Optimization Techniques

Dual Theory

Dual of a Primal

LPP (Primal) $\longrightarrow$ Dual (LPP)

Opt (Max | Min)

$$\begin{cases} \text{Opt } z = c^T x \\ Ax \le \ge = b \\ x \text{ - unrest.} \\ x \ge 0. \end{cases}$$

1. How to write dual of a given Primal.
2. What is the relation between a dual & Primal.
3. What is the Economic Interpretation of a dual.

Shadow price

"unit worth of resource"

$x_1 + 2x_2 \le 50$ unit

Writing Dual from Primal -1

How to Write Dual of a Primal

Primal (LPP) $\longrightarrow$ Dual (LPP)

$\downarrow$

Dual (Dual)

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 7 \end{bmatrix}_{2\times 3}$$

$\downarrow$

$$A^T = \begin{bmatrix} 2 & 1 \\ 3 & 2 \\ 4 & 7 \end{bmatrix}_{3\times 2}$$

$$A = (a_{ij})_{m\times n}$$

$$A^T = (a_{ij})_{n\times m}$$

$$\& \ (A^T)^T = (a_{ij})_{m\times n}$$

$$= A$$

Rules for Constructing the Dual Problem

Prerq.: Write Primal into standard form;

Standard Primal Problem	→	Dual Problem
$\text{Max } Z = c_1x_1 + \cdots + c_nx_n$	→	$\text{Min } Z^* = b_1y_1 + b_2y_2 + \cdots + b_my_m$
s.t. $AX = b$, $X \geq 0$		s.t. $A^TY \geq C^T$
$A = (a_{ij})_{m \times n}$		Y – unrestricted in sign.
$b = (b_1, \ldots, b_m)^T$		where $Y = (y_1, \ldots, y_m)^T$

Rules for Constructing the Dual Problem

Prerq.: Write Primal into standard form;

Standard Primal Problem $\longrightarrow$ Dual Problem

$$\text{Min } Z^* = b_1 y_1 + b_2 y_2 + \cdots + b_m y_m$$

$$\& \begin{cases} \text{Min } Z = c^T X \\ \text{s.t. } AX = b \\ \quad\quad X \geq 0 \end{cases} \longrightarrow \text{Dual Problem} \begin{cases} \text{Max } Z^* = b^T Y \\ \text{s.t. } A^T Y \leq c \end{cases}$$

; Y-unrestricted in sign

Remark

a) For every primal constraint, there is a dual variable.

b) For every primal variable, there is a dual constraint.

c) The coefficient of dual variables in the constraints are same as the coefficients of primal variables except that they are transposed.

$A \rightarrow$ Primal

$A^T \rightarrow$ Dual

d) The no. of dual constraints = No. of primal variables

no. of dual variables = No. of primal constraints

e) The objective coefficients of the primal variables becomes the right hand side constants of dual constraints; & R.H.S. of constraints of the dual constraints becomes the objective coefficients of the primal problem.

Writing Primal in standard form

$\text{Opt } z = c^T x$

$Ax = b \; ; \; x \geq 0$

Standard Primal Objective	Dual Objective	Constraints	Variables
Maximization	Minimization	$\geq$	Unrestricted
Minimization	Maximization	$\leq$	Unrestricted

Example Write dual of the given Primal: Max. $Z = 5x_1 + 12x_2 + 4x_3$

s.t. $x_1 + 2x_2 + x_3 \le 10$

$2x_1 - x_2 + 3x_3 = 8$

$x_1, x_2, x_3 \ge 0$

Sol^n: Write Primal into standard form:

$$\text{Max } Z = \underbrace{5x_1 + 12x_2 + 4x_3}$$

$$\text{s.t. } x_1 + 2x_2 + x_3 + s_1 = 10 \quad ; y_1$$

$$2x_1 - x_2 + 3x_3 = 8 \quad ; y_2$$

$$x_1, x_2, x_3, s_1 \ge 0$$

$$A = \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & s_1 \end{array} \\ \begin{pmatrix} 1 & 2 & 1 & 1 \\ 2 & -1 & 3 & 0 \end{pmatrix} \end{array}$$

x_1, x_2, x_3, s_1, s_2

Dual:

$$b_1 y_1 + b_2 y_2$$

$$\text{Min } Z^* = 10y_1 + 8y_2$$

s.t.

$$y_1 + 2y_2 \geq 5$$
$$2y_1 - y_2 \geq 12$$
$$y_1 + 3y_2 \geq 4$$
$$y_1 \geq 0$$

; y_2 - unrestricted

$$A^T = \begin{pmatrix} 1 & 2 \\ 2 & -1 \\ 1 & 3 \\ 1 & 0 \end{pmatrix}$$

$$A^T Y \geq C$$

$$\begin{pmatrix} 1 & 2 \\ 2 & -1 \\ 1 & 3 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \geq \begin{pmatrix} 5 \\ 12 \\ 4 \\ 0 \end{pmatrix}$$

Example: Write dual of the Primal

Minimize $Z = 15x_1 + 12x_2$

s.t. $x_1 + 2x_2 \geq 3$

$2x_1 - 4x_2 \leq 5$

$x_1, x_2 \geq 0$

Primal

$\text{Min } Z = C^T X$

$AX = b$

$X \geq 0$

Dual

$\text{Max } W = b^T Y$

$\underbrace{A^T Y}_{Y\text{-unrestricted}} \leq C$

Sol^n: Primal:

