FINM2002 Derivatives
FINM6041 Applied Derivatives
Lecture 5 - Binomial Option Pricing Model

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Hull *et al.*: Chapters 12

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Review of Previous Lecture

- Options basics
- Option price boundaries
- Put-call parity

Lecture Overview

- In today's class
  - Intuition behind the Binomial Option Pricing Model
    - Arbitrage-free replication approach (optional)
    - Delta-hedging approach (optional)
    - Traditional approach (optional)
    - Risk-neutral approach
  - Binomial tree (risk-neutral) approach
  - Value European and American options

1. Arbitrage-free replication approach

- How to value the options?
  - Two assets that provide the same cash flow must logically have the same price today
  - Arbitrage-free, also called Law of One Price
- Basic approach
  - Create a portfolio that “replicates” the payoff of option
  - The replicating portfolio can be formed from any assets
  - But usually it is comprised of stocks (short or long positions) and risk-free bonds (lending or borrowing)
  - The present value of the replicating portfolio is the price of the option today

1. Arbitrage-free replication approach

- How to create a portfolio using stock and risk-free bond to replicate the payoff to an option?
  - Option payoff is dependent on the value of the underlying stock
  - We can observe the current price of the underlying stock and bonds
  - Payoff of bond is fixed over time
  - We need a model of how stock price might move from now through to the expiry of the option.
- Model the "evolution" of stock price
  - The Binomial Model assumes that stock prices follow a multiplicative Binomial process over discrete periods.
  - Black-Scholes Model use a continuous-time stochastic differential equation to model stock prices ("geometric Brownian motion")

1. Arbitrage-free replication approach

- The Binomial Model
  - Assumes that stock prices follow a multiplicative Binomial process over discrete periods
  - The current stock price is S. In one period's time, the stock price will take on one of two values. It starts at S, then either:
    - S_u if moves up, or
    - S_d if moves down
  - This is what is meant by Binomial (only two events can occur)
    - Easier to build some intuition about what determines option values
    - Avoids the complexities of continuous-time mathematics
    - Widely employed in practice.
- Once we know the evolution of stock price, we can build a "replicating portfolio" to price the option written on this stock.

1. Arbitrage-free replication approach

- Assume a stock currently worth \$10 will be worth either \$13.5 or \$7.4 next period one year later. What is the value of a call (f) with a \$7 strike price if the risk free rate is 5% p.a.?

$$
\begin{array}{ccc}
 & & \begin{aligned} S_u &= 13.5 \\ f_u &= 6.5 \end{aligned} \\
 & \nearrow & \\
\begin{aligned} S &= 10 \\ X &= 7 \\ f &= ? \end{aligned} & & \\
 & \searrow & \\
 & & \begin{aligned} S_d &= 7.4 \\ f_d &= 0.4 \end{aligned} \\
\text{Time 0} & & \text{Time 1}
\end{array}
$$

- f_u, f_d are the payoffs from option when stock price moves up or down. $\mathrm{Max}(S_T - X, 0)$ for call options.

1. Arbitrage-free replication approach

- At Time 0 today, we construct a two-piece portfolio comprising:
  - Δ shares in the underlying stock, and
  - $\$B$ invested in the risk-free bond
- We will cleverly choose Δ and B so that the portfolio value "replicates" the call option payoff.
  - At Time 1, no matter the stock price goes up or down, the payoff on the replicating portfolio equals the payoff on call option
  - If we can do this, then the construction cost of the replicating portfolio must equal the price of the option.
- Why risk-free bond?
  - It's easier to just think of this as the existence of a bank at which investors can borrow or lend as at the risk-free interest rate.
  - Hence, if \$1 is invested in the bank account and left there for T years, it will grow to $\$1e^{rT}$.

1. Arbitrage-free replication approach

- Notations
  - Stock Price (S)
  - Option strike price (X)
  - Current option price on Stock (f)
  - Option duration on Stock (T)
  - Stock price at time T if it moves up (S_u)
  - Stock price at time T if it moves down (S_d)
  - Payoff from option when stock price moves up (f_u)
  - Payoff from option when stock price moves down (f_d)

$$
\begin{tikzcd}
 & S_u,\ f_u \\
\mathrm{S,\ X,\ f} \arrow[ur] \arrow[dr] & \\
 & S_d,\ f_d
\end{tikzcd}
$$

Time 0 Time 1

1. Arbitrage-free replication approach

- Payoff from replicating portfolio equals option payoff at Time 1

$$S_u\Delta + e^{rT}B = f_u$$
$$S_d\Delta + e^{rT}B = f_d$$

- Two equations with two unknowns (Δ and B)

$$\Delta = \frac{f_u - f_d}{S_u - S_d}$$
$$B = \frac{S_u f_d - S_d f_u}{(S_u - S_d)e^{rT}}$$

