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Summary

- Forced oscillations w/ damping:

$$m\ddot{x} = -kx - \beta\dot{x} + F(t) \;\longrightarrow\; \ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = f(t) \qquad \begin{cases} \omega_0 = \sqrt{k/m} \\ 2\gamma = \beta/m \\ f = F/m \end{cases}$$

- Sol.'s of form: $x(t) = x_p(t) + x_h(t)$

x_p = "particular sol.'n"

x_h = "homog. sol.'n" $\longrightarrow \ddot{x}_h + 2\gamma\dot{x}_h + \omega_0^2 x_h = 0$.

- Important example: $f(t) = f_0 \cos \Omega t$.

$$x(t) \longrightarrow z(t) \in \mathbb{C}$$

$$\ddot{z} + 2\gamma\dot{z} + \omega_0^2 z = f_0 e^{i\Omega t}.$$

$$\rightarrow z(t) = z_0 e^{i\Omega t}$$

$$z_0 = \frac{f_0}{\omega_0^2 - \Omega^2 + 2i\gamma\Omega}$$

$$= a(\Omega)\, e^{i\delta(\Omega)} f_0$$

$$a(\Omega) = \frac{1}{\sqrt{(\omega_0^2 - \Omega^2)^2 + 4\gamma^2\Omega^2}}, \qquad \tan\delta(\Omega) = \frac{2\gamma\Omega}{\Omega^2 - \omega_0^2}$$

- Eqn. we want to solve: $\ddot{x} + 2\gamma\dot{x} + \omega_0^2 z = f_0 \cos \Omega t$.

$\Rightarrow$ a particular sol.'n is:

$$\boxed{x_p(t) = \mathrm{Re}\, z(t) = a(\Omega) f_0 \cos(\Omega t + \delta(\Omega))}$$

$\rightarrow$ study properties of sol.'n as a fn. of drive freq. Ω.

Focus on the case $\gamma < \omega_0$.

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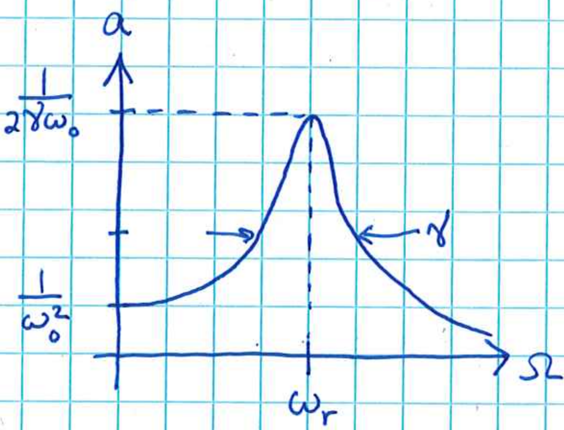
· "amplitude" of oscillation $a(\Omega)$.

$$a(\Omega) = \frac{1}{\sqrt{(\omega_0^2-\Omega^2)^2+4\gamma^2\Omega^2}}$$ → when $\gamma \ll \omega_0$, response strongest (amplitude largest) when $\Omega \simeq \omega_0$.

More precisely: max $a(\Omega)$ from $\frac{da}{d\Omega} = 0$.

$$\Rightarrow \Omega = \sqrt{\omega_0^2 - 2\gamma^2} \equiv \omega_r$$

for $\gamma \ll \omega_0$, $\omega_r \simeq \omega_0$.

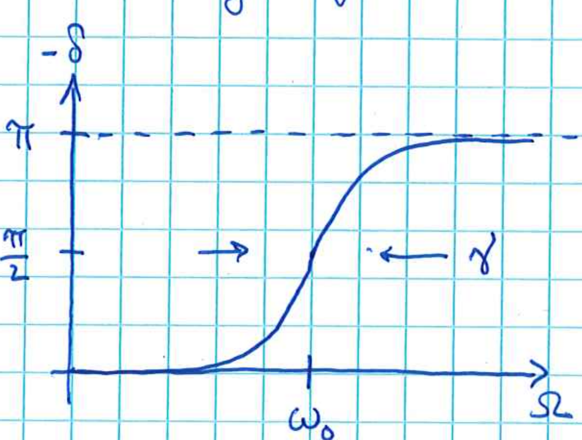


· ω_r = "resonance frequency"

→ freq. at which amp. of forced oscillations is largest.

