Non Linear Programming Problem

NLPP → Non Linear Programing Problem

Remark. The optimal solution can be found anywhere, depending on problem, it may exist on boundary of feasible region, or even at interior point; so we don't have general technique to solve all NLPP with one method.

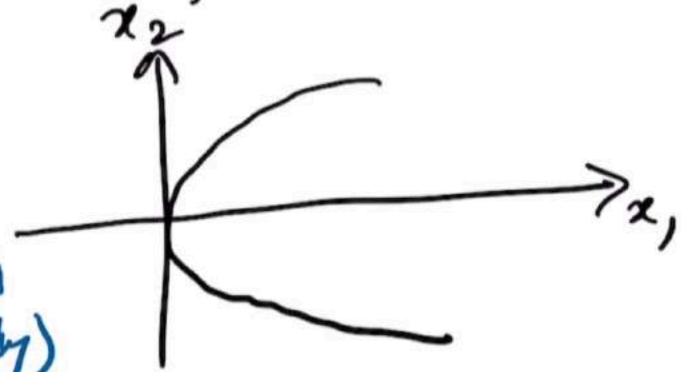
General NLPP : opt $f(x)$; s.t. $g_i(x) \leq, \geq, = b$; X - unrestricted or restricted

- $f(x)$ or $g_i(x)$ or both are Non-Linear . ; NLPP.

1) Max $f(X) = x_1^2 + x_2^2 + 2x_2$ Without Constraints

$f(x) = y^2 - x$

$f(X) = x_2^2 - x_1$



2) Max $f(X) = x_1^2 + x_2^2$ s.t.

$x_1 + x_2 \geq 4$; $2x_1 + x_2 \geq 5$; $x_1, x_2 \geq 0$

With Constraints (with inequality)

3) Min $f(X) = 2x_1^2 - 24x_1 + 2x_2^2 - 8x_2 + 2x_3^2 - 12x_3 + 200$

s.t. $x_1 + x_2 + x_3 = 11$; $x_1, x_2, x_3 \geq 0$

(with constraints ; with equality constraints)

4) Quadratic forms (without constraint)

$f(X) = x_1^2 + x_2^2 + 5x_1x_2$

NLPP

Without constraint

Quadratic forms. ✓

Non-Quadratic forms. ✓

With constraint

Inequality constraints → KKT conditions

Nec. & Suff.

Equality Constraints → Lagrange's Multiplies

Search Techniques. → Trial & Optimizie.

Problem.

A manufacturing company produces two products: Radios and Tv sets. Sales price relationship for these two products are given below:

Products		Quantity Demanded		Unit price
Radio		1500-5P_1		P_1
Tv		3800-10P_2		P_2

The total cost functions for these two products are given by $200x_1+0.1x_1^2$ and $300x_2+0.1x_2^2$ respectively. The production takes place on two assembly lines. Radio sets are assembled on Assebly line 1 and TV sets are assembled on Assembly line II. Because of the limitations of the assembly line capacities, the daily production is limited to nomore than 80 radio sets and 60 TV sets. The production of both types of products require electronic components. The production of each of these sets requires five units and six units of electronic equipment components respectively. The electronic components are supplied by another manufacturer, and the supplyis limited to 600 units per day. The company has 160 employees, the labor supply amounts to 160 man-days. The production of one unit of radio set requires 1 man-day of labor, whereas 2 man-days of labor are required for a TV set. How many units of radio and TV sets should the company produce in order to maximize the total profit. Formulate the problem as a non-linear Programming problem.

Assumption ✓

whatever is produced is sold in market.

let x_1 & x_2 → quantities of radio sets & TV sets respectively

✓ $x_1 = 1500 - 5p_1$

✓ $x_2 = 3800 - 10p_2$

or

$p_1 = 300 - 0.2x_1$

$p_2 = 380 - 0.1x_2$

C_1, C_2 → total cost of production of these units of radio sets & TV sets resp.

$C_1 = 200x_1 + 0.1x_1^2$; $C_2 = 300x_2 + 0.1x_2^2$

Revene ; $R = p_1x_1 + p_2x_2 = (300 - 0.2x_1)x_1 + (380 - 0.1x_2)x_2$

$= 300x_1 - 0.2x_1^2 + 380$

units of electronic equipment components respectively. The electronic components are supplied by another manufacturer, and the supplyis limited to 600 units per day. The company has 160 employees, the labor supply amounts to 160 man-days. The production of one unit of radio set requires 1 man-day of labor, whereas 2 man-days of labor are required for a TV set. How many units of radio and TV sets should the company produce in order to maximize the total profit. Formulate the problem as a non-linear Programming problem.

or

$p_1 = 300 - 0.2x_1$

$p_2 = 380 - 0.1x_2$

$c_1, c_2 \rightarrow$ total cost of production of these units of radio sets & TV sets resp.

$C_1 = 200x_1 + 0.1x_1^2$; $C_2 = 300x_2 + 0.1x_2^2$

Revene ; $R = p_1x_1 + p_2x_2 = (300 - 0.2x_1)x_1 + (380 - 0.1x_2)x_2$

$= 300x_1 - 0.2x_1^2 + 380x_2 - 0.1x_2^2$.

Total Profit $Z = R - C_1 - C_2 = 100x_1 - 0.3x_1^2 + 80x_2 - 0.2x_2^2 \leftarrow$ Non-linear objective.

s.t.

$$\left.\begin{aligned} 5x_1 + 6x_2 &\le 600 \\ x_1 + 2x_2 &\le 160 \\ 0 \le x_1 \le 80 \; ; \; 0 \le x_2 &\le 60 \end{aligned}\right\}$$

NLPP.

formulation !!

Non-Linear Programming Problem (Without Constraint)

1. Quadratic form
2. Non-Quadratic form

Quadratic Form:

$$f(x) = c_{11}x_1^2 + c_{22}x_2^2 + \cdots + c_{nn}x_n^2 + c_{1n}x_1x_n$$
$$+ c_{12}x_1x_2 + c_{13}x_1x_3 + \cdots + c_{1n}x_1x_n$$
$$+ \cdots \cdots$$
$$+ c_{n-1,n}\, x_{n-1}x_n$$

$$= X^T A X \quad ; \quad \text{where } X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$A = (a_{ij})_{n \times n} \to$ square matrix

$a_{ii} = c_{ii}$; $\underbrace{a_{ij} = a_{ji}}\, = \underbrace{\dfrac{c_{ij}}{2}}$; $i \neq j$

Ex. $f(X) = x_1^2 + 2x_2^2 - 7x_3^2 - 4x_1x_2 + 6x_1x_3 - 5x_2x_3$

$= X^T A X$

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \qquad A = \begin{array}{c} \begin{array}{ccc} & a_{12} & a_{13} \end{array} \\ \begin{pmatrix} 1 & -2 & 3 \\ -2 & 2 & -5/2 \\ 3 & -5/2 & -7 \end{pmatrix}_{3\times 3} \end{array}$$

a_{21} a_{31} a_{32} a_{23}

$$X^T A X = f(X)$$

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Example 2 : $f(x) = \underline{2x_1^2} - \underline{x_2^2} + \underline{4x_1x_2} + \underline{x_3^2}$

$$A = \begin{bmatrix} 2 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3} \quad ; \quad \underbrace{f(x) = X^T A X}$$

Non-Quadratic form

$H(x) \rightarrow$ Hessian Matrix

$$H(x) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & & & \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}$$

$f(x)$:

$$X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}^T$$

'f' $\rightarrow$ Partial order derivative continuous.

$$\underbrace{\frac{\partial^2 f}{\partial x_1 \partial x_2} = \frac{\partial^2 f}{\partial x_2 \partial x_1}}$$

Ex: $f(x) = 2 + 2x_1 + 3x_2 - x_1^2 - x_2^2$

$$\frac{\partial f}{\partial x_1} = 2 - 2x_1$$

$$\frac{\partial f}{\partial x_2} = 3 - 2x_2$$

$$H(x) = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}_{2 \times 2}$$

Some More Examples of Quadratic & Non-Quadratic forms & Associated Matrix.

