Simplex Method

Transition of Algebraic (BFS) / Graphical Method to Simplex method

LPP. ⟶

Mathematically, the general linear programming problem (LPP) may be stated as:

$$
\begin{aligned}
\text{Maximize or Minimize} \quad & Z = c_1x_1 + c_2x_2 + \ldots + c_nx_n \\
\text{subject to} \quad & a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \ (\leq, =, \geq) \ b_1 \\
& a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \ (\leq, =, \geq) \ b_2 \\
& \vdots \\
& a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \ (\leq, =, \geq) \ b_m \\
& x_1, x_2, \ldots, x_n \geq 0
\end{aligned}
$$

Solution Methods

1. Graphical Solution (Corner Point).
2. Algebraic Method/ Finding Basic Solution | BFS. (degenerate / Non-deg.)
3. Simplex method and its Variants. Big M | Two Phase -

Problem: Solve the given LPP;

Maximize $Z = 2x_1 + 3x_2$

$2x_1 + x_2 \le 4$

$x_1 + 2x_2 \le 5$ and $x_1, x_2 \ge 0$

$s_1, s_2 \ge 0$

$AX = b$

slack/surplus

$(A)_{2\times 4}$

$^4C_2 = 6$ cases

Simplex Method.

$AX = b$ (*)

$BX_B = b$

$X_B = B^{-1}b$

$2x_1 + x_2 + s_1 = 4$

$x_1 + 2x_2 + s_2 = 5$

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Solution:

Using Basic Solution / Algebraic Method

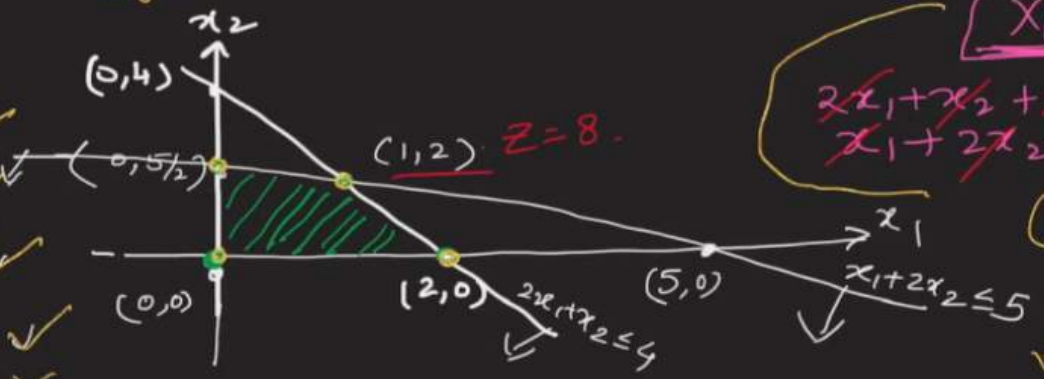
Case	Basic Variables	Non Basic Variables	Solution (X_B) x_1, x_2, s_1, s_2
(i)	s_1, s_2	$x_1 = x_2 = 0$	$(0, 0, 4, 5)$
(ii)	x_1, s_1	$x_2 = s_2 = 0$	$(5, 0, -6, 0)$
(iii)	x_1, s_2	$x_2 = 0 = s_1$	$(2, 0, 0, 3)$
(iv)	x_2, s_1	$x_1 = s_2 = 0$	$(0, \frac{5}{2}, \frac{3}{2}, 0)$
(v)	x_2, s_2	$x_1 = s_1 = 0$	$(0, 4, 0, -3)$
(vi)	x_1, x_2	$s_1 = s_2 = 0$	$(1, 2, 0, 0)$

^nC_m

Non deg.

BFS → Corner Pts.

Using Graphical Method



B.V. s_1, s_2 ; N.B.V. $x_1 = x_2 = 0$

$B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

(*) $I X_B = b$

$X_B = I^{-1}b = b \ge 0$

BFS.

Feasibility Criteria

Optimality Criteria

Simplex Table

Eg: $\text{Max } Z = 2x_1 + 3x_2$

s.t. $x_1 - x_2 \leq 2$

$-3x_1 + x_2 \leq 4$

$x_1, x_2 \geq 0$

B

Standard LPP:

$\text{Max } Z = 2x_1 + 3x_2 + 0 \cdot s_1 + 0 \cdot s_2$

$x_1 - x_2 + s_1 = 2$

$-3x_1 + x_2 + s_2 = 4$

$x_1, x_2, s_1, s_2 \geq 0$

$-3x_1 + x_2 \le 4$

$x_1, x_2 \ge 0$

$-3x_1 + x_2 + s_2 = 4$

$x_1, x_2, s_1, s_2 \ge 0$

Cost of B.V. C_B	Basic Variable B.V.	x_1	x_2	s_1	s_2	X_B (Solution)
	z					value of $z = 0$
0	s_1	1	−1	1	0	$2 = x_{B_1}$
0	s_2	−3	1	0	1	$4 = x_{B_2}$

scalars.

$$A = \begin{array}{c} \begin{array}{cccc} a_1 & a_2 & a_3 & a_4 \end{array} \\ \begin{pmatrix} 1 & -1 & 1 & 0 \\ -3 & 1 & 0 & 1 \end{pmatrix} \\ \begin{array}{cccc} x_1 & x_2 & s_1 & s_2 \end{array} \end{array}_{2\times 4}$$

$$B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_{2\times 2}$$

$B X_B = b$

$X_B = b$

$$\begin{pmatrix} 1 \\ -3 \end{pmatrix} = a_1 = 1\begin{pmatrix} 1 \\ 0 \end{pmatrix} + (-3)\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} -1 \\ 1 \end{pmatrix} = a_2 = (-1)\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 1\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = a_3 = 1\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 0\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = a_4 = 0\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 1\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$-3x_1 + x_2 \le 4$

$x_1, x_2 \ge 0$

$-3x_1 + x_2 + s_1 = 4$

$x_1, x_2, s_1, s_2 \ge 0$

Cost of B.V. C_B	Basic Variable B.V.	x_1	x_2	s_1	s_2	X_B (Solution)
	Z	-2	-3	0		Value of $Z = 0$
0	s_1	1	-1	1	0	$2 = x_{B_1}$
0	s_2	-3	1	0	1	$4 = x_{B_2}$

scalars

$$A = \begin{array}{c} \begin{array}{cccc} a_1 & a_2 & a_3 & a_4 \end{array} \\ \begin{pmatrix} 1 & -1 & 1 & 0 \\ -3 & 1 & 0 & 1 \end{pmatrix} \\ \begin{array}{cccc} x_1 & x_2 & s_1 & s_2 \end{array} \end{array} \quad 2\times 4$$

$$B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad 2\times 2$$

$B X_B = b$

$X_B = b$

$Z_j - C_j$: Net Evaluation | Relative Evaluation

$$Z_j - C_j = C_B \text{ (scalars)} - C_j$$

$$Z_1 - C_1 = \begin{pmatrix} 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -3 \end{pmatrix} - 2 = 0 - 2 = -2$$

$$Z_2 - C_2 = \begin{pmatrix} 0 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} - 3 = 0 - 3 = -3$$

Simplex Table Layout

optimality criteria !!

LPP.

$X \geq 0$

B.V.	x_1	x_2	---	s_n	X_B (Sol^n.)
Z	$z_j - c_j$				value of Z
s_1	scalars.				x_{B_1} ≥ 0
x_1					x_{B_2} ≥ 0
x_2					x_{B_3} ≥ 0

Feasibility !

Simplex Method (Solving LPP)

Steps	Procedures
1.	Write LPP in standard form.
2.	Make first Simplex table with initial BFS.
3.	Check $z_j - c_j \geq 0$ (Objective - Maximization) $\forall j$ → Optimal or $z_j - c_j \leq 0$ (Objective - Minimization) $\forall j$ → Optimal
4.	If step 3 does not hold; find new BFS; where Entering and Leaving variable $z_j - c_j$ most −ve (Max.) $z_j - c_j$ most +ve (Min) $\frac{x_{Br}}{\alpha_{rj}} = \min_i \left\{ \frac{x_{Bi}}{\alpha_{ij}} ; \alpha_{ij} > 0 \right\}$
5.	Make next simplex table using new BFS | Apply Row operation
6.	Repeat Step 3 onwards.

Status

optimal

Optimality

feasibility

Problem: Use Simplex method to solve

$$\text{Max } Z = 4x_1 + 3x_2$$

$$\text{s.t.} \quad x_1 + x_2 \le 8$$

$$2x_1 + x_2 \le 10$$

$$x_1, x_2 \ge 0$$

		4	3	0	0	
C_B	B.V.	x_1	x_2	s_1	s_2	X_B (Solution)
Z-Row:	Z	-4	-3	0	0	0
0	s_1	1	1	1	0	8
0	s_2	2	1	0	1	10

$Z_j - C_j$

Solution:

Write LPP into standard form; introduce slack variables;

$$\text{Max } Z = 4x_1 + 3x_2 + 0.s_1 + 0.s_2$$

$$\text{s.t.} \quad x_1 + x_2 + s_1 = 8$$

$$2x_1 + x_2 + s_2 = 10$$

$$x_1, x_2, s_1, s_2 \ge 0$$

$$\underset{2\times 4}{(A)}X = b$$

$$A = \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & s_1 & s_2 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{pmatrix}_{2\times 4} \end{array} \qquad X = \begin{pmatrix} x_1 \\ x_2 \\ s_1 \\ s_2 \end{pmatrix} \qquad b = \begin{pmatrix} 8 \\ 10 \end{pmatrix}$$

B.V. $= s_1, s_2$ $\quad AX = b \Rightarrow BX_B = b$

$B = \begin{pmatrix} 1 & 0 \end{pmatrix}$ $\quad (B)_{\cdot} X_B = b$

$$Z_j - C_j = \underbrace{C_B}_{\text{zero}}(\text{scalars}) - C_j = -C_j$$

$$Z_1 - C_1 = (0 \;\; 0)\begin{pmatrix} 1 \\ 2 \end{pmatrix} - 4 = -4$$

$$Z = 4(x_1) + 3(x_2) + 0(s_1) + 0(s_2)$$

$$= 0 + \underbrace{0(8) + 0(10)}_{0}$$

$$= 0$$

Problem: Use Simplex method to solve

$$\text{Max } Z = 4x_1 + 3x_2$$

$$\text{s.t. } x_1 + x_2 \le 8$$

$$2x_1 + x_2 \le 10$$

$$x_1, x_2 \ge 0$$

Solution:

Write LPP into standard form; introduce slack variables;

$$\text{Max } Z = 4x_1 + 3x_2 + 0 \cdot s_1 + 0 \cdot s_2$$

$$\text{s.t. } x_1 + x_2 + s_1 = 8$$

$$2x_1 + x_2 + s_2 = 10$$

$$x_1, x_2, s_1, s_2 \ge 0$$

C_B scalars

Enter

Remove

Pivot element

	B.V.	x_1	x_2	s_1	s_2	X_B	
	Z	-4	-3	0	0	0	: R_1
	s_1	1	1	1	0	8	: R_2
	s_2	$\boxed{2}$	1	0	1	10	: R_3
	Z	0	-1	0	2	20	R_1
0	s_1	0	$1/2$	1	$-1/2$	3	R_2
4	x_1	1	$1/2$	0	$1/2$	5	R_3

$$Z_1 - C_1 = (0 \;\; 4)\begin{pmatrix} 0 \\ 1 \end{pmatrix} - 4 = 4 - 4 = 0$$

$$Z_2 - C_2 = (0 \;\; 4)\begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} - 3 = (0 + 2) - 3 = -1$$

$$Z_3 - C_3 = (0 \;\; 4)\begin{pmatrix} 1 \\ 0 \end{pmatrix} - 0 = (0 + 0) - 0 = 0$$

$$\text{Max } Z = 4x_1 + 3x_2$$

s.t. $x_1 + x_2 \le 8$

$2x_1 + x_2 \le 10$

$x_1, x_2 \ge 0$

Solution:

Write LPP into standard form; introduce slack variables;

$$\text{Max } Z = 4x_1 + 3x_2 + 0 \cdot s_1 + 0 \cdot s_2$$

s.t. $x_1 + x_2 + s_1 = 8$

$2x_1 + x_2 + s_2 = 10$

$x_1, x_2, s_1, s_2 \ge 0$

Enter

Remove

B.V.	x_1	x_2	s_1	s_2	X_B	
Z	-4	-3	0	0	0	: R_1
s_1	1	1	1	0	8	: R_2
s_2	$\boxed{2}$	1	0	1	10	: R_3
Z	1	-1	0	2	20	R_1
s_1	0	$\boxed{1/2}$	1	$-1/2$ ✓	3 ✓	R_2 ✓
x_1	1	$1/2$	0	$1/2$ ✓	5 ✓	R_3
Z	1	0	2	1	26	R_1 ✓
x_2	0	1	2	-1	6	R_2
x_1	1	0	-1	1	2	R_3

$Z_j - C_j \ge 0$

$R_1 \to R_1 + 1(2R_2)$

✓ $R_2 \to 2R_2$

$R_3 \to R_3 - \frac{1}{2}(2R_2)$

$R_3 - R_2$

$Z_j - C_j \ge 0$ optimal

$Z_{max} = 26$ at $x_1 = 2$; $x_2 = 6$

Problem: Use Simplex Method to solve

$$\text{Minimize } Z = x_1 - 3x_2 + 2x_3$$

$$\text{s.t. } 3x_1 - x_2 + 2x_3 \le 7$$
$$-2x_1 + 4x_2 \le 12$$
$$-4x_1 + 3x_2 + 8x_3 \le 10$$
$$x_1 \ge 0;\ x_2 \ge 0;\ x_3 \ge 0$$

Solution:

Method I:

Write LPP into standard form.

$$\text{Min } Z = x_1 - 3x_2 + 2x_3 + 0.s_1 + 0.s_2 + 0.s_3$$

$$\text{s.t. } 3x_1 - x_2 + 2x_3$$

$$\hat{Z} = Z_0 - \theta_j (Z_j - C_j)$$

improved, previous

$\theta_j \ge 0$

Method I.