- The option price is same as replicating portfolio at Time 0

$$f = S\Delta + B$$

1. Arbitrage-free replication approach

- Assume a stock currently worth \$10 will be worth either \$13.5 or \$7.4 next period one year later. What is the value of a call (f) with a \$7 strike price if the risk free rate is 5% p.a.?

$$
\begin{array}{lcr}
 & & S_u = 13.5 \\
 & \nearrow & f_u = 6.5 \\
S = 10 & & \\
X = 7 & & \\
f = ? & & \\
 & \searrow & S_d = 7.4 \\
 & & f_d = 0.4
\end{array}
$$

Time 0 Time 1

1. Arbitrage-free replication approach

- Using the numbers from the example:

$$\Delta = \frac{f_u - f_d}{S_u - S_d} = \frac{6.5 - 0.4}{13.5 - 7.4} = 1$$

$$B = \frac{S_u f_d - S_d f_u}{(S_u - S_d)e^{rT}} = \frac{13.5 \times 0.4 - 7.4 \times 6.5}{(13.5 - 7.4)e^{0.05 \times 1}} = -\$6.66$$

- Note
    - Δ won't always be one. It depends on the *moneyness* of the option
    - If B is negative, that just means short bond (borrow money)
    - The replicating portfolio to value a call option is a long position in the stock with a short position in the bond (borrow money)
    - The replicating portfolio to value a put option is a short position in the stock and a long position in the bond (lend money)

1. Arbitrage-free replication approach

- Let's prove that the replication is indeed correct
- If we buy one share and borrow \$6.66, then this strategy should generate payoffs that replicate a call option.
- If stock prices rises:

$$1 \times 13.5 - 6.66e^{0.05\times 1} = \$6.5 = f_u$$

- If stock price falls:

$$1 \times 7.4 - 6.66e^{0.05\times 1} = \$0.4 = f_d$$

- Since this portfolio generates payoffs that precisely replicate the call option, the option price must equal the construction cost of the portfolio (else arbitrage opportunity).

1. Arbitrage-free replication approach

- Back at Time 0, the construction cost of this portfolio is:

$$f = S\Delta + B = 1 \times 10 - 6.66 = \$3.34$$

- This is the arbitrage-free price of the call option.
- Our ability to replicate the option payoff enforces a particular price on the option (specifically, the construction cost of the replicating portfolio).

1. Arbitrage-free replication approach - Extension 1

- In the previous example, we assume S_u and S_d directly
- We can also write S_u and S_d as a fraction of S

$$S_u = uS$$
$$S_d = dS$$

- Where $u > 1$ and $0 \leq d < 1$
  - In the up state, the stock price increases by $(u - 1)$
  - In the down state, the stock price decreases by $(d - 1)$
- In the previous example numerical example, we have

$$S = \$10, S_u = \$13.5, S_d = \$7.4$$

- We can rewrite as $u = 1.35$, and $d = 0.74$
  - In the up state, the stock price increases by 35%
  - In the down state, the stock price decreases by 26%

1. Arbitrage-free replication approach - Extension 1

- With $S_u = uS$ and $S_d = dS$, we can rewrite the formula

$$\Delta = \frac{f_u - f_d}{S_u - S_d} = \frac{f_u - f_d}{(u-d)S}$$

$$B = \frac{S_u f_d - S_d f_u}{(S_u - S_d)e^{rT}} = \frac{uf_d - df_u}{(u-d)e^{rT}}$$

$$f = S\Delta + B = \frac{f_u - f_d}{(u-d)} + \frac{uf_d - df_u}{(u-d)e^{rT}}$$

- We can also show (we will come back later)

$$f = e^{-rT}[af_u + (1-a)f_d] \qquad \text{where} \quad a = \frac{e^{rT} - d}{u - d}$$

1. Arbitrage-free replication approach - Extension 2

- What is the probabilities of price goes up (down) over period?
- It is irrelevant!
- We managed to value the call option without using the probabilities of up/down movements. This fact is crucial (it motivates our use of risk-neutral pricing approach).
- Likewise, there was no mention of the expected return on the stock or the expected return on the call option.
- The option price ($3.34) is driven solely by the cost of the replicating strategy.

