# Chapter 5: Determinants

## Chapter 5.1: The Properties of Determinants

- The determinant of a square matrix, denoted det(A), det A, or |A| is a single scalar value.
  - For 2 by 2 matrices:
    $$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$
  - For 3 by 3 matrices
    $$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei + bfg + cdh - ceg - bdi - afh$$
  - And so on...
- The determinant is zero if and only if the matrix is singular.
  When A is invertible, $\det(A^{-1}) = \frac{1}{\det(A)}$
  - For 2 by 2 matrices:
    $$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ has inverse } A^{-1} = \frac{1}{\det(A)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$
- The product of the pivots is the determinant.
  - For a 2 by 2 matrix:
    $$\det A = ad - bc = a(d - \tfrac{c}{a}b)$$
- The determinant of a matrix can be found in three ways
  1. Multiply the n pivots (times 1 or -1) (Pivot formula)
  2. Add up n! (times 1 or -1) ("big" formula)
  3. Combine n smaller determinants (times 1 or -1) (Cofactor formula)
  - The (times 1 or -1) comes from the following rule:
    The determinant changes signs when two rows (or two columns) are exchanged.
  - The identity matrix has determinant +1. Exchange the 2 rows and det P = -1.
    Half of all permutations are even (det P = 1) and half are odd (det P = -1)
    Starting from I, half of P's involve an even number of exchanges and half involve an odd number
    $$\det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1 \quad \text{and} \quad \det \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = -1$$
- Another rule is linearity:
  $\det(cA) = c^n \det(A)$ for an n by n matrix

- Applications of determinants:
  - Determinants give $A^{-1}$ and solutions to $Ax = b$ (this formula is called Cramer's Rule)
  - When the edges of a box are the rows of A, the volume is $|\det A|$
  - For n special numbers $\lambda$, called eigenvalues, the determinant of $A - \lambda I$ is zero.

The Properties of the Determinant

- Determinants have three basic properties

1. The determinant of the n by n identity matrix is one

$$\begin{vmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{vmatrix} = 1 \quad \text{and} \quad \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

2. The determinant changes signs when two rows are exchanged

$$\begin{vmatrix} c & d \\ a & b \end{vmatrix} = - \begin{vmatrix} a & b \\ c & d \end{vmatrix} \quad \text{(both sides equal } bc - ad\text{)}$$

- We can find the determinant of any permutation matrix by counting the number of row exchanges, $k$. Then $\det P = (-1)^k$

3. The determinant is a linear function of each row separately (all other rows remain fixed)
   - If the first row is multiplied by $t$, the determinant is also multiplied by $t$. If the first rows of 2 matrices are added, the determinants are added.
   - This rule only applies when the other rows remain the same

$$\begin{vmatrix} ta & tb \\ c & d \end{vmatrix} = t \begin{vmatrix} a & b \\ c & d \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} a + a' & b + b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$$

   - Combining multiplication and addition, we can get linear combinations in one row
   - This does not mean $\det(2I) = 2\det(I)$. To get $2I$, we multiply both rows by 2 and the factor 2 comes out both times

$$\begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 2^2 = 4 \quad \text{and} \quad \begin{vmatrix} t & 0 \\ 0 & t \end{vmatrix} = t^2$$

   - This is like area and volume. Expanding a rectangle by 2 increases the area by 4.

4. If two rows of A are equal, then $\det A = 0$

$$\begin{vmatrix} a & b \\ a & b \end{vmatrix} = 0$$

   - This follows from rule 2. Exchanging the two equal rows changes the sign of the determinant, yet the matrix has not changed. $\det A = -\det A$ implies $\det A = 0$

5. Subtracting a multiple of a row from another row leaves $\det A$ unchanged.

$$\begin{vmatrix} a & b \\ c - \ell a & d - \ell b \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

   - This follows from rule 3

$$\begin{vmatrix} a & b \\ c - \ell a & d - \ell b \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a & b \\ -\ell a & -\ell b \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + 0 = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

- Elimination steps don't change the determinant. Since permuting the rows only inverts the sign of the determinant, $\det A = \pm \det U$

6. A matrix with a row of zeroes has $\det A = 0$

$$\begin{vmatrix} 0 & 0 \\ c & d \end{vmatrix} = 0$$

- Add some other row in the matrix to the zero row. The determinant has not changed and there are now 2 equal rows so $\det A = 0$

7. If $A$ is triangular then $\det A = a_{11}a_{22}\cdots a_{nn}$ = product of diagonal entries

$$\begin{vmatrix} a & b \\ 0 & d \end{vmatrix} = ad \quad \text{and} \quad \begin{vmatrix} a & 0 \\ c & d \end{vmatrix} = ad$$

