Q1	Q2	Q3	Q4	Q5	Q6	Q7	Total (out of 25)
3.0	3.0	3.0	3.0	3.0	3.0	4.0	22.0

hl2248_ipynb

May 3, 2023

[1]:
```python
import numpy as np

def syntheticY(n):
    """
    Synthetic rank-2 n x n matrix
    """
    U = np.vstack((np.arange(n)/n, (1-np.arange(n)/n)**2)).T
    W = np.vstack((np.arange(n)/n, (1-np.arange(n)/n)**2)).T
    return U @ W.T

def bernoulli(n,p):
    """
    Bernoulli sampling.
    Produces a set of index pairs I = {(i1,i2)} where
    i1,i2 = 0,...,n-1, and each pair is sampled with
    probability p
    """
    I1, I2 = np.meshgrid(np.arange(n), np.arange(n))
    I1 = I1.flatten()
    I2 = I2.flatten()
    I = []
    for i1,i2 in zip(I1,I2):
        if (np.random.rand()<p):
            I.append([i1,i2])
    return np.array(I)

def Loss(U,W,Y,I):
    """
    Loss function (2), where
    U, W are n x r factor matrices,
    Y is a n x n data matrix,
    and I is a m x 2 matrix of index pairs
    """
    L = 0
    for i1,i2 in I:
        L = L + (U[i1] @ W[i2] - Y[i1,i2])**2
    return L / I.shape[0]

def initial(n,r):
    """
    Initial guess of the matrix factors
```

```
    """
    U0 = np.vstack((np.identity(r), np.zeros((n-r,r))))
    W0 = np.vstack((np.identity(r), np.zeros((n-r,r))))
    return U0, W0


def test_pointwise_gradient_1():
    """
    Unit tests for the pointwise gradient
    """
    Y = syntheticY(4)
    U,W = initial(4,2)
    vu = np.array([0,-1.125])
    vw = np.array([-1.125,0])
    assert np.array_equal(pointwise_gradient(U,W,Y,0,1), [vu, vw])

def test_pointwise_gradient_2():
    Y = syntheticY(4)
    U,W = initial(4,2)
    vu = np.array([0,0])
    vw = np.array([-0.5,0])
    assert np.array_equal(pointwise_gradient(U,W,Y,0,2), [vu, vw])

def test_pointwise_gradient_3():
    Y = syntheticY(4)
    U,W = initial(4,2)
    vu = np.array([-0.5,0])
    vw = np.array([0,0])
    assert np.array_equal(pointwise_gradient(U,W,Y,2,0), [vu, vw])


def test_full_gradient_1():
    """
    Unit tests for the full gradient
    """
    Y = syntheticY(4)
    U,W = initial(4,2)
    Gu = np.array([[0,-1.125], [0,     0], [0,0], [0,0]])
    Gw = np.array([[0,     0], [-1.125,0], [0,0], [0,0]])
    assert np.array_equal(full_gradient(U,W,Y, np.array([[0,1]])), [Gu, Gw])

def test_full_gradient_2():
    Y = syntheticY(4)
    U,W = initial(4,2)
    Gu = np.array([[0,    0], [-9/16,0], [0,    0], [0,0]])
    Gw = np.array([[0,-9/16], [0,    0], [-1/4,0], [0,0]])
    assert np.array_equal(full_gradient(U,W,Y, np.array([[1,0], [0,2]])),
 ↪[Gu, Gw])

def test_full_gradient_3():
    Y = syntheticY(4)
```

```
    U,W = initial(4,2)
    Gu = np.array([[0,-9/32], [-9/32,0], [-1/8,0], [0,0]])
    Gw = np.array([[0,-9/32], [-9/32,0], [0,   0], [0,0]])
    assert np.array_equal(full_gradient(U,W,Y, np.array([[1,0], [0,1],
 ↪[2,0], [0,0]])), [Gu, Gw])
```

1 Question 1

[2]:
```
def pointwise_gradient(U,W,Y,i1,i2):
    # compute formulas (5) and (6) in the assignment
    vu = 2*(U[i1,:] @ W[i2,:].T - Y[i1,i2])* W[i2,:]
    vw = 2*(U[i1,:] @ W[i2,:].T - Y[i1,i2])* U[i1,:]
    return vu,vw
```

✓

2 Question 2

[3]:
```
def full_gradient(U,W,Y,I):
    # initialise Gu and Gw to same dimensions as U and W
    Gu = np.zeros((U.shape[0],U.shape[1]))
    Gw = np.zeros((U.shape[0],U.shape[1]))
    # iterate over samples
    for i in range(I.shape[0]):
        i1 = I[i,0]
        i2 = I[i,1]
        # compute formulas (5) and (6) in the assignment
        [vu,vw] = pointwise_gradient(U,W,Y,i1,i2)
        # increment ith row of Gu and Gw
        Gu[i1,:] = Gu[i1,:]+vu
        Gw[i2,:] = Gw[i2,:]+vw
    Gu=Gu/I.shape[0]
    Gw=Gw/I.shape[0]
    return Gu, Gw
```

✓

3 Question 3

[4]:
```
from matplotlib import pyplot as plt
def gd(U0, W0, Y, I, Itest, t=1,eps=1e-6, K=100):
    #initialise starting values and arrays for iterations
    U = U0
    W = W0
    LossVals = np.zeros(0)
    kdagg = Loss(U,W,Y,Itest)
    for k in range(K):
        # save loss for log plot
        LossVals = np.append(LossVals,Loss(U,W,Y,Itest))
        # is loss sufficiently accurate?
        if (LossVals[k] < eps):
            print("converged in "+str(k)+" iterations")
```

```
            # if so exit iteration loop
            break
        if (LossVals[k] > 100):
            print('loss value at iteration '+str(k)+' is too big, stop algorithm t='+str(t))
            break
        # if not then calulate a sufficient update for U and W
        Gu,Gw = full_gradient(U,W,Y,I)
        U = U - t*Gu
        W = W - t*Gw
    print('the loss value at '+str(k)+' was '+str(LossVals[k])+' for t='+str(t))
    karray = np.arange(0,k+1)
    # plot k iterations on x-axis and loss values for each k on y-axis
    plt.plot(karray, LossVals, label = 't='+str(t)+'')
    # set the y-axis to log scale
    plt.yscale('log')
    # return U and W which minimise the loss function to sufficient accuracy
    return U,W
```

4 Question 4

Ran 3 times and then computed the average for reproduceability (Note: using different testing and training sets for each run)

[5]:

```
# function to create I, Itest
def createIs(Iext):
    # shuffle an array of indicies of length Iext
    ind = np.arange(Iext.shape[0])
    np.random.shuffle(ind)
    # shuffle data pairs according to suffled indexes
    Irand = Iext[ind]
    # select first mu pairs to be the testing data set
    Itest = Irand[:mu,:]
    # select the remaining pairs to be the I data set
    I = Irand[Itest.shape[0]:Irand.shape[0]]
    return I, Itest
```

[6]:

```
# initialising all variables required in description of question

np.random.seed(0)
n=32
p=1
r = 2
mu = 20
Y = syntheticY(n)
Iext = bernoulli(n,p)
U0,W0=initial(n,r)
I, Itest = createIs(Iext)
K = 5000
```

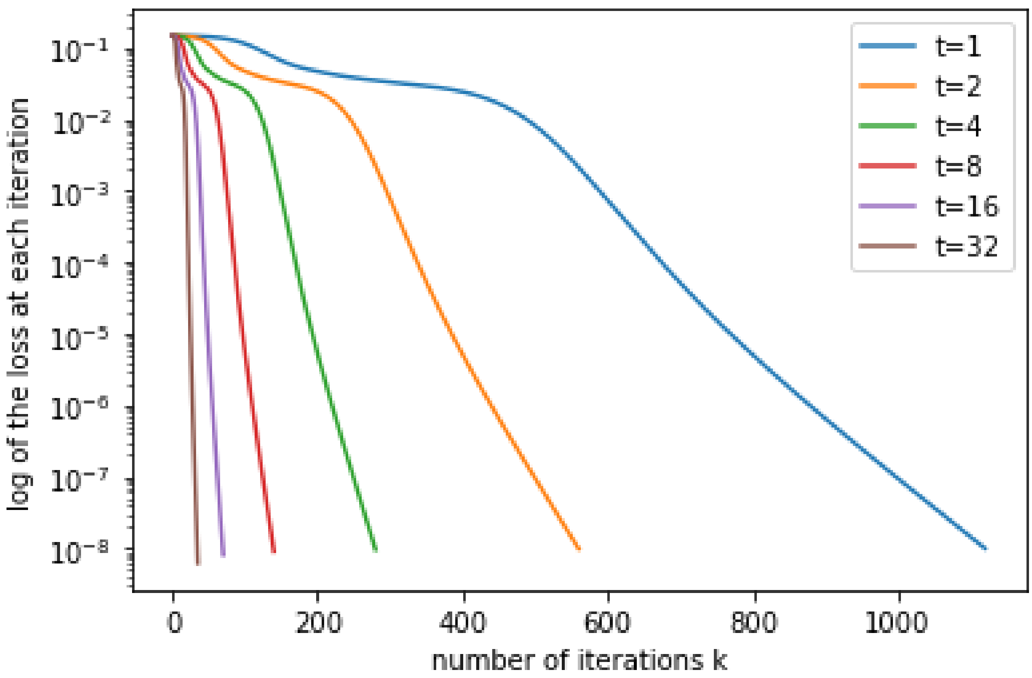
```
eps = 1e-8
t = [1,2,4,8,16,32]
```

[7]:

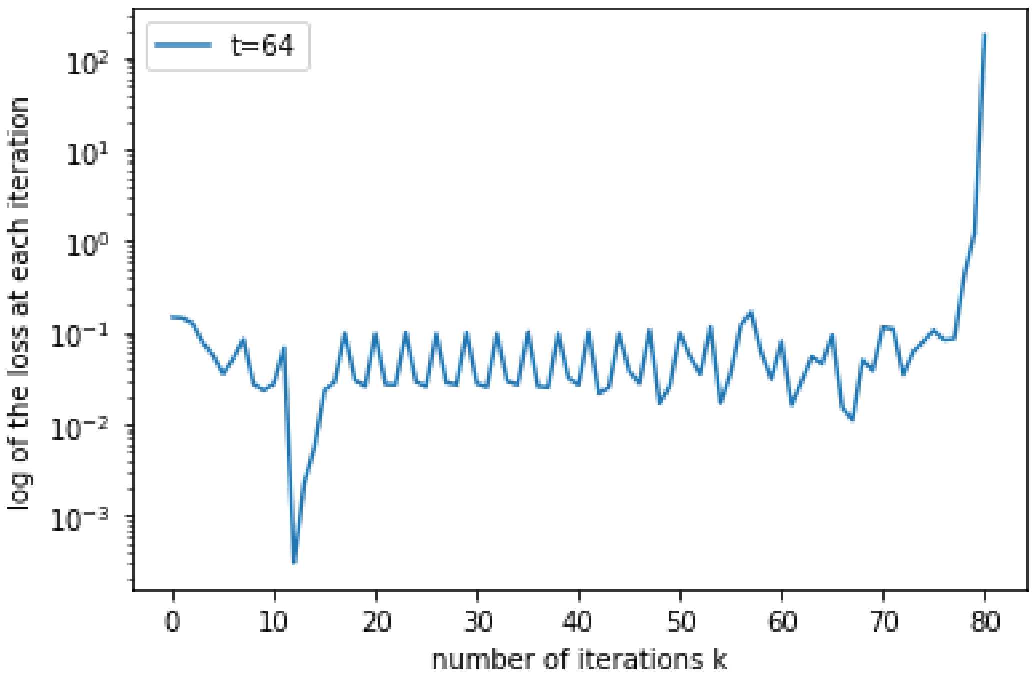
```
# 1st run

# plotting log of loss against iterations k
frobnormsave = []
for i in t:
    U,W = gd(U0, W0, Y, I, Itest, i,eps, K)
    # calcuting the frobenius norm and saving it for each t
    frobnorm = np.linalg.norm(U @ W.T - Y)/np.linalg.norm(Y)
    frobnormsave = np.append(frobnormsave,frobnorm)
# titles for plots
plt.ylabel('log of the loss at each iteration')
plt.xlabel('number of iterations k')
plt.title('Gradient Descent (GD) method for Matrix Completion (run 1)')
plt.legend()
plt.show()
# plotting t = 64 seperately due to different resolution
U,W = gd(U0, W0, Y, I, Itest, 64,eps, K)
frobnorm = np.linalg.norm(U @ W.T - Y)/np.linalg.norm(Y)
frobnormsave = np.append(frobnormsave,frobnorm)
plt.ylabel('log of the loss at each iteration')
plt.xlabel('number of iterations k')
plt.title('Gradient Descent (GD) method for Matrix Completion (run 1)')
plt.legend()
plt.show()
```

✓

```
converged in 1120 iterations
the loss value at 1120 was 9.980461676684319e-09 for t=1
converged in 560 iterations
the loss value at 560 was 9.853979004444943e-09 for t=2
converged in 280 iterations
the loss value at 280 was 9.60203115704594e-09 for t=4
converged in 140 iterations
the loss value at 140 was 9.102122493833053e-09 for t=8
converged in 70 iterations
the loss value at 70 was 8.116977336146316e-09 for t=16
converged in 35 iterations
the loss value at 35 was 6.190982801618545e-09 for t=32
```