(i) $f(X) = x_1^2 + 2x_2^2 - 7x_3^2 - 4x_1x_2 + 8x_1x_3$; $A \to$ Symmetric matrix

$= X^T A X = (x_1\ x_2\ x_3)\, A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$A = \begin{bmatrix} 1 & -2 & 4 \\ -2 & 2 & 0 \\ 4 & 0 & -7 \end{bmatrix}_{3\times 3}$ $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

(ii) $f(x) = 5x_1^2 - 7x_2$; $H(x) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix}_{2\times 2} = \begin{bmatrix} 10 & 0 \\ 0 & 0 \end{bmatrix}$

✓ $\frac{\partial f}{\partial x_1} = 10x_1$

✓ $\frac{\partial f}{\partial x_2} = -7$

(iii) $\boxed{\text{Remark : } H = 2A}$

from (i) : $f(X) = x_1^2 + 2x_2^2 - 7x_3^2 - 4x_1x_2 + 8x_1x_3$

Some More Examples of Quadratic & Non-Quadratic forms & Associated Matrix.

i) $f(X) = x_1^2 + 2x_2^2 - 7x_3^2 - 4x_1x_2 + 8x_1x_3$; $A \to$ Symmetric matrix

$= X^T A X = (x_1\ x_2\ x_3)\, A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$A = \begin{bmatrix} 1 & -2 & 4 \\ -2 & 2 & 0 \\ 4 & 0 & -7 \end{bmatrix}_{3\times 3}$ $\quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ (*)

ii) $f(x) = 5x_1^2 - 7x_2$; $H(X) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix}_{2\times 2} = \begin{bmatrix} 10 & 0 \\ 0 & 0 \end{bmatrix}$

$\frac{\partial f}{\partial x_1} = 10x_1$

$\frac{\partial f}{\partial x_2} = -7$

iii) $\boxed{\text{Remark : } H = 2A}$

from (i) : $f(X) = x_1^2 + 2x_2^2 - 7x_3^2 - 4x_1x_2 + 8x_1x_3$

$\frac{\partial f}{\partial x_1} = 2x_1 - 4x_2 + 8x_3$ ✓

$\frac{\partial f}{\partial x_2} = 4x_2 - 4x_1$ ✓

$\frac{\partial f}{\partial x_3} = -14x_3 + 8x_1$ ✓

$H(X) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_1 \partial x_3} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \frac{\partial^2 f}{\partial x_2 \partial x_3} \\ \frac{\partial^2 f}{\partial x_3 \partial x_1} & \frac{\partial^2 f}{\partial x_3 \partial x_2} & \frac{\partial^2 f}{\partial x_3^2} \end{bmatrix}_{3\times 3} = \begin{bmatrix} 2 & -4 & 8 \\ -4 & 4 & 0 \\ 8 & 0 & -14 \end{bmatrix} = 2A$ (**)

$\boxed{H = 2A}$

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To find Maxima | Minima of Non-linear Problem

$f(X)$ — Quad form / Non-Quad. form

Step 1: Find Stationary points, that is $\boxed{x^*}$ s.t. $\nabla f(x) = 0$

or $\dfrac{\partial f}{\partial x_1} = \dfrac{\partial f}{\partial x_2} = \cdots = \dfrac{\partial f}{\partial x_n} = 0$

Step 2: Check the matrix corresponding to Quadratic & Non-Quadratic forms definitness.

Remark: $H(x) = 2A$

Step 3: x^* is a point of relative minimum if matrix is positive definite

x^* is a point of relative maximum if matrix is negative definite

x^* is a saddle point if matrix is Indefinite.

Definitness of a Matrix / Quadratic form

Positive Definite : The Quadratic form is +ve definite if $f(x) > 0$ for all $x \neq 0$

$f(x) = \underbrace{x_1^2 + 5x_2^2 + 7x_3^2} > 0$; $X \neq 0$; $x_1 \neq 0$; $x_2 \neq 0$; $x_3 \neq 0$

Positive Semi-Definite : The Quadratic form is +ve semidefinite if $f(x) \geq 0$ for all x and there exist atleast one non-zero vector $x \neq 0$ for which $f(x) = 0$.

Eg: $f(x) = (x_1 - x_2)^2 + 2x_3^2$; $X = (1, 1, 0) \neq 0$; $f(x) \geq 0$

$f(x) = 0$

Negative Definite : If $f(x) < 0$; for all $x \neq 0$

Negative Semi-Definite : If $f(x) \leq 0$; for all $x \neq 0$; $\exists$ atleast one non zero $x \neq 0$ s.t. $f(x) = 0$.

Indefinite : If none of above hold, it is negative definite.

$$\boxed{\text{Matrix Minor Test}}$$

Matrix Minor Test

To check Definitness of Matrix 'A'; $A = [a_{ij}]_{n}$

Positive Definite: $D_1 = a_{11} > 0$; $D_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0$; .... $D_n = \begin{vmatrix} a_{11} & a_{12} \cdots a_{1n} \\ \vdots & \\ a_{n1} \cdots & a_{nn} \end{vmatrix} > 0$

Positive Semi-Definite: $\underline{D_1 > 0}$; $D_i \geq 0$; $i = 2, 3, \ldots, n$.

Negative Definite: $\underline{D_1 < 0}$; $\underline{D_2 > 0}$; $D_3 < 0$; ... $(-1)^i D_i > 0$

Negative Semi-Definite: $\underline{D_1 < 0 \;;\; D_2 \geq 0 \;;\; D_3 \leq 0, \ldots\ldots}$

Indefinite: None of above

(i) $f(x) = x_1^2 + 2x_2^2 - 7x_3^2 - 4x_1x_2 + 8x_1x_3$; $A = \begin{pmatrix} 1 & -2 & 4 \\ -2 & 2 & 0 \\ 4 & 0 & -7 \end{pmatrix}$ ✓

$D_1 = 1 > 0$; $D_2 = \begin{vmatrix} 1 & -2 \\ -2 & 2 \end{vmatrix} = 2 - 4 = -2 < 0$

$D_3 = \begin{vmatrix} 1 & -2 & 4 \\ -2 & 2 & 0 \\ 4 & 0 & -7 \end{vmatrix} = -18 < 0$; $\underbrace{D_1 > 0 ;\ D_2 < 0 ;\ D_3 < 0}_{\text{Indefinite form.}}$

(ii) $f(x) = 5x_1^2 - 7x_2$

$H = \begin{bmatrix} 10 & 0 \\ 0 & 0 \end{bmatrix}$; $D_1 = 10 > 0$

$D_2 = \begin{vmatrix} 10 & 0 \\ 0 & 0 \end{vmatrix} = 0$; $\underbrace{D_1 > 0}$; $\underbrace{D_2 = 0}$

+ve Semi-definite

(iii) $A = \begin{bmatrix} 4 & 2 & 0 \\ 2 & 9 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

$D_1 = 4 > 0$; $D_2 = \begin{vmatrix} 4 & 2 \\ 2 & 9 \end{vmatrix} = 32 > 0$; $D_3 = \begin{vmatrix} 4 & 2 & 0 \\ 2 & 9 & 0 \\ 0 & 0 & 2 \end{vmatrix} = 64 > 0$; +ve Definite

$\boxed{H, \text{ or } \underline{A} = \frac{H}{2}}$ → Definitness.

