Sample Spaces, collection of events, probability measure

- Sample space Ω: set of all possible outcomes of an experiment. Comes in n-tuples where n represents number of repeated trials.
- Collection of events $\mathcal{F}$: subset of state space to which we assign a probability.
- Probability measure: function that assigns a probability to each event. $P : F \to \mathbb{R}$.
  - Range is $[0, 1]$.
- Axioms
  - $P(\Omega) = 1 \text{ and } P(\emptyset) = 0$
  - For pairwise disjoint events $A_1, A_2, \ldots,$ $P(A_1 \cup A_2 \cup \ldots) = P(A_1) + P(A_2) + \ldots$

Sampling

Uniform sampling

If the sample space Ω has finitely many elements and each outcome is equally likely, then for any event $A \subset \Omega$ we have

$$P(A) = \frac{\#A}{\#\Omega} \tag{1}$$

where # means the "cardinality" of the set.

- uniform sampling: each outcome is equally likely
- Binomial coeff

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \tag{2}$$

Sampling with Replacement, order matters

- ex: sample K distinct marked balls from N balls in a box, **with** Replacement

$$\begin{gathered} \Omega = \{1, 2, 3, \ldots, N\}^K \\ \|\Omega\| = N^K \\ P(\text{none of the balls is marked } 1) = \frac{(N-1)^K}{N^K} \end{gathered} \tag{3}$$

- ex: sample K distinct marked balls from N balls in a box, **without** Replacement

$$\begin{gathered} \Omega = \{(i_1, i_2, \ldots, i_K) \mid i_1, \ldots, i_K \in \{1, 2, \ldots, N\}, \text{distinct} \\ \|\Omega\| = \binom{N-1}{K} \\ P(\text{none of the balls is marked } 1) = \frac{\binom{N-1}{K}}{\binom{N}{K}} = \frac{N-K}{N} \end{gathered} \tag{4}$$

Order

- order matters: $A_n^k = \frac{n!}{(n-k)!}$
- order doesn't matter: $\binom{n}{k} = C_n^k = \frac{n!}{k!(n-k)!}$

Infinite Sample Spaces

discrete

$$\Omega = \{\infty, 1, 2, ...\} \tag{5}$$

continuous

$$\begin{gathered} P([a', b']) = \frac{\text{length of } [a', b']}{\text{length of}[a, b]} \\ \text{single point, or sets of points: } P(\{x\}) = P(\cup_{i=1}^{\infty} \{x_i\}) = 0 \end{gathered} \tag{6}$$

- Complements: $P(A) = 1 - P(A^C)$

Conditioinal Probability, Law of Total Prob., Bayes' Theorem, Independence

Conditional prob.

$$P(A|B) = \frac{|A \cap B|}{|B|} \Rightarrow P(AB) = P(B)P(A|B) \tag{7}$$

(new sample space is B, total number of outcomes is $A \cap B$)

Law of total probability:

Given partitions $B_1, B_2, ...$ of Ω,

$$P(A) = \sum_i P(A|B_i)P(B_i) \tag{8}$$

Bayes' Theorem:

Given events A, B, P(A) and P(B) >0,

$$P(B_i|A) = \frac{P(A|B_i)P(B_i)}{P(A)} \tag{9}$$

Considering the law of total prob., the generalized form, when B_i are partitions, is given as:

$$P(B_i|A) = \frac{P(A|B_i)P(B_i)}{\sum_j P(A|B_j)P(B_j)} \tag{10}$$

Independence:

$$P(AB) = P(A)P(B) \Leftrightarrow P(B|A) = P(B) \tag{11}$$

Note: By virtue of conventions, we write $A \cap B$ as AB in Probability.
If A,B,C,D are independent, it follows that $P(ABCD) = P(A)P(B)P(C)P(D)$; however, the inverse is not always true.

- Independence of Random Variables (messy as hell...)

Given 2 random variables

$$\begin{gathered} X_1 \in \{x_{11}, x_{12}, x_{13}, ..., x_{1m}\} \\ X_2 \in \{x_{21}, x_{22}, x_{23}, ..., x_{2n}\} \\ \text{Random variables X\_1 and X\_2 are independent} \Leftrightarrow \\ P(X_1 = x_{1i}, X_2 = x_{2j}) = P(X_1 = x_{1i}) P(X_2 = x_{2j}) \end{gathered} \tag{12}$$

Need to check n*m equations to verify independence.

Conditional Independence:

For events $A_1, A_2, ..., A_n, B$, any set of events in A: A_{i1}, A_{i2}, A_{i3}, they are conditionally independent given B if

$$P(A_{i1} A_{i2} A_{i3} | B) = P(A_{i1} | B) * P(A_{i2} | B) * P(A_{i3} | B) \tag{13}$$

Independent Trials, Distributions

Bernoulli dirtribution:

a single trial, with success probability p, and failure probability 1-p. Prameter being the success probability.

$$X \sim \text{Ber}(p) \Rightarrow P(X = x) = p^x * (1 - p)^{1-x}, x \in \{0, 1\} \tag{14}$$

Binomial Distribution:

multiple independent Bernoulli trials, with success probability p, and failure probability 1-p. Parameters being the number of trials n and the success probability p.

$$X \sim \text{Bin}(n, p) \Rightarrow P(X = k) = \binom{n}{k} p^k * (1 - p)^{n-k}, k \in \{0, 1, ..., n\} \tag{15}$$

Geometric distribution:

multiple independent Bernoulli trials with success probability p, while stoping the experiment at the first success.

$$X \sim \text{Geom}(p) = p * (1 - p)^{k-1}, k \in \{1, 2, ...\} \tag{16}$$

Hypergeometric distribution:

There are N objects of type A, and $N_A - N$ objects of type B. Pick n objects without replacement. Denote number of A objects we picked as k. Parameters are N, N_A, n.

$$\begin{gathered} P(X = k) = \frac{\binom{N_A}{k}\binom{N - N_A}{n - k}}{\binom{N}{n}} \\ \text{choose k from N\_A, choose n-k from N-N\_A, divide by total number of ways to choose n from N} \end{gathered} \tag{17}$$