$\text{Min } Z = 15x_1 + 12x_2 + 0.s_1 + 0.s_2$

s.t. $x_1 + 2x_2 - s_1 = 3$: y_1

$2x_1 - 4x_2 + s_2 = 5$: y_2

$x_1, x_2, s_1, s_2 \geq 0$

Dual

$\text{Max } W = 3y_1 + 5y_2$

$y_1 + 2y_2 \leq 15$

$2y_1 - 4y_2 \leq 12$

$-y_1 + 0.y_2 \leq 0$

$0.y_1 + y_2 \leq 0$

y_1, y_2 - unrestricted

Example: Write dual of

Primal: Maximize $Z = 5x_1 + 6x_2$

s.t. $x_1 + 2x_2 = 5$

$-x_1 + 5x_2 \geq 3$

$4x_1 + 7x_2 \leq 8$

x_1 - unrestricted; $x_2 \geq 0$

$x_1 = x_1' - x_2''$; $x_1' \geq 0$; $x_2'' \geq 0$

Primal: Max $Z = 5(x_1') - 5x_2'' + 6x_2 + 0.s_1 + 0.s_2$

s.t. $x_1' - x_2'' + 2x_2 = 5$ y_1

$-x_1' + x_2'' + 5x_2 - s_1 = 3$ y_2

$4x_1' - 4x_2'' + 7x_2 + s_2 = 8$ y_3

$x_1', x_2'', x_2, s_1, s_2 \geq 0$

A

Dual: $\text{Min } W = 5y_1 + 3y_2 + 8y_3$

s.t.

$$y_1 - y_2 + 4y_3 \geq 5$$
$$-y_1 + y_2 - 4y_3 \geq -5$$
$$2y_1 + 5y_2 + 7y_3 \geq 6$$
$$0y_1 - y_2 + 0.y_3 \geq 0$$
$$0y_1 + 0y_2 + 1.y_3 \geq 0$$

A^T

$\Rightarrow -y_2 \geq 0 \Rightarrow y_2 \leq 0$

$\Rightarrow y_3 \geq 0$

y_1, y_1, y_3 – unrestricted

Dual: $\text{Min } W = 5y_1 + 3y_2 + 8y_3$

s.t.

$$y_1 - y_2 + 4y_3 \geq 5$$
$$-y_1 + y_2 - 4y_3 \geq -5$$
$$2y_1 + 5y_2 + 7y_3 \geq 6$$

$y_1 - y_2 + 4y_3 \leq 5 \Leftarrow$

; y_1 – unrestricted

$y_2 \leq 0$ ✓

$y_3 \geq 0$ ✓

Dual: $\text{Min } W = 5y_1 + 3y_2 + 8y_3$

s.t. $y_1 - y_2 + 4y_3 = 5$

$2y_1 + 5y_2 + 7y_3 \geq 6$

; y_1 – unrestricted

$y_2 \leq 0$

$y_3 \geq 0$

Writing Dual From Primal using Table

Rules for Constructing Dual Problem

Maximization Problem	⟷	Minimization Problem
Constraints		**Variables**
$\geq$		≤ 0
$\leq$		≥ 0
$=$		unrestricted
Variables		**Constraints**
≥ 0		$\geq$
≤ 0		$\leq$
Unrestricted		$=$

Problem 1 Write dual of

Primal: Maximize $Z = -5x_1 + 2x_2$

s.t. $-x_1 + x_2 \le -2 \quad : y_1$

$2x_1 + 3x_2 \le 5 \quad : y_2$

$x_1, x_2 \ge 0$

Dual: Minimization $Z^* = -2y_1 + 5y_2$

s.t. $-y_1 + 2y_2 \boxed{\ge} -5$

$y_1 + 3y_2 \boxed{\ge} 2$

$y_1 \boxed{\ge 0} \qquad y_2 \boxed{\ge 0}$

Primal

for Constructing Dual Problems

Maximization Problem	⟷	Minimization Problem
Constraints		Variables
$\ge$		≤ 0
$\le$		≥ 0
$=$		unrestricted
Variables		Constraints
≥ 0		$\ge$
≤ 0		$\le$
Unrestricted		$=$

Primal: opt (Max/Min

$Z = c^T x$

s.t Ax (sign) b

x (sign)

Dual: opt (Min/Max)

$Z^* = b^T y$

s.t. $A^T y$ (sign) C

Problem 2: Minimize $Z = 6x_1 + 3x_2 + 0.x_3$ | Write Dual

s.t.

$$6x_1 - 3x_2 + x_3 \geq 2 \quad y_1$$
$$3x_1 + 4x_2 + x_3 \geq 5 \quad y_2$$
$$x_1, x_2, x_3 \geq 0$$

Dual: Maximization $Z^* = 2y_1 + 5y_2$

s.t.

$$6y_1 + 3y_2 \leq 6$$
$$-3y_1 + 4y_2 \leq 3$$
$$y_1 + y_2 \leq 0$$

$$y_1 \geq 0 \; ; \; y_2 \geq 0$$

Maximization Problem	Minimization Problem
Constraints	Variables
$\geq$	≤ 0
$\leq$	≥ 0
$=$	unrestricted
Variables	Constraints
≥ 0	$\geq$
≤ 0	$\leq$
Unrestricted	$=$

Primal

Dual

Problem 3: Max $Z = x_1 + x_2 + 3x_3$

Write Dual

s.t.

$$2x_1 + x_2 + 4x_3 = 7$$
$$4x_1 - x_2 \geq 5$$
$$x_1 + x_2 \leq 6$$
$$x_1 \geq 0 \; ; \; x_2 - \text{unrestricted} \; ; \; x_3 \leq 0$$

Primal $\longrightarrow$ Dual

Maximization Problem	$\longleftrightarrow$	Minimization Problem
Constraints		Variables
$\geq$	$\longrightarrow$	≤ 0
$\leq$	$\longrightarrow$	≥ 0
$=$	$\longrightarrow$	unrestricted
Variables		Constraints
≥ 0	$\longrightarrow$	$\geq$
≤ 0	$\longrightarrow$	$\leq$
Unrestricted	$\longrightarrow$	$=$

Dual:

$$\text{Min } Z^* = 7y_1 + 5y_2 + 6y_3$$

s.t.

$$2y_1 + 4y_2 + y_3 \geq 1$$
$$y_1 - y_2 + y_3 = 1$$
$$4y_1 + 0 \cdot y_2 + 0 \cdot y_3 \leq 3$$

y_1 - unrestricted sign ; $y_2 \leq 0$ $\quad y_3 \geq 0$

Weak Duality Theorem

Weak Duality Theorem

Saturday, 11 April 2020 12:56 PM

Theorem: Consider Primal (LPP) $\text{Max } Z = C^T X$ s.t. $AX \leq b$, $X \geq 0$ (*)

and Dual LPP) $\text{min } W_0 = b^T W$ s.t. $A^T W \geq C$; $W \geq 0$ (**).

