Chapter 2: Solving Linear Equations

Chapter 2.1: Vectors and Linear Equations

- The central problem of linear algebra is to solve a system of linear equations.
- A simple example:

$$\begin{cases} x - 2y = 1 : l_1 \\ 3x + 2y = 11 : l_2 \end{cases}$$

  $(x,y) = (3,1)$,
  that is $(3,1) \in l_1$
  and $(3,1) \in l_2$
  - See Fig 2.1.1

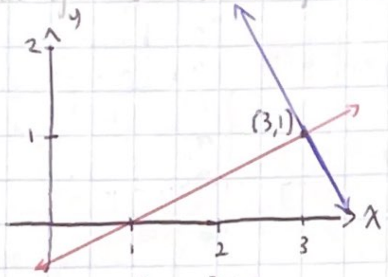
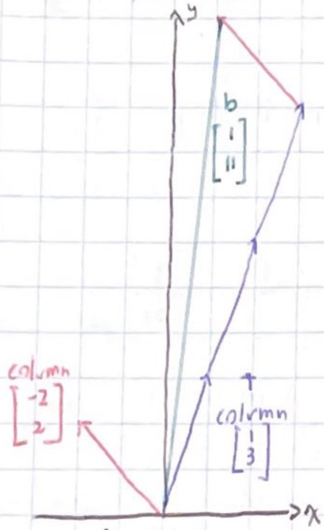


fig 2.1.1

- However, as a matrix, we see 2 different viewpoints
  - As rows: we see 2 simultaneous equations whose solution is one which satisfies both equations.
  - As columns: we instead see a vector equation (see fig 2.1.2)

$$\underbrace{x\begin{bmatrix}1\\3\end{bmatrix} + y\begin{bmatrix}-2\\2\end{bmatrix} = \begin{bmatrix}1\\11\end{bmatrix}}_{\text{"Find the linear combination"}} \Rightarrow \underbrace{\begin{bmatrix}1 & -2\\3 & 2\end{bmatrix}}_{\text{"A"}}\underbrace{\begin{bmatrix}x\\y\end{bmatrix}}_{\text{"x"}} = \underbrace{\begin{bmatrix}1\\11\end{bmatrix}}_{\text{"b"}} \Rightarrow 3\begin{bmatrix}1\\3\end{bmatrix} + 1\begin{bmatrix}-2\\2\end{bmatrix} = \begin{bmatrix}1\\11\end{bmatrix}$$



Sig 2.1.2

- We view the 2 equations as a linear combination problem. Find the coefficients x and y which generate the vector (1, 11) from the column vectors.
- The matrix generated by combining the column vectors is called the <u>coefficient matrix</u>.

$$A = \begin{bmatrix}1 & -2\\3 & 2\end{bmatrix}$$

- Finally we can view this system as a matrix equation:

$$Ax = b \Rightarrow \begin{bmatrix}1 & -2\\3 & 2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}1\\11\end{bmatrix}$$

$$\begin{bmatrix}1 & -2\\3 & 2\end{bmatrix}\begin{bmatrix}3\\1\end{bmatrix} = \begin{bmatrix}1\\11\end{bmatrix}$$

- This chapter addresses the use of matrices to solve n equations for n unknowns with concepts such as <u>matrix multiplication</u> and <u>matrix inversion</u>.
- Four steps needed to understand elimination using matrices.
  1. Elimination goes from A to triangular U by a sequence of matrix steps E_{ij}
  2. The inverse matrices E_{ij}^{-1} in reverse order bring U back to the original A
  3. In matrix language that reverse language is $A = LU$ = (lower triangle)(upper triangle)
  4. Elimination succeeds if A is invertible (It may need row exchanges)
- The most used algorithm (LU decomposition) uses these steps

Three Equations in Three Unknowns

- Given the equations

$$\begin{cases} x + 2y + 3z = 6 \\ 2x + 5y + 2z = 4 \\ 6x - 3y + z = 2 \end{cases}$$

- As a row, we see 3 planes which, in this case, intersect at a point.
- As a column we see the following vector equation:

$$x\begin{bmatrix}1\\2\\6\end{bmatrix} + y\begin{bmatrix}2\\5\\-3\end{bmatrix} + z\begin{bmatrix}3\\2\\1\end{bmatrix} = \begin{bmatrix}6\\4\\2\end{bmatrix} \Rightarrow \begin{bmatrix}1 & 2 & 3\\2 & 5 & 2\\6 & -3 & 1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}6\\4\\2\end{bmatrix}$$

- Since our column vectors are linearly independent, any vector b in $\mathbb{R}^3$ can be generated.
  - In this case $(x,y,z) = (0,0,2)$ produces $(6,4,2)$

The Matrix Form of the Equations

- With our above example our coefficient matrix is:

$$A = \begin{bmatrix}1 & 2 & 3\\2 & 5 & 2\\6 & -3 & 1\end{bmatrix} \Rightarrow Ax = \begin{bmatrix}1 & 2 & 3\\2 & 5 & 2\\6 & -3 & 1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}$$

- We can perform the multiplication in 2 ways
  - Rows:

$$Ax = \begin{bmatrix}(\text{row } 1)\cdot x\\(\text{row } 2)\cdot x\\(\text{row } 3)\cdot x\end{bmatrix}$$

  - Columns

$$Ax = x(\text{column } 1) + y(\text{column } 2) + z(\text{column } 3)$$

- Example with 2 matrices: A and I

$$Ax = \begin{bmatrix}1 & 0 & 0\\1 & 0 & 0\\1 & 0 & 0\end{bmatrix}\begin{bmatrix}4\\5\\6\end{bmatrix} = \begin{bmatrix}4\\4\\4\end{bmatrix}, \quad Ix = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}4\\5\\6\end{bmatrix} = \begin{bmatrix}4\\5\\6\end{bmatrix}$$

  - I is the identity matrix, a matrix with 1s on the "main diagonal"; multiplying a matrix by it leaves the matrix unchanged
    - $Ix = x$

Matrix Notation

- We use the notation:

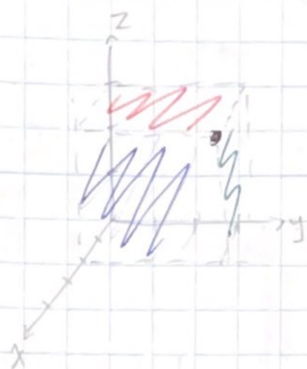
$$A = \begin{bmatrix}a_{11} & a_{12} & a_{13}\\a_{21} & a_{22} & a_{23}\\a_{31} & a_{32} & a_{33}\end{bmatrix}$$

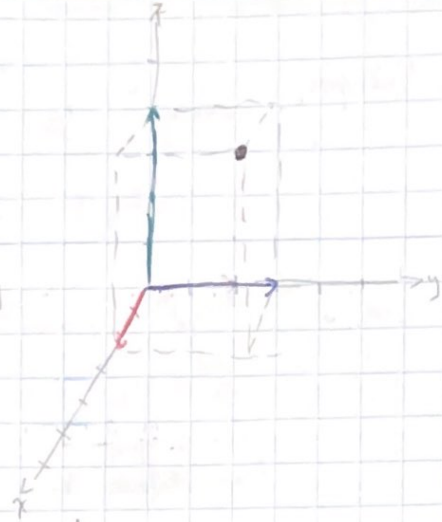
$a_{\text{row column}}$

- For an $m \times n$ matrix,
  - i (row) goes from 1 to m (inclusive)
  - j (column) goes from 1 to n (inclusive)

Problem set 2.1

1.





3.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 4 \end{bmatrix}$$

- All but the solutions have changed.

4.

$$\begin{cases} x+y+3z = 6 & ① \\ x-y+z = 4 & ② \end{cases}$$

let $t = z$

$①+②: 2x+4t = 10$

$2x = 10-4t$

$x = 5-2t$

sub $x = 5-2t$, $t = z$ into ②

$5-2t-y+t = 4$

$5-t-y = 4$

$y = 1-t$

$(x,y,z) = (5-2t, 1-t, t)$

sub $z = t = 2$

$(x,y,z) = (1,-1,2)$

sub $z = t = 0$

$(x,y,z) = (5,1,0)$

sub $z = t = 1$

$(x,y,z) = (3,0,1)$

7.

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 2 & 3 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \text{ or } \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 2 & 3 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ c \end{bmatrix} \Rightarrow c = 10$$

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$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 3 \\ 2 \end{bmatrix}$$

$\Downarrow$

$$\begin{cases} x+y+z+t = 3 \\ y+z+t = 3 \\ z+t = 3 \\ t = 2 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 0 \\ z = 1 \\ t = 2 \end{cases}$$

9.

a) $$\begin{bmatrix} 1 & 2 & 4 \\ -2 & 3 & 1 \\ -4 & 1 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 20 \\ 5 \\ 0 \end{bmatrix}$$

b) $$\begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 3 \\ 5 \end{bmatrix}$$

10.

$$Ax = 2\begin{bmatrix} 1 \\ -2 \\ -4 \end{bmatrix} + 2\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 20 \\ 5 \\ 0 \end{bmatrix}$$

11.

a) $\begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 14 \\ 22 \end{bmatrix}$ b) $\begin{bmatrix} 3 & 6 \\ 6 & 12 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ c) $\begin{bmatrix} 1 & 2 & 4 \\ 2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 9 \\ 7 \end{bmatrix}$

12.

a) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} z \\ y \\ x \end{bmatrix}$ b) $\begin{bmatrix} 2 & 1 & 3 \\ 1 & 2 & 3 \\ 3 & 3 & 6 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ c) $\begin{bmatrix} 2 & 1 \\ 1 & 2 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 6 \end{bmatrix}$

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a) An $m \times n$ matrix multiplies a vector with n components to produce a vector with m components

b) The planes from the m equations $Ax=b$ are in n-dimensional space.
The combination of columns of A is in m dimensional space

14.

$\begin{bmatrix} 2 & 3 & 1 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 8 \end{bmatrix}$

15.

a) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} = I$

b) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix} = P$

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a) $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ -x \end{bmatrix} = R$

b) $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ -y \end{bmatrix} = R^2$

17.

$\underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}}_{P}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} y \\ z \\ x \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} y \\ z \\ x \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$

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$E\begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \Rightarrow \underbrace{\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}}_{E}\begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 7 \end{bmatrix}$

19.

$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}}_{E}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ z+x \end{bmatrix}, \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}}_{E^{-1}}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ z-x \end{bmatrix}$

$E\begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 8 \end{bmatrix} \Rightarrow E^{-1}\begin{bmatrix} 3 \\ 4 \\ 8 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$

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$P_1 x = b \Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix}$

$P_2 x = b \Rightarrow \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ y \end{bmatrix}$

$P_1\begin{bmatrix} 5 \\ 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 5 \\ 7 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$

$P_2\begin{bmatrix} 5 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

21.

