

# FINM2002 Derivatives FINM6041 Applied Derivatives

## Lecture 6 – Black-Scholes Option Pricing Model

Yichao Zhu

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Hull *et al.*: Chapters 13



Australian  
National  
University

# Review of Previous Lecture

- Option pricing methods in a binomial framework
    - Arbitrage-free replication approach
    - Delta-hedging approach
    - Traditional approach
    - Risk-neutral approach
  - Binomial tree (risk-neutral) approach
  - European and American options
- 
- Progress test and assignment
  - Teaching arrangement for week 7
    - No in-person lecture and workshop (tutorials as usual)
    - Lecture 7 will be pre-recorded and posted on Echo360



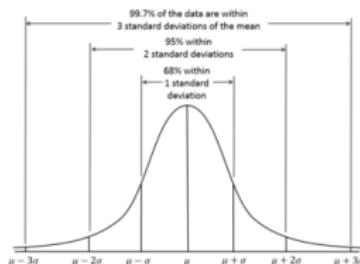
# Lecture Overview

- In today's class
  - The distribution of stock prices
  - The distribution of stock returns (optional)
  - Assumptions of Black-Scholes
  - Intuition behind Black-Scholes
  - The Black-Scholes Model
  - The Black-Scholes Model with dividend
  - Implied Volatility



# 1. The Distribution of Stock Prices

- Normal Distribution: Recap STAT1008/7055
  - Random variable is a variable whose possible values are numerical outcomes of some random event/process/phenomenon
  - Continuous random variable can take any value within an interval
  - Probability density function (PDF) defines the distribution of the possible values of the continuous random variable
  - Normal distribution is the most popular continuous distribution, with a probability distribution  $X \sim \Phi(\mu, \sigma)$



- We can calculate probabilities associated with the values of  $X$ , such as  $Pr(X < a)$  or  $Pr(a < X < b)$

# 1. The Distribution of Stock Prices

- Normal Distribution (continued)

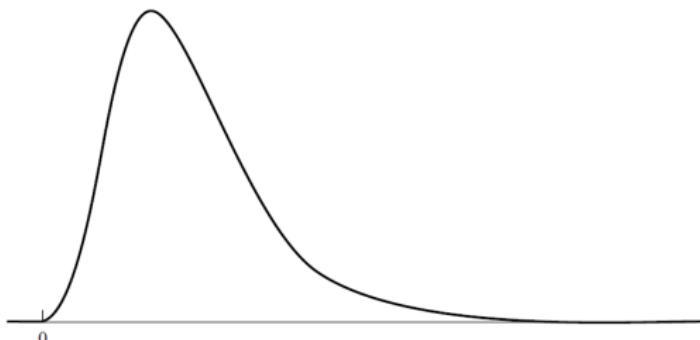
- Transform a normal distribution to standard normal distribution  $Z \sim \Phi(0, 1)$
- $Z = (X - \mu)/\sigma$ , and  $Pr(X < a) = Pr[Z < (a - \mu)/\sigma]$
- Use the Z-table. For example,
  - $Pr(Z < 0.66) = N(0.66) = 0.7454$
  - $Pr(Z < 1.96) = 0.9750$
  - $Pr(Z > 1.96) = 1 - Pr(Z < 1.96) = 0.0250$

$z$	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7122	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830



# 1. The Distribution of Stock Prices

- Lognormal distribution:



- Black-Scholes Model

- Assumes that stock prices follow a **lognormal** distribution
- Stock prices bounded on the downside by zero, but unlimited upside
- If there is no dividend, stock prices follow a random walk

# 1. The Distribution of Stock Prices

- The natural logarithm of a lognormal distr. is normally distributed
- Black-Scholes assumes that  $S_T$  is lognormal
- Therefore  $\ln S_T$  is normal

$$\ln S_T \sim \Phi \left[ \ln S_0 + \left( \mu - \frac{\sigma^2}{2} \right) T, \sigma\sqrt{T} \right]$$

