Table of Eigenvalues and Eigenvectors

Symmetric $A^T = A$	real λ's	orthogonal $x_i^T x_j = 0$
Orthogonal $Q^T = Q^{-1}$	all $\lvert\lambda\rvert = 1$	orthogonal $\bar{x}_i^T x_j = 0$
Skew-Symmetric $A^T = -A$	imaginary λ's	orthogonal $\bar{x}_i^T x_j = 0$
Complex Hermitian $\bar{A}^T = A$	real λ's	orthogonal $\bar{x}_i^T x_j = 0$
Positive Definite $x^T A x > 0 \; \forall x \neq 0$	all $\lambda > 0$	orthogonal since $A^T = A$
Markov $m_{ij} > 0$, $\sum_{i=1}^{n} m_{ij} = 1$	$\lambda_{max} = 1$	Steady state $x > 0$
Similar $B = M^{-1} A M$	$\lambda(B) = \lambda(A)$	$x(B) = M^{-1} x(A)$
Projection $P = P^2 = P^T$	$\lambda = 1; 0$	column space; nullspace
Plane Rotation	$e^{i\theta}$ and $e^{-i\theta}$	$x = (1, i)$ and $(1, -i)$
Reflection $I - 2uu^T$	$\lambda = -1; 1, \dots, 1$	u; whole plane $u_\perp$
Rank One uv^T	$\lambda = v^T u; 0, \dots, 0$	u; whole plane $v^\perp$
Inverse A^{-1}	$1/\lambda(A)$	Same eigenvectors
Shift $A + cI$	$\lambda(A) + c$	Same eigenvectors
Stable powers $A^n \to 0$	all $\lvert\lambda\rvert < 1$	any eigenvectors
Stable exponential $e^{At} \to 0$	all $Re(\lambda) < 0$	any eigenvectors
Cyclic Permutation row 1 of I last	$\lambda_k = e^{2\pi i k/n}$	$x_k = (1, \lambda_k, \dots, \lambda_k^{n-1})$
Tridiagonal $-1, 2, -1$ on diagonals	$\lambda_k = 2 - 2\cos\frac{k\pi}{n+1}$	$x_k = (\sin\frac{k\pi}{n+1}, \sin\frac{2k\pi}{n+1}, \dots)$
Diagonalizable $A = S\Lambda S^{-1}$	diagonal of Λ	columns of S are independent
Symmetric $A = Q\Lambda Q^T$	diagonal of Λ (real)	columns of Q are orthonormal
Schur $A = QTQ^{-1}$	diagonal of T	columns of Q if $A^T A = AA^T$
Jordan $J = M^{-1} A M$	diagonal of J	each block gives $x = (0, \dots, 1, \dots, 0)$
Rectangular $A = U\Sigma V^T$	$rank(A) = rank(\Sigma)$	eigenvectors of $A^T A$, AA^T in V, U

Chapter 7: Linear Transformations

Chapter 7.1: The Idea of a Linear Transformation

- When the matrix A multiplies a vector v, it "transforms" into another vector Av. In goes v, out comes Av. $T(v) = Av$
  - A transformation is very much like a function. To find a specific output, we simply evaluate Av.
  - The deeper goal is to see all v's at once. We are transforming the entire space V when we multiply all vectors v.
- Start with a matrix A. It transforms vector v into Av and vector w into Aw. Then we know what it does to $u = v + w$; it equals $Av + Aw$. Matrix multiplication $T(v) = Av$ gives a linear transformation.
- A transformation T assigns an output $T(v)$ to each input v in V. It is linear if it meets the below requirements for all v and w

a) $T(v + w) = T(v) + T(w)$ b) $T(cv) = cT(v)$ for all c

- We combine these requirements into one.

  $T(cv + dw) = cT(v) + dT(w)$

- A linear transformation is very restrictive. A transformation $T(v) = v + u_0$ adding a constant vector u_0 is not linear:

  $T(v + w) = v + w + u_0 \neq T(v) + T(w)$

- $T(v) = v$ is the identity transformation ($A = I$). The input space V is the same as the output space W.
- The linear-plus-shift transformation $T(v) = Av + u_0$ is called "affine". Computers work with these because images must be able to move.
- Example
  - Choose a fixed vector $a = (1, 3, 4)$ and let $T(v)$ be the dot product $a \cdot v$

    The input is $v = (v_1, v_2, v_3)$ the output is $T(v) = v_1 + 3v_2 + 4v_3$

    This a linear transformation with $\mathbb{R}^3 \to \mathbb{R}^1$
- Example
  - The length transformation $T(v) = \|v\|$ is not linear.
  - Both requirements are false
    a) $T(v + w) \leq T(v) + T(w)$ (inequality, not equality)
    b) $T(cv) \neq cT(v)$ for negative c
- Example
  - T is the transformation that rotates every vector by 30°.
    Its domain and range are both $\mathbb{R}^2$.
  - T is linear: The sum of rotations is the rotation of a sum.

Lines to Lines, Triangles to Triangles

- fig 7.1.1 shows the line from v to w in the input space and the line from $T(v)$ to $T(w)$ in the output space. Linearity tells us that every point on the input line goes on the output line. More than that, equally spaced points go to equally spaced points on the output.
- The second figure goes up a dimension. Now we have three points v_1, v_2, v_3 mapped to outputs $T(v_1), T(v_2), T(v_3)$. Equally space points along the triangle stay equally spaced. The middle point $v = \frac{1}{3}(v_1 + v_2 + v_3)$ goes to the middle point $T(v) = \frac{1}{3}(T(v_1) + T(v_2) + T(v_3))$.
- Linearity extends to n vectors