Let X be the feasible solution of the Primal and let W be the feasible solution of dual then $C^T X \leq b^T W$ —— (1)

Remark: Primal (Min) & Dual (Max.) then (1) becomes

$$C^T X \geq b^T W$$

Proof: → Given X be the feasible solution of the primal LPP

(*) must be satisfied

$$AX \le b$$

$(AB)^T = B^T A^T$

Take transpose on both sides;

$$X^T A^T = (AX)^T \le b^T$$

Now Post Multiply by W; on both sides

$$X^T A^T W \le b^T W \quad \text{——— (2)}$$

Given W be the feasible solution of the dual; then it satisfies (**); $A^T W \ge C$

Take transpose on both sides;

$[\because (A^T)^T = A$

$$W^T A = (A^T W)^T \ge C^T$$

Now Post Multiply by X; on both sides

$$W^T A X \ge C^T X \quad \text{——— (3)}$$

Now ; from ② & ③ we get

$$\underbrace{c^T X \le W^T A X}_{\text{from ③}} = \underbrace{X^T A^T W \le b^T W}_{\text{from ②}}$$

We know $\underline{W^T A X} = \underline{X^T A^T W}$
$= \text{scalar}$

$\Rightarrow c^T X \le b^T W$; which proves weak duality Theorem

To prove (Δ) for better understanding

$W^T A X = X^T A^T W =$ Scalars.

Primal: $\text{Max } Z = C^T X = (c_1 \cdots c_n)\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = c_1x_1 + \cdots + c_nx_n$

constraint: $AX \leq b$

Dual: $\text{Min } W_0 = b^T W$

constraint $A^T W \geq C$

LPP:

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}_{m \times n}, \quad X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}_{n \times 1} \quad C = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}_{n \times 1} \quad b = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}_{m \times 1}$$

$$W = \begin{pmatrix} w_1 \\ \vdots \\ w_m \end{pmatrix}_{m \times 1}$$

$$A^T = (\quad)_{n \times m}; \quad X^T = (\quad)_{1 \times n}, \quad C^T = (\quad)_{1 \times n}$$

$$b^T = (\quad)_{1 \times m}; \quad W^T = (\quad)_{1 \times m}$$

$$W = \begin{pmatrix} w_1 \\ \vdots \\ w_m \end{pmatrix}_{m\times 1}$$

$$A^T = \underbrace{(\quad)_{n\times m}}; \quad \underbrace{x^T = (\quad)_{1\times n}}, \quad c^T = (\quad)_{1\times n}$$

$$b^T = (\quad)_{1\times m} \quad ; \quad \underline{W^T = (\quad)_{1\times m}}$$

$$\underline{W^T A X = X^T A^T W = \text{scalar}}$$

$$W^T A X = (\quad)_{1\times m} \underbrace{(\quad)_{m\times n}} \underbrace{(\quad)_{n\times 1}}$$

$$\Downarrow$$

$$= (\qquad)_{1\times n} (\quad)_{n\times 1} = (\cdot)_{1\times 1} = \text{scalar}$$

$$X^T A^T W = (\quad)_{1\times n} \underbrace{(\quad)_{n\times m}} \underbrace{(\quad)_{m\times 1}}$$

$$= (\cdot)_{1\times 1} = \text{scalar}$$

Dual Theory

How to find Solution of Dual problem from Primal problem

(LPP) ⟷ (LPP)

Weak Duality Thm.

Theorem: Consider Primal (LPP) Max $Z = C^T X$ s.t. $AX \le b$, $X \ge 0$ (*)

and Dual LPP) min $W_0 = b^T W$ s.t. $A^T W \ge C$; $W \ge 0$ (**)

Let X be the feasible solution of the Primal and let W be the feasible solution of dual then $C^T X \le b^T W$

Max → Min

Remark: Primal (Min) & Dual (Max.) then ① becomes

$$C^T X \ge b^T W$$

Min ≥ Max

Example: Find the optimal solution of primal LPP

$$\text{Max } Z = 2x_1 + x_2$$

s.t. $x_1 + x_2 \le 1$

$x_1 \ge 0$

$x_2 \ge 0$

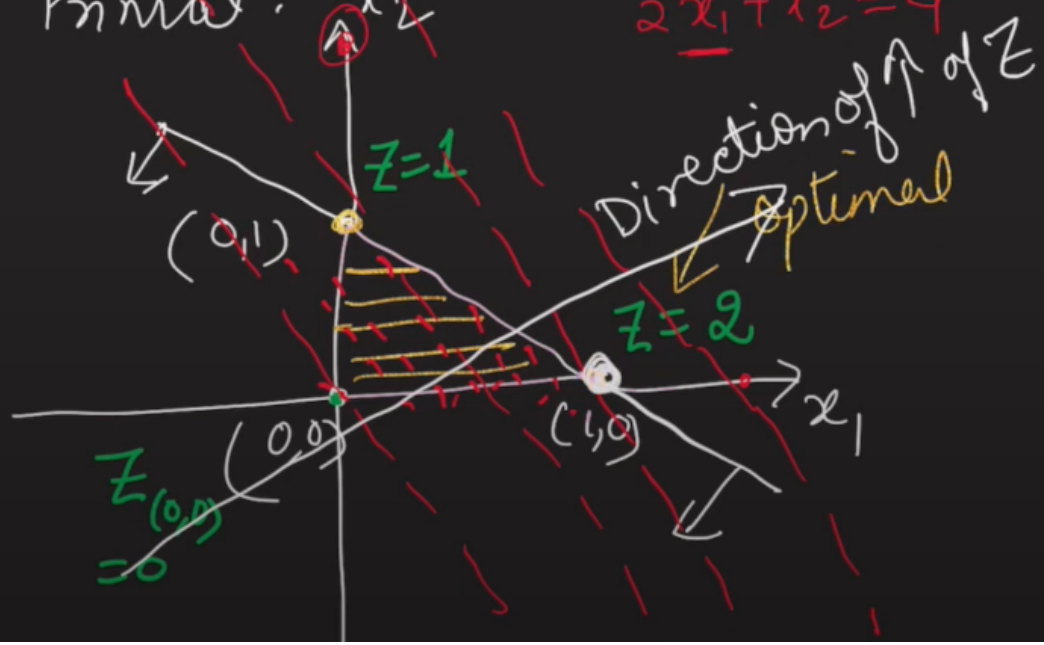
Write the dual and find its optimal solution.

