Analysis & Design of Algorithms (CS302)

Gomory-Hu Tree Algorithm

Problem Description

Given a graph G(V, E), a *k-cut* is a partition of the vertices V into k sets $S_1, S_2, \ldots, S_k$. Any edge (u,v) $\in$ E with u $\in S_i$ and v $\in S_j$ (i $\neq$ j) is said to be crossing the cut and is a cut edge. The *capacity* of a k-cut is the sum of weights of all the cut edges.

Minimum K-Cut Problem: Given a graph G(V, E), find the minimum weight set of edges E′ $\subseteq$ E such that removing E′ from G leaves k connected components i.e. find the k-cut with the minimum capacity.

Algorithm

MINIMUM K-CUT WITH GOMORY-HU TREE (G(V, E), w: E → R^+, k)

1. Compute the Gomory-Hu Tree of G, call it T
2. Pick the k − 1 lightest edges in the T, call them e_1, e_2, ..., e_{k-1}
3. Return the union of the corresponding cuts, C, in G,
   $C = C_{e1} \cup C_{e2} \cup \ldots \cup C_{ek-1}$

Gomory-Hu Tree

Definition: Given an undirected weighted graph G(V, E, c), a cut-tree T(V, F, w) is a tree with edge-set F and capacities w that fulfills the following properties:

1. Each Edge in T corresponds to min-cut between two adjacent vertices in a tree.
2. Equivalent Flow Tree: For any pair of vertices s, t $\in$ V, maximum s-t flow in G is equal to the maximum s-t flow in T.
3. Cut Property: A minimum s-t cut in T is also a minimum s-t cut in G.

Gomory-Hu Tree Construction Algorithm

The algorithm maintains a partition of V, (sets $S_1, \ldots, S_t$), and a spanning tree T on the vertex set $\{S_1, \ldots, S_t\}$.

Initially, there exists only the set $S_1 = V$.

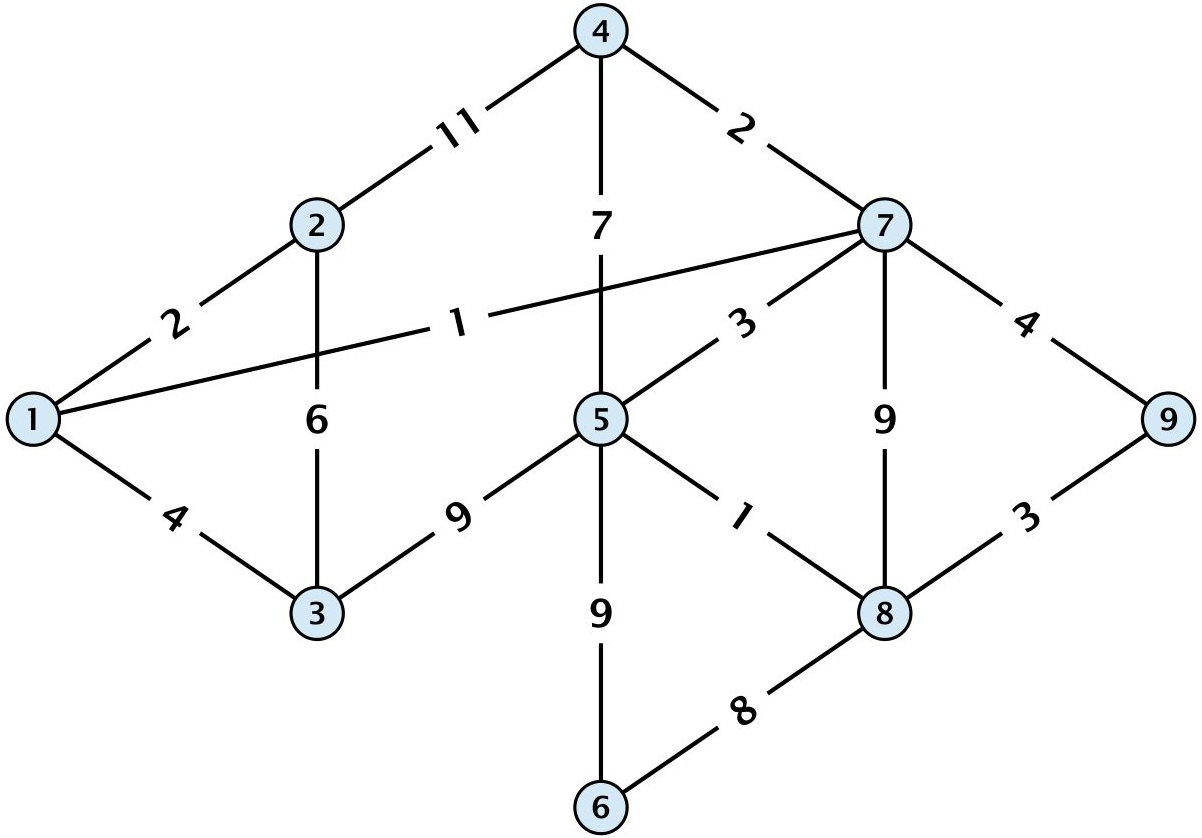
Then the algorithm performs $n - 1$ split-operations:

- In each such split-operation it chooses a set S_i with $|S_i| \geq 2$ and splits this set into two non-empty parts X and Y.
- S_i is then removed from T and replaced by X and Y.
- X and Y are connected by an edge, and the edges that before the split were incident to S_i are attached to either X or Y.

Details of the Split-operation

- Select S_i that contains at least two nodes a and b.
- Compute the connected components of the forest obtained from the current tree T after deleting S_i. Each of these components corresponds to a set of vertices from V.
- Consider the graph H obtained from G by contracting these connected components into single nodes.
- Compute a minimum a-b cut in H. Let A and B denote the two sides of this cut.
- Split S_i in T into two sets/nodes $X = S_i \cap A$ and $Y = S_i \cap B$ and add edge {X, Y} with capacity equal to that of the cut.
- Replace an edge $\{S_i, S_j\}$ with $\{X, S_j\}$ if $S_j \in A$ and by $\{Y, S_j\}$ if $S_j \in B$.

Example



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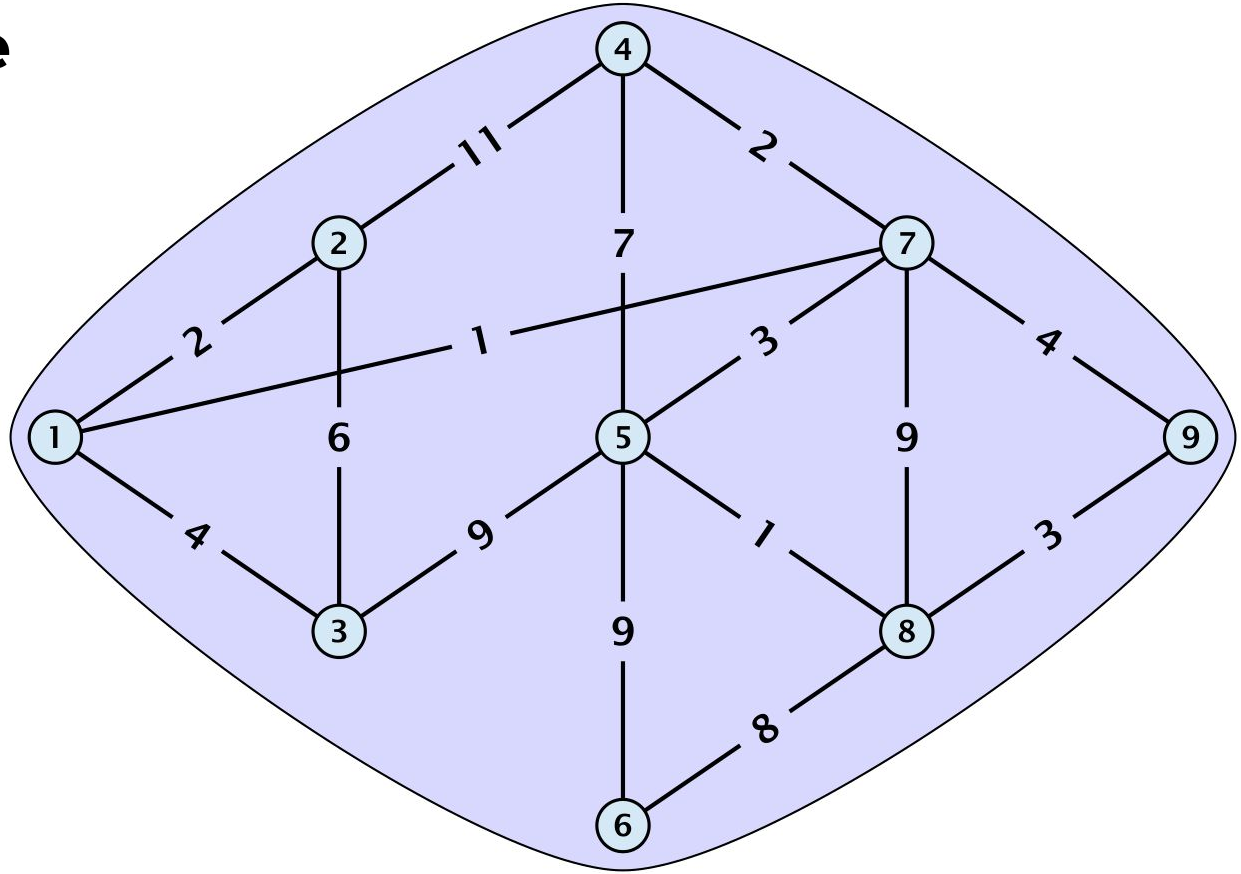

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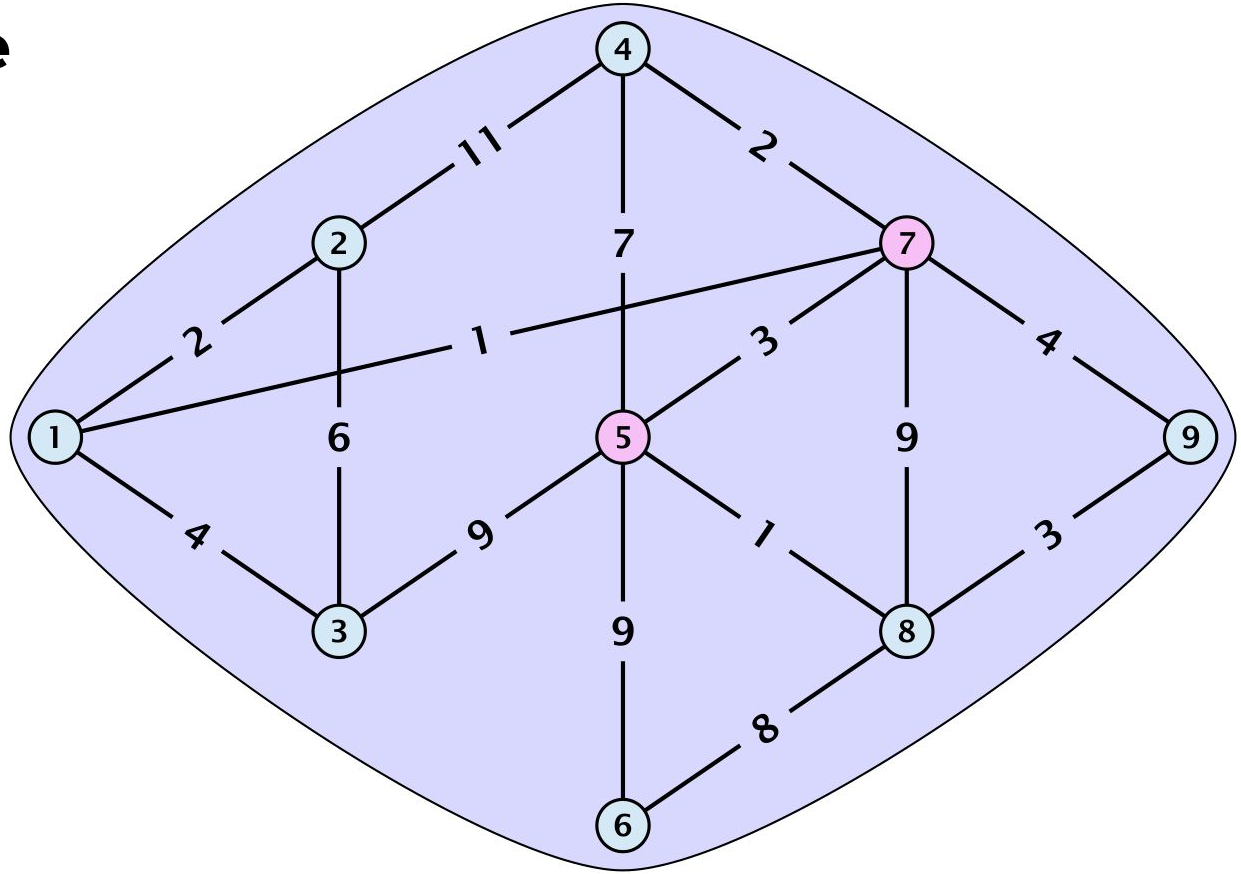

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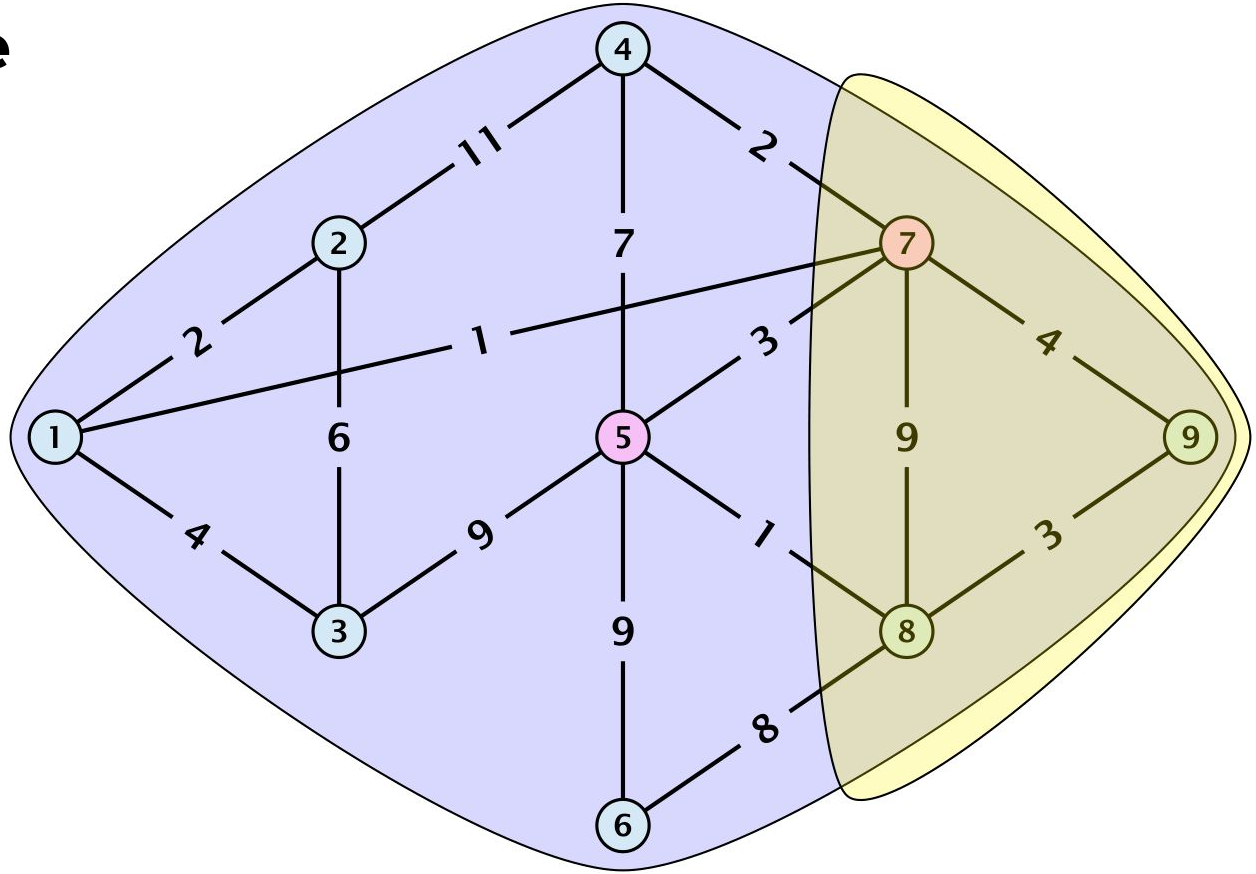

