Integer Programming Problem

Integer Programming Problem (IPP)

Optimize (Minimize/Maximize) $Z = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n$

Subject to

$$a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n \leq, \geq, = b_1$$
$$a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n \leq, \geq, = b_2$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$a_{m1} x_1 + a_{m2} x_2 + \ldots + a_{mn} x_n \leq, \geq, = b_m$$

$x_1, x_2, \ldots\ldots, x_n \geq 0,$

and $x_1, x_2, \ldots\ldots, x_n$ are integers (some or all)

LPP: Obj fn. } Linear
constraints
non-ve restrictions

IPP

Types of Integer Programming Problem

1. Pure IPP : when $x_i \in \mathbb{Z} \quad \forall i$ (when all decision variables are integers)

2. Mixed IPP : $x_i \in \mathbb{Z}$ for some i (when some decision variables are integers)

3. Zero-one IPP : $x_i \in \mathbb{Z} \quad \forall i$; $x_i \in \{0, 1\}$

Problem:

A hotel operating 24 hours has the following minimum requirement of waiters:

Period	1	2	3	4	5	6
Time	7–11am	11–3pm	3–7pm	7–11pm	11–3am	3–7am
Minimum no. of waiters required	6	12	8	16	5	3

A waiter reports to the hotel manager at the beginning of the period and continues to work for 8 hours. (consecutive)
Formulate the problem as an IPP to minimize the total number of waiters required.

Sol^n:→ Let $x_1, x_2, \ldots, x_6$ be the no. of waiters required at i^{th} period (join)

$$\min Z = x_1 + \cdots + x_6$$

s.t.

$$x_1 + x_2 \geq 12$$
$$x_2 + x_3 \geq 8$$
$$x_3 + x_4 \geq 16$$
$$x_4 + x_5 \geq 5$$
$$x_5 + x_6 \geq 3$$
$$x_6 + x_1 \geq 6$$

$$x_i \geq 0 \; ; \; x_i \in \mathbb{Z} \quad \forall i$$

This is an example of Pure IPP

Problem: (Mixed IPP)

A company manufactures three products P1, P2 and P3 which yield per unit profits of Rs. 200, Rs. 400, and Rs. 300 respectively. Each of these products is processed on three different machines. The time required on each machine per unit of the product is given below:

Product	Time required (hours per unit)		
	Machine 1	Machine 2	Machine 3
P1	30	20	10
P2	40	10	30
P3	20	20	20
Time available (Per hour)	600	400	800

How many products of each type should be produced to maximize profit? Fractional Units of product P1 and P2 is not allowed.

Solution :→ Let x_1, x_2, x_3 be the units of products P1, P2, P3 respectively

$$\text{Max } Z = 200x_1 + 400x_2 + 300x_3$$

s.t.

$$30x_1 + 40x_2 + 20x_3 \le 600$$
$$20x_1 + 10x_2 + 20x_3 \le 400$$
$$10x_1 + 30x_2 + 20x_3 \le 800$$

; $x_i \ge 0$; $\forall i$

x_1, x_2 — integers

(zero-one IPP)

Problem:

Four items are considered for loading on an airplane, which has a capacity to load upto 25 tonnes. The weights and the values of the items are as follows:

Item:	**A**	**B**	**C**	**D**
Weight (Tonnes):	**2**	**7**	**5**	**3**
Value (Quantity per unit)	**10**	**36**	**25**	**14**

Which items, and what quantities should be loaded on the plane so to maximize the value of the cargo transported? Formulate the problem as an IPP.

Note : The above is known as Knapsack Problem- An IPP which has only one constraint is known as Knapsack Problem

Solution:→ Let x_1, x_2, x_3, x_4 be the no. of items A, B, C, D resp. that are loaded on the plane.

$$\text{Max } Z = 10x_1 + 36x_2 + 25x_3 + 14x_4$$

s.t.

$$2x_1 + 7x_2 + 5x_3 + 3x_4 \leq 25$$

$$x_i = 0 \text{ or } 1 \quad (j = 1, 2, 3, 4)$$

(do not load) → (load)

Methods to Solve Integer Programming Problem (IPP)

1. Branch Bound Techniques (IPP with two decision variable)

2. Fractional Cut Method (Gomory's Constraint) (IPP with two or more decision variable)

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Branch and Bound Technique - Integer Programing Problem

Friday, 27 March 2020 7:45 PM

Integer Programming Problem (IPP)

Optimize (Minimize/Maximize) $Z = C^T X$

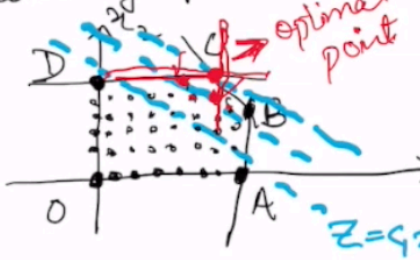
Subject to $AX \le, \ge, = b$

$X = (x_1, x_2) \ge 0$ and x_1, x_2 are integers

LPP: $\left\{ \begin{array}{l} \text{opt } Z = C^T X \\ \text{s.t. } AX \le, \ge, = b \\ X \ge 0 \end{array} \right\}$

IPP: x_1, x_2 are integers

Step 1: Solve LPP with graphical method



$C = (x_1, x_2) \in \mathbb{Z}$

Z at O
Z at A
Z at B
Z at C
Z at D

Max. or Min

Branch and Bound Technique:

Step 1. Solve the associated LPP with graphical method
If the decisions variables are integers then it is the optimal solution for IPP too, if not move to step 2.

