Angular momentum of a rigid body

$\vec{L}$ in non-inertial frame

$$\vec{L} = \sum m(\vec{r} \times \vec{v}) = \sum m\left[\vec{\Omega} r^2 - \vec{r}\left(\vec{\Omega} \cdot \vec{r}\right)\right]$$

$$L_i = \boxed{I_{ij}\Omega_j} \qquad \vec{L} = I * \vec{\Omega}$$

If $(x_1 x_2 x_3)$ are principal axis, $L_1 = I_1\Omega_1$, $L_2 = I_2\Omega_2$, $L_3 = I_3\Omega_3$

Free motion of a rigid body

angular momentum is conserved if no external torque. Motion in inertial COM frame is simplier.

- *ex motion of a symmetric top* $I_1 = I_2 = I_3 = I, \quad \tilde{I} = I\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$\vec{L} = I\vec{\Omega} \to \dot{\vec{L}} = 0 \Rightarrow \dot{\vec{\Omega}} = 0$ Uniform rotation about fixed axis paralle to $\vec{L}$

- *ex rigid rotor* $I_1 = I_2 = \sum m x_3^2, \quad I_3 = 0$

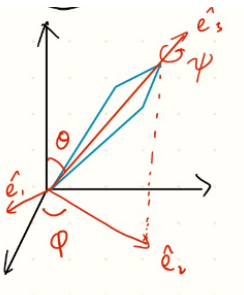
$\vec{L} = I\vec{\Omega}, \quad \vec{\Omega} \perp x_3$ by geometry We have $\dot{\vec{\Omega}} = 0 \Rightarrow$ Motion is unif in plane perp to $\vec{\Omega}$ and that it stays in that plane.

- *ex asymmetric top* $I_1 = I_2 = I_\perp \neq I_3 \Rightarrow \tilde{I} = \begin{pmatrix} I_\perp & 0 & 0 \\ 0 & I_\perp & 0 \\ 0 & 0 & I_3 \end{pmatrix}$ x_3 is symm. axis, for any orthogonal axes

Rigid body EOM

$$\begin{cases} \dot{\vec{p}} = \vec{F} \\ \dot{\vec{L}} = \vec{K} \text{ torque} \end{cases}$$

Euler angles: ψ spin, θ nutation, φ precession



$(\theta \in [0, \pi], \varphi \in [0, 2\pi], \psi \in [0, 2\pi])$ in turns of rotation $R = R(\hat{z}, \varphi)R\left(\widehat{X}, \theta\right)R\left(\hat{Z}, \psi\right)$

The lagrangian in Euler angles

- First: $T = \frac{1}{2}\left(I_1\Omega_1^2 + I_2\Omega_2^2 + I_3\Omega_3^2\right)$
- Rotation in components:

$$\Omega_1 = \dot{\varphi}\sin\theta\sin\psi + \dot{\theta}\cos\psi$$

$$\Omega_2 = \dot{\varphi}\sin\theta\cos\psi - \dot{\theta}\sin\psi$$

$$\Omega_3 = \dot{\varphi}\cos\theta + \dot{\psi}$$

- $T = \frac{1}{2}I_1\left(\dot\varphi \sin\theta \sin\psi + \dot\theta\cos\psi\right)^2 + \frac{1}{2}I_2\left(\dot\varphi\sin\theta\cos\psi - \dot\theta\sin\psi\right)^2 + \frac{1}{2}I_3\left(\dot\varphi\cos\theta + \dot\psi\right)^2$
- $L\left(\theta, \varphi, \psi, \dot\theta, \dot\varphi, \dot\psi\right) = T - U$

Free motion of symmetric top in Euler angles

$I_1 = I_2 = I_\perp \Rightarrow \quad T = \frac{1}{2}I_\perp\left(\dot\theta^2 + \dot\varphi^2\sin^2\theta\right) + \frac{1}{2}I_3\left(\dot\varphi\cos\theta + \dot\psi\right)^2$