2. Delta-hedging approach

- Replication approach
  - Construct a strategy consisting of stock and bond
  - Replicate the option payoff
- Delta-hedging approach
  - An alternative, but essentially equivalent method
  - Construct a strategy consisting of option and stock
  - Such that the value of this portfolio is constant, no matter which direction the stock moves

- Delta-hedging approach
- Use same Binomial model of stock price movements
- Construct a portfolio consisting of:
  - Δ shares in the underlying stock, and
  - One short option position (e.g., write a call option)
- Let V_u and V_d denote the value of this portfolio at the end of the next period if share price moves up and down respectively.
- Irrespective of whether stock price moves up or down, the final portfolio value will be the same
- The present value of such risk-free investment equals the initial investment of the portfolio

- Value of this portfolio is constant (risk-free)

$$
\begin{aligned}
V_u &= V_d \\
S_u\Delta - f_u &= S_d\Delta - f_d \\
\Delta &= \frac{f_u - f_d}{S_u - S_d} = \frac{f_u - f_d}{(u-d)S}
\end{aligned}
$$

- Recall the initial portfolio was constructed as Δ shares and one short option, then we can compute the option price at Time 0

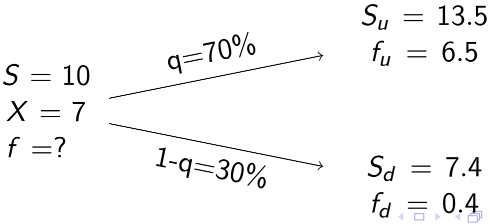
$$
\begin{aligned}
S\Delta - f &= V_u e^{-rT} = V_d e^{-rT} \\
f &= \frac{f_u - f_d}{(u-d)} + \frac{uf_d - df_u}{(u-d)e^{rT}}
\end{aligned}
$$

- Exactly the same option price as in the Replication approach
- Both approaches give the same and correct answer

3. Traditional approach

- Fundamental valuation principle
  - An asset's value equals its expected future cashflows discounted to present value at an appropriate discount rate.
- In Foundations of Finance, you valued many assets as follows:
  - Draw a timeline and schedule the expected cashflows from the asset,
  - Consider how risky/uncertain the cashflows are and choose a discount rate (a required rate of return) that reflects the riskiness,
  - For stocks, use expected stock return $E(r)$ as the discount rate.
  - Discount all cashflows to present value and add them up to a value.
- This approach is used to value stocks, bonds, projects, companies.
- Let's try to use this familiar approach to value a call option
- Traditional approach is not required in this class, but it helps you to understand the risk-neutral approach

- Assume a stock currently worth \$10 will be worth either \$13.5 or \$7.4 next period one year later. What is the value of a call (f) with a \$7 strike price if the risk free rate is 5% p.a.?
- In the replication approach, we show that the probability of stock price moves up or down is irrelevant to the option pricing
- Let's (randomly) assume the probability of stock price goes up is $q = 70\%$, and price goes down is $1 - q = 30\%$
- We call q the physical/actual/real probability



3. Traditional approach

- To value an option using traditional approach, we need to consider:
  - The timing of the option payoff (discount time): expiration period T
  - The expected payoff (cash flow)
    - There is a 70% chance stock price goes up and option pays off \$6.50
    - and a 30% chance stock price goes down and option pays off \$0.40

$$\text{Expected Option Payoff} = 70\% \times 6.5 + 30\% \times 0.4 = \$4.67$$

  - The discount rate that represents the risk of option?

3. Traditional approach

- For stocks, we can use CAPM to compute $E(r)$ as discount rate

$$E(r_s) = r_f + \beta_S[E(r_m) - r_f]$$

- The discount rate that represents the risk of option?
- Cox and Rubinstein (1985) show that the appropriate rate of return for a call option is:

$$E(r_c) = r_f + \Omega[E(r_s) - r_f]$$

- This is a one-period static model
- r_f is the discrete risk-free rate, $E(r_s)$ is the expected stock return
- Ω is the elasticity of the option price (i.e., the % change in option price for a 1% change in underlying stock value)

3. Traditional approach

- Given $S = \$10, S_u = \$13.5, S_d = \$7.4, u = 1.35, d = 0.74$
  $f_u = \$6.5, f_d = \$0.4, q = 70\%, 1 - q = 30\%, r_f = 1 - e^{5\%} = 5.13\%$
- Step 1, expected stock return

$$\begin{aligned} E(r_s) &= q \times (u - 1) + (1 - q) \times (d - 1) \\ &= 70\% \times 0.35 + 30\% \times (-0.26) = 0.167 \end{aligned}$$

- Step 2, elasticity of option price

$$\Omega = \frac{(f_u - f_d)/f}{(S_u - S_d)/S} = \frac{(f_u - f_d)/f}{(u - d)S/S} = \frac{S}{f}\Delta = \frac{10}{f}$$