Max: Enter $Z_j - C_j$ most −ve

$\hat{Z} \ge Z_0$

Min: Enter $Z_j - C_j$ most +ve

$\hat{Z} \le Z_0$

Method 2

Convert Min Problem to Max Problem

$$\text{Min } Z = -\text{Max}(-Z)$$

or

$$\text{Max } Z = -\text{Min}(-Z)$$

$$A = \begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 \end{array} \\ \begin{pmatrix} 3 & -1 & 2 & 1 & 0 & 0 \\ -2 & 4 & 0 & 0 & 1 & 0 \\ -4 & 3 & 8 & 0 & 0 & 1 \end{pmatrix}_{3\times 6} \end{array}$$

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{3\times 3}$$

$$X_B = \underset{\substack{\downarrow \\ \text{Identity}}}{B^{-1}} b = b = \begin{pmatrix} 7 \\ 12 \\ 10 \end{pmatrix}$$

Min

Entering most +ve $Z_j - C_j$

C_B	B.V.	x_1	x_2	x_3	s_1	s_2	s_3	X_B	Ratio
	Z	-1	3	-2	0	0	0	0	
0	s_1	3	-1	2	1	0	0	7	
0	s_2	-2	$\boxed{4}$	0	0	1	0	12	$\frac{12}{4} = 3$
0	s_3	-4	3	8	0	0	0	10	$\frac{10}{3} = 3.3\ldots$

$$Z_j - C_j = C_B(B^{-1}a_j) - C_j$$

$$= -C_j$$

$\begin{pmatrix} 0 & 0 & 1 \end{pmatrix}_{3\times 3}$ Identity

Min

Entering most +ve $Z_j - C_j$

C_B	B.V.	x_1	x_2	x_3	s_1	s_2	s_3	X_B
	Z	−1	3	−2	0	0	0	0
0	s_1	3	−1	2	1	0	0	7
0	s_2	−2	$\boxed{4}$	0	0	1	0	12
0	s_3	−4	3	8	0	0	0	10
		1/2	0	−2	0	−3/4	0	−9
	s_1	5/2	0	2	1	1/4	0	10
	x_2	−1/2	1	0	0	1/4	0	3
	s_3	−5/2	0	8	0	−3/4	1	1

Pivot element

Ratio

$\frac{12}{4} = 3$ III

$\frac{10}{3} = 3.3\ldots$ IV

$Z_j - C_j = C_B(\text{scalar}) - C_j$

$B^{-1}a_j$

I

II $= -C_j$

III

IV

$R_1 \to R_1 - 3(R_3/4)$

$R_2 \to R_2 + 1(R_3/4)$

$R_3 \to R_3/4$

$R_4 \to R_4 - 3(R_3/4)$

$R_1 \to 0 - 3(12/4)$

$= 0 - 3(3) = -9$

0	s_1	3	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	
0	s_2	−2	$\boxed{4}$	0	0	1	0	12	$\frac{12}{4}=3$ III
0	s_3	−4	3	8	0	0	0	10	$\frac{10}{3}=3.3..$ IV
	z	1/2	0	−2	0	−3/4	0	−9	
	s_1	$\boxed{5/2}$	0	2	1	1/4	0	10	
	x_2	−1/2	1	0	0	1/4	0	3	
	s_3	−5/2	0	8	0	−3/4	1	1	
	z	0	0	−12/5	−1/5	−8/10	0	−11	
−1	x_1	1	0	4/5	2/5	1/10	0	4	
3	x_2	0	1	2/5	1/5	3/10	0	5	
0	s_3	0	0	10	2/5	−1/2	1	11	

Pivot element

Remove

$R_1 \to R_1 - 3(R_3/4)$

$R_2 \to R_2 + 1(R_3/4)$

$R_3 \to R_3/4$

$R_4 \to R_4 - 3(R_3/4)$

$R_1 \to R_1 - \frac{1}{2}\left(\frac{2}{5}R_2\right)$

$R_2 \to \frac{2}{5}R_2$

$R_3 \to R_3 + \frac{1}{2}\left(\frac{2}{5}R_2\right)$

$R_4 \to R_4 + \frac{5}{2}\left(\frac{2}{5}R_2\right)$

$Z_j - C_j \le 0$ (Min)

optimal; $Z_{min} = -11$

Method-2 (Convert Minimization into Maximization)

$\text{Min } Z = -\text{Max}(-Z)$

Write LPP into standard form.

$$\text{Min } Z = x_1 - 3x_2 + 2x_3 + 0.s_1 + 0.s_2 + 0.s_3$$

or

$$-\text{Max}(-Z) = x_1 - 3x_2 + 2x_3 + 0.s_1 + 0.s_2 + 0.s_3$$

$$\boxed{\text{Max}(-Z) = -x_1 + 3x_2 - 2x_3 - 0.s_1 - 0.s_2 - 0.s_3}$$

$$\text{s.t.} \quad \begin{cases} 3x_1 - x_2 + 2x_3 + s_1 = 7 \\ -2x_1 + 4x_2 + s_2 = 12 \\ -4x_1 + 3x_2 + 8x_3 + s_3 = 10 \\ x_1, x_2, x_3, s_1, s_2, s_3 \geq 0 \end{cases}$$

$$A = \begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 \end{array} \\ \begin{pmatrix} 3 & -1 & 2 & 1 & 0 & 0 \\ -2 & 4 & 0 & 0 & 1 & 0 \\ -4 & 3 & 8 & 0 & 0 & 1 \end{pmatrix}_{3\times 6} \end{array}$$

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$X_B = B^{-1}b = \begin{pmatrix} 7 \\ 12 \\ 10 \end{pmatrix} \geq 0 \quad \text{BFS}$$

Enter most -ve

$\begin{pmatrix} \\ 10 \end{pmatrix}$ BFS.

C_B	B.V.	x_1	x_2	x_3	s_1	s_2	s_3	X_B
	$-Z$	1	-3	+2	0	0	0	0
0	s_1	3	-1	2	1	0	0	7
0	s_2	-2	$\boxed{4}$	0	0	1	0	12
0	s_3	-4	3	8	0	0	1	10
	$-Z$	-1/2	0	2	0	3/4	0	9
	s_1	$\boxed{5/2}$	0	2	1	1/4	0	10
	x_2	-1/2	1	0	0	1/4	0	3
	s_3	-5/2	0	8	0	-3/4	1	1
	$-Z$	0	0	12/5	1/5	8/10	0	11
	x_1	1	0	4/5	2/5	1/10	0	4
	x_2	0	1	2/5	1/5	3/10	10	5
	s_3	0	0	10	2/5	-1/2	1	11

max.

$Z_j - C_j \geq 0$

$-Z = 11$

$\boxed{Z = -11}$

Simplex Method - Unbounded Solution

Theorem: Let there exist a BFS to the given LPP. If for entering variable; $z_j - c_j$ must -ve the corresponding scalar $\alpha_{ij} \leq 0$. (consider the case of Maximization), then there does not exist any optimum value to LPP; we say it has unbounded solution.

$$\min \left\{ \frac{x_{Bi}}{\alpha_{ij}} ; \alpha_{ij} > 0 \right\}$$

Problem: Solve LPP using Simplex Method

$$\text{Max } Z = 4x_1 + x_2$$
$$\text{s.t.} \quad x_1 - x_2 \le 1$$
$$-2x_1 + x_2 \le 2 \quad ; \quad x_1, x_2 \ge 0$$

Sol^n: Write LPP into standard form

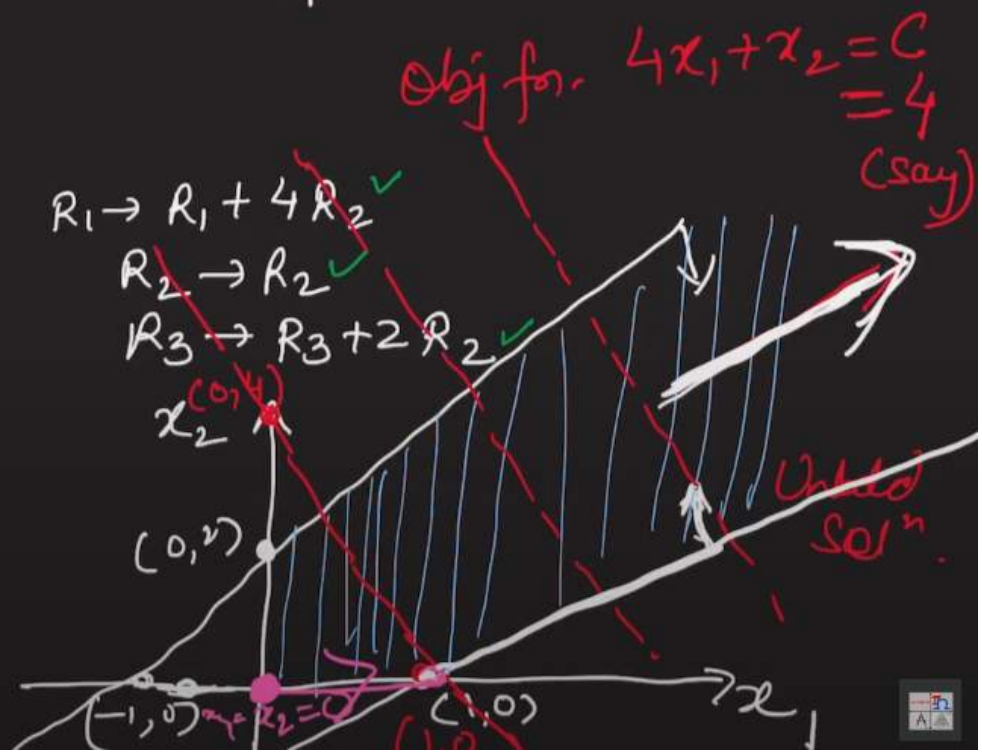
B.V.	x_1	x_2	s_1	s_2	X_B
	-4	-1	0	0	0
s_1	$\boxed{1}$	-1	1	0	1
s_2	-2	1	0	1	2
	0	-5	4	0	4
x_1	1	-1	1	0	1
s_2	0	-1	2	1	4

$x_1 = 0$
$x_2 = 0$

$R_1 \to R_1 + 4R_2$
$R_2 \to R_2$
$R_3 \to R_3 + 2R_2$

We have Entering Variable but do not have leaving variable; so this gives us

$\boxed{\text{Unbounded Solution}}$



Problem: Use Simplex method to solve the Reddy Mikks Model:

Maximize $Z = 5x_1 + 4x_2$

$$6x_1 + 4x_2 \le 24,\ x_1 + 2x_2 \le 6,\ -x_1 + x_2 \le 1,\ x_2 \le 2,\ x_1, x_2 \ge 0$$

Solution: Write the LPP into standard form

Maximize $Z = 5x_1 + 4x_2 + 0.s_1 + 0.s_2 + 0.s_3 + 0.s_4$

s.t. $6x_1 + 4x_2 + s_1 = 24$

$x_1 + 2x_2 + s_2 = 6$

$-x_1 + x_2 + s_3 = 1$

$x_2 + s_4 = 2$; $x_1, x_2, s_1, s_2, s_3, s_4 \ge 0$

B.V.	x_1	x_2	s_1	s_2	s_3	s_4	X_B (Sol.)		Min Ratio
Z	-5	-4	0	0	0	0	0	Z-row	
s_1	6	4	1	0	0	0	24	R_1-row	24/6 = 4 — Min.
s_2	1	2	0	1	0	0	6	R_2-row	6/1 = 6
s_3	-1	1	0	0	1	0	1	R_3-row	
s_4	0	1	0	0	0	1	2	R_4-row	

$x_2 + s_4 = 2$; $x_1, x_2, s_1, s_2, s_3, s_4 \geq 0$

$\min\left\{ \frac{x_{Bi}}{\alpha_{ij}} ; \alpha_{ij} > 0 \right\}$

B.V.	x_1	x_2	s_1	s_2	s_3	s_4	X_B (Sol.)		Min Ratio
Z	-5	-4	0	0	0	0	0	Z-row	
s_1	6	4	1	0	0	0	24	R_1-row	$24/6 = 4$ → Min.
s_2	1	2	0	1	0	0	6	R_2-row	$6/1 = 6$
s_3	-1	1	0	0	1	0	1	R_3-row	
s_4	0	1	0	0	0	1	2	R_4-row	
Z	0	-2/3	5/6	0	0	0	20	Z-row	
x_1	1	2/3	1/6	0	0	0	4	R_1-row	
s_2	0	4/3	-1/6	1	0	0	2	R_2-row	
s_3	0	5/3	1/6	0	1	0	5	R_3-row	
S_4	0	1	0	0	0	1	2	R_4-row	
Z	0	0	3/4	1/2	0	0	21	Z-row	
x_1	1	0	1/4	-1/2	0	0	3	R_1-row	
x_2	0	1	-1/8	3/4	0	0	3/2	R_2-row	
s_3	0	0	3/8	-5/4	1	0	5/2	R_3-row	
s_4	0	0	1/8	-3/4	0	1	1/2	R_4-row	

$R_1 \to R_1 + 5(R_2/6)$

$R_2 \to R_2/6$

$R_3 \to R_3 - 1(R_2/6)$

$R_4 \to R_4 + 1(R_2/6)$

$R_5 \to R_5$

Here $z_j - c_j \geq 0$; Maximization LPP.

