Essay II: An Abstract Concept of Entropy and Its use in Inductive Logic

Summary. The aim is to construct an abstract concept of entropy and, with its help, a concept of degree of confirmation, applicable to any system of N elements of any kind which are characterized by quantitative magnitudes. As a preliminary for this aim, the following concepts have been defined for a simple classification system: degree of order $\mathfrak{o}^*$ and degree of confirmation c^* (§2), degree of disorder d^* and entropy S^* (§3). Then the abstract concept S^{**} for elements with quantitative magnitudes is defined. The definition is given first in a more general form with weights assigned to the elements (§11) and then in a simpler form without weights (§12). On the basis of S^{**}, we define the degree of disorder d^{**}, the degree of order $\mathfrak{o}^{**}$, the probability density ϑ, the measure function m^{**}, and finally the degree of confirmation c^{**} (§13). These definitions are such that the relations between these concepts are analogous to the relations between the corresponding concepts for a simple classification (§§2, 3). However, in distinction to the earlier c^*, c^{**} is applicable to our present systems involving quantitative magnitudes. Theorems of inductive logic concerning ϑ and c^{**} are developed in greater detail for a schema with only one magnitude (§§14, 15).

11
The abstract concept of entropy

Summary. We use here the concepts explained in §1. Each element a_i $(i = 1, \ldots, N)$ is represented by its phase point b_i in the μ-space. We assign to each phase point b_i an environment e_i such that every point in the μ-space belongs to the environment of the nearest phase point ((11-6), fig. 1). By the use of these environments Boltzmann's cell system becomes unnecessary. Thus the D^{st} and the D^{ind} drop out and we have to deal only with D^{prec}. But our definitions will refer to the precise propositions p^{prec}, i.e. to the systems of values u_{ij}, rather than to the D^{prec}, because not all of the p^{prec} are expressible by D^{prec}. A modified version S'_B of Boltzmann's S_B is defined in terms of the environments (11-9); S'_B is a continuous function of the values u_{ij}. Our abstract concept of entropy S^{**} is defined (11-10) in analogy to Boltzmann's concept. For greater generality, this definition permits the assignment of different weights to the elements. (The second part of this section involving weights may be skipped, because later we shall use chiefly the simple schema without weights dealt with in §12.)

We go back to the concepts explained in §1. They refer to a system of N elements a_i $(i = 1, \ldots, N)$, each characterized by the values of n magnitudes ϕ_j $(j = 1, \ldots, n)$, with $\phi_j(a_i) = u_{ij}$. The nature of the elements is not specified; they need not be molecules or other physical particles; they may be entities of any kind whatever. Each element a_i is represented by its phase point b_i in the n-dimensional μ-space. R^μ is the total range, i.e., the region of admitted phase points. It has a finite volume V^μ. We make the assumption:

(11-1) No two distinct elements have exactly the same values of all n magnitudes. Hence distinct elements have distinct phase points.

This assumption is certainly fulfilled if the elements are physical bodies or particles at a given time t_1 and the magnitudes include the coordinates of position at t_1 (as in the Boltzmann case).

Boltzmann[1] considers the state of a gas body g at a given time t. g consists of N molecules, each characterized by n magnitudes ϕ_j. This is a special case of the kind explained in §1; but Boltzmann transforms it by the introduction of a cell system into a classification system (as discussed in §§2 and 3). For each magnitude ϕ_j, its interval of admitted values is divided into small intervals of equal

length Δ_j. Hereby the μ-space is divided into a system Ω^μ of *cells* of equal volume $v^\mu = \Delta_1 \ldots \Delta_n$. Let K be the number of these cells within the total range R^μ; then:

$$v^\mu = \frac{V^\mu}{K}.$$

These μ-cells Q_j ($j = 1, \ldots, K$) are analogous to the cells Q_j in the classification system (§2).

The introduction of the system of cells is a very ingenious device by which Boltzmann transforms the problem of defining entropy within the framework of the kinetic theory into a problem concerning the simple schema of a K-fold classification. While in the kinetic theory (in its classical form) the possible states constitute an infinite and even non-denumerable set, in the K-schema the number of D^{st} and even that of D^{ind} is finite.

Let f_j be the density in Q_j, i.e., the number of molecules per unit of μ-volume:

$$f_j =_{Df} \frac{N_j}{v^\mu}.$$

Boltzmann defines his entropy concept, which we denote by 'S_B,' for a given D_i^{st} with cell numbers N_j as follows:

(11-2) $$S_B(D_i^{st}) =_{Df} -k \sum_{j=1}^{K} [f_j \ln f_j] v^\mu,$$

where $k = 1.38 \times 10^{-16}$ erg/°C.

We have, as is customary, assumed that the cells Q_j in the μ-space have equal volumes v^μ. However, this assumption is not necessary. We could use cells Q_j with different volumes v_j and replace 'v^μ' in the definition of S_B by 'v_j' and define 'f_j' by 'N_j/v_j.' Suppose that two cells Q_j' and Q_j'' with equal densities $f' = f''$ are merged into one new cell Q_j with $v_j = v_j' + v_j''$ (v_j' and v_j'' need not be equal) and $N_j = N_j' + N_j''$. Then in Q_j, $f = f' = f''$. The two terms $(f' \ln f')v_j'$ and $(f'' \ln f'')v_j''$ in the $\sum$ for the first system are replaced for the second system by one term $(f \ln f)v_j$, which is equal to the sum of the two terms. Thus $\sum$ and S_B remain unchanged.

The definition of our abstract concept S^{**} will be made analogous to Boltzmann's definition for S_B. It will differ from the latter only in the way of dividing R^μ into parts. Boltzmann chooses an arbitrary number K of parts (cells) of equal volume v^μ; then the

[1] L. Boltzmann, *Vorlesungen ueber Gastheorie*, Teil I (Leipzig: J. A. Barth, 1896).

density f_j is N_j/v^μ. We choose instead a fixed equal number of phase points in each part, namely one; that is, we assign to each phase point b_i a part of R^μ containing it, which we call the *environment* e_i of b_i. The volumes v_i of different environments will in general be different. Then the density in e_i could be taken as $1/v_i$ (this will later be modified).

We shall soon give a definition of the environments e_i for a given system. The choice of the particular definition is not important, but it is essential that any system of environments e_i fulfill the following conditions:

(11-3) **(a)** Every e_i $(i = 1, \ldots, N)$ is a subset of R^μ.
(b) For every e_i, its volume $v_i > 0$.
(c) The environments of two distinct phase points do not overlap.
(d) Almost every point of R^μ (i.e., every point with the possible exception of a set of points of measure 0) belongs to one of the N environments; hence $\sum_{i=1}^{N} v_i = V^\mu$.

Let x and x' be two points in the μ-space; let u_j $(j = 1, \ldots, n)$ be the coordinates of x and u'_j those of x'. We define the *distance* D in the usual pythagorean form:

(11-4) $$D^2(x, x') =_{\mathrm{Df}} \sum_{j=1}^{n} (u_j - u'_j)^2.$$

From (11-1)

(11-5) The distance of the phase points of two distinct elements is > 0.

We now define the environments in terms of distances:

(11-6) The *environment* e_i of the phase point $b_i =_{\mathrm{Df}}$ the set of those points X of R^μ for which the distance between X and b_i is smaller than the distance between X and any other of the N phase points.

The *diagram* (fig. 1) represents a two-dimensional μ-space. The rectangular area within the heavy line represents the total range R^μ. It is divided into smaller areas representing the environments; each contains a dot representing one of the N phase points b_i. In

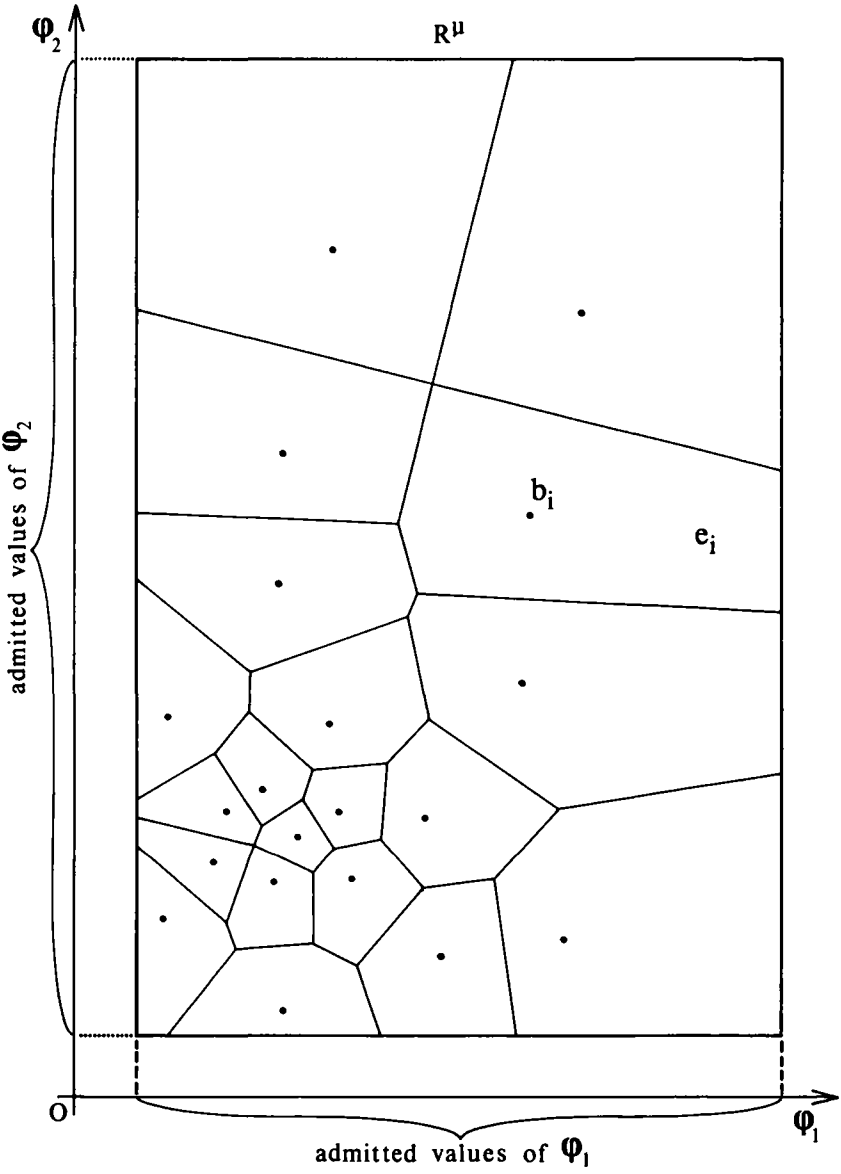

Figure 1. Environments in a 2-dimensional phase space.

accordance with the definition (11-6), in this diagram a segment marking the boundary between the environments e_i of b_i and e_i' of b_i' is part of the straight line which is the perpendicular bisector of the segment b_ib_i'. Analogously, in the n-dimensional μ-space, the boundaries between environments are parts of $(n-1)$-dimensional hyperplanes which bisect perpendicularly the segments between two phase points. Consequently, the set of all boundary points (which are the points equidistant from the two nearest phase points) has measure 0; thus the requirement (d) in (11-3) is fulfilled. It is easily seen that the same holds for (a), (b), and (c); (b) follows from (11-5) because the number N of phase points is finite.