Example Find stationary point and classify this as point of Maxima or Minima

$$f(x) = 2 + 2x_1 + 3x_2 - x_1^2 - x_2^2$$

Sol^n: I. $\frac{\partial f}{\partial x_1} = \frac{\partial f}{\partial x_2} = 0$ To find stationary point

$$\begin{cases} \frac{\partial f}{\partial x_1} = 2 - 2x_1 \\ \frac{\partial f}{\partial x_2} = 3 - 2x_2 \end{cases}$$

$\Rightarrow \frac{\partial f}{\partial x_1} = \frac{\partial f}{\partial x_2} = 0$

$\therefore 2 - 2x_1 = 0 \Rightarrow x_1 = 1$

$3 - 2x_2 = 0 \Rightarrow x_2 = 3/2$

$\therefore \underbrace{X^* = (x_1, x_2) = (1, 3/2)}_{\text{stationary point.}}$

II. $$H = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$$

$\underbrace{D_1 = -2 < 0}$; $\underbrace{D_2 = \begin{vmatrix} -2 & 0 \\ 0 & -2 \end{vmatrix} = 4 > 0}$ $\therefore$ Negative Definite

$\therefore (1, 3/2)$ is a point of local Maximum; $f(1, 3/2) = \frac{21}{4}$

Example: Identify relative Maxima or Minima for

$$f(x) = 25x_1^2 - 8x_1x_2 + x_2^2$$

$$A = \begin{bmatrix} 25 & -4 \\ -4 & 1 \end{bmatrix} \; ; \quad \frac{\partial f}{\partial x_1} = 50x_1 - 8x_2$$

$$\frac{\partial f}{\partial x_2} = -8x_1 + 2x_2$$

$$\frac{\partial f}{\partial x_1} = 0 = \frac{\partial f}{\partial x_2} \Rightarrow x^* = (0,0)$$

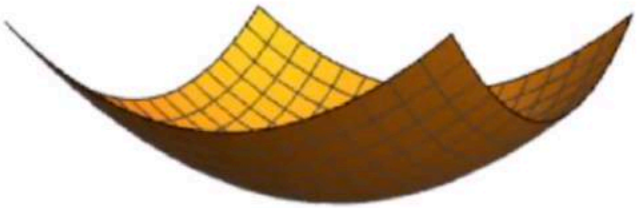
$$\downarrow$$

$$D_1 = 25 > 0$$

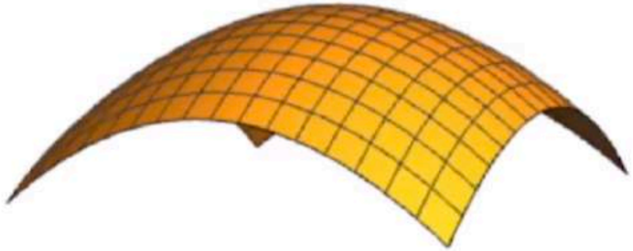
$$D_2 = \begin{vmatrix} 25 & -4 \\ -4 & 1 \end{vmatrix} = 25 - 16 = 9 > 0$$

+ve Definite

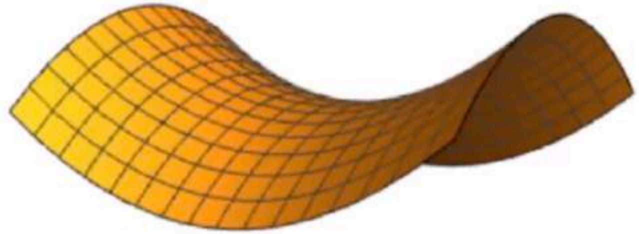
$\boxed{\therefore \text{ relative minima, } x^* \quad f(x^*) = 0.}$



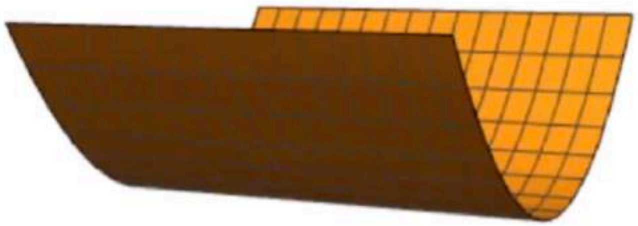
(a) A positive-definite form.



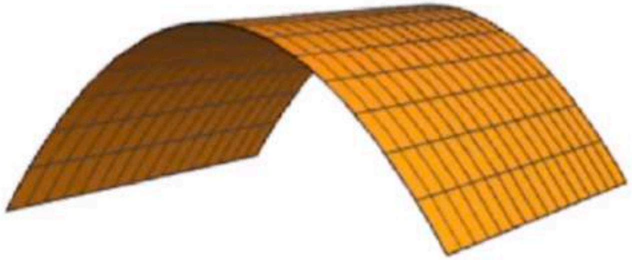
(b) A negative-definite form.



(c) An indefinite form.



(d) A positive semi-definite form.



(e) A negative semi-definite form.

Non-Linear Programming Problem (With Equality Constraint)

Lagrange Multiplier Method

Consider the NLPP;

$$\text{Opt } f(x) \quad \text{s.t.} \quad g_i(x) = 0 \; ; \quad i = 1, 2, \dots, m \; ; \; X = (x_1, x_2, \dots, x_n)^T$$

[Here $f(x)$ or $g_i(x)$ or both Non-linear]

[Recall: Optimizing Non-Linear Objective fn. without constraint]

Opt: $f(x)$

for eg. $f(x) = x_1^2 + 2x_2^2 - 2x$

Quad Non-Quad.

- stationary point.

$$\frac{\partial f}{\partial x_i} = 0 \quad ; \quad \underbrace{\frac{\partial f}{\partial x_1} = \frac{\partial f}{\partial x_2} = 0}$$

$$X^* = (\quad)$$

- Matrix → −ve, → +ve }

Define the Lagrange function $L(X, \lambda)$; as

$$L \equiv L(X, \lambda) = f(X) + \sum_{i=1}^{m} \underline{\lambda_i g_i(X)}$$

where $\lambda_i \rightarrow$ Lagrange multipliers.

[for Eg: opt $\underline{f(x) = 2x_1^2 + x_2^2 + 10x_1}$ s.t. $x_1 + x_2 = 20$
$2x_1 - x_2 = 6$
$x_1, x_2 \geq 0$]

$g_1(x) : x_1 + x_2 - 20 = 0$

$g_2(x) : 2x_1 - x_2 - 6 = 0$

$$L(X, \lambda) = (2x_1^2 + x_2^2 + 10x_1) + \lambda_1(x_1 + x_2 - 20) + \lambda_2(2x_1 - x_2 - 6)$$

Necessary Condition (Stationary Point) $L(x,\lambda) = f(x) + \sum_{i=1} \lambda_i g_i(x)$

$$\frac{\partial L}{\partial x_j} = 0 \text{ and } \frac{\partial L}{\partial \lambda_i} = 0 \quad \begin{bmatrix} x^* = \\ \lambda^* = \end{bmatrix} \quad \begin{matrix} j = 1, \ldots, n \\ i = 1, \ldots, m \end{matrix}$$

Sufficient condition (Bordered Hessian Matrix)

$\boxed{\text{Note.}}$ The necessary condition become sufficient conditions for a maximum (minimum) if objective function is concave (convex) and constraints are of equality type

Bordered Hessian Matrix

$$H^B = \begin{bmatrix} O & P \\ P^T & Q \end{bmatrix}_{(m+n)\times(m+n)}$$

$f(x)$

$g_i(x)$

$L = f + \lambda_1 g_1(x) + \cdots + \lambda_m g_m(x)$

$(x_1, \ldots, x_n)$

O is $m \times m$ zero matrix

$$P = \begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_n} \\ \frac{\partial g_2}{\partial x_1} & \cdots & \frac{\partial g_2}{\partial x_n} \\ \vdots & & \\ \frac{\partial g_m}{\partial x_1} & \cdots & \frac{\partial g_m}{\partial x_n} \end{bmatrix} ; \; P^T =$$

$$Q = \begin{bmatrix} \frac{\partial^2 L}{\partial x_1^2} & \frac{\partial^2 L}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 L}{\partial x_1 \partial x_n} \\ \frac{\partial^2 L}{\partial x_2 \partial x_1} & \frac{\partial^2 L}{\partial x_2^2} & \cdots & \frac{\partial^2 L}{\partial x_2 \partial x_n} \\ \vdots & & & \\ \frac{\partial^2 L}{\partial x_n \partial x_1} & \frac{\partial^2 L}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 L}{\partial x_n^2} \end{bmatrix}$$

Algorithm

$\frac{\partial L}{\partial x_j} = 0$; $\frac{\partial L}{\partial \lambda_i} = 0$

consider (x^*, λ^*) → as stationary point for the function $L(x, \lambda)$.