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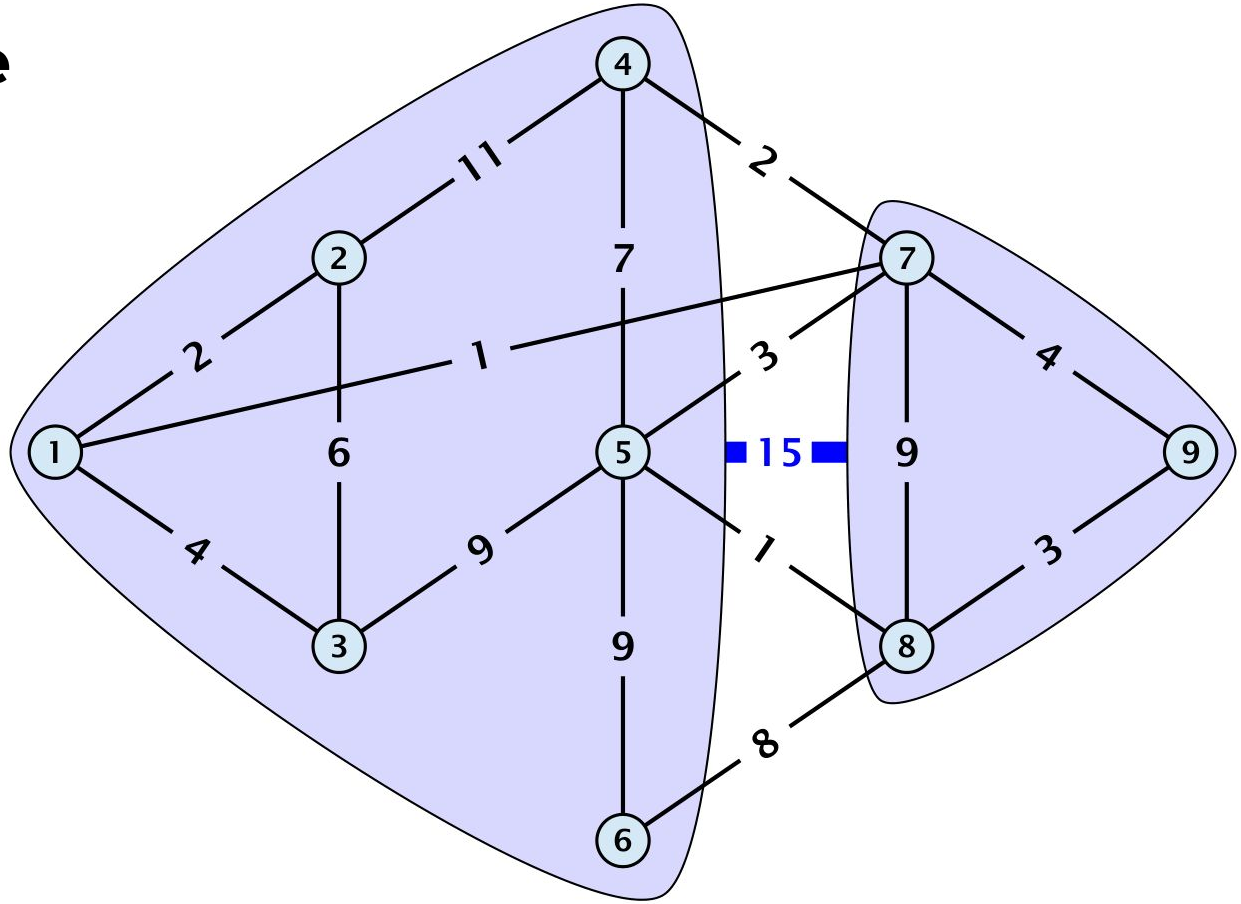

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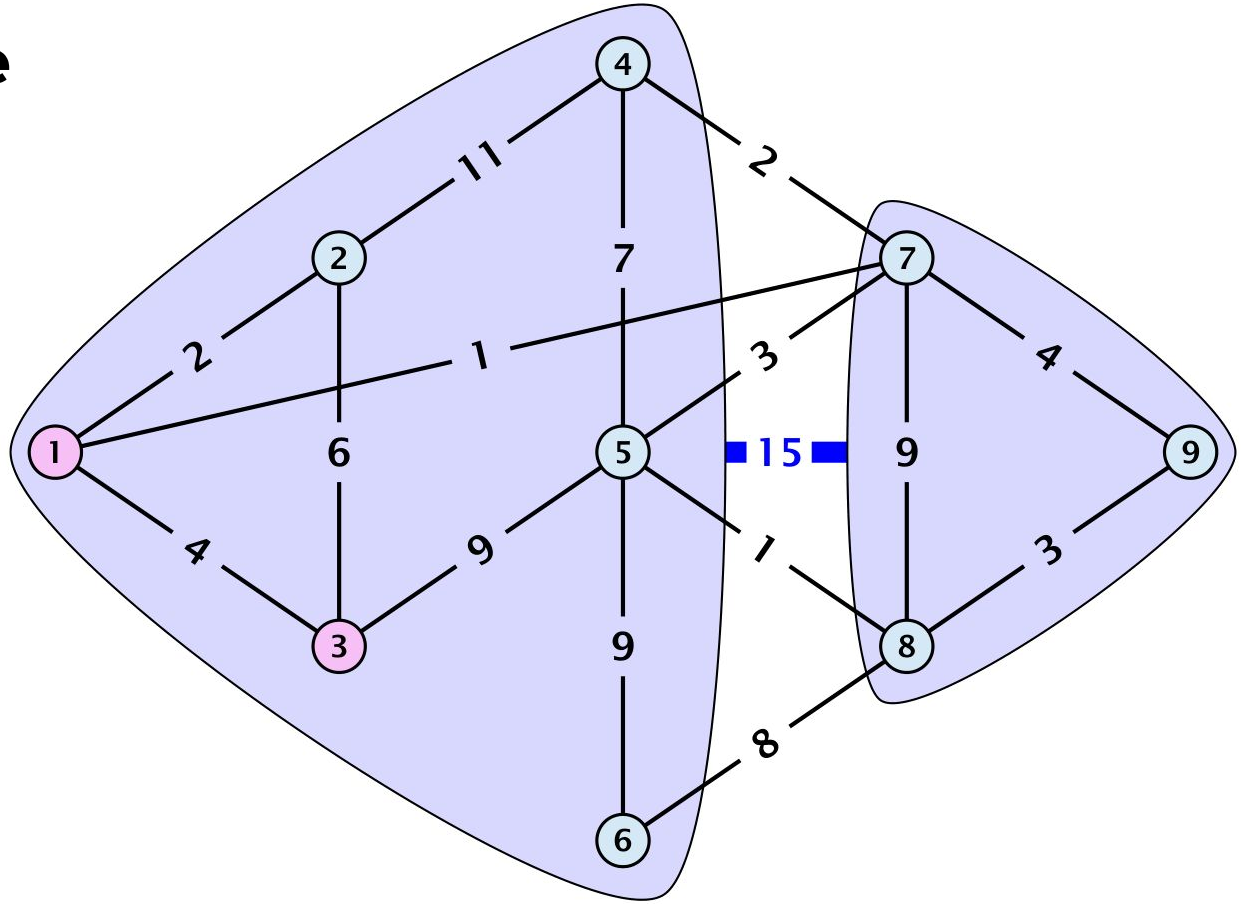

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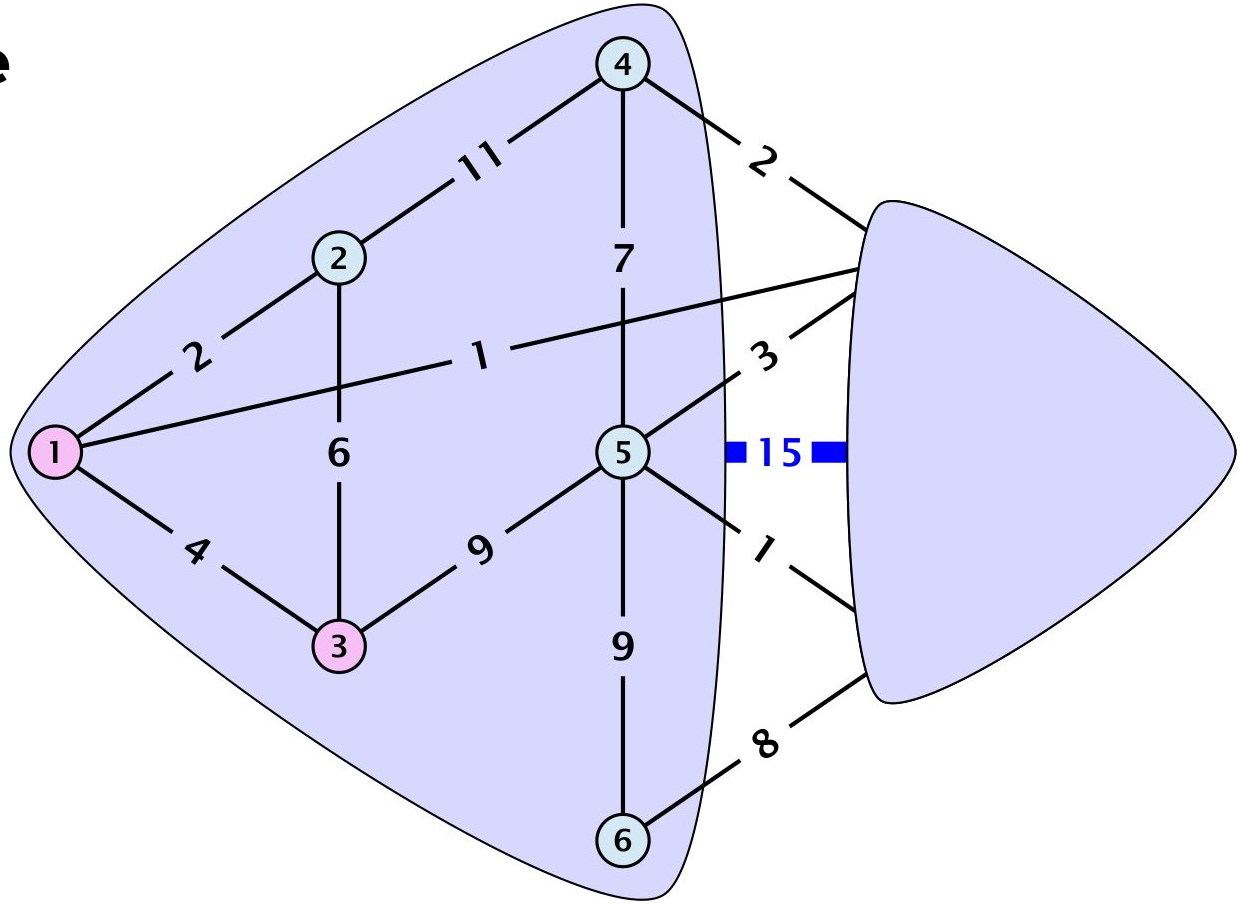

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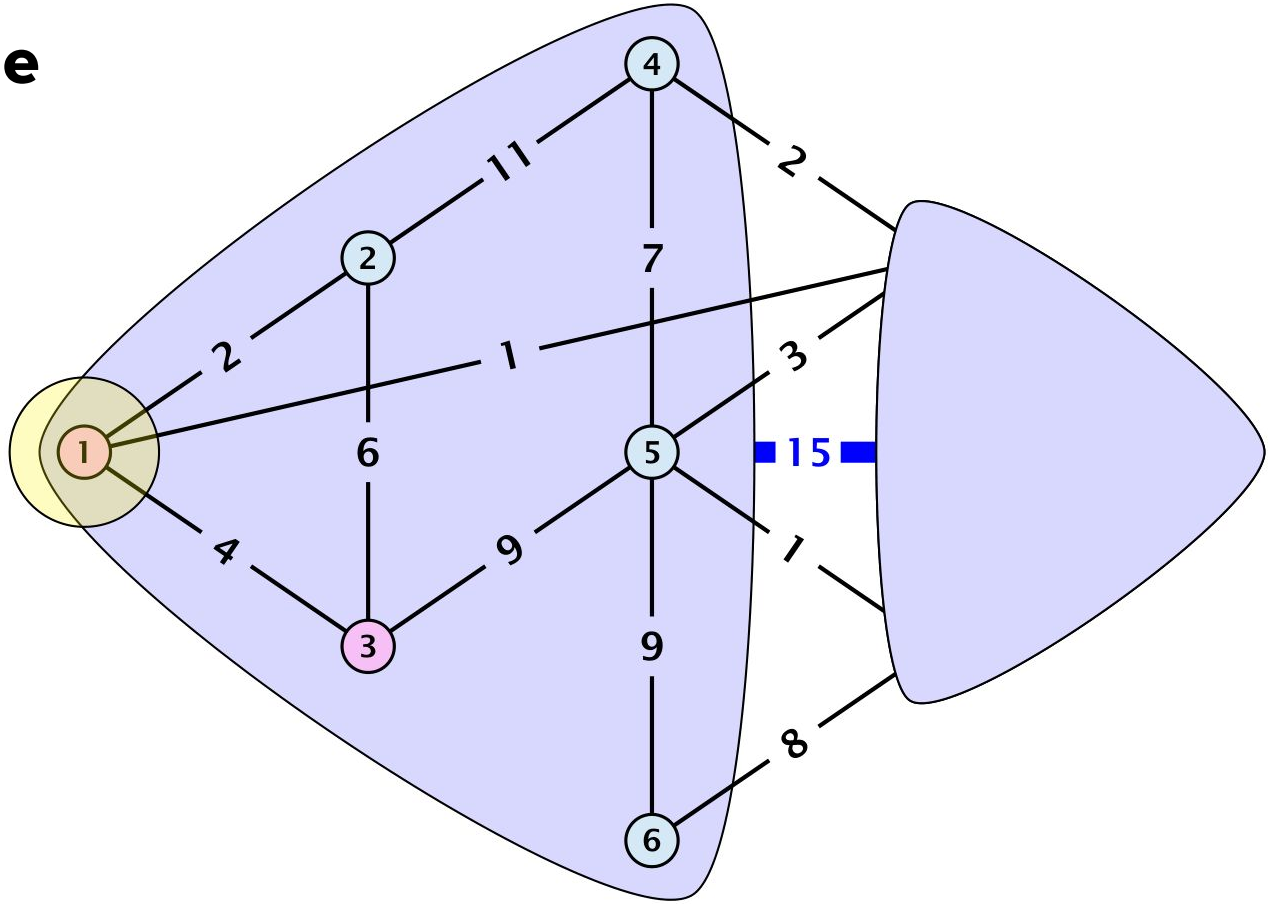

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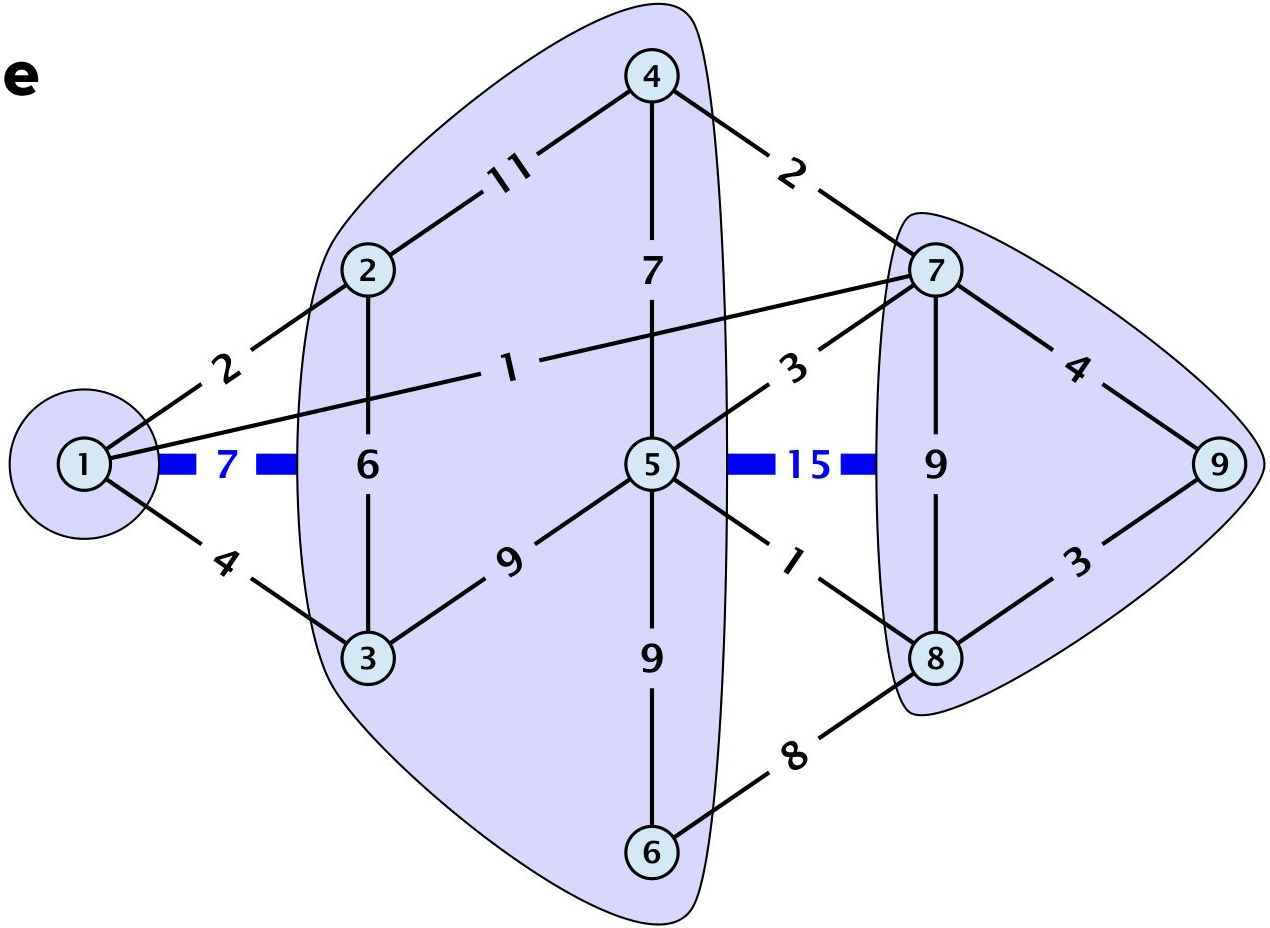

