3F3 Lab Report.

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Definition of Global Variables

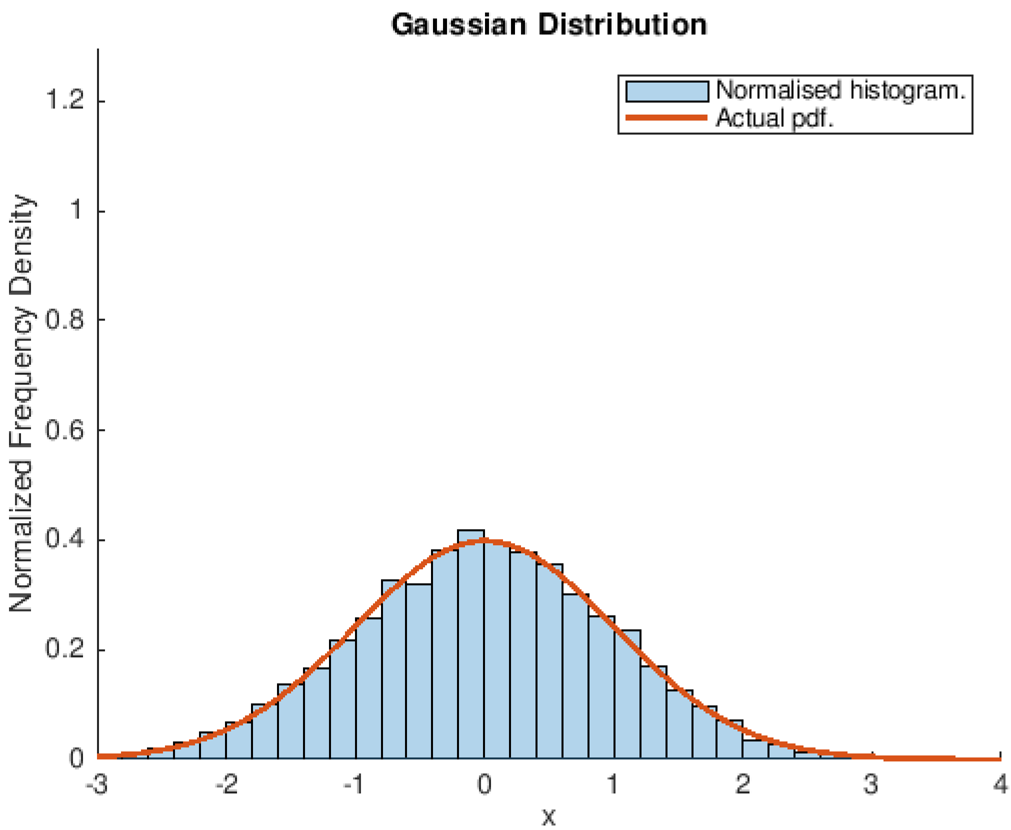
```
N = 10000;

global graph_max t;

graph_max = 1.3;
t = linspace(-3, 4, 100);
```

2.1 Uniform and normal Variables

```
gaussian_values = randn(N, 1);
gaussian_pdf = @(x) exp(-x.^2 / 2) / sqrt(2 * pi);
uniform_values = rand(N, 1) - 0.5;
uniform_pdf = @(x) 1 * (abs(x) < 0.5);
```

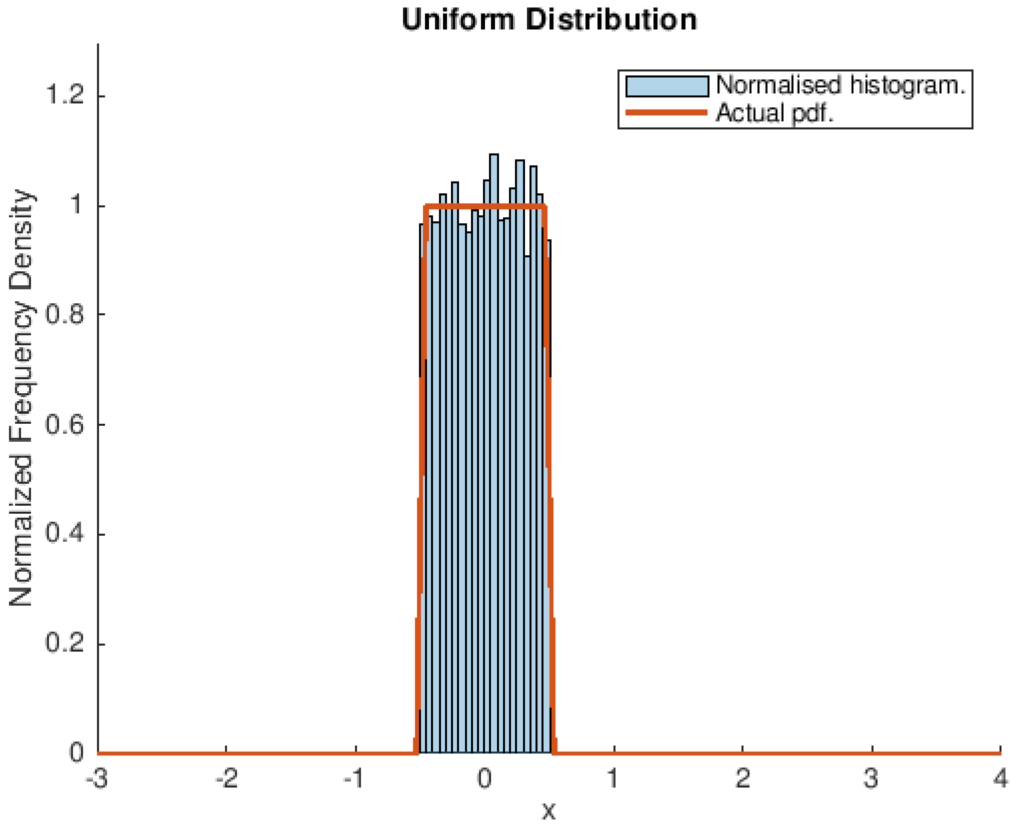
Scaled histogram of gaussian random numbers overlaid on the gaussian probability density function:

```
figure;
plotHistogram(gaussian_values, gaussian_pdf);
title('Gaussian Distribution');
```



