Paper Review: Efficient reallocation under additive and responsive preferences.

Abstract

In most multi-agent systems, reallocating resources among the agents of the system in such a manner that no agent is worse than before reallocation of resources is a huge and important problem. While finding an arbitrary *Pareto Optimal* solution is trivial, determining whether the given allotment is *Pareto Optimal* or not is difficult and sometimes intractable. Before mentioned problem can also be rephrased as finding an allocation such that all agents at least weakly prefer new allotment and at least one agent strongly prefers new allotment. In the paper two types of preference relation are studied: a) additive cardinal utilities over objects and b) ordinal preferences over objects and their additive preference over each object is independent of other objects and for such preference relation, characterization and polynomial-time algorithm for necessary and possible *Pareto Optimality* are given. For additive cardinal utilities, the paper presents computational hardness results and polynomial-time algorithms to test *Pareto Optimality* under different restrictions.

Introduction

Generally, in all multi-agent systems, reallocating resources to achieve better outcomes is always a central concern and a well-known method of doing so is trying to generate a sequence of *Pareto Improvements* to finally converge to a *Pareto Optimal* outcome. These improvements should be such that after reallocation of resources all agents should be as happy as before and at least one agent should be strictly happier than before. Such improvements are sought after for firstly, they uphold the 'individual rationality' property for each agent that is no agent is at loss after the exchange of objects and secondly, it leads to an increase in welfare for almost all social welfare metrics (utilitarian or egalitarian).

Throughout the paper, it is assumed that each agent is initially endowed a set of objects, or an initial assignment of objects is done, and our goal is to determine whether it is *Pareto Optimal* and if not finding a *Pareto Improvement* which satisfies individual rationality leading to a *Pareto Optimal* assignment. It is easy to come up with *Pareto Optimal* assignments from scratch. For scenarios in which additive preference is present, assign each object to any one agent who desires it most and for scenarios in which ordinal preference is followed then we assume an agent prefers a superset of objects always against a subset and thus assigning all the objects to a single agent is also a *Pareto Optimal* assignment.

Finding a *Pareto Optimal* assignment that respects individual rationality in cases where the initial endowment is present is equivalent to testing *Pareto Optimality* of the initial assignment. If testing *Pareto Optimality* is computationally difficult then finding *individually rational* and *Pareto Optimal* assignment is also computationally difficult, so they have concentrated on testing *Pareto Optimality* of an assignment, if they can test it efficiently then algorithms to compute *individually rational* and *Pareto Optimal* assignments are given.

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For cardinal additive preference, it is assumed that each agent has a greater than or equal to 0 valuation for each object, and valuation for a set of objects is the sum of valuation of each object in the set. A weak monotonicity assumption is made by assuming all valuations to be non-negative. For ordinal preference, we assume that valuation is additively separable, and if the agent prefers to exchange object o for o' where o is in the set of objects S, then the agent will prefer to exchange o for o' in any other set of objects containing o too. Due to a possible scenario that an agent is indifferent towards an object, we assume strict monotonicity to test *Pareto Optimality*.

Some Definitions and Notations

1. Let set of agents be denoted by $N = \{1 \dots. n\}$
2. Let set of objects be denoted by $O = \{1 \; \dots. m\}$
3. An assignment $p = \big(p(1) \dots p(n)\big)$ is a partitioning of O into n subsets, where $p(i)$ denotes the set of objects assigned/endowed to agent i.
4. For agents expressing cardinal utility function u_i over O, it is assumed that for each agent its utility for each object is a non-negative rational number. Also, as utility is additive $u_i(O') = \sum_{o \,\in\, O'} u_i(o)$ for each $i \in N$ and $O' \subseteq O$. The vector $u = (u_1, \dots \dots, u_n)$ consisting of the utility function of agents is referred to as *utility function profile.*
5. For agents expressing ordinal preference over objects, it is assumed that each agent i expresses only rational preference relation $\succcurlyeq_i \; over \; O$ i.e., it is complete, reflexive and transitive. $\sim_i$ represents weak preference and $\succ_i$ represents strong preference. We divide O into m_i equivalence classes denoted by $E_i^1 \dots \dots E_i^{m_i}$ such that agent i is indifferent between objects belonging to the same class and strictly prefers objects from the class $E_i^k \; over \; E_i^l$ if $k < l$. The preference relation profile $\succcurlyeq = (\succcurlyeq_i, \dots \dots, \succcurlyeq_n)$ specifies for each agent i its preference relation $\succcurlyeq_i$ over objects. A utility function u_i is said to be consistent with $\succcurlyeq_i$ if $u_i(o) \geq u_i(o') \; iff \; o \succcurlyeq_i \; o'$. Set of all utility functions consistent with $\succcurlyeq_i$ is denoted by $U(\succcurlyeq_i)$. We will denote set of all utility function profiles $u = (u_1 \dots \dots u_n)$ such that $u_i \in U(\succcurlyeq_i)$ for each $i \in N$ by $U(\succcurlyeq)$.
6. χ is used to denote the set of all possible assignments.
7. An assignment $p \in \chi$ is said to be *individually rational* for an initial assignment $e \in \chi$ if $u_i\big(p(i)\big) \geq u_i\big(e(i)\big)$ holds for every agent i.
8. An assignment $p \in \chi$ is said to be *Pareto Dominated* by another $q \in \chi$ if (a) for every agent $i \in N$, $u_i\big(q(i)\big) \geq u_i\big(p(i)\big)$ is true and (b) for at least one agent $i \in N$, $u_i\big(q(i)\big) > u_i\big(p(i)\big)$ is valid.