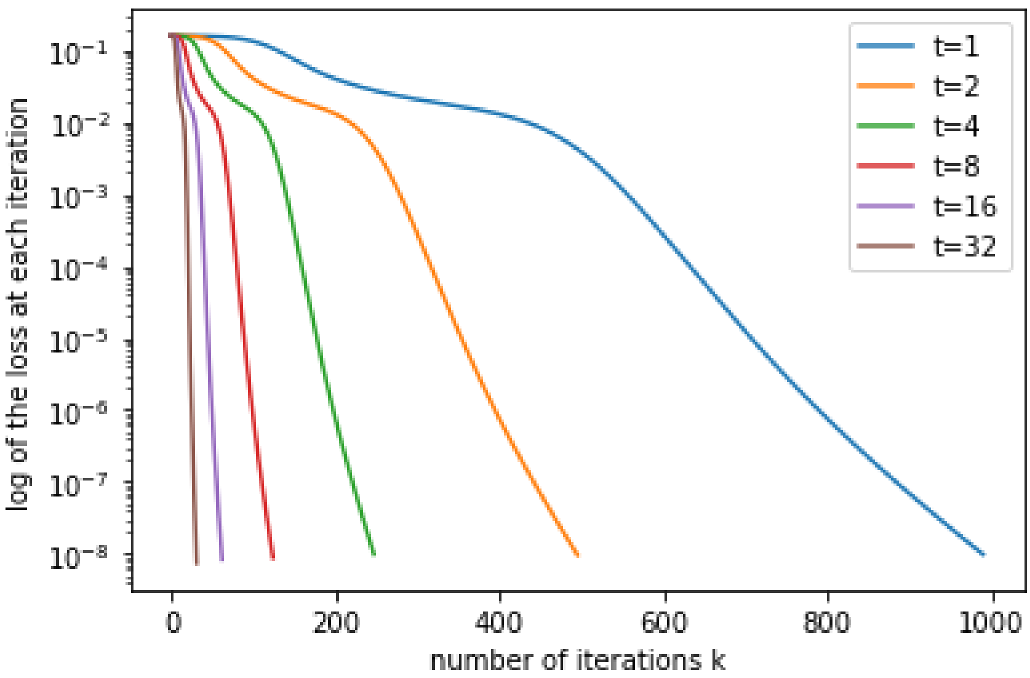

```
loss value at iteration 80 is too big, stop algorithm t=64
the loss value at 80 was 179.00657256646565 for t=64
```



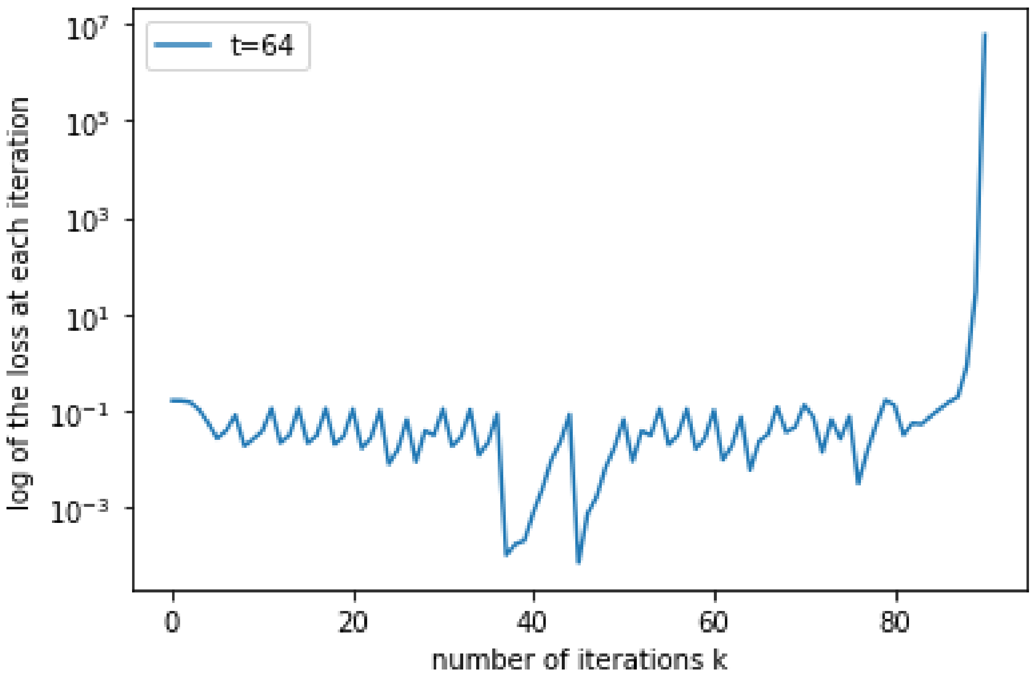
[8]:
```
# second run
I, Itest = createIs(Iext)
frobnormsave1 = []
```

```python
for i in t:
    U,W = gd(U0, W0, Y, I, Itest, i,eps, K)
    # calcuting the frobenius norm and saving it for each t
    frobnorm = np.linalg.norm(U @ W.T - Y)/np.linalg.norm(Y)
    frobnormsave1 = np.append(frobnormsave,frobnorm)
# titles for plots
plt.ylabel('log of the loss at each iteration')
plt.xlabel('number of iterations k')
plt.title('Gradient Descent (GD) method for Matrix Completion (run 2)')
plt.legend()
plt.show()
# plotting t = 64 seperately due to different resolution
U,W = gd(U0, W0, Y, I, Itest, 64,eps, K)
frobnorm = np.linalg.norm(U @ W.T - Y)/np.linalg.norm(Y)
frobnormsave1 = np.append(frobnormsave,frobnorm)
plt.ylabel('log of the loss at each iteration')
plt.xlabel('number of iterations k')
plt.title('Gradient Descent (GD) method for Matrix Completion (run 2)')
plt.legend()
plt.show()
```

```
converged in 989 iterations
the loss value at 989 was 9.895551369911352e-09 for t=1
converged in 495 iterations
the loss value at 495 was 9.63206813686197e-09 for t=2
converged in 247 iterations
the loss value at 247 was 9.926279041326018e-09 for t=4
converged in 124 iterations
the loss value at 124 was 8.888184955487948e-09 for t=8
converged in 62 iterations
the loss value at 62 was 8.41468041533058e-09 for t=16
converged in 31 iterations
the loss value at 31 was 7.421879312266183e-09 for t=32
```



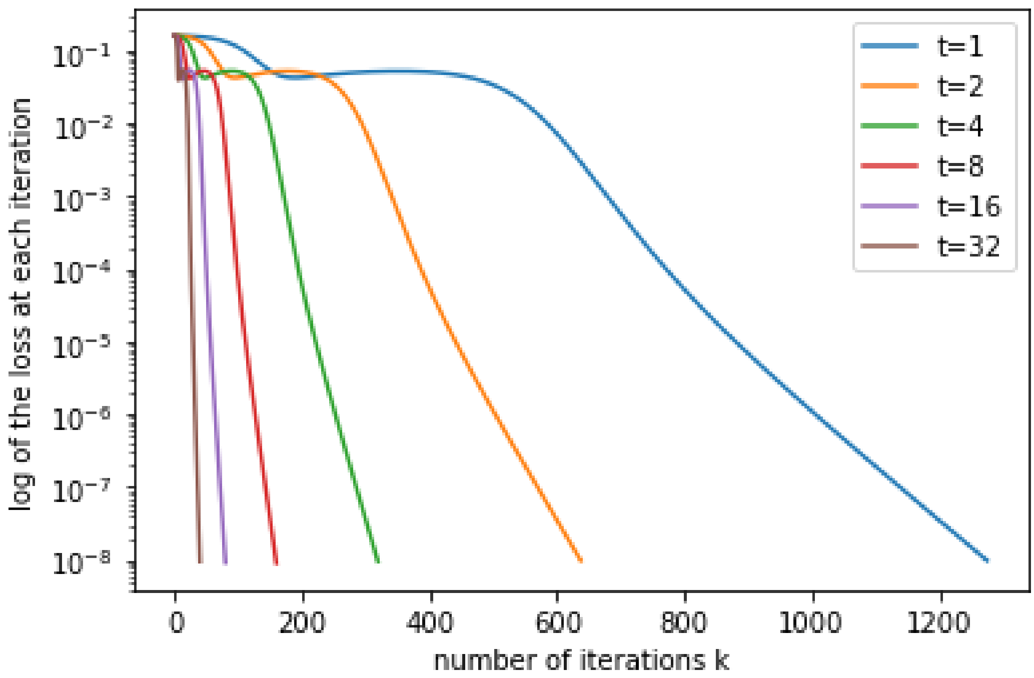

```
loss value at iteration 90 is too big, stop algorithm t=64
the loss value at 90 was 5951762.95178904 for t=64
```



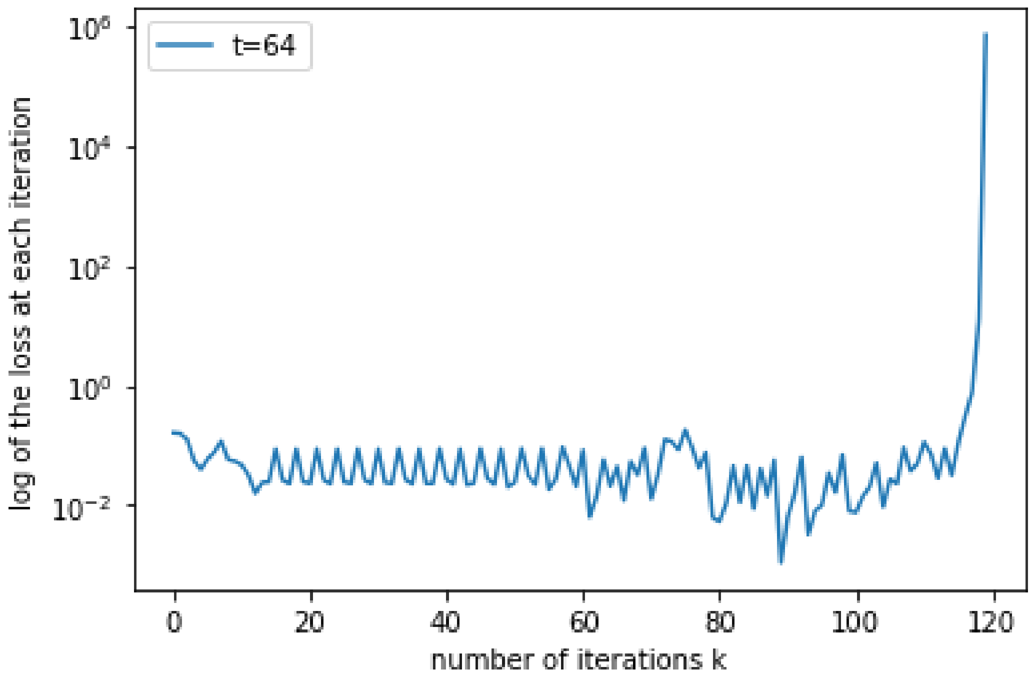
[9]:
```
# third run
I, Itest = createIs(Iext)
frobnormsave2 = []
```

```python
for i in t:
    U,W = gd(U0, W0, Y, I, Itest, i,eps, K)
    # calcuting the frobenius norm and saving it for each t
    frobnorm = np.linalg.norm(U @ W.T - Y)/np.linalg.norm(Y)
    frobnormsave2 = np.append(frobnormsave,frobnorm)
# titles for plots
plt.ylabel('log of the loss at each iteration')
plt.xlabel('number of iterations k')
plt.title('Gradient Descent (GD) method for Matrix Completion (run 3)')
plt.legend()
plt.show()
# plotting t = 64 seperately due to different resolution
U,W = gd(U0, W0, Y, I, Itest, 64,eps, K)
frobnorm = np.linalg.norm(U @ W.T - Y)/np.linalg.norm(Y)
frobnormsave2 = np.append(frobnormsave,frobnorm)
plt.ylabel('log of the loss at each iteration')
plt.xlabel('number of iterations k')
plt.title('Gradient Descent (GD) method for Matrix Completion (run 3)')
plt.legend()
plt.show()
```

```
converged in 1274 iterations
the loss value at 1274 was 9.855748448498271e-09 for t=1
converged in 637 iterations
the loss value at 637 was 9.882654551517296e-09 for t=2
converged in 319 iterations
the loss value at 319 was 9.602505517800192e-09 for t=4
converged in 160 iterations
the loss value at 160 was 9.051379183888803e-09 for t=8
converged in 80 iterations
the loss value at 80 was 9.232691054394484e-09 for t=16
converged in 40 iterations
the loss value at 40 was 9.557535793971314e-09 for t=32
```




```
loss value at iteration 119 is too big, stop algorithm t=64
the loss value at 119 was 692005.8325373644 for t=64
```



Average Frobenius norm across all runs for written pdf.

