Chapter 3: Vector Spaces and Subspaces

Chapter 3.1: Spaces of Vectors

- A vector space is a set of whose elements, often called vectors, may be added together and scaled. Within this space, the operations of vector addition and scalar multiplication must satisfy certain axioms*
- The vector spaces $\mathbb{R}^1, \mathbb{R}^2, \mathbb{R}^3, \ldots \mathbb{R}^n$, $n \in \mathbb{N}$ contain all column vectors with n components. The components of said vectors are real numbers, hence the $\mathbb{R}$. Complex vector spaces are denoted as $\mathbb{C}, \mathbb{C}^2, \mathbb{C}^3, \ldots, \mathbb{C}^n$
- The vector space $\mathbb{R}^2$ is represented by an xy plane. Each vector gives the x and y coordinates of a point on the plane $v=(x,y)$
- Similarly $\mathbb{R}^3$ corresponds to (x,y,z) in three dimensional space and $\mathbb{R}^1$ is a line.
- Above $\mathbb{R}^3$, it is hard to visualize spaces geometrically but algebraically everything is the same.
- From the two operations (vector addition and scalar multiplication), we produce linear combinations.
  - Important, all the operations go on "inside the vector space" - The resultant vector stays within the original vector space.
- *The 8 axioms for a vector space V.
  1. Associativity of vector addition: $u+(v+w)=(u+v)+w$
  2. Commutivity of vector addition: $u+v=v+u$
  3. Identity element of vector addition: $\exists 0 \in V$ s.t $v+0=v \;\forall v \in V$
  4. Inverse elements of vector addition: $\forall v \in V, \exists -v \in V$ s.t. $v+(-v)=0$
  5. Compatability of scalar multiplication with field multiplication: $(ab)v=a(bv)$
  6. Identity element of scalar multiplication: $1v=v$, where 1 is the multiplicative identity
  7. Distributivity of scalar multiplication with respect to vector addition. $a(u+v)=au+av$
  8. Distributivity of scalar multiplication with respect to field addition - $(a+b)v=av+bv$
- Other vector spaces:
  - M: The vector space of all 2×2 matrices
  - F: The vector space of all real functions $f(x)$
  - Z: The vector space containing only the zero vector.
- The function space F is infinite-dimensional. A smaller function space is P or P_n, containing all polynomials $a_0+a_1x+\ldots+a_nx^n$ of degree n.
- The space Z is zero-dimensional, containing just the zero vector
- Every space needs and has its own zero vector: $(0,0)$, $(0,0,0)$, the zero function, the zero matrix, etc

Subspaces

- A subspace is simply a subset of the vectors in a vector space and closed under its two operations
- For example, a plane through the origin in 3D is a subspace of $\mathbb{R}^3$.
  - Adding or scaling vectors in the plane, stay in the plane
  - The vectors in the plane are still in $\mathbb{R}^3$ (they have 3 components)
- A subspace of a vector space is a set of vectors (including 0) that is closed under scaler multiplication and vector addition. If v and w are vectors in the subspace and c is any scalar, then:
  - $v + w$ is in the subspace
  - cv is in the subspace.
- A consequence is that all linear combinations remain in the subspace
- All subspaces must contain the zero vectors
  - If we choose $c = 0$, then $cv = 0$, which means the zero vector must be contained.
- Lines and planes through the origin are subspaces.
- The entire space and the zero vector space are trivial subspaces.
- A subspace containing v and w must contain all linear combinations $cv + dw$

The column space of A.

- The most important subspaces are tied directly to a matrix A. We are trying to solve $Ax = b$. If A is not invertible, the system is solvable for some b and not solvable for other b. We want to describe the possible b for which $Ax = b$ is solvable. These b's form the column space of A. Also called the image.
- Ax is a linear combination of the columns of A. To get every possible b, we need to use every possible x, ie. every possible linear combinations of the column vectors of A.
- The column space is denoted as $C(A)$.
- The system $Ax = b$ is solvable if and only if b is in the column space of A.
- Suppose A is an m by n matrix. It's columns have m components, so they belong to $\mathbb{R}^m$. The column space of A is a subspace of $\mathbb{R}^m$
- Instead, we could start with an arbitrary set of vectors S in a vector space V. To get a subspace SS of V, we take all combinations of the vectors in that set. SS will always be the smallest subspace containing S. SS is the span of S, containing all combinations of vectors in S.
- For example:

$$I \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow C(I) = \mathbb{R}^2 \qquad A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \Rightarrow C(A) = \mathbb{R}^1$$

Problem Set 3.1

10

a) $(b_1, b_2, b_3) + (a_1, a_2, a_3)$

$(b_1, b_2, b_3) + ($

$-1 \quad 1 \quad 1 \quad + (1 \; 1 \; 2) = (-1 \; 3, -1$

a) T

b) F

c)

13.

$P_0: x + y - 2z = 0$

take $v = (1, 1, 1)$,

$w = (2, 2, 2)$

$v + w = (3, 3, 3)$

$(3) + (3) - 2(3) = 0$

$0 = 0$ ✓

14

a) $\mathbb{R}^2$ itself, Z, and lines through the origin

b) D itself, Z, $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, ... $\begin{bmatrix} 0 & b \\ a & 0 \end{bmatrix}$

15.

a) line, plane,

b) point, line

n.

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, ad-bc ≠ 0, If we apply c=0, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is singular

b)

$\underbrace{\begin{bmatrix} 3 & 6 \\ 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}}_{\text{singular}} = \begin{bmatrix} 4 & 7 \\ 2 & 3 \end{bmatrix}$ → not singular (det ≠ 0)

19.

A: line, B: plane, C: line

20.

a) b must be a scalar multiple of $(1, 2, -1)$

b)

1	2	-1	1	2	-1
4	9	-4	4	9	-4

$-8 \quad -4 \quad 9$

$-9 \quad -4 \quad 8$

$\vec{n} = (1, 0, 1)$

b must be on the plane

$x + z = 0$

21.

A and C have the same column span

23.

unless it is linearly dependent.

It means the columns of A and b are linearly dependent.

Therefore b can be expressed as a linear combination of the columns of A.

28.

$\begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

29. $\mathbb{R}^9$

30.

S+T:

because S, T are closed

Addition $(s_1 + t_1) + (s_2 + t_2) = \overbrace{(s_1 + s_2)}^{S} + \overbrace{(t_1 + t_2)}^{T}$

Scalar multiplication: $c(s_1 + t_1) = \underbrace{cs_1}_{S} + \underbrace{ct_1}_{T}$

Chapter 3.2 : The Nullspace of A : Solving $Ax = 0$

- The nullspace of A, denoted as N(A), consists of all solutions of $Ax = 0$, where x is in $\mathbb{R}^n$
  - For invertible matrices, the only solution is $x = 0$.
  - For singular matrices, there exist nonzero solutions to $Ax = 0$
- The nullspace is a subspace of $\mathbb{R}^n$
  - suppose x and y are in the nullspace ($Ax = 0$ and $Ay = 0$)
  - Then $Ax + Ay = A(x + y) = 0 + 0 = 0$
  - Then $A(cx) = c(Ax) = c(0) = 0$
  - Therefore $x + y$ and cx are also in the nullspace, making it a subspace
- Also called the kernel.
- The solution have n components, so the nullspace is a subspace of $\mathbb{R}^n$
- If the right side b is not zero, then the solutions of $Ax = b$ do NOT form a subspace, since $Ax = b$, $b \neq 0$ does not contain $x = 0$ as a solution (i.e. it can't be a subspace)

- Example:

Given $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \Rightarrow Ax = 0 \Rightarrow \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} x_1 + 2x_2 = 0 \\ 3x_1 + 6x_2 = 0 \end{cases} \Rightarrow \begin{cases} x_1 + 2x_2 = 0 \\ 0 = 0 \end{cases}$

$\therefore$ the nullspace of A is all vectors $x = (x_1, x_2)$ such that $x_1 + 2x_2$

- we can take a single solution and consider all its multiples to be in the nullspace.

