CALCULUS II - TB2-CW

UP930537

Answer the following problems and justify your answers.

Problem 1. Let $f(x) = x$ for $-\pi < x < \pi$ and extended periodically by $f(x+2\pi) = f(x)$.

(1) (6 marks) Is f even, odd, or neither?

(2) (10 marks) Find the Fourier representation

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)$$

of f making a suitable choice of L. Give an explicit description of the coefficients a_n and b_n.

(3) (6 marks) Determine the value that the Fourier series you found converges to when $x = 0$, when $x = \pi$, and when $x = \pi/2$.

(4) (6 marks) Use the results above to show that

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots = \frac{\pi}{4}.$$

(5) (6 marks) Use Parseval's Identity to show that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

(6) (6 marks) Show that

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}.$$

Hint: Write

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} + \sum_{n=1}^{\infty} \frac{1}{(2n)^2}$$

and use the result from (5).

Problem 2. The orbit of a planet around the sun is typically elliptical and so understanding ellipses is important. In this exercise we fix real numbers $a, b > 0$ and refer to the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

which is an ellipse in the $X - Y$ plane. Further, the vector function $\mathbf{r}\colon [0, 2\pi] \to \mathbb{R}^3$ given by $\mathbf{r}(t) = \langle a\cos t, b\sin t, 0\rangle$ is a parametrisation of the ellipse, traced once.

(1) (6 marks) Verify that $\mathbf{r}(t)$ is a point whose x and y coordinates satisfy the equation of the ellipse.

(2) (6 marks) Using the equation for the ellipse set up a double integral expressing the area of the ellipse.

(3) (6 marks) Using any method you like evaluate the integral from (2) to obtain the area of the ellipse.
(4) (6 marks) Compute the curvature $\kappa(t)$ of the ellipse.
(5) (6 marks) Set up an integral that computes the circumference of the ellipse. Do not try to solve this integral. It is proven the integral cannot be solved in terms of elementary functions. This is an example of a notoriously complicated class of integrals known as elliptic integrals.

Problem 3. For each of the following claims determine if it is true or false. If you claim it is true, then provide a proof. If you claim it is false, then provide a counterexample.

(1) (6 marks) If two differentiable functions $f, g\colon [-1,1] \to \mathbb{R}$ are related by $g(-x) = f(x)$ for all $x \in [-1,1]$, then the lengths of their graphs are equal.
(2) (6 marks) The curvature of a circle of radius K is $1/K$.
(3) (6 marks) If the curvature of $\mathbf{r}$ at $t = 1$ is 1, then the curvature of $\mathbf{r}$ at $t = 2$ is 2.
(4) (6 marks) If $\mathbf{r}$ is a curve and $\mathbf{r}' \neq 0$ for all t, then the curve given by $\mathbf{r}_1(t) = \mathbf{r}(t)/|\mathbf{r}'(t)|$ is a reparametrisation of $\mathbf{r}$.
(5) (6 marks) If a curve $\mathbf{r}$ has the property that $|\mathbf{r}|$ is a constant, then $\mathbf{r}'$ is perpendicular to $\mathbf{r}$ for all t

Solutions.

Problem 1.

