Summary

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- $S[q(t)] = \int_{t_1}^{t_2} dt \; L(q,\dot{q},t)$ actional functional.

  $q = (q_1, q_2, \ldots, q_n)$ generalized coord.'s, $n = \#$ of DOF.

  L = Lagrangian.

- physical paths $q(t) \rightarrow \delta S = 0$ (Hamilton's principle)

$$\delta S = 0 \Rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i}$$ (Euler-Lagrange eqn.'s)

→ n second-order diff. eq.'s for $q_i(t)$.

Q: What is L?

→ use symmetry to constrain form of L.

- Inertial frames

→ postulate there exist priveleged frames of reference s.t. the following properties hold:

(i) space is homogeneous (no preferred pt. in space).

(ii) space is isotropic (no preferred direction).

(iii) time is homogeneous (no preferred time).

Such frames are called "inertial frames".

→ now explore consequences for L.

Start w/ the case of a single free particle. 01/29/24

use Cartesian coord.'s: $q \to \vec{r}$, $\dot{q} \to \vec{v}$., $L = L(\vec{r}, \vec{v}, t)$.

(i) homog. of space:

$$L(\vec{r}, \vec{v}, t) = L(\vec{r} + \vec{a}, \vec{v}, t)$$

for arbitrary translation $\vec{a}$.

$\Rightarrow$ $L(\vec{r}, \vec{v}, t) = L(\vec{v}, t)$; i.e., L indep. of $\vec{r}$.

More formally: take $\vec{a} = \vec{\varepsilon}$ to be a small translation & expand to first order:

$$L(\vec{r} + \vec{\varepsilon}, \vec{v}, t) \simeq L(\vec{r}, \vec{v}, t) + \frac{\partial L}{\partial \vec{r}} \cdot \vec{\varepsilon} = L(\vec{r}, \vec{v}, t)$$

$$\Rightarrow \quad \frac{\partial L}{\partial \vec{r}} \cdot \vec{\varepsilon} = 0.$$

since $\vec{\varepsilon}$ is arbitrary, this means $\frac{\partial L}{\partial \vec{r}} = 0$ & hence L is indep. of $\vec{r}$

(ii) isotropy

$L(\vec{v}, t)$ can only depend on dot products $\vec{v} \cdot \vec{v} = v^2$

$$\Rightarrow \quad L(\vec{v}, t) = L(v^2, t).$$

(iii) homog. of time:

$L(v^2, t) = L(v^2)$ (same argument as (i)).

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So, we have found that, in an inertial frame (which we defined in terms of symmetry properties), the Lagrangian for a free particle is $L = L(v^2)$.

Q: What does this mean for the motion in such a frame?

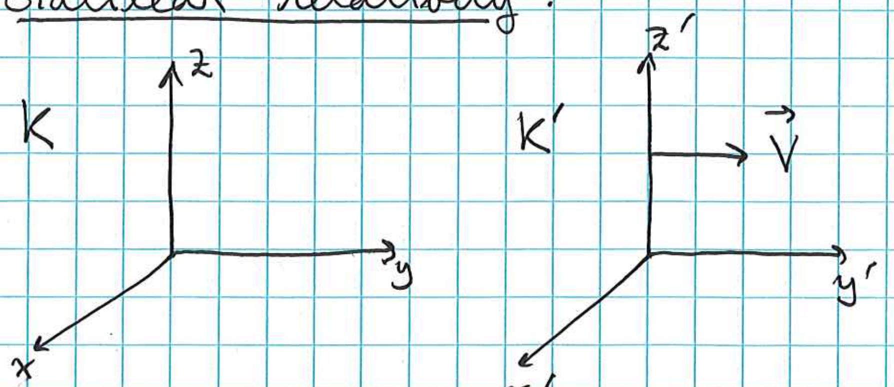
→ E.-L. eqn.'s: $$\frac{d}{dt}\frac{\partial L}{\partial \vec{v}} = \cancel{\frac{\partial L}{\partial \vec{r}}}^{\,=0} \qquad \left(\text{since } L = L(v^2)\right)$$

$$\Rightarrow \frac{\partial L}{\partial \vec{v}} = \text{const. in time.}$$

$$\Rightarrow \vec{v} = \text{const.}$$, since L depends only on $\vec{v}$.

⇒ In an inertial frame, free particles move w/ const. velocity $\vec{v}$ (law of inertia).

• We can further constrain the form of L ~~faro~~ for a free particle by invoking the principle of Galilean relativity:



If K = inertial frame & K' is another frame moving w/ const. velocity $\vec{V}$ w.r.t K, then K' = inertial frame. & all laws of physics same in K & K'

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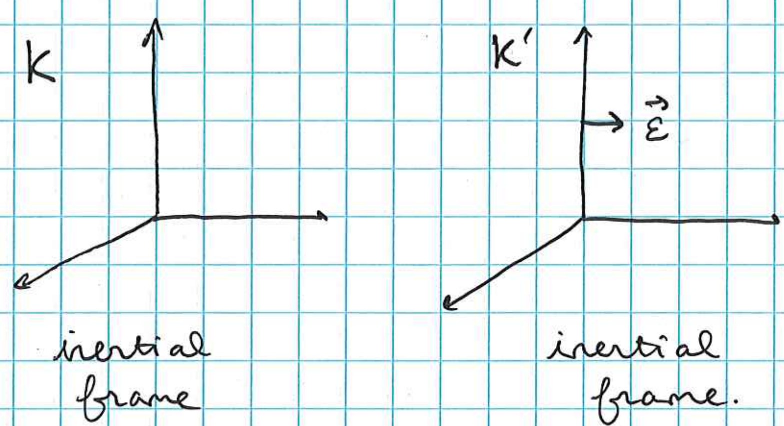
$\Rightarrow$ preferred <u>class</u> of frames (inertial) but no single preferred frame.

