Small Oscillations

- Motion near a point of stable equilibrium.

DOF= 1 (one dimension)

- For a system of DOF = 1, with potential $U(q)$:
  - **stable equilibrium** at $U(q)_{\text{min}}$, upward parabola, where $F = -\frac{\mathrm{d}U}{\mathrm{d}q} = 0$
    - restoring force for small displacements $q - q_0$ is $F = -\frac{\mathrm{d}U(q-q_0)}{\mathrm{d}q}$
- **Unstable equilibrium** at $U(q)_{\text{max}}$, downward parabola, where $F = -\frac{\mathrm{d}U}{\mathrm{d}q} = 0$ as well.
- Consider small deviation from point of stable equilibrium, we use taylor expansion to show that it is really a small displacement. that is,

$$U(q) \approx U(q_0) + \frac{\mathrm{d}U(q_0)}{\mathrm{d}q}(q - q_0) + \frac{\mathrm{d}^2U(q_0)}{2\,\mathrm{d}q^2}(q - q_0)^2 + \ldots$$
$$\text{while } \frac{\mathrm{d}U(q_0)}{\mathrm{d}q}(q - q_0) = 0 \tag{1}$$

letting $x = q - q_0$, we have

$$\begin{cases} U(x) = U(q_0) + \left(\frac{1}{2}\right)\frac{\mathrm{d}^2U(q_0)}{\mathrm{d}q^2}x^2 \\ \text{putting into the form of } U(x) = U(x_0) + \left(\frac{1}{2}\right)kx^2. \end{cases}$$
$$\Rightarrow \boxed{k = \frac{\mathrm{d}^2U(q_0)}{\mathrm{d}q^2} > 0} \tag{2}$$

we get KE, while choosing $U(q_0) = 0$:

$$T = \frac{1}{2}a(q)^2\dot{q}^2 = \frac{1}{2}a(q_0 + x)\dot{x}^2 \approx \frac{1}{2}m\dot{x}^2, \overset{m=a(q_0)}{\Rightarrow}$$
$$\boxed{L = T - U = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2} \tag{3}$$

EOM for DOF = 1 small Oscillations

using EL on Equation 3, we can get the EOM for one dimensional small Oscillations:

$$m\ddot{x} = -kx$$
$$\Rightarrow \ddot{x} + \omega_0^2 x = 0, \text{where } \boxed{\omega_0 = \sqrt{\frac{k}{m}} \text{ freq of osc.}} \tag{4}$$

by magic of ODE, EOM reduces down to:

$$\boxed{x(t) = C_1\cos(\omega_0 t) + C_2\sin(\omega_0 t)}$$
$$\text{where } C_1, C_2 \text{ are constants} \tag{5}$$

by trig magic, this could also be written as

$$x(t) = a\cos(\omega_0 t + \alpha),$$
$$\text{where } \begin{cases} a = \sqrt{C_1^2 + C_2^2} & \text{amplitude of oscillation} \\ \omega_0 & \text{frequency of oscillation} \\ \tan\alpha = C_2/C_1 & \text{phase at t=0} \end{cases} \quad (6)$$

energy for 1D small Oscillation
checking $\frac{\partial L}{\partial t} = 0 \Rightarrow$ energy-conservation:

$$\begin{aligned} E = T + U &= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 \\ &= \frac{1}{2}ma^2\omega_0^2, [\text{ constant}] \end{aligned} \quad (7)$$

Damped 1D oscillation, and Complex representation

- when there is damping (friction, resistence, etc) $F_{\text{fric}} = -\beta\dot{x}$, the EOM becomes:

$$\begin{gathered} \ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0, \\ \text{where } 2\gamma = \frac{\beta}{m}, \omega_0 = \sqrt{\frac{k}{m}} \end{gathered} \quad (8)$$

with ansatz $x(t) = e^{rt}, \dot{x} = re^{rt}, \ddot{x} = r^2 e^{rt}$, the solution to Equation 8 is:

$$\begin{gathered} r^2 + 2\gamma r + \omega_0^2 = 0, \\ \text{which has solution } r_+, r_- = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2} \\ \Rightarrow x(t) = C_1 e^{r_+ t} + C_2 e^{r_- t}, \end{gathered} \quad (9)$$

notice the r subscripts here: r_+, r_-

underdamped, overdamped, and critically damped
Recall from your ODE class...

