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$\hat{L}$înear $\hat{A}$lgebra

Via Introduction to Linear Algebra: Fourth Edition

Chapter 1: Introduction to Vectors

- At the heart of linear algebra lies two vector operations: Addition ($v+w$) and linear combinations ($cv+dw$, $c,d \in \mathbb{R}$)
  - Example:

$$cv + wd = c\begin{bmatrix}1\\1\end{bmatrix} + d\begin{bmatrix}2\\3\end{bmatrix} = \begin{bmatrix}c+2d\\c+3d\end{bmatrix}$$

- The vector v and its scaler multiple cv lie on the same line when w is not on that line (i.e. they are linearly independent), $cv+dw$ span the whole 2D plane.

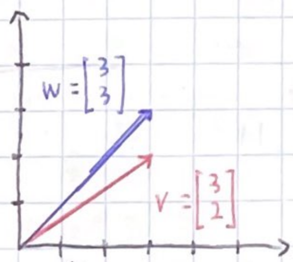


fig 1.1.1

Chapter 1.1: Vectors and Linear Combinations

Introduction

- Vectors in this book are represented as columns:

$$v = \begin{bmatrix}v_1\\v_2\end{bmatrix}, \quad v_1 = \text{First component}, \quad v_2 = \text{Second component}$$

- Vector Addition:

$$v = \begin{bmatrix}v_1\\v_2\end{bmatrix}, \quad w = \begin{bmatrix}w_1\\w_2\end{bmatrix}, \quad v+w = \begin{bmatrix}v_1+w_1\\v_2+w_2\end{bmatrix}$$

- Scalar Multiplication:

$$v = \begin{bmatrix}v_1\\v_2\end{bmatrix}, \quad 2v = \begin{bmatrix}2v_1\\2v_2\end{bmatrix}, \quad -v = \begin{bmatrix}-v_1\\-v_2\end{bmatrix}$$

- Zero vector
  - The sum of v and $-v$ is the zero vector $\mathbf{0} = \begin{bmatrix}0\\0\end{bmatrix}$ and it is not the same as 0 scalar.

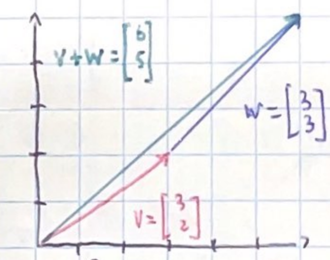


fig 1.1.2

Linear Combinations

- The vector sum $cv + dw$ is a <u>Linear Combination</u> of v and w
- Vectors can be visualized on a cartesian plane, although it should be noted the position of a vector is not part of its identity. Vectors can be "moved" as long as its magnitude and direction remain the same.
  - See Fig 1.1.1 and Fig 1.1.2

Vectors in Three Dimensions

- A vector with 2 components correspond to a point in the Cartesian Plane. (It is the **position vector** of that point)
- Similarly, a vector in three dimensions represents a point in 3D space.

  $v = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$, $w = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$, $v + w = \begin{bmatrix} 3 \\ 4 \\ 3 \end{bmatrix}$; See fig 1.1.3

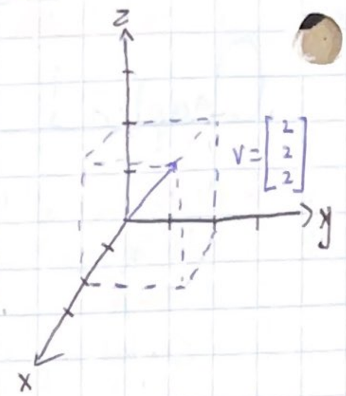


fig 1.1.3

- To save space, a column vector may be written as $v = (v_1, v_2, v_3, \ldots)$. Note this is <u>not</u> the same as a row vector $v = [v_1, v_2, v_3, \ldots]$
- Given 3 non-zero <u>linearly independent</u> vectors u, v, w,
  - The combinations cu fill a line
  - The combinations $cu + dv$ fill a plane
  - The combinations $cu + dv + ew$ fill a three-dimension space
- A set of vectors is <u>linearly independent</u> if no vector in the set can be expressed as a linear combination of the others

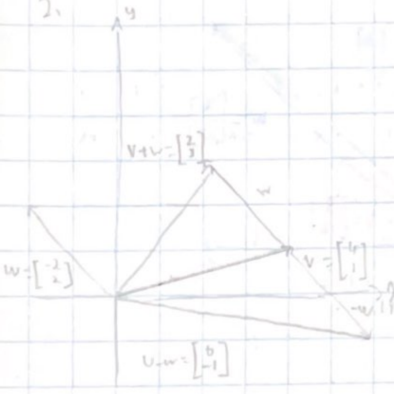
Problem Set 1.1

1.