· width of resonance set by damping γ.

· "Phase lag" of osc. $\delta(\Omega)$.



$$\tan\delta(\Omega) = \frac{2\gamma\Omega}{\Omega^2-\omega_0^2}$$

· $-\pi < \delta \leq 0$ & response "lags" behind applied force

· as $\Omega \to 0$, response nearly in phase. as $\Omega \to \infty$, perfectly out of phase ($\delta \to -\pi$).

General sol.'n has homog. part: 02/28/24

$$x(t) = \underbrace{a(\Omega) f_0 \cos(\Omega t + \delta(\Omega))}_{\text{particular}} + \underbrace{a_0 e^{-\gamma t} \cos(\omega t + \alpha)}_{\text{homog., } a_0 \,\&\, \alpha \text{ determined by initial conditions}}$$

After sufficient time $t \gg 1/\gamma$, homog. part decays away & system "forgets" initial cond.'s. Homog. part also called "transient sol.'n".

$$\Rightarrow \quad x(t) \xrightarrow[t \gg 1/\gamma]{} a(\Omega) f_0 \cos(\Omega t + \delta(\Omega)).$$

Power absorbed by oscillator:

$$dW = F\,dx$$

$$\Rightarrow \quad P = \dot{W} = F\dot{x} = mf\dot{x}$$

avg. power per cycle $\left(T = \frac{2\pi}{\Omega}\right)$:

$$\overline{P} = \frac{1}{T}\int_0^T dt \; mf\dot{x}$$

$$= \frac{-mf_0^2 \Omega^2 a(\Omega)}{2\pi} \underbrace{\left[\int_0^T dt \; \cos(\Omega t) \sin(\Omega t + \delta)\right]}_{= \frac{\pi \sin\delta}{\Omega}}$$

$$= -\frac{1}{2} m f_0^2 a(\Omega) \Omega \sin\delta$$

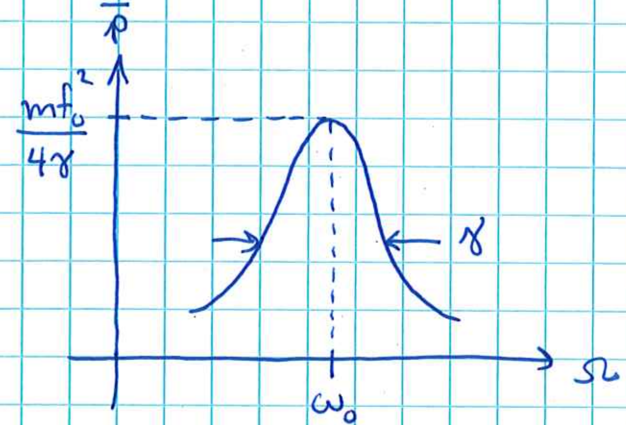
$$\sin\delta = -2\gamma\Omega a(\Omega) \quad \Rightarrow \quad \overline{P}(\Omega) = \gamma m f_0^2 \Omega^2 a^2(\Omega).$$

Interesting to consider absorption near resonance: (02/28/24)

$$\Omega = \omega_0 + \varepsilon \quad \Rightarrow \quad a(\Omega) \simeq \frac{1}{\sqrt{4\omega_0^2\varepsilon^2 + 4\gamma^2\omega_0^2}} = \frac{1}{2\omega_0}\frac{1}{\sqrt{\varepsilon^2+\gamma^2}}$$

$$\Rightarrow \bar{P}(\varepsilon) \simeq \frac{\gamma m f_0^2}{4(\varepsilon^2+\gamma^2)}$$

Lorentzian line shape.

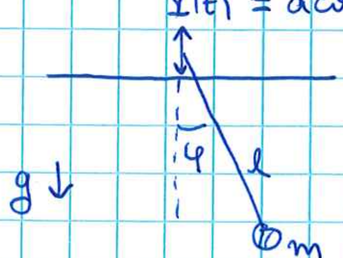


Parametric resonance (Bonus topic - will not be tested).

· Consider the problem of oscillatory system in which external action corresponds to a time variation of the parameters.

