Probability

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① **Mutually exclusive events:-** (Disjoint events)
If A and B are two events.
$A \cap B = \phi$.

② **Equally Likely events:-**
Both events occurs same times.

③ **exhaustive set of events:-**
A set $\{A1, A2, ____, AK\}$ is called exhaustive if at least one of the events from the set must occur. Thus,

$$\{A1 \cup A2 \cup ____ \cup AK\} = S \text{ (Sample Space)}$$

(i) $P(\phi) = 0$
(ii) If event A implies event B, then $P(A) \leq P(B)$
(iii) $P(\bar{A}) = 1 - P(A)$
(iv) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$.

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Conditional Probability :- The probability of an event A, given that the event B has occured, is called Conditional probability of A. It is denoted by $P(A/B)$.

$P(A/B)$: Probability of A given that B has occured

$$\Rightarrow \boxed{P(A/B) = \frac{P(A \cap B)}{P(B)} \quad ; P(B) \neq 0}$$

Multiplication rule of probability :-

The probability of simultaneous happening of two events A and B is defined as

$\Rightarrow P(A \cap B) = P(A) \cdot P(B/A) \quad ; P(A) \neq 0.$

$\Rightarrow P(A \cap B) = P(B) \cdot P(A/B) ; P(B) \neq 0.$

For 3 events,

$$\Rightarrow \boxed{P(A_1 \cap A_2 \cap A_3) = P(A_1) \cdot P(A_2/A_1) \cdot P(A_3/A_1 \cap A_2).}$$

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Independent Events :- Events are said to be independent to each other, if happening of one of them is not affected by and does not affect the happening of any one of the others.

$$\Rightarrow \boxed{P(A/B) = P(A)} \text{ and } P(B/A) = P(B)$$

Multiplication theorem for independent events :-

$\Rightarrow P(A \cap B) = P(A) \cdot P(B)$ (Pairwise independent)

Mutually independent if

$P(A_i \cap A_j) = P(A_i) \cdot P(A_j) \quad ; \quad i \neq j$ — (i)

$P(A_i \cap A_j \cap A_k) = P(A_i) \cdot P(A_j) \cdot P(A_k) \quad ; \quad i \neq j \neq k$ — (ii)

|

$P(A_1 \cap A_2 \text{ — } \cap A_n) = P(A_1) \cdot P(A_2) \cdot \text{ — } P(A_n)$

No. of combinations, we get

$\Rightarrow {}^nC_2 + {}^nC_3 + \text{ — } + {}^nC_n$

$\Rightarrow ({}^nC_0 + {}^nC_1 + {}^nC_2 + {}^nC_3 + \text{ — } + {}^nC_n) - ({}^nC_0 + {}^nC_1)$

$\Rightarrow 2^n - 1 - n$

Disjoint / Mutually exclusive	$\neq$ Independent Events
Let $P(A), P(B) > 0$	$P(A/B) = P(A)$
$A \cap B = \phi$; $P(A \cap B) = 0$	$P(A \cap B) = P(A) \cdot P(B)$
$\Rightarrow P(A/B) = 0$	
$\Rightarrow P(B/A) = 0$	

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(ii) <u>Continuous</u>:- which does not assume the finite values.

e.g. weight of students in a class.

$100 \Rightarrow \{40 < \quad < 70\}$

There can be infinite values of wt. b/w 40 and 70.

→ <u>Probability Distribution of a random variable</u>:-

(i) <u>Discrete</u>:- $X = \{x_1, x_2, \ldots, x_n\}$

Random var.

$p_1 \quad p_2 \quad \ldots \quad p_n$

$p_i = P(X = x_i)$; $\forall\ i = 1, \ldots, n$

This function is called probability mass function, (p.m.f.)

1. $p_i \geq 0$
2. $\sum_{i=1}^{n} p_i = 1$

$(x_i, p_i) \rightarrow$ prob. distribution of r.v. X.

where

$x_i \rightarrow$ r.v.

$p_i \rightarrow$ p.m.f. of X.

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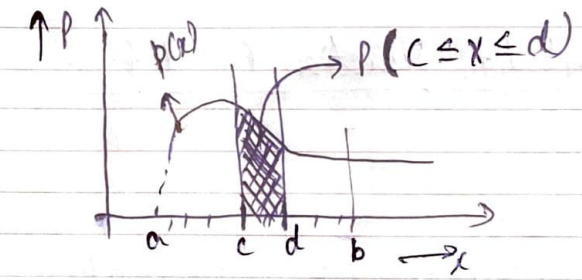
<u>Probability Distribution of a Continuous R.V.</u>:-

Let X be a Cont. r.v. taking values in the interval $[a, b]$. A function $f(x)$ or $P(x)$ is said to be prob. density function if it satisfies the following properties:-

(i) $P(x) \geq 0\ \forall x \in [a, b]$. $(\ominus\ x \in [-\infty, \infty))$

(ii) for any distinct nos. c and d in $[a, b]$.

$P(c \leq X \leq d)$ = Area under the curve b/w ordinates at $x = c$ and $x = d$



(iii) Total area under the curve is 1.

i.e. $P(a \leq X \leq b) = 1$

$$P(c \leq X \leq d) = \int_c^d P(x) \cdot dx$$

and

$$P(X = a) = P(a \leq X \leq a) = \int_a^a P(x) \cdot dx = 0$$

Prob. of a Cont. r.v. at a single point = 0 always

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Since, $P(X=a)=0$

$$P(a \le X \le b) = P((a < X \le b) \cup (X=a))$$

Disjoint

$[P(A \cup B) = P(A) + P(B)$ for disjoint sets$]$

$\Rightarrow P(a < X \le b) + \cancel{P(X=a)} \to 0$ ← at singlept.

$\Rightarrow P(a \le X \le b) = P(a < X \le b)$
$= P(a \le X < b)$
$= P(a < X < b)$

Cummulative distribution function (c.d.f.) :-

$F(x) = P(X \le x)$

$F(3) = P(X \le 3)$
$F(3) = P(X < 3) \Rightarrow$ not in general.

