Optimization Techniques

What is Optimization

- It is a theory and subject related to finding optimal points (referring to maxima or minima.
- Exploring various methods/algorithms that can be applied to mathematical models to get the solution.
- Best solution for a given situation or a problem.

  For example;

  - Minimum or maximum of a function
  - Transportation Problem (Lowest cost/or Highest profit)
  - Network and Scheduling Problems

- Optimization in Linear | Non-Linear Functions;

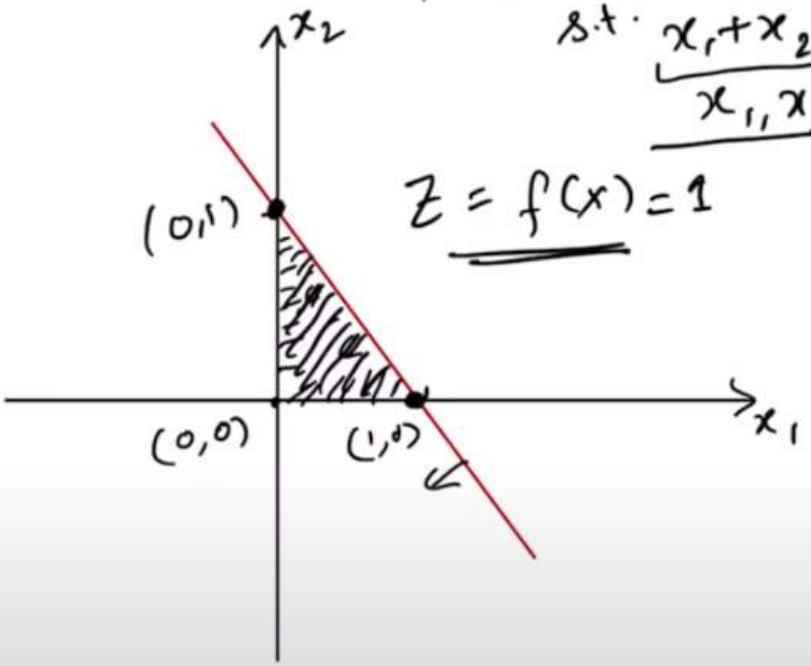
Linear :

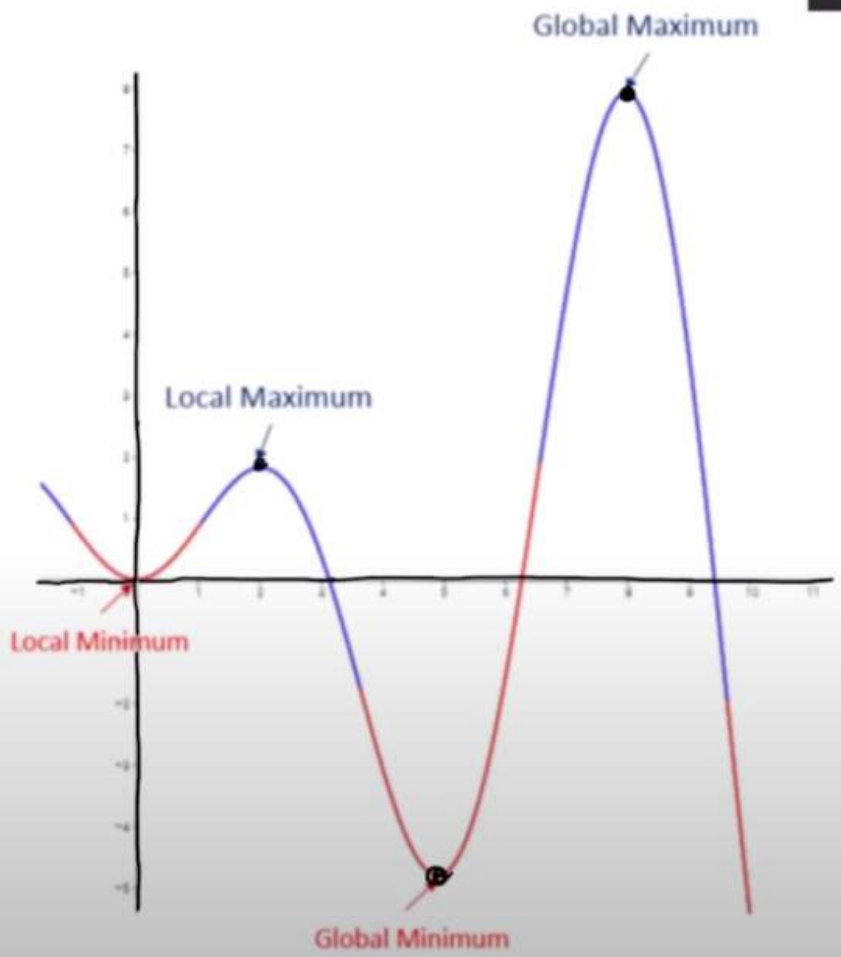
$$\text{Opt } f(x) = x_1 + x_2$$

$$\text{s.t. } x_1 + x_2 \leq 1$$

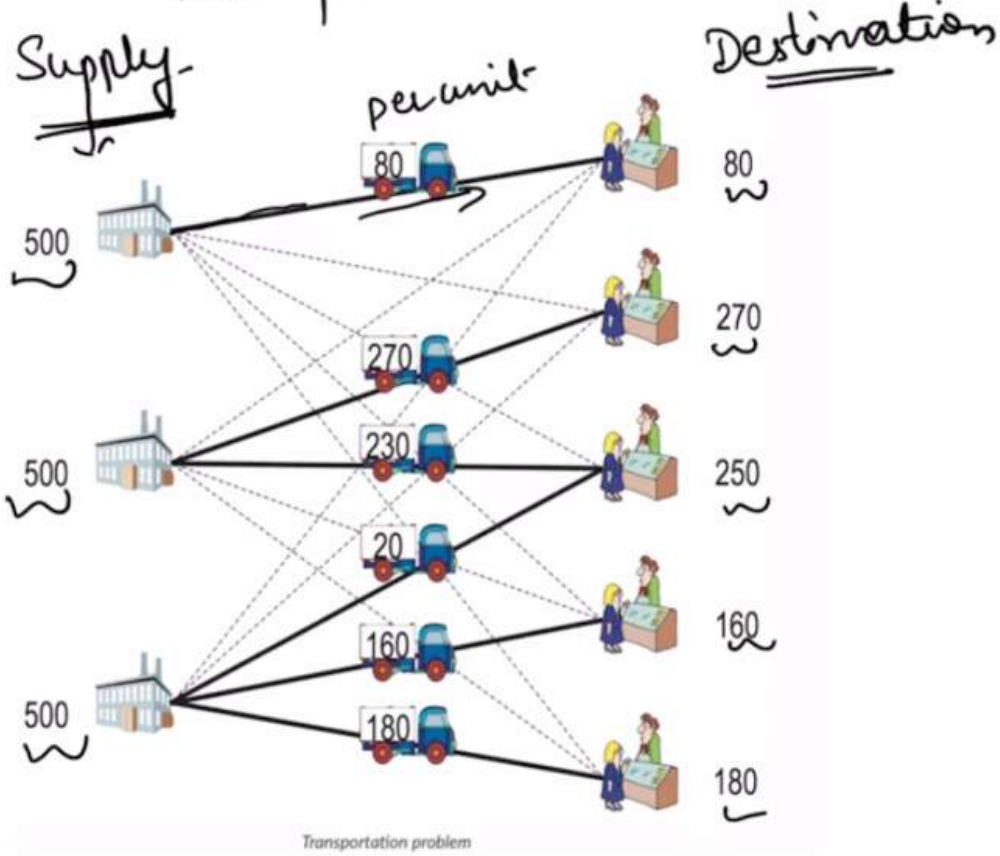
$$x_1, x_2 \geq 0$$

$$Z = f(x) = 1$$





Transportation Problem (Network)



Source: https://scipbook.readthedocs.io/en/latest/intro.html#transportation-problem

Role of Mathematical Formulation/Model in Optimization

Mathematical Formulation or mathematical model allows us to relate the real-life problem with mathematical variables, tools and solution strategies.

- It allow us to choose from alternatives available in situation.
- The best mathematical model gives us the best (optimal) results.

Situation-1 (To understand Alternatives)

Imagine that I have a 5-week business commitment between Fayetteville (FYV) and Denver (DEN).

- I can fly out of FYV on Mondays and return on Wednesdays.
- A regular round-trip costs $400, but a 20% discount is granted if the round-trip dates span a weekend.
- A one-way ticket in either direction costs 75% of the regular price.

How should I buy the tickets for the 5-week period?

Three alternatives are available:

1. Buy five regular FYV-DEN-FYV for departure on Monday and return on Wednesday

$ 400 x 5 = $ 2000

2. Buy one FYV-DEN, four DEN-FYV-DEN that spans weekend, and one DEN-FYV.

(.75 x $400) + (.8 x 400) + (.75 x 400) = $1880.

3. Buy one FYV-DEN-FYV (Round) (cover Monday first week & Wednesday of last week), & four DEN-FYV-DEN to cover remaining legs & spans weekend.

= 5 x (.8 x 400) = $ 1600

Situation-2 (To understand Model)

Consider forming a maximum-area rectangle out of a piece of wire of length L inches. What should be the best width and height of the rectangle?

Restrictions in this situation:

1. Width of rectangle + Height of rectangle=Half of length of the wire
2. Width and Height cannot be negative.

Algebraically, Model is

$$\text{Maximize } z = wh$$

Subject to $$2(w + h) = L$$

$$w, h \geq 0$$

General Model in Optimization Techniques:

Maximize or Minimize **Objective Function**

Subject to

Constraints

OR

Optimize $z = f(x)$

Subject to

constraints $g_i(x)$

- $f(x)$, $g_i(x)$ all are linear $\forall i$ $\hookrightarrow$ Linear P P (LPP)
- If $f(x)$ or $g_i(x)$ or both are non-linear programing NLPP !!

Formulation of LPP

Example: A Company manufactures two types of chips, A and B. The unit selling price for A and B is Rs. 15 and Rs. 25. Profits are proportional to the selling prices. To manufacture A, the Company has to invest 3 Hours of skilled labor, 2 Hrs. of unskilled labor and 1 unit of raw material. This data for B is 4, 3 Hours and 2 units respectively. The Company has 100 Hours of skilled labor, 70 Hrs. of unskilled labor and 30 units of raw material. The Company has to produce at least 3 units of chip A and 5 units of chip B. Formulate the problem as a LPP.

Linear Programming Problem

$A \geq 3$

$B \geq 5$

Company	Rs	SL	UL	RM
A	(15) x_1	3	2	1
B	(25) x_2	4	3	2
Availability		100	70	30

Objective fn.

Max.

constraints

Let x_1 : Units of chip A, company produced (No. of chips)

x_2 : Units of chip B; - - - - (No. of chips)

$$\text{Max } Z = 15x_1 + 25x_2$$

subject to

$3x_1 + 4x_2 \leq 100 \rightarrow$ hours.

hours $\leftarrow 2x_1 + 3x_2 \leq 70 \rightarrow$ "

" units. $\leftarrow x_1 + 2x_2 \leq 30 \rightarrow$ units.

$x_1 \geq 3$; $x_2 \geq 5$.

Formulation of Linear Programming Problem

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Formulation:

Let $x_1 \rightarrow$ No. of chips (A) company produced
$x_2 \rightarrow$ " " " (B) " "

] Decision Variables

$c_1 x_1 + c_2 x_2$

$$\text{Max } Z = 15x_1 + 25x_2$$

$\longrightarrow$ Objective function

$a_{11} x_1 + a_{12} x_2 \leq$

$$\text{s.t. } 3x_1 + 4x_2 \leq 100$$
$$2x_1 + 3x_2 \leq 70$$
$$x_1 + 2x_2 \leq 30$$
$$x_1 \geq 3 \; ; \; x_2 \geq 5$$

$\longrightarrow$ resource | availability | Demand

Restriction
Decision

} constraints

Mathematically, the general linear programming problem (LPP) may be stated as:

Linear

$$
\begin{aligned}
\text{Maximize or Minimize} \quad & Z = c_1x_1 + c_2x_2 + \ldots + c_nx_n \\
\text{subject to} \quad & a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \ (\leq, =, \geq) \ b_1 \\
& a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \ (\leq, =, \geq) \ b_2 \\
& \vdots \\
& a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \ (\leq, =, \geq) \ b_m \\
& x_1, x_2, \ldots, x_n \geq 0
\end{aligned}
$$

← objective fn.

constraints

Non-ve Restriction

In formulating the LPP as a mathematical model we shall follow the following four steps.

1. Identify the **decision variables** and assign symbols to them (eg x, y, z,... or x_1, x_2, x_2, ...). These decision variables are those quantities whose values we wish to determine.
2. Identify the set if constraints and express them in terms of inequalities involving the decision variables.
3. Identify the objective function and express it is terms of the decision variables.
4. Add the *non-negativity condition.*

PROBLEM 1:

A pharmaceutical company produces three types of medicines M1, M2, M3. For manufacturing these medicines, threefold types of ingredients L1, L2, L3 are required. One M1 requires 3 units of L1, and 2 units of L2, One M2 requires 2 units of L1, 3 units of L2, and 4 units of L3; One M3 requires 4 units of L1 and 3 units of L3. The company has a stock of 40 units of L1, 30 units of L2, and 45 units of L3. The profit per unit for M1, M2, M3 are Rs. 8, Rs. 13, Rs. 10 respectively. Formulate the problem as an LPP so that company maximize the profit .

Solution: Formulation: Let x_i be units of medicines produced. M_i ; $1 \le i \le 3$

	M1	M2	M3	Availability/Stock
L1	3	2	4	40
L2	2	3	0	30
L3	0	4	3	45
Profit	8	13	10	

$$\text{Max } Z = 8x_1 + 13x_2 + 10x_3$$

s.t.

$$3x_1 + 2x_2 + 4x_3 \le 40$$
$$2x_1 + 3x_2 \le 30$$
$$4x_2 + 3x_3 \le 45,$$
$$x_1, x_2, x_3 \ge 0$$

Formulation of Linear Programming Problem

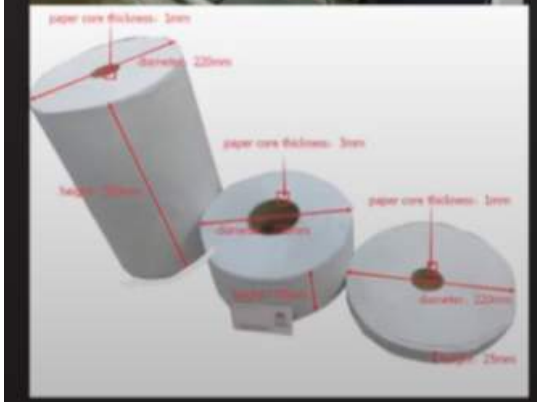
Friday, 15 May 2020 6:13 PM

PROBLEM:

The Pacific Paper Company produces paper rolls with a standard width of 20 feet each. Special customer orders with different widths are produced by slitting the standard rolls. Typical orders (which may vary daily) are summarized in the following table:

Order	Desired Width(feet)	Desired number of rolls
1	5	150
2	7	200
3	9	300

In practice, an order is filled by setting the knives to the desired widths. Find the minimum number of standard rolls to be slit according to the given data. Formulate the problem as an LPP model so as to minimize the total wastages.



Trim-loss problems!!

Now, data

Order	Desired Width	Desired No. of rolls
✓ 1	5 ✓	150
2	7	200
3	9	300

✓

20 feet = 4 × 5 feet

14 + 5 = 19 feet

Width (ft)	p_1	p_2	
5	4	1	
7	0	2	
9	0	0	
Wastage	0	1	

x_1 p_1 p_2 x_2 p_3 p_4 p_5 p_6 x_6

Required Width	1	2	3	4	5	6	Min. Number of Rolls
5	0	2	2	4	1	0	150
7	1	1	0	0	2	0	200
9	1	0	1	0	0	2	300
Trim loss per unit length	4	3	1	0	1	2	

← 20 feet →

Formulation:

Let xi be the number of standard rolls to be slit according to the setting i; i = 1,2,3,4,5,6

Minimize $Z = \overset{4}{x_1} + \overset{3}{x_2} + \cdots + \overset{2}{x_6}$ subject to

$2x_2 + 2x_3 + 4x_4 + x_5 \geq 150$ (5–ft rolls)

$x_1 + x_2 + 2x_5 \geq 200$ (7–ft rolls)

$x_1 + x_3 + 2x_6 \geq 300$ (9–ft rolls)

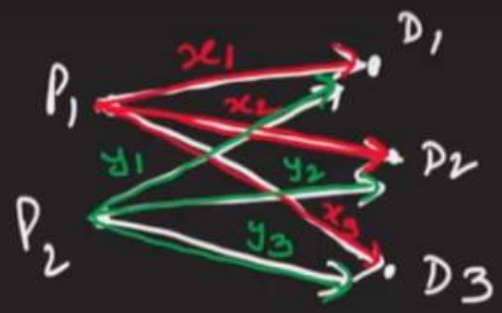
$x_1, x_2, \cdots, x_6 \geq 0$

$\text{Min } Z = 4x_1 + 3x_2 + x_3 + 0 \cdot x_4 + 0 \cdot x_5 + 2 \cdot x_6$

Trim-Loss!!

PROBLEM :

A dairy firm has two milk plants, daily milk production of 6 million litres and 9 million litres respectively. Each day, the firm must fulfil the needs of its three distribution centres which have the milk requirement of 7, 5, 3 million litres, respectively. Cost of transportation of 1 million litres of milk from each plant to each distribution centre is given in hundreds of rupees below. Formulate the LP model to minimize the transportation cost.



Transportation table

Decision Variable :

x_{ij} : units | commodity supplied from i^{th} plant to j^{th} destination

$1 \le i \le 2 \; ; \; 1 \le j \le 3$

$x_{11} \rightarrow$

$x_{12} \rightarrow$

$x_{21} \rightarrow$

Dist. Centre

Plant	1	2	3	Supply
1	2	3	11	6
2	1	9	6	9
Demand	7	5	3	

$x_{11} \rightarrow$

$x_{12} \rightarrow$

$x_{21} \rightarrow$

Dist. Centre

Plant	1	2	3	Supply
1	2 (x_{11})	3 (x_{12})	11 (x_{13})	6
2	1 (x_{21})	9 (x_{22})	6 (x_{23})	9
Demand	7	5	3	

ith plant to jth destination

$1 \le i \le 2 \; ; \; 1 \le j \le 3$

Minimize $Z =$

$$\left.\begin{aligned} &2x_{11} + 3x_{12} + 11x_{13} \\ &+ x_{21} + 9x_{22} + 6x_{23} \end{aligned}\right] \text{obj fn.}$$

s.t.

Constraints

$$\left.\begin{aligned} x_{11} + x_{12} + x_{13} &\le 6 \\ x_{21} + x_{22} + x_{23} &\le 9 \end{aligned}\right] \text{supply}$$

$$\left.\begin{aligned} x_{11} + x_{21} &\ge 7 \\ x_{12} + x_{22} &\ge 5 \\ x_{13} + x_{23} &\ge 3 \end{aligned}\right] \text{Demand}$$

Non-ve restriction $\{$ $x_{ij} \ge 0 \quad \forall i, j$

PROBLEM:

A ship has three cargo hold, forward, aft and centre; the capacity limits are:

Forward	2000 tonnes	100,000 m^3
Centre	3000 tonnes	135,000 m^3
Aft	1500 tonnes	30,000 m^3

The following cargoes are offered; the ship owners may accept all or any part of each commodity:

Commodity	Amount	Volume per tonne(m^3)	Profit per tonne (Rs.)
A	6000	60	60
B	4000	50	80
C	2000	25	50

In order to preserve the trim of the ship, the weight in each hold must be proportional to the capacity in tonnes. The objective is to maximize the profit. Formulate the linear Programming model.

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C	2000	25	50

In order to preserve the trim of the ship, the weight in each hold must be proportional to the capacity in tonnes. The objective is to maximize the profit. Formulate the linear Programming model.

$A \le 6000$

$x_{1A} + x_{2A} + x_{3A} \le 6000$

Solution:

Let x_{1A}, x_{2A}, x_{3A} be weight of the commodity A to be accommodated in forward, Centre and aft respectively.

Let x_{1B}, x_{2B}, x_{3B} be weight of the commodity B to be accommodated in forward, Centre and aft respectively.

Let x_{1c}, x_{2c}, x_{3c} be weight of the commodity C to be accommodated in forward, Centre and aft respectively.

Maximize $Z = 60(x_{1A} + x_{2A} + x_{3A}) + 80(x_{1B} + x_{2B} + x_{3B}) + 50(x_{1c} + x_{2c} + x_{3c})$

$x_{1A} + x_{2A} + x_{3A} \le 6000$

$x_{1B} + x_{2B} + x_{3B} \le 4000$

$x_{1c} + x_{2c} + x_{3c} \le 2000$

(Commodity constr.)

$x_{1A} + x_{1B} + x_{1c} \le 2000$

$x_{2A} + x_{2B} + x_{2c} \le 3000$

$x_{3A} + x_{3B} + x_{3c} \le 1500$

(Weight Capacity)

$60\,x_{1A} + 50\,x_{1B} + 25\,x_{1c} \le 100000$

$60\,x_{2A} + 50\,x_{2B} + 25\,x_{2c} \le 135000$

$60\,x_{3A} + 50\,x_{3B} + 25\,x_{3c} \le 30000$

$x_{iA}, x_{iB}, x_{ic} \ge 0.$

PROBLEM:

A bus operating in a particular city for 24x7, the schedule for 24 hours has the following minimum requirement of buses. The new buses can start plying at a particular period given below:

Period	Time	Minimum no. of buses required
1	12 am–4 am	4
2	4 am–8 am	8
3	8 am–12 noon	10
4	12 noon–4 pm	7
5	4 pm–8 pm	12
6	8 pm–12 midnight	4

To carry out the required operation, each bus should operate 8 successive hours a day only. Formulate the problem as an LPP to minimize the total number of buses required.

