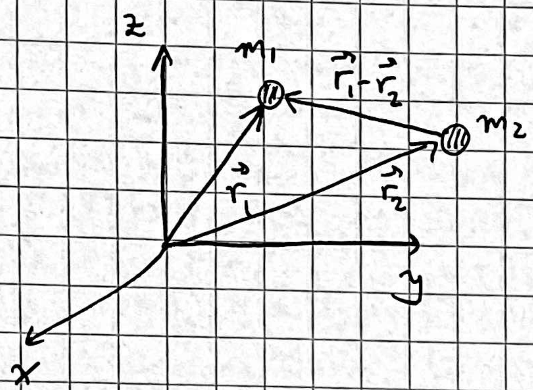
Two-body problem

02/12/24



$U = U(|\vec{r}_1 - \vec{r}_2|)$

→ form consistent with homogeneity & isotropy of space

Ex: Gravitational potential:

$$U(|\vec{r}_1 - \vec{r}_2|) = -\frac{\alpha}{|\vec{r}_1 - \vec{r}_2|}.$$

- problem with 6 DOF.

→ symmetry will reduce this to a problem in <u>one</u> DOF & allow for complete solution.

The Lagrangian:

$$L = \frac{1}{2} m_1 \dot{\vec{r}}_1^{\,2} + \frac{1}{2} m_2 \dot{\vec{r}}_2^{\,2} - U(|\vec{r}_1 - \vec{r}_2|).$$

Conservation of total momentum suggests separation into motion of COM + motion relative to COM:

$$\rightarrow \begin{cases} \vec{r} = \vec{r}_1 - \vec{r}_2 \\ \vec{R} = \dfrac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{M}, \quad M = m_1 + m_2 \end{cases}$$

& inverse transform'n:

$$\begin{cases} \vec{r}_1 = \vec{R} + \dfrac{m_2}{M} \vec{r} \\ \vec{r}_2 = \vec{R} - \dfrac{m_1}{M} \vec{r} \end{cases}$$

plugging transform.'n into L: 02/12/24

$$L = \frac{1}{2} m_1 \left(\dot{\vec{R}} + \frac{m_2}{M}\dot{\vec{r}}\right)^2 + \frac{1}{2} m_2 \left(\dot{\vec{R}} - \frac{m_1}{M}\dot{\vec{r}}\right)^2 - U(r), \quad r = |\vec{r}|$$

$$= \frac{1}{2}\underbrace{(m_1+m_2)}_{M}\dot{\vec{R}}^2 + \cancel{\frac{m_1 m_2}{M}\dot{\vec{R}}\cdot\dot{\vec{r}}} - \cancel{\frac{m_1 m_2}{M}\dot{\vec{R}}\cdot\dot{\vec{r}}} + \frac{1}{2}\underbrace{\frac{(m_1 m_2^2 + m_2 m_1^2)}{M^2}}_{m_1 m_2 \frac{(m_1+m_2)}{M^2}}\dot{\vec{r}}^2 - U(r)$$

$$= \frac{m_1 m_2}{M}$$

$\equiv \mu$ "reduced mass".

$$\Rightarrow \quad L = \underbrace{\frac{1}{2} M \dot{\vec{R}}^2}_{L_{COM}} + \underbrace{\frac{1}{2}\mu\dot{\vec{r}}^2 - U(r)}_{L_{rel.}}$$

→ we find L separates into a COM part (L_{com}) & a relative part ($L_{rel.}$)

EOM for $\vec{R}$: $\quad \frac{d}{dt}\frac{\partial L}{\partial \dot{\vec{R}}} = \frac{\partial L}{\partial \vec{R}}$

$$\Rightarrow \quad \frac{d}{dt}(M\dot{\vec{R}}) = 0 \quad \text{or} \quad M\dot{\vec{R}} = \text{const.}$$

As expected based on considerations of translation symm. & cons. of total momentum, we find COM moves w/ const. velocity

⇒ By going to COM frame we can remove this motion entirely

In COM frame:

$$L = \frac{1}{2}\mu\dot{\vec{r}}^2 - U(r)$$

→ effective one-body problem!

particle of mass μ moving in central potential $U(r)$.

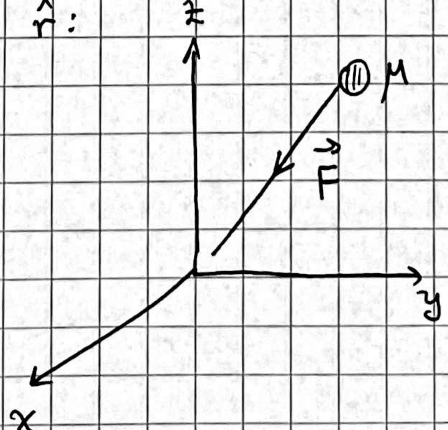
$\Rightarrow$ 6 DOF → 3 DOF.