Step 2. Do the branching with respect to any decision variable having non-integral value. Suppose X_1 is non-integral then do branching as : the first branch has constraint $X_1 \le [X_1]$ and other branch has constraint as $X_1 \le [X_1] + 1$.

Step 3. Repeat step 2, until all decision variables are integers.

Step 2: $x_1 \notin \mathbb{Z}$ or $x_2 \notin \mathbb{Z}$
or x_1 and $x_2 \notin \mathbb{Z}$

Suppose $x_1 \notin \mathbb{Z}$; $x_1 \in$ Rational No.

$\boxed{x_1 = 5/2 ; x_2 = 4}$

$x_1 \le 2$ $x_2 \ge 3$

Let $(x_1 = 0, x_1 = 1)$, $x_2 = 2$

$(x_2 = 3, x_2 = 4, \ldots)$

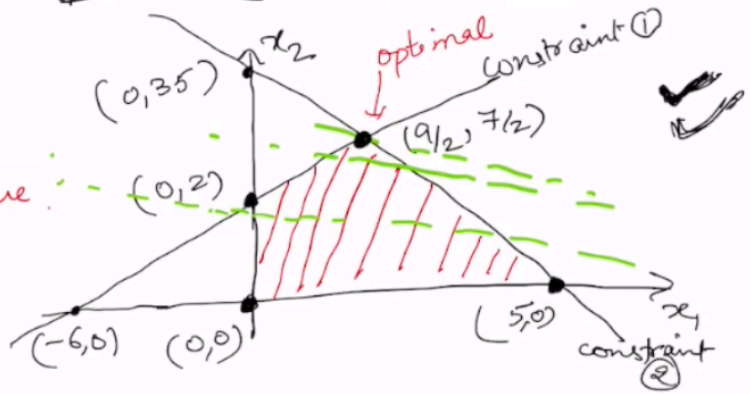
Example 1: Max $Z = 7x_1 + 9x_2$

s.t. $-x_1 + 3x_2 \le 6$ — (1)

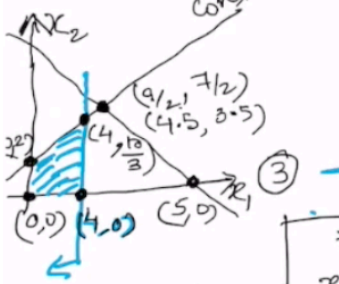
$7x_1 + x_2 \le 35$ — (2)

$x_1, x_2 \ge 0$; and are integers

Solve this IPP using Branch & Bound Technique.



Solution:



optimal solution of associated LPP

$Z = 7(9/2) + 9(7/2) = 63$

$x_1 = 9/2$; $x_2 = 7/2$

$x_1 \le [9/2]$ $\qquad$ $x_1 \ge [\frac{9}{2}] + 1$

$x_1 \le 4$ $\qquad$ $x_1 \ge 5$ — (4)

$Z = 58$, $x_1 = 4$; $x_2 = 10/3$

$Z = 35$, $x_1 = 5$; $x_2 = 0$

$x_1, x_2 \in \mathbb{Z}$

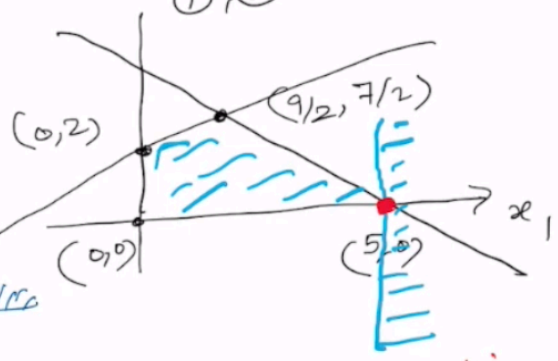
as $x_1 \le 4$, cut (intersect) the constraint (1); so put $x_1 = 4$ in constraint (1) & $x_2 = 10/3$

