An Exact Compliance Matrix and Consistent Load Vector for a Beam on Elastic Foundation

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The exact compliance matrix for a beam of length L and rigidiy EI on an elastic foundation of stiffness k can be expressed as:

$$\begin{bmatrix} w_1 \\ \varphi_1 \\ w_2 \\ \varphi_2 \end{bmatrix} = \frac{2\lambda}{k(sh^2 - s^2)} \begin{bmatrix} [sh \cdot ch - s \cdot c] & -\lambda\left[sh^2 + s^2\right] & [sh \cdot c - s \cdot ch] & -2\lambda\,[sh \cdot s] \\ & 2\lambda^2\,[sh \cdot ch + s \cdot c] & 2\lambda\,[sh \cdot s] & 2\lambda^2\,[sh \cdot c + s \cdot ch] \\ & & [sh \cdot ch - s \cdot c] & \lambda\left[sh^2 + s^2\right] \\ symmetric & & & 2\lambda^2\,[sh \cdot ch + s \cdot c] \end{bmatrix} \begin{bmatrix} F_1 \\ M_1 \\ F_2 \\ M_2 \end{bmatrix}$$

where $s = \sin(\lambda L)$, $c = \cos(\lambda L)$, $sh = \sinh(\lambda L)$, $ch = \cosh(\lambda L)$ and $\lambda = \sqrt[4]{\frac{k}{4EI}}$. This matrix can be inverted to give the stiffness matrix S_{ij}.

The above expressions are based on the following solution for the 4th order differential equation [1]:

$$w(x) = w_0 F_1(\lambda x) + \frac{1}{\lambda}\varphi_0 F_2(\lambda x) - \frac{1}{\lambda^2 EI} M_0 F_3(\lambda x) + \frac{1}{\lambda^3 EI} Q_0 F_4(\lambda x)$$

where w_0, φ_0, M_0 and Q_0 are the displacement, rotation, bending moment and shear force, respectively, at x=0 and

$$\begin{aligned} F_1(x) &= \cosh(\lambda x)\cos(\lambda x) \\ F_2(x) &= \frac{1}{2}\big(\cosh(\lambda x)\sin(\lambda x) + \sinh(\lambda x)\cos(\lambda x)\big) \\ F_3(x) &= \frac{1}{2}\sinh(\lambda x)\sin(\lambda x) \\ F_4(x) &= \frac{1}{4}\big(\cosh(\lambda x)\sin(\lambda x) - \sinh(\lambda x)\cos(\lambda x)\big) \end{aligned}$$

The consistent load vector $\underline{F}$ for a distributed load of the form $q(x) = q_1(1 - \frac{x}{L}) + q_2\frac{x}{L}$ can then be obtained from the virtual work expression:

$$\begin{aligned} \underline{F}^T \delta\underline{a} &= \int_0^L \left[q_1(1 - \frac{x}{L}) + q_2\frac{x}{L}\right] [\delta w_{w0}(x) + \delta w_{\varphi 0}(x) + \delta w_{wL}(x) + \delta w_{\varphi L}(x)]\, dx \\ &= q_1\left[I_{11}\delta w_0 + I_{12}\delta\varphi_0 + I_{13}\delta w_L + I_{14}\delta\varphi_L\right] + q_2\left[I_{21}\delta w_0 + I_{22}\delta\varphi_0 + I_{23}\delta w_L + I_{24}\delta\varphi_L\right] \end{aligned}$$

where $\delta \underline{a} = [\delta w_0 \; \delta\varphi_0 \; \delta w_L \; \delta\varphi_L]^T$ are the virtual displacements and rotations at the beam ends,