Let H^B be the corresponding Bordered Hessian Matrix.

Then x^* is a

(i) Maximum Point ; if starting with principal minor of order $(2m+1)$, the last $(n-m)$ principal minors of H^B form an alternating sign pattern starting with $(-1)^{m+n}$.

(ii) Minimum Point ; if starting with principal minor of order $(2m+1)$, the last $(n-m)$ principal minors of H^B have the sign of $(-1)^m$.

(either +ve , −ve)
depend.

Problem: Use the Lagrange Multiplier method to solve NLPP

$$\text{Opt } f(x) = 2x_1^2 + x_2^2 + 3x_3^2 + 10x_1 + 8x_2 + 6x_3 - 100$$

$$\text{s.t.} \quad x_1 + x_2 + x_3 = 20$$

$$x_1, x_2, x_3 \geq 0$$

Sol^n:

$$L(X, \lambda) = f(X) + \lambda\, g(X)$$

$$= \underbrace{2x_1^2 + x_2^2 + 3x_3^2 + 10x_1 + 8x_2 + 6x_3 - 100}_{} + \lambda\left(\underbrace{x_1 + x_2 + x_3 - 20}_{}\right)$$

The Necessary conditions for stationary Point:

$$\frac{\partial L}{\partial x_1} = 4x_1 + 10 + \lambda = 0 \;; \quad \frac{\partial L}{\partial \lambda} = x_1 + x_2 + x_3 - 20 = 0$$

$$\frac{\partial L}{\partial x_2} = 2x_2 + 8 + \lambda = 0 \;;$$

Solve these together;

$$\frac{\partial L}{\partial x_3} = 6x_3 + 6 + \lambda = 0 \;;$$

$$\frac{\partial L}{\partial x_2} = 2x_2 + 8 + \lambda = 0 \; ; \quad (2)$$

$$\frac{\partial L}{\partial x_3} = 6x_3 + 6 + \lambda = 0 \; ; \quad (3)$$

Solve these together;

from (1) $\quad x_1 = -\frac{(10+\lambda)}{4}$ ✓ ; put these in (4)

(2) $\quad x_2 = -\frac{(8+\lambda)}{2}$ ✓

(3) $\quad x_3 = -\frac{(6+\lambda)}{6}$ ✓

$$\underbrace{-\left(\frac{10+\lambda}{4}\right) + \left(-\left(\frac{8+\lambda}{2}\right)\right) - \left(\frac{6+\lambda}{6}\right) - 20 = 0}$$

$$\boxed{\lambda = -30}$$

$$X^* = (5, 11, 4)$$

λ

$$H^B = \begin{bmatrix} O & P \\ P^T & Q \end{bmatrix}$$

$O \rightarrow m \times m$ zero matrix
$\hookrightarrow$ no. of constraint

$$P = \begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} & \frac{\partial g_1}{\partial x_3} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$

$$= \left[\begin{array}{c|ccc} 0 & 1 & 1 & 1 \\ \hline 1 & 4 & 0 & 0 \\ 1 & 0 & 2 & \\ 1 & & & \end{array}\right]$$

$$Q = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

where
$$L = 2x_1^2 + x_2^2 + 3x_3^2 + 10x_1 + 8x_2 + 6x_3 - 100 + \lambda(x_1 + x_2 + x_3 - 20)$$

$$\frac{\partial L}{\partial x_1} = \underline{4}x_1 + 10 + \lambda$$ w.r.t. x_1

$$\frac{\partial L}{\partial x_2} = 2x_2 + 8 + \lambda$$ w.r.t x_2

$$\frac{\partial L}{\partial x_3} = 6x_3 + 6 + \lambda$$ w.r.t. $\underline{x_3}$

$n \to$ no. of variables $(x_1, x_2\ x_3)$; $n = 3$

$m \to$ no. of constraints ; $m = 1$

$(2m+1) = (2 \cdot 1 + 1) = 3$

$$D_3 = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 4 & 0 \\ 1 & 0 & 2 \end{vmatrix} = -6 \quad ; \quad D_4 = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 4 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 6 \end{vmatrix} = -44$$

Both D_3 & D_4 have sign of $(-1)^m$;

x^* is a point of Minimum ;

$f(x^*) = f(5, 11, 4) = 281$

Problem : Solve NLPP

Opt $Z = 4x_1^2 + 2x_2^2 + x_3^2 - 4x_1x_2$ s.t. $x_1 + x_2 + x_3 = 15$; $2x_1 - x_2 + 2x_3 = 20$

Sol^n:

$$L = f(x) + \lambda_1 g_1(x) + \lambda_2 g_2(x)$$

$$= (4x_1^2 + 2x_2^2 + x_3^2 - 4x_1x_2) + \lambda_1(x_1 + x_2 + x_3 - 15) + \lambda_2(2x_1 - x_2 + 2x_3 - 20)$$

$$\frac{\partial L}{\partial x_1} = 8x_1 - 4x_2 - \lambda_1 - 2\lambda_2 = 0$$

$$\frac{\partial L}{\partial x_2} = 4x_2 - 4x_1 - \lambda_1 + \lambda_2 = 0$$

$$\frac{\partial L}{\partial x_3} = 2x_3 - \lambda_1 - 2\lambda_2 = 0$$

$$\frac{\partial L}{\partial \lambda_1} = x_1 + x_2 + x_3 - 15 = 0$$

$$\frac{\partial L}{\partial \lambda_2} = 2x_1 - x_2 + 2x_3 - 20 = 0$$

$$X^* = (x_1, x_2, x_3) = \left(\frac{33}{9}, \frac{10}{3}, 8\right)$$ stationary pt.

$$\lambda^* = (\lambda_1, \lambda_2) = \left(\frac{40}{9}, \frac{52}{9}\right)$$

;

$$H^B = \left[\begin{array}{cc|ccc} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & -1 & 2 \\ \hline 1 & 2 & 8 & -4 & 0 \\ 1 & -1 & -4 & 4 & 0 \\ 1 & 2 & 0 & 0 & 2 \end{array}\right]$$

Note $n = 3$; $m = 2$

$\therefore (2m+1) = 5$;

$\therefore |H^B| = 90 > 0$

$\therefore X^*$ is a Minimum Point.