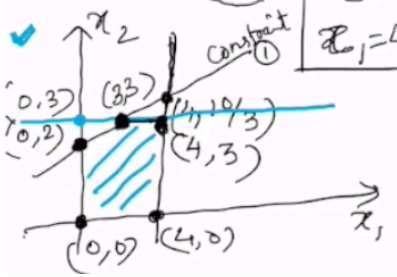
$x_2 \le 3$ $\qquad$ $x_2 \ge 4$

$Z = 55$, $x_1 = 4$; $x_2 = 3$

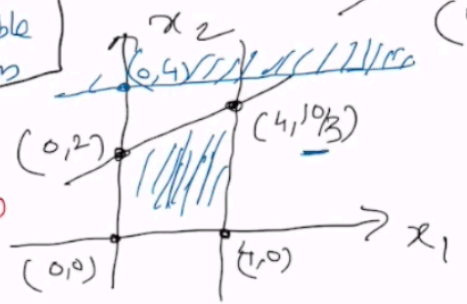
No feasible solution

Draw feasible region including constraint (1), (2) & (4)





optimal solution of IPP



Optimal solution

$Z_{max} = 55$

$(x_1, x_2) = (4, 3)$.

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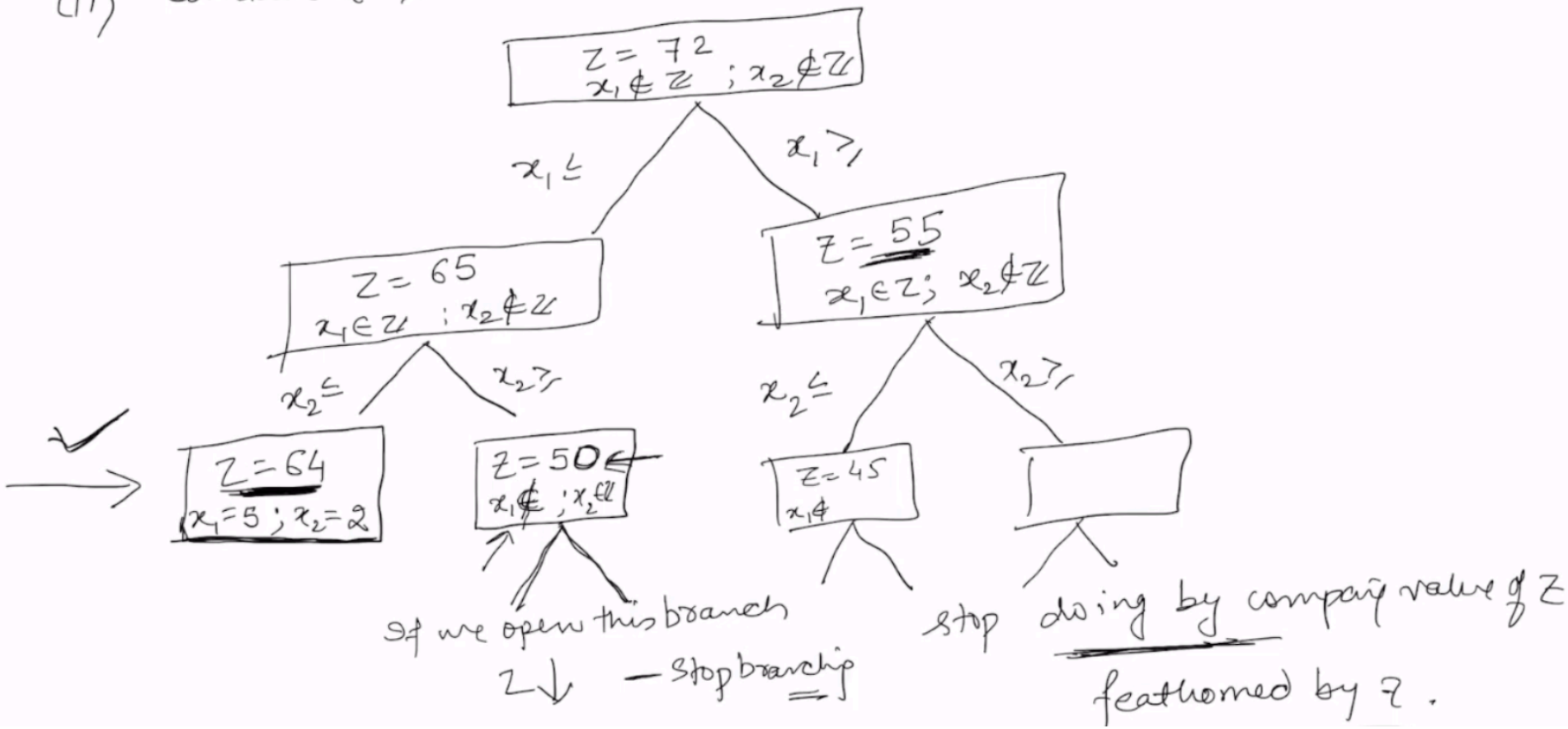
Remark: (i) $Z_{max}(LPP) \geq Z_{max}(IPP)$

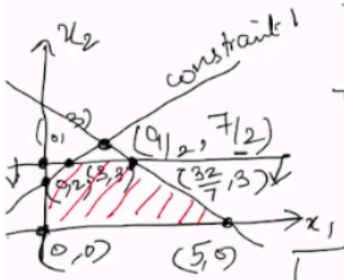
(In last e.g. $63 \geq 55$)