  $v = c_1 v_1 + c_2 v_2 + \dots + c_n v_n$

  Linearity $T(v) = c_1 T(v_1) + c_2 T(v_2) + \dots c_n T(v_3)$

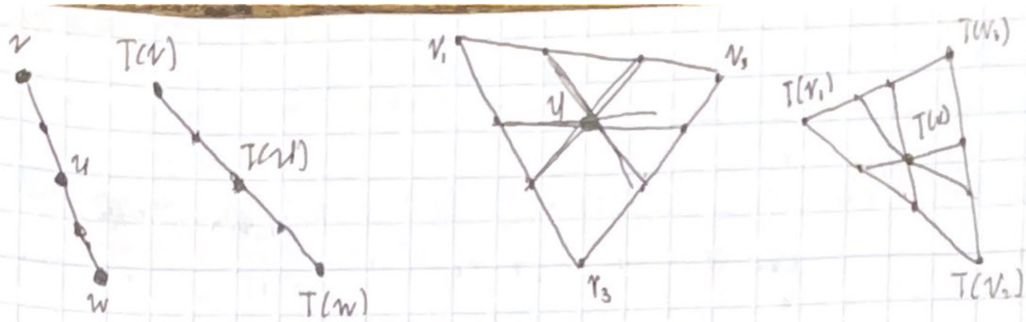


Fig 7.1.1

- Some terms for Linear Transformations
  - Range of T = Set of all outputs T(v); corresponds to the column space.
  - Kernel of T: Set of all inputs for which T(v) = 0; corresponds to nullspace.
- The range is in the output space W and the Kernel is in the input space V.

Examples of Transformations (mostly linear)

- Example:
  - Project every 3-dimensional vector onto the xy space. Then $T(x,y,z) = (x,y,0)$.
  - The range is that plane. The kernel is the z-axis.
  - This projection is linear
- Example
  - Project every 3 dimensional vectors onto the horizontal plane $z=1$,
    $T(x,y,z) = (x,y,1)$, is not linear because $T(0) \neq 0$
- Example
  - Suppose A is an invertible matrix. The Kernel of T is the zero vector; the range W equals the domain V. The inverse transformation T^{-1} multiplies the input by A^{-1}
    $T^{-1}(T(v)) = v$ matches the matrix multiple $A^{-1}(Av)$
- All linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^n$ are produced by matrices

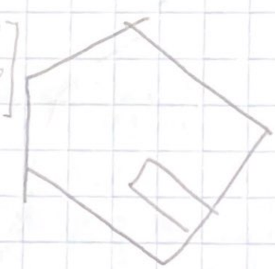
Linear Transformations of the Plane

- House Matrix

$$H = \begin{bmatrix} -6 & -6 & -7 & 0 & 7 & 6 & 6 & -3 & -3 & 0 & 0 & -6 \\ -7 & 2 & 1 & 8 & 1 & 2 & -7 & -7 & -2 & -2 & -7 & -7 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} \cos 35^\circ & -\sin 35^\circ \\ \sin 35^\circ & \cos 35^\circ \end{bmatrix}$$



$$A = \begin{bmatrix} .7 & .3 \\ .3 & .7 \end{bmatrix}$$

Problem Set 7.1

1.

$T(v+0) = T(v) + T(0)$

$T(v) = T(v) + T(0)$

$T(0) = 0$

$T(0v) = 0T(v)$

$T(0) = 0T(v)$

$T(0) = 0$

4.

a) $S(T(v)) = S(v) = v$

b)

$S(T(v_1) + T(v_2))$

$= S(T(v_1)) + S(T(v_2))$

10

11.

a), c) $\mathbb{R}^n \to \mathbb{R}^m$

3.

a)

$T(v+w) = T(v) + T(w)$

$(v_2 + w_2, v_1 + w_1) = (v_2 + w_2, v_1 + w_1)$

d)

$T(0) \neq 0$, no

7.

a) $T(T(v)) = T(-v) = -(-v) = v$

b) $T(T(v)) = T(v + (1,1)) = v + (2,2)$

9.

$T(T(v)) = T(v_2, v_3, v_1) = (v_3, v_1, v_2)$

$T^{100}(v) = T(v) = (v_2, v_3, v_1)$

13.

$T(M) = AM$

$T(cM) = A(cM) = cAM$

$T(M_1 + M_2) = A(M_1 + M_2) = AM_1 + AM_2$

e)

$T(cv + dw) = cT(v) + dT(w)$

$T((cv_1 + dw_1, cv_2 + dw_2) = cT(v_1, v_2) + dT(w_1, w_2)$

$(v_1 + dw_1 - (v_2 + dw_2 = (v_1 - (v_2 + dw_1 - dw_2$

Yes

Chapter 7.2 : The Matrix of a Linear Transformation

- The next pages assign a matrix to every linear transformation T. For column vectors, the input v is in $V = \mathbb{R}^n$ and the output $T(v)$ is in $W = \mathbb{R}^m$. The matrix A for the transformation will be m by n. Our choice of basis in V and W will decide A.
- The standard basis vectors for $\mathbb{R}^n$ and $\mathbb{R}^m$ are the columns of I.
- When V and W are not $\mathbb{R}^n$ and $\mathbb{R}^m$, they still have bases. Each choice of basis leads to a matrix for T. When the input basis is different from the output basis, the matrix for $T(v) = v$ will not be I - It will be a "change of basis" matrix.
- Suppose you know $T(v_1), \dots, T(v_n)$ for basis vectors $v_1, \dots, v_n$. Then linearity produces $T(v)$ for every other input vector v.
- Example 1
    - Suppose T transforms $v_1 = (1,0)$ to $T(v_1) = (2,3,4)$ and $v_2 = (0,1)$ goes to $T(v_2) = (5,5,5)$. If T is linear from $\mathbb{R}^2$ to $\mathbb{R}^3$ then its "standard matrix" is 3 by 2. These outputs $T(v_1)$ and $T(v_2)$ go into its columns.

$$A = \begin{bmatrix} 2 & 5 \\ 3 & 5 \\ 4 & 5 \end{bmatrix} \quad T(v_1 + v_2) = T(v_1) + T(v_2) \quad \begin{bmatrix} 2 & 5 \\ 3 & 5 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$$

- Example 2
    - If we have the transformation $T(v) = dv/dt$, it is a linear transformation.

$$T(cv + dw) = c\frac{dv}{dt} + d\frac{dw}{dx}$$

    - This linearity and the derivative of the basis vectors $1, x, x^2, x^3$ allow us to find the derivative of polynomials like $5 + 2x - 2x^2 + 2x^3$

