

FINM2002 Derivatives FINM6041 Applied Derivatives

Lecture 2 – Forwards and Futures

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Hull *et al.*: Chapters 3 & 5



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Review of Previous Lecture

- Forwards and futures contracts
- Settlement procedures for futures
 - Closing-out before expiry
 - Settlement at expiry (physical and cash)
- Types of Market Traders



Lecture Overview

- In today's class
 - Interest rates compounding
 - Short selling
 - Assumptions on the market
 - The Price of Forward and Futures (different underlying assets)
 - The Value of Forward
 - Hedge
 - Cross Hedging and Optimal Hedge Ratio
 - Hedging Using Index Futures



1. Interest rates compounding

- Assume \$10,000 investment earns 15% interest over the next year
- The ending value depends on the compound frequencies

Annual Compounding: $FV = \$10,000 \times (1 + 0.15)^1 = \$11,500$

Semi-Annual Compounding: $FV = \$10,000 \times \left(1 + \frac{0.15}{2}\right)^{2 \times 1} = \$11,556.25$

Quarterly Compounding: $FV = \$10,000 \times \left(1 + \frac{0.15}{4}\right)^{4 \times 1} = \$11,586.50$

Monthly Compounding: $FV = \$10,000 \times \left(1 + \frac{0.15}{12}\right)^{12 \times 1} = \$11,607.55$

Daily Compounding: $FV = \$10,000 \times \left(1 + \frac{0.15}{365}\right)^{365 \times 1} = \$11,617.98$

Continuous Compounding: $FV = \$10,000 \times e^{0.15 \times 1} = \$11,618.34$



1. Interest rates compounding

- Interest compounding is important in this course
- We will mainly use **continuous compounding**
- For example
 - \$100 grows to $\$100e^{rT}$ when invested at a continuously compounded rate r for time T
 - \$100 received at time T discounts to $\$100e^{-rT}$ at time zero when the continuously compounded discount rate is r



1. Interest rates compounding

Future Value	Present Value
$FV = PV(1 + r)^T$	$PV = \frac{FV}{(1+r)^T}$
$FV = PVe^{rT}$	$PV = FVe^{-rT}$

- Convert annual nominal rate to continuously compounded rate:

$$1 \times e^{R_c \times 1} = (1 + R_{\text{effective}})^1 = \left(1 + \frac{R_m}{m}\right)^m$$

$$\Rightarrow R_c = \ln\left(1 + R_{\text{effective}}\right) = m \times \ln\left(1 + \frac{R_m}{m}\right)$$

- where m is number of times the interest rate is compounded in a year, $\ln(\cdot)$ is the natural logarithm function, R_c is a rate of interest with continuous compounding, and R_m is the equivalent rate with compounding m times per annum.



1. Interest rates compounding

- Example 1: If the nominal interest rate is 10% p.a. compounded weekly, what is the continuously compounded rate?

$$R_c = 52 \times \ln \left(1 + \frac{10\%}{52} \right) = 9.99\%$$

- Example 2: If the nominal interest rate is 10% p.a. compounded quarterly, what is the continuously compounded rate?

$$R_c = 4 \times \ln \left(1 + \frac{10\%}{4} \right) = 9.877\%$$

1. Interest rates compounding

- Example 3: If the nominal interest rate is **1% per month compounded monthly**, what is the continuously compounded rate?
- Approach 1: compute directly

$$R_c = 12 \times \ln \left(1 + \frac{1\% \times 12}{12} \right) = 11.940\%$$

- Approach 2: Converting from nominal rate to effective rate first:

$$R_{\text{effective}} = (1 + 1\%)^{12} - 1 = 12.68\%$$

- and then convert to continuously compounded rate:

$$R_c = \ln(1 + R_{\text{effective}}) = \ln(1 + 12.68\%) = 11.940\%$$



2. Short Selling

- Short selling (shorting): sell an asset that is not owned
 - Only available for some investment assets, most often stocks
 - Borrow from broker or lender and sell in the market as usual
 - At a later date, need to purchase the equivalent asset and pay back the one borrowed
 - The lender cannot be better off or worsen off
 - Borrower must pay dividends and other benefits that would have accrued to the lender
 - Likewise, if the borrowed asset is gold, the lender also needs to pay storage costs of the gold to the borrower
 - Short seller profits from price fall
 - Margin account with the broker is required, so that possible adverse movements (increases) in the price of shorted security are covered
 - Incur lending fee, but assume no lending fee in this course