- Step 3, discount rate for the call option

$$E(r_c) = r_f + \Omega[E(r_s) - r_f] = 0.0513 + \frac{10}{f}(0.167 - 0.0513)$$

3. Traditional approach

- If we use the traditional valuation approach, we take the expected option payoff and discount it to PV
- Step 4 option price

$$f = \frac{\text{Expected Option Payoff}}{1 + E(r_c)} = \frac{q \times f_u + (1-q) \times f_d}{1 + E(r_c)}$$
$$= \frac{70\% \times 6.5 + 30\% \times 0.4}{1 + [0.0513 + \frac{10}{f}(0.167 - 0.0513)]}$$

We can solve

$$f = \$3.34$$

- The option price is the same as previous approaches
- Notice that we randomly assumed q
- The result remains the same with other q (such as 0.8 or 0.6)

3. Traditional approach

- Actually, we can solve f analytically from equations step 1 to 4

$$f = \frac{f_u - f_d}{(u-d)} + \frac{uf_d - df_u}{(u-d)(1+r_f)}$$

- q is canceled out during the derivation and becomes irrelevant!
- It can be translated into continuous version

$$f = \frac{f_u - f_d}{(u-d)} + \frac{uf_d - df_u}{(u-d)e^{rT}}$$

- This is the same as the Replication and Delta-hedging approaches

4. Foundations for risk-neutral approach

- So far we learned three approaches to price options
- We can (nearly) always use Replication or Delta-hedging
- Traditional approach does work but is too complicated!
  - Discount rate for the option depends on the current share price
  - Discount rate keeps changing from period to period
- Why do we go through all these three approaches?
  - In Replication or Delta-hedging approaches,
    - We did not use the real-world probabilities (q) of up/down movements in stock price
    - We did not use the expected return on the stock $E(r_S)$, or the appropriate discount rate for the option $E(r_C)$.
  - In Traditional approach
    - We did use q, $E(r_S)$ and $E(r_C)$, but they turned out irrelevant
    - Even a wrong q will give us correct answer!

4. Foundations for risk-neutral approach

- In fact, the probabilities q of up/down stock price movements are so irrelevant, that we can get the correct option value (\$3.34) no matter what probabilities we use (even if we use the wrong ones).
- Then why don't we assume a probability that's easiest for us to compute the option price?
- A risk-neutral world in which investors are risk neutral and risk free rate is the appropriate discount rate for all asset.
- This is the fundamental intuition for risk-neutral approach

4. Foundations for risk-neutral approach

- Valuation in the real world
- Asset pricing is difficult because investors are risk averse
  - Risk averse investors don't like risk.
  - It doesn't mean that they won't take on a risky investment,
  - but they want to be compensated by a higher expected return.
  - Expected payoff is calculated using real-world probabilities, and then discounted by real-world discount rate.

$$E(r_s) = r_f + \beta_S[E(r_m) - r_f] \qquad E(r_c) = r_f + \Omega[E(r_s) - r_f]$$

$$S = \frac{E(\text{Payoff})_S}{1 + E(r_s)} \qquad f = \frac{E(\text{Payoff})_f}{1 + E(r_c)}$$

4. Foundations for risk-neutral approach

- Valuation in a risk-neutral world
- Risk-neutral investors, they would not demand extra compensation for bearing risk.
- The appropriate discount rate for stocks and options would be the risk-free rate.

$$E(r_s) = r_f + \cancel{\beta_S[E(r_m) - r_f]} \qquad E(r_c) = r_f + \cancel{\Omega[E(r_s) - r_f]}$$

Therefore

$$S = \frac{E^*(\text{Payoff})_S}{1 + r_f} \qquad f = \frac{E^*(\text{Payoff})_f}{1 + r_f}$$

- However, we can't use the real-world probabilities q to compute the expected payoff
- Find the probabilities that would apply if investors are risk-neutral

4. Foundations for risk-neutral approach

- Find the risk-neutral probabilities p
- Such that the discount rate for stock cash flow is risk-free rate

$$
\begin{aligned}
S_0 &= E^*(S_T)e^{-rT} \\
S &= [pS_u + (1-p)S_d]e^{-rT} \\
&= [pu + (1-p)d]Se^{-rT}
\end{aligned}
$$

- Therefore we can solve

$$p = \frac{e^{rT} - d}{u - d}$$

- p is the probability of an up movement in a risk-neutral world
- $1 - p$ is the probability of a down movement in a risk-neutral world

5. Risk-neutral approach

- We can then compute the option price

$$f = e^{-rT} E^{*}[\text{Option payoff}]$$
$$f = e^{-rT}[pf_u + (1-p)f_d]$$

- Let's try our previous example

$$\begin{aligned}
p &= \frac{e^{rT} - d}{u - d} = \frac{e^{0.05\times 1} - 0.74}{1.35 - 0.74} = 0.5103 \\
1 - p &= 0.4897 \\
f &= e^{-rT}[pf_u + (1-p)f_d] \\
&= e^{0.05\times 1}[0.5103 \times 6.5 + 0.4897 \times 0.4] \\
&= \$3.34
\end{aligned}$$

- We get the same option price as other approaches

5. Risk-neutral approach

- Risk-neutral approach to price option

$$f = e^{-rT}[pf_u + (1-p)f_d]$$
$$p = \frac{e^{rT} - d}{u - d}$$
$$u = S_u/S$$
$$d = S_d/S$$

- Summary
  - We can derive the same formula for the option price through four different approaches: Replication, Delta-hedging, Traditional, and risk-neutral approaches.
  - Each approach begins with different intuition but reaches same conclusion.

