Transportation Problem

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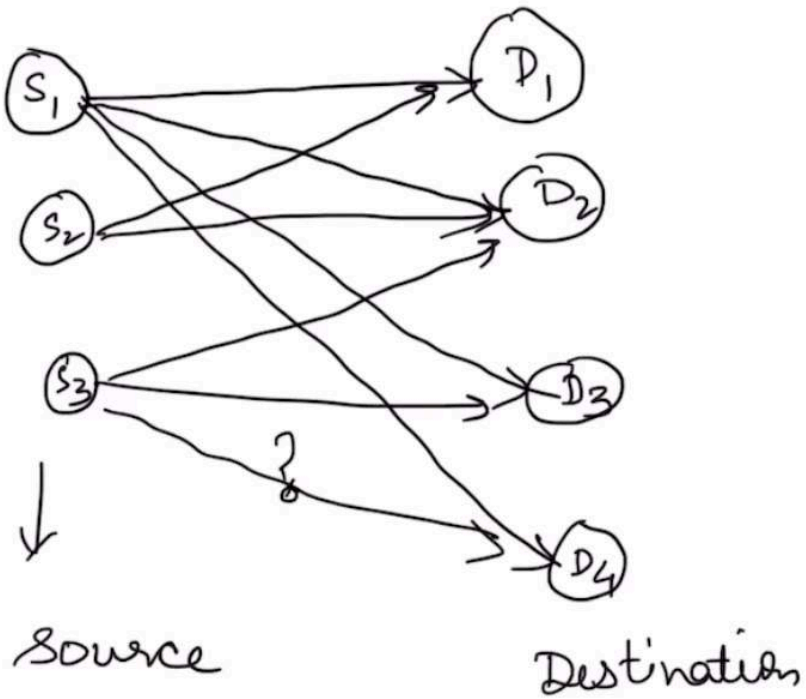


Example (Transportation Problem)

A dairy firm has two milk plants, daily milk production of 6 million litres and 9 million litres respectively. Each day, the firm must fulfil the needs of its three distribution centres which have the milk requirement of 7, 5, 3 million litres, respectively. Cost of transportation of 1 million litres of milk from each plant to each distribution centre is given in hundreds of rupees below. Formulate the LP model to minimize the transportation cost.

Plant\Distribution Centers	D1	D2	D3	Supply
P1	2 (x_{11})	3 (x_{12})	11 (x_{13})	6
P2	1 (x_{21})	9 (x_{22})	6 (x_{23})	9
Demand	7	5	3	

Formulation: Let x_{ij} be the units of commodity supplied from i^{th} plant to j^{th} destination

$1 \le i \le 2$
$1 \le j \le 3$

$$\text{Minimize } Z = 2x_{11} + 3x_{12} + 11x_{13} + x_{21} + 9x_{22} + 6x_{23}$$

s.t.

$$\left.\begin{aligned} x_{11} + x_{12} + x_{13} &\le 6 \\ x_{21} + x_{22} + x_{23} &\le 9 \end{aligned}\right\} \text{(source constraints)}$$

$$\left.\begin{aligned} x_{11} + x_{21} &\ge 7 \\ x_{12} + x_{22} &\ge 5 \\ x_{13} + x_{23} &\ge 3 \end{aligned}\right\} \text{(Demand constraints)}$$

$x_{ij} \ge 0 \quad \forall i$

(So T.P. can be Max. also) $\left[\begin{array}{l} \text{Max } Z = \sum\sum x_{ij} c_{ij} \\ \text{s.t. } \sum x_{ij} = a_i \\ \sum x_{ij} = b_j \; ; \; x_{ij} \geq 0 \end{array}\right]$

General Transportation Problem

$$\begin{array}{c|cccc|c} & D_1 & \cdots & & D_n & \\ \hline S_1 & x_{11} & \cdots & & x_{1n} & a_1 \\ \vdots & & & & & \vdots \\ S_m & x_{m1} & \cdots & & x_{mn} & a_m \\ \hline & b_1 & b_2 & \cdots & b_n & \end{array}$$

$1 \leq i \leq m$

$1 \leq j \leq n$

$$\text{Minimize } Z = \sum_{j=1}^{n} \sum_{i=1}^{m} x_{ij} c_{ij}$$

; c_{ij} is per unit cost of ith plant to j^{th} destination, sources

x_{ij} : units of commodity

s.t.

$$\left.\begin{array}{l} x_{11} + x_{12} + \cdots + x_{1n} \leq a_1 \\ x_{21} + x_{22} + \cdots + x_{2n} \leq a_2 \\ \vdots \\ x_{m1} + \cdots + x_{mn} \leq a_m \end{array}\right\} \text{source constraints}$$

$$\left.\begin{array}{l} x_{11} + x_{21} + \cdots + x_{m1} \geq b_1 \\ \vdots \\ x_{1n} + \cdots + x_{mn} \geq b_n \end{array}\right\} \text{Demand constraints}$$

$$x_{ij} \geq 0$$

Or

$$\text{Minimize } Z = \sum_{j=1}^{n} \sum_{i=1}^{m} x_{ij} c_{ij}$$

$$\sum_{j=1}^{n} x_{ij} \leq a_i \qquad (i = 1, \ldots, m) \rightarrow \text{source constraint}$$

$$\sum_{i=1}^{m} x_{ij} \geq b_j$$

$$x_{ij} \geq 0$$

Standard T.P.

$$\min Z = \sum_{j} \sum_{i} x_{ij} c_{ij}$$

s.t.

$$\left.\begin{array}{l} \sum_{j=1}^{n} x_{ij} = a_i \\ \sum_{i=1}^{m} x_{ij} = b_j \end{array}\right\}$$

$$x_{ij} \geq 0$$

Remarks:

1. Necessary and sufficient condition to have feasible solution is total supply = total demand.

$\sum a_i = \sum b_j$ | T.P. → Balanced T.P.

if $\sum a_i \neq \sum b_j$ then T.P. → Unbalanced T.P.

2. There is always an existence of Optimum Solution to transportation problem.

T.P is LPP

feasible infeasible

we always have feasible

3. The number of Basic variables are m+n-1.

no. of sources

no. of destinations !!

For proofs, watch video 58 MathPod

Theorem: (Existence of feasible solution) A necessary and sufficient condition for the existence of a feasible solution to the general transportation problem is $\underbrace{\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j}$ (*) where $a_i \to i^{th}$ supply, $b_j \to j^{th}$ demand

Proof: Necessary: Assume that T.P. has a feasible solution.

$$\text{Min } Z = \sum_{j=1}^{n}\sum_{i=1}^{m} x_{ij} c_{ij}$$

$$\text{s.t. } \sum_{j=1}^{n} x_{ij} = a_i \ \forall i \longrightarrow$$

$$\sum_{i=1}^{m} x_{ij} = b_j \ \forall j \longrightarrow$$

$$x_{ij} \geq 0 \ \forall i,j$$

then we need to prove $\sum_i a_i = \sum_j b_j$

$\sum_{j=1}^{n} x_{ij} = a_i$ (source constraint) ; $\sum_{i=1}^{m} x_{ij} = b_j$ (destination)

$$\sum_{j=1}^{n}\sum_{i=1}^{m} x_{ij} = \sum_{i=1}^{m} a_i \quad ; \quad \sum_{j=1}^{n}\sum_{i=1}^{m} x_{ij} = \sum_{j=1}^{n} b_j$$

$$\Rightarrow \boxed{\sum a_i = \sum b_j}$$

Sufficient: Given $\sum_i a_i = \sum_j b_j = \beta$ (say)

To show; there exist a feasible solution; Choose $x_{ij} = \frac{a_i b_j}{\beta} \ \forall i,j$

→ source constraint;

$$\sum_{j=1}^{n} x_{ij} = \sum_{j=1}^{n} \frac{a_i b_j}{\beta} = \frac{a_i}{\beta}\sum_{j=1}^{n} b_j = \frac{a_i}{\beta}(\beta) = a_i$$

→ destination constraint

$$\sum_{i=1}^{m} x_{ij} = \sum_{i=1}^{m} \frac{a_i b_j}{\beta} = \frac{b_j}{\beta}\sum_{i=1}^{m} a_i = \frac{b_j}{\beta}\cdot\beta = b_j$$

Eg:

	D_1	D_2	D_3	Supply
S_1	x_{11}	x_{12}	x_{13}	a_1
S_2	x_{21}	x_{22}	x_{23}	a_2
Demand	b_1	b_2	b_3	

$x^* = (x_{11}, x_{12}, x_{13}, \ldots x_{23})$

source constr.

$x_{11} + x_{12} + x_{13} = a_1$

$x_{21} + x_{22} + x_{23} = a_2$

add

& $x_{11} + x_{21} = b_1$

$x_{12} + x_{22} = b_2$

$x_{13} + x_{23} = b_3$

add

$\Rightarrow x_{11} + x_{12} + x_{13} + x_{21} + x_{22} + x_{23} = a_1 + a_2 = \sum_{i=1}^{2} a_i$ (1)

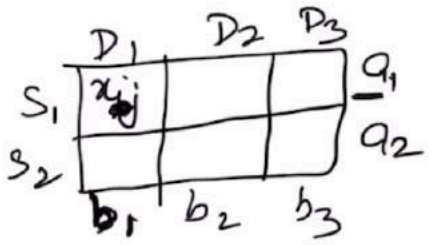
$x_{11} + x_{21} + x_{12} + x_{22} + x_{13} + x_{23} = b_1 + b_2 + b_3 = \sum_{j=1}^{3} b_j$ (2)

$\Rightarrow$ L.h.s of (1) = L.h.s of (2)

R.h.s " " = R.h.s " " (2)

$\sum a_i = \sum b_j$

Corollary: (Existence of an Optimum Solution) There always exists an optimum solution to a T.P.