3. Assumptions On The Market

- In this lecture, we make the following assumptions
 - There is no transaction costs
 - All investors subject to same tax rate
 - Borrow and lend money at the same risk-free rate
 - Take advantage of arbitrage opportunities as they occur



4. The Price of Forward and Futures

- Arbitrage pricing approach
- The payoff from a forward / futures contract can be replicated by
 - The underlying asset
 - Risk-free bonds
- The price of a forward / futures contract should be the same as the replication portfolio
- Otherwise arbitrage opportunity arises
- Industry practice to price futures contract as if forward contracts



4. The Price of Forward and Futures

- Assume an asset that pays no dividend and incurs no storage cost
- Consider the following trading strategy
 - Borrow S_0 today at an interest rate of r
 - Buy one asset at price S_0
 - Hold it for a period of T
 - Enter a forward contract today to sell one asset at forward price F
 - Sell one asset at forward price F at time T
 - Payback the loan at $S_0 e^{rT}$ at time T
- The forward price is $F = S_0 e^{rT}$



4. The Price of Forward and Futures

- The cash flow table
- Arbitrage relationship between spot and forward contracts

Position	Today T_0	T
Borrow cash S_0 and repay at T	S_0	$-S_0 e^{rT}$
Buy one asset at S_0 and hold	$-S_0$	S_T
Enter forward to sell at F	0	$F - S_T$
Net cash flow	0	$F - S_0 e^{rT}$

- $F = S_0 e^{rT}$ to prevent arbitrage



4. The Price of Forward and Futures

- Example:

- Consider a four-month forward contract to buy a zero-coupon bond
- i.e. $T = 4/12$
- The current price of the bond is $S_0 = \$930$
- The risk-free rate is $r = 6\%$ p.a. continuously compounded
- The forward price

$$F_0 = S_0 e^{rT} = 930 e^{0.06 \times \frac{4}{12}} = \$948.79$$



4. The Price of Forward and Futures

- In general if:

$$F_0 > S_0 e^{rT}$$

- Arbitrageurs can make a riskless profit from buying the asset and entering into a short forward contract on the asset. This strategy is financed by borrowing funds at the risk free rate.

$$F_0 < S_0 e^{rT}$$

- Arbitrageurs can make a riskless profit by shorting the asset and entering into a long forward contract. The excess funds are invested at the risk-free rate of interest until they are needed to buy back the asset.



4.1. The Price of Forward and Futures: known income

- Price of forward and futures, **on assets with known income**
- Consider a forward contract on an asset that will provide a perfectly predictable cash income to the holder. e.g.
 - Stocks paying known dividends
 - Coupon bonds
 - D is the **present value** of the income

$$F_0 = (S_0 - D)e^{rT}$$



4.1. The Price of Forward and Futures: known income

- Example

- A long position in a forward contract to buy a coupon bond
- The contract matures in 9 months, i.e. $T = 9/12 = 0.75$
- The current price of this bond is $S_0 = \$900$
- Assume a coupon payment \$40 is expected after 4 months
- The four-month interest rate is 3% p.a.
- The nine-month interest rate is 4% p.a.
- All interest rates are continuously compounded
- What is the forward price today?



4.1. The Price of Forward and Futures: known income

$$F_0 = (S_0 - D)e^{rT}$$



$$D = 40e^{-0.03 \times \frac{4}{12}} = 39.60$$

9-month risk-free rate is different from 4-month rate

$$F_0 = (900 - 39.60)e^{0.04 \times 0.75} = \$886.60$$



4.2. The Price of Forward and Futures: known yield

- The price of forward and futures, **on assets with known yield**
- Income on asset is often expressed as yield rather than dollar value
 - A percentage of the asset's price when income is paid
 - d as the average yield per annum on an asset during the life of a forward contract with continuous compounding

$$F_0 = S_0 e^{(r-d)T}$$



4.2. The Price of Forward and Futures: known yield

- Example
 - a six-month forward contract on an asset, $T = 0.5$
 - The asset is expected to pay a dividend equals to 2% of the current asset price over a six-month period
 - The current asset price is $S_0 = \$25$
 - The risk-free rate r is 10% p.a. continuously compounded
- Note: the given yield is 2% per half year
- Convert it to continuously compounded annual rate

$$d = R_c = m \ln\left(1 + \frac{R_m}{m}\right) = 2 \ln\left(1 + \frac{2\% \times 2}{2}\right) = 3.96\%$$