Convex Functions

Recall: $\boxed{\text{Convex Set}}$

S is said to be a convex sets; if any $x_1, x_2 \in S$ then

$(1-\alpha) x_1 + \alpha x_2 \in S \quad \forall \alpha;$

$0 \le \alpha \le 1$

or $\alpha_1 x_1 + \alpha_2 x_2 \in S \quad \forall \alpha_1, \alpha_2 \geq 0$

$\alpha_1 + \alpha_2 = 1.$

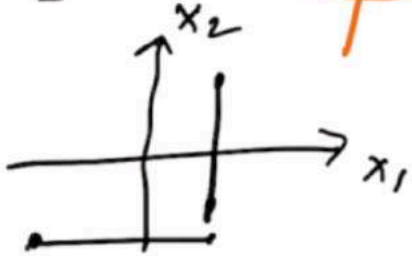
$\downarrow$

$\alpha_1 = 1 - \alpha_2$

For Example:

1. $S = \{ (x_1, x_2) : x_1^2 + x_2^2 \le 1 \}$

2. $S = \mathbb{R}^2$

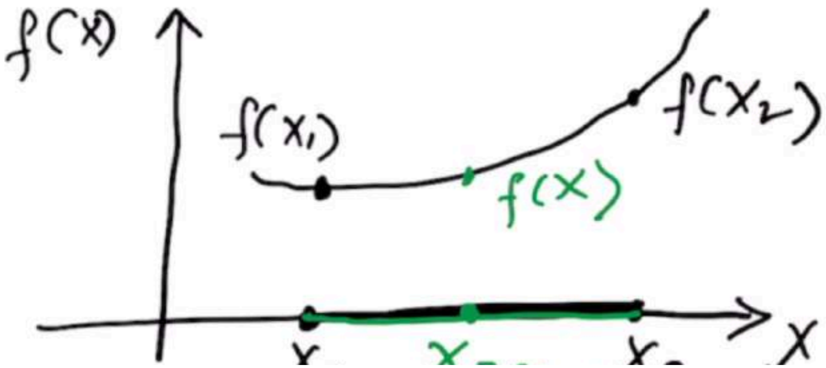


3. $S = \{$

Definition: **Convex Function**. Let S be a convex set in $\mathbb{R}^n$. A function $f(x)$ defined on S is said to be convex function, if for any pair of points $x_1, x_2 \in S$; for all α; $0 \le \alpha \le 1$

$$f\big((1-\alpha)x_1 + \alpha x_2\big) \le (1-\alpha) f(x_1) + \alpha f(x_2)$$

Geometrically; $f(X)$ is convex if for any two points x_1, x_2; the chord joining the points $(x_1, f(x_1))$ & $(x_2, f(x_2))$ is above $f(X)$ where $X = (1-\alpha)x_1 + \alpha x_2$; $0 \le \alpha \le 1$.



1. $f(x)$ is strictly convex if $f((1-\alpha)x_1 + \alpha x_2) < (1-\alpha) f(x_1) + \alpha f(x_2)$

2. $f(x)$ is concave (or strictly concave) if $-f(x)$ is convex (or strictly convex)

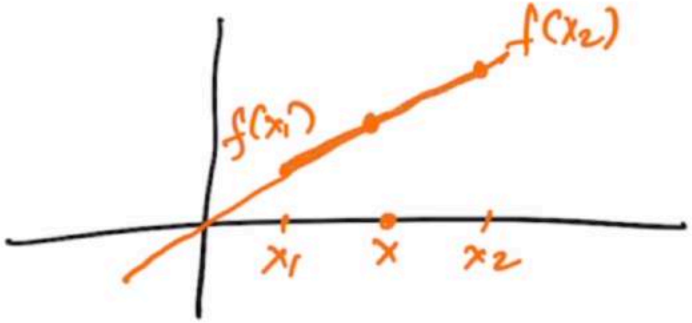
or

$$f((1-\alpha)x_1 + \alpha x_2) \geq (1-\alpha) f(x_1) + \alpha f(x_2)$$ Concave fn.

$$f((1-\alpha)x_1 + \alpha x_2) > (1-\alpha) f(x_1) + \alpha f(x_2)$$ strict concave fn.

3. A linear function is convex as well as concave.

$f(x) = x$



Results

Proposition 1. The sum of two convex function is convex.

Proposition 2 The $f(x)$ is convex in $\mathbb{R}^n$ if X^TAX is positive semi-definite where $f(x) = X^TAX$.

and $f(x)$ is strictly convex if $f(x) = X^TAX$ is positive definite.

Corollary:

1. $f(x)$ is convex $\Leftrightarrow$ its Hessian matrix is positive semidefinite
2. $f(x)$ is strictly convex $\Leftrightarrow$ its Hession matrix is positive definite.

Theorem 1. Let $f(x)$ be a convex function defined over a convex set $S \subseteq \mathbb{R}^n$. Then the local minimum is global minimum of $f(x)$ over S.

Theorem 2. The feasible solution of CNLPP is a convex set.

NLPP → CNLPP

↳ Lagrange's

$f(x)$ s.t. $g_i(x) = 0$

$g_i(x) \leq 0$

KKT.

Min $f(x)$

s.t. $g_i(x) \leq 0$; $i = 1, 2, -, m$

$x \geq 0$

$f(x)$ and $g_i(x)$ are convex fns.

Problem

$f(x) = x_1 + x_2$

s.t. $x_1^2 + x_2^2 \leq 1$; $g_1(x) = x_1^2 + x_2^2 - 1 \leq 0$

$x_1^2 \leq x_2$; $g_2(x) = x_1^2 - x_2 \leq 0$

KKT.

$x \geq 0$

$f(x)$ and $g_i(x)$ are convex fns.

Problem

CNLPP

$f(x) = x_1 + x_2$

s.t. $x_1^2 + x_2^2 \leq 1$; $g_1(x) = x_1^2 + x_2^2 - 1 \leq 0$

$x_1^2 \leq x_2$; $g_2(x) = x_1^2 - x_2 \leq 0$

$f(x) \rightarrow$ convex

$g_1(x) \rightarrow$ strict convex

$g_2(x) \rightarrow$ convex

$H(X)$ (correspondig to $g_1(x)$) $= \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix}$ ✓

$\underbrace{\frac{\partial g_1}{\partial x_1} = 2x_1 \; ; \; \frac{\partial g_1}{\partial x_2} = 2x_2}$; $H = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

$D_1 = 2 > 0$; $D_2 = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix}$ ...

Unimodal Functions

Introduction to Unimodal Function & Search Techniques in Optimization.

Objectives : 1) Define Unimodal function

2) Introduction to search Techniques

LPP: Opt $f(x) = f(x_1, x_2, \dots, x_n)$

s.t. $g_i(x) \le, \ge, = b_i$

$x \ge 0$

Simplex Methods

$\underbrace{f(x) ; g_i(x)}_{\text{Linear}}$

NLPP: opt $f(x)$ without constraint

Opt $f(x)$ s.t. $g_i(x) \le 0$ (or ≥ 0) $\to$ La

$f(x)$ or $g_i(x)$ or both NLPP.

$\boxed{\text{Note}}$ • Objective function can depend on single variable ; or on multi variable

• For simplicity definitions, written below, are for single variable.

$\boxed{\text{Recall}}$

Monotonic Function : A function $f(x)$ is monotonic (either increasing or decreasing) if for any two points x_1 and x_2 with $x_1 \leq x_2$, it follows

$f(x_1) \leq f(x_2)$ Monotonically increasing

$f(x_1) \geq f(x_2)$ Monotonically decreasing

Monotonic Function : A function $f(x)$ is monotonic (either increasing or decreasing) if for any two points x_1 and x_2 with $x_1 \le x_2$, it follows

$$f(x_1) \le f(x_2) \qquad \text{Monotonically increasing}$$

$$f(x_1) \ge f(x_2) \qquad \text{Monotonically decreasing}$$

Unimodal Function : A function $f(x)$ is unimodal on the interval $a \le x \le b$ iff it is monotonic on either side of the single optimal point x^* in the interval.