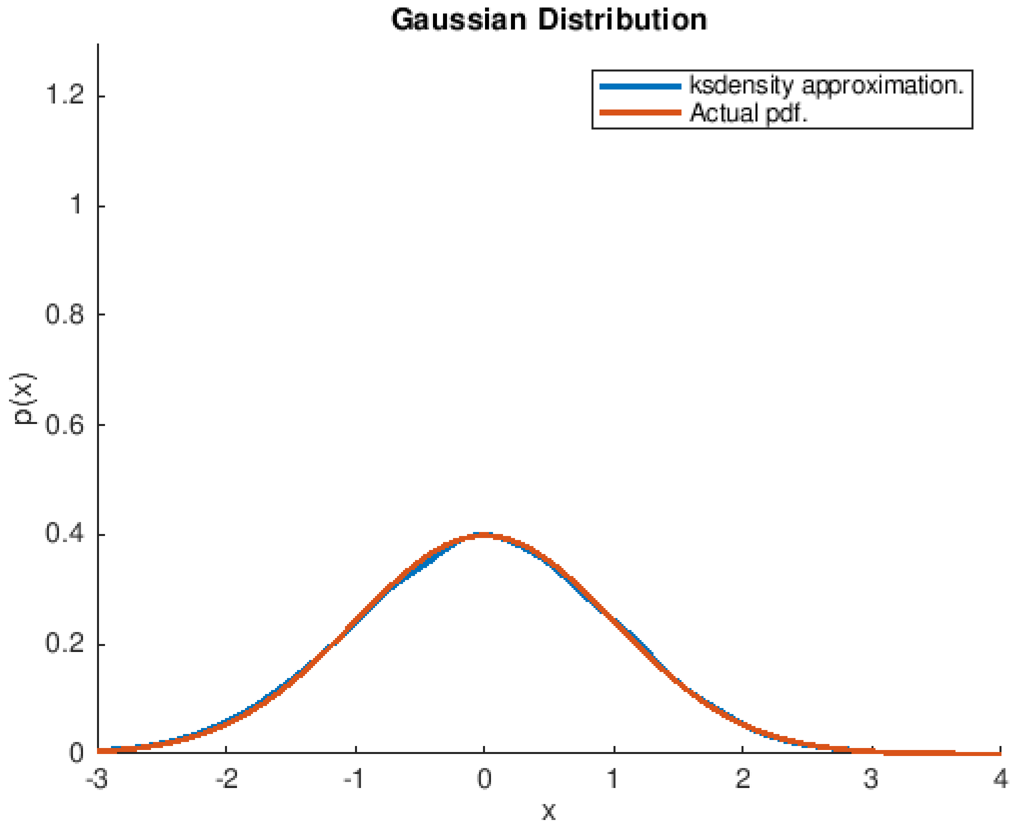
Scaled histogram of uniform random numbers overlaid on the uniform probability density function:

```
figure;
plotHistogram(uniform_values, uniform_pdf);
title('Uniform Distribution');
```



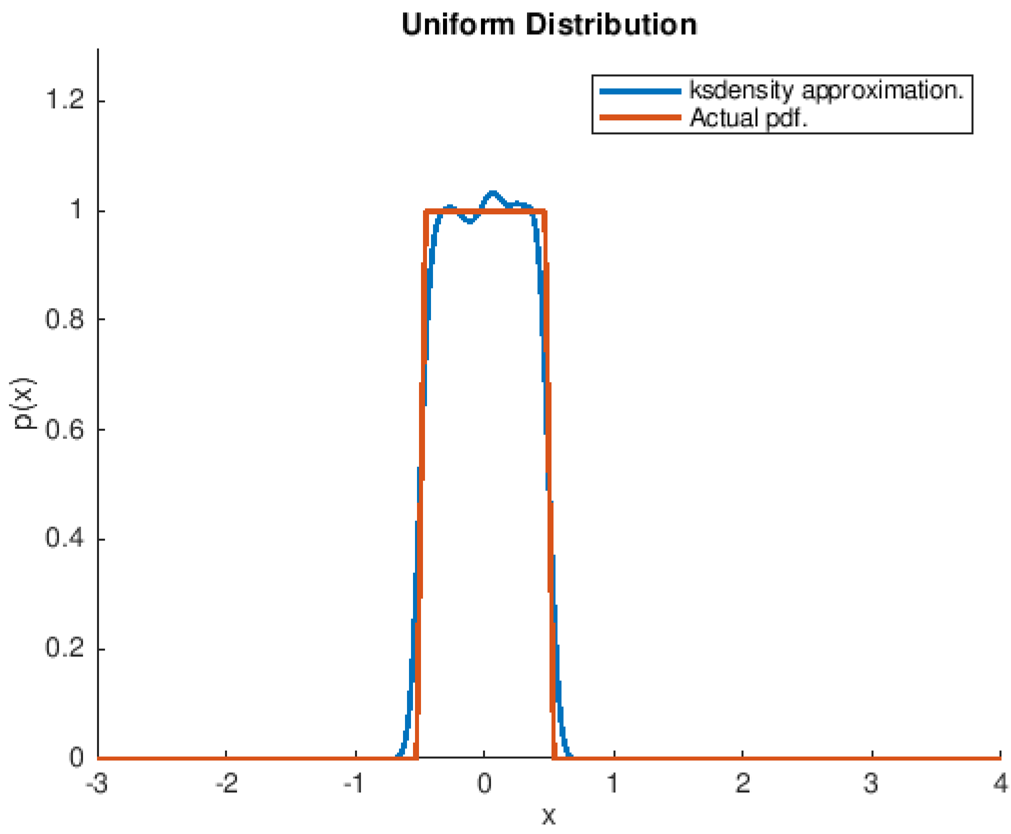
Kernel density estimate for Gaussian random numbers overlaid on the gaussian probability density function:

```
figure;
plotKsdensity(gaussian_values, gaussian_pdf);
title('Gaussian Distribution');
```