- Here the input space contains all linear combinations of 1, x, x^2, and x^3. These specific functions are the basis for the space of cubic polynomials
- To find the kernel, we solve $\frac{dv}{dx}=0$, which is $v=C$. This gives us a nullspace of T that is one-dimensional.
- To find the range, we look at the outputs $T(v)$, which are all quadratic polynomials
- The input space has dimension 4 and the output space 3, so the "derivative matrix" will be 3 by 4.
- The range of T is a three dimensional subspace, so the matrix will have rank 3. The kernel is of dimension one, so $3+1=4$ is the dimension of the input space

- Example 3
  - The integral is the inverse of the derivative, so the transformation T^{-1} represents this
    $$\int_0^x dt = x, \quad \int_0^x t\,dt = \frac{1}{2}x^2, \quad \int_0^x t^2 dt = \frac{1}{3}x^3$$
  - By linearity, the integral of $B+Cx+Dx^2$ is $T^{-1}(w) = Bx + \frac{1}{2}Cx^2 + \frac{1}{3}Dx^3$. Input space = quadratics, Output space = cubics
  - Integration takes W back to V. Its matrix will be 4 by 3
  - Range of T^{-1}: The outputs $Bx + \frac{1}{2}Cx^2 + \frac{1}{3}Dx^3$ are cubics with no constant term
  - Kernel of T^{-1}: The output is 0 iff $B=C=D=0$. The nullspace is $Z=\{0\}$
  - Fundamental theorem! $3+0$ is the dimension of the input space W for T^{-1}.

Matrices for the Derivative and Integral

- The matrix form of derivative T:
$$A=\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$
  - Why is A the correct matrix? Because multiplying by A agrees with transforming by T. The derivative of $v=a+bx+cx^2+dx^3$ is $T(v) = b+2cx+3dx^3$
$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} b \\ 2c \\ 3d \end{bmatrix}$$

- Similarly, the matrix form of the integral T^{-1}
$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}\begin{bmatrix} B \\ C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ B \\ \frac{1}{2}C \\ \frac{1}{3}D \end{bmatrix}$$
  - We'll call this matrix A^{-1}, although you'll notice A is rectangular. So this integral matrix is a one-sided inverse of the derivative matrix.
  - Integrating then differentiating gives us $AA^{-1}=I$. The opposite direction would lose the constant term before integrating. The integral of the derivative of 1 is 0

Construction of the Matrix

- Suppose T transforms n-dimensional space V to m-dimensional space W. We choose a basis $v_1, \dots, v_n$ for V and $w_1, \dots, w_m$ for W. The matrix A will be m by n. To find the first column of A, apply T to the first basis vector v_1: $T(v_1)$. $T(v_1)$ will be in W.

  $T(v_1)$ is a combination $a_{11}w_1 + \dots + a_{m1}w_m$ of the output basis for W

  The numbers $a_{11}, \dots, a_{m1}$ go into the first column of A. Transforming v_1 to $T(v_1)$ matches multiplying $(1, 0, \dots, 0)$ by A. It yields that first column of the matrix
- When T is the derivative and the first basis vector is 1, its derivative $T(v_1) = 0$, so the first column of the derivative matrix is all zeros.
- For the integral, the first basis function is again 1. Its integral is the second basis function x, so the first column of A^{-1} was $(0, 1, 0, 0)$
- The jth column of A is found by applying T to the jth basis vector v_j

  $T(v_j)$ = combination of basis vectors of W $= a_{1j}w_1 + \dots a_{mj}w_m$

  - These numbers $a_{1j}, \dots, a_{mj}$ go into column j of A. The matrix is constructed to get the basis vectors right.
- Example 4
  - If the bases change, T is the same but the matrix A is different. Suppose we reorder the basis to $x, x^2, x^3, 1$ for the cubics in V. Keep the original basis $1, x, x^2$ for the quadratics in W. The derivative of the first basis vector $v_1 = x$ is $w_1 = 1$, so we get

    $$A_{new} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix}$$ = matrix for the derivative T when the bases change to $x, x^2, x^3, 1$ and $1, x, x^2$

  - Reordering the basis vectors of V, we reorder the columns of A.

Products AB match Transformations TS

- Example
  - T rotates every vector by the angle θ. Here $V = W = \mathbb{R}^2$. Find A
    The standard basis for $\mathbb{R}^2$ are $\hat{i} = (1, 0)$ and $\hat{j} = (0, 1)$. Apply T to $\hat{i}$ gets us $(\cos\theta, \sin\theta)$ and doing the same for $\hat{j}$ gets us $(-\sin\theta, \cos\theta)$. This gives A:

    $$\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = A$$
- Example
  - Suppose T projects every plane vector onto the 45° line. Find its matrix for two different choices of the basis.

- We start with the basis v_1, v_2 which point in the $45°$ and $135°$ directions respectively.
  $T(v_1)$ projects onto itself: $T(v_1) = v_1$, so we get $a_1 = (1, 0)$
  $T(v_2)$ projects to 0: $T(v_2)$, so we get $a_2 = (0,0)$
  $P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ when V and W have basis vectors at $45°$ and $135°$
- If we choose the standard basis vectors $(1,0)$ and $(0,1)$,
  Both vectors project to $(\frac{1}{2}, \frac{1}{2})$. So we get
  $P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

- Matrix multiplication is defined as such because it gives the correct output which represents the composition of transformations T and S as the product of their matrices A and B.
- The linear transformation TS starts with any vector u in U, goes to $S(u)$ in V and then to $T(S(u))$ in W. The matrix AB starts with any x in $\mathbb{R}^p$, goes to Bx in $\mathbb{R}^n$ and then to ABx in $\mathbb{R}^m$

  $TS: U \to V \to W \qquad AB: (m \text{ by } n)(n \text{ by } p) = (m \text{ by } p)$

- Example
  - S rotates the plane by θ and so does T. Then TS rotates by 2θ.
    We have $S = T$ and $A = B$
    $$T^2 = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}^2 = \begin{bmatrix} \cos^2\theta - \sin^2\theta & -2\sin\theta\cos\theta \\ 2\sin\theta\cos\theta & \cos^2\theta - \sin^2\theta \end{bmatrix} = \begin{bmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix}$$