$\begin{bmatrix} \frac{\sqrt{2}}{2} & - \\ \frac{\sqrt{2}}{2} & - \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}$

and

$\begin{bmatrix} - & -\frac{\sqrt{2}}{2} \\ - & \frac{\sqrt{2}}{2} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}$

$R = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$

22.

$\begin{bmatrix} 1 & 4 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$

plane $\perp$ vector $\Rightarrow (1, 4, 5)$

1D

Chapter 2.2: The Idea of Elimination

- Elimination is a systematic way to solve linear equations!
- For example:

$$\begin{cases} x-2y=1 \\ 3x+2y=11 \end{cases} \Rightarrow \begin{cases} x-2y=1 \\ 8y=8 \end{cases}$$ gives us $y=1 \Rightarrow x=3$

- Elimination produces an upper triangular system, which is the goal.
  - That system is quickly solved via substitution from the bottom up (back substitution)
  - If instead we form a lower triangular system, substitution would start from the top down (forward substitution)
- In the example above, to eliminate x, we subtracted a multiple of equation 1 from equation 2.
  - How was the multiplier (3) found?
    The first pivot was 1 (the coefficient of x) and the second equation had a coefficient of 3 for the x. Therefore the multiplier is 3.
- If instead we had the equations:

$$\begin{cases} 4x-8y=4 \\ 3x+2y=11 \end{cases}$$

*the nth pivot is the entry of a_{nn} after performing elimination $n-1$ times

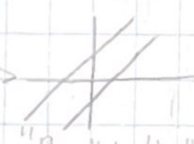
our pivot would be 4 and the multiplier $\ell=\frac{3}{4}$

- Pivot*: First nonzero entry in the row performing the elimination. $\ell_{ij}=\frac{\text{entry to eliminate in row } i}{\text{pivot in row } j}$
- Multiplier: (entry to eliminate) / (pivot)
- To solve n equations, we want n pivots

Breakdown of Elimination

- Sometimes, when performing elimination, a pivot may not exist (all zeros)
  - If we end with $0x=c$, $c\neq 0$, there are no solutions
  - On the other hand, $0x=0$ implies infinite solutions
- From our viewpoints (No Solutions)
  - Rows

$$\begin{cases} x-2y=1 \\ 3x-6y=11 \end{cases} \Rightarrow$$

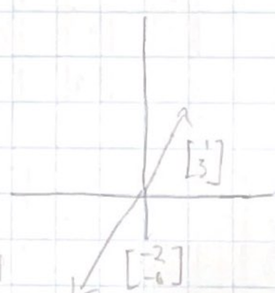


"Parallel lines"

  - Columns

$$x\begin{bmatrix}1\\3\end{bmatrix}+y\begin{bmatrix}-2\\-6\end{bmatrix}=\begin{bmatrix}1\\11\end{bmatrix}$$

Collinear columns cannot form certain vectors



- From our viewpoints (Infinite Solutions)
  - Rows

$$\begin{cases} x-2y=1 \\ 3x-6y=3 \end{cases}$$

"coincident lines"

  - Columns

$$x\begin{bmatrix}1\\3\end{bmatrix}+y\begin{bmatrix}-2\\-6\end{bmatrix}=\begin{bmatrix}1\\3\end{bmatrix}$$

(1,3) lies in the span of (1,3) and (-2,-6)

- Case 3: Temporary failure
  - Given

$$\begin{cases} 0x+2y=4 \\ 3x-2y=5 \end{cases} \Rightarrow$$ we can simply exchange rows to get an upper triangular matrix $$\Rightarrow \begin{cases} 3x-2y=5 \\ 2y=4 \end{cases}$$

Three Equations in Three Unknowns

- Gaussian Elimination
- Combine a coefficient matrix and right side to form augmented matrix

Problem Set 2.2

12.

$$\begin{cases} 2x+3y+z=8 \\ 4x+7y+5z=20 \\ -2y+2z=0 \end{cases} \xrightarrow[\text{into } R_2]{R_2-2R_1} \begin{cases} 2x+3y+z=8 \\ y+3z=4 \\ -2y+2z=0 \end{cases} \xrightarrow[\text{into } R_3]{R_3+2R_2} \begin{cases} 2x+3y+z=8 \\ y+3z=4 \\ 8z=8 \end{cases} \rightarrow \begin{cases} 2x+3y+z=8 \\ y+3z=4 \\ z=1 \end{cases} \rightarrow \begin{cases} x=2 \\ y=1 \\ z=1 \end{cases}$$

21.

$$\begin{cases} 2x+y=0 \\ x+2y+z=0 \\ y+2z+t=0 \\ z+2t=5 \end{cases} \rightarrow \begin{cases} 2x+2y+2z+t=0 \\ x+2y+2z+2t=5 \\ y+2z+t=0 \\ z+2t=5 \end{cases} \xrightarrow[\text{into } R_2]{R_2-\frac{1}{2}R_1} \begin{cases} 2x+2y+2z+t=0 \\ y+z+\frac{3}{2}t=5 \\ y+2z+t=0 \\ z+2t=5 \end{cases}$$

$$\xrightarrow[\text{into } R_3]{R_2-R_3} \begin{cases} 2x+2y+2z+t=0 \\ y+z+\frac{3}{2}t=5 \\ -z+\frac{1}{2}t=5 \\ z+2t=5 \end{cases} \xrightarrow[\text{into } R_4]{R_4+R_3} \begin{cases} 2x+2y+2z+t=0 \\ y+z+\frac{3}{2}t=5 \\ -z+\frac{1}{2}t=5 \\ \frac{5}{2}t=10 \end{cases} \rightarrow \begin{cases} x=-1 \\ y=2 \\ z=-3 \\ t=4 \end{cases}$$

$$\begin{cases} 2x-y=0 \\ -x+2y-z=0 \\ -y+2z-t=0 \\ -z+2t=5 \end{cases} \xrightarrow[\text{into } R_1]{R_1+R_3} \begin{cases} 2x-2y+2z+t=0 \\ -x+2y-z=0 \\ -y+2z-t=0 \\ -z+2t=5 \end{cases} \xrightarrow[\text{into } R_2]{R_2+R_4} \begin{cases} 2x-2y+2z+t=0 \\ -x+2y-2z+2t=5 \\ -y+2z-t=0 \\ -z+2t=5 \end{cases} \quad \begin{cases} x=-\frac{25}{7} \\ y=\frac{10}{7} \\ z=\frac{5}{7} \\ t=\frac{20}{7} \end{cases}$$

$$\xrightarrow[\text{into } R_2]{R_2+\frac{1}{2}R_1} \begin{cases} 2x-2y+2z+t=0 \\ y-z+\frac{5}{2}t=5 \\ -y+2z-t=0 \\ -z+2t=5 \end{cases} \xrightarrow[\text{into } R_3]{R_2+R_3} \begin{cases} 2x-2y+2z+t=0 \\ y-z+\frac{5}{2}t=5 \\ z+\frac{3}{2}t=5 \\ -z+2t=5 \end{cases} \xrightarrow[\text{into } R_4]{R_3+R_4} \begin{cases} 2x-2y+2z+t=0 \\ y-z+\frac{5}{2}t=5 \\ z+\frac{3}{2}t=5 \\ \frac{7}{2}t=10 \end{cases}$$

Chapter 2.3 Elimination Using Matrices

- Example System:

"$Ax=b$"

$$\begin{cases} 2x_1+4x_2-2x_3=2 \\ 4x_1+9x_2-3x_3=8 \\ -2x_1-3x_2+7x_3=10 \end{cases} \text{ is equivalent to } \begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}$$

  - An $n \times n$ square matrix multiplied by a vector in n-dimensional space
- Ax is a combination of the columns of A
- The ith component of Ax is the dot product of the ith row of A and x
  - The ith component of $Ax = \sum_{j=1}^{n} a_{ij} x_j$
- Example:

$$\begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3\cdot 2+4\cdot 1 \\ 5\cdot 2+6\cdot 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 16 \end{bmatrix}$$

The Matrix Form of One Elimination Step

- We want a way to represent an elimination step as a matrix
- From our example:

$$\begin{bmatrix} 2 & 4 & 2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}$$

- From our algorithm, we want subtract twice the first row from the second row.

$$b = \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix} \Rightarrow b = \begin{bmatrix} 2 \\ 4 \\ 10 \end{bmatrix}$$

- We can use an elimation matrix to represent this

$$E = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow E \begin{bmatrix} 2 & 4 & 2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & 2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 2 \\ 0 & 1 & -7 \\ -2 & -3 & 7 \end{bmatrix}$$

  - To generate an elimination/elementary matrix, start with the identity matrix and change one of its zeroes to $-\ell$.
- The purpose of E_{ij} is to produce a zero in the (i,j)th position
- Start with A. Apply E's to produce zeros to get a upper triangular U.
- The vector x remains the same - The solution does not change.
- $Ax = b \Rightarrow$ Apply E on both sides $\Rightarrow \underbrace{EA}_{\text{new matrix}}x = Eb$

Matrix Multiplication

- Matrix multiplication is:
  - Associative: $A(BC) = (AB)C$
  - Non-commutative: $AB \neq BA \;\forall A, B$
- We can keep our viewpoint, that for EA, E acts on the columns of A in the same way it would act on a column vector
  - $AB = A[b_1\ b_2\ b_3] = [Ab_1, Ab_2, Ab_3]$
- The (i,j)th entry of the product is the dot product of the ith row of the first matrix and the jth column of the second matrix
- The number of columns of A must equal the number of rows of B and the output has the number of rows of A and the number of columns of B
- Example

red • blue

$$\begin{bmatrix} \boxed{1 \quad 2 \quad 4} \\ 7 \quad -2 \quad 5 \\ 3 \quad 3 \quad -1 \end{bmatrix} \begin{bmatrix} 2 & -4 & -7 \\ 6 & -2 & 5 \\ 1 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 18 & -12 & -1 \\ 7 & -29 & -64 \\ 23 & -17 & -8 \end{bmatrix}$$

The Matrix P_{ij} for a Row Exchange

- To subtract row j from row i, we use E_{ij}
- To exchange/permute rows, we use another matrix, P_{ij}
- To get P_{ij}, we simply exchange the row of the identity matrix

$$P_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \Rightarrow P_{23}\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 3 & 3 \\ 2 & 2 & 2 \end{bmatrix}$$

- Multiplying by P_{ij} swaps the ith and jth components of a column vector, so it works on matrices as well.

