- Projection
  - A plane through the origin is a vector space, other planes are affine spaces, aka "flats". We want to project three-dimensional vectors onto planes. Start with a plane through the origin, whose normal vector is n. The vectors in the plane satisfy $n^Tv=0$. The usual projection onto the plane is $I-nn^T$
  - In homogeneous coordinates, this becomes 4 by 4

$$\text{projection onto the plane } n^Tv=0 \quad P=\begin{bmatrix} & & & 0 \\ & I-nn^T & & 0 \\ & & & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

  - Now, project onto a flat $n^T(v-v_0)=0$, with v_0 as a point on the plane. First we translate v_0 to the origin by T_-, project it along the n direction, and translate back along the row vector v_0:

$$\text{Projection onto a flat} \quad T_-PT_+=\begin{bmatrix} I & 0 \\ -v_0 & 1 \end{bmatrix}\begin{bmatrix} I-nn^T & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} I & 0 \\ v_0 & 1 \end{bmatrix}$$

- Reflection
  - A reflection simply moves a point in the direction of a projection, but twice as far
  - Instead of $I-nn^T$, we have $I-2nn^T$

$$\text{Reflection across a flat} \quad T_-RT_+=\begin{bmatrix} I & 0 \\ -v_0 & 1 \end{bmatrix}\begin{bmatrix} I-2nn^T & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} I & 0 \\ v_0 & 1 \end{bmatrix}$$

- The projection matrix P gives a parallel projection. All points move parallel to n until they reach the plane. Another choice in computer graphics is a perspective projection, in which objects get smaller with distance.

Chapter 10: Complex Vectors and Matrices

Chapter 10.1: Complex Numbers

- $i=\sqrt{-1}$
- $Re(a+bi)=a$, $Im(a+bi)=b$
- $\bar{z}_1\bar{z}_2=\overline{z_1z_2}$
- $\bar{z}_1+\bar{z}_2=\overline{z_1+z_2}$
- If $Ax=\lambda x$ and A is real, then $A\bar{x}=\bar{\lambda}\bar{x}$
- The number $z=a+bi$ is also $z=r\cos\theta+ir\sin\theta=re^{i\theta}$
- Powers and multiplication is easy with complex numbers
- $z^n=r^n(\cos n\theta+i\sin n\theta)$
- Set $w=e^{2\pi i/n}$. The nth powers of $1, w, w^2, \dots, w^{n-1}$ all equal 1

Chapter 10.2: Hermitian and Unitary Matrices

- We introduce a new matrix: a conjugate transpose. Transpose the matrix then take the conjugate of each entry.
- One reason to go $\bar{z}$ is that the length^2 of a real vector is $x_1^2 + x_2^2 + \ldots$, but this is not the case for a complex vector. $|z| \neq z_1^2 + z_2^2 + \ldots + z_n^2$. Then the vector $\begin{bmatrix} 1 \\ i \end{bmatrix}$ would have length 0 — not good. Instead of $(a+bi)^2$, we want a^2+b^2, the modulo squared. This is $(a+bi)(a-bi)$.
- For each component, we want z_j times $\bar{z}_j$, which is $|z_j|^2 = a_j^2 + b_j^2$. That comes from $\bar{z}^T z$.

  Length squared $[\bar{z}_1 \ldots \bar{z}_n]\begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} = |z_1|^2 + \ldots + |z_n|^2$. This is $\bar{z}^T z = \|z\|^2$
- Now, $(1, i)$ has length^2 $1^2 + (i)(-i) = 2$. Then it has a length $\sqrt{2}$ and the zero vector is the only vector with length 0
- $\bar{z}^T = z^H$. $[\,]^H$ denotes the conjugate transpose of vectors and matrices. H for Hermitian ("Hermeeshan")
- Example

$$A = \begin{bmatrix} 1 & i \\ 0 & 1+i \end{bmatrix} \Rightarrow A^H = \begin{bmatrix} 1 & 0 \\ -i & 1-i \end{bmatrix}$$

Complex Inner Products

- For real vectors, $\|v\|^2 = v \cdot v = v^T v$. We want this property to hold. That is, $z \cdot z = \|z\|^2$. To make that happen, instead of $v^T v$, we do $z^H z$, and we define the inner product as such.
- For real or complex vectors u and v

$$\langle u, v \rangle = u \cdot v = u^H v = [\bar{u}_1 \ldots \bar{u}_n]\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \bar{u}_1 v_1 + \ldots + \bar{u}_n v_n$$