Kernel density estimate for Uniform random numbers overlaid on the uniform probability density function:

```
figure;
plotKsdensity(uniform_values, uniform_pdf);
title('Uniform Distribution');
```



The `ksdesnity` method is effective if the pdf is smooth (for example a guassian pdf) but is unsuitable for pdf's including discontinuities which it overly smooths. This is because the `ksdensity`'s use of gaussian kernals causes it to act a a low pass filter and therefore cannot be used with pdf's containing high frequencies (i.e. discontinuities).

```
figure;
hold on;

n = 10.^[2,3,4];
axes = 0 * n;
```

For a uniform distribution the histogram count data (the heights of unormalised bars on a histogram plot) will be multinomially distributed. The probability p_j of a sample $x^{(i)}$ lying in a bin is equal to the width of the bin divided by the total width of the distribution, therefore $p_j = 1/J$ where $J$is the number of bins.

For a histogram of Nsamples of a uniform random variable.

The mean bar height $\mu = NJ$.

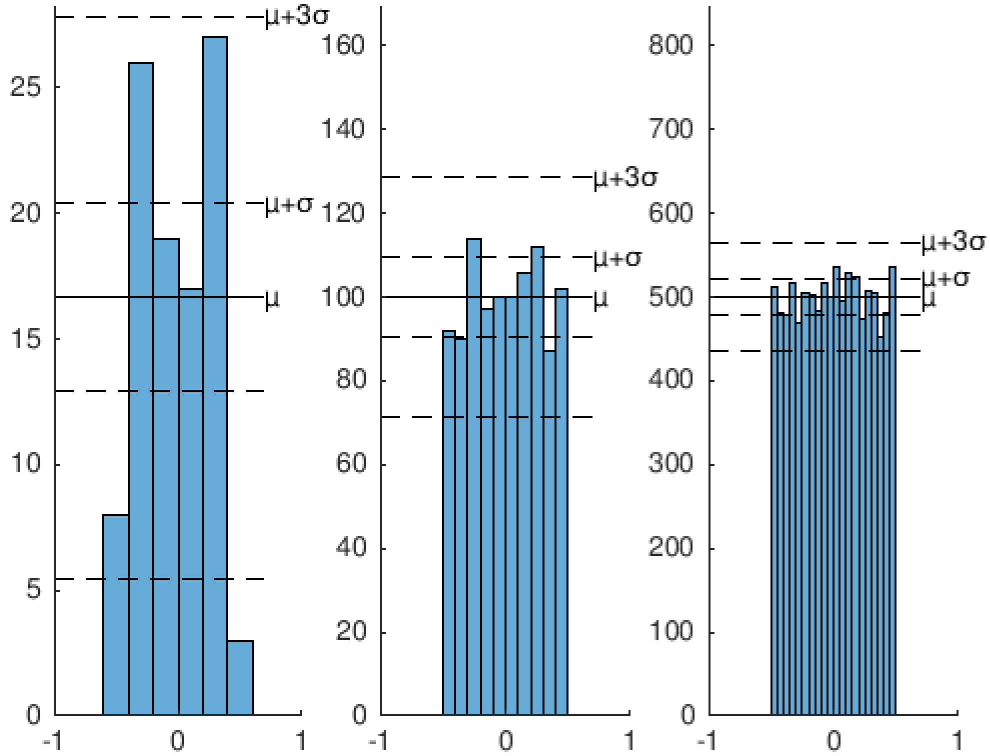
The variance in bar height $\sigma^2 = NJ(1 - J)$.

```
for i = 1:length(n)
    axes(i) = subplot(1,length(n),i);
```

```
    h = histogram(rand(n(i), 1) - 0.5);
    mean = n(i) / h.NumBins;
    std_dev = sqrt(mean * (1 - 1 / h.NumBins));

    line_heights = mean + std_dev * [3; 1; -1; -3];

    line([-1, 0.7], repelem(line_heights, 1, 2), ...
        'LineStyle', '--', ...
        'Color', 'black');

    line([-1, 0.7], [mean mean], 'Color', 'black');

    text([0.7 0.7], line_heights(1:2), ["μ+3σ"; "μ+σ"]);
    text(0.7, mean, "μ");

    set(gca, 'Box', 'off');
    ylim([0 mean * 1.7]);

    fprintf('For N = %i the mean of bar heights μ = %g and the variance in bar heights
        n(i), mean, std_dev^2);
end
```

```
For N = 100 the mean of bar heights μ = 16.6667 and the variance in bar heights σ² = 13.8889
For N = 1000 the mean of bar heights μ = 100 and the variance in bar heights σ² = 90
For N = 10000 the mean of bar heights μ = 500 and the variance in bar heights σ² = 475
```

```
hold off;
```



The histogram bars are centred on the mean (μ) and the variance of the bar heights (σ^2) increase linearly with N. Therefore, the standard deviation of bar height as a fraction of the total number of samples decreases which can be seen from the graph above as the bars are much more consistant in height for $N = 10^4$ compared to $N = 10^2$.