Mathematically: $$\left.\begin{aligned} \vec{r} &= \vec{r}\,' + \vec{V}t \\ t &= t' \end{aligned}\right\}\ \text{Galilean transform.'n}$$

$\rightarrow$ maps free particle solutions to free particle solutions.

· Principle of Gal. relativity + prop.'s of inertial frames fix the Lagrangian for a free particle:



suppose velocity $\vec{\varepsilon}$ = small.

$$\left.\begin{aligned} \vec{r} &= \vec{r}\,' + \vec{\varepsilon}t \\ t &= t' \\ \vec{v} &= \vec{v}\,' + \vec{\varepsilon} \end{aligned}\right\}\ \text{Gal. transform.'n.}$$

K: $\quad L = L(v^2) \quad$ (by prop. of inertial frame).

K': $\quad L' = L(v'^2) \quad$ " "

$$\simeq L(v^2 - 2\vec{v}\cdot\vec{\varepsilon})$$

$$\simeq L(v^2) - \frac{\partial L}{\partial v^2}(2\vec{v}\cdot\vec{\varepsilon})$$

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- The difference btwn. ~~Lagrangians~~ Lagrangians in frames K & K' is thus:

$$L' - L \simeq -\frac{\partial L}{\partial v^2}(2\vec{v}\cdot\vec{\varepsilon}).$$

- According to Gal. princ. of relativity, EOM in K & K' must be same. $\Rightarrow$ $L' - L = \frac{d}{dt}f(\vec{r},t)$. (HW 1 & Lec. 2)

$$\Rightarrow \quad \vec{v}\cdot\left(-2\frac{\partial L}{\partial v^2}\vec{\varepsilon}\right) = \frac{d}{dt}f(\vec{r},t). \qquad (*).$$

- Now, dep. of df/dt on $\vec{v}$ is linear:

$$\frac{d}{dt}f(\vec{r},t) = \frac{\partial f}{\partial \vec{r}}\cdot\vec{v} + \frac{\partial f}{\partial t} \equiv \vec{\alpha}\cdot\vec{v} + \beta$$

w/ $\vec{\alpha} = \vec{\alpha}(\vec{r},t)$ & $\beta = \beta(\vec{r},t)$.

- Comparing to $(*)$, we conclude $\frac{\partial L}{\partial v^2}$ = const. fn. of v^2.

$$\Rightarrow \quad L \sim v^2$$

& we write $\boxed{L = \frac{1}{2}mv^2}$ (free particle).

→ in our approach, this defines the "mass" m of a particle.

Q: Is m arbitrary?

→ see end of notes for discussion of "principle of additivity"

Useful to record v^2 in various coord.'s: 01/29/24

$$d\vec{s} = \vec{v}\, dt \quad \Rightarrow \quad ds^2 = v^2 dt^2$$

$$\Rightarrow \quad v^2 = \frac{(ds^2)}{(dt^2)}$$

that is, to obtain v^2 in various coord. systems, ~~it does~~ we may simply write down ds^2 & then formally divide by dt^2

(i) Cartesian: $ds^2 = dx^2 + dy^2 + dz^2$

$$\Rightarrow v^2 = \cancel{d}\dot{x}^2 + \dot{y}^2 + \dot{z}^2$$

(ii) Cylindrical: $ds^2 = dr^2 + r^2 d\varphi^2 + dz^2$

$$\Rightarrow v^2 = \dot{r}^2 + r^2\dot{\varphi}^2 + \dot{z}^2$$

(iii) Polar: $ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\varphi^2$

$$\Rightarrow v^2 = \dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\, \dot{\varphi}^2$$

Interactions btwn. particles:

$$L = \underbrace{\sum_{a=1}^{N} \frac{1}{2} m_a v_a^2}_{\text{"kinetic energy" } (T)} - \underbrace{U(\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_N)}_{\text{"potential energy" } (U)}$$

a = particle label
N = # of part.

$$\Rightarrow L = T - U.$$

Now consider the E-L. eqn's: 01/29/24

$$\frac{d}{dt}\frac{\partial L}{\partial \vec{v}_a} = \frac{\partial L}{\partial \vec{r}_a} \qquad (a=1,\dots,N)$$

$$\Rightarrow \quad m\dot{\vec{v}}_a = \underbrace{-\frac{\partial}{\partial \vec{r}_a} U(\vec{r}_1,\dots,\vec{r}_N)}_{\vec{F}_a,\ \text{force on particle } a}.$$

(→ Newton's 2nd law)

Q: Is m arbitrary?

→ A system of N non-interacting particles should be understood as the limit of a system in which particles are far separated s.t. their interactions may be ignored

$$\Rightarrow \quad L_{\text{non-interacting}} = \lim L = \sum_{a=1}^{N} L_a = \sum_{a=1}^{N} \frac{1}{2} m_a v_a^2$$

⇒ EOM only invariant under rescaling of L, not individual L_a.

⇒ ratios of particle masses are physically meaningful. overall rescaling ⟷ choice of mass unit