Xiong_Yinjiang_HW3

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1 Linear Model Selection and Regularization

1.Generate a data set with $p = 20$ features, $n = 1000$ observations, and an associated quantitative response vector generated according to the model

$$Y = X\beta + \epsilon$$

where β has some elements that are exactly equal to zero.

[19]:
```
import numpy as np
import itertools
import pandas as pd
import matplotlib.pyplot as plt
import statsmodels.api as sm
from sklearn.model_selection import train_test_split
from sklearn.linear_model import LinearRegression
from sklearn.metrics import mean_squared_error
from mlxtend.feature_selection import SequentialFeatureSelector as SFS
from sklearn.model_selection import GridSearchCV
from sklearn.linear_model import Ridge
from sklearn.linear_model import Lasso
from sklearn.linear_model import ElasticNet
np.random.seed(24)
```

[2]:
```
# first, we randomly generate 20 features with 1000 observations
X = []
for _ in range(20):
    X.append(np.random.normal(np.random.choice(100,1), 1, 1000))
X = np.array(X)
```

[3]:
```
# then, we randomly generate 20 random betas
Beta = []
for _ in range(20):
    Beta.append(10 * np.random.random())

# let's generate 5 random indices and replace them with zero
zero_index = []
```

```
for n in np.random.choice(20,5):
    Beta[n] = 0
    zero_index.append(n)

# check beta
Beta = np.array(Beta)
Beta = Beta.reshape(-1,1)
Beta

# note that we have kept the beta = 0 information in zero_index
```

[3]:
```
array([[0.        ],
       [7.41212581],
       [8.09098401],
       [3.68270568],
       [0.        ],
       [4.53731565],
       [9.0219442 ],
       [0.71447831],
       [2.10138621],
       [6.86903874],
       [0.        ],
       [1.96584887],
       [9.85123182],
       [6.30804308],
       [0.42835617],
       [8.25425396],
       [0.        ],
       [8.11969394],
       [7.67387209],
       [0.        ]])
```

[4]:
```
# finally, create Y
# use broadcasting
Y = X * Beta
# generate error: with mean = 0, sd = 10
error = np.random.normal(0,10,1000)
Y = np.sum(Y, axis=0) + error
```

2.Split your data set into a training set containing 100 observations and a test set containing 900 observations.

[5]:
```
# create a dataframe containing Y and X
Xt = X.transpose()
df = pd.DataFrame({'Y':Y})
for n in range(0,20):
    df['X{}'.format(n)] = Xt[:,n]
```

```
df.head(5)
```

[5]:

	Y	X0	X1	X2	X3	X4	\
0	4707.633622	33.771182	95.932867	67.736543	52.220935	12.539658	
1	4667.831605	33.894806	94.071022	66.757592	50.726081	12.705394	
2	4709.137224	35.487894	95.383519	67.634304	51.974740	13.004285	
3	4677.981952	34.046731	93.676851	68.747795	50.745773	11.456581	
4	4708.148487	36.323789	93.901247	68.072361	50.108320	11.592697	

	X5	X6	X7	X8	...	X10	X11	\
0	20.860180	42.636533	12.114676	72.888851	...	26.157619	75.434748	
1	19.889749	40.640765	12.149863	70.991400	...	27.466977	75.355861	
2	21.134597	41.584655	12.629108	71.200583	...	27.244833	73.746079	
3	23.047611	41.068100	10.311666	73.056294	...	28.034128	73.104750	
4	22.659707	42.416513	11.340568	72.633341	...	26.507698	74.296530	

	X12	X13	X14	X15	X16	X17	\
0	54.385360	35.092312	62.661170	59.300054	28.214382	56.909681	
1	54.587197	34.028896	60.410975	60.129402	26.831438	58.587753	
2	54.553417	34.193719	61.728457	61.885028	27.077562	57.150222	
3	53.587076	33.125989	62.532450	59.693219	27.240666	55.608992	
4	55.103520	35.013668	61.938490	59.159034	27.542115	57.361038	

	X18	X19
0	32.762930	11.025487
1	35.306703	11.370518
2	34.742751	9.815837
3	36.655743	12.631524
4	33.358290	11.258816

[5 rows x 21 columns]

[6]:
```
X = df.drop(['Y'], axis=1)
Y = df['Y']
X_train, X_test, y_train, y_test = train_test_split(X, Y, test_size = 0.9)
```

[7]:
```
# check X_train shape
X_train.shape
```

[7]: (100, 20)

3.Perform best subset selection on the training set, and plot the training set MSE associated with the best model of each size. For which model size does the training set MSE take on its minimum value?