(a) Since $v = (1,2,3)$ and $w = (3,6,9)$ are linearly dependent, the combinations $cv + dw$ span only a line.

(b) Since $v = (1,0,0)$ and $w = (0,2,3)$ are linearly independent, the combinations $cv + dw$ span a plane

2.



3.

$$\begin{cases} v + w = \begin{bmatrix} 5 \\ 1 \end{bmatrix} & ① \\ v - w = \begin{bmatrix} 1 \\ 5 \end{bmatrix} & ② \end{cases}$$

$① + ②: \; 2v = \begin{bmatrix} 6 \\ 6 \end{bmatrix}$

$v = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$

Sub $v = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$ into ②

$\begin{bmatrix} 3 \\ 3 \end{bmatrix} - w = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$

$w = \begin{bmatrix} 3 \\ 3 \end{bmatrix} - \begin{bmatrix} 1 \\ 5 \end{bmatrix}$

$= \begin{bmatrix} 2 \\ -2 \end{bmatrix}$

$(v, w) = \left(\begin{bmatrix} 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \end{bmatrix}\right)$

5.

$u + v + w = (0,0,0)$

Since $\exists\,(c,d,e) \neq (0,0,0),\ c,d,e \in \mathbb{R}$ s.t. $cu + dv + ew = 0$, the vector set $\{u, v, w\}$ are linearly dependent and span a plane

6.

$cv + dw = (3,3,-6)$

$c(1,-2,1) + d(0,1,-1) = (3,3,-6)$

$(c, -2c + d, c - d) = (3,3,-6)$

$$\begin{cases} c = 3 \\ -2c + d = 3 \\ c - d = -6 \end{cases}$$

$(c, d) = (3, 9)$

Chapter 1.2 : Lengths and Dot Products

- The dot product or inner product of vectors $v=(v_1, v_2)$ and $w=(w_1, w_2)$ is defined as

angle between the two vectors "tail-to-tail"

$$v \cdot w = v_1 w_1 + v_2 w_2 = |v||w|\cos\theta$$

- If, for any 2 non-zero vectors, their dot product is zero, the two vectors are perpendicular. For example

$$\begin{bmatrix} 4 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 2 \end{bmatrix} = (4)(-1) + (2)(2) = 0$$

- The dot product is commutative, that is, $v \cdot w = w \cdot v$
- Example Application : Engineering

  Given a seesaw with weights as such:

The seesaw will remain balanced because the dot product $(4)(-1)+(2)(2)=0$.
The vector of weights $(w_1, w_2) = (4,2)$ and the vector of distances $(v_1, v_2) = (-1, 2)$
The weight multiplied by distance give the "moments", a quantity proportional to the torque generated. If the moments equal zero, the seesaw remains balanced.

- Example Application : Economics

  We have 3 goods to buy and sell. Their prices are (p_1, p_2, p_3) - The price vector.
  The quantities we buy or sell is (q_1, q_2, q_3) – positive when selling, negative when buying.
  Their dot product in three dimensions is the total income:

  $p_1 q_1 + p_2 q_2 + p_3 q_3$ = total income $= (p_1, p_2, p_3) \cdot (q_1, q_2, q_3)$

- Dot product (General form) :

  Given $v=(v_1, v_2, v_3, \dots v_n)$, $w=(w_1, w_2, w_3, \dots w_n)$, $n \in \mathbb{N}$

$$v \cdot w = \sum_{i=1}^{n} v_i w_i$$

Lengths and unit vectors

- A vector's dot product with itself generates a scalar equal to the magnitude of the vector squared.
- The length of a vector $v = (v_1, v_2, v_3, \dots, v_n)$ is $\|v\| = \sqrt{\sum_{i=1}^{n} v_i^2}$
- A Unit Vector is a vector with magnitude 1, often in the direction of another vector
- For any vector v, $v \neq 0$, we can get a unit vector in its direction by dividing the vector by its length. $\hat{v} = \frac{1}{\|v\|} v$