Choose $x = (-2, 1) \Rightarrow$ The nullspace of A, $N(A)$, contains all multiples of $s = (-2, 1)$

- We can find these "special solutions" by setting a component of x that is not a pivot to zero
  - Example:

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \overset{\text{not free (don't set to 0)}}{x} + \underbrace{2y + 3z}_{\text{free (set to 0)}} = 0 \Rightarrow s_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \; s_2 = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

- Consider

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix} \quad B = \begin{bmatrix} A \\ 2A \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 8 \\ 2 & 4 \\ 6 & 16 \end{bmatrix} \quad C = \begin{bmatrix} A & 2A \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 3 & 8 & 6 & 16 \end{bmatrix}$$

$N(A) = 0$

$N(B) = 0$ b/c the first 2 rows lead to the nullspace. Adding additional rows do not widen the nullspace

$N(C)$:

$$C = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 3 & 8 & 6 & 16 \end{bmatrix} \text{ becomes } U = \begin{array}{cccc} \begin{bmatrix} 1 & 2 & 2 & 4 \\ 0 & 2 & 0 & 4 \end{bmatrix} \\ \uparrow\;\uparrow\;\uparrow\;\uparrow \\ \text{pivot columns, free columns} \end{array} \Rightarrow Ux = 0 \Rightarrow \begin{bmatrix} x_1 + 2x_2 + 2x_3 + 4x_4 \\ 2x_2 + 4x_4 \end{bmatrix}$$

To get special solutions, set the free columns to 1 and 0

$$s_1 = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \; s_2 = \begin{bmatrix} 0 \\ -2 \\ 0 \\ 1 \end{bmatrix} \Rightarrow N(C) \text{ is all linear combinations of } s_1 \text{ and } s_2$$

- The nullspace becomes easier to see when A reaches reduced row echelon form (its pivot column becomes I), using the matrix C above:

$$U = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 0 & 2 & 0 & 4 \end{bmatrix} \text{ becomes } R = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix} \Rightarrow \begin{cases} x_1 + 2x_3 = 0 \\ x_2 + 2x_4 = 0 \end{cases}$$

- A nullspace of Z implies the columns of A are independent

$N(A) = Z \iff$ the columns of A are linearly independent.

Solving $Ax=0$ by Elimination

- Even when A is rectangular, we still use elimination
- The steps are still the same:
  1. Forward substitution gets A to triangular U or reduced R.
  2. Back substitution in $Ux = 0$ or $Rx = 0$ produces x.
- Pivots are still no zero and the column below the pivots are zero but if a column does not produce a pivot, we simply move to the next column

- Given:

$$A=\begin{bmatrix}1&1&2&3\\2&2&8&10\\3&3&10&13\end{bmatrix} \text{ Apply } E_{32}E_{21} \Rightarrow A=\begin{bmatrix}1&1&2&3\\0&0&4&4\\0&0&4&4\end{bmatrix} \text{ Applying } E_{33} \Rightarrow A=\begin{bmatrix}1&1&2&3\\0&0&4&4\\0&0&0&0\end{bmatrix}$$

our second pivot is 4 (in column 3) only 2 pivots

only pivot variables are x_1 and x_3 (column 1 and 3 have pivots)

Only free variables are x_2 and x_4 (column 2 and 4 have no pivots)

- Giving the free variables any values then solving for the pivot variables gives us special solutions

$x_1+x_2+2x_3+3x_4=0$ set $x_2=1$ and $x_4=0$, then by back substitution $x_3=0$ $x_1=-1$

$4x_3+4x_4=0$ Then set $x_2=0$ and $x_4=1$, then by back substitution $x_3=-1$ and $x_1=-1$

The nullspace is the linear combination of these 2 solutions

$$x=x_2\begin{bmatrix}-1\\1\\0\\0\end{bmatrix}+x_4\begin{bmatrix}-1\\0\\-1\\1\end{bmatrix}=\begin{bmatrix}-x_2-x_4\\x_2\\-x_4\\x_4\end{bmatrix}$$

- Example: Find the nullspace of U

$$\begin{array}{c}U=\begin{bmatrix}1&5&7\\0&0&9\end{bmatrix}\\ \quad\text{pivot}_1\quad\text{pivot}_2\end{array}$$ Pivots: x_1, x_3 free: x_2 U as a system $\begin{cases}x_1+5x_2+7x_3=0\\ 9x_3=0\end{cases}$

set $x_2=1$

$\therefore N(U)=c(-5,1,0) \leftarrow \begin{cases}x_1+5(1)+7x_3=0\\ x_3=0\end{cases} \Rightarrow \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}-5\\1\\0\end{bmatrix}$

OR convert U to reduced row echelon form (RREF)

$$U=\begin{bmatrix}1&5&7\\0&0&9\end{bmatrix}\Rightarrow\begin{bmatrix}1&5&7\\0&0&1\end{bmatrix}\rightarrow\begin{bmatrix}1&5&0\\0&0&1\end{bmatrix}\rightarrow\begin{cases}x_1+5x_2=0\\x_3=0\end{cases}\rightarrow \text{Set } x_2=1 \Rightarrow x=\begin{bmatrix}-5\\1\\0\end{bmatrix}$$

pivot column

Echelon Matrices

- Forward elimination goes from A to U. It acts by row operations, including row exchanges. It goes to the next column when no pivot is available in the current column. The m by n "Staircase" U is an echelon matrix.
- For example, here is a 4 x 7 echelon matrix with three pivots

$$U=\begin{bmatrix}p&x&x&x&x&x&x\\0&p&x&x&x&x&x\\0&0&0&0&0&p&x\\0&0&0&0&0&0&0\end{bmatrix}$$

- Three pivots x_1, x_2, x_6
- Four free variables x_3, x_4, x_5, x_7
- Four vectors as the basis of N(U)

- The column space of U is $C(U)=c(1,1,1,0), c\in\mathbb{R}$
- The nullspace of this matrix is a subspace of $\mathbb{R}^7$. It is the linear combinations of the four special solutions - one for each free variable.
  - Set one free variable to 1 and the others to zero. Then solve for the pivot variables
  - Doing this for all free variables gets us the four vectors which act as the basis of N(U)

- If A has more columns than rows ($n>m$), there is at least one free variable, ergo there is at least one special solution, ergo $N(A) \neq Z$
   - Suppose A has more unknowns than equations ($n>m$), then there must be nonzero solutions. There must be free column's.
   - There must be at least $n-m$ free variables, since the number of pivots cannot exceed m (There can only be 1 pivot per row). When there is a free variable, it can be set to one and a nonzero solution to $Ax=0$ is created
- The nullspace is a subspace; Its "dimension" is the number of free variables

The Reduced Row Echelon Matrix R

- From an echelon matrix U we can go one more step"

  Given:

$$U=\begin{bmatrix}1&1&2&3\\0&0&4&4\\0&0&0&0\end{bmatrix}\xrightarrow[\text{into } R_2]{R_2\div 4}\begin{bmatrix}1&1&2&3\\0&0&1&1\\0&0&0&0\end{bmatrix}\xrightarrow[\text{into } R_1]{R_1-2R_2}\begin{bmatrix}1&1&0&1\\0&0&1&1\\0&0&0&0\end{bmatrix}$$

   - The pivots themselves equal one and there are zeros above and below the pivots. (i.e. the pivot rows contain I)

$$R=\text{rref}(U)=\begin{bmatrix}1&1&0&1\\0&0&1&1\\0&0&0&0\end{bmatrix}$$

- If A is invertible, then its rref is the identity matrix I.
- To find special solutions:

  in column 2 (with plus signs)

  1. Set $x_2=1$ and $x_4=0$ and solve $Rx=0$ for x_1 and x_3 $(-1, 1, 0, 0)$
  2. Set $x_2=0$ and $x_4=1$ and solve $Rx=0$ for x_1 and x_3 $(-1, 0, -1, 1)$

  in column 4 (with plus signs)

   - By reversing signs we can directly read off the special solutions
- Example:

  Create a 3×4 matrix whose special solutions to $Ax=0$ are s_1 and s_2

$$s_1=\begin{bmatrix}-3\\1\\0\\0\end{bmatrix}\qquad s_2=\begin{bmatrix}-2\\0\\-6\\1\end{bmatrix}$$

  Pivots: columns 1 and 3

  Free variables: columns 2 and 4

$$R=\begin{bmatrix}1&3&0&2\\0&0&1&6\\0&0&0&0\end{bmatrix}$$

  Pivots

  The free columns 2 and 4 will be combinations of the pivot columns

   - R can be multiplied on the left by any invertible matrix without changing its nullspace

Problem Set 3.2

1.