Discrete :-

$F(x) = P(X=0) + P(X=1) + \cdots + P(X=x)$
$\Rightarrow F(x) = p(0) + p(1) + \cdots + p(x)$

$\Rightarrow F(x) = F(x-1) + p(x)$

$\Rightarrow \boxed{p(x) = F(x) - F(x-1)}$

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Continuous r.v. :-

$F(x) = P(X \le x)$

$$\boxed{F(x) = \int_{-\infty}^{x} p(x) \cdot dx}$$

$$\boxed{\frac{d F(x)}{dx} = f(x)}$$

$F(X) = P(X < x)$ — (1) ??

For discrete

$P(X < x) \ne P(X \le x)$
in general

$\therefore$ Eqn (1) doesn't hold

For continuous :-

$P(X \le x) = P(X < x)$
$\therefore F(x) = P(X \le x)$
$= P(X < x)$
always holds.

Properties of c.d.f. :-

$F(x) = P_r(X \le x)$

$= \int_{-\infty}^{x} p(x) \cdot dx$ → r.v. → p.d.f.

prop. of p.d.f.
(i) $p_i \ge 0$
(ii) $\sum p_i = 1$ (for discrete)
$\int_{-\infty}^{\infty} p(x) \cdot dx = 1$ (for cont.)

(i) $0 \le F(x) \le 1 ; \forall x$ $\quad (\because F'(x) = p(x) \ge 0)$

(ii) $F(x)$ is non-decreasing function, i.e. if $x < y$, then $F(x) \le F(y)$.

(iii) $F(\infty) = 1$; $F(-\infty) = \int_{-\infty}^{-\infty} p(x) \cdot dx = 0$.

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(iv) $F(x)$ is a continuous function of x on the right and may have countable no. of discontinuities.

(v) $F'(x) = f(x)$ at all pt. of x where $F(x)$ is differentiable.

Expectation:- Let X be any random variable and $\varphi(x)$ be any function of X. Then expectation of $\varphi(x)$ is denoted by $E(\varphi(x))$ and is defined by

$$E(\varphi(x))$$

D.R.V: $\sum_x \varphi(x) \cdot f(x)$

C.R.V: $\int_{-\infty}^{\infty} \varphi(x) \cdot f(x) \cdot dx$

If $\varphi(x) = x$

$\Rightarrow \bar{x} = \text{mean } (E(x))$

D.R.V: $\sum_x x \cdot f(x)$

C.R.V: $\int_{-\infty}^{\infty} x \cdot f(x) \cdot dx$

$$\boxed{E(x) = \bar{x}}$$

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$$\text{Variance} = E(X-\bar{X}) = E(X^2 - 2\bar{X}X + \bar{X}^2)$$
$$= E(X^2) - 2E(\bar{X}X) + E(\bar{X}^2)$$
$$= E(X^2) - 2\bar{X}\bar{X} + \bar{X}^2$$
$$= E(X^2) - \bar{X}^2$$
$$= E(X^2) - (E(X))^2$$

$$\boxed{\int_0^{\infty} x^{n-1} e^{-ax}\, dx = \frac{\Gamma n}{a^n} \qquad \Gamma n = (n-1)!}$$

Functions of Random variable:-

① For discrete variable:-

$P(X = x_i) = p(x_i)$, $i = 1, 2, 3, \ldots$

Let $Y = \varphi(X)$.

then:

$P(Y = y_i) = p(y_i) = p(\varphi(x_i))$

such that $y_i = \varphi(x_i)$, $i = 1, 2, 3, \ldots$

② For continuous variable:-

$X \rightarrow$ continuous r.v.

Let $Y = \varphi(X)$, then Y is also a r.v.

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Let us assume $\psi(x)$ is either monotonically ↑ or ↓ function of X, then the pdf of Y can be obtained with the formula,

$$\Rightarrow \boxed{g(y) = f(\psi^{-1}(y)) \left| \frac{dx}{dy} \right|}$$

<u>Alternative method:-</u> Using c.d.f.
Applied when function is not monotonic on the given interval. e.g. $Y = X^2$.

Let $G(y)$ is c.d.f. of Y and $F(x)$ ——— X.

e.g. $G(y) = P(Y \le y)$

$\Rightarrow$ ——— $= P(3X+1 \le y)$

$\Rightarrow$ ——— $= P(X \le \frac{y-1}{3})$

$\Rightarrow G(y) = F(\frac{y-1}{3})$

Using relation, $f(x) = F'(x)$

$\Rightarrow g(y) = G'(y) = \frac{d}{dy}\left(F\left(\frac{y-1}{3}\right)\right)$

$\Rightarrow g(y) = f\left(\frac{y-1}{3}\right)\left(\frac{1}{3}\right)$

$\Rightarrow g(y) = 2\left(\frac{y-1}{3}\right)\left(\frac{1}{3}\right)$ $\quad [f(x) = 2x]$

$\Rightarrow g(y) = \frac{2}{9}(y-1)$

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The rth moment of a r.v. X about a pt. a is denoted μ_r' and is defined as:

$$\mu_r' = E(X-a)^r$$

If we take $a = 0$ → moment about origin
$\mu_r' = E(X)^r$

If a → expected value of X → Central moments

$\mu_r = E(X-\mu)^r$ where $\mu = E(X)$

If $r = 0$, $\mu_0 = 1$
for $r = 1$, $\mu_1 = 0$
for $r = 2$, $\mu_2 = E(X-\mu)^2$ → measure of dispersion also called as variance of r.v. X.

for $r = 3$, $\mu_3 = E(X-\mu)^3$ → measure of asymmetry i.e. skewness

for $r = 4$, $\mu_4 = E(X-\mu)^4$ → moment of measure of Kurtosis i.e. whether the values are concentrated around the mean or are scattered. This defines the peakness of curve.

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Markov's Inequality :-

To find the probabilities of extreme events.
To find the upper bound of probability when expected value is given.

If $X \geq 0$ and $a > 0$, then $\boxed{P(X \geq a) \leq \frac{E(X)}{a}}$

Chebyshev's Inequality :-

stronger + more informative

Requirements :-
1. R.V. X, with finite mean μ and variance σ^2.
2. If variance is small, then X is unlikely to be too far from the mean.

$$\Rightarrow \boxed{P(|X - \mu| \geq c) \leq \frac{\sigma^2}{c^2}}$$

Discrete Uniform Function :-

A r.v. X is said to have a discrete uniform distribution over the range $[1, n]$, if its pmf is expressed as :

$$P(X = x) = \begin{cases} 1/n & \text{for } x = 1, 2, \ldots, n \\ 0 & \text{, otherwise} \end{cases}$$

$n \Rightarrow$ parameter of distribution and lies in the set of all +ve integers.