$\Omega_\perp = L_z/I_\perp, \quad \Omega_3 = L_z\cos\theta/I_3$ E-L ->

$$\theta : \frac{\mathrm{d}}{\mathrm{d}t}I_\perp\dot\theta = I_\perp\sin\theta\cos\theta\,\dot\varphi^2 - I_3\dot\varphi\sin\theta\left(\dot\varphi\cos\theta + \dot\psi\right)$$

$$\varphi : \frac{\mathrm{d}}{\mathrm{d}t}\left(I_\perp\dot\varphi\sin^2\theta + I_3\cos\theta\left(\dot\varphi\cos\theta + \dot\psi\right)\right) = 0$$

$$\psi : \frac{\mathrm{d}}{\mathrm{d}t}I_3\left(\dot\varphi\cos\theta + \dot\psi\right) = 0$$

choosing $\hat{z}$ along the angular momentum, we have $L_3 = L_z\cos\theta = I_3\Omega_3 = I_3\left(\dot\varphi\cos\theta + \dot\psi\right)$
$\Rightarrow \dot{L}_3 = \text{const} \Rightarrow \theta = \text{const} \quad \Omega_3 = \frac{L_z\cos\theta}{I_3} \quad \dot\varphi = \frac{L_3}{I_\perp\cos\theta} = \frac{L_z}{I_\perp} = \text{const}$

- *ex heavy symmertic top with one pt fixed* By paralle axis thm, $I'_{ij}I_{ij} + M\left(l^2\delta_{ij} - l_i l_j\right)$

  $\Rightarrow I'_\perp = I_\perp + Ml^2, \quad I'_3 = I_3, \quad U = mgZ = Mgl\cos\theta$

  $\Rightarrow L = T - U = \frac{1}{2}I'_\perp\left(\dot\theta^2 + \dot\varphi^2\sin^2\theta\right) + \frac{1}{2}I_3\left(\dot\psi + \dot\varphi\cos\theta\right)^2 = Mgl\cos\theta$
  E-L :

$$L_z = p_\varphi = \left(I'_\perp\sin^2\theta + I_3\cos^2\theta\right)\dot\varphi \quad \text{const}$$

$$L_3 = p_\psi = I_3\left(\dot\psi + \varphi\cos\theta\right) \quad \text{const}$$

Considering energy conservation

$$E = T + U \Rightarrow \quad \underbrace{E - \frac{L_3^2}{2I_3} - Mgl}_{E'} = \frac{1}{2}I'_\perp\dot\theta^2 + \underbrace{\frac{1}{2I'_\perp}\frac{\left(L_z - L_3\cos\theta\right)^2}{\sin^2\theta} - Mgl(1 - \cos\theta)}_{U_{\text{eff}}(\theta)}$$

effective 1 dof problem. recognizing

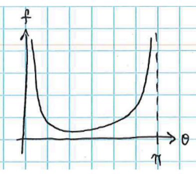
$$\dot\theta = \frac{\mathrm{d}\theta}{\mathrm{d}t} \Rightarrow \quad t = \int\frac{d\theta}{(\sqrt{2[E - U_{\text{eff}}(\theta)]/I'_\perp}}$$

Considering U_eff: when $\theta = 0, L_z = L_3$when $\theta \approx 0 \Rightarrow \quad U_{\text{eff}} \approx \left(\frac{L_3^2}{8I'_\perp} - \frac{Mgl}{2}\right)\theta^2$

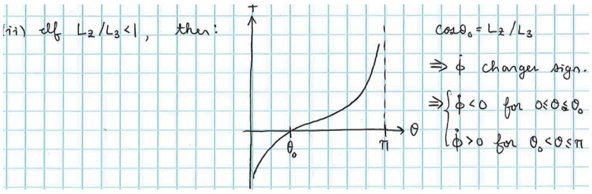
Motion about $\theta = 0$ stable if $L_3^2 > 4I'_\perp Mgl \Rightarrow \Omega_3^2 > 4I'_\perp Mgl/I_3^2$, or stable if sping ab. symm. axis is fast enough.

- Nutuation: cosider $\dot\varphi = \frac{L_3}{I'_\perp}\frac{(L_z/L_3) - (\cos\theta)}{\sin^2\theta} = \frac{L_3}{I'_\perp}f(\theta)$

(i) if $L_z/L_3 > 1$, then:



$\Rightarrow \dot\phi > 0.$



considering the sign and trends of $f(\theta)$ given constrains on theta, we can differentiate different nutation motion. If θ_0 in graph 2 is out of range, the nutation is smooth; if θ_0 is in range, the nutation is oscillatory(will change sign and spin in spiral.); if θ_0 is on the endpoint of our constrained range, the nutation is spiky and "not smooth" at points.