2.2 Functions of Random Variables

For normally distributed $X \sim \mathrm{N}(x|0, 1)$ random variables, take $y = f(X) = aX + b$.

$$p(y) = \frac{p(x)}{\frac{dy}{dx}}\Bigg|_{x=f^{-1}(x)}$$

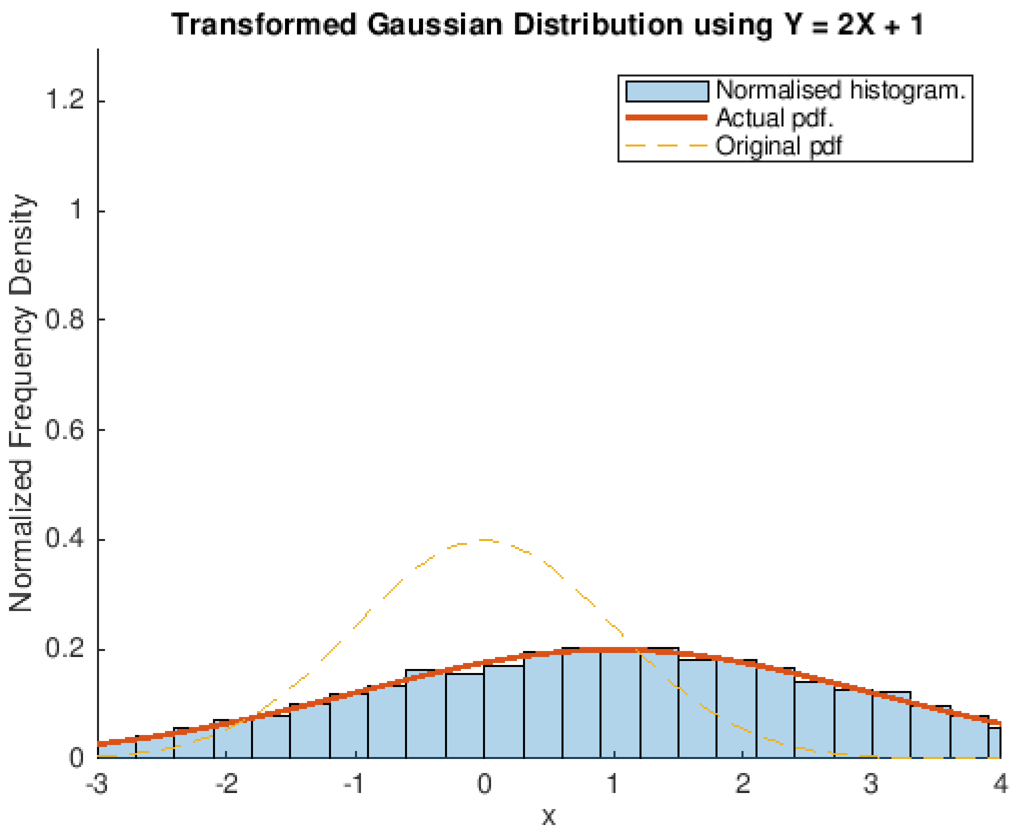
$f(x) = ax + b$ and so $f^{-1}(y) = \frac{y-b}{a}$

$$p(y) = \frac{p\left(\frac{y-b}{a}\right)}{a}$$

Therefore $Y \sim \mathrm{N}(y|b, a^2)$.

```
a = 2;
```

```
b = 1;

transformed_gaussian_values = a * gaussian_values + b;
transformed_gaussian_pdf = @(x) exp(-(x - b).^ 2 / (2 * a^2)) / (sqrt(2 * pi) * a);

figure
plotHistogram(transformed_gaussian_values, transformed_gaussian_pdf);
hold on;
plot(t, gaussian_pdf(t), '--', 'DisplayName', 'Original pdf');
title(sprintf('Transformed Gaussian Distribution using Y = %gX + %g', a, b));
hold off;
```



The graph shows histogramed values of Y overlayed with a guassian kernal $\mu = 1$ and $\sigma = 2$. The histogrammed results strongly follow the anylitically calculated pdf.

Now take $X \sim p(x) = \mathrm{N}(x|0, 1)$ and $Y = f(X) = X^2$

$f(x) = x^2$ and so $\frac{dy}{dx} = 2x$ and $f^{-1}(y) = \pm\sqrt{y}$. Note $f^{-1}(y)$ is multi valued

with $f_1^{-1}(y) = \sqrt{y}$ and $f_2^{-1}(y) = -\sqrt{y}$.