Equation 9 has the following 3 cases, each with different physical interpretation:

1. underdamped:

$$\gamma < \omega_0 \Rightarrow 2 \text{ complex roots: } \begin{cases} r_\pm = -\gamma \pm i\sqrt{\omega_0^2 - \gamma^2} \\ = -\gamma \pm i\omega \\ \omega = \sqrt{\omega_0^2 - \gamma^2} \end{cases} \quad (10)$$

   The EOM is thus a linear combination of two complex expoentials:

$$\begin{aligned} x(t) &= e^{-\gamma t}\left(C_1 e^{i\omega t} + C_2 e^{-i\omega t}\right) \\ &= e^{-\gamma t}(A\cos(\omega t) + B\sin(\omega t)) \\ &\text{-- where } \begin{cases} A = C_1 + C_2 \\ B = i(C_1 - C_2) \end{cases} \\ &= ae^{-\gamma t}\cos(\omega t + \alpha) \\ &a, \alpha \text{ are constants} \end{aligned} \quad (11)$$

"The solution is a damped oscillation with frequencyω, and amplitude expoentially decaying with time."

2. Overdameped

$$\gamma > \omega \Rightarrow x(t) = c_1 e^{-\gamma + \sqrt{\gamma^2 - \omega^2} t} + c_2 e^{-\gamma - \sqrt{\gamma^2 - \omega^2} t} \qquad 12$$

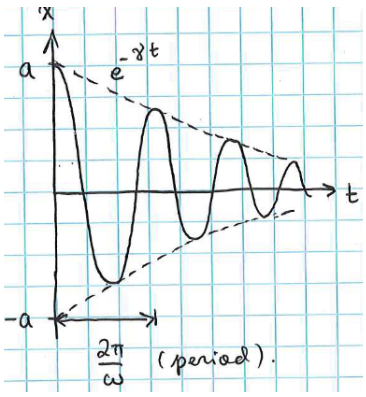
When

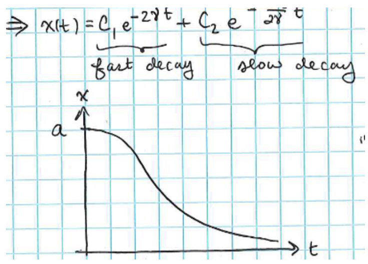
$$\gamma \gg \omega_0, \Rightarrow \begin{cases} \gamma + \sqrt{\gamma^2 - \omega_0^2} \approx 2\gamma \\ \gamma - \sqrt{\gamma^2 - \omega^2} = \frac{\omega^2}{2\gamma} \end{cases} \qquad 13$$

$$x(t) = c_1 e^{-2\gamma t} + c_2 e^{(-\omega_0^2/2\gamma)t}$$

3. Critically damped

$$\gamma = \omega_0 \Rightarrow x(t) = c_1 e^{-\gamma t} + c_2 t e^{-\gamma t} \qquad 14$$





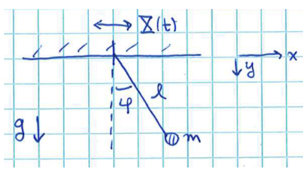
Forced Oscillations

When external force (F) is applied to the system, the largrangian becomes

$$L = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2 + F(t) x$$

$$\text{EL} \Rightarrow \ddot{x} + \omega_0^2 x = \frac{F(t)}{m}, \text{where } \omega_0 = \sqrt{\frac{k}{m}} \qquad 15$$