PROBLEM 2: (Scheduling Problem)

A bus operating in a particular city for 24x7, the schedule for 24 hours has the following minimum requirement of buses. The new buses can start plying at a particular period given below:

	Period		Time	Minimum no. of buses required
$x_1 \rightarrow$	1	$\rightarrow (x_1+x_6)$	12 am–4 am	4
$x_2 \rightarrow$	2	$\rightarrow (x_1+x_2)$	4 am–8 am	8
$x_3 \rightarrow$	3	$\rightarrow (x_2+x_3)$	8 am–12 noon	10
$x_4 \rightarrow$	4	$\rightarrow (x_4+x_3)$	12 noon–4 pm	7
$x_5 \rightarrow$	5	$\rightarrow (x_5+x_4)$	4 pm–8 pm	12
$x_6 \rightarrow$	6	$\rightarrow (x_6+x_5)$	8 pm–12 midnight	4

To carry out the required operation, each bus should operate 8 successive hours a day only. Formulate the problem as an LPP to minimize the total number of buses required.

Solution: Let $x_1, x_2, \ldots, x_6$ be the no. of buses joined / started plying at i^{th} period $1 \le i \le 6$

Minimize $Z = x_1 + x_2 + x_3 + \cdots + x_6 \longrightarrow$ obj fn.

s.t.

$$\left.\begin{aligned} x_1+x_2 &\geq 8 \\ x_2+x_3 &\geq 10 \\ x_3+x_4 &\geq 7 \\ x_4+x_5 &\geq 12 \end{aligned}\right\} \quad \left.\begin{aligned} x_5+x_6 &\geq 4 \\ x_6+x_1 &\geq 4 \end{aligned}\right] \text{ constraints}$$

$$x_i \geq 0$$

PROBLEM: The super bazar in a city daily needs between 32 and 40 workers depending on the time of day. The rush hours are between Noon and 2 pm. The table indicates the number of workers needed at various hours when the bazar is open.

Time period	No. of workers needed
9 am-11 am	32
11 am-1 pm	40
1 pm-3 pm	35
3 pm-5 pm	33

The super bazar now employs 34 full-time workers, but needs a few part time workers also. A part time worker must put in exactly 4 hours per day, but can start any time between 9 am and 1 pm. Full time workers work from 9 am to 5 pm but are allowed an hour for lunch (half of the full time workers eat at 12 noon, the other half at 1 pm). The management of the super bazar limits part-time hours to a maximum of 50 percent of the day's total requirement.

Part-timers earn Rs. 48 per day on the average, while full timers earn Rs. 140 per day in salary and benefits on the average. The management wants to set a schedule that would minimize total daily manpower costs. Formulate the problem as an LPP model.

Solution:

Decision Variables:

Let y represent the number of full-time workers needed in Super Bazar, and let x_j represents the part-time workers starting at 9 am, 11 am, and 1 pm respectively. ($j=1,2,3$).

Objective Function:

Minimize (Manpower Cost): $140\,y + 48\,(x_1+x_2+x_3)$

subject to

constraints:

$y + x1 \geq 32$	(9 am-11 am need)
$(1/2)y + x_1 + x_2 \geq 40$	(11 am-1 pm need)
$(1/2)y + x_2 + x_3 \geq 35$	(1 pm-3 pm need)
$y + x_3 \geq 33$	(3 pm-5 pm need)
$y \leq 34$	(Full timers available)

$4(x_1 + x_2 + x_3) \leq 0.50(32+40+35+33)$ (Part timers hour cannot exceed 50% of total hours required each day which is the sum of the workers needed each hour)

$y, x_1, x_2, x_3 \geq 0$

Q2. The NDMC has two plants, each of which produces and supplies two products: milk and butter. Each plant can work up to 16 Hrs a day. In plant A, it takes 3 Hrs to prepare and pack 1000 liters of milk and 1 Hr to prepare and pack 100 Kg of butter. In plant B, these figures are 2 Hrs. and 1.5 Hrs. In plant A (B) it cost Rs.15,000 (18,000) to prepare and pack 1000 liter of milk and Rs. 28,000 (26,000) for 100 Kg. of butter daily. The NDMC is obliged to produce daily at least 10000 liters of milk and 800 kg. of butter. Formulate the problem as a LPP.

NDMC

Plant 1 → Milk, Butter

Plant 2 → Milk, Butter

Time $\leq$ 16 hrs $\leq$ 16 hrs

Method 1 :

1 unit of milk = 1000 lt of milk

1 unit of butter = 100 kg butter

x_1, x_2 units of milk in Plant 1, 2

y_1, y_2 units of butter in Plant 1, 2

$$\text{Minimize } Z = 15000x_1 + 18000x_2 + 28000y_1 + 26000y_2$$

$$3x_1 + y_1 \leq 16$$

$$2x_2 + (1.5)y_2 \leq 16$$

$$x_1 + x_2 \geq 10$$

$$y_1 + y_2 \geq 8 \quad ; \quad x_i \geq 0, \; y_i \geq 0 \; ; \; i = 1, 2$$

Method 2

1 unit of milk = 1 lt of milk

1 unit of butter = 1 kg of butter

$x_i \rightarrow$ units of milk in plant i, $1 \leq i \leq 2$

$y_i \rightarrow$ --- butter ---

$$\text{Min } Z = 15x_1 + 18x_1 + 280y_1 + 260y_2$$

s.to

$$\frac{3}{1000}x_1 + \frac{y_1}{100} \leq 16$$

$$\frac{2x_2}{1000} + \frac{1.5}{100}y_2 \leq 16$$

$$x_1 + x_2 \geq 10000$$

$$y_1 + y_2 \geq 800$$

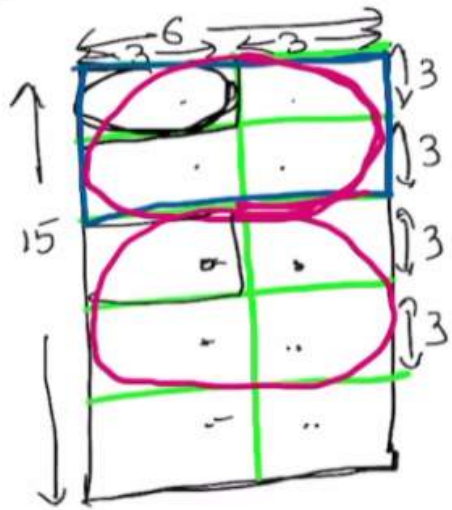
$$x_i \geq 0, \; y_i$$

(Trim-loss Problem)

Q3. Material Science department of TIET needs circular metallic plates of diameter 3 cm and 6 cm to perform experiments on heat treatment studies and requires minimum 2500 and 1500 units respectively. These are to be cut from parent metallic sheets of dimension 6 x 15 cm^2. Formulate the problem as a LPP so that minimum number of parent metallic sheets are used.

Sol^n:

	P_1	P_2	P_3
3 cm	10	6	2
6 cm	0	1	2



Let x_i no. of circular metallic plates cutted on pattern P_i; $1 \le i \le 3$

Minimize $Z = x_1 + x_2 + x_3$

s.t.

$$10x_1 + 6x_2 + 2x_3 \geq 2500$$
$$0x_1 + 1.x_2 + 2.x_3 \geq 1500$$
$$x_1, x_2, x_3 \geq 0$$

Q4. A manufacturing company has a contract to deliver 100, 250, 190, 140, 220 and 110 home windows over the next six months. Production cost per window varies by period and is estimated to be \$50, \$45, \$55, \$48, \$52 and \$ 50 over next six months. To take advantage of fluctuations in manufacturing cost, company can produce more windows than needed in a given month and hold the extra units for delivery in later months. This will incur a storage cost at rate of \$8 per window per month. Formulate the problem as a linear programming problem.

Let $x_i \rightarrow$ no. of units of windows produced in the ith month.

$n_i \rightarrow$ no. of units of windows left at end of month i.

$$\text{Minimize } Z = \underbrace{50x_1 + 45x_2 + 55x_3 + 48x_4 + 52x + 50x_6}_{\text{Production cost}} + \underbrace{8(n_1 + n_2 + \cdots + n_6)}_{\text{storage cost}}$$

s.t.

$x_1 = 100 + n_1$ (month 1)

$n_1 + x_2 = 250 + n_2$ (month 2)

$n_2 + x_3 = 190 + n_3$ (month 3)

$n_3 + x_4 = 140 + n_4$ (month 4)

$n_4 + x_5 = 220 + n_5$ (" 5)

$\underbrace{n_5 + x_6 = 110}_{\text{last month}}$ (month 6)

$x_i \geq 0$; $n_i \geq 0$

Q5. John must work at least 20 hours a week. To supplement his income while attending school, he has the opportunity to work in two retail stores. In store 1, he can work between 5 and 12 hours a week and in store 2, he can work between 6 and 10 hours. Both stores pay the same hourly wage. In deciding, how many hours to work in each store, John wants to base his decision on work stress, which are 8 and 6, respectively. Assume that total stress is proportional to total number of hours, determine how many hours John should work in each store?

Store1 $5 \le \quad \le 12$; ≥ 20

Stor2 $6 \le \quad \le 10$

x_1: No. of hours John work in store 1

x_2: - - - - - - John work in store 2

$$\underbrace{\text{Minimize}}_{} \; Z = \underbrace{8x_1 + 6x_2}_{}$$

s.t. $x_1 \ge 5$, $x_1 \le 12$, $x_2 \ge 6$, $x_2 \le 10$

$5 \le x_1 \le 12$ (store 1)

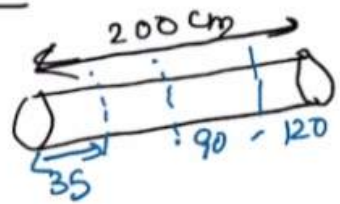
$6 \le x_2 \le 10$ (store 2)

$x_1 + x_2 \ge 20$; $x_1 \ge 0$; $x_2 \ge 0$

Q6. A metal slitting company cuts master rolls with width 200 centimeters into subrolls of small width. Customer specify that they need subrolls of different widths given in the following table:

Trim Loss Problem

Width of subroll (in cm)	Numbers required
35	100
80	90
90	400
120	800



The objective is to use a minimum number of master rolls to satisfy set of customers' orders. Formulate the problem as linear programming problem.

Sol^n: Possible Patterns from given width (Total 200 cm)

Width of subrolls	p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8	No. req.
35	5	3	2	0	0	1	0	3	100
80	0	0	0	1	1	2	0	1	90
90	0	1	0	1	0	0	2	0	400
120	0	0	1	0	1	0	0	0	800
Wastage	25								

Let x_i be no. of subrolls on pattern p_i

$$\text{Min } Z = x_1 + x_2 + x_3 + \cdots + x_8$$

$$\text{s.t. } 5x_1 + 3x_2 + 2x_3 + x_6 + 3x_8 = 100$$

$$x_4 + x_5 + 2x_6 + x_8 = 90$$

$$x_2 + x_4 + 2x_7 = 400$$

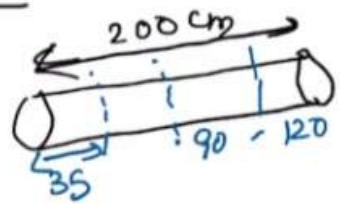
$$x_3 + x_5 = 800$$

$$x_i \geq 0$$

Q6. A metal slitting company cuts master rolls with width 200 centimeters into subrolls of small width. Customer specify that they need subrolls of different widths given in the following table:

Trim Loss Problem

Width of subroll (in cm)	Numbers required
35	100
80	90
90	400
120	800



The objective is to use a minimum number of master rolls to satisfy set of customers' orders. Formulate the problem as linear programming problem.

Sol^n: Possible Patterns from given width (Total 200 cm)

Width of subrolls	p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8	No. req.
35	5	3	2	0	0	1	0	3	100
80	0	0	0	1	1	2	0	1	90
90	0	1	0	1	0	0	2	0	400
120	0	0	1	0	1	0	0	0	800
Wastage	25								

Let x_i be no. of subrolls on pattern p_i

$$\text{Min } Z = x_1 + x_2 + x_3 + \cdots + x_8$$

$$\text{s.t. } 5x_1 + 3x_2 + 2x_3 + x_6 + 3x_8 = 100$$

$$x_4 + x_5 + 2x_6 + x_8 = 90$$

$$x_2 + x_4 + 2x_7 = 400$$

$$x_3 + x_5 = 800$$

$$x_i \geq 0$$

Q7. A company has two grades of inspectors, I and II, who are to be assigned for a quality control inspection. It is required that at least 2000 pieces be inspected per 8 hour day. Grade I inspectors can check pieces at the rate of 50 per hour with an accuracy of 97%. Grade II inspectors can check pieces at the rate of 40 per hour with an accuracy of 95%. The wage rate of grade I inspector is Rs. 4.50 per hour and that of grade II is Rs. 2.50 per hour. Each time an error is made by an inspector, the cost to the company is Rs. 2.00. The company has available for inspection job, 10 grade I and 5 grade II inspectors. Formulate the problem (DO NOT SOLVE) to minimize the total cost of inspection.

	Grade 1	Grade II
→	Rs 4.5	Rs 2.5
✓	50 per hour (97%)	40 per hour (95%)
	Error cost Rs 2	Rs 2

Let x_1 no. of grade I inspector

x_2 - - - - II inspector -

$$\text{Min } Z = \underbrace{(4.5)x_1 + (2.5)x_2}_{\text{cost per hour}} + \underbrace{(2 \times 0.03 \times 50)x_1}_{\text{cost of error of Grade 1}} + \underbrace{(2 \times 0.05 \times 40)x_2}_{} \quad \text{--- 2}$$

s.t.

$$8 \times 50 x_1 + 8 \times 40 x_2 \geq 2000$$

$$x_1 \leq 10$$

$$x_2 \leq 5 \quad ; \quad x_1, x_2 \geq 0$$

Q8. A company produces two types of hats. Type 1 requires twice as much labour time as type 2 alone. If all labour time is dedicated to type 2 alone, the company can produce a total of 400 hats of type 2 per day. Respective market limits for two types are 150 and 200 per day. The profit is Rs. 8 per type 1 hat and Rs. 5 per type 2 hat. Find the optimum number of hats to be prepared so that it maximizes company's profits.

Type 1	Type 2
'2t'	't'
x	all time (400 hats)
≤ 150	≤ 200
Profits Rs 8	Rs 5

Maximize $Z = 8x_1 + 5x_2$

where x_1 No. of Hats of type 1

x_2 - - - - - - type 2.

s.t.

$x_1 \leq 150$

$x_2 \leq 200$

$x_1 \geq 0 \quad ; \quad x_2 \geq 0$

	'2t'	't'
	X	all time (400 hats)
	≤ 150	≤ 200
Profits	Rs 8	Rs 5

400 hats of type 2 —— 't' time (complete time available)

1 hat $\longrightarrow \frac{t}{400}$ time

x_2 hats $\longrightarrow \left(\frac{t}{400}\right) x_2$ time

1 hat of type 1 $\longrightarrow 2\left(\frac{t}{400}\right) = \frac{2t}{400}$

x_1 hats of type 1 $\longrightarrow \frac{2t x_1}{400}$

$$\frac{2t x_1}{400} + \frac{t x_2}{400} \leq t$$

where x_1 No. of Hats of type 1

x_2 ------ type 2.

s.t.

$x_1 \leq 150$

$x_2 \leq 200$

$2x_1 + x_2 \leq 400$

$x_1 \geq 0 \quad ; \quad x_2 \geq 0$

Q9. Reddy Mikks produces both interior and exterior paints from two raw materials M_1 and M_2. The following table provide the data

	Exterior paint (tons) (x_1)	Interior paint (tons) (x_2)	Daily availability (tons)
Raw material M_1	6	4	24
Raw Material M_2	1	2	6
Daily Profit/ton (1000 $)	5	4	

A market survey indicates that daily demand for interior paint cannot exceed that for exterior paint by more than 1 ton. Also, the maximum daily demand for interior paint is 2 tons. Determine the optimum product mix of interior and exterior paints that maximizes the total daily profit.

Let $x_1 \rightarrow$ tons of Exterior paint

$x_2 \rightarrow$ - - - Interior paint

$$6x_1 + 4x_2 \leq 24$$

$$x_1 + 2x_2 \leq 6$$

	(x_1)	(x_2)	availability (tons)
Raw material M_1	6	4	24
Raw Material M_2	1	2	6
Daily Profit/ton (1000 \$)	5	4	

A market survey indicates that daily demand for interior paint cannot exceed that for exterior paint by more than 1 ton. Also, the maximum daily demand for interior paint is 2 tons. Determine the optimum product mix of interior and exterior paints that maximizes the total daily profit.

Let $x_1 \rightarrow$ tons of Exterior paint

$x_2 \rightarrow$ - - - Interior paint

$$x_2 \leq x_1 + 1$$

$$\text{Max } Z = 5x_1 + 4x_2$$

$$6x_1 + 4x_2 \leq 24$$

$$x_1 + 2x_2 \leq 6$$

$$-x_1 + x_2 \leq 1$$

$$x_2 \leq 2$$

$$x_1 \geq 0 \; ; \; x_2 \geq 0$$

Graphical Method

Methods For Solving Linear Programming Problem

Wednesday, 20 May 2020 2:00 PM

Mathematically, the general linear programming problem (LPP) may be stated as:

$$
\begin{aligned}
\text{Maximize or Minimize} \quad & Z = c_1x_1 + c_2x_2 + \ldots + c_nx_n \quad \leftarrow \text{obj fn.} \\
\text{subject to} \quad & a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \ (\leq, =, \geq) \ b_1 \\
& a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \ (\leq, =, \geq) \ b_2 \\
& \vdots \\
& a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \ (\leq, =, \geq) \ b_m \quad \text{constraints} \\
& x_1, x_2, \ldots, x_n \geq 0 \quad \rightarrow \text{Non-ve restriction}
\end{aligned}
$$

Solution Methods

1. Graphical Solution
2. Algebraic Method/ Finding Basic Solution
3. Simplex method and its Variants.

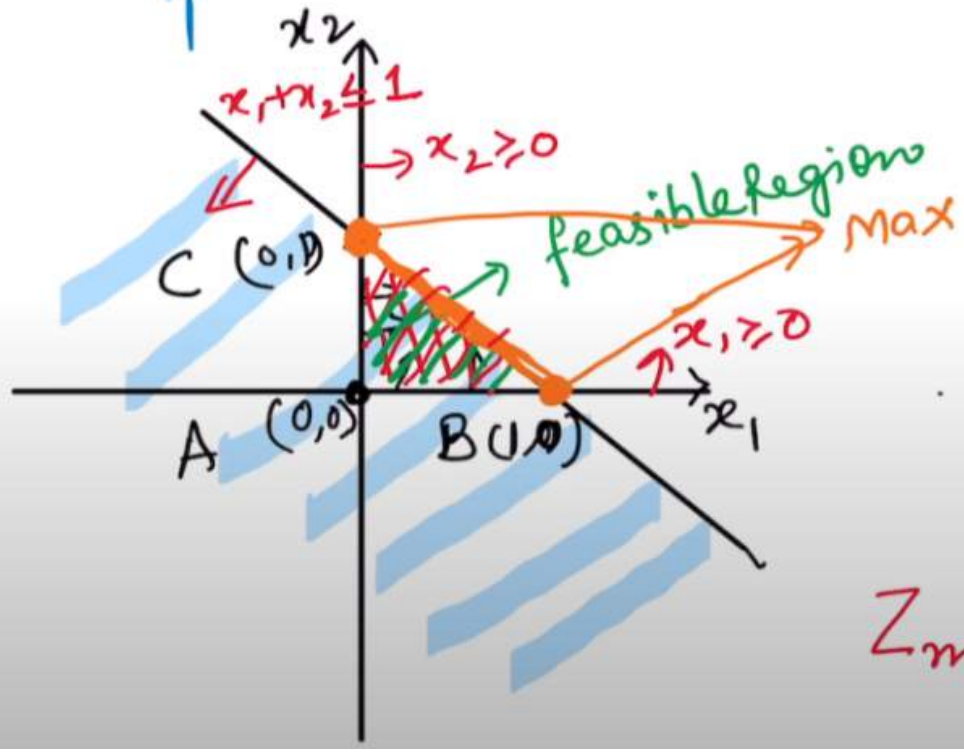
Graphical Solution of LPP

- Unique Solution
- Two Solution | Infinite solution | Alternate Solution
- Unbounded Solution (LPP, that is, objective function value 'Z' is not bounded and region is unbounded)
- Bounded LPP, with unbounded region.
- Infeasible solution (No solution)

Problem: Using the Graphical method solve the following LPP;

$$\begin{cases} \text{Maximize } Z = x_1 + x_2 \\ \text{subject to} \\ x_1 + x_2 \le 1, \quad x_1, x_2 \ge 0 \end{cases}$$

Solution:



$$\cancel{x_1}^{0} + x_2 = 1 \rightarrow$$

$$x_1 + \cancel{x_2}_{0} = 1$$

$$\left.\begin{aligned} Z_{\text{at } A} &= 0+0 = 0 \\ Z_{\text{at } B} &= 1+0 = 1 \checkmark \text{ Max} \\ Z_{\text{at } C} &= 0+1 = 1 \checkmark \end{aligned}\right\}$$

$Z_{max} = 1$ at (x_1, x_2)

