Small Oscillations

- Motion near a point of stable equilibrium.

DOF= 1 (one dimension)

- For a system of DOF = 1, with potential $U(q)$:
  - **stable equilibrium** at $U(q)_{\min}$ where $F = -\frac{\mathrm{d}U}{\mathrm{d}q} = 0$
    - restoring force for small displacements $q - q_0$ is $F = -\frac{\mathrm{d}U(q-q_0)}{\mathrm{d}q}$
- **Unstable equilibrium** at $U(q)_{\max}$ where $F = -\frac{\mathrm{d}U}{\mathrm{d}q} = 0$ as well.
- Consider small deviation from point of stable equilibrium, we use taylor expansion to show that it is really a small displacement. that is,

$$\begin{gathered} U(q) \approx U(q_0) + \frac{\mathrm{d}U(q_0)}{\mathrm{d}q}(q - q_0) + \left(\frac{1}{2}\right)\frac{\mathrm{d}^2U(q_0)}{\mathrm{d}q^2}(q - q_0)^2 + \dots \\ \text{while } \frac{\mathrm{d}U(q_0)}{\mathrm{d}q}(q - q_0) = 0 \end{gathered} \tag{1}$$

letting $x = q - q_0$, we have

$$\begin{cases} U(x) = U(q_0) + \left(\frac{1}{2}\right)\frac{\mathrm{d}^2U(q_0)}{\mathrm{d}q^2}x^2 \\ \text{also } U(x) = \left(\frac{1}{2}\right)kx^2. \end{cases} \Rightarrow \boxed{k = \frac{\mathrm{d}^2U(q_0)}{\mathrm{d}q^2} > 0} \tag{2}$$

we get KE, while choosing $U(q_0) = 0$:

$$\begin{gathered} T = \frac{1}{2}a(q)^2\dot{q}^2 = \frac{1}{2}a(q_0 + x)\dot{x}^2 \approx \frac{1}{2}m\dot{x}^2, \text{letting } m = a(q_0) \\ \Rightarrow L = T - U = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 \end{gathered} \tag{3}$$

EOM for DOF = 1 small Oscillations

using EL on Equation 3, we can get the EOM for one dimensional small Oscillations:

$$\begin{gathered} m\ddot{x} = -kx \\ \Rightarrow \ddot{x} + \omega_0^2 x = 0, \text{where } \boxed{\omega_0 = \sqrt{\frac{k}{m}} \text{ freq of osc.}} \end{gathered} \tag{4}$$

by magic of ODE, EOM reduces down to:

$$\begin{gathered} \boxed{x(t) = C_1\cos(\omega_0 t) + C_2\sin(\omega_0 t)} \\ \text{where } C_1, C_2 \text{ are constants} \end{gathered} \tag{5}$$

by trig magic, this could also be written as

$$\begin{gathered} x(t) = a\cos(\omega_0 t + \varphi), \\ \text{where } \begin{cases} a = \sqrt{C_1^2 + C_2^2} & \text{amplitude of oscillation} \\ \omega_0 & \text{frequency of oscillation} \\ \tan\varphi = C_2/C_1 & \text{phase at t=0} \end{cases} \end{gathered} \tag{6}$$

energy for 1D small Oscillation
checking $\frac{\partial L}{\partial t} = 0 \Rightarrow$ energy-conservation:

$$\begin{aligned} E = T + U &= \frac{1}{2} m\dot{x}^2 + \frac{1}{2} kx^2 \\ &= \frac{1}{2} ma^2\omega_0^2, [\text{ constant}] \end{aligned} \tag{7}$$

Damped 1D oscillation, and Complex representation
[*I dont like the how the subscripts are used in this lecture but I guess this is what we are stuck with.*]

- when there is damping (friction, resistence, etc) $F_{\text{fric}} = -\beta\dot{x}$, the EOM becomes:

$$\begin{gathered} \ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0, \\ \text{where } 2\gamma = \frac{\beta}{m}, \omega_0 = \sqrt{\frac{k}{m}} \end{gathered} \tag{8}$$

with ansatz $x(t) = e^{rt}, \dot{x} = re^{rt}, \ddot{x} = r^2 e^{rt}$, the solution to Equation 8 is:

$$\begin{gathered} r^2 + 2\gamma r + \omega_0^2 = 0, \\ \text{which has solution } r_+, r_- = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2} \\ \Rightarrow x(t) = C_1 e^{r_+ t} + C_2 e^{r_- t}, \end{gathered} \tag{9}$$

notice the r subscripts here: r_+, r_-

underdamped, overdamped, and critically damped
Recall from your ODE class...

Equation 9 has the following 3 cases, each with different physical interpretation:

1. underdamped:

$$\gamma < \omega_0 \Rightarrow 2 \text{ complex roots: } \begin{cases} r_\pm = -\gamma \pm i\sqrt{\omega_0^2 - \gamma^2} = -\gamma \pm i\omega \\ \omega = \sqrt{\omega_0^2 - \gamma^2} \end{cases} \tag{10}$$

The EOM is thus a linear combination of two complex expoentials:

$$\begin{aligned} x(t) &= e^{-\gamma t}(C_1 e^{i\omega t} + C_2 e^{-i\omega t}) \\ &= e^{-\gamma t}(A\cos(\omega t) + B\sin(\omega t)) \\ &\text{-- said Euler,where } \begin{cases} A = C_1 + C_2 \\ B = i(C_1 - C_2) \end{cases} \\ &= ae^{-\gamma t}\cos(\omega t + \alpha) \\ &a, \alpha \text{ are constants} \end{aligned} \tag{11}$$

The solution is a damped oscillation with frequency ω, and amplitude expoentially decaying with time.