Super Dense Coding

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1 Superdense Coding

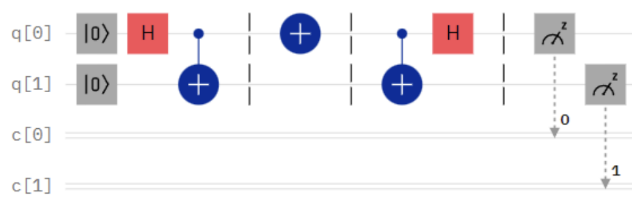

Figure 1: Original super-dense coding circuit

The purpose of super dense coding is to send classical bit information with less number of qubits. This can be achieved by applying simple yet non-trivial principles of quantum mechanics.
The most elementary example of this phenomenon is sending 2 bits of classical information with 1 qubit only. Alice wants to send Bob one of 00,01,10,11 by givig Bob 1 qubit.
The steps are dependent on the value of the classical bits to be sent. Alice and Bob each have one of the qubit of the entangled state

$$|\psi\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

If '00' is to be sent, Alice does not apply function to her qubit. To send '01', she applies phase flip Z to her qubit.To send '10', Alice applies the NOT gate, X, to her own qubit. Finally, to send '11', she applies iY gate to her qubit. The resulting state is

$$00 : |\psi\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}} \equiv |\beta_{00}\rangle$$
$$01 : |\psi\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}} \equiv |\beta_{10}\rangle$$
$$10 : |\psi\rangle = \frac{|10\rangle + |01\rangle}{\sqrt{2}} \equiv |\beta_{01}\rangle$$
$$11 : |\psi\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}} \equiv |\beta_{11}\rangle$$

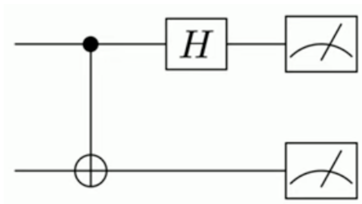

Figure 2: Bell state measure circuit

These four are called the *Bell states*, and form the orthonormal basis. Therefore, these can be easily measured with a quantum measurement circuit. Thus we were able to send 2 classical bits information with only 1 qubit.

1.1 3 Qubit Case

Alice and Bob share a 3-qubit entangled state. We can use the GHZ state for the distribution of the qubits:

$$|\psi\rangle_{GHZ} = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$$

Here, qubits 1 and 2 are with Alice, and qubit 3 is with Bob.

Alice has 3 classical bits of information, c_1, c_2, c_3, that she wants to transmit to Bob.

1.1.1 Step 1

1. - If $c_1 = 1$, apply the X gate to q_1.
   - If $c_1 = 0$, do nothing.
2. - If $c_2 = 1$, apply the X gate to q_2.
   - If $c_2 = 0$, do nothing.
3. - If $c_3 = 1$, apply the Z gate to q_2.
   - If $c_3 = 0$, do nothing.

1.1.2 Step 2

Measure the two qubits q_1 and q_2. This measurement will collapse her two qubits into one of the four Bell states, and the result can be represented by 2 classical bits, m_1 and m_2. These will then be sent to Bob via classical communication. With this information in hand, bob will perform the following conditional operations on his qubit q_3.

1. If $m_1 = 1$, Bob applies an X gate.
2. If $m_2 = 1$, Bob applies a Z gate.

1.1.3 Step 3

Now measure the qubit q_3 and store it in the register m_3.

Thus we have successfully retracted the 3 classical bits from these 3 qubits using the above described protocol.