$R_1 \to R_1 + \frac{2}{3}\left(\frac{3}{4}R_3\right)$

$R_2 \to R_2 - \frac{2}{3}\left(\frac{3}{4}R_3\right)$

$R_3 \to \frac{3}{4}R_3$

Z	-5	-4	0	0	0	0	0	Z-row
s_1	6	4	1	0	0	0	24	R_1-row
s_2	1	2	0	1	0	0	6	R_2-row
s_3	-1	1	0	0	1	0	1	R_3-row
s_4	0	1	0	0	0	1	2	R_4-row
Z	0	-2/3	5/6	0	0	0	20	Z-row
x_1	1	2/3	1/6	0	0	0	4	R_1-row
s_2	0	4/3	-1/6	1	0	0	2	R_2-row
s_3	0	5/3	1/6	0	1	0	5	R_3-row
S_4	0	1	0	0	0	1	2	R_4-row
Z	0	0	3/4	1/2	0	0	21	Z-row
x_1	1	0	1/4	-1/2	0	0	3	R_1-row
x_2	0	1	-1/8	3/4	0	0	3/2	R_2-row
s_3	0	0	3/8	-5/4	1	0	5/2	R_3-row
s_4	0	0	1/8	-3/4	0	1	1/2	R_4-row

Optimal Value, Z= 21 at Optimal Solution (x_1,x_2)= (3,3/2)

Min Ratio

$24/6 = 4 \rightarrow$ Min.

$6/1 = 6$

$R_1 \rightarrow R_1 + 5(R_2/6)$

$R_2 \rightarrow R_2/6$

$R_3 \rightarrow R_3 - 1(R_2/6)$

$R_4 \rightarrow R_4 + 1(R_2/6)$

$R_5 \rightarrow R_5$

Here $Z_j - C_j \geq 0$; Maximization LPP.

$R_1 \rightarrow R_1 + \frac{2}{3}\left(\frac{3}{4}R_3\right)$

$R_2 \rightarrow R_2 - \frac{2}{3}\left(\frac{3}{4}R_3\right)$

$R_3 \rightarrow \frac{3}{4}R_3$

$R_4 \rightarrow R_4 - \frac{5}{3}\left(\frac{3}{4}\right)R_3$

$R_5 \rightarrow R_5 - \frac{3}{4}R_3$

s_3	2	3	0	0	1	10
Z	0	0	0	2	0	200
s_1	8/5	0	1	-1/5	0	30
x_2	2/5	1	0	1/5	0	20
s_3	4/5	0	0	-3/5	1	30

$Z_j - C_j \geq 0$; Optimal soln, $Z_{max} = 200$ at $x_1 = 0$; $x_2 = 20$

Complete solution

One Solution

$X_1 = (x_1, x_2, s_1, s_2, s_3)$
$= (0, 20, 30, 0, 30)$

1

Note: For alternate solution ; enter a Non-Basic variable with $Z_j - C_j = 0$ in optimal simplex table ; and find the corresponding leaving variable.

B.V.	x_1	x_2	s_1	s_2	s_3	X_B
Z	0	0	0	2	0	200
x_1	1	0	5/8	-1/8	0	150/8
x_2	0	1	-1/4	1/4	0	25/2
s_3	0	0	-1/2	-1/2	1	15

$R_1 \to R_1$

$R_2 \to \frac{5}{8} R_2$

$R_3 \to R_3 - 2/5 \left(5/8 R_2\right)$

$R_4 \to R_4 - 4/5 \left(5/8 R_2\right)$

$X_2 = \left(\frac{150}{8}, \frac{25}{2}, 0, 0, 15\right)$

$Z_{max} = 200$ at $x_1 = \frac{150}{8}$; $x_2 = 25/2$

2

Take convex linear combination to get Infinite Solutions

x_1	1	0	5/8	1/8	0	150/8
x_2	0	1	-1/4	1/4	0	25/2
s_3	0	0	-1/2	-1/2	1	15

$R_3 \to R_3 - \frac{2}{5}\left(\frac{5}{8} R_2\right)$

$R_4 \to R_4 - \frac{4}{5}\left(\frac{5}{8} R_2\right)$

$X_2 = \left(\frac{150}{8}, \frac{25}{2}, 0, 0, 15\right)$

$\boxed{Z_{max} = 200 \text{ at } x_1 = \frac{150}{8};\ x_2 = 25/2}$

2

Take convex linear combination to get Infinite Solutions

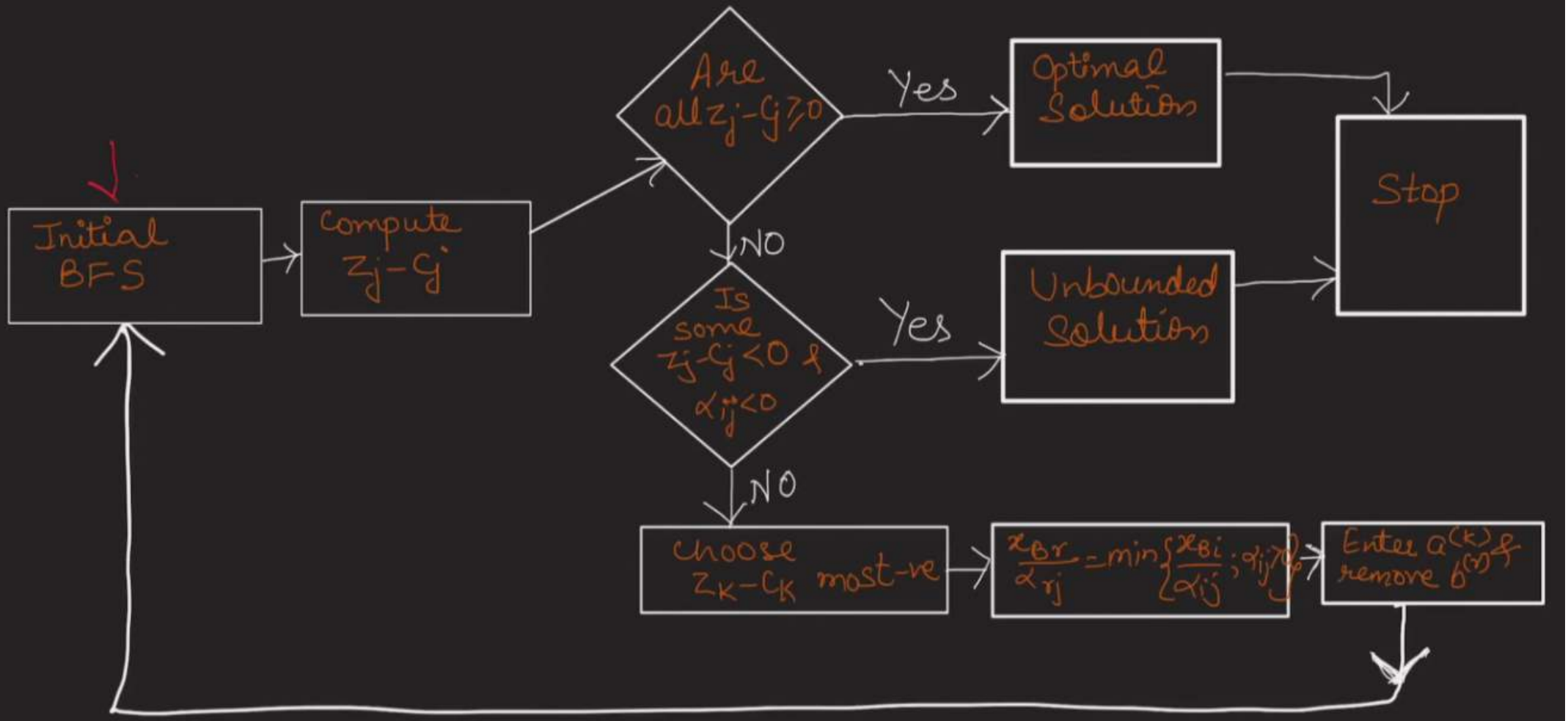
$$X = \lambda_1 X_1 + \lambda_2 X_2 \quad ; \quad \lambda_1 + \lambda_2 = 1 \quad ; \lambda_i \geq 0$$

$$= \lambda_1 (0, 20, 30, 0, 30) + \lambda_2 \left(\frac{150}{8}, \frac{25}{2}, 0, 0, 15\right)$$

$\lambda_1 = 1/2, \lambda_2 = 1/2$

$\lambda_1 = \frac{1}{4}, \lambda_2 = 3/4$

Summary of Simplex Method and Transition to Big-M and Two-Phase Method



Big M Method

When we solve an LPP using Simplex method we start with a ready BFS, with basis matrix as Identity matrix. If the Identity matrix is missing then we introduce an artificial Variable a_i for ready BFS and start the algorithm/simplex method.

LPP: Opt (Max | Min) $Z = C^T X$

(in standard form) $AX = b$; $b \geq 0$; $(A)_{m \times n}$

$X \geq 0$

where $A = (a_1, a_2, \ldots, a_n)$; a_i's → column of A

$B = (b_1, b_2, \ldots, b_m)$; b_j's → column of Basis Matrix

Now for initial BFS ; we select B as basis matrix and selecting Basic variables and non Basic Variables

→ $AX = b \Rightarrow BX_B = b$

$\Rightarrow X_B = B^{-1} b = b$ (Noting B = Identity Matrix)

$\therefore X_B = b \geq 0$ → Act as Initial BFS

(feasible region) Z_{max} $Z_{min} \to -\infty$

Important formulas for Simplex Method

1. $X_B = B^{-1} b$; $B \to$ Basis Matrix corresponding to selected Basic Variable.

2. scalars $= \alpha^j = \underbrace{B^{-1} a_j}$; where $\alpha^j \to$ scalars corresponding to j^{th} variable.

3. $Z = C_B X_B$

4. $Z_j - C_j = C_B (B^{-1} a_j) - C_j$

5. Note, for B.V. '$Z_j - C_j = 0$'.

6. For N.B.V. '$Z_j - C_j$' $\to$ calculate

$\boxed{B^{-1}}$

Eg: Max $Z = 4x_1 + 3x_2$

s.t. $x_1 + x_2 \le 8$

$2x_1 + x_2 \le 10$

$x_1, x_2 \ge 0$

Write LPP into standard form

Max $Z = 4x_1 + 3x_2 + 0 \cdot s_1 + 0 \cdot s_2$

s.t. $x_1 + x_2 + S_1 = 8$ ✓

$2x_1 + x_2 + S_2 = 10$ ✓

all var. ≥ 0

$B = I$

BFS)

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{pmatrix}_{2\times 4}$$

$$\underline{B} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_{2\times 2}$$

B.V. = (s_1, s_2)

Simplex Table:

B.V.	x_1	x_2	s_1	s_2	X_B
s_1	.				8
s_2	.				10

scalars.

$\uparrow$ $\underline{B^{-1}b}$

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Can We apply Simplex Method :

Problem 1 : $\text{Max } Z = x_1 + 2x_2 + 3x_3$

s.t. $x_1 + 2x_2 + 3x_3 \leq 10$

$x_1 + x_2 \leq 5$

$x_1, x_2, x_3 \geq 0$

$$A = \begin{pmatrix} 1 & 2 & 3 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{pmatrix}$$

Yes

Problem 2 : $\text{Max } Z = 4x_1 + 5x_2 + 9x_3 + 11x_4$

s.t. $x_1 + x_2 + x_3 + x_4 \leq 15$

$7x_1 + 5x_2 + 3x_3 + 2x_4 \leq 120$

$3x_1 + 5x_2 + 10x_3 + 15x_4 \leq 100$

$x_1, x_2, x_3, x_4 \geq 0$

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 7 & 5 & 3 & 2 & 0 & 1 & 0 \\ 3 & 5 & 10 & 15 & 0 & 0 & 1 \end{pmatrix}$$

Problem 3 : $\text{Max } Z = 4x_1 + x_2 + 3x_3 + 5x_4$

s.t. $4x_1 - 6x_2 - 5x_3 - 4x_4 \geq -20$

$3x_1 - 2x_2 + 4x_3 + x_4 \leq 10$

$8x_1 - 3x_2 + 3x_3 + 2x_4 \leq 20$

$x_1, x_2, x_3, x_4 \geq 0$

Problem 4 : $\text{Max } Z = 107x_1 + x_2 + 2x_3$

Problem 3: $\text{Max } Z = 4x_1 + x_2 + 3x_3 + 5x_4$

s.t.

$$4x_1 - 6x_2 - 5x_3 - 4x_4 \geq -20$$
$$3x_1 - 2x_2 + 4x_3 + x_4 \leq 10$$
$$8x_1 - 3x_2 + 3x_3 + 2x_4 \leq 20$$
$$x_1, x_2, x_3, x_4 \geq 0$$

Multiply by -1

$$A = \begin{pmatrix} -4 & 6 & 5 & 4 & 1 & 0 & 0 \\ 3 & -2 & 4 & 1 & 0 & 1 & 0 \\ 8 & -3 & 3 & 2 & 0 & 0 & 1 \end{pmatrix}$$

$$X_B = \begin{pmatrix} 20 \\ 10 \\ 20 \end{pmatrix} \geq 0 \quad \text{Yes} \rightarrow \text{BFS}$$

Problem 4:

$\text{Max } Z = 107x_1 + x_2 + 2x_3$

s.t.