9. An assignment is *Pareto Optimal* if it is not *Pareto Dominated* by another assignment.

10. For cardinal utilities, the utilitarian social welfare metric of an assignment p is defined as $SW(p) = \sum_{i \in N} u_i\big(p(i)\big)$

Additive Utilities

Under this heading, we assume that each agent expresses a cardinal utility function $u_i\ over\ O$, where $u_i(o) \geq 0$ for all $i \in N\ and\ o \in O$

A. **Testing *Pareto Optimality for hard cases***

Lemma 1: If there exists a polynomial-time algorithm to compute a *Pareto Optimal and individually rational* assignment, then there exists a polynomial-time algorithm to test *Pareto Optimality.*

Proof: Assuming there is a polynomial-time algorithm A that can compute *individually rational and Pareto Optimal* assignment and an assignment p for which *Pareto Optimality* needs to be tested. Then to do so we can use algorithm A to generate a new assignment q such that it is individually rational over p and *Pareto Optimal.* To satisfy individual rationality, $u_i\big(q(i)\big) \geq u_i\big(p(i)\big)$ should be valid for all $i \in N$. If p is *Pareto Optimal* then, $u_i\big(q(i)\big) = u_i\big(p(i)\big)$ for all $i \in N$ will be valid. However, if there exists $i \in N$ such that $u_i\big(q(i)\big) > u_i\big(p(i)\big)$, it signifies that p is not *Pareto Optimal.*

A Decision Problem *P* is said to belong to *coNP* complexity class if and only if its complement ($\overline{\mathrm{P}}$) is in the *NP* complexity class. It is also defined as a complexity class of problems in which for an instance to be no-instance it should get verified in polynomial time by a polynomial-time algorithm.

***Theorem 1*:** Under additive preferences, testing *Pareto Optimality* of a given assignment is *weakly coNP-complete*, even for n =2 even if the induced ordinal preferences over individual objects are the same.

Proof: Testing *Pareto Optimality* is in *coNP* as it is possible to test whether an assignment is *Pareto Dominated* in polynomial time. To prove completeness, we will reduce *PARTITION* problem to *Testing Pareto Optimality* problem.

An instance of *PARTITION* problem is described as a set of t elements $E = \{\, e_1 \dots \dots e_t\}$ and integer weights w(e_j) for each element in E such that $\sum_{e_i \in E} w(e_i) = 2M.$ The problem is to decide whether a balanced partition S exists for E such that $\sum_{e_i \in S} w(e_i) = \sum_{e_i \in E \setminus S} w(e_i) = M$.

For reduction, we assume a set of $t+1$ objects $\{g^+, g_1, \dots \dots, g_t\}$ and two agents $\{1,2\}$. Utility function for agent 1 is defined as $u_1(g^+) = M$ and $u_1(g_i) = w(e_i)$ for all $i \in \{1 \dots \dots t\}$. Utility function for agent 2 is defined as $u_2(g^+) = M + \varepsilon$, with $0 < \varepsilon < 1$ and $u_2(g_i) = w(e_i)$ for all $i \in \{1 \dots \dots t\}$. Then it can be easily checked that in the assignment in which agent 1 gets g^+ and agent 2 gets all g_j objects is *Pareto Optimal* if and only if there is no balanced partition of E.

Example:

Suppose $E = \{e_1, e_2, e_3\}$ and $M = 3$.