The Augmented Matrix

- Elimination performs the same row operations to A and to b ($EAx = Eb$), so we simply include b as another column of the coefficient matrix
- Starting from

$$\begin{bmatrix} 2 & 4 & -2 & 2 \\ 4 & 9 & -3 & 8 \\ -2 & -3 & 7 & 10 \end{bmatrix} \Rightarrow E_{21}\begin{bmatrix} 2 & 4 & -2 & 2 \\ 4 & 9 & -3 & 8 \\ -2 & -3 & 7 & 10 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \overset{-l = -\frac{4}{2} = -2}{-2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 4 & -2 & 2 \\ 4 & 9 & -3 & 8 \\ -2 & -3 & 7 & 10 \end{bmatrix} = \begin{bmatrix} 2 & 4 & -2 & 2 \\ 0 & 1 & 1 & 4 \\ -2 & -3 & 7 & 10 \end{bmatrix}$$

  - $E_{32}E_{31}E_{21}A$ is a triangular matrix
- We can now express elimination as a sequence of prefix multiplications on A. We can also merge operations in a prefix manner.
  - Example

$$P_{32}E_{21} = \text{eliminate entry at (2,1) and exchange row 2 \& 3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -l & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ -l & 1 & 0 \end{bmatrix}$$

Problem Set 2.3

1.

a) $\begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 7 & 1 \end{bmatrix}$ c) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

2.

$$E_{32}E_{21}b = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 7 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 7 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -5 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -5 \\ -35 \end{bmatrix}$$

$$E_{21}E_{32}b = \begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -7 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -5 \\ 0 \end{bmatrix}$$

Row 3 feels no effect from row 2

3.

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 4 & 6 & 1 \\ -2 & 2 & 0 \end{bmatrix} \Rightarrow E_{21}A = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 4 & 6 & 1 \\ -2 & 2 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ -2 & 2 & 0 \end{bmatrix}$$

5. $\begin{bmatrix} - & - & - \\ - & - & - \\ - & - & 17 \end{bmatrix}$

$$E_{31}(E_{21}A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ -2 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 4 & 0 \end{bmatrix}$$

$$E_{32}(E_{31}E_{21}A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 4 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -2 \end{bmatrix}$$

$$E_{32}E_{31}E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 10 & -2 & 1 \end{bmatrix}$$

4.

$$E_{32}E_{31}E_{21}b = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 10 & -2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \\ 10 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \\ 10 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -5 \end{bmatrix}$$

6.

$$\begin{bmatrix} a & b & c \\ a & b & c \\ a & b & c \end{bmatrix} = A \Rightarrow E_{21}A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} a & b & c \\ a & b & c \\ a & b & c \end{bmatrix} = \begin{bmatrix} a & b & c \\ 0 & 0 & 0 \\ a & b & c \end{bmatrix} \Rightarrow E_{31}E_{21}A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} a & b & c \\ 0 & 0 & 0 \\ a & b & c \end{bmatrix}$$

7.

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -7 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 7 & 0 & 1 \end{bmatrix} = E^{-1} \quad EE^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -7 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 7 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

8.

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad M^* = \begin{bmatrix} 1 & 0 \\ -l & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ -al+c & -bl+d \end{bmatrix}$$

$$\det(M^*) = a(-bl+d) - b(-al+c)$$
$$= -abl + ad + abl - bc$$
$$= ad - bc$$

$$M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$$

10.

$$\begin{cases} X = 2Y \\ X + Y = 33 \end{cases} \Rightarrow \begin{matrix} X - 2Y = 0 \\ X + Y = 33 \end{matrix}$$

11

$$\left[\begin{array}{cc|c} 1 & 2 & 20 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{array}\right] \quad \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 1 & 1 & 33 \end{array}\right]$$

$$\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\left[\begin{array}{cc|c} 1 & -2 & 0 \\ 1 & 1 & 33 \end{array}\right] = \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 3 & 33 \end{array}\right] \Rightarrow \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 22 \\ 11 \end{bmatrix}$$

17.

$$\begin{bmatrix} 1 & 1 & 1 & 4 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 14 \end{bmatrix} = A \Rightarrow E_{12}A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 & 4 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 14 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & 1 & 3 & 4 \\ 1 & 3 & 9 & 14 \end{bmatrix}$$

$$E_{31}(E_{12}A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & 1 & 3 & 4 \\ 1 & 3 & 9 & 14 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & 1 & 3 & 4 \\ 0 & 2 & 8 & 10 \end{bmatrix}$$

$$E_{32}(E_{31}E_{12}A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & 1 & 3 & 4 \\ 0 & 2 & 8 & 10 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 2 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

19.

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad Q = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$PQ = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$QP = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$PP = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

26.

$$\begin{bmatrix} 1 & 4 \\ 2 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Apply $E_{21} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$ to both sides

$$\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 2 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} x+4y \\ -y \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -7 \\ 2 \end{bmatrix}$$

chapter 2.4: Rules for Matrix Operations

- Scaling a Matrix
  - Works the exact same as a vector; "distribute" the constant to all entries
- Adding Matrices
  - Added entry by entry, much like vectors
- Multiplying matrices
  - Refer to chapter 2.3: Matrix Multiplication

The Laws for Matrix Operations

- Addition
  - $A+B=B+A$ (commutative law)
  - $c(A+B)=cA+cB$ (distributive law)
  - $A+(B+C)+(A+B)+C=(A+C)+B$ (associative law)
- Multiplication
  - $C(A+B)=CA+CB$ (distributive prefix)
  - $(A+B)C=AC+BC$ (distributive postfix)
  - $A(BC)=(AB)C$ (associative law)

- $AI = IA$; All square matrices commute with I and cI.
- When $A = B = C$ = square matrix, then its exponents follow the same rules as we would expect.

$$A^p = \overbrace{AAA \ldots A}^{p \text{ factors}} \qquad (A^p)(A^q) = A^{p+q} \qquad (A^p)^q = A^{pq}$$

  - These rules still hold when p or q are zero or negative.

    A^{-1} is the inverse matrix, and $A^{-1}A = A^0 = I$. A will not always be invertible

Block Matrices and Block Multiplication

- Matrices can be cut into blocks (smaller matrices). This often happens naturally
- Example

$$\left[\begin{array}{cc|cc|cc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ \hline 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{array}\right] = \begin{bmatrix} I & I & I \\ I & I & I \end{bmatrix}$$

- If we have B with the same dimensions and the block sizes match, we can add A+B block at a time.
- For multiplication, block multiplication (blocks x blocks) is allowed when their shapes permit
  - If all submatrix products are defined, block multiplication is allowed.
  - Generally:

    Given an (mxp) matrix with q row partitions and s column partitions and a (p x n) matrix with s row partitions and r column partitions that are compatible with the partitions of A, C=AB can be formed blockwise, as an (mxn) matrix with q row partitions and r column partitions

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} B_{11} \\ B_{21} \end{bmatrix} = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} \\ A_{21}B_{11} + A_{22}B_{21} \end{bmatrix}$$

- Example (special case):
  - Let the blocks of A be its n columns. Let the blocks of B be its n rows

$$A = \begin{bmatrix} - & - & - & \ldots \\ - & - & - & \ldots \\ - & - & - & \ldots \end{bmatrix} \qquad AB = \begin{bmatrix} a_1 & a_2 & a_3 & \ldots & a_n \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} a_1b_1 + \ldots a_nb_n \end{bmatrix}$$

$$a_1 \quad a_2 \quad a_3 \ldots$$

$$\left[\begin{array}{c|c} 1 & 4 \\ 1 & 5 \end{array}\right]\left[\begin{array}{cc} 3 & 2 \\ \hline 1 & 0 \end{array}\right] = \begin{bmatrix} 1 \\ 1 \end{bmatrix}\begin{bmatrix} 3 & 2 \end{bmatrix} + \begin{bmatrix} 4 \\ 5 \end{bmatrix}\begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 2 \\ 3 & 2 \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ 5 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 2 \\ 8 & 2 \end{bmatrix}$$

- Example (Elimination by Blocks):
  - Suppose the first column of A contains 1, 3, 4.

$$A = \begin{bmatrix} 1 & \cdots \\ 3 & \cdots \\ 4 & \cdots \end{bmatrix},$$ then our elimination matrices would be $E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$

  - The block idea is to combine the matrices into one, which clear (2,1) and (3,1)

$$E = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$$

  - Suppose A has 4 blocks A, B, C, D we can perform elimination in blocks, using inverse matrices. The $D - CA^{-1}B$ we get as the last block is called Schur complement

$$\begin{bmatrix} I & 0 \\ -CA^{-1} & I \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A & B \\ 0 & D - CA^{-1}B \end{bmatrix}$$

Problem set 2.4

1.

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

a) $BA = \begin{bmatrix} 3 & 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 & 3 \end{bmatrix}$ b) $AB = \begin{bmatrix} 5 & 5 & 5 \\ 5 & 5 & 5 \\ 5 & 5 & 5 \end{bmatrix}$ c) $ABD = \begin{bmatrix} 5 & 5 & 5 \\ 5 & 5 & 5 \\ 5 & 5 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 15 \\ 15 \\ 15 \end{bmatrix}$

6.

$$(A+B)^2 = \begin{bmatrix} 2 & 2 \\ 3 & 0 \end{bmatrix}^2 = \begin{bmatrix} 2 & 2 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 10 & 4 \\ 6 & 6 \end{bmatrix}$$

$$A^2 + B^2 = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 2 \\ 3 & 0 \end{bmatrix}$$

$$(A+B)(A+B) = (A+B)A + (A+B)B$$
$$= AA + BA + AB + B^2$$

$$BA + AB = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 8 & 2 \\ 3 & 6 \end{bmatrix}$$

11.

c) $\begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} =$

12.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \Rightarrow b = c = 0$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\begin{bmatrix} 0 & a \\ 0 & c \end{bmatrix} = \begin{bmatrix} c & d \\ 0 & 0 \end{bmatrix} \quad d = a$$

17.

a) $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$ b) $A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$ c) $A = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ 2 & 1 & \frac{2}{3} \\ 3 & \frac{3}{2} & 1 \end{bmatrix}$

18

a) $A = \begin{bmatrix} c_1 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & c_3 \end{bmatrix}$ b) $A = \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ c) $A = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{21} & a_{22} & a_{32} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ d) $A = \begin{bmatrix} a & b & c \\ a & b & c \\ a & b & c \end{bmatrix}$

20.

$A = \begin{bmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $A^2 = \begin{bmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$A^3 = (A^2)A = \begin{bmatrix} 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 8 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

25.

