Assignment1 Solutions

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1 Linear Algebra

1.1 Problem 1

Necessity condition
Let's assume that $\{|k\rangle\}$ is an orthonormal basis for the Hilbert space $\mathcal{H}$ containing $|a\rangle$ and $|b\rangle$. Then we can write $\langle a|$ and $|b\rangle$ as

$$|b\rangle = \sum_i d_i |k_i\rangle, \;\; \langle a| = \sum_i c_i^* \langle k_i| \quad where \;\; c_i, d_i \in \mathcal{C}$$

Taking inner product with the orthonormal basis,

$$\langle k_i|b\rangle = d_i, \;\; \langle a|k_i\rangle = c_i^*$$

The right side of the given equation simplifies to

$$\sum_i \langle a|k_i\rangle\langle k_i|b\rangle = \sum_i c_i^* d_i$$

The left side of the given equation simplifies to

$$\begin{aligned} \langle a|b\rangle &= (\sum_i c_i^* \langle k_i|)(\sum_j d_j |k_j\rangle) \\ &= \sum_i \sum_j c_i^* d_j \delta_{ij} \\ &= \sum_i c_i^* d_i \end{aligned} \tag{1}$$

Hence, if $\{|k\rangle\}$ is an orthonormal basis of the Hilbert space $\mathcal{H}$, the above condition in necessary.
Sufficiency condition
We have been given the equation and we have to prove that this condition is sufficient to show that $\{|k\rangle\}$ is an orthonormal basis for the Hilbert space $\mathcal{H}$ containing $|a\rangle$ and $|b\rangle$.

$$\langle a|b\rangle = \sum_i \langle a|k_i\rangle\langle k_i|b\rangle \tag{2}$$

Post-multiplying both sides by $|a\rangle$,

$$\langle a|a\rangle|b\rangle = \langle a|a\rangle \sum_i |k_i\rangle\langle k_i|b\rangle$$

As $|a\rangle$ is an arbitrary vector in the Hilbert space, we can take its modulus to be non-zero, leaving us with

$$|b\rangle = \sum_i |k_i\rangle\langle k_i|b\rangle \quad \forall b \in \mathcal{H}$$

Thus, $\{|k\rangle\}$ form a basis of the Hilbert space $\mathcal{H}$.

1.2 Problem 2

We need to prove that log is a non-unique function, i.e., for every matrix B, there are multiple matrices A that satisfy the matrix relation $e^A = B$.

Since $e^x = e^{x+i2\pi n} \quad \forall n \in \mathcal{Z}$, we can extend this relation to matrices as following,

$$e^A = e^{A+i2\pi n\mathcal{K}} = B \ \ \forall n \in \mathcal{Z}$$

where $\mathcal{K}$ is a matrix with the same dimension as the matrix A and $k_{ij} \in \{0, 1\}$ with at least one element being non-zero. Therefore, log is a non-unique.

1.3 Problem 3

problem i).

By observation, we can see that the given matrix is the tensor product of $\begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, i.e., $\begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Eigenvalues of $\begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$ are {1,5} and eigenvalues of $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ are {-1,1}. Therefore, the eigenvalues of the given matrix will be $\{1 \times 1, 1 \times (-1), 5 \times 1, 5 \times (-1)\}$, i.e., $\{1, -1, 5, -5\}$.

problem ii).

By observation, we can see that the given matrix is

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \bigotimes \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$$

Therefore, the eigenvalues of the given matrix will be $\{1 \times 1, 1 \times (-1), 5 \times 1, 5 \times (-1)\}$, i.e., $\{1, -1, 5, -5\}$.

problem iii).
By observation, we can see that the given matrix is

$$\begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix} \bigotimes \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$$

Therefore, the eigenvalues of the given matrix will be $\{1 \times 1, 1 \times 5, 5 \times 1, 5 \times 5\}$, i.e., $\{1, 5, 5, 25\}$.

2 Quantum Mechanics

2.1 Problem 1

Given $|\psi\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$ and unitary operator $H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, we need $|\psi^{'}\rangle = H|\psi\rangle$. Writing H in terms of Pauli's matrices,

$$H = \frac{1}{\sqrt{2}}(\sigma_x + \sigma_z)$$

Thus we can apply the X-gate and Z-gate on $|\psi\rangle$.

$$\begin{aligned} |\psi\prime\rangle &= \frac{1}{2}(\sigma_x + \sigma_z)(|0\rangle - |1\rangle) \\ &= \frac{1}{2}(|1\rangle - |0\rangle + |0\rangle - (-|1\rangle)) \\ &= |1\rangle \end{aligned} \tag{3}$$

2.2 Problem 2

Calculating the probablity distribution $|\psi\rangle$ and $|\psi^{'}\rangle$
For $|\psi\rangle$
P($|0\rangle$)=$||\langle 0|\psi\rangle||^2 = \frac{1}{2}$,
P($|1\rangle$)=$||\langle 1|\psi\rangle||^2 = \frac{1}{2}$
Post-measurement state:
$|\psi_{pm}\rangle = |0\rangle$ if $|0\rangle$ is measured.
$|\psi_{pm}\rangle = |1\rangle$ if $|1\rangle$ is measured.

For $|\psi^{'}\rangle$
P($|0\rangle$)=$||\langle 0|\psi^{'}\rangle||^2 = 0$,
P($|1\rangle$)=$||\langle 1|\psi^{'}\rangle||^2 = 1$
Post-measurement state:
$|\psi^{'}_{pm}\rangle = |1\rangle$