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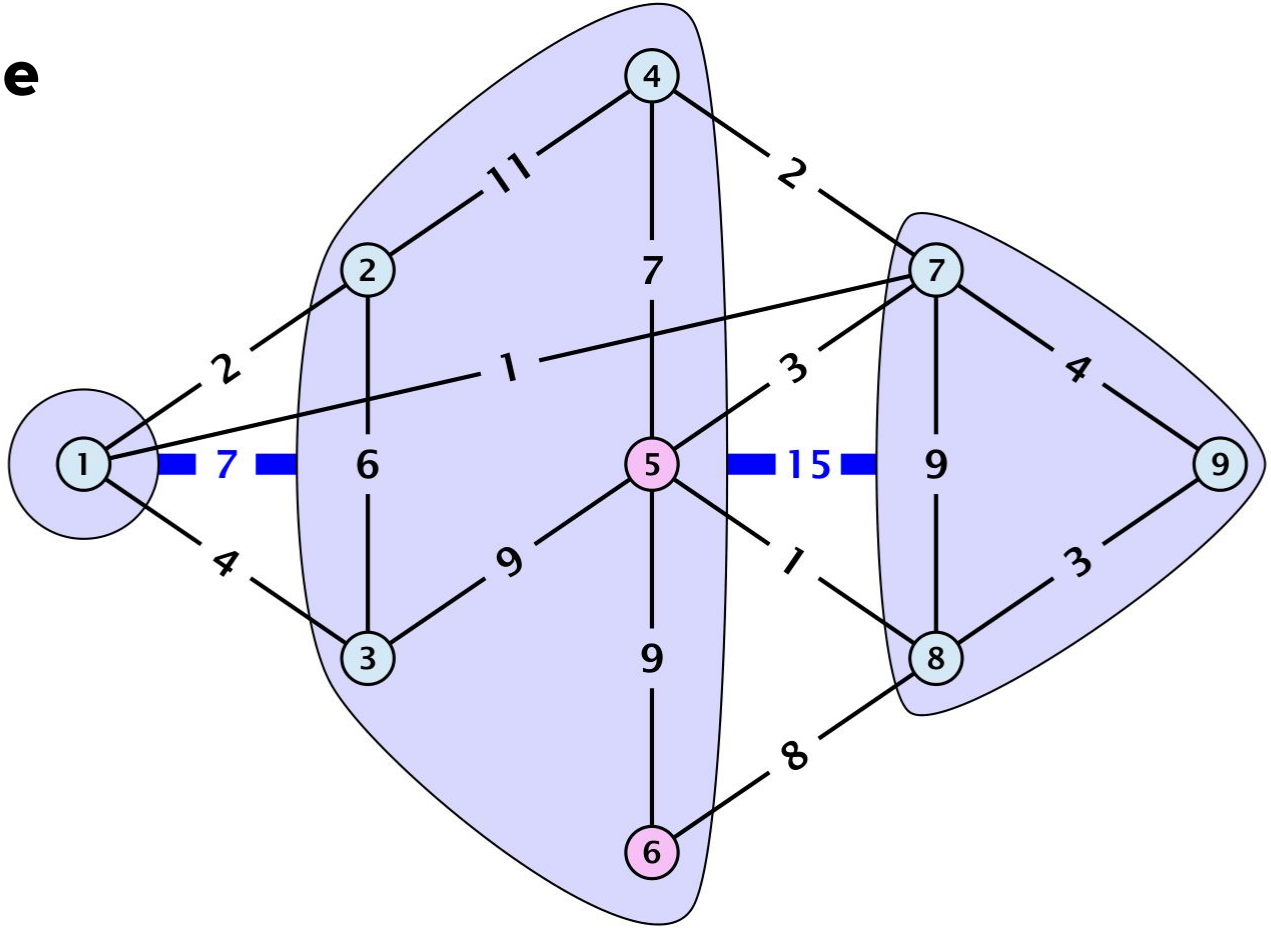

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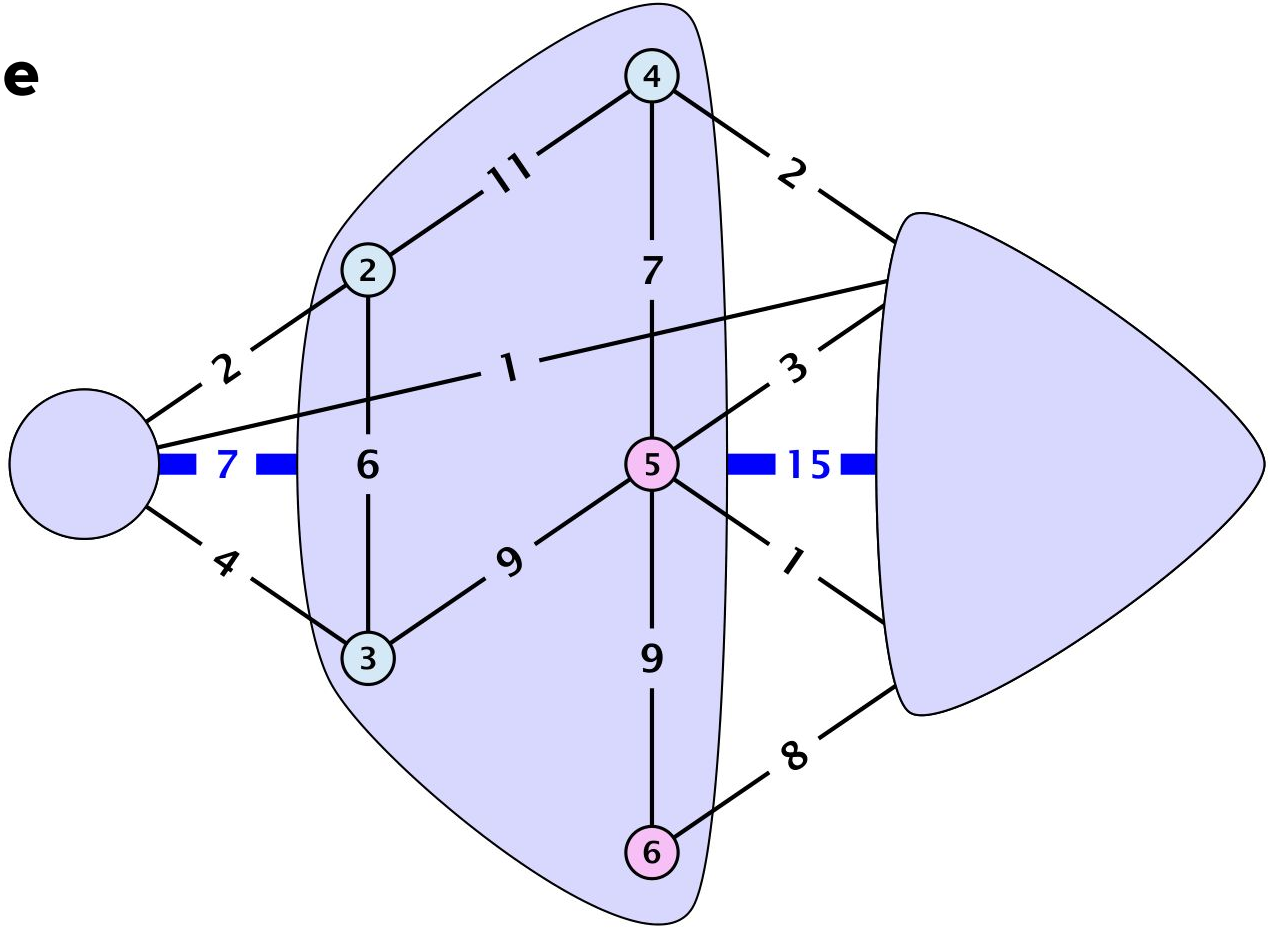

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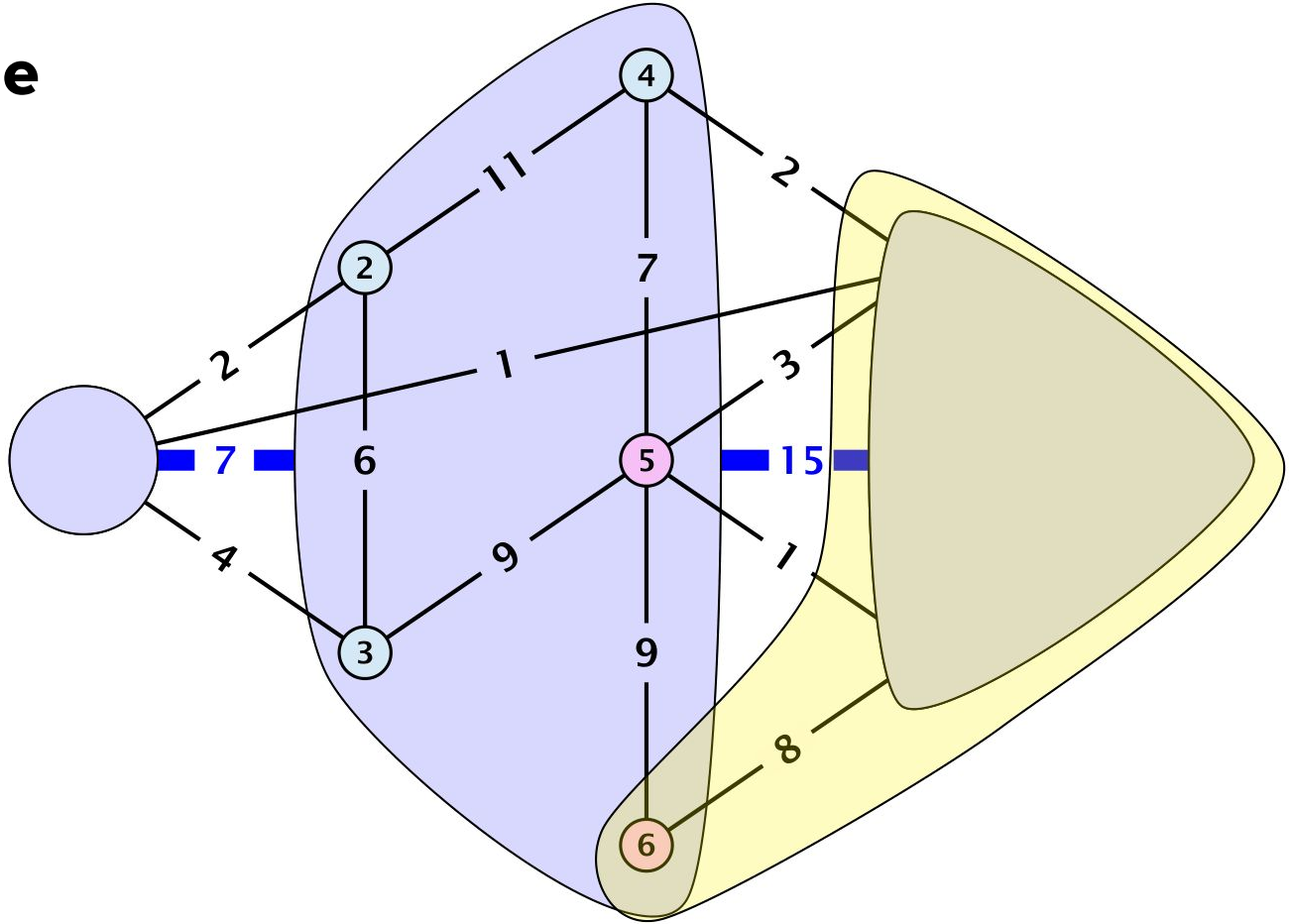

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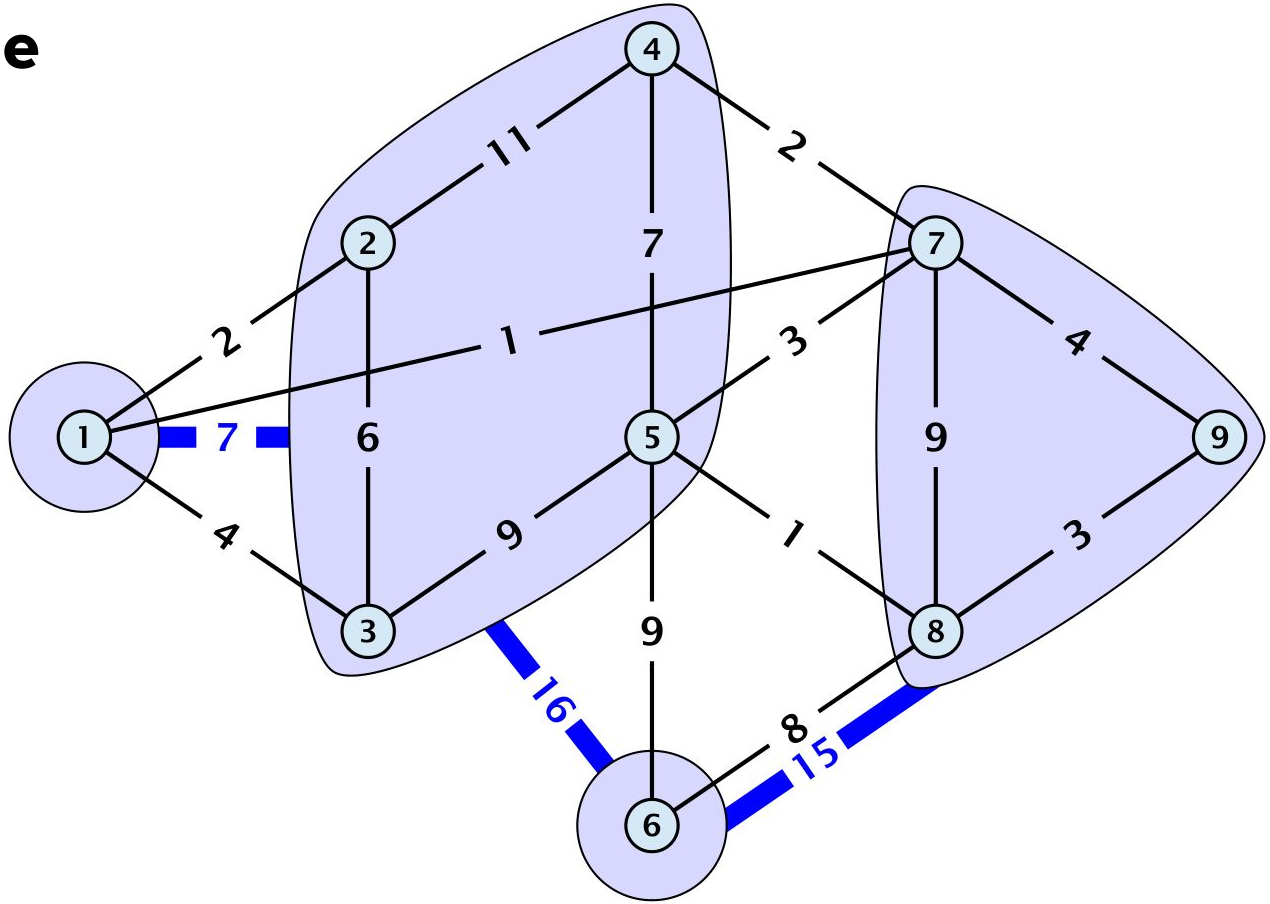