Ex: (Simple pendulum w/ oscillating support)

$Y(t) = a\cos\Omega t$



$$\begin{cases} x = l\sin\varphi \\ y = Y + l\cos\varphi \end{cases} \qquad \begin{cases} \dot{x} = l\dot{\varphi}\cos\varphi \\ \dot{y} = \dot{Y} - l\dot{\varphi}\sin\varphi \end{cases}$$

$$\Rightarrow T = \frac{1}{2}m(\dot{x}^2+\dot{y}^2) = \frac{1}{2}m\Big(l^2\dot{\varphi}^2 + \underbrace{\dot{Y}^2}_{\text{may be omitted from } L} \underbrace{- 2l\dot{Y}\dot{\varphi}\sin\varphi}\Big)$$

$$= \underbrace{\frac{d}{dt}\left(2l\dot{Y}\cos\varphi\right)}_{\text{may be omitted from } L} - 2l\ddot{Y}\cos\varphi$$

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$$\Rightarrow \quad T = \frac{1}{2} m l^2 \dot{\varphi}^2 - ml\ddot{Y}\cos\varphi$$

$$= \frac{1}{2} m l^2 \dot{\varphi}^2 + mla\Omega^2 \cos\Omega t \cos\varphi \qquad Y(t) = a\cos\Omega t.$$

$$\& \quad U = -mg(l\cos\varphi + Y)$$

Y may be dropped from L.

$$\Rightarrow \quad L = \frac{1}{2} m l^2 \dot{\varphi}^2 + mgl\left(1 + \frac{a\Omega^2}{g}\cos\Omega t\right)\cos\varphi.$$

φ ~~…~~ near zero: $\quad L = \frac{1}{2} m l^2 \dot{\varphi}^2 - \frac{1}{2} mgl\left(1 + \frac{a\Omega^2}{g}\cos\Omega t\right)\varphi^2$

$$\begin{pmatrix} \cos\varphi \simeq 1 - \frac{1}{2}\varphi^2 \\ \sin\varphi \simeq \varphi \end{pmatrix}$$

E-L. eqn: $\quad \frac{d}{dt}\frac{\partial L}{\partial \dot{\varphi}} = \frac{\partial L}{\partial \varphi}$

$$\Rightarrow \quad ml^2\ddot{\varphi} = -mgl\left(1 + \frac{a\Omega^2}{g}\cos\Omega t\right)\varphi.$$

$$\Rightarrow \quad \ddot{\varphi} + \omega^2(t)\varphi = 0, \qquad \omega^2(t) = \omega_0^2\left(1 + \frac{a\Omega^2}{g}\cos\Omega t\right)$$

$\to$ problem of harmonic oscillator w/ t-dep. natural freq.

The problem we'll consider is thus:

$$\ddot{x} + \omega^2(t)\, x = 0 \qquad (*).$$

& we'll suppose $\omega(t)$ is periodic $\omega(t+T) = \omega(t)$, $T = \frac{2\pi}{\Omega}$.

<u>Q</u>: For what values of Ω, if any, can we obtain resonance phenomena?, i.e., large amplitude oscillations.

$\to$ "parametric resonance".

· In fact, one can show (see Landau & Lifshitz §27) that the sol.'s to (*) are in general either oscillatory in time <u>or</u> have the form:

$$x(t) = (\text{exponentially growing}) \times (\text{oscillatory})$$

→ most interesting case, corresponds to parametric resonance.

· illustrate this w/ specific example:

$$\omega^2(t) = \omega_0^2 (1 + h \cos \Omega t).$$

& suppose $h \ll 1$.

EOM: $$\ddot{x} + \omega_0^2 (1 + h \cos \Omega t) x = 0.$$

$$\Rightarrow \ddot{x} + \omega_0^2 x = -h \omega_0^2 \cos(\Omega t)\, x. \qquad (**)$$

[c.f. $\ddot{x} + \omega_0^2 x = f_0 \cos \Omega t$, forced oscillations.

In the present case, appearance of x on RHS dramatically changes behavior.]

<u>Idea:</u> We expect ~~resonance etc. occur~~ strongest response when system oscillating near natural freq $\approx \omega_0$. So suppose $x \sim \cos \omega_0 t$. Then RHS of (**) is:

$$\cos(\Omega t) \cos(\omega_0 t) \sim \cos[(\Omega + \omega_0) t] + \cos[(\Omega - \omega_0) t]$$

→ this will drive further oscillations at ω_0 if $\Omega \approx 2\omega_0$, so that $\Omega - \omega_0 \approx \omega_0$.