In order to make our schema more general, we shall admit the possibility that to each element a_i a positive real number w_i is assigned as its *weight*. For example, the assignment of weight 3 to a_1 and of the weight 5 to a_2 would mean that, in the calculation of densities in the regions of phase space and then of the entropy and of derivative magnitudes, we are to proceed as if there were not one element with the characteristics of a_1 but three, and not one but five with those of a_2; or six like a_1 and ten like a_2, etc. Thus all that matters are the ratios of the weights, not their absolute values. We normalize the weights in such a way that the mean weight is 1; hence:

(11-7) $$\sum_{i=1}^{N} w_i = N.$$

Then we define density thus:

(11-8) $$f_i =_{Df} w_i/v_i.$$

The sum in the definition of S_B will therefore be replaced by

$$\sum_{i=1}^{N} \left[\frac{w_i}{v_i} \ln \left(\frac{w_i}{v_i}\right)\right] v_i = \sum_i \left[w_i \ln \frac{w_i}{v_i}\right].$$

While Boltzmann's definitions of S_B referred to cell numbers N_j and were applied to D^{st}, our definitions for S^{**} and other concepts will refer only to the environment volumes v_i. Thus we have here no longer to do with D^{st} or D^{ind}, but only with D^{prec}. A D^{prec} is supposed to be a description or statement, formulated in the language, say, of physics, which gives for each element a_i and each function ϕ_j the value $\phi_j(a_i) = u_{ij}$. However, a language system cannot possibly contain expressions for all real numbers of a given interval; therefore it cannot contain D^{prec} for all possible value sets

u_{ij}. Since these value sets are the arguments of our basic concepts, we shall draw the practical consequence from this limitation of all language systems. We shall not refer any longer to D^{prec} but instead to the *precise propositions*, p^{prec}, i.e., the possible value sets u_{ij} themselves. In any given language system, some p^{prec} can be expressed by D^{prec} while all the others are inexpressible. Thus we shall henceforth speak of all p^{prec} for a given system of N elements a_i and n functions ϕ_j without regard to expressibility, just as a theorem of classical mathematics refers to all real numbers without discrimination between those which are expressible in a given language system and those which are not.

For the special case of Boltzmann's theory for a homogeneous gas, we take simply $w_i = 1$. Then we obtain the following definition for an alternative S'_B to Boltzmann's S_B:

$$\textbf{(11-9)} \qquad S'_B(p^{\text{prec}}) =_{\text{Df}} k \sum_i \ln v_i.$$

In distinction to S_B, this concept S'_B is not based on an arbitrary cell system and is a continuous function of the u_{ij}.

Since the abstract concept is not restricted to gas molecules but intended to be applicable to arbitrary elements, we may replace in its definition Boltzmann's gas constant k by an arbitrary constant A (>0). For our version S^{**} we choose $A = 1/\ln 2$, as we did for S^* (3-23). One might consider to insert into the definition an additive constant, possibly dependent on N and V^μ. We shall, however, not do so, for the following reason. With respect to the definitions to be given in §13, the insertion would result in the appearance of a constant factor in the definitions of o^{**}; but the definition of ϑ would remain unchanged because of the normalization condition (13-13), and hence also the definitions of m^{**} and c^{**}. Therefore we define the *abstract entropy concept* S^{**} as follows:

$$\textbf{(11-10)} \qquad S^{**}(p_k^{\text{prec}}) =_{\text{Df}} \sum_{i=1}^{N} \left[w_i \operatorname{Log} \frac{v_i}{w_i} \right]$$

(where 'Log' denotes the logarithm to base 2).

$S^{**}(p_k^{\text{prec}})$ is a continuous function of the N values v_i, and each of these values in turn is a continuous function of the nN values u_{ij}. [The latter would not be the case if a phase point could not only approach another one at any arbitrarily small distance, but could finally coincide with it; this, however, is excluded by our assumption (11-1).]

The entropy $S^{**}(p_k^{\text{prec}})$ may be regarded as the sum of N unequal parts, each attributed to one element. We define the *partial*

entropy of an element a_i with respect to a given p_k^{prec} as:

(11-11) $$S_{\text{el}}^{**}(a_i, p_k^{\text{prec}}) =_{\text{Df}} w_i \operatorname{Log} \frac{v_i}{w_i}.$$

Then we have:

(11-12) $$S^{**}(p_k^{\text{prec}}) = \sum_i S_{\text{el}}^{**}(a_i, p_k^{\text{prec}}).$$

Let the region R_j be a part of R^μ consisting of N_j environments e_{ji} $(i = 1, \ldots, N_j)$ with volumes v_{ji} and weights w_{ji}, with respect to a given p_k^{prec}. Then the *partial entropy* of R_j with respect to p_k^{prec} is defined as follows:

(11-13) $$S_{\text{part}}^{**}(R_j, p_k^{\text{prec}}) =_{\text{Df}} \sum_{i=1}^{N_j} \left[w_{ji} \operatorname{Log} \frac{v_{ji}}{w_{ji}} \right].$$

If R^μ consists of K non-overlapping regions R_j $(j = 1, \ldots, K)$ we have:

(11-14) $$S^{**}(p_k^{\text{prec}}) = \sum_{j=1}^{K} S_{\text{part}}^{**}(R_j, p_k^{\text{prec}}).$$

If R_j is a region as above, its volume is

(11-15) $$V_j = \sum_{i=1}^{N_j} v_{ji},$$

and its weight is

(11-16) $$W_j = \sum_{i=1}^{N_j} w_{ji}.$$

Now suppose that R_j is such that the density f has the same value throughout R_j; in other words, the v_{ji} are proportional to the weights w_{ji}; say for every i from 1 to m_j, $v_{ji} = Cw_{ji}$. Then $V_j = \sum Cw_{ji} = CW_j$. Hence $v_{ji}/w_{ji} = C = V_j/W_j$. Thus with (11-13) and (11-16):

(11-17) $$S_{\text{part}}^{**}(R_j, p_k^{\text{prec}}) = W_j \operatorname{Log} \frac{V_j}{W_j}.$$

We denote the mean environment volume by $\bar{v}$:

(11-18) $$\bar{v} =_{\text{Df}} \frac{V^\mu}{N}.$$

S^{**} has its maximum if the density f_i is the same throughout R^μ. In this case R^μ is like the R_j in (11-17), with $V_j = V^\mu$ and

$W_j = N$ (11-6). Hence:

$$S^{**}_{\max} = N \operatorname{Log} \frac{V^{\mu}}{N} = N \operatorname{Log} \bar{v}. \tag{11-19}$$

Suppose that with respect to a given p_k^{rec}, R^{μ} can be divided into K regions R_j $(j = 1, \ldots, K)$ with volumes V_j and weights W_j such that within each of these regions the density f is the same for all environments. Then with (11-14) and (11-17):

$$S^{**}(p_k^{\text{rec}}) = \sum_{j=1}^{K} \left[W_j \operatorname{Log} \frac{V_j}{W_j} \right]. \tag{11-20}$$

12
A Simpler Schema Without Weights

Summary. The schema is simplified by abandoning the weights. Then S^{**} is simply the sum of the logarithms of the volumes v_i of the environments e_i (12-2)(a). We compare Boltzmann's S_B, which neglects differences of density within the cells, with our version S'_B, which reflects such differences. It is shown that, if the density is constant within each cell, both functions have the same value.

The assignment of weights to the elements seems desirable only for special purposes. In general, it seems sufficient to use a simpler schema, dropping the weights. This schema may be regarded as a special case of the former one, with all $w_i = 1$. Thus from (11-7):

(12-1) $$f_i = 1/v_i.$$

The definition (4-10) of *the abstract entropy concept* S^{**} becomes now:

(12-2) (a) $$S^{**}(p^{\text{prec}}) =_{\text{Df}} \sum_{i=1}^{N} \text{Log}\, v_i.$$

(b) $$= \text{Log} \prod_{i=1}^{N} v_i.$$

In the same way we obtain the following definitions from (11-11), (11-13), etc.: The partial entropy of an element:

(12-3) $$S^{**}_{\text{el}}(a_i, p^{\text{prec}}) =_{\text{Df}} \text{Log}\, v_i.$$

The partial entropy of a region R_j consisting of N_j environments e_{ji} with volumes v_{ji}:

(12-4) $$S^{**}_{\text{part}}(R_j, p^{\text{prec}}_k) =_{\text{Df}} \sum_{i=1}^{N_j} \text{Log}\, v_{ji}.$$

The theorems (11-12) and (11-14) remain valid, of course.

The case of a region R_j with constant density is simpler here than previously. Here R_j consists of N_j environments of equal volume v_j, and hence has volume $V_j = N_j v_j$. The earlier W_j (11-16) is here N_j. Thus the partial entropy of R_j is now (from (11-17)):

(12-5) $$S^{**}_{\text{part}}(R_j, p^{\text{prec}}_k) = N_j \text{Log}\, v_j.$$

(This also follows directly from (12-4).)

Let R^μ consist of K regions R_j $(j = 1, \ldots, K)$ such that in each region R_j all N_j environments have the same volume $v_j = V_j/N_j$. Then from (12-5) (in analogy to (11-20)):

$$\textbf{(12-6)}\ \textbf{(a)}\quad S^{**}(p_k^{\text{prec}}) = \sum_{j=1}^{K}\left[N_j \operatorname{Log}\frac{V_j}{N_j}\right],$$

$$\textbf{(b)}\qquad = \sum_{j=1}^{K}[N_j \operatorname{Log} v_j].$$

In the special case that all $V_j = V^\mu/K$, we have:

$$\textbf{(12-7)}\qquad S^{**}(p_k^{\text{prec}}) = \sum_{j=1}^{K}\left[N_j \operatorname{Log}\frac{V^\mu}{N_j K}\right],$$

$$= -\sum_j [N_j \operatorname{Log} N_j] + N \operatorname{Log}\frac{V^\mu}{K}.$$

Analogously, for our version S'_B (11-9) of Boltzmann's concept:

$$\textbf{(12-8)}\qquad S'_B(p^{\text{prec}}) = -k\left\{\sum_j [N_j \ln N_j] - N \ln\frac{V^\mu}{K}\right\}.$$

Boltzmann's cells are regions with equal volume $v^\mu = V^\mu/K$. Thus, comparing (12-8) with (11-2), we see that if the phase points in each cell are uniformly distributed, i.e., have equal environments, then S'_B and S_B (the latter taken for the D^{st} corresponding to the given p^{prec}) are exactly equal. If the cell numbers N_j remain the same but the phase points in at least one of the cells are unevenly distributed, then S'_B reflects this fact by having a somewhat smaller value than in the former case. On the other hand S_B remains the same, because it depends merely upon the cell numbers and disregards the distributions within the cells. Thus S'_B reflects more faithfully local changes in the physical state.