$a_1 > b_1$

$x_{ij} \le \min\{\text{supply}, \text{demand}\}$

$a_1 < b_1$

Proof: We know; for feasible solution

$$\sum_i a_i = \sum_j b_j$$

and $$0 \le x_{ij} \le \min(a_i, b_j)$$

$\downarrow$

decision variables are bounded

$\Rightarrow$ feasible region is closed & bound

By Fundamental thm. of LPP

$\Rightarrow$ optimal solution exist.

Theorem (Basic Feasible Solution)

The no. of basic variables of the T.P. at any stage of feasible solution is $m+n-1$.

Pf: $\sum_{j=1}^{n} x_{ij} = a_i$ $\qquad$ $\sum_{i=1}^{m} x_{ij} = b_j$

sum all source $\qquad$ sum all demand leaving the last

$$\sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij} = \sum_{i=1}^{m} a_i \quad (1) \qquad \sum_{j=1}^{n-1}\sum_{i=1}^{m} x_{ij} = \sum_{j=1}^{n-1} b_j \quad (2)$$

$$\sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij} - \sum_{j=1}^{n-1}\sum_{i=1}^{m} x_{ij} = \sum_{i=1}^{m} a_i - \sum_{j=1}^{n-1} b_j$$

$$\sum_{i=1}^{m} x_{in} = b_n$$

$$\boxed{x_{1n} + x_{2n} + \cdots + x_{mn} = b_n}$$

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$$

	D_1	D_2	D_3
S_1			
S_2			

$$\left.\begin{aligned} x_{11}+x_{12}+x_{13} &= a_1 \\ x_{21}+x_{22}+x_{23} &= a_2 \end{aligned}\right] \text{Source Const.}$$

$$\left.\begin{aligned} x_{11}+x_{21} &= b_1 \\ x_{12}+x_{22} &= b_2 \\ x_{13}+x_{23} &= b_3 \end{aligned}\right] \text{Demand}$$

$\rightarrow$ leave last constraint

$$x_{11}+x_{12}+x_{13}+x_{21}+x_{22}+x_{23} = a_1+a_2 \quad (1)$$

sum all demand constraint leaving the last

$$x_{11}+x_{21}+x_{12}+x_{22} = b_1+b_2$$

We know $\sum a_i = \sum b_j \quad (2)$

$$a_1+a_2 = b_1+b_2+b_3$$

$$\cancel{x_{11}}+\cancel{x_{12}}+x_{13}+\cancel{x_{21}}+\cancel{x_{22}}+\cancel{x_{23}} = \cancel{x_{11}}+\cancel{x_{21}}+\cancel{x_{12}}+\cancel{x_{22}}+b_1+b_2+b_3$$

$$\Rightarrow x_{13}+x_{23} = b_3$$

L.D.

$$\text{Min } Z = \sum_j \sum_i x_{ij} c_{ij}$$

$$\sum_{j=1}^{n} x_{ij} = a_i \quad ; 1 \le i \le m$$

$$\sum_{i=1}^{m} x_{ij} = b_j \quad ; 1 \le j \le n$$

Total constraints $m+n$

Total variables mn

$$A = \begin{pmatrix} & \\ & \end{pmatrix}_{m+n,\, mn}$$

$m+n \rightarrow$ B.V. ✗

$m+n-1 \rightarrow$ B.V.

Rank $A \leq m+n$

Rank $A = m+n-1$

To solve Transportation Problem (Minimization)

Minimize $z = \sum_{j=1}^{n} \sum_{i=1}^{m} c_{ij} x_{ij}$

s.t. $\sum_{j=1}^{n} x_{ij} = a_i \quad (i=1,\dots,m)$

$\sum_{i=1}^{m} x_{ij} = b_j \quad (j=1,\dots,n)$

$x_{ij} \geq 0$

Consider m-sources and n-destinations in a T.P.

and $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$ (It is a Balanced T.P.) ← (*)

that is; supply = demand

Remark: Suppose Given T.P. is unbalanced

Eg:

	D_1	D_2	D_3	Availability
S_1				20
S_2				10
Demand	10	20	20	$\sum a_i \neq \sum b_j$

$\sum b_j = 50$, $\sum a_i = 30$ → Unbalanced T.P.

→ if Demand > supply then add dummy supply

if. Supply > Demand → demand

→ Dummy source

	D_1	D_2	D_3	
S_1				20
S_2				10
S_3	0	0	0	20
	10	20	20	50 = 50

→ Now it is Balanced.

Example 1: Solve the T.P with 3 sources and 2 destinations and with cost per unit of supply is given in table with availability & demand.

	D_1	D_2	↓ Availability
S_1	2	3	20
S_2	1	5	10
S_3	7	4	30
Demand →	10	50	$\boxed{60}$ ← Balanced.

Possible Solution: Formulation of above T.P.:

$$\text{Min } Z = 2x_{11} + 3x_{12} + x_{21} + 5x_{22} + 7x_{31} + 4x_{32}$$

s.t.

$$\left.\begin{array}{ll} \left.\begin{array}{l} x_{11} + x_{12} = 20 \\ x_{21} + x_{22} = 10 \\ x_{31} + x_{32} = 30 \end{array}\right] & \text{source} \\ \left.\begin{array}{l} x_{11} + x_{21} + x_{31} = 10 \\ x_{12} + x_{22} + x_{32} = 50 \end{array}\right] & \text{Destinations} \end{array}\right\} \text{5-constraints}$$

$x_{ij} \geq 0$

It is LPP

$\text{Min } Z = CX$
s.t. $AX = b$
$X \geq 0$

In Matrix form above constraints are written as;

$$\begin{array}{c} \begin{array}{cccccc} x_{11} & x_{12} & x_{21} & x_{22} & x_{31} & x_{32} \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix} \end{array} \begin{pmatrix} x_{11} \\ x_{12} \\ x_{21} \\ x_{22} \\ x_{31} \\ x_{32} \end{pmatrix} = \begin{pmatrix} 20 \\ 10 \\ 30 \\ 10 \\ 50 \end{pmatrix}$$

or $AX = b$

B.V.

$m+n-1$
$3+2-1$
$= 4$

Normal simplex method requires a lot of iterations

Suppose we solve this problem with two-phase,

Add artificial variables in all constraints; (in this problem; we need to $a_1, a_2, \ldots, a_5$ in all constraints (resp)

Phase-I

B.V.	x_{11}	x_{12}	x_{21}	x_{22}	x_{31}	x_{32}	$a_1 \ldots a_5$	XB
a_1								20
a_2								10
a_3								30
a_4								10
a_5								50

↓ Here to find to feasible soln; we require atleast 5-iterations

Simplex format

Phase-II

B.V.

x_{11}
x_{31}
x_{32}
x_{21}

→ feasible soln.

To solve any balanced T.P. ;

Step I : Find Initial BFS

Step II : Check optimality ; $Z_{ij} - C_{ij}$

(for minimization T.P. $Z_{ij} - C_{ij} \leq 0$

or Maximization T.P. $Z_{ij} - C_{ij} \geq 0$

or

we convert Max. T.P. in Min T.P and follow minimization criteria

	$Z_{ij} - C_{ij}$
x_{11}	
x_{21}	
x_{31}	

Step I :

Find Initial BFS

1. North-West Corner Rule (NW-Rule)
2. Least-Cost Method (LCM)
3. Vogel's Approximation Method (VAM)

Step II : $Z_{ij} - C_{ij}$ (dual of T.P. (Given))

North-West Corner Method

Find initial BFS of T.P. given in following table

	D_1	D_2	D_3	D_4	
S_1	10	2	20	11	15
S_2	12	7	9	20	25
S_3	4	14	16	18	10
	5	15	15	15	

Solution:

	D_1	D_2	D_3	D_4	Availability
S_1	10 / 5	2 / 10	20	11	~~15~~ 10
S_2	12	7 / 5	9 / 15	20 / 5	~~25~~ ~~20~~ 5
S_3	4	14	16	18 / 10	10
Demand	5	~~15~~ 5	15	~~15~~ 10	50

North-west cell

$x_{11} = ?$

$S_1 \to D_1$

This is a Balanced T.P.

$0 \le x_{ij} \le \min\{a_i, b_j\}$

$$\boxed{X_B = B.V.} = (x_{11}, x_{12}, x_{22}, x_{23}, x_{24}, x_{34}) = (5, 10, 5, 15, 5, 10)$$

B.V. $\to m+n-1 = 3+4-1 = 6$

$$N.B.V. = (x_{13}, x_{14}, x_{21}, x_{31}, x_{32}, x_{33}) = (0, \dots, 0)$$

$(m-1)(n-1) = 2(3) = 6$

$$\text{Min } Z = \sum\sum x_{ij} c_{ij}$$

$$\boxed{\begin{aligned} Z &= C_B X_B \\ &= (5)(10) + (10)(2) + (5)(7) + (15)(9) + 5(20) + 10(18) \\ &= 520 \end{aligned}}$$

Recall: Total variables in T.P. mn

Total B.V. $= m+n-1$

Total Non B.V. $= mn - (m+n-1)$

$= (m-1)(n-1)$

$\therefore$ No. of BFS

$BFS \le {}^{mn}C_{m+n-1}$

In usual LPP

$\boxed{BFS \le {}^{n}C_{m}}$

More Examples of North West Corner Method

: In all following questions find initial BFS using north-west corner method.