The Identity Transformation and the Change of Basis Matrix

- When each output $T(v_j) = v_j$ is the same as w_j, the matrix is just I.
- Suppose the basis is different. Then $T(v_1) = v_1$ is a combination of w's.
  The combinations $m_{11}w_1 + \dots + m_{n1}w_n$ gives us the first column of the matrix M.
- Identity transformation:
  - When the outputs $T(v_j) = v_j$ are combinations $\sum_{i=1}^{n} m_{ij} w_i$, the "change of basis" matrix is M
  - The basis is changing but the vectors are not: $T(v) = v$
- Example
  - The input basis is $v_1 = (3,7)$, $v_2 = (2,5)$. The output basis is $w_1 = (1,0)$, $w_2 = (0,1)$.
    Then the matrix M is easy to compute:
    $T(v_1) = (3,7) = 3w_1 + 7w_2$
    $T(v_2) = (2,5) = 2w_1 + 5w_2 \implies M = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}$
- Example
  - The input basis is $\{(1,0),(0,1)\}$ and the output basis is $\{(3,7),(2,5)\}$
    $T(v_1) = (1,0) = 5w_1 - 7w_2$
    $T(v_2) = (0,1) = -2w_1 + 3w_2 \implies M = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}^{-1} = \begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}$
    $[w_1 \ \ w_2]\begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix} = [v_1 \ \ v_2]$ is $MM^{-1} = I$

Wavelet Transform = Change to Wavelet Basis

- Wavelets are little waves. They have different lengths and they are localized at different places. The first basis vector is actually not a wavelet, but the very useful matrix of all 1s. This example show "Haar Wavelets"

Haar Basis $w_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ $w_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$ $w_3 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$ $w_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$

- These vectors are orthogonal. w_3 is localized in the second half and w_4 is localized in the second half. The wavelet transform finds the coefficients c_1, c_2, c_3, c_4 when the input signal $v = (v_1, v_2, v_3, v_4)$ is expressed in the wavelet basis:

Transform v to c $\quad v = c_1 w_1 + c_2 w_2 + c_3 w_3 + c_4 w_4 = Wc$

- The coefficients c_3 and c_4 tell us about details in the first and second half of v. The coefficient c_1 is the average.
- Why do we want to change the basis? Think of v_1, v_2, v_3, and v_4 as the intensities of a signal. We want to compress this signal but if we keep only 5% of the standard basis coefficients, we lose 95% of the signal. But if we choose a better basis of w's, 5% of the basis vectors can combine to form a signal very close to the original.
  - One good basis vector would be $(1,1,1,1)$. This can represent the constant background of an image. A short wave like $(0,0,1,-1)$ represents a detail at the end of our signal.

Input $v \xrightarrow[\text{[lossless]}]{} $ coefficients $c \xrightarrow[\text{[lossy]}]{}$ compressed $\hat{c} \xrightarrow[\text{[reconstruct]}]{}$ compressed $\hat{v}$

  - Ideally, we don't need the compression step and transform gives us $c = W^{-1}v$ and the reconstruction gives us $v = Wc$. In true signal processing, we have $\hat{v} = W\hat{c}$
- Example
  - For a vector $(6,4,5,1) = v$, its wavelet coefficients are $c = (4,1,1,2)$

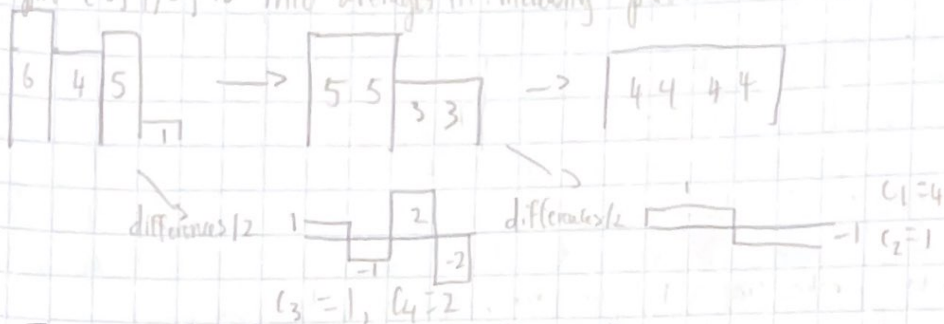
$$\begin{bmatrix} 6 \\ 4 \\ 5 \\ 1 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & 1 \\ 1 & -1 & 0 & -1 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \\ 1 \\ 2 \end{bmatrix}$$

  - The coefficients are $c = W^{-1}v$. W^{-1} is easy to find because W's are orthogonal. They are not orthonormal, so we need to rescale

$$W^{-1} = \begin{bmatrix} \frac{1}{4} & & & \\ & \frac{1}{4} & & \\ & & \frac{1}{2} & \\ & & & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

  - The $\frac{1}{4}$'s in the first row of $c = W^{-1}v$ means that $c_1 = 4$ is the average of $6,4,5,1$

- Example
  - You can pick off the details in c_3 and c_4, and the coarse details in c_2 and the average in c_1
  - Split $(6,4,5,1)$ into averages in the wavy partitions



Fourier Transform (DFT) = Change to Fourier Basis

- The DFT involves complex numbers (powers of $e^{2\pi i/n}$). But if we choose $n=4$, the matrices are small and the only complex numbers are i and $i^3=-i$

Fourier basis w_1 to w_4 in the columns of F

$$F=\begin{bmatrix}1&1&1&1\\1&i&i^2&i^3\\1&i^2&i^4&i^6\\1&i^3&i^6&i^9\end{bmatrix}$$

- The first column is the flat wave $(1,1,1,1)$. It represents the average signal or the direct current. It is a wave of zero frequency.
- The third column is $(1,-1,1,-1)$, which alternates at the highest frequency
- The Fourier transform decomposes the signal into waves at equally spaced frequencies

Problem set 7.2

1.

$Sv_1=0 \Rightarrow b_1=(0,0,0,0)$

$Sv_2=0 \Rightarrow b_2=(0,0,0,0)$

$Sv_3=2 \Rightarrow b_3=(2,0,0,0)$

$Sv_4=6x \Rightarrow b_4=(0,6,0,0)$

$$B=\begin{bmatrix}0&0&2&0\\0&0&0&6\\0&0&0&0\\0&0&0&0\end{bmatrix}$$

2.