Solution:

$Z = 2x_1 + x_2 = \text{Constant} = (4) \text{ say}$

$2x_1 + x_2 = 4$

Primal:



Example: Find the optimal solution of primal LPP

$$\text{Max } Z = 2x_1 + x_2$$

s.t. $x_1 + x_2 \le 1 \rightarrow y_1$

$x_1 \ge 0$

$x_2 \ge 0$

Write the dual and find its optimal solution.

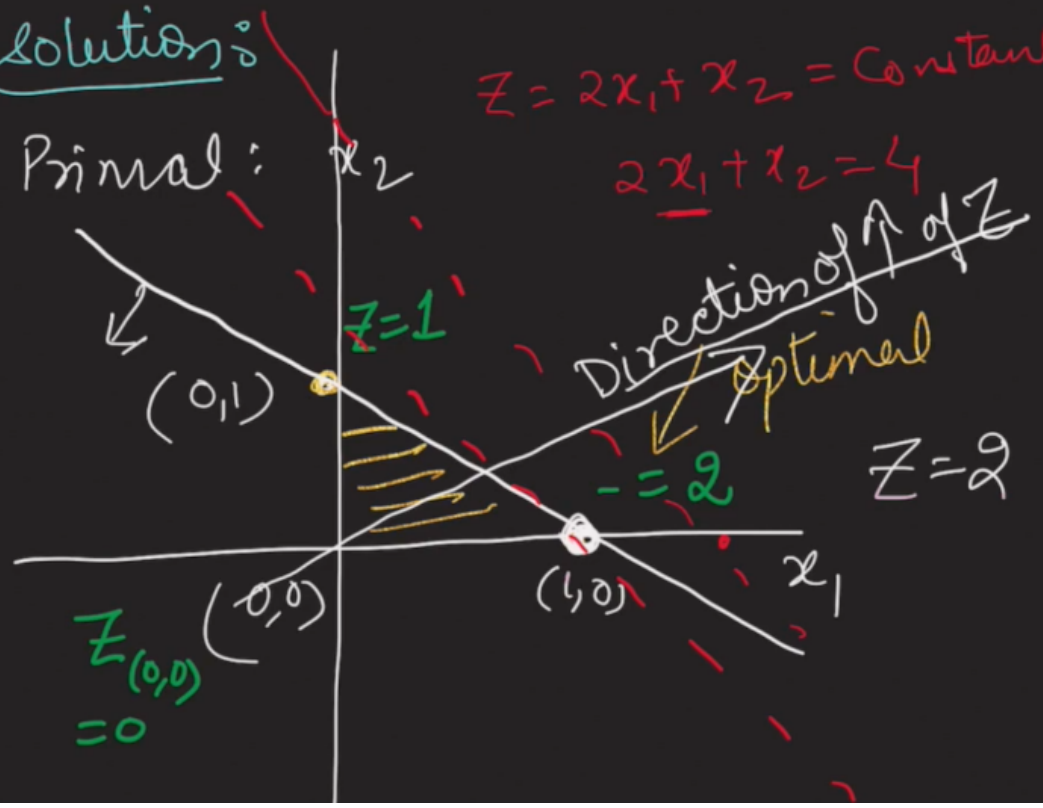
$$\begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \le 1$$

Solution:

Primal:

$Z = 2x_1 + x_2 = \text{Constant} = (4) \text{ say}$

$2x_1 + x_2 = 4$



Dual:

$$\text{Min } W = y_1$$

$y_1 \ge 2$

$y_1 \ge 1$

$y_1 \ge 0$

$y_1 \ge 2$; $\text{Min } W = y_1$

$\therefore W = 2$

To find Dual variable values, we will use the relation

$$\boxed{Y^T = C_B B^{-1}}$$

Primal: $\text{Max } Z = CX$

s.t. $AX = b$

$X \geq 0$

Dual: $\text{Min} = b^T Y$

$A^T Y \geq C$

Y - unrestricted

Note: $Z_j - C_j \rightarrow$ Net - evaluation

Max's (optimal)

$$Z_j - C_j \geq 0$$

$$C_B (B^{-1} A_j) - C_j \geq 0$$

$$\underbrace{(C_B B^{-1}) A_j}_{} - C_j \geq 0$$

$$\underbrace{(C_B B^{-1})}_{Y^T} A_j \geq C_j$$

$$(AB)^T = B^T A^T$$

Problem: Write the dual of the given Primal (LPP)

Primal

$$\text{Max } Z = 4x_1 + 3x_2$$

$$\text{s.t.} \quad x_1 + x_2 \le 8 \quad \checkmark$$

$$2x_1 + x_2 \le 10 \quad \checkmark$$

$$x_1, x_2 \ge 0$$

Solve Primal; then find optimal solution of dual.

Solution: Primal standard form

First Simplex

B.V.	x_1	x_2	s_1	s_2	X_B
Z	-4	-3	0	0	0
s_1	1	1	1	0	8
s_2	2	1	0	1	10

After iterations entering & leaving criteria we reach to optimal solution.

Problem: Write the dual of the given Primal (LPP)

Primal

$$\text{Max } Z = 4x_1 + 3x_2$$

$$\text{s.t. } x_1 + x_2 \le 8 \quad w_1$$

$$2x_1 + x_2 \le 10 \quad w_2$$

$$x_1, x_2 \ge 0$$

Solve Primal; then find optimal solution of dual.

Dual

$$\text{Min } W = 8w_1 + 10w_2$$

$$\text{s.t. } w_1 + 2w_2 \ge 4$$

$$w_1 + w_2 \ge 3$$

$$w_1, w_2 \ge 0$$

Solution: Primal standard form

First Simplex

B.V.	x_1	x_2	s_1	s_2	X_B
Z	-4	-3	0	0	0
s_1	1	1	1	0	8
s_2	2	1	0	1	10

After iterations entry & leaving criteria we reach to optimal solution.