- The mean of  $\ln S_T$

$$\ln S_0 + \left( \mu - \frac{\sigma^2}{2} \right) T$$

- The volatility (standard deviation) of  $\ln S_T$

$$\sigma\sqrt{T}$$

- $\mu$  is the expected return on the stock
- $\sigma$  is the volatility of the stock price



# 1. The Distribution of Stock Prices

- Example: find the probability distribution of the stock price in six months, for a stock with a current price of \$50, an expected return of 20% p.a, and a volatility of 30% p.a.
- The probability distribution of the stock price in six months:

$$\ln S_T \sim \Phi \left[ \ln S_0 + \left( \mu - \frac{\sigma^2}{2} \right) T, \sigma\sqrt{T} \right]$$

- We know that  $S_0 = \$50$ ,  $\mu = 0.2$ ,  $\sigma = 0.3$  and  $T = 0.5$

$$\ln S_T \sim \Phi \left[ \ln 50 + \left( 0.2 - \frac{0.3^2}{2} \right) \times 0.5, 0.3 \times \sqrt{0.5} \right]$$

- The probability distribution

$$\ln S_T \sim \Phi [3.990, 0.212]$$



# 1. The Distribution of Stock Prices

- Application:  $Pr(S_T < X)$

$$\ln S_T \sim \Phi [3.990, 0.212]$$

- The probability of stock price falls below \$70 in six months?

$$\begin{aligned} Pr(S_T < 70) &= Pr[\ln S_T < \ln(70)] \\ &= Pr[Z < (\ln(70) - \mu^*)/\sigma^*] \\ &= Pr[Z < (\ln(70) - 3.990)/0.212] \\ &= Pr(Z < 1.22) \\ &= 0.8888 \end{aligned}$$

- The probability of stock price goes above \$70 in six months?

$$Pr(S_T > 70) = 1 - Pr(S_T < 70) = 1 - 0.8888 = 0.1112$$



# 1. The Distribution of Stock Prices

- Generalize

$$\begin{aligned} \Pr(S_T < X) &= \Pr[\ln S_T < \ln X] \\ &= \Pr[Z < (\ln X - \mu^*)/\sigma^*] \\ &= \Pr[Z < \frac{\ln X - (\ln S_0 + (\mu - \sigma^2/2) T)}{\sigma\sqrt{T}}] \\ &= \Pr[Z < -\frac{\ln S_0 + (\mu - \sigma^2/2) T - \ln X}{\sigma\sqrt{T}}] \\ &= \Pr[Z < -\frac{\ln(S_0/X) + (\mu - \sigma^2/2) T}{\sigma\sqrt{T}}] \end{aligned}$$

$$\begin{aligned} \Pr(S_T > X) &= 1 - \Pr(S_T < X) = \Pr(S_T < -X) \\ &= \Pr[Z < \frac{\ln(S_0/X) + (\mu - \sigma^2/2) T}{\sigma\sqrt{T}}] \end{aligned}$$



## 2. The Distribution of Stock Return

- The distribution of stock return (optional)
- Assume  $x$  is the stock return, continuously compounded p.a., realized between 0 and  $T$

$$S_T = S_0 e^{xT}$$

$$x = \frac{1}{T} \ln \frac{S_T}{S_0}$$

- It follows a normal distribution

$$x \sim \Phi \left( \mu - \frac{\sigma^2}{2}, \frac{\sigma}{\sqrt{T}} \right)$$



## 2. The Distribution of Stock Return

- Derivation

$$x = \frac{1}{T} \ln \frac{S_T}{S_0} = \frac{1}{T} \ln S_T - \frac{1}{T} \ln S_0$$

- Recall

$$\ln S_T \sim \Phi \left[ \ln S_0 + \left( \mu - \frac{\sigma^2}{2} \right) T, \sigma\sqrt{T} \right]$$