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Here, $E(X) = \frac{1}{n} \sum i = \frac{1}{n} \frac{n(n+1)}{2} = \frac{n+1}{2}$

$E(X^2) = \frac{1}{n} \sum i^2 = \frac{1}{n} \frac{n(n+1)(2n+1)}{6} = \frac{(n+1)(2n+1)}{6}$

$\Rightarrow V(X) = E(X^2) - E(X)^2 = \frac{n^2 - 1}{12}$

Moment Generating function (MGF) :-

$$M_X(t) = E(e^{tX}) = \sum e^{tX} \cdot f(x)$$
$$= \sum e^{tX} \cdot \left(\frac{1}{n}\right)$$
$$= \frac{e^t (1 - e^{xt})}{n(1 - e^t)}$$

$$M_X'(t) = \frac{1}{n} \sum_{x=1}^{n} e^{tX} \cdot x \Big|_{t=0}$$
$$= \frac{1}{n} \sum_{x=1}^{n} x = \frac{n+1}{2}$$

$$M_X^2(t) = \frac{1}{n} \sum_{x=1}^{n} e^{tX} \cdot x^2 \Big|_{t=0} = \frac{1}{n} \sum_{x=1}^{n} x^2$$

$$\Rightarrow M_X^2(t) = \frac{(n+1)(2n+1)}{6}$$

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Binomial Distribution :-

1. All trials are independent.
2. Number (n) of trials is finite.
3. The probability (p) of successes is same for each trial.

$$P(x) = {}^nC_x \, p^x \, q^{n-x} \qquad (p+q=1)$$

Ques Show that $P(x)$ is p.m.f.

$$\Rightarrow \sum_{x=0}^{n} P(x) = \sum_{x=0}^{n} {}^nC_x \, p^x \cdot q^{n-x}$$

$$= q^n + {}^nC_1 \, p^1 q^{n-1} \longleftarrow + p^n$$

$$= (p+q)^n$$

$$= (1)^n = 1$$

also $P(x) \geq 0$

So, it is a p.m.f.

Ques Find M.G.F. of Binomial dist^n

$$\Rightarrow M.G.F. = M_X(t) = E(e^{tx})$$

$$\Rightarrow \text{———} = \sum_{x=0}^{n} e^{xt} \cdot P(x)$$

$$\Rightarrow \text{———} = \sum_{x=0}^{n} e^{xt} \cdot {}^nC_x \, p^x \cdot q^{n-x}$$

$$\Rightarrow \text{———} = \sum_{x=0}^{n} {}^nC_x \, (p \cdot e^t)^x \, q^{n-x}$$

$$\Rightarrow \boxed{\text{———} = (p \cdot e^t + q)^n}$$

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Characteristic function :-

$$\Rightarrow \varphi_X(t) = E(e^{ixt})$$

$$\Rightarrow \boxed{\text{———} = (p \cdot e^{it} + q)^n}$$

Probability Generating function :- (PGF)

$$\Rightarrow Z_X(t) = E(z^X)$$

$$\Rightarrow \text{———} = \sum_{x=0}^{n} z^x \cdot {}^nC_x \, p^x q^{n-x}$$

$$\Rightarrow \text{———} = \sum_{x=0}^{n} {}^nC_x \, (pz)^x \, q^{n-x}$$

$$\Rightarrow \boxed{\text{———} = (pz+q)^n}$$

Expected value (Mean) of Binomial Distribution :-

$$E(X) = \sum_{x=0}^{n} x \, {}^nC_x \, p^x \cdot q^{n-x}$$

$$= \sum_{x=0}^{n} \cancel{x} \cdot \frac{n!}{\cancel{x!}\,(x-1)!\,(n-x)!} \, p \cdot p^{x-1} q^{n-x}$$

$$= \sum_{x=0}^{n} (np) \frac{(n-1)!}{(x-1)!\,(n-x)!} \cdot p^{x-1} q^{n-x}$$

$$= np \sum_{x=0}^{n} {}^{n-1}C_{x-1} \, p^{x-1} q^{n-x}$$

$$\Rightarrow E(X) = np\,(p+q)^{n-1}$$

$$\Rightarrow \boxed{E(X) = np}$$

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For finding value of $E(x^2)$, we will calculate $E(x(x-1))$

$$E(x(x-1)) = \sum_{x=0}^{n} x(x-1)\, {}^nC_x\, p^x \cdot q^{n-x}$$

$$= \sum_{x=0}^{n} \cancel{x(x-1)} \frac{n!}{\underset{(x-2)!}{\cancel{x!}}\,(n-x)!} \cdot p^x \cdot q^{n-x}$$

$$= \sum_{x=0}^{n} \frac{n(n-1).(n-2)!}{(x-2)!\,(n-x)!}\, p^2 \cdot p^{x-2} \cdot q^{n-x}$$

$$= n(n-1)p^2 \cdot \sum_{x=0}^{n} \text{———}$$

$$\Rightarrow E(x(x-1)) = n(n-1)p^2 \cancel{(p+q)^{n-2}} 1$$

$$\Rightarrow E(x^2) = E(x(x-1)) + E(x)$$

$$= n(n-1)p^2 + np$$
$$= n^2p^2 - np^2 + np$$
$$= n^2p^2 + np(1-p)$$
$$= n^2p^2 + np(q)$$

$$\Rightarrow \boxed{E(x^2) = n^2p^2 + npq}$$

$$\Rightarrow \text{variance } (\sigma^2) = E(x^2) - (E(x))^2$$
$$= n^2p^2 + npq - \cancel{(np)^2}$$

$$\Rightarrow \boxed{\sigma_x^2 = npq}$$

when n is very large and p is very small.

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Poisson Distribution :-

A discrete R.V. x which has the following pmf

$$p(x) = \frac{\lambda^x \cdot e^{-\lambda}}{x!} \; ; x = 0, 1, 2, \ldots \infty$$

is called poisson variate and its distribution is called poisson distribution.

