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Part 2: Lack of Optimality of K-means

The dataset is:

$$x_1 = 1, \quad x_2 = 2, \quad x_3 = 3, \quad x_4 = 4$$

We use $K = 2$, and initialize centroids as:

$$\mu_1 = 2, \quad \mu_2 = 4$$

Iteration 1: Assign points to nearest centroids

Distance of $x_1 = 1$: $|1-2| = 1$, $|1-4| = 3 \rightarrow$ assign to μ_1

Distance of $x_2 = 2$: $|2-2| = 0$, $|2-4| = 2 \rightarrow$ assign to μ_1

Distance of $x_3 = 3$: $|3-2| = 1$, $|3-4| = 1 \rightarrow$ Tie, assign to smaller index, so assign to μ_1

Distance of $x_4 = 4$: $|4-2| = 2$, $|4-4| = 0 \rightarrow$ assign to μ_2

Thus:

$$C_1 = \{1, 2, 3\}, \quad C_2 = \{4\}$$

Iteration 2: Update centroids

- For cluster $C_1 = \{1, 2, 3\}$: new centroid $= \frac{1+2+3}{3} = 2$
- For cluster $C_2 = \{4\}$: new centroid $= 4$

So centroids remain :

$$\mu_1 = 2 \quad , \quad \mu_2 = 4$$

This is a stable solution (no change in assignments)

Distortion for this clustering

$$J = (1-2)^2 + (2-2)^2 + (3-2)^2 + (4-4)^2 = 1+0+1+0 = 2$$

But is this globally optimal?

consider another partition :

$$C_1 = \{1, 2\} \quad , \quad C_2 = \{3, 4\}$$

- centroids : $\mu_1 = 1.5$, $\mu_2 = 3.5$
- Distortion :

$$(1-1.5)^2 + (2-1.5)^2 + (3-3.5)^2 + (4-3.5)^2 = 0.25 + 0.25 + 0.25 + 0.25$$
$$= 1$$

This is smaller than the $J = 2$ we got before

Explanation:

Since J= 1 < Jconv= 2, the solution found by k-means (with the given initialization) is not globally optimal.

k-means is an alternating minimization method that guarantees a non-increasing SSE at each step but only convergence to a local fixed point. Starting from $\mu_1 = 2$, $\mu_2 = 4$, the algorithm converges to the clustering {1, 2, 3}, {4} with objective value Jconv = 2. The tie at x_3 causes it to be assigned to cluster 1, keeping the centroid at 2 and preventing movement to the globally optimal clustering {1, 2}, {3, 4}, which achieves J = 1.

This demonstrates that k-means can converge to a suboptimal local solution rather than the global optimum, due to its sensitivity to initialization and the nonconvexity of its objective.