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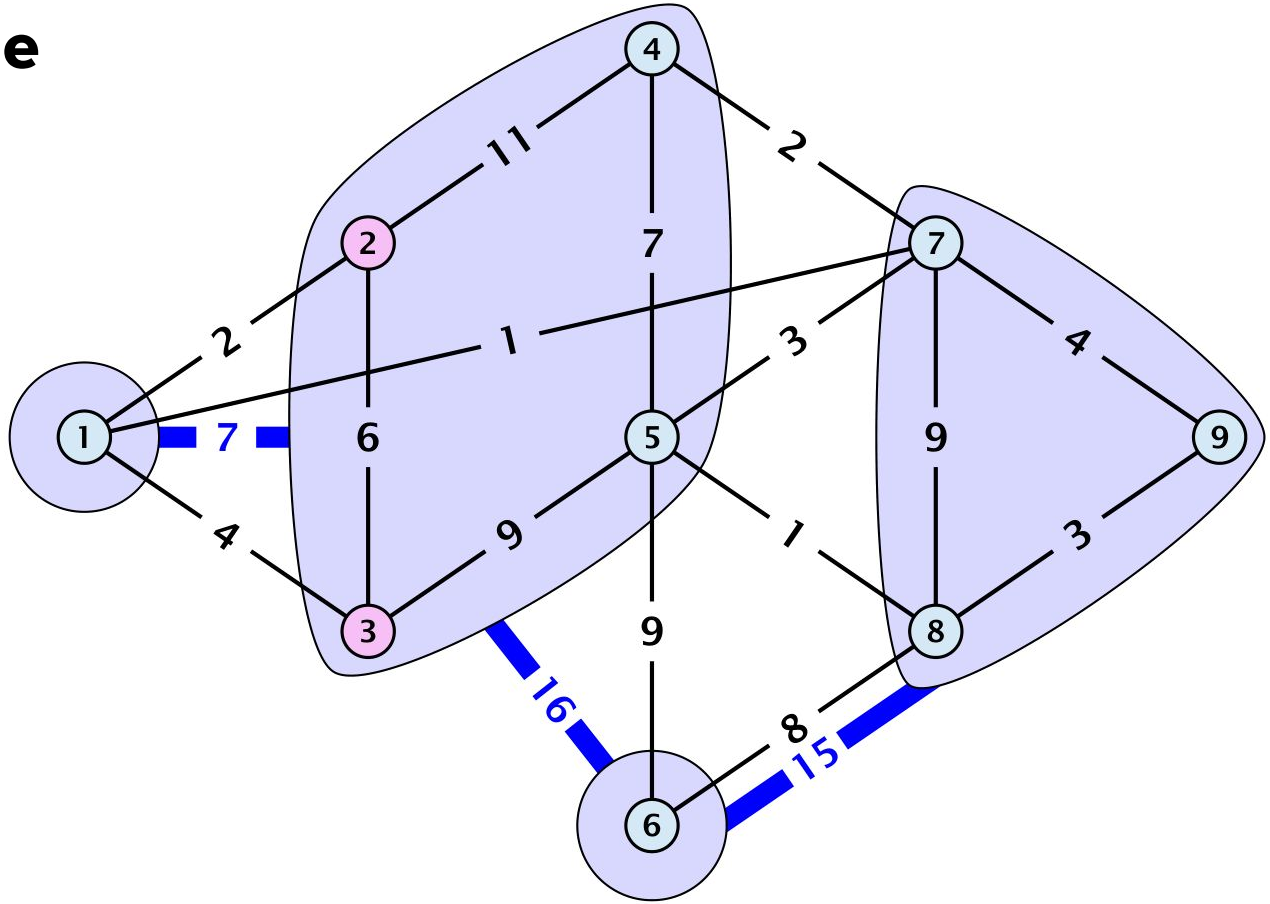

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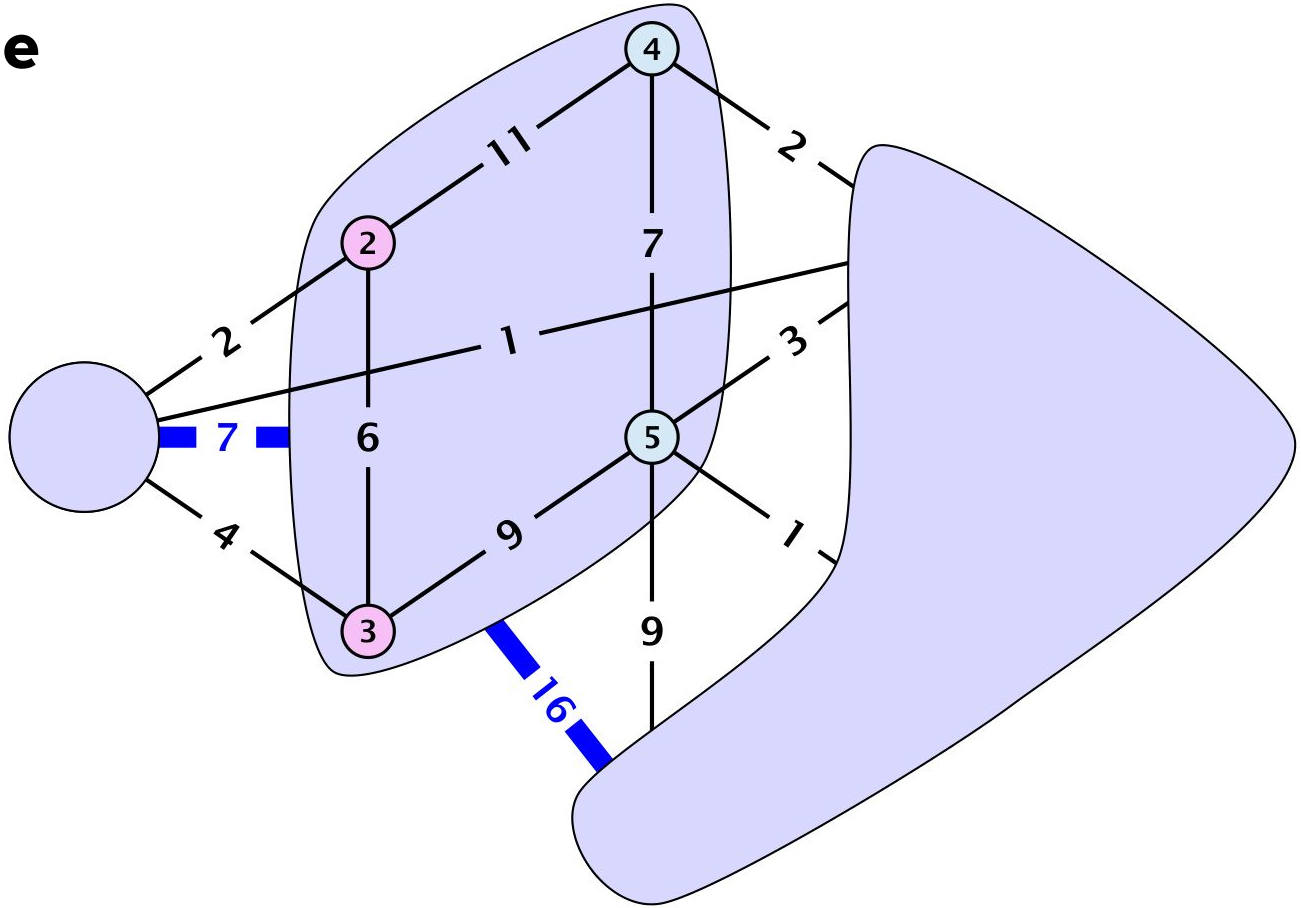

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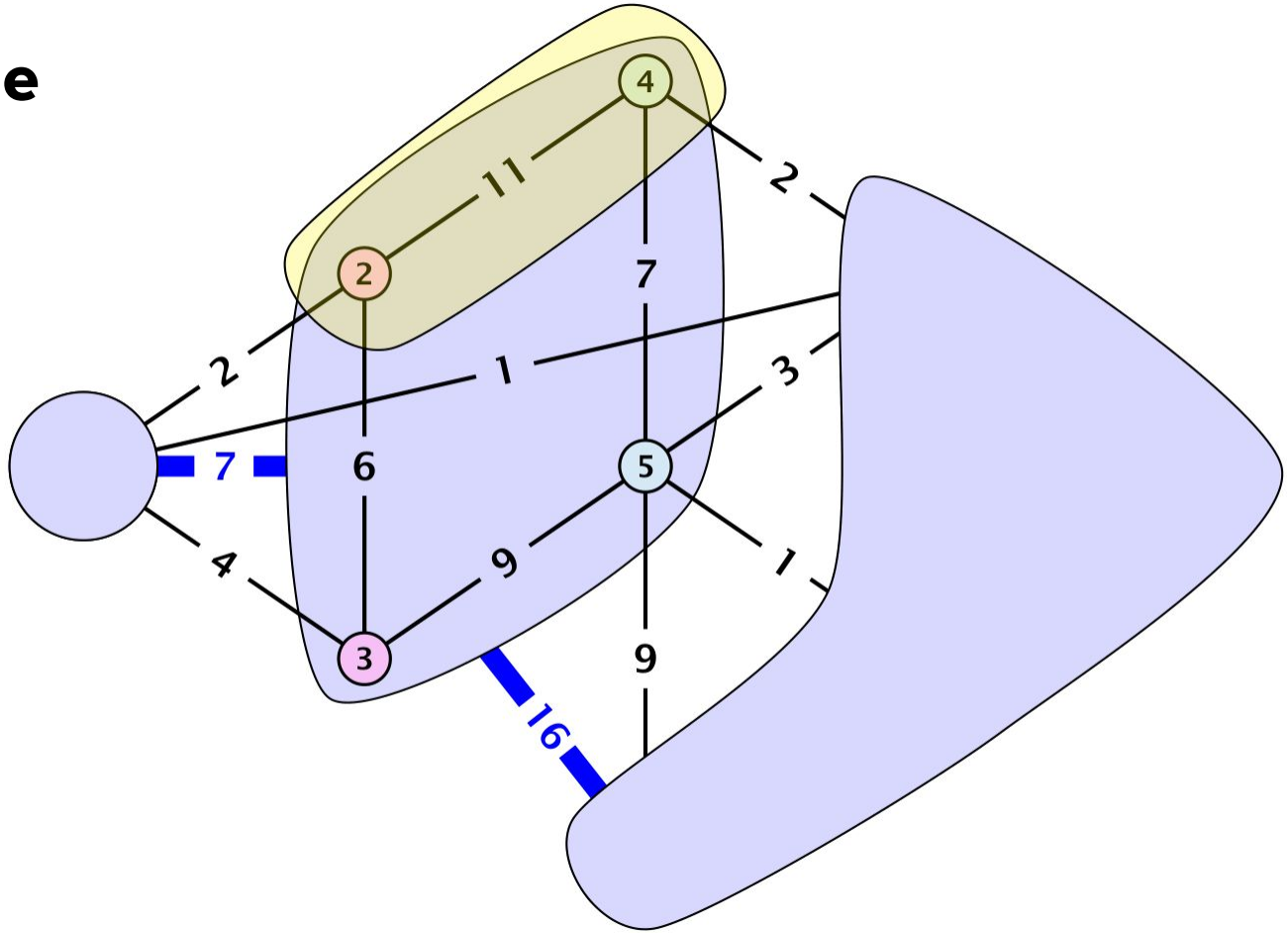

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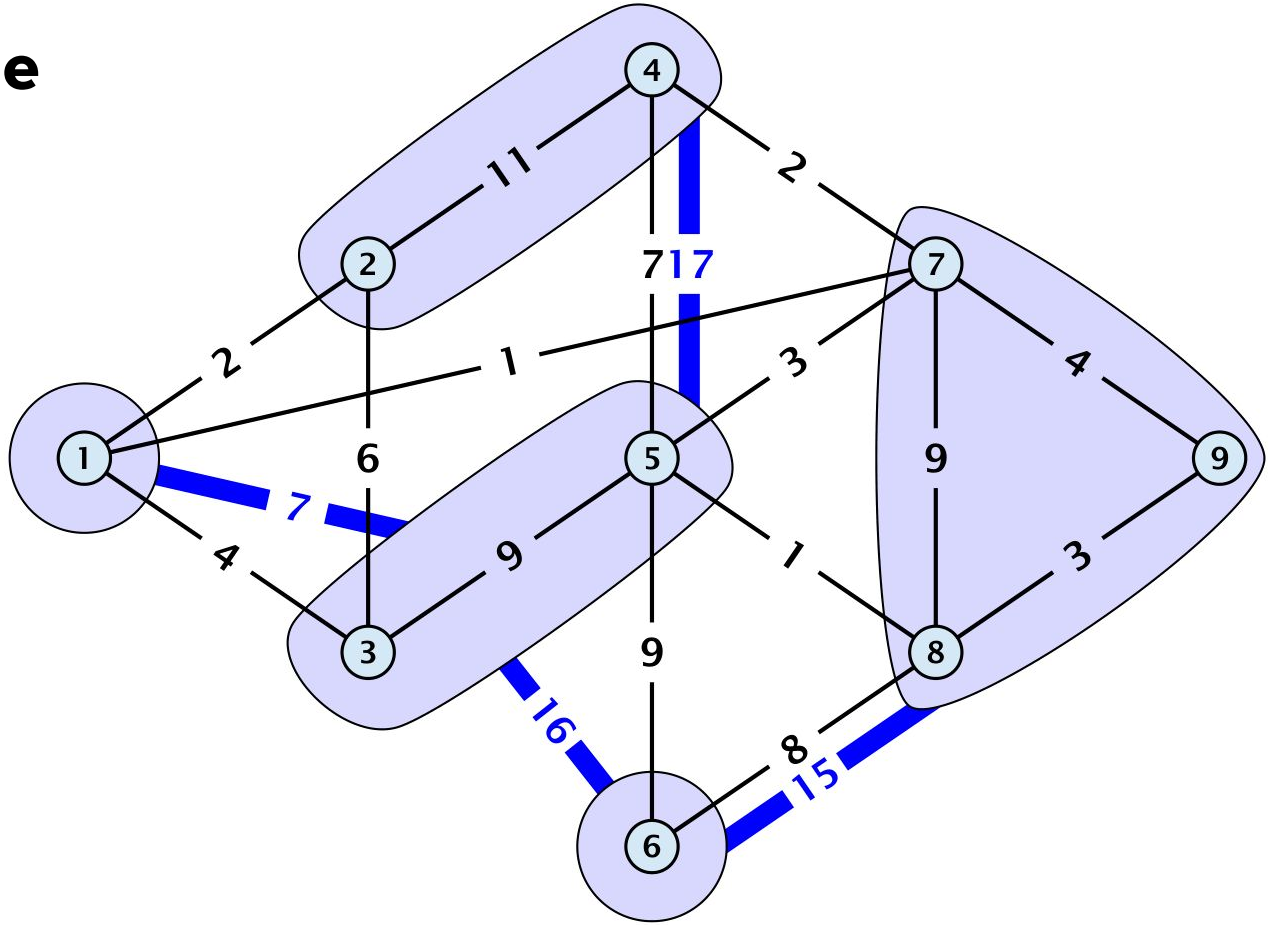

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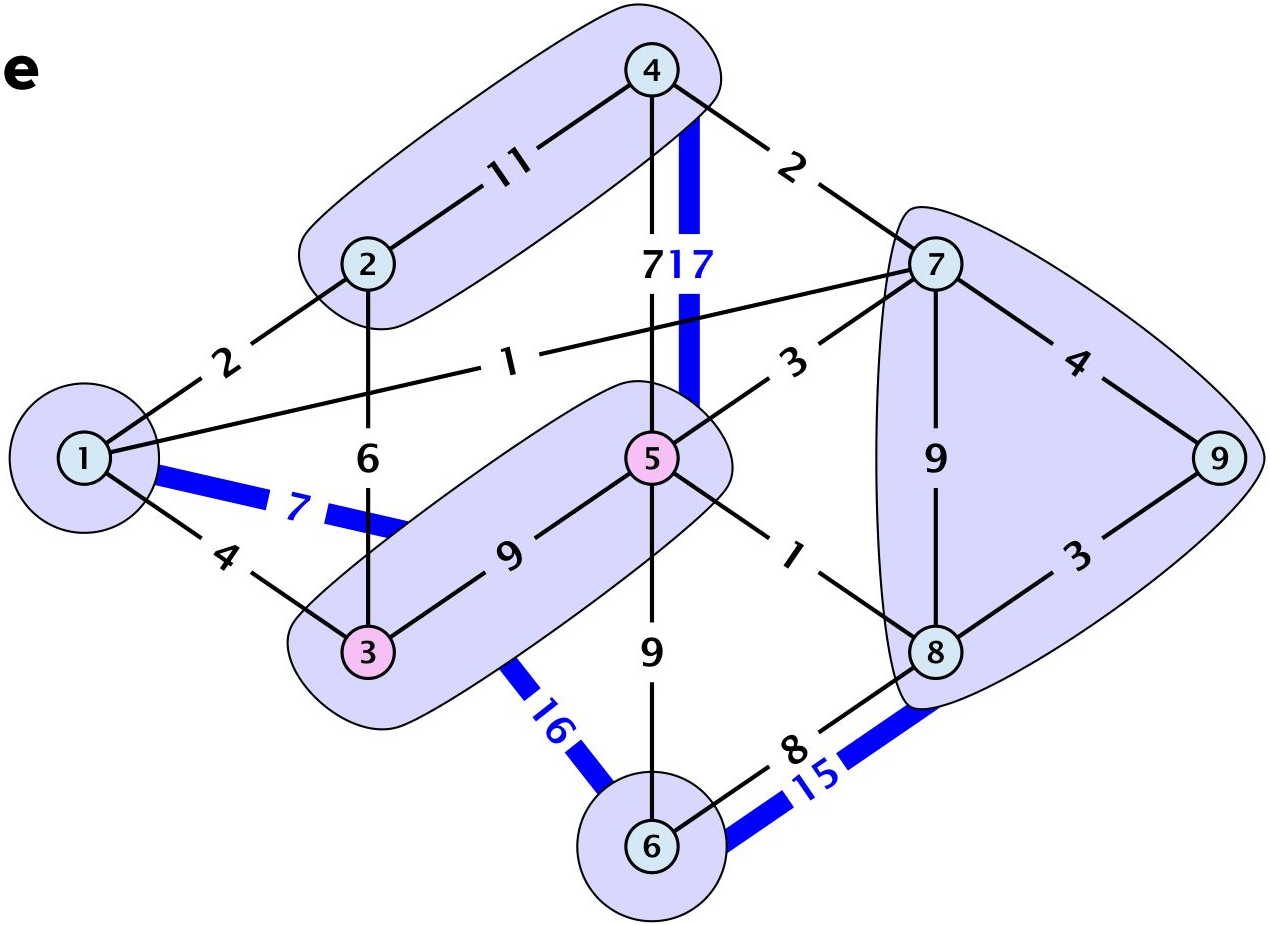