Ques show that Poisson distribution is limiting case of Binomial distribution under the following conditions

1. $n \to \infty$
2. $p \to 0$
3. $np \Rightarrow \lambda$ (finite)

Proof:- $\lim_{n\to\infty} p(x) = \lim_{x\to\infty} {}^nC_x\, p^x\, q^{n-x}$

$$\Rightarrow \lim_{n\to\infty} \frac{n!}{x!\,(n-x)!}\, p^x \cdot (1-p)^{n-x}$$

$$\Rightarrow \lim_{n\to\infty} \frac{n(n-1)\text{———}\cancel{(n-x)!}}{x!\,\cancel{(n-x)!}} \left(\frac{\lambda}{n}\right)^x \cdot \left(1-\frac{\lambda}{n}\right)^{n-x}$$

$$\Rightarrow \lim_{n\to\infty} \frac{\cancel{n^x}\left(1\left(1-\frac{1}{n}\right)(\text{———})\left(1-\frac{(x-1)}{n}\right)\right) \cdot \frac{\lambda^x}{\cancel{n^x}} \cdot \left(1-\frac{\lambda}{n}\right)^{n-x}}{x!}$$

$$\Rightarrow \lim_{n\to\infty} \frac{1}{x!}\, \lambda^x \left(1-\frac{\lambda}{n}\right)^n \left(1-\frac{\lambda}{n}\right)^{-x}$$

$$\Rightarrow \frac{\lambda^x}{x!} \lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n \lim_{n\to\infty}\cancel{\left(1-\frac{\lambda}{n}\right)^{-x}} \to 1$$

$$\Rightarrow \left(\frac{\lambda^x}{x!}\right)\left(e^{-\lambda}\right) = \boxed{\frac{\lambda^x \cdot e^{-\lambda}}{x!}}$$

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Mean of Poisson distribution:-

$$P(x) = \frac{\lambda^x e^{-\lambda}}{x!} \; ; x = 0, 1, 2, \ldots$$

$$\text{mean} = E(x) = \sum_x x \cdot P(x) = \sum x \cdot \frac{\lambda^x \cdot e^{-\lambda}}{\cancel{x} \cdot (x-1)!}$$

$$\Rightarrow \text{Mean} = e^{-\lambda} \cdot \lambda \boxed{\sum \frac{\lambda^{x-1}}{(x-1)!}}$$

$$\Rightarrow \text{Mean} = \cancel{e^{-\lambda}} \cdot \lambda \cdot \cancel{e^{\lambda}}$$

$$\Rightarrow \boxed{\text{Mean } (E(x)) = \lambda}$$

$$E(x(x-1)) = \sum x \cdot (x-1) \cdot \frac{\lambda^x \cdot e^{-\lambda}}{x!}$$

$$= \sum \frac{\cancel{x(x-1)} \cdot \lambda(\lambda) \cdot \lambda^{x-2} \cdot e^{-\lambda}}{\cancel{x(x-1)}(x-2)!}$$

$$= \lambda(\lambda) \cdot e^{-\lambda} \sum \frac{\lambda^{x-2}}{(x-2)!}$$

$$= \lambda \cdot (\lambda) \cancel{e^{-\lambda}} \cdot \cancel{e^{\lambda}}$$

$$\Rightarrow E(x(x-1)) = \lambda^2$$

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$$E(x^2) = E(x(x-1)) + E(x)$$

$$\boxed{E(x^2) = \lambda^2 + \lambda}$$

$$\Rightarrow var(x) = E(x^2) - (E(x))^2$$

$$\Rightarrow \text{———} = \cancel{\lambda^2} + \lambda - \cancel{\lambda^2}$$

$$\Rightarrow \boxed{var(x) = \lambda}$$

$$\Rightarrow \sigma_x^2 = \lambda$$

$$\Rightarrow S.D. (\sigma_x) = \sqrt{\lambda}$$

Ques Prove poisson distribution is pmf.

Ans -

$$\Rightarrow \sum P(x) = \sum \frac{\lambda^x \cdot e^{-\lambda}}{x!}$$

$$= e^{-\lambda} \sum \frac{\lambda^x}{x!}$$

$$= \cancel{e^{-\lambda}} \cdot \cancel{e^{\lambda}}$$

$$\Rightarrow \sum P(x) = 1$$

So, poisson distribution is a pmf.

Ques MGF of poisson.

Ans -

$$M_x(t) = E(e^{xt}) = \sum e^{xt} \cdot P(x)$$

$$= \sum e^{xt} \cdot \frac{\lambda^x \cdot e^{-\lambda}}{x!}$$

$$= e^{-\lambda} \sum \frac{e^{xt} \cdot \lambda^x}{x!}$$

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$$\Rightarrow \text{—} = e^{-\lambda} \sum \frac{(\lambda e^t)^x}{x!}$$

$$\cancel{= e^{-\lambda} e^{\lambda e^t}}$$

$$= e^{-\lambda} \cdot e^{\lambda e^t}$$

$$= e^{\lambda e^t - \lambda}$$

$$\Rightarrow \boxed{M.G.F. = e^{\lambda(e^t - 1)}}$$

Characteristic Function:-

$$\psi_x(t) = E(e^{ixt}) = e^{\lambda(e^{it} - 1)}$$

Probability Generating Function:-

$$Z_x(t) = E(z^x) = e^{\lambda(z-1)}$$

Mean and variance by M.G.F.:-

$$E(x) = \frac{d}{dt}(M_x(t))_{t=0}$$

$$E(x^2) = \frac{d^2}{dt^2}(M_x(t))_{t=0}$$

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Exponential Distribution:-

A continuous random variable x which has the following pdf.

$$f(x) = \lambda \cdot e^{-\lambda x} \quad ; \lambda > 0 \ , \ 0 < x < \infty$$

is called exponential variable and its distribution is called exponential distribution.