- For a proof, eliminate until rref is reached (this does not change the determinant), and factor out each pivot so that $\det A = a_{11}a_{22}a_{33}\cdots a_{nn}\det I$

8. If $A$ is singular then $\det A = 0$. If $A$ is invertible then $\det A \neq 0$
   - If $A$ is invertible then it must have $n$ pivots. If pivots are zero (singular), then its determinant is also zero

9. The determinant of $AB$ is $\det A$ times $\det B$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix}\begin{vmatrix} p & q \\ r & s \end{vmatrix} = \begin{vmatrix} ap+br & aq+bs \\ cp+dr & cq+ds \end{vmatrix}$$

- When $B = A^{-1}$, then from this rule we get $\det A^{-1} = \frac{1}{\det A}$
  $AA^{-1} = I$ so $\det(A)\det(A^{-1}) = \det I = 1$
- Proof: When $|B| \neq 0$, consider the ratio $D(A) = |AB|/|B|$.
  This ratio has properties 1, 2, and 3 and then $D(A)$ is the determinant.
  $|A| = |AB|/|B| \Rightarrow |AB| = |A||B|$

10. The transpose $A^T$ has the same determinant as $A$.
   - If $A$ is singular, so is $A^T$, so $\det A = \det A^T = 0$
   - If $A$ is invertible, it has the factorization $PA = LU$. Transposing both sides, $A^TP^T = U^TL^T \Rightarrow |A^T| = \frac{|U^T||L^T|}{|P^T|}$, $\det L = \det L^T = 1$ (both have 1's on diagonal). $\det U = \det U^T$ (same diagonal), $\det P = \det P^T$ (permutations have $P^TP = I$), so $L, U, P$ have the same determinants as $L^T, U^T, P^T$ so $\det A = \det A^T$.

- Every rule above also applies to columns, since $\det A = \det A^T$

Problem Set 5.1

1.

$\det A = \frac{1}{2}$

$\det(2A) = 2^4\det(A) = 16(\frac{1}{2}) = 4$

$\det(-A) = (-1)^4\det(A) = \frac{1}{2}$

$\det(A^2) = \det(AA) = \det(A)\det(A) = (\frac{1}{2})(\frac{1}{2}) = \frac{1}{4}$

$\det(A^{-1}) = \frac{1}{\det A} = 2$

2.

$\det(A) = -1$

$\det(\frac{1}{2}A) = \frac{1}{2}^3 \det A = \frac{1}{8}(-1) = -\frac{1}{8}$

$\det(-A) = (-1)^3\det A = 1$

$\det(A^2) = (\det A)(\det A) = 1$

$\det(A^{-1}) = -1$

3.

a) False

b) True

c) False

d) True

4.

$J_3 = \underbrace{P_{13}}_{\text{one exchange}} I$

$\therefore \det(J_3) = -1$

$J_4 = \underbrace{P_{14} P_{23}}_{\text{two exchanges}} I$

$\therefore \det(J_4) = 1$

5.

$\det(J_n) = (-1)^n, n \in \mathbb{N}, n \geq 3$

8.

a)

$|Q^T Q| = |I|$

$|Q^T||Q| = 1$

$|Q|^2 = 1$

$|Q| = \pm 1$

9.

$\det A = 1$

$\det B = 2$

$\det C = 0$

13.

$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

$\det A = 1$

19.

$\det U = 6$

$\det U^{-1} = \frac{1}{6}$

$\det U^2 = 36$

30.

$A^{-1} = \begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

## Chapter 5.2: Permutations and Cofactors

- A computer finds the determinant from the pivots. This section explores the two other formulas: the "big" formula and the cofactor formula
- Example

$$A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \quad \text{has } \det A = 5$$

we can compute the determinant in three ways:

1. The product of the pivots $2 \cdot \frac{3}{2} \cdot \frac{4}{3} \cdot \frac{5}{4} = 5$
2. The "big" formula has $4! = 24$ terms. Only 5 terms are nonzero

   $\det A = 16 - 4 - 4 - 4 + 1 = 5$

3. The numbers $2, -1, 0, 0$ in the first row multiply their cofactors $4, 3, 2, 1$ from the other rows. That gives $2 \cdot 4 - 1 \cdot 3 = 5$. Those cofactors are 3 by 3 determinants. Cofactors use the rows and column not used by the entry in the first row.