$\boxed{\text{OR}}$ If x^* is the single minimum point of $f(x)$ in the range $a \le x \le b$; then $f(x)$ is unimodal on the interval iff for any two points x_1 and x_2 ;

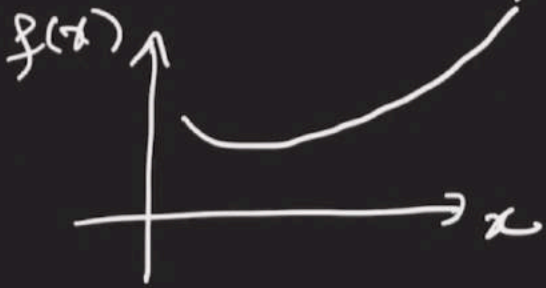
$$x^* \le x_1 \le x_2 \implies f(x^*) \le f(x_1) \le f(x_2)$$

and

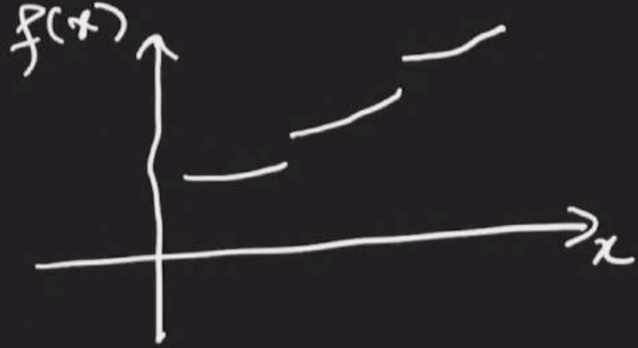
$$x^* \ge x_1 \ge x_2 \implies f(x^*) \le f(x_1) \le f(x_2).$$