Mean:-

$$E(x) = \int_{-\infty}^{\infty} x \cdot f(x) \cdot dx = \int_0^{\infty} x \cdot f(x) \cdot dx.$$

$$= \int_0^{\infty} x \cdot \lambda \cdot e^{-\lambda x} \cdot dx.$$

$$= \lambda \int_0^{\infty} x \cdot e^{-\lambda x} \cdot dx = \cancel{\lambda} \frac{\Gamma 2}{\lambda^{\cancel{2}}}$$

$$\boxed{\text{Mean} = \frac{1}{\lambda}}$$

$$E(x^2) = \int_0^{\infty} x^2 \cdot \lambda \cdot e^{-\lambda x} \cdot dx = \lambda \frac{\Gamma 3}{\lambda^3}$$

$$\Rightarrow E(x^2) = \frac{2}{\lambda^2}$$

$$\Rightarrow \text{Variance } (\sigma_x^2) = E(x^2) - (E(x))^2$$

$$\Rightarrow \boxed{\sigma_x^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}}$$

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M.G.F :-

$$M_X(t) = E(e^{xt}) = \int_{-\infty}^{\infty} e^{xt} \cdot f(x) \cdot dx.$$

$$= \int_0^{\infty} e^{-xt} \cdot \lambda \cdot e^{-\lambda x} \cdot dx.$$

$$= \lambda \int_0^{\infty} e^{-x(\lambda - t)} \cdot dx$$

$$= \lambda \left[\frac{e^{-x(\lambda - t)}}{-(\lambda - t)} \right]_0^{\infty}$$

$$= -\lambda \left(0 - \frac{1}{\lambda - t} \right)$$

$$\Rightarrow \boxed{M_X(t) = \frac{\lambda}{-t + \lambda}}$$

Characteristic function :-

$$\varphi_X(t) = E(e^{itx}) = \frac{\lambda}{it - \lambda}$$

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Rectangular or Uniform Distribution :-

A continuous r.v. is said to follow a continuous uniform or rectangular distribution over a interval (a, b), if its p.d.f. is given by

$$f(x) = \begin{cases} K & ; \ a < x < b \\ 0 & ; \ \text{otherwise} \end{cases}$$

where, $f(x)$ is p.d.f. $\int_{-\infty}^{\infty} f(x) \cdot dx = 1$

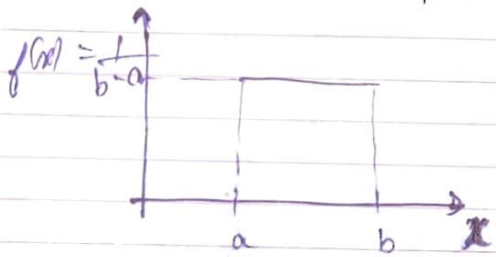
$$\Rightarrow \int_a^b K \cdot dx = 1$$

$$\Rightarrow K \left[x \right]_a^b = 1$$

$$\Rightarrow K(b - a) = 1$$

$$\Rightarrow \boxed{K = \frac{1}{b - a}}$$

$$f(x) = \begin{cases} \frac{1}{b-a} & ; \ a < x < b \\ 0 & ; \ \text{otherwise} \end{cases}$$



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Mean :-

$\Rightarrow E(x) = \int_{-\infty}^{\infty} x \cdot f(x) \cdot dx \quad \left[\Rightarrow f(x) = \begin{cases} \frac{1}{b-a}, & a<x<b \\ 0 & ; \text{o/w} \end{cases} \right]$

$\Rightarrow E(x) = \int_a^b x \cdot \frac{1}{b-a} \cdot dx = \frac{1}{b-a} \int_a^b x \cdot dx$

$= \left[\frac{x^2}{2}\right]_a^b \left(\frac{1}{b-a}\right) = \left(\frac{b^2-a^2}{2}\right)\left(\frac{1}{b-a}\right)$

$\Rightarrow \boxed{E(x) = \left(\frac{b+a}{2}\right)}$

$\Rightarrow E(x^2) = \int_{-\infty}^{\infty} x^2 \cdot f(x) \cdot dx = \frac{1}{b-a} \int_a^b x^2 \cdot dx$

$\Rightarrow E(x^2) = \frac{1}{3}\left(\frac{1}{b-a}\right)\left[x^3\right]_a^b = \frac{1}{3(b-a)}(b^3-a^3)$

$\Rightarrow \boxed{E(x^2) = \frac{b^2+ab+a^2}{3}}$

$\Rightarrow Var(x) \ (\sigma_x^2) = E(x^2) - (E(x))^2$

$= \frac{b^2+ab+a^2}{3} - \frac{(b+a)^2}{4}$

$= \frac{4b^2+4ab+4a^2-(3b^2+3a^2+6ab)}{12}$

$\Rightarrow \boxed{\sigma_x^2 = \frac{a^2+b^2-2ab}{12} = \frac{(a-b)^2}{12}}$

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M.G.F :-

$M_x(t) = E(e^{xt}) = \int_{-\infty}^{\infty} e^{xt} \cdot f(x) \, dx$

$= \int_a^b (e^{xt}) \cdot \left(\frac{1}{b-a}\right) \cdot dx$

$= \left(\frac{1}{b-a}\right)\left[\frac{e^{xt}}{t}\right]_a^b = \left(\frac{1}{b-a}\right)\left[\frac{e^{bt}-e^{at}}{t}\right]$

$\cancel{= \left(\frac{1}{b-a}\right)}$

$\Rightarrow \boxed{M_x(t) = \frac{e^{bt}-e^{at}}{t(b-a)}}$

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Normal Distribution :-

A Continuous r.v. which has the following p.d.f

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}} \; ; \; -\infty < x < \infty, \; -\infty < \mu < \infty, \; \sigma > 0$$

is called normal distribution.
written as, $X \sim N(\mu, \sigma^2)$

$$E(x) = \int_{-\infty}^{\infty} x \cdot f(x) \cdot dx = \int_{-\infty}^{\infty} x \cdot \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}} \cdot dx$$

Let $\frac{x-\mu}{\sigma} = z \Rightarrow x = \mu + \sigma z \; ; \; dx = \sigma dz$

$$\Rightarrow E(x) = \frac{1}{\cancel{\sigma}\sqrt{2\pi}} \int_{-\infty}^{\infty} (\mu + \sigma z) \cdot e^{-\frac{z^2}{2}} \cdot \cancel{\sigma} \cdot dz$$

$$\Rightarrow E(x) = \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{\infty} \mu \cdot e^{-\frac{z^2}{2}} \cdot dz + \cancel{\sigma \int_{-\infty}^{\infty} z \cdot e^{-\frac{z^2}{2}} \cdot dz} \right]$$