- The forward price

$$F_0 = S_0 e^{(r-d)T} = 25 e^{(10\% - 3.96\%) \times 0.5} = \$25.77$$



4.3. The Price of Forward and Futures: stock indices

- The price futures **on stock indices**
- A stock index can be viewed as an investment asset paying a dividend yield
- The price of futures on a stock index

$$F_0 = S_0 e^{(r-d)T}$$

- d is the dividend yield on the portfolio represented by the index.
- S is the points of index times a factor conversion of dollar per point



4.3. The Price of Forward and Futures: stock indices

- Example
 - A 1-year futures contract on the ASX S&P200
 - The stocks underlying the index provide a dividend yield of 5% p.a. continuously compounded
 - Risk-free rate is 10% p.a. continuously compounded
 - The current ASX S&P200 index is 3529
 - The dollar value of each point is \$25
- The price of this futures contract in terms of points

$$F_0 = S_0 e^{(r-d)T}$$

$$F_0 = 3529 e^{(0.10 - 0.05)} = 3710$$

- The dollar price of this futures contract

$$F_0 = 3710 \times \$25 = \$92,750$$



4.4. The Price of Forward and Futures: currencies

- The price of forward and futures, **on foreign currency**
- Consider the foreign currency as a security paying a dividend yield
- Foreign currency earns the foreign risk-free rate $r_{foreign}$
- Thus, the continuous dividend yield is the foreign risk-free rate
- S_0 spot price in dollars of one unit of foreign currency
 - e.g. the price for one US dollar is 1.5 Australian dollars
- F_0 forward/futures price in dollars of one unit of foreign currency

$$F_0 = S_0 e^{(r - r_{foreign})T}$$



4.5. The Price of Forward and Futures: commodities

- Investment assets
 - Held by many people purely for investment purposes
 - Examples: stocks, bonds and gold
- Consumption assets
 - Held primarily for consumption
 - (usually) not for investment purposes
 - Examples: commodities such as copper, oil, corn and meat
- When we use arbitrage arguments to price the forward and futures
 - Easy for investment assets
 - Difficult or infeasible for consumption assets



4.5. The Price of Forward and Futures: commodities

- The price of forward and futures, on commodities as investment assets (such as gold, silver)
- If there is no income or storage cost

$$F_0 = S_0 e^{rT}$$

- If Q is the present value of all of the storage costs minus all income during the life of the forward

$$F_0 = (S_0 + Q) e^{rT}$$

- If q is the percentage storage costs minus the percentage income during the life of the forward

$$F_0 = S_0 e^{(r+q)T}$$



4. The Price of Forward and Futures

- Cost of carry (optional)
 - Cost of holding a physical quantity of the commodity
 - e.g. for wheat the cost of carry is the storage cost; for gold it comes from storage and security costs
 - Some underlying assets have a negative cost of carry
 - e.g. holding a stock index provides the benefit of receiving dividends



5. The Value of Forward

- The value of derivatives is “derived” from underlying asset
- The price of derivatives \neq its value
- The initial value of a forward/futures contract at the time it is first entered/negotiated is **zero**
- At a later stage it may have a positive or negative value



5. The Value of Forward

- Define the following notations
- Assume we entered a forward contract some time ago
 - The initial forward price we negotiated is $F_0 = S_0 e^{r_0 T_0}$
 - F_0 is also called the delivery price of this contract
- Next, we come to today, what is the value of our contract?
 - If someone wants to negotiate a new forward contract today
 - $F_1 = S_1 e^{r_1 T_1}$ is the current forward price
 - $T(T_1)$ is the delivery time (in years) from today
 - $r(r_1)$ is the T -year risk-free interest rate, continuously compounded
 - f is today's **value** of our previous forward contract



5. The Value of Forward

- The value of a long forward contract

$$f = (F_1 - F_0)e^{-rT}$$

- The value of a short forward contract

$$f = (F_0 - F_1)e^{-rT}$$

- Additional notes

- The **present value** from today to delivery date, so need discount
- Some textbooks/slides denote $K = F_0$ and $F = F_1$



5. The Value of Forward

- Example

- Enter long position in a forward on a non-dividend-paying stock
- Delivery date is in 6 months, $T_0 = 6/12$
- Current stock price is $S_0 = \$30$
- The risk-free rate is $r_0 = 5\%$ p.a. with continuous compounding
- What is the forward price for this contract?
- What is the initial value of this forward contract? ¹

¹I put time subscripts to each variable just to make it less confusing when we compute forward value on a later time. You are encouraged to do so, but it is not required as long as you get the correct answer.