Optimal simplex

C_B	B.V.	x_1	x_2	s_1	s_2	
	Z	0	0	2	1	26
3	x_2	0	1	2	-1	6
4	x_1	1	0	-1	1	2

$Z_{max} = 26$ at $x_1 = 2$; $x_2 = 6$

B.V. (x_2, x_1); Basis $B = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ $\quad B^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$

$$W^T = C_B B^{-1} \rightarrow \text{Primal table}$$

$$(w_1 \ w_2) = (3 \ 4)\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$$

$$= (2 \ 1)$$

$$\boxed{\begin{array}{l} W = 26 \\ w_1 = 2;\ w_2 = 1 \end{array}}$$

Example: → Primal

$$\text{Max } Z = -2x_1 - x_2$$

s.t.

$$3x_1 + x_2 = 3 \longrightarrow a_1$$

$$4x_1 + 3x_2 \geq 6 \longrightarrow s_1 \to \text{surplus} + a_2$$

$$x_1 + 2x_2 \leq 3 \longrightarrow s_2 \to \text{slack}$$

$$x_1, x_2 \geq 0$$

Dual: $\text{Min } 3w_1 + 6w_2 + 3w_3$

s.t.

$$3w_1 + 4w_2 + w_3 \geq -2$$

$$w_1 + 3w_2 + 2w_3 \geq -1$$

w_1 – unrestricted

$$w_2 \leq 0$$

$$w_3 \geq 0$$

Initial table (Primal)

Big M
Two-phase

surplus Initial identity

B.V.	x_1	x_2	s_1	a_1	a_2	s_2	X_B
Z	$2-7M$	$1-4M$	M	0	0	0	$-9M$
a_1	3	1	0	1	0	0	3
a_2	4	3	-1	0	1	0	6
s_2	1	2	0	0	0	1	3

Last table

C_B	B.V.	x_1	x_2	s_1	a_1	a_2	s_2	X_B
		0	0	$1/5$	$M-3/5$	$M-1/5$	0	$-12/5$
-2	x_1	1	0	$1/5$	$3/5$	$-1/5$	0	$3/5$
-1	x_2	0	1	$-3/5$	$-4/5$	$3/5$	0	$6/5$
0	s_2	0	0	1	1	-1	1	0

$$W^T = (w_1 \ w_2 \ w_3)$$

$$= C_B B^{-1}$$

$$= (-2 \ -1 \ 0) \begin{pmatrix} 3/5 & -1/5 & 0 \\ -4/5 & 3/5 & 0 \\ 1 & -1 & 1 \end{pmatrix}$$

$$= (-2/5 \ \ -1/5 \ \ 0)$$

$$\text{Min } W = -12/5$$

at W1=-2/5 , W2=-1/5, W3=0

How to find Solution of Dual problem from Primal problem

Method 1: $W^T = C_B^T B^{-1}$ → Primal optimal

Method 2: Direct Reading $(z_j - c_j)$ of optimal primal problem.

- Primal Problem; slack variables; optimal table; $z_j - c_j$ of slack variables represents the dual variables of the corresponding dual problem.
- If Primal Problem involve surplus variables then multiply by (-1) to $z_j - c_j$ of surplus variable to get the corresponding dual variable value.

Example: → **Primal** $\quad$ Max $Z = -2x_1 - x_2$

Dual variables

w_1 → $\quad 3x_1 + x_2 = 3$

w_2 → $\quad 4x_1 + 3x_2 \geq 6 \longrightarrow s_1$

w_3 → $\quad x_1 + 2x_2 \leq 3 \longrightarrow s_2$

s.t.

$$x_1, x_2 \geq 0$$

Dual: Min $3w_1 + 6w_2 + 3w_3$

s.t.

$$3w_1 + 4w_2 + w_3 \geq -2$$
$$w_1 + 3w_2 + 2w_3 \geq -1$$
$$w_1 - \text{unrestricted}$$
$$w_2 \leq 0$$
$$w_3 \geq 0$$

Initial table (Primal)

B.V.	x_1	x_2	s_1 (surplus)	a_1 (artificial)	a_2 (artificial)	s_2 (slack)	X_B
Z	$2-7M$	$1-4M$	M	0	0	0	$-9M$
a_1	3	1	0	1	0	0	3
a_2	4	3	-1	0	1	0	6
s_2	1	2	0	0	0	1	3

Last table

C_B	B.V.	x_1	x_2	s_1	a_1	a_2	s_2	X_B
		0	0	$1/5$	$M-2/5$	$M-1/5$	0	$-12/5$
		1	0	$1/5$	$3/5$	$-1/5$	0	$3/5$
-2	x_1	0	1	$-3/5$	$-4/5$	$3/5$	0	$6/5$
-1	x_2	0	0	1	1	-1	1	0
0	s_2							

$Z_{max} = -12/5$; $x_1 = 3/5$, $x_2 = 6/5$

Method 1:

$$(w_1\ w_2\ w_3) = C_B^T B^{-1}$$

$$= (-2\ \ -1\ \ 0)\begin{pmatrix} 3/5 & -1/5 & 0 \\ -4/5 & 3/5 & 0 \\ 1 & -1 & 1 \end{pmatrix}$$

$$= (-2/5\ \ -1/5\ \ 0)$$

$W_{min} = -12/5$.

Method 2: Look at $z_j - c_j$ in optimal primal table; of initial starting B.V.