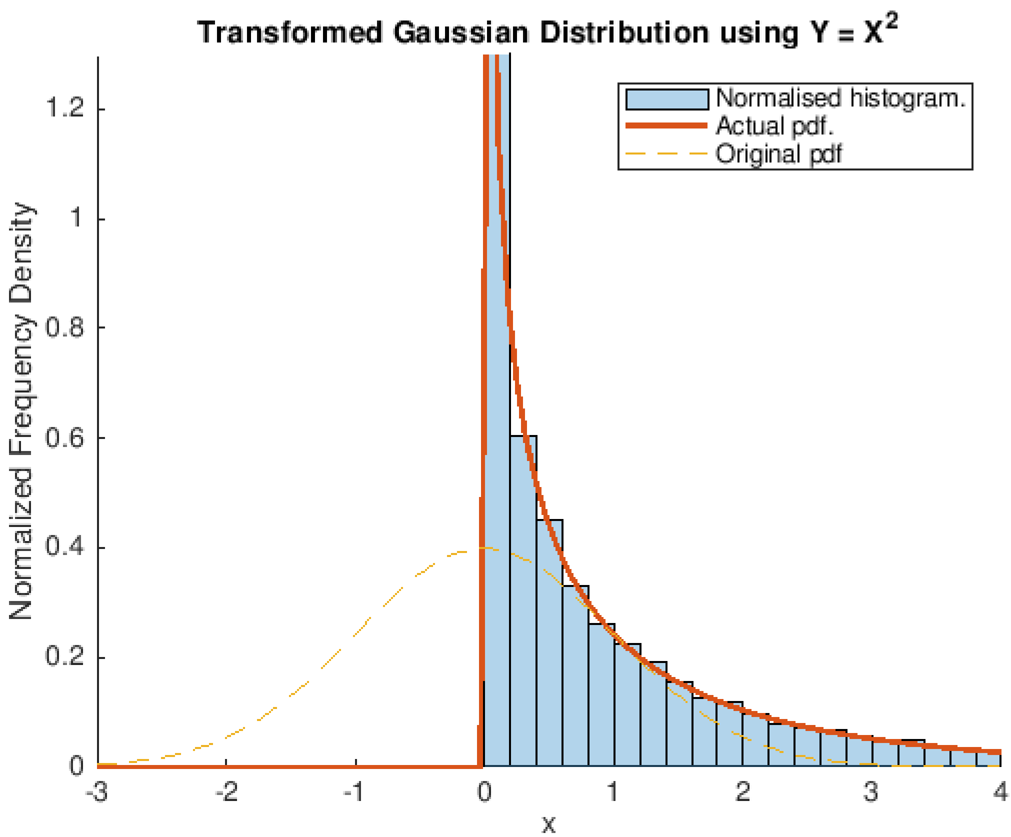
$$p_Y(y) = \left.\frac{p_X(x)}{\left|\frac{dy}{dx}\right|}\right|_{x=f_1{}^{-1}(x)} + \left.\frac{p_X(x)}{\left|\frac{dy}{dx}\right|}\right|_{x=f_2{}^{-1}(x)}$$

$$p_Y(y) = \frac{p_X(\sqrt{y})}{|2\sqrt{y}|} + \frac{p_X(-\sqrt{y})}{|-2\sqrt{y}|}$$

$P_Y(y) = \frac{p_X(\sqrt{y})}{\sqrt{y}}$ because $p_X(x)$ is an even function.

Y is a chi-squared distribution with one degree of freedom.

```
transformed_gaussian_values = gaussian_values.^2;
transformed_gaussian_pdf = @(x) exp(-x / 2) ./ (sqrt(x * 2* pi)) .* heaviside(x);

figure
plotHistogram(transformed_gaussian_values, transformed_gaussian_pdf);
hold on;
plot(t, gaussian_pdf(t), '--', 'DisplayName', 'Original pdf');
title('Transformed Gaussian Distribution using Y = X^2');
hold off;
```



The graph shows histogramed values of Y overlayed with plot of $\frac{e^{-x/2}}{\sqrt{2\pi x}}$. The histogrammed results strongly follow the anylitically calculated pdf.

2.3 Inverse CDF Method

For an exponential distribution with $\mu = 1$ the PDF $p_Y(y)$, CDF $F_y(y)$ and inverse CDF $F_Y{}^{-1}(x)$ are given by:

$$p_Y(y) = e^{-y},\ y \geq 0.$$

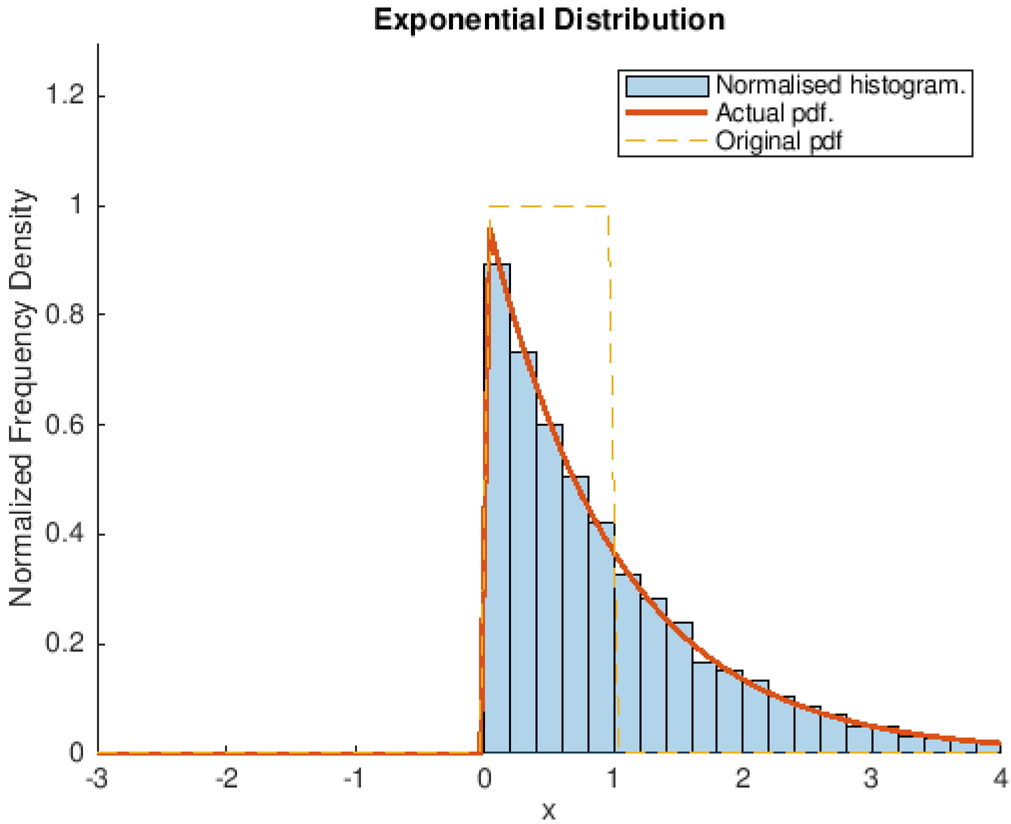
$$F_Y(y) = \int_{-\infty}^{y} p_Y(y)\,dy = \int_0^y e^{-y}\,dy = 1 - e^{-y}$$