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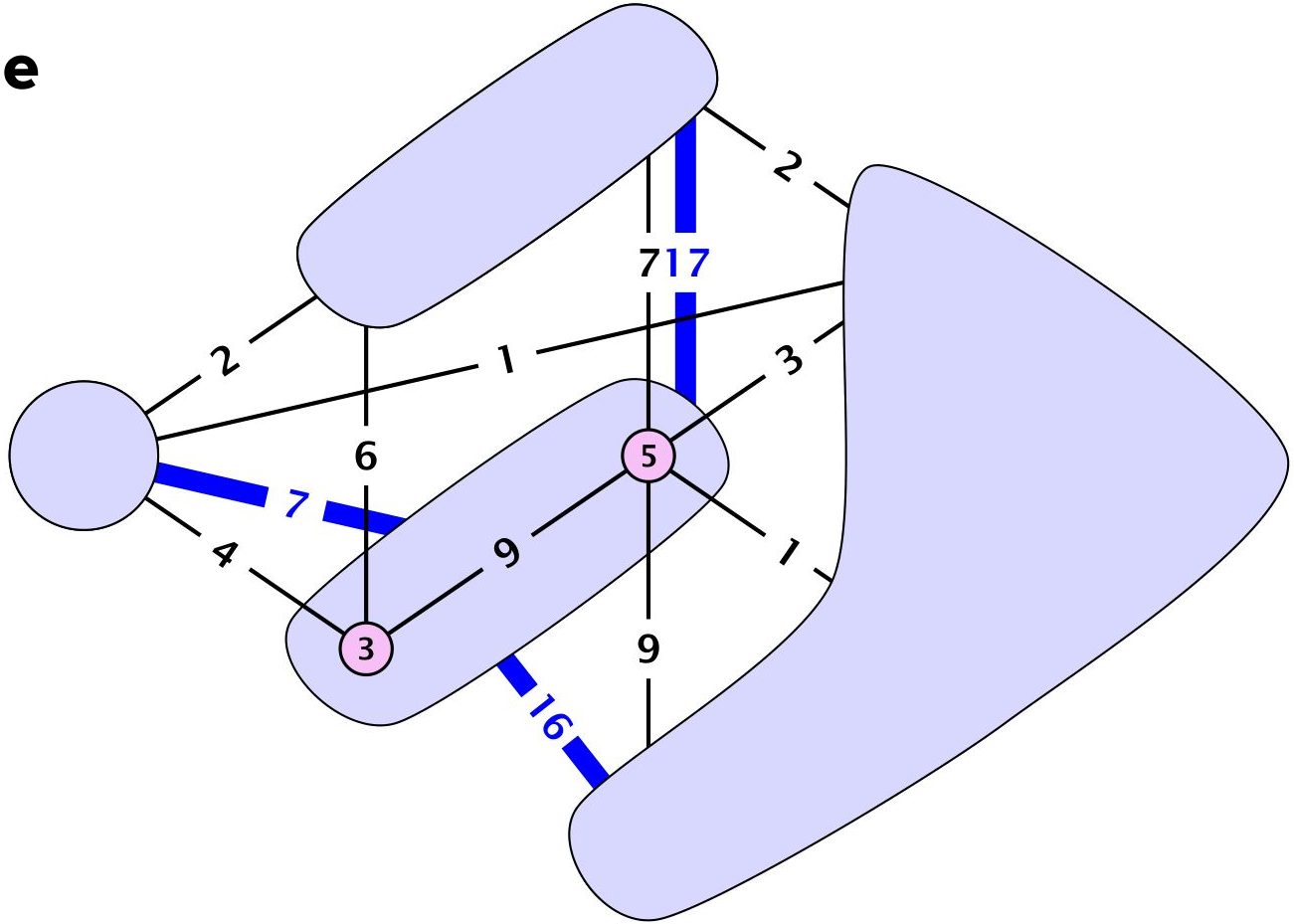

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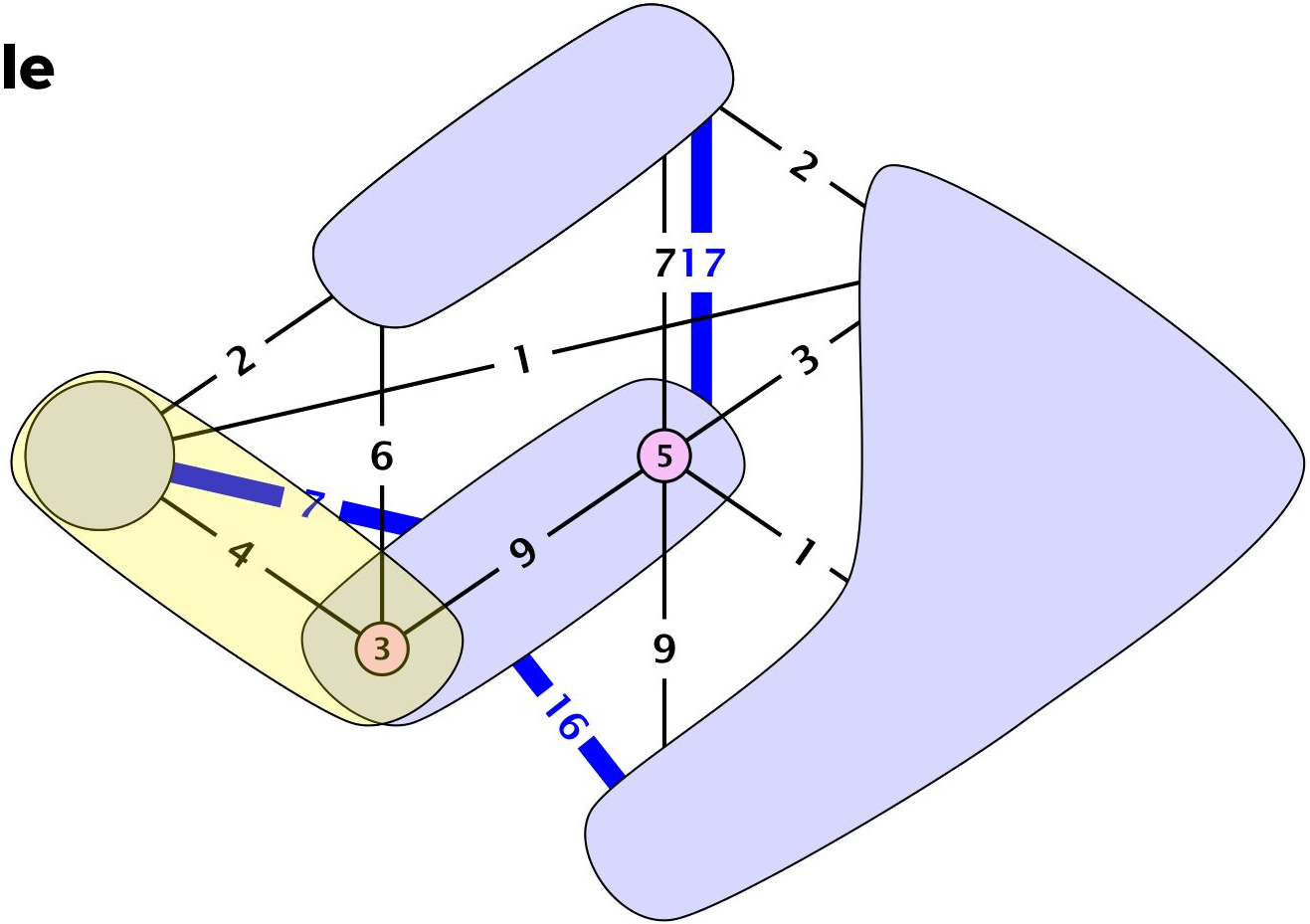

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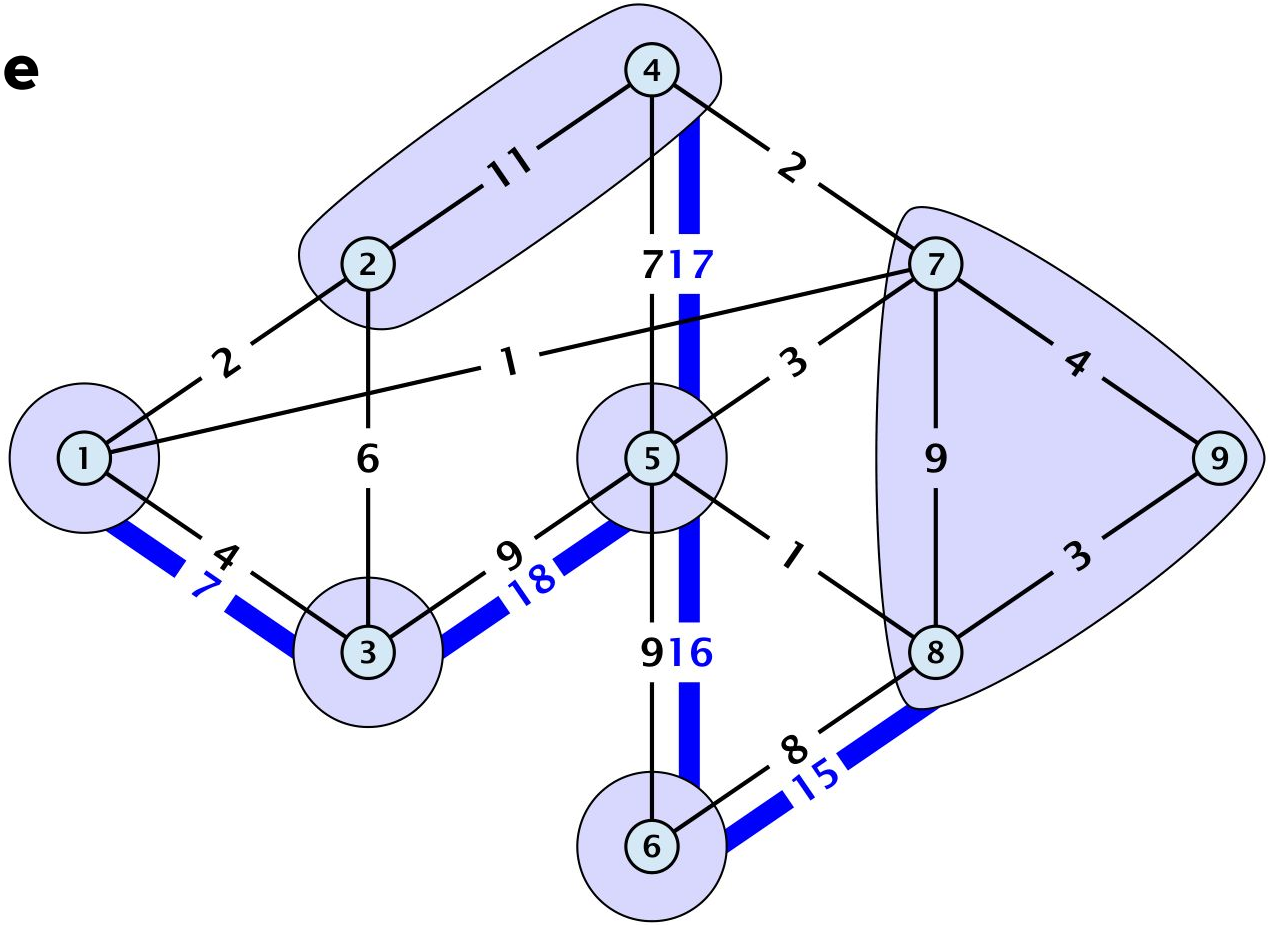

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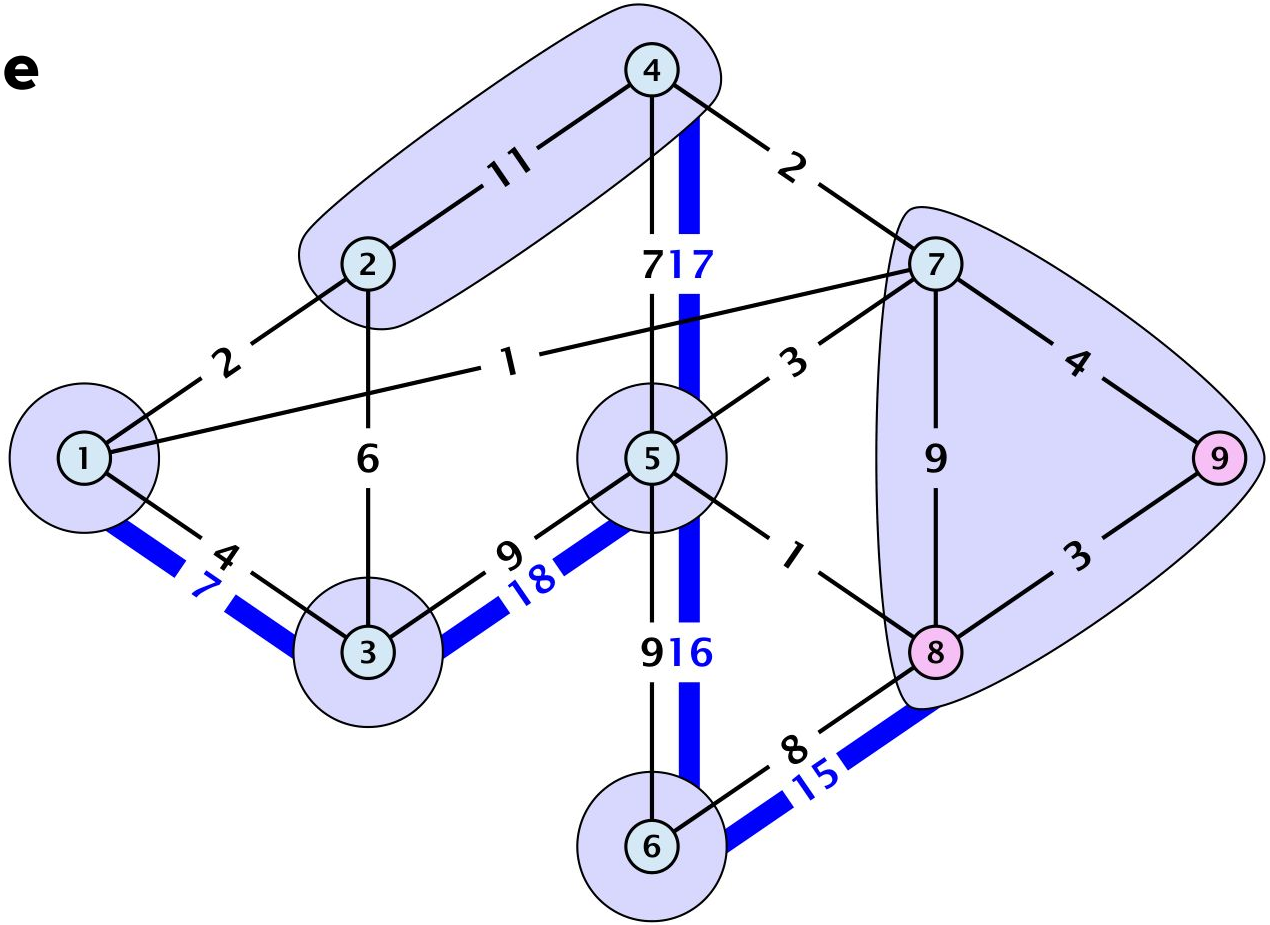