As 'a_1' is first B.V. in start $\Rightarrow$

As 'a_2' is second B.V. in start $\Rightarrow$

As 's_2' is third B.V. in start $\Rightarrow$

Now look $z_j - c_j$ for these variable in optimal table

$z_j - c_j$ (for a_1) $= M - 2/5$

$z_j - c_j$ (for a_2) $= M - 1/5$

$z_j - c_j$ (for s_2) $= 0$ (*)

from (*)

w_1 : $(z_j - c_j) - c_j = (M - 2/5) - M = -2/5$

w_2 : $(z_j - c_j) - c_j = (M - 1/5) - M = -1/5$

w_3 : $(z_j - c_j) - c_j = 0 - 0 = 0$

Also Note;

$z_j - c_j$ (slack variable in optimal simplex table gives corresponding dual variable)

& $z_j - c_j$ (surplus variable in optimal simplex table gives corresponding dual variable after multiplying by -1)

In above question s_1 is surplus & it comes in 2nd constraint so w_2 is corresponding dual variable and $\Rightarrow$ $w_2 = -1/5$

and

s_2 is slack variable & it comes in 3rd constraint so third dual variable can be [illegible] $z_j - c_j$ that is

optimal table of primal problem.

Method 3. Complementary slackness conditions

Condition 1: In the optimal solution of system (Primal), if the ith variable is strictly positive (or it is a basic variable) then the corresponding i^{th} constraint in dual problem is satisfied as an equality constraint or i^{th} constraint slack / surplus variable is zero.

Condition 2: In the optimal solution of Primal, if the ith constraint is satisfied as strict inequality (slack | surplus appear in basis) then i^{th} variable of its dual vanishes.

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Example 1: LPP (Primal) $\text{Max } Z = 4x_1 + 10x_2$

$$\text{s.t.} \quad 2x_1 + x_2 \leq 50$$
$$2x_1 + 5x_2 \leq 100$$
$$2x_1 + 3x_2 \leq 90 \quad ; \quad x_1, x_2 \geq 0$$

Solve LPP using simplex method and find the optimal solution of dual using the optimal solution of primal by applying CSC.

Solution: Using simplex method; we can find optimal solution of Primal;

solution of dual using the optimal solution of P

by applying CSC.

solution: Using simplex method; we can find optimal solution of Primal; optimal simplex table of primal is given below:

B.V.	x_1	x_2	s_1	s_2	s_3	X_B (Solution)
Z	0	0	0	2	0	200
x_1	1	0	5/8	-1/8	0	150/8
x_2	0	1	-1/4	1/4	0	25/2
s_3	0	0	-1/2	-1/2	1	15

optimal solution of Primal:

Max $Z = 200$ at $(x_1, x_2, s_1, s_2, s_3) = \left(\frac{150}{8}, \frac{25}{2}, 0, 0, 15\right)$

Dual :

B.V. N.B.V.

$$\text{Min } Y = 50y_1 + 100y_2 + 90y_3$$

s.t. $2y_1 + 2y_2 + 2y_3 \geq 4 \Rightarrow 2y_1 + 2y_2 + 2y_3 - s_1' = 4$

$y_1 + 5y_2 + 3y_3 \geq 10 \Rightarrow y_1 + 5y_2 + 3y_3 - s_2' = 10$ (Δ)

$y_1, y_2, y_3 \geq 0$

$s_1', s_2' \geq 0$

optimal solution of dual:

$$(y_1, y_2, y_3, s_1', s_2') = (y_1, y_2, 0, 0, 0) \quad (*)$$

$$(x_1, x_2, s_1, s_2, s_3)$$

Now substitute (*) in (Δ) we find, after solving

$2y_1 + 2y_2 = 4$

$y_1 + 5y_2 = 10$

$\therefore$ optimal solution of dual is

$$= (0, 2, 0, 0, 0)$$

$$\underline{\text{Min } Y = 200}$$

Example 2: Find the optimal solution of

(Primal)

$$\text{Max } Z = 2x_1 + x_2 \quad \text{s.t.}$$

$$x_1 + 2x_2 \le 10$$

$$x_1 + x_2 \le 6$$

$$x_1 - x_2 \le 2$$

$$x_1 - 2x_2 \le 1 \; ; \quad x_1, x_2 \ge 0$$

by solving its dual. Use CSG to write the optimal solution of Primal

(Dual)

$$\text{Min } Y = 10y_1 + 6y_2 + 2y_3 + y_4$$

s.t.

$$y_1 + y_2 + y_3 + y_4 \ge 2$$

$$2y_1 + y_2 - y_3 - 2y_4 \ge 1$$

$$y_1, y_2, y_3, y_4 \ge 0$$

Solution: We solve dual using BigM | Two-Phase method; Let the optimal table is;

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optimal solution of dual :

$Y_{min} = 10$ at $(y_1, y_2, y_3, y_4, s_1', s_2') = (0, 3/2, 1/2, 0, 0, 0)$

optimal primal solution using CSC;

$(x_1, x_2, s_1, s_2, s_3, s_4) = (x_1, x_2, s_1, \underline{0}, \underline{0}, s_4)$

convert Primal problem constraint into equality;

Optimal primal solution is

$$(x_1, x_2, s_1, s_2, s_3, s_4) = (x_1, x_2, s_1, \underline{0, 0}, s_4)$$

convert Primal problem constraint into equality;

Primal: $\text{Max } Z = 2x_1 + x_2$

s.t. $x_1 + 2x_2 + s_1 = 10$

$$\left.\begin{aligned} x_1 + x_2 + \underline{s_2} &= 6 \\ x_1 - x_2 + \underline{s_3} &= 2 \end{aligned}\right] \Rightarrow \left.\begin{aligned} x_1 + x_2 &= 6 \\ x_1 - x_2 &= 2 \end{aligned}\right\} \Rightarrow x_1 = 4,\ x_2 = 2$$