(1) Looking at the point $x = \frac{\pi}{2}$ for example, which is in the range $-\pi < x < \pi$:
$f(-\frac{\pi}{2}) = -\frac{\pi}{2}$, so this satsfies $f(-x) = -f(x)$, therefore f is odd
(2) $L = \pi$
$b_n = \frac{2}{\pi}\int_0^{\pi} xsin(\frac{n\pi x}{\pi})dx$
$=\frac{2}{\pi}\int_0^{\pi} xsin(nx)dx$
$= \frac{2}{\pi}[-\frac{x}{n}cos(nx) + \frac{1}{n^2}sin(nx)]_0^{\pi}$
$= \frac{2}{\pi}[(-\frac{\pi}{n}cos(\pi x) + \frac{1}{n^2}sin(\pi x)) - (-\frac{0}{n}cos(0x) + \frac{1}{n^2}sin(0x))]$
$= \frac{2}{\pi}[(-\frac{\pi}{n}(-1) + \frac{1}{n^2}sin(0)) - (-\frac{0}{n}(1) + \frac{1}{n^2}(0))]$
$= \frac{2}{\pi}[\frac{\pi}{n} - 0]$
$= \frac{2}{\pi}[\frac{\pi}{n}]$
$= \frac{2\pi}{n\pi}$
$= \frac{2}{n}$
This is the coefficent of b_n. The coefficents of a_n and $\frac{a_0}{2}$ will be equal to zero since the function $f(x)$ is odd. $a_n = 0, \forall n \geqslant 0$, because looking at the formula $a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos(\frac{n\pi x}{L})dx$, we have an odd function $f(x)$ being multiplied by an even function $\cos(\frac{n\pi x}{L})$, so the result will be even, and therefore zero in this case. So the Fourier Series will be:
$f(x) = \sum_{n=1}^{\infty} \frac{2}{n}\sin(nx)$
(3) $x = 0$:
$f(0) = \sum_{n=1}^{\infty} \frac{2}{n}sin(0) = \sum_{n=1}^{\infty} \frac{2}{n}(0) = 0$
$x = \pi$:
$f(\pi) = \sum_{n=1}^{\infty} \frac{2}{n}sin(n\pi) = \sum_{n=1}^{\infty} \frac{2}{n}(0) = 0$
$x = \frac{\pi}{2}$:

$f(\frac{\pi}{2}) = \sum_{n=1}^{\infty} \frac{2}{n} sin(n\frac{\pi}{2}) = \frac{2}{1} sin(\frac{\pi}{2}) + \frac{2}{2} sin(2\frac{\pi}{2}) + \frac{2}{3} sin(3\frac{\pi}{2}) + \frac{2}{4} sin(4\frac{\pi}{2}) + \frac{2}{5} sin(5\frac{\pi}{2}) + \frac{2}{6} sin(6\frac{\pi}{2}) + \frac{2}{7} sin(7\frac{\pi}{2}) + \frac{2}{8} sin(8\frac{\pi}{2}) + \frac{2}{9} sin(9\frac{\pi}{2}) + \frac{2}{10} sin(10\frac{\pi}{2}) + \frac{2}{11} sin(11\frac{\pi}{2}) + ...$

$= \frac{2}{1}(1) + \frac{2}{2}(0) + \frac{2}{3}(-1) + \frac{2}{4}(0) + \frac{2}{5}(1) + \frac{2}{6}(0) + \frac{2}{7}(-1) + \frac{2}{8}(0) + \frac{2}{9}(1) + \frac{2}{10}(0) + \frac{2}{11}(-1) + ...$

$= 2 - \frac{2}{3} + \frac{2}{5} - \frac{2}{7} + \frac{2}{9} - \frac{2}{11} + ...$

$= \frac{\pi}{2}$

(4) This series is very similar to the previous series. The only difference is that the 2 is replaced with 1. So we have the series:

$f(x) = \sum_{n=1}^{\infty} \frac{1}{n} \sin(nx)$ instead of $g(x) = \sum_{n=1}^{\infty} \frac{2}{n} \sin(nx)$ where $g(x)$ is the series from (3)

The series here is 2 times smaller than the previous series, so if

$g(\frac{\pi}{2}) = \sum_{n=1}^{\infty} \frac{2}{n} \sin\left(n\frac{\pi}{2}\right) = \frac{\pi}{2}$

Then that means:

$f(\frac{\pi}{2}) = \sum_{n=1}^{\infty} \frac{1}{n} \sin\left(n\frac{\pi}{2}\right) = \frac{1}{1} sin(\frac{\pi}{2}) + \frac{1}{2} sin(2\frac{\pi}{2}) + \frac{1}{3} sin(3\frac{\pi}{2}) + \frac{1}{4} sin(4\frac{\pi}{2}) + \frac{1}{5} sin(5\frac{\pi}{2}) + \frac{1}{6} sin(6\frac{\pi}{2}) + \frac{1}{7} sin(7\frac{\pi}{2}) + \frac{1}{8} sin(8\frac{\pi}{2}) + \frac{1}{9} sin(9\frac{\pi}{2}) + \frac{1}{10} sin(10\frac{\pi}{2}) + \frac{1}{11} sin(11\frac{\pi}{2}) + ...$