- The mean of  $x$

$$\frac{1}{T} \times [\ln S_0 + \left( \mu - \frac{\sigma^2}{2} \right) T] - \frac{1}{T} \ln S_0 = \mu - \frac{\sigma^2}{2}$$

- The standard deviation of  $x = \sigma\sqrt{T} \times \frac{1}{T} = \frac{\sigma}{\sqrt{T}}$ . Note  $\frac{1}{T} \ln S_0$  is a constant, which does not affect the standard deviation of  $x$

$$x \sim \Phi \left( \mu - \frac{\sigma^2}{2}, \frac{\sigma}{\sqrt{T}} \right)$$



### 3. Assumptions of Black-Scholes

- Stock prices follow lognormal distribution with constant  $\mu$  and  $\sigma$
- No transaction costs or taxes
- All securities are perfectly divisible
- No dividends on the stock during the life of the option
- No riskless arbitrage opportunities
- Security trading takes place continuously in time
- Investors can borrow and lend at the same risk-free rate
- The short-term risk-free rate is constant

## 4. Intuition Behind Black-Scholes

- Full derivation of Black-Scholes will be covered in advanced course
- Intuition in this course
- Construct a perfectly hedged portfolio, using a position in the option and a position in the underlying stock
- If there are no arbitrage opportunities, the return for the portfolio must be the risk-free rate
- The reason we can set up this riskless portfolio is that the stock price and option price are both affected by the same underlying uncertainty, i.e. stock price movements
- “Continuously revised delta hedging”



## 4. Intuition Behind Black-Scholes

- In any short period of time, price of a call (put) option is perfectly positively (negatively) correlated with price of underlying stock
  - In this perfectly hedged portfolio, the gain or loss from the stock position will be completely offset by the option position
  - The total value of the portfolio at the end is certain (no risk)
  - Therefore the return of this portfolio in any short period of time must be the risk-free rate
  - Key element in the Black-Scholes Model for option price



## 5. The Black-Scholes Model

- Black-Scholes for European option on non-dividend paying stock

$$c = S_0 N(d_1) - X e^{-rT} N(d_2)$$

$$p = X e^{-rT} N(-d_2) - S_0 N(-d_1)$$

Where:

$$N(d) = \Pr(Z < d)$$

$$d_1 = \frac{\ln(S_0/X) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln(S_0/X) + (r - \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$= d_1 - \sigma\sqrt{T}$$



## 5. The Black-Scholes Model

$c$  = European call price

$p$  = European put price

$S_0$  = The stock price

$X$  = The strike price

$T$  = Time to expiration

$\sigma$  = Volatility of the stock price

$r$  = Risk-free rate



## 5. The Black-Scholes Model

- What are  $d_1$ ,  $d_2$  and  $N(x)$ ?
- $d_1$  and  $d_2$  are just preliminary calculations to simplify formula
- $N(d) = \Pr(Z < d)$  represents the probability under standard normal distribution
- $N(d_2)$  and  $N(-d_2)$  tell the probability of option being exercised
- $N(d_1)$  and  $N(-d_1)$  tell the degree of moneyness if the option will be exercised



## 5. The Black-Scholes Model

- Prove Put-Call Parity in Black-Scholes Model
- European options on non-dividend paying stock

$$c = S_0 N(d_1) - X e^{-rT} N(d_2)$$

$$p = X e^{-rT} N(-d_2) - S_0 N(-d_1)$$

$$= X e^{-rT} [1 - N(d_2)] - S_0 [1 - N(d_1)]$$

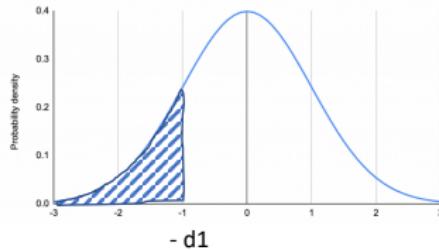
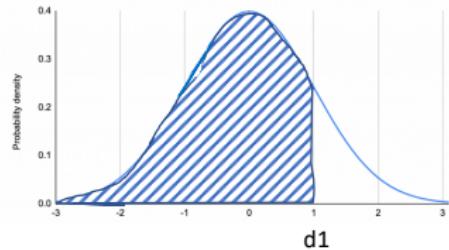
$$= S_0 N(d_1) - X e^{-rT} N(d_2) + X e^{-rT} - S_0$$