$$x_1 - 2x_2 + s_4 = 1$$

$s_1 = 2$; $s_4 = 1$

optimal solution of primal: $(x_1, x_2, s_1, s_2, s_3, s_4)$

$$= (\underline{4, 2, 2, 0, 0, 1})$$

$$\text{Max } Z = 2(4) + 2 = 10$$

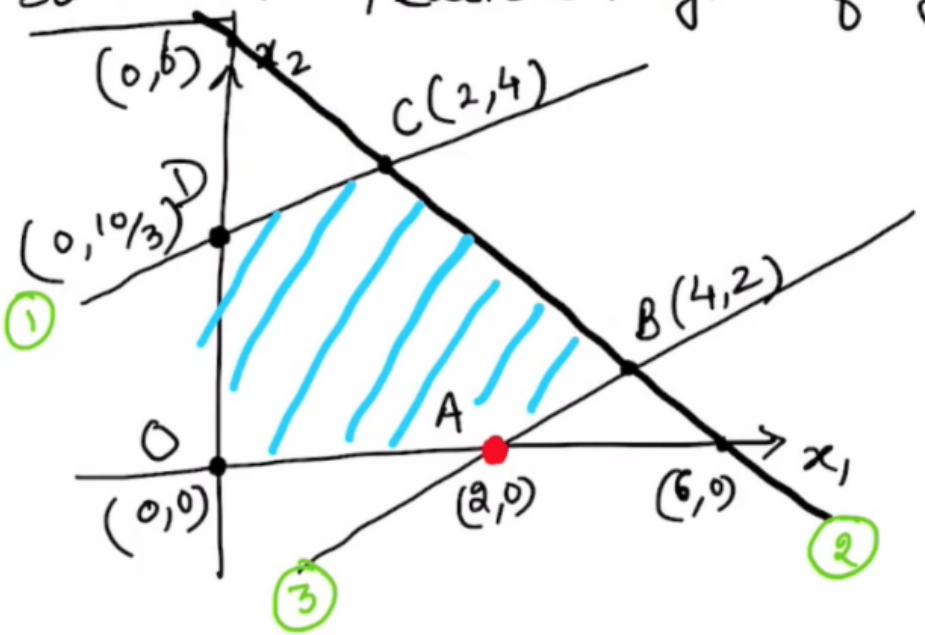
Example 3: Solve the given LPP with graphical method and then apply CSC to find optimal solution of dual.

$$\text{Min } Z = -x_1 + 2x_2$$

$$\text{s.t.} \quad -x_1 + 3x_2 \le 10 \;(1) \;;\quad x_1 + x_2 \le 6 \;(2) \;;\quad x_1 - x_2 \le 2 \;(3)$$

$$x_1, x_2 \ge 0 \quad (*)$$

Solution: Feasible region of given constraints;



Optimal solution:

$$Z_{min} = -2 \quad \text{at } A(2,0)$$

complete solution;

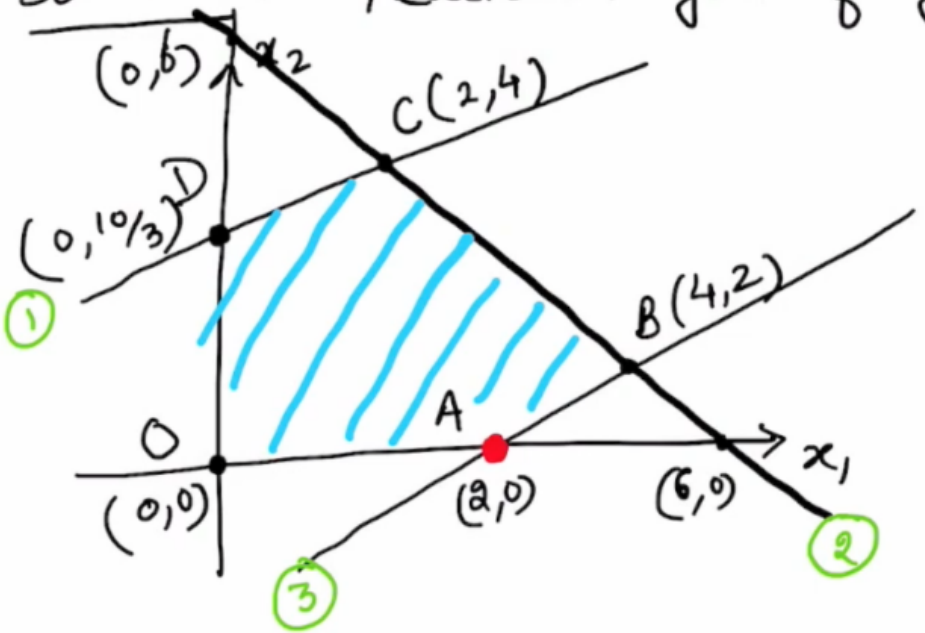
$$(x_1, x_2, s_1, s_2, s_3) = (2, 0, 12, 4, 0)$$

B.V.

N.B.V.

$x_1, x_2 \geq 0$

Solution: Feasible region of given constraints;



Optimal solution:

$Z_{min} = -2$ at $A(2,0)$

complete solution;

$(x_1, x_2, s_1, s_2, s_3) = (2, 0, 12, 4, 0)$

B.V. B.V. N.B.V.

Dual: $\text{Max } Y = -10y_1 - 6y_2 - 2y_3$

s.t. $y_1 - y_2 - y_3 \leq -1$

$-3y_1 - y_2 + y_3 \leq 2$; $y_1, y_2, y_3 \geq 0$

sub. & solve

Dual optimal solution;

$(y_1, y_2, y_3, s_1', s_2')$

$= (0, 0, y_3, 0, s_2')$

$(y_1, y_2, y_3, s_1', s_2') = (0, 0, 1, 0, 1)$; $\text{Max } Y = -2$

Example 1: Use dual simplex method to solve the given LPP.

$$\text{Max } Z = -2x_1 - x_2$$

$$\text{s.t.} \quad 2x_1 - x_2 - x_3 \geq 3$$

$$x_1 - x_2 + x_3 \geq 2$$

$$x_1, x_2, x_3 \geq 0$$

Solution: Introduce surplus variables;

$$\text{Max } Z = -2x_1 - x_2 + 0.x_3 + 0.s_1 + 0.s_2$$

$$\text{s.t.} \quad \left.\begin{aligned} 2x_1 - x_2 - x_3 - s_1 &= 3 \\ x_1 - x_2 + x_3 - s_2 &= 2 \end{aligned}\right\} \text{multiply by } -1$$

$$x_1, x_2, x_3, s_1, s_2 \geq 0$$

$$\text{Max } Z = -2x_1 - x_2$$

$$\left.\begin{aligned} -2x_1 + x_2 + x_3 + s_1 &= -3 \\ -x_1 + x_2 - x_3 + s_2 &= -2 \\ \text{all var.} &\geq 0 \end{aligned}\right]$$