$= (0,1), (1,0)$

$= (1/2, 1/2)$

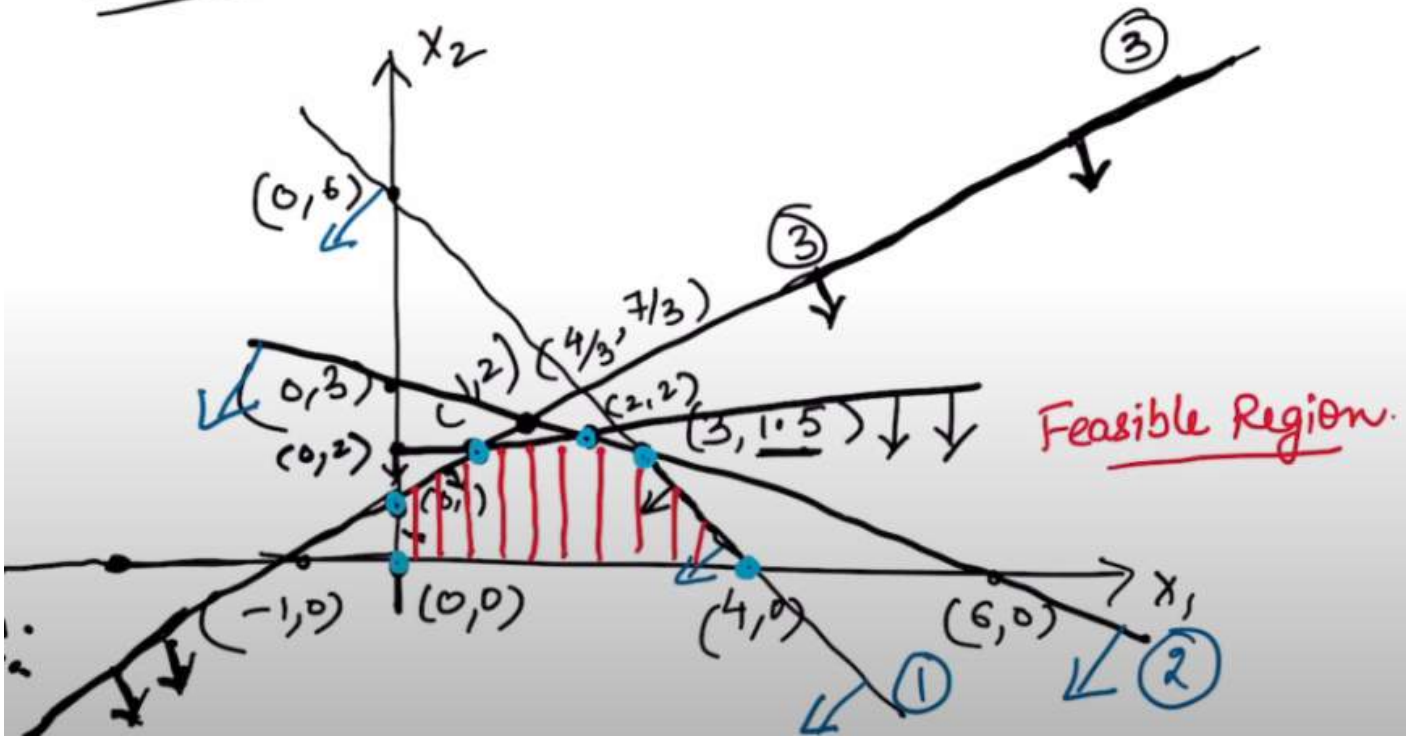
Problem: Using the Graphical method solve the following LPP:

Maximize $Z = 5x_1 + 4x_2$

subject to

$$6x_1 + 4x_2 \le 24, \quad x_1 + 2x_2 \le 6, \quad -x_1 + x_2 \le 1, \quad x_2 \le 2, \quad x_1, x_2 \ge 0$$

Solution:



I: $6x_1 + 4x_2 = 24$

$x_1 = 0$; $x_2 = 6$

$x_1 = 4$; $x_2 = 0$

II: $x_1 + 2x_2 = 6$

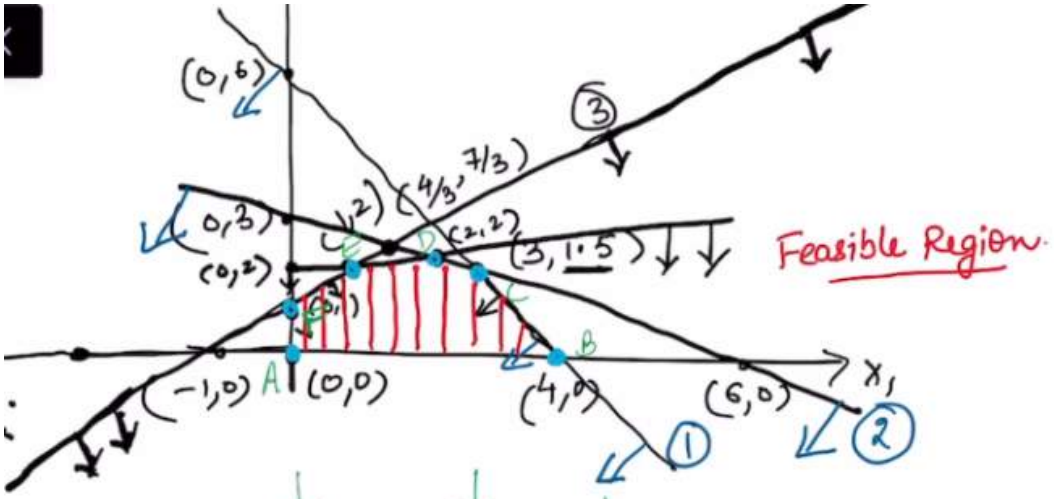
$x_1 = 0$; $x_2 = 3$

$x_1 = 6$ $x_2 = 0$

III. $-x_1 + x_2 = 1$

$x_1 = 0$; $x_2 = 1$

$x_1 = -1$ $x_2 = 0$



$x_1 = 0$; $x_2 = 6$

$x_1 = 4$; $x_2 = 0$

II: $x_1 + 2x_2 = 6$

$x_1 = 0$; $x_2 = 3$ ✓

$x_1 = 6$ $x_2 = 0$ ✓

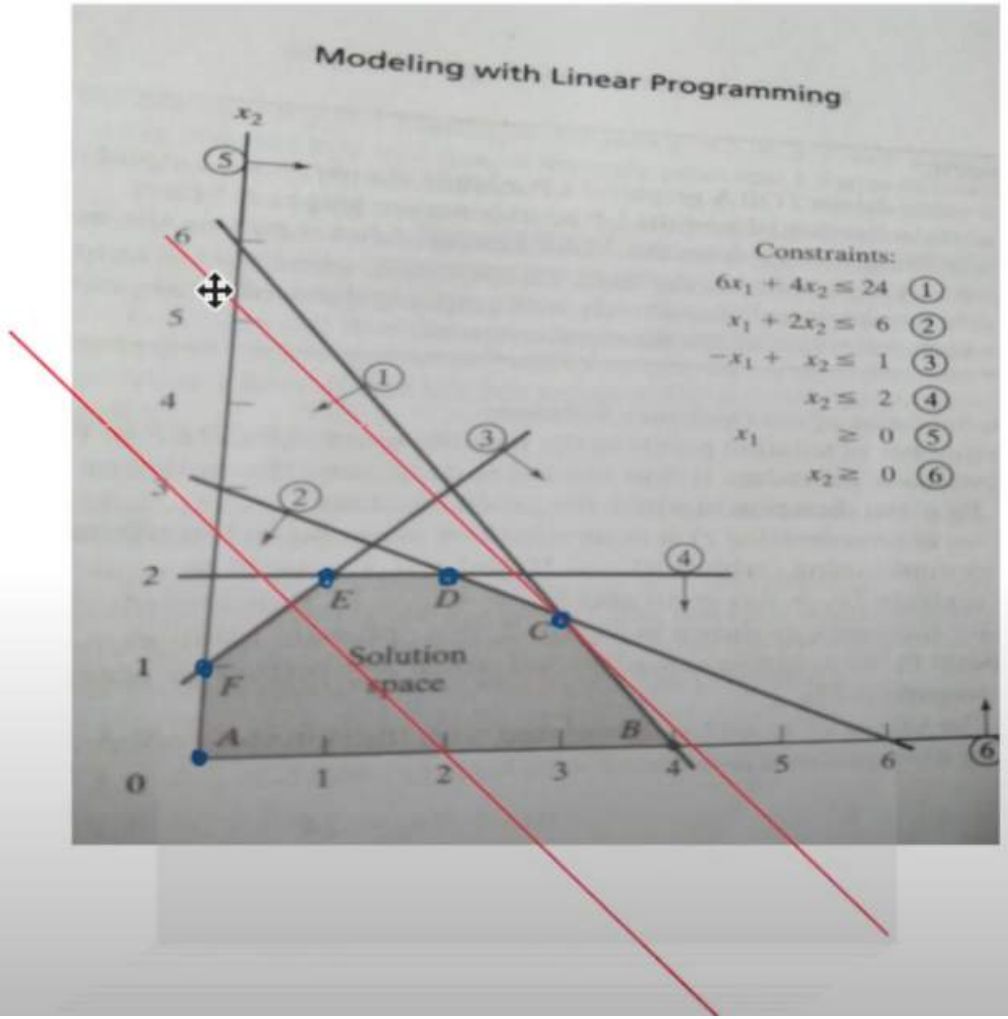
III. $-x_1 + x_2 = 1$

$x_1 = 0$; $x_2 = 1$ ✓

$x_1 = -1$ $x_2 = 0$

Corner Point	(x_1, x_2)	$Z = 5x_1 + 4x_2$
A	(0, 0)	0 ✓
B	(4, 0)	20 ✓
C	(3, 1.5)	21 **(Maximum)**
D	(2, 2)	18
E	(1, 2)	13
F	(0, 1)	4

Optimum, Z = 21 at (3,1.5)



$Max\ Z = 5x_1 + 4x_2$

$Z = 5x_1 + 4x_2 = \text{Constant}$

$Z_1 = 5x_1 + 4x_2 = 10$

$Z_1 = 5x_1 + 4x_2 = 20$

objective fn. lin

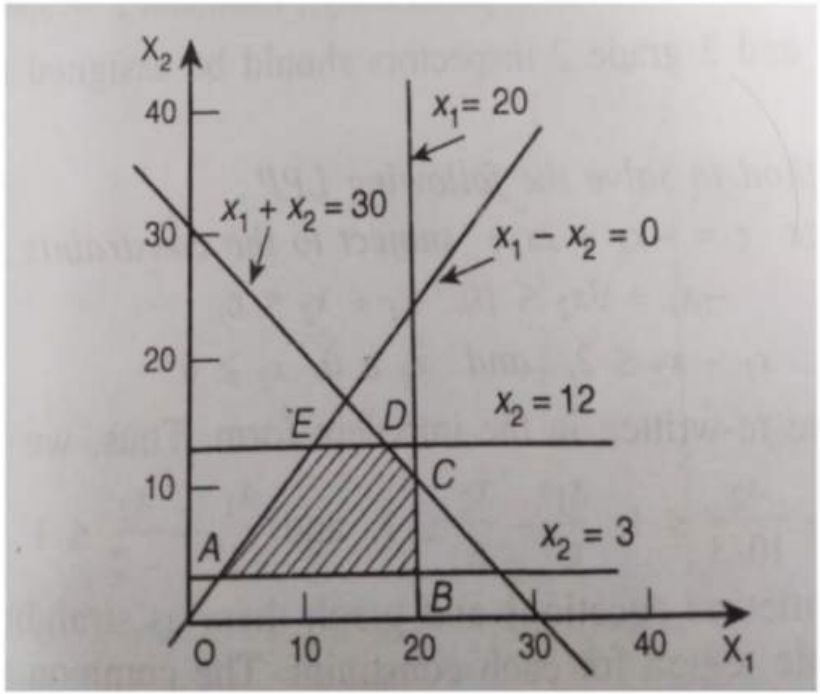
Red Arrow shows the direction of objective function. In this case it shows the direction of increase as objective function is of maximization.

Problem 3. Use graphical method to solve the LPP

Minimize $Z = 2x_1 + 3x_2$

subject to

$x_1 + x_2 \le 30,\ x_1 - x_2 \ge 0,\ x_2 \ge 3,\ 0 \le x_1 \le 20,\ 0 \le x_2 \le 12$



Corner Point	(x_1, x_2)	Z
A	(3, 3)	15 (Minimum)
B	(20, 3)	49
C	(20, 10)	70
E	(12, 12)	60
D	(18, 12)	72

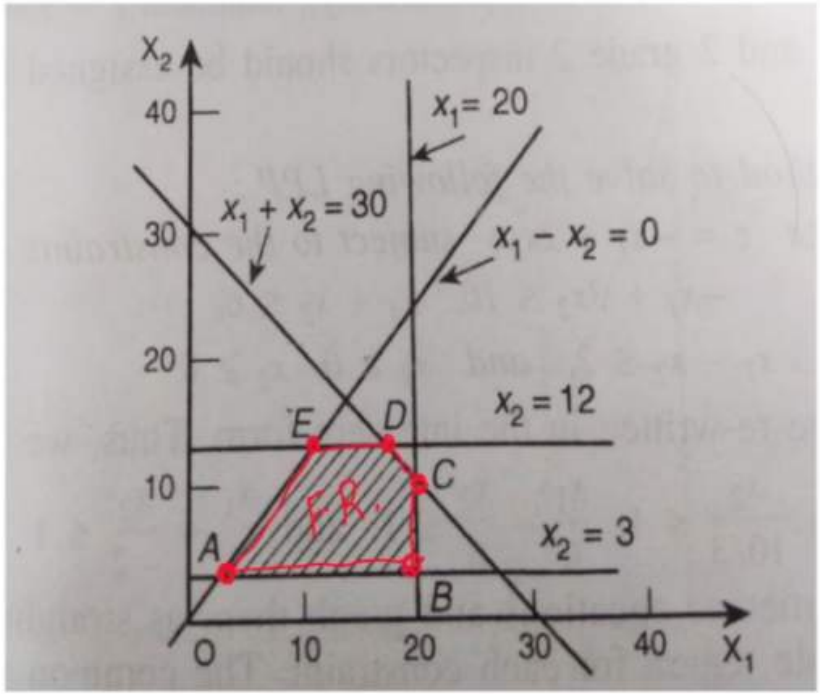
Optimum, $Z_{minimum} = 15$ at (3, 3)

Problem 3. Use graphical method to solve the LPP

Minimize $Z = 2x_1 + 3x_2$

subject to

$x_1 + x_2 \le 30$, $x_1 - x_2 \ge 0$, $x_2 \ge 3$, $0 \le x_1 \le 20$, $0 \le x_2 \le 12$



Corner Point	(x_1, x_2)	Z
A	(3, 3)	15 (Minimum)
B	(20, 3)	49
C	(20, 10)	70
E	(12, 12)	60
D	(18, 12)	72

UNIQUE SOLUTION

Optimum, $Z_{minimum} = 15$ at (3, 3)

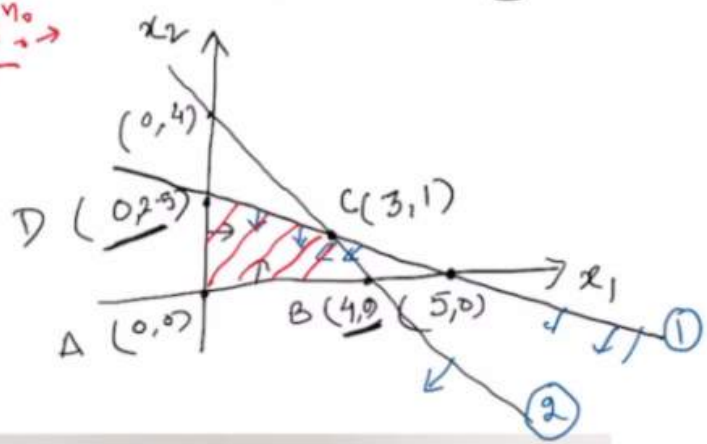
Problem 4: Use graphical method to solve the LPP

Maximize $Z = 2x_1 + 4x_2$

subject to

$x_1 + 2x_2 \le 5$, $x_1 + x_2 \le 4$, $x_1, x_2 \ge 0$.

Sol^n:



Method 1: objective fn. at each corner point.

$Z_{at\,A} = 0$

$Z_{at\,B} = 8$

$Z_{at\,C} = 10$

$Z_{at\,D} = 10$

optimal value

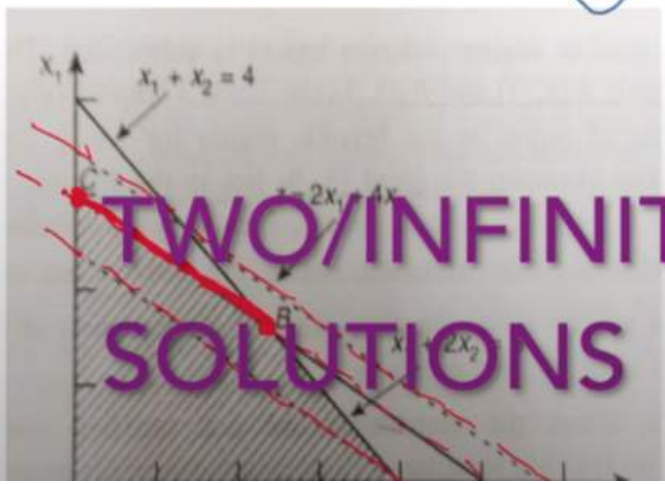
Max.

optimal value (of Z) = 10

Optimal solution; $(x_1, x_2) = (3,1)$ or $(0, 2.5)$

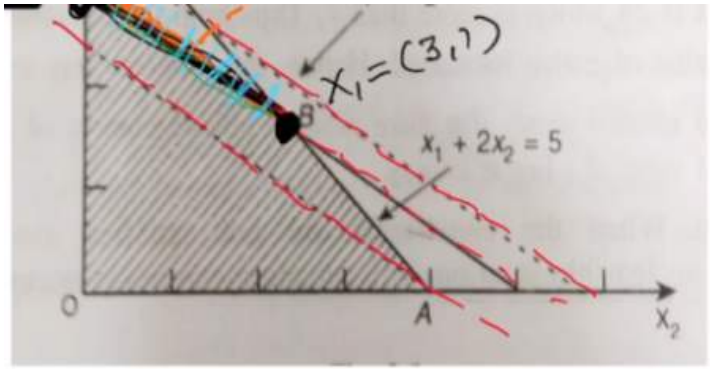
Method 2:

objective fn. $Z = 2x_1 + 4x_2 = C = 8$ (say)



TWO/INFINITE/ALTERNATE OPTIMAL SOLUTIONS

So from objective function line; line segement BC will give no optimal solution.



So from objective function line; line segement BC will give us optimal solution

Note :→

I. $\text{Max } Z = 2x_1 + 4x_2$; $x_1 + 2x_2 \le 5$; $x_1 + x_2 \le 4$

$2(x_1 + 2x_2) \le 2(5)$

$\text{Max } Z = 2x_1 + 4x_2 \le 10$

II. $Z_{max} = 10$ at $X = \alpha_1 X_1 + \alpha_2 X_2$; $\alpha_1 \ge 0$, $\alpha_2 \ge 0$, $\alpha_1 + \alpha_2 = 1$

$= \alpha_1 (3,1) + \alpha_2 (0, 5/2)$

$\alpha_1 = 0$; $\alpha_2 = 1$; $X = (0, 5/2)$

$\alpha_1 = 1$; $\alpha_2 = 0$; $X = (3,1)$

$0 < \alpha_1 < 1$; $0 < \alpha_2 < 1$; $X = \frac{1}{3}(3,1) + \frac{2}{3}(0, 5/2)$

$\alpha_1 = 1/3$, $\alpha_2 = 2/3$

$= (1, 1/3) + (0, 5/3)$

$Z = 2x_1 + 4x_2 \Big|_{(1,2)} = 2 + 8 = 10$

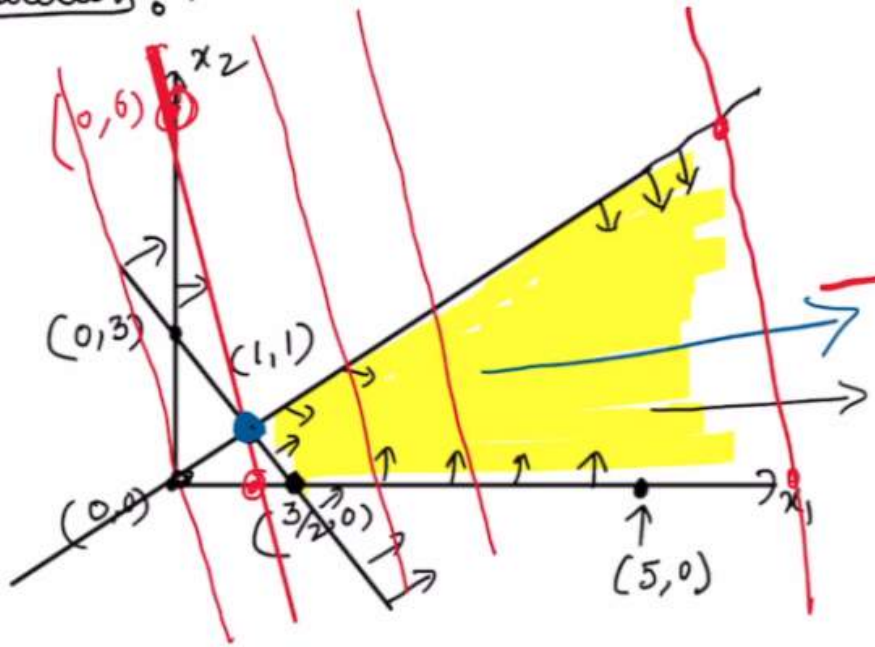
$= (1, 2)$

Problem : (Unbounded Solution/ Unbounded Region) Use graphical method to solve the LPP: Maximize $Z = 6x_1 + x_2$

subject to

$$2x_1 + x_2 \geq 3, \quad x_1 - x_2 \geq 0, \quad x_1, x_2 \geq 0$$

Solution :→



Direction of increase of Z

- Unbounded Region ✓
- Unbounded Solution
  $Z \to \infty$
  x_1

x_1, x_2
(5, 0)