$= 1(1) + \frac{1}{2}(0) + \frac{1}{3}(-1) + \frac{1}{4}(0) + \frac{1}{5}(1) + \frac{1}{6}(0) + \frac{1}{7}(-1) + \frac{1}{8}(0) + \frac{1}{9}(1) + \frac{1}{10}(0) + \frac{1}{11}(-1) + ...$

$= 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + ...$

$= \frac{\pi}{4}$

(5) Parsevals Identity states $\frac{1}{2L}\int_{-L}^{L} [f(x)]^2 dx = \frac{a_0 2}{4} + \sum_{n=1}^{\infty} \frac{\left\{a_n^2 + b_n^2\right\}}{2}$

Using the fourier series for $f(x) = x$, from (2) we know that $a_0 = 0$, $a_n = 0, \forall n \geqslant 0$ and $b_n = \frac{2}{n}$

LHS:

$\frac{1}{2\pi}\int_{-\pi}^{\pi} (x)^2 dx$

$= \frac{1}{2\pi}[\frac{1}{3}x^3]_{-\pi}^{\pi}$

$= \frac{1}{6\pi}[x^3]_{-\pi}^{\pi}$

$= \frac{1}{6\pi}(\pi^3 - (-\pi)^3)$

$= \frac{1}{6\pi}(\pi^3 + \pi^3)$

$= \frac{1}{6\pi}(2\pi^3)$

$= \frac{2\pi^3}{6\pi}$

$= \frac{\pi^2}{3}$

RHS:

$\frac{(0)^2}{4} + \sum_{n=1}^{\infty} \frac{\left\{(0)^2 + (\frac{2}{n})^2\right\}}{2}$

$= \sum_{n=1}^{\infty} \frac{(\frac{2}{n})^2}{2}$

$= \sum_{n=1}^{\infty} \frac{4}{n^2} \cdot \frac{1}{2}$

$= \sum_{n=1}^{\infty} \frac{4}{2n^2}$

$= \sum_{n=1}^{\infty} \frac{2}{n^2}$

$= 2\sum_{n=1}^{\infty} \frac{1}{n^2}$

LHS=RHS:

$2\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{3}$

$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{3} \cdot \frac{1}{2}$

$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

(6) $\sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} + \sum_{n=1}^{\infty} \frac{1}{(2n)^2}$

$\sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} + \sum_{n=1}^{\infty} \frac{1}{4n^2}$
$\sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} + \frac{1}{4}\sum_{n=1}^{\infty} \frac{1}{n^2}$
From (5), we know that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$, so we can substitute this in:
$\frac{\pi^2}{6} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} + \frac{1}{4}(\frac{\pi^2}{6})$
$\frac{\pi^2}{6} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} + \frac{\pi^2}{24}$
$\frac{\pi^2}{6} - \frac{\pi^2}{24} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}$
$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{6} - \frac{\pi^2}{24}$
$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}$

Problem 2.

(1) $\mathbf{r}(t) = \langle a\cos t, b\sin t, 0\rangle$, so to verify that the x and y coordinates satisfy the ellipse, let $x = a\cos t$ and $y = b\sin t$. Substitute them into the equation for the ellipse:
$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
$\frac{(a\cos t)^2}{a^2} + \frac{(b\sin t)^2}{b^2} = 1$
$\frac{a^2\cos^2 t}{a^2} + \frac{b^2\sin^2 t}{b^2} = 1$
$\cos^2 t + \sin^2 t = 1$
This is true, therefore the x and y coordinates do satisfy the equation of the ellipse.