Q1:

	D_1	D_2	D_3	Availability
S_1	2	1	4	10
S_2	6	3	2	15
S_3	4	2	3	20
Demand	13	18	14	45

→ Balanced

$\Sigma a_i = \Sigma b_i$

Solution:

	D_1	D_2	D_3		
S_1	(10) [2]	[1]	[4]	10	
S_2	(3) [6]	(12) [3]	[2]	~~15~~	12
S_3	[4]	(6) [3]	(14) [3]	~~20~~	14
	~~13~~ 3	~~18~~ 6	14		

(10) [2]		
(3) [6]	(12) [3]	
	(6) [3]	(14) [3]

$3 + 3 - 1 = 5$

$m + n - 1$

B.V. $= (x_{11}, x_{21}, x_{22}, x_{32}, x_{33})$

5 B.V.

$$\boxed{Z = 20 + 18 + 36 + 18 + 42}$$

at Initial BFS.

Q2:

	D_1	D_2	D_3	
S_1	2	1	4	10
S_2	6	3	2	20
S_3	4	2	3	10
	10	15	15	40

Solution:

	D_1	D_2	D_3		
S_1	(10)			10	
S_2		(15)	(5)	~~20~~	5
S_3			(10)	10	
	10	15	~~15~~ 10		

Here note; no. of Basic cells is one less than required no. which is $m+n-1 = 5$

∵ one step we have exhausted supply/demand both

⇒ This is a degenerate BFS

	D_1	D_2	D_3		
S_1	(10)	(0)		10	0
S_2		(15)	(5)	20	5
S_3			(10)	10	
	10	15	15		

5 B.V.

or

	D_1	D_2	D_3	
S_1	(10)			10
S_2	(0)	(15)	(5)	20
S_3			(10)	10
	~~10~~ 0	15	15	

Q3:

	D_1	D_2	D_3	
S_1	2	1	4	10
S_2	6	3	2	10
S_3	4	2	3	10
	10	10	10	30

Solution:

	D_1	D_2	D_3	
S_1	(10)			10
S_2		(10)		10
S_3			(10)	10
	~~10~~	10	10	30

B.V. 2 less than $m+n-1$

count we got 3 → B.V.

so this is degenerate

so one of possible B.V. soluti

(10)	(0)		~~10~~	0
	(10)	(0)	~~10~~	0
		(10)	10	
10	10	10		

Least Cost Method (LCM)

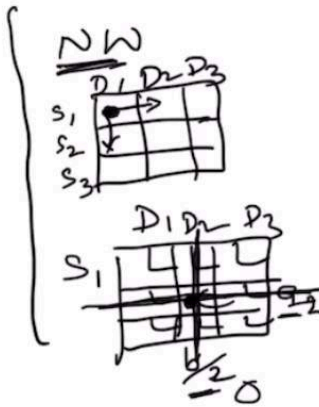
In LCM, we take into account the minimum unit cost.

Step 1: Determine the smallest cost in the cost matrix of T.P. and corresponding to smallest cost do the allocation

$$x_{ij} = \min\{a_i, b_j\}$$

Step 2: If $x_{ij} = a_i$; then cross ith row ;
otherwise if $x_{ij} = b_j$; then cross jth column ;
identify remaining a_i or b_j whichever be the case.

Step 3: Repeat step 1 & step 2.



1. Find initial BFS of T.P. using LCM

	D_1	D_2	D_3	Supply
O_1	7	1	4	10
O_2	6	3	2	15
O_3	4	5	3	20
Demand →	13	18	14	$\boxed{\Sigma a_i = \Sigma b_j}$

<u>Solution</u>: <u>Step 1</u>: find <u>least</u> cost in the matrix of T.P.

$$x_{ij} = \min\{a_i, b_j\}$$

	D_1	D_2	D_3	
O_1	[7]	10 [1]	[4]	10
O_2	[6]	1 [3]	14 [2]	~~15~~ 1
O_3	13 [4]	7 [5]	[3]	20
	13	~~18~~ ~~8~~ 7	14	

$B.V. = \{x_{12}, x_{22}, x_{23}, x_{31}, x_{32}\}$

$m + n - 1 \rightarrow B.V.$

$3 + 3 - 1 = 5$

all other variables are non-B.V.

$$Z_{min}(IBFS) = \underbrace{(10)(1) + (1)(3) + (14)(2) + (13)(4) + 7(5)}_{C_B X_B}$$

2. Find the initial BFS using LCM. (Breaking Tie)

	D_1	D_2	D_3	Supply
O_1	2	1	4	10
O_2	6	3	2	15
O_3	4	2	3	20
Demand	13	18	14	$\boxed{\Sigma a_i = \Sigma b_j}$

There is a tie in minimum cost; so to break tie; we allocate max x_{ij} (allowed in tie cells)

so either {14, 8}

so we choose x_{23} cell first ∵ of cost tie.

Solution:

	D_1	D_2	D_3	
O_1	2	1 (10)	4	10
O_2	6	3	2 (14)	15
O_3	4	2 (8)	3	20
	13	~~18~~ 8	14	

	D_1	D_2	D_3	
O_1	2	1 (10)	4	10
O_2	6 (1)	3	2 (14)	~~15~~ 1
O_3	4 (12)	2 (8)	3	~~20~~ 12
	~~13~~ 1	~~18~~ 8	~~14~~	

B.V. $= (x_{12}, x_{21}, x_{23}, x_{31}, x_{32})$

$= (10, 1, 14, 12, 8)$

3. Find initial BFS using LCM to T.P.

Degenerate BFS

	D_1	D_2	D_3	D_4	Supply
S_1	10	2	20	11	15
S_2	12	7	9	20	25
S_3	4	14	16	18	10
Demand	5	15	15	15	

$\boxed{\sum a_i = \sum b_j}$

Solution:

	D_1	D_2	D_3	D_4		
S_1	10	2 (15)	20	11	~~15~~	
S_2	12	7	9 (15)	20 (10)	~~25~~	10
S_3	4 (5)	14	16	18 (5)	~~10~~	5
	5	~~15~~	~~15~~	15		

$B.V. = m+n-1$

$= 3+4-1$

$= 6$

But we have got 5 B.V.'s.

$\boxed{Z = C_B X_B}$

Degenerate B.F.S :→ any one of B.V.' takes the value zero.

~~To avoid~~ To know Deg. B.V.

	D_1	D_2	D_3	D_4		
S_1	10	2 (15)	20	11	~~15~~	
S_2	12	7 (0)	9 (15)	20 (10)	~~25~~	10
S_3	4 (5)	14	16	18 (5)	~~10~~	5
	5	~~15~~	15	15		

← we have identified the deg. BFS.

4. Obtain initial BFS using LCM to the T.P.

	D_1	D_2	D_3	D_4	Capacity
O_1	1	2	3	4	6
O_2	4	3	2	0	8
O_3	0	2	2	1	10
Demand	4	6	8	6	

Tie.

Solution:

	D_1	D_2	D_3	D_4		
O_1	1	2 (6)	3 (0)	4	~~6~~	0
O_2	4	3	2 (2)	0 (6)	~~8~~	2
O_3	0 (4)	2	2 (6)	1	~~10~~	6
	~~4~~	~~6~~	8	~~6~~		

$B.V. = (x_{11}, x_{12}, x_{23}, x_{24}, x_{31}, x_{33})$

(degenerate) 6 0 2 6 4 6

$Z = 6(2) + 0(3) + 2(2) + 6(0) + 4(0) + 6(2)$

$= 12 + 4 + 12 = \underline{28}$

Vogel's Approximation Method (VAM) (Penalty Method)

Penalty: The difference of cost per unit; that occur as penalty if least cost cell is not selected.

Step 1: Find the cells having smallest and next to smallest cost in each row and each column; and write the difference (called penalty) along the side of the table in row/column penalty.

Step 2: Select row/column with maximum penalty and find cell that has least cost in selected row/column. Allocate x_{ij} in that cell.

- If there is a tie in penalty choose least cost available; if there is a tie in least cost then allocate maximum possible x_{ij} (where $x_{ij} = \min(a_i, b_j)$)

Step 3: Adjust supply/demand and repeat step 1 & step 2.

Eg:

	D_1	D_2	D_3	D_4	Penalty
S_1	4	6	8	10	(2)
S_2	5	7	4	6	(1)
S_3	2	6	3	1	(1)
Penalty =	(2)	(0)	(1)	(5)	

2	1	3		(5)
0				(1)
3				(2)
(5)	(3)	(1)		

$x_{ij} =$

Example 1 :→ Find initial BFS using VAM to given T.P.