(ii) Consider a situation;

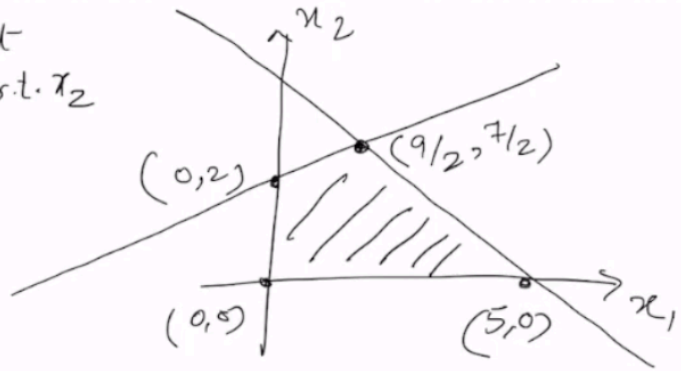


Alternate Solution of Example 1:

Here we start branching w.r.t. x_2



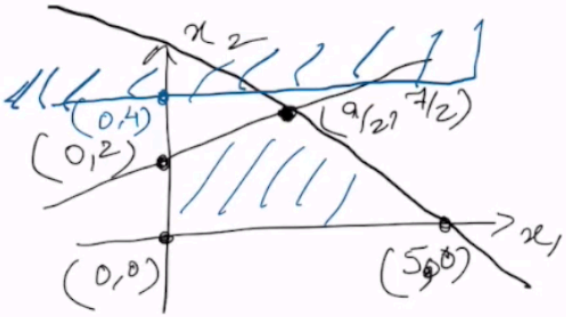
$$\boxed{\begin{array}{l} Z = 63 \\ x_1 = 9/2 \;;\; x_2 = 7/2 \end{array}}$$



$x_2 \le 3$ $\qquad$ $x_2 \ge 4$

$$\boxed{\begin{array}{l} Z = 59 \\ x_1 = 32/7 \;;\; x_2 = 3 \end{array}}$$

$$\boxed{\text{No Feasible solution}}$$



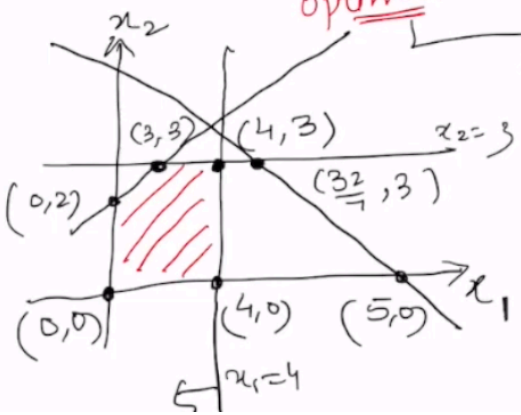
$x_1 \le 4$ $\qquad$ $x_1 \ge 5$

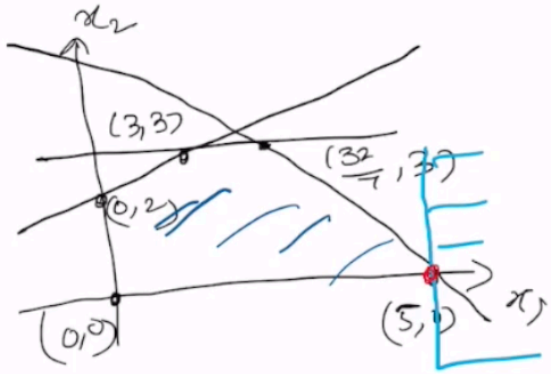
$$\boxed{\begin{array}{l} Z = 55 \\ x_1 = 4 \;;\; x_2 = 3 \end{array}}$$

$$\boxed{\begin{array}{l} Z = 35 \\ x_1 = 5 \;;\; x_2 = 0 \end{array}}$$

optimal

compare





$Z = 55$ (optimal value)

at $(x_1, x_2) = (4, 3)$

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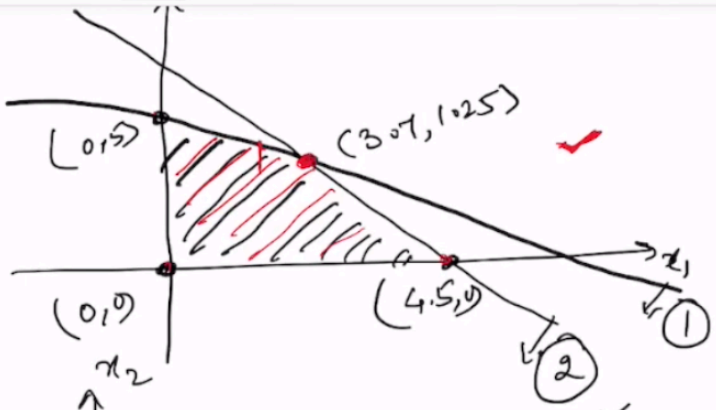
Example 2: Minimize $Z = -5x_1 - 4x_2$