(2) First, convert to the $u - v$ plane:
Let $u = \frac{x}{a}$, $v = \frac{y}{b}$.
We now have $u^2 + v^2 = 1$, which is a circle of radius 1 centered at the origin.
To set up the double integral, it will look like this:
$Area = \iint_R \mathbf{J}\cdot dA$, where $\mathbf{J}$ is the Jacobian.
$\mathbf{J} = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \begin{vmatrix} f_u & g_u \\ f_v & g_v \end{vmatrix} = |f_u g_v - g_u f_v|$
Rearranging, we have $f = x = au$, $g = y = bv$. Working out the Jacobian:
$\mathbf{J} = \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix} = |ab - 0| = ab$
Now the double integral is:
$Area = \iint_R ab\cdot dA$
Converting to Polar coordinates, these will be the limits of integration:
$0 \le r \le 1$
$0 \le \theta \le 2\pi$
The final double integral expressing the area of the ellipse in terms of polar coordinates will be:
$\iint z(r,\theta)\cdot r\cdot dr\cdot d\theta = \int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=1} ab\cdot r\cdot dr\cdot d\theta$

(3) Evaluating the integral from (2):
$\int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=1} ab\cdot r\cdot dr\cdot d\theta$
$= \int_{\theta=0}^{\theta=2\pi}\left[\frac{1}{2}abr^2\right]_{r=0}^{r=1} d\theta$
$= \int_{\theta=0}^{\theta=2\pi}(\frac{1}{2}ab - 0)d\theta$
$= \int_{\theta=0}^{\theta=2\pi}\frac{1}{2}ab\cdot d\theta$

$= \left[\frac{1}{2}ab\theta\right]_{\theta=0}^{\theta=2\pi}$
$= \frac{1}{2}ab(2\pi) - \frac{1}{2}ab(0)$
$= \pi ab$

Checking this is the area of the ellipse using the parametrisation in the $u - v$ plane:

In the $u - v$ plane, the area was a circle with radius $r = 1 = a = b$, centered at the origin. The area of a circle is equal to πr^2

$Area = \pi ab$

(4) The paramertisation of the ellipse is given by $\mathbf{r}(t) = \langle a\cos t, b\sin t, 0\rangle$, and the formula for curvature is given by $\kappa(t) = \frac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$.

Working out the derivatives of $\mathbf{r}(t)$:
$\mathbf{r}(t) = \langle a\cos t, b\sin t, 0\rangle$
$\mathbf{r}'(t) = \langle -a \text{ sin } t, b \text{ cos } t, 0\rangle$
$\mathbf{r}''(t) = \langle -a \text{ cos } t, -b \text{ sin } t, 0\rangle$

Working out the cross product of $\mathbf{r}'(t)$ and $\mathbf{r}''(t)$ using cofactor expansion:
$|\mathbf{r}'(t) \times \mathbf{r}''(t)|$
$= |\langle -a \text{ sin } t, b \text{ cos } t, 0\rangle \times \langle -a \text{ cos } t, -b \text{ sin } t, 0\rangle|$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a\text{sin}t & b\text{cos}t & 0 \\ -a\text{cos}t & -b\text{sin}t & 0 \end{vmatrix}$$

$$= \mathbf{i}\begin{vmatrix} b\text{cos}t & 0 \\ -b\text{sin}t & 0 \end{vmatrix} - \mathbf{j}\begin{vmatrix} -a\text{sin}t & 0 \\ -a\text{cos}t & 0 \end{vmatrix} + \mathbf{k}\begin{vmatrix} -a\text{sin}t & b\text{cos}t \\ -a\text{cos}t & -b\text{sin}t \end{vmatrix}$$

$= \mathbf{i}(0) - \mathbf{j}(0) + \mathbf{k}(ab\text{sin}^2 t + ab\text{cos}^2 t)$
$= \mathbf{k}(ab(\text{sin}^2 t + \text{cos}^2 t))$
$= \mathbf{k}(ab)$

$|\mathbf{r}'(t) \times \mathbf{r}''(t)| = |\langle 0, 0, ab\rangle| = \sqrt{(0)^2 + (0)^2 + (ab)^2} = \sqrt{(ab)^2} = ab$, since $a, b > 0$