[8]:
```
def best_subset (X, y):
    res = {}
```

```
    for k in range (1,6):
        for each in itertools.combinations(X, k):
            X_full = sm.add_constant(X[list(each)])
            lm = sm.OLS(y, X_full).fit()
            mse = lm.mse_resid
            if k not in res.keys() or mse < res[k][1]:
                res[k] = [each, mse]
    return res
```

[20]:
```
result = best_subset(X_train, y_train)
```

```
//anaconda3/lib/python3.7/site-packages/numpy/core/fromnumeric.py:2389:
FutureWarning: Method .ptp is deprecated and will be removed in a future
version. Use numpy.ptp instead.
  return ptp(axis=axis, out=out, **kwargs)
```

[10]:
```
result
```

[10]:
```
{1: [('X18',), 509.8681335035226],
 2: [('X12', 'X18'), 420.6992067860652],
 3: [('X12', 'X17', 'X18'), 359.85352401386166],
 4: [('X12', 'X15', 'X17', 'X18'), 302.3414129350472],
 5: [('X9', 'X12', 'X15', 'X17', 'X18'), 269.5070301760166]}
```

[11]:
```
sfs = SFS(LinearRegression(), k_features=5, forward=True, scoring =␣
 ↪'neg_mean_squared_error', cv=0)
sfs.fit(X_train, y_train)
sfs.k_feature_names_
```

[11]:
```
('X9', 'X12', 'X15', 'X17', 'X18')
```

[12]:
```
# since the computational load for best subset selection is so high, we can use␣
 ↪forward selection to sustitute best subset
# this is due to we are convinced that all the features were independently and␣
 ↪randomly generated
result_n = {}
for n in range (1,21):
    sfs = SFS(LinearRegression(), k_features=n, forward=True, scoring =␣
 ↪'neg_mean_squared_error', cv=0)
    sfs.fit(X_train, y_train)
    result_n[len(sfs.k_feature_names_)] = - sfs.k_score_
result_n
```

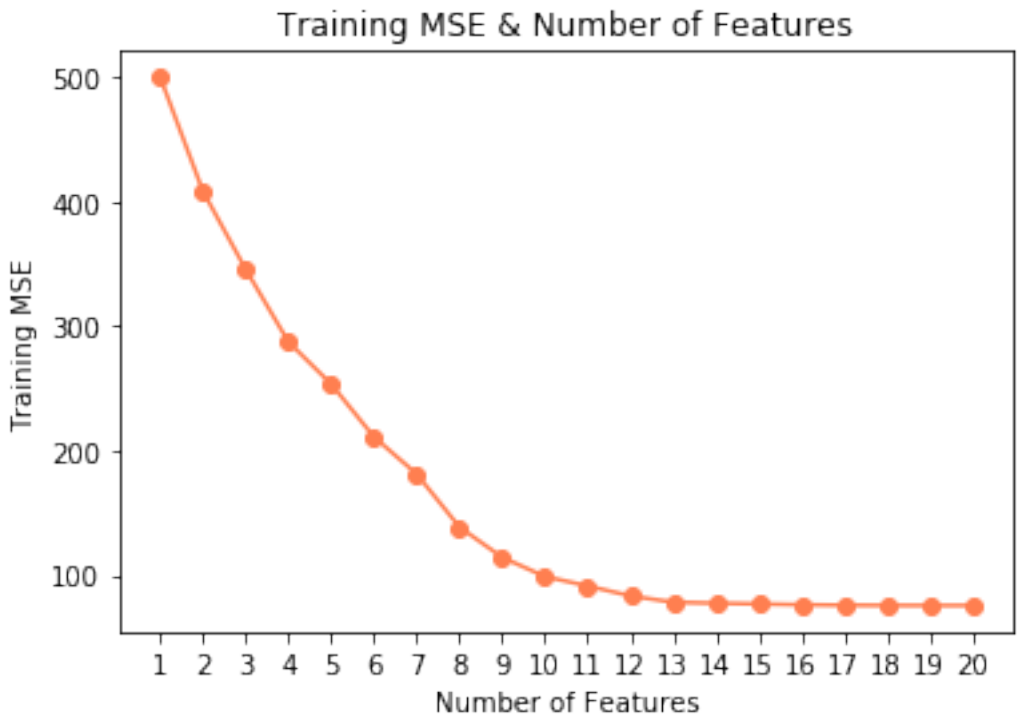
[12]:
```
{1: 499.670770833451,
 2: 408.0782305824839,
 3: 345.4593830533083,
 4: 287.2243422882952,
```

```
 5: 253.33660836545536,
 6: 210.960909775253,
 7: 181.1896416015844,
 8: 138.63140592406535,
 9: 114.30884282186533,
 10: 98.90705973623922,
 11: 91.33150126120103,
 12: 83.37011237866305,
 13: 78.26657924587751,
 14: 77.48658605515386,
 15: 76.91620595014322,
 16: 76.27295046607294,
 17: 75.88961818249493,
 18: 75.73377362717241,
 19: 75.66394279406703,
 20: 75.66166894726723}
```