- Example: Simple pendulum with moving pivot



$$\begin{cases} x = X + l \sin \varphi \\ y = l \cos \varphi \end{cases} \Rightarrow \begin{cases} \dot{x} = \dot{X} + l \dot{\varphi} \cos \varphi \\ \dot{y} = -l \dot{\varphi} \sin \varphi \end{cases} \qquad 16$$

$$\Rightarrow L = T - U$$

$$L = \frac{1}{2} m l^2 \dot{\varphi}^2 - mgl(1 - \cos \varphi) - ml\ddot{X} \sin \varphi$$

$$\text{Expand ab. } \varphi = 0 \Rightarrow L = \frac{1}{2} m l^2 \dot{\varphi}^2 - \frac{1}{2} mgl\varphi^2 - ml\ddot{X}\varphi \qquad 17$$

$$\text{EL} \Rightarrow \boxed{\ddot{\varphi} + \omega_0^2 \varphi = -\frac{\ddot{X}}{l} \text{, where } \omega_0 = \sqrt{\frac{g}{l}}}$$

reintroducing damping via external forcing

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = f(t), f(t) = \frac{F(t)}{m} \quad 18$$

When damping $f(t) = f_0 \cos(\Omega t)$, solution via complex number:

$$\ddot{z} + 2\gamma\dot{z} + \omega_0^2 = f_0 e^{i\Omega t}$$

$$\text{ansatz } z(t) = z_0 e^{i\Omega t} \Rightarrow z_0 = \frac{f_0}{\omega_0^2 + 2i\gamma\Omega + \Omega_0^2}$$

$$\boxed{z_0 = a(\Omega)\cos(\Omega t + \delta(\Omega)) f_0} \text{ is a partcular solution,where} \quad 19$$

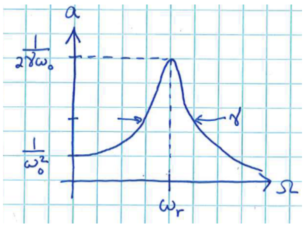
$$\begin{cases} a(\Omega) = \frac{1}{\sqrt{\left(\omega_0^2 - \Omega^2\right)^2 + (2\gamma\Omega)^2}} \\ \delta(\Omega) = \arctan\left(2\gamma\frac{\Omega}{\omega_0^2 - \Omega^2}\right) \end{cases}$$

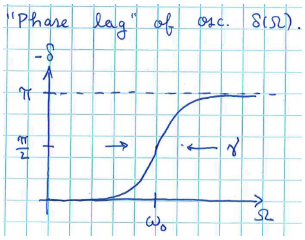
We can study the properties of the system by looking at the amplitude and phase of the solution.

- Amplitude:

$$a_{(\Omega)} = \frac{1}{\sqrt{\left(\omega_0^2 - \Omega^2\right)^2 + (2\gamma\Omega)^2}} \quad 20$$

, when $\gamma \ll \omega_0$, response strongest and amplitude largest when $\omega_r = \omega_0$.





- Phase lag: $\tan\delta(\Omega) = 2\gamma\frac{\Omega}{\Omega^2 - \omega_0^2}$

  in phase as $\Omega \to 0$, and out of phase as $\Omega \to \omega_0$.

- Genral solution to sinusoidal forcing:

$$x(t) = a(\Omega) f_0 \cos(\Omega t + \delta(\Omega)) + a_0 e^{-\gamma t}\cos(\omega t + \alpha)$$
$$\xrightarrow{t > \frac{1}{r}} a(\Omega) f_0 \cos(\Omega t + \delta(\Omega)) \quad 21$$

  Forgets initial condition after time.

- Power obsorbed by oscillation

  $p = F\dot{x} = mf\dot{x}$

  Avg power of oscillation

$$P_{\text{avg}} = \frac{1}{T}\int_0^T mf\dot{x}\,\mathrm{d}t = -\frac{1}{2} m f_0 a(\Omega)\Omega \sin\delta(\Omega) \quad 22$$

$$\text{simplifies to } P_{\text{avg}}(\Omega) = \gamma m f_0^2 \Omega^2 a_{(\Omega)}^2$$

  Absorption around resonance frequency $\Omega = \omega_0 + \varepsilon$ is maximum:

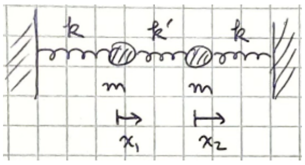
$$P = \frac{\gamma m f_0^2}{4(\varepsilon^2 + \gamma^2)} \approx \frac{m f_0^2}{4\gamma} \tag{23}$$