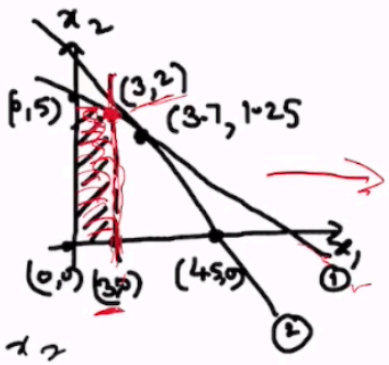
s.t. $x_1 + x_2 \le 5$

$10x_1 + 6x_2 \le 45$; $x_1, x_2 \ge 0$ - integers.

LPP →



Solution :→



$\boxed{Z = -23.75 \quad x_1 = 3.7 ; x_2 = 1.25}$

$x_1 \le 3$ | $x_1 \ge 4$

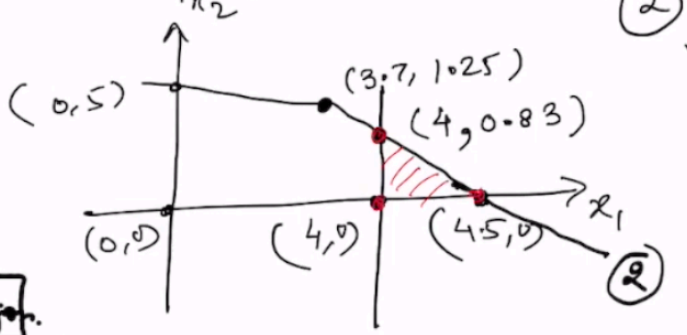
$\boxed{Z = -23 \quad x_1 = 3 ; x_2 = 2}$

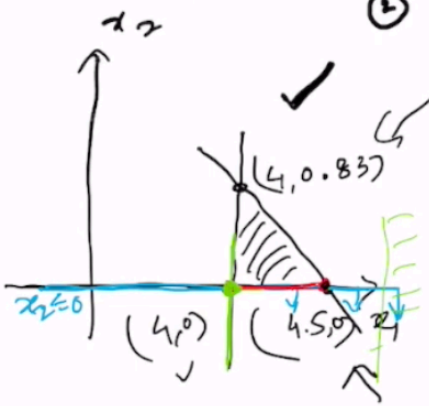
$\boxed{Z = -23.33 \quad x_1 = 4 , x_2 = .83}$

$x_2 \le 0$ | $x_2 \ge 1$

$\boxed{Z = -22.5 \quad x_1 = 4.5 ; x_2 = 0.}$

$\boxed{\text{NO Feasible Solution Region.}}$

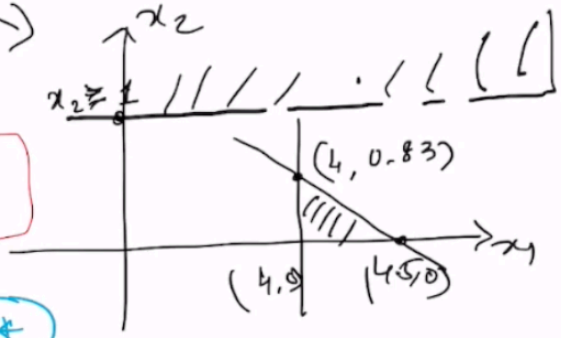




$x_1 \le 4$ | $x_1 \ge 5$

$\boxed{Z = -20 \quad x_1 = 4 ; x_2 = 0}$

$\boxed{\text{No F. Solns.}}$



$\boxed{\text{optimal solution } Z = -23 \text{ at } x_1 = 3 ; x_2 = 2}$

Remark: $Z_{LPP}(\text{min}) \le Z_{IPP}(\text{min})$ — (*)

e.g. $-23.75 \le -23$ →

Gomoary's Cut Method for IPP-Integer Programming Problem (or **LIP**-Linear Integer Programming)

Max|Min

LPP: Optimize $C^T X$ subject to $AX = b$, $x >= 0$

IPP or LIPP: Optimize $C^T X$ subject to $AX = b$, $X >= 0$ and integer(s).

Example

Consider IPP,

$z^*: =$	Max $z = 5x1 + 8x2$

s.t.