$$14x_1 + x_2 - 6x_3 + 3x_4 = 7$$
$$16x_1 + x_2 - 6x_3 \leq 5$$
$$3x_1 - x_2 - x_3 \leq 0$$
$$x_1, x_2, x_3, x_4 \geq 0$$

$$A = \begin{array}{c} \begin{pmatrix} 14 & 1 & -6 & 3 & 0 & 0 \\ 16 & 1 & -6 & 0 & 1 & 0 \\ 3 & -1 & -1 & 0 & 0 & 1 \end{pmatrix} \\ \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & s_1 & s_2 \end{array} \end{array}$$

(in 2nd constraint eqn) (in third)

$B = \text{Identity}$

Problem 4:

$$\text{Max } Z = 107x_1 + x_2 + 2x_3$$

s.t

$$14x_1 + x_2 - 6x_3 + 3x_4 = 7$$
$$16x_1 + x_2 - 6x_3 \le 5$$
$$3x_1 - x_2 - x_3 \le 0$$
$$x_1, x_2, x_3, x_4 \ge 0$$

$x_1, x_2, x_3, x_4 \ge 0$

$X_B = \begin{pmatrix} 10 \\ 20 \end{pmatrix} \ge 0$ Yes → BFS

$$A = \begin{array}{c} \begin{pmatrix} 14 & 1 & -6 & 3 & 0 & 0 \\ 16 & 1 & -6 & 0 & 1 & 0 \\ 3 & -1 & -1 & 0 & 0 & 1 \end{pmatrix} \\ \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & s_1 & s_2 \end{array} \end{array}$$

(in 2nd constraint eqn) (in third eqn)

B = Identity

Yes

C_B	B.V.	x_1	x_2	x_3	x_4	s_1	s_2	X_B
	Z	-107	-1	-2	0	0	0	0
0	x_4	$14/3$	$1/3$	$-6/3$	1	0	0	$7/3$
0	s_1	16	1	-6	0	1	0	5
0	s_2	3	-1	-1	0	0	1	0

When to Transit for Simplex to Big-M / Two-Phase Method

Problem: Max $Z = 10x_1 + 20x_2$

s.t. $2x_1 + x_2 = 1$

$x_1 + 2x_2 = 5$; $x_1, x_2 \geq 0$

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \qquad B \neq I \ ?$$

Problem: Min $Z = x_1 + x_2 + 2x_3$

s.t. $2x_1 + x_2 + x_3 \geq 4$ ✓

$x_1 + 7x_2 \geq 7$ ✓

$x_1, x_2 \geq 0$

$$A = \begin{pmatrix} 2 & 1 & 1 & -1 & 0 \\ 1 & 7 & 0 & 0 & -1 \end{pmatrix} \quad \times$$

$B = -I$

$X_B = B^{-1} b = -b$

$= -\begin{pmatrix} 4 \\ 7 \end{pmatrix}$

Problem: Min $Z = x_1 + x_2 + x_3$

s.t. $x_1 - 3x_2 + 4x_3 = 5$

$x_1 - 2x_2 \leq 3$

$2x_2 + x_3 \geq 4$

$x_1 \geq 0$; $x_2 \geq 0$; x_3 - unrestricted in sign

$x_1, x_2 \geq 0$

Problem: Min $Z = x_1 + x_2 + x_3$

s.t. $x_1 - 3x_2 + 4x_3 = 5$

$x_1 - 2x_2 \leq 3$ ✓

$2x_2 + x_3 \geq 4$ ✓

$x_1 \geq 0$; $x_2 \geq 0$; x_3 - unrestricted in sign

or Min $Z = x_1 + x_2 + x_3' - x_3''$

s.t. $x_1 - 3x_2 + 4x_3' - 4x_3'' = 5$

$x_1 - 2x_2 + s_1 = 3$

$2x_2 + x_3' - x_3'' - s_2 = 4$

$x_1, x_2, x_3', x_3'', s_1, s_2 \geq 0$

where $x_3 = x_3' - x_3''$

$B = -I$

$X_B = B^{-1}b = -b$

$= -\begin{pmatrix} 4 \\ 7 \end{pmatrix}$

$$A = \begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3' & x_3'' & s_1 & s_2 \end{array} \\ \begin{pmatrix} 1 & -3 & 4 & -4 & 0 & 0 \\ 1 & -2 & 0 & 0 & 1 & 0 \\ 0 & 2 & 1 & -1 & 0 & -1 \end{pmatrix} \end{array}$$

B -

Introduction & Use of Artificial Variables

LPP: While applying simplex method; we start with basic variables that gives initial BFS. Infact; no computations is required to construct first simplex table.

'$\le$', $B = I$.

Eg: $\text{Min } Z = -x_1 - x_2$

$$\text{s.t. } x_1 + 2x_2 \le 1$$
$$2x_1 - x_2 + x_3 = 5$$
$$x_1, x_2, x_3 \ge 0$$

$$A = \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & s_1 \end{array} \\ \begin{pmatrix} 1 & 2 & 0 & 1 \\ 2 & -1 & 1 & 0 \end{pmatrix} \\ \begin{array}{cccc} a_1 & a_2 & a_3 & a_4 \end{array} \end{array}$$

B.V. = (s_1, x_3)

$$B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

C_B	B.V.	x_1	x_2	x_3	s_1	X_B
	Z	1	1	0	0	0
	s_1	1	2	0	1	1
	x_3	2	-1	1	0	5
		$B^{-1}a_1$			$B^{-1}a_4$	$B^{-1}b$

Eg: $\text{Min } Z = 2x_1 + x_2$

s.t. $3x_1 + x_2 = 3$

$4x_1 + 3x_2 \geq 6$

$x_1 + 2x_2 \leq 3$

$x_1, x_2 \geq 0$

$\text{Min } Z = 2x_1 + x_2 + 0 \cdot s_1 + 0 \cdot s_2$

$\Rightarrow 3x_1 + x_2 = 3$

$\Rightarrow 4x_1 + 3x_2 - s_1 = 6$

$\Rightarrow x_1 + 2x_2 + s_2 = 3$

all var. ≥ 0

or

$$\left.\begin{aligned} 3x_1 + x_2 + a_1 &= 3 \\ 4x_1 + 3x_2 - s_1 + a_2 &= 6 \\ x_1 + 2x_2 + s_2 &= 3 \end{aligned}\right]$$

$$A = \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & s_1 & s_2 \end{array} \\ \begin{pmatrix} 3 & 1 & 0 & 0 \\ 4 & 3 & -1 & 0 \\ 1 & 2 & 0 & 1 \end{pmatrix} \end{array} \quad 3 \times 4$$

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$B.V. = (a_1, a_2, s_2)$$

Introduce artificial variable(s) for ready identity as submatrix of coefficient matrix in LPP with high penalty/Cost (-M) in Maximization Problem and with penalty/ Cost (M) in Minimization Problem.

$M > 0$

Consider cost of artificial variable with following rule.

$$C_a = \begin{cases} M & ; \text{ if LPP objective function is of Minimization} \\ -M & ; \text{ if LPP objective function is of Maximization.} \end{cases}$$

Cost of artificial variable

In general Big-M can give following cases:

- Case 1. If no artificial variable remain in the basis (in the last simplex table) then the system is consistent.
- Case 2. If artificial variable remain in the basis (in last simplex table) with positive value then the system is inconsistent/ have no solution. → Infeasible solution
- Case 3. If artificial variable remain in the basis (in last simplex table) with zero value (zero level) then the system is consistent. (That is the system is degenerate) (a Basic variable with zero value)

Problem 1 : Use Big-M method to solve the LPP given below

Minimize $Z = 2x_1 + 8x_2$

s.t.

$$5x_1 + 10x_2 = 150,\quad x_1 \le 20,\quad x_2 \ge 14,\quad x_1, x_2 \ge 0$$

$5x_1 + 10x_2 = 150$; $x_1 + s_1 = 20$; $x_2 - s_2 = 14$; $x_1, x_2, s_1, s_2 \ge 0$

$$A = \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & s_1 & s_2 \end{array} \\ \begin{pmatrix} 5 & 10 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} \end{array}$$

$B \ne I$?

Solution: Write the LPP into standard form by introducing slack/surplus variables (s_j) with cost 0 and artificial variables (a_i) with cost M.

Minimize $Z = 2x_1 + 8x_2 + 0.s_1 + 0.s_2 + M.a_1 + M.a_2$

s.t.

$$5x_1 + 10x_2 + a_1 = 150$$
$$x_1 + s_1 = 20$$
$$x_2 - s_2 + a_2 = 14$$
$$x_1, x_2, s_1, s_2, a_1, a_2 \ge 0$$

$$B = I_3 = \begin{array}{c} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ \begin{array}{ccc} a_1 & s_1 & a_2 \end{array} \end{array}$$

Solution: Write the LPP into standard form by introducing slack/surplus variables (s_j) with cost 0 and artificial variables (a_i) with cost M.

$(M>0)$

$B = I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $\begin{array}{ccc} a_1 & s_1 & a_2 \end{array}$

Minimize $Z = 2x_1 + 8x_2 + 0.s_1 + 0.s_2 + M.a_1 + M.a_2$

s.t.

$$5x_1 + 10x_2 + a_1 = 150$$
$$x_1 + s_1 = 20$$
$$x_2 - s_2 + a_2 = 14$$
$$x_1, x_2, s_1, s_2, a_1, a_2 \geq 0$$

Now $A = \begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & s_1 & s_2 & a_1 & a_2 \end{array} \\ \begin{pmatrix} 5 & 10 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$; $B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{3\times3}$

$B.V. = (a_1, s_1, a_2)$

		2	8	0	0	M	M		
C_B	B.V.	x_1	x_2	s_1	s_2	a_1	a_2	X_B (Sol.)	Ratio for Leaving
	Z	5M-2	11M-8	0	-M	0	0	164M	
M	a_1	5	10	0	0	1	0	150	
0	s_1	1	0	1	0	0	0	20	
M	a_2	0	1	0	-1	0	1	14	
	Z	5M-2	0	0	10M-8	0	$-11M+8$	112+10M	
M	a_1	5	0	0	10	1	-10	10	

$z_j - c_j = C_B(B^{-1}A_j) - c_j$

$z_j - c_j \leq 0$ Min (optimal)

$M>0$

$$A = \begin{pmatrix} 5 & 10 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}; \quad B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{3\times 3}$$

$$B.V. = (a_1, s_1, a_2)$$

		2	8	0	0	M	M			
C_B	B.V.	x_1	x_2	s_1	s_2	a_1	a_2	X_B (Sol.)	Ratio for Leaving	
	Z	5M-2	11M-8	0	-M	0	0	164M		R_1
M	a_1	5	10	0	0	1	0	150	$\frac{150}{10}=15$	R_2
0	s_1	1	0	1	0	0	0	20		R_3
M	a_2	0	1	0	-1	0	1	14	$\frac{14}{1}=14$	R_4
	Z	5M-2	0	0	10M-8	0	-11M+8	112+10M		$R_1 \to R_1 - (11M-8)R_4$
M	a_1	5	0	0	10	1	-10	10		$R_2 \to R_2 - 10R_4$
0	s_1	1	0	1	0	0	0	20		$R_3 \to R_3$
8	x_2	0	1	0	-1	0	1	14		$R_4 \to 1$
	Z	2	0	0	0	X	X	120		
0	s_2	1/2	0	0	1	X	X	1		
0	s_1	1	0	1	0	X	X	20		
8	x_2	1/2	1	0	0	X	X	15		
	Z	0	0	0	-4	X	X	116		
2	x_1	1	0	0	2	X	X	2		

$z_j - c_j = C_B(B^{-1}A_j) - c_j$

$z_j - c_j \le 0$ Min (optimal)

$M > 0$

Solution: Write the LPP into standard form by introducing slack/surplus variables (s_j) with cost 0 and artificial variables (a_i) with cost -M.

Maximize $Z = -x_1 + x_2 + 0.s_1 + 0.s_2 - M.a_1$

s.t. $x_1 + x_2 + s_1 = 1$

$2x_1 + 3x_2 + s_2 + a_1 = 6$

$x_1, x_2, s_1, s_2, a_1 \geq 0$

Here $A = \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & s_1 & s_2 & a_1 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 2 & 3 & 0 & -1 & 1 \end{pmatrix} \end{array}$

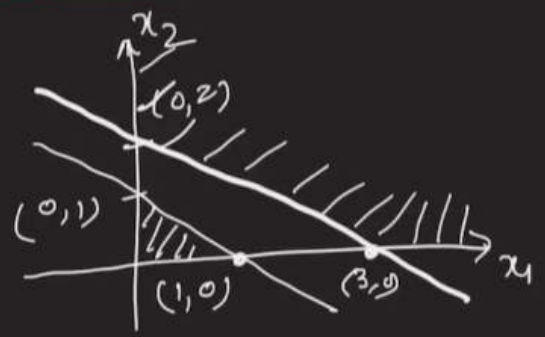
Use these columns in B

$\therefore$ B.V. = (s_1, a_1)

$z_j - c_j \geq 0$ optimal ✗

Entry $z_j - c_j$ most -ve

C_B	B.V.	x_1	x_2	s_1	s_2	a_1	X_B (Sol.)	Ratio for Leaving
	Z	-2M+1	-3M-1	0	M	0	-6M	
0	s_1	1	1	1	0	0	1	=
-M	a_1	2	3	0	-1	1	6	$6/3 = 2$
	Z	M+2	0	3M+1	M	0	-3M+1	
1	x_2	1	1	1	0	0	1	
-M	a_1	-1	0	-3	-1	1	3	



No common Region

$\therefore$ Infeasible solution.

Conclusion: No Solution of above LPP; Infeasible solution.