S^{**} has its *maximum* if all environments have the same volume $\bar{v} = V^\mu/N$. Hence with (12-2)(a), in agreement with (11-19):

$$\textbf{(12-9)}\qquad S^{**}_{\max} = N \operatorname{Log} \bar{v}.$$

The problem of the *minimum* of S^{**} is more complicated. We saw earlier that z and therefore S^* and S_B have their minimum values if all elements belong to the same cell. With respect to a p^{prec}, there are no cells. But it is easily seen that here S^{**} is smaller the closer the phase points are crowded together. Let us assume that all phase points are compressed into a dense cloud occupying only a very small part of R^μ. Then the environments of the phase points in the

interior part R_1 of the cloud, which constitute a great majority, say N_1, have very small volumes v_1. We assume that these volumes are equal, say $= v_1$. Outside of R_1, in the remaining region R_2 with volume $V_2 = V^\mu - V_1$, are only the phase points of the external fringe of the cloud. We assume that their number N_2 is only a very small part of the total; hence $r_2 = N_2/N$ is a very small fraction. Because of their relatively small number it does not matter much for the total S^* whether their v_i-values are equal or not. Therefore we may, for an approximate result, take them as being equal, say $= v_2$. Then $V_2 = N_2 v_2$.

According to (12-5), the partial S^{**} for R_1 is $N_1 \operatorname{Log} v_1 = N(1-r_2) \operatorname{Log} v_1$. That for R_2 is $N_2 \operatorname{Log} v_2 = Nr_2 \operatorname{Log} (V_2/Nr_2); \cong Nr_2 \operatorname{Log} (V^\mu/Nr_2)$, because $V_2 \cong V^\mu$. Hence for the total S^{**}:

(12-10) **(a)** $$S^{**} \cong N\left\{\operatorname{Log} v_1 + r_2 \operatorname{Log} \frac{V^\mu}{Nv_1} + r_2 \operatorname{Log} \frac{1}{r_2}\right\},$$

(b) $$\cong N \operatorname{Log} v_1.$$

The cruder approximation (b) is obtained by dropping the terms with the factor r_2.

Thus S^{**} is the smaller, the smaller v_1. If the cloud of phase points is further compressed into half its previous volume, then S^{**}, according to (12-10)(b), decreases by N. Now we must distinguish two cases.

1. According to our assumptions, every $v_i > 0$ (11-3)(b). First we consider the case that for given N, n, and V^μ there is no positive lower bound for v_1 (although, of course, in any given p^{prec} there is a positive minimum v_i). Then there is no positive lower bound for S^{**}.

2. Suppose the general schema is such that there is a minimum admissible distance between two phase points and hence a minimum possible v_i, say $v_{\min}$, in other words a minimum cloud volume $Nv_{\min}$. Then there is a minimum for S^{**}; with (12-10)(b):

(12-11) $$S^{**}_{\min} \cong N \operatorname{Log} v_{\min}.$$

[We considered above a system, say p_1^{prec}, in which all phase points are concentrated in one cloud. Let us now consider systems p_k^{prec} $(k = 1, 2, \ldots)$ with a small number k of separate clouds of equal volume, all with the same v_1 as before. We shall find that $S^{**}(p_k^{\text{prec}})$ slightly increases with k, and therefore that among these systems (with given N, n, V^μ, and v_1) p_1^{prec} has the smallest S^{**}. For the sake of simplicity, let us assume that each of the k clouds has

the shape of an n-dimensional hypercube and that the phase points are arranged in each cloud in an n-dimensional cubic lattice with distance D. Hence $v_1 = D^n$. Let the number of phase points along each edge of a hypercube be m. (We assume that N/k is sufficiently large so that the difference between it and the nearest nth power of an integer, viz. m^n, may be neglected; and that m is considerably larger than $2n$.) Then each cloud contains m^n phase points. Hence $N = km^n$ and $m = \sqrt[n]{\frac{N}{k}}$. The number of interior phase points in each cloud is $(m-2)^n = m^n\left(1-\frac{2}{m}\right)^n \cong m^n\left[1-\frac{2n}{m}\right]$, where $\frac{2n}{m}$ is a small fraction. r_1 is the ratio of interior phase points to all phase points, which is equal to this ratio in each cloud; hence $r_1 \cong 1-\frac{2n}{m}$. Then $r_2 = 1-r_1 \cong \frac{2n}{m} \cong \frac{2n\sqrt[n]{k}}{\sqrt[n]{N}}$. Thus r_2 increases slowly with k. Suppose that N, n, V^μ, and v_1 are given. We see from (12-10)(b) that S** is in the first approximation independent of r_2 and hence of k; in other words, dividing the cloud into k separate parts does not make much difference for the value of S^{**}. (12-10)(a) shows that S^{**} increases with r_2 (since $r_2 \operatorname{Log}\frac{1}{r_2}$ increases with r_2 for $r_2 < e^{-1} \approx 0.368$) and hence with k. Thus S^{**} increases slightly if the cloud is divided into separate parts.]

13
Degree of Confirmation

Summary. We defined earlier (§2 and §3) for a classification schema, first the degree of order o^*, then the degree of disorder $d^* = 1/o^*$, and finally the entropy $S^{**} = \text{Log } d^*$ (+ const.). Now, for the quantitative schema, we proceed analogously in the inverse direction. We have defined S^{**}; we now define d^{**} such that $\text{Log } d^{**} \simeq S^{**}$, and $o^{**} = 1/d^{**}$. For the schema without weights (§12), we have simply $o^{**} = \prod_i (1/v_i)$ (13-9). In order to obtain a concept of degree of confirmation c^{**} for quantitative systems analogous to the earlier c^*, we define the probability density ϑ as proportional to o^{**}, then the measure function m^{**} for a proposition p as the integral of ϑ over the range of p in the γ-space, and finally, in the usual way, $c^{**}(h, e)$ as $m^{**}(e \cdot h)/m^{**}(e)$.

Earlier (§3), in the case of a simple classification system, we defined the entropy S^* on the basis of the degree of disorder d^*. Now we have defined the entropy S^{**} for a system involving quantitative magnitudes, without making use of any concept of degree of disorder. However, we can now find a concept d^{**}, the degree of disorder for a quantitative system, by requiring that the relation between S^{**} and d^{**} be analogous to that between S^* and d^*. On the basis of d^{**}, we define the degree of order as its reciprocal and then the degree of confirmation in analogy to the earlier procedure.

S^* was defined in such a way that it was equal to Log d^*, aside from an additive constant dependent upon N and K (see (3-27)). In the case of S^{**}, we might analogously add a constant dependent on N and n. However, the value of this constant is inessential, since it would not influence the value of the degree of confirmation (see (13-17) below). Therefore we shall omit the constant. Thus we wish d^{**} to be such that

$$\begin{aligned} \text{Log } d^{**}(p_k^{\text{prec}}) &= S^{**}(p_k^{\text{prec}}), \\ &= \sum_{i=1}^{N} \left[w_i \text{ Log } \frac{v_i}{w_i} \right] \quad (11\text{-}10) \\ &= \text{Log} \prod_i \left(\frac{v_i}{w_i} \right)^{w_i}. \end{aligned}$$

Therefore we define the *degree of disorder*:

(13-1) $$d^{**}(p_k^{\text{prec}}) =_{\text{Df}} \prod_{i=1}^{N} \left(\frac{v_i}{w_i} \right)^{w_i}.$$

d^{**} has, like S^{**}, its maximum if the density is the same throughout R^{μ}. Therefore, with (11-19):

(13-2) $$d^{**}_{\max} = \bar{v}^N, \quad \text{where} \quad \bar{v} = V^{\mu}/N.$$

For the schema without weights (i.e., $w_i = 1$, §12):

(13-3) $$d^{**}(p_k^{\text{prec}}) =_{\text{Df}} \prod_i v_i.$$

If r^{μ} consists of K regions $R_j (j = 1, \ldots, K)$ such that in R_j all N_j environments have the same volume v_j, we have:

(13-4) $$d^{**}(p_k^{\text{prec}}) = \prod_{j=1}^{K} v_j^{N_j}.$$

If all phase points are concentrated in a cloud of small volume V_c with equal density, then approximately (from (12-10)(b)):

(13-5) $$d^{**} \cong v_1^N, \quad \text{where} \quad v_1 = \frac{V_c}{N}.$$

If the schema is such that there is a minimum $v_{\min}$ for the possible environment volumes v_i, then according to (13-5):

(13-6) $$d^{**}_{\min} \cong v_{\min}^N.$$

If there is no positive lower bound for the possible v_i, then d^{**} has no minimum, and its greatest lower bound is 0.

Originally we defined disorder as the reciprocal of order (3-1). Thus now we take o^{**} as the reciprocal of d^{**}:

(13-7) $$o^{**}(p_k^{\text{prec}}) =_{\text{Df}} \frac{1}{d^{**}(p^{\text{prec}})}.$$

Hence we have for a system with weights:

(13-8) $$o^{**}(p_k^{\text{prec}}) = \prod_{i=1}^{N} \left(\frac{w_i}{v_i}\right)^{w_i},$$

and for a system without weights:

(13-9) $$o^{**}(p_k^{\text{prec}}) = \prod_{i=1}^{N} \frac{1}{v_i}.$$

From (13-2):

(13-10) $$o^{**}_{\min} = \left(\frac{1}{\bar{v}}\right)^N, \quad \text{where} \quad \bar{v} = V^\mu/N.$$

If there is a minimum $v_{\min}$ of the possible v_i-values, then

(13-11) $$o^{**}_{\max} = \left(\frac{1}{v_{\min}}\right)^N.$$

Otherwise there is no maximum of o^{**}. If the volume of concentration V_c is compressed to $1/m$ of its previous value, o^{**} is multiplied by approximately m^N (from (13-5)).

Now we come to the problem of defining a concept of degree of confirmation c^{**} for our quantitative systems. c^{**} will be based in the customary way on a measure function m^{**} which represents the initial degree of confirmation. For classification systems (§2) we defined m^* as proportional to the degree of order o^*. For the quantitative systems, we have defined a degree of order o^{**}, and therefore we shall proceed here in a similar way. However, in the present case the situation is somewhat more complicated. The number of possible D^{ind} in a classification schema with given N and K was finite; therefore we could simply ascribe values of m^* to the D^{ind} in such a way that the sum of all these values was 1. This is not possible for the p^{prec} in a quantitative schema, because they form a continuum. Therefore we must ascribe to them values of an *m-density function* ϑ such that the integral of ϑ over the whole continuum is 1. We shall take ϑ to be proportional to o^{**}.

Previously we represented each of the N elements a_i, characterized by the n values $u_{i1}, \ldots, u_{in}$, by a phase point in the n-dimensional μ-space (molecule-space). Following Gibbs, we introduce now a second phase-space, the so-called *γ-space* (gas-space). In this Nn-dimensional space the whole system of N elements, characterized by the Nn values $u_{11}, \ldots, u_{in}$, $u_{21}, \ldots, u_{2n}, \ldots, u_{N1}, \ldots, u_{Nn}$, in other words, the p^{prec}, is represented by one point. We shall use 'U' as a variable for an ordered set of Nn values $u_{11}, \ldots, u_{Nn}$ as coordinates and hence for a point in the γ-space. The total range R^μ in the μ-space determines the total range R^γ in the γ-space. Let V^γ be the (Nn-dimensional) volume of R^γ; then $V^\gamma = (V^\mu)^N$.

