Brief Theory of Probability, Part 1
Survey of main ideas and equations

Contents

1 Vector algebra

1.a Coordinate Transformation

1.a.a cylindical

$$x = \rho \cos \varphi$$
$$y = \rho \sin \varphi$$
$$z = z$$

reverse

$$\rho = \sqrt{x^2 + y^2}$$
$$\cos \varphi = \frac{x}{\rho}$$
$$\sin \varphi = \frac{y}{\rho}$$

1.a.b spherical

$$x = \rho \sin \varphi \cos \theta$$
$$y = \rho \sin \varphi \sin \theta$$
$$z = \rho \cos \varphi$$

reverse

$$\rho = \sqrt{x^2 + y^2 + z^2}$$
$$\cos\varphi = \frac{z}{\rho}$$
$$\cos\theta = \frac{x}{r}$$
$$\sin\theta = \frac{y}{r}$$

1.b Dot product

- commutative
- positive definite
- distributive
- cauchy-schwarz inequality

1.c cross product

- anticommutative $\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$
- distributive $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} + \vec{w}$
- scalar mulipication
- triple scalar product $\vec{u} \cdot (\vec{v} \times \vec{w}) = (\vec{u} \times \vec{v} \cdot \vec{w})$
- triple vector product $\vec{a} \times \left(\vec{b} \times \vec{c}\right) = \left(\vec{b} \cdot \vec{a}\right)\vec{c} - (\vec{c} \cdot \vec{a})\vec{b}$

---

2 Vector calculus

2.a Are length

- Def: Given a curve $\vec{r}(u) = (x(u), y(u), z(u))$ for $a \leq t \leq b$ the length of the curve S, as a function of time is given by

$$\boxed{\begin{gathered} S(t) = \int_a^t \left\|r(\dot{u})\right\| \mathrm{d}u \\ \text{where } \|\dot{r}(u)\| = \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2} \end{gathered}}$$

- Curvature:

$$K(t) = \frac{\left\|\dot{T}(t)\right\|}{\|\dot{r}(t)\|} = \frac{\|(\dot{r}(t) \times \ddot{r}(t))\|}{(\|\dot{r}(t)\|)^3}, \text{where } T(t) = \frac{\dot{r}(t)}{\|\dot{r}(t)\|}$$

2.b Line integration

- for curve $\vec{r}(t) = (x(t), y(t))$

$$\int_C f(x(t), y(t)) \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2} \,\mathrm{d}t$$

- center of mass $(\overline{x}, \overline{y}, \overline{z})$, where

$$\begin{cases} \overline{x} = \left(\frac{1}{M}\right) \int_C \rho(x,y,z) x ds \\ \overline{y} = \left(\frac{1}{M}\right) \int_C y\rho(x,y,z) ds \\ \overline{z} = \left(\frac{1}{M}\right) \int_C z\rho(x,y,z) ds \end{cases}$$

- Work done by force F along curve, $\vec{r}(t)$, which can be generalized into the formula for line integration,

$$W = \int_C F \cdot \mathrm{d}\vec{r} = \int_C \vec{F} \cdot \vec{T} \, \mathrm{d}s = \boxed{\int_a^b F[x(t), y(t)] \cdot (\dot{r}(t)) \, \mathrm{d}t}$$

- When vector field $\vec{F} = \vec{F}(x,y,z) = (P,Q,R)$,

$$\int_C \vec{F} \cdot \mathrm{d}\vec{r} = \int_C P dx + Q dy + R dz$$

2.c Surface integration

- Parametric representation of surface:

$$\begin{cases} x = x(u,v) \\ y = y(u,v) \\ z = z(u,v) \end{cases}$$

- Use normal vector at a point (u_0, v_0) of surface to represent tangent plane.

$$\vec{r_v} = \frac{\partial \vec{r}}{\partial v}(u_0, v_0), \vec{r_u} = \frac{\partial \vec{r}}{\partial u}(u_0, v_0)$$
$$\vec{N} = \vec{r_u} \times \vec{r_v}$$

- Surface area of a surface S with $(u,v) \in D$

$$A(S) = \iint_D \|\vec{r_u} \times \vec{r_v}\| \, \mathrm{d}u \, \mathrm{d}v$$

2.d Jacobian

- Def: Given a transformation $(u,v) \in D \longrightarrow [x(u,v), y(u,v)] \in S$, the Jacobian is given by

$$J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} \equiv \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$$

- Jacobian in coordinate transformation

$$\iint_S f(x,y) \, \mathrm{d}A = \iint_D f(x(u,v), y(u,v)) \; |J(u,v)| \, \mathrm{d}u \, \mathrm{d}v$$