$z_j - c_j \geq 0$ optimal ; $a_1 = 3$

Inconsistent

$Z = -3M+1$

Problem 3: Use Big M method to solve the following LPP

$$\text{Max } Z = -2x_1 - x_2$$

$$\text{s.t.} \quad 3x_1 + x_2 = 3$$
$$4x_1 + 3x_2 \geq 6$$
$$x_1 + 2x_2 \leq 3$$
$$x_1, x_2 \geq 0$$

Solution: Write above LPP in standard form; after introducing slack/surplus artificial variable.

$$A = \begin{array}{c} \begin{pmatrix} 3 & 1 & 0 & 0 & 1 & 0 \\ 4 & 3 & -1 & 0 & 0 & 1 \\ 1 & 2 & 0 & 1 & 0 & 0 \end{pmatrix} \\ \begin{array}{cccccc} x_1 & x_2 & s_1 & s_2 & a_1 & a_2 \end{array} \end{array}$$

$$\text{Max } Z = -2x_1 - x_2 + 0.s_1 + 0.s_2 - Ma_1 - Ma_2$$

$$\text{s.t.} \quad 3x_1 + x_2 + a_1 = 3$$
$$4x_1 + 3x_2 - s_1 + a_2 = 6$$
$$x_1 + 2x_2 + s_2 = 3 \quad ; \quad \text{all var.} \geq 0$$

C_B	B.V.	x_1	x_2	s_1	a_1	a_2	s_2	X_B	Ratio
	Z	$-7M+2$	$-4M+1$	M	0	0	0	$-9M$	
$-M$	a_1	$\boxed{3}$	1	0	1	0	0	3	3/3
$-M$	a_2	4	3	-1	0	1	0	6	6/4
0	s_2	1	2	0	0	0	1	3	3/1
		0	$-\frac{5}{3}M+\frac{1}{3}$	M	$\frac{4M}{3}-\frac{2}{3}$	0	0	$-2M-2$	Ratio
-2	x_1	1	$1/3$	0	$1/3$	0	0	1	$1/1/3 = 3$

Minimum

C_B	B.V.	x_1	x_2	s_1	a_1	a_2	s_2		Ratio
	Z	$-7M+2$	$-4M+1$	M	0	0	0	$-9M$	
$-M$	a_1	$\boxed{3}$	1	0	1	0	0	3	$3/3$
$-M$	a_2	4	3	-1	0	1	0	6	$6/4$
0	s_2	1	2	0	0	0	1	3	$3/1$
		0	$\frac{-5}{3}M+\frac{1}{3}$	M	$\frac{4M}{3}-\frac{2}{3}$	0	0	$-2M-2$	Ratio
-2	x_1	1	$1/3$	0	$1/3$	0	0	1	$1/1/3=3$
$-M$	a_2	0	$5/3$	-1	$-4/3$	1	0	2	$6/5$
0	s_2	0	$\boxed{5/3}$	0	$-1/3$	0	1	2	$6/5$
		0	0	M	$M-3/5$	0	$M-1/5$	$-12/5$	
-2	x_1	1	0	0	$2/5$	0	$-1/5$	$3/5$	
$-M$	a_2	0	0	-1	-1	1	-1	0	
-1	x_2	0	1	0	$-1/5$	0	$3/5$	$6/5$	

Tie in the minimum Ratio

In above table; $Z_j - C_j \geq 0$ $\therefore$ optimality has arrived.
artificial variable is in the final basis but it is a zero level.
$\therefore$ the system is feasible.

$\therefore$ $Z_{max} = -12/5$; $x_1 = 3/5$; $x_2 = 6/5$

Two-Phase Method for Solving LPP

- Simplex Method
- Big-M

→ artificial variables

$B = I$

Row-operations

Phase I → constraint feasible

Phase II → optimization

Phase I: Introduce additional objective function in place of the given objective function, the new LPP formed is called Auxiliary LPP. The auxiliary LPP is

or Maximize $Z^* = -a_1 - a_2 - \cdots - a_r$

Minimize $Z^* = a_1 + a_1 + \ldots\ldots\ldots + a_r$ (the sum of artificial variables)

Subject to the constraints of the original LPP, and $a_i \geq 0$.

Auxiliary LPP.

feasible

$Z^* = 0$

$\sum a_i = 0$

$a_i = 0 \ \forall i$

$a_i = 0 \rightarrow$ deg. BFS.

$a_i > 0$

Solve the above LPP with usual simplex algorithm. The following three cases may arise.

Case 1. Minimize $Z^* > 0$, and artificial variable remain in the basis. Then the LPP has no solution or have Infeasible solution.

Case 2. Minimize $Z^* = 0$, and no artificial variable remain in the basis. Then the LPP has feasible solution and go to Phase II.

Case 3. Minimize $Z^* = 0$, and artificial variable remain in the basis at zero level. Then the LPP has feasible solution and go to Phase II.

or Maximize $Z = -a_1 - a_2 - \dots - a_r$

Minimize $Z^* = a_1 + a_1 + \dots\dots\dots + a_r$ (the sum of artificial variables)

Subject to the constraints of the original LPP, and $a_i \geq 0$.

Auxiliary LPP.

Solve the above LPP with usual simplex algorithm. The following three cases may arise.

- Case 1. Minimize $Z^* > 0$, and artificial variable remain in the basis. Then the LPP has no solution or have Infeasible solution.
- Case 2. Minimize $Z^* = 0$, and no artificial variable remain in the basis. Then the LPP has feasible solution and go to Phase II.

  Case 3. Minimize $Z^* = 0$, and artificial variable remain in the basis at zero level. Then the LPP has feasible solution and go to Phase II.

Phase II: Use the last table of Phase-I and find new z-row with original objective function. Do simplex iteration if required to find optimal

(Max $Z^* < 0$)

- Case 1. Minimize $Z^* > 0$, and artificial variable remain in the basis. Then the LPP has no solution or have Infeasible solution.
- Case 2. Minimize $Z^* = 0$, and no artificial variable remain in the basis. Then the LPP has feasible solution and go to Phase II.

  Max $Z^* = 0$
- Case 3. Minimize $Z^* = 0$, and artificial variable remain in the basis at zero level. Then the LPP has feasible solution and go to Phase II.

  Max $Z^* = 0$ (deg.) BS.

Phase II: Use the last table of Phase-I and find new z-row with original objective function. Do simplex iteration if required to find optimal solution.

$z_j - c_j \geq 0$ (Max.)

Enter most -ve $z_j - c_j$

$z_j - c_j \leq 0$ (Min

Problem: Use two-Phase method to solve

Max $Z = -x_1 + x_2$

s.t. $x_1 + x_2 \le 1$

$2x_1 + 3x_2 \ge 6$; $x_1, x_2 \ge 0$

Phase I

Solution: → Write LPP into standard form

$$\text{Max } Z = -x_1 + x_2 + 0.s_1 + 0.s_2$$

s.t. $x_1 + x_2 + s_1 = 1$

$2x_1 + 3x_2 - s_2 = 6$; all var. ≥ 0

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 2 & 3 & 0 & -1 \end{pmatrix}$$

B ≠ I

Auxiliary LPP

$$\text{Min } Z^* = a_1 + 0.s_i + 0.x_i$$

s.t. $x_1 + x_2 + s_1 = 1$

$2x_1 + 3x_2 - s_2 + a_1 = 6$; $x_1, x_2, s_1, s_2, a_1 \ge 0$

apply simplex iteration

$Z_j - C_j \le 0$

C_B	B.V.	x_1	x_2	s_1	s_2	a_1	X_B
	Z^*	2	3	0	-1	0	6
0	s_1	1	$\boxed{1}$	1	0	0	1
1	a_1	2	3	0	-1	1	6
	Z^*	-1	0	-3	-1	0	3
0	x_2	1	1	1	0	0	1
1	a_1	-1	0	-3	-1	1	3

∴ Infeasible Sol^n

No. Sol^n.

Min $Z^* > 0$

$a_1 = 3 > 0$

Problem: Use two-phase method to solve LPP;

Minimize $Z = 2x_1 + 8x_2 + 0 \cdot s_1 + 0 \cdot s_2$

s.t. $5x_1 + 10x_2 = 150$

$x_1 \le 20$

$x_2 \ge 14$; $x_1, x_2 \ge 0$

Solution Write LPP into standard form by introducing slack / surplus and artificial variables!

Phase I: Auxiliary LPP

Minimize $Z^* = a_1 + a_2$

s.t. $5x_1 + 10x_2 + a_1 = 150$

$x_1 + s_1 = 20$

$x_2 - s_2 + a_2 = 14$

$x_1, x_2, s_1, s_2, a_1, a_2 \ge 0$

$z_j - c_j \le 0$

$R_1 \to R_1 - 11(R_4)$

$R_2 \to R_2 - 10R_4$

$R_3 \to R_3$

$R_4 \to R_4$

C_B	B.V.	x_1	x_2	s_1	s_2	a_1	a_2	X_B
	Z^*	5	11	0	-1	0	0	164
1	a_1	5	10	0	0	1	0	150
0	s_1	1	0	1	0	0	0	20
1	a_2	0	$\boxed{1}$	0	-1	0	1	14
	Z^*	5	0	0	10	0	-10	10
	a_1	5	0	0	$\boxed{10}$	1	-10	10
	s_1	1	0	1	0	0	0	20
	x_2	0	1	0	-1	0	1	14
	Z^*	0	0	0	0	-1	-1	0
	s_2	1/2	0	0	1	1/10	-1	1
	s_1	1	0	1	0	0	0	20
	x_2	1/2	1	0	0	1/10	0	15

$z_j - c_j \le 0 \ \forall j$

Phase-I

Min $Z^* = 0$; artificial → Non B.V.

Phase II

$R_1 \to R_1 - R_2$

$R_2 \to R_2/10$

$R_3 \to R_3$

$R_4 \to R_4 + 1(R_2/10)$

	B.V.	x_1	x_2	s_1	s_2	X_B
		2	8	0	0	
		2	0	0	0	120
0	s_2	1/2	0	0	1	1
0	s_1	1	0	1	0	20
8	x_2	1/2	1	0	0	15

Solution Write LPP into standard form by introducing slack / surplus and artificial variables!

Phase I: Auxiliary LPP

$$\text{Minimize } Z^* = a_1 + a_2$$

$$\text{s.t.} \quad 5x_1 + 10x_2 + a_1 = 150$$
$$x_1 + s_1 = 20$$
$$x_2 - s_2 + a_2 = 14$$
$$x_1, x_2, s_1, s_2, a_1, a_2 \geq 0$$

$z_j - c_j \leq 0$

$R_1 \to R_1 - 11(R_4)$
$R_2 \to R_2 - 10R_4$
$R_3 \to R_3$
$R_4 \to R_4$

	B.V.	x_1	x_2	s_1	s_2	a_1	a_2	X_B
	Z^*	5	0	0	10	0	−10	10
	a_1	5	0	0	$\boxed{10}$	1	−10	10
	s_1	1	0	1	0	0	0	20
	x_2	0	1	0	−1	0	1	14
	Z^*	0	0	0	0	−1	−1	0
	s_2	1/2	0	0	1	1/10	−1	1
	s_1	1	0	1	0	0	0	20
	x_2	1/2	1	0	0	1/10	0	15

$z_j - c_j \leq 0 \ \forall j$

Phase-I

Min $Z^* = 0$; artificial → Non B.V.

Phase II

$R_1 \to R_1 - R_2$
$R_2 \to R_2/10$
$R_3 \to R_3$
$R_4 \to R_4 + 1(R_2/10)$

	B.V.	x_1	x_2	s_1	s_2	X_B
		2	8	0	0	
	Z	2	0	0	0	120
0	s_2	$\boxed{1/2}$	0	0	1	1
0	s_1	1	0	1	0	20
8	x_2	1/2	1	0	0	15
	Z	0	0	0	−4	116
	x_1	1	0	0	+2	2
	s_1	0	0	1	−2	18
	x_2	0	1	0	−1	14

Optimal

$Z_{min} = 116$
at $x_1 = 2$; $x_2 = 14$

Problem: Use two phase method to solve the LPP

$$\text{Max } Z = -2x_1 - x_2 + 0.x_3 + 0.x_4$$

s.t. $3x_1 + x_2 = 3$

$4x_1 + 3x_2 - x_3 = 6$

$x_1 + 2x_2 + x_4 = 3$; $x_i \geq 0$

Solution: Write LPP into standard form; (involving artificial var.)