The values of the function $o^{**}(p^{\text{prec}})$ will also be assigned to the phase points U representing the p^{prec}. Thus we shall have a function $o^{**}(U)$ in the γ-space. The density function δ to be defined may likewise be regarded as a function $\delta(U)$ in the γ-space.

Now we define δ as proportional to o^{**}:

(13-12) **(a)** For a point U in R^{γ}, $\delta(U) =_{\mathrm{Df}} Co^{**}(U)$.

(b) For a point U outside of R^{γ}, $\delta(U) =_{\mathrm{Df}} 0$.

The proportionality factor C, possibly dependent upon N, n, and V^{μ}, but not on U, is uniquely determined by the normalization requirement previously mentioned:

(13-13) $$\int_{R^{\gamma}} \delta(U)\, dU = 1,$$

where $dU = du_{11} \ldots du_{Nn}$ is a volume differential in the γ-space, and the integral runs over R^{γ} (or over the whole γ-space; this makes no difference because of (13-12)(b)). With (13-12a):

(13-14) $$C \int_{R^{\gamma}} o^{**}(U)\, dU = 1,$$

hence:

(13-15) $$C = \frac{1}{\displaystyle\int_{R^{\gamma}} o^{**}(U)\, dU}.$$

From (13-12)(a):

(13-16) For any U in R^{γ}, $\delta(U) =_{\mathrm{Df}} \dfrac{o^{**}(U)}{\displaystyle\int_{R^{\gamma}} o^{**}(U)\, dU}$.

We shall now prove two results (13-17) and (13-18) concerning the *invariance of* δ under certain conditions. The results show likewise the invariance of m^{**} and c^{**}, because these functions will be defined on the basis of δ. The results hold both for the schema with weights and for that without weights.

(13-17) If $S^{**\prime} = S^{**} + B$, where the additive constant B depends at most on N, n, and V^{μ}, and if $d^{**\prime}$, $o^{**\prime}$, and δ' are defined on $S^{**\prime}$ as d^{**}, o^{**}, and δ were on S^{**}, then δ' and δ coincide.

Proof. $\mathrm{Log}\, d^{**\prime} = S^{**\prime} = S^{**} + B$. Hence $d^{**\prime} = 2^{S^{**}+B} = 2^{B} d^{**}$. Hence $o^{**\prime} = o^{**}/2^{B}$. The factor $1/2^{B}$ cancels out in (13-16), thus δ remains unchanged.

(13-18) δ is invariant with respect to any linear transformation of any of the magnitudes ϕ_j.

Proof. Consider, as an example, ϕ_1. Let $\phi_1'(a_i) =_{\text{Df}} A_1\phi_1(a_i) + A_0$. This means a change in the scale of ϕ_1 with the new unit being $1/A_1$ times the original unit and the zero point shifted by the amount A_0. The difference in ϕ_1 for any two phase points of elements in the μ-space is hereby multiplied by A_1. Therefore v_i is replaced by $v_i' = A_1 v_i$. [In the earlier diagram (fig. 1, in §11), the change may be represented either by shifting the zero point of ϕ_1 by A_0 to the left and taking a new unit $1/A_1$ times the original unit, or by leaving the scale on the ϕ_1-axis intact but shifting the whole diagram to the right and extending it A_1 times in the ϕ_1-direction.] Thus in (11-10) $w_i \operatorname{Log} \frac{v_i}{w_i}$ is replaced by $w_i \operatorname{Log} \frac{A_1 v_i}{w_i} = w_i \operatorname{Log} \frac{v_i}{w_i} + w_i \operatorname{Log} A_1$. Therefore S^{**} is increased by $\sum (w_i \operatorname{Log} A_1) = N \operatorname{Log} A_1$ (from (11-6)). The same result holds for the schema without weights. Hence the assertion with (13-17).

It is important that the invariance (13-18) holds for any one of the magnitudes ϕ_j separately. Thus δ is independent of the more or less arbitrary relations which may hold between the units chosen for different magnitudes. δ is still affected by a change in the scale form of a magnitude ϕ_j, i.e., by a nonlinear transformation. This seems inevitable.

The definition (13-16) for δ can, if desired, be applied to the general schema with weights by using for o^{**} the definition (13-8). We shall, however, *develop our inductive method only for the simple schema without weights,* using (13-9).

The environment volumes $v_1, \ldots, v_N$ in the μ-space depend upon the set of values $U = \{u_{11}, \ldots, u_{Nn}\}$ and may therefore be regarded as functions of $U: v_1(U), \ldots, v_N(U)$. We introduce for abbreviation the function sign $\prod$:

(13-19) $$\prod(U) =_{\text{Df}} \prod_{i=1}^{N} \frac{1}{v_i(U)}.$$

For any subset α of the γ-space, we put

(13-20) $$I(\alpha) =_{\text{Df}} \int_\alpha \prod(U)\, dU, \quad = \int_\alpha \prod_{i=1}^{N} \frac{1}{v_i(U)}\, dU.$$

From (13-16) and (13-9):

(13-21) For any U in R^γ, $\delta(U) = \frac{\prod(U)}{I(R)}$.

Let p be any proposition in the given schema, i.e., concerning the n elements and the n magnitudes ϕ_i. By the *range of* p we understood the set of those p^{prec} in which p holds, or the set R_p of the corresponding phase points in the γ-space. [For the reasons given in §11, we refer here to propositions and not to sentences, as in (Prob.).]
Then we define:

(13-22) $$m^{**}(p) =_{\text{Df}} \int_{R_p} \delta(U)\, dU.$$

Hence

(13-23) $$m^{**}(p) = \frac{I(R_p)}{I(R^\delta)}.$$

We define c^* in the customary way (see (2-9)):

(13-24) $$c^{**}(h, e) =_{\text{Df}} \frac{m^{**}(e \cdot h)}{m^{**}(e)}$$

Hence

(13-25) $$c^{**}(h, e) = \frac{I(R_{e \cdot h})}{I(R_e)}.$$

Suppose that the evidence e is such that for one (or several) of the magnitudes u_{ij} only a finite number of values are compatible with e. Then $I(R_e) = 0$ and, for any h, $I(R_{e \cdot h}) = 0$. Thus (13-25) would yield 0/0; this means that the given definitions are not applicable in this case. The following procedure leads still to a definite c^{**}-value in many cases of this kind. Let $e_{I(\alpha)}$, where α is a subset of R_e, be formed by modifying (13-20) as follows: in the multiple integral, the integral for each magnitude u_{ij} which is confined in e to a finite number of values is replaced by a sum running through these values, and the corresponding differential in dU is dropped; if only one value of a magnitude is compatible with e, this value is substituted in $\Pi(U)$ and no sum is applied. Then the following definitions are used; they are modifications of (13-21) and (13-25), respectively:

(13-26) For any U in R_e, $^e\delta(U) =_{\text{Df}} \frac{\prod(U)}{^eI(R_e)}$.

(13-27) $$c^{**}(h, e) =_{\text{Df}} \frac{^eI(R_{e \cdot h})}{^eI(R_e)}.$$

This procedure will be applied in the next section.

14
Application to a schema with one magnitude

Summary. The concepts defined in the preceding section are now applied to the following case. The N elements are characterized by the values of only one magnitude ϕ_1. The evidence e specifies the values $c_1, \ldots, c_{N-1}$ of ϕ_1 for $a_1, \ldots, a_{N-1}$, respectively. The value u for a_N is unknown. Theorems about the following values are stated: $o^{**} = \prod_i (1/v_i)$, now taken as a function $\prod(u)$, and a modified form $\prod'(u)$; the probability density $\delta(u)$ and a modified form ${}^e\delta(u)$; c^{**}, with respect to the given evidence, for various hypotheses concerning u, among them the hypothesis that u lies within the interval $R_k = (c_k, c_{k+1})$.

We shall now apply the concept of degree of confirmation c^{**} defined in the preceding section to a schema with $n = 1$, i.e., with only one magnitude ϕ_1. Let the interval of admitted values of ϕ_1 be the open interval (A, B).

Throughout our further discussion (§§14, 15), we presuppose a fixed evidence e which specifies fixed values of ϕ_1 for all elements except the last one, a_N:

(14-1) $\quad \phi_1(a_1) = c_1;\ \phi_1(a_2) = c_2;\ \ldots;\ \phi_1(a_{N-1}) = c_{N-1}.$

The ϕ_1-values for distinct elements are distinct from one another (11-1) and from A and B. But we assume that the difference between two ϕ_1-values, or between a ϕ_1-value and A or B may be arbitrarily small. We assume further that the values $c_1, \ldots, c_{N-1}$ are in ascending order:

(14-2) $\quad A < c_1 < c_2 \ldots c_i < c_{i+1} \ldots c_{N-2} < c_{N-1} < B.$

Thus the only value left undetermined in e is $\phi_1(a_N)$. We shall use for this value the variable u, which is restricted as follows:

(14-3) **(a)** $\quad A < u < B.$
(b) $\quad u \neq c_1, \ldots, c_{N-1}.$

On the basis of the given e, U is the value system $\{c_1, \ldots, c_{N-1}, u\}$, where only the last value can vary. Therefore, as an argument expression for a function, we shall often simply write 'u' instead of '$c_1, \ldots, c_{N-1}, u$' or 'U' e.g. '$v_i(u)$,' '$\Pi(u)$,' '$\delta(u)$,' etc. Thus (6-19)

with (6-9) now becomes:

(14-4) $$o^{**}(u) = \Pi(u) = \prod_{i=1}^{N} \frac{1}{v_i(u)}.$$

The μ-space here is one-dimensional and the γ-space N-dimensional. However, although R^γ is an N-dimensional region in the γ-space, R_e is only a one-dimensional subset of R^γ. The first $N-1$ coordinates in R_e are fixed as $c_1, \ldots, c_{N-1}$, and R_e is simply the segment (A, B) of the Nth coordinate. The N-dimensional volume of R_e is 0, hence $I(R_e) = 0$. Therefore, to avoid the result 0/0, we must apply here the procedure explained at the end of the preceding section. We form ${}^eI(\alpha)$ by dropping in (13-20) the first $N-1$ integrals and differentials, leaving only those with u; no sums are needed here, since for each a_i ($i = 1$ to $N-1$) one value of ϕ_1 is fixed in e. Thus we have:

(14-5) For any subset α of R_e,

$$ {}^eI(\alpha) =_{\mathrm{Df}} \int_{u \text{ in } \alpha} \Pi(u)\, du$$
$$= \int \left(\prod_{i=1}^{N} \frac{1}{v_i(u)} \right) du$$

Then (13-26) and (13-27) can be applied. To the values $c_1, \ldots, c_{N-1}$ we add two fictitious ones, for simplifying formulas: c_0 for the mirror image of c_1 with respect to A, and c_N for the mirror image of c_{N-1} with respect to B (see the diagram; note that the coordinate of a_N is not c_N, but rather u):

(14-6) **(a)** $c_0 =_{\mathrm{Df}} 2A - c_1$.
(b) $c_N =_{\mathrm{Df}} 2B - c_{N-1}$.