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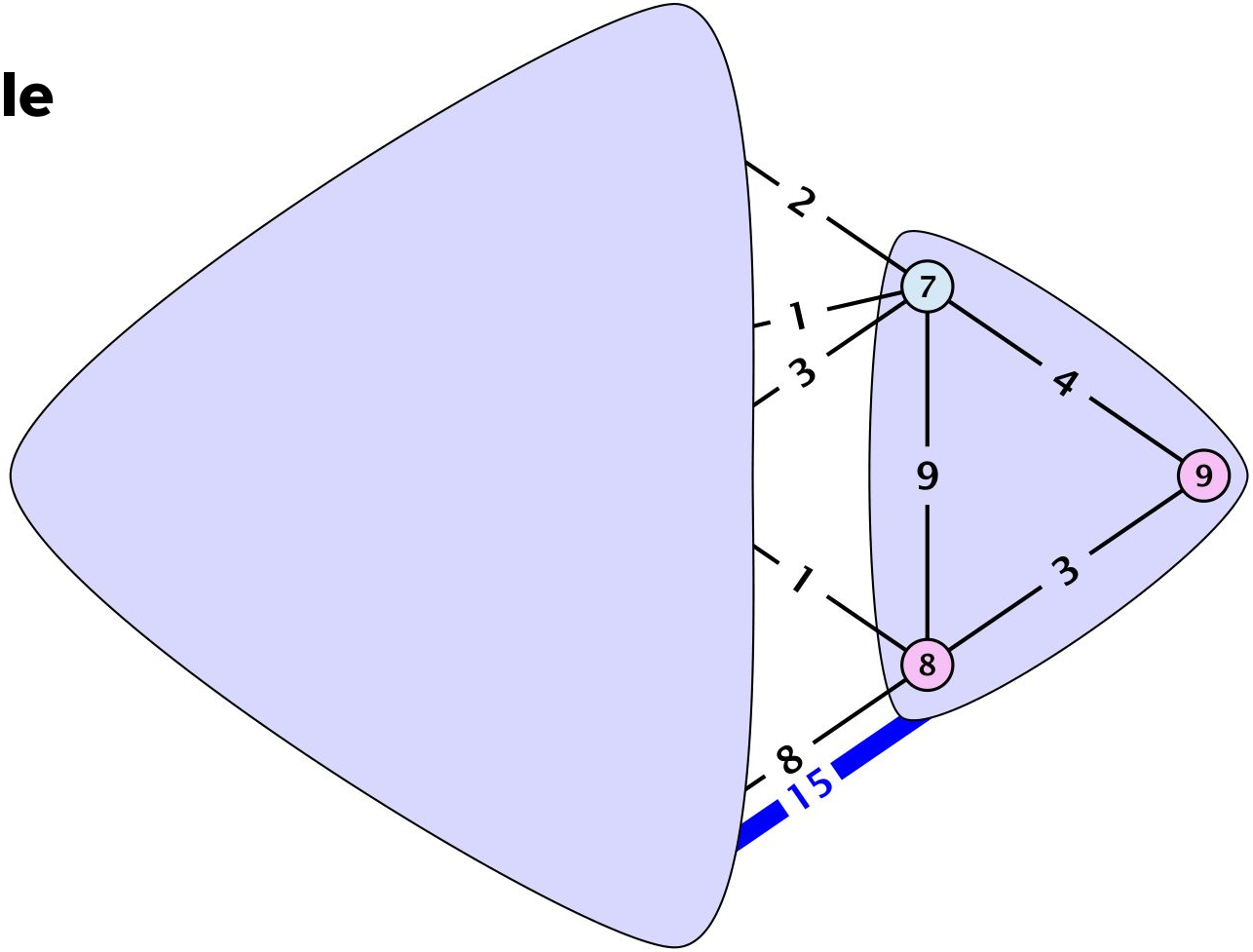

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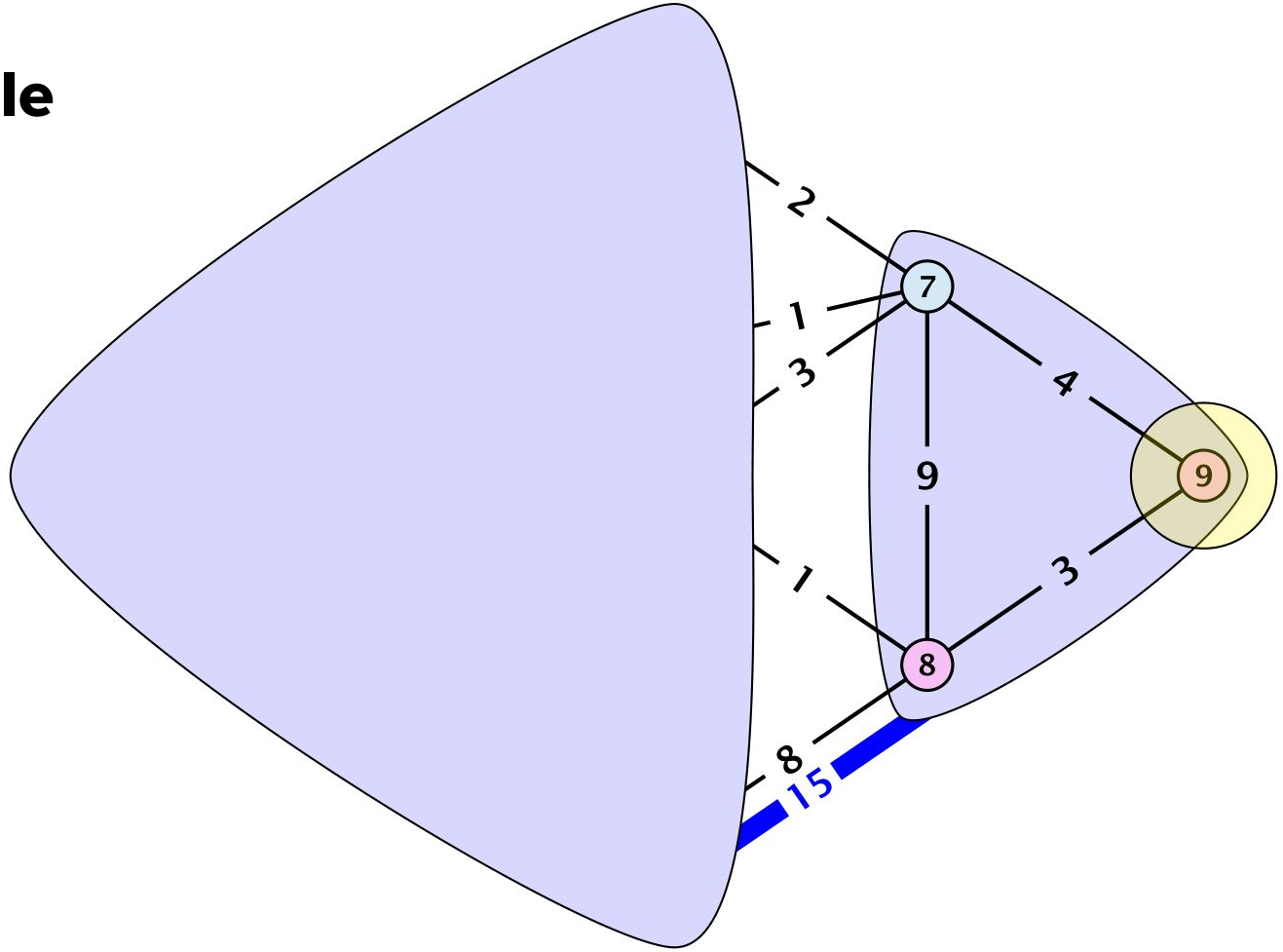

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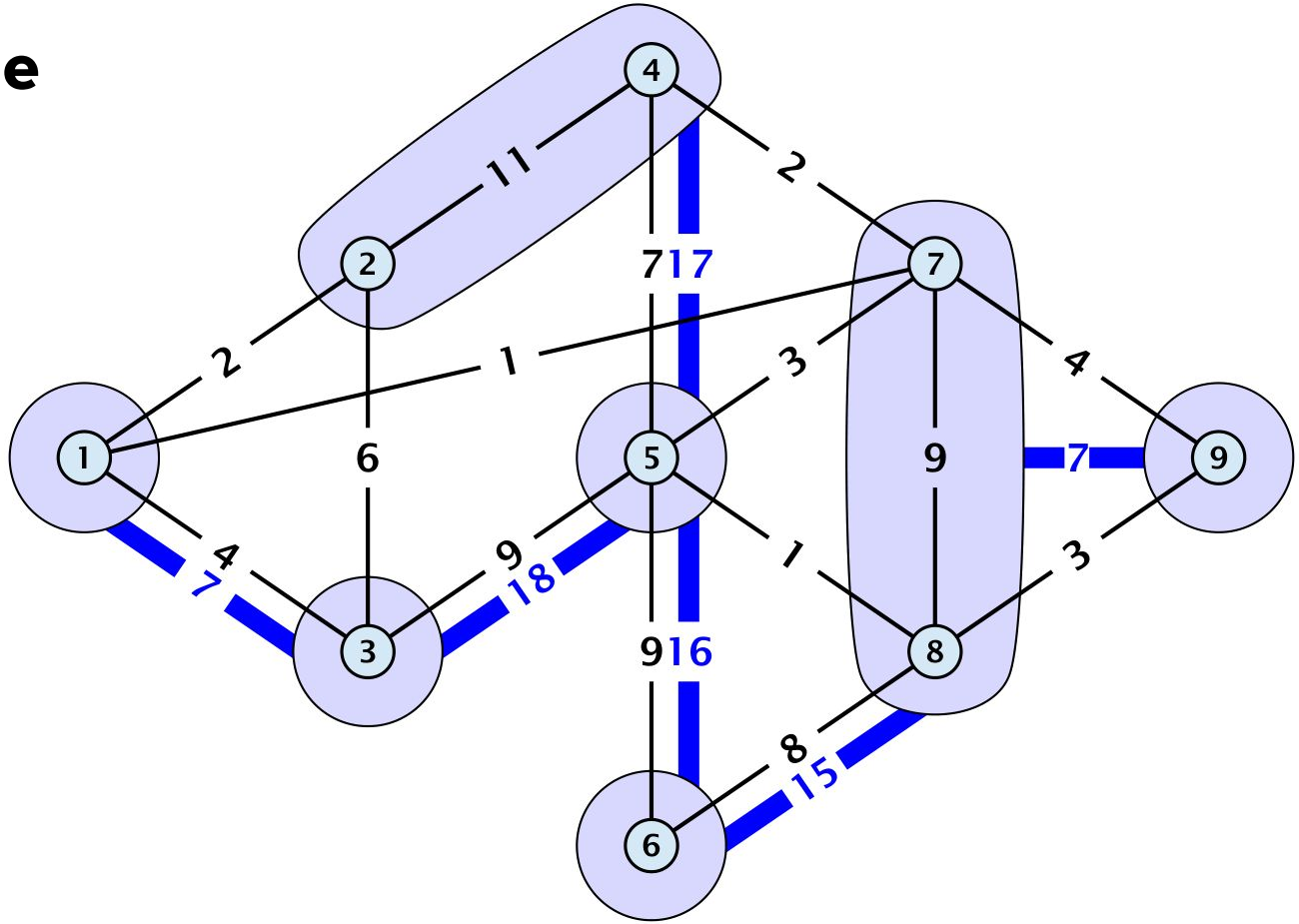

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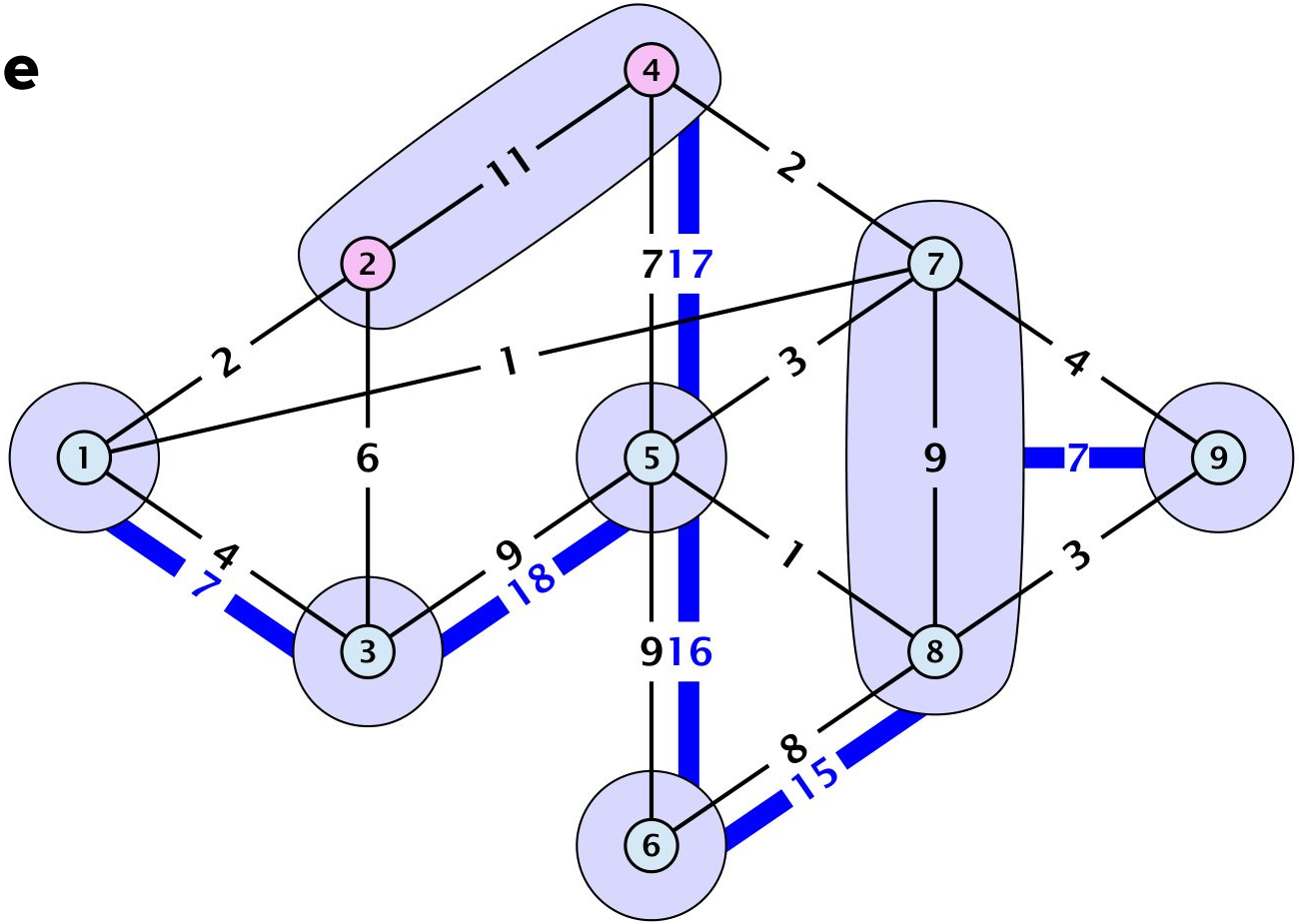

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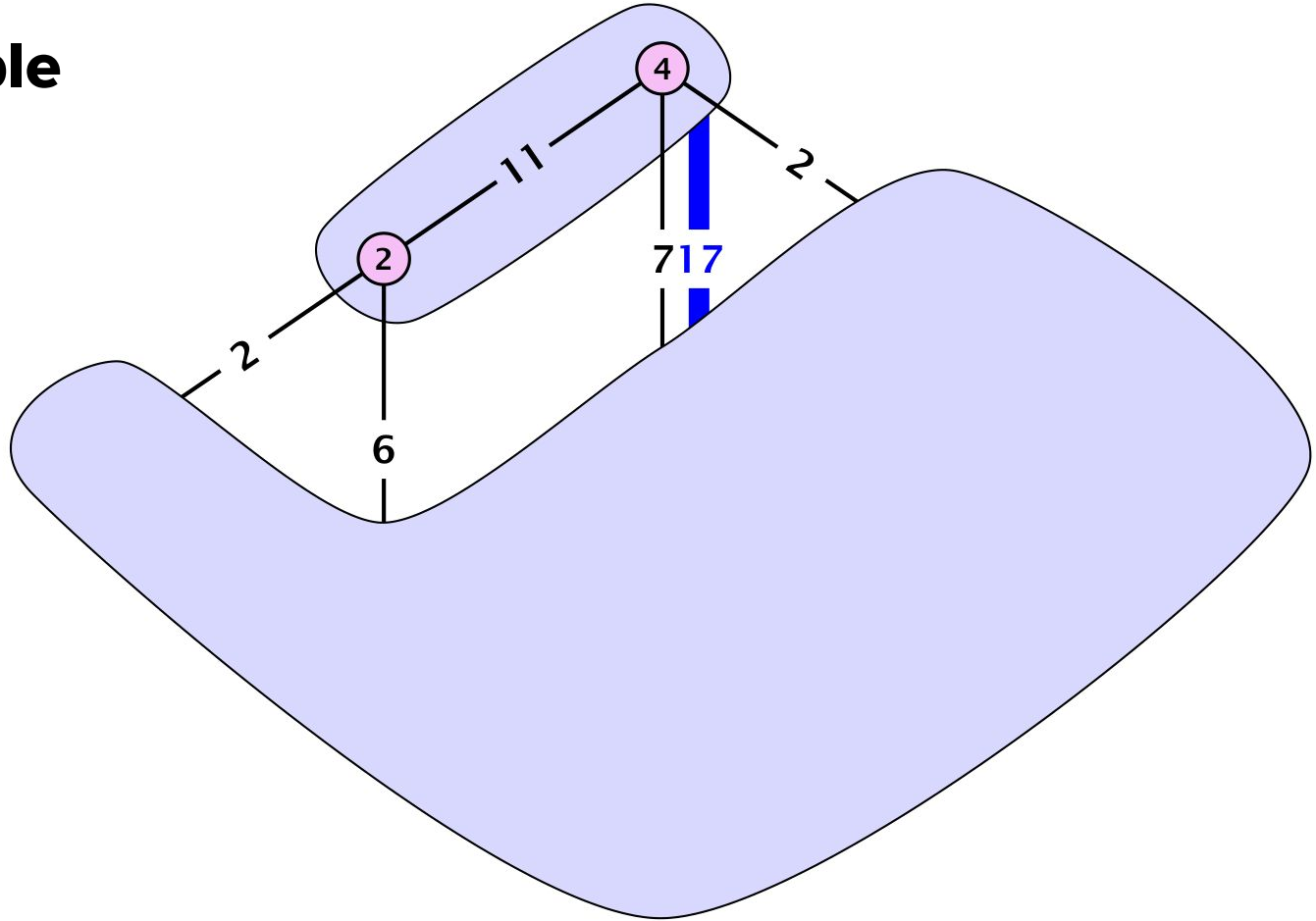