	D_1	D_2	D_3	D_4	Supply
S_1	19	30	50	10	7
S_2	70	30	40	60	9
S_3	40	8	70	20	18
Demand	5	8	7	14	

Solution :

$x_{ij} = \min\{a_i, b_j\}$

	D_1	D_2	D_3	D_4		
S_1	19 (5)	30	50	10 (2)	~~7~~	2
S_2	70	30	40 (7)	60 (2)	9	
S_3	40	8 (8)	70	20 (10)	~~18~~	~~10~~
	~~5~~	~~8~~	~~7~~	~~14~~ ~~4~~ 2		

Supply-demand

Penalties (Difference of least & next least)

9	9	40	40
10	20	20	20
12	20	50	—

Penalties (Difference of least & next least cost)

21	22↑	10	10
21↑	—	10	10
—	—	10	10
—	—	10	50↑

$Z_{\text{at IBFS (VAM)}}$

$= 5(19) + 2(10) + 7(40) + 2(60) + 8(8) + 10(20)$

Example 2: Find initial BFS to T.P. using VAM

Mills

Sources	M_1	M_2	M_3	M_4	
S_1	3	1	7	4	300
S_2	2	6	5	9	400
S_3	8	3	3	2	500
	250	350	400	200	1200

← Balanced $\sum a_i = \sum b_j$

$m+n-1$ Basic Cells.

solution:→

3	1 (300)	7	4	~~300~~	(2)	(3)	—	—
2 (250)	6	5 (150)	9	150 ✓ ~~400~~	(3)	(1)	(1)	(1)
8	3 (50)	3 (250)	2 (200)	~~500~~ ~~300~~ 250 ✓	(1)	(1)	(1)	(0)
~~250~~	~~350~~ ~~50~~	400	~~200~~					

Penalty:

(1)	(2)	(2)	(2)
—	(2)	(2)	(2)
—	(3)	(2)	(7) ↑
—	(3) ↑	(2)	—

$x_{12} = 300$; $x_{21} = 250$; $x_{22} = 150$;
$x_{32} = 50$; $x_{33} = 250$
$x_{34} = 200$

$Z = 300(1) + 250(2) + 150(5) + 50(3) + 250(3) + 200(2)$

at Ip

Example 3: Find initial BFS (degenerate) from the T.P.; using VAM

	D1	D2	D3	
S1	2	1	4	10
S2	6	3	2	20
S3	4	2	3	10
	10	15	15	

Solution:

10	2	1	4	~~10~~
	6	5 3	15 2	~~20~~ 5
	4	10 2	3	10
	~~10~~	15	15	

Penalty:

(2)↑	(1)	(1)
—	(1)↑	(1)↑

Penalty

(1)	—
(1)	(1)
(1)	(1)

In this e.g.

The no. of Basic cells < $m+n-1$

$3+3-1=5$

$\Rightarrow$ BFS is degenerate

To identify all Basic Variable (including degenerate)

					Penalty
10	2	0 1	4	~~0~~ ~~10~~	(1)(5)
	6	5 3	15 2	~~20~~ 5	(1)(1)(1)
	4	10 2	3	10	(1)(1)(1)
	10	~~5~~ 15	15		

Penalty

(2)↑	(1)	(1)
—	(1)	(1)
—	(1)	(1)

$x_{11}=10$; $x_{12}=0$

$x_{22}=5$; $x_{23}=15$

$x_{32}=10$

$\therefore$ We got $m+n-1$ B.V.

$Z = 10(2)+5(3)+15(2)+10(2)+0(1)$

Unbalanced Transportation Problem

Saturday, 11 April 2020 9:18 PM

T.P. Minimize $Z = \sum_j \sum_i c_{ij} x_{ij}$

s.t. $\sum_{j=1}^{n} x_{ij} = a_i \quad ; 1 \le i \le m$

$\sum_{i=1}^{m} x_{ij} = b_j \quad ; 1 \le j \le n$

$x_{ij} \geq 0$

	D_1	D_2	---	D_n	Supply
S_1					a_1
S_2					a_2
$\vdots$					$\vdots$
S_m					a_m
Demand	b_1	b_2	----	b_n	$\boxed{\Sigma a_i = \Sigma b_j}$

Balanced T.P. : $\sum_i a_i = \sum_j b_j$ i.e. Total supply = Total Demand

Unbalanced T.P. : $\sum_i a_i \neq \sum_j b_j$

so either $\sum_i a_i > \sum_j b_j$ or $\sum_i a_i < \sum_j b_j$

Supply > Demand Supply < Demand

s.t. $\sum_{j=1}^{n} x_{ij} = a_i \quad ; 1 \le i \le m$

$\sum_{i=1}^{m} x_{ij} = b_j \quad ; 1 \le j \le n$

$x_{ij} \geq 0$

s_m | a_m

Demand $b_1 \quad b_2 \quad \cdots \quad b_n$ | $\boxed{\sum a_i = \sum b_j}$

Balanced T.P. : $\sum_i a_i = \sum_j b_j$ i.e. Total supply = Total Demand

Unbalanced T.P. : $\sum_i a_i \neq \sum_j b_j$

so either $\sum_i a_i > \sum_j b_j$ or $\sum_i a_i < \sum_j b_j$

Supply > Demand (1) Supply < Demand (2)

Case (1) : We add a dummy Demand with cost involved as '0'

Case (2) : We add a dummy source (—— d)

Find the initial BFS to the T.P. using

Example: Find the initial BFS to the T.P. using

(i) NW

(ii) LCM

(iii) VAM

	D1	D2	D3	D4	Supply
S1	20	25	28	31	200
S2	32	28	32	41	180
S3	18	35	24	32	110
Demand	150	40	180	170	

$\boxed{\sum a_i \neq \sum b_j}$

solution:

(i) Initial BFS using NW

Demand > Supply

	D1	D2	D3	D4	Supply
S1	20: 150	25: 40	28: 10	31	~~200~~ ~~50~~ ~~10~~
S2	32	28	32: 170	41: 10	~~180~~ 10
S3	18	35	24	32: 110	110
Dummy Source S4	0	0	0	0: 50	50
Demand	~~150~~	~~40~~	~~180~~ ~~170~~	~~170~~ 160	

$\sum a_i = \sum b_j$ if the T.P. is balanced then we can always find feasible

(ii) Initial BFS using LCM of Unbalanced T.P;

add dummy supply;

	D1	D2	D3	D4	
S_1	20	25 (40)	28 (160)	31	~~200~~ ~~160~~
S_2	32	28	32 (10)	41 (170)	180
S_3	18 (100)	35	24 (10)	32	~~110~~ ~~10~~
S_4	0 (50)	0	0	0	50
	~~150~~ ~~100~~	~~40~~	~~180~~ ~~170~~ ~~10~~	170	

→ Dummy supply

$$Z = \sum x_{ij} c_{ij}$$

$$= 14,810$$

(iii) Initial BFS using VAM of the given Unbalanced T.P.

	D1	D2	D3	D4		Penalties
S1	40 [20]	[25]	40 [28]	120 [31]	~~160~~ 40 ~~200~~ ✓	(5) (5) (5) (3) (3)
S2	[32]	40 [28]	140 [32]	[31]	180 ✓	(4) (4) (4) (4) (4) Tie.
S3	110 [18]	[35]	[24]	[32]	~~110~~ ✓	(6) (6) — — —
Dummy	[0]	[0]	[0]	50 [0]	~~50~~ ✓	(0) — — — —
	~~150~~ ~~40~~	40 ✓	~~180~~ ✓ 140	~~170~~ ✓ ~~120~~		

Penalties

(18)	(25)	(24)	(31)
(2)	(3)	(4)	(1)
(12)	(3)	(4)	(0)
—	(3)	(4)	(10)
—	(3)	(4)	—

Tie.

$$Z = 40(20) + 40(28) + 120(31) + 40(28) + 140(32) + 110(18) + 50(0)$$

$$= 13,220$$

Dual of a transportation Problem and its Application to Zj-Cj

Saturday, 11 April 2020 10:05 PM

① What is Dual of T.P.

② How dual variables are associated with $Z_j - C_j$

[$Z_j - C_j \geq 0$ for Max (optimality condition]

$Z_j - C_j \leq 0$ for Min ("

m-sources; n-destination

T.P. (Primal)

Minimize $Z = \sum_{j=1}^{n} \sum_{i=1}^{m} x_{ij} C_{ij}$

s.t. $\sum_{j=1}^{n} x_{ij} = a_i \; ; \; 1 \leq i \leq m$

$\sum_{i=1}^{m} x_{ij} = b_j \; ; \; 1 \leq j \leq n$

$x_{ij} \geq 0$

Dual:

Maximize $W = \sum a_i u_i + \sum b_j v_j$

s.t.

$u_i + v_j \leq C_{ij}$

u_i, v_j are unrestricted in sign

Eg: consider a 2-source and 3-destination T.P.

$$\begin{array}{c|ccc|c} & D_1 & D_2 & D_3 & \text{supply} \\ \hline S_1 & c_{11} & c_{12} & c_{13} & a_1 \\ S_2 & c_{21} & c_{22} & c_{23} & a_2 \\ \text{Demand} & b_1 & b_2 & b_3 & \end{array}$$

x_{ij} : units of commodity transported from S_i to D_j

Primal T.P:

Min Z =

s.t.

$$\left.\begin{aligned} x_{11} + x_{12} + x_{13} &= a_1 \\ x_{21} + x_{22} + x_{23} &= a_2 \\ x_{11} + x_{21} &= b_1 \\ x_{12} + x_{22} &= b_2 \\ x_{13} + x_{23} &= b_3 \end{aligned}\right\}$$

2-sources

5-constraits.

3-destination

five dual var.

$$x_{ij} \geq 0$$

(constraints $AX = b$)

$$\begin{array}{c} \begin{array}{cccccc} x_{11} & x_{12} & x_{13} & x_{21} & x_{22} & x_{23} \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix} \end{array} \begin{pmatrix} x_{11} \\ x_{12} \\ x_{13} \\ x_{21} \\ x_{22} \\ x_{23} \end{pmatrix}$$

(constraints $A^T Y = C^T$)

$$\underbrace{\begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 = u_1 \\ y_2 = u_2 \\ y_3 = v_1 \\ y_4 = v_2 \\ y_5 = v_3 \end{pmatrix}} = \begin{pmatrix} c_{11} \\ c_{12} \\ c_{13} \\ c_{21} \\ c_{22} \\ c_{23} \end{pmatrix}$$

after mult. of matrices

$$\begin{pmatrix} u_1 + v_1 \\ u_1 + v_2 \\ u_1 + v_3 \\ u_2 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \end{pmatrix} = \begin{pmatrix} c_{11} \\ c_{12} \\ \vdots \\ c_{23} \end{pmatrix}$$

symbol | inequality

Application of Dual variables

Recall: Simplex Method

	B.V.	$Z_j - C_j$			X_B
	s_1	0	0	$B^{-1}A_j$	•
foreg.	x_1	0	0		•
	x_2	0			•

$$Z_j - C_j = C_B(B^{-1}A_j) - C_j$$

?