Auxiliary LPP is

$$\text{Min } Z^* = a_1 + a_2$$

s.t. $3x_1 + x_2 + a_1 = 3$

$4x_1 + 3x_2 - x_3 + a_2 = 6$; all var. ≥ 0

$x_1 + 2x_2 + x_4 = 3$

Phase I (Last table of Phase I)

B.V.	x_1	x_2	x_3	x_4	a_1	a_2	X_B	
					2	0	0	
Z^*	0	0	1	1	2/5	0	3/5	$\to R_1$
x_1	1	0	0	-1/5	-1	1	0	$\to R_2$
a_2	0	0	-1	$\boxed{-1}$	8/5	-1/5	0	$\to R_3$
x_2	0	1	0				6/5	

Phase II:

$z_j - c_j \geq 0$

C_B	B.V.	-2 x_1	-1 x_2	0 x_3	0 x_4	X_B
		0	0	1/5	0	-12/5
-2	x_1	1	0	1/5	0	3/5
0	x_4	0	0	1	1	0
-1	x_2	0	1	-3/5	0	6/5

Optimal Solution

$Z_{max} = -12/5$

at $x_1 = 3/5$

$x_2 = 6/5$

$x_3 = 0$

$x_4 = 0$

Problem: Solve using two-phase method

(Redundant)

$$\text{Max } Z = x_1 - 2x_2 + 3x_3$$

s.t.

$$x_1 + x_2 + x_3 = 6$$
$$-x_1 + x_2 + 2x_3 = 4$$
$$2x_2 + 3x_3 = 10$$
$$x_3 + x_4 = 2$$

all var. ≥ 0

Solution: Write LPP into Auxiliary LPP

$$\text{Min } Z^* = a_1 + a_2 + a_3$$

s.t.

$$x_1 + x_2 + x_3 + a_1 = 6$$
$$-x_1 + x_2 + 2x_3 + a_2 = 4$$
$$2x_2 + 3x_3 + a_3 = 10$$
$$x_3 + x_4 = 2$$

all var. ≥ 0

Optimal solution
$Z_{max} = 4$
at
$x_1 = 2$
$x_2 = 2$
$x_3 = 2$

Phase I (Last table)

B.V. | x_1 | x_2 | x_3 | x_4 | a_1 | a_2 | a_3 | X_B

B.V.	x_1	x_2	x_3	x_4	a_1	a_2	a_3	X_B
Z^*	0	0	0	0	-2	-2	0	0
x_1	1	0	0	1/2	1/2	-1/2	0	2
x_2	0	1	0	3/2	1/2	1/2	0	2
~~a_3~~	~~0~~	~~0~~	~~0~~	~~$\boxed{0}$~~	~~-1~~	~~-1~~	~~1~~	~~0~~
x_3	0	0	1	1	0	0	0	2

Phase II

CB	B.V.	x_1	x_2	x_3	x_4	X_B
		0	0	0	1/2	4
1	x_1	1	0	0	1/2	2
-2	x_2	0	1	0	3/2	2
3	x_3	0	0	1	1	2

Big M & Two Phase Method (Unbounded Solution)

$M > 0$

Question: Solve the LPP using Big M - method

$$\text{Max } Z = 3x_1 + 5x_2$$

$$\text{s.t.} \quad x_1 - 2x_2 \le 6$$
$$x_1 \le 10$$
$$x_2 \ge 1$$
$$x_1, x_2 \ge 0$$

$$A = \begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & s_1 & s_2 & s_3 & a_1 \end{array} \\ \begin{pmatrix} 1 & -2 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & -1 & 1 \end{pmatrix} \end{array}$$

Sol^n: → Standard form;

$$\text{Max } Z = 3x_1 + 5x_2 + 0.s_1 + 0.s_2 + 0.s_3 - Ma_1$$

$$\text{s.t.} \quad x_1 - 2x_2 + s_1 = 6$$
$$x_1 + s_2 = 10$$
$$x_2 - s_3 + a_1 = 1$$
$$x_1, x_2, s_1, s_2, s_3, a_1 \ge 0$$

C_B	B.V.	x_1	x_2	s_1	s_2	s_3	a_1	X_B	
	Z	-3	$-M-5$	0	0	M	0	$-M$	
0	s_1	1	-2	1	0	0	0	6	R_1
0	s_2	1	0	0	1	0	0	10	R_2
$-M$	a_1	0	$\boxed{1}$	0	0	-1	1	1	R_3
		-3	0	0	0	-5	5	5	
0	s_1	1	0	1	0	-2	2	8	
0	s_2	1	0	0	1	0	0	10	
5	x_2	0	1	0	0	-1	1	1	

Unbounded solution!!

We have an entering varible but not a leaving variabl;e.

Solve LPP using Two-Phase Method

$$\text{Max } Z = 3x_1 + 5x_2$$

s.t.

$$x_1 - 2x_2 \le 6$$
$$x_1 \le 10$$
$$x_2 \ge 1$$
$$x_1, x_2 \ge 0$$

Solution: Auxiliary LPP

$$\text{Min } Z = a_1 + 0.x_1 + 0.x_2 + 0.s_i$$

s.t.

$$x_1 - 2x_2 + s_1 = 6$$
$$x_1 + s_2 = 10$$
$$x_2 - s_3 + a_1 = 1$$
$$x_1, x_2 \ge 0;\ s_1, s_2, s_3, a_1 \ge 0$$

Phase I

$z_j - c_j$

C_B	B.V.	x_1	x_2	s_1	s_2	s_3	a_1	X_B
	Z	0	1	0	0	-1	0	1
0	s_1	1	-2	1	0	0	0	6
0	s_2	1	0	0	1	0	0	10
1	a_1	0	$\boxed{1}$	0	0	-1	1	1
		0	0	0	0	0	-1	0
0	s_1	1	0	1	0	-2	0	8
0	s_2	1	0	0	1	0	0	10
0	x_2	0	1	0	0	-1	1	1

Phase II

	B.V.	x_1	x_2	s_1	s_2	s_3	X_B
	Z	-3	0	0	0	-5	5
0	s_1	1	0	1	0	-2	8
0	s_2	1	0	0	1	0	10
5	x_2	0	1	0	0	-1	1

$z_j - c_j$ with most -ve

$z_j - c_j \ge 0$ opt. max.

Tie in Entering Variable

Example. Solve the given LPP using Simplex Method

$$\text{Max } Z = 4x_1 + 4x_2 + 3x_3 + 0.s_1 + 0.s_2 + 0.s_3$$

s.t.
$$2x_1 + x_2 + x_3 \le 4$$
$$2x_1 + 3x_2 + 3x_3 \le 5$$
$$x_1 + x_2 - x_3 \le 3$$
$$x_i \ge 0$$

Or

$$2x_1 + x_2 + x_3 + s_1 = 4$$
$$2x_1 + 3x_2 + 3x_3 + s_2 = 5$$
$$x_1 + x_2 - x_3 + s_3 = 3$$
$$x_i \ge 0;\ s_i \ge 0$$

Solution: Write LPP into standard form; involve slack variables; we then make simplex table as

C_B	B.V.	x_1	x_2	x_3	s_1	s_2	s_3	X_B (Sol^n)
	Z	-4	-4	-3	0	0	0	0
0	s_1	2	1	1	1	0	0	4
0	s_2	2	3	3	0	1	0	5
0	s_3	1	1	-1	0	0	1	3

$$Z_j - C_j = C_B(B^{-1}A_j) - C_j$$

Entering: $Z_j - C_j$ with most -ve

Optimality; $Z_j - C_j \ge 0$

Tie in Entering v.

involve slack variables ; we then make simplex table as

C_B	B.V.	x_1	x_2	x_3	s_1	s_2	s_3	X_B	Entre x_1 Min θ_1	Enter x_2 Min θ_2
	Z	-4	-4	-3	0	0	0	0		
0	s_1	2	1	1	1	0	0	4	$\boxed{2}$	4
0	s_2	2	3	3	0	1	0	5	2.5	$\boxed{\frac{5}{3}=1.6}$
0	s_3	1	1	-1	0	0	1	3	3	3

$z_j - c_j =$

Entering : $z_j - c_j$ with most -ve

Optimality ; $z_j - c_j \geq 0$

Tie in Entering variable

To Break Tie in Entering Variable

Way 1. Make arbitrary choice to select entering variable

Way 2. Choose $\theta_j (z_j - c_j)$ most -ve ;

$\theta_j \rightarrow$ Min Ratio

$z_j - c_j \rightarrow$ Net- evaluation

$$\theta = \min \left\{ \frac{x_{Bi}}{\alpha_{ij}} ; \alpha_{ij} > 0 \right\}$$

$$\hat{Z} = Z_0 - \theta_j (z_j - c_j)$$

- If x_1 enter into basis

$$\hat{Z} = 0 - 2(-4) = 8$$

- If x_2 enters into basis

$$\hat{Z} = 0 - (1.6)(-4) = 6.4$$

We choose x_2 as entering variable as $\theta_j(z_j - c_j)$ is most-ve.

(Remark. For Minimization problem; $\theta_j(z_j - c_j)$ most +ve should be entering variable)

$\theta_j(z_j - c_j) \rightarrow$ most +ve

We have chosen 2 there that is x1 since min ratio is 2 , as it gives better value of z and we will reach solution earlier.

Tie in Leaving Variable

Example. Solve the LPP using simplex Method

$$\text{Max } Z = 2x_1 + x_2$$

$$\text{s.t. } 4x_1 + 3x_2 \le 12$$

$$4x_1 + x_2 \le 8$$

$$4x_1 - x_2 \le 8$$

$$x_1, x_2 \ge 0$$

Solution: The standard form of above LPP is

$$\text{Max } Z = 2x_1 + x_2 + 0.s_1 + 0.s_2 + 0.s_3$$

$$\text{s.t. } 4x_1 + 3x_2 + s_1 = 12$$

$$4x_1 + x_2 + s_2 = 8$$

$$4x_1 - x_2 + s_3 = 8$$

$$x_1, x_2, s_1, s_2, s_3 \ge 0$$

Solution: The standard form of above LPP is

$$\text{Max } Z = 2x_1 + x_2 + 0.s_1 + 0.s_2 + 0.s_3$$

s.t.

$$\begin{cases} 4x_1 + 3x_2 + s_1 = 12 \\ 4x_1 + x_2 + s_2 = 8 \\ 4x_1 - x_2 + s_3 = 8 \\ x_1, x_2, s_1, s_2, s_3 \geq 0 \end{cases}$$

B.V.	x_1	x_2	s_1	s_2	s_3	X_B	Min Ratio
Z	-2	-1	0	0	0	0	
s_1	4	3	1	0	0	12	3
s_2	4	1	0	1	0	8	2
s_3	4	-1	0	0	1	8	2

Tie

Req. BFS.

$$X = (\underbrace{x_1, x_2}_{\text{N.B.V.}}, \underbrace{s_1, s_2, s_3}_{\text{B.V.}})$$

$$= (0, 0, 12, 8, 8)$$

$$\hat{X} = (\underbrace{\quad}_{\text{B.V.}}, \underbrace{0, 0}_{\text{N.B.V}})$$

atleast one of B.V. $= 0$

B.V.	x_1	x_2	s_1	s_2	s_3	X_B	Min Ratio
Z	-2	-1	0	0	0	0	
s_1	4	3	1	0	0	12	3
s_2	4	1	0	1	0	8	2
s_3	4	-1	0	0	1	8	2

(Min Ratio Tie) Tie in Leaving variable → Leads to Degenerate BFS

Option 1: Make arbitrary selection for leaving variable

Option 2: To break tie;

$$\frac{\text{Elements of } \boxed{\text{first}} \text{ column (tied variables) of unit Matrix}}{\text{Corresponding elements of entering column with Key element}} = \left\{\frac{0}{4}, \frac{0}{4}\right\} = \{0, 0\}$$

$$\frac{\text{Elements of second column (tied variables) of unit Matrix}}{\text{Corresponding elements of entering column with Key element}} = \left\{\frac{1}{4}, \frac{0}{4}\right\} = \left\{\frac{1}{4}, 0\right\}$$

Min

and

so on. . . . .

B.V.	x_1	x_2	s_1	s_2	s_3	X_B	Min Ratio	
Z	-2	-1	0	0	0	0		R_1
s_1	4	3	1	0	0	12	3	R_2
s_2	4	1	0	1	0	8	2	R_3
s_3	$\boxed{4}$	-1	0	0	1	8	2	R_4
	0	$-3/2$	0	0	$1/2$	4		
s_1	0	4	1	0	-1	4		
s_2	0	$\boxed{2}$	0	1	-1	0		
x_1	1	$-1/4$	0	0	$1/4$	2		
	0	0	0	$3/4$	$-1/4$	4		
s_1	0	0	1	-2	$\boxed{1}$	4		
x_2	0	1	0	$1/2$	$-1/2$	0		
x_1	1	0	0	$1/8$	$1/8$	2		

← Deg. BFS.

Max.

$Z_j - C_j \geq 0$

$\rightarrow R_1 \rightarrow R_1 + 2(R_4/4)$

$\rightarrow R_2 \rightarrow R_2 - R_4$

$\rightarrow R_3 \rightarrow R_3 - R_4$

$R_4 \rightarrow R_4/4$

s_1	0	4	1	0	−1	4
s_2	0	$\boxed{2}$	0	1	−1	0
x_1	1	−1/4	0	0	1/4	2
	0	0	0	3/4	−1/4	4

← Deg. BFS.

$\rightarrow R_1 \rightarrow R_1 + (-1/4)$

$\rightarrow R_2 \rightarrow R_2 - R_4$

$\rightarrow R_3 \rightarrow R_3 - R_4$

$R_4 \rightarrow R_4/4$

s_1	0	0	1	−2	$\boxed{1}$	4
x_2	0	1	0	1/2	−1/2	0
x_1	1	0	0	1/8	1/8	2
	0	0	1/4	1/4	0	5

s_3	0	0	1	−2	1	4 ✓
x_2	0	1	1/2	−1/2	0	2 ✓
x_1	1	0	−1/8	3/8	0	3/2 ✓

Temporary deg.

$Z_{max} = 5$;

$x_1 = 3/2$; $x_2 = 2$

$s_1 = 0 = s_2$; $s_3 = 4$

If Basic Variable has value zero at end,
Then it is a degenerate BFS.

If there is a tie in basic variable at end,
Then it is called non degenerate BFS.

Tie in leaving variable (or <u>Minimum Ratio</u>) leads to Degenerate BFS.

