Summary

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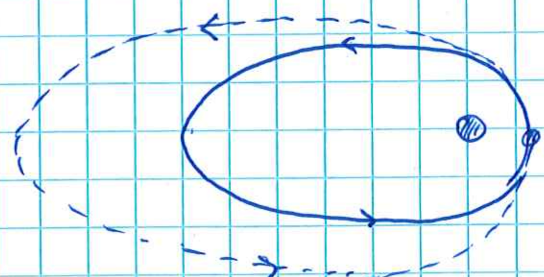
· Kepler $U = -\frac{\alpha}{r}$. shape of orbits:

$$r(\varphi) = \frac{p}{1+e\cos\varphi} \qquad \begin{cases} p = \dfrac{L_z^2}{\mu\alpha} \\ e = \sqrt{1+\dfrac{2EL_z^2}{\mu\alpha^2}} \end{cases}$$

Application: orbital transfer.

→ Q: How does a satellite change from one orbit to another?



· orbit characterized by eccentricity e & latus-rectum p, which are determined by energy E & ang. mom. L_z.

· one way to change orbit is via thrust (= rapid impulse)

→ instantaneous (approximately) change in velocity

$$(E, L_z) \longrightarrow (E', L_z')$$

$$\Rightarrow (e, p) \longrightarrow (e', p')$$

If thrust occurs when satellite is at angle φ_0, then

$$\frac{p}{1+e\cos\varphi_0} = \frac{p'}{1+e'\cos(\varphi_0-\delta)}$$

possible change in orientation of orbit

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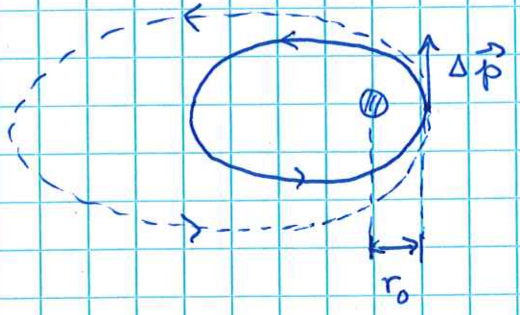
Ex: (Tangential thrust at perigee).



- perigee = pt of closest approach to Earth.
- assume tangential thrust

$\Rightarrow$ orientation of orbit unchanged $(\delta = 0)$

$$\Rightarrow \text{ at } \varphi = 0: \quad \frac{p}{1+e} = \frac{p'}{1+e'} \quad (*).$$

- Let v = speed just before thrust & v' = speed right after.

Let $\lambda = v'/v$. We have:

$$L_z = \mu r_0 v$$

$$\rightarrow L_z' = \mu r_0 v' = \lambda \mu r_0 v = \lambda L_z$$

$$\Rightarrow p' = \lambda^2 p.$$

- From $(*)$:

$$\frac{\cancel{p}}{1+e} = \frac{\lambda^2 \cancel{p}}{1+e'}$$

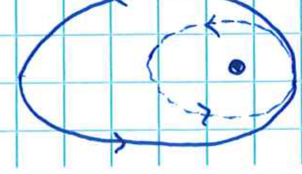
$$\Rightarrow e' = \lambda^2(1+e) - 1$$

= eccentricity of new orbit in terms of $\lambda = v'/v$.

(i) If ~~v~~ $\lambda > 1$ (forward thrust) then $e' > e$

For λ large enough, $e' > 1$ & new orbit unbounded.

(ii) If $\lambda < 1$ (backward thrust) then $e' < e$:



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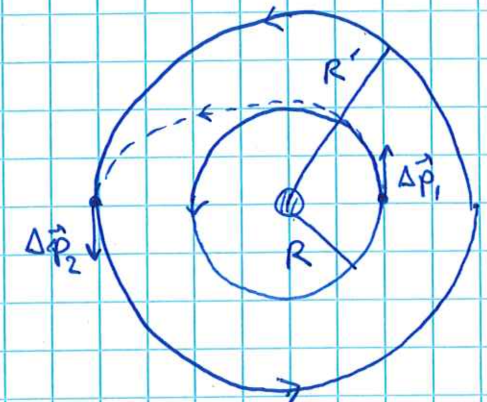
<u>Ex</u>: (Changing btwn. circular orbits)

- Transfer btwn. two circular orbits w/ radii R & R' can be accomplished by sequence of boosts.



<u>Q</u>: what are required thrust factors λ_1 & λ_2? (Given R & R').