$y=Ax+B$

3.

$$A=\begin{bmatrix}0&1&0&0\\0&0&2&0\\0&0&0&3\\0&0&0&0\end{bmatrix} \Rightarrow A^2=\begin{bmatrix}0&0&2&0\\0&0&0&6\\0&0&0&0\\0&0&0&0\end{bmatrix}$$

5.

$$A=\begin{bmatrix}0&1&1\\1&0&0\\0&1&1\end{bmatrix}$$

$$Av=\begin{bmatrix}0&1&1\\1&0&0\\0&1&1\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix}=\begin{bmatrix}2\\1\\2\end{bmatrix}$$

7.

$$\begin{bmatrix}3\\0\\2\end{bmatrix}$$

10.

$$A=\begin{bmatrix}1&0&0\\1&1&0\\1&1&1\end{bmatrix}$$

$T(v_1-v_2)=w_1$

11.

$$A^{-1}=\begin{bmatrix}1&-1&0\\0&1&-1\\0&0&1\end{bmatrix}$$

$T^{-1}(w_1)=v_1$

$T^{-1}(w_2)=v_2-v_1$

$T^{-1}(w_3)=v_3-v_2$

13.

a) $\begin{bmatrix}-1&0\\0&-1\end{bmatrix}$

b)

14.

a) $\begin{bmatrix}2&1\\5&3\end{bmatrix}$

b) $\begin{bmatrix}3&-1\\-5&2\end{bmatrix}$

20.

$(0,0,1)$

a) $w_2=-(x^2-1)=(0,1,0)$

b) $w_3=\frac{1}{2}(x^2-x)=(1,0,0)$

$y=4w_1+5w_2+6w_3$

21.

$T(v)=v$

$T(1)=1=w_1-w_3=(1,0,-1)$

$T(x)=x=$

22.

$4 = A + aB + a^2C$

$6 = A + bB + b^2C$

$6 = A + cB + c^2C$

$$\begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix} = A$$

$\det A = bc^2 + ab^2 + a^2c - a^2b - ac^2 - b^2c$

24.

$T(v) = aT(v_1) + bT(v_2) + cT(v_3)$

$= a\lambda_1 v_1 + b\lambda_2 v_2 + c\lambda_3 v_3$

$T(v_1) = \lambda_1 v_1$

$$A = \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{bmatrix}$$

34.

$(7,5,3,1) = (6,0,1,2) + (1,-1,1,-1)$

$= (4,4,4,4) + (2,2,-2,-2) + (1,-1,1,-1)$

$= 4(1,1,1,1) + 2(1,1,-1,-1) + (1,-1,1,-1)$

$$\begin{bmatrix} 4 \\ 2 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & 1 \\ 1 & -1 & 0 & -1 \end{bmatrix}$$

Chapter 7.3: Diagonalization and the Pseudoinverse

- This section produces better matrices by choosing better bases.
  - When the goal is a diagonal matrix, one way is a basis of eigenvectors
  - The other way is two bases (the input and output bases are different). The left and right singular vectors are orthonormal basis vectors for the 4 fundamental subspaces. They come from SVD.
- By reversing those input and output bases we find the "pseudoinverse" of A. This matrix A^+ sends $\mathbb{R}^m$ back to $\mathbb{R}^n$ and the column space back to the row space
- All our great factorizations of A can be viewed as a change of basis; we'll focus on two:
  1. $S^{-1}AS = \Lambda$ when the input and output bases are the eigenvectors of A
  2. $U^{-1}AV = \Sigma$ when those bases are eigenvectors of A^TA and AA^T
- In Λ, the bases are the same. Then $m=n$ and the matrix A must be square. Some square matrices cannot be diagonalized by any S because they don't have n independent eigenvectors
- In Σ, the input and output bases are different. The matrix A can be rectangular. The bases are orthonormal because A^TA and AA^T are symmetric. Then $U^{-1} = U^T$ and $V^{-1} = V^T$. Every matrix A is allowed and A has the diagonal form Σ
- The eigenvector basis is orthonormal only when $A^TA = AA^T$ (a "normal" matrix). That includes symmetric, antisymmetric and orthogonal matrices. In this case the singular values of Σ are the absolute values $\sigma_i = |\lambda_i|$. The two diagonalizations are the same when $A^TA = AA^T$ except for possible factors -1 and $e^{i\theta}$.
- The factorization $A = QR$ chooses only one new basis. That is the orthogonal output basis given by Q. The input uses the standard basis given by I. The output basis matrix appears on the left and the input basis appears on the right, in $A = QRI$. We start with input basis = output basis, that will produce S and S^{-1}

Similar Matrices: A and $S^{-1}AS$ and $W^{-1}AW$

- Begin with a square matrix and one basis. The input space V is $\mathbb{R}^n$ and the output space W is also $\mathbb{R}^n$. The standard basis vectors are the columns of I. The linear transformation T is "multiplication by A".
- The change from A to Λ comes from a change of basis; Eigenvalue matrix from eigenvector basis.
- When you change the basis for V, the matrix changes from A to AM. Because V is the input space, M goes on the right (to come first). When you change the basis for W, the new matrix is $M^{-1}A$. We are working with the output space so M^{-1} is on the left.
- If you change both bases in the same way, the new matrix is $M^{-1}AM$. The good basis vectors are the eigenvectors of A, when the matrix becomes $S^{-1}AS=\Lambda$

  When the basis contains the eigenvectors $x_1, \ldots, x_n$, the matrix for T is Λ.