- The standard unit vectors along the x, y, and z-axis are $\hat{i}, \hat{j}, \hat{k}$
- In the xy plane, a unit vector making an angle "θ" with the positive x-axis is given as $u = (\cos\theta, \sin\theta)$
  - When $\theta = 0°$, the horizontal vector u is $\hat{i}$.
  - When $\theta = 90°$, the vertical vector u is $\hat{j}$.
  - For any θ, $u \cdot u = 1$ because $\cos^2\theta + \sin^2\theta = 1$
- Since $u \cdot v = |u||v|\cos\theta$, the angle between 2 vectors is given as

$$\theta = \cos^{-1}\left(\frac{u \cdot v}{|u||v|}\right)$$

- Starting from $\frac{u \cdot v}{|u||v|}$, which is $-1 \le \frac{u \cdot v}{|u||v|} \le 1$,

$$\frac{|u \cdot v|}{|u||v|} \le 1$$

$$\boxed{|u \cdot v| \le |u||v|}$$

which is the Cauchy - Bunyakovsky - Schwartz Inequality for Euclidean Vectors.
In Euclidean Space, this becomes

$$\left(\sum_{i=1}^{n} u_i v_i\right)^2 \le \left(\sum_{i=1}^{n} u_i^2\right)\left(\sum_{i=1}^{n} v_i^2\right)$$

- Taking the dot product of $u = (a, b)$ and $v = (b, a)$ is $2ab$. The lengths of both u and v are $\sqrt{a^2 + b^2}$. The Cauchy-Swartz Inequality says $2ab \le a^2 + b^2$

$$2ab \le a^2 + b^2 \Rightarrow ab \le \frac{a^2+b^2}{2} \Rightarrow \text{Let } x = a^2, y = b^2, \boxed{\sqrt{xy} \le \frac{x+y}{2}}$$

which is the famous AM-GM Inequality.

- Triangle Inequality $\boxed{\|u + v\| \le \|u\| + \|v\|}$

Problem Set 1.2

1.

$u \cdot v = (-0.6)(3) + (0.8)(4) = \frac{7}{5}$

2.

$\|u\| = \sqrt{(-0.6)^2 + (0.8)^2}$
$= 1$

$\|v\| = \sqrt{(3)^2 + (4)^2}$
$= 5$

$\|w\| = \sqrt{(8)^2 + (6)^2}$
$= 10$

3.

$\hat{u} = \frac{1}{\|u\|} u$

$= \frac{1}{\sqrt{(0.6)^2 + (0.8)^2}} u$

$= u$

4.

a.

$v \cdot (-v)$
$= \begin{bmatrix} 3 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} -3 \\ -4 \end{bmatrix}$
$= (3)(-3) + (4)(-4)$
$= -25$

b.

$(v + w) \cdot (v - w)$
$= v \cdot v + w \cdot w$
$= \begin{bmatrix} 3 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 4 \end{bmatrix} + \begin{bmatrix} 8 \\ 6 \end{bmatrix} \cdot \begin{bmatrix} 8 \\ 6 \end{bmatrix}$
$= (3)(3) + (4)(4) + (8)(8) + (6)(6)$
$= 125$

5.

$u_1 = \frac{1}{\|v\|} v$

$= \frac{1}{\sqrt{(3)^2 + (4)^2}} v$

$= \frac{1}{5} \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

$= \left(\frac{3}{5}, \frac{4}{5}\right)$

$U_1 = \left(-\frac{4}{5}, \frac{3}{5}\right)$

7.

b.

$v = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}, \quad w = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$

$$\theta = \cos^{-1}\left(\frac{v \cdot w}{\|v\|\|w\|}\right)$$

$$= \cos^{-1}\left(\frac{(2)(2) + (2)(-1) + (-1)(2)}{\sqrt{(2)^2 + (2)^2 + (-1)^2}\sqrt{(2)^2 + (-1)^2 + (2)^2}}\right) = 90°$$

12.

$$v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad w = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

Set $v \cdot (w - cv) = 0$

$v \cdot w - c\, v \cdot v = 0$

$(1,1) \cdot (1,5) - c\,(1,1) \cdot (1,1) = 0$

$(1)(1) + (1)(5) - c(1)(1) - c(1)(1) = 0$

$6 - 2c = 0$

$c = 3$

Let $v = (v_1, v_2)$, $w = (w_1, w_2)$

Set $v \cdot (w - cv) = 0$

$v \cdot w - c\, v \cdot v = 0$

$c\, v \cdot v = v \cdot w$

$$c = \frac{v \cdot w}{v \cdot v}$$

16.