Note the force $\vec{F} = -\frac{\partial U}{\partial \vec{r}} = -\frac{dU}{dr}\frac{\vec{r}}{r}$

Check: $F_x = -\frac{\partial U}{\partial x} \underset{\substack{U=U(r)\\ r=r(x,y,z)}}{=} -\frac{dU}{dr}\frac{\partial r}{\partial x} \underset{r=\sqrt{x^2+y^2+z^2}}{=} -\frac{dU}{dr}\frac{x}{\sqrt{x^2+y^2+z^2}} = -\frac{dU}{dr}\frac{x}{r}$ ✓

Same goes for y & z, giving formula above

$\Rightarrow$ magnitude of force depends only on r & is directed along $\hat{r}$:



We will use symmetry to simplify yet further: 02/12/24

Recall $U = U(r) \Rightarrow \vec L = \vec r \times \vec p$ is conserved as a consequence of rotational symmetry.

We can also demonstrate this explicitly:

$$\dot{\vec L} = \dot{\vec r} \times \vec p + \vec r \times \dot{\vec p}$$

$$= \mu\,\cancel{(\vec v \times \vec v)}^{=0} + \vec r \times \vec F \qquad \text{EOM} \quad \dot{\vec p} = \vec F = -\frac{\partial U}{\partial \vec r} = -\frac{dU}{dr}\frac{\vec r}{r}$$

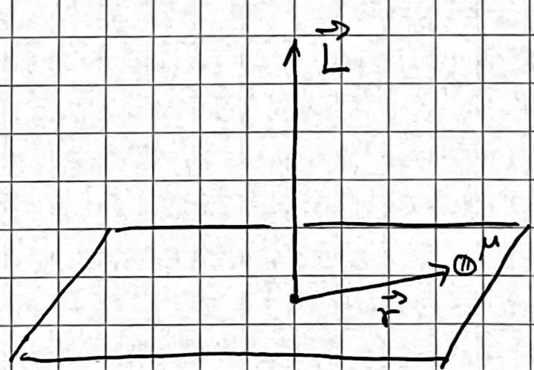
$$= \vec r \times \left(-\frac{dU}{dr}\frac{\vec r}{r}\right)$$

$$= -\frac{dU}{dr}\frac{1}{r}\,\cancel{(\vec r \times \vec r)}^{0}$$

$$= 0 \quad \checkmark$$

Now, $\vec L$ = const. vector & $\vec L = \vec r \times \vec p$ means $\vec r \perp$ to fixed direction of $\vec L$ throughout the motion.

Hence, $\vec r \in$ plane $\perp$ $\vec L$ $\Rightarrow$ motion is in 2D:



Use this to express motion in terms of 2 DOF

$\Rightarrow$ cons. of $\vec L$ takes us from 3 DOF $\to$ 2 DOF.

Choose polar coord.'s (r, φ) ~~in~~ in the 2D plane of the motion: 02/12/24

$$\cancel{T = \tfrac{1}{2}\mu v^2 = }$$

$$T = \frac{1}{2}\mu v^2 = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\varphi}^2\right)$$

$$\Rightarrow \boxed{L = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\varphi}^2\right) - U(r)}$$

$\Rightarrow$ symmetry led to a reduction of a 6 DOF problem to a problem in 2 DOF!

- Now consider E-L. eqn for φ:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\varphi}} = \frac{\partial L}{\partial \varphi} \quad \} \quad \varphi = \text{cyclic coord.}$$

$$\rightarrow \frac{d}{dt}\left(\mu r^2 \dot{\varphi}\right) = 0$$

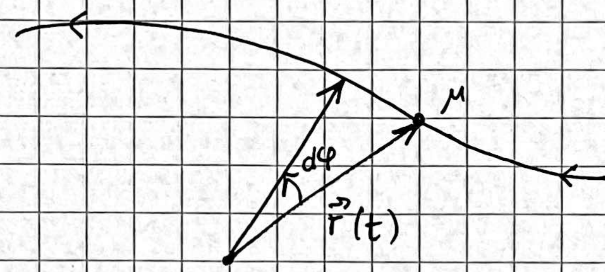
$$\Rightarrow \mu r^2 \dot{\varphi} = \text{const.}$$

$\rightarrow$ this is not a new conservation law, it is just the conservation of the magnitude of the ang. mom. $\vec{L}$.

- Choosing the z-axis to be $\perp$ ~~plane~~ to the plane of motion, we may write:

$$L_z = \mu r^2 \dot{\varphi} = \text{const.} \qquad \left(L_z = \hat{z}\cdot\vec{L} = |\vec{L}|\right).$$

Geometrical interpretation: 02/12/24



$dA = \frac{1}{2} r^2 d\varphi$ = differential area swept out by particle moving through angle $d\varphi$.

$$\Rightarrow \frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\varphi}{dt} = \frac{1}{2\mu} L_z$$

$L_z = \mu r^2 \dot{\varphi}$

$$\Rightarrow L_z = 2\mu \dot{A}, \quad \dot{A} = \text{"areal velocity"}$$

Conservation of L_z thus implies areal velocity = const.

$\Rightarrow$ radius vector sweeps out equal areas in equal times (Kepler's second law).