Lab 11 Solution

Harsh Agrawalla

Problem 1

Estimate the parameters (μ, σ^2) of a normal distribution $\mathcal{N}(\mu, \sigma^2)$ using the **Newton-Raphson method**, and compare with their closed-form MLEs.

Solution:

```
# Simulate data from N(mu, sigma^2), where mu is 5 and sigma = 2

set.seed(123)
x <- rnorm(100, mean = 5, sd = 2)

# No. of data points
n <- length(x)
```

```
# Log-likelihood derivatives

loglik_grad <- function(mu, sigma2, x) {
  dmu <- sum(x - mu) / sigma2
  dsigma2 <- -n / (2 * sigma2) + sum((x - mu)^2) / (2 * sigma2^2)
  return(c(dmu, dsigma2))
}
```

The above function '**loglik_grad**' returns the gradient of the log likelihood function for a particular value of μ and σ^2 .

```
# Setting up the Hessian matrix

loglik_hessian <- function(mu, sigma2, x) {
```

```r
  # Partial double derivative of log likelihood wrt mu
  h11 <- -n / sigma2
  # Non diagonal elements of the Hessian matrix
  h12 <- -2 * sum(x - mu) / sigma2^2
  h21 <- h12  # Symmetric
  # Partial double derivative of log likelihood wrt sigma^2
  h22 <- n / (2 * sigma2^2) - sum((x - mu)^2) / (sigma2^3)

  return(matrix(c(h11, h12, h21, h22), nrow = 2, byrow = TRUE))
}
```

The above function '**loglik_hessian**' returns the Hessian matrix of the log likelihood function for a particular value of μ and σ^2 .

```r
# Newton-Raphson to estimate both mu and sigma^2
newton_raphson_mle <- function(x, mu_init = 0, sigma2_init = 1,
                               tol = 1e-8, max_iter = 1000) {

  # Suitable initial choices for mu and sigma^2
  mu <- mu_init
  sigma2 <- sigma2_init

  for (i in 1:max_iter) {
    grad <- loglik_grad(mu, sigma2, x)
    hess <- loglik_hessian(mu, sigma2, x)

    # Hessian_inverse %*% Gradient(theta_old)
    step <- solve(hess, grad)

    # Updating parameters as per
    # ...theta_new = theta_old - Hessian_inverse %*% Gradient(theta_old)
    mu_new <- mu - step[1]
    sigma2_new <- sigma2 - step[2]

    # Stopping condition
    if (sqrt((mu_new - mu)^2 + (sigma2_new - sigma2)^2) < tol) {
      break
    }

    mu <- mu_new
    sigma2 <- sigma2_new
```

```r
  }

  return(c(mu = mu, sigma2 = sigma2))
}
```

The above function uses the Newton-Raphson Method to find the MLEs of the parameters and returns the estimated parameter values with a tolerance of 10^{-6}.

```r
# Closed-form MLEs
mu_mle <- mean(x)
sigma2_mle <- var(x)

# Closed-form Estimates
cat("  mu       =", mu_mle)
```

```
  mu       = 5.180812
```

```r
cat("  sigma^2 =", sigma2_mle)
```

```
  sigma^2 = 3.332931
```

The above are the closed form MLEs of the parameters μ and σ^2.

```r
# Run Newton-Raphson
mu_init <- 5
sigma2_init <- 4
mle_estimates <- newton_raphson_mle(x, mu_init, sigma2_init)

# Newton-Raphson Estimates:
cat("  mu       =", mle_estimates["mu"])
```

```
  mu       = 5.180812
```

```r
cat("  sigma^2 =", mle_estimates["sigma2"])
```

```
  sigma^2 = 3.299602
```

The above are the MLEs of the parameters μ and σ^2, using the Newton-Raphson method.

```r
# Comparing the results obtained

# Absolute Error
abs_error_mu <- abs(mle_estimates["mu"] - mu_mle)
abs_error_sigma2 <- abs(mle_estimates["sigma2"] - sigma2_mle)

# Relative Error
rel_error_mu <- abs_error_mu / abs(mu_mle)
rel_error_sigma2 <- abs_error_sigma2 / abs(sigma2_mle)

# Squared Error
squared_error_mu <- (mle_estimates["mu"] - mu_mle)^2
squared_error_sigma2 <- (mle_estimates["sigma2"] - sigma2_mle)^2

# Total report
comparison_error <- data.frame(
  row.names = c("Closed Form", "Newton-Raphson", "Absolute Error",
               "Relative Error", "Squared Error"),

  mu = c(mu_mle, mle_estimates["mu"], abs_error_mu,
         rel_error_mu, squared_error_mu),

  sigma2 = c(sigma2_mle, mle_estimates["sigma2"], abs_error_sigma2,
             rel_error_sigma2, squared_error_sigma2)
)

print(comparison_error)
```

```
                         mu      sigma2
Closed Form    5.180812e+00 3.332931321
Newton-Raphson 5.180812e+00 3.299602008
Absolute Error 5.329071e-15 0.033329313
Relative Error 1.028617e-15 0.010000000
Squared Error  2.839899e-29 0.001110843
```

From the above values of errors, we can see that the MLEs obtained by Newton-Raphson are quite close to that of the closed form MLEs of μ and σ^2.