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1

Recall that cauchy-riemann equation: $\partial_x u = \partial_y v, \quad \partial_y u = -\partial_x v$ is a necessary condition for a complex function to be holomorphic. let $f(z) = (x+iy)^3$. Expanding, we have:

$$\begin{aligned} f(x,y) &= x^3 + 3ix^2y - 3xy^2 - iy^3 \\ &= (x^3 - 3xy^2) + i(3x^2y - y^3) \\ &\Rightarrow \begin{cases} u(x,y) = x^3 - 3xy^2 \\ v(x,y) = 3x^2y - y^3 \end{cases} \end{aligned}$$

check if the Cauchy-Riemann equations hold:

$$\partial_x u = 3x^2 - 3y^2, \quad \partial_y u = -6xy$$

$$\partial_x v = 6xy, \quad \partial_y v = 3x^2 - 3y^2$$

$$\Rightarrow \partial_x u = \partial_y v, \quad \partial_y u = -\partial_x v$$

Thus the function satisfies the cauchy-riemann equations and is holomorphic.

2

$$\begin{aligned} f(x,y) &= e^x(\cos y + i \sin y) \\ &= e^x \cos y + ie^x \sin y \\ &\Rightarrow \begin{cases} u(x,y) = e^x \cos y \\ v(x,y) = e^x \sin y \end{cases} \end{aligned}$$

$$\Rightarrow \begin{cases} \partial_x u = e^x \cos y & \partial_y u = -e^x \sin y \\ \partial_y v = e^x \cos y & \partial_x v = e^x \sin y \end{cases}$$

$$\Rightarrow \partial_x u = \partial_y v, \quad \partial_y u = -\partial_x v$$

Cauchy-Riemann equations hold at any point $z \in \mathbb{C}$

3

let $z = x + iy$. the function becomes

$$\begin{aligned} f(x,y) &= \frac{1}{x+iy} = \frac{x-iy}{(x+iy)(x-iy)} = \frac{x-iy}{x^2+y^2} \\ &= \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2} \\ &\Rightarrow \begin{cases} u(x,y) = \frac{x}{x^2+y^2} \\ v(x,y) = -\frac{y}{x^2+y^2} \end{cases} \end{aligned}$$

$$\begin{cases} \partial_x u = \frac{-x^2+y^2}{(x^2+y^2)^2} & \partial_y u = -\frac{2xy}{(x^2+y^2)^2} \\ \partial_x v = \frac{2xy}{(x^2+y^2)^2} & \partial_y v = \frac{y^2-x^2}{(x^2+y^2)^2} \end{cases}$$

$$\Rightarrow \partial_x u = \partial_y v, \quad \partial_y u = -\partial_x v$$

It is obvious that the four partial differentiations exist and are continuous.

Cauchy-Riemann equations hold at any point $z \in \mathbb{C}$.

$\boxed{\text{Thus the function is holomorphic on } (\mathbb{C}\ ).)}$

4

we propose the following parametrization of the contour C: $z = e^{it}, \mathrm{d}z = ie^{it}\,\mathrm{d}t$, for $t \in (0, 2\pi)$

The integration becomes:

$$\iint_C z^{-n}\,\mathrm{d}z = \int_0^{2\pi} \left(e^{it}\right)^{-n} ie^{it}\,\mathrm{d}t = \int_0^{2\pi} ie^{i(1-n)t}\,\mathrm{d}t$$

Noticing $\int e^{iNt}\,\mathrm{d}t = \frac{1}{iN}e^{iNt}$ if $N \neq 0$, the integral becomes

$$\boxed{i\left[\frac{1}{i(1-n)}e^{i(1-n)t}\right]_0^{2\pi} = 0}$$

5

. Evaluate the contour integral

$$\int_C (z^3 + e^z)dz$$

where C is a the portion of the unit circle centered at the origin, and C connects the point $(0,1)$ to the point $(1,0)$.

- solution:

We parametrize the unit circle as $z = e^{it}, \mathrm{d}z = ie^{it}\,\mathrm{d}t$, for t from $\frac{\pi}{2}$ to 0

The integration becomes

$$\int_{\frac{\pi}{2}}^{0} \left(e^{3it} + e^{e^{it}}\right) ie^{it}\,\mathrm{d}t = \int_0^{\frac{\pi}{2}} -ie^{4it}\,\mathrm{d}t + \int_0^{\frac{\pi}{2}} -ie^{e^{it}+it}\,\mathrm{d}t$$

$$= -i\left[\frac{e^{4it}}{4i}\right]_0^{\frac{\pi}{2}} - i\left(-ie^{e^{it}}\Big|_0^{\frac{\pi}{2}}\right)$$

$$= e - e^{e^{\pi/2\ i}} - \frac{1}{4}e^{2\pi i} + \frac{1}{4}$$

$$\boxed{= e - e^{i}}$$