- With complex vectors, $u^H v$ is not necessarily $v^H u$. In fact, $v^H u = \bar{v}_1 u_1 + \ldots + \bar{v}_n u_n$ is the complex conjugate of $u^H v$
- Example
  - Given $u = \begin{bmatrix} 1 \\ i \end{bmatrix}$ and $v = \begin{bmatrix} i \\ 1 \end{bmatrix}$, $u^H v = [1 \ -i]\begin{bmatrix} i \\ 1 \end{bmatrix} = 0$. The vectors u and v are orthogonal
  - The inner product of Au with v equals the inner product of u with $A^H v$:

    A^H = "adjoint" of A $\quad (Au)^H v = u^H (A^H v)$
- The rule for the transpose of products remain: $(AB)^H = B^H A^H$

Hermitian Matrices

- Hermitian matrices are complex square matrices equal to their conjugate transpose. $A = A^H$; $a_{ij} = \overline{a_{ji}}$. Every real symmetric matrix is Hermitian.
- Example:

$$A = \begin{bmatrix} 2 & 3-3i \\ 3+3i & 5 \end{bmatrix}$$ is Hermitian.

- If $A = A^H$ and z is any vector, the number $z^H A z$ is real
  - Proof: $z^H A z$ is 1 by 1 (i.e. a number)
    $(z^H A z)^H = z^H A^H (z^H)^H = z^H A z$
  - $z^H A z$ equals its own conjugate transpose and is 1 by 1, therefore it must be real
  - This number $z^H A z$ often represents energy
  - with the previous matrix

$$[\bar{z}_1 \ \bar{z}_2]\begin{bmatrix} 2 & 3-3i \\ 3+3i & 5 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \underbrace{\overbrace{2\bar{z}_1 z_1}^{\text{real}} + \overbrace{5\bar{z}_2 z_2}^{\text{real}}}_{\text{diagonal}} + \underbrace{\overbrace{(3-3i)\bar{z}_1 z_2 + (3+3i) z_1 \bar{z}_2}^{\text{conjugates} \therefore \text{real}}}_{\text{off diagonal}}$$

- Every eigenvalue of a Hermitian matrix is real
  - Proof: suppose $Az = \lambda z$. Multiply both sides by z^H to get $z^H A z = \lambda z^H z$.
    $z^H A z$ is real and so is $z^H z$. Therefore, λ is also real
- The eigenvectors of a Hermitian Matrix can be chosen orthonormal (when they correspond to different eigenvalues). If $Az = \lambda z$ and $Ay = \beta y$ then $y^H z = 0$
  - Proof: Multiply $Az = \lambda z$ on the left by y^H and multiply $y^H A^H = \beta y^H$ on the right by z

    $y^H A z = \lambda y^H z$ and $y^H A^H z = \beta y^H z$

  - The left sides are equal because $A = A^H$. Therefore the right sides are equal. Since $\lambda \neq \beta$, the factor $y^H z$ must be 0 and the eigenvectors y and z are orthogonal
- The eigenvector matrix can be chosen orthonormal, with complex entries. We call such a matrix Unitary

Unitary Matrices

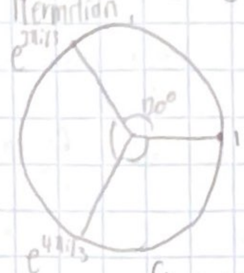
- A Unitary matrix U is a (complex) square matrix that has orthonormal columns. U is the complex equivalent of Q

  Unitary Matrix $\quad U = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1-i \\ 1+i & -1 \end{bmatrix}$

- The matrix test for real orthonormal columns is $Q^T Q = I$. When Q^T multiplies Q, the zero inner products appear off diagonal. In the complex case, for this to happen, we use the conjugate transpose $U^H U = I$
- Every matrix U with orthonormal columns has $U^H U = I$
  If U is square, it is a unitary matrix. Then $U^H = U^{-1}$
- Suppose U (with orthonormal columns) multiplies any z. The vector length stays the same, because $z^H U^H U z = z^H z$. If z is an eigenvector of U we learn something more: The eigenvalues of unitary (and orthogonal) matrices all have absolute value $|\lambda| = 1$.

  If U is unitary then $\|Uz\| = \|z\|$. Therefore $Uz = \lambda z$ leads to $|\lambda| = 1$

- Example:
  - The 3 by 3 Fourier Matrix is in fig 10.2.1. Is it Hermitian and or unitary? F is certainly symmetric, but does not equal its conjugate transpose. It is not Hermitian.