0

odd function

$$\Rightarrow E(x) = \frac{2}{\sqrt{2\pi}} (\mu) \int_{0}^{\infty} e^{-\frac{z^2}{2}} \cdot dz$$

Let $\frac{z^2}{2} = p$

$\Rightarrow \frac{\cancel{2}z}{\cancel{2}} dz = dp$

$\Rightarrow z \cdot dz = dp$

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$$\Rightarrow E(x) = \frac{2}{\sqrt{2\pi}} \mu \int_{0}^{\infty} e^{-p} \cdot \frac{dp}{\sqrt{2p}}$$

$$\Rightarrow E(x) = \frac{\cancel{2}\mu}{\cancel{\sqrt{2}}\pi \cancel{\sqrt{2}}} \int_{0}^{\infty} p^{-1/2} \cdot e^{-p} \cdot dp$$

$$\Rightarrow E(x) = \frac{\mu}{\sqrt{\pi}} \Gamma\tfrac{1}{2} = \cancel{\frac{\mu\sqrt{\pi}}{\sqrt{\pi}}}$$

$$\Rightarrow \boxed{E(x) = \frac{\mu}{\cancel{\sqrt{\pi}}} \cancel{\sqrt{\pi}} = \mu}$$

$$\Rightarrow E(x^2) = \mu^2 + \sigma^2$$

$$\Rightarrow \sigma_x^2 = \cancel{\mu^2} + \sigma^2 - \cancel{(\mu)^2}$$

$$\Rightarrow \boxed{\sigma_x^2 = \sigma^2}$$

M.G.F. :-

$$\Rightarrow f(x) = \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

$$\Rightarrow M_x(t) = E(e^{xt}) = \int_{-\infty}^{\infty} e^{xt} \cdot f(x) dx.$$

$$\Rightarrow M_x(t) = \int_{-\infty}^{\infty} e^{xt} \cdot \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \cdot dx$$

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Let $\frac{x-\mu}{\sigma} = z$

$\Rightarrow \frac{dx}{\sigma} = dz$

$\Rightarrow dx = \sigma dz$

$$\Rightarrow M_x(t) = \frac{1}{\cancel{\sigma}\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{(\mu+\sigma z)t} \cdot e^{-\frac{z^2}{2}} \cdot \cancel{\sigma} dz$$

$$\Rightarrow M_x(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\mu t} \cdot e^{\sigma z t} \cdot e^{\frac{-z^2}{2}} \cdot dz$$

$$= \frac{e^{\mu t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left((z^2 - 2\sigma z t) + (\sigma t)^2 - (\sigma t)^2\right)} \cdot dz$$

$$= \frac{e^{\mu t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z-\sigma t)^2} \cdot e^{\frac{\sigma^2 t^2}{2}} \cdot dz$$

$$= \frac{e^{\mu t + \frac{\sigma^2 t^2}{2}}}{\sqrt{2\pi}} \times 2 \int_{0}^{\infty} e^{-\frac{1}{2}(z-\sigma t)^2} \cdot dz$$

Let $\frac{1}{2}(z-\sigma t)^2 = \theta$

$\Rightarrow \frac{1}{\cancel{2}} \times \cancel{2}(z-\sigma t) \cdot dz = d\theta$

$\Rightarrow dz = \frac{d\theta}{\sqrt{2\theta}}$

$$\Rightarrow M_x(t) = \frac{\cancel{2} e^{\mu t + \frac{\sigma^2 t^2}{2}}}{\sqrt{2\pi}} \int_{0}^{\infty} \frac{e^{-\theta}}{\sqrt{2\theta}} \cdot d\theta$$

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$$\Rightarrow M_x(t) = \frac{\cancel{2} \cdot e^{\mu t + \frac{\sigma^2 t^2}{2}}}{\cancel{2}\sqrt{\pi}} \int_{0}^{\infty} \theta^{-1/2} \cdot e^{-\theta} \cdot d\theta$$

$$\Rightarrow M_x(t) = \frac{e^{\mu t + \frac{\sigma^2 t^2}{2}}}{\cancel{\sqrt{\pi}}} \cdot \cancel{\sqrt{\pi}}$$

$$\Rightarrow M_x(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}}$$

For Standard Normal Distribution:-

X is normally distributed with mean μ and variance σ^2. Let us transform the variable x to z, where $z = \frac{x-\mu}{\sigma}$, then,

$$\Rightarrow E(z) = \frac{E(x)-\mu}{\sigma} = \frac{\mu-\mu}{\sigma} = 0$$

$$\Rightarrow \boxed{E(z) = 0}$$

$$\Rightarrow V(z) = V\left(\frac{x-\mu}{\sigma}\right) = \frac{1}{\sigma^2} V(x-\mu)$$

$$\Rightarrow \text{———} = \frac{1}{\sigma^2} V(x)$$

$$\Rightarrow \text{———} = \frac{1}{\cancel{\sigma^2}} \cdot \cancel{\sigma^2} = 1$$

$$\Rightarrow \boxed{V(z) = 1}$$

z → Standard normal variate

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MGF of standard Normal variate (z):-

$$z = \frac{x-\mu}{\sigma}$$

$$M_z(t) = E(e^{zt})$$
$$= E\left(e^{\left(\frac{x-\mu}{\sigma}\right)t}\right)$$
$$= E\left(e^{xt/\sigma - \frac{\mu t}{\sigma}}\right)$$
$$= E\left(e^{\frac{xt}{\sigma}} \cdot e^{-\frac{\mu t}{\sigma}}\right)$$
$$= e^{-\frac{\mu t}{\sigma}} E\left(e^{\frac{xt}{\sigma}}\right)$$
$$= e^{-\mu t/\sigma} \cdot M_x(t/\sigma)$$
$$= e^{-\mu t/\sigma} \cdot e^{\mu t/\sigma + \frac{\sigma^2 t^2}{2\sigma^2}}$$
$$= e^{-\cancel{\mu t/\sigma} + \cancel{\frac{\mu t}{\sigma}} + \frac{t^2}{2}}$$