We can see that the MSE monotonically decreases as the number of predictors increases. The model that includes all the variables have the lowest MSE.

4.Plot the test set MSE associated with the best model of each size.

[13]:
```
feature_num =␣
 ↪['1','2','3','4','5','6','7','8','9','10','11','12','13','14','15','16','17','18','19','20']
MSE = []
for value in result_n.values():
    MSE.append(value)
plt.plot(feature_num, MSE, marker='o', color='coral', label='MSE')
plt.xlabel('Number of Features')
plt.ylabel('Training MSE')
plt.title('Training MSE & Number of Features');
```



5.For which model size does the test set MSE take on its minimum value? Comment on your results

[14]:
```
result_test = {}
for n in range (1,21):
    sfs = SFS(LinearRegression(), k_features=n, forward=True, scoring =␣
 ↪'neg_mean_squared_error', cv=0)
    sfs.fit(X_train, y_train)
    lm = LinearRegression().fit(X_train[list(sfs.k_feature_names_)], y_train)
    result_test[sfs.k_feature_names_] = mean_squared_error(lm.
 ↪predict(X_test[list(sfs.k_feature_names_)]), y_test)
result_test
```

[14]:
```
{('X18',): 689.5460159571702,
 ('X12', 'X18'): 591.2393811718593,
 ('X12', 'X17', 'X18'): 531.2964732098586,
 ('X12', 'X15', 'X17', 'X18'): 461.35689807785906,
 ('X9', 'X12', 'X15', 'X17', 'X18'): 409.9729576218823,
 ('X2', 'X9', 'X12', 'X15', 'X17', 'X18'): 347.25063341145824,
 ('X2', 'X6', 'X9', 'X12', 'X15', 'X17', 'X18'): 270.4397176916692,
 ('X1', 'X2', 'X6', 'X9', 'X12', 'X15', 'X17', 'X18'): 202.4318215626645,
 ('X1',
  'X2',
  'X6',
```

```
 'X9',
 'X12',
 'X13',
 'X15',
 'X17',
 'X18'): 155.23704729305297,
('X1',
 'X2',
 'X6',
 'X9',
 'X11',
 'X12',
 'X13',
 'X15',
 'X17',
 'X18'): 162.9525230580034,
('X1',
 'X2',
 'X6',
 'X9',
 'X11',
 'X12',
 'X13',
 'X15',
 'X16',
 'X17',
 'X18'): 173.86069977498698,
('X1',
 'X2',
 'X5',
 'X6',
 'X9',
 'X11',
 'X12',
 'X13',
 'X15',
 'X16',
 'X17',
 'X18'): 160.51457171053048,
('X1',
 'X2',
 'X3',
 'X5',
 'X6',
 'X9',
 'X11',
 'X12',
```

```
 'X13',
 'X15',
 'X16',
 'X17',
 'X18'): 139.36621585309246,
('X0',
 'X1',
 'X2',
 'X3',
 'X5',
 'X6',
 'X9',
 'X11',
 'X12',
 'X13',
 'X15',
 'X16',
 'X17',
 'X18'): 141.50144237019364,
('X0',
 'X1',
 'X2',
 'X3',
 'X5',
 'X6',
 'X9',
 'X11',
 'X12',
 'X13',
 'X14',
 'X15',
 'X16',
 'X17',
 'X18'): 141.89538526960015,
('X0',
 'X1',
 'X2',
 'X3',
 'X4',
 'X5',
 'X6',
 'X9',
 'X11',
 'X12',