$A = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = a_1\begin{bmatrix} 1 & 0 & 0 \end{bmatrix} + a_2\begin{bmatrix} 0 & 1 & 0 \end{bmatrix} + a_3\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} a_1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & a_2 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & a_3 \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}$

26.

$AB = \begin{bmatrix} 1 & 0 \\ 2 & 4 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 3 & 3 & 0 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}\begin{bmatrix} 3 & 3 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 4 \\ 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 0 \\ 6 & 6 & 0 \\ 6 & 6 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 4 & 8 & 4 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 0 \\ 10 & 14 & 4 \\ 7 & 8 & 1 \end{bmatrix}$

27.

$AB = \begin{bmatrix} x & x & x \\ 0 & x & x \\ 0 & 0 & x \end{bmatrix}\begin{bmatrix} x & x & x \\ 0 & x & x \\ 0 & 0 & x \end{bmatrix} = \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix}\begin{bmatrix} x & x & x \end{bmatrix} + \begin{bmatrix} x \\ x \\ 0 \end{bmatrix}\begin{bmatrix} 0 & x & x \end{bmatrix} + \begin{bmatrix} x \\ x \\ x \end{bmatrix}\begin{bmatrix} 0 & 0 & x \end{bmatrix}$

$= \begin{bmatrix} x^2 & x^2 & x^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & x^2 & x^2 \\ 0 & x^2 & x^2 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & x^2 \\ 0 & 0 & x^2 \\ 0 & 0 & x^2 \end{bmatrix} = \begin{bmatrix} x^2 & 2x^2 & 3x^2 \\ & x^2 & 2x^2 \\ & & x^2 \end{bmatrix}$

$AB = \begin{bmatrix} x & x & x \\ 0 & x & x \\ 0 & 0 & x \end{bmatrix}\begin{bmatrix} x & x & x \\ 0 & x & x \\ 0 & 0 & x \end{bmatrix} = \begin{bmatrix} x^2 & 2x^2 & 3x^2 \\ 0 & x^2 & 2x^2 \\ 0 & 0 & x^2 \end{bmatrix}$

28.

(1)

$$\begin{bmatrix} & A & \end{bmatrix}\begin{bmatrix} b_1 & b_2 & b_3 & b_4 \end{bmatrix} = \begin{bmatrix} Ab_1 & Ab_2 & Ab_3 & Ab_4 \end{bmatrix}$$

(2)

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}\begin{bmatrix} B \end{bmatrix} = a_1 B + a_2 B$$

32.

$$Ax_1 + Ax_2 + Ax_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$Ax = A\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$Ax_1 \quad Ax_2 \quad Ax_3$

29.

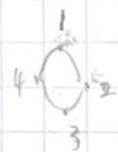
$$E = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix} \Rightarrow EA = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 & 0 \\ -2 & 0 & 1 \\ 8 & 5 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 3 \end{bmatrix}$$

$$D = \begin{bmatrix} 0 & 1 \\ 5 & 3 \end{bmatrix},\ C = \begin{bmatrix} -2 \\ 8 \end{bmatrix},\ b = [1, 0]$$

$$D - cb/a = \begin{bmatrix} 0 & 1 \\ 5 & 3 \end{bmatrix} - \begin{bmatrix} -2 \\ 8 \end{bmatrix}[1 \ 0]\frac{1}{2} = \begin{bmatrix} 0 & 1 \\ 5 & 3 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} -2 & 0 \\ 8 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 5 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ -4 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}$$

35.

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} = A$$



$$A^2 = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \\ 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \end{bmatrix}$$

Graphs

- Given the adjacency matrix of a graph (above), A, A^k is another matrix whose (i,j)th entry is the count of k-long paths from i to j
- Additionally, with words, the (i,j)th entry is the number of words of length $k+1$ starting with the ith letter and ending with the jth.

Chapter 2.5: Inverse matrices

- Suppose A is a square matrix, we look for an inverse matrix A^{-1} such that $AA^{-1} = I$. A^{-1} undoes A on a matrix: $A^{-1}Ax = x$:
  $Ax = b \Rightarrow A^{-1}Ax = A^{-1}b \Rightarrow x = A^{-1}b$
- Invertibility
  - An $n \times n$ (square) matrix A is called invertible if there exists an $n \times n$ matrix A^{-1} such that $AA^{-1} = A^{-1}A = I_n$, where I_n is the $n \times n$ identity matrix
  - A^{-1} does not always exist and A is called singular if it does not exist

- Rules for A^{-1}
  1. The inverse (A^{-1}) exists if and only if elimination produces n pivots (row exchanges allowed)
  2. The matrix A cannot have multiple different inverses. Suppose $BA=I$ and $AC=I$, then $B(AC)=(BA)C \Rightarrow BI=IC \Rightarrow B=C$. A left-inverse (prefix multiplication) and right-inverse (postfix multiplication) must be the same matrix
  3. If A is invertible, the one and only solution to $Ax=b$ is $x=A^{-1}b$
  4. Suppose there is a nonzero vector x such that $Ax=0$, then A cannot have an inverse. No matrix can bring 0 back to x. This implies the columns of A must be linearly independent.

     A is invertible $\Leftrightarrow$ $Ax=0$ can only have the zero solution $A^{-1}0=0$

  5. A matrix is invertible if and only if the determinant is non-zero.
     For a 2x2 matrix, $\det A = ad-bc$. For a 2x2 matrix:

$$A^{-1}=\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\det A}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

  6. A diagonal matrix has an inverse provided no diagonal entries are zero

$$A=\begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{bmatrix} \Rightarrow A^{-1}=\begin{bmatrix} \frac{1}{d_1} & & \\ & \ddots & \\ & & \frac{1}{d_n} \end{bmatrix}$$

The Inverse of a Product AB

- The sum of a product $A+B$ does not have an inverse, much in the same way that an "inverse" of $a+b=0$ does not make sense.
- The product AB has an inverse if and only if the two factors A and B are themselves invertible (and the same size)
  - Note that A^{-1} and B^{-1} are in the reverse order

$$(AB)^{-1}=B^{-1}A^{-1} \qquad ABB^{-1}A^{-1}=AIA^{-1}=IAA^{-1}=II=I$$

  - To see why the order is reversed, try multiplying $B^{-1}A^{-1}$ and AB (above)
- This rule generalizes to multiple multiplications

  $(ABC)^{-1}=C^{-1}B^{-1}A^{-1}$

- If we view matrices as transformations upon another, to undo them, we need to apply the inverse transformations in the inverse order.

- For an elimination matrix

$$E=\begin{bmatrix}1&0&0\\-5&1&0\\0&0&1\end{bmatrix} \Rightarrow E^{-1}=\begin{bmatrix}1&0&0\\5&1&0\\0&0&1\end{bmatrix} \Rightarrow EE^{-1}=E^{-1}E=I$$

eliminates the "-5" from E

- Example: Suppose:

$$F=\begin{bmatrix}1&0&0\\0&1&0\\0&-4&1\end{bmatrix} \Rightarrow F^{-1}=\begin{bmatrix}1&0&0\\0&1&0\\0&4&1\end{bmatrix}, \text{ then } FE=\begin{bmatrix}1&0&0\\-5&1&0\\20&-4&1\end{bmatrix}, \; E^{-1}F^{-1}=(FE)^{-1}=\begin{bmatrix}1&0&0\\5&1&0\\0&4&1\end{bmatrix}$$

FE contains 20 but its inverse $E^{-1}F^{-1}$ does not.
E subtracts 5 times row 1 from row 2, then F subtracts 4 times the new row 2.
In FE, row 3 is affected by row 1. In the reverse order $E^{-1}F^{-1}$, row 3 feels no effect from row 1.

Calculating A^{-1} by Gauss-Jordan Elimination

- A^{-1} might not be explicitly needed.
  - Elimination goes directly to x
- Elimination is also the way to calculate A^{-1}
- To do this, we use Gauss-Jordan Elimination
  1. We start with an augmented matrix $[A \; I]$

$$[K \; I]=\left[\begin{array}{ccc|ccc}2&-1&0&1&0&0\\-1&2&-1&0&1&0\\0&-1&2&0&0&1\end{array}\right]$$

  2. To goal is to get K to reduced row echelon form (get A to I) using elementary row operations E and P.

$$[K \; I]=\left[\begin{array}{ccc|ccc}2&-1&0&1&0&0\\-1&2&-1&0&1&0\\0&-1&2&0&0&1\end{array}\right] \quad \left(\text{Apply } E_{21}=\begin{bmatrix}1&0&0\\\frac{1}{2}&1&0\\0&0&1\end{bmatrix} \text{ to K and I}\right)$$

$$=\left[\begin{array}{ccc|ccc}2&-1&0&1&0&0\\0&\frac{3}{2}&-1&\frac{1}{2}&1&0\\0&-1&2&0&0&1\end{array}\right] \quad \left(\text{Apply } E_{32}=\begin{bmatrix}1&0&0\\0&1&0\\0&\frac{2}{3}&1\end{bmatrix} \text{ to K and I}\right)$$