The *diagram* (fig. 2) shows the one-dimensional μ-space. The total range R^μ is the interval (A, B), represented by a heavy line. We introduce a notation '$R_.$' for some open subinterval and '$D_.$' for

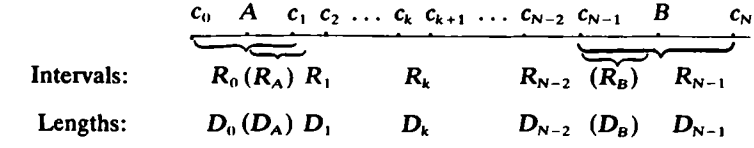

Figure 2. Fixed points and intervals given by e.

their lengths:

(14-7) **(a)** For $i = 0, \ldots, N-1$, $R_i =_{\mathrm{Df}} (c_i, c_{i+1})$.
(b) $R_A =_{\mathrm{Df}} (A, c_1)$.
(c) $R_B =_{\mathrm{Df}} (c_{N-1}, B)$.

(14-8) **(a)** For $i = 0, \ldots, N-1$, $D_i =_{\mathrm{Df}} c_{i+1} - c_i$.
(b) $D_A =_{\mathrm{Df}} c_1 - A = \frac{1}{2} D_0$.
(c) $D_B =_{\mathrm{Df}} B - c_{N-1} = \frac{1}{2} D_{N-1}$.

We further introduce '${}^e v_i$' ($i = 1, \ldots, N-1$), where 'e' refers to the given evidence, as auxiliary terms for our calculations. ${}^e v_i$ is, so to speak, the volume of the fictitious environment which the phase point a_i would have on the basis of the evidence e alone, i.e., if the elements $a_1, \ldots, a_{N-1}$ as described in e were the only elements. In this fictitious case, according to (11-5), the environment corresponding to a_i would be the interval $\left(\frac{c_{i-1} + c_i}{2}, \frac{c_i + c_{i+1}}{2}\right)$. (Because of (14-6), this holds also for $i = 1$ and $i = N-1$.) Therefore we define:

(14-9) For $i = 1, \ldots, N-1$, ${}^e v_i =_{\mathrm{Df}} \frac{1}{2}(c_{i+1} - c_{i-1}) = \frac{1}{2}(D_{i-1} + D_i)$.

In terms of these fictitious volumes we define:

(14-10)
$$\Pi_e =_{\mathrm{Df}} \prod_{i=1}^{N-1} \frac{1}{{}^e v_i} = \frac{2^{N-1}}{\prod_{i=1}^{N-1} (D_{i-1} + D_i)}.$$

Now we shall study the various possibilities for $\phi_1(a_N) = u$. First we consider the case that u lies in a given interval R_k, i.e., $c_k < u < c_{k+1}$ ($k = 1, \ldots, N-2$). In this case, for every i from 1 to $N-1$, except for $i = k$ and $i = k+1$, $v_i(u) = {}^e v_i$. But instead of the two fictitious volumes ${}^e v_k$ and ${}^e v_{k+1}$, we have here three actual ones, viz. $v_k(u) = \frac{1}{2}(u - c_{k-1})$, $v_N(u) = \frac{1}{2}(c_{k+1} - c_k) = \frac{1}{2} D_k$, and $v_{k+1}(u) = \frac{1}{2}(c_{k+2} - u)$; see the diagram (fig.3).

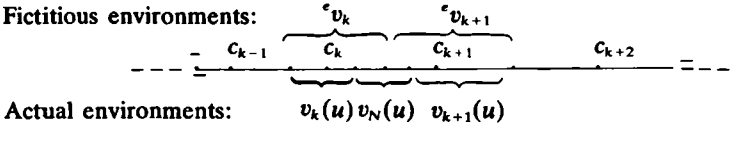

Figure 3. Fictitious and actual environments.

Thus $\Pi(u)$ can now be obtained as a modification of Π_e (14-10):

(14-11) For any u in R_k $(k=1,\ldots,N-2)$,

$$\Pi(u)=\Pi_e=\frac{{}^e v_k \cdot {}^e v_{k+1}}{v_k(u)\cdot v_N(u)\cdot v_{k+1}(u)}$$

$$=\Pi_e=\frac{2(c_{k+1}-c_{k-1})(c_{k+2}-c_k)}{(u-c_{k-1})(c_{k+1}-c_k)(c_{k+2}-u)}$$

$$=2\Pi_e\cdot\frac{(D_{k-1}+D_k)(D_k+D_{k+1})}{(u-c_{k-1})D_k(c_{k+2}-u)}.$$

Since $\Pi(u)$ and the various values of ${}^eI(\alpha)$ contain the factor $2\Pi_e$, it is convenient to put:

(14-12) $$\Pi'(u)=_{\mathrm{Df}}\frac{\Pi(u)}{2\Pi_e},$$

(14-13) $${}^eI'(\alpha)=_{\mathrm{Df}}\frac{{}^eI(\alpha)}{2\Pi_e}.$$

We shall find that ${}^eI'(R_k)$ depends only upon the quotients of the lengths of contiguous intervals, not upon these lengths themselves. Therefore it is convenient to have a notation for these quotients:

(14-14) **(a)** For $i=0,\ldots,N-2$, $\quad q_i^+=_{\mathrm{Df}}\frac{D_{i+1}}{D_i}$;

(b) For $i=1,\ldots,N-1$, $\quad q_i^-=_{\mathrm{Df}}\frac{D_{i-1}}{D_i}$;

(c) $q_A^+=_{\mathrm{Df}}\frac{D_1}{D_A}=2q_0^+$.

(d) $q_B^-=_{\mathrm{Df}}\frac{D_{N-2}}{D_B}=2q_{N-1}^-$.

The following results hold for any R_k $(k=1,\ldots,N-2)$. From (14-11)

(14-15) **(a)** For u in R_k, $\Pi'(u)$

$$=\frac{(D_{k-1}+D_k)(D_k+D_{k+1})}{D_k}\cdot\frac{1}{(u-c_{k-1})(c_{k+2}-u)},$$

(b) $$=D_k(1+q_k^-)(1+q_k^+)\cdot\frac{1}{(u-c_{k-1})(c_{k+2}-u)}.$$

From (14-5):

(14-16) For any interval (u', u'') within R_k,

$$^eI'((u', u'')) = D_k(1+q_k^-)(1+q_k^+)\int_{u'}^{u''} \frac{du}{(u-c_{k-1})(c_{k+2}-u)}.$$

Now we shall use the following indefinite integral:[1]

(14-17) If $W = b^2 - ac > 0$, then

$$\int \frac{dx}{ax^2+2bx+c} = \frac{1}{2\sqrt{W}} \ln\left[C\frac{ax+b-\sqrt{W}}{ax+b+\sqrt{W}}\right].$$

The integral in (14-16) is

$$\int_{u'}^{u''} \frac{du}{-u^2+(c_{k-1}+c_{k+2})u - c_{k-1}c_{k+2}}.$$

Hence $W = \frac{1}{4}(c_{k+2}-c_{k-1})^2 = \frac{1}{4}(D_{k-1}+D_k+D_{k+1})^2 > 0$. Therefore, with (14-17), the integral is

$$\frac{1}{D_{k-1}+D_k+D_{k+1}} \ln\left[\frac{(u''-c_{k-1})(c_{k+1}-u')}{(c_{k+2}-u'')(u'-c_{k-1})}\right].$$

Hence:

(14-18) $$^eI'((u', u'')) = \frac{(1+q_k^-)(1+q_k^+)}{(1+q_k^-+q_k^+)} \ln\left[\frac{(u''-c_{k-1})(c_{k+1}-u')}{(c_{k+2}-u'')(u'-c_{k-1})}\right].$$

For R_k itself, with $u' = c_k$ and $u'' = c_{k+1}$, the argument of ln in (14-18) becomes

$$\frac{(D_{k-1}+D_k)(D_k+D_{k+1})}{D_{k-1}D_{k+1}};$$

hence:

(14-19) $$^eI'(R_k) = \frac{(1+q_k^-)(1+q_k^+)}{(1+q_k^-+q_k^+)} \ln\left[\left(1+\frac{1}{q_k^-}\right)\left(1+\frac{1}{q_k^+}\right)\right].$$

The results just stated hold for any one of the interior subintervals R_k $(k = 1, \ldots, N-2)$. Now we study the two *marginal intervals*. First we consider the case that u lies within R_A, i.e., $A < u < c_1$. Here we have to replace in Π_e the fictitious volume $^ev_1 = \frac{1}{2}(D_0+D_1)$

[1] W. Gröbner and N. Hofreiter, *Integraltafeln*, (Wien and Innsbruck: Springer-Verlag, 1949), 1:1(7b).

(14-9) by the two actual volumes $v_N(u) = \frac{1}{2}(u + c_1) - A = \frac{1}{2}(u - c_0)$ and $v_1(u) = \frac{1}{2}(c_2 - u)$. Therefore, in analogy to (14-11):

(14-20) For any u in R_A,

(a) $$\Pi(u) = \Pi_e \cdot \frac{{}^e v_1}{v_N(u) \cdot v_1(u)}.$$

(b) $$\Pi'(u) = \frac{D_0 + D_1}{(u - c_0)(c_2 - u)}.$$

With (14-5) and (14-13):

(14-21) For any interval (u', u'') within R_A,

$${}^e I'((u', u'')) = (D_0 + D_1) \int_{u'}^{u''} \frac{du}{(u - c_0)(c_2 - u)}.$$

By (14-17), this integral is

$$\frac{1}{c_2 - c_0} \ln \left[\frac{(u'' - c_0)(c_2 - u')}{(c_2 - u'')(u' - c_0)} \right].$$

Hence:

(14-22) $${}^e I'((u', u''')) = \ln \left[\frac{(u'' - c_0)(c_2 - u')}{(c_2 - u'')(u' - c_0)} \right].$$

Hence for R_A itself, with $u' = A$, $u'' = c_1$, and $c_0 = 2A - c_1$ (14-6)(a):

(14-23) $$\begin{aligned} {}^e I'(R_A) &= \ln \left[\frac{(c_1 - c_0)(c_2 - A)}{(c_2 - c_1)(A - c_0)} \right], \\ &= \ln \frac{2(D_A + D_1)}{D_1} = \ln \left[2\left(1 + \frac{1}{q_A^+}\right) \right]. \end{aligned}$$

Analogously for the interval R_B:

(14-24) For any u in R_B,

$$\Pi'(u) = \frac{D_{N-2} + D_{N-1}}{(c_N - u)(u - c_{N-2})}.$$

(14-25) For any interval (u', u'') within R_B,

$${}^e I'((u', u'')) = \ln \left[\frac{(c_N - u')(u'' - c_{N-2})}{(u' - c_{N-2})(c_N - u'')} \right].$$

(14-26) $${}^e I'(R_B) = \ln \frac{2(D_{N-2} + D_B)}{D_{N-2}} = \ln \left[2\left(1 + \frac{1}{q_B^-}\right) \right].$$

${}^eI'$ for R_e, i.e., for the whole interval (A, B), is the sum of its values for the N subintervals:

(14-27)
$$
\begin{aligned}
&{}^eI'(R_A)+{}^eI'(R_B)+\sum_{k=1}^{N-2}{}^eI'(R_k),\\
&\quad=\ln\left[2\left(1+\frac{1}{q_A^+}\right)\right]+\ln\left[2\left(1+\frac{1}{q_B^-}\right)\right]\\
&\quad+\sum_{k=1}^{N-2}\left[\frac{(1+q_k^-)(1+q_k^+)}{(1+q_k^-+q_k^+)}\ln\left[\left(1+\frac{1}{q_k^-}\right)\left(1+\frac{1}{q_k^+}\right)\right]\right].
\end{aligned}
$$

For any interval α within (A, B), let $[\alpha]$ be the proposition that the coordinate u of a_N belongs to α. From (13-27):

(14-28) For any intervals α and β within (A, B), β being within α,

$$c^{**}([\beta],[\alpha])=\frac{{}^eI(\beta)}{{}^eI(\alpha)}=\frac{{}^eI'(\beta)}{{}^eI'(\alpha)}.$$

We have determined the values of ${}^eI'$ for (A, B) (14-27), R_k $(k=1,\ldots,N-2)$ (7-19), R_A (14-23), R_B (14-26), and arbitrary subintervals of R_k, R_A, or R_B ((14-18), (14-22), (14-25)). Thus $c^{**}([R_k], e)$, and for any interval β within R_k, $c^{**}([\beta], e)$ and $c^{**}([\beta],[R_k])$, etc. can be determined by (14-28).