Euler equations

set body frame $(X, Y, Z) = (\hat{e}_1^0, \hat{e}_2^0, \hat{e}_3^0$, space frame $(x_1, x_2, x_3) = (\hat{e}_1, \hat{e}_2, \hat{e}_3)$ Set any vector $\vec{A} = \sum A_i^0 \hat{e}_i^0 = \sum A_i \hat{e}_i$ By magic of vec analysis,

$$\left(\frac{\mathrm{d}\vec{A}}{\mathrm{d}t}\right)_{\text{Space}} = \left(\frac{\mathrm{d}\vec{A}}{\mathrm{d}t}\right)_{\text{Body}} + \vec{\Omega} \times \vec{A}_{\text{Space}}$$

When applied to $\left(\frac{\mathrm{d}\vec{L}}{\mathrm{d}t}\right)_{\text{Space}} = \vec{\mathrm{K}} = \left(\frac{\mathrm{d}\vec{L}}{\mathrm{d}t}\right)_{\text{body}} + \vec{\Omega} \times \vec{L}$, recognizing $L_i = I_i \Omega_i$:

$$\begin{aligned} I_1 \dot{\Omega}_1 + (I_3 - I_2)\Omega_2\Omega_3 &= K_1 \\ I_2 \dot{\Omega}_2 + (I_1 - I_3)\Omega_3\Omega_1 &= K_2 \\ I_3 \dot{\Omega}_3 + (I_2 - I_1)\Omega_1\Omega_2 &= K_3 \end{aligned}$$

$K_i = 0$ if $\vec{L}$ is conserved on i axis.

- *ex symmetric top* $I_1 = I_2 = I, \vec{K} = 0 \quad \left(\dot{\Omega}_1 + \frac{I_3 - I_1}{I_\perp}\Omega_2\Omega_3 = 0; \dot{\Omega}_2 + \frac{I_1 - I_3}{I_\perp}\Omega_3\Omega_1 = 0; \dot{\Omega}_3 = 0\right)$
  let $\omega = ((I_3 - I_\perp)/(I_\perp))\Omega_3 \Rightarrow \boxed{\left(\Omega_1 = A\cos\omega t; \Omega_2 = -\frac{1}{\omega}\dot{\Omega}_1 = +A\sin\omega t\right)}$

Motion in non-inertial frame

- Set non-inertial frame with velocity $\vec{V}(t)$, $\vec{A} = \dot{\vec{V}}$, $\quad \vec{v} = \vec{v}' + \vec{V}(t)$ where $\vec{v}'$ is velocity w.r.t. non-inertial frame.

lagrangian $L' = \frac{1}{2}mv'^2 - m\vec{r}' \cdot \vec{A} - U$, using E-L eq: $m\dot{\vec{v}}' = -\frac{\partial U}{\partial \vec{r}'} - m\vec{A}$

- *ex pendulum in acc. car* $\quad m\ddot{\vec{r}} = \vec{T} + m\vec{g} - m\vec{A}$,

  finding equil. angle: $\vec{T} = -m\left(\vec{g} - \vec{A}\right) = -m\vec{g}_{\text{eff}}$, then use geometry between $\vec{g}, -\vec{A} \Rightarrow \tan\varphi_0 = \frac{A}{g}$. Oscillation freq. $\omega = \sqrt{g_{\text{eff}}/l}$

Motion in rotating frame

Set rotation with $\vec{\Omega}$, $L = \frac{1}{2}mv^2 + m\vec{v} \cdot \left(\vec{\Omega} \times \vec{r}\right) + \frac{1}{2}m\left(\vec{\Omega} \times \vec{r}\right)^2 - m\vec{r} \cdot \vec{A} - U$

Using E-L, $\boxed{m\dot{\vec{v}} = -\frac{\partial U}{\partial \vec{r}} - m\vec{A} + 2m\left(\vec{v} \times \vec{\Omega}\right) + m\vec{\Omega} \times \left(\vec{r} \times \vec{\Omega}\right) + m\vec{r} \times \dot{\vec{\Omega}}}$