(i) Let weights be $\{1,2,3\}$ then as can be seen there exists a balanced partition $\{e_1, e_2\}$ *and* $\{e_3\}$ such that the sum of their weights is 3. So, if we consider above mentioned initial assignment that agent 1 gets g^+ and its utility is 3 and agent 2 gets $\{g_1, g_2, g_3\}$ and its utility is 6. We can see a *Pareto Improvement* which is also *individually rational* such that agent 1 gives g^+ in exchange for $\{g_3\}$ then utility of agent 1 remains the same but utility of agent 2 becomes $1 + 2 + 3 + \varepsilon = 6 + \varepsilon$, thus increasing and therefore above-mentioned assignment is not *Pareto Optimal*.

(ii) Let weights be $\{2,2,2\}$ then as can be seen there does not exist a balanced partition. So, if we consider above mentioned initial assignment that agent 1 gets g^+ and its utility is 3 and agent 2 gets $\{g_1, g_2, g_3\}$ and its utility is 6. We can see that for agent 1 to be *individually rational*, it needs to take 2 objects from agent 2 in exchange for g^+, which decreases the utility of agent 2 and is therefore not *individually rational*, and therefore above-mentioned assignment is *Pareto Optimal* assignment.

Corollary 1: Computing *individually rational* and *Pareto Optimal* assignment is *weakly NP-hard* for n = 2.

B. Testing *Pareto Optimality* for tractable cases

Here paper describes conditions under which the problem of computing *individually rational* and *Pareto Optimal* assignments can be solved in polynomial-time.

1. Constant number of agents and small utilities.

Lemma 2: If there is a constant number of agents and the utilities are all integers, then the set of all vectors of utilities that correspond to an assignment can be computed in pseudo-polynomial-time.

Proof:

Consider the below algorithm where 0^k $denotes$ $0, \dots\dots, 0$ k $times$.

1. L = $\{(0^n)\}$
2. for k = 1 to m do
   a. $L' = \{l + (0^{i-1}, u_i(o_j), 0^{n-i}) | i \in N ; l \in L\}$
   b. L = L'
3. End for
4. return L

If W is the maximal social welfare that is possible, then, at any iteration of the algorithm, the no of vectors in L can never exceed $(W+1)^2$. Hence time complexity is $O(W^2 . n . m)$. Also $W \leq \sum_{i,j} u_i(o_j)$ and since n is constant, the algo runs in pseudo-polynomial-time.

Using the Mathematical Induction method, it can be proved on for k, a vector of utilities $l = (v_1, \dots\dots, v_n)$ can be achieved by assigning objects $o_1, \dots, o_k$ to the agents if and only if $l \in L$ after $o_1, \dots, o_k$ objects have been considered. It is trivial that the above statement is true at the start of the algorithm when no objects have been considered. Now assuming the above to be true for kth iteration, then $l' \in L$ after k+1th iteration if l' is obtained from l by adding $u_i(o_k)$ to the utility of some agent i, i.e., if $l = (v_1, \dots\dots, v_n)$ can be achieved by assigning $o_1 \dots\dots o_{k+1}$ objects.

Theorem 2: If there is a constant number of agents and the utilities are all integers, then there exists a pseudo-polynomial-time algorithm to compute a *Pareto Optimal* and *individually rational* assignment.

Proof: We apply the algorithm discussed in **Lemma 2,** but we also keep track of partial assignment that supports each $l \in L$, so every time we append $l + (0^{i-1}, u_i(o_j), 0^{n-i})$ to L', we store the partial assignment for l and then adding o_j as an object of agent i. If several partial assignments correspond to the same utility vector, then we randomly choose one. At the end of the algorithm, we receive a list of utility vectors and corresponding object assignments. For each utility vector $l \in L$, we check whether a l' exists such that *Pareto Dominates* l, if exists then we remove l. This takes at most $O(|L|^2)$ time and remaining vectors in L are *Pareto Optimal.*

2. Lexicographic utilities

A utility function is Lexicographic if for each agent $i \in N$ and each object $o \in O$, $u_i(o) > \sum_{o \succ_i o'} u_i(o')$ with the condition that $\sum_{o \in \emptyset} u_i(o) = 0$, which implies that $u_i(o) > 0$ for each o. Example an agent with utilities (11,6,3,1).

For *Lexicographic* utilities, to test *Pareto Optimality* of an assignment p, we construct a graph called envy graph of p. The vertices of this graph are one vertex for each object $o \in O$. For each vertex associated with object o, set of edges are (o, o') for any object $o' \in O - \{o\}$ such that $o' \succcurlyeq_i o$, where i is the agent to whom object o is allotted in assignment p.