$$v = (\underbrace{1, 1, 1, \ldots, 1}_{9 \text{ times}})$$

$\|v\| = \sqrt{(1)^2 + (1)^2 + (1)^2 + (1)^2 + (1)^2 + (1)^2 + (1)^2 + (1)^2 + (1)^2}$

$= \sqrt{9}$

$= 3$

$$u = \frac{1}{\|v\|} v$$

$= \frac{1}{3} v$

$$= (\underbrace{\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}, \ldots \tfrac{1}{3}}_{9 \text{ times}})$$

Let $V = (1, -1, 1, -1, 1, -1, 1, -1, 0)$

$$\|V\| = \sqrt{\underbrace{(1)^2 + (-1)^2 \cdots + (-1)^2}_{8 \text{ times}} + (0)^2}$$

$= \sqrt{8} = 2\sqrt{2}$

$$U = \frac{1}{\|V\|} V$$

$= \frac{\sqrt{2}}{4} V$

$$= (\underbrace{\tfrac{\sqrt{2}}{4}, -\tfrac{\sqrt{2}}{4}, \tfrac{\sqrt{2}}{4}, -\tfrac{\sqrt{2}}{4}, \ldots -\tfrac{\sqrt{2}}{4}}_{8 \text{ times}}, 0)$$

17.

$$\cos\alpha = \frac{v \cdot i}{\|v\|\|i\|}$$

$$= \frac{(1,0,-1) \cdot (1,0,0)}{\sqrt{(1)^2 + (0)^2 + (-1)^2}\sqrt{(1)^2 + (0)^2 + (0)^2}}$$

$$= \frac{1}{\sqrt{2}}$$

18.

$\|v\|^2 + \|w\|^2 = \|v + w\|^2$

$\|(4,2)\|^2 + \|(-1,2)\|^2 = \|(3,4)\|^2$

$(4)^2 + (2)^2 + (-1)^2 + (2)^2 = (3)^2 + (4)^2$

$25 = 25$

21

$\|u + v\| \leq \|u\| + \|v\|$

$\|u + v\|^2 \leq (\|u\| + \|v\|)^2$

$(u + v) \cdot (u + v) \leq \|u\|^2 + 2\|u\|\|v\| + \|v\|^2$

$u \cdot u + 2u \cdot v + v \cdot v \leq u \cdot u + 2\|u\|\|v\| + v \cdot v$

$2u \cdot v \leq 2\|u\|\|v\|$

$u \cdot v \leq \|u\| \|v\|$,

which is the Cauchy-Schwarlz Inequality

22.

$|v \cdot w| \leq |v||w|$

$|v \cdot w|^2 \leq |v|^2|w|^2$

$(v_1w_1 + v_2w_2)^2 \leq (v_1^2 + v_2^2)(w_1^2 + v_2^2)$

$v_1^2w_1^2 + 2v_1w_1v_2w_2 + v_2^2w_2^2 \leq v_1^2w_1^2 + v_1^2w_2^2 + v_2^2w_1^2 + v_2^2w_2^2$

$2v_1w_1v_2w_2 \leq v_1^2w_2^2 + v_2^2w_1^2$

$0 \leq v_1^2w_2^2 + v_2^2w_1^2 - 2v_1w_1v_2w_2$

$0 \leq (v_1w_2 - v_2w_1)^2$

27.

$|v - w|^2 = (v - w) \cdot (v - w)$

$= v \cdot v - 2v \cdot w + w \cdot w$

$= |v|^2 - 2v \cdot w + |w|^2$

$= |v|^2 + |w|^2 - 2|v||w|\cos\theta$

$= 5^2 + 3^2$

$= 34$

Chapter 1.3: Matrices

- Example 1:

Given $u = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $v = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$, $w = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Their linear combinations in 3D space are:

$$c\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + d\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + e\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} c \\ d-c \\ e-d \end{bmatrix}$$

The above combination can be written as a matrix multiplied by a vector

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} c \\ d \\ e \end{bmatrix}}_{x} = \begin{bmatrix} c \\ d-c \\ e-d \end{bmatrix}$$

$$Ax = \begin{bmatrix} u & v & w \end{bmatrix} \begin{bmatrix} c \\ d \\ e \end{bmatrix} = cu + dv + ew$$

This is more than a simple rewrite. Instead of viewing it as the scalars c, d, and e multiplying the vectors, now the matrix is multiplying the scalars.
The matrix A acts on the vector x.
The result of Ax is a combination b of the columns of A.
Rewriting with new variables:

$$Ax = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = b$$

$$\boxed{A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \quad \text{"difference matrix"}}$$

The input vector is x and the output vector is b.
This specific matrix is called a "difference matrix" because its output consists of the differences of the input vector x.
Computing $Ax = b$ with the squares

$$Ax = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ 9 \end{bmatrix} = \begin{bmatrix} 1-0 \\ 4-1 \\ 9-4 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$$

- When multiplying a matrix and column vector, the output is another column vector, when the ith entry is the dot product of the ith row of the matrix and the input matrix.