We are at row echelon form, which is where Gaussian would stop for back substitution

$$=\left[\begin{array}{ccc|ccc}2&-1&0&1&0&0\\0&\frac{3}{2}&-1&\frac{1}{2}&1&0\\0&0&\frac{4}{3}&\frac{1}{3}&\frac{2}{3}&1\end{array}\right] \quad \left(\text{Apply } E_{23}=\begin{bmatrix}1&0&0\\0&1&\frac{3}{4}\\0&0&1\end{bmatrix} \text{ to K and I}\right)$$

$$=\left[\begin{array}{ccc|ccc}2&-1&0&1&0&0\\0&\frac{3}{2}&0&\frac{3}{4}&\frac{3}{2}&\frac{3}{4}\\0&0&\frac{4}{3}&\frac{1}{3}&\frac{2}{3}&1\end{array}\right] \quad \left(\text{Apply } E_{12}=\begin{bmatrix}1&\frac{2}{3}&0\\0&1&0\\0&0&1\end{bmatrix} \text{ to K and I}\right)$$

$$=\left[\begin{array}{ccc|ccc}2&0&0&\frac{3}{2}&1&\frac{1}{2}\\0&\frac{3}{2}&0&\frac{3}{4}&\frac{3}{2}&\frac{3}{4}\\0&0&\frac{4}{3}&\frac{1}{3}&\frac{2}{3}&1\end{array}\right] \xrightarrow{\text{divide each row by pivot}} \left[\begin{array}{ccc|ccc}1&0&0&\frac{3}{4}&\frac{1}{2}&\frac{3}{4}\\0&1&0&\frac{1}{2}&1&\frac{1}{2}\\0&0&1&\frac{1}{4}&\frac{1}{2}&\frac{3}{4}\end{array}\right]$$

$$\underbrace{\qquad\qquad}_{K^{-1}}$$

- If A cannot be reduced to I, then A is singular

- Through elementary row operations, we get A to I,
  $\underbrace{E_n \ldots E_3 E_2 E_1}_{A^{-1}} A = I$, $\underbrace{E_n \ldots E_3 E_2 E_1}_{A^{-1}} I = A^{-1} I = A^{-1}$
  - Through the augmented identity matrix, we keep track of the product of the row operation, which gets us the inverse matrix.
- Terminology
  - Symmetric: A symmetric matrix is a square matrix where $a_{ij} = a_{ji}$; it is symmetric across its main diagonal
  - Band Matrix: A matrix whose non-zero entries are confined to a diagonal band comprising the main diagonals and its surrounding diagonals
  - Tridiagonal: A band matrix whose band with is 3 (the diagonal consist of the main diagonal and a surrounding diagonal of each side)
  - Dense Matrix: A matrix with no zero entries. Generally, the inverse of a band matrix is a dense matrix.
- The product of the pivots of K is $(2)(\frac{3}{2})(\frac{4}{3}) = 4$, which is the determinant of K. The computation of K^{-1} involves division by the determinant, which is why it must be non zero.
- Gauss-Jordan is relatively expensive. We must solve n equations for its n columns
  - To solve $Ax = b$ without A^{-1}, we deal with one column b to find column x.
  - Instead we could simply perform elementary row operations on $[A\ b]$ and perform back substitution upon reaching U.
  - The cost to solve for the n columns of A^{-1} is only multiplied by 3 compared to Gaussian since we have 3 augmented columns (I) instead of one (b).
  - A^{-1} takes n^3 steps whereas solving for x directly is $\frac{n^3}{3}$

Singular vs Invertible

- A^{-1} exists if and only if A has a full set of n pivots (pivot test).
  - Elimination solves all equations $Ax_i = I_i$. The columns x_i go into A^{-1}. Then $AA^{-1} = I$ and A^{-1} is at least a right-inverse
  - Elimination is really a sequence of multiplications by E, P, and D^{-1}. D^{-1} divides by the pivots.
    $(D^{-1} \ldots E \ldots P) A = I \Rightarrow (D^{-1} \ldots E \ldots P) = A^{-1}$. (Left-inverse)
  - The right-inverse equals the left inverse.
- Reasoning about the pivot test (by contradiction)
  1. If A doesn't have n pivots, elimination leads to a zero row.
  2. Those elimination steps are taken by an invertible matrix M. Then a row of MA is 0.
  3. If $AC = I$ is possible, then $MAC = M$. The zero row of MA times C gives a zero row in M.
  4. An invertible matrix M can't have a zero row. Therefore A must have n pivots.

- If L is lower triangular with 1's on the diagonal, so is L^{-1}
- A triangular matrix is invertible if and only if no diagonal entries are zero

$$[L\ I] = \left[\begin{array}{ccc|ccc}1&0&0&1&0&0\\3&1&0&0&1&0\\4&5&1&0&0&1\end{array}\right] \Rightarrow E_{21}[L\ I] = \left[\begin{array}{ccc|ccc}1&0&0&1&0&0\\0&1&0&-3&1&0\\4&5&1&0&0&1\end{array}\right]$$

$$\Rightarrow E_{31}E_{21}[L\ I] = \left[\begin{array}{ccc|ccc}1&0&0&1&0&0\\0&1&0&-3&1&0\\0&5&1&-4&0&1\end{array}\right] \Rightarrow E_{32}E_{31}E_{21}[L\ I] = \begin{bmatrix}1&0&0\\0&1&0\\0&-5&1\end{bmatrix}\left[\begin{array}{ccc|ccc}1&0&0&1&0&0\\0&1&0&-3&1&0\\0&5&1&-4&0&1\end{array}\right]$$

$$E_{32}E_{31}E_{21}[L\ I] = \left[\begin{array}{ccc|ccc}1&0&0&1&0&0\\0&1&0&-3&1&0\\0&0&1&11&-5&1\end{array}\right] = [I\ L^{-1}] \Rightarrow L^{-1} = \begin{bmatrix}1&0&0\\-3&1&0\\11&-5&1\end{bmatrix}$$

Problem Set 2.5

1.

$$[A\ I] = \left[\begin{array}{cc|cc}0&3&1&0\\4&0&0&1\end{array}\right] \Rightarrow P_{12}[A\ I] = \left[\begin{array}{cc|cc}4&0&0&1\\0&3&1&0\end{array}\right] \Rightarrow D^{-1}P_{12}[A\ I] = \left[\begin{array}{cc|cc}1&0&0&\frac{1}{4}\\0&1&\frac{1}{3}&0\end{array}\right]$$

$$D^{-1}P_{12}[A\ I] = A^{-1}[A\ I] = [A^{-1}A\ \ A^{-1}I] = [I\ A^{-1}] = \left[\begin{array}{cc|cc}1&0&0&\frac{1}{4}\\0&1&\frac{1}{3}&0\end{array}\right] \Rightarrow A^{-1} = \begin{bmatrix}0&\frac{1}{4}\\\frac{1}{3}&0\end{bmatrix}$$

$$B = \begin{bmatrix}2&0\\4&2\end{bmatrix} \Rightarrow B^{-1} = \frac{1}{\det B}\begin{bmatrix}2&0\\-4&2\end{bmatrix} = \frac{1}{4}\begin{bmatrix}2&0\\-4&2\end{bmatrix} = \begin{bmatrix}\frac{1}{2}&0\\-1&\frac{1}{2}\end{bmatrix}$$

$$[C\ I] = \left[\begin{array}{cc|cc}3&4&1&0\\5&7&0&1\end{array}\right] \Rightarrow E_{21}[C\ I] = \begin{bmatrix}1&0\\-\frac{5}{3}&1\end{bmatrix}\left[\begin{array}{cc|cc}3&4&1&0\\5&7&0&1\end{array}\right] = \left[\begin{array}{cc|cc}3&4&1&0\\0&\frac{1}{3}&-\frac{5}{3}&1\end{array}\right]$$

$$E_{12}E_{21}[C\ I] = \begin{bmatrix}1&-12\\0&1\end{bmatrix}\left[\begin{array}{cc|cc}3&4&1&0\\0&\frac{1}{3}&-\frac{5}{3}&1\end{array}\right] = \left[\begin{array}{cc|cc}3&0&21&-12\\0&\frac{1}{3}&-\frac{5}{3}&1\end{array}\right] \Rightarrow D^{-1}E_{12}E_{21}[C\ I] = \begin{bmatrix}\frac{1}{3}&0\\0&3\end{bmatrix}\left[\begin{array}{cc|cc}3&0&21&-12\\0&\frac{1}{3}&-\frac{5}{3}&1\end{array}\right]$$

$$D^{-1}E_{12}E_{21}[C\ I] = C^{-1}[C\ I] = [C^{-1}C\ \ C^{-1}I] = [I\ C^{-1}] = \left[\begin{array}{cc|cc}1&0&7&-4\\0&1&-5&3\end{array}\right] \Rightarrow C^{-1} = \begin{bmatrix}7&-4\\-5&3\end{bmatrix}$$

2.

$$P = \begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix} \Rightarrow P^{-1}P = I \Rightarrow \overbrace{\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}}^{P^{-1}}\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix} = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

3.

$$\begin{bmatrix}10&20\\20&50\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix} \Rightarrow \left[\begin{array}{cc|c}10&20&1\\20&50&0\end{array}\right] \Rightarrow E_{21}\left[\begin{array}{cc|c}10&20&1\\20&50&0\end{array}\right] = \begin{bmatrix}1&0\\-2&1\end{bmatrix}\left[\begin{array}{cc|c}10&20&1\\20&50&0\end{array}\right] = \left[\begin{array}{cc|c}10&20&1\\0&10&-2\end{array}\right] \Rightarrow \begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}\frac{1}{2}\\-\frac{1}{5}\end{bmatrix}$$

4.

$$\left[\begin{array}{cc|cc}1&2&1&0\\3&6&0&1\end{array}\right] = [A\ I] \Rightarrow E_{21}[A\ I] = \begin{bmatrix}1&0\\-3&1\end{bmatrix}\left[\begin{array}{cc|cc}1&2&1&0\\3&6&0&1\end{array}\right] = \left[\begin{array}{cc|cc}1&2&1&0\\0&0&-3&1\end{array}\right]$$

6.

A is invertible $\Rightarrow$ for $AB = AC \Rightarrow A^{-1}AB = A^{-1}AC \Rightarrow IB = IC \Rightarrow B = C$

7.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{11}+a_{21} & a_{12}+a_{22} & a_{13}+a_{23} \end{bmatrix} \Rightarrow A \text{ does not produce n pivots because elimination needs to a zero row}$$

$\Rightarrow A$ is singular

a)

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{11}+a_{21} & a_{12}+a_{22} & a_{13}+a_{23} \end{bmatrix} \Rightarrow Ax = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow E_{31}Ax = E_{31}b$$

b)

$b = (a, b, a+b)$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{11}+a_{21} & a_{12}+a_{22} & a_{13}+a_{23} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \Rightarrow 0 = -1$$

8.

$$\begin{bmatrix} a_1 & b_1 & a_1+b_1 \\ a_2 & b_2 & a_2+b_2 \\ a_3 & b_3 & a_3+b_3 \end{bmatrix} = A \Rightarrow \text{set } Ax = 0 \Rightarrow \begin{bmatrix} a_1 & b_1 & a_1+b_1 \\ a_2 & b_2 & a_2+b_2 \\ a_3 & b_3 & a_3+b_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow x_1 \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} + x_2 \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} + x_3 \begin{bmatrix} a_1+b_1 \\ a_2+b_2 \\ a_3+b_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow x = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ is a non-zero solution $\Rightarrow A$ is not invertible

9.