$|\mathbf{r}'(t)|^3 = |\langle -a \text{ sin } t, b \text{ cos } t, 0\rangle|^3$
$= (\sqrt{(-a\text{sin}t)^2 + (b\text{cos}t)^2})^3$
$= (\sqrt{a^2\text{sin}^2 t + b^2\text{cos}^2 t})^3$
$= (a^2\text{sin}^2 t + b^2\text{cos}^2 t)^{\frac{3}{2}}$

Substituting into the curvature formula:
$\kappa(t) = \frac{ab}{(a^2\text{sin}^2 t + b^2\text{cos}^2 t)^{\frac{3}{2}}}$

(5) Starting with the equation for the ellipse, and solving for y :
$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
$y^2 = b^2(1 - \frac{x^2}{a^2})$
$y = \sqrt{b^2(1 - \frac{x^2}{a^2})}$
$y = ^{+}_{-} b\sqrt{1 - \frac{x^2}{a^2}}$

The formula for arc length is given by $\int_a^b \sqrt{1 + (f'(x))^2}dx$. We can subsitute y' into this equation, and integrate between $x = 0$, and $x = a$. It's above the x axis, so we will only need the positive solution to y.

Working out y' using the chain rule:
$y = b\sqrt{1 - \frac{x^2}{a^2}} = b(1 - \frac{x^2}{a^2})^{\frac{1}{2}}$
$y' = b\cdot \frac{1}{2}\cdot \frac{-2x}{a^2}(1 - \frac{x^2}{a^2})^{-\frac{1}{2}}$

$y' = \frac{-bx}{a^2}(1-\frac{x^2}{a^2})^{-\frac{1}{2}}$
$y' = \frac{-bx}{a^2(1-\frac{x^2}{a^2})^{\frac{1}{2}}}$
$y' = \frac{-bx}{a^2\sqrt{1-\frac{x^2}{a^2}}}$
Substituting y' into the Arc length formula:
$\int_0^a \sqrt{1+(\frac{-bx}{a^2\sqrt{1-\frac{x^2}{a^2}}})^2}dx$
$= \int_0^a \sqrt{1+\frac{b^2x^2}{a^4(1-\frac{x^2}{a^2})}}dx$

The arc length between the limits 0 and $a$will be $\frac{1}{4}$ of the circumference of the ellipse, so the integral needs to be multipled by 4. The circumference will be:
$C = 4\int_0^a \sqrt{1+\frac{b^2x^2}{a^4(1-\frac{x^2}{a^2})}}dx$

Now letting $x = a\sin\theta$, $\frac{dx}{d\theta} = a\cos\theta$, so $dx = a\cos\theta d\theta$. The limits of integration need to change:
When $x = 0$:
$a\sin\theta = 0$
$\sin\theta = 0$
$\theta = \sin^{-1}(0) = 0$
When $x = a$:
$a\sin\theta = a$
$\sin\theta = 1$
$\theta = \sin^{-1}(1) = \frac{\pi}{2}$
So now the integral will be between $\theta = 0$ and $\theta = \frac{\pi}{2}$.
$C = 4\int_0^{\frac{\pi}{2}} \sqrt{1+\frac{b^2a^2\sin^2\theta}{a^4(1-\frac{a^2\sin^2\theta}{a^2})}}a\cos\theta d\theta$

To simplify this integral further we can say $a\cos\theta = \sqrt{a^2\cos^2\theta}$ and substitute this in:
$C = 4\int_0^{\frac{\pi}{2}} \sqrt{1+\frac{b^2a^2\sin^2\theta}{a^4(1-\sin^2\theta)}}\sqrt{a^2\cos^2\theta}d\theta$
Simplifying the denominator first:
$C = 4\int_0^{\frac{\pi}{2}} \sqrt{1+\frac{b^2\sin^2\theta}{a^2(1-\sin^2\theta)}}\sqrt{a^2\cos^2\theta}d\theta$
Now multiplying by the $\sqrt{a^2\cos^2\theta}$:
$C = 4\int_0^{\frac{\pi}{2}} \sqrt{a^2\cos^2\theta+\frac{b^2\sin^2\theta a^2\cos^2\theta}{a^2(1-\sin^2\theta)}}d\theta$
Since $\sin^2\theta+\cos^2\theta = 1$, $\cos^2\theta = 1-\sin^2\theta$. We can now substitute this in as well:
$C = 4\int_0^{\frac{\pi}{2}} \sqrt{a^2\cos^2\theta+\frac{b^2\sin^2\theta a^2\cos^2\theta}{a^2\cos^2\theta}}d\theta$
$C = 4\int_0^{\frac{\pi}{2}} \sqrt{a^2\cos^2\theta+b^2\sin^2\theta}d\theta$
This is the integral that computes the circumference of the ellipse.