$$\Rightarrow \boxed{M_z(t) = e^{t^2/2}}$$

Mean and variance by M.G.F.:-

$$M_x(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}}$$

$$E(X) = \frac{d}{dt}(M_x t)_{t=0} = \left(\left(e^{\mu t + \frac{\sigma^2 t^2}{2}}\right)(\mu + \sigma^2 t)\right)_{t=0}$$

$$\Rightarrow E(X) = (e^0)(\mu + 0) = \mu$$

$$\Rightarrow \boxed{E(X) = \mu}$$

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$$E(X^2) = \mu^2 + \sigma^2$$

$$\Rightarrow \sigma_x^2 = E(X^2) - (E(X))^2$$

$$\Rightarrow \text{—} = \cancel{\mu^2} + \sigma^2 - \cancel{\mu^2}$$

$$\Rightarrow \boxed{\sigma_x^2 = \sigma^2}$$

Area under Curve:- Bell shaped

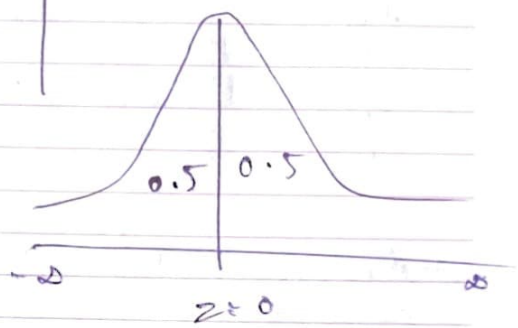
$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} ; \quad -\infty < x < \infty$$

$\mu = 0$

$\sigma = 1$

$$z = \frac{x-\mu}{\sigma}$$

$$f(z) = \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{z^2}{2}} ; \quad -\infty < z < \infty$$



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Moments :-

① Moments about origin :-

$\mu_r' = E(x-0)^r$

$\mu_0' = E(x-0)^0 = 1$

$\Rightarrow \boxed{\mu_1' = E(x) = \text{mean}}$

$\Rightarrow \mu_2' = E(x^2)$ — D.R.V. $\sum x^2 \cdot f(x)$; C.R.V. $\int_{-\infty}^{\infty} x^2 \cdot f(x)\, dx$

$\Rightarrow \mu_3' = E(x^3)$

② Moment about mean :-

$\mu_r = E(x-\bar{x})^r$

$\mu_0 = E(x-\bar{x})^0 = 1$

$\mu_1 = E(x-\bar{x}) = E(x) - E(\bar{x})$

$\Rightarrow \boxed{\mu_1 = \bar{x} - \bar{x} = 0}$

$\Rightarrow \mu_2 = E(x-\bar{x})^2 = E(x^2 - 2x\cdot\bar{x} + \bar{x}^2)$

$\Rightarrow$ —— $= E(x^2) - 2\bar{x}E(x) + \bar{x}^2$

$\Rightarrow$ —— $= E(x^2) - 2\bar{x}^2 + x^2$

$\Rightarrow$ —— $= E(x^2) - \bar{x}^2$

$\Rightarrow \mu_2 = E(x^2) - (E(x))^2 = \mu_2' - (\mu_1')^2$

$\mu_3 = E(x-\bar{x})^3 = \mu_3' - 3\mu_2'\mu_1' + 2(\mu_1')^3$

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③ Moment about any other point :-

$\Rightarrow \mu_r'' = E(x-A)^r$

$\Rightarrow \mu_1'' = E(x-A)$

$\Rightarrow \mu_2'' = E(x-A)^2$

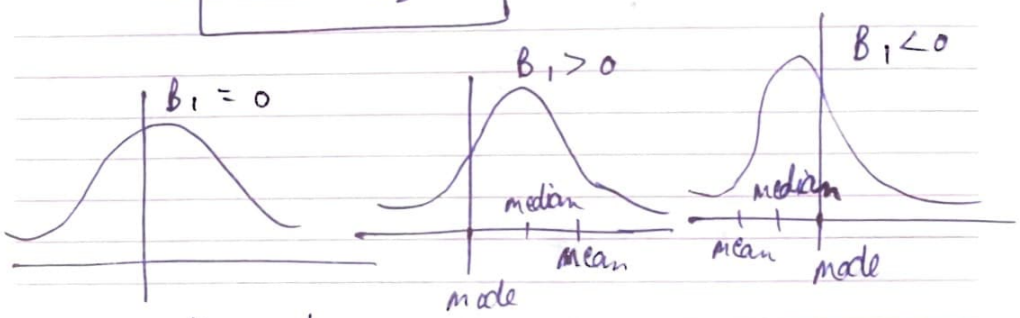
$\Rightarrow \mu_3'' = E(x-A)^3$

$\Rightarrow \mu_4'' = E(x-A)^4$

Skewness and Kurtosis :-

Skewness :- mean lack of ~~[illegible]~~ symmetry

$$\Rightarrow \boxed{\beta_1 = \frac{\mu_3^2}{\mu_2^3}}$$



mean = median = mode
Symmetrical
means it will normal distribution

positively skewed

Negatively skewed.

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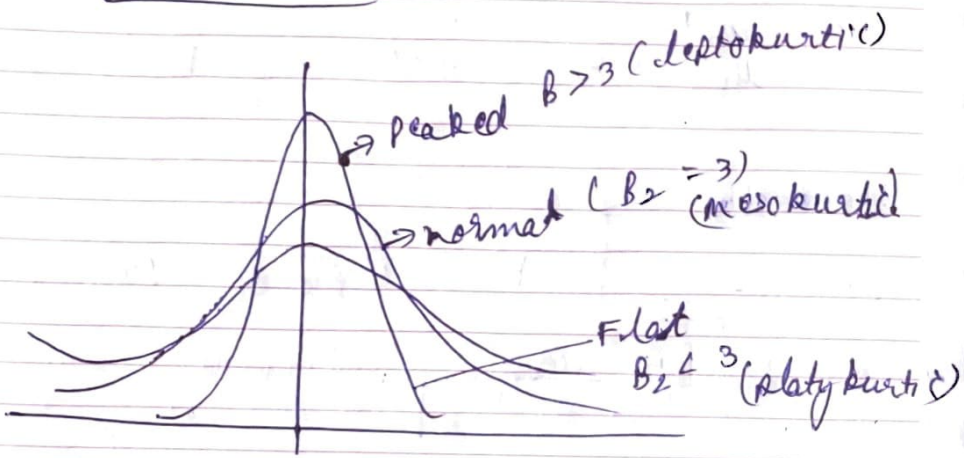
Kurtosis:- The extent to which a distribution is ~~[illegible]~~ peaked or flat called kurtosis.

$$\Rightarrow \boxed{\beta_2 = \frac{\mu_4}{\mu_2^2}}$$



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2-D Random Variable:-

Let there are 2 functions $X = X(s)$ and $Y = Y(s)$ for sample space S. So that these assign a real no. to each element $s \in S$.