A is invertible $\Rightarrow \underbrace{P_{12} A}_{\text{both invertible}} = B \Rightarrow B^{-1} = (P_{12}A)^{-1} \Rightarrow B^{-1} = A^{-1}P_{12}^{-1}$

10.

$$[B\ I] = \left[\begin{array}{cccc|cccc} 3 & 2 & 0 & 0 & 1 & 0 & 0 & 0 \\ 4 & 3 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 6 & 5 & 0 & 0 & 1 & 0 \\ 0 & 0 & 7 & 6 & 0 & 0 & 0 & 1 \end{array}\right] \Rightarrow E_{21}[B\ I] = \left[\begin{array}{cccc|cccc} 3 & 2 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & -\frac{4}{3} & 1 & 0 & 0 \\ 0 & 0 & 6 & 5 & 0 & 0 & 1 & 0 \\ 0 & 0 & 7 & 6 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\Rightarrow E_{43}E_{21}[B\ I] = \left[\begin{array}{cccc|cccc} 3 & 2 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & -\frac{4}{3} & 1 & 0 & 0 \\ 0 & 0 & 6 & 5 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{6} & 0 & 0 & -\frac{7}{6} & 1 \end{array}\right] \Rightarrow E_{12}E_{43}E_{21}[B\ I] = \left[\begin{array}{cccc|cccc} 3 & 0 & 0 & 0 & 9 & -6 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & -\frac{4}{3} & 1 & 0 & 0 \\ 0 & 0 & 6 & 5 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{6} & 0 & 0 & -\frac{7}{6} & 1 \end{array}\right]$$

$$\Rightarrow E_{34}E_{12}E_{43}E_{21}[B\ I] = \left[\begin{array}{cccc|cccc} 3 & 0 & 0 & 0 & 9 & -6 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & -\frac{4}{3} & 1 & 0 & 0 \\ 0 & 0 & 6 & 0 & 0 & 0 & 36 & -30 \\ 0 & 0 & 0 & \frac{1}{6} & 0 & 0 & -\frac{7}{6} & 1 \end{array}\right] \Rightarrow D^{-1}E_{34}E_{12}E_{43}E_{21}[B\ I] = \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 3 & -2 & 0 & 0 \\ 0 & 1 & 0 & 0 & -4 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 6 & -5 \\ 0 & 0 & 0 & 1 & 0 & 0 & -7 & 6 \end{array}\right]$$

$$B^{-1} = \begin{bmatrix} 3 & -2 & 0 & 0 \\ -4 & 3 & 0 & 0 \\ 0 & 0 & 6 & -5 \\ 0 & 0 & -7 & 6 \end{bmatrix}$$

11.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

12.

$C = AB \Rightarrow CB^{-1} = ABB^{-1} \Rightarrow CB^{-1} = AI = A \Rightarrow A^{-1} = BC^{-1}$

13.

$M = ABC \Rightarrow A^{-1}M = A^{-1}ABC \Rightarrow A^{-1}M = BC \Rightarrow A^{-1}MC^{-1} = BCC^{-1} \Rightarrow A^{-1}MC^{-1} = B \Rightarrow B^{-1} = CM^{-1}A$

14.

$$B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} A \Rightarrow B^{-1} = A^{-1} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^{-1} \Rightarrow B^{-1} = A^{-1} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

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$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} ad-bc & 0 \\ 0 & -bc+ad \end{bmatrix}$$

23.

$$[A\ I] = \left[\begin{array}{ccc|ccc} 2 & 1 & 0 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \Rightarrow E_{21}[A\ I] = \left[\begin{array}{ccc|ccc} 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & \frac{3}{2} & 1 & -\frac{1}{2} & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \quad A^{-1} = \begin{bmatrix} \frac{3}{4} & -\frac{1}{2} & \frac{1}{4} \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ \frac{1}{4} & -\frac{1}{2} & \frac{3}{4} \end{bmatrix}$$

$$\Rightarrow E_{32}E_{21}[A\ I] = \left[\begin{array}{ccc|ccc} 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & \frac{3}{2} & 1 & -\frac{1}{2} & 1 & 0 \\ 0 & 0 & \frac{4}{3} & \frac{1}{3} & -\frac{2}{3} & 1 \end{array}\right] \Rightarrow E_{23}E_{32}E_{21}[A\ I] = \left[\begin{array}{ccc|ccc} 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & \frac{3}{2} & 0 & -\frac{3}{4} & \frac{3}{2} & -\frac{3}{4} \\ 0 & 0 & \frac{4}{3} & \frac{1}{3} & -\frac{2}{3} & 1 \end{array}\right]$$

$$\Rightarrow E_{12}E_{23}E_{32}E_{21}[A\ I] = \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & \frac{3}{2} & -1 & \frac{1}{2} \\ 0 & \frac{3}{2} & 0 & -\frac{3}{4} & \frac{3}{2} & -\frac{3}{4} \\ 0 & 0 & \frac{4}{3} & \frac{1}{3} & -\frac{2}{3} & 1 \end{array}\right] \Rightarrow D^{-1}E_{12}E_{23}E_{32}E_{21}[A\ I] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{4} & -\frac{1}{2} & \frac{1}{4} \\ 0 & 1 & 0 & -\frac{1}{2} & 1 & -\frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{4} & -\frac{1}{2} & \frac{3}{4} \end{array}\right]$$

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a) True b) False $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ c) True

30.

$$A = \begin{bmatrix} 2 & c & c \\ c & c & c \\ 8 & 7 & c \end{bmatrix} \Rightarrow \det A = 2c^2 + 8c^2 + 7c^2 - 8c^2 - c^3 - 14c$$
$$= -c^3 + 9c^2 - 14c$$
$$= -c(c^2 - 9c + 14c)$$
$$= -c(c-2)(c-7) \Rightarrow c \neq 0, 2, 7$$

Chapter 2.6: Elimination = Factorization: A = LU

- Lower-upper (LU) factorization/decomposition factors a matrix as the product of a lower triangular and upper triangular matrix.
- Performing Gaussian elimination takes A to U. If we take the inverse of the product of the elementary row operations used in elimination, we get a lower triangular matrix L. Hence A = LU
- For example:

$$E_{21}A = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 6 & 8 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 5 \end{bmatrix} = U \text{ (from A to U)}$$

$$\underbrace{E_{21}^{-1}\overbrace{E_{21}A}^{U}}_{L} = E_{21}^{-1}U = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 6 & 8 \end{bmatrix} = A \text{ (from U to A)}$$

Hence:

$$A = LU \Rightarrow \begin{bmatrix} 2 & 1 \\ 6 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 5 \end{bmatrix}$$

- L will be the inverse of all elementary row operations during elimination

$$E_n \dots E_3 E_2 E_1 A = U \Rightarrow A = \underbrace{E_1^{-1} E_2^{-1} E_3^{-1} \cdots E_n^{-1}}_{L} U = \underbrace{(E_n \dots E_3 E_2 E_1)^{-1}}_{L} U$$

Explanations and Examples

- Every inverse matrix E^{-1} is lower triangular. Its off-diagonal entry is l_{ij} to offset the subtraction produced by $-l_{ij}$. The main diagonals of E and E^{-1} have 1's.
- Each multiplier l_{ij} goes directly into its ij position unchanged
- Example 1:

$$A = LU \Rightarrow \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & & \\ \frac{1}{2} & 1 & \\ 0 & \frac{2}{3} & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 0 & \frac{3}{2} & 1 \\ 0 & 0 & \frac{4}{3} \end{bmatrix}$$

- Example 2

$$A = LU \Rightarrow \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- When can we predict zeroes in L and U?
  - When a row of A starts with zeroes, so does that row of L
  - When a column of A starts with zeroes, so does that column of U.
  - If a row starts with zero, an elimination step is not needed. If a column starts with zeroes, they survive into U
- The pivot rows are the same rows as U. When computing the third row of, we subtract multiples of earlier rows of U (not the rows of A):

  Row 3 of U = (Row 3 of A) $- l_{31}$(Row 1 of U) $- l_{32}$(Row 2 of U)

  Rewrite the equation:

  (Row 3 of A) $= l_{31}$(Row 1 of U) $+ l_{32}$(Row 2 of U) + 1(Row 3 of U)

  $$= \begin{bmatrix} l_{31} & l_{32} & 1 \end{bmatrix} U$$

  Then (Row 3 of A) = (row 3 of L)U $\Rightarrow A = LU$
- The LU factorization is unsymmetrical because U has the pivots on the main diagonal whereas L has 1's. To fix this, divide U by a diagonal matrix D that contains the pivots

  Split U into $\begin{bmatrix} d_1 & & \\ & d_2 & \\ & & \ddots \\ & & & d_n \end{bmatrix} \begin{bmatrix} 1 & \frac{u_{12}}{d_1} & \frac{u_{13}}{d_1} & \cdot \\ & 1 & \frac{u_{23}}{d_2} & \cdot \\ & & \ddots & \cdot \\ & & & 1 \end{bmatrix}$

  Now we write A as the product of 3 matrices: **A = LDU**

$$\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 8 \\ 0 & 5 \end{bmatrix} \text{ further splits into } \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$$

One Square System = Two Triangular Systems

- The matrix L contains a record of gaussian elimination. It holds the multipliers for the pivot rows before subtracting them from lower rows
- We need L as soon as there is a right side b. L and U are determined entirely by the left side (the matrix A). On the right side we use U^{-1} and L^{-1}.
  1. Factor into L and U via elimination on A.
  2. Solve via forward substitution on b using L then back substitution for x using U).
     - First we apply forward elimination to the right side using the multipliers stored in L. This changes b to a new right side c. We are really solving $Lc = b$
     - Then we solve $Ux = c$ via back substitution.
     - The original system $Ax = b$ is factored into 2 triangular systems

  Solve $Lc = b$ and then Solve $Ux = c$

$Lc=b \Rightarrow c = L^{-1}b$

L is the inverse of the elementary row operations, so L^{-1} is applying the same operations to the right side.

- Example:

$$\underbrace{\begin{matrix} u+2v=5 \\ 4u+9v=21 \end{matrix}}_{\text{"}Ax=b\text{"}} \quad \text{becomes} \quad \underbrace{\begin{matrix} u+2v=5 \\ v=1 \end{matrix}}_{\text{"}Ux=c\text{"}} \quad U = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \quad L = E_{21}^{-1} = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}$$

$L^{-1} = E_{21}$

$Lc = b \Rightarrow$ the lower triangular system $\begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} c \end{bmatrix} = \begin{bmatrix} b \end{bmatrix} = \begin{bmatrix} 5 \\ 21 \end{bmatrix} \Rightarrow c = L^{-1}B = \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ 21 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$

$Ux = c \Rightarrow$ the upper triangular system $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \end{bmatrix} = \begin{bmatrix} c \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix} \Rightarrow x = U^{-1}c = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

The cost of Elimination

- Elimination on A requires about $\frac{1}{3}n^3$ multiplications and $\frac{1}{3}n^3$ subtractions
- To solve the right side (b to c to x), it takes n^2 multiplications and n^2 subtractions
- For a band matrix of width w, the cost of factoring (elimination) takes nw^2 and solving takes $2nw$ operations.

