{2})\,\omega_0^2.$$

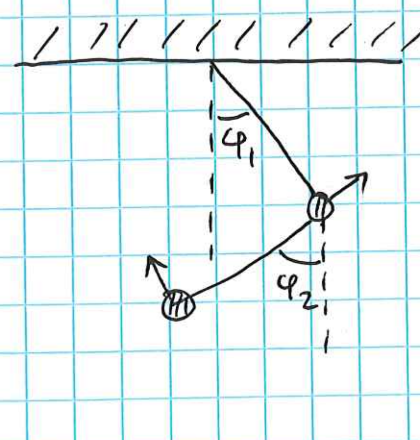
ω_-: $$\begin{pmatrix} 2\omega_-^2 - 2\omega_0^2 & \omega_-^2 \\ \omega_-^2 & \omega_-^2 - \omega_0^2 \end{pmatrix} \begin{pmatrix} a_1^{(-)} \\ a_2^{(-)} \end{pmatrix} = 0$$

$$\omega_-^2 = (2 - \sqrt{2})\,\omega_0^2 \Rightarrow \begin{pmatrix} a_1^{(-)} \\ a_2^{(-)} \end{pmatrix} = C_- \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}$$



ω_+: $$\begin{pmatrix} 2\omega_+^2 - 2\omega_0^2 & \omega_+^2 \\ \omega_+^2 & \omega_+^2 - \omega_0^2 \end{pmatrix} \begin{pmatrix} a_1^{(+)} \\ a_2^{(+)} \end{pmatrix} = 0.$$

$$\omega_+^2 = (2 + \sqrt{2})\,\omega_0^2 \Rightarrow \begin{pmatrix} a_1^{(+)} \\ a_2^{(+)} \end{pmatrix} = C_+ \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix}$$



03/11/24

Normal coordinates

· normal mode analysis suggests a change of coord.'s in terms of simpler "elementary" motions

· Change in notation: $A_{i\alpha} = a_i^{(\alpha)}$

→ collect the $\alpha = 1, \ldots, n$ normal modes into $n \times n$ matrix $A_{i\alpha}$

· Solution of EOM suggests the change of coord.'s:

$$\{x_i\}_{i=1,\ldots,n} \rightarrow \{Q_\alpha\}_{\alpha=1,\ldots,n}$$

according to $$x_i = \sum_{\alpha=1}^{n} A_{i\alpha} Q_\alpha$$

(linear transform.$^{\text{n}}$).

· In the Lagrangian:

$$\begin{cases} T = \frac{1}{2} \sum_{ij} m_{ij} \dot{x}_i \dot{x}_j = \frac{1}{2} \sum_{\alpha\beta} \dot{Q}_\alpha \dot{Q}_\beta \left(\sum_{ij} A_{i\alpha} m_{ij} A_{j\beta} \right) \\ U = \frac{1}{2} \sum_{ij} k_{ij} x_i x_j = \frac{1}{2} \sum_{\alpha\beta} Q_\alpha Q_\beta \left(\sum_{ij} A_{i\alpha} k_{ij} A_{j\beta} \right) \end{cases}$$

· Now, $A_{i\alpha}$ (& $A_{j\beta}$) defined according to $\sum_j (\omega_\beta^2 m_{ij} - k_{ij}) A_{j\beta} = 0$

→ multiply by $A_{i\alpha}$ & sum over i:

$$\sum_{ij} \omega_\beta^2 A_{i\alpha} m_{ij} A_{j\beta} = \underbrace{\sum_{ij} A_{i\alpha} k_{ij} A_{j\beta}}_{\text{plug into } U}$$

03/11/24

$$\Rightarrow U = \frac{1}{2} \sum_{\alpha\beta} Q_\alpha Q_\beta \omega_\beta^2 \sum_{ij} A_{i\alpha} m_{ij} A_{j\beta}$$

$$\Rightarrow L = T - U = \frac{1}{2} \sum_{\alpha\beta} \left(\dot{Q}_\alpha \dot{Q}_\beta - \omega_\beta^2 Q_\alpha Q_\beta \right) \underbrace{\left(\sum_{ij} A_{i\alpha} m_{ij} A_{j\beta} \right)}_{\text{can be chosen } = \delta_{\alpha\beta}}$$

(see below for proof)

$$\Rightarrow \boxed{L = \frac{1}{2} \sum_{\alpha=1}^{n} \left(\dot{Q}_\alpha^2 - \omega_\alpha^2 Q_\alpha^2 \right)}$$

Q_α = "normal coord.'s".