5. Risk-neutral approach

- Summary (continue)
  - We noticed that the real-world probabilities (q) of up/down movements are irrelevant to pricing an option.
  - We cleverly chose the risk-neutral probability p that is consistent with investors being risk neutral.
  - We can then value the option by:
    - Calculate the risk-neutral probability p
    - Calculate the expected option payoff using p
    - Discount the expected payoff to present value using risk-free rate
  - We get the right option value, with a lot less effort

5. Risk-neutral approach

- Other notes
  - Please do not infer that investors are risk neutral. They most certainly are not.
  - We choose risk-neutral approach because it's easy and yield the correct option price
  - For the purpose of the option valuation, we merely use a set of probabilities that are consistent with a risk-neutral world.
  - Doing this gives us the correct option price – it is the same price we get if we replicate or delta hedge.

6. Application of risk-neutral approach: One period

$$
\begin{array}{ccc}
 & & S_u = 22 \\
 & \overset{p}{\nearrow} & f_u = 1 \\
S = 20 & & \\
X = 21 & & \\
f = ? & & \\
 & \underset{(1-p)}{\searrow} & S_d = 18 \\
 & & f_d = 0
\end{array}
$$

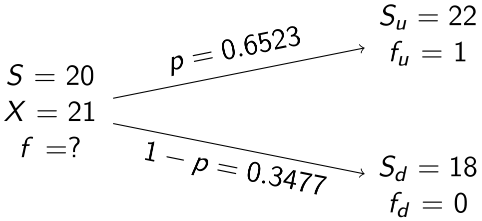
- We know the following for the stock and call option

$$r = 12\%, \quad T = 3 \text{ months} = 0.25,$$
$$u = S_u/S = 22/20 = 1.1, \quad d = S_d/S = 18/20 = 0.9$$

- The risk-neutral probability of an up movement in the stock price:

$$p = \frac{e^{rT} - d}{u - d} = \frac{e^{12\% \times 0.25} - 0.9}{1.1 - 0.9} = 0.6523$$

6. Application of risk-neutral approach: One period



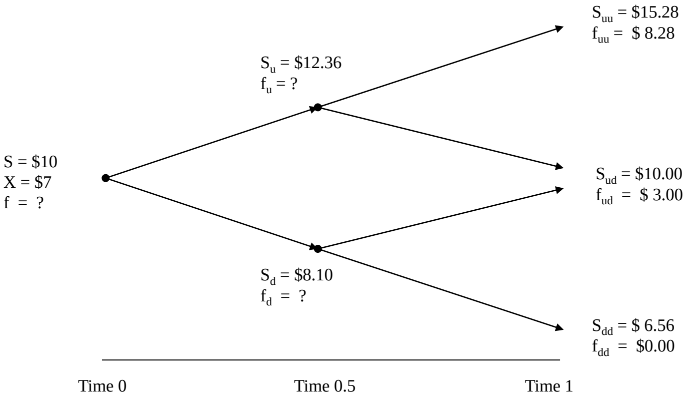
- The value of the option is:

$$
\begin{aligned}
f &= e^{-rT}[pf_u + (1-p)f_d] \\
&= e^{-0.12\times 0.25}(0.6523 \times 1 + 0.3477 \times 0) = \$0.633
\end{aligned}
$$

7. Application of risk-neutral approach: Multi-period

- (One-step) Binomial Tree risk-neutral approach
- Multi-step Binomial Tree risk-neutral approach
- Example
  - Current stock price S = \$10
  - Price a call option with $X = \$7$, expires in 12 months
  - With a twp-step tree, each step is 6 months ($T = 0.5$)
  - The risk-free interest rate is 5% p.a., continuously compounded
  - At the end of each period, stock price either goes up by 23.6% or down by 19%. Which means $u = 1.236, d = 0.81$