Oscillations DOF>1

For a system with n DOF: $q = (q_1, q_2, ..., q_n), \mathrm{PE} = U(q)$

- Stable equilibrium $\frac{\partial U(q)}{\partial q_i}\big|_{q=0}$

Example: Oscillation with 2 mass and 3 springs



$L = \frac{1}{2}m\dot{x_1} + \frac{1}{2}m\dot{x_2} - \frac{1}{2}kx_1^2$
$-\frac{1}{2}kx_2^2 - \frac{1}{2}k'(x_1 - x_2)^2$

EOM:

$$\begin{gathered} M \cdot \ddot{\vec{x}} = -K\vec{x} \quad \text{, where } M = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}, \\ \vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, K = \begin{pmatrix} k + k' & -k' \\ -k' & k + k' \end{pmatrix} \end{gathered} \tag{24}$$

ansatz: $\vec{x} = \mathrm{Re}\left[\vec{a}e^{i\omega t}\right]$ Then the EOM eq becomes solving the eigenvalue problem:

$$\begin{gathered} \det(\omega^2 M - K) = 0 \\ \Rightarrow \begin{cases} \omega_-^2 = \frac{k}{m} \\ \omega_+^2 = \frac{k+2k'}{m} \end{cases} \begin{cases} \overrightarrow{x_-} = a_- \binom{1}{1} \cos(\omega\_t + \delta\_) \\ \overrightarrow{x_+} = a_+ \binom{1}{-1} \cos(\omega_+ t + \delta_+) \end{cases} \end{gathered} \tag{25}$$

with constants $a_-, a_+, \delta_-, \delta_+$.

New Coords

$$\begin{gathered} \begin{cases} Q_1 = \sqrt{\frac{m}{2}}(x_1 + x_2) \\ Q_2 = \sqrt{\frac{m}{2}}(x_1 - x_2) \end{cases} \\ \Rightarrow L = \frac{1}{2}\left(\dot{Q_1}^2 + \dot{Q_2}^2\right) - \frac{1}{2}(\omega_-^2 Q_1^2 + \omega_+^2 Q_2^2) \\ \overset{\text{E-L}}{\Rightarrow} \ddot{Q}_1 = -\omega_-^2 Q_1, \ddot{Q}_2 = -\omega_+^2 Q_2 \end{gathered} \tag{26}$$

Decoupled oscillators with coords Q_1, Q_2.

General Coords

for general coords q_i, let $x_i = q_i - q_i^{(0)}$

$$\begin{gathered} U(q) = \frac{1}{2}\sum_{i,j} k_{ij} x_i x_j, \quad k_{ij} = k_{ji} = \frac{\partial^2 U(q)}{\partial q_i \partial q_j} \text{ symmetric matrix} \\ T = \frac{1}{2}\sum_{i,j} m_{ij} \dot{x_i} \dot{x_j}, \quad m_{ij} = m_{ji} = a_{ij}\left(q^{(0)}\right) \end{gathered} \tag{27}$$

the largrangian, in Matix form:

$$L = \frac{1}{2}\dot{\vec{x}}^T \cdot M \cdot \dot{\vec{x}} - \frac{1}{2}\vec{x}^T \cdot K\vec{x} \overset{\text{EL}}{\Longrightarrow} (\omega^2 M - K) \cdot \vec{a} = 0 \qquad 28$$

$\Rightarrow \det(\omega^2 M - K) = 0$ Solving the det for omega gives the normal freq (Eigenvalues)of system ω_α^2 . plug in Evalue into Equation 28 for eigenvec(normal modes) $\overrightarrow{a^\alpha}$ of system.