 'X13',
 'X14',
 'X15',
```

```
 'X16',
 'X17',
 'X18'): 142.40940833761636,
('X0',
 'X1',
 'X2',
 'X3',
 'X4',
 'X5',
 'X6',
 'X8',
 'X9',
 'X11',
 'X12',
 'X13',
 'X14',
 'X15',
 'X16',
 'X17',
 'X18'): 137.86452108442697,
('X0',
 'X1',
 'X2',
 'X3',
 'X4',
 'X5',
 'X6',
 'X8',
 'X9',
 'X11',
 'X12',
 'X13',
 'X14',
 'X15',
 'X16',
 'X17',
 'X18',
 'X19'): 137.03646625796628,
('X0',
 'X1',
 'X2',
 'X3',
 'X4',
 'X5',
 'X6',
 'X7',
 'X8',
```

```
 'X9',
 'X11',
 'X12',
 'X13',
 'X14',
 'X15',
 'X16',
 'X17',
 'X18',
 'X19'): 137.68837233638956,
('X0',
 'X1',
 'X2',
 'X3',
 'X4',
 'X5',
 'X6',
 'X7',
 'X8',
 'X9',
 'X10',
 'X11',
 'X12',
 'X13',
 'X14',
 'X15',
 'X16',
 'X17',
 'X18',
 'X19'): 137.745095532462}
```

'X0','X1','X2','X3','X4','X5','X6','X8','X9','X11','X12','X13','X14','X15','X16','X17','X18','X19' compose the model that produces the lowest MSE.

6.How does the model at which the test set MSE is minimized compare to the true model used to generate the data? Comment on the coefficient sizes.

[15]:
```
# let's use the model with the smallest MSE to fit our entire data set
lm = LinearRegression().
 ↪fit(X[['X0','X1','X2','X3','X4','X5','X6','X8','X9','X11','X12','X13','X14','X15','X16','X17'
 ↪Y)
lm.coef_
```

[15]:
```
array([ 0.15306324,  7.39989836,  7.96820064,  3.18239778, -0.25892266,
        4.21550617,  9.40403319,  2.74606066,  7.26016452,  1.61416783,
        9.90378175,  6.73563248,  0.19929759,  8.57069405, -0.39267453,
        8.26137323,  7.49143639,  0.01921556])
```

The model estimated using best subset selection is Y = 0.153X0 + 7.40X1 + 7.97X2 + 3.18X3 + -0.26X4 + 4.22X5 + 9.40X6 + 2.75X8 + 7.26X9 + 1.61X11 + 9.90X12 + 6.74X13 + 0.20X14 + 8.57X15 -0.39X16 + 8.28X17 + 7.49X18 + 0.02X19 + error
The true model that we generate is: Y = 7.41X1 + 8.09X2 + 3.68X3 + 4.54X5 + 9.02X6 + 0.71X7 + 2.10X8 + 6.87X9 + 1.97X11 + 9.85X12 + 6.31X13 + 0.43X14 + 8.25X15 + 8.12X17 + 7.67X18
For the predictors missed or added by the estimated model, they all have small coefficients. The overlapped coefficients are very similar from the two models.

7.Create a plot displaying

$$\sqrt{\sum_{j=1}^{p}(\beta_j - \hat{\beta}_j^r)^2}$$

for a range of values of r, where $\hat{\beta}_j^r$ is the jth coefficient estimate for the best model containing r coefficients. Comment on what you observe. How does this compare to the test MSE plot?