a) $A=\begin{bmatrix}1&2&2&4&6\\1&2&3&6&9\\0&0&1&2&3\end{bmatrix}\xrightarrow[\text{into } R_2]{R_2-R_1}\begin{bmatrix}1&2&2&4&6\\0&0&1&2&3\\0&0&1&2&3\end{bmatrix}\xrightarrow[\text{into } R_3]{R_2-R_3}\begin{bmatrix}1&2&2&4&6\\0&0&1&2&3\\0&0&0&0&0\end{bmatrix}$

Pivot variables: x_1, x_3

Free variables: x_2, x_4, x_5

$\begin{cases}x_1+2x_2+2x_3+4x_4+6x_5=0\\ x_3+2x_4+3x_5=0\end{cases}$

Set $(x_2,x_4,x_5)=(1,0,0)$

$\begin{cases}x_1+2(1)+2x_3+4(0)+6(0)=0\\ x_3+2(0)+3(0)=0\end{cases}$

$\begin{cases}x_1+2+2x_3=0\\ x_3=0\end{cases}$

$x_1+2+2(0)=0$

$x_1=-2$

$s_1=(-2,0,0,0,0)$

Set $(x_2,x_4,x_5)=(0,1,0)$

$\begin{cases}x_1+2(0)+2x_3+4(1)+6(0)=0\\ x_3+2(1)+3(0)=0\end{cases}$

$\begin{cases}x_1+2x_3+4=0\\ x_3+2=0\end{cases}$

$\begin{cases}x_1+2(-2)+4=0\\ x_3=-2\end{cases}$

$s_2=(0,0,-2,1,0)$

Set $(x_2,x_4,x_5)=(0,0,1)$

$\begin{cases}x_1+2(0)+2x_3+4(0)+6(1)=0\\ x_3+2(0)+3(1)=0\end{cases}$

$\begin{cases}x_1+2x_3+6=0\\ x_3+3=0\end{cases}$

$\begin{cases}x_1+2(-3)+6=0\\ x_3=-3\end{cases}$

$s_3=(0,0,-3,0,1)$

$N(A)=x_1(-2,0,0,0,0)+x_2(0,0,-2,1,0)+x_3(6,0,-6,0,1)$

$=(-2x_1+6x_3,\ 0,\ -2x_2-6x_3,\ x_2,\ x_3),\ x_1,x_2,x_3\in\mathbb{R}$

b) $B=\begin{bmatrix}2&4&2\\0&4&4\\0&8&8\end{bmatrix}\xrightarrow[\text{into } R_3]{R_2-2R_3}\begin{bmatrix}2&4&2\\0&4&4\\0&0&0\end{bmatrix}\xrightarrow[R_2\div 4]{R_1\div 2}\begin{bmatrix}1&2&1\\0&1&1\\0&0&0\end{bmatrix}\xrightarrow[\text{into } R_1]{R_1-2R_2}\begin{bmatrix}1&0&-1\\0&1&1\\0&0&0\end{bmatrix}$

Pivot variables: x_1, x_2

Free variables: x_3

Set $x_3=1$

$\begin{cases}x_1-(1)=0\\ x_2+(1)=0\end{cases}\Rightarrow s_1=(1,-1,1)$

$N(A)=x_1s_1=(x_3,-x_3,x_3)$

2.

3. free columns

4.

$U=\begin{bmatrix}1&2&2&4&6\\0&0&1&2&3\\0&0&0&0&0\end{bmatrix}\xrightarrow[\text{into } R_1]{R_1-2R_2}\begin{bmatrix}1&2&0&0&0\\0&0&1&2&3\\0&0&0&0&0\end{bmatrix}$

5. 6. n

a)

$A=\begin{bmatrix}-1&3&5\\-2&6&10\end{bmatrix}\xrightarrow[\text{into } R_2]{R_2-2R_1}\begin{bmatrix}-1&3&5\\0&0&0\end{bmatrix}=U$

$LU=\begin{bmatrix}1&0\\2&0\end{bmatrix}\begin{bmatrix}-1&3&5\\0&0&0\end{bmatrix}=\begin{bmatrix}-1&3&5\\-2&6&10\end{bmatrix}$

pivot: x_1

Free: x_2, x_3

Set $(x_2,x_3)=(1,0)$

$-x_1+3(1)+5(0)=0$

$x_1=3$

$x=(3,1,0)$

Set $(x_2,x_3)=(0,1)$

$-x_1+3(0)+5(1)=0$

$x_1=5$

$x=(5,0,1)$

plane

b)

$B=\begin{bmatrix}-1&3&5\\-2&6&7\end{bmatrix}\xrightarrow[\text{into } R_2]{R_2-2R_1}\begin{bmatrix}-1&3&5\\0&0&-3\end{bmatrix}$

pivot: x_1, x_3

free: x_2

Set $x_2=1$

$\begin{cases}-x_1+3x_2+5x_3=0\\ -3x_3=0\end{cases}$

$\begin{cases}-x_1+3(1)+5(0)=0\\ x_3=0\end{cases}\rightarrow x=(3,1,0)$

9.

a) false $\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$ 1 pivot

b) true.
An invertible matrix has n pivots.
Therefore there are not free variables

c) false

d) true

11.

a)
$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

b)
$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

15.

$n-r$, m, M

12.

a)
$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

17.

$A = \begin{bmatrix} 1 & -3 & -1 \end{bmatrix}$

17. $(1, 0, 1)$

18.

$(3, 1, 0)$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

direction vectors

21.

$$\begin{bmatrix} 1 & 0 & -2 & -3 \\ 0 & 1 & -2 & -1 \end{bmatrix}$$

22.

$$\begin{bmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

23.

$$\left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{2} & 1 \\ 1 & 3 & -2 & 1 \\ 5 & 1 & -3 & 0 \end{array}\right]$$

24.

$$\begin{matrix} 0 \\ 1 \\ 1 \end{matrix}$$

Chapter 3.3: The Rank and Row Reduced Form

- The numbers m and n give the size of the matrix, but not necessarily the true size of the linear system. An equation like $0=0$ should not count.
  If there are two identical rows in A, they would disappear
- The true size of A is given by its rank.
  - The rank r is the number of pivots.
- This definition is computational.

$$A = \begin{bmatrix} 1 & 1 & 2 & 4 \\ 1 & 2 & 2 & 5 \\ 1 & 3 & 2 & 6 \end{bmatrix}$$

  - Column 1 & 3 are multiples
  - Column 4 is a combination of column 1 and 2
  - 2 pivots $\rightarrow$ rank(A) = rank(U) = 2

$$U = \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- The rank is the dimension of the vector space spanned by its columns.
  This corresponds to the maximal number of linearly independent columns of A, which is also equal to the dimension of the vector space spanned by its rows.
  - The column rank (dimension of column span) always equals the row rank (dimension of row span)
- The nullity is, similarly, the dimension of the nullspace of A, which corresponds to the number of free columns of A.
- The rank-nullity theorem states that rank(A) + nullity(A) = n = number of columns.

Rank One

- Matrices of rank one only have one pivot. When elimination produces zero in the first column, it produces zero in all the columns. Every row is a multiple of the pivot row.
- The column space of a rank one matrix is one-dimensional
- For example

$$A=\begin{bmatrix}1&3&10\\2&6&20\\3&9&30\end{bmatrix}\longrightarrow R=\begin{bmatrix}1&3&10\\0&0&0\\0&0&0\end{bmatrix}$$

  - The columns are all multiples of $u=(1,2,3)$
  - $A=\begin{bmatrix}u & 3u & 10u\end{bmatrix}$
  - We can write A as an outer product:

$v^T=[1\ 3\ 10]$

$$uv^T=\begin{bmatrix}1\\2\\3\end{bmatrix}[1\ 3\ 10]=\begin{bmatrix}1&3&10\\2&6&20\\3&9&30\end{bmatrix}$$