### The Pivot Formula

- Elimination leaves the pivots $d_1, \dots, d_n$ on the diagonal of the upper triangular $U$. If no row exchanges are involved, multiply those pivots to find the determinant.

$$\det A = (\det L)(\det U) = (1)(d_1 d_2 \cdots d_n)$$

- For the factorization $PA = LU$, the determinant of $P$ is either $+1$ or $-1$, so

$$(\det P)(\det A) = (\det L)(\det U) \Rightarrow \det A = \pm(d_1 d_2 \dots d_n)$$

- When $A$ has fewer than $n$ pivots, $\det A = 0$ and $A$ is singular.
- Example 1:

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \quad PA = \begin{bmatrix} 4 & 5 & 6 \\ 0 & 2 & 3 \\ 0 & 0 & 1 \end{bmatrix} \quad \det A = -(4)(2)(1) = -8$$

P has one exchange

• Example 2

$$\begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & & & & \\ -\frac{1}{2} & 1 & & & \\ & -\frac{2}{3} & 1 & & \\ & & \ddots & \ddots & \\ & & & -\frac{n-1}{n} & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & & & \\ & \frac{3}{2} & -1 & & \\ & & \frac{4}{3} & -1 & \\ & & & \ddots & \ddots \\ & & & & \frac{n+1}{n} \end{bmatrix}$$

$\det A = (\det L)(\det U) = (1)(2)(\frac{3}{2})(\frac{4}{3})\dots(\frac{n+1}{n}) = n+1$

◦ The first pivots depend only on the upper left corner of the original matrix A. This is the rule for all matrices without row exchanges.

• The first k pivots come from the k by k submatrix $A_k$ in the top corner of the original matrix A. The determinant of that corner submatrix is $d_1 d_2 \dots d_k$

$A_1 = [2] \Rightarrow \det A_1 = d_1 = 2$ $\quad A_2 = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ $\det A_2 = d_1 d_2 = 3$

$A_3 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \Rightarrow \det A_3 = d_1 d_2 d_3 = 4$

• Elimination deals with the corner matrix $A_k$ while starting on the whole matrix. We assume no row exchanges — then $A = LU$ and $A_k = L_k U_k$. Dividing one determinant by the previous gives us the latest pivot $d_k$:

The kth pivot is $d_k = \frac{d_1 d_2 \dots d_k}{d_1 d_2 \dots d_{k-1}} = \frac{\det A_k}{\det A_{k-1}}$

## The Big Formula for Determinants

• We want to derive a single explicit formula for the determinant from the entries $a_{ij}$.

• The formula has n! terms.

• For a 3 by 3 matrix

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei + bfg + cdh - ceg - bdi - afh$$

• Each term has one entry from each row and one from each column. The order of the permuted columns tells us the sign. For example: the column order 1, 3, 2 is negative, and 3, 1, 2 is positive.

◦ Down the main diagonal, the column order 1, 2, 3, 4 is always positive. That order is the identity permutation.

• Start with n=2. The goal is to reach ad-bc in a systematic way

$[a\ b] = [a\ 0] + [0\ b]$ and $[c\ d] = [c\ 0] + [0\ d]$

Now apply linearity: 0 b/c dependent rows

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & 0 \\ c & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & 0 \\ c & 0 \end{vmatrix} + \begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & 0 \end{vmatrix} + \begin{vmatrix} 0 & b \\ 0 & d \end{vmatrix}$$

$$\begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & 0 \end{vmatrix} = ad\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + bc\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = ad - bc$$

- Now try $n=3$; we again split into permutation matrices

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & & \\ & a_{22} & \\ & & a_{33} \end{vmatrix} + \begin{vmatrix} & a_{12} & \\ & & a_{23} \\ a_{31} & & \end{vmatrix} + \begin{vmatrix} & & a_{13} \\ a_{21} & & \\ & a_{32} & \end{vmatrix} + \begin{vmatrix} a_{11} & & \\ & & a_{23} \\ & a_{32} & \end{vmatrix} + \begin{vmatrix} & a_{12} & \\ a_{21} & & \\ & & a_{33} \end{vmatrix} + \begin{vmatrix} & & a_{13} \\ & a_{22} & \\ a_{31} & & \end{vmatrix}$$

  - We get 27 determinants, only 6 of which are non zero (3 positions per row: $3\times3\times3$)
  - There are $3! = 6$ ways to order the columns, so 6 determinants
  - The determinants are non zero only when the non zero terms come from different columns.
  - The 6 permutations of $(1,2,3)$ are:
    $(1,2,3),(2,3,1),(3,1,2),(1,3,2),(2,1,3),(3,2,1)$
  - The last 3 are odd permutations (1 exchange) and the other 3 are even, (0 or 2 exchanges). When the column sequence is $(\alpha,\beta,\omega)$, we have chosen entries $a_{1\alpha}a_{2\beta}a_{3\omega}$ and the column sequence comes with a plus or minus sign.