[11]:
```
print('Average Frobenius norm for increasing t^i from 1-64')
for i in range(7):
    print((1/3)*(frobnormsave[i] + frobnormsave1[i] + frobnormsave2[i]))
```

```
0.0001896422060749921
0.00018777262845073606
0.0001839843243524339
0.0001762156444120005
0.0001599516027254508
0.0001249989603203094
48.63534797649846
```

5 Question 5

[16]:
```
def sag(U0, W0, Y, I,Itest, t=1, eps=1e-6, K=100):
    #initialise matrices
    U = U0
    W = W0
    m = I.shape[0]
    Vu_tilde = np.zeros((m,U.shape[1]))
    Vw_tilde = np.zeros((m,U.shape[1]))
    Gu_tilde = np.zeros((U.shape[0],U.shape[1]))
    Gw_tilde = np.zeros((U.shape[0],U.shape[1]))
    LossValsS = np.zeros(0, dtype=float)
    for k in range(K):
        LossValsS = np.append(LossValsS,Loss(U,W,Y,Itest))
        if (LossValsS[k] < eps):
            print("converged in "+str(k)+" iterations")
            # if so exit iteration loop
            break
        if (LossValsS[k] > 100):
            print('loss value at iteration '+str(k)+' is too big, stop algorithm for t='+str(t))
            break
        # sample i uniformly at random
        i = np.arange(m)
        np.random.shuffle(i)
        i = i[0]
        # calculate the corresponding ith pair of I using random i
        i1 = I[i,0]
        i2 = I[i,1]
        # subtract old gradients
        Gu_tilde[i1,:] = Gu_tilde[i1,:] - (1/m)*Vu_tilde[i,:]
        Gw_tilde[i2,:] = Gw_tilde[i2,:] - (1/m)*Vw_tilde[i,:]
        # compute new gradients
        [Vu_tilde[i,:],Vw_tilde[i,:]] = pointwise_gradient(U,W,Y,i1,i2)
        # Add new gradients
        Gu_tilde[i1,:] = Gu_tilde[i1,:] + (1/m)*Vu_tilde[i,:]
        Gw_tilde[i2,:] = Gw_tilde[i2,:] + (1/m)*Vw_tilde[i,:]
        # Update iterates
```

```
        U = U-t*Gu_tilde
        W = W-t*Gw_tilde
    # create an array for the k iterations
    karray = np.arange(0,k+1)
    # plot k iterations on x-axis and loss values for each k on y-axis
    plt.plot(karray, LossValsS, label = 't='+str(t)+'')
    # set the y-axis to log scale
    plt.yscale('log')
    print('the loss value at '+str(k)+' was '+str(LossValsS[k])+' for␣
 ↪t='+str(t))
    # return U and W which minimise the loss function to sufficient accuracy
    return U,W
```

6 Question 6

Ran 3 times for reproducability in same proceedure as question 4, with different maximum iterations

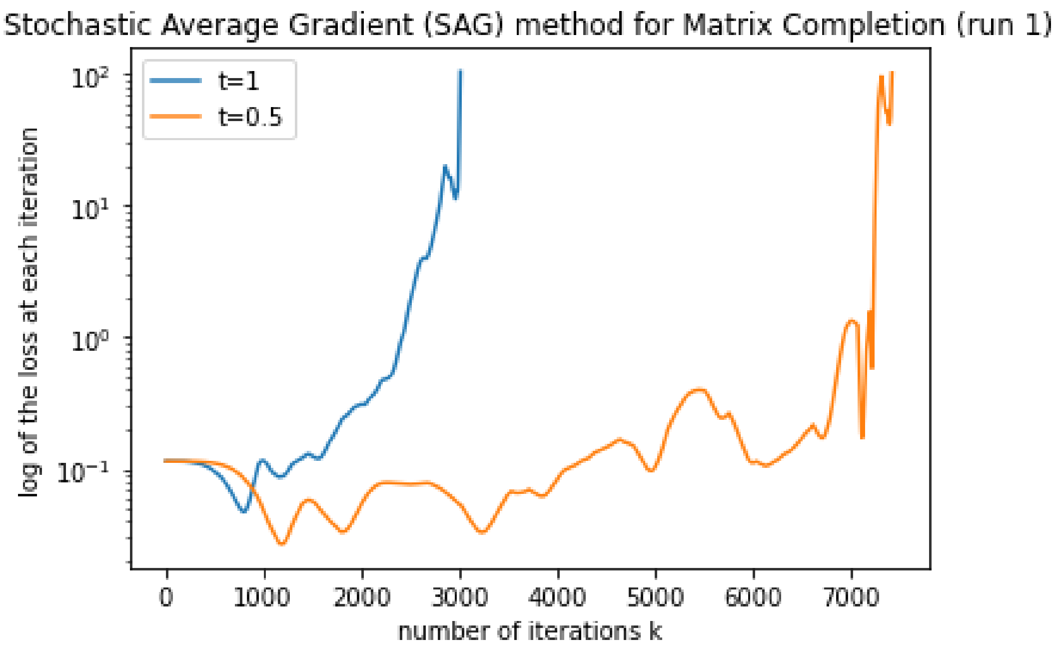
[33]:

✓

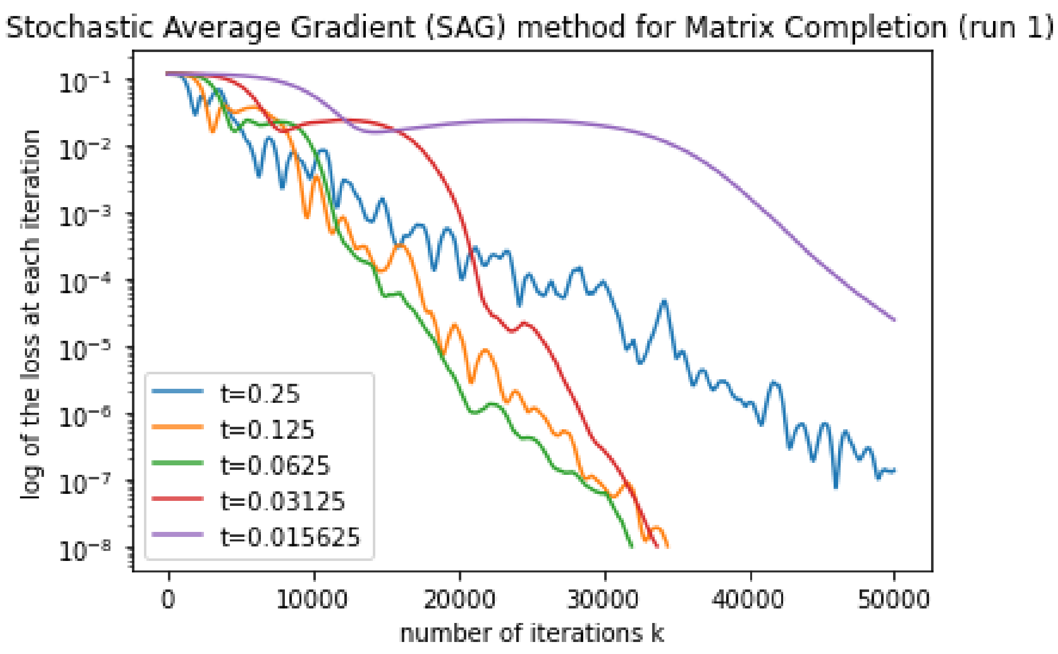
```
#the number of iterations is changed from question 4
K = 50000
eps = 1e-8
# plotting 1, 1/2 seperately due to different resolution

# RUN 1
# create a different permutation for I and Itest
I, Itest = createIs(Iext)
frobnormsave = []
t = [1,1/2]
for i in t:
    U,W = sag(U0, W0, Y, I, Itest, i,eps, K)
    frobnorm = np.linalg.norm(U @ W.T - Y)/np.linalg.norm(Y)
    frobnormsave = np.append(frobnormsave,frobnorm)
plt.ylabel('log of the loss at each iteration')
plt.xlabel('number of iterations k')
plt.title('Stochastic Average Gradient (SAG) method for Matrix Completion␣
 ↪(run 1)')
plt.legend()
plt.show()
# plotting t values that have similar resolution that dont blow up!
t = [1/4,1/8,1/16,1/32,1/64]

for i in t:
    U,W = sag(U0, W0, Y, I, Itest, i,eps, K)
    # calcuting the frobenius norm and saving it for each t
    frobnorm = np.linalg.norm(U @ W.T - Y)/np.linalg.norm(Y)
    frobnormsave = np.append(frobnormsave,frobnorm)
# titles for plots
plt.ylabel('log of the loss at each iteration')
plt.xlabel('number of iterations k')
```

```
plt.title('Stochastic Average Gradient (SAG) method for Matrix Completion␣
  ↪(run 1)')
plt.legend()
plt.show()
```

```
loss value at iteration 3012 is too big, stop algorithm for t=1
the loss value at 3012 was 104.47399315541789 for t=1
loss value at iteration 7427 is too big, stop algorithm for t=0.5
the loss value at 7427 was 102.24347451991909 for t=0.5
```

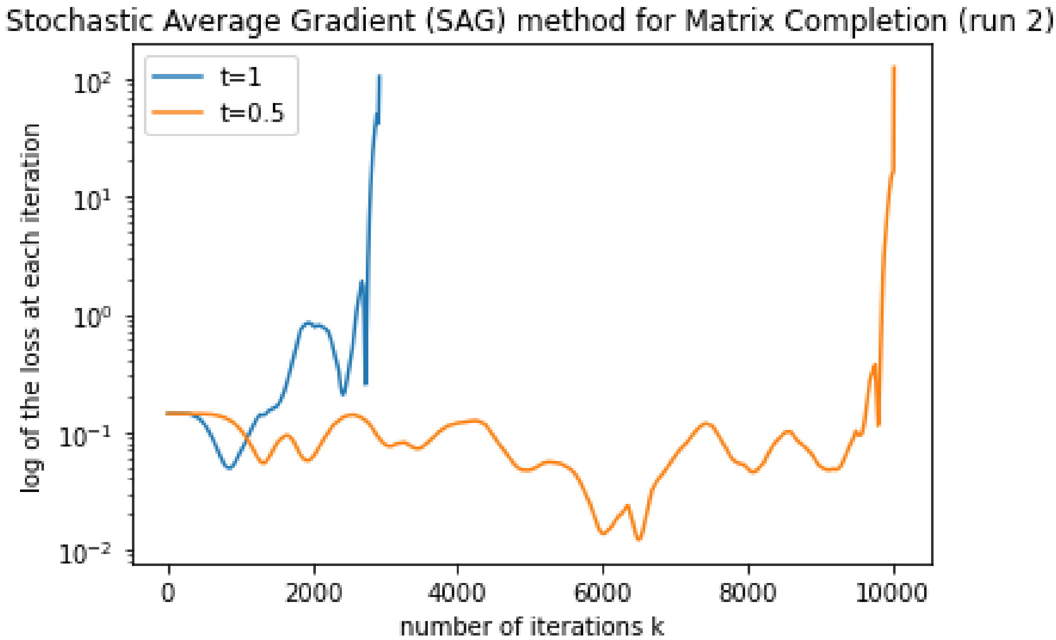



```
the loss value at 49999 was 1.3622219276664165e-07 for t=0.25
converged in 34397 iterations
the loss value at 34397 was 9.997063446770463e-09 for t=0.125
converged in 31923 iterations
the loss value at 31923 was 9.987824776294165e-09 for t=0.0625
converged in 33684 iterations
the loss value at 33684 was 9.994760637315859e-09 for t=0.03125
the loss value at 49999 was 2.427565972114296e-05 for t=0.015625
```

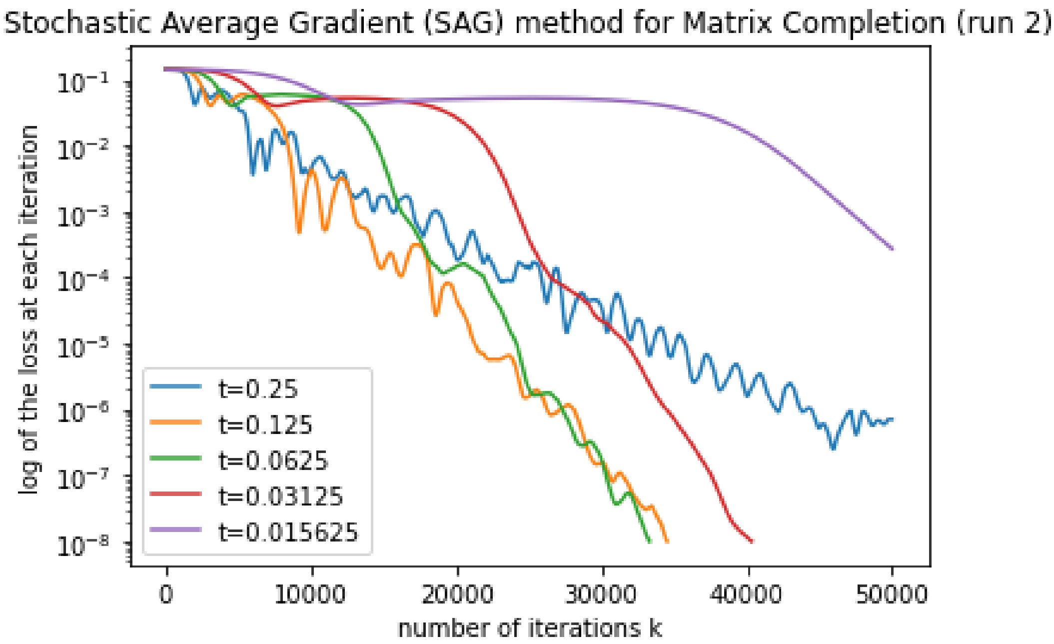


[34]:
```
# RUN 2
# create a different permutation for I and Itest
I, Itest = createIs(Iext)
frobnormsave1 = []
t = [1,1/2]
for i in t:
    U,W = sag(U0, W0, Y, I, Itest, i,eps, K)
    frobnorm = np.linalg.norm(U @ W.T - Y)/np.linalg.norm(Y)
    frobnormsave1 = np.append(frobnormsave,frobnorm)
plt.ylabel('log of the loss at each iteration')
plt.xlabel('number of iterations k')
plt.title('Stochastic Average Gradient (SAG) method for Matrix Completion␣
 ↪(run 2)')
plt.legend()
plt.show()
# plotting t values that have similar resolution that dont blow up!
t = [1/4,1/8,1/16,1/32,1/64]
for i in t:
    U,W = sag(U0, W0, Y, I, Itest, i,eps, K)
    # calcuting the frobenius norm and saving it for each t
    frobnorm = np.linalg.norm(U @ W.T - Y)/np.linalg.norm(Y)
    frobnormsave1 = np.append(frobnormsave,frobnorm)
# titles for plots
plt.ylabel('log of the loss at each iteration')
plt.xlabel('number of iterations k')
plt.title('Stochastic Average Gradient (SAG) method for Matrix Completion␣
 ↪(run 2)')
plt.legend()
plt.show()
```

```
loss value at iteration 2918 is too big, stop algorithm for t=1
the loss value at 2918 was 104.1794905692511 for t=1
loss value at iteration 10027 is too big, stop algorithm for t=0.5
the loss value at 10027 was 124.38840082334471 for t=0.5
```