$x_1 + x_2 <= 6,$	Ist constraint
$5x_1 + 9x_2 <= 45,$	2nd constr.
$x_1, x_2 >= 0$, and integer	

LPP

• Graphical

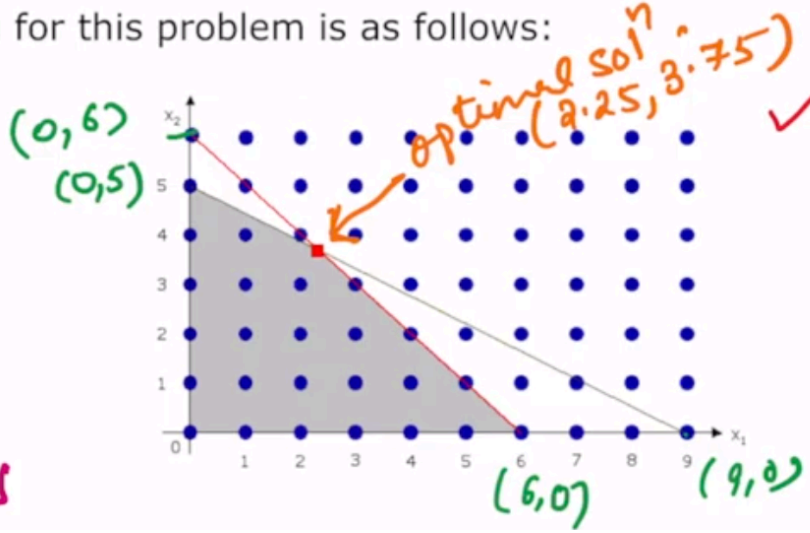
The optimal solution to the linear programming problem (relaxing the integer condition) is

X = (2.25, 3.75) and the final Simplex Tableau for this problem is as follows:

Simplex Method

BV	x_1	x_2	s_1	s_2	X_B
z	0	0	1.25	0.75	41.25
x_1	1	0	2.25	-0.25	2.25
x_2	0	1	-1.25	0.25	3.75

$x_1 = 2.25$; $x_2 = 3.75$



Procedure for Solving IPP using Gomory's Cut / or Gomory's Constraint Method

Solve IPP;

- Simplex Method ; find optimal solution

[• Branch & Bound
↓
(x_1, x_2)

- check optimal solutions $\in \mathbb{Z} \rightarrow$ integers
- If not ; Add a constraint- 'Gomory' constraint'
- Use Post Optimality | Sensitivity Analysis Max (Assume
- Use Dual simplex Method.

B.V.	optimal $z_j - c_j \geq 0$	X_B
x_1		
x_2		
x_{G_1}		

≥ 0

feasibility

May be

Find Gomory Constraint

Consider optimal simplex table

B.V.	x_1	x_2	x_3	x_4	X_B
Z	$Z_{11}-C_{11}$	$Z_{12}-C_{12}$	$Z_{13}-C_{13}$	$Z_{14}-C_{14}$	Z_0
x_1	y_{11}	y_{12}	y_{13}	y_{14}	$x_{B1}=I_{10}+f_{10}$
x_3	y_{21}	y_{22}	y_{23}	y_{24}	$x_{B2}=I_{20}+f_{20}$

x_{B1} ; $x_{B2} \notin \mathbb{Z}$

x_{B1} = Integral + fractional

Eg: $\frac{11}{2} = 5 + 1/2$

(say) $y_{24} = -\frac{5}{8} = -1 + 3/8$

$\boxed{\text{Note}}$ fractional part ≥ 0

→ choose the largest fractional value of B.V., let it be f_{K0}.

$$y_{11}x_1 + y_{12}x_2 + y_{13}x_3 + y_{14}x_4 = x_{B1}$$

↓ decompose

$$(I_{11}+f_{11})x_1 + (I_{12}+f_{12})x_2 + (I_{13}+f_{13})x_3 + (I_{14}+f_{14})x_4 = I_{10} + f_{10}$$

$$(I_{11}x_1 + I_{12}x_2 + I_{13}x_3 + I_{14}x_4) = f_{10} - f_{11}x_1 - f_{12}x_2 - f_{13}x_3 - f_{14}x_4$$

→ Choose the largest fractional value of B.V., let it be f_{K0}.

Note: fractional part

$$y_{11}x_1 + y_{12}x_2 + y_{13}x_3 + y_{14}x_4 = x_{B1}$$

↓ decompose

$$(I_{11}+f_{11})x_1 + (I_{12}+f_{12})x_2 + (I_{13}+f_{13})x_3 + (I_{14}+f_{14})x_4 = I_{10} + f_{10}$$

$$(I_{11}x_1 + I_{12}x_2 + I_{13}x_3 + I_{14}x_4) = f_{10} - f_{11}x_1 - f_{12}x_2 - f_{13}x_3 - f_{14}x_4$$

$$\boxed{f_{10} - f_{11}x_1 - f_{12}x_2 - f_{13}x_3 - f_{14}x_4 \le 0}$$ ← Gomory's constraint

on contary

$$f_{10} - f_{11}x_1 - f_{12}x_2 - f_{13}x_3 - f_{14}x_4 = h > 0$$

$$\therefore f_{10} = \underset{>0}{h} + (f_{11}x_1 + \cdots + f_{14}x_4) \quad > 1$$