[16]:
```
Beta
```

[16]:
```
array([[0.        ],
       [7.41212581],
       [8.09098401],
       [3.68270568],
       [0.        ],
       [4.53731565],
       [9.0219442 ],
       [0.71447831],
       [2.10138621],
       [6.86903874],
       [0.        ],
       [1.96584887],
       [9.85123182],
       [6.30804308],
       [0.42835617],
       [8.25425396],
       [0.        ],
       [8.11969394],
       [7.67387209],
       [0.        ]])
```

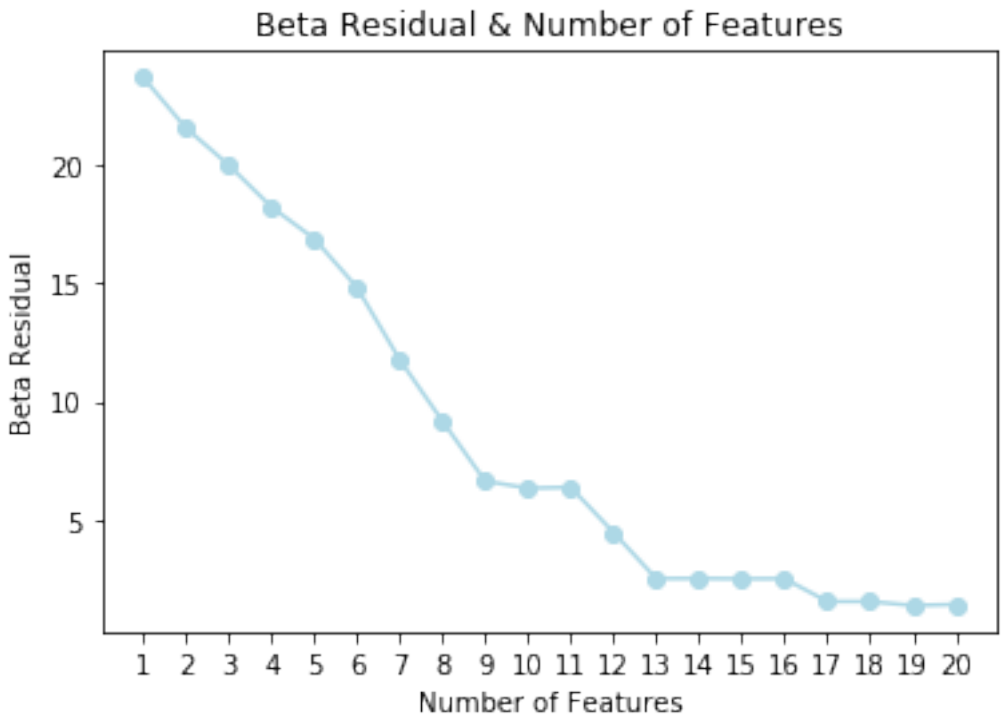
[17]:
```
# get the coefficient for each r
Beta_n = Beta.ravel()
coef_hat = {}
for key, value in result_test.items():
    lm = LinearRegression().fit(X[list(key)], Y)
    coef_hat[key] = lm.coef_

# compute the square root of sum of beta residual squared for each model
```

```
Beta_res = {}
for key, value in coef_hat.items():
    zero = np.zeros(20)
    for n in range(len(value)):
        zero[int(df.columns.get_loc(key[n])) - 1] = value[n]
    Beta_res[len(key)] = (((Beta_n - zero) ** 2).sum()) ** 0.5
Beta_res
# the key represents how many features are included in the model
```

[17]:
```
{1: 23.729779786514,
 2: 21.590407945846284,
 3: 20.017667712533186,
 4: 18.241414276066724,
 5: 16.90109050027116,
 6: 14.840306986506496,
 7: 11.77776854843948,
 8: 9.176623860557708,
 9: 6.632625661480814,
 10: 6.332061782435096,
 11: 6.369344481609852,
 12: 4.460600790391436,
 13: 2.504324563157582,
 14: 2.5108489418470814,
 15: 2.4966200932697684,
 16: 2.5028397896612766,
 17: 1.5350109630372342,
 18: 1.5344099584444775,
 19: 1.366216585737508,
 20: 1.4105674478420807}
```

[18]:
```
Beta_rss = []
for value in Beta_res.values():
    Beta_rss.append(value)
plt.plot(feature_num, Beta_rss, marker='o', color='lightblue')
plt.xlabel('Number of Features')
plt.ylabel('Beta Residual')
plt.title('Beta Residual & Number of Features');
```



The graph suggests an overall downward trend in Beta residuals, but if we check the data computed above, we can find that 19 features yield the lowest Beta residual. Note that the true model contains 15 features, and the model with 15 features has indeed very low Beta residual (second lowest). The reason why the 14-feature model is the lowest is possibly due to the size of training set being too small and test set being too large, along with the irreducible error.