Problem 3.

(1) The claim is true.
$g(-x)$ is a reflection of $f(x)$ in the $y$axis, so their length will be equal for all $x \in [-1,1]$
Looking at the case where $f = g = 1$ on the interval $[-1,1]$:
At $x = 1$: $g(-1) = -1$, $f(1) = 1$

At $x = -1$:$g(-(-1)) = 1,\ f(-1) = -1$
The length of $g$will be:
$\sqrt{(1 - -1)^2 + (-1 - 1)^2}$
$= \sqrt{(2)^2 + (-2)^2}$
$= \sqrt{4 + 4}$
$= \sqrt{8}$
$= 2\sqrt{2}$
The length of f will be:
$\sqrt{(-1 - 1)^2 + (-1 - 1)^2}$
$= \sqrt{(-2)^2 + (-2)^2}$
$= \sqrt{4 + 4}$
$= \sqrt{8}$
$= 2\sqrt{2}$
This will be the same for any value given to f and $g$in the interval $[-1, 1]$, therefore the lengths of their graphs are equal.

(2) The following claim is true. The curvature of a circle of radius K is $1/K$. This can be shown by taking the parametrisation of a circle and computing the curvature
This is the position vector for the circle of radius K:
$\mathbf{r}(t) = \langle K\cos t, K\sin t, 0\rangle$
$\mathbf{r}'(t) = \langle -K\text{sin } t, K\text{cos } t, 0\rangle$
$\mathbf{r}''(t) = \langle -K\cos t, -K\sin t, 0\rangle$
The formula for curvature is given by $\kappa(t) = \frac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$
$|\mathbf{r}'(t)\times\mathbf{r}''(t)| = |\langle -K\text{sin } t, K\text{cos } t, 0\rangle \times \langle -K\cos t, -K\sin t, 0\rangle|$
$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -K\text{sin } t & K\text{cos } t & 0 \\ -K\cos t & -K\sin t & 0 \end{vmatrix}$
$= \mathbf{i}\begin{vmatrix} K\text{cos}t & 0 \\ -K\text{sin}t & 0 \end{vmatrix} - \mathbf{j}\begin{vmatrix} -K\text{sin}t & 0 \\ -K\text{cos}t & 0 \end{vmatrix} + \mathbf{k}\begin{vmatrix} -K\text{sin}t & K\text{cos}t \\ -K\text{cos}t & -K\text{sin}t \end{vmatrix}$
$= \mathbf{i}(0) - \mathbf{j}(0) + \mathbf{k}(K^2\text{sin}^2t + K^2\text{cos}^2t)$
$= \mathbf{k}(K^2)$
$= |\langle 0, 0, K^2\rangle|$
$= \sqrt{(K^2)^2}$
$= K^2$, it has to be positive, since the radius cannot be less than zero.
So $|\mathbf{r}'(t)\times\mathbf{r}''(t)| = K^2$
$|\mathbf{r}'(t)|^3 = |\langle -K\ \text{sin } t, K\ \text{cos } t, 0\rangle|^3$
$= (\sqrt{(-K\text{sin } t)^2 + (K\text{cos } t)^2})^3$
$= (\sqrt{K^2\text{sin}^2t + K^2\text{cos}^2t})^3$
$= (\sqrt{K^2})^3$
$= (K^2)^{\frac{3}{2}}$
$= K^3$
Substiuting these into the formula for the curvature:
$\kappa(t) = \frac{K^2}{K^3} = \frac{1}{K}$
Therefore the curvature of a circle of radius K is $1/K$.