From (13-26):

(14-29) For any u in R_e,

$${}^e\delta(u)=\frac{\Pi'(u)}{{}^eI'(R_e)}.$$

Thus the m^{**}-density ${}^e\delta$ (with respect to e) is proportional to $\Pi'(u)$. We shall now study the shape of the function $\Pi'(u)$ or of a curve representing it; that will give us a picture of the shape of ${}^e\delta$.

$\Pi'(u)$ within R_k is given by (14-15). Put $f(u)=_{\mathrm{Df}}(u-c_{k-1})\times(c_{k+2}-u)$. f is 0 for $u=c_{k-1}$ and $u=c_{k+2}$. f has its maximum at the midpoint of c_{k-1} and c_{k+2}; $f_{\max}=\frac{1}{4}(c_{k+2}-c_{k-1})^2=\frac{1}{4}(D_{k-1}+D_k+D_{k+1})^2=W_k$. f is represented by a parabola with vertical axis (the dotted line in the *diagram* (fig. 4); the maximum point just mentioned is its vertex. $\Pi'(u)$ is proportional to $\frac{1}{f(u)}$. The latter function is still determined by c_{k-1} and c_{k+2} alone. It is indicated in the diagram by the solid and dashed line. $\Pi'(u)$ is

furthermore influenced by the values c_k and c_{k+1} in two ways. First, $\frac{1}{f(u)}$ is multiplied by $D_k(1+q_k^-)(1+q_k^+)$. Since this is a constant factor, the resulting function may still be regarded as represented by the same curve in the diagram (with a different scale). Second, the positions of c_k and c_{k+1} determine the interval $R_k = (c_k, c_{k+1})$ and thereby that segment of the curve which represents $\Pi'(u)$ in R_k. This segment is the solid part of the curve in the diagram. [The remaining dashed parts do *not* represent $\Pi'(u)$ in the two adjacent intervals; for each interval, $\Pi'(u)$ is to be determined by a new construction of the kind described.]

(14-30) If the midpoint $\frac{1}{2}(c_{k-1}+c_{k+2})$ belongs to R_k, then at this point

For a curve representing $\Pi'(u)$ in R_A, according to (14-20)(b), a similar construction would be made over the base (c_0, c_2); analogously for R_B (14-24).

Thus the total curve for $\Pi'(u)$ over (A, B) consists of the N segments described for the intervals $R_A, R_1, \ldots, R_{N-2}, R_B$. The question arises whether these segments join together at their ends or whether there could be jumps at the boundary points. We shall see that there are no jumps, in other words, the function $\Pi'(u)$ is continuous in the following sense. Strictly speaking, the coordinate u of a_N cannot coincide with any of the values c_k (14-3)(b), and $\Pi'(u)$ is therefore not defined for these values. However, since $\Pi'(u)$ is defined for all other values within (A, B), we can determine its limits for any value c_k. Then we find the following:

(14-31) The limits of $\Pi'(u)$ for u approaching a given c_k $(k = 1, \ldots, N-1)$ from below and from above are both $= \frac{D_{k-1}+D_k}{D_{k-1}D_k}$. Henceforth we shall take this value as the value of Π' for c_k. Then Π' is everywhere continuous.

Proof. (1) For $k = 2, \ldots, N-2$. The limit at c_k from above is found from $\Pi'(u)$ in R_k (14-15) by simply taking $u = c_k$. Hence it is

$$\frac{D_{k-1}+D_k}{D_{k-1}D_k}.$$

(2) $\Pi'(u)$ in R_{k-1} is, in analogy to (14-15), equal to $\frac{(D_{k-2}+D_{k-1})(D_{k-1}+D_k)}{D_{k-1}} \cdot \frac{1}{(u-c_{k-2})(c_{k+1}-u)}$. The limit at c_k from

$\prod'(u)$ has its minimum, which is

$$\frac{4D_k(1+q_k^-)(1+q_k^+)}{(D_{k-1}+D_k+D_{k+1})^2}=\frac{4(1+q_k^-)(1+q_k^+)}{D_k(1+q_k^-+q_k^+)^2}.$$

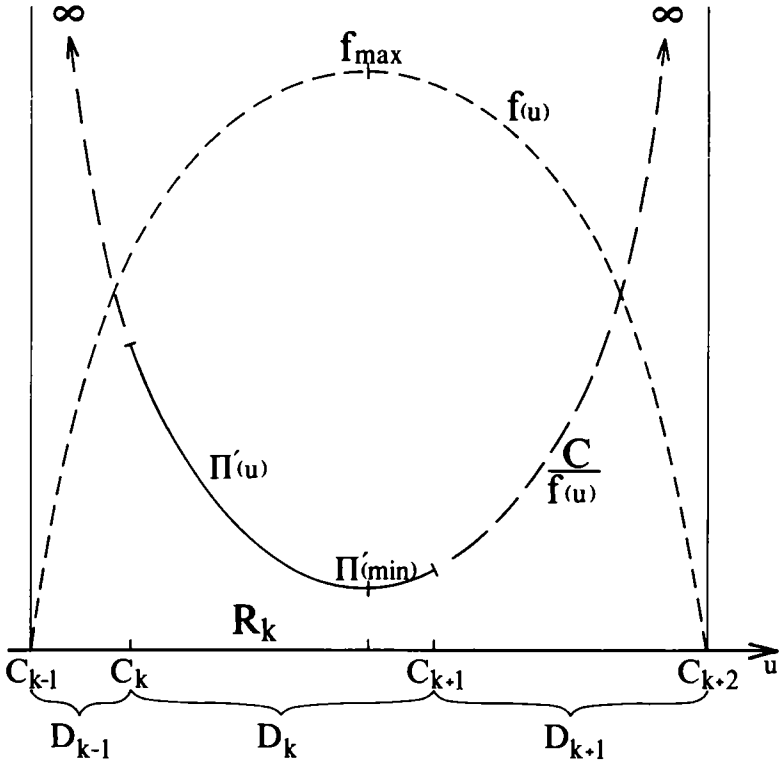

Figure 4. Curve for $\Pi'(u)$ in R_k.

below is obtained by taking here $u = c_k$. The result is the same as in (1).

(3) The limit at c_1 from above is obtained as in (1), with $k = 1$. Hence it is $\frac{D_0 + D_1}{D_0 D_1}$.

(4) The limit at c_1 from below is obtained from (14-20)(b) by taking $u = c_1$. The result is the same as under (3).

(5) Similarly, we find that both limits at c_{N-1} are $\frac{D_{N-2} + D_{N-1}}{D_{N-2} D_{N-1}}$.

From (14-31):

$$\Pi'(c_{k+1}) = \frac{D_k + D_{k+1}}{D_k D_{k+1}}.$$

Hence

$$\frac{\Pi'(c_k)}{\Pi'(c_{k+1})} = \frac{1 + \frac{D_k}{D_{k-1}}}{1 + \frac{D_k}{D_{k+1}}}$$

This is $\gtreqless 1$, according as $D_{k-1} \gtreqless D_{k+1}$. Therefore:

(14-32) $\Pi'(c_k) \gtreqless \Pi'(c_{k+1})$ $(k = 1, \ldots, N-1)$ according as $D_{k-1} \gtreqless D_{k+1}$.

This means that Π', and hence also ${}^e\delta$ (14-29), is higher at that end of an interval with the smaller neighbor. This is as it should be, because the probability for the phase point of a new element should be higher where the concentration of old phase points is higher.

(14-33) Let the intervals R_k and R_m $(k, m = 1, \ldots, N-2)$ be such that the lengths of R_k and its neighbor intervals are proportional to those of R_m and its neighbor intervals (i.e., $D_{m-1} = CD_{k-1}$, $D_m = CD_k$, $D_{m+1} = CD_{k+1}$, with an arbitrary C), and let u_k in R_k and u_m in R_m be corresponding values (i.e., $u_m - c_m = C(u_k - c_k)$ Then the following holds:

(a) $D_k \Pi'(u_k) = D_m \Pi'(u_m)$.
(b) $D_k \, {}^e\delta(u_k) = D_m \, {}^e\delta(u_m)$.
(c) ${}^eI'(R_k) = {}^eI'(R_m)$.
(d) $c^{**}([R_k], e) = c^{**}([R_m], e)$.

Proof. From the assumptions, $q_k^- = q_m^-$ and $q_k^+ = q_m^+$ (14-14). Therefore

(a) from (14-15)(b), since $u_m - c_{m-1} = C(u_k - c_{k-1})$ and $c_{m+2} - u_m = C(c_{k+2} - u_k)$.

(b) from (a) with (14-29).

(c) from (14-19).

(d) from (c) with (14-28).

We see from (14-33)(a) that, within two intervals with proportional surroundings, $\Pi'(u)$ and therefore ${}^e\delta$ at corresponding points are inversely proportional to the length of the interval; in other words, it is the more probable for the new element a_N to be represented at this point, the smaller the interval is. This is as it should be. In the classification schema (§2), it was the more probable for a new element to belong to a given cell, the greater the number of old phase points in this cell (see (2-12)); this means, in terms of a quantitative system, the nearer the old phase points in the given region are to each other.

The result (14-33)(d) is important. It says that if two of the interior e-intervals have proportional surroundings, then it is just as probable for the new element a_N to belong to the one as to the other, irrespective of the lengths of the intervals. This result may at first be surprising, but it is easily seen that it is just what should be expected. Suppose that $D_m = 2D_k$. Then, according to (b), the probability density at corresponding points is in R_k twice that in R_m. Therefore, since the length of R_k is one half that of R_m, the probabilities are equal.

The following result concerns the most important special case of (14-33)(c). We shall make use of it in the study of regions with constant interval length (in the next section).

(14-34) Suppose that R_k $(k = 1, \ldots, N-2)$ is equal in length to its neighbor intervals, i.e., $D_{k-1} = D_k = D_{k+1}$. Then:

(a) $\Pi'(u)$ for c_k and for c_{k+1} is $\dfrac{2}{D_k}$. (From (14-31)).