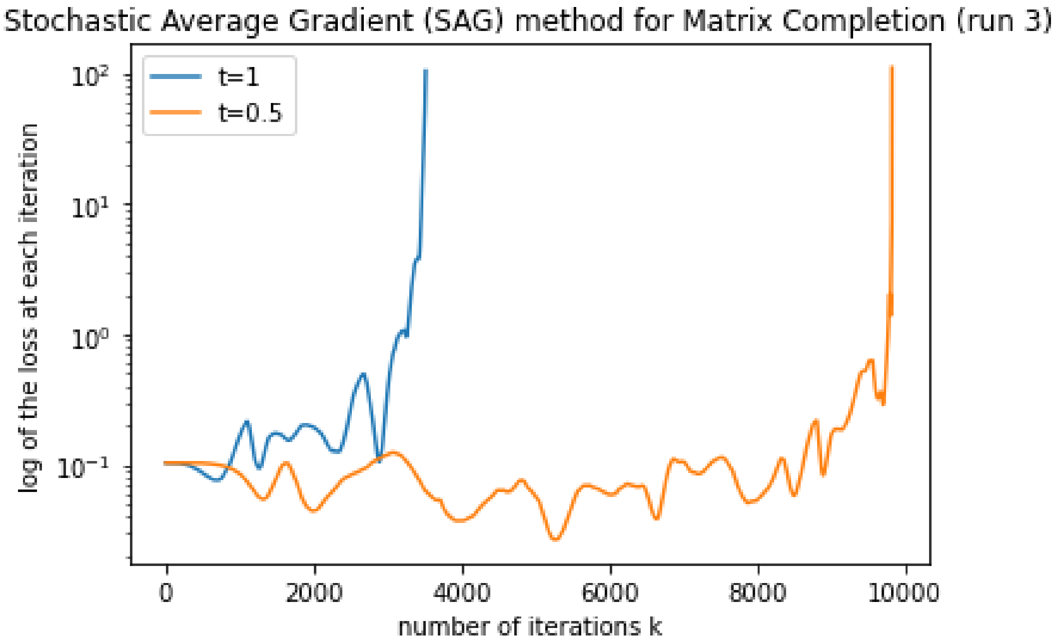



```
the loss value at 49999 was 7.064275549208246e-07 for t=0.25
converged in 34510 iterations
the loss value at 34510 was 9.99179256299337e-09 for t=0.125
converged in 33303 iterations
the loss value at 33303 was 9.99564911667753e-09 for t=0.0625
converged in 40340 iterations
the loss value at 40340 was 9.99986032126327e-09 for t=0.03125
the loss value at 49999 was 0.0002754742488160346 for t=0.015625
```

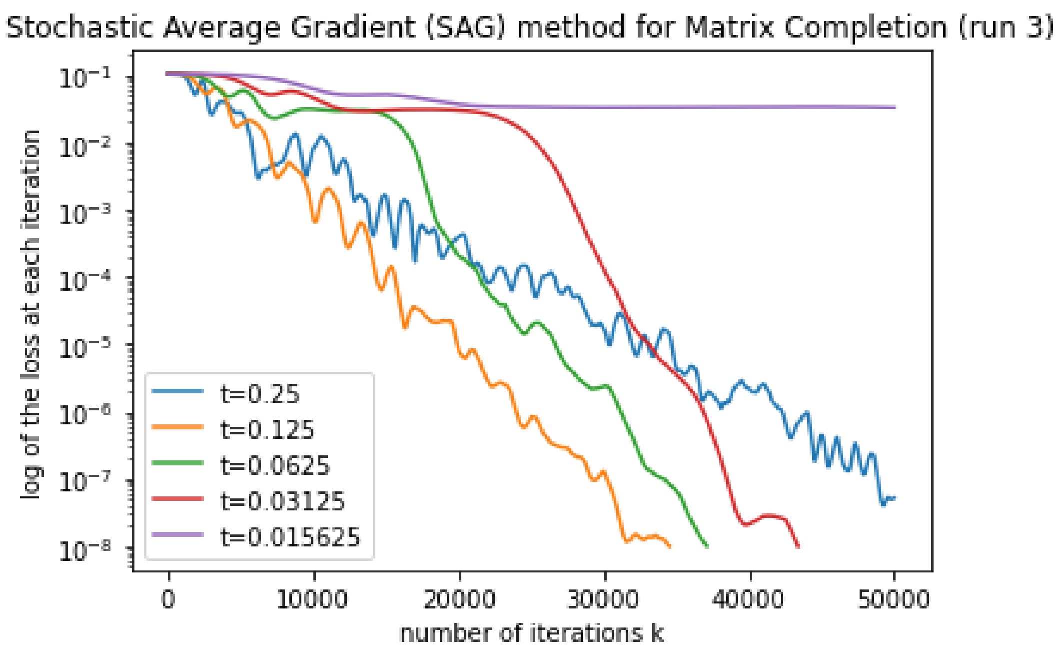


[36]:
```
# RUN 3
# create a different permutation for I and Itest
I, Itest = createIs(Iext)
frobnormsave2 = []
t = [1,1/2]
for i in t:
    U,W = sag(U0, W0, Y, I, Itest, i,eps, K)
    frobnorm = np.linalg.norm(U @ W.T - Y)/np.linalg.norm(Y)
    frobnormsave2 = np.append(frobnormsave,frobnorm)
plt.ylabel('log of the loss at each iteration')
plt.xlabel('number of iterations k')
plt.title('Stochastic Average Gradient (SAG) method for Matrix Completion␣
 ↪(run 3)')
plt.legend()
plt.show()
# plotting t values that have similar resolution that dont blow up!
t = [1/4,1/8,1/16,1/32,1/64]
for i in t:
    U,W = sag(U0, W0, Y, I, Itest, i,eps, K)
    # calcuting the frobenius norm and saving it for each t
    frobnorm = np.linalg.norm(U @ W.T - Y)/np.linalg.norm(Y)
    frobnormsave2 = np.append(frobnormsave,frobnorm)
# titles for plots
plt.ylabel('log of the loss at each iteration')
plt.xlabel('number of iterations k')
plt.title('Stochastic Average Gradient (SAG) method for Matrix Completion␣
 ↪(run 3)')
plt.legend()
plt.show()
```

```
loss value at iteration 3513 is too big, stop algorithm for t=1
the loss value at 3513 was 104.94929343487327 for t=1
loss value at iteration 9815 is too big, stop algorithm for t=0.5
the loss value at 9815 was 110.90323747086575 for t=0.5
```




```
the loss value at 49999 was 5.209258271153829e-08 for t=0.25
converged in 34545 iterations
the loss value at 34545 was 9.99776905129805e-09 for t=0.125
converged in 37125 iterations
the loss value at 37125 was 9.998713746375011e-09 for t=0.0625
converged in 43394 iterations
the loss value at 43394 was 9.990650752428566e-09 for t=0.03125
the loss value at 49999 was 0.032885275624277566 for t=0.015625
```



✓

[38]:
```python
print('Average Frobenius norm for increasing i with 1/t^i from 1 to 1/64')
for i in range(7):
    print((1/3)*(frobnormsave[i] + frobnormsave1[i] + frobnormsave2[i]))
```

```
Average Frobenius norm for increasing i with 1/t^i from 1 to 1/64
30.030491415878288
4251.20587331164
0.0008124153580489851
0.00033638404888243603
0.00024946534498733946
0.00021932800859761044
0.00951639951635317
```

7 Question 7

This will be done in steps: * Loading in the data * Initialising variables specified in question * Running GD and SGD x2 and finding L_{Low} for respective runs * Finding the t value which gives the fastest convergence (x2)

[17]:
```python
# initialising all variables required in description of question

#loading the data and storing it as a matrix called Y
np.random.seed(1)

data=np.load('CW.npz')
Y=data['Y']
n=Y.shape[0]
mu = 20
p=1/3
```

✓

✓

```
r=4
Iext = bernoulli(n,p)
U0,W0=initial(n,r)
I, Itest = createIs(Iext)
K = 5000
eps = 1.0e-8
t = [1,2,4,8,16,32,64]
```

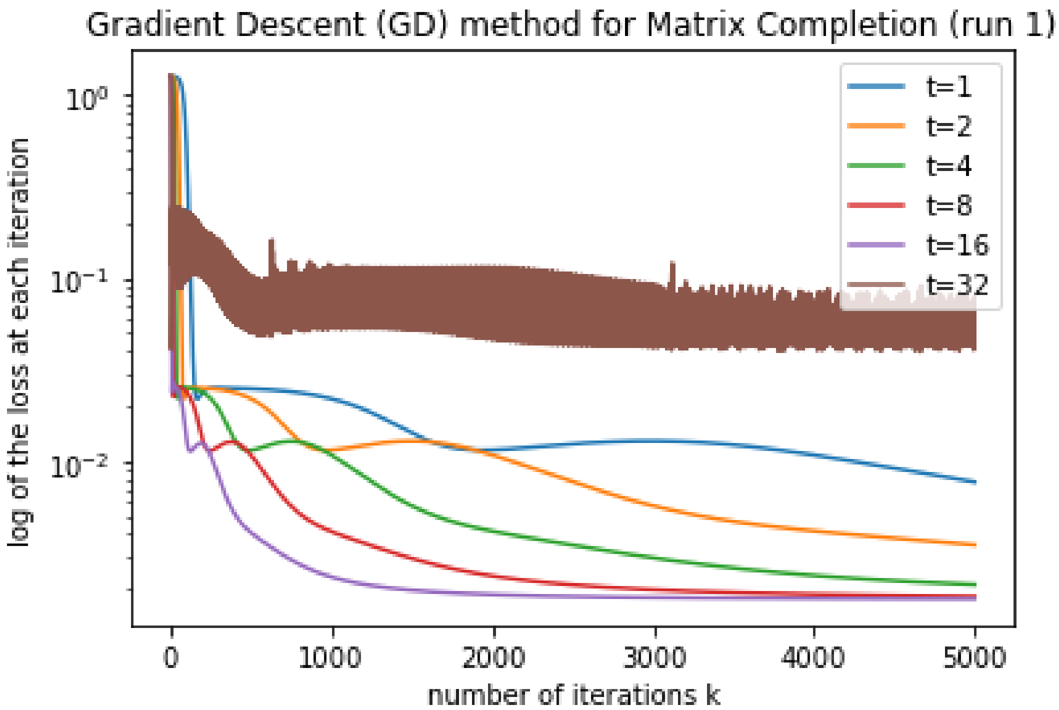
Running GD two times to check for reducability in results (Note: using different testing and training sets)

[51]:

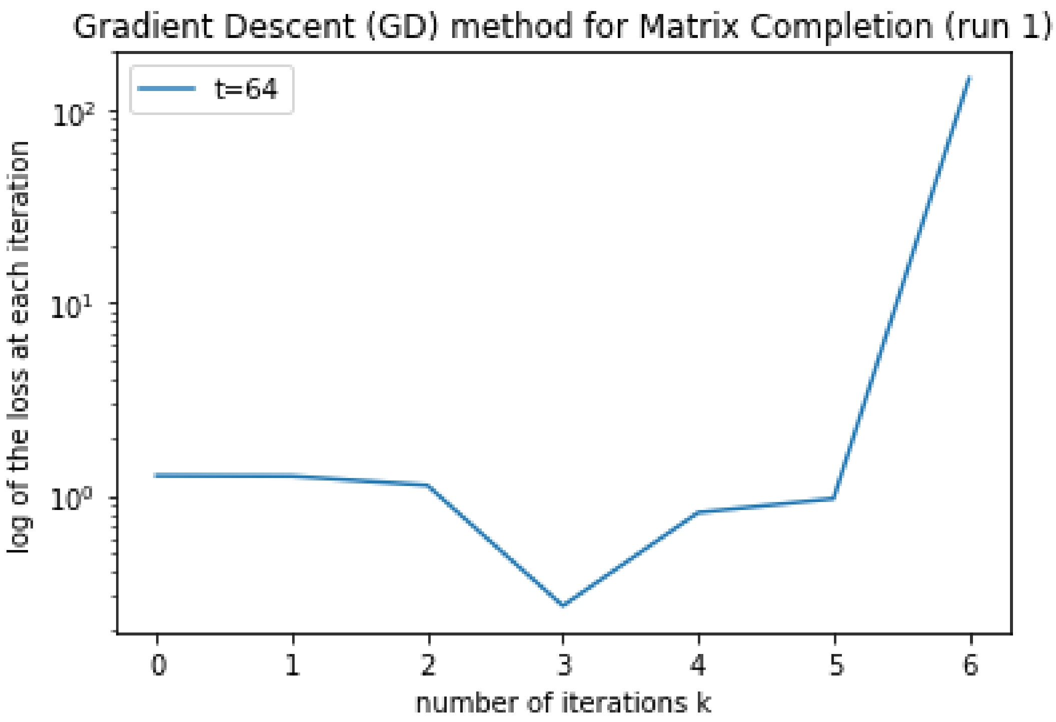
```
## NOTE: THIS TAKES 25 MINS TO RUN!

# Run 1 of GD
frobnormsave = []
t = [1,2,4,8,16,32]
for i in t:
    U,W = gd(U0, W0, Y, I, Itest, i,eps, K)
    # calcuting the frobenius norm and saving it for each t
    frobnorm = np.linalg.norm(U @ W.T - Y)/np.linalg.norm(Y)
    frobnormsave = np.append(frobnormsave,frobnorm)
# titles for plots
plt.ylabel('log of the loss at each iteration')
plt.xlabel('number of iterations k')
plt.title('Gradient Descent (GD) method for Matrix Completion (run 1)')
plt.legend()
plt.show()
# plotting 64 seperately due to different resolution
U,W = gd(U0, W0, Y, I, Itest, 64,eps, K)
frobnorm = np.linalg.norm(U @ W.T - Y)/np.linalg.norm(Y)
frobnormsave = np.append(frobnormsave,frobnorm)
plt.ylabel('log of the loss at each iteration')
plt.xlabel('number of iterations k')
plt.title('Gradient Descent (GD) method for Matrix Completion (run 1)')
plt.legend()
plt.show()

# Run 2 of GD
# change I and Itest data sets
I1, Itest1 = createIs(Iext)
frobnormsave1 = []
t = [1,2,4,8,16,32]
for i in t:
    U,W = gd(U0, W0, Y, I1, Itest1, i,eps, K)
    # calcuting the frobenius norm and saving it for each t
    frobnorm = np.linalg.norm(U @ W.T - Y)/np.linalg.norm(Y)
    frobnormsave1 = np.append(frobnormsave,frobnorm)
# titles for plots
plt.ylabel('log of the loss at each iteration')
plt.xlabel('number of iterations k')
plt.title('Gradient Descent (GD) method for Matrix Completion (run 2)')
```

```
plt.legend()
plt.show()
# plotting 64 seperately due to different resolution
U,W = gd(U0, W0, Y, I1, Itest1, 64,eps, K)
frobnorm = np.linalg.norm(U @ W.T - Y)/np.linalg.norm(Y)
frobnormsave1 = np.append(frobnormsave,frobnorm)
plt.ylabel('log of the loss at each iteration')
plt.xlabel('number of iterations k')
plt.title('Gradient Descent (GD) method for Matrix Completion (run 2)')
plt.legend()
plt.show()

# calculating the average Frobenius norm values for increasing t
print('Average Frobenius norm for increasing t^i from 1-64')
for i in range(7):
    print((1/2)*(frobnormsave[i] + frobnormsave1[i]))
```

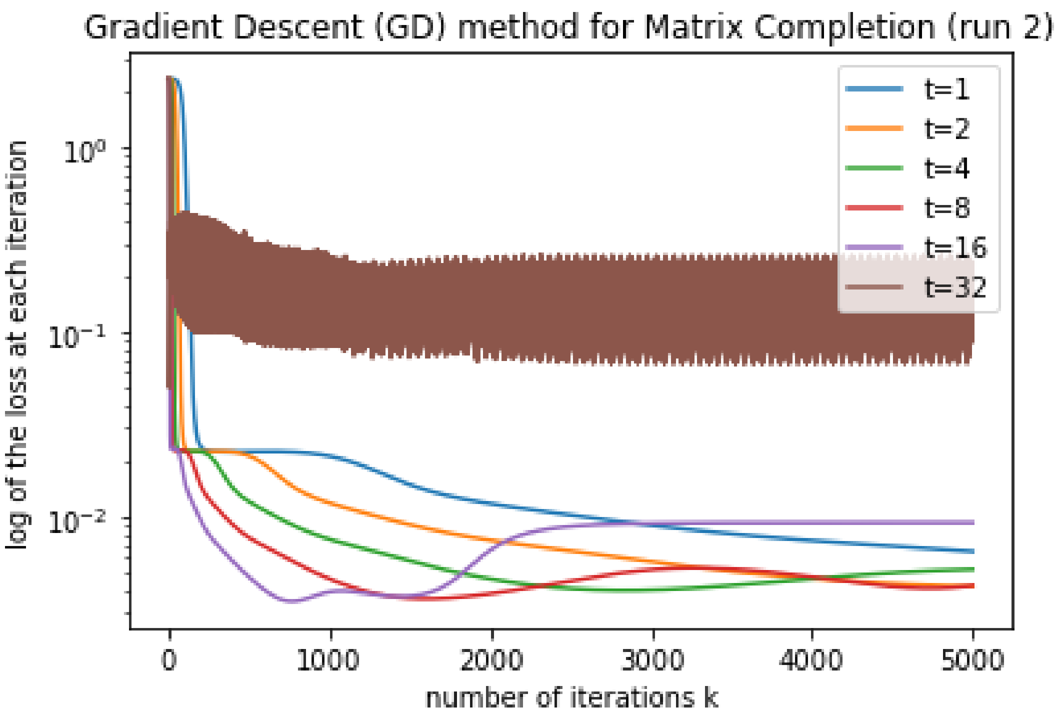
```
the loss value at 4999 was 0.007716006328542054 for t=1
the loss value at 4999 was 0.0034954883750474295 for t=2
the loss value at 4999 was 0.002121139037511279 for t=4
the loss value at 4999 was 0.0018294204513026788 for t=8
the loss value at 4999 was 0.0017730281702907648 for t=16
the loss value at 4999 was 0.07910263113667576 for t=32
```



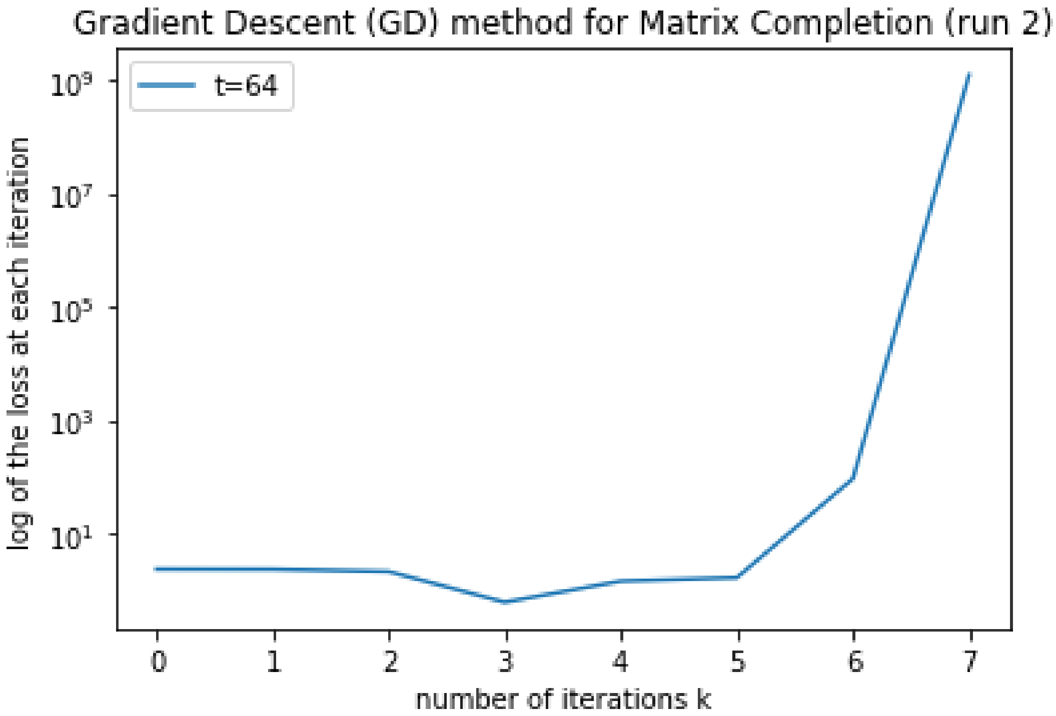

```
loss value at iteration 6 is too big, stop algorithm t=64
the loss value at 6 was 146.33155631767244 for t=64
```




```
the loss value at 4999 was 0.006460579511994191 for t=1
the loss value at 4999 was 0.004180110885865918 for t=2
the loss value at 4999 was 0.005159881789417569 for t=4
the loss value at 4999 was 0.004224643562368216 for t=8
the loss value at 4999 was 0.00925924386519562 for t=16
the loss value at 4999 was 0.23430910573206884 for t=32
```



```
loss value at iteration 7 is too big, stop algorithm t=64
the loss value at 7 was 1210047796.0507398 for t=64
```




```
Average Frobenius norm for increasing t^i from 1-64
0.05472871773064081
0.0436990171453183
0.044913686277408896
0.04557739892247233
0.04575025533508791
0.9583462110744541
38.51327187418235
```

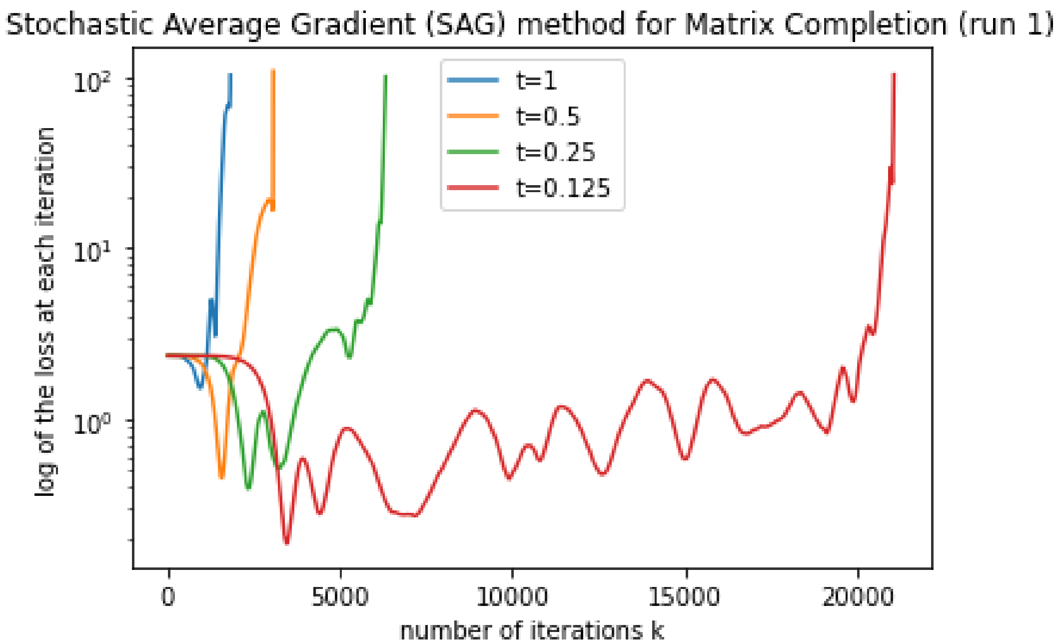
Running SAG two times to check for reducability in results (Note: using different testing and training sets)

[53]:
```python
# change number of iterations for this test
K = 50000
eps = 1e-8
# plotting 1, 1/2,1/4,1/8 seperately due to different resolution

# RUN 1 of SAG
frobnormsave = []
t = [1,1/2,1/4,1/8]
for i in t:
    U,W = sag(U0, W0, Y, I, Itest, i,eps, K)
    frobnorm = np.linalg.norm(U @ W.T - Y)/np.linalg.norm(Y)
```