Solve LPP without Involving Artificial Variables

Simplex Method

- Simplex Method (Slack Variables Only)
- Big-M (Slack | surplus variables; artificial variables)
- Two-Phase Method

LPP
- Feasible
  - Unique
  - Alternate | Infinite
  - Unbounded
- Infeasible

Problem: Solve the LPP using two-phase Method

$$\text{Min } Z = 4x_1 + 6x_2 + 5x_3$$

$$\text{s.t.} \quad 2x_1 + 4x_2 + 3x_3 \geq 32$$

$$x_1 + 2x_2 + 4x_3 \geq 28$$

$$x_1, x_2, x_3 \geq 0$$

Solution: Write the above in standard form

$$\text{Min } Z = 4x_1 + 6x_2 + 5x_3 + 0.s_1 + 0.s_2$$

$$\text{s.t.} \quad 2x_1 + 4x_2 + 3x_3 - s_1 = 32$$

$$x_1 + 2x_2 + 4x_3 - s_2 = 28$$

$$x_1, x_2, x_3, s_1, s_2 \geq 0$$

Coefficient Matrix

$$A = \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & s_1 & s_2 \end{array} \\ \begin{pmatrix} 2 & 4 & 3 & -1 & 0 \\ 1 & 2 & 4 & 0 & -1 \end{pmatrix} \end{array}$$

Solution: Write the above in standard form

$$\text{Min } Z = 4x_1 + 6x_2 + 5x_3 + 0.s_1 + 0.s_2$$

s.t.

$$2x_1 + 4x_2 + 3x_3 - s_1 = 32$$
$$x_1 + 2x_2 + 4x_3 - s_2 = 28$$
$$x_1, x_2, x_3, s_1, s_2 \geq 0$$

Coefficient Matrix

$$A = \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & s_1 & s_2 \end{array} \\ \begin{pmatrix} 2 & 4 & 3 & -1 & 0 \\ 1 & 2 & 4 & 0 & -1 \end{pmatrix} \end{array}$$

Introducing artificial variable; the auxiliary LPP is;

$$\text{Min } Z^* = a_1 + a_2$$

s.t.

$$2x_1 + 4x_2 + 3x_3 - s_1 + a_1 = 32$$
$$x_1 + 2x_2 + 4x_3 - s_2 + a_2 = 28$$
$$x_1, x_2, x_3, s_1, s_2, a_1, a_2 \geq 0$$

$$\text{Min } z^* = a_1 + a_2 + 0 \cdot x_i + 0 \cdot s_j$$

s.t.

$$2x_1 + 4x_2 + 3x_3 - s_1 + a_1 = 32$$
$$x_1 + 2x_2 + 4x_3 - s_2 + a_2 = 28$$
$$x_1, x_2, x_3, s_1, s_2, a_1, a_2 \geq 0$$

Phase I.

B.V.	x_1	x_2	x_3	s_1	s_2	a_1	a_2	X_B
z^*	3	6	7	-1	-1	0	0	60
a_1	2	4	3	-1	0	1	0	32
a_2	1	2	$\boxed{4}$	0	-1	0	1	28
	5/4	5/2	0	-1	3/4	0	-7/4	11
a_1	5/4	$\boxed{5/2}$	0	-1	3/4	1	-3/4	11
x_3	1/4	1/2	1	0	-1/4	0	1/4	7
z^*	0	0	0	0	0	-1	-1	0
x_2	1/2	1	0	-2/5	3/10	2/5	-3/10	22/5
x_3	0	0	1	1/5	-2/5	-1/5	2/5	24/5

Original objective fn

$$\text{Min } Z = 4x_1 + 6x_2 + 5x_3$$

Phase II

C_B	B.V.	x_1	x_2	x_3	s_1	s_2	X_B
	Z	-1	0	0	-7/5	-4/5	252/5
6	x_2	1/2	1	0	-2/5	3/10	22/5
5	x_3	0	0	1	1/5	-2/5	24/5

$z_j - c_j \leq 0$; Opt.

$$\boxed{x_1 = 0;\ x_2 = 22/5;\ x_3 = 24/5 \qquad Z_{min} = 252/5}$$

{Unique Solution}

If non Basic Variables is not 0, and basic variable is 0, then it is a unique solution.

Problem: Solve the LPP ; without using artificial variables

$$\text{Min } Z = 4x_1 + 6x_2 + 5x_3$$

$$\text{s.t. } 2x_1 + 4x_2 + 3x_3 \geq 32$$

$$x_1 + 2x_2 + 4x_3 \geq 28$$

$$x_1, x_2, x_3 \geq 0$$

$$\boxed{AX = b} \quad ; \; b \geq 0$$

Coeff^n Var. R.H.S.

$$\Downarrow$$

$$\underline{B} X_B = b$$

$$X_B = B^{-1} b$$

$$= b \quad ; \; B = I$$

$$\downarrow$$

$$X_B \rightarrow \text{Feasible.}$$

Coefficient matrix

$$\begin{array}{ccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & b \\ \hline 2 & 4 & 3 & -1 & 0 & 32 \\ 1 & 2 & 4 & 0 & -1 & 28 \end{array}$$

$$R_2 \begin{bmatrix} 1 & 2 & 4 & 0 & -1 & \vdots & 28 \end{bmatrix}$$

$X_B \rightarrow$ Feasible.

$R_1 \rightarrow R_1 - R_2$

$$\approx \left[\begin{array}{ccccc|c} 1 & 2 & -1 & -1 & 1 & 4 \\ 1 & 2 & 4 & 0 & -1 & 28 \end{array}\right]$$

$R_2 \rightarrow R_2 - R_1$

$$\approx \left[\begin{array}{ccccc|c} 1 & 2 & -1 & -1 & 1 & 4 \\ 0 & 0 & 5 & 1 & -2 & 24 \end{array}\right]$$

$R_2 \rightarrow R_2/5$

$$\approx \left[\begin{array}{ccccc|c} 1 & 2 & -1 & -1 & 1 & 4 \\ 0 & 0 & 1 & 1/5 & -2/5 & 24/5 \end{array}\right]$$

$R_1 \rightarrow R_1 + R_2$

$$\approx \begin{array}{c} \begin{bmatrix} 1 & 2 & 0 & -4/5 & 3/5 & 44/5 \\ 0 & 0 & 1 & 1/5 & -2/5 & 24/5 \end{bmatrix} \\ \begin{array}{cccccc} \uparrow & & \uparrow & & & \\ x_1 & & x_3 & & & \end{array} \end{array}$$

$R_2 \to R_2 - R_1$

$$\approx \left[\begin{array}{ccccc|c} 1 & 2 & -1 & -1 & 1 & 4 \\ 0 & 0 & 5 & 1 & -2 & 24 \end{array}\right] \approx \left[\begin{array}{ccccc|c} 1 & 2 & -1 & -1 & 1 & 4 \\ 0 & 0 & 1 & 1/5 & -2/5 & 24/5 \end{array}\right]$$

$R_1 \to R_1 + R_2$

$$\approx \left[\begin{array}{ccccc|c} 1 & 2 & 0 & -4/5 & 3/5 & 44/5 \\ 0 & 0 & 1 & 1/5 & -2/5 & 24/5 \end{array}\right]$$

$\underline{\text{Min } z} = 4x_1 + 6x_2 + 5x_3$

C_B	B.V.	x_1	x_2	x_3	s_1	s_2	X_B
	z	0	2	0	$-11/5$	$2/5$	$296/5$
4	✓ x_1	1	$\boxed{2}$	0	$-4/5$	$3/5$	$44/5$
5	✓ x_3	0	0	1	$1/5$	$-2/5$	$24/5$
		-4	0	0	$-7/5$	$-1/5$	$252/5$
	x_2	$1/2$	1	0	$-2/5$	$3/10$	$22/5$
	s_3	0	0	1	$1/5$	$-2/5$	$24/5$

$\underline{z_j - c_j \le 0}$

$x_1 = 0$; $x_2 = 22/5$; $x_3 = 24/5$

$z_{min} = 252/5$

Construct Simplex Table Without Simplex iteration/ Row Operations for a given set of Basic Variables/ Using B^{-1} inverse

Example. Consider the LPP

$$\text{Max } Z = 4x_1 + x_2 + 3x_3 + 5x_4$$

$$\text{s.t. } 4x_1 - 6x_2 - 5x_3 + 4x_4 \geq -20$$

$$3x_1 - 2x_2 + 4x_3 + x_4 \leq 11$$

$$8x_1 - 3x_2 + 3x_3 + 2x_4 \leq 23$$

$$x_1, x_2, x_3, x_4 \geq 0$$

Construct simplex table using Basic variables as (s_1, x_4, x_2); where s_1 is slack variable in first constraint.

Solution: Writing above into standard form

$$\text{Max } Z = 4x_1 + x_2 + 3x_3 + 5x_4$$

$$\text{s.t. } -4x_1 + 6x_2 + 5x_3 - 4x_4 + s_1 = 20$$

$$3x_1 - 2x_2 + 4x_3 + x_4 + s_2 = 11$$

$$8x_1 - 3x_2 + 3x_3 + 2x_4 + s_3 = 23 \quad ; \quad x_1, x_2, x_3, s_1, s_2, s_3 \geq 0$$

where s_1 is slack variable in first constraint.

Solution: Writing above into standard form

$$\text{Max } Z = 4x_1 + x_2 + 3x_3 + 5x_4$$

$$\text{s.t.} \quad -4x_1 + 6x_2 + 5x_3 - 4x_4 + s_1 = 20$$
$$3x_1 - 2x_2 + 4x_3 + x_4 + s_2 = 11$$
$$8x_1 - 3x_2 + 3x_3 + 2x_4 + s_3 = 23$$

; $x_1, x_2, x_3, x_4, s_1, s_2, s_3 \geq 0$

$$A = \begin{array}{c} \begin{array}{ccccccc} A_1 & A_2 & A_3 & A_4 & A_5 & A_6 & A_7 \end{array} \\ \begin{pmatrix} -4 & 6 & 5 & -4 & 1 & 0 & 0 \\ 3 & -2 & 4 & 1 & 0 & 1 & 0 \\ 8 & -3 & 3 & 2 & 0 & 0 & 1 \end{pmatrix} \\ \begin{array}{ccccccc} x_1 & x_2 & x_3 & x_4 & s_1 & s_2 & s_3 \end{array} \end{array}$$

B.V.	x_1	x_2	x_3	x_4	s_1	s_2	s_3	X_B (Solution)
		0		0	0 ✓			
s_1		0		0	1			
→ x_4		0		1	0			
x_2		1		0	0			

Result: Basic variables, corresponds to identity column and their $Z_j - C_j = 0$.

Now Looking at formula's

$$Z_j - C_j = C_B(B^{-1}A_j) - C_j$$

C_B	B.V.	x_1	x_2	x_3	x_4	s_1	s_2	s_3	X_B (Solution)
	Z		0		0	0			$C_B X_B$
0	s_1		0		0	1			
5	x_4		0		1	0			
1	x_2		1		0	0			

$$A = \begin{array}{c} \begin{pmatrix} -4 & 6 & 5 & -4 & 1 & 0 & 0 \\ 3 & -2 & 4 & 1 & 0 & 1 & 0 \\ 8 & -3 & 3 & 2 & 0 & 0 & 1 \end{pmatrix} \\ \begin{array}{ccccccc} x_1 & x_2 & x_3 & x_4 & s_1 & s_2 & s_3 \end{array} \end{array}$$

Result: Basic variables, corresponds to identity column and their $Z_j - C_j = 0$.

Now Looking at formula's

$\alpha^j = \text{scalars} = B^{-1} A_j$; $Z_j - C_j = C_B(B^{-1}A_j) - C_j$

$X_B = B^{-1} b$; $Z = C_B X_B$

RHS.

$B \rightarrow$ Basis Matrix ; $B \cdot V_1 = (s_1 \ x_4 \ x_2) \Rightarrow$

$$B = \begin{array}{c} \begin{pmatrix} 1 & -4 & 6 \\ 0 & 1 & -2 \\ 0 & 2 & -3 \end{pmatrix} \\ \begin{array}{ccc} s_1 & x_4 & x_2 \end{array} \end{array}$$

$$B^{-1} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & -3 & 2 \\ 0 & -2 & 1 \end{pmatrix}$$

$$B^{-1} = \frac{1}{|B|} \cdot \text{adj}\, B.$$

$$BB^{-1} = \text{Identity}.$$

scalars corresponding to x_1

$$\alpha^1 = B^{-1} A_1 = \begin{pmatrix} 1 & 0 & 2 \\ 0 & -3 & 2 \\ 0 & -2 & 1 \end{pmatrix} \begin{pmatrix} -4 \\ 3 \\ 8 \end{pmatrix} = \begin{pmatrix} -4+0+16 \\ 7 \\ 2 \end{pmatrix} = \begin{pmatrix} 12 \\ 7 \\ 2 \end{pmatrix}$$

$$\begin{array}{c} \quad A_1 \quad A_2 \quad A_3 \quad A_4 \quad A_5 \quad A_6 \quad A_7 \\ A = \begin{pmatrix} -4 & 6 & 5 & -4 & 1 & 0 & 0 \\ 3 & -2 & 4 & 1 & 0 & 1 & 0 \\ 8 & -3 & 3 & 2 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

$$Z_1 - C_1 = C_B^T (B^{-1} A_1) - C_1$$

cost of x_1 - variable

$$= (0 \;\; 5 \;\; 1) \begin{pmatrix} 12 \\ 7 \\ 2 \end{pmatrix} - 4$$

$$= (0 + 35 + 2) - 4 = 37 - 4 = 33$$

$= (0 + 35 + \alpha)$

scalars corresponding to x_3 & $z_3 - c_3$

$$\alpha^3 = \underbrace{\begin{bmatrix} 1 & 0 & 2 \\ 0 & -3 & 2 \\ 0 & -2 & 1 \end{bmatrix}}_{B^{-1}A_3} \begin{bmatrix} 5 \\ 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 11 \\ -6 \\ -5 \end{bmatrix}$$

$$z_3 - c_3 = (0 \;\; 5 \;\; 1) \begin{pmatrix} 11 \\ -6 \\ -5 \end{pmatrix} - 3 = -38$$

scalars corresponding to s_2 & $z_6 - c_6$

$$\alpha^6 = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -3 & 2 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ -2 \end{bmatrix}$$

$$z_6 - c_6 = (0 \;\; 5 \;\; 1) \begin{pmatrix} 0 \\ -3 \\ -2 \end{pmatrix} - 0$$

$$= -17$$

scalars corresponding to s_3 & $z_7 - c_7$

$$\alpha^7 = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -3 & 2 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$$

$$z_7 - c_7 = (0 \;\; 5 \;\; 1) \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} - 0$$

$$= 11$$

Now $\underline{\underline{X_B}} = B^{-1} b$

$$= \begin{bmatrix} 1 & 0 & 2 \\ 0 & -3 & 2 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 20 \\ 11 \\ 23 \end{bmatrix} = \begin{bmatrix} 66 \\ 13 \\ 1 \end{bmatrix}$$

$$Z = C_B X_B$$

$$= \begin{bmatrix} 0 & 5 & 1 \end{bmatrix} \begin{bmatrix} 66 \\ 13 \\ 1 \end{bmatrix} = 0 + 65 + 1 = 66$$