7. Two-Step Binomial Tree

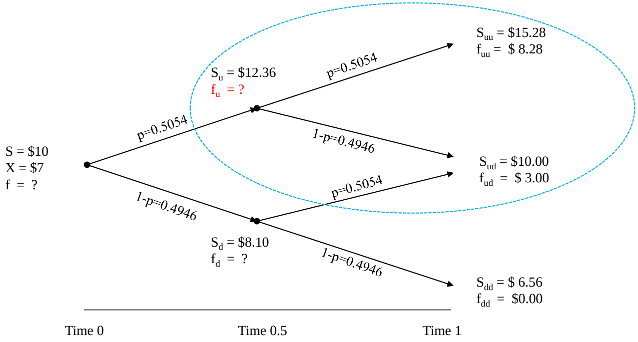


- Calculate the risk-neutral probability p

$$p = \frac{e^{rT} - d}{u - d} = \frac{e^{0.05 \times 0.5} - 0.81}{1.236 - 0.81} = 0.5054$$
$$1 - p = 0.4946$$

- I will demonstrate two ways to use risk-neutral approach to value this option in a multi-step tree:
  - Method (i) involves focusing on each branch of the tree and valuing the option on that branch.
  - Method (ii) can only be used for European options and essentially values the option with one calculation.

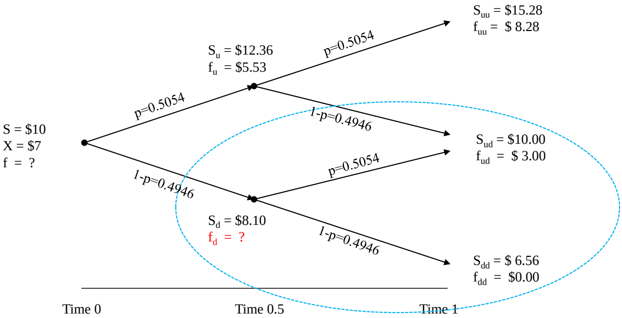
- Method (i)
  - Start at expiry (right-side of tree)
  - Focus on one "branch" at a time
  - Work from right to left
  - We will start with the upper-right branch and calculate the call option value one-step (six months) back



- Upper-right branch

$$
\begin{aligned}
f_u &= e^{-rT}[pf_{uu} + (1-p)f_{ud}] \\
&= e^{-0.05\times0.5}(0.5054 \times 8.28 + 0.4946 \times 3) \\
&= \$5.53
\end{aligned}
$$

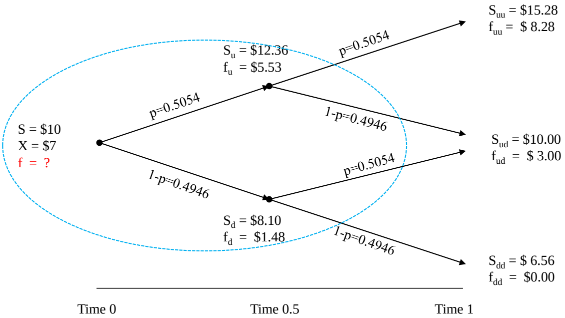
7. Two-Step Binomial Tree



- Lower-right branch

$$
\begin{aligned}
f_d &= e^{-rT}[pf_{ud} + (1-p)f_{dd}] \\
&= e^{-0.05\times 0.5}(0.5054 \times 3 + 0.4946 \times 0) \\
&= \$1.48
\end{aligned}
$$

7. Two-Step Binomial Tree

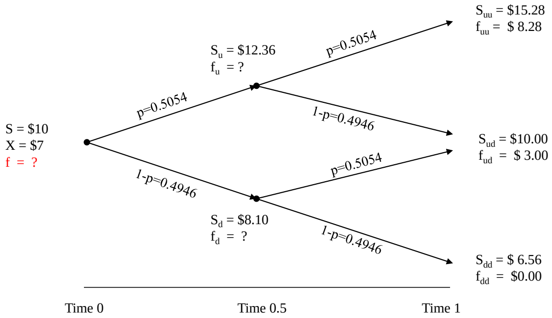


- Far-left branch

$$
\begin{aligned}
f &= e^{-rT}[pf_u + (1-p)f_d] \\
&= e^{-0.05\times 0.5}(0.5054 \times 5.53 + 0.4946 \times 1.48) \\
&= \$3.44
\end{aligned}
$$

7. Two-Step Binomial Tree

- Method (ii)
  - In the case of European options, actually there is no need to work backwards through the tree
  - Focus on the option payoff at each terminal node
    - With a two-step tree, there are three finishing nodes
    - They generate option payoffs of \$8.28, \$3 and \$0
  - Calculate the "path probability" of getting that payoff
    - To get to the \$8.28 payoff, we need two up steps
    - To get to the \$3 payoff, we need one up step and one down step
    - To get to the \$0 payoff, both steps must be down
  - Calculate how many paths lead to that payoff



$$
\begin{aligned}
f &= e^{-rT}[p \times p \times f_{uu} + (1-p)(1-p) \times f_{dd} + 2 \times p \times (1-p) \times f_{ud}] \\
&= e^{-0.05 \times 1}[0.5054^2 \times 8.28 + 0 + 2 \times 0.5054 \times 0.4946 \times 3] \\
&= \$3.44
\end{aligned}
$$