$x_1 - x_2 \geq 0$
$5 - 0 \geq 0$ ✓

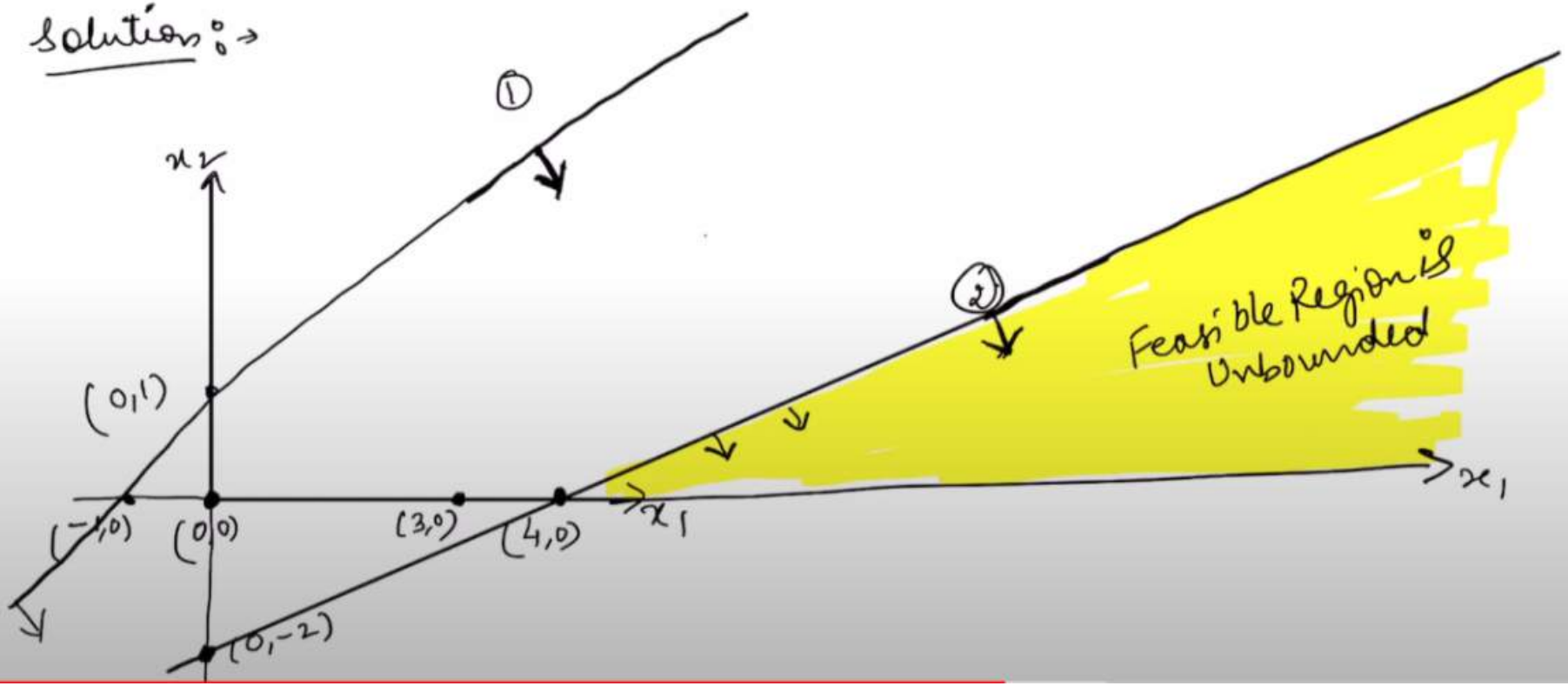
$Z = 6x_1 + x_2 = C =$ Constant
$6x_1 + x_2 = 0$ ✓
$6x_1 + x_2 = 6$ ✓
$6x_1 + x_2 = 12$

Problem : (Unbounded Region/ Bounded Solution) Use graphical method to solve the LPP

Maximize $Z = 2x_2 - x_1$

subject to $x_1 - x_2 \geq -1$, $-0.5x_1 + x_2 \leq -2$, $x_1, x_2 \geq 0$

Solution :→



Problem : (Unbounded Region/ Bounded Solution) Use graphical method to solve the LPP

Maximize $Z = 2x_2 - x_1$

subject to $x_1 - x_2 \geq -1$, $-0.5x_1 + x_2 \leq -2$, $x_1, x_2 \geq 0$

objective fn. $-x_1 + 2x_2 = C$

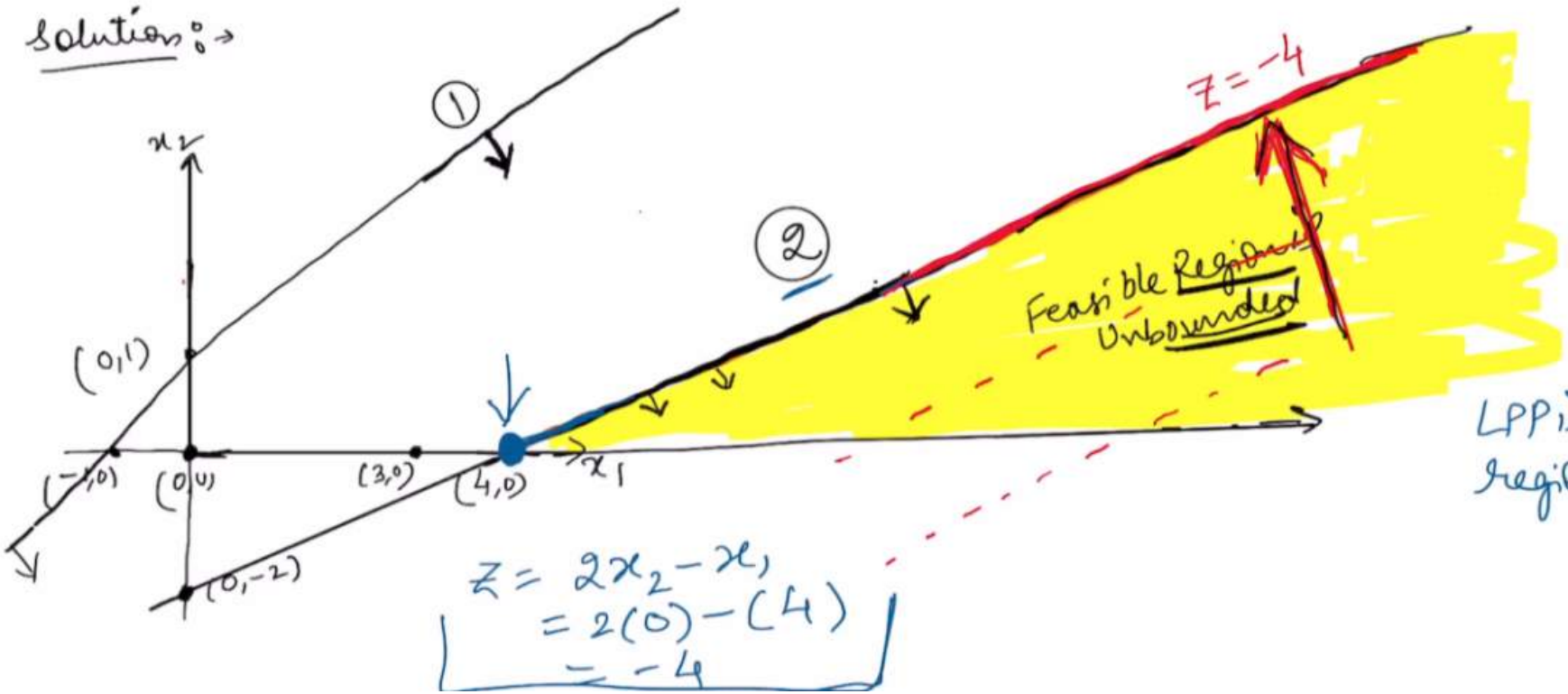
constraint ② $-0.5x_1 + x_2 \leq -2$

$-\frac{x_1}{2} + x_2 \leq -2$

$-x_1 + 2x_2 \leq -4$

$C = -4$

Solution :→



$Z = 2x_2 - x_1$
$= 2(0) - (4)$
$= -4$

Max $Z = -4$

⇓

LPP is bounded; region is unbounded.

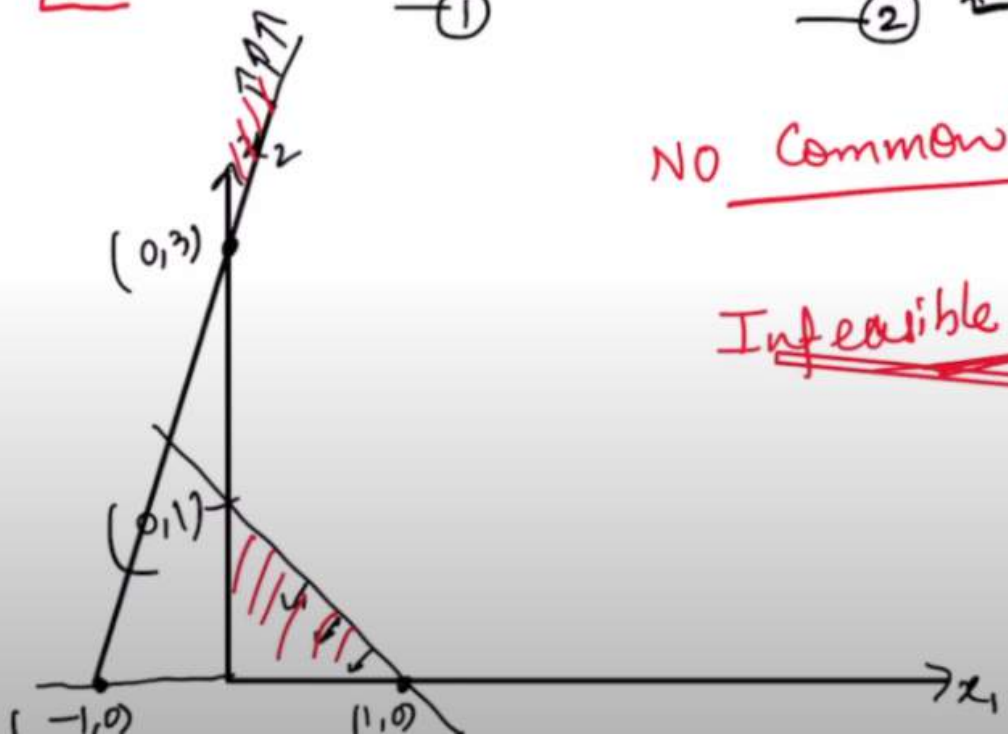
Problem: (Infeasible Solution) Use graphical method to solve the LPP

Maximize $Z = x_1 + x_2$

subject to

$$\left[\; x_1 + x_2 \le 1, \quad -3x_1 + x_2 \ge 3, \quad x_1, x_2 \ge 0 \;\right]$$

Feasible Region



NO Common Region.

Infeasible Solution | No. Solution

Basic Solutions

Basic Solutions- Linear Programming Problem

Slack and Surplus variables

$\leq$, $\geq$, $=$

slack surplus No change

A variable added to the '≤' type constraints to achieve equality is called a slack variable. A variable subtracted from the '≥' type constraints to achieve equality is called a surplus variable.

L.H.S. ≤ 4 L.H.B. $+ S_1 = 4$

Example 1. Converting the inequality constraint $x_1 + 3x_2 + x_3 \leq 4$ into equality constraint as

L.H.S. ≤ 4

$x_1 + 3x_2 + x_3 + s_1 = 4$ where s_1 is slack variable, and $s_1 \geq 0$

L.H.S $+ s_1 = 4$

$s_1 > 0$

Example 2. Converting the inequality constraint $x_1 + 3x_2 + x_3 \geq 4$ into equality constraint as

L.H.S. ≥ 4

$x_1 + 3x_2 + x_3 - s_2 = 4$ where s_2 is surplus variable, and $s_2 \geq 0$

surplus

Basic Solution (BS)

Matrix form

Consider a system "AX=b", $(A)_{m \times n}$, of m equations with n variables (n > m), and Put (n−m) variables to zero and the resulting system $B X_B = b$ solution is called the Basic Solution.

LPP: $\text{Max } Z = x_1 + x_2$

s.t. $x_1 + x_2 \le 1$

$x_1 \ge 0;\ x_2 \ge 0$

or $\text{Max } Z = x_1 + x_2 + 0 \cdot s_1$

s.t. $x_1 + x_2 + s_1 = 1$

$x_1 \ge 0;\ x_2 \ge 0,\ s_1 \ge 0$

Now; $x_1 + x_2 + s_1 = 1$

$$\begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ s_1 \end{pmatrix} = 1$$

$$A \quad X \quad = b$$

$$(A)_{1 \times 3} X = b$$

$m = 1$ $\quad n = 3$

$n - m = 2$

Matrix form

Consider a system "AX=b", $(A)_{m \times n}$, of m equations with n variables (n > m), and Put (n−m) variables to zero and the resulting system $B X_B = b$ solution is called the Basic Solution.

LPP: Max $Z = x_1 + x_2$

s.t. $x_1 + x_2 \le 1$

$x_1 \ge 0;\ x_2 \ge 0$

or Max $Z = x_1 + x_2 + 0 \cdot s_1$

s.t. $x_1 + x_2 + s_1 = 1$

$x_1 \ge 0;\ x_2 \ge 0,\ s_1 \ge 0$

Case (i) $x_1 = x_2 = 0$; $s_1 = 1$

(ii) $x_1 = s_1 = 0$; $x_2 = 1$

(iii) $x_2 = s_1 = 0$; $x_1 = 1$

x_1	x_2	s_1
0	0	1
0	1	0
1	0	0

Notice

$x_1 + x_2 + s_1 = 1 \rightarrow (1 \times 3)$

$(1/2 + 1/2 + 0) = 1$

Infinite sol^n

$n - m \rightarrow$ zero

Problem : Find all Basic Solution of the LPP; (Optimal)

? Maximize $Z = 2x_1 + 3x_2$

$$\left\{ \begin{array}{l} 2x_1 + x_2 \le 4 \\ x_1 + 2x_2 \le 5 \text{ and } x_1, x_2 \ge 0 \end{array} \right\}$$

Solution:

$$2x_1 + x_2 + s_1 = 4$$
$$x_1 + 2x_2 + s_2 = 5$$

$$AX = b$$

$$\begin{array}{c} \begin{pmatrix} 2 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{pmatrix} \\ \begin{array}{cccc} x_1 & x_2 & s_1 & s_2 \end{array} \end{array} \begin{pmatrix} x_1 \\ x_2 \\ s_1 \\ s_2 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$$

$$\underbrace{(A)_{2\times 4}}\ X = b$$

$$n - m = 0$$
$$4 - 2$$

$$\Downarrow$$

$$B(X_B) = b$$

Basis Matrix

Case (i) $x_1 = x_2 = 0$ — Non-Basic Variable

$s_1 = 4$
$s_2 = 5$

$s_1 = 4$
$s_2 = 5$

Basic Variable: $s_1 \quad s_2$

$x_1 \quad x_2 \quad s_1 \quad s_2$
$(0, 0, 4, 5)$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$$

$$\begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$$

B

Basic Variable

Problem : Find all Basic Solution of the LPP; (optimal)

$n - m$
$4 - 2 = 2$

? Maximize $Z = 2x_1 + 3x_2$

$$\begin{cases} 2x_1 + x_2 \le 4 \\ x_1 + 2x_2 \le 5 \end{cases} \text{ and } \boxed{\begin{matrix} s_1, s_2 \geq 0 \\ x_1, x_2 \geq 0 \end{matrix}}$$

Solution:

$$2x_1 + x_2 + s_1 = 4$$
$$x_1 + 2x_2 + s_2 = 5$$

Case	Basic Variables	Non Basic Variables	Solution (X_B) x_1, x_2, s_1, s_2
(i)	s_1, s_2	$x_1 = x_2 = 0$	$(0, 0, 4, 5)$
(ii)	x_1, s_1	$x_2 = s_2 = 0$	$(5, 0, -6, 0)$
(iii)	x_1, s_2	$x_2 = 0 = s_1$	$(2, 0, 0, 3)$
(iv)	x_2, s_1	$x_1 = s_2 = 0$	$(0, \frac{5}{2}, \frac{3}{2}, 0)$
(v)	x_2, s_2	$x_1 = s_1 = 0$	$(0, 4, 0, -3)$
(vi)	x_1, x_2	$s_1 = s_2 = 0$	$(1, 2, 0, 0)$

Basic Solutions

Feasible

BFS

$$\boxed{{}^nC_m} = {}^4C_2 = \frac{4!}{2!(4-2)!} = 6$$

Basic Solution (BS)

Rank (A) = m ; m → L.I.

Consider a system "AX=b", $(A)_{m\times n}$, of m equations with n variables (n > m), and Put (n−m) variables to zero and the resulting system $B X_B = b$ solution is called the Basic Solution.

$(B)_{m\times m}$; $|B| \neq 0$

Basic Feasible Solution (BFS) and infeasible basic solution

If the values of all the variables in the basic solution are non-negative, it is called a basic feasible solution (BFS). A basic solution having at least one variable with a negative value is called an infeasible basic solution.

- Optimal B.S.
- B.S.

$BFS \leq B.S. \leq {}^nC_m$

- Degenerate BS | BFS

LPP: $x_i \geq 0$; $s_i \geq 0$

- BFS
- $A X = b$ → $X = (X_B, X_N)$ (B.V., Non B.V.)

X ⇒ all decision Variable (n)

$B X_B = b$

Basis Matrix → $(B)_{m\times m} (X_B)_{m\times 1} = (b)_{m\times 1}$

$X_B = B^{-1} \cdot b$

$X_N = 0$

$X = (X_B, X_N)$

$X_B = (x_1, x_2)$

$X_N = (s_1, s_2)$

2×4

$X = (x_1, x_2, s_1, s_2)$

x_1, x_2 = ? → X_B → B.V.

s_1, s_2 = 0 → X_N → Non B.V.

Basic Solution (BS)

slack/surplus

Rank (A) = m]

Consider a system "AX=b", $(A)_{mxn}$, of m equations with n variables (n > m), and Put (n−m) variables to zero and the resulting system $B X_B = b$ solution is called the Basic Solution.

$x_i \geq 0$

Basic Feasible Solution (BFS)

If the values of all the variables in the basic solution are non-negative, it is called a basic feasible solution (BFS).

infeasible basic solution

A basic solution having at least one variable with a negative value is called an infeasible basic solution.

Degenerate and non-degenerate basic feasible solutions

A BFS with at least one basic variable assuming the value zero is called a degenerate BFS; otherwise it is called a non-degenerate BFS.

Optimal BFS: BFS that optimize the objective function value.

Eg: $X = (\overbrace{x_1, x_2}^{B.V.}, \overbrace{s_1, s_2}^{N.B.V.})$

$= (2, 3, 0, 0)$

$\neq 0 \quad \neq 0$

or $= (-2, 3, 0, 0)$

or $= (0, 5, 0, 0)$

≥ 0

degenerate BFS

OBJECTIVES

1. Connection of Graphical & Algebraic Method

   Connection of Corner Point & BFS

2. Vertices $\leq$ BFS $\leq$ BS $\leq$ ${}^nC_m$
   (Corner Point)

   $(A)_{m \times n}$ ${}^nC_m$

   Rank $= m$

   B.V. $= m$; N.B.V $= n-m$

3. Optimal Solution using basic Solution

Problem. Write all basic solution to the following system and find optimal solution. Maximize $Z = 2x_1 + 3x_2$

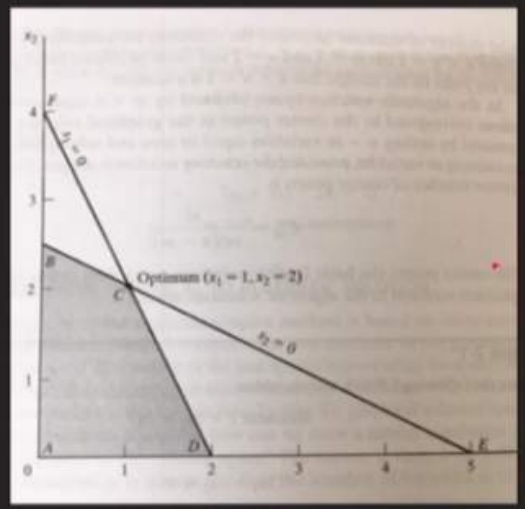
$$\begin{cases} 2x_1 + x_2 \le 4 \\ x_1 + 2x_2 \le 5 \quad \text{and} \quad x_1, x_2 \ge 0 \end{cases}$$

Solution. Convert inequalities into equality by adding slack/surplus variable

$$\begin{cases} 2x_1 + x_2 + s_1 = 4 \\ x_1 + 2x_2 + s_2 = 5 \quad \text{and} \quad x_1, x_2, s_1, s_2 \ge 0 \end{cases}$$

Non Basic Variables	Basic Variables	Basic solution	Associated corner point	Feasibility	Objective Value
$(x_1, x_2) = 0$	(s_1, s_2)	(4, 5)	A	Yes	0
$(x_1, s_1) = 0$	(x_2, s_2)	(4, -3)	E	No	-
$(x_1, s_2) = 0$	(x_2, s_1)	(2.5, 1.5)	B	Yes	7.5
$(x_2, s_1) = 0$	(x_1, s_2)	(2, 3)	D	Yes	4
$(x_2, s_2) = 0$	(x_1, s_1)	(5, -6)	E	No	-
$(s_1, s_2) = 0$	(x_1, x_2)	(1, 2)	C	Yes	8 (Optimum)

Graphical and algebraic solution connection



$$\text{Max } Z = 2x_1 + 3x_2 + 0 \cdot s_1 + 0 \cdot s_2$$

$$= 2 \cdot 0 + 3(2.5) + 0 \cdot 1.5 + 0$$

$$= 15 = 7.5$$

optimal!!