$$\det A = a_{11}a_{22}a_{33}\begin{vmatrix} 1 & & \\ & 1 & \\ & & 1 \end{vmatrix} + a_{12}a_{23}a_{31}\begin{vmatrix} & 1 & \\ & & 1 \\ 1 & & \end{vmatrix} + a_{13}a_{21}a_{32}\begin{vmatrix} & & 1 \\ 1 & & \\ & 1 & \end{vmatrix} +$$

$$a_{11}a_{23}a_{32}\begin{vmatrix} 1 & & \\ & & 1 \\ & 1 & \end{vmatrix} + a_{12}a_{21}a_{33}\begin{vmatrix} & 1 & \\ 1 & & \\ & & 1 \end{vmatrix} + a_{13}a_{22}a_{31}\begin{vmatrix} & & 1 \\ & 1 & \\ 1 & & \end{vmatrix}$$

- The above goes for any $n$ by $n$ matrix.
  - There are $n!$ permutations of the columns $(1,2,\dots,n)$. Taking $(\alpha,\beta,\dots,\omega)$ takes $a_{1\omega}, a_{2\beta}, \dots a_{n\omega}$. The determinant contains the product $a_{1\alpha}a_{2\beta}\dots a_{n\omega}$ times $+1$ or $-1$, which comes from the permutation matrix.

$$\det A = \text{Sum over all } n! \text{ column permutations } P = (\alpha,\beta,\dots,\omega)$$

$$= \sum (\det P) a_{1\alpha}a_{2\beta}\dots a_{n\omega}$$

## Determinant by Cofactors

- For a 3 by 3 matrix, linearity becomes clear if you factor $a_{11}, a_{12}$, or $a_{13}$ that comes from the first row.

$$\det A = a_{11}\underbrace{(a_{22}a_{33} - a_{23}a_{32})} + a_{12}\underbrace{(a_{23}a_{31} - a_{21}a_{33})}_{\text{cofactors}} + a_{13}\underbrace{(a_{21}a_{32} - a_{22}a_{31})}$$

- These cofactors are 2 by 2 determinants coming from submatrices in row 2 and 3
  - The first row contributes the factors $a_{11}, a_{12}, a_{13}$. The lower rows contribute the cofactors $C_{11}, C_{12}, C_{13}$.
  - The cofactor of $a_{11}$ is $C_{11} = a_{22}a_{33} - a_{23}a_{32}$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & & \\ & a_{22} & a_{23} \\ & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} & a_{12} & \\ a_{21} & & a_{23} \\ a_{31} & & a_{33} \end{vmatrix} + \begin{vmatrix} & & a_{13} \\ a_{21} & a_{22} & \\ a_{31} & a_{32} & \end{vmatrix}$$

  - We're still choosing one entry per row and column. Since each entry in the first row takes up a row and column, that leaves a 2 by 2 submatrix as the cofactor

- We need to watch signs. The 2 by 2 determinant that goes with $a_{12}$ looks like $a_{21}a_{33} - a_{23}a_{31}$, but in the cofactor $C_{12}$, its sign is reversed. Then $a_{12}C_{12}$ is the correct 3 by 3 determinant.
- The sign pattern for cofactors along the first row is plus-minus-plus-minus etc. You cross out row $i$ and column $j$ to get a submatrix $M_{ij}$ of size $(n-1)$ by $(n-1)$. Multiply the determinant of $M$ by $(-1)^{i+j}$ to get the cofactor
- The cofactors along row 1 are $C_{ij} = (-1)^{i+j} \det M_{ij}$
  The cofactor expansion is $\det A = a_{11}C_{11} + a_{12}C_{12} + \dots + a_{1n}C_{1n}$
- This is possible for any row, not just row 1. The entries $a_{ij}$ in that row have cofactors $C_{ij}$. These are determinants of order $n-1$ multiplied by $(-1)^{i+j}$. Since $a_{ij}$ accounts for row $i$ and column $j$, the submatrix $M_{ij}$ throws out row $i$ and column $j$.

$$A = \begin{vmatrix} \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ & & a_{43} & \end{vmatrix} \qquad \text{Signs } (-1)^{i+j} = \begin{vmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{vmatrix}$$

- The determinant is the dot product of any row $i$ of $A$ with its cofactors using other rows.