(3) This claim is false. This can be shown using the counterexample $\mathbf{r}(t) = \langle\cos t, \sin t, 0\rangle$, and computing the curvature at $t = 1$ and $t = 2$.

$\mathbf{r}(t) = \langle \cos t, \sin t, 0\rangle$
$\mathbf{r}'(t) = \langle -\sin t, \cos t, 0\rangle$
$\mathbf{r}''(t) = \langle -\cos t, -\sin t, 0\rangle$
$|\mathbf{r}'(t) \times \mathbf{r}''(t)| = |\langle -\sin\ t, \cos\ t, 0\rangle \times \langle -\cos t, -\sin t, 0\rangle|$
$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin\ t & \cos\ t & 0 \\ -\cos t & -\sin t & 0 \end{vmatrix}$
$= \mathbf{i}\begin{vmatrix} \cos t & 0 \\ -\sin t & 0 \end{vmatrix} - \mathbf{j}\begin{vmatrix} -\sin t & 0 \\ -\cos t & 0 \end{vmatrix} + \mathbf{k}\begin{vmatrix} -\sin t & \cos t \\ -\cos t & -\sin t \end{vmatrix}$
$= \mathbf{i}(0) - \mathbf{j}(0) + \mathbf{k}(\sin^2 t + \cos^2 t)$
$= |\langle 0, 0, 1\rangle|$
$= \sqrt{1^2}$
$= 1$
$|\mathbf{r}'(t)|^3 = |\langle -\sin t, \cos t, 0\rangle|^3$
$= (\sqrt{(-\sin t)^2 + (\cos t)^2})^3$
$= (\sqrt{\sin^2 t + \cos^2 t})^3$
$= (\sqrt{1})^3$
$= (1)^{\frac{3}{2}}$
$= 1$
So $\kappa(t) = \frac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \frac{1}{1} = 1$
Computing the curvature:
At $t = 1 : \kappa(1) = 1$
At $t = 2 : \kappa(2) = 1$
Since the curvature at $t = 2$ does not equal 2, the claim is false.

(4) The claim is false. The counterexample $\mathbf{r}(t) = \langle \cos t, \sin t, 0\rangle$ can be used to show this.
$\mathbf{r}(t) = \langle \cos t, \sin t, 0\rangle$
$\mathbf{r}'(t) = \langle -\sin t, \cos t, 0\rangle$
$|\mathbf{r}'(t)| = \sqrt{\sin^2(t) + \cos^2(t)} = \sqrt{1} = 1$
Computing $\mathbf{r}_1(t)$, given by the formula $\mathbf{r}_1(t) = \frac{\mathbf{r}(t)}{|\mathbf{r}'(t)|}$
$\mathbf{r}_1(t) = \frac{\mathbf{r}(t)}{|\mathbf{r}'(t)|} = \frac{\langle \cos t, \sin t, 0\rangle}{1} = \langle \cos t, \sin t, 0\rangle$
$\mathbf{r}_1(t) = \mathbf{r}(t)$,so $\mathbf{r}_1$ is not a reparametrisation of $\mathbf{r}$. Therefore the claim is false.

(5) The claim is true
For$|\mathbf{r}|$to be a constant, $\mathbf{r}(t)$ must not depend on t. So $\mathbf{r}$can be written like this:
$\mathbf{r}(t) = \langle a, b, c\rangle$ where a, b and c are constants. And $\mathbf{r}$ does not involve t.
$\mathbf{r}'(t) = \langle 0, 0, 0\rangle$
For two vectors to be perpendicular, their dot product must be equal to zero.
Computing the dot product of $\mathbf{r}$ and $\mathbf{r}'$:
$\mathbf{r}\cdot \mathbf{r}' = \langle a, b, c\rangle \cdot \langle 0, 0, 0\rangle$
$= a(0) + b(0) + c(0)$
$= 0$
So $\mathbf{r}'$ is perpendicular to $\mathbf{r}$ for all t.