- The nullspace is easy to visualize with a rank one matrix
  That equation $u(v^Tx)=0$ leads us to $v^Tx=0$ i.e. the vector v^T is orthogonal (perpendicular) to x. All vectors in the nullspace are on a plane perpendicular to v^T
- Now with numbers:
  pivot variable: x_1
  free variables: x_2, x_3
  Since R is already in rref, we can simply read the solutions off
  $s_1=(-3,1,0)$, $s_2=(-10,0,1)$

$$N(A)=c_1s_1+c_2s_2 \text{ (plane) } \xrightarrow[\text{Standard}]{\text{Convert to}}$$

$$\begin{matrix}-3&1&0&-3&1&0\\-10&0&1&-10&0&1\end{matrix}$$

$$\begin{matrix}1&0&0\\0&-3&-10\end{matrix}$$

$\vec{n}=(1,3,10)=u$

$\therefore$ N(A) is the plane $x+3y+10z=0$,
$\therefore$ the nullspace is the plane normal to the row vector $(1,3,10)$

The Pivot columns

- The pivot columns of $R=\text{rref}(A)$ have 1's in their pivots and 0's elsewhere. All together, the r pivot columns contain an $r\times r$ identity matrix. It sits above $m-r$ rows of zeroes
- The numbers of the pivot columns are in the list Pivcol
- The pivot columns of A are not immediately obvious from A itself, but A and R have the same pivot columns

$$A=\begin{bmatrix}1&3&0&2&-1\\0&0&1&4&-3\\1&3&1&6&-4\end{bmatrix}\longrightarrow R=\begin{bmatrix}1&3&0&2&-1\\0&0&1&4&-3\\0&0&0&0&0\end{bmatrix}$$

Pivcol $=(1,3)$

  - The column space of A and R are different

- The r pivot columns of A are also the first r columns of E^{-1}

$$A=\begin{bmatrix} 1 & 3 & 0 & 2 & -1 \\ 0 & 0 & 1 & 4 & -3 \\ 1 & 3 & 1 & 6 & -4 \end{bmatrix} \quad E=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \Rightarrow E^{-1}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

  - When we compute $A=E^{-1}R$, the r by r identity matrix inside R picks out the first r columns of E^{-1} as columns of $A=E^{-1}R$
- The pivot columns are not combinations of earlier (leftward) columns. The free columns are instead combinations of earlier columns. These combinations are the special solution

The Special Solutions

- Each special solution to $Ax=0$ and $Rx=0$ has one free variable equal to 1. The others are all set to zero. These solutions come directly from the RREF R.

$$Rx = \begin{bmatrix} 1 & 3 & 0 & 2 & -1 \\ 0 & 0 & 1 & 4 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$$

  - Notice that the first & second entries in each free column corresponds to the values of the pivot variables in special solutions

1. Set $(x_2, x_4, x_5)=(1,0,0)$ $\quad s_1=(-3,1,0,0,0)$
2. Set $(x_2, x_4, x_5)=(0,1,0)$ $\quad s_2=(-2,0,-4,1,0)$
3. Set $(x_2, x_4, x_5)=(0,0,1)$ $\quad s_3=(1,0,3,0,1)$

- The nullspace matrix N contains the three special solutions as columns. As each column corresponds to a solution x that makes $Ax=0$, it should be no suprise that $AN=0=$ zero matrix.

Null space matrix
$n-r=5-2=3$ special soltions
by rank-nullity theorem

$$N=\begin{bmatrix} -3 & -2 & 1 \\ 1 & 0 & 0 \\ 0 & 4 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

  - Whereas R had the identity matrix (2×2) in its pivot columns, N has the identity matrix (3×3) in its free rows
- The special solutions for R are really easy to find
  - Suppose that the first r columns are the pivot columns. R now looks like

$$R=\begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}\begin{matrix} \} \ r \text{ pivot rows} \\ \} \ m-r \text{ zero rows} \end{matrix}$$

r pivot columns $\quad n-r$ free column

- The pivot variables in the $n-r$ special solutions come by changing F to $-F$

$$N=\begin{bmatrix} -F \\ I \end{bmatrix}\begin{matrix} r \text{ pivot variables} \\ n-r \text{ free variables} \end{matrix} \Rightarrow \text{then } RN=0 \Rightarrow \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}\begin{bmatrix} -F \\ I \end{bmatrix} = \begin{bmatrix} -IF + FI \\ 0 \end{bmatrix} = \begin{bmatrix} -F+F \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- The columns of N solve $Rx=0$

$$Rx=0 \Rightarrow \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_{pivot} \\ x_{free} \end{bmatrix} = 0 \Rightarrow I x_{pivot} + F x_{free} = 0 \Rightarrow I x_{pivot} = -F x_{free}$$

- In each special solution, the free variables are a column of I. Then the pivot variables a column of $-F$.
- The idea still holds even when the pivot columns and free columns are mixed together.

Problem Set 3.3

2

a)

$$A = \begin{bmatrix} 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 \end{bmatrix} \to R = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

b)

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \end{bmatrix} \xrightarrow[\text{into } R_2]{R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 3 & 4 & 5 & 6 \end{bmatrix} \xrightarrow[\text{into } R_3]{R_3 - 3R_1} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & -2 & -4 & -6 \end{bmatrix} \xrightarrow[\text{into } R_3]{R_3 - 2R_2} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow[\text{into } R_2]{R_2 \times -1} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[\text{into } R_1]{R_1 - 2R_2} \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R$$

3.

$$B = [A \ A] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 3 \\ 2 & 4 & 6 & 2 & 4 & 6 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 2 & 4 & 6 & 2 & 4 & 6 \\ 0 & 0 & 3 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[R_2 \div 3]{R_1 \div 2}$$

$$\longrightarrow \begin{bmatrix} 1 & 2 & 3 & 1 & 2 & 3 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[\text{into } R_1]{R_1 - 3R_2} \begin{bmatrix} 1 & 2 & 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

4

$$R = \begin{bmatrix} F & I \\ 0 & 0 \end{bmatrix} \} \ m-r \text{ zero rows}$$

$$N = \begin{bmatrix} I \\ -F \end{bmatrix}$$

8

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \\ 4 & 8 & 16 \end{bmatrix}$$

9. line, $n-1$, $m \times (n-1)$

16.

10

$$A = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \end{bmatrix}$$

15.

$$AB = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 2 & 1 & 4 \\ 3 & 1.5 & 6 \end{bmatrix} = \begin{bmatrix} 8 & 4 & 16 \\ 16 & 8 & 32 \end{bmatrix}$$

rank A = 1   rank B = 1   rank AB = 1

Chapter 3.4: The complete solution to $Ax = b$

- We augment b onto A into the matrix [A b]
- Given the matrix:

$$\begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 1 & 3 & 1 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 7 \end{bmatrix} \Longrightarrow [A \ b] = \left[\begin{array}{cccc|c} 1 & 3 & 0 & 2 & 1 \\ 0 & 0 & 1 & 4 & 6 \\ 1 & 3 & 1 & 6 & 7 \end{array}\right]$$

- This ensure all operations on A also act on b.
- In the above example, we can subtract row 1 and row 2 from row 3

$$\left[\begin{array}{cccc|c} 1 & 3 & 0 & 2 & 1 \\ 0 & 0 & 1 & 4 & 6 \\ 1 & 3 & 1 & 6 & 7 \end{array}\right] \xrightarrow[\text{into } R_3]{R_3 - R_2 - R_1} \left[\begin{array}{cccc|c} 1 & 3 & 0 & 2 & 1 \\ 0 & 0 & 1 & 4 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] = [R \ d]$$

  - The third equation has become $0 = 0$. This means equation 3 = equation 1 + equation 2

One Particular Solution

- For the above example, we can get an easy solution by setting the free variables $x_2 = x_4 = 0$. Then the two nonzero equations give $x_1 = 1$ and $x_3 = 6$. our Solution is $(1, 0, 6, 0)$
- For a solution to exist, zero rows in R must be zero in d as well $(0 = 0)$ Since I is in the pivot rows and columns of R, the pivot variables in our one particular solution come from d once we set all free variables to zero.
- Note that this is one solution out of infinitely many.
- We write the complete solution for $Ax = b$ as

$$x = x_p + x_n = \begin{bmatrix} 1 \\ 0 \\ 6 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ -4 \\ 1 \end{bmatrix}$$

  - The sum of a particular solution and the null space vector
  - $Ax = b$ is solved by one particular solution x_p
  - $Ax = 0$ is solved by the nullspace

$$Ax = \underbrace{Ax_p}_{b} + \underbrace{Ax_n}_{0}$$

$$Ax = b$$

- If A is an invertible square matrix, x_p is the only solution $A^{-1}b$, while the nullspace is 0 because an invertible matrix has n pivots. Then, our complete solution is $x = A^{-1}b + 0 = A^{-1}b$. Invertible square matrices are a special case because their nullspace is 0
- Example:

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ -2 & -3 \end{bmatrix} \Rightarrow [A \ b] = \left[\begin{array}{cc|c} 1 & 1 & b_1 \\ 1 & 2 & b_2 \\ -2 & -3 & b_3 \end{array}\right] \xrightarrow[\text{into } R_2]{R_2 - R_1} \left[\begin{array}{cc|c} 1 & 1 & b_1 \\ 0 & 1 & b_2 - b_1 \\ -2 & -3 & b_3 \end{array}\right] \xrightarrow[\text{into } R_3]{R_3 + 2R_1} \left[\begin{array}{cc|c} 1 & 1 & b_1 \\ 0 & 1 & b_2 - b_1 \\ 0 & -1 & b_3 + 2b_1 \end{array}\right]$$

$$\xrightarrow[\text{into } R_3]{R_3 + R_2} \left[\begin{array}{cc|c} 1 & 1 & b_1 \\ 0 & 1 & b_2 - b_1 \\ 0 & 0 & b_3 + b_1 + b_2 \end{array}\right] \xrightarrow[\text{into } R_1]{R_1 - R_2} \left[\begin{array}{cc|c} 1 & 0 & 2b_1 - b_2 \\ 0 & 1 & b_2 - b_1 \\ 0 & 0 & b_3 + b_2 + b_1 \end{array}\right] \quad x = x_p + x_n = \begin{bmatrix} 2b_1 - b_2 \\ b_2 - b_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$b_1 + b_2 + b_3$ must $= 0$

- The matrix has full column rank $(r=n)$. Every column has a pivot.
  The matrix is reduced to
  $$R=\begin{bmatrix} I \\ 0 \end{bmatrix}=\begin{bmatrix} n \text{ by } n \text{ identity matrix} \\ m-n \text{ rows of zeroes} \end{bmatrix}$$
- Every matrix with full column rank has these properties
  1. All columns of A are pivot columns
  2. There are no free variables or special solutions (nullity $=0$)
  3. The nullspace contains only the zero vector $x=0$
  4. If $Ax=b$ has a solution (it might not), then it only has one

The Complete Solution

- The other extreme case is full row rank. Then $Ax=b$ has one or infinitely many solutions.
- In this case the matrix must have dimensions $m \le n$. If $r=m$ (every row has a pivot), then it has full row rank.
- Example:
  $$\begin{cases} x+y+z=3 \\ x+2y-z=4 \end{cases} \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 2 & -1 & 4 \end{array}\right]=A \quad (\text{rank } A=m=2)$$
  - These 2 equations are two planes in 3D. They are not parallel so the intersect at a line. Our particular solution x_p will be a point on the line. Adding the nullspace vectors will move us along the line. Then $x=x_p+x_n$ will give us the complete solution (See fig 3.4.1)

$$A \to R \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 2 & -1 & 4 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & -2 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 3 & 2 \\ 0 & 1 & -2 & 1 \end{array}\right]=[R \; d]$$

Particular Solution:
Set $x_3=0$, (free)
then $x_1=2$, $x_2=1$
$x_p=(2,1,0)$

nullspace solutions
Set $x_3=1$ (free) for $\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$
then $x_1=-3$, $x_2=2$
$x_n=s_1=(-3,2,1)$

$$x=x_p+x_n=\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}+c\begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$$

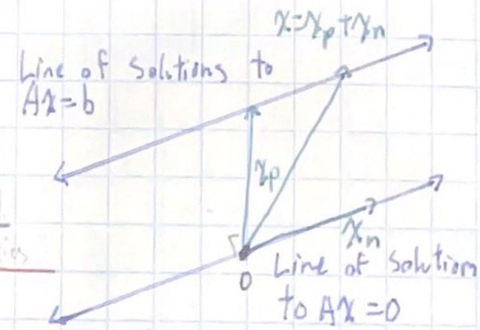

Fig 3.4.1

- If $m<n$, the equation $Ax=b$ is underdetermined.
- Every matrix with full row rank has these properties
  1. All rows have pivots, and R has no zero rows
  2. $Ax=b$ has a solution for every right side b
  3. The column space is the whole space $\mathbb{R}^m$
  4. There are $n-r=n-m$ special solutions in the nullspace of A

- The four possibilities for linear equations:
  1. $r=m$ and $r=n$
     - Square and invertible
     - $Ax=b$ has one solution
  2. $r=m$ and $r<n$
     - Short and wide
     - $Ax=b$ has infinite solutions
  3. $r<m$ and $r=n$
     - Tall and thin
     - $Ax=b$ has 0 or 1 solution
  4. $r<m$ and $r<n$
     - Not full rank
     - $Ax=b$ has 0 or infinite solutions

Problem Set 3.4

1.

$$\begin{bmatrix} A & b \end{bmatrix} = \left[\begin{array}{cccc|c} 2 & 4 & 6 & 4 & 4 \\ 2 & 5 & 7 & 6 & 3 \\ 2 & 3 & 5 & 2 & 5 \end{array}\right] \xrightarrow[R_3-R_1]{R_2-R_1} \left[\begin{array}{cccc|c} 2 & 4 & 6 & 4 & 4 \\ 0 & 1 & 1 & 2 & -1 \\ 0 & -1 & -1 & -2 & 1 \end{array}\right] \xrightarrow[\text{into } R_3]{R_2+R_3} \left[\begin{array}{cccc|c} 2 & 4 & 6 & 4 & 4 \\ 0 & 1 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Column space: $\mathbb{R}^2$

plane formed by $(2,0,0)$ and $(4,1,0)$

$$\begin{array}{c|ccc} 2 & 0 & 0 \\ 4 & 1 & 0 \end{array} \qquad \begin{array}{ccc} 2 & 0 & 4 \\ 4 & 1 & 0 \end{array}$$

$$\begin{array}{ccc} 0 & 0 & 2 \\ 0 & 0 & 0 \end{array}$$

$n=(0,0,2)$
$=2(0,0,1)$

$\Pi: z=0$

Nullspace: $\mathbb{R}^2$

s_1: Set $(x_3, x_4)=(1,0)$

$s_1=(-1,-1,1,0)$

s_2: Set $(x_3, x_4)=(0,1)$

$s_2=(2,-2,0,1)$

$x_n=c_1s_1+c_2s_2$

$$=c_1\begin{bmatrix}-1\\-1\\1\\0\end{bmatrix}+c_2\begin{bmatrix}2\\-2\\0\\1\end{bmatrix}$$

Particular Solution:

Set $x_3=x_4=0$

$x_p=(4,-1,0,0)$

$x=x_p+x_n$

$$=\begin{bmatrix}4\\-1\\0\\0\end{bmatrix}+c_1\begin{bmatrix}-1\\-1\\1\\0\end{bmatrix}+c_2\begin{bmatrix}2\\-2\\0\\1\end{bmatrix}$$

$$\begin{bmatrix} U & c \end{bmatrix} = \left[\begin{array}{cccc|c} 2 & 4 & 6 & 4 & 4 \\ 0 & 1 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1 \div 2} \left[\begin{array}{cccc|c} 1 & 2 & 3 & 2 & 2 \\ 0 & 1 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow[\text{into } R_1]{R_1-2R_2} \left[\begin{array}{cccc|c} 1 & 0 & 1 & -2 & 4 \\ 0 & 1 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

2.

$$\begin{bmatrix} A & b \end{bmatrix} = \left[\begin{array}{ccc|c} 2 & 1 & 3 & b_1 \\ 6 & 3 & 9 & b_2 \\ 4 & 2 & 6 & b_3 \end{array}\right] \xrightarrow[\text{into } R_3]{R_3-2R_1} \left[\begin{array}{ccc|c} 2 & 1 & 3 & b_1 \\ 6 & 3 & 9 & b_2 \\ 0 & 0 & 0 & b_3-2b_1 \end{array}\right] \xrightarrow[\text{into } R_2]{R_2-3R_1} \left[\begin{array}{ccc|c} 2 & 1 & 3 & b_1 \\ 0 & 0 & 0 & b_2-3b_1 \\ 0 & 0 & 0 & b_3-2b_1 \end{array}\right]$$

$$\begin{cases} b_2-3b_1=0 \\ b_3-2b_1=0 \end{cases}$$ these 2 planes put b in the column space, which is a line.