(b) Within R_k, $\Pi'(u)$ has its minimum at the midpoint $\frac{1}{2}(c_k + c_{k+1})$; this minimum is $\dfrac{16}{9D_k}$, hence $\frac{8}{9}$ of its value at the end points c_k and c_{k+1}. (From (14-30), since $q_k^- = q_k^+ = 1$.)

(c) ${}^eI'(R_k) = \frac{8}{3} \ln 2$ $(= 1.84840)$. (From (14-19).)

In accordance with (14-33)(a) and (c), the values of Π' in (14-34)(a) and (b) are inversely proportional to D_k, and the value of ${}^eI'$ in (14-34)(c) is independent of D_k.

15
Regional systems

Summary. A special case of the schema with one magnitude is investigated, in which the intervals R_k between the values given in the evidence are such that they can be divided into several regions, each consisting of intervals of equal lengths. Again theorems for $\prod'(u)$, $^{e}\delta(u)$, and c^{**} are given, some of them of surprising simplicity. They are illustrated by a numerical example ((15-8) and (15-29)) and a curve diagram (fig. 5).

Within the schema with one magnitude, developed in the preceding section, we shall now investigate in greater detail systems of a special kind, which we call *regional systems*. The intervals R_A, $R_1, \ldots, R_{N-2}$, R_B given by the evidence e (14-7) are now assumed to such that (A, B) can be divided into p regions ($p \geqq 1$) each consisting of at least three intervals for R_A and R_B. The first and last intervals within a region are called its marginal intervals, the others its interior intervals.

We make the following assumption on D_A and D_B in the regional system. (This will simplify the results, see the remark on (15-9) below.)

(15-1) **(a)** $D_A = \frac{1}{2}D_1$; hence $D_0 = D_1$.

(b) $D_B = \frac{1}{2}D_{N-2}$; hence $D_{N-1} = D_{N-2}$.

We shall first determine the course of $\prod'(u)$. (14-34)(a) and (b) yield:

(15-2) If R_k is any *interior interval* in any region then:

(a) $$\prod'(c_k) = \prod'(c_{k+1}) = \frac{2}{D_k}.$$

(b) At the center of R_k, $\prod'$ has its minimum value $\frac{16}{9D_k} = \frac{8}{9}\prod'(c_k)$

From (14-34)(a) and (15-1)(a): $\prod'(c_1) = \frac{2}{D_1}$. This is the minimum of $\prod'$ in R_A, since c_1 is the midpoint of c_0 and c_2 (see the

remark following (14-30)). According to (14-20)(b) generally, $\prod'(A)=\frac{D_0+D_1}{D_A(D_A+D_1)}$; therefore in the present system, with (15-1)(a), it is $\frac{8}{3D_1}$. The situation in R_B is analogous.

(15-3) **(a)** $\prod'(c_1)=\frac{2}{D_1}$.

(b) $\prod'(A)=\frac{8}{3D_1}=\frac{4}{3}\prod'(c_1)$.

(c) $\prod'(c_{N-1})=\frac{2}{D_{N-2}}$.

(d) $\prod'(B)=\frac{8}{3D_{N-2}}=\frac{4}{3}\prod'(c_{N-1})$.

Now we consider the two *marginal intervals* at the boundary between two regions. Let c_m be the boundary point between two regions. Then R_{m-1} is the last interval of the first region and R_m is the first interval of the second region. Then $D_{m-2}=D_{m-1}\neq D_m=D_{m+1}$. Let us assume that $D_{m-1}<D_m$. We put $q=D_m/D_{m-1}$; then $q>1$. (As an example, see the relation between the first and the second regions in the later numerical example (15-8) and fig. 5. Here $c_m=c_5$ and $q=6$. If the first marginal interval is larger than the second, as at c_9 in the example, the results are of course analogous.) We see from (14-32) that $\prod'(c_{m-1})>\prod'(c_m)>\prod'(c_{m+1})$. More specifically, we find from (15-2)(a), since R_{m-2} and R_{m+1} are interior intervals, the following results:

(15-4) **(a)** $\prod'(c_{m+1})=\prod'(c_{m+2})=\frac{2}{D_{m+1}}=\frac{2}{D_m}$.

(b) $\prod'(c_{m-2})=\prod'(c_{m-1})=\frac{2}{D_{m-2}}=\frac{2}{D_{m-1}}=q\prod'(c_{m+1})$.

From (14-31):

(15-5) $$\prod'(c_m)=\frac{D_{m-1}+D_m}{D_{m-1}D_m}=\frac{1+q}{D_m}=\frac{q+1}{2}\prod'(c_{m+1}).$$

Hence:

(15-6) $$\prod'(c_{m-1})-\prod'(c_m)=\prod'(c_m)-\prod'(c_{m+1})=\frac{q-1}{2}\prod'(c_{m+1}).$$

Thus at all boundary points between intervals of the second region (c_{m+1}, c_{m+2}, etc.), $\prod'$ has the same value, which is relatively low. At all boundary points within the first region, $\prod'$ has q times that value. At the boundary point c_m between the two regions, the value of $\prod'$ is just the mean of those two values. [If $q > 2$, $\prod'$ decreases throughout R_{m-1}, and also in the greater part of R_m; in R_m it has a minimum at a point near c_{m+1} (the mid point of c_{m-1} and c_{m+2}, (14-30)); this minimum is only a little less than its value at c_{m+1}.] The values of $\prod'$ for the points mentioned are shown in the following table:

(15-7) Table for $\prod'(u)/\prod'(c_{m+}g_1)$:

u	generally	for $q = 9$	for $q = 1000$
c_{m-2}	q	9	1000
midpoint	$\frac{8}{9}q$ (min.)	8	888.9
c_{m-1}	q	9	1000
c_m	$\frac{q+1}{2}$	5	500.5
c_{m+1}	1	1	1
midpoint	$\frac{8}{9}$ (min.)	$\frac{8}{9}$	$\frac{8}{9}$
c_{m+2}	1	1	1

Thus the overall picture of the curve for $\prod'$, and likewise that for ${}^e\delta$, in the regional system is as follows. The value of $\prod'$ is the same at all interior boundary points c_k in a region; it is inversely proportional to the length of the intervals of the region. At the midpoint of any two of those points it drops to $\frac{8}{9}$ of the value at the points; this is its minimum in the interval. At the boundary point between two regions the value of $\prod'$ is the arithmetic mean of the values at the interior boundary points in the regions. The value at A is $\frac{4}{3}$ of that at the nearest boundary point; likewise the value at B.

(15-8) *Numerical example.* $N = 14$. Let the values of A, $c_1, \ldots, c_{13}$, B be such that there are three regions with boundaries at c_5 and c_9 (see fig. 5) and with interval lengths $D = 1$, 6, and 2, respectively. Then we find the following values, which are represented by the curve for $\prod'(u)$ in fig. 5. This curve also represents ${}^e\delta(u)$ (with a different scale), since the latter is proportional to $\prod'(u)$ (14-29); we shall later find: ${}^e\delta(u) = 0.0396 \prod'(u)$ (15-29).

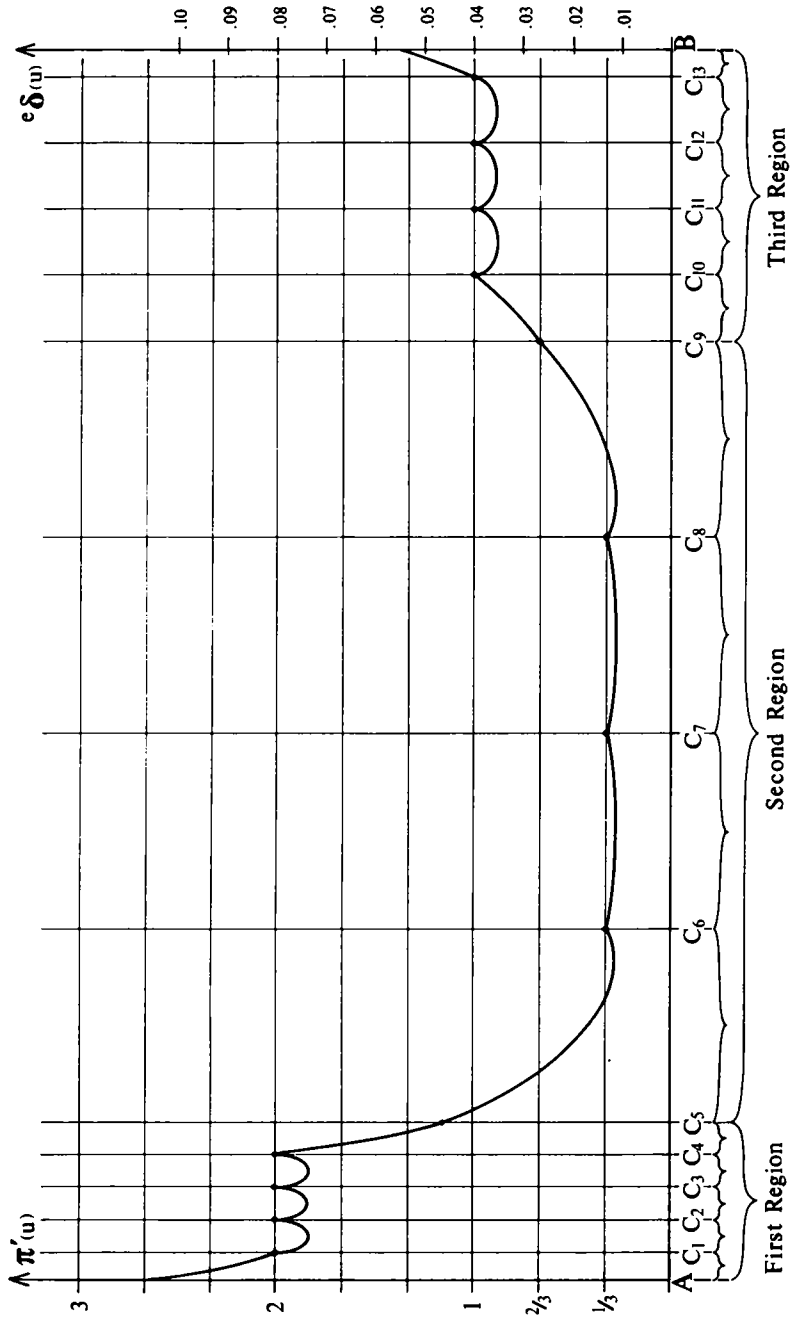

Figure 5. Curve for $\prod'(u)$ and $^e\delta(u)$.

	Regions first	second	third
Length D of intervals (except D_A and D_B)	1	6	2
D_A and D_B (15-1)	$D_A = \frac{1}{2}$		$D_B = 1$
Π' for interior boundary points:			
$\Pi'_b = \frac{2}{D}$ (15-2)(*a*)	2	$\frac{1}{3}$	1
Π' for midpoints of interior intervals			
$= \frac{16}{9D} = \frac{8}{9}\Pi'_b$ (15-2)(*b*)	$\frac{16}{9}$	$\frac{8}{27}$	$\frac{8}{9}$
Π' for the two end points			
$= \frac{8}{3D} = \frac{4}{3}\Pi'_b$ (15-3)(*b*)	for A: $\frac{8}{9}$		for B: $\frac{4}{3}$

Now we shall determine the values of ${}^eI'$ for the intervals in the regional system.