$$\boxed{Z_{max} = 8 \text{ at } x_1 = 1, x_2 = 2}$$

Problem. Write all basic solution to the following system and find optimal solution. Maximize $Z = 2x_1 + 3x_2$

$$2x_1 + x_2 \le 4$$
$$x_1 + 2x_2 \le 5 \quad \text{and} \quad x_1, x_2 \ge 0$$

Solution. Convert inequalities into equality by adding slack/surplus variable

$$2x_1 + x_2 + s_1 = 4$$
$$x_1 + 2x_2 + s_2 = 5 \quad \text{and} \quad x_1, x_2, s_1, s_2 \ge 0$$

$$\begin{cases} 2x_1 + x_2 + s_1 = 4 \\ x_1 + 2x_2 + s_2 = 5 \end{cases}$$

$2 + 2(0) + s_2 = 5$

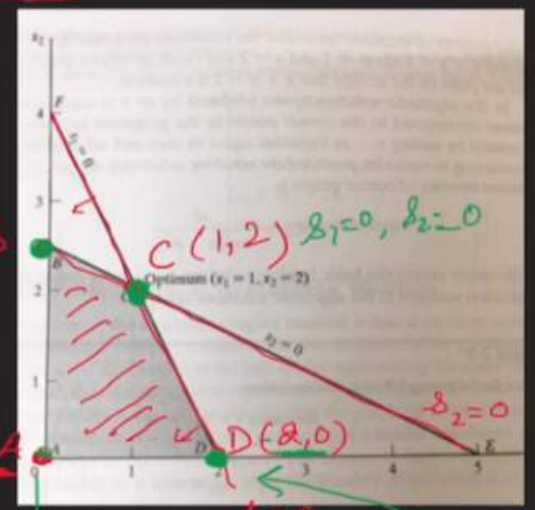
$s_2 = 3$

Corner — BFS

Graphical and algebraic solution connection

BFS

Non Basic Variables	Basic Variables	Basic solution	Associated corner point	Feasibility	Objective Value
(x_1, x_2)	(s_1, s_2)	(4, 5)	A	Yes	0
(x_1, s_1)	(x_2, s_2)	(4, -3)	F	No	-
(x_1, s_2)	(x_2, s_1)	(2.5, 1.5)	B	Yes	7.5
(x_2, s_1)	(x_1, s_2)	(2, 3)	D	Yes	4
(x_2, s_2)	(x_1, s_1)	(5, -6)	E	No	-
(s_1, s_2)	(x_1, x_2)	(1, 2)	C	Yes	8 (Optimum)



$$\begin{cases} s_1 = 4 \\ s_2 = 5 \end{cases}$$

Non-degen B.V. $\neq 0$

degenerate BFS

4 BFS | 4 Corner Points

vertices corner pt $\le$ BFS $\le$ BS $\le$ ${}^nC_m$

Problem: Find Basic Solution, Identify BFS; degenerate BS; Show correspondence with graphical corner points:

$$\text{Max } Z = 2x_1 + x_2$$
$$\text{s.t. } x_1 + x_2 + x_3 = 3$$
$$x_1 - x_2 + x_4 = 0$$
$$x_i \geq 0;\ i = 1,2,3,4$$

$(A)_{2\times 4}$

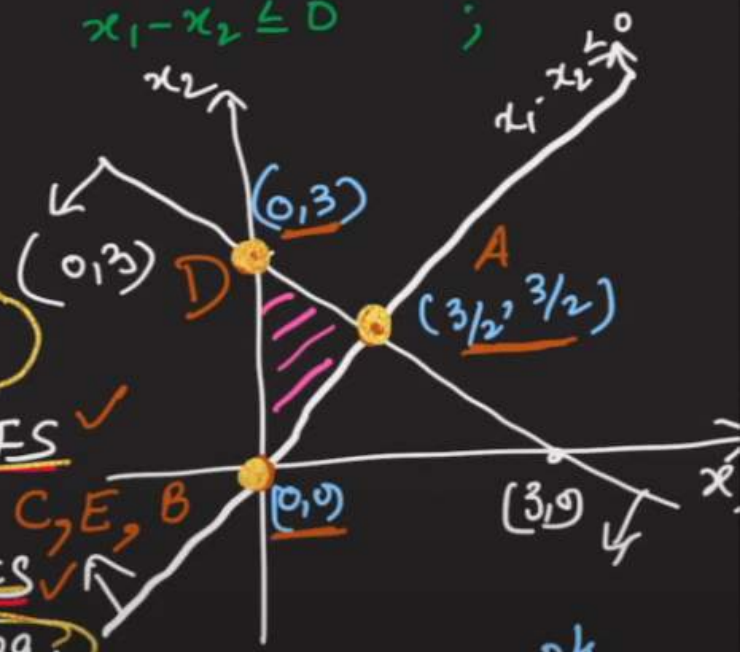
$\text{Max } Z = 2x_1 + x_2$

$x_1 + x_2 \leq 3$

$x_1 - x_2 \leq 0$;

Solution: Basic solution

	B.V.	N.B.V.	Solution x_1, x_2, x_3, x_4		
1.	x_1, x_2	$x_3 = x_4 = 0$	A $(3/2, 3/2, 0, 0)$ ✓	BFS ✓	Non deg.
2.	x_1, x_3	$x_2 = x_4 = 0$	B $(0, 0, 3, 0)$ ✓	BFS	deg. BFS ✓
3.	x_1, x_4	$x_2 = x_3 = 0$	$(3, 0, 0, -3)$ ✗		
4.	x_2, x_3	$x_1 = x_4 = 0$	C $(0, 0, 3, 0)$ ✓	BFS	deg. BFS ✓
5.	x_2, x_4	$x_1 = x_3 = 0$	D $(0, 3, 0, 3)$ ✓	BFS	Non deg.
6.	x_3, x_4	$x_1 = x_2 = 0$	E $(0, 0, 3, 0)$ ✓	BFS	deg. BFS ✓



3-Corner pts

3 Vertices < 5 = BFS < $6 = {}^4C_2$

Problem. Determine the optimum solution for following LP by enumerating all the basic solution.

Maximize $Z = 2x_1 - 4x_2 + 5x_3 - 6x_4$

$$x_1 + 4x_2 - 2x_3 + 8x_4 \le 2$$
$$-x_1 + 2x_2 + 3x_3 + 4x_4 \le 1$$
$$x_1, x_2, x_3, x_4 \ge 0$$

$$A = \begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & s_1 & s_2 \end{array} \\ \begin{pmatrix} 1 & 4 & -2 & 8 & 1 & 0 \\ -1 & 2 & 3 & 4 & 0 & 1 \end{pmatrix} \end{array}_{2\times 6}$$

Coefficient Matrix

$$AX = b \qquad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ s_1 \\ s_2 \end{pmatrix} \qquad b = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$${}^6C_2 = 15$$

Solution: Introduce slack variables in the constraints.

Cases	B.V.	Non-B.V.	Solution $(x_1,x_2,x_3,x_4,s_1,s_2)$	Value of Z
1	x_1, x_2	$x_3 = x_4 = s_1 = s_2 = 0$	(0, 1/2, 0, 0, 0, 0)	-2
2	x_1, x_3	$x_2 = x_4 = s_1 = s_2 = 0$	(8, 0, 3, 0, 0, 0)	31 (Optimal)
3	x_1, x_4	$x_2 = x_3 = s_1 = s_2 = 0$	(0, 0, 0, 1/4, 0, 0)	-1.5
4	x_1, s_1	$x_2 = x_3 = x_4 = s_2 = 0$	(-1, 0, 0, 0, 3, 0)	Not a BFS × (BS)
5	x_1, s_2	$x_2 = x_3 = x_4 = s_1 = 0$	(2, 0, 0, 0, 0, 3)	4
6	x_2, x_3	$x_1 = x_4 = s_1 = s_2 = 0$	(0, 1/2, 0, 0, 0, 0)	-2
7	x_2, x_4	$x_1 = x_3 = s_1 = s_2 = 0$	Not a part of BS	Linear dependent columns
8	x_2, s_1	$x_1 = x_3 = x_4 = s_2 = 0$	(0, 1/2, 0, 0, 0, 0)	-2
9	x_2, s_2	$x_1 = x_3 = x_4 = s_1 = 0$	(0, 1/2, 0, 0, 0, 0)	-2
10	x_3, x_4	$x_1 = x_2 = s_1 = s_2 = 0$	(0, 0, 0, 1/4, 0, 0)	-1.5
11	x_3, s_1	$x_1 = x_2 = x_4 = s_2 = 0$	(0, 0, 1/3, 0, 8/3, 0)	5/3=1.6
12	x_3, s_2	$x_1 = x_2 = x_4 = s_1 = 0$	(0, 0, -1, 0, 0, 4)	Not a BFS
13	x_4, s_1	$x_1 = x_2 = x_3 = s_2 = 0$	(0, 0, 0, 1/4, 0, 0)	-1.5
14	x_4, s_2	$x_1 = x_2 = x_3 = s_1 = 0$	(0, 0, 0, 1/4, 0, 0)	-1.5
15	s_1, s_2	$x_1 = x_2 = x_3 = x_4 = 0$	(0, 0, 0, 0, 2, 1)	0

4 cases ×

same corner pt.

vertices ≤ BFS ≤ BS ≤ ${}^nC_m$]
comb. ||
15

$AX = b$

B.V. | N.B.V.

x_2, x_4 $\qquad$ $x_1 = x_3 = s_1 = s_2 = 0$

$$\begin{vmatrix} 4 & 8 \\ 2 & 4 \end{vmatrix} = 16 - 16 = 0$$

$$B X_B = b \Rightarrow X_B = B^{-1} b$$

$$\begin{pmatrix} 4 & 8 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} x_2 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\begin{bmatrix} 4x_2 + 8x_4 = 2 \\ 2x_2 + 4x_4 = 1 \end{bmatrix}$$

$B \neq 0$

Problem. Consider the following LPP : Maximize $Z = 2x_1 + 3x_2 + 5x_3$

$$6x_1 + 7x_2 - 9x_3 \geq 4$$
$$x_1 + x_2 + 4x_3 = 10$$
$$x_1,\ x_3 \geq 0,\ x_2\text{-unrestricted in sign}$$

Show that a basic solution cannot include both x_2^+ and x_2^- simultaneous where $x_2 = x_2^+ - x_2^-$

Solution: Introduce Surplus variable in first constraint.

Cases	B.V.	Non-B.V.	Solution	Value of Z
1	x_1, x_2^+	$x_2^- = x_3 = s_1 = 0$	(66/13, 64/13, 0, 0, 0)	24.9
2	x_1, x_2^-	$x_2^+ = x_3 = s_1 = 0$	(66/13, -64/13, 0, 0, 0)	
3	x_1, x_3	$x_2^+ = x_2^- = s_1 = 0$	(-106/15, 0, 0, 64/15, 0)	
4	x_1, s_1	$x_2^+ = x_2^- = x_3 = 0$	(10, 0, 0, 0, 64)	20
5	x_2^+, x_2^-	Should not be a choice of BS		
6	x_2^+, x_3	$x_1 = x_2^- = s_1 = 0$	(0, 106/37, 0,66/37, 0)	17.5
7	x_2^+, s_1	$x_1 = x_2^- = x_3 = 0$	(0, 0, 10, 0, 66)	30 (Optimal)
8	x_2^-, x_3	$x_1 = x_2^+ = s_1 = 0$	(0, -106/37, 0, 66/37, 0)	
9	x_2^-, s_1	$x_1 = x_2^+ = x_3 = 0$	(0, 0, -10, 0, 66)	
10	x_3, s_1	$x_1 = x_2^+ = x_2^- = 0$	(0, 0, 0, 10/4, 106/4)	12.5

Optimal solution is found by usual manner

$$6x_1 + 7x_2^+ - 7x_2^- - 9x_3 - s_1 = 4$$
$$x_1 + \underbrace{x_2^+ - x_2^-} + 4x_3 = 10$$

$x_B = B^{-1}b$

$$AX = b \Rightarrow BX_B = b$$

$$\begin{pmatrix} 6 & 7 & -7 & -9 & -1 \\ 1 & 1 & -1 & 4 & 0 \end{pmatrix}_{2\times 5} \begin{pmatrix} x_1 \\ x_2^+ \\ x_2^- \\ x_3 \\ s_1 \end{pmatrix} = \begin{pmatrix} 4 \\ 10 \end{pmatrix}$$

L.D.

$^5C_2 = 10$ cases.

B.V. $= x_2^+, x_2^-$

$$B = \begin{vmatrix} 7 & -7 \\ 1 & -1 \end{vmatrix} = 0$$

$BS \leq {}^nC_m$

Problems 1: Write the LPP into standard form

$$\text{Max } Z = x_1 + x_2 - 2x_3$$

s.t. $x_1 + x_2 + x_3 \le 15$

$2x_1 - x_2 + x_3 \le -10$

$x_1 + 2x_3 = 10$

$x_1, x_2 \ge 0$; x_3 – unrestricted in sign

Solution: $x_1, x_2 \ge 0$ ✓

$x_3 = x_3' - x_3''$; $x_3' \ge 0$, $x_3'' \ge 0$

(or $x_3 = 7 = 7 - 0$, $x_3 = -7 = 0 - 7$)

LPP:

$$\text{Max } Z = x_1 + x_2 - 2(x_3' - x_3'')$$

s.t. $x_1 + x_2 + x_3' - x_3'' \le 15$

$2x_1 - x_2 + x_3' - x_3'' \le -10$

$x_1 + 2(x_3' - x_3'') = 10$

$x_1 \ge 0$; $x_2 \ge 0$; $x_3' \ge 0$; $x_3'' \ge 0$

LPP: $\text{Max } Z = x_1 + x_2 - 2x_3' + 2x_3''$

s.t. $x_1 + x_2 + x_3' - x_3'' \le 15$

$-2x_1 + x_2 - x_3' + x_3'' \ge 10$

$x_1 + 2x_3' - 2x_3'' = 10$

all var. ≥ 0 $\uparrow \ge 0$

Standard LPP

$$\text{Max } Z = x_1 + x_2 - 2x_3' + 2x_3'' + 0.s_1 + 0.s_2$$

s.t.

$x_1 + x_2 + x_3' - x_3'' + s_1 = 15$

$-2x_1 + x_2 - x_3' + x_3'' - S_2 = 10$

$x_1 + 2x_3' - 2x_3'' = 10$

$x_1, x_2, x_3', x_3'', s_1, s_2 \ge 0$

Example: Write the LPP into standard form

Minimize $Z = 4x_1 - 2x_2 + 6x_3$

s.t. $2x_1 - x_2 \le 6$

$x_1 + 4x_2 + 7x_3 \ge 7$

$x_1 - 6x_3 \le -8$

$x_1 \ge 0$; $x_2 \ge 2$; x_3 – unrestricted in sign.

Solution:

$x_2 \ge 2 \Rightarrow x_2 - 2 \ge 0$

$\Rightarrow x_2' = x_2 - 2 \ge 0$ ✓

$\Rightarrow x_2' \ge 0$; $x_2' = x_2 - 2$

or $x_2 = x_2' + 2$

x_3 – unrestricted in sign

$x_3 = x_3' - x_3''$; $x_3' \ge 0$; $x_3'' \ge 0$

x_3 – unrestricted

$x_3' = \begin{cases} x_3 ; & x_3 \ge 0 \\ 0 ; & x_3 < 0 \end{cases}$ $\quad x_3'' = \begin{cases} 0 & x_3 \ge 0 \\ -x_3 & x_3 < 0 \end{cases}$

$x_3' - x_3'' = \begin{cases} x_3 & x_3 \ge 0 \\ x_3 & x_3 < 0 \end{cases}$

e.g. $x_3 = 7 = 7 - 0$

$x_3' = 7 \ge 0$

$x_3'' = 0 \ge 0$

$x_3 = -7 = 0 - 7$

$x_3' = 0 \ge 0$

$x_3'' = 7 \ge 0$

$$x_1 - 6(x_3' - x_3'') \le -8$$

Or

$$\text{Min } Z = 4x_1 - 2x_2' - 4 + 6x_3' - 6x_3''$$

s.t.

$$2x_1 - x_2' \le 8$$
$$x_1 + 4x_2' + 7x_3' - 7x_3'' \ge -1$$
$$x_1 - 6x_3' + 6x_3'' \le -8$$

$$x_1, x_2', x_3', x_3'' \ge 0$$

Or

$$\text{Min } Z = 4x_1 - 2x_2' - 4 + 6x_3' - 6x_3''$$

s.t.

$$2x_1 - x_2' \le 8$$
$$-x_1 - 4x_2' - 7x_3' + 7x_3'' \le 1$$
$$-x_1 + 6x_3' - 6x_3'' \ge 8$$
$$x_1, x_2', x_3', x_3'' \ge 0$$

$\Rightarrow$

or

$$\text{Min } Z = 4x_1 - 2x_2' - 4 + 6x_3' - 6x_3''$$

s.t.

$$2x_1 - x_2' + S_1 = 8$$
$$-x_1 - 4x_2' - 7x_3' + 7x_3'' + S_2 = 1$$
$$-x_1 + 6x_3' - 6x_3'' - S_3 = 8$$
$$x_1, x_2', x_3', x_3'', S_1, S_2, S_3 \ge 0$$

Example: Write LPP into standard form

$$\text{Max } Z = x_1 + x_2$$

s.t. $|4x_1 + x_2| \le 6$

$7x_1 - x_2 \ge 5$

$x_1 \ge 0 \; ; \; x_2 \ge 0$

Solution:

$$\text{Max } Z = x_1 + x_2$$

s.t. $-6 \le 4x_1 + x_2 \le 6$ ✓

$7x_1 - x_2 \ge 5$ ✓

$x_1 \ge 0 \; ; \; x_2 \ge 0$

$$\text{Max } Z = x_1 + x_2 + 0 \cdot s_1 + 0 \cdot s_2 + 0 \cdot s_3$$

$4x_1 + x_2 \ge -6 \Rightarrow -4x_1 - x_2 + s_1 = 6$

$4x_1 + x_2 \le 6$ ✓ $\Rightarrow 4x_1 + x_2 + s_2 = 6$

$7x_1 - x_2 \ge 5$ ✓ $\Rightarrow 7x_1 - x_2 - s_3 = 5$

$x_1 \ge 0 \; ; \; x_2 \ge 0 \; ; \; s_1, s_2, s_3 \ge 0$

standard for

Example: Write LPP into standard Form

$$\text{Max } Z = x_1 + 2x_2 - x_3$$

s.t. $x_1 + x_2 - x_3 \le 5$

$-x_1 + 2x_2 + 3x_3 \ge -4$

$2x_1 + 3x_2 - 4x_3 \ge 3$

$x_1 + x_2 + x_3 = 2$

$x_1 \ge 0$; $x_2 \ge p$; x_3 - unrestricted in sign

Solution

$x_1 \ge 0$

$x_2 \ge p \Rightarrow x_2 - p \ge 0$

$\therefore x_2' = x_2 - p$; $x_2 = x_2' + p$; $x_2' \ge 0$

$x_3 = x_3' - x_3''$; $x_3' \ge 0$; $x_3'' \ge 0$

$$\text{Max } Z = x_1 + 2(x_2' + p) - x_3' - x_3''$$

s.t. $x_1 + (x_2' + p) - (x_3' - x_3'') \le 5$

$-x_1 + 2(x_2' + p) + 3(x_3' - x_3'') \ge -4$

$2x_1 + 3(x_2' + p) - 4(x_3' - x_3'') \ge 3$

$x_1 + (x_2' + p) + (x_3' - x_3'') = 2$

$x_1, x_2', x_3', x_3'' \ge 0$

OR

After introducing slack/surplus variable

$$\text{Max } Z = x_1 + 2(x_2' + p) - x_3' - x_3''$$

s.t.

$x_1 + x_2' - x_3' + x_3'' + s_1 = 5 - p$

$x_1 - 2x_2' - 3x_3' + 3x_3'' + s_2 = 4 + 2p$

$2x_1 + 3x_2' - 4x_3' - 4x_3'' - s_3 = 3 - 3p$

$x_1 + x_2' + x_3' - x_3'' = 2 - p$

$x_1, x_2', x_3', x_3'' \ge 0$

Solution

$x_1 \geq 0$

$$\begin{cases} x_2 \geq p \Rightarrow \underline{x_2 - p} \geq 0 \\ \quad \therefore x_2' = x_2 - p \;;\; x_2 = x_2' + p \;;\; x_2' \geq 0 \\ x_3 = x_3' - x_3'' \;;\; x_3' \geq 0 \;;\; x_3'' \geq 0 \end{cases}$$

After introducing slack/surplus variable

$$\text{Max } Z = x_1 + 2(x_2' + p) - x_3' - x_3''$$

s.t.

$$x_1 + x_2' - x_3' + x_3'' + s_1 = 5 - p$$
$$x_1 - 2x_2' - 3x_3' + 3x_3'' + s_2 = 4 + 2p$$
$$2x_1 + 3x_2' - 4x_3' - 4x_3'' - s_3 = 3 - 3p$$
$$x_1 + x_2' + x_3' - x_3'' = 2 - p$$
$$x_1, x_2', x_3', x_3'' \geq 0$$

R.H.S.

$5 - p \geq 0 \;;\; 4 + 2p \geq 0 \;;\; 3 - 3p \geq 0 \;;\; 2 - p \geq 0$

$p \leq 5 \;;\; 2p \geq -4 \;;\; 3 \geq 3p \;;\; p \leq 2$

$p \geq -2 \;;\; p \leq 1$

$p \geq -2 \;;\; p \leq 1 \quad \therefore \; p \in [-2, 1]$

Matrix

Mathematical Backgrounds-Simplex Method

- Matrices & Determinants
  Linear equations
  Vectors
  Hyperplanes
  Convex Sets/ Convex Hull
  Simplex
  Some basic results for LPP
  Simplex Algorthim main theorems/ Fundamental Theorem of LPP

Definition: A matrix is a rectangular array of mn members, real or complex, arranged in m-rows and n-columns.

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}_{m \times n}$$

e.g. $A = \begin{bmatrix} 4 & 5 & 6 \\ 2 & 0 & 3 \end{bmatrix}_{2 \times 3}$ or $A = \begin{pmatrix} 4 & 5 & 6 \\ 2 & 0 & 3 \end{pmatrix}_{2 \times 3}$

- $m = n$; $A \rightarrow$ square Matrix
- $1 \times n$; row-Matrix
- $m \times 1$; column matrix
- $a_{ij} = 0 \ \forall i, j$ Null Matrix

  $\forall$ → for all
- A square matrix; principle diagon

$A = \begin{bmatrix} 1 & 2 & 3 & 0 & 7 \end{bmatrix}$

Row Matrix

$A = \begin{pmatrix} 2 \\ 3 \\ 4 \\ 6 \end{pmatrix}_{4 \times 1}$

$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{3 \times 3}$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}_{3\times 3}$$

$$\begin{bmatrix} 1 \end{bmatrix}_{1\times 1} \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times 2} \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}_{4\times 4}$$

- $A = (a_{ij})_{m\times n}$; $B = (b_{ij})_{m\times n}$

$A = B$ iff $a_{ij} = b_{ij}$ $\forall i, j$

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 4 \\ 6 & 3 \end{bmatrix} \qquad A \neq B$$

$$A = \begin{bmatrix} 1 & 4 & 5 \\ 2 & 3 & 6 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 4 & 5 \\ 2 & 3 & 6 \end{bmatrix} \qquad A = B.$$

Transpose

$A = (a_{ij})_{m \times n}$; interchange row & column.