2 Application Exercises

[313]:
```
gss_train = pd.read_csv('gss_train.csv')
gss_test = pd.read_csv('gss_test.csv')
gss_train.head(5)
```

[313]:

	age	attend	authoritarianism	black	born	childs	colath	colrac	colcom	\
0	21	0	4	0	0	0	1	1	0	
1	42	0	4	0	0	2	0	1	1	
2	70	1	1	1	0	3	0	1	1	
3	35	3	2	0	0	2	0	1	0	
4	24	3	6	0	1	3	1	1	0	

	colmil	...	zodiac_GEMINI	zodiac_CANCER	zodiac_LEO	zodiac_VIRGO	\
0	1	...	0	0	0	0	
1	0	...	0	0	0	0	
2	0	...	0	0	0	0	

```
3       1  ...              0              0          0             0
4       0  ...              0              0          0             0

   zodiac_LIBRA  zodiac_SCORPIO  zodiac_SAGITTARIUS  zodiac_CAPRICORN  \
0             0               0                   0                 0
1             0               0                   0                 0
2             0               0                   0                 0
3             0               1                   0                 0
4             0               1                   0                 0

   zodiac_AQUARIUS  zodiac_PISCES
0                0              0
1                0              0
2                0              0
3                0              0
4                0              0

[5 rows x 78 columns]
```

1.Fit a least squares linear model on the training set, and report the test MSE.

[318]:
```
X_train = gss_train.drop(['egalit_scale'], axis=1)
X_test = gss_test.drop(['egalit_scale'], axis=1)
y_train = gss_train['egalit_scale']
y_test = gss_test['egalit_scale']
lm = LinearRegression().fit(X_train, y_train)
MSE = mean_squared_error(lm.predict(X_test), y_test)
print('The test MSE for linear model is', MSE)
```

```
The test MSE for linear model is 63.213629623014995
```

2.Fit a ridge regression model on the training set, with λ chosen by 10-fold cross-validation. Report the test MSE.

[327]:
```
parameters = {'alpha':[1e-15, 1e-10, 1e-8, 1e-4, 1e-3, 1e-2, 1, 5, 10, 20]}
ridge_regressor = GridSearchCV(Ridge(), parameters,␣
 ↪scoring='neg_mean_squared_error', cv=10)
ridge_regressor.fit(X_train, y_train)
print('The best alpha value is', ridge_regressor.best_params_)
print('The training MSE is', -ridge_regressor.best_score_)
```

```
The best alpha value is {'alpha': 20}
The training MSE is 61.148463225383004
```

[325]:
```
MSE = mean_squared_error(ridge_regressor.predict(X_test), y_test)
print('The test MSE for ridge regression is', MSE)
```

```
The test MSE for ridge regression is 62.298944127974785
```

3.Fit a lasso regression on the training set, with λ chosen by 10-fold cross-validation. Report the test MSE, along with the number of non-zero coefficient estimates.

[334]:
```
parameters = {'alpha':[1e-15, 1e-10, 1e-8, 1e-4, 1e-3, 1e-2, 1, 5, 10, 20]}
lasso_regressor = GridSearchCV(Lasso(), parameters,␣
 ↪scoring='neg_mean_squared_error', cv=10)
lasso_regressor.fit(X_train, y_train)
print('The best alpha value is', lasso_regressor.best_params_)
print('The training MSE is', -lasso_regressor.best_score_)
```

```
The best alpha value is {'alpha': 0.01}
The training MSE is 61.37196490960359
```

[329]:
```
MSE = mean_squared_error(lasso_regressor.predict(X_test), y_test)
print('The test MSE for lasso regression is', MSE)
```

```
The test MSE for lasso regression is 62.37909680839509
```

[341]:
```
lasso = Lasso(alpha=0.01)
lasso.fit(X_train, y_train)
print('There are',(lasso.coef_ != 0).sum(),'non-zero coefficient estimates in␣
 ↪the lasso regression.')
```

```
There are 65 non-zero coefficient estimates in the lasso regression.
```

4.Fit an elastic net regression model on the training set, with α and λ chosen by 10-fold cross-validation. That is, estimate models with $\alpha = 0, 0.1, 0.2, \ldots, 1$ using the same values for λ across each model. Select the combination of α and λ with the lowest cross-validation MSE. For that combination, report the test MSE along with the number of non-zero coefficient estimates.

