Lab 6 Solution

Problem 1:

To find the expectation of N(0,1) distributions using importance sampling from the family of N(0, σ^2) .

The importance distribution is $N(0, \sigma^2)$ for some $\sigma > 0$ is optimal when its variance of corresponding importance estimator is minimum. It has the minimum variance when $\int_{-\infty}^{\infty} \frac{h(x)^2 f(x)^2}{g(x)} dx$ is minimum. On solving, we get that the above integral minimizes for $\sigma = \sqrt{2}$ and hence we have the importance distribution as $N(0, 2)$.

```
# Density of standard normal
standard_normal_pdf <- function(x) {
  dnorm(x, mean = 0, sd = 1)
}

# Density of the importance distribution
importance_density <- function(x) {
  dnorm(x, mean = 0, sd = sqrt(2))
}

# Function to estimate the expectation using importance sampling
importance_sampling_normal <- function(n_samples) {
  # Generate samples from N(0, sqrt(2))
  # ... the optimal importance sampling estimater amongst N(0,sigma^2)
  samples <- rnorm(n_samples, mean = 0, sd = sqrt(2))

  # Calculate weights
  weights <- standard_normal_pdf(samples)/importance_density(samples)

  # Estimate the expectation
  expectation <- mean(samples * weights)
```

```
  return(expectation)
}
```

The above code snippet contains the function that estimates the mean of $N(0,1)$ with the help of the importance distribution $N(0,2)$. It takes the number of samples to be generated as argument and returns the estimated mean.

```
# Number of samples
n_samples <- 1e4

# Function Call
estimated_expectation <- importance_sampling_normal(n_samples)

# Displaying the results
cat("Estimated expectation of N(0,1) distribution:", estimated_expectation)
```

```
Estimated expectation of N(0,1) distribution: 0.002775674
```

From the above result we can see that the estimated mean is quite close to the real mean 0.

Problem 2

To estimate the variance of Gamma(2,4) distribution and variance of the corresponding importance sampling estimator.

```
importance_sampling_gamma_variance <- function(n_samples) {
  # Parameters for original Gamma(2, 4) distribution
  k_orig <- 2
  theta_orig <- 4

  # Parameters for importance Gamma(4, 4) distribution
  k_imp <- 4
  theta_imp <- 4

  # Generate samples from importance density (Gamma(4, 4))
  samples <- rgamma(n_samples, shape = k_imp, rate = theta_imp)

  # Generate samples from importance density (Gamma(3,4))
  samples_3 <- rgamma(n_samples, shape = k_imp - 1, rate = theta_imp)
```

```
  # Calculate weights: f(x)/g(x),
  # ...where f(x) is target density and g(x) is importance density
  weights_4 <- dgamma(samples, shape = k_orig, rate = theta_orig) /
    dgamma(samples, shape = k_imp, rate = theta_imp)

  weights_3 <- dgamma(samples, shape = k_orig, rate = theta_orig) /
    dgamma(samples, shape = k_imp - 1, rate = theta_imp)

  # Estimate variance of Gamma(2,4)
  variance_estimate <- mean(samples^2 * weights_4) -
    (mean(samples_3 * weights_3))^2

  return(variance_estimate)
}
```

The above code snippet has a function that returns the estimated variance of $Gamma(2,4)$ distribution using the importance sampling technique to find the variance.

```
# Number of samples
n_samples <- 1e4

# Function Call
estimated_variance <- importance_sampling_gamma_variance(n_samples)
estimated_variance
```

```
[1] 0.1253061
```

We can see that the expected value of variance is close to that of the theoretical value of the variance $\frac{1}{8}$, in this case.

```
# Estimating the variance of the estimator
k <- 1e3
sample_variances <- numeric(k)
for (i in 1:k){
  sample_variances[i] <- importance_sampling_gamma_variance(n_samples)
}

# Variance of estimator
var(sample_variances)
```

```
[1] 2.442833e-05
```

We can also see that the variance of the estimator is approximately zero.