From (14-34)(c):

(14-9) If R_k $(k = 1, \ldots, N-2)$ is an interior interval in any region, then ${}^eI'(R_k) = \frac{8}{3} \ln 2$ $(= 1.84840)$. We denote this value by 'C.'

This result holds for $R_2, \ldots, R_{N-3}$ independently of the choice of D_A and D_B. We make the assumption (15-1) in order to assure the validity of (15-9) also for R_1 and R_{N-2}. By the same assumption, with (14-23) and (14-26):

(15-10) (a) ${}^eI'(R_A) = {}^eI'(R_B) = \ln 3$ $(= 1.0986)$.

(b) $$= \tfrac{3}{8} \cdot \frac{\ln 3}{\ln 2} \cdot C = 0.5944C.$$

Let R_{m-1} and R_m again be two contiguous marginal intervals as described above, with $D_m > D_{m-1}$ and $q = D_m/D_{m-1} > 1$. For R_{m-1}, we have $q^-_{m-1} = 1$ and $q^+_{m-1} = q$. Therefore, with (14-19):

(15-11) $$ {}^eI'(R_{m-1}) = \frac{2(1+q)}{(2+q)} \ln \left[2\left(1 + \frac{1}{q}\right) \right]. $$

q may have any value > 1; it is always finite, since $D_{m-1} > 0$ (14-4).

(15-12) (a) $\lim_{q \to 1} {}^eI'(R_{m-1}) = \frac{8}{3} \ln 2 = C.$

(b) $\lim_{q \to \infty} {}^eI'(R_{m-1}) = 2 \ln 2 = \frac{3}{4}C.$

(a) is obvious, since for $q = 1$ we should have $D_{m-1} = D_m$; thus the two intervals R_{m-1} and R_m would not be marginal but interior. The possible values of ${}^eI'(R_{m-1})$ are between the above two limits:

(15-13) $$\tfrac{3}{4}C < {}^eI'(R_{m-1}) < C.$$

For R_m, we have $q_m^- = \dfrac{1}{q}$ and $q_m^+ = 1$. Hence with (14-19):

(15-14) $${}^eI'(R_m) = \frac{2(1+q)}{1+2q} \ln [2(1+q)].$$

(15-15) $$\lim_{q \to 1} {}^eI'(R_m) = \tfrac{8}{3} \ln 2 = C.$$

(15-16) $${}^eI'(R_m) > C.$$

As $q \to \infty$, ${}^eI'(R_m)$ has no finite limit, but it is always finite.

The results are again analogous, of course, if the first marginal interval is larger than the second.

${}^eI'(R_e)$ is the sum of the ${}^eI'$-values of the N intervals in (A, B). Since there are p regions, there are $2p$ marginal intervals and $N-2p$ interior intervals. Let $c_{m1}, \ldots, c_{m_{p-1}}$ be the $p-1$ boundary points between the regions. Let $R_{m_j'}$ $(j = 1, \ldots, p-1)$ be the smaller and $R_{m_j''}$ the larger of the two marginal intervals adjacent to c_{m_j}. Let $q_j = D_{m_j''}/D_{m_j'}$; hence $q_j > 1$. Then, from (14-27) and (15-9):

(15-17) $$\begin{aligned} {}^eI'(R_e) = (N-2p)C + {}^eI'(R_A) + {}^eI'(R_B) \\ + \sum_{j=1}^{p-1} [{}^eI'(R_{m_j'}) + {}^eI'(R_{m_j''})]. \end{aligned}$$

Now we shall assume that N is large in relation to p. Then only a small fraction of the intervals are marginal. If q_j is small, the ${}^eI'$-values of the marginal intervals do not deviate very much from C, and hence ${}^eI'(R_e)$ is near to NC. But with increasing q_j, ${}^eI'(R_{m_j''})$ grows beyond any bound, as we saw from (15-14). Therefore we shall now examine the situation for large q_j; since we aim only at an approximation for ${}^eI'(R_e)$, and the number of marginal intervals is small in relation to the total number N of intervals, a rough approximation for the marginal intervals will suffice for our purpose.

(15-18) Approximations for marginal intervals with large q_j (from (15-11) and (15-14)):

(a) $${}^{e}I'(R_{m_j'}) \cong 2\left(1-\frac{1}{q_j}\right)\left[\ln 2+\ln\left(1+\frac{1}{q_j}\right)\right].$$

(b) $${}^{e}I'(R_{m_j''}) \cong \left(1+\frac{1}{2q_j}\right)\left[\ln 2+\ln q_j+\ln\left(1+\frac{1}{q_j}\right)\right].$$

(c) $$\begin{aligned}&{}^{e}I'(R_{m_j'})+{}^{e}I'(R_{m_j''})\\ &\cong 3\left(1-\frac{1}{2q_j}\right)\left[\ln 2+\ln\left(1+\frac{1}{2q_j}\right)\right]\ln q_j\\ &\cong 3\left(1-\frac{1}{2q_j}\right)\ln 2+\frac{3}{q_j}+\left(1+\frac{1}{2q_j}\right)\ln q_j.\end{aligned}$$

We put:

$$x_j =_{\mathrm{Df}} \mathrm{Log}\, q_j; \quad \text{hence} \quad q_j = 2^{x_j};$$
$$X_j =_{\mathrm{Df}} \frac{{}^{e}I'(R_{m_j'})+{}^{e}I'(R_{m_j''})}{C}-2.$$

Then, from (15-18)(c):

(15-19) Approximations for very large q_j:

(a) $${}^{e}I'(R_{m_j'})+{}^{e}I'(R_{m_j''}) \cong \ln q_j + 3\ln 2.$$

(b) $$\cong (x_j+3)\ln 2.$$

(c) $$X_j \cong \tfrac{3}{8}(x_j+3)-2 = \tfrac{1}{8}(3x_j-7).$$

(15-20) A few values of X_j as examples (from (15-19)(c)):

x_j	q_j	X_j
10	$2^{10}(\cong 10^3)$	2.88
20	$2^{20}(\cong 10^6)$	6.63
30	$2^{30}(\cong 10^9)$	10.38

From (15-10)(b):

(15-21) $$\frac{{}^{e}I'(R_{\mathrm{A}})+{}^{e}I'(R_{\mathrm{B}})}{C}-2=\frac{3}{4}\cdot\frac{\ln 3}{\ln 2}-2=-0.8112.$$

Thus we obtain the following approximation for (15-17):

(15-22) $$ {}^{e}I'(R_e) \cong C\left[N + \sum_{j=1}^{p-1} X_j - 0.8112\right]. $$

We see from (15-20) that X_j grows only slowly with q_j. Even for a very large q_j, say 10^9, the corresponding X_j, about 10, may be neglected in (15-22) if N is large, say $> 10^4$. Thus:

(15-23) If N is large in relation to p and not many values of q_j are large (say $> 10^3$), then ${}^{e}I'(R_e) \cong NC$.

Thus, under the conditions stated in (15-23), we obtain the following approximate values of the m^{**}-density ${}^{e}\delta$ with respect to e, and of c^{**} in the regional system.

From (14-29):

(15-24) For any u in (A, B), $${}^{e}\delta(u) \cong \frac{\prod'(u)}{NC}.$$

Thus ${}^{e}\delta(u)$ in the regional system can now be determined with the help of the formulas for $\prod'(u)$ ((14-15), (14-20)(*b*), (14-24)).

From (14-28):

(15-25) For any interval β within (A, B),

$$ c^{**}([\beta], e) \cong \frac{{}^{e}I'(\beta)}{NC}. $$

The following results are special cases of this. With (15-9):

(15-26) If R_k $(k = 1, \ldots, N-2)$ is an interior interval in any region,

$$ c^{**}([R_k], e) \cong \frac{1}{N}. $$

From (15-10)(b):

(15-27) $$ c^{**}([R_A], e) = c^{**}([R_B], e) \cong \frac{3 \ln 3}{8N \ln 2} \cong \frac{0.5944}{N}. $$

(15-28) Let α be the smaller and β the larger of two contiguous

marginal intervals. Let $q = D_\beta/D_\alpha$ $(q>1)$. Then:

(a) $\dfrac{3}{4N} < c^{**}([\alpha], e) < \dfrac{1}{N}$. (From (15-13).)

(b) $c^{**}([\alpha], e) \cong \dfrac{3(q+1)}{4(q+2)N} \cdot \dfrac{\ln\left[2\left(1+\dfrac{1}{q}\right)\right]}{\ln 2}$,

$$\cong \frac{3(q+1)}{4(q+2)N}\left[1+\operatorname{Log}\left(1+\frac{1}{q}\right)\right].$$

(From (15-11).)

(c) For large q: $\cong \dfrac{3}{4N}\left(1-\dfrac{1}{q}\right)\left[1+\operatorname{Log}\left(1+\dfrac{1}{q}\right)\right]$.

(From (15-8)(a).)

(d) For every large q: $\cong \dfrac{3}{4N}$.

(e) $c^{**}([\beta], e) > \dfrac{1}{N}$. (From (15-16).)

(f) $c^{**}([\beta], e) \cong \dfrac{3(q+1)}{4(2q+1)N}[1+\operatorname{Log}(q+1)]$.

(From (15-14).)

(g) For large q: $\cong \dfrac{3}{8N}\left(1+\dfrac{1}{2q}\right)[1+\operatorname{Log}(q+1)]$.

(h) For very large q: $\cong \dfrac{3}{8N}[1+\operatorname{Log} q]$.

(15-29) *Numerical example.* On the basis of the data in the example (15-8), we obtain the following values of $^eI'$ with the help of (15-10), (15-11), (15-14), and (15-19).

Interval β	$^eI'(\beta)$	$c^{**}([\beta], e)$
R_A	1.099	0.0435
R_B	1.099	0.0435
R_4	1.483	0.0587
R_5	2.842	0.1126
R_8	2.376	0.0941
R_9	1.569	0.0621
sum	10.468	0.4145
Eight interior intervals, each	1.8484	0.0732
together	14.787	0.5856
Sum total	$\Sigma = 25.255$	1.0001

$^{e}I'$ for R_e, i.e., for the entire interval (A, B), is the sum of the values given for the fourteen intervals; this is $\Sigma = 25.255$. [We see that $NC = 25.878$ is a fair approximation for Σ, in accordance with (15-23); the approximation is not good because $N = 14$ is not large in relation to $p = 3$.] According to (14-28), $c^{**}([\beta], e) = {}^{e}I'(\beta)/\Sigma = 0.0396\, {}^{e}I'(\beta)$. These values of c^{**} are listed above in the last column. They give the degree of confirmation, on the evidence e, for the hypothesis that $\phi_1(a_N)$ lies in the interval β.

According to (14-29), the density ${}^{e}\delta(u) = \prod'(u)/\Sigma = 0.0396 \prod'(u)$. Therefore the curve in fig. 5 also represents ${}^{e}\delta(u)$. With the values of $\prod'$ from (15-8), we obtain the following values of ${}^{e}\delta$:

Points	$\prod'$	${}^{e}\delta$
Interior boundary points, first region	2	0.080
second region	$\frac{1}{3}$	0.013
third region	1	0.040
A	$\frac{8}{3}$	0.106
B	$\frac{4}{3}$	0.053