- Namely,

$$m\dot{\vec{v}} = -\frac{\partial U}{\partial \vec{r}} + \vec{F}_{\text{cor}} + \vec{F}_{\text{cent}}$$

$$\vec{F}_{\text{Cor}} = 2m\left(\vec{v} \times \vec{\Omega}\right), \quad \vec{F}_{\text{cent}} = m\vec{\Omega} \times \left(\vec{r} \times \vec{\Omega}\right) = m\left(\vec{\Omega} \times \vec{r}\right) \times \vec{\Omega}$$

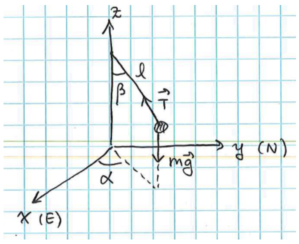
- *ex free fall on earth, centrifugal force* $\vec{F} = \vec{g}_0 + m\Omega^2 R \sin\theta \hat{\rho} \Rightarrow \vec{g}_{\text{eff}} = \vec{g}_0 + \Omega^2 R \sin\theta \hat{\rho}$
- *ex free fall, coriolis force* $\dot{\vec{v}} = \vec{g} + 2\vec{v} \times \vec{\Omega}, \quad \vec{\Omega} = \Omega \sin\theta \hat{y} + \Omega \cos\theta \hat{z}$

In components,

$$\vec{v_x} = 2\Omega\left(v_y \cos\theta - v_z \sin\theta\right)$$
$$\vec{v_y} = -2\Omega v_x \cos\theta$$
$$\vec{v_z} = 2\Omega v_x \sin\theta - g$$

Free fall EOM: $\vec{R} = \int v \, \mathrm{d}r$, consider $\vec{v} = \vec{v_1} + \vec{v_2} = -\vec{g} + 2\vec{v_1} \times \vec{\Omega} + 2\vec{v_2} \times \vec{\Omega}$ where approximately, $\vec{v_2} = 2(\vec{v_0} - gt\hat{z}) \times \vec{\Omega}$. If no initial velocity, integrating velocity in x components gives, $x(t) = \frac{1}{3} g\Omega\left(\frac{2h}{g}\right)^{3/2} \sin\theta$

- *ex foucaults pendulum* EOM



$$\vec{r} = l \sin\beta \cos\alpha \hat{x} + l \sin\beta \sin\alpha \hat{y} + (l - l\cos\beta)\hat{z}$$
$$\vec{T} = -T \sin\beta \cos\alpha \hat{x} - T \sin\beta \sin\alpha \hat{y} + T \cos\beta \hat{z}$$
$$\vec{\Omega} = \Omega \sin\theta \hat{y} + \Omega \cos\theta \hat{z}$$

$$\begin{cases} T = mg \\ m\ddot{x} = T_x + 2m\hat{x} \cdot \left(\dot{\vec{r}} \times \vec{\Omega}\right) = -\frac{mgx}{l} + 2m\Omega\dot{y}\cos\theta \\ m\ddot{y} = -\frac{mgy}{l} - 2m\Omega\dot{x}\cos\theta \end{cases}$$

letting $\omega^2 = \frac{g}{l}, \Omega_z = \Omega\cos\theta$, $\boxed{\eta = x + iy = e^{i\gamma t}}$

$$\ddot{x} + \omega^2 x = 2\Omega_z \dot{y}, \ddot{y} + \omega^2 y = -2\Omega_z \dot{x}$$
$$\gamma = -\Omega_z \pm \sqrt{\omega^2 - \Omega_z^2}$$
$$\eta(t) = a e^{-i\Omega_z t} \cos\omega t$$
$$\Rightarrow \begin{cases} x = a\cos\Omega_z t \cos\omega t \\ y = a \sin\Omega_z t \cos\omega t \end{cases}$$

Hamiltonian Mechanics

$H(q, p, t) = \sum_{j=1}^{n} p_j \dot{q}_j - L(q, \dot{q}, t)$ 1D: $H = \frac{p^2}{2m} + U(x)$

- Hamilton's equation $\dot{q}_i = \frac{\partial H}{\partial p_i} \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}$