C_B	B.V.	x_1	x_2	x_3	x_4	s_1	s_2	s_3	X_B (Solution)
	Z	33	0	38	0	0	-17	11	66
0	s_1	12	0	11	0	1	0	2	66
5	x_4	7	0	-6	1	0	-3	2	13
1	x_2	2	1	-5	0	0	-2	1	1

$$= \begin{bmatrix} 0 & 5 & 1 \end{bmatrix} \begin{bmatrix} 66 \\ 13 \\ 1 \end{bmatrix} = 0 + 65 + 1 = 66$$

C_B	B.V.	x_1	x_2	x_3	x_4	s_1	s_2	s_3	X_B (Solution)
	Z	33	0	38	0	0	-17	11	66
0	s_1	12	0	11	0	1	0	2	66
5	x_4	7	0	-6	1	0	-3	2	13
1	x_2	2	1	-5	0	0	-2	1	1

Max.

$z_j - c_j \geq 0$

$\boxed{\text{Optimal}}$

Unbounded solution!!

An Interesting Example (Simplex Method)

Question Two consecutive simplex table of a LPP are, given below,

B.V.	x_1	x_2	x_3	x_4	x_5	X_B
Z	A	−1	3	0	0	
B.V. x_4	B	C	D	I	0	6
B.V. x_5	−1	2	E	0	1	1

Table 1

B.V.	x_1	x_2	x_3	x_4	x_5	X_B
Z	0	−4	J	K	0	
B.V. x_1	G	2/3	2/3	1/3	0	F
B.V. x_5	H	8/3	−1/3	1/3	1	3

Table 2.

Find Missing enteries A, B, C, D, E, F, G, H, I, J, K.

Similarly in table 2, x_1 column will have scalars corresponding to identity.

B.V.	x_1	x_2	x_3	x_4	x_5	X_B
Z	A	-1	3	0	0	
x_4	$\boxed{B}$	C	D	1	0	6
x_5	-1	2	E	0	1	1

Table 1

$I = 1$
$G = 1$
$H = 0$

Z	0	-4	J	K	0	
x_1	1	$2/3$	$2/3$	$1/3$	0	F
x_5	0	$8/3$	$-1/3$	$1/3$	1	3

Table

B → Leading / Pivot ele

B.V.	x_1	x_2	x_3	x_4	x_5	x_B	
Z	A	−1	3	0	0		R_1
x_4	$\boxed{B}$	C	D	1	0	6	R_2
x_5	−1	2	E	0	1	1	R_3

Table 1

$I = 1$
$G = 1$
$H = 0$

Z	0	−4	J	K	0	
x_1	1	2/3	2/3	1/3	0	F
x_5	0	8/3	−1/3	1/3	1	3

Table 2

$R_1 \to R_1 - A(R_2/B)$

$R_2 \to R_2/B$

$R_3 \to R_3 + 1\left(\frac{R_2}{B}\right)$

B → Leading / Pivot element.

so comparing first & table table w.r.t. row-operations.

$R_2 \to (R_2/B)$

$\frac{B}{B} = 1$; $\frac{C}{B} = 2/3$; $\frac{D}{B} = 2/3$; $\frac{1}{B} = \frac{1}{3}$; $\frac{6}{B} = F$

$1 = 1$; $C = 2$; $D = 2$ $B = 3$; $F = 2$

B.V.	x_1	x_2	x_3	x_4	x_5	X_B
Z	A	-1	3	0	0	
x_4	3	2	2	1	0	6
x_5	-1	2	E	0	1	1

Table 1

Z	0	-4	J	K	0	
x	1	$2/3$	$2/3$	$1/3$	0	2
x_5	0	$8/3$	$-1/3$	$1/3$	1	3

Table 2

$$R_1 \to R_1 - A\left(\frac{R_2}{B}\right)$$

$$= R_1 - A\left(\frac{R_2}{3}\right)$$

$$0 = A - A\left(\frac{3}{3}\right)$$

$$0 = A - A$$

✓

$$-4 = -1 - A\left(\frac{2}{3}\right)$$

$$-4 = -1 - \frac{2A}{3}$$

$$-3 = -\frac{2A}{3}$$

$$-9 = -2A$$

$$\boxed{A = 9/2}$$ ✓

$$J = 3 - A\left(\frac{2}{3}\right)$$

$$= 3 - A(2/3)$$

$$J = 3 - (9/2)(2/3)$$

$$= 3 - 3$$

$$\boxed{J = 0}$$

$$K = 0 - A\left(\frac{1}{3}\right)$$

$$K = -\frac{9}{2}\left(\frac{1}{3}\right)$$

$$= -3/2$$

$$K = -3/2$$

$A = 1/2$

B.V.	x_1	2	x_3	x_4	x_5	X_B
Z	9/2	−1	3	0	0	
x_4	3	2	2	1	0	6
x_5	−1	2	E	0	1	1

Table 1

Z	0	−4	0	−3/2	0	
x	1	2/3	2/3	1/3	0	2
x_5	0	8/3	−1/3	1/3	1	3

Table 2

$$\hat{R}_3 \to R_3 + \frac{R_2}{B}$$

$$\to R_3 + \frac{R_2}{3}$$

$$-1/3 = E + \frac{2}{3} \Rightarrow E = -1/3 - 2/3 = \frac{-3}{3} = -1$$

$$\boxed{E = -1}$$

An Interesting Example (Simplex Method)

Problem: Solve the following system of simultaneous linear equations by using the Simplex method.

$$\left.\begin{aligned} x_1 + x_2 &= 1 \\ 2x_1 + x_2 &= 3 \end{aligned}\right\}$$

$$\left[\begin{aligned} &\text{Max}/\text{Min } Z = C^T X \\ &\text{s.t. } AX \le, \ge, = b \\ &\underbrace{X \ge 0} \end{aligned}\right.$$

Solution: → As x_1, x_2 is not given as non-ve decision variables; so convert these unrestricted decision variables into non-ve restricted variables.

$$x_1 = x_1' - x_1'' \quad ; \quad x_1' \ge 0 \quad ; \quad x_2' \ge 0$$

$$x_2 = x_2' - x_2'' \quad ; \quad x_2' \ge 0$$

$x_2 = x_2' - x_2$

$$\therefore \left[\begin{array}{l} x_1' - x_1'' + x_2' - x_2'' = 1 \\ 2x_1' - 2x_1'' + x_2' - x_2'' = 3 \end{array}\right] \rightarrow A = \begin{pmatrix} 1 & -1 & 1 & -1 \\ 2 & -2 & 1 & -1 \end{pmatrix}$$

Introduce artificial variables ; $a_1 \geq 0$; $a_2 \geq 0$; each having cost -1 ; and so applying two-phase method.

$$\therefore \text{Max } Z = 0 \cdot x_1 + 0 \cdot x_2 - a_1 - a_2$$

$$\text{or Min } Z = 0 \cdot x_1 + 0 \cdot x_2 + a_1 + a_2)$$

Consider or Big-M

$$\text{Max } Z = -Ma_1 - Ma_2$$

$$\text{or Min } Z = Ma_1 + Ma_2)$$

C_B	B.V.	x_1'	x_1''	x_2'	x_2''	a_1	a_2	x_B
	Z	-3	3	-2	2	0	0	-4
-1	a_1	$\boxed{1}$	-1	1	-1	1	0	1
-1	a_2	2	-2	1	-1	0	1	3
	Z	0 ✓	0	1 ✓	-1	3	0	-1
	x_1	1	-1	1	-1	1	0	1 ✓
	a_2	0 ✓	0	-1 ✓	$\boxed{1}$	-2	1	1 ✓
	Z	0	0	0	0	1	1	0
	x_1	1	-1	0	0	-1	1	2
	x_2''	0	0	-1	1	-2	1	1

s.t.

$$x_1' - x_1'' + x_2' - x_2'' + a_1 =$$

$$2x_1' - 2x_1'' + x_2' - x_2'' + a_2 =$$

$R_1 \to R_1 + 3R_2$

$R_2 \to R_2$

$R_3 \to R_3 - 2R_2$

$z_j - c_j \geq 0$

$R_1 \to R_1 + R_3$

$R_2 \to R_2 + R_3$ ✓

$R_3 \to R_3$

Phase I.

$Z = 0$ $\longrightarrow$ Phase II. Consider LPP (original obj fn.)

$$\underbrace{x_1' = 2 \; ; \; x_2'' = 1}$$

$$x_1'' = 0 \; ; \; x_2' = 0$$

$$\begin{cases} x_1 = x_1' - x_1'' = 2 - 0 = 2 \\ x_2 = x_2' - x_2'' = 0 - 1 = -1 \end{cases}$$

$$\underbrace{x_1 = 2 \; ; \; x_2 = -1}$$

Use Simplex Method to obtain the inverse of the matrix

Max | Min

Opt $Z = C^T X$

s.t. $AX \leq, \geq, = b$

$X \geq 0$

Problem Use Simplex method to obtain the inverse of the matrix

$$A = \begin{pmatrix} 3 & 2 \\ 4 & -1 \end{pmatrix}$$

Sol^n: consider a system of equations;

$$\begin{pmatrix} 3 & 2 \\ 4 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \end{pmatrix} \quad ; x_1, x_2 \geq 0$$

$$A \quad X \quad = b$$

dummy column.

Sol^n: consider a system of equations;

column.

$$\begin{pmatrix} 3 & 2 \\ 4 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \end{pmatrix} \quad ; x_1, x_2 \geq 0$$

$$A \quad X \quad = b \rightarrow$$

Introduce Artificial var.

C_B	B.V.	x_1	x_2	a_1	a_2	X_B
	Z'	-7	-1	0	0	-10
-1	a_1	$\boxed{3}$	2	1	0	4
-1	a_2	4	-1	0	1	6

or $\left[\begin{array}{l} \text{Min } Z = a_1 + a_2 \\ \text{Max } Z' = -a_1 - a_2 \end{array}\right.$

$$\begin{pmatrix} 3 & 2 & 1 & 0 \\ 4 & -1 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$$

or $\underline{\text{Min } Z} = Ma_1 + Ma_2$

or $\underline{\text{Max } Z'} = -Ma_1 - Ma_2$]

	B.V.	x_1	x_2	a_1	a_2	
	z'	-7	-1	0	0	-10
-1	a_1	$\boxed{3}$	2	1	0	4
-1	a_2	4	-1	0	1	6
		0	$11/3$	$4/3$	0	$-2/3$
0	x_1	1	$2/3$	$1/3$	0	$4/3$
-1	a_2	0	$\boxed{-11/3}$	$-4/3$	1	$2/3$
	z'	0	0	1	1	0
0	x_1	1	0	$1/11$	$2/11$	$16/11$
0	x_2	0	1	$4/11$	$-3/11$	$-2/11$

or $\boxed{\begin{array}{l} \text{Min } Z = a_1 + a_2 \\ \text{Max } Z' = -a_1 - a_2 \end{array}}$

$$\begin{pmatrix} 3 & 2 & 1 & 0 \\ 4 & -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 4 \\ -6 \end{pmatrix}$$

or $\underline{\text{Min } Z} = Ma_1 + Ma_2$

or $\underline{\text{Max } Z'} = -Ma_1 - Ma_2$

$$A = \begin{pmatrix} 3 & 2 \\ 4 & -1 \end{pmatrix} = ?$$

$$A^{-1} = \begin{pmatrix} & \end{pmatrix}$$

		0	11/3	4/3	0	-2/3
0	x_1	1	2/3	1/3	0	4/3
-1	a_2	0	$\boxed{-11/3}$	-4/3	1	2/3
	Z'	0	0	1	1	0
0	x_1	1	0	1/11	2/11	16/11
0	x_2	0	1	4/11	-3/11	-2/11

or $\underline{\text{Min } Z} = Ma_1 + Ma_2$

or $\underline{\text{Max } Z'} = -Ma_1 - Ma_2$

$$A = \begin{pmatrix} 3 & 2 \\ 4 & -1 \end{pmatrix} = ?$$

$$A^{-1} = \begin{pmatrix} & \\ & \end{pmatrix}$$

$$\begin{matrix} A & I \\ \downarrow & \downarrow \\ I & A^{-1} \end{matrix}$$

$$A^{-1} = \begin{pmatrix} 1/11 & 2/11 \\ 4/11 & -3/11 \end{pmatrix}$$