↓

fractional ; $0 < f_{10} < 1$

Problem: Find the optimal solution of IPP

$$\text{Max } Z = x_1 + 2x_2$$

$$\text{s.t. } x_1 + x_2 \le 7$$

$$2x_1 \le 11 \quad ; \quad x_1, x_2 \ge 0 \; ; \text{ integers}$$

$$2x_2 \le 7$$

6 ≥ '

Step 1: Find optimal solⁿ using Simplex; associated LPP

Step 1: Find optimal ... we get;

C_B	B.V.	x_1	x_2	s_1	s_2	s_3	X_B
	Z	-1	-2	0	0	0	0
0	s_1	1	1	1	0	0	7
0	s_2	2	0	0	1	0	11
0	s_3	0	$\boxed{2}$	0	0	1	7
	Z	-1	0	0	0	1	7
0	s_1	$\boxed{1}$	0	1	0	$-1/2$	$7/2$
0	s_2	2	0	0	1	0	11
2	x_2	0	1	0	0	$1/2$	$7/2$
	Z	0	0	1	0	$1/2$	$21/2$
1	x_1	1	0	1	0	$-1/2$	$7/2$
0	s_2	0	0	-2	1	1	4
2	x_2	0	1	0	0	$1/2$	$7/2$

optimal

feasibility

$x_1 = \frac{7}{2} = 3 + 1/2 =$ Integral + fractional ≥ 0

$x_2 = \frac{7}{2} = 3 + 1/2 =$ Integral + fractional ≥ 0

2nd constraint (corresponding to x_2)

$$0 \cdot x_1 + 1 \cdot x_2 + 0 \cdot s_1 + 0 \cdot s_2 + 1/2 \cdot s_3 = 7/2$$

$= 3 + 1/2$

$\Downarrow$

Gomary's constraint

$$\boxed{f_{10} - f_{11}x_1 - f_{12}x_2 \cdots \leq 0}$$

solution

$$\therefore \quad \frac{1}{2} - \frac{1}{2} \cdot s_3 \leq 0 \longleftarrow \text{Gomary's Constraint}$$

Gomary's constraint is constructed against constraint which has max fractional part.
Here, x1 and x2 's fractional part is 1/2(same).
So, we can close any of them.

	x_2	0	1	0	0	1/2	7/2
	Z	0	0	1	0	1/2	21/2
1	x_1	1	0	1	0	−1/2	7/2
0	s_2	0	0	−2	1	1	4
2	x_2	0	1	0	0	1/2	7/2

optimal

feasibility

$$\boxed{f_{10} - f_{11}x_1 - f_{12}x_2 \cdots \le 0}$$

solution

Add.

$$\therefore \quad \frac{1}{2} - \frac{1}{2} \cdot s_3 \le 0 \longleftarrow \text{Gomory's Constraint}$$

$$-\frac{1}{2} \cdot s_3 \le -1/2$$

or $\boxed{\frac{1}{2} s_3 \geq 1/2}$

$$\frac{1}{2} s_3 - x_{G_1} = 1/2$$

$$\boxed{-\frac{1}{2} s_3 + x_{G_1} = -1/2}$$

Next Step: After finding Gomary Constraint; Add this constraint into optimal Simplex table.

B.V.	x_1	x_2	s_1	s_2	s_3	x_{G_1}	X_B
Z	0	0	1	0	1/2	0	21/2
x_1	1	0	1	0	−1/2	0	7/2
s_2	0	0	−2	1	1	0	4
x_2	0	1	0	0	1/2	0	7/2
x_{G_1}	0	0	0	0	−1/2	1	−1/2

Apply Dual Simplex method.

$$-\frac{1}{2} \cdot s_3 \le -1/2$$

or $\boxed{\frac{1}{2} s_3 \geq 1/2}$

$$\frac{1}{2} s_3 - x_{G_1} = 1/2$$

$$\boxed{-\frac{1}{2} s_3 + x_{G_1} = -1/2}$$

Next Step: After finding Gomary constraint; Add this constraint into optimal Simplex table.

B.V.	x_1	x_2	s_1	s_2	s_3	x_{G_1}	X_B
Z	0	0	1	0	1/2	0	21/2
x_1	1	0	1	0	-1/2	0	7/2
s_2	0	0	-2	1	1	0	4
x_2	0	1	0	0	1/2	0	7/2
x_{G_1}	0	0	0	0	$\boxed{-1/2}$	1	-1/2

Apply Dual simplex method.