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \dots + a_{in}C_{in}$$

Each cofactor $C_{ij}$ (order $n-1$, without $i$ and column $j$)

$$\boxed{C_{ij} = (-1)^{i+j} \det M_{ij}}$$

- A determinant of order $n$ is a combination of determinants of order $n-1$. We can recursively break it down until we reach order 1 (determinant of 1 by 1 matrix $= a$)
- Example 6

$$\begin{vmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ & & -1 & 2 \end{vmatrix} = 2\begin{vmatrix} 2 & -1 & \\ -1 & 2 & -1 \\ & -1 & 2 \end{vmatrix} - (-1)\begin{vmatrix} -1 & -1 & \\ & 2 & -1 \\ & -1 & 2 \end{vmatrix} = 2\begin{vmatrix} 2 & -1 & \\ -1 & 2 & -1 \\ & -1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix}$$

## Problem Set 5.2

1.

$$\det A = (1)(1)(1) + (2)(2)(3) + (3)(3)(2) - (3)(1)(3) - (3)(2)(1) - (1)(2)(2)$$
$$= 1 + 12 + 18 - 9 - 6 - 4$$
$$= 12$$

3.

$$\det A = x\begin{vmatrix} 0 & x \\ 0 & x \end{vmatrix} + x\begin{vmatrix} 0 & x \\ 0 & x \end{vmatrix} + x\begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix}$$
$$= 0$$

5.

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

11.

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\quad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 0 & 0 \end{bmatrix}$

$C = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$ $\quad D = \begin{bmatrix} 0 & 42 & -35 \\ 0 & -21 & 14 \\ -3 & 6 & -3 \end{bmatrix}$

$$AC = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} d & -c \\ -b & a \end{bmatrix} = \begin{bmatrix} ad - b^2 & -ac + ab \\ cd - bd & -c^2 + ad \end{bmatrix}$$

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a)

$$E_n = 1\underbrace{\begin{vmatrix} 1 & 1 & & \\ 1 & 1 & 1 & \\ & 1 & \ddots & \ddots \\ & & \ddots & 1 \end{vmatrix}}_{n-1} - 1\underbrace{\begin{vmatrix} 1 & 1 & & \\ & 1 & 1 & \\ & 1 & \ddots & \ddots \\ & & \ddots & 1 \end{vmatrix}}_{n-1}$$

$$= \underbrace{\begin{vmatrix} 1 & 1 & & \\ 1 & 1 & 1 & \\ & 1 & \ddots & \ddots \\ & & \ddots & 1 \end{vmatrix}}_{n-1} - \underbrace{\begin{vmatrix} 1 & 1 & & \\ 1 & 1 & 1 & \\ & 1 & \ddots & \ddots \\ & & \ddots & 1 \end{vmatrix}}_{n-2} + \underbrace{\begin{vmatrix} & 1 & & \\ & 1 & 1 & \\ & 1 & \ddots & \ddots \\ & & \ddots & \end{vmatrix}}_{0}$$

$$= E_{n-1} - E_{n-2}$$

28.

$$D = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$$

$$= (1)(5)(9) + (2)(6)(7) + (3)(4)(8) - (3)(5)(7) - (2)(4)(9) - (1)(6)(8)$$
$$= 0$$

## Chapter 5.3 : Cramer's Rule, Inverses, and Volumes

- Cramer's Rule solves $Ax = b$. A neat idea gives the first column $x_1$. Replacing the first column of $I$ by $x$ gives a matrix with determinant $x_1$. When you multiply by $A$, the first column becomes $Ax$ which is $b$. The other columns are copied from $A$.

Key Idea
$$\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix}\begin{bmatrix} x_1 & 0 & 0 \\ x_2 & 1 & 0 \\ x_3 & 0 & 1 \end{bmatrix} = \begin{bmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{bmatrix} = B_1$$

Take determinants of the three matrices
$$(\det A)(x_1) = \det B_1 \quad \text{or} \quad x_1 = \frac{\det B_1}{\det A}$$

- We follow the same principle to find all components of $x$
$$x_1 = \frac{\det(A_1)}{\det(A)}, \quad x_2 = \frac{\det(A_2)}{\det(A)}, \quad \dots, \quad x_n = \frac{\det(A_n)}{\det(A)}$$
Where $A_i$ denotes the matrix created by replacing the $i$th column of $A$ with $b$

- Example
$$\begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} \Rightarrow x_1 = \frac{\det\begin{bmatrix} 2 & 4 \\ 4 & 6 \end{bmatrix}}{\det\begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}} = \frac{-4}{-2} = 2, \quad x_2 = \frac{\det\begin{bmatrix} 3 & 2 \\ 5 & 4 \end{bmatrix}}{\det\begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}} = \frac{2}{-2} = -1$$