Note $T = 1$ here

7. Two-Step Binomial Tree

- European options (so far)
  - We can use it to price both European call and put
  - The holder can only exercise the option on the expiry date
  - This makes valuing the option easy because we know precisely when the payoff will occur (on the expiry date if option is ITM)
  - The Binomial tree approach is important for non-European options (e.g., American options, or more exotic options)
    - Despite considerable effort, mathematicians have not yet been able to derive a formula (like Black Scholes) to value American options
    - This is a problem because most traded options are American-style
    - we have to rely on numerical approximation methods
    - The Binomial tree approach is easily adjusted to handle the early-exercise feature of American options

8. American options

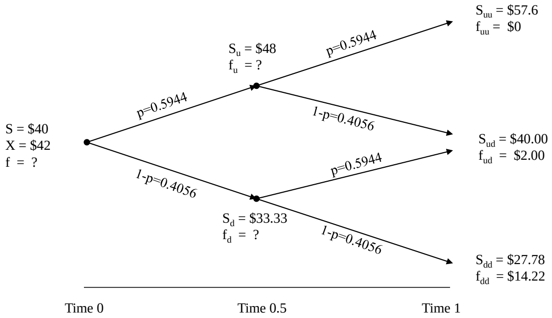
- We learned that an American call option on a non-dividend paying stock should never be exercised early
- Hence, the price of such an American call will equal the price of a corresponding European call
- However, we cannot conclude that an American put option should never be exercised early
- Therefore, we need a way to value American put options

8. American put option

- When will you exercise an American put?
  - Recall that an American put option's lower bound at any time τ leading up to expiry is $X – S_\tau$
  - If the value of an American put ever falls below its $X – S_\tau$, the holder should exercise early
  - The ability of the holder to exercise the put before expiry is valuable and such value should be reflected in option price
  - For this reason, an American put option will be slightly more expensive than an otherwise-equivalent European put option

- Example
  - The current stock price $S = \$40$
  - This stock does not pay dividends
  - Price an American put with $X = \$42$, expires in 12 months
  - With a two-step tree, each step is 6 months ($T = 0.5$)
  - The risk-free rate is $r = 10\%$ p.a., continuously compounded
  - In each branch of the tree, share price will either rise by 20% or fall by 16.67% ($u = 1.20$, $d = 0.8333$)

8. American put option

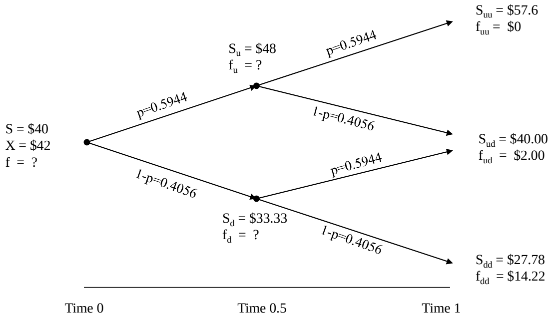


- Plot the binomial tree and calculate p

$$p = \frac{e^{rT} - d}{u - d} = \frac{e^{0.1\times 0.5} - 0.8333}{1.20 - 0.8333} = 0.5944$$

$$1 - p = 0.4056$$

8. American put option



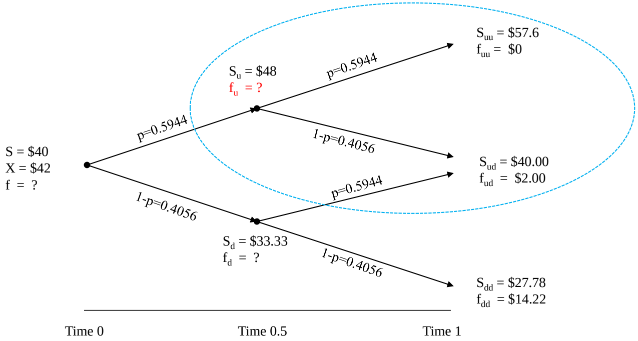
- If it was a European put option

$$\begin{aligned} f &= e^{-rT}[p \times p \times f_{uu} + (1-p)(1-p) \times f_{dd} + 2 \times p \times (1-p) \times f_{ud}] \\ &= e^{-0.1\times 1}[0.5944^2 \times 0 + 0.4056^2 \times 14.22 + 2 \times 0.5944 \times 0.4056 \times 2] \\ &= \$2.99 \end{aligned}$$