  - To find column 1 of the matrix, input the first basis vector x_1. The transformation multiplies by A. The output is $Ax_1=\lambda x_1$, giving us a basis vector $(\lambda_1, 0, \ldots, 0)$.
- Example
  - Project onto the line $y=-x$. $(1,0)$ projects to $(.5,-.5)$. $(0,1)$ projects to $(-.5,.5)$
    1. Standard matrix: project standard basis $A=\begin{bmatrix} .5 & -.5 \\ -.5 & .5 \end{bmatrix}$
    2. Find the diagonal matrix Λ in the eigenvector basis
       eigenvectors are $x_1=(1,-1)$ and $x_2=(1,1)$
       Projection of $x_1 = x_1 \Rightarrow$ basis of $(1,0)$ $\quad \Lambda=\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$
       Projection of $x_2 = 0 \Rightarrow$ basis of $(0,0)$
    3. Find a third matrix B using another basis $v_1=w_1=(2,0)$ and $v_2=w_2=(1,1)$
       w_1 is not an eigenvector, so B will not be diagonal
       Projection of $v_1=(1,-1)=w_1-w_2 \Rightarrow$ basis of $(1,-1)$ $\quad B=\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$
       Projection of $v_2=(0,0) \Rightarrow$ basis of $(0,0)$
  - Another way to find B: use W^{-1} and W to change between the standard basis and the w's. Those change of basis matrices are representing the identity transformation. The product of transformations is just ITI. The product of matrices is $W^{-1}AW$. B is similar to A.
- For any basis $w_1, \ldots, w_n$, we find B in 3 steps. Change the input basis to the standard basis with W. The matrix in the standard matrix is A. Change the output basis back to the w's with W^{-1}. Then $B=W^{-1}AW$ represents ITI
- A change of basis produces a similarity transformation to $W^{-1}AW$ in the matrix.

  $B_{w\text{'s to }w\text{'s}} = W^{-1}_{\text{standard to }w\text{'s}} \, A_{\text{standard}} \, W_{w\text{'s to standard}}$

- Example
  - Continuing with the previous example

$$W=\begin{bmatrix}2 & 1\\ 0 & 1\end{bmatrix} \Rightarrow W^{-1}=\begin{bmatrix}\frac{1}{2} & -\frac{1}{2}\\ 0 & 1\end{bmatrix} \quad A=\begin{bmatrix}\frac{1}{2} & -\frac{1}{2}\\ -\frac{1}{2} & \frac{1}{2}\end{bmatrix}$$

Rewrites A (in standard basis) with W as the basis

$$B=W^{-1}AW=\begin{bmatrix}\frac{1}{2} & -\frac{1}{2}\\ 0 & 1\end{bmatrix}\begin{bmatrix}\frac{1}{2} & -\frac{1}{2}\\ -\frac{1}{2} & \frac{1}{2}\end{bmatrix}\begin{bmatrix}2 & 1\\ 0 & 1\end{bmatrix}=\begin{bmatrix}1 & 0\\ -1 & 0\end{bmatrix}$$

The Singular Value Decomposition (SVD)

- Now the input basis can differ from the output basis $u_1, \dots, u_m$.
- Again, the best matrix is diagonal (now m by n). To achieve this diagonal Σ_1, each input vector v_j must transform into a multiple of the output vector u_j. That multiple is the singular value σ_j.

$$\text{SVD} \qquad Av_j = \begin{cases}\sigma_j u_j & \text{for } j \le r\\ 0 & \text{for } j > r\end{cases} \quad \text{with orthonormal bases}$$

- A and Σ_1 represent the same transformation. $A=U\Sigma_1 V^T$ uses the standard bases for $\mathbb{R}^n$ and $\mathbb{R}^m$. The diagonal Σ_1 uses the input basis of v's and the output bases of u's. The orthogonal matrices V and U give the basis changes. They represent the identity transformations (in $\mathbb{R}^n$ and $\mathbb{R}^m$)

$$\Sigma_{1\,v\text{'s to } u\text{'s}} = U^{-1}_{\text{standard to } u\text{'s}}\, A_{\text{standard}}\, V_{v\text{'s to standard}}$$

Polar Decomposition

- Every complex number has the polar form $re^{i\theta}$. A non-negative number r multiplies a number on the unit circle. Think of these numbers as 1 by 1 matrices: $r \ge 0$ corresponds to a positive semidefinite matrix, call it H, and $e^{i\theta}$ corresponds to an orthogonal matrix Q. The polar decomposition extends this factorization to matrices: orthogonal times semidefinite. $A=QH$
- Every real square matrix can be factored into $A=QH$, where Q is orthogonal and H is symmetric positive semidefinite.

$$A=U\Sigma_1 V^T = UV^T V\Sigma V^T = (UV^T)(V\Sigma_1 V^T) = QH$$

  - UV^T: the product of orthogonal matrices is also orthogonal
  - $V\Sigma_1 V^T$: It is positive semidefinite because its eigenvalues are in Σ_1. If A is invertible, so are Σ_1 and H. H is the symmetric positive definite square root of A^TA ($H^2 = V\Sigma_1^2 V^T = A^TA$)
- There is also a $A=KQ$ in the reverse direction. Q is the same but $K=U\Sigma_1 U^T$. This is the symmetric positive definite square root of AA^T

- Example

$$A = \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{2} & \\ & 2\sqrt{2} \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = U \Sigma_1 V^T$$

$$Q = UV^T, \quad H = V\Sigma_1 V^T \text{ or } H = Q^{-1}A = Q^T A$$

$$Q = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}, \quad H = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{3}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{3}{\sqrt{2}} \end{bmatrix}$$

- In mechanics, Q is a rotation and H is the stretching factor
  - The eigenvalues of H are the singular values of A; they give the stretching factors
  - The eigenvectors of H are the eigenvectors of A^TA. They give the stretching directions.
  - Then Q rotates those axes
- $A = QH$ splits $Av_i = \sigma_i u_i$ into 2 steps: H multiplies v_i by σ_i. Q rotates v_i into u_i

The Pseudo Inverse

- By choosing good bases, A multiplies v_i in the row space to give $\sigma_i u_i$ in the column space. A^{-1} must do the opposite: $A^{-1}u = v/\sigma$. The singular values of A^{-1} are $1/\sigma$, just as the eigenvalues of A^{-1} are $1/\lambda$. The u's are in the row space of A^{-1} and the v's are in the column space
- A matrix that multiplies u_i to produce v_i/σ_i does exist and is the pseudoinverse A^+.

$$\text{Pseudoinverse: } A^+ = V\Sigma_1^+ U^T = \underset{n \text{ by } n}{\begin{bmatrix} v_1 & \cdots & v_r & \cdots & v_n \end{bmatrix}} \underset{n \text{ by } m}{\begin{bmatrix} \sigma_1^{-1} & & \\ & \ddots & \\ & & \sigma_r^{-1} \end{bmatrix}} \underset{m \text{ by } m}{\begin{bmatrix} u_1 & \cdots & u_r & \cdots & u_m \end{bmatrix}^T}$$