- Cramer's Rule requires the computation of $n+1$ determinants, each of which is $n!$ terms each $(n+1)!$ terms!! Inefficient but still an explicit formula

- Example 2
  - Finding the inverse matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$ can be split into $$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$ and $$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

We need 5 determinants:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \text{ and } \begin{vmatrix} 1 & b \\ 0 & d \end{vmatrix}, \begin{vmatrix} a & 1 \\ c & 0 \end{vmatrix}, \begin{vmatrix} 0 & b \\ 1 & d \end{vmatrix}, \begin{vmatrix} a & 0 \\ c & 1 \end{vmatrix}$$

The last four are $d, -c, -b, a$ (They are the cofactors)

$$x_1 = \frac{d}{|A|}, \; x_2 = -\frac{c}{|A|}, \; y_1 = -\frac{b}{|A|}, \; y_2 = \frac{a}{|A|} \quad \text{then } A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

- When the right side is a column of the identity matrix, the determinant of each $A_i$ in Cramer's Rule is a cofactor.
  - For the first column of a 3 by 3 matrix

$$\begin{vmatrix} 1 & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{vmatrix} = C_{11} \quad \begin{vmatrix} a_{11} & 1 & a_{13} \\ a_{21} & 0 & a_{23} \\ a_{31} & 0 & a_{33} \end{vmatrix} = C_{12} \quad \begin{vmatrix} a_{11} & a_{12} & 1 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & 0 \end{vmatrix} = C_{13}$$

* These 3 determinants solve the first column of A (i.e. vertically), while the identity matrix column moves horizontally)

  - These cofactors go in their transpose (j, i) position.* So we transpose the cofactor matrix (this is called the adjugate matrix of A, adj(A)), and we divide by the determinant

- Formula for $A^{-1}$:

$$(A^{-1})_{ij} = \frac{C_{ji}}{\det A} \quad \text{and} \quad A^{-1} = \frac{C^T}{\det A} = \frac{\operatorname{adj}(A)}{\det(A)},$$ where adj denotes the adjugate.

- In solving $AA^{-1} = I$, the columns of I lead to the columns of $A^{-1}$. Then Cramer's Rule using b = columns of I gives the formula for $A^{-1}$.
- Direct Proof: Multiply A by $C^T$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} = \begin{bmatrix} \det A & 0 & 0 \\ 0 & \det A & 0 \\ 0 & 0 & \det A \end{bmatrix}$$

  - The reason we get zeros for expansions like: $a_{21}C_{11} + a_{22}C_{12} + a_{23}C_{13} = 0$
  - This is the cofactor rule for a new matrix $A^*$ with the second row of A is copied into its first row, and so $\det A^* = 0$ b/c there are two identical rows.

$$AC^T = (\det A)I$$
$$A^{-1} = \frac{C^T}{\det A}$$

- Example:

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$ has inverse $$A^{-1} = \frac{C^T}{1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

"sum matrix"

# Area of a triangle

- Determinants allow us to compute the area of a triangle.
- The area of a triangle with corners $(x_1, y_1)$, $(x_2, y_2)$, and $(x_3, y_3)$ have the area

$$\text{Area} = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \qquad \text{Area} = \frac{1}{2}\begin{vmatrix} x_1 & y_1 \\ x_3 & y_2 \end{vmatrix} \quad \text{when } (x_3, y_3) = (0,0)$$

- The 3 by 3 determinant can be broken down by cofactors

$$\text{Area} = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{1}{2}x_1\begin{vmatrix} y_2 & 1 \\ y_3 & 1 \end{vmatrix} - \frac{1}{2}y_1\begin{vmatrix} x_2 & 1 \\ x_3 & 1 \end{vmatrix} + \frac{1}{2}\begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix}$$

$$= \frac{1}{2}x_1y_2 - \frac{1}{2}x_1y_3 - \frac{1}{2}x_2y_1 + \frac{1}{2}x_3y_1 + \frac{1}{2}x_2y_3 - \frac{1}{2}x_3y_2$$

$$= \frac{1}{2}(x_1y_2 - x_2y_1) + \frac{1}{2}(x_2y_3 - x_3y_2) + \frac{1}{2}(x_3y_1 - x_1y_3)$$

- The 3 by 3 determinant is split into two 2 by 2 determinants just as triangles can be split into 3 triangles with a point at $(0,0)$ (see fig 5.3.1)