Fourier Matrix $F = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{2\pi i/3} & e^{4\pi i/3} \\ 1 & e^{4\pi i/3} & e^{2\pi i/3} \end{bmatrix}$

fig 10.2.1: The cube roots of 1 go into the Fourier Matrix $F = F_3$

  - F is unitary. The squared length of every column is $\frac{1}{3}(1+1+1) = 1$. The first column is orthogonal to the rest: $1 + e^{2\pi i/3} + e^{4\pi i/3} = 0$
  - From the figure on the left, it is really obvious that summing columns 2 and 3 ends up with 0.
  - Column 2 and 3 are orthogonal, so F is unitary

$$F_2^H F_3 = \frac{1}{3}(1 \cdot 1 + e^{-2\pi i/3} e^{4\pi i/3} + e^{-4\pi i/3} e^{2\pi i/3})$$
$$= \frac{1}{3}(1 + e^{2\pi i/3} + e^{-2\pi i/3})$$
$$= 0$$

  - When we multiply by F, we are computing the Discrete Fourier Transform, and multiplying by F^{-1} computes the inverse transform. Since F is unitary and square, $F^{-1} = F^H$

Real vs Complex

- Length: $\|x\|^2 = x_1^2 + \dots + x_n^2 \longleftrightarrow$ Length: $\|z\|^2 = |z_1|^2 + \dots |z_n|^2$
- Transpose: $(A^T)_{ij} = A_{ji} \longleftrightarrow$ conjugate transpose: $(A^H)_{ij} = \bar{A}_{ji}$
- Product rule: $(AB)^T = B^T A^T \longleftrightarrow$ product rule: $(AB)^H = B^H A^H$
- Dot product: $x^T y = x_1 y_1 + \dots + x_n y_n \longleftrightarrow$ inner product: $u^H v = \bar{u}_1 v_1 + \dots + \bar{u}_n v_n$
- Orthogonality: $x^T y = 0 \longleftrightarrow v^H v = 0$
- $A = Q \Lambda Q^{-1} = Q \Lambda Q^T$ (real Λ) $\longleftrightarrow A = U \Lambda U^{-1} = U \Lambda U^H$ (real Λ)
- $(Qx)^T (Qy) = x^T y$ and $\|Qx\| = \|x\| \longleftrightarrow (Ux)^H (Uy) = x^H y$ and $\|Uz\| = \|z\|$

Chapter 10.3: The Fast Fourier Transform

- We want to multiply quickly by F and F^{-1}. This is achieved by the Fast Fourier Transform. An ordinary product Fc uses n^2 multiplications (F has n^2 entries). The FFT only needs $\frac{1}{2} n \log_2 n$
- Fourier's idea is to represent f as a sum of harmonics $c_k e^{ikx}$. The function is seen in frequency space through the coefficients c_k instead of physical space with values $f(x)$. FFT provides fast passage between c and f

Roots of Unity and the Fourier Matrix

- Equations of degree n have n roots (counting repetition). This is the Fundamental Theorem of Algebra, and to make it true, we need to allow complex mots
- This section focuses on $z^n = 1$. The solutions are the "nth roots of unity". They are n evenly spaced points around the unit circle in the complex plane
- For $z^8 = 1$, we have points spaced at $\frac{1}{8}(360°) = 45°$ degrees. The solutions are numbers $w = e^{i\theta} = e^{i2\pi/8}$, and the powers $w^2, w^3, \dots w^8$. This is shown in fig 10.3.1

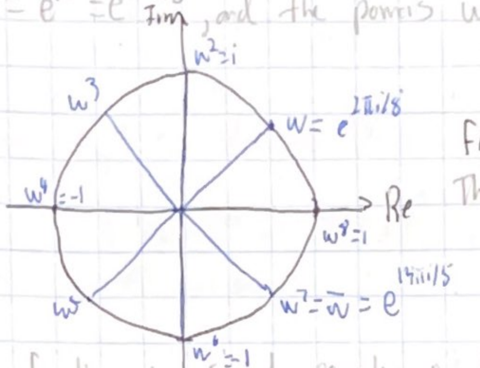


Fig 10.3.1
The eight solutions to $z^8 = 1$

- The fourth roots of 1 are also in this figure. They are $i, -1, -i$, and 1. The angle is now $90°$. The idea behind FFT is to go from an 8 by 8 matrix to 4 by 4 to 2 by 2. By exploiting the connections between F_8 and F_4 and F_8 and F_{16}, we can make multiplication by F_{1024} very fast.
- The Fourier matrix for $n=4$ is

Fourier Matrix $n=4$
$$F = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & w & w^2 & w^3 \\ 1 & w^2 & w^4 & w^6 \\ 1 & w^3 & w^6 & w^9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \end{bmatrix}$$