Linear Equations

- Originally, we viewed the equation $Ax=b$ with b as the unknown.
- Now, we view b as known and x as unknown. That is, we want to solve for x.
  - Instead of: compute the linear combination $x_1u + x_2v + x_3w$ to find b.
  - Which combination of $u, v,$ and w produces a particular vector b.
- From the previous example:

$$\begin{aligned} x_1 &= b_1 \\ x_2 - x_1 &= b_2 \\ x_3 - x_2 &= b_3 \end{aligned} \Rightarrow \begin{aligned} x_1 &= b_1 \\ x_2 &= b_1 + b_2 \\ x_3 &= b_1 + b_2 + b_3 \end{aligned} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

- Most linear systems are not easy to solve. The above example was in a particular arrangement (it is lower triangular).
- We can say matrix A is invertible: From b, we can recover x.

The Inverse Matrix

- The "inverse matrix" of the aforementioned difference matrix is the sum matrix

$$\boxed{S = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}}. \text{ That is, } Ax=b \Rightarrow x = Sb \Rightarrow A^{-1} = S,\ S = A^{-1}$$

- For example, given $x = (1,2,3)$

$$Ax = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}_{b} \Rightarrow Sb = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}}_{S} \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}_{b} = \underbrace{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}_{x}$$

S and A are inverse matrices

Cyclic Differences

- Modifying the difference matrix slightly, we get the cyclic difference matrix C.

$$\boxed{C = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}} \Rightarrow Cx = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 - x_3 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix} = b$$

- This matrix is not triangular, so it is not easy to solve for x when given b.
  - There are either 0 or infinitely many solutions
  - Case $Cx = 0$

$$\begin{bmatrix} x_1 - x_3 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ is solved by } x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} c \\ c \\ c \end{bmatrix}$$

  - Case $Cx = (1,3,5)$

$$\begin{bmatrix} x_1 - x_3 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$$ has no solutions. Geometrically, this means no combination of $(1,-1,0)$, $(0,1,-1)$, and $(-1,0,1)$ will create $(1,3,5)$; they are linearly dependent and $(1,3,5)$ does not lie on the spanned plane.

Linear Independence and Dependence

- A set of vectors is said to be <u>linearly independent</u> is there exists no nontrivial linear combination of the vectors that equals the zero vector. If there exists nontrivial solutions, the vectors are said to be <u>linearly dependent</u>
- Linear independence implies that no vector in the set can be expressed as a linear combination of the others.
- A set of n linearly independent vectors in $\mathbb{R}^n$ will span $\mathbb{R}^n$.
  - For example, three linearly independent vectors in $\mathbb{R}^3$ will span $\mathbb{R}^3$.
  - If one vector can be expressed as a linear combination of the two others, the set can effectively "reduced" to two and other a plane in $\mathbb{R}^3$ can be spanned.
- For the <u>cyclic difference matrix</u>, since the vectors are linearly dependent $(w=-u-v)$, the vectors lie on one plane (coplanar).
  - This explains why a solution is not guaranteed for $Cx=b$
- Iff the columns of an n by n square matrix are <u>independent</u> (forms a spanning set of $\mathbb{R}^n$):
  - $Ax=0$ has 1 solution (trivial). <u>A is invertible</u>. $Ax=b$ has 1 solution
- Iff the columns are <u>dependent</u>
  - $Ax=0$ has multiple solutions. <u>A is singular</u>. $Ax=b$ may have solutions

Triangular Matrices

- A triangular matrix is a special case of a square matrix.
  - A lower triangular matrix has all entries above the main diagonal as zero
  - An upper triangular matrix has all entries below the main diagonal as zero

$$L=\begin{bmatrix} l_{1,1} & & & 0 \\ l_{2,1} & l_{2,2} & & \\ \vdots & & \ddots & \\ l_{n,1} & l_{n,2} & \cdots & l_{n,n} \end{bmatrix}, \quad U=\begin{bmatrix} u_{1,1} & u_{1,2} & \cdots & u_{1,n} \\ & u_{2,2} & & u_{2,n} \\ & & \ddots & \vdots \\ & & & u_{n,n} \end{bmatrix}$$

Problem Set 1.3

1.