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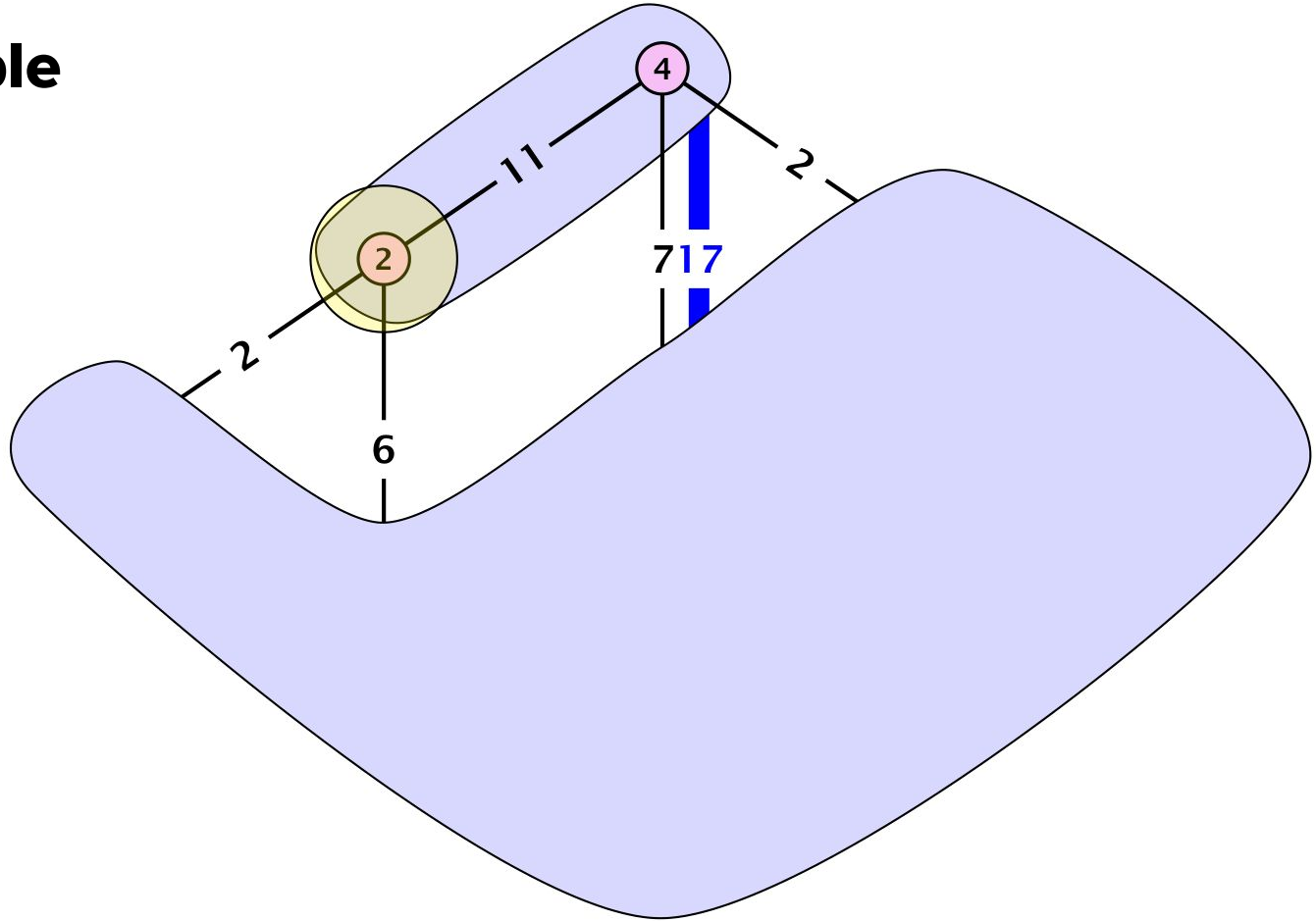

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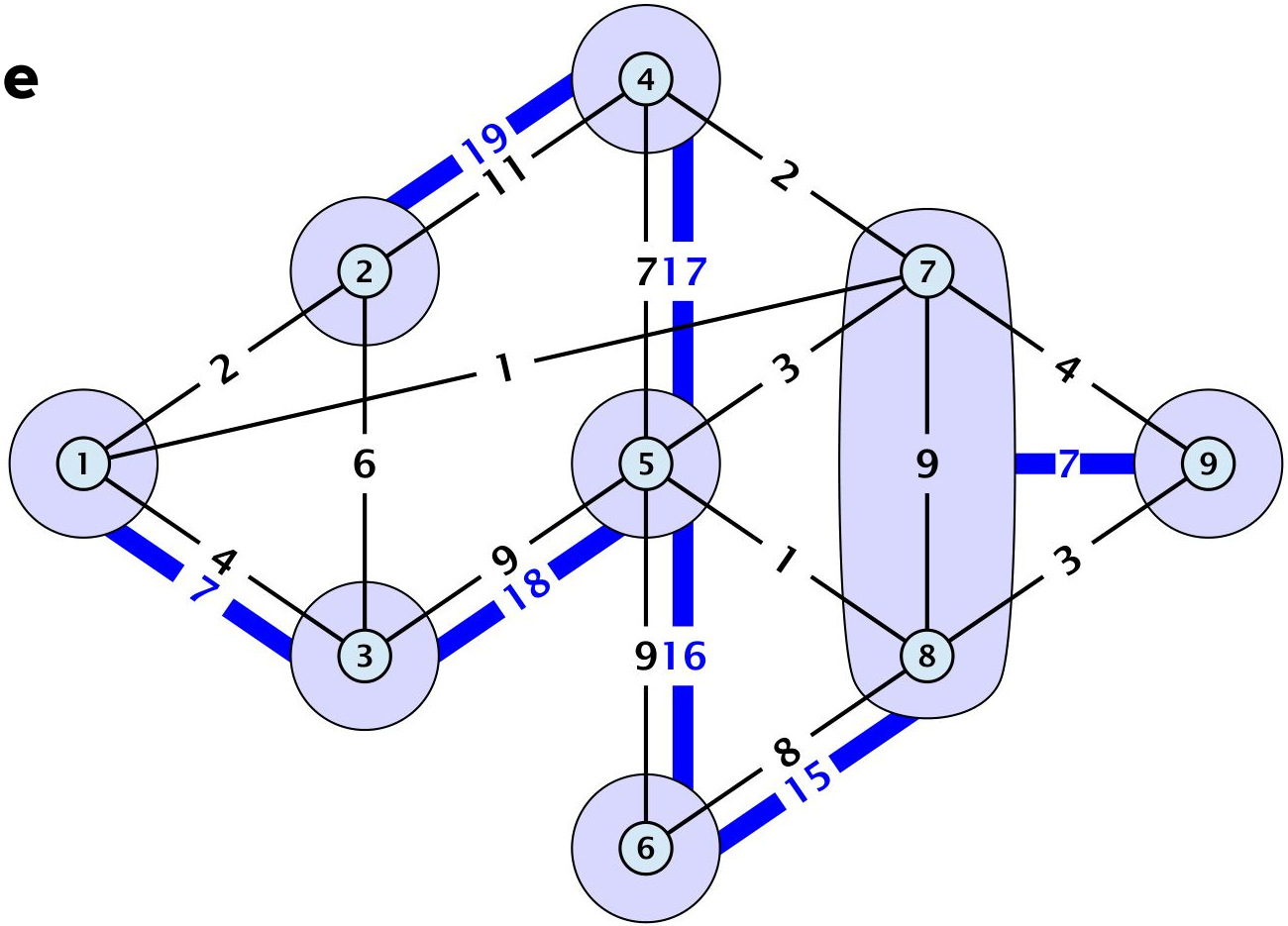

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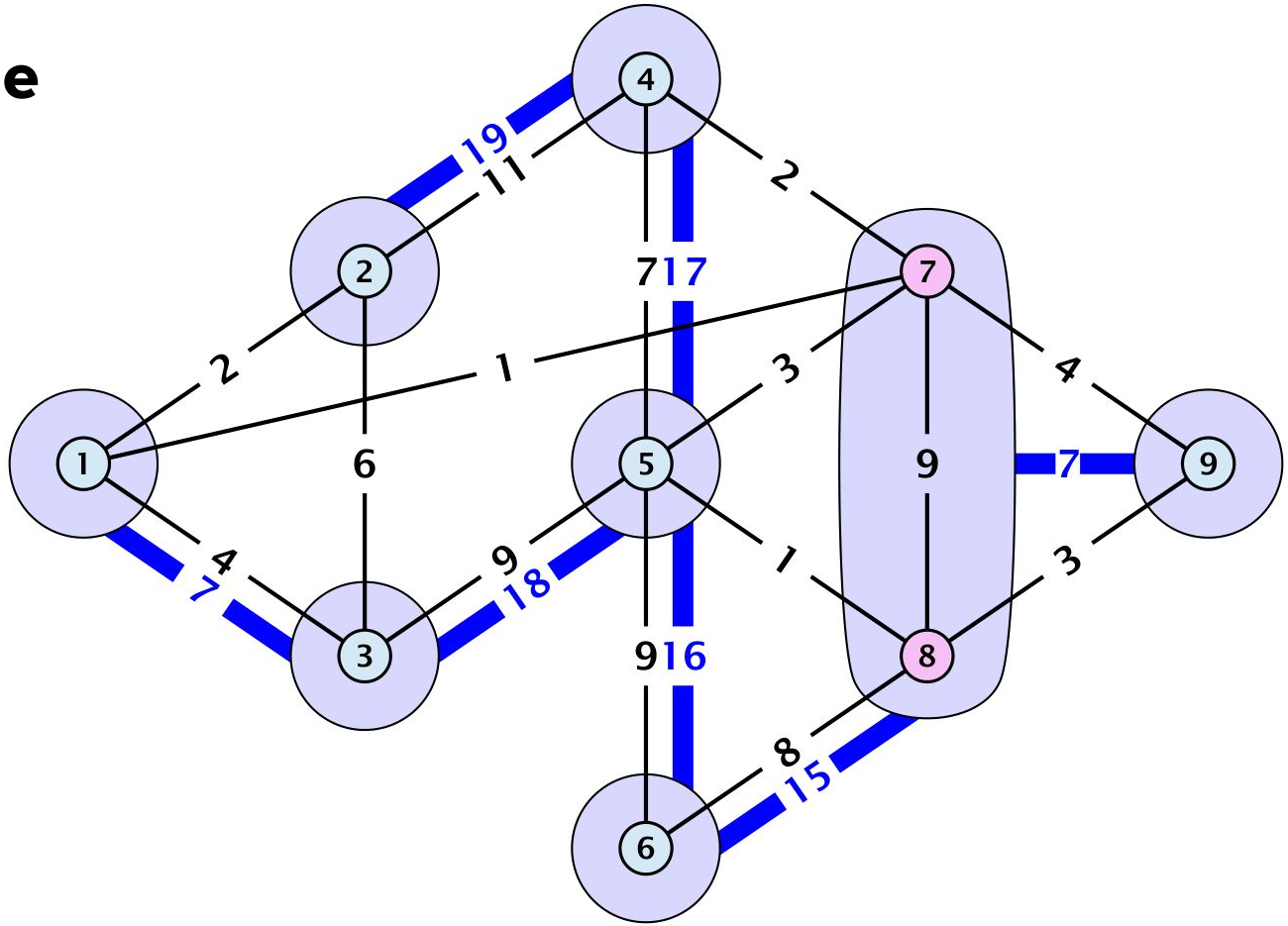

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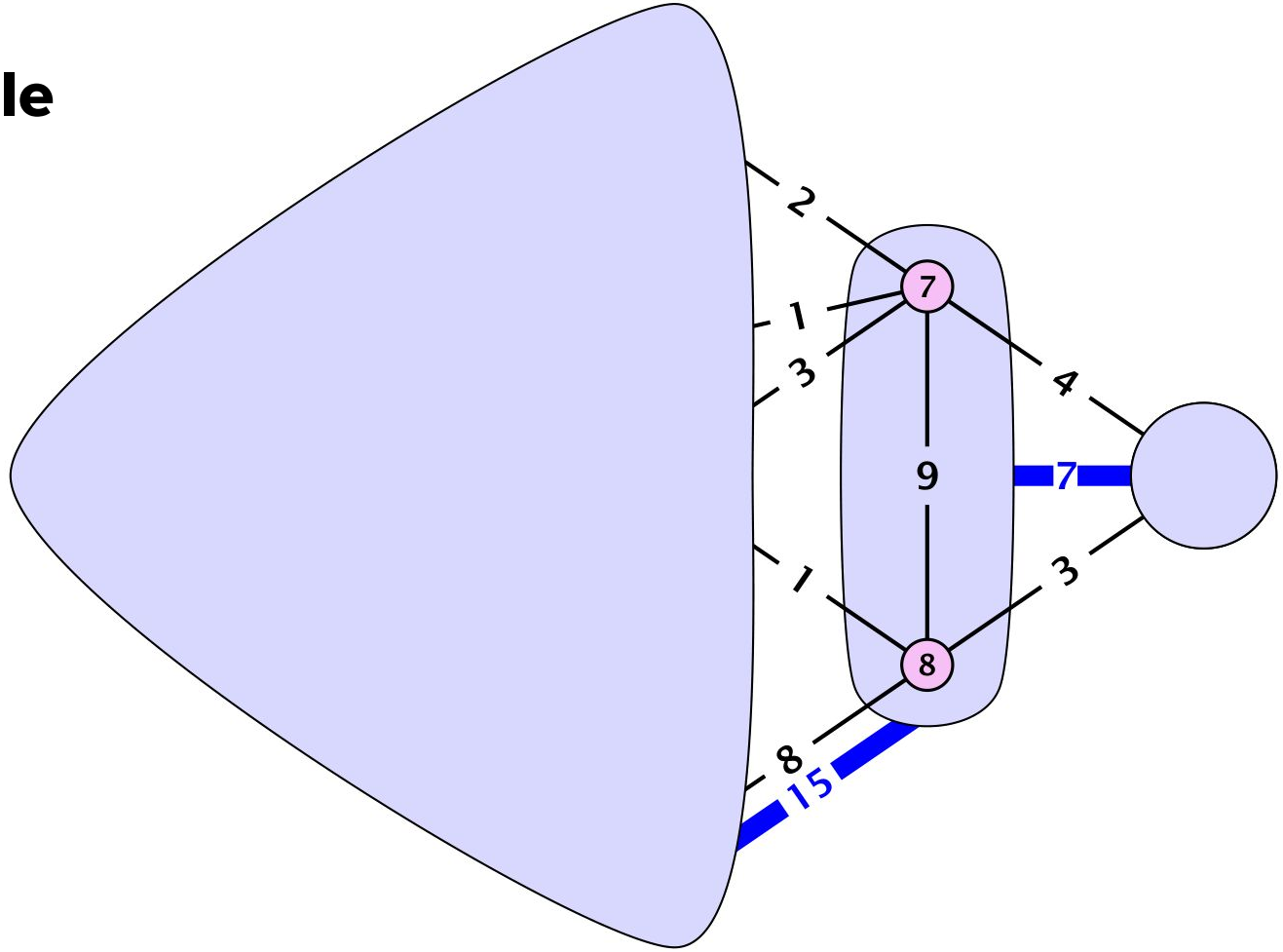

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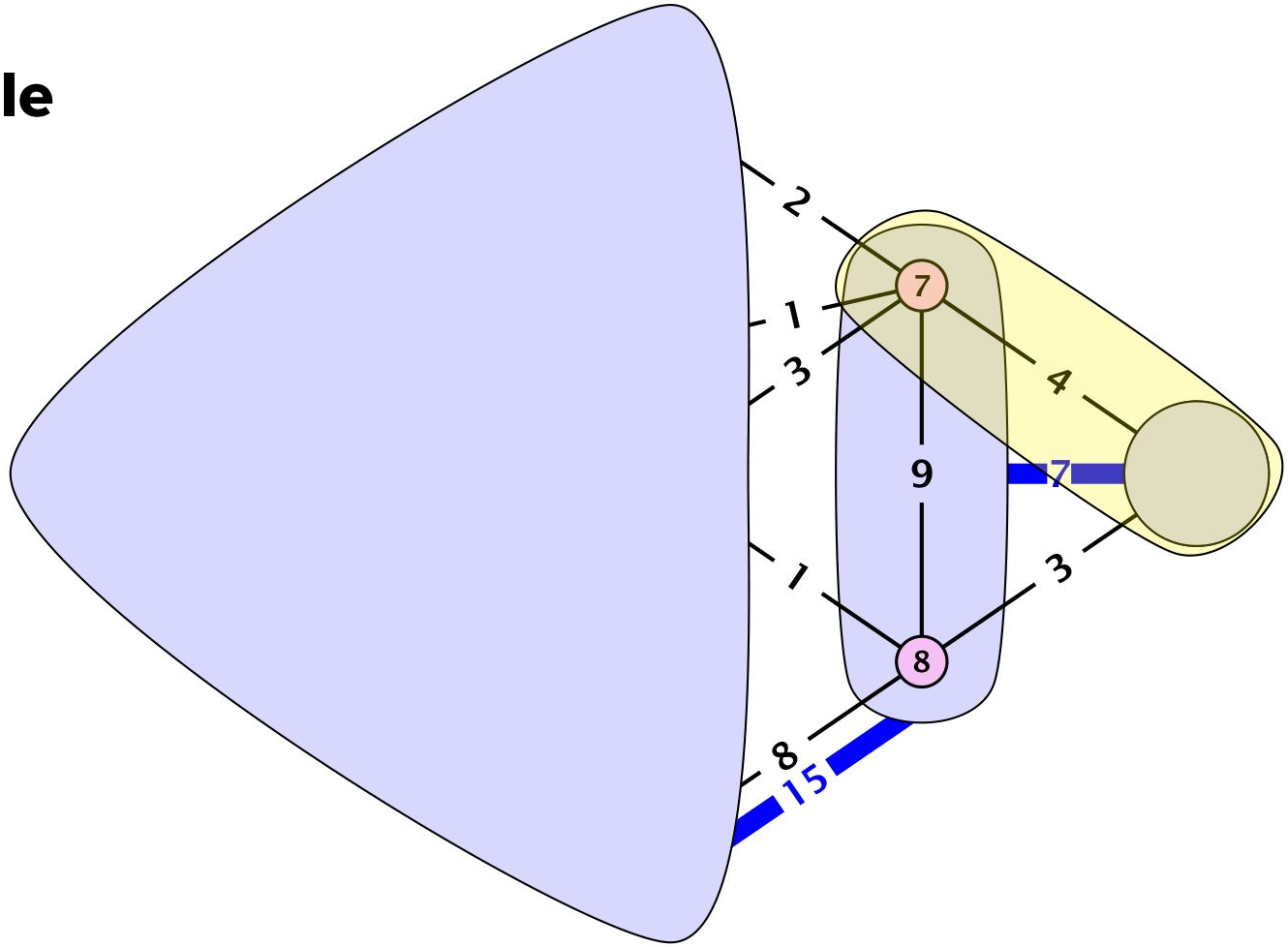