Problem Set 2.6

1.

$L = E_{21}^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} x = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$

3.

$$U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \Rightarrow Lc = b \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ 11 \end{bmatrix} \Rightarrow \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 2 \end{bmatrix}$$

$$\Rightarrow Ux = c \Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ & 1 & 2 \\ & & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 2 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \\ 1 \end{bmatrix}$$

5.

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 6 & 3 & 5 \end{bmatrix} \Rightarrow E_{31}A = \overbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}}^{E} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 6 & 3 & 5 \end{bmatrix} = \underbrace{\overbrace{\begin{bmatrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 0 & 0 & 5 \end{bmatrix}}^{REF}}_{U} \quad L = E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$$

$$A = LU \Rightarrow \begin{bmatrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 6 & 3 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 0 & 0 & 5 \end{bmatrix}$$

6.

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \Rightarrow L = E_{21}^{-1} E_{32}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

7.

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 2 & 2 \\ 3 & 4 & 5 \end{bmatrix} \Rightarrow E_{21}A = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 2 & 2 & 2 \\ 3 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 3 & 4 & 5 \end{bmatrix} \Rightarrow E_{31}E_{21}A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 3 & 4 & 5 \end{bmatrix}$$

$$\Rightarrow E_{31}E_{21}A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 4 & 2 \end{bmatrix} \Rightarrow E_{32}E_{31}E_{21}A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 4 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = U$$

$$L = E_{21}^{-1} E_{31}^{-1} E_{32}^{-1} = \begin{bmatrix} 1 & & \\ 2 & 1 & \\ 3 & 2 & 1 \end{bmatrix} \Rightarrow A = LU = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 2 & 2 \\ 3 & 4 & 5 \end{bmatrix}$$

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$$E_{32}E_{31}E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -c & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -b & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -c & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ -b & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ ac-b & -c & 1 \end{bmatrix}$$

11.

$$A = \begin{bmatrix} 2 & 4 & 8 \\ 0 & 3 & 9 \\ 0 & 0 & 7 \end{bmatrix} = U \Rightarrow L = I \Rightarrow D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 7 \end{bmatrix} \Rightarrow U = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

12.

$$A = \begin{bmatrix} 2 & 4 \\ 4 & 11 \end{bmatrix} \Rightarrow L = E_{21}^{-1} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \Rightarrow U = \begin{bmatrix} 2 & 4 \\ 0 & 3 \end{bmatrix}$$

$$A = LDU \Rightarrow \begin{bmatrix} 2 & 4 \\ 4 & 11 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \Rightarrow U \text{ and } L \text{ are symmetrical}$$

13.

$$A = \begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix}$$

15.

$$Lc=b \Rightarrow \begin{bmatrix}1&0\\4&1\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix}=\begin{bmatrix}2\\11\end{bmatrix} \Rightarrow \begin{bmatrix}c_1\\c_2\end{bmatrix}=\begin{bmatrix}2\\3\end{bmatrix} \qquad \begin{bmatrix}1&0\\4&1\end{bmatrix}\begin{bmatrix}2&4\\0&1\end{bmatrix}=\begin{bmatrix}2&4\\8&17\end{bmatrix}$$

$$Ux=c \Rightarrow \begin{bmatrix}2&4\\0&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}2\\3\end{bmatrix} \Rightarrow \begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}-5\\3\end{bmatrix} \qquad \begin{bmatrix}2&4\\8&17\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}2\\11\end{bmatrix}=$$

19.

$$A=\begin{bmatrix}1&1&0\\1&2&1\\0&1&2\end{bmatrix} \Rightarrow E_{21}A=\begin{bmatrix}1&0&0\\-1&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&1&0\\1&2&1\\0&1&2\end{bmatrix}=\begin{bmatrix}1&1&0\\0&1&1\\0&1&2\end{bmatrix} \Rightarrow E_{32}E_{21}A=\begin{bmatrix}1&0&0\\0&1&0\\0&-1&1\end{bmatrix}\begin{bmatrix}1&1&0\\0&1&1\\0&1&2\end{bmatrix}$$

$$E_{32}E_{21}A=\begin{bmatrix}1&1&0\\0&1&1\\0&0&1\end{bmatrix}=U \Rightarrow L=E_{21}^{-1}E_{32}^{-1}=\begin{bmatrix}1&0&0\\1&1&0\\0&1&1\end{bmatrix}=$$

$$A=LU \Rightarrow \begin{bmatrix}1&1&0\\1&2&1\\0&1&2\end{bmatrix}=\begin{bmatrix}1&0&0\\1&1&0\\0&1&1\end{bmatrix}\begin{bmatrix}1&1&0\\0&1&1\\0&0&1\end{bmatrix}$$

Chapter 2.7: Transposes and Permutations

- The transpose of A, denoted by A^T. The columns of A^T are the rows of A.
  - When A is an $m\times n$ matrix, its transpose is $n\times m$.

    If $A=\begin{bmatrix}1&2&3\\0&0&4\end{bmatrix}$ then $A^T=\begin{bmatrix}1&0\\2&0\\3&4\end{bmatrix}$

  - The matrix "flips over" its main diagonal. The entry in row i, column j of A^T comes from row j, column i of the original A.

    $(A^T)_{ij}=A_{ji}$

- The transpose of a lower triangular matrix is upper triangular.
- The transpose of A^T is A. That is $(A^T)^T=A$.
- Rules
  - Sum: $(A+B)^T=A^T+B^T$
  - Product: $(AB)^T=B^TA^T$.
  - Inverse: $(A^{-1})^T=(A^T)^{-1}$
- Proof of the product identity

  Start with $(Ax)^T=x^TA^T$, where x is a column vector.

  Ax combines the columns of A while x^TA^T combines the rows of A^T. This combination is the same for both cases

$$\begin{bmatrix}a_1&a_2&a_3\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=x_1a_1+x_2a_2+x_3a_3=\begin{bmatrix}x_1&x_2&x_3\end{bmatrix}\begin{bmatrix}a_1\\a_2\\a_3\end{bmatrix}=x_1a_1+x_2a_2+x_3a_3$$

  Generalizing to AB:

  $(AB)^T=\begin{bmatrix}Ab_1&Ab_2&\cdots\end{bmatrix}^T$ gives $\begin{bmatrix}b_1^TA^T\\ \vdots\end{bmatrix}$ which is B^TA^T.

- The reverse order rule extends to multiple factors

  If $A = LDU$ then $A^T = U^T D^T L^T$

- Then apply this product rule to $A^{-1}A = I$,
  $(A^{-1}A)^T = I^T \Rightarrow A^T(A^{-1})^T = I \Rightarrow$ A^T and $(A^{-1})^T$ are inverses
  - We can invert the transpose or transpose the inverse.
  - A^T is invertible if and only if A is invertible.

The meaning of Inner Products

- Instead of using dot notation, we can use matrices to notate dot products
- Given 2 vectors of size n (x and y):
  - $x^T y \Rightarrow (1 \times n)(n \times 1)$ = dot/inner product (scalar)
  - $x y^T \Rightarrow (n \times 1)(1 \times n)$ = outer product (matrix)
    - The outer product is a matrix whose (i,j)th entry is the product of the ith entry of the first vector and the jth entry of the second.
    - Given

$$u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix}, \quad v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

$$u \otimes v = uv^T = \begin{bmatrix} u_1v_1 & u_1v_2 & \cdots & u_1v_n \\ u_2v_1 & u_2v_2 & \cdots & u_2v_n \\ \vdots & \vdots & \ddots & \vdots \\ u_mv_1 & u_mv_2 & \cdots & u_mv_n \end{bmatrix}$$

    - The dot product is the trace, or sum of the main diagonal of the outer product.
    - The outer product is not commutative.

- Thus we may also define the transpose as the matrix that makes these two inner products equal for every x and y.

  $(Ax)^T y = x^T(A^T y) \Rightarrow$ Inner product of Ax and y = Inner product of x and $A^T y$

- Changing the difference matrix to a derivative $A = \frac{d}{dt}$, its transpose is $(\frac{dx}{dt}, y) = (x, -\frac{dy}{dt})$. Then the inner product changes to an integral

$$x^T y = (x,y) = \int_{-\infty}^{\infty} x(t)y(t)\,dt$$

  Transpose rule $(Ax)^T y = x^T(A^T y)$:

$$\int_{-\infty}^{\infty} \frac{dx}{dt} y(t)\,dt = \int_{-\infty}^{\infty} x(t)\left(-\frac{dy}{dt}\right)dt$$ shows A^T (integration by parts)

- The "transpose" of the derivative is

Symmetric Matrices

- A symmetric matrix is a matrix equal to its transpose: $A = A^T$ or $a_{ji} = a_{ij}$
- The inverse of a symmetric matrix is also symmetric, when it is invertible.
  $(A^{-1})^T = (A^T)^{-1} = A^{-1}$
- We produce symmetric matrices by multiplying any matrix R by R^T

Symmetric Products R^TR and RR^T and LDL^T

- choose any matrix R, probably rectangular. Then the product R^TR is automatically a square symmetric matrix.
  - The (i,j)th entry of R^TR is the dot product of the ith row of R^T (the ith column of R) with the jth column of R. The (j,i)th entry is the same dot product (jth column $\cdot$ ith column).
  - RR^T is also symmetric, although it is a different matrix.

$$R=\begin{bmatrix}-1 & 1 & 0\\ 0 & -1 & 1\end{bmatrix} \Rightarrow R^T=\begin{bmatrix}-1 & 0\\ 1 & -1\\ 0 & 1\end{bmatrix} \Rightarrow RR^T=\begin{bmatrix}2 & -1\\ -1 & 2\end{bmatrix} \text{ and } R^TR=\begin{bmatrix}1 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 1\end{bmatrix}$$

  - R^TR is n by n, while RR^T is m by m. Even if $m=n$, it is not very likely $R^TR=RR^T$
- Symmetric matrices in elimination:
  - $A=A^T$ makes elimination faster
  - The upper triangular U is likely not symmetrical. The symmetry is in the triple product $A=LDU$
  - The diagonal matrix D divides the pivots in U, so L and U both have 1's on their main diagonals
  - When A is symmetric, $A=LDU$ becomes $A=LDL^T$
  - If $A=A^T$ is factored into LDU with no row exchanges, then U is exactly L^T.

$$\begin{bmatrix}1 & 2\\ 2 & 7\end{bmatrix}=\begin{bmatrix}1 & 0\\ 2 & 1\end{bmatrix}\begin{bmatrix}1 & 2\\ 0 & 3\end{bmatrix}=\underbrace{\begin{bmatrix}1 & 0\\ 2 & 1\end{bmatrix}}_{L}\begin{bmatrix}1 & 0\\ 0 & 3\end{bmatrix}\underbrace{\begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix}}_{L^T}$$

  - Note the transpose of $LDL^T \Rightarrow (LDL^T)^T=(L^T)^TD^TL^T=LDL^T$.
  - This reduces the amount of operations from $\frac{n^3}{3}$ to $\frac{n^3}{6}$, and also reduces storage needed.