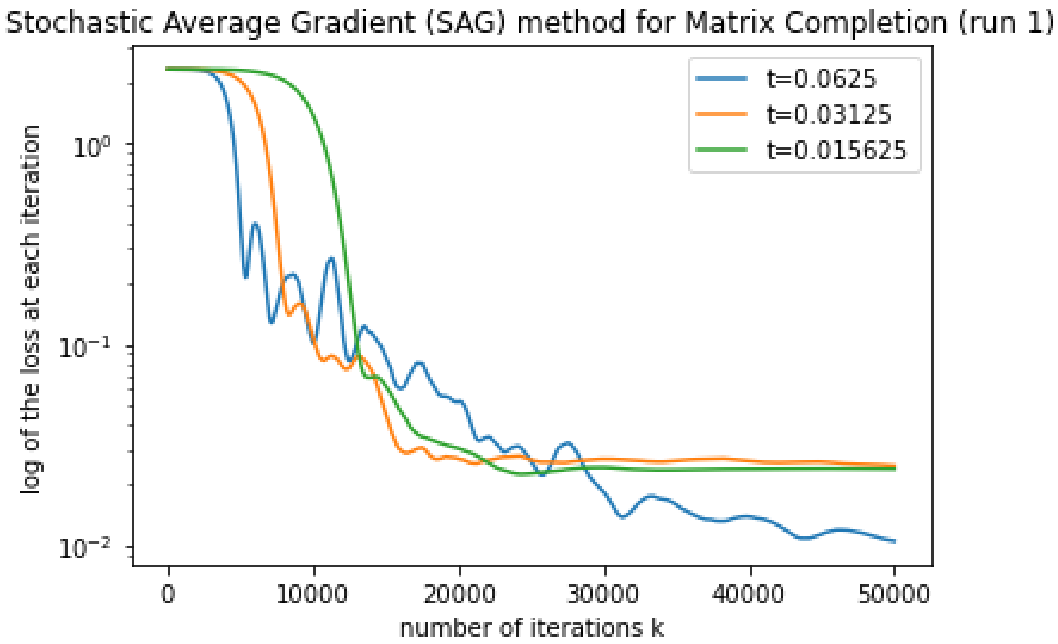
```
    frobnormsave = np.append(frobnormsave,frobnorm)
plt.ylabel('log of the loss at each iteration')
plt.xlabel('number of iterations k')
plt.title('Stochastic Average Gradient (SAG) method for Matrix Completion␣
 ↪(run 1)')
plt.legend()
plt.show()
# plotting t values that have similar resolution that dont blow up!
t = [1/16,1/32,1/64]
for i in t:
    U,W = sag(U0, W0, Y, I, Itest, i,eps, K)
    # calcuting the frobenius norm and saving it for each t
    frobnorm = np.linalg.norm(U @ W.T - Y)/np.linalg.norm(Y)
    frobnormsave = np.append(frobnormsave,frobnorm)
# titles for plots
plt.ylabel('log of the loss at each iteration')
plt.xlabel('number of iterations k')
plt.title('Stochastic Average Gradient (SAG) method for Matrix Completion␣
 ↪(run 1)')
plt.legend()
plt.show()

# RUN 2 of SAG

frobnormsave1 = []
t = [1,1/2,1/4,1/8]
for i in t:
    U,W = sag(U0, W0, Y, I1, Itest1, i,eps, K)
    frobnorm = np.linalg.norm(U @ W.T - Y)/np.linalg.norm(Y)
    frobnormsave1 = np.append(frobnormsave,frobnorm)
plt.ylabel('log of the loss at each iteration')
plt.xlabel('number of iterations k')
plt.title('Stochastic Average Gradient (SAG) method for Matrix Completion␣
 ↪(run 2)')
plt.legend()
plt.show()
# plotting t values that have similar resolution that dont blow up!
t = [1/16,1/32,1/64]
for i in t:
    U,W = sag(U0, W0, Y, I1, Itest1, i,eps, K)
    # calcuting the frobenius norm and saving it for each t
    frobnorm = np.linalg.norm(U @ W.T - Y)/np.linalg.norm(Y)
    frobnormsave1 = np.append(frobnormsave,frobnorm)
# titles for plots
plt.ylabel('log of the loss at each iteration')
plt.xlabel('number of iterations k')
plt.title('Stochastic Average Gradient (SAG) method for Matrix Completion␣
 ↪(run 2)')
plt.legend()
plt.show()
```

```
# compute the average frobenius norm

print('Average Frobenius norm for increasing t^i from 1-1/64')
for i in range(7):
    print((1/2)*(frobnormsave[i] + frobnormsave1[i]))
```

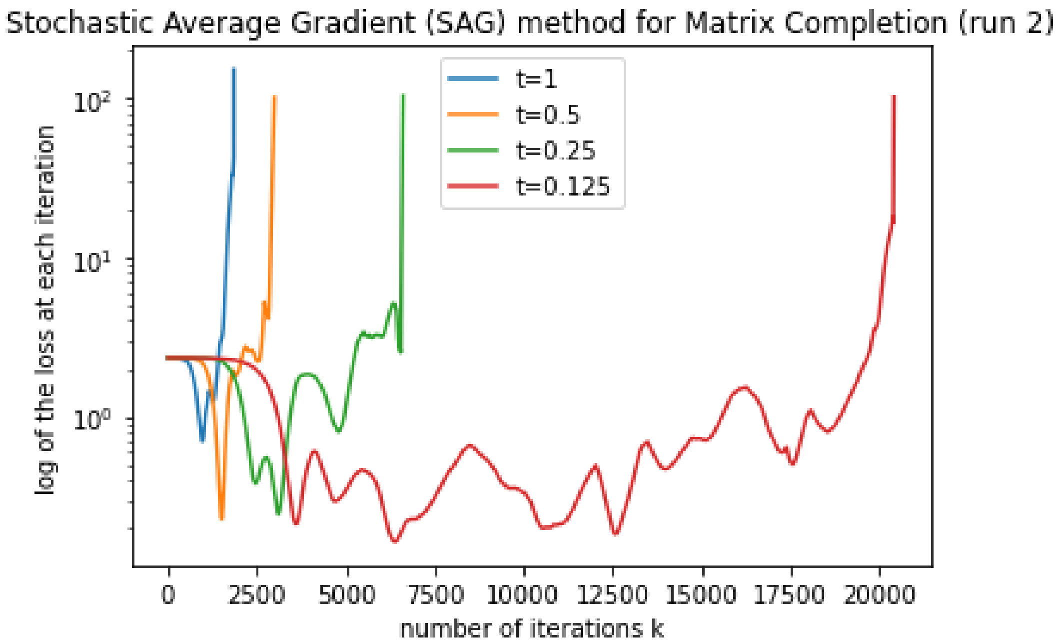
```
loss value at iteration 1815 is too big, stop algorithm for t=1
the loss value at 1815 was 103.6848892458178 for t=1
loss value at iteration 3060 is too big, stop algorithm for t=0.5
the loss value at 3060 was 108.99659077211541 for t=0.5
loss value at iteration 6324 is too big, stop algorithm for t=0.25
the loss value at 6324 was 101.11156288481202 for t=0.25
loss value at iteration 21087 is too big, stop algorithm for t=0.125
the loss value at 21087 was 104.28025909932894 for t=0.125
```



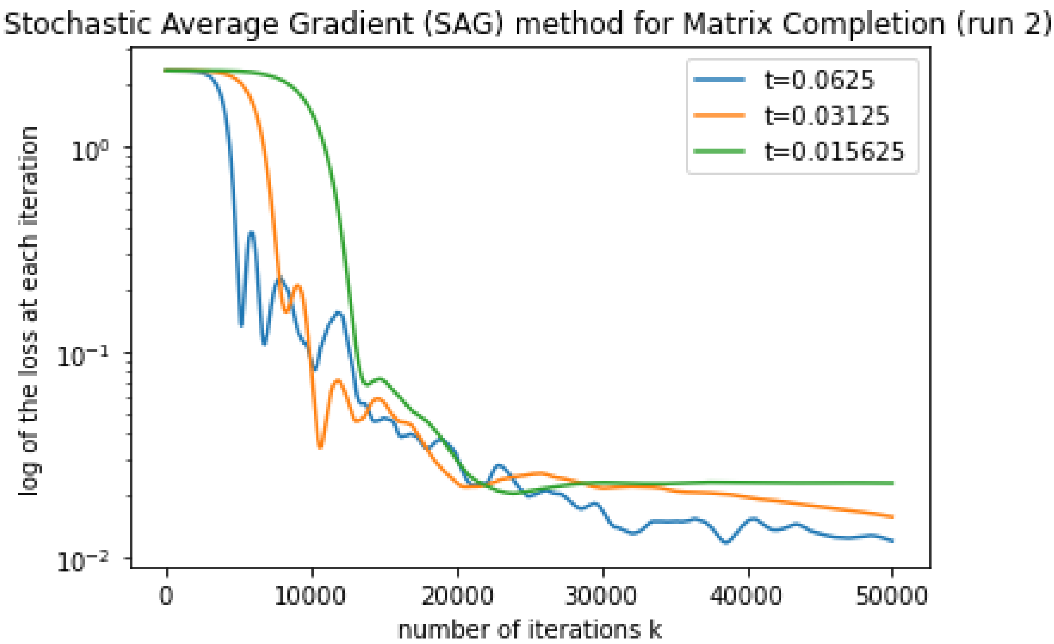

```
the loss value at 49999 was 0.010504701650727657 for t=0.0625
the loss value at 49999 was 0.02487297843880441 for t=0.03125
the loss value at 49999 was 0.0240016076722070б for t=0.015625
```




```
loss value at iteration 1850 is too big, stop algorithm for t=1
the loss value at 1850 was 151.19790475715965 for t=1
loss value at iteration 2990 is too big, stop algorithm for t=0.5
the loss value at 2990 was 100.88114550279529 for t=0.5
loss value at iteration 6608 is too big, stop algorithm for t=0.25
the loss value at 6608 was 103.4857859985356 for t=0.25
loss value at iteration 20409 is too big, stop algorithm for t=0.125
the loss value at 20409 was 101.58999986196989 for t=0.125
```



```
the loss value at 49999 was 0.012054233821322136 for t=0.0625
the loss value at 49999 was 0.015759654120829227 for t=0.03125
the loss value at 49999 was 0.022970488966494073 for t=0.015625
```




```
Average Frobenius norm for increasing t^i from 1-1/64
13.313392580177505
22.991319575510236
28.57295953683959
390057.6869718698
0.07702741990466157
0.1005363547446034
0.11762976857525714
```

For SG * Run 1: $L_{low} = 0.0017730281702907648$ * Run 2: $L_{low} = 0.004180110885865918$ ✓

For SAG * Run 1: $L_{low} = 0.010504701650727657$ * Run 2: $L_{low} = 0.012054233821322136$ too large

Proceed to re-run tests with $\epsilon = 2 * L_{low}$ and changed maximum iterations where nessaccery

(NOTE: Originally set $K = 10000$, but tests took 1hr to run… Setting $K = 3000$ is sufficient to obtain the quickest converging t)

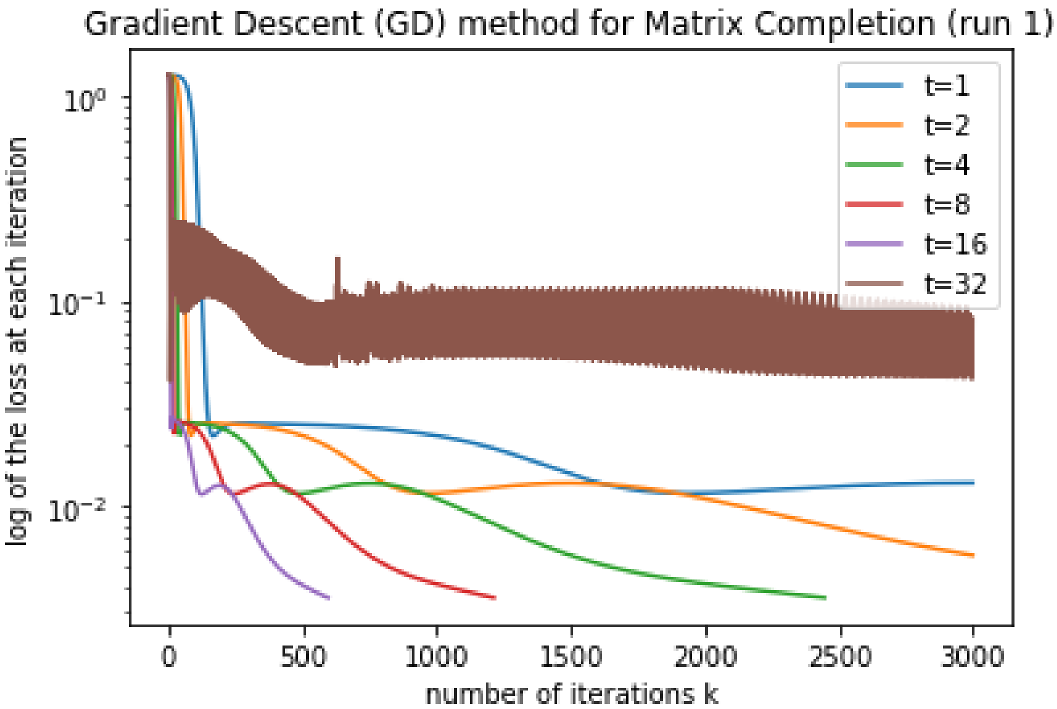
[94]: ✓

```
## NOTE: THIS TAKES A LONG TIME TO RUN!
K=3000
# Run 1 of SG1 for eps = 2*0.0017730281702907648
eps = 2*0.0017730281702907648
frobnormsave = []
t = [1,2,4,8,16,32]
for i in t:
    U,W = gd(U0, W0, Y, I, Itest, i,eps, K)
    # calcuting the frobenius norm and saving it for each t
    frobnorm = np.linalg.norm(U @ W.T - Y)/np.linalg.norm(Y)
```