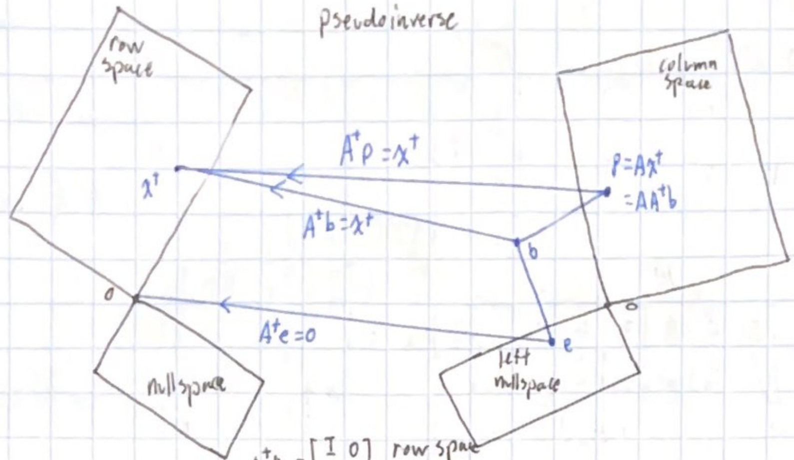
- The pseudoinverse A^+ is an n by m matrix. If A^{-1} exists, then A^+ is the same as A^{-1}.
  - In that case $m = n = r$ and we are inverting $U\Sigma_1 V^T$ to get $V\Sigma_1^{-1}U^T$.
- If $r < m$ or $r < n$, then A has no two-sided inverse, but a pseudoinverse A^+ with the same rank r

$$A^{+}u_i = \frac{1}{\sigma_i}v_i \text{ for } i \le r \text{ and } A^{+}u_i = 0 \text{ for } i > r$$

- The vectors $u_1, \ldots, u_r$ in the column space of A go back to $v_1, \ldots, v_r$ in the row space.
- The other vectors $u_{r+1}, \ldots, u_m$ are in the left nullspace, and A^+ sends them to 0
- Notice the pseudoinverse Σ_1^+ of the diagonal matrix. Each σ is replaced by σ^{-1}. The product $\Sigma_1^+\Sigma_1$ is as near to identity as we can get it (It is partly I and partly 0) We get r 1's. We take the reciprocal of Σ_1 and transpose it.

$$\Sigma_1^+\Sigma_1 = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \Sigma_1^+\Sigma_1 \text{ is a projection matrix}$$

- The pseudoinverse is the n by m matrix that makes AA^+ and A^+A into projection pseudoinverse



$A^+A = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$ row space / nullspace

Fig 7.3.1: Ax^+ in the column space goes back to $A^+Ax^+ = x^+$

- Trying for $AA^{-1} = A^{-1}A = I$
  - AA^+ = Projection matrix onto the column space of A
  - A^+A = Projection matrix onto the row space of A
- Example
  - Find the pseudoinverse of $A = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}$. A has 1 singular value $\sqrt{10}$

    $A^+ = V\Sigma_1^{+}U^T = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} \sqrt{10}/10 & 0 \\ 0 & 0 \end{bmatrix}\frac{1}{\sqrt{5}}\begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}$
  - A^+ also has rank 1. Its column space is the row space of A. When A takes $(1,1)$ in the row space to $(4,2)$ in the column space, A^+ does the reverse: $A^+(4,2) = (1,1)$.
  - Every rank 1 matrix is a columns times a row. With unit vectors u and v, that is $A = \sigma u v^T$. Then the best inverse is $A^+ = vu^T/\sigma$. The product AA^+ is uu^T, the projection onto the line through u. The product A^+A is vv^T.
- Application to least squares
  - The equation $A^TA\hat{x} = A^Tb$ assumes A^TA is invertible
  - A may have dependent columns (rank $< n$) and there are many solutions to $A^TA\hat{x} = A^Tb$. One solution is $x^+ = A^+b$ from the pseudoinverse
  - We can check that A^TAA^+b is A^Tb because Fig 7.3.1 shows that $e = b - AA^+b$ is the part of b in the left nullspace. Any vector in the nullspace of A can be added to give another solution, but x^+ will be the shortest

The shortest least squares solution to $Ax = b$ is $x^+ = A^+b$

Problem Set 7.3

1.

a) $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$

$A^TA = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} = \begin{bmatrix} 10 & 20 \\ 20 & 40 \end{bmatrix}$

$\lambda = 0, 50$

case $\lambda = 0$

$(A^TA - 0I)x_1 = 0$

$\begin{bmatrix} 10 & 20 \\ 20 & 40 \end{bmatrix} x_1 = 0$

$x_1 = (2, -1)$

$v_1 = \left(\frac{2\sqrt{5}}{5}, -\frac{\sqrt{5}}{5}\right)$

case $\lambda = 50$:

$(A^TA - 50I)x_2 = 0$

$\begin{bmatrix} -40 & 20 \\ 20 & -10 \end{bmatrix} x_2 = 0$

$x_2 = (1, -2)$

$v_2 = \left(\frac{\sqrt{5}}{5}, \frac{2\sqrt{5}}{5}\right)$

$Av_1 = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}\begin{bmatrix} \frac{2\sqrt{5}}{5} \\ -\frac{\sqrt{5}}{5} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\sigma_1 = 0 \quad u_1 = (0, 0)$

Choose v_1 so $v_1 \perp v_2$

$Av_2 = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}\begin{bmatrix} \frac{\sqrt{5}}{5} \\ \frac{2\sqrt{5}}{5} \end{bmatrix} = \begin{bmatrix} \sqrt{5} \\ 3\sqrt{5} \end{bmatrix}$

$\sigma_2 = 5\sqrt{2}, \quad u_2 = \left(\frac{\sqrt{10}}{10}, \frac{3\sqrt{10}}{10}\right)$