- $\frac{1}{2}F$ is unitary, so $(\frac{1}{2}F^H)(\frac{1}{2}F) = I$
- The columns of F give $F^H F = 4I$. Its inverse is $\frac{1}{4}F^H$, which is $F^{-1} = \frac{1}{4}\bar{F}$
- The inverse changes from $w = i$ and $\bar{w} = -i$. That takes us from F to $\bar{F}$. The FFT gives a quick way to multiply by F and by F^{-1}.
- The unitary matrix is $U = F/\sqrt{n}$. We avoid the $\sqrt{n}$ and put n outside F^{-1}.
- The main point is to multiply F times the Fourier coefficients c_0, c_1, c_2, c_3

4-point Fourier Series
$$\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix} = Fc = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & w & w^2 & w^3 \\ 1 & w^2 & w^4 & w^6 \\ 1 & w^3 & w^6 & w^9 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

- The input is four complex coefficients c_0, c_1, c_2, c_3. The output is four function values y_0, y_1, y_2, y_3. The first output $y_0 = c_0 + c_1 + c_2 + c_3$ is the value at the Fourier Series at $x = 0$. The second output is the value of the series $\sum c_k e^{ikx}$ at $x = 2\pi/4$. Likewise y_2 and y_3 are the outputs of $\sum c_k e^{ikx}$ at $x = 4\pi/4$ and $6\pi/6$

$$y_1 = c_0 + c_1 e^{i2\pi/4} + c_2 e^{i4\pi/4} + c_3 e^{i6\pi/4} = c_0 + c_1 w + c_2 w^2 + c_3 w^3.$$

- These are finite Fourier Series. They contain $n=4$ terms and are evaluated at 4 equally space points in $[0, 2\pi)$

- The period then repeats for the next 2π.
- j and k will be zero-indexed for the rest of the section
- The n by n Fourier Matrix contains powers of $e^{2\pi i/n}$:

$$F_n c = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & w & w^2 & \cdots & w^{n-1} \\ 1 & w^2 & w^4 & \cdots & w^{2(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & w^{n-1} & w^{2(n-1)} & \cdots & w^{(n-1)^2} \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_{n-1} \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_{n-1} \end{bmatrix} = y \qquad (F_n)_{ij} = w^{ij}, \text{ where } i, j$$

- F_n is symmetric but not Hermitian. Its columns are orthogonal and $\bar{F}_n F_n = nI$. Then $F_n^{-1} = \bar{F}_n / n$. The inverse contains powers of $\bar{w}_n = e^{-2\pi i/n}$.
- When we multiply c by F_n, we sum the series at n points. When we multiply y by F_n^{-1}, we find the coefficients c from function values y. The matrix F passes from "frequency space" to "physical space."
- Many authors prefer to use $\omega = e^{-2\pi i/N}$, the complex conjugate of w. Then F becomes $\bar{F}$.
- When a function $f(x)$ has period 2π, and we change x to $e^{i\theta}$, the function is defined around the unit circle (where $z = e^{i\theta}$). Then the DFT from y to c is matching n values of this $f(z)$ by a polynomial $p(z) = c_0 + c_1 z + \dots + c_{n-1} z^{n-1}$
- Interpolation: Find $c_0, \dots, c_{n-1}$ so that $p(z) = f(z)$ at n points $z = 1, \dots, w^{n-1}$
- The Fourier matrix is the Vandermonde Matrix for interpolation at those n points

One Step of the Fast Fourier Transform

- We want to multiply F and c as quickly as possible. Usually this takes n^2 multiplications. Since F follows a special pattern ($(F_n)_{ij} = w^{ij}$), we can factor it in a way that produces many zeroes. This is the FFT
- We connect F_n to two copies of $F_{n/2}$. For example, with F_4

$$F_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \end{bmatrix} \text{ and } \begin{bmatrix} F_2 & \\ & F_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & & \\ 1 & i^2 & & \\ & & 1 & 1 \\ & & 1 & i^2 \end{bmatrix}$$

- We can factorize F_4 as the following:

$$F_4 = \begin{bmatrix} 1 & & 1 & \\ & 1 & & i \\ 1 & & -1 & \\ & 1 & & -i \end{bmatrix} \begin{bmatrix} 1 & 1 & & \\ 1 & i^2 & & \\ & & 1 & 1 \\ & & 1 & i^2 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & & 1 & \\ & 1 & & \\ & & & 1 \end{bmatrix}$$

  - The last matrix is a permutation. It puts the even c's (c_0 and c_2) ahead of the odd c's (c_1 and c_3). The middle matrix performs half sized matrix transforms F_2 and F_2 on the evens and odds. The left matrix combines the two half-sized outputs in a way that correctly produces $y = F_4 c$.