8. American put option

- For an American put, the value of the option at the final nodes is the same as for European put
- Calculate option value at each earlier node on the tree (use the risk-neutral approach formula as before)
- At each of the earlier node, we compare
  - The option value we computed (f_{node})
  - The payoff from early exercise ($X - S_{node}$)
- Value of the option is the greater of the above two in each node

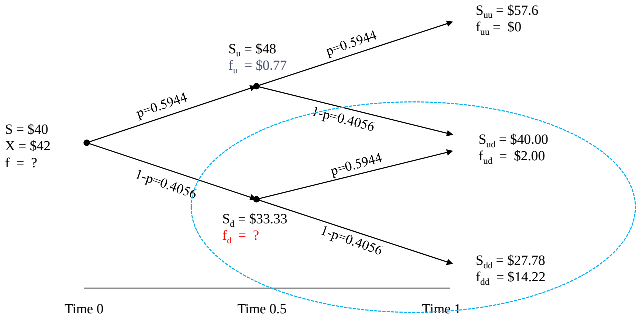
8. American put option



- Upper-right branch

$$f_u = e^{-rT}[pf_{uu} + (1-p)f_{ud}] = e^{-0.1\times0.5}(0.5944 \times 0 + 0.4056 \times 2)$$
$$= \$0.77$$
$$X - S_u = 42 - 48 < 0 < \$0.77$$

- We do not exercise the put early at this node. Hence $f_u = \$0.77$.

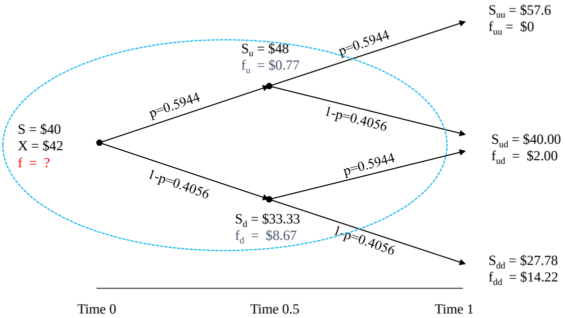


- Lower-right branch

$$f_d = e^{-rT}[pf_{ud} + (1-p)f_{dd}] = e^{-0.1\times 0.5}(0.5944 \times 2 + 0.4056 \times 14.22)$$
$$= \$6.62$$
$$X - S_d = 42 - 33.33 = \$8.67 > \$6.62$$

- We exercise the put early at this node. Hence $f_d = \$8.67$.

8. American put option



- Far-left branch

$$f = e^{-rT}[pf_u + (1-p)f_d] = e^{-0.1\times 0.5}(0.5944 \times 0.77 + 0.4056 \times 8.67)$$
$$= \$3.78$$

$$X - S = 42 - 40 = \$2 < \$3.78$$

- We do not exercise the put at Time 0. Hence $f = \$3.78$.

9. Choosing u and d

- Up until now we have assumed values for u and d. In practice, u and d are determined from the stock price volatility, σ. One way of matching the volatility is to set:

$$u = e^{\sigma\sqrt{\Delta t}}$$
$$d = 1/u$$

  - where σ is the volatility and Δt is the length of the time step. This is the approach used by Cox, Ross, and Rubinstein (1979).
  - For the purposes of this course, if in the exam I express volatility in terms of per annum, then use the above formula. If I say the stock price is expected to go up and down by a certain percentage over the next two periods, etc, then use the methodology from the previous slides for determining u and d.

9. Choosing u and d

- For example, given that $\sigma = 0.30$ $p.a.$ and the time step in the binomial tree is 6 months ($\Delta t = 0.5$)

$$u = e^{\sigma\sqrt{\Delta t}} = e^{0.3\sqrt{0.5}} = 1.2363$$
$$d = 1/u = 1/1.2363 = 0.8089$$

10. Key takeaways

- The traditional approach to value assets is never used to value derivative securities
- We have easier ways to value options (e.g., replication, delta hedging or risk-neutral approach)
- The risk-neutral approach is abstract but very easy to implement:
  - We just need to calculate the risk-neutral probability (p) and use it to calculate the expected payoff to the option
  - We can then discount option payoffs to present value using the risk-free rate of interest
- The Binomial model, in conjunction with the risk-neutral approach, is able to numerically approximate the price of virtually any option
- For American options, we can capture the value of being able to exercise early. On each branch of the tree, we simply check to see whether the payoff from exercising immediately beats the value from holding onto the put option a little longer.

10. Conclusion

- In today's class
  - Intuition behind the risk-neutral approach
  - Use Binomial tree (risk-neutral) approach to value European and American options
- Next week, Black-Scholes Option Pricing Model.
- Please review normal distribution and how to use Z-table in foundation statistics course