In T.P.

A Text Mode Lasso Select Insert Space

x_2]

In T.P.

So we know

T.P:

$$z_{ij} - c_{ij} = C_B (B^{-1} A_j) - c_{ij}$$

Dual variable

$$= (u_1\ u_2 \cdots u_m\ v_1 \cdots v_n)\ A_j - c_{ij}$$

$W = C_B B^{-1}$

$$= (u_1\ u_2 \cdots u_m\ v_1 \cdots v_n) \begin{pmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} - c_{ij}$$

$\rightarrow$ i^{th} source

$\rightarrow$ j^{th} dest.

multiply

$$= (u_i + v_j) - c_{ij}$$

$$\boxed{z_{ij} - c_{ij} = (u_i + v_j) - c_{ij}}$$

multiply

$$= (u_i + v_j) - c_{ij}$$

$$\boxed{z_{ij} - c_{ij} = (u_i + v_j) - c_{ij}}$$

<u>Result</u>: For Basic variables ; $z_{ij} - c_{ij} = 0$

$$\Rightarrow z_{ij} = c_{ij}$$

$$\Rightarrow u_i + v_j = c_{ij}$$

<u>To Find</u> For <u>Non. B.V.</u>

$$z_{ij} - c_{ij} = (u_i + v_j) - c_{ij} = ?$$

For min (optimal) $z_{ij} - c_{ij} \leq 0$; if this does not hold then we look for entering & leaving variable

Optimal Solution of a Transportation Problem

Saturday, 11 April 2020 11:00 PM

Points to Remember

(i) Balanced | Unbalanced T.P.

(ii) Initial BFS (NW, LCM, VAM)

(iii) Degenerate | Non-Degenerate BFS (Count of BFS = $m+n-1$)

(iv) Dual variables u_i, v_j

(v) $z_{ij} - c_{ij} = u_i + v_j - c_{ij}$

[For B.V. $z_{ij} - c_{ij} = 0$
$\Rightarrow u_i + v_j = c_{ij}$]

(vi) In primal T.P.; we have total constraints $m+n$;
one of these constraint is L.D.; so $m+n-1$ constraints are L.I.
and hence we find $m+n-1$ B.V.'s

$\Rightarrow$ one of dual variable can arbitrarily choose any value
(we may assign one dual variable any value +ve, -ve, or zero)

Problem: Find the optimal solution of the T.P.; and find initial BFS using VAM.

Distribution Centres

Mills	D1	D2	D3	D4	
1	14	56	48	27	13
2	82	35	21	81	19
3	99	31	71	63	16
	7	14	21	6	48

Min problem

Solution:

Apply VAM to find Initial BFS

					Penalty
	7 (14)	(56)	0 (48)	6 (27)	~~13~~ 6 0	(13) (21) (8)
	(82)	(35)	19 (21)	(81)	19	(14) (14) (14)
	(99)	14 (31)	2 (71)	(63)	~~16~~ 2	(32) (32) (40)
	~~7~~	~~14~~	21	~~6~~		

Penalty (68) (4) (27) (36)
— (4) (27) (36)
— (4) (27) —

$Z = 1235$ at IBFS

To check $Z_{ij} - C_{ij}$

7 (14)	(56)	0 (48)	6 (27)
(82)	(35)	19 (21)	(81)
(99)	14 (31)	2 (71)	(63)

Solution :

Apply VAM to find Initial BFS

					Penalty
	14 (7)	56	48 (0)	27 (6)	~~13~~ 6 0 (13) (21) (8)
	82	35	21 (19)	81	19 (14) (14) (14)
	99	31 (14)	71 (2)	63	~~16~~ 2 (32) (32) (40)
	~~7~~	~~14~~	21	~~6~~	

Penalty (68) (4) (27) (36)
— (4) (27) (36)
— (4) (27) —

$Z = 1235$ at IBFS

To check $Z_{ij} - C_{ij}$

	D_1	D_2	D_3	D_4	
S_1	14 (7)	56	48 (0)	27 (6)	$u_1 =$
S_2	82	35	21 (19)	81	$u_2 =$
S_3	99 (14)	31	71 (2)	63	$u_3 =$
	$v_1 =$	$v_2 =$	$v_3 =$	$v_4 =$	

For B.V. $\boxed{u_i + v_j = c_{ij}}$

$u_1 + v_1 = c_{11} = 14$ — (1)

$u_1 + v_3 = c_{13} = 48$ — (2)

$u_1 + v_4 = c_{14} = 27$ — (3)

$u_2 + v_3 = c_{23} = 21$ — (4)

$u_3 + v_2 = c_{32} = 31$ — (5)

$u_3 + v_3 = c_{33} = 71$ — (6)

(As one of dual var. is arb.; so assign any value to u_i or v_j — any one variable)

for easy calculation; let us assign ; $u_3 = 0$

$\Rightarrow$ from (5)

$u_3 = 0 \Rightarrow v_2 = 31$

from (6)

$v_3 = 71$

from (2)

$u_1 = -23$

continuing ; we find all u_i's v_j's

$u_1 = -23$; $u_2 = -50$; $u_3 = 0$;

$v_1 = 37$; $v_2 = 31$; $v_3 = 71$; $v_4 = 50$

$v_1 = 37$; $v_2 = 31$; $v_3 = 41$, $v_4 = 59$

For Non-B.V. : $\boxed{u_i + v_j - c_{ij} = z_{ij} - c_{ij}}$

$$u_1 + v_2 - c_{12} = (-23) + (31) - 56 = -48$$
$$u_2 + v_1 - c_{21} = -95$$
$$u_2 + v_2 - c_{22} = -54$$
$$u_2 + v_3 - c_{23} = -81$$
$$u_3 + v_1 - c_{31} = -62$$
$$u_3 + v_4 - c_{34} = -13$$

All $z_{ij} - c_{ij} \leq 0$ for Non-B.V.

so $z_{ij} - c_{ij} \leq 0$; current T.P. is optimal

$$Z = \sum\sum x_{ij} c_{ij} = 1235 \text{ (optimal)}$$

Remark: If in (*) some of $z_{ij} - c_{ij}$ is +ve; then we require entering & leaving procedure; we need to study Loop in T.P.

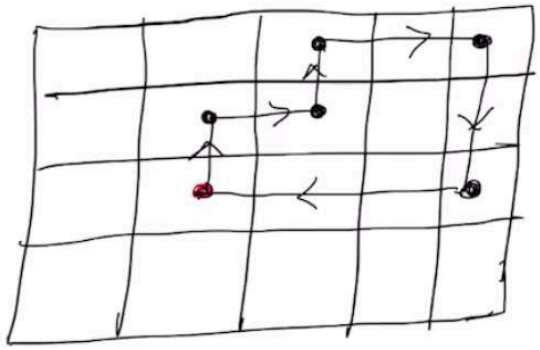
Definition (Loop) :→ In a T.P, (or Transportation table), an ordered set of four or more cells is said to form a loop if

(i) any two adjacent cells in the ordered set lie in same row or in the same column, and

(ii) any three or more adjacent cells in the ordered set do not lie in the same row or the same column. The first cell of the set is set to follow the last cell ; each cell appear only once in ordered set.

Eg: T.P. consider a loop = { (3,2), (2,2), (2,3), (1,3), (1,5), (3,5) }

these are positions in Transp. Table



↑ row ← → column.

Total no. of cells in loop are even.

Example: → Solve the T.P. using UV-Method (or Modi Method), use NW-Method to find Initial BFS

	D1	D2	D3	D4	
S1	14	56	48	27	13
S2	82	35	21	81	19
S3	99	31	71	63	16
	7	14	21	6	

$\boxed{\text{Balanced}}$ $\Sigma a_i = \Sigma b_j$

$[z_{ij} = u_i + v_j]$

Solution: Initial BFS using North West Method

14 (7)	56 (6)	48	27	~~13~~ 6
82	35 (8)	21 (11)	81	~~19~~ ~~11~~
99	31	71 (10)	63 (6)	16
~~7~~	~~14~~ 8	~~21~~ 10	6	

14 (7)	56 (6)	48	27	$u_1 = 0$
82	35 (8)	21 (11)	81	$u_2 = -21$
99	31	71 (10)	63 (6)	$u_3 = 29$

$v_1 = 14$ $v_2 = 56$ $v_3 = 42$ $v_4 = 34$

$$Z_{NW} = 14(7) + 56(6) + 35(8) + 21(11) + 71(10) + 63(6)$$

(Initial) $= 1913$

For B.V. | Basic Cells

$u_i + v_j = c_{ij}$ (Put $u_1 = 0$)

$u_1 + v_1 = 14$, $u_2 + v_3 = 21$

$u_1 + v_2 = 56$, $u_3 + v_3 = 71$

Look here ⇓ $z_{ij} - c_{ij}$ is not ≤ 0 $\forall\, i,j$ (for Non-B.V.)