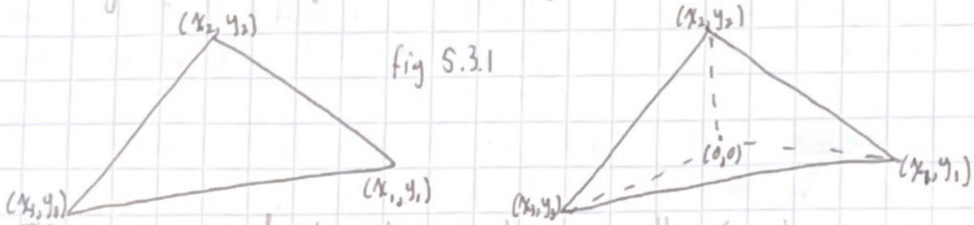



fig 5.3.1

- If $(0,0)$ is outside the triangle, two of the special areas can be negative — but the sum is still correct. The real problem is to explain the special area $\frac{1}{2}(x_1y_2 - x_2y_1)$
- If $\frac{1}{2}(x_1y_2 - x_2y_1)$ is the area of a triangle, then $x_1y_2 - x_2y_1$ is the area of a parallelogram
- Proof that a parallelogram starting from $(0,0)$ has are = 2 by 2 determinant
  - The area has the same properties as the determinant
    1. When $A = I$, the parallelogram is the unit square, $\det I = 1$
    2. When the rows are exchanged, the determinant reverses sign. The absolute value (the positive area) remains the same.
    3. If row 1 is multiplied by $t$, the area is also multiplied by $t$. Suppose a new row $(x_1', y_1')$ is added to $(x_1, y_1)$. The area is then added

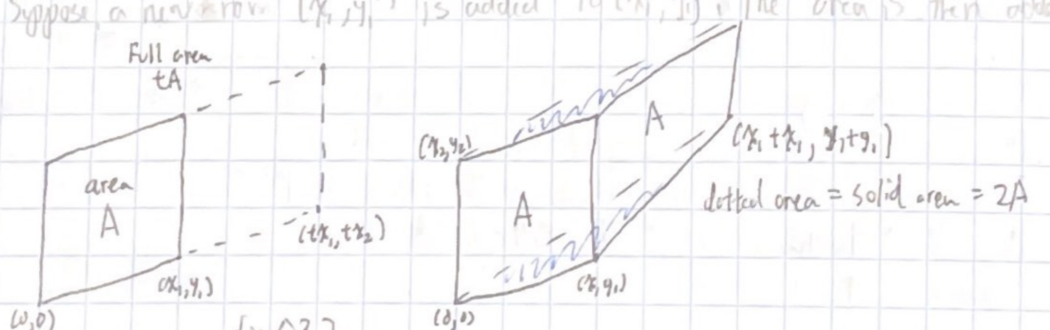
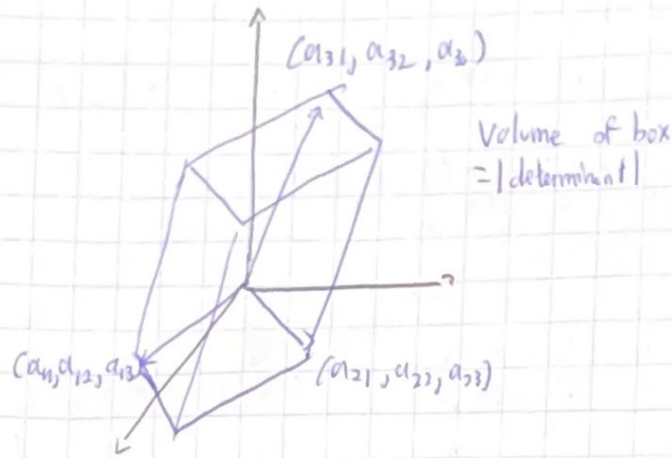



fig 5.3.2

- The $n$ edges going out from the origin are given by the rows of an $n$ by $n$ square matrix.

- The volume of that parallelepiped is the absolute value of $\det A$.
- The 3 rules for determinants are also obeyed by volumes




- Example
  - Suppose a rectangular box has side lengths $r$, $s$, and $t$. The diagonal matrix with entries $r, s, t$ produces those 3 sides. Then $\det A = rst$
- Example
  - In calculus, to integrate over a circle, we might use polar coordinates. $x = r\cos\theta$, $y = r\sin\theta$. The area of a "polar box" is a determinant $J$ times $dr\,d\theta$

$$J = \begin{vmatrix} \partial x/\partial r & \partial x/\partial\theta \\ \partial y/\partial r & \partial y/\partial\theta \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

The determinant is the $r$ in the differential $dA = r\,dr\,d\theta$.