A^T ;

$$A = \begin{bmatrix} 2 & 3 & 5 \\ 1 & 0 & 2 \end{bmatrix}_{2 \times 3}$$

$$A^T = \begin{bmatrix} 2 & 1 \\ 3 & 0 \\ 5 & 2 \end{bmatrix}_{3 \times 2}$$

$$A = \begin{bmatrix} 1 & 4 & 5 \\ 2 & 3 & 6 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 4 & 5 \\ 2 & 3 & 6 \end{bmatrix} \qquad \underline{\underline{A = B}}.$$

- <u>Operations</u>:

Addition

Multiplication

$A = (a_{ij})_{m \times n}$

$B = (b_{ij})_{m \times n}$

$A + B = (C_{ij})_{m \times n}$

$\underline{\underline{C_{ij}}} = a_{ij} + b_{ij}$

$$A = \begin{bmatrix} \overset{a_{11}}{1} & \overset{a_{12}}{2} & \overset{a_{13}}{3} \\ 4 & 5 & 6 \end{bmatrix} \qquad B = \begin{bmatrix} \overset{b_{11}}{1} & \overset{b_{12}}{0} & \overset{b_{13}}{2} \\ 2 & 3 & 4 \end{bmatrix}$$

$$\underline{\underline{A+B}} = \begin{bmatrix} 2 & 2 & 5 \\ 6 & 8 & 10 \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \end{bmatrix}$$

- **Multiplication**

$A = (a_{ij})_{m \times n}$, $B = (b_{jk})_{n \times p}$

col of A = Row of B

AB → defined

$AB = C = (c_{ik})$

Eg: $A = \begin{bmatrix} 4 & 5 & 6 \\ 1 & 0 & 2 \end{bmatrix}_{2 \times 3}$ $B = \begin{bmatrix} 1 & 0 \\ 3 & 2 \\ 0 & 1 \end{bmatrix}_{3 \times 2}$

$$AB = \begin{bmatrix} 4+15+0 & 0+10+6 \\ 1+0+0 & 0+0+2 \end{bmatrix}_{2 \times 2}$$

$$= \begin{bmatrix} 19 & 16 \\ 1 & 2 \end{bmatrix}_{2 \times 2}$$

Determinant :

Let A be a square matrix, then determinant of A (or $|A|$ or $\det A$) is defined as

$$|A| = \det A = \sum_{j=1}^{n} \left((-1)^{i+j} \det M_{ij}\right) a_{ij} \; ; \quad i\text{- fixed}$$

where $A = (a_{ij})_{n \times n}$ and M_{ij} = Minor of a_{ij}

$$A = \begin{array}{c} \begin{array}{ccc} & & a_{13} \end{array} \\ \begin{bmatrix} 4 & 0 & 2 \\ 1 & 0 & 3 \\ 2 & 4 & 6 \end{bmatrix}_{3\times 3} \end{array} \quad i = 1$$

$$= 4\left((-1)^2(-12)\right) - 0\left(1\right) + 2(4)$$

$$= 48 - 0 + 8 = 56$$

Elementary transformations of a Matrix

An elementary row (or column) transformation of a matrix is one of the following types:

- Type 1. Interchange of any two rows (or columns)
- Type 2. Multiply elements of a row (column) by a non-zero scalar.
- Type 3. Adding to any row (or column), a scalar multiple of another row (or column)

$IA = A$

$I, A = A$

$$\boxed{BA = I}$$

$B \to$ act as A^{-1}.

Example: Find inverse of $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$ by elementary row-operations.

$$A = I\,A$$
$$\downarrow \qquad \downarrow$$
$$I \qquad B.A \quad ; \quad B = A^{-1}$$

$$[A : I] = \left[\begin{array}{ccc:ccc} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 3 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

Interchip $R_1 \leftrightarrow R_2$

Identity

$$\left[\begin{array}{ccc:ccc} 1 & 2 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 3 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$R_3 \to R_3 - 3R_1$

$1 - 3(2) = 1 - 6 = -5$

$$\left[\begin{array}{ccc:ccc} 1 & 2 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & -5 & -8 & 0 & -3 & 1 \end{array}\right]$$

$R_1 \to R_1 - 2R_2, \; R_3 \to R_3 + 5R_2$

$$\left[\begin{array}{ccc:ccc} 1 & 0 & -1 & -2 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ & & & & & \end{array}\right]$$

$R_3 \to \frac{1}{2}$,

Rank of a Matrix ; denoted by rank(A) or $\rho(A)$.

Rank of a matrix is the order of 'higher order non-vanishing Minor'.

$$A = \begin{pmatrix} 1 & -3 & 4 & 7 \\ 9 & 1 & 2 & 0 \end{pmatrix}_{2\times 4}$$

$$\rho(A) = 2$$

atleast

$$A' = \begin{pmatrix} -3 & 4 \\ 1 & 2 \end{pmatrix}$$

$$|A'| \neq 0$$

$$|A'| = -6-4$$

$$\neq 0$$

$$\begin{pmatrix} 1 & 4 \\ 9 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 4 & 7 \\ 2 & 0 \end{pmatrix}$$

$$\vdots$$

$\neq 0$!

Linear Equation : LPP

opt z =

$$\left\{\begin{array}{l} 2x + 3y + z = 4 \\ x - y - z = 1 \\ 2x + 4y = 5 \end{array}\right\} \longrightarrow A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & -1 & -1 \\ 2 & 4 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}_{3\times 1} = \begin{bmatrix} 4 \\ 1 \\ 5 \end{bmatrix}_{3\times 1}$$

coeff^n Matrix

$$AX = b$$

$$\underline{X = A^{-1} b}$$

$(A)_{n\times n}$	$(A)_{3\times 3}$
$e(A) = n$	$e(A) \leq 3$

$e(A) = 3$ Unique $\longrightarrow$

$e(A) < 3$ Infinite $\longrightarrow$

Vectors

Vectors: A vector in R^n is an ordered tuple of n-real numbers.

$\rightarrow a = (a_1, a_2, \ldots, a_n)$; $a_i \rightarrow$ components of a.

$n=1$

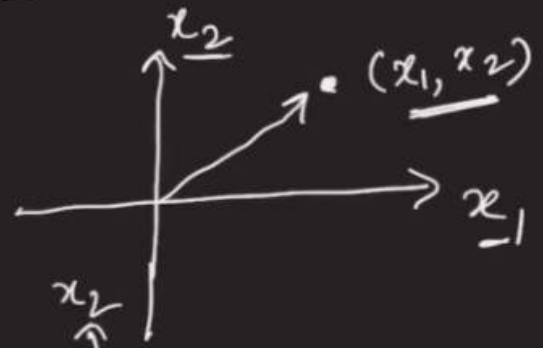
R^1: Real line

$a = (a_1)$; $a_1 \in R$.

$n=2$;

R^2: plane

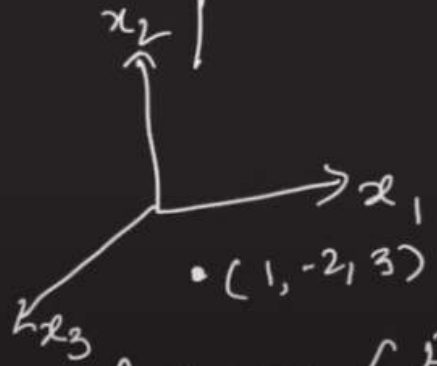
(xy-plane)



$a \in R^2$

$a = (a_1, a_2)$; $a_1, a_2 \in R$.

$n=3$; R^3:



$a = (a_1, a_2, a_3)$

$(1 \quad -2 \quad 3)$ $a_i \in R$.

$\vdots$

"n" R^n: $\rightarrow$ n-dimensional space (Euclidean space)

$a \in R^n$; $a = (a_1, a_2, \ldots, a_n)$; $a_i \in R$.

Properties:

- Equal Vector ; $a = (a_1, a_2, \dots, a_n)$; $b = (b_1, b_2, \dots, b_n)$

  $a = b$ iff $a_i = b_i$.

- Sum | difference of vector

  $$a \pm b = (a_1 \pm b_1, a_2 \pm b_2, \dots, a_n \pm b_n)$$

- multiplication of a vector by a scalar

  $$\lambda a = \lambda (a_1, \dots, a_n)$$
  $$= (\lambda a_1, \dots, \lambda a_n)$$

  and $\lambda(a+b) = \lambda a + \lambda b$;

  $\lambda(\mu a) = \lambda \mu a$

$\mathbb{R}^3$

eg. $a = (2, 1, 3)$

$4a = 4(2, 1, 3)$

$= (8, 4, 12)$

- Row and Column Vector : An n-vector $a = (a_1, a_2, \dots, a_n)$ is called a row vector.

  and $a^T = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$ is called a column vector.

- Linear Combination of vectors

Consider $\{a_1, a_2, \ldots, a_k\}$ is a set of k-vectors from $\mathbb{R}^n$

and $\{\lambda_1, \lambda_2, \ldots, \lambda_k\}$ is set of scalars

then $\lambda_1 a_1 + \lambda_2 a_2 + \cdots + \lambda_k a_k$ is a l.c. of vectors $a_1, \ldots, a_k$.

Eg: $a_1 = (2, 3) \to \mathbb{R}^2$, $a_2 = (1, 5) \to \mathbb{R}^2$, $a_3 = (4, 0) \to \mathbb{R}^2$

$\lambda_1 = 2$, $\lambda_2 = 4$, $\lambda_3 = 1$

$$(\quad) \to \mathbb{R}^2 = \lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3 = 2(2,3) + 4(1,5) + 1(4,0)$$

$$= (4, 6) + (4, 20) + (4, 0)$$

$$= (12, 26) \in$$

Linear dependent | Linear Independent Vectors

A set of vectors $a_1, a_2, \ldots, a_k \in \mathbb{R}^n$; for $\lambda_1, \ldots, \lambda_k$ - scalars

L.D. ; $\lambda_1 a_1 + \cdots + \lambda_k a_k = 0$ holds for some $\lambda_i \neq 0$ (or atleast one i)

L.I. ; $\lambda_1 a_1 + \cdots + \lambda_k a_k = 0$ holds only if $\lambda_i = 0$ for all i.

Eg: $a_1 = (1,2)$, $a_2 = (2,4)$ $\longrightarrow$ L.D.

$$\lambda_1 a_1 + \lambda_2 a_2 = 0$$

$\lambda_1 = 2$; $\lambda_2 = -1$

$$2(1,2) + (-1)(2,4) = 0 = (0,0)$$

$$(2,4) + (-2,-4)$$

$$(0,0)$$

Eg: consider $a_1 = (1,0,0) = e_1$ $\longrightarrow$ $\mathbb{R}^3$

$a_2 = (0,1,0) = e_2$

$a_3 = (0,0,1) = e_3$

$\mathbb{R}^1$

$e_1 = (1)$

$\mathbb{R}^2$

$e_1 = (1,0)$

$e_2 = (0,1)$

$$\alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3 = (0,0,0)$$

$$\alpha_1(1,0,0) + \alpha_2(0,1,0) + \alpha_3(0,0,1) = 0$$

$$\alpha_1 = \alpha_2 = \alpha_3 = 0$$

a_1, a_2, a_3 $\hookrightarrow$ L.I. vector.

Basis Set

A L.I. set of vectors $a_1, a_2, \ldots, a_k$ of R^n, having the property that any vector of R^n can be expressed as linear combination of these vector, is called a basis set for R^n.

$$a_1 = (1,0), \quad a_2 = (0,1)$$

$$(2,4) = \lambda_1 a_1 + \lambda_2 a_2$$
$$= 2(1,0) + 4(0,1)$$
$$= (2,0) + (0,4) = (2,4)$$

If the determinant is zero, they are linearly dependent,

Any other value of determinant (like here) they are Linearly independent.

$$(2,4) = \lambda_1 a_1 + \lambda_2 a_2$$
$$= 2(1,0) + 4(0,1)$$
$$= (2,0) + (0,4) = (2,4)$$

L.I.

4 $a_1 = (1,2)$, $a_2 = (5,6)$ $\quad \lambda_1 = 0 \; \lambda_2 = 0$;

$$\lambda_1(1,2) + \lambda_2(5,6) = (0,0)$$
$$(\lambda_1, 2\lambda_1) + (5\lambda_2, 6\lambda_2) = (0,0)$$
$$(\lambda_1 + 5\lambda_2, \; 2\lambda_1 + 6\lambda_2) = (0,0)$$

$$\lambda_1 + 5\lambda_2 = 0 \Rightarrow 2\lambda_1 + 10\lambda_2 = 0$$
$$2\lambda_1 + 6\lambda_2 = 0 \Rightarrow 2\lambda_1 + 6\lambda_2 = 0$$
$$4\lambda_2 = 0$$
$$\Rightarrow \lambda_2 = 0$$

$$\therefore \lambda_1 = -5\lambda_2$$
$$\lambda_1 = -5(0)$$
$$\lambda = 0$$

Alternate

$$\begin{vmatrix} 1 & 5 \\ 2 & 6 \end{vmatrix} = 6 - 10 \neq 0$$

or

$$\begin{vmatrix} 1 & 2 \\ 5 & 6 \end{vmatrix} = 6 - 10 \neq 0$$

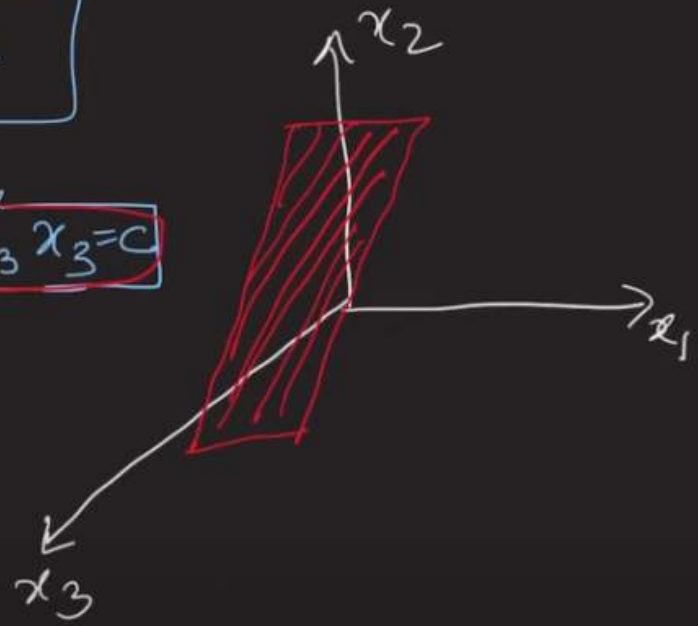
Hyperplanes

R^1 : $\boxed{ax = b}$ $\Rightarrow x = \frac{b}{a}$

$a \neq 0$

R^2 : $\boxed{ax + by = c}$ or $\boxed{a x_1 + b x_2 = c}$

R^3 : $\boxed{ax + by + cz = d}$ or $\boxed{a_1 x_1 + a_2 x_2 + a_3 x_3 = c}$



$\vdots$ $(a_1, \ldots, a_n)$

$\vdots$ $(x_1, \ldots, x_n)$

R^n : $(a_1 x_1 + \cdots + a_n x_n) = d.$

n-dimensional plane $\rightarrow$ Hyperplane

R^n: $(a_1x_1 + \cdots + a_nx_n) = d$

x_3

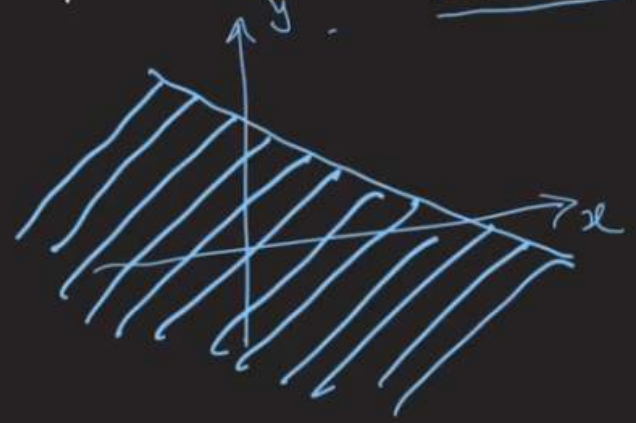
n-dimensional plane → Hyperplane

Note: In LPP; Objective function is;

$$Z = c_1x_1 + c_2x_2 + \cdots + c_nx_n = \beta = \begin{matrix}\max\\ \min\end{matrix}$$

R^2

$2x + 3y \leq 6$



Closed and Open Half Spaces

In R^n;

$a_1x_1 + \cdots + a_nx_n < d$

or

$a_1x_1 + \cdots + a_nx_n > d$

open half spaces

$a_1x_1 + \cdots + a_nx_n \leq d$

$a_1x_1 + \cdots + a_nx_n \geq d$

closed half.

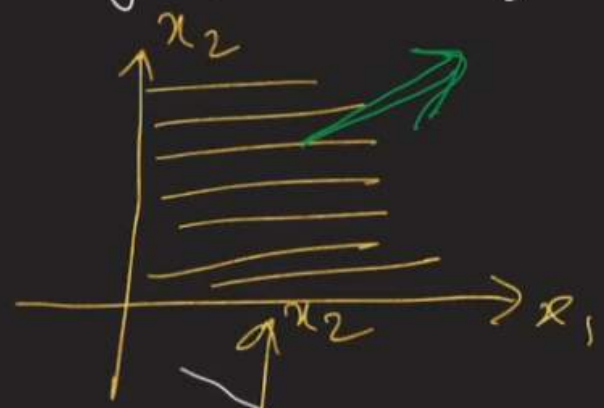
Polyhedron/
Polytope

Polytope | Polyhedron

The intersection of a finite no. of closed half spaces ; is called a polyhedron.

If polyhedon is bounded then it is called a polytope.

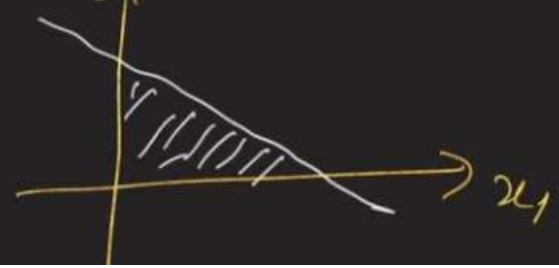
- polytope is closed and bounded having finite no. of corner points.



$$S = \{ (x_1, x_2) \; ; \; x_1 \geq 0 \; ; x_2 \geq 0 \}$$

Polyhedron

$$S = \{ (x_1, x_2) : x_1 + x_2 \leq 1, \; x_1 \geq 0 ; x_2 \geq 0 \}$$



Convex Sets

Convex Sets

Sets : A set is a collection of well-defined objects, called <u>elements</u>.

<u>Tabular form</u>

$S = \{1, 2, 3, \ldots\}$

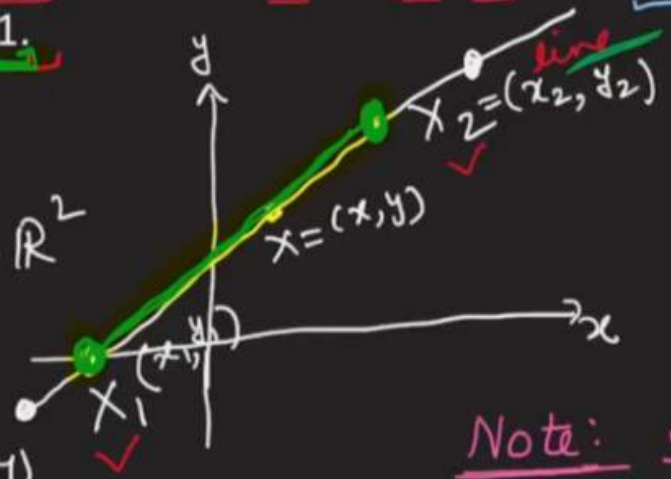
Set-builder form.

$S = \{x : x \in N\}$ → Natural Nos.

Eg: $S = \{(x_1, x_2) ; \underbrace{x_1^2 + x_2^2 = r^2}\}$

Convex Set:

A set S is said to be convex if for any two points X_1 and $X_2 \in S$, their line joining $X=\lambda X_1+(1-\lambda)X_2 \in S$, $0 \le \lambda \le 1$.