$$= c + X e^{-rT} - S_0$$

$$c + X e^{-rT} = p + S_0$$



## 5. The Black-Scholes Model



$$N(-d_1) = 1 - N(d_1)$$

$$N(-d_2) = 1 - N(d_2)$$



## 5. The Black-Scholes Model

- Verify Black-Scholes with extreme stock price values
- When stock prices  $S_0$  become very large
  - A call option is almost certain to be exercised,  $c = S_0 - Xe^{-rT}$
  - In Black-Scholes, both  $d_1$  and  $d_2$  become large and positive, so both  $N(d_1)$  and  $N(d_2)$  become close to 1

$$c = S_0 N(d_1) - X e^{-rT} N(d_2)$$

$$d_1 = \frac{\ln(S_0/X) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

- Becomes  $c = S_0 - Xe^{-rT}$  as stated above



## 5. The Black-Scholes Model

- When stock prices  $S_0$  become very large (continued)
  - A put option would not be exercised, price approaches zero
  - In Black-Scholes, both  $N(-d_1)$  and  $N(-d_2)$  are close to zero
  - $p = Xe^{-rT}N(-d_2) - S_0N(-d_1)$  becomes  $p = 0$
- When stock prices  $S_0$  become very small
  - Price of call option approaches zero. Both  $d_1$  and  $d_2$  become large and negative.  $N(d_1)$  and  $N(d_2)$  very close to zero.  $c = 0$
  - Price of put option approaches  $Xe^{-rT} - S_0$ . Both  $N(-d_1)$  and  $N(-d_2)$  close to one,  $p = Xe^{-rT} - S_0$



## 5. The Black-Scholes Model

- Example (call)
  - Suppose the current stock price  $S_0 = \$10.5$
  - There is no dividend
  - The volatility of stock  $\sigma = 30\%$  p.a.
  - The risk-free rate  $r = 5\%$  p.a., continuously compounded
  - A two-year European call option on this stock
  - The strike price  $X = \$8$
  - Use Black-Scholes model to price this call option
- Applying the Black-Scholes model is a three-step process
  - Calculate  $d_1$  and  $d_2$
  - Look up the probabilities  $N(d_1)$  and  $N(d_2)$  in the Z-table
  - Plug all inputs into the Black-Scholes formula



## 5. The Black-Scholes Model

- Step 1: Calculate  $d_1$  and  $d_2$

$$\begin{aligned}d_1 &= \frac{\ln(S_0/X) + (r + \sigma^2/2)T}{\sigma\sqrt{T}} \\&= \frac{\ln(10.5/8) + (0.05 + 0.3^2/2)2}{0.3\sqrt{2}} \\&= 1.0888 \approx 1.09\end{aligned}$$

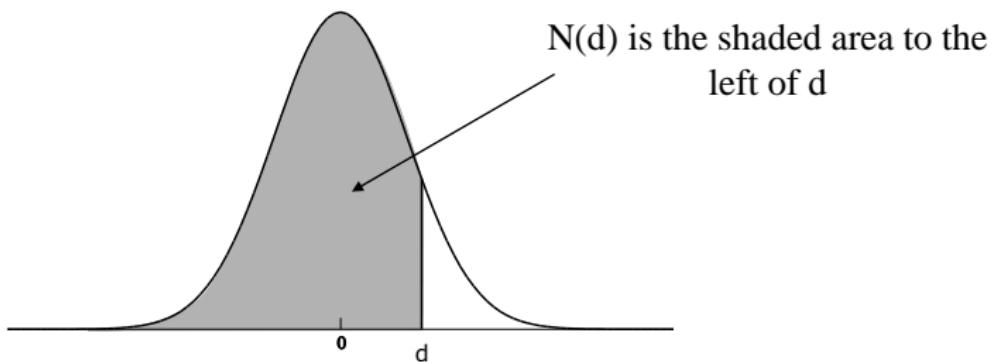
$$\begin{aligned}d_2 &= d_1 - \sigma\sqrt{T} \\&= 1.0888 - 0.3\sqrt{2} \\&= 0.6645 \approx 0.66\end{aligned}$$