Permutation Matrices

- The transpose plays a special role for a permutation matrix. The matrix P has a single 1 in every row and column. Then P^T is also a permutation matrix – maybe the same or maybe different.
- Any product P_1P_2 is also a permutation matrix
- A permutation matrix P has the rows of I in any order.
- For 3x3 matrices we have:

$$I=\begin{bmatrix}1 & & \\ & 1 & \\ & & 1\end{bmatrix} \quad P_{21}=\begin{bmatrix} & 1 & \\ 1 & & \\ & & 1\end{bmatrix} \quad P_{31}=\begin{bmatrix} & & 1\\ & 1 & \\ 1 & & \end{bmatrix} \quad P_{32}=\begin{bmatrix}1 & & \\ & & 1\\ & 1 & \end{bmatrix}$$

$$P_{32}P_{21}=\begin{bmatrix} & 1 & \\ & & 1\\ 1 & & \end{bmatrix} \quad P_{21}P_{32}=\begin{bmatrix} & & 1\\ 1 & & \\ & 1 & \end{bmatrix}$$

- Every individual P is its own inverse.

- There are $n!$ permutation matrices of order n.
- P^{-1} is also a permutation matrix.
- P^{-1} is always the same as P^T
  - $PP^T = P^TP = I$

The $PA = LU$ Factorization with row Exchanges

- $A = LU$ is awesome, but doesn't always work i.e. when we need row exchanges,
  Then $A = (E^{-1} \ldots P^{-1} \ldots E^{-1} \ldots P^{-1} \ldots)U$
- Every exchange is carried out by a P_{ij} and inverted by that P_{ij}.
  We compress all those exchanges into a single Permutation matrix P
- Now the question is where to collect the P_{ij}'s.
  - We can either do all the exchanges before or after the elimination.
  - Exchanging before gives $PA = LU$
  - Exchanging after gives a permutation matrix P_1 in between ($A = L_1P_1U_1$)
  This is because the pivot rows are in a strange order after elimination, so P_1 puts U_1 in the correct position
- If $P = I$ (no exchanges needed), then $A = LU$

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 1 \\ 2 & 7 & 9 \end{bmatrix} \to PA = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 2 & 7 & 9 \end{bmatrix} \to \underset{l_{31} = 2}{\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 3 & 7 \end{bmatrix}} \to \underset{l_{32} = 3}{\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{bmatrix}}$$

Hence

$$PA = LU \Rightarrow \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 1 \\ 2 & 7 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{bmatrix}$$

Problem Set 2.7

1.

$$A = \begin{bmatrix} 1 & 0 \\ 9 & 3 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 1 & 9 \\ 0 & 3 \end{bmatrix} \to A^{-1} = \begin{bmatrix} 1 & -3 \\ 0 & \frac{1}{3} \end{bmatrix}$$

2.

$(AB)^T$

$$\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}$$

$$B^TA^T = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}$$

5.
a)

$$x^TAy = \begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 5 \end{bmatrix} = 5$$

7.

$A = A^T$, $B = B^T$

$(AB)^T = B^TA^T = BA$ False

a

$$\begin{bmatrix} & 1 & \\ 1 & & \\ & & 1 \end{bmatrix}\begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix} = \begin{bmatrix} & 1 & \\ & & 1 \\ 1 & & \end{bmatrix}$$

$$\begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix}\begin{bmatrix} & 1 & \\ 1 & & \\ & & 1 \end{bmatrix} = \begin{bmatrix} & & 1 \\ 1 & & \\ & 1 & \end{bmatrix}$$

11

$$A=\begin{bmatrix}0&0&6\\1&2&3\\0&4&5\end{bmatrix} \Rightarrow PA=\begin{bmatrix}&1&\\&&1\\1&&\end{bmatrix}\begin{bmatrix}0&0&6\\1&2&3\\0&4&5\end{bmatrix}=\begin{bmatrix}1&2&3\\0&4&5\\0&0&6\end{bmatrix}$$

$$P_1AP_2=\begin{bmatrix}1&&\\&&1\\&1&\end{bmatrix}\begin{bmatrix}0&0&6\\1&2&3\\0&4&5\end{bmatrix}\begin{bmatrix}&&1\\&1&\\1&&\end{bmatrix}=\begin{bmatrix}1&&\\&&1\\&1&\end{bmatrix}\begin{bmatrix}6&0&0\\3&2&1\\5&4&0\end{bmatrix}=\begin{bmatrix}6&0&0\\5&4&0\\3&2&1\end{bmatrix}$$

16.

a) $(A^2-B^2)^T=(A^2)^T-(B^2)^T=.$

b $[(A+B)(A-B)]^T=(A-B)^T(A+B)^T=(A^T-B^T)(A^T+B^T)=(A-B)(A+B)$

17.

a) $\begin{bmatrix}1&1\\1&1\end{bmatrix}$ b) $\begin{bmatrix}0&1\\1&0\end{bmatrix}$

27.

$$A=\begin{bmatrix}0&1&1\\1&0&1\\2&3&4\end{bmatrix} \Rightarrow P=\begin{bmatrix}&1&\\1&&\\&&1\end{bmatrix} \Rightarrow PA=\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}\begin{bmatrix}0&1&1\\1&0&1\\2&3&4\end{bmatrix}=\begin{bmatrix}1&0&1\\0&1&1\\2&3&4\end{bmatrix}$$

Get PA to U $\Rightarrow E_{31}PA=\begin{bmatrix}1&0&0\\0&1&0\\-2&0&1\end{bmatrix}\begin{bmatrix}1&0&1\\0&1&1\\2&3&4\end{bmatrix}=\begin{bmatrix}1&0&1\\0&1&1\\0&3&2\end{bmatrix}$

$$E_{32}E_{31}PA=\begin{bmatrix}1&0&1\\0&1&1\\0&-3&2\end{bmatrix}\begin{bmatrix}1&0&1\\0&1&1\\0&3&2\end{bmatrix}=\begin{bmatrix}1&0&1\\0&1&1\\0&0&-1\end{bmatrix}=U$$

$$L=(E_{32}E_{31})^{-1}=E_{31}^{-1}E_{32}^{-1}=\begin{bmatrix}1&0&0\\0&1&0\\2&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&3&1\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\2&3&1\end{bmatrix}$$

$$PA=LU\Rightarrow\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}\begin{bmatrix}0&1&1\\1&0&1\\2&3&4\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\2&3&1\end{bmatrix}\begin{bmatrix}1&0&1\\0&1&1\\0&0&-1\end{bmatrix}$$

$$A=\begin{bmatrix}1&2&0\\2&4&1\\1&1&1\end{bmatrix} \Rightarrow P=\begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix} \Rightarrow PA=\begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}\begin{bmatrix}1&2&0\\2&4&1\\1&1&1\end{bmatrix}=\begin{bmatrix}1&1&1\\1&2&0\\2&4&1\end{bmatrix}$$

"$R_1 \longleftrightarrow R_3$"

Get PA to U! $E_{21}PA=\begin{bmatrix}1&0&0\\-1&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&1&1\\1&2&0\\2&4&1\end{bmatrix}=\begin{bmatrix}1&1&1\\0&1&-1\\2&4&1\end{bmatrix}$

$$E_{31}E_{21}PA=\begin{bmatrix}1&0&0\\0&1&0\\-2&0&1\end{bmatrix}\begin{bmatrix}1&1&1\\0&1&-1\\2&4&1\end{bmatrix}=\begin{bmatrix}1&1&1\\0&1&-1\\0&2&-1\end{bmatrix} \Rightarrow E_{32}E_{31}E_{21}PA=\begin{bmatrix}1&0&0\\0&1&0\\0&-2&1\end{bmatrix}\begin{bmatrix}1&1&1\\0&1&-1\\0&2&-1\end{bmatrix}=\begin{bmatrix}1&1&1\\0&1&-1\\0&0&1\end{bmatrix}$$

$$L=(E_{32}E_{31}E_{21})^{-1}=E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}=\begin{bmatrix}1&0&0\\1&1&0\\2&2&1\end{bmatrix} \Rightarrow$$

$$PA=LU \Rightarrow \begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}\begin{bmatrix}1&2&0\\2&4&1\\1&1&1\end{bmatrix}=\begin{bmatrix}1&0&0\\1&1&0\\2&2&1\end{bmatrix}\begin{bmatrix}1&1&1\\0&1&-1\\0&0&1\end{bmatrix}$$

24.

$$A=\begin{bmatrix}0&1&2\\0&3&8\\2&1&1\end{bmatrix} \Rightarrow E_{12}A=\begin{bmatrix}1&0&0\\-3&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}0&1&2\\0&3&8\\2&1&1\end{bmatrix}=\begin{bmatrix}0&1&2\\0&0&2\\2&1&1\end{bmatrix} \quad P=\begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix} \Rightarrow P^{-1}=\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}$$

$$PE_{12}A=\begin{bmatrix}2&1&1\\0&1&2\\0&0&2\end{bmatrix} \Rightarrow L=E_{12}^{-1}=\begin{bmatrix}1&0&0\\3&1&0\\0&0&1\end{bmatrix} \Rightarrow PL^{-1}A=U \Rightarrow A=LP^{-1}A$$

$$A=\begin{bmatrix}1&0&0\\3&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}\begin{bmatrix}2&1&1\\0&1&2\\0&0&2\end{bmatrix}$$

$$\begin{bmatrix}x\\y\\z\end{bmatrix} \Rightarrow \begin{bmatrix}&&1\\1&1&\\&1&\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}z\\x\\y\end{bmatrix}$$

$\theta=\cos^{-1}($ $=128$

31

$$\begin{bmatrix}1&50\\40&1000\\2&50\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}x_1+50x_2\\40x_1+1000x_2\\2x_1+50x_2\end{bmatrix}=A$$

Sub $(x_1,x_2)=(1,1)$

$$A=\begin{bmatrix}51\\1040\\52\end{bmatrix} \Rightarrow A^Ty=\begin{bmatrix}51&1040&52\end{bmatrix}\begin{bmatrix}200\\3\\3000\end{bmatrix}$$

$$=194820$$