## The Cross Product

- In three dimensions, a cross product between 2 vectors is defined as

$$a \times b = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2b_3 - a_3b_2)i + (a_3b_1 - a_1b_3)j + (a_1b_2 - b_1a_2)k$$

This vector is perpendicular to both $u$ and $v$. The cross product of $b \times a = -(a \times b)$
  - The above formula is a very useful mnemonic, but is not particularly legal, since $a$ and $b$ are scalars and $i$, $j$, and $k$ are vectors
- Properties
  1. $v \times u$ reverses rows 2 and 3 in the determinant and so it equals $-(u \times v)$
  2. The cross product $u \times v$ is perpendicular to $u$ and $v$.
  3. The cross product of any vector with itself is $0$ (2 equal rows)

  $\|u \times v\| = \|u\|\|v\|\sin\theta$ $\quad\quad$ $\|u \cdot v\| = \|u\|\|v\||\cos\theta|$
- The magnitude $u \times v$ is the area of the parallelogram with sides $u$ and $v$
- Example

$u = (1, 1, 1)$, $v = (1, 1, 2)$

$$\begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{vmatrix} = i\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} - j\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} + k\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = i - j = (1, -1, 0)$$

# Triple Product = Determinant = Volume

- A triple product is a specific product of 3 3-dimensional vectors
  - Scalar triple product: $(u \times v) \cdot w$
  - Vector triple product: $(u \times v) \times w$
- The scalar triple product is a determinant and gives the volume of the parallelepiped formed by $u$, $v$, and $w$.

$$(u \times v) \cdot w = \begin{vmatrix} w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$ (two row exchanges)

## Problem Set 5.3

1, a)

$$x_1 = \frac{\begin{vmatrix} 1 & 5 \\ 2 & 4 \end{vmatrix}}{\begin{vmatrix} 2 & 5 \\ 1 & 4 \end{vmatrix}} = \frac{-6}{3} = -2 \qquad x_2 = \frac{\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}}{\begin{vmatrix} 2 & 5 \\ 1 & 4 \end{vmatrix}} = \frac{3}{3} = 1$$

$$x = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

2, a)

$$y = \frac{\begin{vmatrix} a & 1 \\ c & 0 \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \frac{-c}{ad-bc}$$

b)

$$x_1 = \frac{\begin{vmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{vmatrix}}{\begin{vmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{vmatrix}} = \frac{3}{4} \qquad x_2 = \frac{\begin{vmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 2 \end{vmatrix}}{\begin{vmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{vmatrix}} = \frac{-2}{4} = -\frac{1}{2} \qquad x_3 = \frac{\begin{vmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & 0 \end{vmatrix}}{\begin{vmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{vmatrix}} = \frac{1}{4}$$

6

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 0 \\ 0 & 7 & 1 \end{bmatrix} \qquad A^{-1} = \frac{C^T}{\det A}$$

$$C = \begin{bmatrix} 3 & 0 & 0 \\ -2 & 1 & -7 \\ 0 & 0 & 3 \end{bmatrix} \Rightarrow C^T = \begin{bmatrix} 3 & -2 & 6 \\ 0 & 1 & 0 \\ 0 & -7 & 3 \end{bmatrix}$$

$$\det A = 1(3) + 2(0) + 0(0) = 3$$

$$A^{-1} = \frac{1}{3}\begin{bmatrix} 3 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & -7 & 3 \end{bmatrix}$$

7,

$\det A = 0$ by cofactor expansion, no

9,

$$AC^T = (\det A)I = I$$

$$C^T = A^{-1}$$

Find inverse & $C^T = A^{-1}$

$$[C^T \; I] \rightarrow [I \; C^{T^{-1}}]$$

$C^{T^{-1}} = (A^{-1})^{-1} = A$

10,

$$AA^{-1} = I$$

$$\underbrace{(\det A)}_{\text{must be both 1 or -1}}\underbrace{(\det A^{-1})}=1$$

15,

$24 \times 4 \times 25 = 2400$

16

$$\begin{vmatrix} 3 & 1 \\ 2 & 4 \end{vmatrix} = 10$$

b)

17,

$$\begin{vmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{vmatrix} = 27 + 1 + 1 - 3 - 3 - 3 = 20$$

36,

$$(x, y, z) \cdot ((1,1,0) \times (1,2,1)) = 0$$

$$\begin{vmatrix} x & 1 & 1 \\ y & 1 & 2 \\ z & 0 & 1 \end{vmatrix} = x + 2z - z - y = 0$$

$$x - y + z = 0$$