- If relying on Z-table to get  $N(d)$ , just round the calculations to two decimal places



## 5. The Black-Scholes Model

- Step 2: Look up the probabilities  $N(d_1)$  and  $N(d_2)$  in the Z-table
- In Z-table,  $Z$  is a standard normal random variable (mean 0, variance 1)
- $N(d)$  is the probability that  $Z$  is less than  $d$



## 5. The Black-Scholes Model

- Step 2: Look up the probabilities  $N(d_1)$  and  $N(d_2)$  in the Z-table.
- Table for  $N(x)$  when  $x \geq 0$
- We find  $N(d_1) = N(1.09) = 0.8621$  and  $N(d_2) = N(0.66) = 0.7454$

$x$	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015



## 5. The Black-Scholes Model

- Step 3: Plug inputs into Black-Scholes formula

$$\begin{aligned}c &= S_0 N(d_1) - X e^{-rT} N(d_2) \\&= 10.50 \times N(1.09) - 8 e^{-0.05 \times 2} N(0.66) \\&= 10.50 \times 0.8621 - 8 \times 0.9048 \times 0.7454 \\&= \$3.66\end{aligned}$$

- The price for this call option is \$3.66

## 5. The Black-Scholes Model

- Example (put)
  - Suppose the current stock price  $S_0 = \$4.5$
  - There is no dividend
  - The volatility of stock  $\sigma = 30\%$  p.a.
  - The risk-free rate  $r = 8\%$  p.a., continuously compounded
  - A 7-month European put option on this stock
  - The strike price  $X = \$4.8$
  - Use Black-Scholes model to price this put option



## 5. The Black-Scholes Model

- Step 1: Calculate  $d_1$  and  $d_2$

$$\begin{aligned}d_1 &= \frac{\ln(S_0/X) + (r + \sigma^2/2)T}{\sigma\sqrt{T}} \\&= \frac{\ln(4.5/4.8) + (0.08 + 0.3^2/2)\frac{7}{12}}{0.3\sqrt{\frac{7}{12}}} \\&= 0.0366 \approx 0.04\end{aligned}$$

$$\begin{aligned}d_2 &= d_1 - \sigma\sqrt{T} \\&= 0.0366 - 0.3\sqrt{\frac{7}{12}} \\&\approx -0.19\end{aligned}$$



## 5. The Black-Scholes Model

- Note, for put options, we need one more step

$$-d_1 = -0.04$$

$$-d_2 = -(-0.19) = 0.19$$

- Step 2: Look up  $N(-d_1)$  and  $N(-d_2)$  in the Z-table
- In this case, we need  $N(-d_1) = N(-0.04)$  and  $N(-d_2) = N(0.19)$



## 5. The Black-Scholes Model

- Table for  $N(x)$  when  $x \leq 0$
- We find  $N(-d_1) = N(-0.04) = 0.4840$

$x$	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
-0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641
-0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
-0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
-0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
-0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
-0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
-0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
-0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
-0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
-0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
-1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
-1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
-1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
-1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
-1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681