5. The Value of Forward

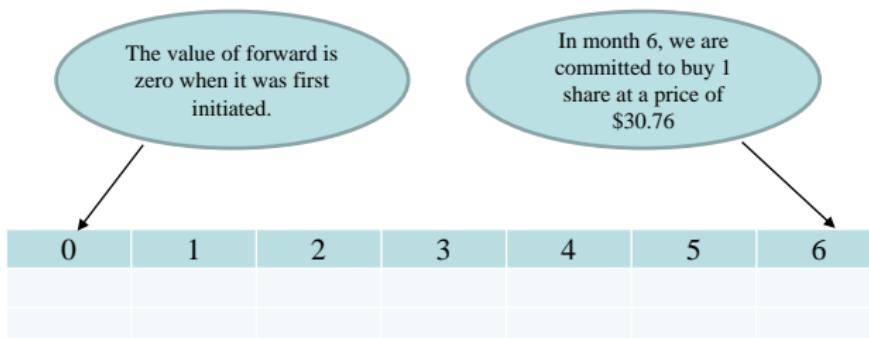
- What is the forward price for this contract?

$$F_0 = S_0 e^{rT}$$

- We know that $S_0 = \$30$, $r_0 = 5\%$, $T_0 = 6/12 = 0.5$

$$F_0 = \$30 e^{5\% \times 0.5} = \$30.76$$

- The initial value of this forward contract is zero.



5. The Value of Forward

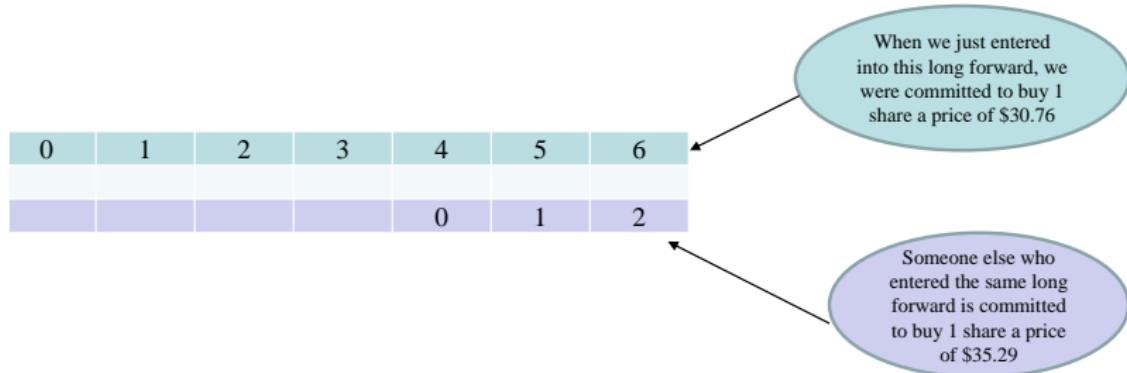
- Four months later
 - Current stock is \$35
 - Risk-free rate remains 5% p.a. with continuous compounding
 - What is the price of the forward contract today?
 - What is the value of your initial long forward?



5. The Value of Forward

- Four months later: $S_1 = \$35$, $r_1 = 5\%$, $T_1 = 2/12$
- The forward price today

$$F_1 = \$35e^{0.05 \times (2/12)} = \$35.29$$



5. The Value of Forward

- The initial forward price $F_0 = \$30.76$
- The current forward price $F_1 = \$35.29$ for the same contract
- Risk-free rate remains 5% p.a. with continuous compounding
- Two months remaining from today to delivery $T = 2/12$
- The value of this long forward contract

$$\begin{aligned}f &= (F_1 - F_0)e^{-rT} \\&= (35.29 - 30.76)e^{-0.05 \times (2/12)} \\&= \$4.49\end{aligned}$$



5. The Value of Forward

- Additional remarks
 - Minor differences between forward and futures
 - Marking-to-the-market
 - If constant interest rates, prices of forward and future are the same
 - In practice, futures are valued as if they were forwards
 - When we talk about **pricing** a forward contract, we really mean determining the correct delivery price F
 - When we talk about **valuing** a forward contract, we really mean calculating f