$$F_Y{}^{-1}(x) = \log\left(\frac{1}{1-x}\right)$$

$$y^{(i)} = F_Y{}^{-1}(x^{(i)}) = \log\left(\frac{1}{1-x^{(i)}}\right)$$

The following MATLAB code calculates samples of the exponential distribution using the inverse CDF method.

```
exponential_pdf = @(x) exp(-x) .* heaviside(x);
inverse_exponental_cdf = @(x) log(1 ./ (1 - x));

exponential_values = inverse_exponental_cdf(uniform_values + 0.5);

figure;
plotHistogram(exponential_values, exponential_pdf);
hold on;
plot(t, uniform_pdf(t - 0.5), '--', 'DisplayName', 'Original pdf');
title('Exponential Distribution');
```



The graph shows histogramed values of exponentially distributed samples generated using the CDF method overlayed with plot of e^{-x}. The histogrammed results strongly follow the exponential pdf.

2.4 Simulation from a 'Difficult' Density

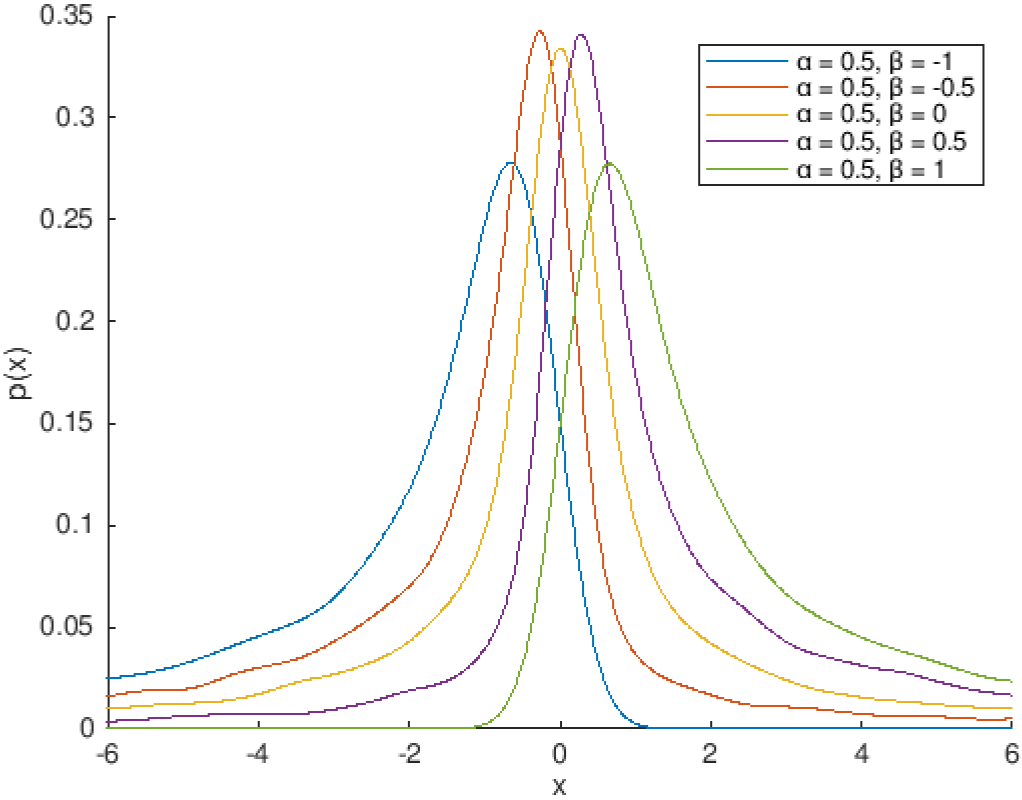
For parameters $\alpha \in (0,2)(\alpha \neq 1)$ and $\beta \in [-1,+1]$, the the function `generate_alpha_stable()` coverts samples from uniform and exponential distributions samples taken from the alpha stable distribution. The function is defined at the end of this document.

```
sample_count = N;
uniform = (rand(sample_count, 1) - 0.5) * pi;
expo = exprnd(1, [sample_count, 1]);

plotdist = @(alpha, beta) plot_alpha_stable(alpha, beta, uniform, expo, 6);
```