For Non-B.V. (Yellow color)

$z_{13} - c_{13} = (u_1 + v_3) - c_{13} = -6$

$= (0 + 42) - 48 = -6$

$z_{14} - c_{14} = (u_1 + v_4) - c_{14} = 7$

$z_{21} - c_{21} = -89$

$z_{24} - c_{24} = -68$

(Initial) = 1913

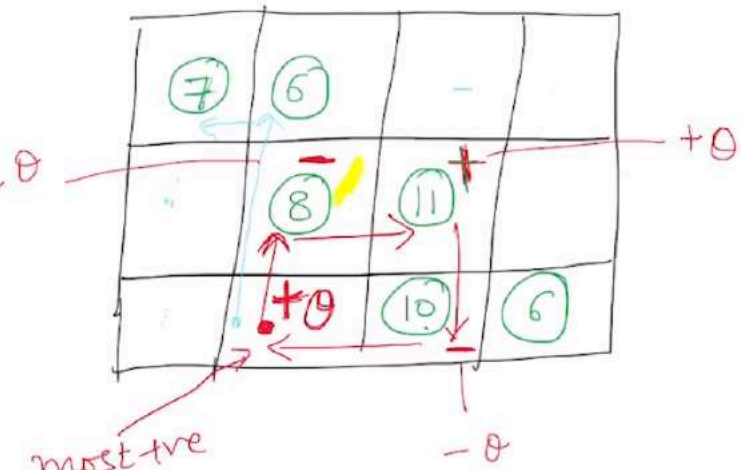
$u_1+v_1=14$, $u_2+v_3=21$
$u_1+v_2=56$, $u_3+v_3=71$
$u_2+v_2=35$, $u_3+v_4=63$

$z_{21}-c_{21}=-89$
$z_{24}-c_{24}=-68$
$z_{31}-c_{31}=-56$
$z_{32}-c_{32}=54$ ← most +ve

(we compare $z_{ij}-c_{ij}$ values of Non-B.V.'s which is most +ve;

Enter: (Loop starts from the allocation where $z_{ij}-c_{ij}$ is most +ve ✓

Remove the cell, which comes with zero allocation in loop.



most +ve cell; start loop from this cell

$\theta = \min\{-\text{ve signed allocations values}\}$

$= \min\{8, 10\} = 8$

7	6			13
		19		19
	8	2	6	16
7	14	21	6	

This cell is now non-Basic cell

Sum allocation Row wise

$\sum x_{ij} = a_i \mid \sum x_{ij} = b_j$

most +ve cell; start loop from this cell

$-\theta$

7	6			13
		19		19
	8	2	6	16
7	14	21	6	

This cell is now non-Basic cell

$\sum x_{ij} = a_i \mid \sum x_{ij} = b_j$

Again check optimality condition at II BFS

For B.V. $u_i + v_j = c_{ij}$

For Non-B.V.

$z_{ij} - c_{ij} = u_i + v_j - c_{ij}$

14, 7	56, 6, 48	48	27, $+\theta$, 61
82	35	21, 19	81
99, 8, −143, −110	31, −54	71, 2	63, 6, −63

$u_1 = 25$

$u_2 = -50$

$u_3 = 0$

$v_1 = -11 \quad v_2 = 31 \quad v_3 = 71 \quad v_4 = 63$

Still $z_{ij} - c_{ij}$ are not ≤ 0

So Again find loop; identify most +ve $z_{ij} - c_{ij}$ cell

Loop cells = { (1, 4), (1, 2), (3, 2), (3, 4) }

cell

$\theta = \min\{6, 6\}$
$= 6$

7	0		6
		19	
	14	2	

Non-Basic Cells.

14 7	56 0	48 θ	27 6
82 -143	35 -56	2 19	81 -129
99 -108	31 14	71 2	63 -67

$u_1 = 0$

$u_2 = -75$

$u_3 = -25$

$v_1 = 14 \quad v_2 = 56 \quad v_3 = 96 \quad v_4 = 27$

For Non-B.V.

$z_{13} - c_{13} = (u_1 + v_3) - c_{13}$
$= 48$ Entry

$z_{21} - c_{21} = -143$

$z_{22} - c_{22} = -56$

$z_{24} - c_{24} = -129$

$z_{31} - c_{31} = -108$

$z_{34} - c_{34} = -67$

Non-Basic Cells

14	56	48	27
7	0	0	6
82	35	2	81
-143	-56	19	-129
99	31	71	63
-108	14	2	-67

$u_1 = 0$

$u_2 = -75$

$u_3 = -25$

$v_1 = 14 \quad v_2 = 56 \quad v_3 = 96 \quad v_4 = 27$

$Z_{13} - C_{13} = (u_1 + v_3) - C_{13}$
$= \underline{48}$ Entry

$Z_{21} - C_{21} = -143$

$Z_{22} - C_{22} = -56$

$Z_{24} - C_{24} = -129$

$Z_{31} - C_{31} = -108$

$Z_{34} - C_{34} = -67$

$\theta = \min\{-\text{ve signed allocation}\}$
$= \min\{0, 2\} = 0$

$\Rightarrow$ the cell contain '0' will change.

Here, a Basic var. is taking value zero
$\Downarrow$
Deg. BFS.

Non-Basic cell

$u_1 = 0$

$u_2 = -46$

$u_3 = 23$

14	56	48	27
7	-ve	0	6
82	35	2	81
-ve	-ve	19	-ve
99	31	71	63
-ve	14	2	-ve

$v_1 = 14 \quad v_2 = 8 \quad v_3 = 48 \quad v_4 = 27$

$\Downarrow$

In this Table; all $Z_{ij} - C_{ij} \le 0$;

$$Z = (7)(14) + 0(48) + 6(27) + 19(2) + 14(31) + 2(71) = 1235$$

Example: Given $x_{13} = 50$ units, $x_{14} = 20$ units; $x_{21} = 55$ units; $x_{31} = 30$ units; $x_{32} = 35$ units and $x_{34} = 25$ units. Is it an optimal solution to the T.P.

				Available units
6	1	9	3	70
11	5	2	8	55
10	12	4	7	90
85	35	50	45	

Required units

If not; modify it to the optimal solution.

Solution:

First iteration

6	1	9 50	3 20	70
11 55	5	2	8	55
10	12	4	7 25	90

Required units

If not ; modify it to the optimal solution.

Solution:

First iteration

B.V. (cells): $u_i + v_j = c_{ij}$

Non Basic cells : $z_{ij} - c_{ij} = u_i + v_j - c_{ij}$

6	1	9	3	70
0	7	50 −	20 +	
11	5	2	8	55
− 55	8	θ + 12	0	
10	12	4	7	90
+ 30	35	9	25 −	
85	35	50	45	

$u_1 = -4$

$u_2 = 1$

$u_3 = 0$

$v_1 = 10 \quad v_2 = 12 \quad v_3 = 13 \quad v_4 = 7$

$\theta = \min\{-\text{ve signed allocation}\}$

$= \min\{50, 25, 55\}$

Second iteration

$u_1 = 0$

$u_2 = -7$

$u_3 = -8$

6	1	9	3
12	19	θ, 25	45
11	5	2	8
30	8	25	-12
10	12	4	7
55	35	-3	-12

$V_1 = 18 \quad V_2 = 20 \quad V_3 = 9 \quad V_4 = 3$

Non-B.V. = 25 in this table

$\theta = \min\{25, 30, 35\}$
$= 25$

Third iteration

Non this is non-B.V.

6	1	9	3
	25	✓	45
11	5	2	8
5		50	
10	12	4	7
80	10		

$V_1 = 18 \quad V_2 = 20 \quad V_3 = 7 \quad V_4 = 0$

Now this is non-B.V.

Third iteration

$u_1 = 0$	6 / -7	1 / 25	9 / -19	3 / 45
$u_2 = 12$	11 / 5	5 / θ	2 / 50	8 / 7
$u_3 = 0$	10 / 80	12 / 10	4 / -14	7 / -4

$V_1 = -1 \quad V_2 = 1 \quad V_3 = -10 \quad V_4 = 3$

Fourth iteration

$u_1 = 0$	-7	25	-11	45
$u_2 = 4$	-8	5	50	-1
$u_3 = 11$	85	5	5	θ / 7

$V_1 = -1 \quad V_2 = 1 \quad V_3 = -2 \quad V_4 = 3$

Fifth iteration

$u_1 = 0$	0	30	-11	40
$u_2 = 4$	-1	5	50	-1
$u_3 = 4$	85	-7	-2	5

$V_1 = 6 \quad V_2 = 1 \quad V_2 = -2 \quad V_3 = 3$

$Z = \sum x_{ij} c_{ij}$

$Z = 1160$

optimal.

Alternative Optimal Solution in Transportation Problem

Wednesday, 15 April 2020 6:06 AM

Points to Know

- How to find optimal solution of a T.P.
  (Initial BFS (NW, LCM, VAM), Optimal solution (UV-Method), LOOP ✓)
  $Z_{ij}-C_{ij} \le 0$ (min T.P.)
- In optimal table; $Z_{ij}-C_{ij} \le 0$ (for every cell)
  Now, if a Non-Basic variable has $Z_{ij}-C_{ij}=0$ in the optimal simplex table then the alternate solution can be obtain (or if $Z_{ij}-C_{ij}=0$ for unoccupied cell, in optimal table).

Problem: → A company manufacturing air-coolers has two plants located at Mumbai and Kolkata with a weekly capacity of 200 units and 100 units respectively. The company supplies air-coolers to its 4 showrooms situated at Ranchi, Delhi, Lucknow and Kanpur which have a demand of 75, 100, 100 and 30 units. The cost of transportation per unit (in Rs.) is given as:

	Ranchi	Delhi	Lucknow	Kanpur
Mumbai	90	90	100	100
Kolkata	50	70	130	85

Plan the minimum transportation cost for above problem.