3.

$$[A \; b] = \left[\begin{array}{ccc|c} 1 & 3 & 3 & 1 \\ 2 & 6 & 9 & 5 \\ -1 & -3 & 3 & 5 \end{array}\right] \xrightarrow{R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 3 & 3 & 1 \\ 0 & 0 & 3 & 3 \\ -1 & -3 & 3 & 5 \end{array}\right] \xrightarrow{R_3 + R_1} \left[\begin{array}{ccc|c} 1 & 3 & 3 & 1 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 6 & 6 \end{array}\right]$$

$$\xrightarrow{R_3 - 2R_2} \left[\begin{array}{ccc|c} 1 & 3 & 3 & 1 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_2 \div 3} \left[\begin{array}{ccc|c} 1 & 3 & 3 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1 - 3R_2} \left[\begin{array}{ccc|c} 1 & 3 & 0 & -2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] = [R \; d]$$

$$x = x_p + x_n = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} + c \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$$

4.

$$[A \; b] = \left[\begin{array}{cccc|c} 1 & 3 & 1 & 2 & 1 \\ 2 & 6 & 4 & 8 & 3 \\ 0 & 0 & 2 & 4 & 1 \end{array}\right] \xrightarrow{R_2 - 2R_1} \left[\begin{array}{cccc|c} 1 & 3 & 1 & 2 & 1 \\ 0 & 0 & 2 & 4 & 1 \\ 0 & 0 & 2 & 4 & 1 \end{array}\right] \xrightarrow{R_3 - R_2} \left[\begin{array}{cccc|c} 1 & 3 & 1 & 2 & 1 \\ 0 & 0 & 2 & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_2 \div 2} \left[\begin{array}{cccc|c} 1 & 3 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1 - R_2} \left[\begin{array}{cccc|c} 1 & 3 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 2 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$x = x_p + x_n = \begin{bmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

7.

$$[A \; b] = \left[\begin{array}{ccc|c} 1 & 3 & 1 & b_1 \\ 3 & 8 & 2 & b_2 \\ 2 & 4 & 0 & b_3 \end{array}\right] \xrightarrow{R_3 - 2R_1} \left[\begin{array}{ccc|c} 1 & 3 & 1 & b_1 \\ 3 & 5 & 2 & b_2 \\ 0 & -2 & -2 & b_3 - 2b_1 \end{array}\right] \xrightarrow{R_2 - 3R_1} \left[\begin{array}{ccc|c} 1 & 3 & 1 & b_1 \\ 0 & -1 & -1 & b_2 - 3b_1 \\ 0 & -2 & -2 & b_3 - 2b_1 \end{array}\right]$$

$$\xrightarrow{R_3 - 2R_2} \left[\begin{array}{ccc|c} 1 & 3 & 1 & b_1 \\ 0 & -1 & -1 & b_2 - 3b_1 \\ 0 & 0 & 0 & b_3 - 4b_2 + 6b_1 \end{array}\right]$$

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a)

$$[A \; b] = \left[\begin{array}{ccc|c} 1 & 2 & 1 & b_1 \\ 2 & 6 & 3 & b_2 \\ 0 & 2 & 5 & b_3 \end{array}\right] \xrightarrow{R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 2 & 1 & b_1 \\ 0 & 2 & 1 & b_2 - 2b_1 \\ 0 & 2 & 5 & b_3 \end{array}\right] \xrightarrow{R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 2 & 1 & b_1 \\ 0 & 2 & 1 & b_2 - 2b_1 \\ 0 & 0 & 4 & b_3 - b_2 + 2b_1 \end{array}\right]$$

$$\xrightarrow{R_3 \div 4} \left[\begin{array}{ccc|c} 1 & 2 & 1 & b_1 \\ 0 & 2 & 1 & b_2 - 2b_1 \\ 0 & 0 & 1 & \frac{1}{4}b_3 - \frac{1}{4}b_2 + \frac{1}{2}b_1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & b_1 - b_2 \\ 0 & 1 & 0 & \frac{5}{8}b_2 - \frac{?}{8}b_1 - \frac{1}{8}b_3 \\ & & 1 & \end{array}\right]$$

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$$A = \begin{bmatrix} 3 & 4 & 1 & 0 \\ 6 & 5 & 2 & 1 \end{bmatrix} \longrightarrow \overset{L}{\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}} \begin{bmatrix} 3 & 4 & 1 & 0 \\ 6 & 5 & 2 & 1 \end{bmatrix} = \overset{U}{\begin{bmatrix} 3 & 4 & 1 & 0 \\ 0 & -3 & 0 & 1 \end{bmatrix}}$$

$$A = \begin{bmatrix} 2 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \xrightarrow{E_{31}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 1 & 1 \\ 0 & 2 \end{bmatrix} \xrightarrow{E_{21}} \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 1 & 1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \\ 0 & 2 \end{bmatrix}$$

$$\xrightarrow{E_{32}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad A = LU \Rightarrow \begin{bmatrix} 2 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{2} & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

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a) $A = \begin{bmatrix} 1 & 5 \\ 2 & 7 \\ 3 & 1 \end{bmatrix}$ b) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 1 \end{bmatrix}$ c) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 5 & 7 & 9 \end{bmatrix}$ d) $\begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$

Chapter 3.5: Independence, Basis, and Dimension

Linear Independence

- The columns of A are linearly independent when the only solution to $Ax = 0$ is $x = 0$
  - That is, no one column vector can be expressed as a linear combination of the others.
  - This also implies that $N(A) = Z$.
  - If there are nonzero (non-trivial) solutions then the vectors are said to be linearly dependent.
  - Examples
    - (1,0) and (0,1) are independent
    - (1,0) and (1, 0.0001) are independent
- Three vectors in $\mathbb{R}^2$ cannot be independent. One way to see this is to write it as a matrix:

  $$\begin{bmatrix} a & c & e \\ b & d & f \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
  - There must be a free variable
- Example 1: The columns of A are dependent.

  $$Ax = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 5 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix} = -3 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} 3 \\ 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
  - The rank is only 2. Independent columns produce full column rank $r = n = 3$
  - In this matrix, the rows are also dependent. For a square matrix, we will show that dependent columns imply dependent rows.
- Suppose we have a matrix with $m < n$ (short and wide). Then the columns must be dependent. There are at least $n - m$ free variables
- If instead we have a matrix with $n < m$ (tall and thin). Elimination will reveal the r pivot columns. It is those r pivot columns that are independent.

Vectors that span a Subspace

- The column space consists of all combinations Ax of the columns of A.
  We now introduce a new term to describe this: <u>Span</u>
- A set of vectors <u>span</u> a space if their linear combinations fill the space
- The columns of a matrix spans its column space.
- Examples:
  - $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ span the entirety of $\mathbb{R}^2$
  - $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $v_3 = \begin{bmatrix} 4 \\ 7 \end{bmatrix}$ also span the full space $\mathbb{R}^2$
  - $w_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $w_2 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$ span a line ($\mathbb{R}^1$) in $\mathbb{R}^2$
- <u>The row space is the space spanned by the rows of a matrix.</u>
  The row space of a matrix is the column space of its transpose.
  <u>row space $= C(A^T)$</u>
- The row space of A is a subspace of $\mathbb{R}^n$ (The rows have n components)
- Example:

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 7 \\ 3 & 5 \end{bmatrix} \qquad \begin{array}{l} C(A) \text{ spans a } \mathbb{R}^2 \text{ plane in } \mathbb{R}^3 \\ C(A^T) \text{ spans a } \mathbb{R}^2 \text{ plane in } \mathbb{R}^2 \end{array}$$

A Basis for a Vector Space

- Two vectors can't span all of $\mathbb{R}^3$ even if they are independent. Four vectors can't be independent even if they span $\mathbb{R}^3$.
- We want enough independent vectors to span the space and not more. Hence the "basis"
- A basis for a vector space is a sequence of vectors that both spans the space and are linearly independent.
  - Every vector v in the space is a combination of the basis vectors
  - Moreover, that combination is unique because because the basis vectors are linearly independent.
    - Suppose $v = a_1 v_1 + \ldots + a_n v_n$ and also $v = b_1 v_1 + \ldots + b_n v_n$. By subtraction $(a_1 - b_1)v + \ldots + (a_n - b_n)v_n = 0$. $a_i - b_i$ must equal zero from the independence of the v's
- The columns of I_n produce the "standard basis" for $\mathbb{R}^n$
- <u>The columns of every n by n invertible matrix give a basis for $\mathbb{R}^n$</u> not pivot column
- The pivot columns of A are a basis for its column space and likewise for the pivot rows and its row space. This also applies to the <u>pivot rows</u> of R
- Given

$$A = \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix}, \quad \text{Basis for column space: } \begin{bmatrix} 2 \\ 3 \end{bmatrix} \quad \text{Basis for row space: } \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