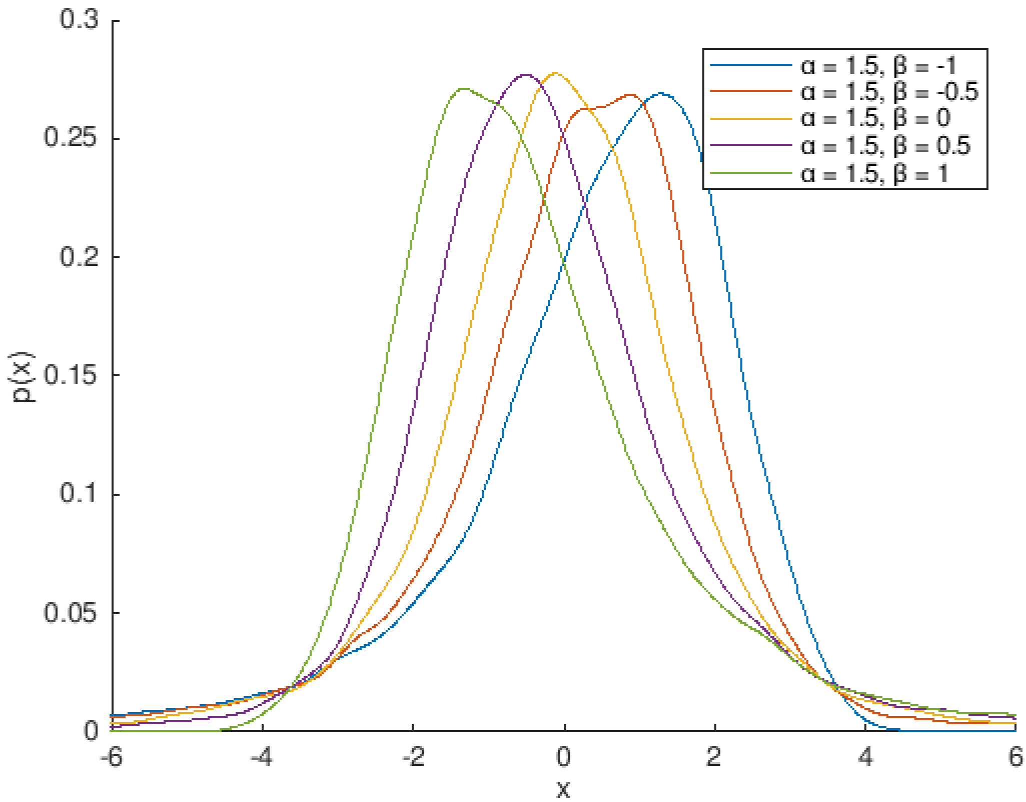
`ksdesnity` estimates of distribution with $\alpha = 0.5, 1.5$ and several values of β.

```
for alpha = [0.5, 1.5]
    figure;
    hold on;
    legend;
    for beta = [-1, -0.5, 0, 0.5, 1]
       plotdist(alpha, beta);
```

```
    end
end
```

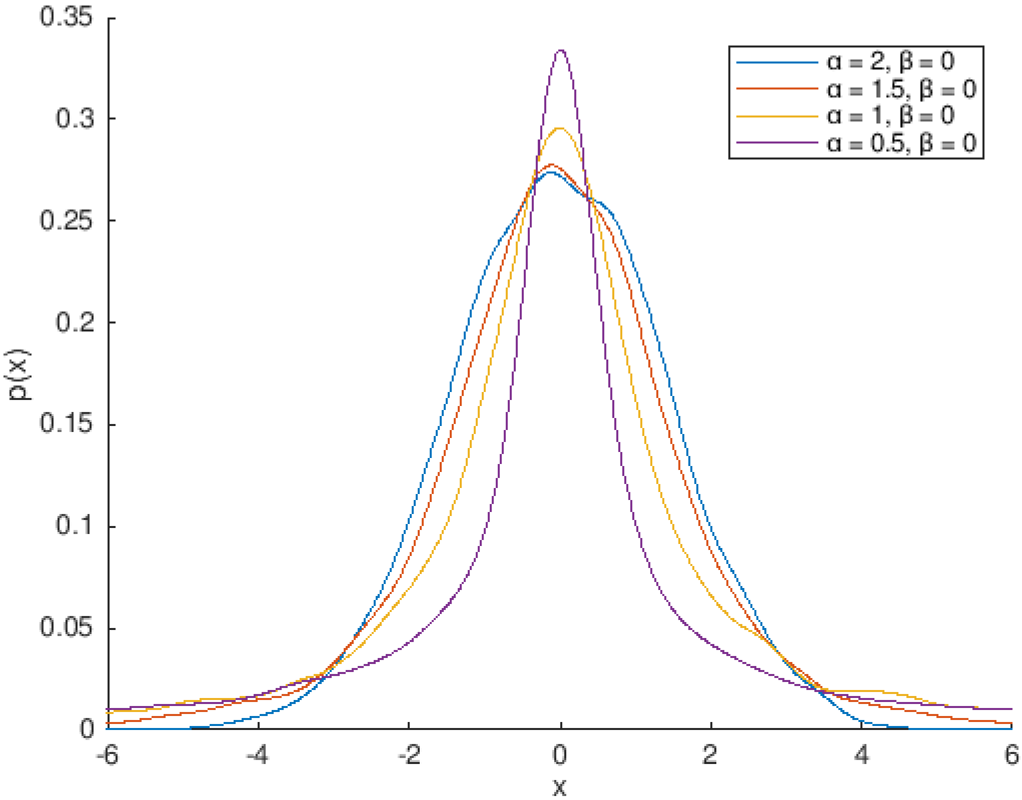


0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
p(x)
-6
-4
-2
0
2
4
6
x
α = 0.5, β = -1
α = 0.5, β = -0.5
α = 0.5, β = 0
α = 0.5, β = 0.5
α = 0.5, β = 1