$0 \le \lambda \le 1$

$X = (1-\lambda)X_1 + \lambda X_2$

$\lambda = 0$; $X = X_1$

$\lambda = 1$; $X = X_2$

$0 < \lambda < 1$;

$X = (1-\lambda)X_1 + \lambda X_2$

Line Joining of Two points:

$S = \mathbb{R}^2$; let $X_1, X_2 \in S = \mathbb{R}^2$

Line joining X_1, X_2 is,

$$\frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1} = \lambda \text{ (say)}$$

$x = x_1 + \lambda(x_2-x_1)$; $y = y_1 + \lambda(y_2-y_1)$

$= x_1(1-\lambda) + \lambda x_2$; $y = (1-\lambda)y_1 + \lambda y_2$

$$X = \begin{pmatrix} x \\ y \end{pmatrix} = (1-\lambda)\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \lambda \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$$

$$X = (1-\lambda)X_1 + \lambda \cdot X_2$$

Note: $S = \mathbb{R}^4$

$X_1 = (a_1, a_2, a_3, a_4)$

$X_2 = (b_1, b_2, b_3, b_4)$ $\in \mathbb{R}^4$

$X = (1-\lambda)X_1 + (\lambda \cdot X_2)$

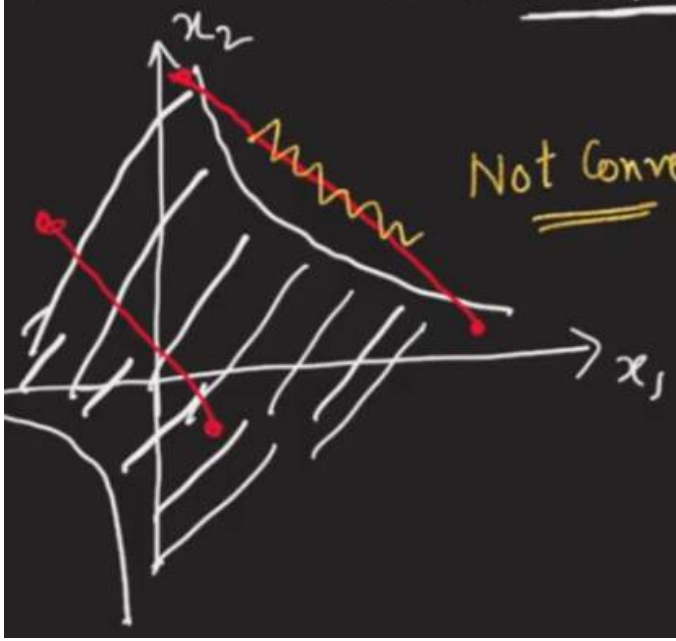
$$= (1-\lambda)\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} + \lambda \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}$$

Convex Set: A set S is called a convex set, if for any two points $x_1, x_2 \in S$; the line joining $\lambda x_1 + (1-\lambda) x_2$ also belongs to S; $0 \le \lambda \le 1$

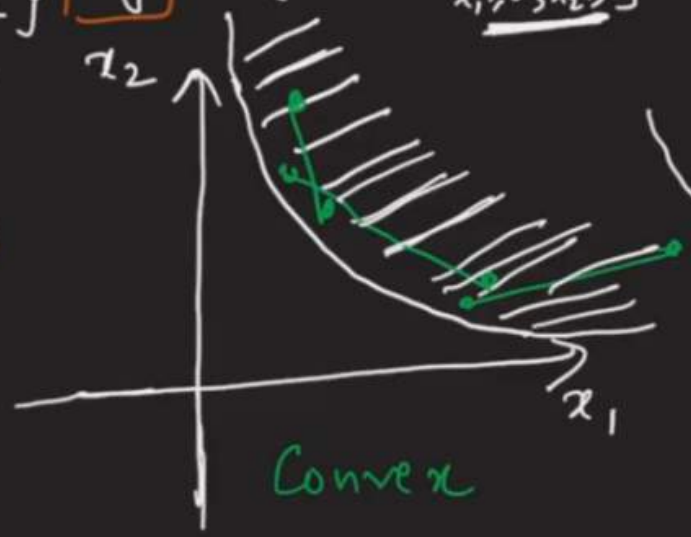
Two ways to check convex sets.

- Graphically
- Algebraically

$\boxed{\text{Eg:}}$ $S = \{ x \in \mathbb{R}^2 : x_1 x_2 \le 4 \}$ where $x = (x_1, x_2)$

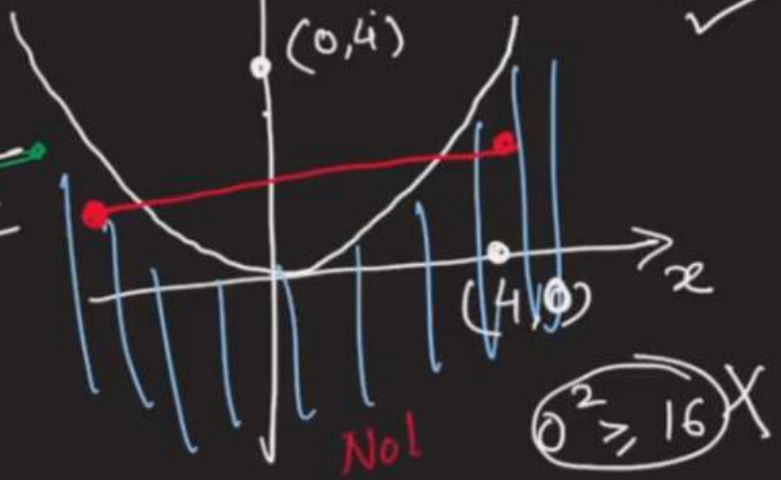


$\boxed{\text{Eg:}}$ $S = \{ x \in \mathbb{R}^2 : x_1 x_2 \ge 4 ;\ x_1 \ge 0 ; x_2 \ge 0 \}$



$\boxed{\text{Eg:}}$ $S = \{ x \in \mathbb{R}^2 : x^2 \ge 4y \}$

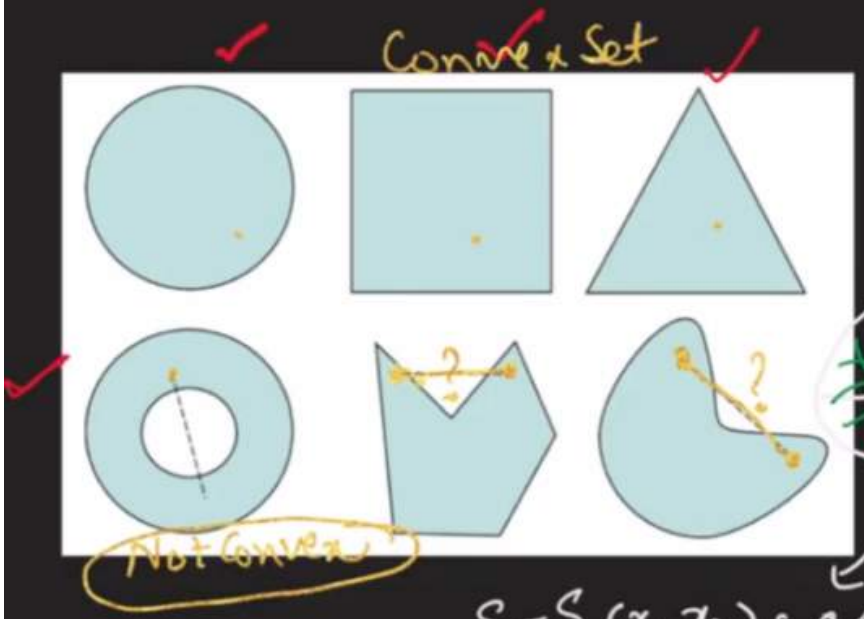
$4^2 \ge 4.0$

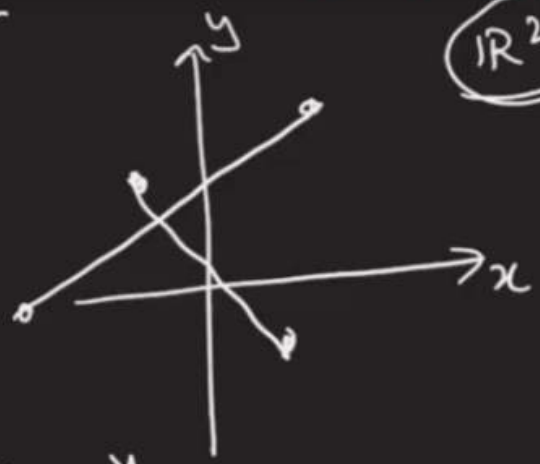


Convex Set in R^1:

Convex Set in R^2:

$[x_1, x_2] \in \mathbb{R}^1$ Convex Set





$\mathbb{R}^2$ whole. Convex Set

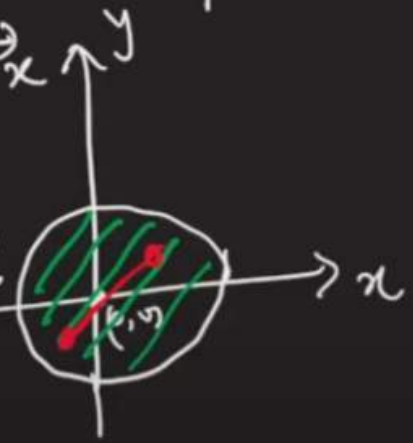
$S \subseteq R^2$

$S = \{(x_1, x_2) : x_1 x_2 \leq 4\}$

Not Convex

$S = \{(x_1, x_2) : x_1^2 + x_2^2 \leq 4\}$

$S = \{(x_1, x_2) : 0 < x_1^2 + x_2^2 \leq 4\}$



Convex Set in R^3:

Question: Show that the set $S = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 \le 4\}$ is convex.

Pf: Take $X_1, X_2 \in S$

$X_1 = (x_1, x_2)$: $x_1^2 + x_2^2 \le 4$ — (1)

$X_2 = (y_1, y_2)$; $y_1^2 + y_2^2 \le 4$ — (2)

To prove

$$\boxed{X = \lambda X_1 + (1-\lambda) X_2} \in S \quad ; \quad 0 \le \lambda \le 1.$$

$$\therefore \; X = \lambda (x_1, x_2) + (1-\lambda)(y_1, y_2)$$

$$= (\lambda x_1 + (1-\lambda) y_1 \; , \; \lambda x_2 + (1-\lambda) y_2)$$

$X \in S$

or To show

$$[\lambda x_1 + (1-\lambda) y_1]^2 + [\lambda x_2 + (1-\lambda) y_2]^2 \le 4 \quad \text{— (3)}$$

L.H.S. of (3) $= \lambda^2 x_1^2 + (1-\lambda)^2 y_1^2 + 2\lambda(1-\lambda) x_1 y_1$

$+ \lambda^2 x_2^2 + (1-\lambda)^2 y_2^2 + 2\lambda(1-\lambda) x_2 y_2$

$= \lambda^2 (x_1^2 + x_2^2) + (1-\lambda)^2 (y_1^2 + y_2^2) + 2\lambda(1-\lambda)[x_1 y_1 + x_2 y_2]$

$\le \lambda^2 \cdot 4 + (1-\lambda)^2 \cdot 4 + 2\lambda(1-\lambda) \cdot 4$

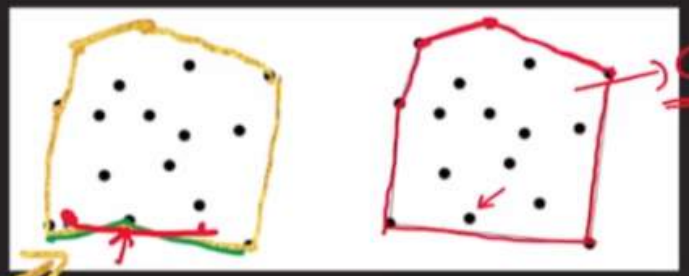
$\le 4(\lambda + (1-\lambda))^2 = 4$

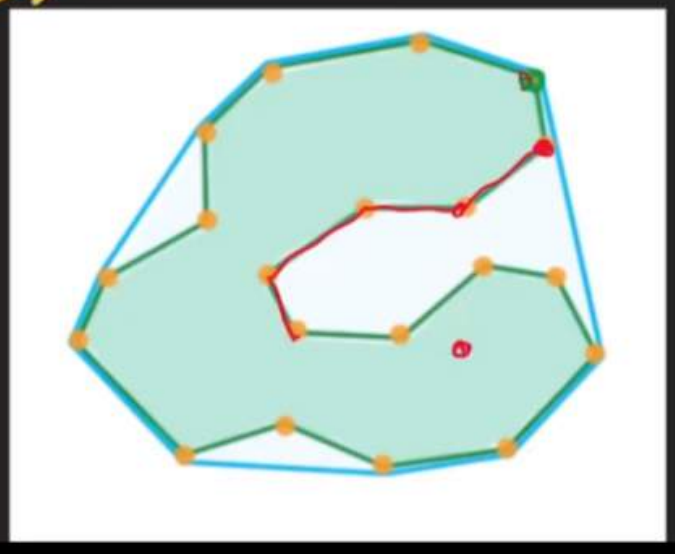
$x_1 y_1 + x_2 y_2 \le 4$ by Cauchy's inequality

Convex Hull

Convex Hull :

Let $S \subseteq \mathbb{R}^n$; then the smallest convex set containing the given set S is called the convex hull of S , denoted by $\mathrm{Conv}(S)$.





Convex Sets:

A set S is said to be convex if for $\boxed{\text{any}}$ two points X_1 and $X_2 \in S$, their line joining

$X = \lambda X_1 + (1-\lambda)X_2 \in S, \quad 0 \le \lambda \le 1.$

- Union of two convex sets is not convex.

$S_1 = \{(x,0) ; x \in R\}$ x-axis

$S_2 = \{(0,y) ; y \in R\}$ y-axis

$\Rightarrow S_1 \cup S_2 = \{(x,0), (0,y) ; x, y \in \mathbb{R}\}$

To show $S_1 \cup S_2$ is convex

$X_1 = (x,0) ; X_2 = (0,y) ;$

$X^* = \frac{1}{2}(x,0) + \frac{1}{2}(0,y)$

$= \left(\frac{x}{2}, \frac{y}{2}\right)$

Result : The intersection of two convex sets is a convex set.

Proof: Let S_1 and S_2 are two convex sets. <u>To prove $S_1 \cap S_2$ is convex set</u>

As S_1 is convex , so for any two points X_1 and $X_2 \in S_1$

we have $X = \lambda X_1 + (1-\lambda) X_2 \in S_1$, for $0 \le \lambda \le 1$

Also as S_2 is convex , so for any two points X_1 and $X_2 \in S_2$

we have $X = \lambda X_1 + (1-\lambda) X_2 \in S_2$, for $0 \le \lambda \le 1$

So for any two points X_1 and $X_2 \in S_1 \cap S_2$ we have to prove

$X = \lambda X_1 + (1-\lambda) X_2 \in S_1 \cap S_2$, which holds true due to above as

$\lambda X_1 + (1-\lambda) X_2 \in S_1$ and $\lambda X_1 + (1-\lambda) X_2 \in S_2$, hence $\lambda X_1 + (1-\lambda) X_2 \in S_1 \cap S_2$

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Also as S_2 is convex, so for any two points X_1 and $X_2 \in S_2$

we have $X = \lambda X_1 + (1-\lambda) X_2 \in S_2$, for $0 \le \lambda \le 1$

So for any two points X_1 and $X_2 \in S_1 \cap S_2$ we have to prove

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Result : The arbitrary intersection of convex sets is convex.

$S_1, S_2, \ldots, S_n$ then $\bigcap_{i=1}^{n} S_i$ is also convex

$\mathbb{R}^2$; $ax + by = c$

$\mathbb{R}^3$; $ax + by + cz = d$

$\mathbb{R}^n$; $a_1x_1 + a_2x_2 + \cdots + a_nx_n = \alpha$

$$\underbrace{(a_1 \cdots a_n)}_{1\times n}\underbrace{\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}}_{n\times 1} = \alpha \Rightarrow \quad X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$a^TX = \alpha$

Result : The hyperplane is a convex set.

Proof: To prove $S = \{X \in R^n$ where $a^TX = \alpha\}$ is convex set.

Take any two points X_1 and $X_2 \in S$ then we have to prove

$X = \lambda X_1 + (1-\lambda) X_2 \in S$, for $0 \le \lambda \le 1$

Now if $X_1 \in S$ then $a^TX_1 = \alpha$

$X_2 \in S$ then $a^TX_2 = \alpha$

$X = \lambda X_1 + (1-\lambda)X_2$

So

$$a^TX = a^T(\lambda X_1 + (1-\lambda) X_2) = a^T\lambda X_1 + a^T(1-\lambda)X_2$$
$$= \lambda a^TX_1 + (1-\lambda) a^TX_2$$
$$= \lambda \alpha + (1-\lambda)\alpha = \alpha$$

Hence $X \in S$, and S is convex.

Result : The half spaces are convex sets.

Proof: To prove $S' = \{X \in R^n$ where $a^TX \le \alpha\}$ is convex set.

Or

To prove $S = \{X \in R^n$ where $a^TX \ge \alpha\}$ is convex set.

$X \in S$

let $X_1, X_2 \in S'$

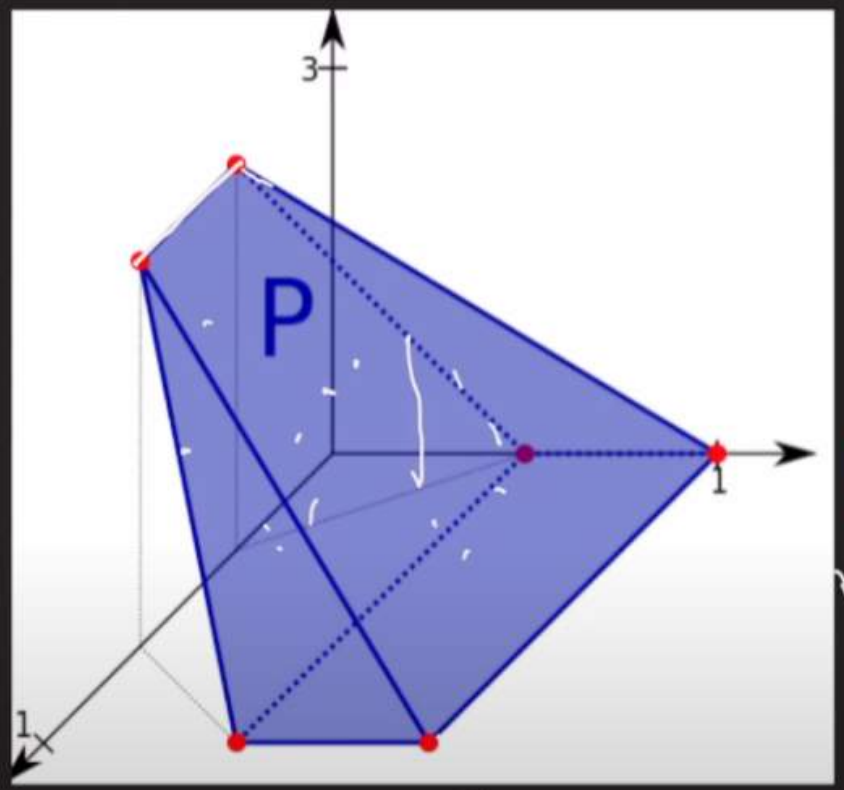
$a^TX_1 \le \alpha$; $a^TX_2 \le \alpha$

$X = \lambda X_1 + (1-\lambda) X_2$; $0 \le \lambda \le 1$

$$a^TX = \lambda a^TX_1 + (1-\lambda) a^TX_2$$
$$\le \lambda\alpha + (1-\lambda)\cdot\alpha$$
$$= \alpha(\lambda + (1-\lambda))$$
$$= \alpha$$

4

Polyhedron/Polytope is Convex Set



Intersection of hyperplane | half spaces.

Convex Set

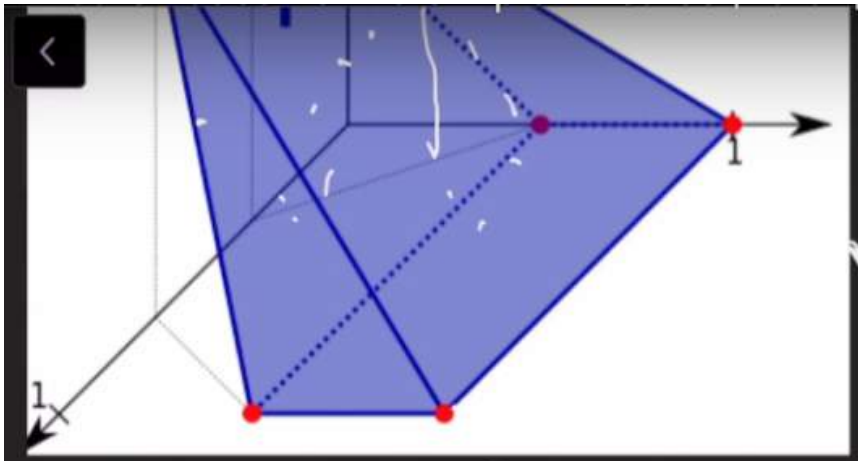
Convex Sets

Result: Feasible Region is Convex Sets.

$S = \{ X \in \mathbb{R}^n ; \; AX = b, \; X \geq 0 \}$

LPP

$$\text{Opt } Z = c^T X$$
$$\text{s.t. } AX = b$$
$$X \geq 0$$



PP $\quad$ Opt $z = c^T x$

s.t. $Ax = b$

$x \geq 0$

Result: Feasible Region is Convex Sets.

$$S = \{ x \in \mathbb{R}^n \; ; \; Ax = b, \; x \geq 0 \}$$

Let $x_1, x_2 \in S$

$Ax_1 = b, \; x_1 \geq 0$

$Ax_2 = b, \; x_2 \geq 0$

$\lambda \leq 1$

$0 \leq 1-\lambda$

$0 \leq \lambda \leq 1$

$$x^* = \lambda x_1 + (1-\lambda) x_2 \; ;$$

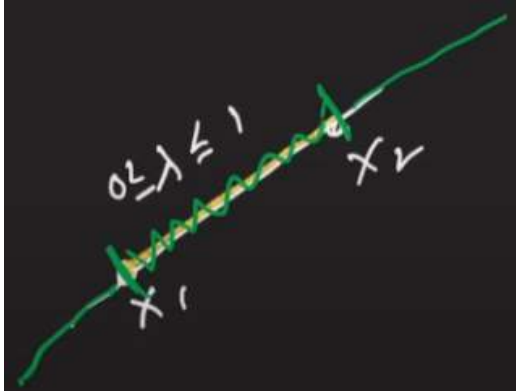
$$Ax^* = \lambda A x_1 + (1-\lambda) A x_2$$

$$= \lambda b + (1-\lambda) b = b$$

$Ax^* = b$

$x^* \geq 0$

$$x^* = \underbrace{\lambda}_{\geq 0} \underbrace{x_1}_{\geq 0} + \underbrace{(1-\lambda)}_{\geq 0} \underbrace{x_2}_{\geq 0}$$



CLC

Convex Set: A set $S \subseteq \mathbb{R}^n$ is called a convex set if for any two points $x_1, x_2 \in S$;

$$\lambda x_1 + (1-\lambda) x_2 \in S \quad ; \quad 0 \le \lambda \le 1.$$

or $\lambda_1 x_1 + \lambda_2 x_2 \in S$; $\lambda_1 + \lambda_2 = 1$; $\lambda_i \ge 0$

$\lambda_2 = 1 - \lambda_1$

Convex Linear Combination of n-points.