Solution :

	R	D	L	K	Supply ↓
M					200
KO					100
→ Demand	75	100	100	30	305 / 300

← This is unbalanced T.P.

So we first balanced above T.P. and find initial BFS using VAM

	R	D	L	K		
M	90 / −20	90 / 75	100 / 95	100 / 30	200	$u_1 = 0$
KO	50 / 75	70 / 25	130 / −50	85 / −5	100	$u_2 = -20$
Dummy supply S_D	0 / −70	0 / −10	0 / 5	0 / 0	5	$u_3 = -100$
	75	100	100	30		

$V_1 = 70,\ V_2 = 90,\ V_3 = 100,\ V_4 = 100$

Using VAM; we find initial BFS; written in ⊙

$m+n-1 \rightarrow$ Basic cells

$3+4-1 = 6$

Apply UV-Method; to check $Z_{ij} - C_{ij}$

(for B.V. $Z_{ij} - C_{ij} = 0$ ↓ $u_i + v_j - C_{ij} = 0$)

M	-20	75	95	30	200	
KO	50 / 75	70 / 25	130 -50	85 -5	100	$u_2 = -20$
Dummy supply S_D	0 -70	0 -10	0 / 5	0 0	5	$u_3 = -100$
	75	100	100	30		

$v_1 = 70, v_2 = 90, v_3 = 100, v_4 = 100$

BFS; ...

$m+n-1 \rightarrow$ Basic cells

$3+4-1 = 6$

Apply UV-Method; to check $z_{ij} - c_{ij}$

(for B.V. $z_{ij} - c_{ij} = 0$ $\Downarrow$ $u_i + v_j - c_{ij} = 0$)

The current Trasp. table gives ~~an~~ optimal solution; All $z_{ij} - c_{ij} \leq 0$;

$$Z = 24,750$$

To find alternate optimal solution

We find a unoccupied cell / a Non-B.V. for which $z_{ij} - c_{ij} = 0$

Identify cell; in above eg.; we have (3, 4) cell is the one, where $z_{ij} - c_{ij} = 0$.

We make a loop, from (3,4)-cell.

	75	95 +	30 −
75	25		
		5 −	θ +

$\theta = \min\{ -ve \text{ signed cells} \}$

$= \min\{5, 30\} = 5$

For this BFS, find u_i, v_j;

-20	75	100	25
75	25	-50	-5
			5
-30	-10	0	

$v_1 = 70$ $v_2 = 90$ $v_3 = 100$ $v_4 = 100$

$u_1 = 0$

$u_2 = -20$

$u_3 = -100$

All $z_{ij} - c_{ij} \le 0$

$\therefore$ Current table is optimo

For this BFS,

-90	75	100	25
75	25	-50	-5
-30	-10	0	5

$v_1 = 70$ $v_2 = 90$ $v_3 = 100$ $v_4 = 100$

$u_1 = 0$

$u_2 = -20$

$u_3 = -100$

All $z_{ij} - c_{ij} \le 0$

$\therefore$ Current table is optimal.

B.V. $= (x_{12}, x_{13}, x_{14}, x_{21}, x_{22}, x_{34}) = (75, 100, 25, 75, 25, 5)$

$Z = 24,750$ ← optimal value.

optimal solution :

Maximization Transportation Problem

Wednesday, 15 April 2020 6:50 AM

To solve Maximization T.P.; we convert the problem first into Minimization Problem.

Method 1: We Multiply all C_{ij} in T.P. by -1 and obtain optimal solution of then formed minimization Problem; multiply by (-1) to value of Z; to obtain Z for maximization T.P.

$$\left[\ \text{Max}(Z) = -\text{Min}(-Z)\ \right]$$

or

Method 2: Convert the given problem into Minimization by replacing each C_{ij} by its difference from the maximum element in the table.

Apply usual process to solve Min. T.P.

To find Z (Maximization T.P.); multiply x_{ij} with original ... optimal table

Problem: (Maximization T.P.)

Let the Transportation Profit table is :

$$\begin{array}{c|ccc|c} & D_1 & D_2 & D_3 & \\ \hline S_1 & 10 & 9 & 8 & 50 \\ S_2 & 2 & 6 & 7 & 40 \\ S_3 & 4 & 4 & 3 & 20 \\ \hline & 10 & 20 & 80 & \boxed{110} \end{array}$$

where in table ; entries are $\underline{c_{ij}}$; which is profit of transporting material from S_i to D_j.

Sol^n:→ Using Method 2; Subtract all entries in Transportation table from maximum entry $\underline{\underline{10}}$.

We get

$$\begin{bmatrix} 0 & 1 & 2 \\ 8 & 4 & 3 \\ 6 & 6 & 7 \end{bmatrix} \longrightarrow$$

Now we solve this T.P. using initial BFS $\underline{\underline{VAM}}$; followed by UV-Method

Now we solve this T.P. using

$$\begin{bmatrix} 0 & 1 & 2 \\ 8 & 4 & 3 \\ 6 & 6 & 7 \end{bmatrix} \longrightarrow$$ initial BFS VAM; followed by UV-Method

After VAM

10	20	20
-7	-2	40
-1	0	20

$u_1 = 0$

$u_2 = 1$

$u_3 = 5$

$v_1 = 0 \quad v_2 = 1 \quad v_3 = 2$

Now $z_{ij} - c_{ij}$ for Non-B.V.

$z_{21} - c_{21} = -7$

$z_{22} - c_{22} = -2$

$z_{31} - c_{31} = -1$

$z_{32} - c_{32} = 0$

As $z_{ij} - c_{ij} \le 0$; current table is optimal.

Now to find Z; we multiply allocations with original c_{ij}.

$$Z = 10(10) + 20(9) + 20(8) + 40(7) + 20(3)$$

$$= 780 \leftarrow \text{optimal value}$$

$x_{11} = 10$; $x_{12} = 20$; $x_{13} = 20$; $x_{23} = 40$; $x_{34} = 20$

optimal solution

Restricted Transportation Problems

Monday, 20 April 2020 1:35 PM

Question: Consider the following minimization transportation problem, where per unit cost corresponding to each source and destination is given in table.

	D1	D2	D3	D4	D5	**Supply**
S1	20	19	14	21	16	**40**
S2	15	20	13	19	16	**60**
S3	18	15	18	20	10	**70**
Demand	**30**	**40**	**50**	**40**	**60**	

Find an optimal solution of above transportation problem with following restrictions:

1. There is no transportation from Source S3 to Destination D5.
2. The Source S1 supplies exactly 20 units to Destination D4.
3. The Destination D2 receives at least 10 units from Source S2.
4. The Source S2 can supply at-most 30 units to Destination D3.

3. The Destination D2 receives at least 10 units from Source S2.
4. The Source S2 can supply at-most 30 units to Destination D3.

Solution: The problem is unbalanced ; so we add a dummy sources.
∴ We add a dummy supply of 50 units.

$x_{35}=0$

M>0

1. There is no transportation from source S_3 to Destination D5.
so increase the cost C_{35} very high

NWC | LCM | VAM

Finding initial BFS using VAM

	D1	D2	D3	D4	D5	Supply
S1	20 (-5)	19 (-9)	14 (-1)	21 (-5)	16 (40)	**40**
S2	− 15 (30)	20 (-10)	13 (20) +	19 (-3)	16 (10)	**60**
S3	+ 18 (0) (2)	15 (40)	18 (30) −	20 (1)	M (-M+21)	**70**
S4	0 (-1)	0 (-6)	0 (-3)	0 (40)	0 (10)	**50**
	30	**40**	**50**	**40**	**60**	
	$v_1=15$	$v_2=10$	$v_3=13$	$v_4=16$	$v_5=16$	

$u_1=0$
$u_2=0$
$u_3=5$
$u_4=-16$

$\theta = \min\{30, 30\}$

	D1	D2	D3	D4	D5	Supply
S1	20 (-5)	19 (-7)	14 (-1)	21 (-5)	16 (40)	**40**
S2	15 (0)	20 (-8)	13 (50)	19 (-3)	16 (10)	**60**
S3	18 (30)	15 (40)	18 (-2)	20 (-1)	M	**70**
S4	0 (-1)	0 (-4)	0 (-3)	0 (40)	0 (10)	**50**
	30	**40**	**50**	**40**	**60**	
	$V_1=15$	$V_2=12$	$V_3=13$	$V_4=16$	$V_5=16$	

$u_1=0$

$u_2=0$

$u_3=3$

$u_4=-16$

$\boxed{u_i+v_j-c_{ij}}$

$Z_{ij}-c_{ij}\le 0$

current table is optimal.

$Z=$

$$Z=\sum\sum c_{ij}x_{ij}$$

$$=40(16)+15(0)+13(50)+16(10)+30(18)+40(15)$$

$$+0(40)+0(10)$$

$$=2590$$

(2) The origin S_1 supplies exactly 20 units to Destination D_4.

Consider $x_{14} = 20$ units ✓

$C_{14} = M$

Start from this problem.