$$
\begin{aligned}
I_{11} &= \int_0^L (1-\tfrac{x}{L})(F_1(\lambda x) - \tfrac{S_{12}}{\lambda^2 EI}F_3(\lambda x) + \tfrac{S_{11}}{\lambda^3 EI}F_4(\lambda x))dx &&= A_{11} - \tfrac{S_{12}}{\lambda^2 EI}A_{13} + \tfrac{S_{11}}{\lambda^3 EI}A_{14} \\
I_{12} &= \int_0^L (1-\tfrac{x}{L})(\tfrac{1}{\lambda}F_2(\lambda x) - \tfrac{S_{22}}{\lambda^2 EI}F_3(\lambda x) + \tfrac{S_{12}}{\lambda^3 EI}F_4(\lambda x))dx &&= \tfrac{1}{\lambda}A_{12} - \tfrac{S_{22}}{\lambda^2 EI}A_{13} + \tfrac{S_{12}}{\lambda^3 EI}A_{14} \\
I_{13} &= \int_0^L (1-\tfrac{x}{L})(-\tfrac{S_{23}}{\lambda^2 EI}F_3(\lambda x) + \tfrac{S_{13}}{\lambda^3 EI}F_4(\lambda x))dx &&= -\tfrac{S_{23}}{\lambda^2 EI}A_{13} + \tfrac{S_{13}}{\lambda^3 EI}A_{14} \\
I_{14} &= \int_0^L (1-\tfrac{x}{L})(-\tfrac{S_{24}}{\lambda^2 EI}F_3(\lambda x) + \tfrac{S_{14}}{\lambda^3 EI}F_4(\lambda x))dx &&= -\tfrac{S_{24}}{\lambda^2 EI}A_{13} + \tfrac{S_{14}}{\lambda^3 EI}A_{14} \\
I_{21} &= \int_0^L \tfrac{x}{L}(F_1(\lambda x) - \tfrac{S_{12}}{\lambda^2 EI}F_3(\lambda x) + \tfrac{S_{11}}{\lambda^3 EI}F_4(\lambda x))dx &&= A_{21} - \tfrac{S_{12}}{\lambda^2 EI}A_{23} + \tfrac{S_{11}}{\lambda^3 EI}A_{24} \\
I_{22} &= \int_0^L \tfrac{x}{L}(\tfrac{1}{\lambda}F_2(\lambda x) - \tfrac{S_{22}}{\lambda^2 EI}F_3(\lambda x) + \tfrac{S_{12}}{\lambda^3 EI}F_4(\lambda x))dx &&= \tfrac{1}{\lambda}A_{22} - \tfrac{S_{22}}{\lambda^2 EI}A_{23} + \tfrac{S_{12}}{\lambda^3 EI}A_{24} \\
I_{23} &= \int_0^L \tfrac{x}{L}(-\tfrac{S_{23}}{\lambda^2 EI}F_3(\lambda x) + \tfrac{S_{13}}{\lambda^3 EI}F_4(\lambda x))dx &&= -\tfrac{S_{23}}{\lambda^2 EI}A_{23} + \tfrac{S_{13}}{\lambda^3 EI}A_{24} \\
I_{24} &= \int_0^L \tfrac{x}{L}(-\tfrac{S_{24}}{\lambda^2 EI}F_3(\lambda x) + \tfrac{S_{14}}{\lambda^3 EI}F_4(\lambda x))dx &&= -\tfrac{S_{24}}{\lambda^2 EI}A_{23} + \tfrac{S_{14}}{\lambda^3 EI}A_{24}
\end{aligned}
$$

and

$$
\begin{aligned}
A_{11} &= sh \cdot s \,/\, (2L\lambda^2) \\
A_{12} &= (s \cdot ch - c \cdot sh) \,/\, (4L\lambda^2) \\
A_{13} &= (1 - c \cdot ch) \,/\, (4L\lambda^2) \\
A_{14} &= (2L\lambda - ch \cdot s - c \cdot sh) \,/\, (8L\lambda^2) \\
A_{21} &= (L\lambda \cdot ch \cdot s + (L\lambda \cdot c - s) \cdot sh) \,/\, (2L\lambda^2) \\
A_{22} &= (-ch \cdot s + (c + 2L\lambda \cdot s) \cdot sh) \,/\, (4L\lambda^2) \\
A_{23} &= (-1 + ch \cdot (c + L\lambda \cdot s) - L\lambda \cdot c \cdot sh) \,/\, (4L\lambda^2) \\
A_{24} &= (ch \cdot (-2L\lambda \cdot c + s) + c \cdot sh) \,/\, (8L\lambda^2)
\end{aligned}
$$

The load vector can then finally be expressed as:

$$
\underline{F} = q_1 \begin{bmatrix} I_{11} \\ I_{12} \\ I_{13} \\ I_{14} \end{bmatrix} + q_2 \begin{bmatrix} I_{21} \\ I_{22} \\ I_{23} \\ I_{24} \end{bmatrix}
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References

[1] Hetényi, M. (1946). Beams on elastic foundation: Theory with applications in the fields of civil and mechanical engineering. University of Michigan Press.