6. Hedge

- Basic principles of hedging
 - Hold (or expected to hold) a position in an asset
 - But don't want to be exposed to movements in asset price
 - Hedge to eliminate (or reduce) the exposure to price movements
 - Enter a derivatives position that provides a payoff which **offsets** the price movements in the exposed underlying asset
- Short hedge
 - Use a short position in futures
 - Hold (or expected to hold) an asset and expects to sell at T
 - Lock in the price to sell.
- Long hedge
 - Use a long position in futures
 - Expect to purchase an asset at T
 - Lock in the price to buy



6. Hedge

- Arguments in favour of hedging
 - Companies should focus on the main business, and use derivatives to minimize (hedge) other risks
 - Companies should use hedge to avoid large adverse movements in the prices of raw materiel or products
- Arguments against hedging
 - Shareholders are usually well diversified and can make their own hedging decisions
 - May not be a good strategy to hedge when competitors do not
 - Difficult to explain a situation where there is a loss on the hedge and could have had a gain on the underlying asset

6. Hedge

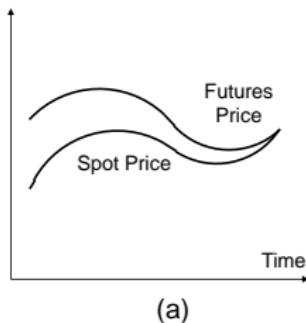
- Hedge is not always perfect and straightforward
 - The asset to be hedged may not be exactly the same as the asset underlying the futures contract
 - Uncertain about the exact date the asset will be bought or sold
 - May have to close-out the futures before delivery period



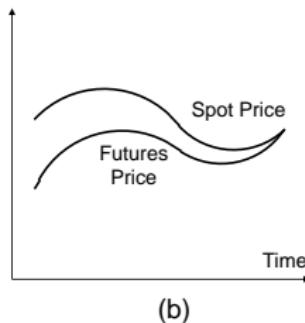
6. Hedge

- Basis risk
 - The risk associated with imperfect hedging
 - In a perfect hedging, basis risk = 0
 - Exact same asset as the underlying asset in the futures contract
 - Hold to delivery date
 - Prior to delivery, the basis may be positive or negative

Basis = Spot price of asset to be hedged - Futures price of contract used



(a)



(b)

7. Cross Hedging and Optimal Hedge Ratio

- Cross hedging
 - When there is no exactly the same futures contract for the asset to be hedged
 - Use futures contract whose underlying asset is closely correlated (in terms of price movement) to the asset to be hedged
 - Example
 - Airlines concern the future price of aviation fuel
 - There are no futures contracts on aviation fuel
 - Use heating oil futures contracts to hedge its exposure

7. Cross Hedging and Optimal Hedge Ratio

- **Hedge ratio** is the ratio of the size of the position taken in futures contracts to the size of the asset with exposure

$$\text{Hedge ratio} = \frac{\text{size of futures}}{\text{size of underlying asset}}$$

$$= \frac{\text{number of contracts (N)} \times \text{size per contract (A)}}{\text{size of underlying asset (P)}}$$

$$h = \frac{N \times A}{P}$$

$$N = h \times \frac{P}{A}$$



7. Cross Hedging and Optimal Hedge Ratio

- Optimal hedge ratio

$$h = \rho \frac{\sigma_S}{\sigma_F}$$

Where

σ_S : is the standard deviation of ΔS , the change in the spot price of underlying asset during the hedging period

σ_F : is the standard deviation of ΔF , the change in the futures price during the hedging period

ρ : is the correlation coefficient between ΔS and ΔF



7. Cross Hedging and Optimal Hedge Ratio

- Optimal hedge ratio

$$h = \rho \frac{\sigma_S}{\sigma_F}$$

- if $\rho = 1$ and $\sigma_S = \sigma_F \rightarrow h = 1$

Futures price mirrors the spot price perfectly

- if $\rho = 1$ and $\sigma_S = 0.5\sigma_F \rightarrow h = 0.5$

Futures price always changes by twice as much as the spot price

Gain/loss in futures will be twice as much as in the spot market



7. Cross Hedging and Optimal Hedge Ratio

- Example
 - Qantas expects to purchase $P = 2,000,000$ gallons of jet fuel
 - In one month $T = 1/12$
 - Decide to use heating oil futures contract as a cross hedge
 - Each futures contract covers $A = 42,000$ gallons of heating oil
 - From historical data, we estimate
 - The std of the price changes in jet fuel is $\sigma_S = 2.63\%$
 - The std of the price changes in futures is $\sigma_F = 3.13\%$
 - The correlation between the two price changes is $\rho = 0.928$
 - What is the optimal hedge ratio?
 - Design the hedging strategy?