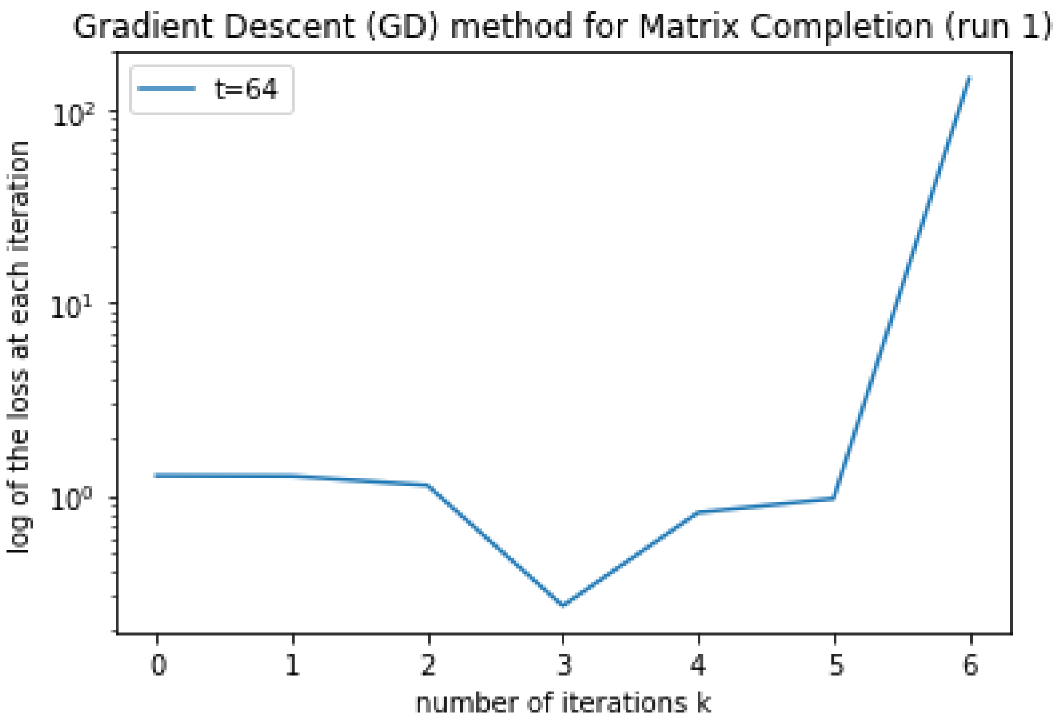
```
    frobnormsave = np.append(frobnormsave,frobnorm)
# titles for plots
plt.ylabel('log of the loss at each iteration')
plt.xlabel('number of iterations k')
plt.title('Gradient Descent (GD) method for Matrix Completion (run 1)')
plt.legend()
plt.show()
# plotting 64 seperately due to different resolution
U,W = gd(U0, W0, Y, I, Itest, 64,eps, K)
frobnorm = np.linalg.norm(U @ W.T - Y)/np.linalg.norm(Y)
frobnormsave = np.append(frobnormsave,frobnorm)
plt.ylabel('log of the loss at each iteration')
plt.xlabel('number of iterations k')
plt.title('Gradient Descent (GD) method for Matrix Completion (run 1)')
plt.legend()
plt.show()

# Run 2 of GD for eps = 2*0.004180110885865918
eps = 2*0.004180110885865918
# change I and Itest data sets
frobnormsave1 = []
t = [1,2,4,8,16,32]
for i in t:
    U,W = gd(U0, W0, Y, I1, Itest1, i,eps, K)
    # calcuting the frobenius norm and saving it for each t
    frobnorm = np.linalg.norm(U @ W.T - Y)/np.linalg.norm(Y)
    frobnormsave1 = np.append(frobnormsave,frobnorm)
# titles for plots
plt.ylabel('log of the loss at each iteration')
plt.xlabel('number of iterations k')
plt.title('Gradient Descent (GD) method for Matrix Completion (run 2)')
plt.legend()
plt.show()
# plotting 64 seperately due to different resolution
U,W = gd(U0, W0, Y, I1, Itest1, 64,eps, K)
frobnorm = np.linalg.norm(U @ W.T - Y)/np.linalg.norm(Y)
frobnormsave1 = np.append(frobnormsave,frobnorm)
plt.ylabel('log of the loss at each iteration')
plt.xlabel('number of iterations k')
plt.title('Gradient Descent (GD) method for Matrix Completion (run 2)')
plt.legend()
plt.show()

# calculating the average Frobenius norm values for increasing t
print('Average Frobenius norm for increasing t^i from 1-64')
for i in range(7):
    print((1/2)*(frobnormsave[i] + frobnormsave1[i]))
```

```
the loss value at 2999 was 0.012925027582220552 for t=1
the loss value at 2999 was 0.005718114045265154 for t=2
converged in 2447 iterations
```

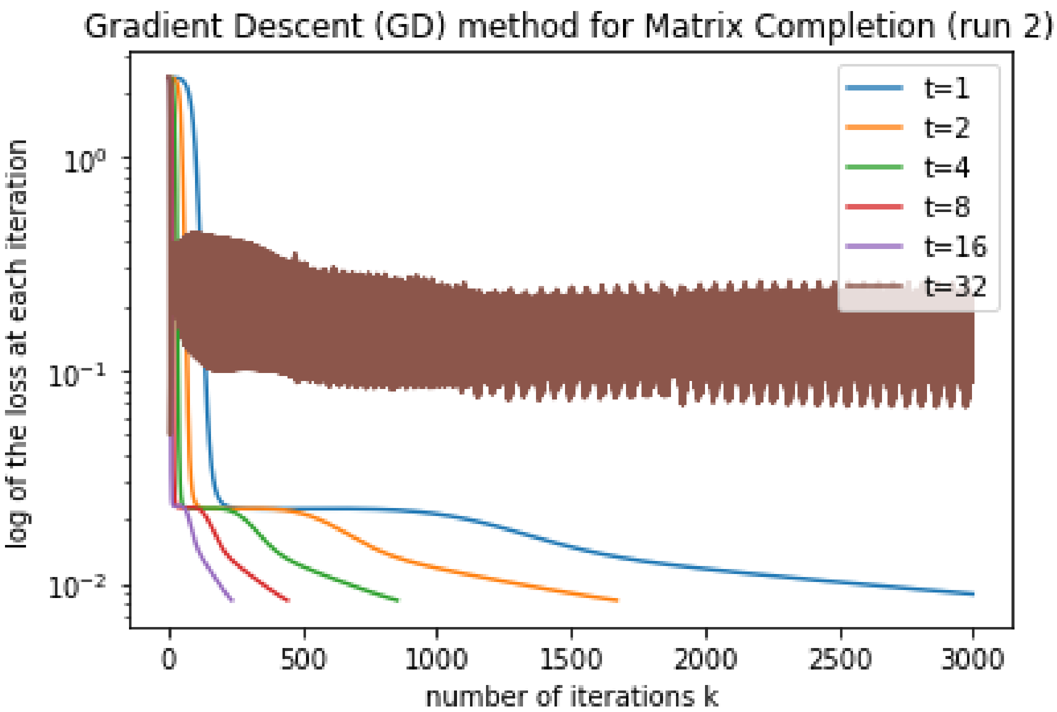
```
the loss value at 2447 was 0.0035457974200612517 for t=4
converged in 1213 iterations
the loss value at 1213 was 0.003545596602725843 for t=8
converged in 595 iterations
the loss value at 595 was 0.0035413436630722197 for t=16
the loss value at 2999 was 0.08187399806301829 for t=32
```



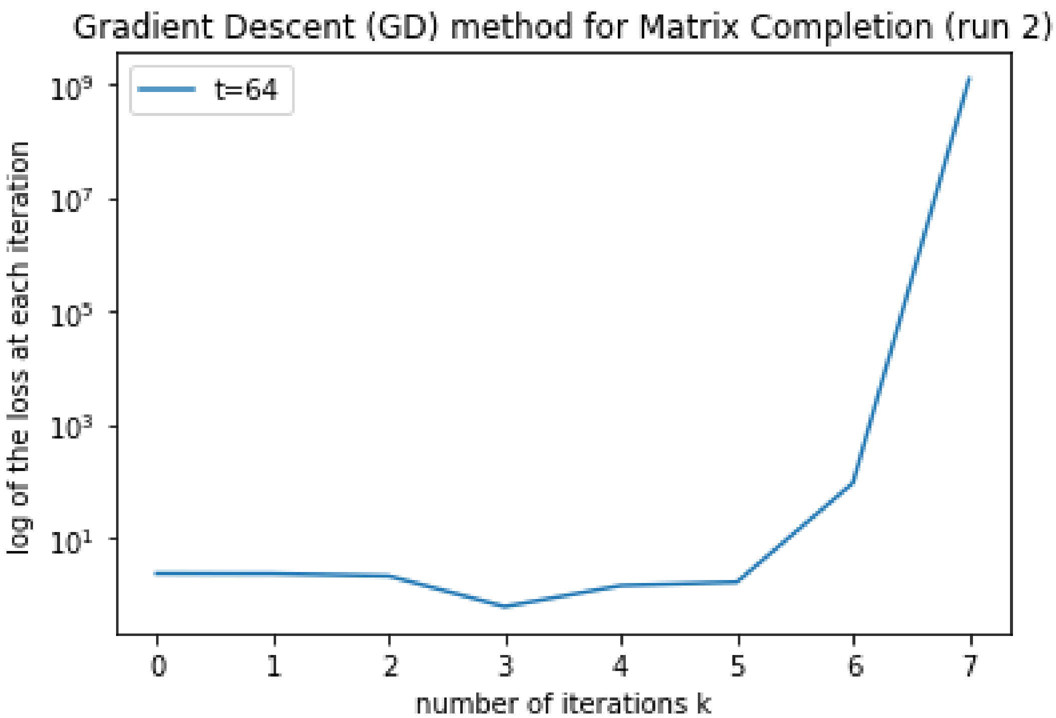

```
loss value at iteration 6 is too big, stop algorithm t=64
the loss value at 6 was 146.33155631767244 for t=64
```




```
the loss value at 2999 was 0.008956469311753552 for t=1
converged in 1671 iterations
the loss value at 1671 was 0.008360026188405274 for t=2
converged in 853 iterations
the loss value at 853 was 0.008358337960403642 for t=4
converged in 443 iterations
the loss value at 443 was 0.008355574872528677 for t=8
converged in 236 iterations
the loss value at 236 was 0.008356740866570738 for t=16
the loss value at 2999 was 0.20436304556563586 for t=32
```




```
loss value at iteration 7 is too big, stop algorithm t=64
the loss value at 7 was 1210047796.0507398 for t=64
```



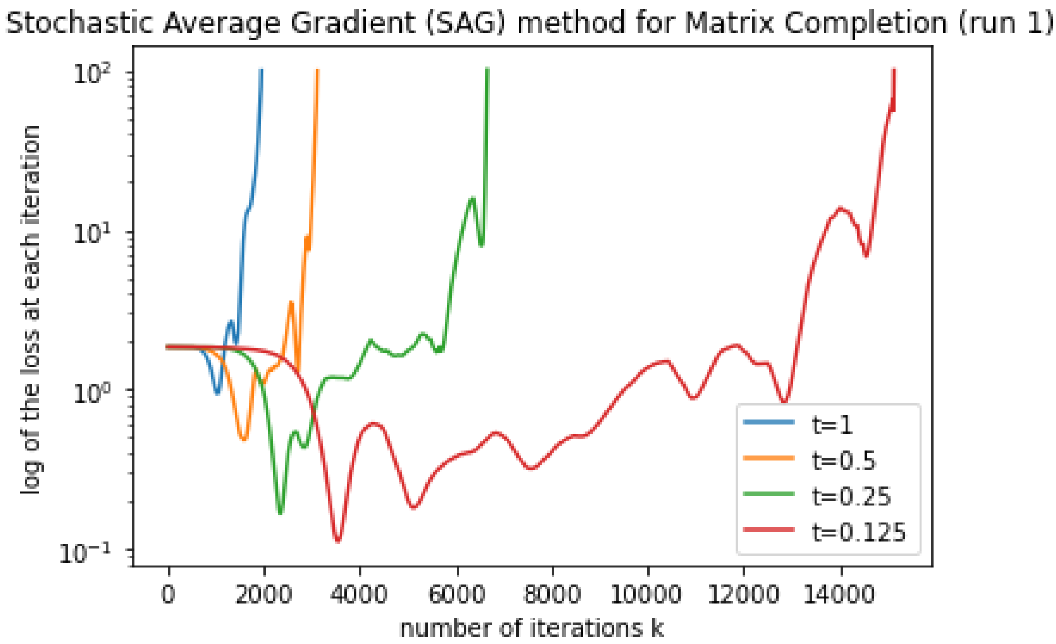

```
Average Frobenius norm for increasing t^i from 1-64
```

```
0.06879717947313183
0.05014335585153289
0.043728538461155486
0.0437412721903246б
0.0437817335923266
0.9325071930242476
38.51327187418235
```