Initial BFS NW | LCM | VAM ✓

	D1	D2	D3	D4	D5	Supply
S1	20	19	14 (20)	~~21~~ M	16	~~40~~ 20
S2	15 (0)	20 (30)	13 (30)	19	16	60
S3	18	15 (10)	18	20	10 (60)	70
S4	0 (30)	0	0	0 (20)	0	50
	30	40	50	~~40~~ 20	60	

⇐ Initial BFS found by VAM

↓ check $z_{ij} - c_{ij}$ for Non-B.V

(u_i, v_j)

$v_1 = 16$ $v_2 = 21$ $v_3 = 14$ $v_4 = 16$ $v_5 = 16$

	D1	D2	D3	D4	D5	Supply
S1	20 (-4)	19 (-2)	14 (20)	M (-M+16)	16 (0)	**20**
S2	15 + (0)	20 − (30)	13 (30)	19 (-5)	16 (-1)	**60**
S3	18 (-8)	15 (10)	18 (-10)	20 (-10)	10 (60)	**70**
S4	0 − (30)	0 θ+ (5)	0 (-2)	0 (20)	0 (0)	**50**
	30	**40**	**50**	**20**	**60**	

$u_1 = 0$

$u_2 = -1$

$u_3 = -6$

$u_4 = -16$

$\theta = \min\{30, 30\}$
$= 30$

$\Downarrow$ New BFS, check u_i, v_j and hence $z_{ij} - c$

S4	0 − (30)	0 θ+ (5)	0 (-2)	0 (20)	0 (0)	**50**
	30	**40**	**50**	**20**	**60**	

$u_4 = -16$

$\theta = \min\{30, 30\}$
$= 30$

New BFS, check u_i, v_j and hence $z_{ij} - c_{ij}$

	D1	D2	D3	D4	D5	**Supply**
S1	20 (-4)	19 θ+ (2)	14 − (20)	M (-M+21)	16 (0)	**20**
S2	15 (30)	20 − (0)	13 + (30)	19 (1)	16 (-1)	**60**
S3	18 (-8)	15 (10)	18 (-10)	20 (-5)	10 (60)	**70**
S4	0 (-5)	0 (30)	0 (-7)	0 (20)	0 (-5)	**50**
	30	**40**	**50**	**20**	**60**	

$v_1 = 16$ $v_2 = 19$ $v_3 = 14$ $v_4 = 19$ $v_5 = 14$

$u_1 = 0$

$u_2 = -1$

$u_3 = -4$

$u_4 = -19$

$\theta = \min\{0, 20\}$
$= 0$

	D1	D2	D3	D4	D5	Supply
S1	20 (-4)	19 (0)	14 (20)	M (-M+19)	16 (-1)	**20**
S2	15 (30)	20 (-2)	13 (30)	19 (-1)	16 (-3)	**60**
S3	18 (-6)	15 (10)	18 (-8)	20 (-5)	10 (60)	**70**
S4	0 (-3)	0 (30)	0 (-5)	0 (20)	0 (-5)	**50**
	30	**40**	**50**	**20**	**60**	

$u_1 = 0$

$u_2 = -1$

$u_3 = -4$

$u_4 = -19$

$v_1 = 16 \quad v_2 = 19 \quad v_3 = 14 \quad v_4 = 19 \quad v_5 = 14$

Now all $z_{ij} - c_{ij} \le 0 \quad \forall\, i,j$; hence the current table is optimal.

$B.V_0 = (x_{12}, x_{13}, x_{21}, x_{23}, x_{31}, x_{35}, x_{42}, x_{44})$ $\rightarrow$ $m+n-1$

$4+5-1 = 8$

$Z = \sum\sum x_{ij} c_{ij} + 20(21)$

$= 2290$

i) Destination D_2 receives atleast 10 units from S_2.

$x_{22} = 10$, C_{22} → original.

⇓ Initial BFS, VAM.

	D1	D2	D3	D4	D5	**Supply**
S1	20	19	14 (40)	21	16	**40**
S2	15 (20)	20 (20)	13 (10)	19	16	**60**
S3	18	15 (10)	18	20	10 (60)	**70**
S4	0 (10)	0	0	0 (40)	0	**50**
	30	**40**	**50**	**40**	**60**	

$\theta = \min\{10, 20\} = 10$

	D1	D2	D3	D4	D5	Supply	
S1	20 (-4)	19 (2)	14 (40)	21 (-5)	16 (0)	**40**	$u_1 = 0$
S2	15 + (20)	20 (20) −	13 (10)	19 (-4)	16 (-1)	**50**	$u_2 = -1$
S3	18 (-8)	15 (10)	18 (-10)	20 (-10)	10 (60)	**70**	$u_3 = -6$
S4	0 − (10)	0 θ + (5)	0 (-2)	0 (40)	0 (0)	**50**	$u_4 = -16$
	30	**30**	**50**	**40**	**60**		
	$v_1 = 16$	$v_2 = 21$	$v_3 = 14$	$v_4 = 16$	$v_5 = 16$		

New BFS

	D1	D2	D3	D4	D5	Supply	
S1	20 (-14)	19 (2)	14 (40)	21 (0)	16 (0)	**40**	$u_1 = 0$
S2	15 (30)	20 (10)	13 (10)	19 (1)	16 (-1)	**50**	$u_2 = -1$

$v_1 = 16$ $v_2 = 21$ $v_3 = 14$ $v_4 = 16$ $v_5 = 16$

New BFS
UV Method.

	D1	D2	D3	D4	D5	Supply
S1	20 (-14)	19 (2)	14 (40)	21 (0)	16 (0)	**40**
S2	15 (30)	20 (10)	13 (10)	19 (1)	16 (-1)	**50**
S3	18 (-8)	15 (10)	18 (-10)	20 (-5)	10 (60)	**70**
S4	0 (-5)	0 (10)	0 (-7)	0 (40)	0 (-5)	**50**
	30	**30**	**50**	**40**	**60**	

$u_1 = 0$
$u_2 = -1$
$u_3 = -6$
$u_4 = -21$

$v_1 = 16$ $v_2 = 21$ $v_3 = 14$ $v_4 = 21$ $v_5 = 16$

	D1	D2	D3	D4	D5	Supply	
S1	20 -ve	19 10	14 30	21 -ve	16 -ve	**40**	$u_1=0$
S2	15 30	20 -ve	13 20	19 -ve	16 -ve	**50**	$u_2=-1$
S3	18 -ve	15 10	18 -ve	20 -ve	10 60	**70**	$u_3=-4$
S4	0 -ve	0 10	0 -ve	0 40	0 -ve	**50**	$u_4=-19$
	30	**30**	**50**	**40**	**60**		
	$v_1=16$	$v_2=19$	$v_3=14$	$v_4=19$	$v_5=14$		

All $z_{ij}-c_{ij} \le 0$; optimal;

$$Z = \sum\sum x_{ij}(c_{ij})$$

$$= \left[19(10) + 14(30) + 15(30) + 13(20) + 15(10) + 10(60) + 0(10) + 0(40)\right] + 10(20)$$

$$= 2270$$

(4) Source S_2 supply at most 30 units to D_3.

$0 \le x_{23} \le 30$ units

$[0, 30] \in x_{23}$

$$\begin{array}{c|c} & D_1 \; - - - - \; D_5 \\ \hline S_1 & \\ S_2 & \longrightarrow \\ S_3 & \\ S_4 & \end{array} \Bigg] \; 60 \checkmark$$

$$\begin{array}{c} \\ S_1 \\ S_2' \\ S_2'' \\ S_3 \\ S_4 \end{array} \begin{array}{c} \begin{array}{ccccc} D_1 & D_2 & D_3 & D_4 & D_5 \end{array} \\ \begin{bmatrix} & & & & \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & M & \cdot & \cdot \\ & & & & \\ & & & & \end{bmatrix} \end{array} \begin{array}{c} \\ \\ \underline{30} \\ \underline{30} \\ \\ \\ \end{array}$$

S_1
S_2 ⟶ 60 ✓
S_3
S_4

$$\begin{array}{c} S_2' \\ S_2'' \\ S_3 \\ S_4 \end{array} \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & M & \cdot & \cdot \\ & & & & \\ & & & & \end{bmatrix} \begin{array}{c} \underline{30} \\ \underline{30} \\ \\ \\ \end{array}$$

✓

	D1	D2	D3	D4	D5	**Supply**
S1	20	19	14	21	16	**40**
$S2^{1}$	15	20	13	19	16	**30**
$S2^{11}$	15	20	M ←	19	16	**30**
S3	18	15	18	20	10	**70**
S4	0	0	0	0	0	**50**

← solve BBFS by VAM

	30	**40**	**50**	**40**	**60**	

Find IBFS using VAM.
UV-Method

	D1	D2	D3	D4	D5	**Supply**
S1	20	19 (20)	14 (20)	21	16	**40**
S2	15	20	13 (30)	19	16	**30**
S2	15 (20)	20 (10)	M	19	16 (60)	**30**
S3	18	15 (10)	18	20	10	**70**
S4	0 (10)	0 (5)	0	0 (40)	0	**50**
	30	**40**	**50**	**40**	**60**	
	$v_1=14$	$v_2=19$	$v_3=14$	$v_4=14$	$v_5=14$	

$u_1=0$

$u_2=-1$

$u_3=1$

$u_4=-4$

$u_5=-14$

New BFS U_i

S1	20	19 (20)	14 (20)	21	16	**40**	$u_1=0$
S2 I	15	20	13 (30)	19	16	**30**	$u_2=-1$
S2 II	15 (30)	20 (0)	M	19	16	**30**	$u_3=1$
S3	18	15 (10)	18	20	10 (60)	**70**	$u_4=-4$
S4	0	0 (10)	0	0 (40)	0	**50**	$u_5=-19$
	30	**40**	**50**	**40**	**60**		
	$v_1=14$	$v_2=19$	$v_3=14$	$v_4=19$	$v_5=14$		

$z_{ij}-c_{ij} \leq 0$ in above table; hence optimal.

$$Z = 19(20) + 14(20) + 30(13) + 15(30) + 20(0) + 15(10) + 10(60) + 0(10) + 0(40)$$

$$= 2250$$

End