7. Cross Hedging and Optimal Hedge Ratio

- The optimal hedge ratio

$$h = \rho \frac{\sigma_S}{\sigma_F} = 0.928 \times \frac{2.63\%}{3.13\%} = 77.77\%$$

- The optimal number of heating oil contracts

$$N = h \times \frac{P}{A} = 77.77\% \times \frac{2,000,000}{42,000} = 37.03$$

- Round to the nearest whole number, 37 contracts
- Since we want to buy the jet fuel (exposed to price increase), we will enter **long** position in 37 contracts of heating oil futures



8. Hedging Using Index Futures

- Use stock index futures to hedge a stock portfolio
 - May want to be out of the market for a short period
 - Hedge systematic risk
 - An application of cross-hedging in stock market



8. Hedging Using Index Futures

- Use stock index futures to hedge a stock portfolio
- The optimal hedge ratio is the beta of portfolio to be hedged

$$h = \beta$$

- The number of index futures contracts to be shorted

$$N = \beta \frac{P}{A}$$

- P is the value of the portfolio
- A is the value of the asset (stock index) underlying one stock index futures contract = index points \times dollar value per point
- Derivation (optional)

$$N = h \times \frac{P}{A} = \rho \frac{\sigma_P}{\sigma_A} \frac{P}{A} = \beta \frac{P}{A}$$



8. Hedging Using Index Futures

- Example
 - The underlying asset of SPI200 futures is S&P/ASX 200 index
 - SPI200 futures contract is valued at AUD\$25 per index point
 - Assume the current S&P/ASX 200 index is 3500 points
 - The value of the portfolio to be hedged is \$5 million.
 - The beta of the portfolio is 1.5.
- What position in SPI200 futures contracts is necessary to hedge the portfolio?



8. Hedging Using Index Futures

- The beta of our portfolio $\beta = 1.5$
- The value of the stock portfolio to be hedged $P = \$5$ million
- The dollar value of a single SPI200 futures contract $A = 3500 \times 25$

$$\begin{aligned} N &= \beta_{\text{portfolio}} \left[\frac{\text{Amount to be hedged}}{\text{Value of one contract}} \right] \\ &= \beta \frac{P}{A} \\ &= 1.5 \times \frac{5 \text{ million}}{3500 \times 25} \\ &= 85.71 \\ &\approx 86 \text{ contracts (short)} \end{aligned}$$

- We need to short 86 contracts in SPI200 futures to hedge our long portfolio position



8. Hedging Using Index Futures

- What is the portfolio's new beta (β^*) after the hedge?
 - Ideally, if all exposure β is perfectly hedged, $\beta^* = 0$
 - But most often, remains β^* unhedged
 - In such case, we only managed to hedge

$$N_{\text{hedged}} = (\beta - \beta^*) \frac{P}{A}$$
$$\beta^* = \beta - N_{\text{hedged}} \frac{A}{P}$$

- In above example, $N_{\text{hedged}} = 86$ contracts, β^* is very close to zero

$$\beta^* = 1.5 - 86 \times \frac{3500 \times 25}{5 \text{ million}} = -0.005$$



8. Hedging Using Index Futures

- Extension 1
 - Instead of (close to) perfect hedge of all β , if we only want to reduce the beta of the portfolio to $\beta^* = 0.75$
 - What position and how many SPI200 futures should we use?
- Shortcut
 - Given we know we can hedge almost 100% with 86 contracts in previous question to reduce beta from 1.5 to 0
 - We can take 43 short futures contract to hedge 50% of the risk, namely reduce beta from 1.5 to 0.75
- Generally

$$N_{\text{hedged}} = (\beta - \beta^*) \frac{P}{A} = (1.5 - 0.75) \frac{5m}{3500 \times 25} = 43(\text{short})$$



8. Hedging Using Index Futures

- Extension 2
- What position is necessary to increase the beta to $\beta^* = 2$?

$$N_{\text{hedged}} = (\beta - \beta^*) \frac{P}{A} = (1.5 - 2) \frac{5m}{3500 \times 25} = -29$$

- Since this is negative, it means we need go **long** position to increase the exposure of our portfolio



9. Conclusion

- In today's class
 - Interest rates compounding
 - The Price of Forward and Futures (different underlying assets)
 - The Value of Forward
 - Hedge
 - Cross Hedging and Optimal Hedge Ratio
 - Hedging Using Index Futures
- Next week
 - Interest rate forward
 - Swaps