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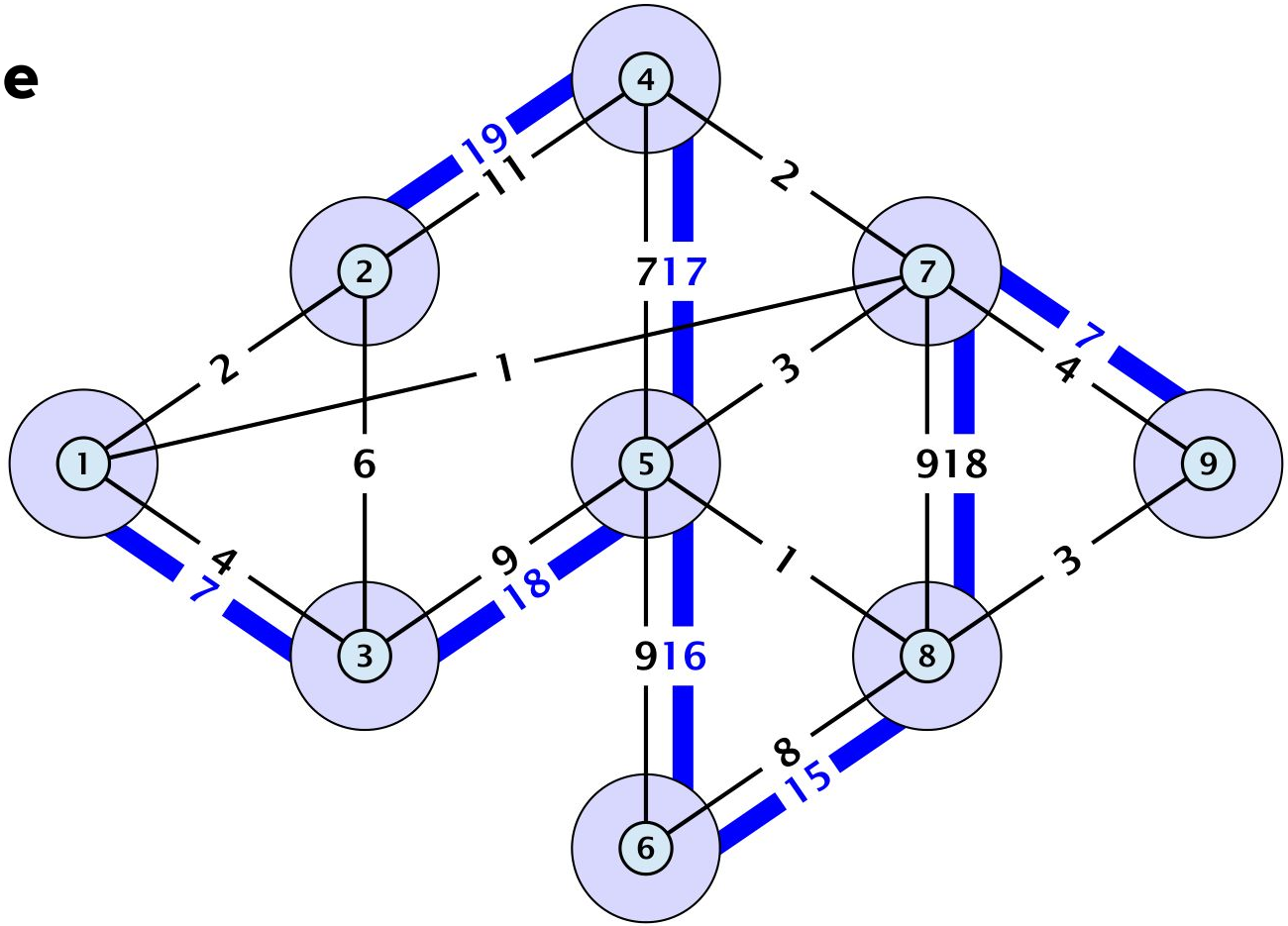


Lemma 1 : For nodes s, t, x $\in$ V we have $f(s, t) \geq \min\{f(s, x), f(x, t)\}$.

Proof :- Suppose this was false and for some nodes s, t, x $\in$ V,

$$f(s, t) < \min\{f(s, x), f(x, t)\}$$

Consider the minimum s-t cut of the graph into node sets S and T. The capacity of the cut is equal to f(s, t). Now,

If x $\in$ S, this is also a x-t cut with capacity f = f(s, t) < f(x, t), which is a contradiction,

Or else x $\in$ T, this is a s-x cut with capacity f = f(s, t) < f(s, x), also a contradiction.

$\therefore$ $f(s, t) \geq \min\{f(s, x), f(x, t)\}$ for all nodes s, t, x $\in$ V.

Lemma 2 : For nodes s, t, $x_1, \ldots, x_k \in V$ we have, $f(s, t) \geq \min\{f(s, x_1), f(x_1, x_2), \ldots, f(x_{k-1}, x_k), f(x_k, t)\}$.

Proof :- Lemma 2, by induction, is an immediate consequence of Lemma 1.

P(m): For nodes s, t, $x_1, \ldots, x_m \in V$ we have, $f(s, t) \geq \min\{f(s, x_1), f(x_1, x_2), \ldots, f(x_{m-1}, x_m), f(x_m, t)\}$

Base case: P(m) is true for m = 1 (lemma 1)

Inductive Step: Let P(m) be true for m = k, then for m = k + 1, we have,

$\min\{f(s, x_1), f(x_1, x_2), \ldots, f(x_k, x_{k+1}), f(x_{k+1}, t)\} = \min\{\min\{f(s, x_1), f(x_1, x_2), \ldots, f(x_k, x_{k+1})\}, f(x_{k+1}, t)\} \leq \min\{f(s, x_{k+1}), f(x_{k+1}, t)\} \leq f(s, t)$

For some set of vertices U, we define δ(U) to be the set of edges with one endpoint in U. Then, for a minimum s-t cut that partitions the vertex set V into S and T, $f(s, t) = c(\delta(S)) = c(\delta(T))$ where c is the *cut function* on the set of vertices.

Definition 1 : Given a finite set E, $f : 2^E \to R$ is *submodular* if for all $A, B \in 2^E$, $f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$.

Definition 2 : Given a finite set E, $f : 2^E \to R$ is *posi-modular* if for all $A, B \in 2^E$, $f(A) + f(B) \geq f(A - B) + f(B - A)$.

Cut function is submodular

Let’s show that the cut function on the vertices of a graph is submodular.

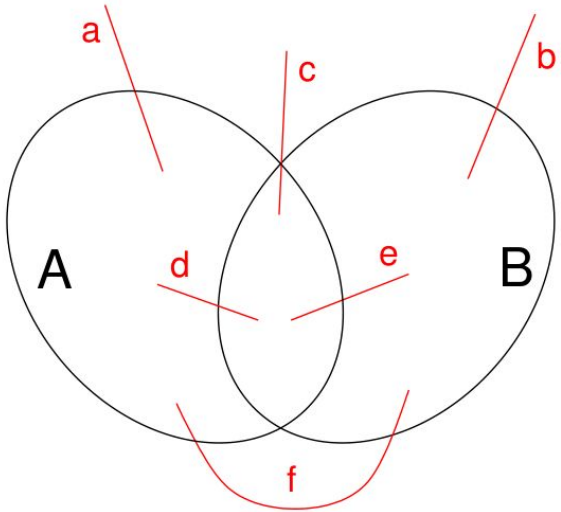
$c(\delta(A)) = a + c + e + f,\ c(\delta(B)) = b + c + d + f$

$c(\delta(A \cup B)) = a + b + c,\ c(\delta(A \cap B)) = c + d + e$

$c(\delta(A)) + c(\delta(B)) = a + b + 2c + d + e + 2f$

$= c(\delta(A \cup B)) + c(\delta(A \cap B)) + 2f$

$\geq c(\delta(A \cup B)) + c(\delta(A \cap B))$



Cut function is posi-modular

Let’s show that the cut function on the vertices of a graph is submodular.

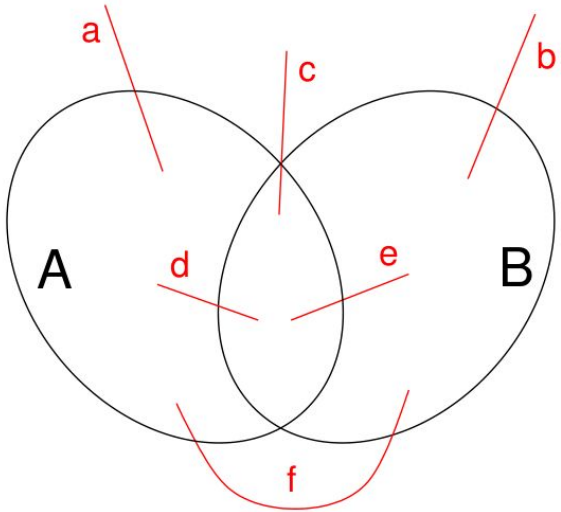
$c(\delta(A)) = a + c + e + f$, $c(\delta(B)) = b + c + d + f$

$c(\delta(A - B)) = a + d + f$, $c(\delta(B - A)) = b + e + f$

$$c(\delta(A)) + c(\delta(B)) = a + b + 2c + d + e + 2f$$

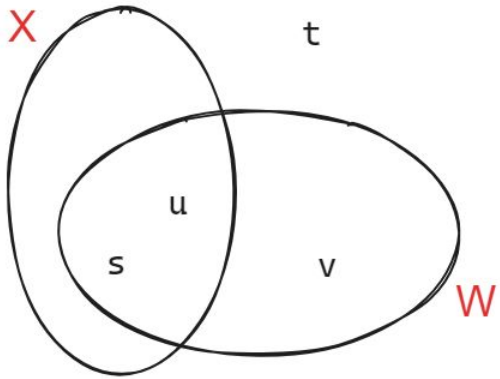
$$= c(\delta(A - B)) + c(\delta(B - A)) + 2c$$

$$\geq c(\delta(A - B)) + c(\delta(B - A))$$



Lemma 3 : Let $\delta(W)$ be an s-t minimum cut in a graph G with respect to a capacity function c. Then for any $u, v \in W$, $u \neq v$, there is a u-v minimum cut $\delta(X)$ where $X \subseteq W$.

Proof :- Let $\delta(X)$ be any u-v minimum cut that crosses W. Suppose without loss of generality that $s \in W$, $s \in X$, and $u \in X$. Then there are two cases to consider:



Case 1 : $t \notin X$, then, since c is submodular,

$\Rightarrow c(\delta(X)) + c(\delta(W)) \geq c(\delta(X \cup W)) + c(\delta(X \cap W))$

Since $X \cup W$ is an s-t cut and $X \cap W$ is an u-v cut,

$\Rightarrow c(\delta(X \cup W)) \geq f(s, t) = c(\delta(W))$ and $c(\delta(X \cap W)) \geq f(u, v) = c(\delta(X))$

$\Rightarrow c(\delta(X \cup W)) = f(s, t) = c(\delta(W))$ and $c(\delta(X \cap W)) = f(u, v) = c(\delta(X))$

$\therefore$ $X \cap W$ is a minimum u-v cut.

Case 2 : $t \in X$. Since c is posi-modular,

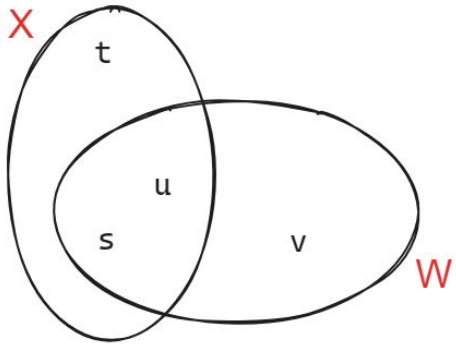
$\Rightarrow c(\delta(X)) + c(\delta(W)) \geq c(\delta(X - W)) + c(\delta(W - X))$

Since X - W is an s-t cut and W - X is an u-v cut,

$\Rightarrow c(\delta(X - W)) \geq f(s, t) = c(\delta(W))$ and $c(\delta(W - X)) \geq f(u, v) = c(\delta(X))$

$\Rightarrow c(\delta(X - W)) = f(s, t) = c(\delta(W))$ and $c(\delta(W - X)) = f(u, v) = c(\delta(X))$

$\therefore$ W - X is a minimum u-v cut.



So, lemma 3 shows that minimum cuts can be uncrossed, a technique that is useful in the construction of the Gomory-Hu Tree.

Theorem: Let T be a Gomory-Hu tree for a graph G = (V, E). Then, for all u, v $\in$ V , let st be the edge on the unique path in T from u to v such that f(s, t) is minimized. Then, f(u, v) = f(s, t) and the cut $\delta(W)$ induced by T − st is a u-v minimum cut in G.

Proof :- Consider the path from u to v in T. We note that if uv = st, then f(u, v) = f(s, t). Otherwise let u, x_1, x_2, ... , x_k, v be the path from u to v in T, then by lemma 2, $f(u, v) \geq \min\{f(u, x_1), f(x_1, x_2), \ldots , f(x_k, v)\} = f(s, t)$. However, we have that $f(u, v) \leq f(s, t)$ since the cut induced by T - st is a valid u-v cut. Thus, we have f(u, v) = f(s, t) and the cut induced by T - st is a minimum u-v cut.

Theorem: The Minimum K-Cut with Gomory-Hu Tree Algorithm has an approximation ratio of 2 − 2/k.

Proof :- Let $E^* \subseteq E$ be an optimal k-cut in G that defines a graph of k connected components, V_1, V_2, ..., V_k. Let E^*_i be the cut that separates the component V_i from the rest of the graph. So, $E^* = \cup E^*_i$.

We can assume without loss of generality that $c(\delta(V_1)) \leq c(\delta(V_2)) \leq \ldots \leq c(\delta(V_k))$. As before, every edge in E^* is incident to two of the connected components. Therefore, each edge will be in exactly two of the cuts $\delta(V_i)$, and therefore, $2c(E^*) = c(\delta(V_1)) + c(\delta(V_2)) + \ldots + c(\delta(V_k))$.

$$c(C) = \sum_{i=1}^{k-1} w(e_i) \leq \sum_{i=1}^{k-1} c(\delta(V_i)) \leq (1 - \tfrac{1}{k}) \sum_{i=1}^{k} c(\delta(V_i)) = 2(1 - \tfrac{1}{k}) c(E^*)$$

where w is the weight function in the Gomory-Hu Tree.

Running Time Analysis

1)**Construction of Gomory-Hu tree**:-

In the process of constructing the Gomory-Hu tree, the algorithm iteratively finds the minimum cut V-1 times, considering each possible pair of vertices. For each iteration, the Ford-Fulkerson algorithm is applied. The time complexity of the Ford-Fulkerson algorithm is O(V^2 * max_flow), where V is the number of vertices.

Therefore, the overall time complexity for constructing the Gomory-Hu tree is given by:

$$T_{\text{Gomory-Hu}} = (V - 1) \cdot O(V^2 \cdot \text{max\_flow})$$

Running Time Analysis

2) Finding k-Min Cut Solution-

The min-K-cut algorithm, following Gomory-Hu tree construction, exhibits a concise time complexity of O(N). **This efficiency arises from the algorithm's focus on finding the path between two vertices in the tree** and determining the minimum weight edges along that path (By Sorting) . Mathematically, the time complexity is succinctly expressed as:-

$$T_{\text{minKcut}} = O(N \log N)$$

N represents the number of vertices considered, aligning directly with the total vertices in the Gomory-Hu tree.

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THANK YOU!