E.-L. eqn.'s: $\frac{d}{dt} \frac{\partial L}{\partial \dot{Q}_\alpha} = \frac{\partial L}{\partial Q_\alpha} \Rightarrow \ddot{Q}_\alpha + \omega_\alpha^2 Q_\alpha = 0.$

$\Rightarrow$ description of dynamics in terms of n <u>decoupled</u> oscillators.

Forced oscillations become simple w/ normal coord.'s:

$$U_{ext}(q_1, q_2, \ldots, q_n, t) = U_{ext}(q_1^{(0)} + x_1, q_2^{(0)} + x_2, \ldots, q_n^{(0)} + x_n, t)$$

small x

$$\simeq U_{ext}(q^{(0)}, t) + \sum_{i=1}^{n} \left. \frac{\partial U_{ext}(q)}{\partial q_i} \right|_{q^{(0)}} x_i$$

$$= \underbrace{U_{ext}(q^{(0)}, t)}_{\text{drop from } L} - \sum_{i=1}^{n} F_i(t) x_i, \quad F_i = -\left. \frac{\partial U_{ext}(q)}{\partial q_i} \right|_{q^{(0)}}$$

$$\Rightarrow L = L_0 + \sum_{i=1}^{n} F_i(t) x_i$$

L_0: free oscillations.

03/11/24

In normal coord.'s $x_i = \sum_{\alpha=1}^{n} A_{i\alpha} Q_\alpha$:

$$L = \frac{1}{2} \sum_\alpha \left(\dot{Q}_\alpha^2 - \omega_\alpha^2 Q_\alpha^2 \right) + \sum_{\alpha=1}^{n} Q_\alpha \Bigg(\underbrace{\sum_{i=1}^{n} A_{i\alpha} F_i(t)}_{\equiv f_\alpha(t)} \Bigg)$$

$$\Rightarrow \quad L = \frac{1}{2} \sum_\alpha \left(\dot{Q}_\alpha^2 - \omega_\alpha^2 Q_\alpha^2 \right) + \sum_{\alpha=1}^{n} f_\alpha(t) Q_\alpha$$

E.-L. eqn.'s: $\ddot{Q}_\alpha + \omega_\alpha^2 Q_\alpha = f_\alpha(t)$.

$\rightarrow$ n decoupled, driven oscillators

Proof of $\sum_{ij} A_{i\alpha} m_{ij} A_{j\beta} = \delta_{\alpha\beta}$:

we have $\sum_{ij} \omega_\beta^2 A_{i\alpha} m_{ij} A_{j\beta} = \sum_{ij} A_{i\alpha} k_{ij} A_{j\beta}$ (1)

now swap $\alpha \leftrightarrow \beta$: $\sum_{ij} \omega_\alpha^2 A_{i\beta} m_{ij} A_{j\alpha} = \sum_{ij} A_{i\beta} k_{ij} A_{j\alpha}$

~~Subtract (1) - (2)~~ i & j are dummy indices, & we rewrite:

$$\sum_{ij} \omega_\alpha^2 A_{j\beta} m_{ji} A_{i\alpha} = \sum_{ij} A_{j\beta} k_{ji} A_{i\alpha}$$

$m_{ji} = m_{ij}$, $k_{ji} = k_{ij}$ $\Rightarrow$ $\sum_{ij} \omega_\alpha^2 A_{j\beta} m_{ij} A_{i\alpha} = \sum_{ij} A_{j\beta} k_{ij} A_{i\alpha}$ (2).

Now subtract (1) - (2):

$$\left(\omega_\beta^2 - \omega_\alpha^2 \right) \sum_{ij} A_{i\alpha} m_{ij} A_{j\beta} = 0$$

For $\alpha \neq \beta$ (& assuming non-degenerate roots):

$$\sum_{ij} A_{i\alpha} m_{ij} A_{j\beta} = 0 \qquad (\alpha \neq \beta)$$

$$\Rightarrow \sum_{ij} A_{i\alpha} m_{ij} A_{j\beta} \propto \delta_{\alpha\beta}$$

For $\alpha = \beta$, we may choose normalization:

$$\sum_{ij} A_{i\alpha} m_{ij} A_{j\alpha} = 1$$

$\blacksquare$