- General motion

$$x_i(t) = \sum_\alpha a_i^\alpha \mathrm{Re}\left[C_\alpha e^{i\omega_\alpha t}\right] \qquad 29$$

- EXAMPLE: Normal freq is given

$$\omega = \left\{0, \sqrt{2}\omega_0, \sqrt{3}\omega_0\right\}.$$

$$\omega = \sqrt{2}\omega_0 \Rightarrow a_1 = -a_3 = -a_2 = ae^{i\delta} \Rightarrow \vec{\theta} = a(1 \ \ -1 \ \ -1)^T \cos\left(\sqrt{2}\omega_0 t + \delta\right) \qquad 30$$

$$\omega = \sqrt{3\omega_0} \Rightarrow a_1 = 0, a_2 = -a_3 = ae^{i\delta} \Rightarrow \vec{\theta} = a(0 \ \ 1 \ \ -1)^T \cos\left(\sqrt{3}\omega_0 t + \delta\right)$$

- EXAMPLE: double pendulum

$$\begin{cases} x_1 = l_1 \sin\varphi_1 \quad y_1 = -l_1 \cos\varphi_1 \\ x_2 = l_1 \sin\varphi_1 + l_2 \sin\varphi_2 \quad y_2 = l_1 \cos\varphi_1 + l_2 \cos\varphi_2 \end{cases}$$

$$\Rightarrow T = \frac{1}{2}m_1 l_1 \dot{\varphi}^2 + \frac{1}{2}m_2\left(l_1^2 \dot{\varphi_1}^2 + l_2^2 \dot{\varphi_2}^2 + 2l_1 l_2 \dot{\varphi}_1 \dot{\varphi}_2 \cos(\varphi_1 - \varphi_2)\right) \qquad 31$$

$$U = -m_1 g l_1 \cos\varphi_1 - m_2 g(l_1 \cos\varphi_1 + l_2 \cos\varphi_2)$$

$$L = \tfrac{1}{2}\begin{pmatrix}\dot{\varphi_1} & \dot{\varphi_2}\end{pmatrix}\begin{pmatrix}(m_1+m_2)l_1^2 & m_2 l_1 l_2 \\ m_2 l_1 l_2 & m_2 l_2^2\end{pmatrix}\begin{pmatrix}\dot{\varphi_1} & \dot{\varphi_2}\end{pmatrix}$$

$$-\tfrac{1}{2}\begin{pmatrix}\varphi_1 & \varphi_2\end{pmatrix}\begin{pmatrix}(m_1+m_2)l_1 g & 0 \\ 0 & m_2 g l_2\end{pmatrix}\begin{pmatrix}\varphi_1 & \varphi_2\end{pmatrix}$$

using $\cos\varphi \approx 1 - \frac{\varphi^2}{2} = \frac{1}{2}\dot{\vec{\varphi}}^T M \cdot \dot{\vec{\varphi}} - \frac{1}{2}\vec{\varphi}^T K \vec{\varphi}$

When $m_1 = m_2 = m, \quad l_1 = l_2 = l \Rightarrow \quad M = ml^2\begin{pmatrix}2 & 1 \\ 1 & 1\end{pmatrix}, K = mgl\begin{pmatrix}2 & 0 \\ 0 & 1\end{pmatrix}$

$$\det((\omega^2 M - K)) = 0 \Rightarrow \omega^2 = \left(2 \pm \sqrt{2}\omega_0^2\right)$$

$$\begin{pmatrix}a_1^- \\ a_2^-\end{pmatrix} = C_-\begin{pmatrix}1 \\ \sqrt{2}\end{pmatrix}, \quad \begin{pmatrix}a_1^+ \\ a_2^+\end{pmatrix} = C_+\begin{pmatrix}1 \\ -\sqrt{2}\end{pmatrix} \qquad 32$$

Normal Coords
$\{x_i\} = \{Q_\alpha\}$, where $x_i = \sum_{\alpha=1}^n A_{i\alpha} Q_\alpha \Rightarrow \sum_j \left(\omega_\alpha^2 m_{ij} - k_{ij} A_{jx}\right) = 0$

$\Rightarrow L = \frac{1}{2}\sum_{\alpha=1}^n \left(\dot{Q}^2{}_\alpha - \omega_\alpha^2 Q_\alpha^2\right) \overset{\text{EL}}{\Longrightarrow} \ddot{Q}_\alpha + \omega_\alpha^2 Q_\alpha$

Motion of Rigid Body