$u_1 = \left(-\frac{3\sqrt{10}}{10}, \frac{\sqrt{10}}{10}\right)$

$\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{10}}{10} & -\frac{3\sqrt{10}}{10} \\ \frac{3\sqrt{10}}{10} & \frac{\sqrt{10}}{10} \end{bmatrix}\begin{bmatrix} 5\sqrt{2} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \frac{\sqrt{5}}{5} & \frac{2\sqrt{5}}{5} \\ \frac{2\sqrt{5}}{5} & -\frac{\sqrt{5}}{5} \end{bmatrix}$

2.

a)

basis $C(A^T) = \{(\frac{\sqrt{5}}{5}, \frac{2\sqrt{5}}{5})\}$

basis $N(A) = \{(\frac{2\sqrt{5}}{5}, -\frac{\sqrt{5}}{5})\}$

basis $C(A) = \{(\frac{\sqrt{10}}{10}, \frac{3\sqrt{10}}{10})\}$

basis $N(A^T) = \{(-\frac{3\sqrt{10}}{10}, \frac{\sqrt{10}}{10})\}$

3.

$Q = UV^T$

$= \begin{bmatrix} \frac{\sqrt{10}}{10} & -\frac{3\sqrt{10}}{10} \\ \frac{3\sqrt{10}}{10} & \frac{\sqrt{10}}{10} \end{bmatrix} \begin{bmatrix} \frac{\sqrt{5}}{5} & \frac{2\sqrt{5}}{5} \\ \frac{2\sqrt{5}}{5} & -\frac{\sqrt{5}}{5} \end{bmatrix}$

$= \begin{bmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$

$H = V\Sigma V^T$

$= \begin{bmatrix} \frac{\sqrt{5}}{5} & \frac{2\sqrt{5}}{5} \\ \frac{2\sqrt{5}}{5} & -\frac{\sqrt{5}}{5} \end{bmatrix} \begin{bmatrix} 5\sqrt{2} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{5}}{5} & \frac{2\sqrt{5}}{5} \\ \frac{2\sqrt{5}}{5} & \frac{\sqrt{5}}{5} \end{bmatrix}$

$= \begin{bmatrix} \sqrt{2} & 2\sqrt{2} \\ 2\sqrt{2} & 4\sqrt{2} \end{bmatrix}$

4.

$A^+ = \begin{bmatrix} \frac{\sqrt{5}}{5} & \frac{2\sqrt{5}}{5} \\ \frac{2\sqrt{5}}{5} & -\frac{\sqrt{5}}{5} \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{10} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{10}}{10} & \frac{3\sqrt{10}}{10} \\ -\frac{3\sqrt{10}}{10} & \frac{\sqrt{10}}{10} \end{bmatrix} = \begin{bmatrix} \frac{1}{50} & \frac{3}{50} \\ \frac{1}{25} & \frac{3}{25} \end{bmatrix}$

$P_{row} = A^+A = \begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{4}{5} \end{bmatrix}$

$P_{col} = AA^+ = \begin{bmatrix} \frac{1}{10} & \frac{3}{10} \\ \frac{3}{10} & \frac{9}{10} \end{bmatrix}$

10.

$A = [3 \quad 4 \quad 0]$

$A^TA = \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} [3 \quad 4 \quad 0]$

$= \begin{bmatrix} 9 & 12 & 0 \\ 12 & 16 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\lambda = 0, 0, 25$

$v_1 = \frac{\sqrt{5}}{5}(-\frac{4}{3}, 1, 0)$

$v_2 = (0, 0, 1)$

$v_3 = \frac{4}{5}(\frac{3}{4}, 1, 0)$

15.

$A^TA\hat{x} = A^Tb$

$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \hat{x} = \begin{bmatrix} \frac{1}{5} & \frac{3}{10} \\ \frac{1}{5} & \frac{3}{10} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{bmatrix} \hat{x} = \begin{bmatrix} \frac{1}{5}b_1 + \frac{3}{10}b_2 \\ 0 \quad 0 \end{bmatrix}$

A^TA is singular

17.

$AA^+p = p$

$AA^+e = 0$

$A^+Ax_r = x_r$

$A^+Ax_n = 0$

18.

$A = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

$A^TA = [3 \quad 4] \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

$= [25]$

$\lambda = 25$

Case $\lambda = 25$

$(A - 25I)x_1 = 0$

$x_1 = (1)$

$v_1 = (1)$

$Av_1 = \begin{bmatrix} 3 \\ 4 \end{bmatrix} [1]$

$= \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

$\sigma_1 = 5, \; u_1 = (\frac{3}{5}, \frac{4}{5})$

choose $u_2 = (\frac{4}{5}, -\frac{3}{5})$

$A = U\Sigma V^T \Rightarrow \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{bmatrix} \begin{bmatrix} 5 \\ 0 \end{bmatrix} [1]$

$A^+ = V\Sigma^+U^T = [1] [\frac{1}{5} \quad 0] \begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{bmatrix}$

$= [\frac{3}{25} \quad \frac{4}{25}]$

$A^+A = [1]$

$AA^+ = \begin{bmatrix} \frac{9}{25} & \frac{12}{25} \\ \frac{12}{25} & \frac{16}{25} \end{bmatrix}$

$x^+ = A^+b$

$= [\frac{3}{25} \quad \frac{4}{25}] \begin{bmatrix} 3 \\ 4 \end{bmatrix} = 1$

$x^+ = A^+b$

$= [\frac{3}{25} \quad \frac{4}{25}] \begin{bmatrix} -4 \\ 3 \end{bmatrix}$

$= 0$

Chapter 8: Applications

Chapter 8.1: Matrices in Engineering

- Engineering problems produce symmetric matrices K, which is often positive definite. This because a) $K = A^TA$ and $K = A^TCA$, and b) the expression $\frac{1}{2}u^TKu$ represents energy, which is never negative. The matrix C, often diagonal, contains positive physical constants
- Our first example: K is the stiffness matrix, and $K^{-1}f$ is the structure's responses to forces f from outside
- Engineering leads to linear algebra in two ways:
  - Direct way: the laws connecting position or velocity are linear. Expressed as matrix eqs.
  - Indirect way: the physical system is continuous; expressed as differential equations. To find accurate solutions, we approximate by finite difference or finite element equations