Then (X, Y) is called 2-D random variable.

(X, Y) is called 2-D discrete r.v. if the possible values of (X, Y) are finite or countably infinite.

(X, Y) is a 2-D continuous r.v. if (X, Y) can assume all values in a subset of X-Y plane.

Joint probability mass function for discrete r.v.:-

If (x, y) is a 2-D discrete r.v., where $x = x_i$ and $Y = y_j$, $i \& j = 1, 2, ____$, then $P(X = x_i, Y = y_j) = p_{ij}$ is called Joint pmf of (x, y) provided

1. $p_{ij} \geq 0$
2. $\sum_{i=1}^{n} \sum_{j=1}^{m} p_{ij} = 1$

The set of triple (x_i, y_j, p_{ij}) is called Joint probability distribution of (X, Y).

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Marginal Probability Mass function of X :-

$P(X=x_i) = \sum_j P_{ij} = P_{i1} + P_{i2} + \text{———}$

The collection of pair (x_i, P_i) is called marginal p.m.f. of X.

Similarly marginal p.m.f. of y :-

$P(Y=y_i) = \sum_i P_{ij} = P_{1j} + P_{2j} + \text{———}$

If X and Y are independent :-

$\Rightarrow \boxed{P_{XY}(x,y) = P_X(x) \cdot P_Y(y)}$

A discrete distribution is called a uniform distribution if $P(X=x_i)$ is constant for all x_i.

Joint Probability Density function for continuous r.v. :-

Let (X,Y) be a 2-D r.v. taking all the values in some given region R of X-Y plane. Then Joint Pdf of (X,Y) is a function $f(x,y)$ satisfying :

1. $f(x,y) \geq 0$ for all $(x,y) \in R$
2. $\iint_R f(x,y) \cdot dx \cdot dy = 1$

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Marginal Pdf :- (Continuous r.v.)

$\Rightarrow \boxed{g(x) = \int_{-\infty}^{\infty} f(x,y) \cdot dy}$

marginal p.d.f. of x.

Marginal p.d.f. of y :-

$\Rightarrow \boxed{h(y) = \int_{-\infty}^{\infty} f(x,y) \cdot dx}$

Conditional Probability distribution :-

① Discrete r.v. :- Same for continuous r.v.

$P(x_i | y_j) = P(X = x_i \mid Y = y_i)$

$= \dfrac{P(x_i, y_j)}{P(y_j)}$; $P(y_j) > 0$

$\Rightarrow P(y_j | x_i) = \dfrac{P(y_j, x_i)}{P(x_i)}$; $P(x_i) > 0$

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Independent Random Variables :-

$P(x_i, y_i) = P(x_i) \cdot P(y_i) \ \forall \ i$ and j

when X, Y is discrete variable.

For Continuous r.v. :-

$f(x, y) = g(x) \cdot h(y) \ \forall \ (x, y)$

Expectation :-

Let $g(x, y)$ is a function of x and y.

For discrete :-

$$\boxed{E(g(x,y)) = \sum_{(x_i, y_i) \in x \times y} \sum g(x_i, y_i) \cdot P(x_i, y_i)}$$

provided given series is convergent.

For Continuous :-

$$\boxed{E(g(x,y)) = \iint g(x, y) \cdot f(x, y) \cdot dx \cdot dy}$$

provided given series is convergent

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Product Moment :-

$\mu'_{r,s} = E(x^r, y^s) \rightarrow (r,s)^{th}$ non-central product moment

$\mu'_{1,1} = E(xy)$

$\mu'_{1,0} = E(x) = \mu_x, \quad \mu'_{0,1} = E(y) = \mu_y$

$$\boxed{\mu_{r,s} = E(x - \mu_x)^r (Y - \mu_Y)^s}$$

$(r,s)^{th}$ Central product moment

If $r = 1, s = 1$,

$$\boxed{\mu_{1,1} = E(xy) - E(x) \cdot E(y)} \rightarrow \text{covariance b/w } x \text{ and } y.$$

If X and Y are independent, then co-variance of X, Y i.e. $cov(X, Y) = 0$.

⇒ **Coefficient of correlation b/w x and y :-**

$$\Rightarrow \boxed{\rho_{X,Y} = \frac{Cov(X,Y)}{sd(X) \cdot sd(Y)} = \frac{\sigma_{XY}}{\sigma_X \cdot \sigma_Y}}$$

$\sigma_X^2 = var(X)$ and $\sigma_Y^2 = var(Y)$

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$$\Rightarrow \boxed{\rho_{XY} = \frac{E(XY) - E(X)E(Y)}{\sqrt{E(X^2) - (E(X))^2}\sqrt{E(Y^2) - (E(Y))^2}}}$$

If X and Y are independent,

$\Rightarrow E(XY) = E(X) \cdot E(Y)$

$\Rightarrow Cov(X, Y) = 0$

$\Rightarrow$ Corr. coeff $\rho = 0$

True

If $P(X = aY + b) = 1$; $a > 0$
then X and Y are perfectly linearly correlated in +ve direction.

If $P(X = aY + b) = 1$; $a < 0$
then X and Y are perfectly linearly correlated in -ve direction.

ρ gives us degree of linear relationship.

If $\rho_{X,Y} = 0 \Rightarrow$ X and Y are uncorrelated

uncorrelated does not mean independent.
But independent means uncorrelated.

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