## 5. The Black-Scholes Model

- Table for  $N(x)$  when  $x \geq 0$
- We find  $N(-d_2) = N(0.19) = 0.5753$

$x$	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319



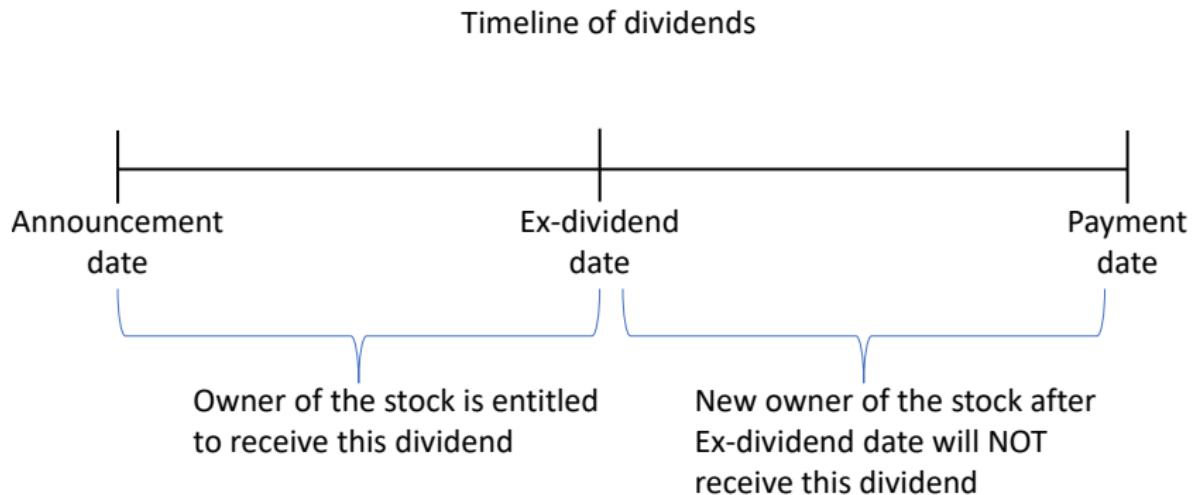
## 5. The Black-Scholes Model

- Step 3: Plug inputs into Black-Scholes formula

$$\begin{aligned} p &= Xe^{-rT}N(-d_2) - S_0N(-d_1) \\ &= (4.8e^{-0.08 \times \frac{7}{12}}) \times 0.5753 - 4.5 \times 0.4840 \\ &= \$0.46 \end{aligned}$$

- The price for this put option is \$0.46

## 6. The Black-Scholes Model with Dividends



## 6. The Black-Scholes Model with Dividends

- Extend the Black-Scholes model to price European option written on stocks paying known/predicted dividend
- Over a short period, dividend to be paid on stock are predictable
- We can consider there are two components in a stock's price  $S_0$ 
  - A risky component ( $S_0 - PV(D)$ ) moves randomly
  - A constant component reserved to pay the predicted dividend
  - This constant component ( $PV(D)$ ) equals the present value of the predicted dividend to be paid during the life of the option
  - We can reduce the stock price to  $S_0 - PV(D)$ , and then price the option as if the stock pays no dividend
  - In other words, replace  $S_0$  with  $S_0^* = S_0 - PV(D)$



## 6. The Black-Scholes Model with Dividends

- Black-Scholes Model with known dividend

$$c = (S_0 - PV(D))N(d_1) - Xe^{-rT}N(d_2)$$

$$p = Xe^{-rT}N(-d_2) - (S_0 - PV(D))N(-d_1)$$

Where:

$$d_1 = \frac{\ln((S_0 - PV(D))/X) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}$$

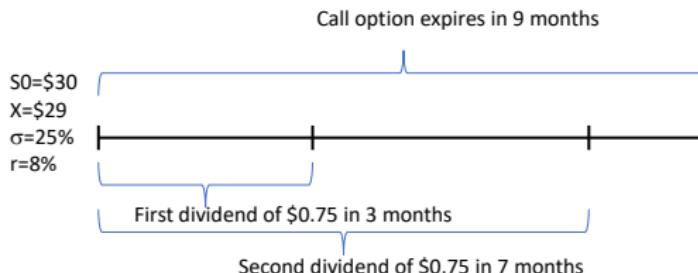
$$d_2 = \frac{\ln((S_0 - PV(D))/X) + (r - \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$= d_1 - \sigma\sqrt{T}$$



## 6. The Black-Scholes Model with Dividends

- Example
  - A 9-months European call option on a stock
  - The stock is expected to pay dividends in three months and seven months, \$0.75 each time
  - The current share price is  $S_0 = \$30$ , strike price  $X = \$29$
  - The stock price volatility is 25% p.a.
  - The risk-free rate is 8% p.a.
  - What is the price of the call option?