Types of functions & Graphs

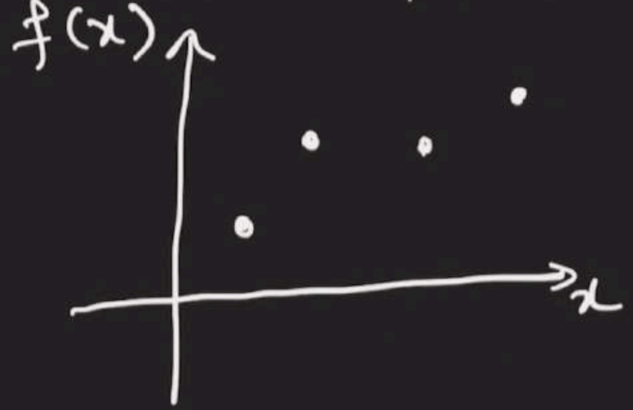
- Continuous function



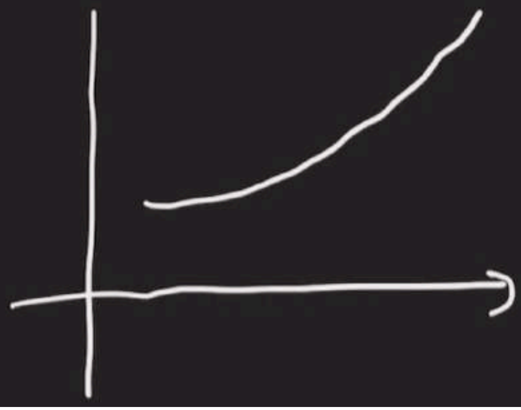
- Discontinuous function

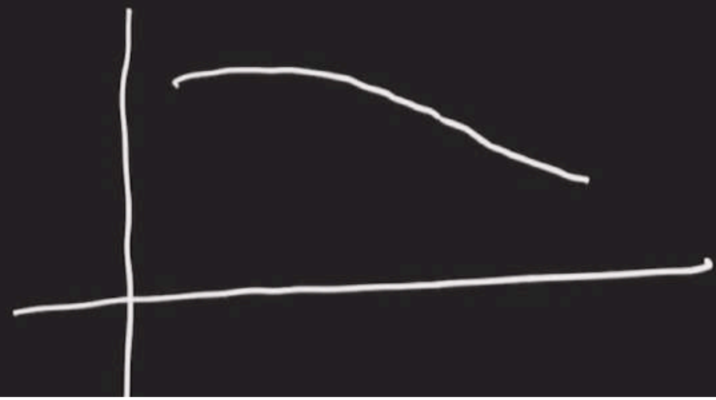


- Discrete Functi

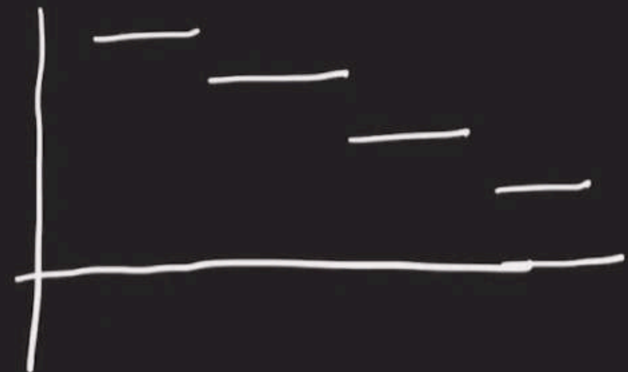


- Monotonically increasing function

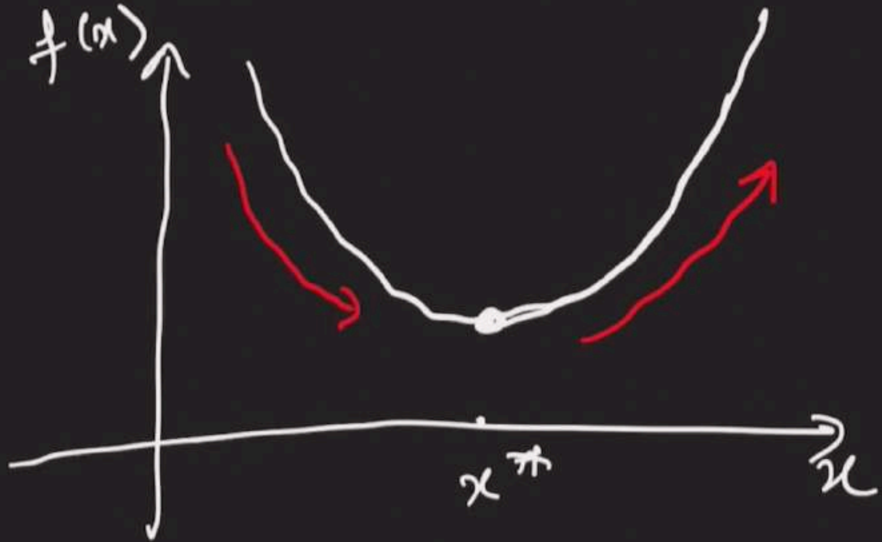




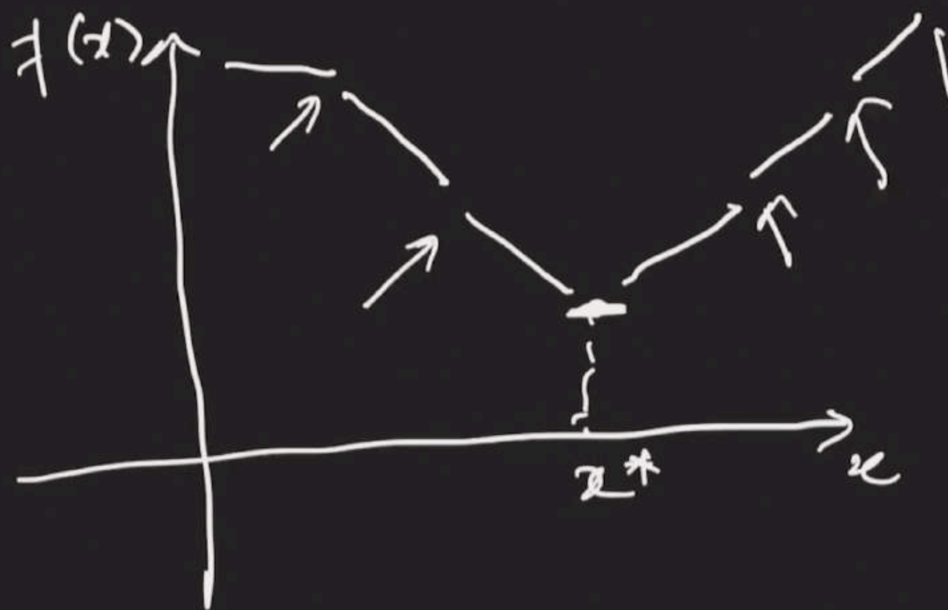
- Monotonic decreasing function



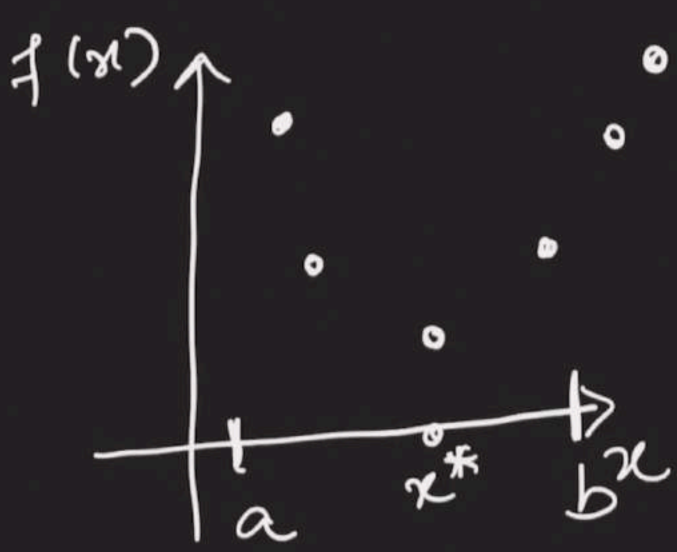
- Continuous Unimodal Function



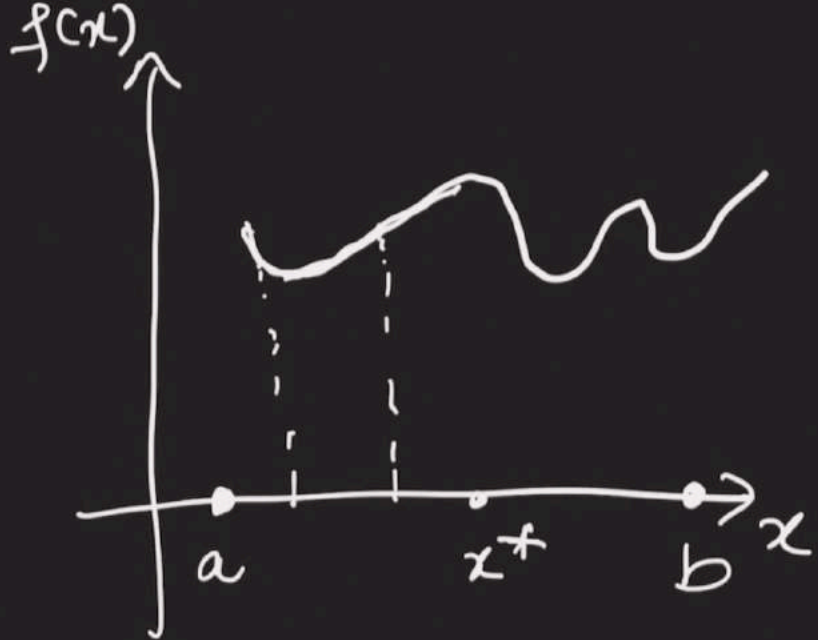
- Discontinuous Unimodal Function ✓



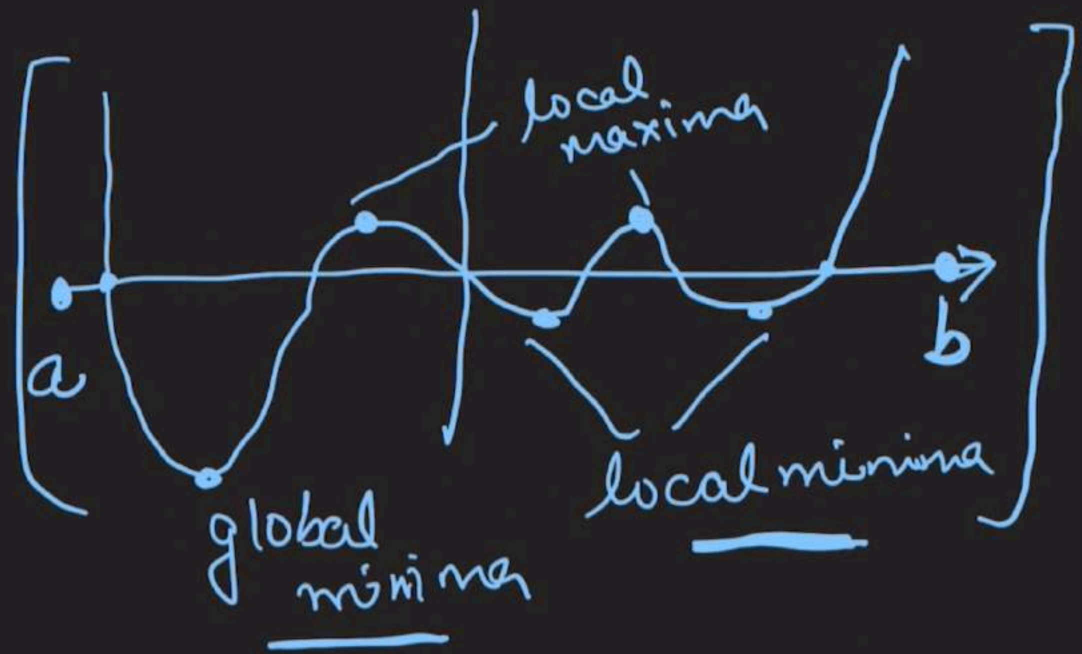
- Discrete Unimodal Function



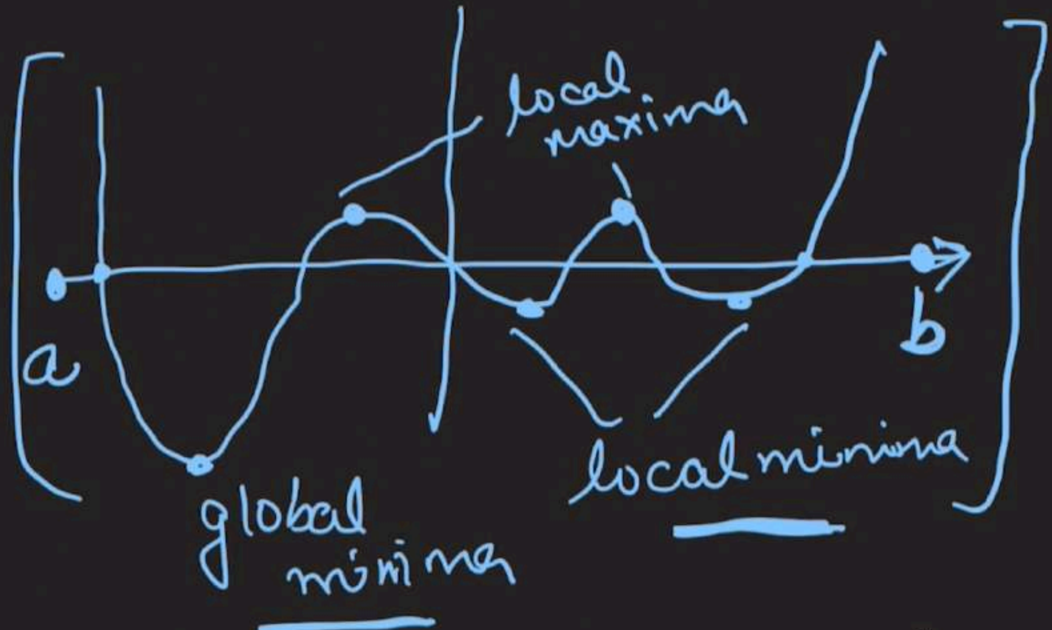
- Non-Unimodal Functi



Remark For a unimodal function, global minimum and local minimum coincide.