- This idea applies for larger n:

$$F_{1024} = \begin{bmatrix} I_{512} & D_{512} \\ I_{512} & -D_{512} \end{bmatrix} \begin{bmatrix} F_{512} & \\ & F_{512} \end{bmatrix} \begin{bmatrix} \text{even-odd} \\ \text{Permutation} \end{bmatrix}$$

  - I is the identity matrix and D is the identity matrix with entries $(1, w, \ldots, w^{511})$
- Here are the algebra formulas meaning the same thing:

(FFT) Set $m = \frac{1}{2}n$. The first m and last m components of $y = F_n c$ combine the half-size transforms $y' = F_m c'$ and $y'' = F_m c''$. The matrix equation shows the step from n to $m = n/2$ as $Iy' + Dy''$ and $Iy' - Dy''$:

$$y_j = y'_j + w_n^j y''_j, \quad j = 0, \ldots, m-1$$
$$y_{j+m} = y'_j - w_n^j y''_j, \quad j = 0, \ldots, m-1 \qquad (5)$$

Split c into c' and c'', transform them by F_m into y' and y'', and reconstruct y.

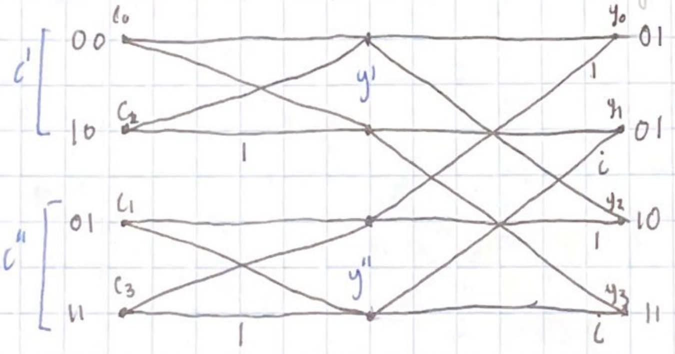
- These formulas come from separating even c_{2k} from odd c_{2k+1}:

$$y_j = \sum_{0}^{n-1} w^{jk} c_k = \sum_{0}^{m-1} w^{2jk} c_{2k} + \sum_{0}^{m-1} w^{j(2k+1)} c_{2k+1} \quad \text{with } m = \tfrac{1}{2}n$$

- The even c's go into $c' = (c_0, c_2, \ldots)$ and the odd c's go into $c'' = (c_1, c_3, \ldots)$. Then comes the transforms $F_m c'$ and $F_m c''$. The key is $w_n^2 = w_m$. This gives $w_n^{2jk} = w_m^{jk}$

Rewrite: $y_j = \sum w_m^{jk} c'_k + (w_n)^j \sum w_m^{jk} c''_k = y'_j + (w_n)^j y''_j$

- For $j \geq m$, the minus sign in (5) comes from factoring out $(w_n)^m = -1$
- The flow graph shows c' and c'' going through the half-size F_2. Those steps are called butterflies. Then the outputs y' and y'' are combined (multiplying y'' by $1, i$, and also by $-1, -i$) to produce $y = F_4 c$.
- This cuts the work almost by half - you see the zeros in the factorization. We then recurse to further decrease time complexity.



The Full FFT by Recursion

- We can further recurse:

$$\begin{bmatrix} F_{512} & \\ & F_{512} \end{bmatrix} = \begin{bmatrix} I & D & & \\ I & -D & & \\ & & I & D \\ & & I & -D \end{bmatrix} \begin{bmatrix} F & & & \\ & F & & \\ & & F & \\ & & & F \end{bmatrix} \begin{bmatrix} \text{pick } 0,4,8 \\ \text{pick } 2,6,10 \\ \text{pick } 1,5,9 \\ \text{pick } 3,7,11 \end{bmatrix}$$

- The total amount of computations for size $n=2^\ell$ is reduced from n^2 to $\frac{1}{2}n\ell$. There are ℓ levels, going down from $n=2^\ell$ to $n=1$. Each level has $n/2$ multiplications from the diagonal D's to reassemble the half-size outputs from the lower level. This gives our final count $\frac{1}{2}n\log_2 n$
- A rule for the order the c's enter the FFT after all the even-odd permutations. Write the numbers 0 to $n-1$ in base 2. Reverse the digits. The complete picture shows the bit-reversed order at the start, the $\ell = \log_2 n$ steps of the recursion, and the final output $y_0, \ldots, y_{n-1}$ which is F_n times c
- The book ends with that fundamental idea, a matrix multiplying a vector. :)

Harry Lu, August 4th 2024, 4:25pm