## 6. The Black-Scholes Model with Dividends

- The price of this call option

$$PV(D) = 0.75e^{-0.08 \times 3/12} + 0.75e^{-0.08 \times 7/12} = 1.451$$

$$d_1 = \frac{\ln((30 - 1.451))/29) + (0.08 + 0.25^2/2)\frac{9}{12}}{0.25\sqrt{\frac{9}{12}}} = 0.31$$

$$d_2 = 0.31 - 0.25\sqrt{\frac{9}{12}} = 0.09$$

$$\begin{aligned} c &= (S_0 - PV(D))N(d_1) - Xe^{-rT}N(d_2) \\ &= (30 - 1.451)N(0.31) - 29e^{-0.08 \times 9/12}N(0.09) \\ &= 28.549 \times 0.6217 - 29 \times 0.9418 \times 0.5359 \\ &= \$3.11 \end{aligned}$$



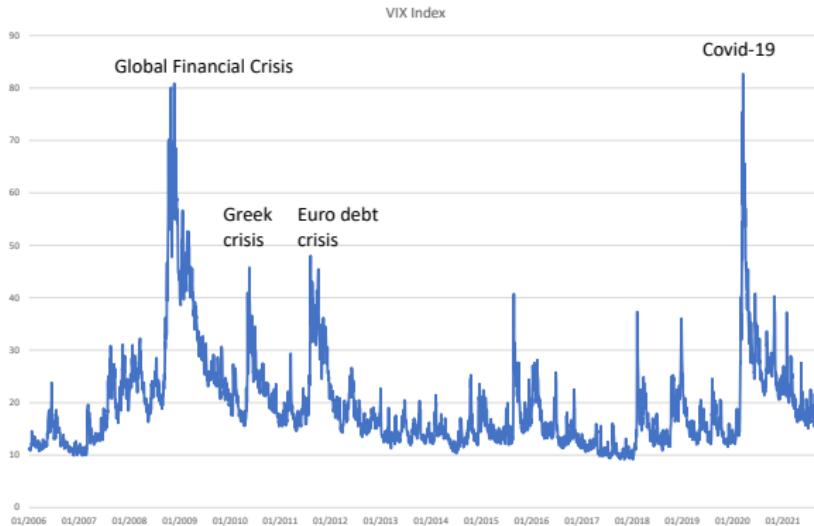
## 7. Implied Volatility

- In practice, the volatility of stock price cannot be observed directly
- Estimated using historical data – people get different estimates
- Implied volatility
  - Volatility implied by an option price observed in the market
  - In option market, we observe  $S_0, X, r, T$ , as well as  $c$  and  $p$
  - We can back out  $\sigma$  – the implied volatility
  - Represents the average market belief about the volatility
  - Forward-looking measure
  - Most accurate for ATM options
  - Unreliable for deep ITM or OTM options (prices insensitive to  $\sigma$ )
  - *Solver* in Excel



## 7. Implied Volatility

- VIX index is the implied volatility of options on S&P 500 Index
- Published by Chicago Board Options Exchange (CBOE)
- Benchmark index to measure the market's expectation of the U.S. equity market volatility in the future. "Fear index".



## 8. Conclusion

- In today's class
  - Black-Scholes model for the basic cases
  - Implied volatility
- Next week (week 7)
  - Options on stock indices and currencies
  - Progress test
  - No in-person lecture and workshop, tutorials as usual
  - Will be pre-recorded and posted on Echo360