- Given

$$R = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \text{Basis for col space: } \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{Basis for row space: } \begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 4 \end{bmatrix}$$

- Given 5 vectors in $\mathbb{R}^7$, how do you find a basis for the space they span?
  - Make them rows of A and eliminate to find the non zero rows of R
  - Make them columns of A and eliminate to find the pivot columns of A (not R)
- All bases for a vector space contain the same number of vectors
  - That number is the "dimension" of the space

Dimension of a Vector Space

- We want to prove that all bases for the same vector space have the same number of vectors.
  - If $v_1, \dots, v_m$ and $w_1, \dots, w_n$ are both bases for the same vector spaces, then $m = n$
  - Suppose that there are more w's than v's. From $n > m$, we want to reach a contradiction
  - If w_1 equals $a_{11}v_1 + \dots + a_{m1}v_m$, this is the first column of a matrix multiplication VA. Each w is a combination of the v's

$$W = \begin{bmatrix} w_1 & w_2 & \dots & w_n \end{bmatrix} = \begin{bmatrix} v_1 & v_2 & \dots & v_n \end{bmatrix} \begin{bmatrix} a_{11} & \cdot & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix} = VA$$

  - We don't know each a_{ij}, but we know the shape of A (m by n).
    The second vector w_2 is also a combination of the v's.
    The key is that A has a row for every V and a column for every W.
    A is a short wide matrix since we assumed $n > m$. So $Ax = 0$ has infinite solutions.
  - $Ax = 0$ gives $VAx = 0$, which is $Wx = 0$. A combination of the w's gives zero, then the w's cannot be a basis.
  - Likewise, assuming $m > n$ and performing the same step leads to the same contradiction.
  - Therefore $m = n$ and the proof is complete
- Dimension: the dimension of a space is the number of vectors in every basis
- The dimension of the column space equals the rank of the matrix

Bases for Matrix Spaces and Function Spaces

- The terms "independence", "bases", and "dimension" are not just restricted to column vectors. It applys to any vector spaces.
  - We can ask if some matrices are independent
  - In differential equation $\frac{d^2y}{dx^2} = y$ has a space of solutions. One basis is $y = e^x$ and $y = e^{-x}$. Counting the basis functions, we get a dimension of 2)
- Matrix Spaces
  - The vector space M contains all 2 by 2 matrices. It's dimension is 4.
    ~~One basis is~~ $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. ~~These matrices are linearly independent~~

- The matrices $\begin{bmatrix}1&0\\0&0\end{bmatrix}$, $\begin{bmatrix}0&1\\0&0\end{bmatrix}$, and $\begin{bmatrix}0&0\\0&1\end{bmatrix}$ are a basis for a subspace, the upper triangular matrices. It has a dimension of three.
- $\begin{bmatrix}1&0\\0&0\end{bmatrix}$, $\begin{bmatrix}1&0\\0&1\end{bmatrix}$ are a basis for diagonal matrices, with dimension 2
- Think about the space of all n by n matrices. One possible basis has a 1 in each position in the matrix, then the dimension is n^2 since there are n^2 positions.
  - The dimension of the whole n by n matrix space is n^2.
  - The dimension of upper triangular matrices is $\frac{1}{2}n^2+\frac{1}{2}n$
  - The dimension of diagonal matrices is n
  - The dimension of symmetric matrices is $\frac{1}{2}n^2+\frac{1}{2}n$
- Function spaces
  - $y''=0$ is solved by any linear function $y=cx+d$. Basis: $\{x,1\}$ ← null space
  - $y''=-y$ is solved by $y=c\sin x+d\cos x$ Basis: $\{\sin x, \cos x\}$
  - $y''=y$ is solved by $y=ce^{x}+de^{-x}$, Basis: $\{e^{x}, e^{-x}\}$
- The dimension is two for the above equations since it is the second derivative.
- The solutions of $y''=2$ don't form a subspace. The right side $b=2$ is not zero
  - A particular solution is $y=x^2$
  - A complete solution is $y=\underbrace{x^2}_{x_p}+\underbrace{cx+d}_{x_n}$
- The space Z contains only the zero vector. The dimension of this set is zero. The basis for Z is the empty set.

Problem Set 3.5

2.

$$V=\begin{bmatrix}1&1&1&0&0&0\\-1&0&0&1&1&0\\0&-1&0&-1&0&1\\0&0&-1&0&-1&-1\end{bmatrix}\xrightarrow[\text{into }R_2]{R_2+R_1}\begin{bmatrix}1&1&1&0&0&0\\0&1&1&1&1&0\\0&-1&0&-1&0&1\\0&0&-1&0&-1&-1\end{bmatrix}\xrightarrow[\text{into }R_3]{R_3+R_2}\begin{bmatrix}1&1&1&0&0&0\\0&1&1&1&1&0\\0&0&1&0&0&1\\0&0&-1&0&-1&-1\end{bmatrix}$$

$$\xrightarrow[\text{into }R_4]{R_4+R_3}\begin{bmatrix}1&1&1&0&0&0\\0&1&1&1&1&0\\0&0&1&0&0&1\\0&0&0&0&-1&0\end{bmatrix}\xrightarrow{R_4\times -1}\begin{bmatrix}1&1&1&0&0&0\\0&1&1&1&1&0\\0&0&1&0&0&1\\0&0&0&0&1&0\end{bmatrix}$$

Columns 1,2,3,5 are independent and form a basis for $\mathbb{R}^4$

3.

The columns of U are independent if the matrix U has full rank (i.e. there are n pivots). If any of a, d, or f equal zero, there would be less than 3 pivots

4.

$$Ux=0\Rightarrow\begin{bmatrix}a&b&c\\0&d&e\\0&0&f\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

Since the matrix has full rank (3 pivots), its nullspace contains only the zero vector.

7

$a_1v_1 + a_2v_2 + a_3v_3 = 0$

$a_1(w_2 - w_3) + a_2(w_1 - w_3) + a_3(w_1 - w_2) = 0$

$a_1w_2 - a_1w_3 + a_2w_1 - a_2w_3 + a_3w_1 - a_3w_2 = 0$

$(a_2w_1 + a_3w_1) + (a_1w_2 - a_3w_2) + (-a_1w_3 - a_2w_3) = 0$

$(a_2 + a_3)w_1 + (a_1 - a_3)w_2 - (a_1 + a_2)w_3 = 0$

Since w_1, w_2, w_3 are independent, the only solution is

$$\begin{cases} a_2 + a_3 = 0 \\ a_1 - a_3 = 0 \\ a_1 + a_2 = 0 \end{cases} \rightarrow \begin{cases} a_2 + a_3 = 0 \\ a_1 = a_3 \\ a_1 + a_2 = 0 \end{cases} \rightarrow \begin{cases} a_1 + a_2 = 0 \\ a_1 = a_3 \rightarrow a_1 + a_2 = 0 \quad (1, 1, 1) \\ a_1 + a_2 = 0 \end{cases}$$

$\therefore v_1, v_2, v_3$ are linearly dependent

11.

a) the $Ax = b$

b) the $A^Ty = c$

c) $\mathbb{R}^3$

d) $\mathbb{R}^3$

12.

13. n, basis and line

a) $\ldots$

b) m

16. a) $(1, 1, 1, 1)$

26.

a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

17.

$\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix} \quad \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \end{bmatrix}$

30.

$\begin{bmatrix} 1 & & -2 \\ & 1 & -1 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & -3 \\ 2 & 1 & -1 \end{bmatrix} \quad \begin{bmatrix} 1 & & -3 \\ 1 & 2 & -4 \end{bmatrix}$

32.

$\frac{dy}{dx} - 2y = 0$

$\frac{dy}{dx} = 2y$

$dy = 2y\,dx$

$\frac{1}{y}dy = 2dx$

$\ln|y| + C_1 = 2x + C_2$

$|y| = e^{2x}e^{C}$

$y = \pm e^{C}e^{2x}$

$= Ae^{2x}$

35.