Consider $x_1, x_2, \ldots, x_n \in S \subseteq \mathbb{R}^n$; Then CLC of these n-pts. is $\lambda_1 x_1 + \cdots + \lambda_n x_n$; $\lambda_i \ge 0$ for all i ; $\lambda_1 + \cdots + \lambda_n = 1$

$$x^* = \lambda_1 x_1 + \cdots + \lambda_n x_n \quad ; \quad \lambda_i \ge 0 \quad ; \quad \sum_{i=1}^{n} \lambda_i = 1$$

$$\boxed{x^* \in S}$$

Vertex or Extreme Points

A point x which cannot be written as a CLC of two distinct points, is called a vertex or Extreme Point or Corner point.

$$x^* = \lambda_1 x_1 + \lambda_2 x_3 \quad ; \quad \lambda_1 = 0, \quad \lambda_2 = 1$$

$$x^* = x_3$$

$$x^* = \frac{1}{2} x_1 + \frac{1}{2} x_2$$

Problem: Find all extreme points of the set

$$S = \{ (x_1, x_2) : x_1 + 2x_2 \geq -2,\ -x_1 + x_2 \leq 4,\ x_1 \leq 4 \}$$

and represent the point $(2,3)$ as the convex linear combination of the extreme points of S.

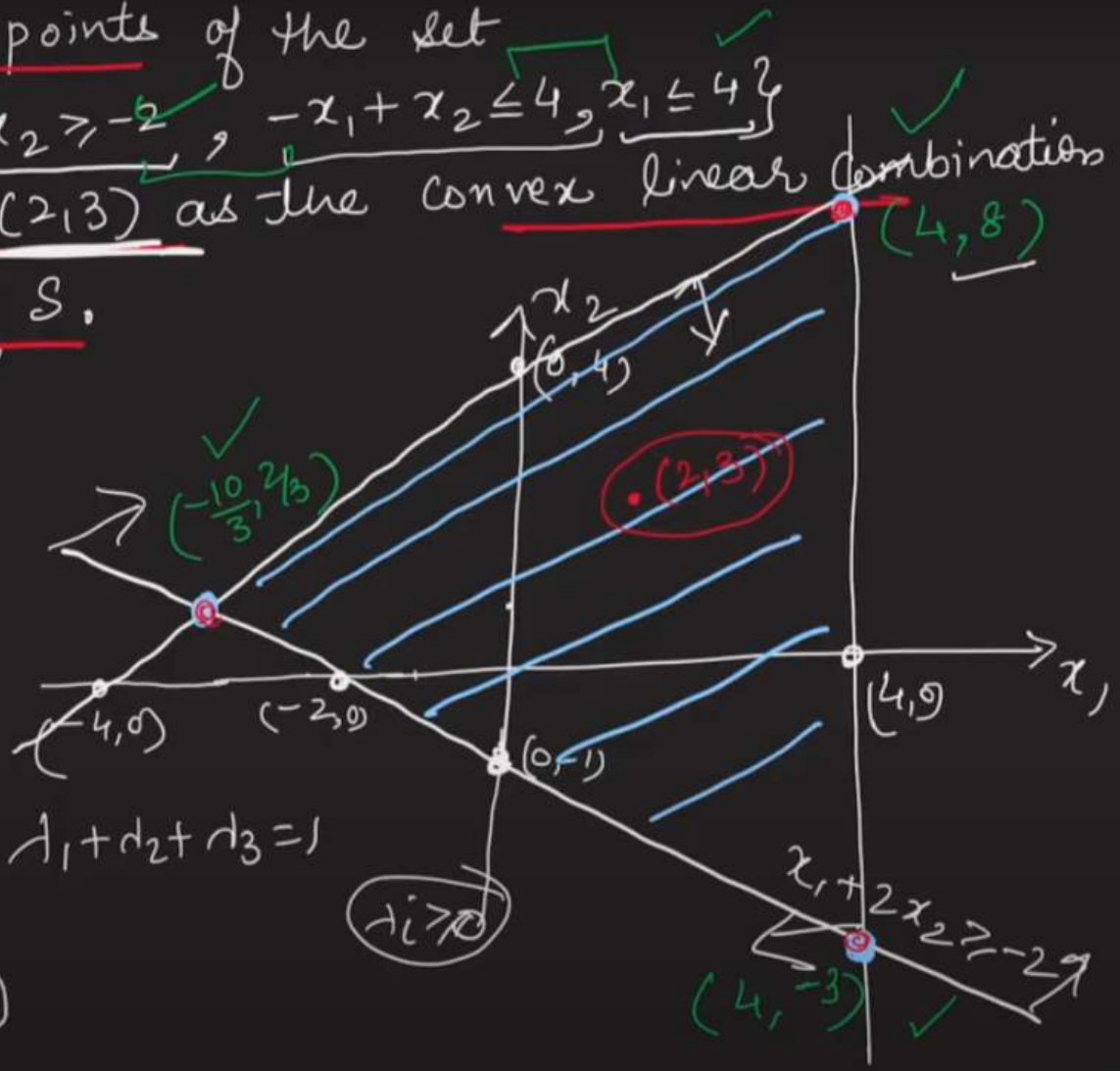
Solution:

$$(2,3) = \lambda_1 \left(-\tfrac{10}{3}, \tfrac{2}{3}\right) + \lambda_2 (4,8) + \lambda_3 (4,-3)$$

$$(2,3) = \left(-\tfrac{10}{3}\lambda_1 + 4\lambda_2 + 4\lambda_3,\ \tfrac{2}{3}\lambda_1 + 8\lambda_2 - 3\lambda_3\right)$$

$$\lambda_1 + \lambda_2 + \lambda_3 = 1$$

$$\lambda_i \geq 0$$

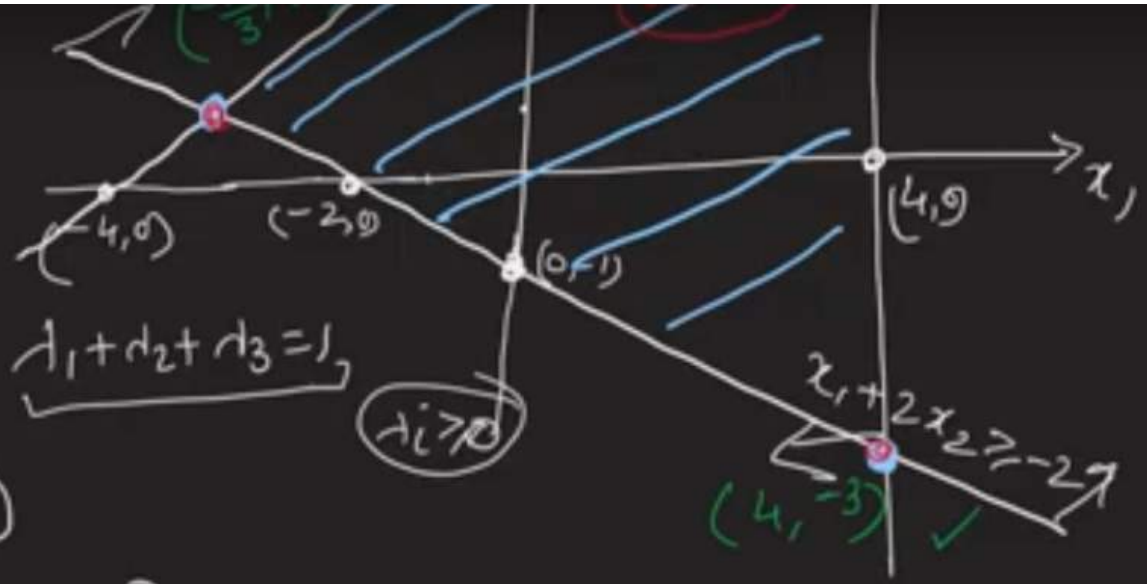


$$(2,3) = \lambda_1\left(-\frac{10}{3}, \frac{2}{3}\right) + \lambda_2(4,8) + \lambda_3(4,-3)$$

$$(2,3) = \left(-\frac{10}{3}\lambda_1 + 4\lambda_2 + 4\lambda_3,\ \frac{2}{3}\lambda_1 + 8\lambda_2 - 3\lambda_3\right)$$

$$\lambda_1 + \lambda_2 + \lambda_3 = 1$$

$$\lambda_i \geq 0$$



$$\left[\begin{aligned} -\frac{10}{3}\lambda_1 + 4\lambda_2 + 4\lambda_3 &= 2 \\ \frac{2}{3}\lambda_1 + 8\lambda_2 - 3\lambda_3 &= -3 \\ \lambda_1 + \lambda_2 + \lambda_3 &= 1 \end{aligned}\right\}$$

$\lambda_1, \lambda_2, \lambda_3$

SIMPLEX

Simplex in R^n:

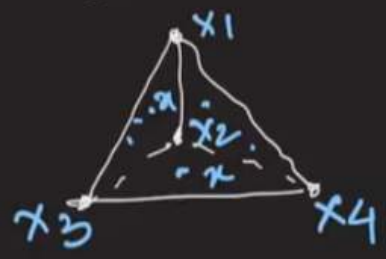
Let $x_1, x_2, \ldots, x_n, x_{n+1}$ be $(n+1)$ points in R^n. Then the convex set spanned by these $(n+1)$ points is called a n-simplex in R^n.

Eg:

0-Simplex • point

1-Simplex line segment

2-Simplex triangle

3-Simplex 

tetrahedron.

* Simplex is a polytope of $n+1$ points.

$$x = \sum_{i=1}^{4} \alpha_i x_i$$

$$\sum_{i=1}^{4} \alpha_i = 1$$

$$; \alpha_i \geq 0$$

Fundamental Theorem of Linear Programming

Statement: If the feasible region of an LPP is a convex polyhedron, then there exists an optimal solution to the LPP and atleast one corner point (/or basic feasible solution) must be optimal.

Proof: Existence, BFS / corner optimal

[$\text{Max } Z = c_1x_1 + \cdots + c_nx_n$]

LPP: $\text{Max } Z = c^T X$ s.t. $AX = b$; $X \geq 0$; $c, X \in \mathbb{R}^n$

$c = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$ $X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

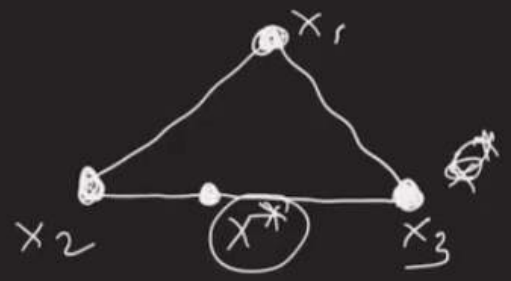
feasible region $S = \{ X \mid AX = b, X \geq 0 \}$

$\because$ S is convex polyhedron, it is non-empty closed, bounded.

$\Rightarrow$ optimal solution must exist.

Let x^* be optimal point of the given LPP ; $Z_0 = Z_{max} = C^T x^*$

$$\boxed{x^* \in S}$$



Now x^* is corner point. ; n.t.p.

On contary ;
assume x^* is not corner point.

As S is polytope it has finite no. of extreme points (closed polyhedron) $x_1, x_2, \ldots, x_K$.

for $x^* \in S$.

$$x^* = \sum_{i=1}^{K} \alpha_i x_i \quad ; \sum_{i=1}^{K} \alpha_i = 1 \; ; \; \alpha_i \geq 0 \; \forall i$$

$$C^T x^* = C^T \sum_{i=1}^{K} \alpha_i x_i = C^T(\alpha_1 x_1 + \cdots + \alpha_K x_K)$$

$$=$$

$\therefore$ any point of S can be expressed as a convex linear combination of extreme points of S

$$= \alpha_1 c^T x_1 + \cdots + \alpha_k c^T x_k$$

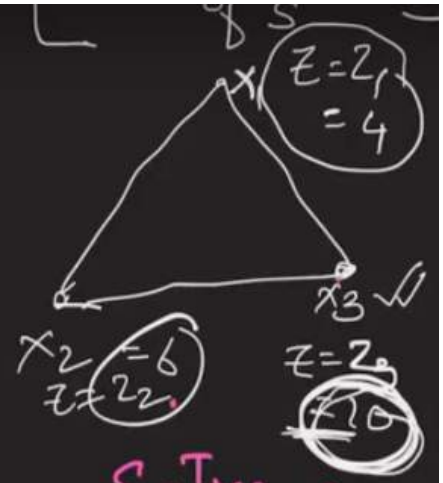
$$\leq \alpha_1 \cdot c^T x_p + \cdots + \alpha_k c^T x_p$$

$$\leq c^T x_p (\underbrace{\alpha_1 + \cdots + \alpha_k}_{1})$$

$$\leq c^T x_p .$$

$$\leq c^T x_p$$

$$(c^T x_p) = z' = \max\{c^T x_i \,;\, i = 1, \ldots, k\}$$



$$c^T x^* \leq c^T x_p$$

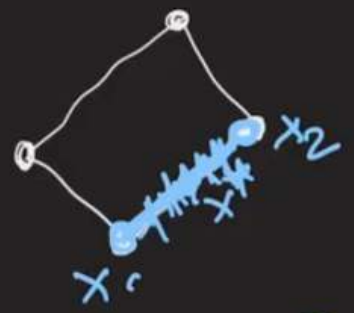
$\in S$ extreme point

optimal $\leftarrow Z_0 \leq Z$

max

A bounded feasible Region <u>always</u> gives us optimal solution.

- Unique
- Infinite



$$\text{Max } Z^* = c^T x_1 = c^T x_2 = 20 \quad (\text{say})$$

$$x^* = \alpha_1 x_1 + \alpha_2 x_2 \; ; \; \alpha_1 + \alpha_2 = 1, \; \alpha_i \geq 0$$

Optimality & Feasibility Criteria for Simplex Method

- Algebraic Method (Basic Feasible Solution) $\longrightarrow$ Simplex Method

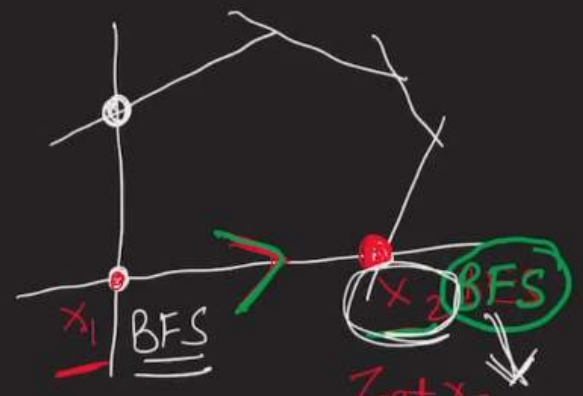
LPP:

$$\text{Opt } Z = c^T x$$
$$AX = b$$
$$x \geq 0$$

$(A)_{m \times n}$

Total no. of BFS $\leq {}^nC_m$

$Z_{at\,BFS} = ?$



$Z_{at\,x_1}$

$Z_{at\,x_2}$

$Z \leq Z_{max}$

$Z \geq Z_{min}$

Theorem: Consider a LPP: Max $Z = C^T X$; $AX = b$; $X \geq 0$. Let X_B be BFS; and If some $Z_j - C_j < 0$ and for that j some $\alpha_{ij} > 0$; then there exists a new BFS $\hat{X}_B$ s.t. $Z(\hat{X}_B) \geq Z(X_B)$

Proof: Given LPP Max $Z = C^T X$ s.t. $AX = b$; $X \geq 0$;

$(A)_{m \times n}$; $\rho(A) = m$

$A = (a_1, \ldots, a_n)$; $a_i \to$ columns of A

[Select a BFS;

$BX_B = b \Rightarrow X_B = B^{-1} b$]

$(B)_{m \times m} = (b_1, b_2, \ldots, b_m)$

L.I. $\to \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}_{2 \times 2}$

$b_1 \quad b_2$

e.g.: $A = \begin{pmatrix} 1 & 3 & 2 & 1 \\ 2 & 4 & 6 & 3 \end{pmatrix}_{2 \times 4}$

$a_1 \quad a_2 \quad a_3 \quad a_4$

For initial Basis matrix B; let $B = [b_1, b_2, \ldots, b_r, b_{r+1}, \ldots, b_m]$ be L.I.

and for above case a_j is a non-basic column.

$$a_j = \alpha_{1j} b_1 + \alpha_{2j} b_2 + \cdots + \alpha_{rj} b_r + \cdots + \alpha_{mj} b_m \quad (\delta)$$

$$\alpha_{rj} \neq 0.$$

To get new $\hat{B}$

$$\alpha_{rj} b_r = a_j - (\alpha_{1j} b_1 + \alpha_{2j} b_2 - \cdots + \alpha_{mj} b_m)$$

$$= a_j - \left(\sum_{\substack{i=1 \\ i \neq r}}^{m} \alpha_{ij} b_i \right)$$

$$b_r = \frac{1}{\alpha_{rj}} \left[a_j - \sum_{\substack{i=1 \\ i \neq r}}^{m} \alpha_{ij} b_i \right]$$

Eg:

$$A = \begin{array}{c} \begin{array}{cccc} \text{L.D.} & & \checkmark & \checkmark \end{array} \\ \begin{bmatrix} 2 & 1 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{bmatrix} \end{array}$$

a_1

$$B = \begin{array}{c} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ b_1 \quad b_2 \end{array}_{2\times 2}$$

$$\hat{B} = \begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ 3 \end{bmatrix} = a_1 = 2\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 3\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\alpha_{rj} \sum_{\substack{i=1 \\ i \neq r}}$$

As B ; $\underline{B X_B = b}$

$$[b_1 \cdots b_r \cdots b_m] \begin{bmatrix} x_{B1} \\ \vdots \\ x_{Bm} \end{bmatrix} = b$$

$$\sum_{\substack{i=1 \\ i \neq r}}^{m} b_i x_{Bi} + b_r x_{Br} = b \qquad (1)$$

by (*)

$$\sum_{\substack{i=1 \\ i \neq r}}^{m} b_i x_{Bi} + \frac{x_{Br}}{\alpha_{rj}} \left(a_j - \sum_{\substack{i=1 \\ i \neq r}}^{m} \alpha_{ij} b_i \right) = b$$

$$\sum_{\substack{i=1 \\ i \neq r}}^{m} \left(x_{Bi} - \frac{x_{Br} \alpha_{ij}}{\alpha_{rj}} \right) b_i + \frac{x_{Br}}{\alpha_{rj}} a_j = b \qquad (2)$$

$$\sum_{\substack{i=1 \\ i \neq r}}^{m} \hat{x}_{Bi} b_i + \hat{x}_{Br} a_j = b$$

$$\hat{x}_B \hat{B} = b$$

$$\hat{x}_B = (\hat{B})^{-1} b.$$

$$\sum_{\substack{i=1 \\ i \neq r}} b_i x_{B_i} + \ldots \quad (1)$$

$$\sum_{\substack{i=1 \\ i \neq r}} \ldots \quad (2)$$

$$\sum_{\substack{i=1 \\ i \neq r}}^{m} \hat{x}_{B_i} b_i + \hat{x}_{B_r} a_j = b$$

$$\hat{x}_B \hat{B} = b$$

$$\hat{x}_B = (\hat{B})^{-1} b \quad ; \quad \hat{x}_B \geq 0$$

For feasibility

$$\Rightarrow x_{B_i} - \frac{x_{B_r}}{\alpha_{rj}} \alpha_{ij} \geq 0$$

and $\frac{x_{B_r}}{\alpha_{rj}} \geq 0$

$$\frac{x_{B_i}}{\alpha_{ij}} \geq \frac{x_{B_r}}{\alpha_{rj}} \Rightarrow$$

Feasibility Criteria

$$\frac{x_{B_r}}{\alpha_{rj}} = \min_i \left\{ \frac{x_{B_i}}{\alpha_{ij}} ; \alpha_{ij} > 0 \right\}$$

$$Z(X_B) = Z(\hat{X}_B)$$

Note;

$$Z = C^T X = C_B^T X_B = \sum_{i=1}^{m} C_{Bi} x_{Bi}$$

$$Z(\hat{X}_B) = \sum_{i=1}^{m} \hat{C}_{Bi} \hat{x}_{Bi}$$

$$\begin{pmatrix} C_{B1} & \cdots & C_{Bm} \end{pmatrix} \begin{pmatrix} x_{B1} \\ \vdots \\ x_{Bm} \end{pmatrix}$$

using from eqn (2)

$$= \sum_{\substack{i=1 \\ i \neq r}}^{m} C_{Bi}\left(x_{Bi} - \frac{x_{Br}}{\alpha_{rj}} \alpha_{ij}\right) - \hat{C}_{Br} \frac{x_{Br}}{\alpha_{rj}}$$

b_r

a_j

$$= \sum_{i=1}^{m} C_{Bi}\left(x_{Bi} - \frac{x_{Br}}{\alpha_{rj}} \alpha_{ij}\right) - C_j \frac{x_{Br}}{\alpha_r}$$

$$= \sum_{i=1}^{m} C_{B_i}\left(x_{B_i} - \frac{x_{Br}}{\alpha_{rj}}\alpha_{ij}\right) - C_j \frac{x_{Br}}{\alpha_{rj}}$$

$$= \underbrace{\sum_{i=1}^{m} C_{B_i} x_{B_i}} - \frac{x_{Br}}{\alpha_{rj}}\left(\sum_{i=1}^{m} C_{B_i}\alpha_{ij} - C_j\right)$$

$$= Z(X_B) - \frac{x_{Br}}{\alpha_{rj}}\underbrace{\left(\sum_{i=1}^{m} C_{B_i}\alpha_{ij} - C_j\right)}$$

$$Z(\hat{X}_B) = Z(X_B) - \frac{x_{Br}}{\alpha_{rj}} Z_j - C_j$$

$\downarrow$ minRatio $\rightarrow \theta_j$

$$\hat{Z}_{new} = Z_{previous} - \underbrace{\theta_j}_{\geq 0}\underbrace{(Z_j - C_j)}_{-ve}$$

$$\hat{Z} \geq Z_{previous}.$$

Max.

- So $\hat{Z} = Z_{previous} - \underset{\geq 0}{\theta_j} (Z_j - C_j)$

For Max: Enter variable with most -ve $Z_j - C_j$

For Min: Enter variable with most +ve $Z_j - C_j$.

Optimality Criteria

$Z_j - C_j \geq 0$	Max
$Z_j - C_j \leq 0$	Min

- Feasibility Criteria

$$\theta_j = \frac{x_{Br}}{\alpha_{ij}} = \text{Min} \left\{ \frac{x_{Bi}}{\alpha_{ij}} ; \alpha_{ij} > 0 \right\}$$

; $x_{Bi} \rightarrow$ BFS. ≥ 0

$$\left.\begin{array}{l} x_{Bi} \geq 0 \\ \alpha_{ij} > 0 \end{array}\right\} \geq 0$$

END