It was inferred during the literature survey that *Pareto Optimality* of an assignment for *Lexicographic* utilities can be tested in polynomial time. So, a simple characterization of a *Pareto Optimal* assignment for *Lexicographic* utilities is presented.

Theorem 3: An assignment p is not *Pareto Optimal* with respect to *Lexicographic* utilities if and only if there exists a cycle in $G(p)$ (envy graph of p) which contains at least one edge corresponding to a strict preference.

Proof: For right-to-left direction, let's assume there exists a cycle C that contains at least one edge corresponding to a strict preference. Then, the exchange of objects along the cycle by agents owning these objects corresponds to a *Pareto Improvement* and thus p is not *Pareto Optimal.*

For left-to-right direction, assume that p is not *Pareto Optimal* and let q_1 be an assignment that *Pareto Dominates* p. For at least one agent i, $q_1(i) \succ_i p(i)$. Therefore, there exists at least one object o_1 in $q_1\,(i) \backslash p(i)$. Let i_1 be the owner of o_1 in p. Since preferences are *Lexicographic,* i_1 must receive an object o_2 in q_1 which is at least as good as o_1 according to its preferences. This will lead to forming of a sequence of agents $i_1, i_2, \dots, i_k$ and a sequence of objects $o_1, o_2, \dots, o_{k+1}$ such that each agent i_j gets object o_{j+1} in q_1 in exchange for the loss of o_j. Since we have a finite set of objects, there must exist $k \; and \; k'$ such that the sequence $o_k \rightarrow o_{k+1} \rightarrow \cdots \rightarrow o_{k'}$ forms a cycle. If there does not exist $l \in [k, k'-1]$ such that $o_{l+1} \succ_{i_l} o_l$ then we consider assignment q_2 derived from q_1 by reassigning every object o_{l+1} to agent i_l for $l \in [k, k'-1]$. It is implied that q_2 is at least as good as q_1 for all agents. Hence q_2 *Pareto Dominates* p. Using the same reasoning we can keep on deriving *Pareto Dominant* assignments but there must exist some finite value t, for which there exists a $l \in [k, k'-1]$ such that $o_{l+1} \succ_i o_l$ for the cycle $o_k \rightarrow o_{k+1} \rightarrow \cdots \rightarrow o_{k'}$ founded in q_t. Otherwise, after a finite number of steps, we would have $q_t(i) = p(i)$ for all $i \in N$, leading to a contradiction with our assumption that q_t *Pareto Dominates* p. Therefore, there must exist a cycle with at least one edge corresponding to a strict preference in graph $G(p)$.

It is evident that envy graph can be constructed in linear time for any assignment p and search for a cycle containing at least one strict preference edge in $G(p)$ can be found out in linear time by applying a graph traversal algorithm for each strict preference edge in $G(p)$. Thus, the complexity of testing *Pareto Optimality* of an assignment p is in linear time for *Lexicographic* utilities.

3. Two Utility Values

Under this condition, each agent uses only two utility values to show their preference over objects. A utility function profile u is bivalued if there exist only two values $\alpha > \beta \geq 0$ such that for every agent i and every object o, $u_i(o) \in \{\alpha, \beta\}$. This signifies that for each agent, the set of objects O is divided into two subsets $E_i^1 = \{o \in O, u_i(o) = \alpha\}$ and $E_i^2 = \{o \in O, u_i(o) = \beta\}$. So for an assignment q, $q^+(i) = q(i) \cap E_i^1$ and $q^-(i) = q(i) \cap E_i^2$

Lemma 3: If an assignment p is *Pareto Dominated* by an assignment q then $|\bigcup_{i\in N} q^+(i)| > |\bigcup_{i\in N} p^+(i)|$

Proof: Done using contradiction, so assume $|\bigcup_{i\in N} q^+(i)| \leq |\bigcup_{i\in N} p^+(i)|$ holds if assignment p is *Pareto Dominated* by an assignment q , so social welfare for assignment q is $SW(q) = |\bigcup_{i\in N} q^+(i)|\alpha + |\bigcup_{i\in N} q^-(i)|\beta = |\bigcup_{i\in N} q^+(i)|(\alpha - \beta) + |O|\beta$. Similarly, $SW(p) = |\bigcup_{i\in N} p^+(i)|(\alpha - \beta) + |O|\beta$. This implies that $SW(p) \geq SW(q)$, but this contradicts the assumption that q *Pareto Dominates* p.