$p(x) = ax^3 + bx^2 + cx + dx$

$a + b + c + d = 0$

$a(x) = x^3 - x^2$

Chapter 3.6: Dimensions of the Four Subspaces

- The main theorem in this chapter connects rank and dimension.
  The rank of a matrix is the number of pivots. The dimension of a subspace is the number of vectors in a basis.
- The rank of A reveals the dimension of all four fundamental subspaces, which are:
  1. The row space is $C(A^T)$, a subspace of $\mathbb{R}^n$
  2. The column space is $C(A)$, a subspace of $\mathbb{R}^m$
  3. The nullspace is $N(A)$, a subspace of $\mathbb{R}^n$
  4. The left nullspace is $N(A^T)$, a subspace of $\mathbb{R}^m$
- For the left nullspace, we solve $A^Ty = 0$, an n by m system.
  The equation can also be written $y^TA = 0^T$.
- The matrices A and A^T are usually different, and so are their column spaces and nullspaces. But these spaces are all connected.
- Part 1 of the fundamental theorem finds the dimensions of the four subspaces
  - One fact stands out: The row space and column space have the same dimension r.
    $\text{Rank}(A) = \dim(C(A)) = \dim(C(A^T))$
  - Another important fact involve the two nullspaces:
    $N(A)$ and $N(A^T)$ have dimensions $n-r$ and $m-r$ respectively, to make up the full n and m with the row and column spaces.
- Part 2 will describe how the four subspaces fit together.

The Four Subspaces for R

- Suppose A is reduced to its row echelon form R. For that special form, the four subspaces are easy to identify.
- We will find a basis for each subspace and check its dimension. Then we watch how the subspaces change (two of them don't) as we go back to A.
- The main point is that the four dimensions are the same for A and R.
- As a specific example:

$$R = \begin{bmatrix} 1 & 3 & 5 & 0 & 7 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{l} \left.\begin{array}{l} \text{pivot row 1} \\ \text{pivot row 2} \end{array}\right\} 2 \\ \end{array}$$

$$\underbrace{\text{pivot column 1} \quad \text{pivot column 4}}_{2}$$

- $\text{Rank}(A) = \dim(C(A)) = \dim(C(A^T))$

1. Row Space
   - The non-zero rows of R form a basis
2. Column space
   - The pivot columns of R form a basis.
   - Every free column is a combination of the pivot columns. These combinations are the special solutions.
3. Null space
   - The nullspace has dimension $n-r = 3$
   - There are 3 free variables
4. Left null space
   - The equation $R^Ty = 0$ looks for combination of the columns of R^T which produce zero. The pivot rows are independent, so to solve the equation, we set the pivot row's coefficient to zero and the zero rows can have any coefficient

- Summary
  - In $\mathbb{R}^n$ the row space and nullspace have dimension r and $n-r$ (adding to n)
  - In $\mathbb{R}^m$ the column space and left nullspace have dimension r and $m-r$ (adding to m)
- So far this is proved for echelon matrices R.

The Four Subspaces for A

- The subspace dimensions for A are the same as R. Why?

A reduces to R $A = \begin{bmatrix} 1 & 3 & 5 & 0 & 7 \\ 0 & 0 & 0 & 1 & 2 \\ 1 & 3 & 5 & 1 & 9 \end{bmatrix}$ Notice $C(A) \neq C(R)$

- An elimination matrix E takes A to R. The invertible matrix E is the product of the elementary matrix that reduce A to R.

$EA = R$ and $A = E^{-1}R$

- Proof:
  1. Row space
     - Same dimension r and same basis.
     - Every row of A is a combination of the rows of R and vice versa. Elimination changes rows but not row spaces.
     - The basis rows for A are the ones that end up as pivot rows
  2. column space
     - The column space of A has dimension r
     - The number of independent columns equals the number of independent rows
     - The same combinations of the columns are zero for A and R. That is, $Ax=0$ exactly when $Rx=0$, even though their column spaces are different. The r pivot columns of both are independent.
  3. Nullspace
     - A has the same nullspace as R
     - The elimination steps don't change the solutions. The special solutions are a basis for this nullspace. There are $n-r$ free variables, so the dimension is $n-r$.
  4. Left nullspace
     - The left nullspace has dimension $m-r$
     - The column space has dimension r. Since A^T is n by m, the whole space is $\mathbb{R}^m$
- Fundamental Theorem of Linear Algebra, Part One
  - The column space and row space both have dimension r
  - The nullspace and left nullspace have dimension $n-r$ and $m-r$ respectively

- Example 1

$A = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$

  - $\dim(C(A)) = r = 1$
  - $\dim(C(A^T)) = r = 1$
  - $\dim(N(A)) = n-r = 3-1 = 2$
  - $\dim(N(A^T)) = m-r = 1-1 = 0$ (Z)

- Example 2

$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix}$

  - $\dim(C(A)) = r = 1$
  - $\dim(C(A^T)) = r = 1$
  - $\dim(N(A)) = n-r = 3-1 = 2$
  - $\dim(N(A^T)) = m-r = 2-1 = 1$

Matrices of Rank One

- When $r=1$, every row is a multiple of the same row, and likewise with columns
- The row space is a line in $\mathbb{R}^n$ and the column space is a line in $\mathbb{R}^m$
- Every rank one matrix is expressible as an outer product uv^T
- The nullspace is the plane perpendicular to v
  $Ax=0 \Rightarrow (uv^T)x=0 \Rightarrow u(v^Tx)=0 \Rightarrow v^Tx=0 \Rightarrow v$ and x are perpendicular
  - It is this perpendicularity of the subspaces that will be part 2 of the Fundamental Theorem

Problem set 3.6

1.

a) $m=7$
$n=9$
$\dim(C(A)) = \dim(C(A^T)) = r = 5$
$\dim(N(A)) = n - r = 9-5=4$
$\dim(N(A^T)) = m-r = 9-7=2$

b) $m=3$, $n=4$
$\dim(C(A)) = \dim(C(A^T)) = r=3$
$C(A) = \mathbb{R}^3$
$\dim(N(A^T)) = m-r = 3-3=0=Z$

2.

$A=\begin{bmatrix}1 & 2 & 4\\ 2 & 4 & 8\end{bmatrix}$

row basis: $\{(1,2,4)\}$
column basis: $\{(1,2)\}$
$\dim(C(A)) = \text{rank}(A) = 1$
$\dim(C(A^T)) = \text{rank}(A) = 1$
$\dim(N(A)) = n-r = 2$
$\dim(N(A^T)) = m-r = 1$

$B=\begin{bmatrix}1 & 2 & 4\\ 2 & 5 & 8\end{bmatrix}$

row basis: $\{(1,2,4),(2,5,8)\}$
column basis: $\{(1,2),(2,5)\}$

4.

a) $\begin{bmatrix}1 & 0\\ 1 & 0\\ 0 & 1\end{bmatrix}$ b) not possible

11.

$r<m$ and $r=n$
$r<m$ and $r<n$
A^Ty is short and wide, which always has solutions

12.

$A = \begin{bmatrix}1\\0\\1\end{bmatrix}\begin{bmatrix}1 & 2 & 0\end{bmatrix} = \begin{bmatrix}1 & 2 & 0\\ 0 & 0 & 0\\ 1 & 2 & 0\end{bmatrix}$

$\dim(C(A^T)) = r = 1$,
then $\dim(N(A)) = n-r = 3-1 = 2$

14. row space, left nullspace

Chapter 4: Orthogonality

Chapter 4.1: Orthogonality of the Four Subspaces

- Two vectors are orthogonal when their dot products are zero: $v^Tw = v \cdot w = 0$
- Orthogonal vectors also satisfy $\|v\|^2 + \|w\|^2 = \|v+w\|^2$
  $(v+w)^T(v+w) = v^Tv + w^Tw + 2v^Tw = \|v\|^2 + \|w\|^2 + 2v^Tw$ must be zero
- The row space is perpendicular to the nullspace. Every row of A is perpendicular to every solution of $Ax=0$
- The column space is perpendicular to the left nullspace. When b is outside the column space, then this nullspace of A^T comes into its own. It contains the error $e=b-Ax$ in the "least-squares" solution, a key application of linear algebra.