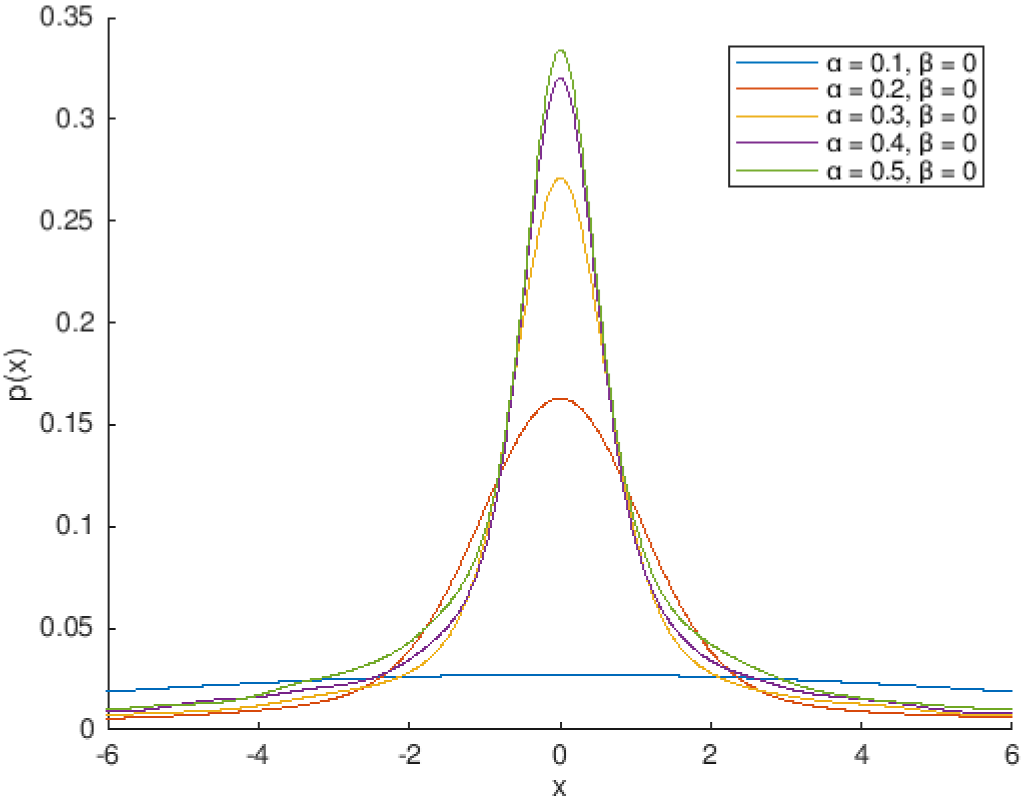


`ksdesnity` estimates of distribution with $\beta = 0$ and several values of α.

```
figure;
hold on;
legend;
for alpha = [2, 1.5, 1, 0.5]
    plotdist(alpha, 0);
end
```




```
figure;
hold on;
legend;
for alpha = [0.1, 0.2, 0.3, 0.4, 0.5]
    plotdist(alpha, 0)
end
```



Parameters α and β define the shape of the distribution. β causes the distribution to be skewed; $\beta > 0$ causes the distribution to be very steep for the left side and then very shallow on the right side, as $\beta \to 1$ the distribution becomes very heavy tailed. The opposite (heavy tail on the left side and sharp cut off on right side) is true for $\beta < 0$.

α controlls the variance of the distribution, $\alpha = 1/2$ creates a very sharp distribution which broadens out as α decreases or increases. In the limit as $\alpha \to 0$ the distribution becomes 0 everywhere (whilst integrating to unity).

Function Definitions

```
function X = generate_alpha_stable(alpha, beta, uniform, expo)
% generate_alpha_stable  Generates an alpha stable distribution.
%   X = generate_alpha_stable(alpha, beta, uniform, expo)
%   alpha and beta are the parameters of the distribution.
%   uniform and expo should be identically sized vectors of uniform and
%       exponentially distributed random numbers which will be used to
%       generate x.
%   X is a vector of random variables of length equal to the length of
%       uniform and expo.

    b = 1/alpha * atan(beta * tan(pi * alpha / 2));
```

```
    s = (1 + beta^2 * tan(pi * alpha /2)^2)^(1 / (2*alpha));


    X = s * sin(alpha * (uniform + b)) ./ ...
        cos(uniform).^(1 / alpha) .* ...
        (cos(uniform  - alpha * (uniform + b)) ./ expo).^((1 - alpha) / alpha);
end

function plot_alpha_stable(alpha, beta, uniform, expo, range)
% plot_alpha_stable  Generate and plot alpha stable distibution
%   in the range specified by range.

    X = generate_alpha_stable(alpha, beta, uniform, expo);

    [f, xi] = ksdensity(X, linspace(-range, range, 1000));

    plot(xi, f,  ...
        'DisplayName', sprintf('α = %g, β = %g', alpha, beta), ...
        'LineWidth', 1);

    ylabel('p(x)');
    xlabel('x');
end

function plotKsdensity (values, pdf)
% plotKsdensity  Plots ksdensity approximation of pdf overlayed
%   with the pdf defined by pdf.

    global graph_max t;
    hold on;

    [f,xi] = ksdensity(values);

    plot(xi, f, 'LineWidth', 2);
    plot(t, pdf(t), 'LineWidth', 2);

    legend('ksdensity approximation.', ...
        'Actual pdf.');
    ylabel('p(x)');
    xlabel('x');

    ylim([0, graph_max]);
    xlim([t(1) t(end)]);

    hold off;
end

function plotHistogram (values, pdf)
% plotHistogram  Plots histogram of values overlayed
%   with the pdf defined by pdf.
    global graph_max t;
    hold on;
```

```
    histogram(values, 'Normalization', 'pdf', 'FaceAlpha', 0.3);

    plot(t, pdf(t), 'LineWidth', 2);

    legend('Normalised histogram.', ...
        'Actual pdf.');

    ylabel('Normalized Frequency Density');
    xlabel('x');
    ylim([0, graph_max]);
    xlim([t(1) t(end)]);

    hold off;
end
```