Z	0	0	1	0	0	1	10
x_1	1	0	1	0	0	-1	4
s_2	0	0	-2	1	0	2	3
x_2	0	1	0	0	0	1	3
s_3	0	0	0	0	1	-2	1

$$\min\left|\frac{Z_j - C_j}{\text{scale} < 0}\right|$$

$\therefore$ $x_1 = 4$; $x_2 = 3$

$Z = 10$

Problem: Find the optimal solution of $\boxed{\text{IPP}}$

$$\text{Max } Z = x_1 + 4x_2$$

$$\text{s.t. } \left.\begin{aligned} 2x_1 + 4x_2 &\le 7 \\ 5x_1 + 3x_2 &\le 15 \end{aligned}\right\}$$

$$x_1, x_2 \ge 0 \quad ; \quad x_i - \text{integers}$$

Solution

Step 1: Find optimal solution of associated LPP. (simplex)

optimal simplex table is

C_B	B.V.	x_1	x_2	s_1	s_2	X_B
	Z	1	0	1	0	7
4	x_2	1/2	1	1/4	0	7/4
0	s_2	7/2	0	−3/4	1	39/4

$\therefore x_1 = 0 \; ; \; x_2 = \frac{7}{4} \; ; \; s_1 = 0 \; ; \; s_2 = \frac{39}{4}$

$= 9 + 3/4$

$\downarrow \quad \downarrow$

$I_2 \quad f_2$

$= 1 + 3/4$

$\downarrow \quad \downarrow$

$I_1 \quad f_1$

$\text{Max } (f_1, f_2) = 3/4$

Step 2: Construct Gomory' constraint

Construct Gomo

0	s_2	7/2	0	−3/4	1	39/4

f_1 f_2 f_2

Max $(f_1, f_2) = 3/4$

Step 2: Construct Gomory constraint

Construct Gomory constraint for s_2

$$x_{G_1} = -f_2 + f_{21}x_1 + f_{22}x_2 + f_{23}s_1 + f_{24}s_2$$

$$x_{G_1} = -\frac{3}{4} + \frac{1}{2}x_1 + 0.x_2 + \frac{1}{4}s_1 + 0.s_2$$

or $$x_{G_1} = -\frac{3}{4} + \frac{1}{2}x_1 + \frac{1}{4}s_1$$

$$-\frac{3}{4} + \frac{1}{2}x_1 + \frac{1}{4}s_1 - x_{G_1} = 0$$

$$\underbrace{\frac{1}{2}x_1 + \frac{1}{4}s_1 - x_{G_1} = 3/4}$$

$$\boxed{-\frac{1}{2}x_1 - \frac{1}{4}s_1 + x_{G_1} = -3/4}$$

$$\frac{7}{2} = 3 + 1/2$$

$$-3/4 = -1 + 1/4$$

Step 3: Add Gomory constraint

B.V.	x_1	x_2	s_1	s_2	x_{G_1}	X_B
	1	0	1	0	0	7
x_2	1/2	1	1/4	0	0	7/4
s_2	7/2	0	-3/4	1	0	39/4
x_{G_1}	-1/2	0	-1/4	0	1	-3/4
Z	0	0	1/2	0	2	11/2
x_2	0	1	0	0	1	1
s_2	0	0	-5/2	1	7	9/2
x_1	1	0	1/2	0	-2	3/2

-ve

Apply Dual simplex table

Integer ?

feasible.

$x_1 = 3/2$; $s_2 = 9/2$

$\frac{9}{2} = 4 + 1/2$

Step 4 Find Gomory's constraint-

Step 4 Find Gomory's constraint-

$\frac{9}{2} = 4 + 1/2 \rightarrow f_1 = 1/2$

$3/2 = 1 + 1/2 \rightarrow f_2 = 1/2$

$\max(f_1, f_2) = 1/2$

$$x_{G_2} = -\frac{1}{2} + \left(0.x_1 + 0.x_2 + \frac{1}{2} s_1 + 0.s_2 + 0.x_{G_1}\right)$$

$$x_{G_2} = -\frac{1}{2} + \frac{1}{2} s_1$$

or $\underbrace{-\frac{1}{2} s_1 + x_{G_2} = -1/2}$

C_B	B.V.	x_1	x_2	s_1	s_2	x_{G_1}	x_{G_2}	X_B
	Z	0	0	1/2	0	2	0	11/2
4	x_2	0	1	0	0	1	0	1
0	s_2	0	0	-5/2	1	7	0	9/2
1	x_1	1	0	1/2	0	-2	0	3/2
0	x_{G_2}	0	0	-1/2	0	0	1	-1/2

Again A1

	Z	0	0	1/2	0	2	0	11/2
4	x_2	0	1	0	0	1	0	1
0	s_2	0	0	-5/2	1	7	0	9/2
1	x_1	1	0	1/2	0	-2	0	3/2
0	x_{G_2}	0	0	-1/2	0	0	1	-1/2
	Z	0	0	0	0	2	1	5
4	x_2	0	1	0	0	1	0	1
0	s_2	0	0	0	1	7	-5	7
1	x_1	1	0	0	0	-2	1	1
0	s_1	0	0	1	0	0	-2	1

Again Apply Dual Simplex Method.

$x_1 = 1 ; x_2 = 1$

$Z = 5$

IPP.

Simplex Method

Post-Opt. Addition of Constraint

Gomory's constraint

Dual Simplex.

The end