- initial orbit $r(\varphi) = R$, $e=0$, $p=R$.
- intermediate orbit $r(\varphi) = \dfrac{p'}{1+e'\cos\varphi}$ s.t. $p' = \lambda_1^2 p$ & $e' = \lambda_1^2(e+1)-1 = \lambda_1^2 - 1$ $(e=0)$

λ_1 = thrust factor for first boost.

→ want $r(\varphi=\pi) = R' = \dfrac{\lambda_1^2 R}{1-(\lambda_1^2-1)} = \dfrac{\lambda_1^2 R}{2-\lambda_1^2}$

$$\Rightarrow \frac{R'}{R} = \frac{\lambda_1^2}{2-\lambda_1^2} \Rightarrow \lambda_1 = \sqrt{\frac{2R'}{R+R'}}$$

- final orbit $r(\varphi) = R'$, $e''=0$, $p''=R'$.

$$\Rightarrow p'' = \frac{p'}{1-e'} \quad \text{\&} \quad p'' = \lambda_2^2 p'$$

$$\lambda_2^2 = \frac{1}{1-e'} = \frac{1}{2-\lambda_1^2} \Rightarrow \lambda_2 = \sqrt{\frac{R+R'}{2R}}$$

Check: what do we expect for ratio of speeds in circular orbits? 02/21/24

(orbit w/ radius R): $\frac{\alpha}{R^2} = \frac{\mu v_R^2}{R}$ (force = centripetal accel.)

$$\Rightarrow v_R = \sqrt{\frac{\alpha}{\mu R}}$$

$$\Rightarrow \frac{v_{R'}}{v_R} = \sqrt{\frac{R}{R'}}$$

→ Does this agree with what we found?

Let $v(\varphi)$ = speed of intermediate orbit (it is a fn. of φ).

We have: $v_{R'} = \lambda_2 v(\varphi=\pi) = \lambda_2 \frac{v(\varphi=\pi)}{v(\varphi=0)} v(\varphi=0) = \lambda_2 \frac{v(\varphi=\pi)}{v(\varphi=0)} \lambda_1 v_R$.

& cons. of ang. mom.: $R v(\varphi=0) = R' v(\varphi=\pi)$

$$\Rightarrow \frac{v_{R'}}{v_R} = \lambda_2 \left(\frac{R}{R'}\right) \lambda_1$$

$$= \sqrt{\frac{R+R'}{2R}} \left(\frac{R}{R'}\right) \sqrt{\frac{2R'}{R+R'}}$$

$$= \sqrt{\frac{R}{R'}} \quad \checkmark.$$

Small oscillations

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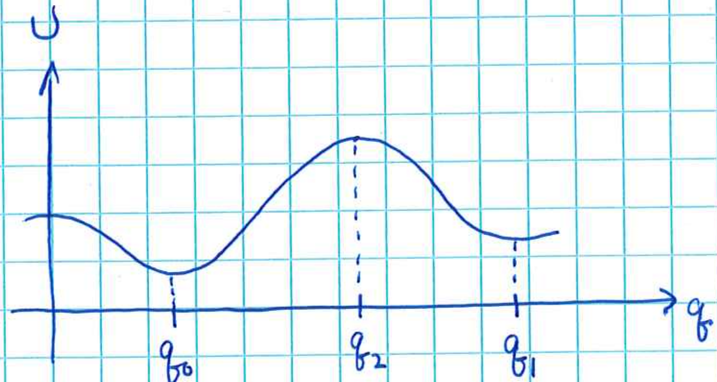
→ motion near a point of stable equilibrium.

One dimension:

Consider system w/ 1 DOF q, moving in potential $U(q)$.



- stable equil. = min. of $U(q)$.

since $F = -\frac{dU}{dq} = 0$ & there is a restoring force for small displacements (e.g., q_0 & q_1).

- unstable equil. = max. of $U(q)$

force pushes away from equil. (e.g., q_2).

- Consider small deviations from pt. of stable equil. q_0:

$$U(q) = U(q_0) + \cancelto{0}{\left.\frac{dU}{dq}\right|_{q_0}} (q-q_0) + \frac{1}{2} \left.\frac{d^2U}{dq^2}\right|_{q_0} (q-q_0)^2 + \dots$$

Let $x = q - q_0$: $\quad U(x) = U(q_0) + \frac{1}{2} k x^2, \quad k = \left.\frac{d^2U}{dq^2}\right|_{q_0} > 0.$

- kinetic energy: $T = \frac{1}{2} a(q) \dot{q}^2 = \frac{1}{2} a(q_0 + x) \dot{x}^2 \simeq \frac{1}{2} m \dot{x}^2, \quad m = a(q_0)$

$$\Rightarrow \boxed{L = T - U = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2} \quad (\text{choosing } U(q_0) = 0).$$