Remark 1) For a unimodal function, global minimum and local minimum coincide.



2) When the function is not unimodal, multiple local optimum are possible and global minimum can be found only by locating all local optima and selecting best.

Global Minima: A function $f(x)$ defined on a set S attains its global minima at a point $x^* \in S$ iff

$$f(x^*) \leq f(x) \quad \forall x \in S.$$

Local Minima: A function $f(x)$ defined on S has a local minima (or relative minima) at a point $x^* \in S$ iff $\exists$ (there exist) an $\epsilon > 0$ s.t.

$$f(x^*) \leq f(x) \quad \forall x \in S$$

satisfying $|x - x^*| < \epsilon$

Search Methods

- Dichotomous Search Method
- Fibonacci Method
- Golden Search Method
- Steepest Descent Method | Gradient Method
- Hooke and Jeeves' Method
- Spendley, Hext and Himsworth's Method
- Nelder and Mead's Method.

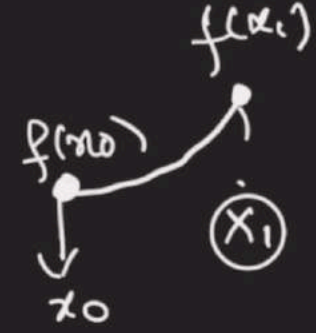
Working of Search Method

opt $f(X)$; x_0

↑

trial interval

Start x_0 ; $f(x_0) \to$ Obj fn. Compute



Find x_1 ; $f(x_1) \to$ compute

$f(x_0) \le f($

$f(x_1) - f(x_0) < \epsilon$?

optimal

Steepest Descent Method

- This is an iterative method, also known as Gradient descent metho[d]

$$\text{Opt } f(x) = x_1^2 + 2x_1x_2 + x_2^2$$

Working : (i) choose initial starting point x_i

(ii) $x_{i+1} = x_i + \lambda_i S_i$

$= x_i + \lambda_i(-\nabla f(x_i))$; where $S_i = -\nabla f(x_i)$

(iii) checking criteria

$\lambda_i \rightarrow$ optimal step length along gradient

a) $|f(x_{i+1}) - f(x_i)| \leq \epsilon$

b) $\left| \dfrac{f(x_{i+1}) - f(x_i)}{f(x_i)} \right| \leq \epsilon$

(ii) $\underline{x_{i+1} = x_i + \lambda_i S_i}$

$f(x_{i+1}) = x_i + \lambda_i(-\nabla f(x_i))$; where $S_i = -\nabla f(x_i)$

(iii) checking criteria $\lambda_i \rightarrow$ optimal step length along gradient

a) $|f(x_{i+1}) - f(x_i)| \leq \epsilon$

b) $\left| \dfrac{f(x_{i+1}) - f(x_i)}{f(x_i)} \right| \leq \epsilon$

c) $|x_{i+1} - x_i| \leq \epsilon$

d) $\left| \dfrac{\partial f}{\partial x} \right|_{x_{i+1}} \leq \epsilon$.

Question: Use steepest Descent method to find minimum $f(x_1, x_2) = x_1^2 - x_1 x_2 + x_2^2$ s.t. error not exceed by 0.05 for function, Initial approximation $X_1 = (1, 1/2)$.

Sol^n: Given $X_1 = (1, 1/2)$; $f(X_1) = f(x_1, x_2) = 3/4$

$$X_2 = X_1 + \lambda_1 (-\nabla f(X_1))$$

$$\nabla f(X) = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2} \right)$$

$$= (2x_1 - x_2, -x_1 + 2x_2)$$

$$\nabla f(x) \Big|_{X_1} = (3/2, 0)$$

$$= (1, 1/2) + \lambda_1 \left(-\left(\frac{3}{2}, 0\right) \right)$$

$$= (1, 1/2)$$

$$= \left(1, \tfrac{1}{2}\right) - \lambda_1 \left(\tfrac{3}{2}, 0\right)$$

$$= \underbrace{\left(1 - \tfrac{3}{2}\lambda_1 \,,\, \tfrac{1}{2}\right)}$$

$$f(X_2) = \left(1 - \tfrac{3}{2}\lambda_1\right)^2 - \left(1 - \tfrac{3}{2}\lambda_1\right)\left(\tfrac{1}{2}\right) + \underbrace{\left(\tfrac{1}{2}\right)^2}$$

fn. of 'λ_1' only

$$\frac{df}{d\lambda_1} = 0 \Rightarrow 2\left(1 - \tfrac{3}{2}\lambda_1\right)\left(-\tfrac{3}{2}\right) + \underbrace{\tfrac{3}{2} \cdot \tfrac{1}{2}} = 0$$

$\Rightarrow \underbrace{\lambda_1 = 1/2}$

$$\therefore \underbrace{x_2} = \left(1 - \frac{3}{2}\lambda_1, 1/2\right) = \left(1 - \frac{3}{2}(1/2), 1/2\right) = \underbrace{\left(\frac{1}{4}, 1/2\right)}$$

$$\underbrace{f(x_2) = f\left(\frac{1}{4}, 1/2\right) = \frac{3}{16}}$$

$$|f(x_{i+1}) - f(x_i)| = |f(x_2) - f(x_1)| < \epsilon$$

$$\left|\left(f(x_2) = \frac{3}{16}\right) - (f(x_1)\right.$$

$$f(x_2) = f\left(\tfrac{1}{4}, \tfrac{1}{2}\right) = \frac{3}{16}$$

$$|f(x_{i+1}) - f(x_i)| = |f(x_2) - f(x_1)| < \epsilon$$

$$\left|\left(f(x_2) = \frac{3}{16}\right) - \left(f(x_1) = \tfrac{3}{4}\right)\right| \not< 0.05$$

$$x_3 = x_2 - \lambda_2 (\nabla f(x_2))$$

$$=$$

$$\nabla f(x) = [2x_1 - x_2, -x_1 + 2x$$

$$\nabla f(x_2) = [0, 3/4]$$

$$x_3 = x_2 - \lambda_2 (\nabla f(x_2))$$

$$= \left(\tfrac{1}{4}, \tfrac{1}{2}\right) - \lambda_2 \left[0, \tfrac{3}{4}\right]$$

$$= \left(\frac{1}{4}, \frac{1}{2} - \frac{3}{4}\lambda_2\right)$$

$$f(x_3) =$$

$$\frac{df}{d\lambda_2} = 0 \Rightarrow \lambda_2 = 1/2$$

$$\therefore x_3 = \left(\frac{1}{4}, \frac{1}{2} - \frac{3}{4}\left(\frac{1}{2}\right)\right) = \left(\frac{1}{4}, \frac{1}{8}\right)$$

$$f(x_3) = 3/64$$

$$\therefore |f(x_3) - f(x_2)| \leq \left|\frac{3}{64} - \frac{3}{16}\right| \leq 0.05$$

$$\therefore f_{min} = 3/64 \ ; \ x_3 = (1/4, 1/8)$$