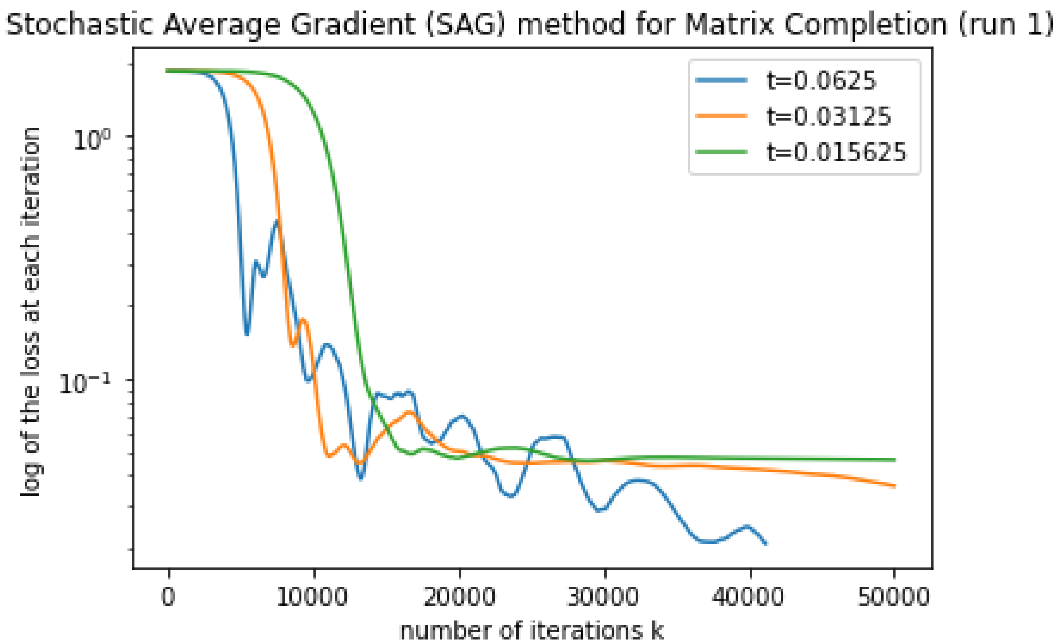
[71]:
```
K = 50000
# plotting 1, 1/2,1/4,1/8 seperately due to different resolution

# RUN 1 of SAG with eps = 2*0.010504701650727657
eps = 2*0.010504701650727657
frobnormsave = []
t = [1,1/2,1/4,1/8]
for i in t:
    U,W = sag(U0, W0, Y, I, Itest, i,eps, K)
    frobnorm = np.linalg.norm(U @ W.T - Y)/np.linalg.norm(Y)
    frobnormsave = np.append(frobnormsave,frobnorm)
plt.ylabel('log of the loss at each iteration')
plt.xlabel('number of iterations k')
plt.title('Stochastic Average Gradient (SAG) method for Matrix Completion␣
↪(run 1)')
plt.legend()
plt.show()
# plotting t values that have similar resolution that dont blow up!
t = [1/16,1/32,1/64]
for i in t:
    U,W = sag(U0, W0, Y, I, Itest, i,eps, K)
    # calcuting the frobenius norm and saving it for each t
    frobnorm = np.linalg.norm(U @ W.T - Y)/np.linalg.norm(Y)
    frobnormsave = np.append(frobnormsave,frobnorm)
# titles for plots
plt.ylabel('log of the loss at each iteration')
plt.xlabel('number of iterations k')
plt.title('Stochastic Average Gradient (SAG) method for Matrix Completion␣
↪(run 1)')
plt.legend()
plt.show()

# RUN 2 of SAG with eps = 2*0.012054233821322136
eps = 2*0.012054233821322136
frobnormsave1 = []
t = [1,1/2,1/4,1/8]
for i in t:
    U,W = sag(U0, W0, Y, I1, Itest1, i,eps, K)
    frobnorm = np.linalg.norm(U @ W.T - Y)/np.linalg.norm(Y)
    frobnormsave1 = np.append(frobnormsave,frobnorm)
plt.ylabel('log of the loss at each iteration')
plt.xlabel('number of iterations k')
```

```
plt.title('Stochastic Average Gradient (SAG) method for Matrix Completion␣
 ↪(run 2)')
plt.legend()
plt.show()
# plotting t values that have similar resolution that dont blow up!
t = [1/16,1/32,1/64]
for i in t:
    U,W = sag(U0, W0, Y, I1, Itest1, i,eps, K)
    # calcuting the frobenius norm and saving it for each t
    frobnorm = np.linalg.norm(U @ W.T - Y)/np.linalg.norm(Y)
    frobnormsave1 = np.append(frobnormsave,frobnorm)
# titles for plots
plt.ylabel('log of the loss at each iteration')
plt.xlabel('number of iterations k')
plt.title('Stochastic Average Gradient (SAG) method for Matrix Completion␣
 ↪(run 2)')
plt.legend()
plt.show()

# compute the average frobenius norm

print('Average Frobenius norm for increasing t^i from 1-1/64')
for i in range(7):
    print((1/2)*(frobnormsave[i] + frobnormsave1[i]))
```

```
loss value at iteration 1952 is too big, stop algorithm for t=1
the loss value at 1952 was 100.38842439345498 for t=1
loss value at iteration 3114 is too big, stop algorithm for t=0.5
the loss value at 3114 was 100.14768436538677 for t=0.5
loss value at iteration 6655 is too big, stop algorithm for t=0.25
the loss value at 6655 was 101.62968008558552 for t=0.25
loss value at iteration 15132 is too big, stop algorithm for t=0.125
the loss value at 15132 was 100.88923423819332 for t=0.125
```

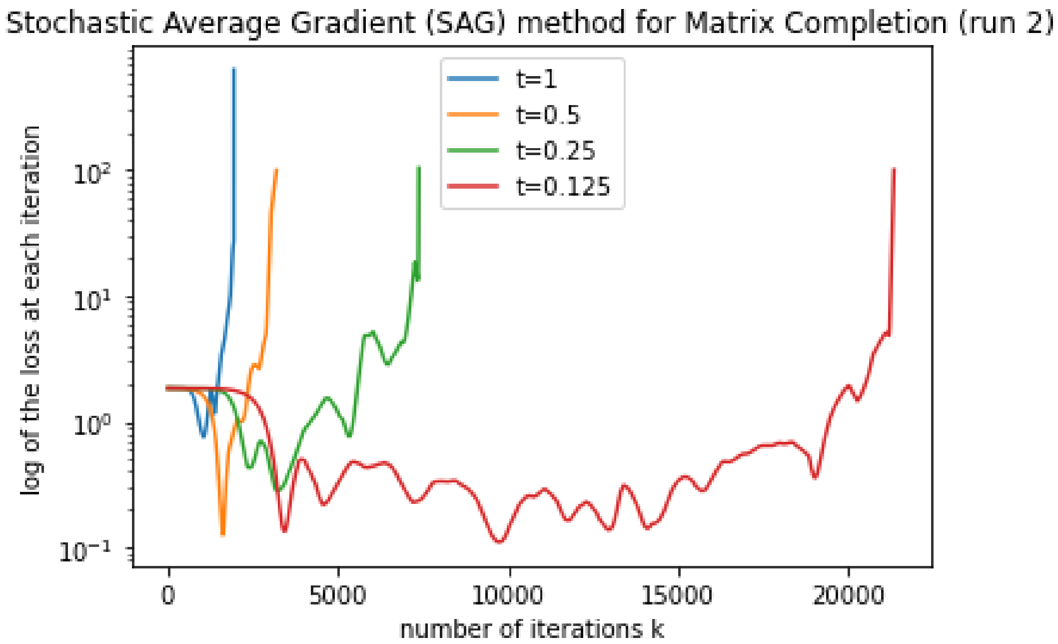



```
converged in 41158 iterations
the loss value at 41158 was 0.021006828863447274 for t=0.0625
the loss value at 49999 was 0.03630205622143094 for t=0.03125
the loss value at 49999 was 0.0464452649352613 for t=0.015625
```

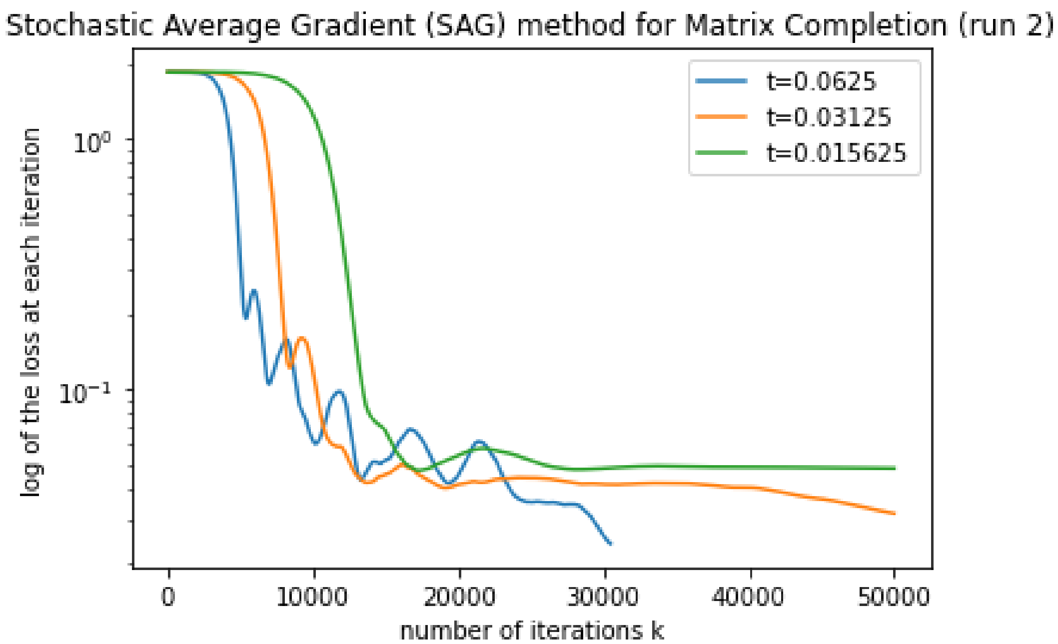



```
loss value at iteration 1937 is too big, stop algorithm for t=1
the loss value at 1937 was 635.7406648075366 for t=1
loss value at iteration 3186 is too big, stop algorithm for t=0.5
```

```
the loss value at 3186 was 100.10698604917111 for t=0.5
loss value at iteration 7380 is too big, stop algorithm for t=0.25
the loss value at 7380 was 104.82293794831662 for t=0.25
loss value at iteration 21348 is too big, stop algorithm for t=0.125
the loss value at 21348 was 101.21249541727884 for t=0.125
```




```
converged in 30472 iterations
the loss value at 30472 was 0.02410557921888988 for t=0.0625
the loss value at 49999 was 0.031830986484798075 for t=0.03125
the loss value at 49999 was 0.0482517152961788 for t=0.015625
```

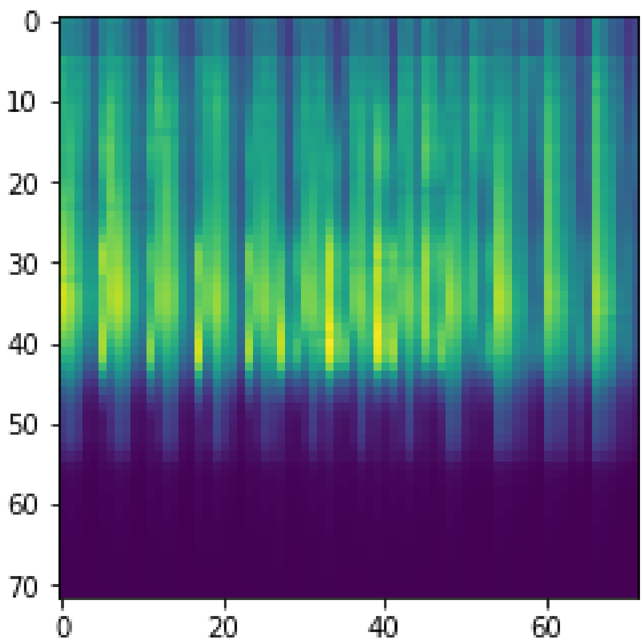


```
Average Frobenius norm for increasing t^i from 1-1/64
54.25976876148113
14.90388584386282
109083.36090683022
11.371580242420599
0.08473476648252666
0.11226283988133913
0.11784878672134419
```

computing images Y and most accurate UW^T computed. Choose $eps = 2 * 0.0017730281702907648$ (since this was the minimum loss found overall described above) and $t = 16$ (quickest average convergence, smallest number of iterations till convergence...) for GD algorithm (accuracy)

[96]:
```python
K=10000
print('This is the image of Y (IMAGE 1)')
# display matrix Y as image
plt.imshow(Y)
plt.show()
# compute GS with computed eps and sufficient K iterations max
UG,WG = gd(U0, W0, Y, I, Itest, 16,2*0.0017730281702907648, K)
print('This is the image of UW.T computed via Matrix Completion using GD␣
↪(IMAGE 2)')
# display UW.T as image
plt.imshow(UG @ WG.T)
plt.yscale('linear')
plt.show()
```

```
This is the image of Y (IMAGE 1)
```



```
converged in 595 iterations
the loss value at 595 was 0.0035413436630722197 for t=16
This is the image of UW.T computed via Matrix Completion using GD (IMAGE 2)
```

```
/tmp/ipykernel_1030203/629753498.py:10: UserWarning: Attempted to set non-
positive top ylim on a log-scaled axis.
Invalid limit will be ignored.
  plt.imshow(UG @ WG.T)
```