2.e Gradient

- Nabla operation:

$$\nabla = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}$$

- Gradient in cartesian Scalar field $f = f(x, y, z)$

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

- Gradient in polar coordinates $f = f(r, \theta)$

$$\nabla f = \vec{e_r}\frac{\partial g}{\partial r} + \vec{e_\theta}\frac{1}{r}\frac{\partial g}{\partial \theta}$$

$$\text{where } \vec{e_r} = \frac{x}{\|x\|} = (\cos\theta, \sin\theta)\vec{e_\theta} = (-\sin\theta, \cos\theta)$$

$$\nabla = \vec{e_r}\partial_r + \vec{e_\theta}\frac{1}{r}\partial_\theta$$

- Gradient in spherical

$$\nabla f = \hat{\rho}\partial_\rho + \hat{\varphi}\frac{1}{\rho}\partial_\varphi + \hat{\theta}\frac{1}{\rho\sin\varphi}\partial_\theta$$

- Gradient of scalar field in spherical coordinates

Let $g(\rho, \phi, \theta) = f(x, y, z)$

$$\begin{cases} x = \rho\sin\phi\cos\theta \\ y = \rho\sin\phi\sin\theta \\ z = \rho\cos\phi \end{cases} \qquad \begin{bmatrix} \partial_\rho g \\ \partial_\phi g \\ \partial_\theta g \end{bmatrix} = \begin{bmatrix} \partial_\rho x & \partial_\rho y & \partial_\rho z \\ \partial_\phi x & \partial_\phi y & \partial_\phi z \\ \partial_\theta x & \partial_\theta y & \partial_\theta z \end{bmatrix} \begin{bmatrix} \partial_x f \\ \partial_y f \\ \partial_z f \end{bmatrix}$$

$$\hat{\rho} = (\partial_\rho x, \partial_\rho y, \partial_\rho z) = \frac{(x, y, z)}{\rho}$$

$$\hat{\phi} = \frac{1}{\rho}(\partial_\phi x, \partial_\phi y, \partial_\phi z)$$

$$\hat{\theta} = \frac{1}{\rho\sin\phi}(\partial_\theta x, \partial_\theta y, \partial_\theta z)$$

$$\Rightarrow \begin{bmatrix} \partial_x f \\ \partial_y f \\ \partial_z f \end{bmatrix} = \begin{bmatrix} \hat{\rho}_1 & \hat{\phi}_1 & \hat{\theta}_1 \\ \hat{\rho}_2 & \hat{\phi}_2 & \hat{\theta}_2 \\ \hat{\rho}_3 & \hat{\phi}_3 & \hat{\theta}_3 \end{bmatrix} \begin{bmatrix} \partial_\rho g \\ \frac{1}{\rho}\partial_\phi g \\ \frac{1}{\rho\sin\phi}\partial_\theta g \end{bmatrix}$$

2.f Divergence

- div of vec field:

3D:

$$\nabla \cdot \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

- Div in polar 2D

$$\vec{U} = U_r\hat{r} + U_\theta\hat{\theta}, \text{where } U_r = U \cdot \hat{r}, U_\theta = U \cdot \hat{\theta}$$

$$\nabla \cdot U = \left(\frac{1}{r}\right)\frac{\partial(rU_r)}{\partial r} + \frac{\partial U_\theta}{\partial \theta}$$

- Div in sphereical coord

$$\vec{U} = U_\rho \hat{\rho} + U_\theta \hat{\theta} + U_\varphi \hat{\varphi},$$

$$\nabla \cdot \vec{U} = \frac{1}{\rho^2} \frac{\partial(\rho^2 U_\rho)}{\partial \rho} + \frac{1}{\rho} \sin\varphi \frac{\partial(U_\theta)}{\partial \theta} + \frac{1}{\rho \sin\varphi} \frac{\partial(U_\theta \sin\varphi)}{\partial \varphi})$$

2.g Green's theorem

$$\int_C P dx + Q dy = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dA = \iint_C \vec{F} \cdot \mathrm{d}\vec{r}$$

2.h Stokes' theorem

- for a surface,

$$\vec{r}(u, v) = (x(u, v), y(u, v), z(u, v))$$

$$\Rightarrow \iint_S \vec{F} \cdot \mathrm{d}\vec{S} = \iint_S \vec{F} \cdot \vec{n} \,\mathrm{d}S = \iint_D \vec{F}(\vec{r}(u, v)) \cdot (\vec{r_u} \times \vec{r_v}) \,\mathrm{d}A$$

- if the surface is a graph of a fucntion $z = g(x, y), (x, y) \in D, \vec{F} = (P, Q, R)$, then

$$\int_S \vec{F} \cdot \mathrm{d}\vec{s} = \iint_D (P, Q, R) \cdot \left(-\partial_x g, -\partial_y g, 1\right) \mathrm{d}A$$

Let $F : R^3 \to R^3$ be a vector field on R^3 , then

$$\int_C \vec{F} \cdot \mathrm{d}\vec{r} = \iint_S \mathrm{curl}\left(\vec{F}\right) \mathrm{d}\vec{s},$$

$$\text{where } \mathrm{curl}\left(\vec{F}\right) = \nabla \times \vec{F}$$