C. *Conservative Pareto Optimality*

An assignment p is *Conservatively Pareto Optimal* if there does not exist another assignment q that *Pareto Dominates* p and $|q(i)| = |p(i)|$ for all $i \in N$. It is applicable in many scenarios where the number of objects initially endowed to each agent needs to be conserved.

Lemma 4: There exists a polynomial-time algorithm to test *Pareto Optimality if and only if there exists a polynomial-time algorithm to test Conservative Pareto Optimality.*

Proof: For left-to-right direction, for each agent and object, we update the utility function from $u_i(o)$ to $u_i(o) + C$, where $C = m * \max_{i\in N, o\in O} u_i(o)$. With the modified utility function, each agent is no more concerned with the objects allotted to it but focuses on the number of objects allotted. Therefore, in any *Pareto Improvement,* no agent will get less number of objects than it received in the original endowment, and hence each agent will get the same number of objects after each reallocation. Hence, it can be inferred that q *Pareto Dominates* p with respect to modified utilities if and only if q Conservatively *Pareto Dominates* p with respect to original utilities. Therefore, by simply modifying the utility function as mentioned above, a polynomial-time algorithm to test *Pareto Optimality* can be used to test *Conservative Pareto Optimality.*

For the right to left direction, let's say there are n agents, m objects, u utility matrix, and an assignment p. Define $(n-1)m$ dummy objects that each agent values at 0. A new assignment q is derived from p by giving $m - |p(i)|$ objects to agent i. Then q is *Conservative Pareto Optimal* for the modified instance if p is *Pareto Optimal* for the original instance.

Ordinal Preference

Under this heading, we consider agents have additive cardinal utilities but only their ordinal preferences over objects are known by the central authority. This may be due to various reasons such as not being known precisely, or the central authority didn't ask for this information. It is assumed that $u_i(o) > 0$ for $i \in N$ and $o \in O$. It is still possible to determine whether a given assignment is *Pareto Optimal* with respect to some or all cardinal utility functions consistent with the ordinal preferences.

An assignment p is *Possibly Pareto Optimal* with respect to preference profile $\succcurlyeq$ if there exists $u \in U(\succcurlyeq)$ such that p is *Pareto Optimal* for u. An assignment p is *Necessarily Pareto Optimal* with respect to preference profile $\succcurlyeq$ if for all $u \in U(\succcurlyeq)$ p is *Pareto Optimal* for u.

Necessary Pareto Optimality implies *Possible Pareto Optimality.* Also, at least one *Necessarily Pareto Optimal* assignment exists in which all objects are given to one agent. Computing *Possibly* or *Necessarily Pareto Optimal* assignment is polynomial-time solvable so let's focus on problems of testing *Possible* and *Necessary Pareto Optimality.*

Theorem 4: A assignment is (1) *Possibly Pareto Optimal* if and only if (2) there exists no cycle in $G(p)$ which contains at least one edge corresponding to a strict preference if and only if (3) it is *Pareto Optimal* under *Lexicographic* utilities.

Proof: We have already proved (2) ⟺ (3) in ***Theorem 3***. We can also infer (3) ⇒ (1) using the definition of *Possibly Pareto Optimal* that if an assignment is *Pareto Optimal* with respect to *Lexicographic* utilities, it is *Possibly Pareto Optimal.*

To show (1) ⇒ (2), Suppose p is not *Pareto Optimal* with respect to *Lexicographic* utilities, then by ***Theorem 3***, $G(p)$ contains a cycle which contains at least one edge corresponding to a strict preference. Now let's consider assignment q which is obtained by exchanging objects along the cycle. If o points to o' in the cycle, then the agent getting o in p now gets o' instead of o. Since the cycle contains at least one edge corresponding to a strict preference, assignment q is a *Pareto Improvement* over p with respect to all utilities consistent with the ordinal preferences. So p is not *Possibly Pareto Optimal.*

Since characterization in ***Theorem 3*** also applies to *Possible Pareto Optimality*, hence *Possible Pareto Optimality* can be tested in linear time.

Conclusion

From a computational point of view, *Pareto Optimality* in resource reallocation under additive utilities and ordinal preferences was studied and the paper came up with various characterization theorems and polynomial-time algorithms to solve problems of testing *Pareto Optimality* under various conditions.

References

Haris Aziz, Péter Biró, Jérôme Lang, Julien Lesca, Jérôme Monnot, Efficient reallocation under additive and responsive preferences.