[349]:
```
alpha = [1e-15, 1e-10, 1e-8, 1e-4, 1e-3, 1e-2, 1, 5, 10, 20]
for n in alpha:
    parameters = {'l1_ratio':[0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1],␣
 ↪'alpha':[n]}
    en_regressor = GridSearchCV(ElasticNet(), parameters,␣
 ↪scoring='neg_mean_squared_error', cv=10)
    en_regressor.fit(X_train, y_train)
    print('The best alpha value is', en_regressor.best_params_)
    print('The training MSE at l1_ratio =', en_regressor.
 ↪best_params_['l1_ratio'], 'and alpha(lambda) =',
          en_regressor.best_params_['alpha'], 'is', -en_regressor.best_score_)
```

```
The best alpha value is {'alpha': 1e-15, 'l1_ratio': 0.1}
The training MSE at l1_ratio = 0.1 and alpha(lambda) = 1e-15 is
62.357446358210225
The best alpha value is {'alpha': 1e-10, 'l1_ratio': 0.1}
The training MSE at l1_ratio = 0.1 and alpha(lambda) = 1e-10 is
62.357446293984346
The best alpha value is {'alpha': 1e-08, 'l1_ratio': 0.1}
```

```
The training MSE at l1_ratio = 0.1 and alpha(lambda) = 1e-08 is
62.357439935622324
The best alpha value is {'alpha': 0.0001, 'l1_ratio': 0.1}
The training MSE at l1_ratio = 0.1 and alpha(lambda) = 0.0001 is
62.298858480534875
The best alpha value is {'alpha': 0.001, 'l1_ratio': 0.1}
The training MSE at l1_ratio = 0.1 and alpha(lambda) = 0.001 is
62.00028310967855
The best alpha value is {'alpha': 0.01, 'l1_ratio': 0.1}
The training MSE at l1_ratio = 0.1 and alpha(lambda) = 0.01 is 61.30512590261391
The best alpha value is {'alpha': 1, 'l1_ratio': 0.1}
The training MSE at l1_ratio = 0.1 and alpha(lambda) = 1 is 65.98844258107927
The best alpha value is {'alpha': 5, 'l1_ratio': 0.1}
The training MSE at l1_ratio = 0.1 and alpha(lambda) = 5 is 76.22947380485463
The best alpha value is {'alpha': 10, 'l1_ratio': 0.1}
The training MSE at l1_ratio = 0.1 and alpha(lambda) = 10 is 81.05727865324738
The best alpha value is {'alpha': 20, 'l1_ratio': 0.1}
The training MSE at l1_ratio = 0.1 and alpha(lambda) = 20 is 85.19674518831349
```

[353]:
```
# from the above CV training MSE, we get the lowest MSE at l1 ration = 0.1 and␣
 ↪alpha(lambda) = 0.01
# now we can fit the model to the test set
en = ElasticNet(l1_ratio=0.1, alpha=0.01)
en.fit(X_train, y_train)
MSE = mean_squared_error(en.predict(X_test), y_test)
print('The test MSE for elastic net regression is', MSE)
print('There are',(en.coef_ != 0).sum(),'non-zero coefficient estimates in the␣
 ↪elastic net regression.')
```

```
The test MSE for elastic net regression is 62.3801287740633
There are 76 non-zero coefficient estimates in the elastic net regression.
```

5.Comment on the results obtained. How accurately can we predict an individual's egalitarianism? Is there much difference among the test errors resulting from these approaches?

All models achieved similar test MSE, with ridge being the best and OLS being the worst, but the margin is within 1. Generally, we get a MSE at around 62, which is not so far from the training data, showing that we are not overfitting. Whether there exist better models deserves a further discussion.