$2s_1+3s_2+4s_3=b$

$b=2(1,1,1)+3(0,1,1)+4(0,0,1)$
$=(2,2,2)+(0,3,3)+(0,0,4)$
$=(2,5,9)$

$$\begin{bmatrix}1&0&0\\1&1&0\\1&1&1\end{bmatrix}\begin{bmatrix}2\\3\\4\end{bmatrix}=\begin{bmatrix}(1,0,0)\cdot(2,3,4)\\(1,1,0)\cdot(2,3,4)\\(1,1,1)\cdot(2,3,4)\end{bmatrix}=\begin{bmatrix}2\\5\\9\end{bmatrix}$$

2.

$$\begin{bmatrix}1&0&0\\1&1&0\\1&1&1\end{bmatrix}\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}=\begin{bmatrix}1\\1\\1\end{bmatrix} \qquad \begin{bmatrix}1&0&0\\1&1&0\\1&1&1\end{bmatrix}\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}=\begin{bmatrix}1\\4\\9\end{bmatrix}$$

$$\begin{bmatrix}y_1\\y_1+y_2\\y_1+y_2+y_3\end{bmatrix}=\begin{bmatrix}1\\1\\1\end{bmatrix} \qquad \begin{bmatrix}y_1\\y_1+y_2\\y_1+y_2+y_3\end{bmatrix}=\begin{bmatrix}1\\4\\9\end{bmatrix}$$

$$y=\begin{bmatrix}1\\0\\0\end{bmatrix} \qquad y=\begin{bmatrix}1\\3\\5\end{bmatrix}$$

S is invertible

3.

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_1 + y_2 \\ y_1 + y_2 + y_3 \end{bmatrix} = \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} B_1 \\ B_2 - y_1 \\ B_3 - y_1 - y_2 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} B_1 \\ B_2 - B_1 \\ B_3 - B_1 - B_2 + B_1 \end{bmatrix} = \begin{bmatrix} B_1 \\ B_2 - B_1 \\ B_3 - B_2 \end{bmatrix}$$

$$S^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} = A$$

The matrix is invertible $\Leftrightarrow$ The column vectors are linearly independent

4.

$$\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 + 4x_2 + 7x_3 \\ 2x_1 + 5x_2 + 8x_3 \\ 3x_1 + 6x_2 + 9x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ or } x = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$w_1 - 2w_2 + w_3 = 0$

These column vectors are linearly dependent and coplanar

$(a,b) \Rightarrow k(c,d)$

$(a,b) \Rightarrow (kc, kd)$

$(b,d) = (kd, d)$
$= d(k,1)$

$(a,c) = (kc, c)$
$= c(k,1)$

7.

Each row is a vector in 3D which is perpendicular to x because their dot product is 0.

8.

$$Ax = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 - x_1 \\ x_3 - x_2 \\ x_4 - x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 + b_1 \\ b_3 + b_2 + b_1 \\ b_4 + b_3 + b_2 + b_1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}}_{S}\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

9.

$$C = \begin{bmatrix} 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

$$Cx = \begin{bmatrix} 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 - x_4 \\ x_2 - x_1 \\ x_3 - x_2 \\ x_4 - x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x = \begin{bmatrix} c \\ c \\ c \\ c \end{bmatrix}$$

10.

$$\Delta z = \begin{bmatrix} -1 & -1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

$$= \begin{bmatrix} z_2 - z_1 \\ z_3 - z_2 \\ -z_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$z = \begin{bmatrix} -b_3 - b_2 - b_1 \\ -b_3 - b_2 \\ -b_3 \end{bmatrix} = \begin{bmatrix} -b_1 - b_2 - b_3 \\ -b_2 - b_3 \\ -b_3 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} -1 & -1 & -1 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{bmatrix}}_{\Delta^{-1}}\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

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$$Cx = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

$$\begin{bmatrix} x_2 \\ x_3 - x_1 \\ x_4 - x_2 \\ -x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -b_4 - b_2 \\ b_1 \\ -b_4 \\ b_3 + b_1 \end{bmatrix} = \begin{bmatrix} & -b_2 & & -b_4 \\ b_1 & & & \\ & & & -b_4 \\ b_1 & & b_3 & \end{bmatrix}$$

$$C^{-1} = \begin{bmatrix} 0 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 1 & 0 \end{bmatrix} B.$$