$$B.V. = (s_1, s_2)$$

$$B = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -\text{Identity matrix}$$

$$B^{-1} = -I$$

$$-x_1 + x_2 - x_3 + s_2 = -2$$

all var. ≥ 0

	C_B	B.V.	x_1	x_2	x_3	s_1	s_2	X_B (Solution)
R_1:		Z	2	1	0	0	0	0
R_2:	0	s_1	$\boxed{-2}$	1	1	1	0	-3
R_3:	0	s_2	-1	1	-1	0	1	-2
			0	2	1	1	0	-3
		x_1	1	$-1/2$	$-1/2$	$-1/2$	0	$3/2$
		s_2	0	$1/2$	$\boxed{-3/2}$	$-1/2$	1	$-1/2$
			0	$5/3$	0	$2/3$	$2/3$	$-10/3$
		x_1	1	$-2/3$	0	$-2/3$	$-1/3$	$5/3$
		x_3	0	$-1/3$	1	$1/3$	$-2/3$	$1/3$

Current table satisfies the optimality criteria $z_j - c_j \geq 0$

remove x_{B_i} with most -ve value

feasibility is disturbed

First: Remove

Second: Enter variable

$R_1 \to R_1 - 2\left(\frac{R_2}{-2}\right)$

$R_2/-2$

$R_3 \to R_3 + \left(R_2/-2\right)$

$$\min\left\{\left|\frac{z_j - c_j}{\alpha_{ij}}\right| ; \alpha_{ij} < 0\right\}$$

$$= \min\left\{\left|\frac{2}{-2}\right|\right\}$$

$$= 1$$

In second table;

$$\min\left\{\left|\frac{z_j - c_j}{\alpha_{ij}}\right| ; \alpha_{ij} < 0\right\}$$

$$= \min\left\{\left|\frac{1}{-3/2}\right|, \left|\frac{1}{-1/2}\right|\right\}$$

optimal

feasible

$$\boxed{\text{Max } Z = -10/3 \; ; \quad \text{at } x_1 = 5/3 \; ; \; x_2 = 0 ; \; x_3 = 1/3}$$

Example: Use dual simplex method to solve

$$\text{Max } -x_1$$

$$\text{s.t.} \quad x_1 - x_2 \geq 3$$

$$-x_1 + x_2 \geq 4 \quad ; \quad x_1, x_2 \geq 0$$

Sol^n:

B.V.	x_1	x_2	s_1	s_2	x_B
z	1	0	0	0	0
s_1	-1	1	1	0	-3
s_2	1	$\boxed{-1}$	0	1	-4
	1	0	0	0	
s_1	0	0	1	1	-7
x_2	-1	1	0	-1	4

$$\min\left\{ \left|\frac{z_j - c_j}{\alpha_{ij}}\right| ; \alpha_{ij} < 0 \right\}$$

$$\min\left\{ \left|\frac{0}{-1}\right| \right\} = 0$$

current table is optimal but it is not feasible (we do not find an entering variable)

The current LPP has Infeasible sol

Example: Solve LPP; using Dual Simplex method

$$\begin{cases} \text{Min } Z = 2x_1 + 3x_2 \\ \text{s.t.} \quad 2x_1 + 2x_2 \le 30 \\ \qquad\quad x_1 + 2x_2 \ge 10 \;;\; x_1, x_2 \ge 0 \end{cases}$$

Or

$$\text{Min } Z = 2x_1 + 3x_2$$
$$\text{s.t.} \quad 2x_1 + 2x_2 + s_1 = 30$$
$$x_1 + 2x_2 - s_2 = 10$$
$$x_1, x_2, s_1, s_2 \ge 0$$

Or

$$\text{Min } Z = 2x_1 + 3x_2 + 0.s_1 + 0s_2$$
$$\text{s.t.} \quad 2x_1 + 2x_2 + s_1 = 30$$
$$-x_1 - 2x_2 + s_2 = -10 \;;\; x_1, x_2, s_1, s_2 \ge 0$$

B.V.	x_1	x_2	s_1	s_2	X_B (Solution)
Z	-2	-3	0	0	0
s_1	2	2	1	0	30
s_2	-1	$\boxed{-2}$	0	1	-10
	$-1/2$	0	0	$-3/2$	15
s_1	1	0	1	1	20
x_2	$1/2$	1	0	$1/2$	5

Infeasible

$$\min\left\{ \left|\frac{-2}{-1}\right|, \left|\frac{-3}{-2}\right| \right\} = \frac{3}{2}$$

optimality

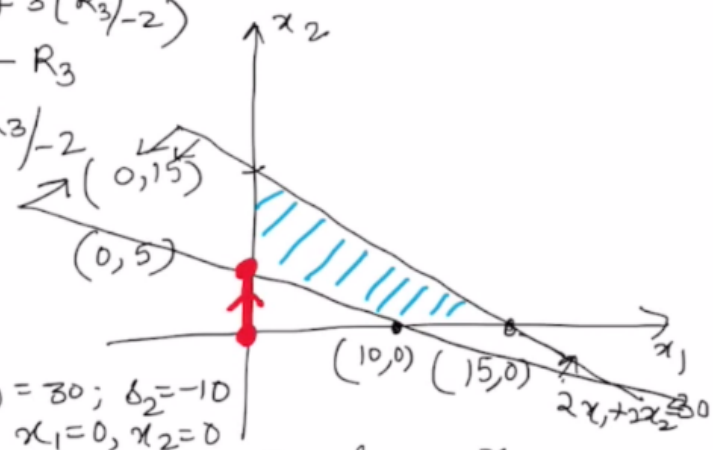
feasibility

$R_1 \to R_1 + 3(R_3/-2)$

$R_2 \to R_2 - R_3$

$R_3 \to R_3/-2$

$$\boxed{Z_{min} = 15 \;;\; \text{at } x_1 = 0 \;;\; x_2 = 5}$$



First Table: $s_1 = 30$; $s_2 = -10$
$x_1 = 0$, $x_2 = 0$

Second Table: $s_1 = 20$; $x_2 = 5$; $s_2 =$, $x_1 =$